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DOCTORAL THESIS

Supersymmetric Field Theories

and

Isomonodromic Deformations

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Declaration of Authorship

I, Fabrizio DEL MONTE, declare that this thesis titled, “Supersymmetric Field Theories and Isomonodromic Deformations” and the work presented in it are my own. Where I have consulted the published work of others, this is always clearly attributed. The thesis is based on the three research papers

- G. Bonelli, F. Del Monte, P. Gavrylenko and A. Tanzini, "$\mathcal{N} = 2^*$ gauge theory, free fermions on the torus and Painlevé VI", Commun. Math. Phys. **377** (2020) no.2, 1381-1419, https://doi.org/10.1007/s00220-020-03743-y;
- G. Bonelli, F. Del Monte, P. Gavrylenko and A. Tanzini, "Circular quiver gauge theories, isomonodromic deformations and W_N fermions on the torus", https://arxiv.org/abs/1909.07990;
- G. Bonelli, F. Del Monte and A. Tanzini, "BPS quivers of five-dimensional SCFTs, Topological Strings and q-Painlevé equations", https://arxiv.org/abs/2007.11596

as well as the still unpublished work

- G. Bonelli, F. Del Monte and A. Tanzini, "Higher genus class S theories and isomonodromy", work in progress

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Abstract

Doctor of Philosophy

Supersymmetric Field Theories and Isomonodromic Deformations

by Fabrizio DEL MONTE

The topic of this thesis is the recently discovered correspondence between supersymmetric gauge theories, two-dimensional conformal field theories and isomonodromic deformation problems. Its original results are organized in two parts: the first one, based on the papers [1], [2], as well as on some further unpublished results, provides the extension of the correspondence between four-dimensional class S theories and isomonodromic deformation problems to Riemann Surfaces of genus greater than zero. The second part, based on the results of [3], is instead devoted to the study of five-dimensional superconformal field theories, and their relation with q-deformed isomonodromic problems.

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Contents

Introduction and Summary

Quantum Field Theory is the framework allowing to describe physical phenomena at all scales, ranging from the astronomical scales of General Relativity, to the everyday physics of conducting metals, all the way to the fundamental physics at the microscopic level of fundamental particles. In all these applications, however, we can see a common thread: while we can perform approximate computations that are often in spectacular agreement with experimental results, we still have a very poor grasp of the nonperturbative and strongly-coupled aspects of Quantum Field Theory. We can predict to incredible accuracy the anomalous magnetic moment of the electron, but we are still unable to give any quantitative argument for confinement in a realistic case.

A lot of progress, however, has been achieved across the last two decades in the understanding of *unrealistic* cases, namely cases with enough symmetries to constrain the problem and have a sounder control of its properties. The most usual example of such system are those in which we have some amount of *unbroken* supersymmetry, and the larger our supersymmetry algebra is, the more detailed informations we are able to ascertain about the system. The underlying philosophy is somewhat similar to that of freshmen learning Newtonian gravity for the first time: they could try to solve Newton's equations for the Solar System, but they would be facing a hopeless task. What one does instead is to consider the two-body problem, which has much more symmetry than the actual physical setting. The difference between the two examples is that, while the two body problem has obviously fewer degrees of freedom than the whole Solar System, supersymmetric QFTs, have instead *more* degrees of freedom than non-supersymmetric ones.

In this thesis, we will focus on such unrealistic cases, in which our supersymmetry algebra has 8 generators, twice the minimal number in four dimensions, which is denoted by $\mathcal{N} = 2$ in four dimensions and by $\mathcal{N} = 1$ in five. More precisely, we will focus on two classes of such theories: four-dimensional "Class S" theories [4], which arise in M-theory from M5-branes wrapping a Riemann Surface with marked points, and five-dimensional superconformal field theories [5] arising from compactification of M-theory on certain noncompact Calabi-Yau manifolds [6]. We will see that the analogy with Newtonian mechanics is actually more accurate than it might seem: the two-body problem is an example of *integrable system* (in fact, it is superintegrable, but let us not dwell on this distinction), and it turns out that the exact solution, first formulated by Seiberg and Witten [7, 8], for the low-energy physics of $\mathcal{N} = 2$ theories, is given by many body classical integrable systems [9, 10]: we will review in the first chapter how the low-energy physics of pure $SU(2)$ $\mathcal{N} = 2$ Super Yang-Mills (SYM) is described by the simple pendulum.

However, such a description in terms of a classical integrable system holds only for the QFT in the deep infrared, and it determines, through the exact low-energy effective action, the space of the supersymmetric vacua of the theory and of the massless excitations around them. A more complete description of the underlying physics should involve following the RG flow leading to such infrared physics, and a natural question is whether one can achieve such an exact description for intermediate energy scales as well: we want to understand what happens to the integrable system under RG flow. It turns out that integrable structures survive to some extent, but the integrable system undergoes deformations parametrized by the dynamically generated energy scale, leading to the so-called Whitham modulations: following this perspective, it was possible already in the '90s to upgrade the Seiberg-Witten low-energy description to include the dependence on the dynamically generated energy scale Λ_{QCD} [11].

While this description has many appealing features, it should be evident that it is not the most physically natural: starting from a UV description of the theory (either purely field-theoretical or coming from String Theory), it provides the exact low-energy effective action through indirect methods, and takes that as a starting point to trace the renormalization group backwards (i.e. from the IR to the UV): everything is parametrized by vevs at zero energy, which are coordinates of the Coulomb branch, and the RG dependence is reconstructed order by order. What would be more natural is instead to start from the UV description of the theory, and to obtain an infrared description by triggering an RG flow in the "correct" direction UV $\rightarrow$ IR, recovering the low-energy description in terms of an integrable system in the infrared. More generally, it is possible to envision, together with relevant deformations triggering RG flows, also exactly marginal ones, that trace out what are called conformal manifold deformations of the theory. In the case of four-dimensional $\mathcal{N} = 2$ "class S" theories that we mentioned above, supersymmetric marginal and relevant deformations can be described geometrically in terms of the Riemann Surface defining the field theory in the M-theoretic compactification. This makes it possible to study this problem in great detail thanks to techniques coming from the world of integrable systems and two-dimensional CFT: the RG/exactly marginal flows lead to integrable equations for the nonperturbative partition function of such theories, that in the prototypical cases are given by famous ODEs from mathematical physics, the Painlevé equations [12, 13], which are the most general nonlinear integrable ODEs of second order. These equations are extremely rich and transcendental, and they are integrable in a highly nontrivial way: for example, their Hamiltonians are time-dependent versions of the original ones, and as such they are not conserved.

More precisely, the nonperturbative partition function of the gauge theory – that requires for its very definition to deform the theory by the so-called Ω-background [14]– turns out to be closely related to a central object in the theory of isomonodromic deformations, called the isomonodromic tau function [15, 16]. We provide a rough picture of this correspondence, that provides important new results both on the physical and mathematical side, in Figure 1.

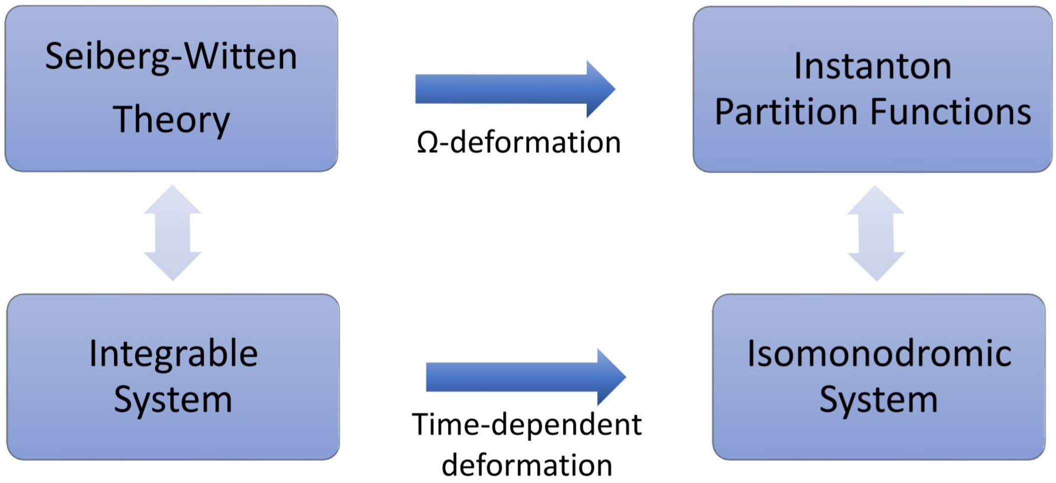


FIGURE 1: The correspondence between 4d $\mathcal{N} = 2$ theories and integrable systems

First and foremost, the differential equation allows to completely determine the full nonperturbative behavior of the partition function once the asymptotic behavior – typically the classical action – is given: knowing one differential equation and the classical limit is enough to obtain the full nonperturbative description of the partition function. Further, the characterization of the partition function as a solution of a differential equation, rather than as arising from a path-integral computation, allows one to study the theory beyond the weakly coupled regime, since the distinction between strong and weak coupling from this point of view is simply that between different asymptotic regimes of the equation's solutions. By using these methods it has also been possible to study some nonlagrangian cases, like the superconformal fixed points called Argyres-Douglas theories [17, 18]. From the mathematical point of view, on the other hand, this relation provides explicit expressions from the gauge theory localization computations [14, 19] of the partition functions of such tau functions, that are notoriously hard functions to study in detail. For example, a long standing problem about the relation between different asymptotic regimes of these tau functions, called the connection problem, was solved in several cases thanks to this correspondence [20, 21]. The physical significance of the connection problem in the gauge theory is that it amounts to study the difference between the low-energy effective action at strong and weak coupling.

Different considerations apply instead for the five-dimensional SCFTs that we mentioned in the beginning. These are isolated superconformal fixed points that behave as supersymmetric gauge theories in the infrared. In five dimensions the gauge coupling is an irrelevant deformation, so that there are no continuous deformations to be studied. However, the string theory realization of these theories, that we recall in section 2.5, provides a rich web of dualities satisfied by such theories: their meaning is that different phases of the same five-dimensional theory will give different gauge theories in the infrared. A special role among these is played *self-dualities*, i.e. sequences of dualities that bring the theory to itself. A more careful study of these self-dualities shows that, while the sequence of dualities does indeed give back the same gauge theory, some parameter (typically the gauge coupling) acquires a discrete shift.

In a very similar fashion to what happened in four dimensions, these shifts can be regarded as – now discrete – time flows for a corresponding discrete integrable system, and again the partition function of such theories on a circle, computable either by localization techniques [22, 23] or by the topological vertex [24] in the context of topological strings, are tau functions for a discretized version of the isomonodromic systems we described above [25, 26], and in the basic cases are solutions of q-difference equations deforming Painlevé equations [27]. Because of this, an analogous diagram to that depicted in Figure 1 holds for five-dimensional theories and discrete integrable systems.

The aim of the thesis is twofold. In the context of class S theories, the correspondence between four-dimensional field theories and isomonodromic deformations had been previously studied only for theories arising from M5-branes wrapping Riemann spheres with marked points, giving linear quiver gauge theories [28, 29]. We will show how to extend this correspondence to the case of a Riemann Surface of genus g, that in general will be a non-lagrangian theory. To do this, we will use techniques from two-dimensional CFT, and the correspondence between four-dimensional supersymmetric QFTs and two-dimensional CFTs known as AGT correspondence.

In the context of five-dimensional SCFTs, we propose a connection between two previously unrelated topics: that of BPS quivers [30, 31], generating the spectrum of BPS states of the theory, and that of q-Painlevé equations, or more generally q-difference integrable equations arising from the same quivers. By carefully studying the example of the $SU(2)$ gauge theory with two flavors, we argue that the topological string partition function for these cases is not just a function, but should be a vector on a lattice defined by the symmetry of BPS states.

Here is an outline of the contents of the thesis, which is subdivided in three parts.

Part I consists of Chapters from 1 to 3, and introduces all the important physical background for our discussion.

In Chapter 1 we recall the Seiberg-Witten solution for the low-energy physics of four-dimensional $\mathcal{N} = 2$ theories, while also introducing some basic notions about integrable systems. Here we illustrate, by considering in detail the example of pure $SU(2)$ gauge theory, how every object in Seiberg-Witten theory can be described in terms of a corresponding quantity in an appropriate integrable system, and provide the dictionary between the two descriptions.

In Chapter 2 we illustrate the different string theory embeddings of the theories of interest: we start by reviewing Witten's brane construction of linear quiver gauge theories, and discuss how this can be generalized in the context of class S theories. We also have a second small integrability detour, where we introduce the notion of Hitchin system, and discuss how this provides a general geometric framework that explains and extends the appearance of integrable systems in Seiberg-Witten theory. Finally, we review the geometric engineering construction of five-dimensional SCFTs on a circle by using M-theory on a local CY_3, and its relation to (p,q) brane webs in type IIB.

In Chapter 3 we get more quantitiative, by introducing the concept of instanton partition function of four- and five-dimensional gauge theories, as well as partition functions in the presence of line and surface defects. We also introduce the AGT correspondence, that identifies the four-dimensional partition functions with conformal blocks of 2d CFTs, and the five-dimensional partition functions with q-deformed conformal blocks. Finally, we introduce the concept of pants decomposition, explaining how this leads to quiver gauge theories in the class S construction.

Part II consists of Chapters from 4 to 7, and contains the original results of this thesis regarding class S theories, isomonodromic deformations and 2d CFT.

In Chapter 4 we introduce Painlevé equations and the concept of isomonodromic deformations. We then show how one can study this class of problems by using braiding of degenerate fields or free fermions in 2d CFT, which correspond to studying surface defects in the four-dimensional QFT. The main statement is that an important object in the theory of isomonodromic deformations, the isomonodromic tau function, coincides (up to some explicit proportionality factors) with a Fourier series constructed from the gauge theory partition function, called the dual partition function. We finally recall the Painlevé-gauge theory correspondence, that studies the identification of isomonodromic tau functions and dual gauge theory partition functions for asymptotically free and nonlagrangian cases.

With Chapter 5, based on [1], begins the exposition of the original results of this thesis. Here we extend the relation between isomonodromic deformations and class S theories to the case of Riemann Surfaces of genus one, by studying in detail the case of $SU(2)$ super Yang-Mills with one adjoint hypermultiplet, the $\mathcal{N} = 2^*$ theory. We show how one has to modify the relation between gauge theory partition functions and isomonodromic tau functions in a highly nontrivial way for it to hold for Riemann Surfaces of $g > 0$, and show how this modification allows to derive two different relations between dual partition functions and tau functions, depending on how we define the Fourier series. We then check in detail that the Painlevé equation relevant to this case reproduces the formulas from the localization computation, and by taking an appropriate limit on this equation, we are also able to provide an exact relation between the UV and IR coupling of the gauge theory. Finally, we argue that the partition function can be expressed as Fredholm determinant of a Cauchy-like operator arising from the pants decomposition of the one-punctured torus, which goes beyond the definition of the partition function as a power series.

In Chapter 6, based on [2], we extend these results to the case of circular quiver gauge theories with $SU(N)$ gauge groups, showing also how there are now N different dual partition functions, related to the tau function in different ways. We also show that the zeros of the partition functions in appropriate fugacities are the appropriate deformation of some old results in integrable systems literature, describing the solution of the dynamical system as zeros of theta functions.

In Chapter 7 we report on still unpublished results, that show the extension of this correspondence to the very general case of Riemann Surfaces with arbitrary genus and number of marked points. To do this, we recall the Hamiltonian construction for isomonodromic deformations in higher genus, and show how these are given by the flatness of a connection on the Teichmüller space of the surface.

Finally, Part III consists only of Chapter 8, based on [3]. In this Chapter we switch from the study of Class S theories to that of five-dimensional SCFTs on a circle and Topological Strings. The constant, however, is the relation to Painlevé equations and isomonodromic deformations, here in their q-deformed incarnation, given by discrete flows defined by the Calabi-Yau manifold engineering the field theory. First we show how these flows naturally reproduce the BPS spectrum of such theories in the known cases, and so provide a general recipe for the study of the BPS spectrum in an appropriate chamber of moduli space. We then argue that the Topological String partition functions on these Calabi-Yau, conjecturally equal to the SUSY index of these theories, are actually vectors on the symmetry lattice of the corresponding discrete system, deriving new equations for such partitions functions that relate partition functions of gauge theory phases in different regions of moduli space. Finally, we show how an appropriate four-dimensional limit allows to recover the structure of the BPS states for the 4d theory obtained by sending the radius of the 5d circle to zero.

Part I

Coulomb Branches and Integrable Systems

Chapter 1

Seiberg-Witten Theory and Integrable Systems

This chapter introduces important notations and concepts to be used throughout the thesis, reviewing the Seiberg-Witten construction and its relation to classical integrable systems.

1.1 (Extended) Supersymmetry Algebra in four dimensions

Since we will be considering theories describing fundamental interactions, we will always deal with relativistic QFTs. This means that the symmetry algebra is given by the Poincaré algebra, with generators $P_\mu, M_{\rho\sigma}$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\sigma},$$

$$[P_\mu, P_\nu] = 0, \qquad [M_{\mu\nu}, P_\rho] = -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu, \tag{1.1}$$

$$[B_a, B_b] = if_{ab}{}^c t_c, \qquad [P_\mu, B_a] = [M_{\mu\nu}, B_a] = 0,$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric, plus eventually discrete symmetries, like for example charge, parity and time reversal C, P, T. It is a classic result by Coleman and Mandula [32, 33] that the only possible Lie algebra extending 1.1 and consistent with certain physical requirements[1] is the direct sum of the above with some internal symmetry algebra with generators t_a and structure constants f_{abc}:

$$[t_a, t_b] = f_{ab}{}^c t_c, \qquad [P_\mu, t_a] = [M_{\mu\nu}, t_a] = 0. \tag{1.2}$$

Of course, by relaxing the assumptions of this theorem we can enlarge the type of symmetries allowed. While some requirements are indeed physically important, some of them can be dropped without abandoning basic physical requirements: Haag, Lopuszanski and Sonius noted that it is possible to

[1]More precisely, the assumptions in the theorem, besides the usual Wightman axioms of QFT, are: finite types of particles at any given energy scale, that all two-particle states undergo some process at almost all energies, and analyticity of scattering amplitudes.

nontrivially extend the Poincaré algebra by considering instead a *graded* Lie algebra. The graded Lie algebra that extends the Poincaré algebra with anticommuting generators is called Super-Poincaré algebra, and the additional (anti-)commutation relations are given by

$$[P_\mu, Q^I_\alpha] = [P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0, \qquad \left\{Q^I_\alpha, Q^J_\beta\right\} = \epsilon_{\alpha\beta} Z^{IJ}, \tag{1.3}$$

$$[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta, \qquad [M_{\mu\nu}, \bar{Q}^I_{\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}}, \tag{1.4}$$

$$\left\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\right\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ}, \qquad \left\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\right\} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}^{IJ}. \tag{1.5}$$

The minimal case, denoted by $\mathcal{N} = 1$, is such that the indices I, J attain only one value, so that in four dimensions there are four supercharges. Since Z^{IJ} is antisymmetric, in this case $Z^{IJ} = 0$ and there is no central charge. When $I, J = 1, 2$, there are eight supercharges, the algebra is denoted by $\mathcal{N} = 2$, and there is one central charge, since $Z^{IJ} = Z\epsilon^{IJ}$. This is the case we will be mostly concerned with. Finally the maximal supersymmetry algebra in four dimensions, $\mathcal{N} = 4$, has sixteen supercharges, and is the largest algebra that does not involve a graviton: any QFT with more than sixteen supercharges in four-dimensions would have to deal with quantum gravity.

In usual, local relativistic QFT, particles are defined as irreducible representations of the Poincaré group. However, as is clear from the Supersymmetry Algebra (1.3), (1.4), (1.5), an irreducible representation of the Poincaré group will not be also a representation of the super Poincaré group. This is because the supercharges and the Lorentz generators do not commute, so that a Supersymmetry transformation, applied to a particle of spin s, will yield particles of spin $s \pm 1/2$. What happens is then that an irreducible representation of the Super-Poincaré group will be a direct sum of irreducible representations of the Poincaré group: in physical terms, it means that we have to work with multiplets, i.e. with collections of particles of different spins. The relevant field content for the $\mathcal{N} = 2$ theories that will be considered in this thesis is:

- **Vector Multiplet:** This (massless) multiplet contains a gauge field, two Weyl fermions and a complex scalar. The fermions and scalar transform under the adjoint representation of the gauge group, while the vector is the gauge connection, as usual;

- **Hypermultiplet:** This multiplet constitutes the matter content of the theory, containing two Weyl fermions and two complex scalars (massless hypermultiplets), or one Dirac fermion and two complex scalars (massive hypermultiplet in a short representation).

1.2 Low-Energy Effective Actions and Prepotentials

Our current goal is to study the Low-Energy Effective Field Theory (LEEFT) of $\mathcal{N} = 2$ theories. A general feature of such theories is that they admit a moduli space of vacua. This is different from what happens without supersymmetry: usually the classical potential can have a manifold of vacua, which are given by its minima, but such manifold is lifted by quantum correction to at most a finite set of points. Because of this, in a generic QFT the scalars in the LEEFT turn out to be massive, the only exception being when we have spontaneous breaking of some symmetry group, and the flat directions of the potential correspond to the presence of massless Goldstone bosons in the spectrum. We will see this is not true in the case of theories with $\mathcal{N} = 2$ supersymmetry.

We will further restrict to the study of such theories in their Coulomb phase, i.e. when the LEEFT is an abelian gauge theory. This is referred to as the "Coulomb branch" of the moduli space, which is the region where the gauge group is broken to its Cartan subgroup:

$$G \to U(1)^r, \qquad\qquad r = rk(G), \tag{1.6}$$

so that the low-energy theory is that of r photons plus neutral matter described by a nonlinear sigma model (NLSM). The matter must be neutral under the low-energy $U(1)$ gauge groups, because if this was not the case any vev acquired by the scalars would Higgs at least some of the vectors, which would not be appearing in the low-energy theory: generically, the low-energy physics is then described by r copies of $\mathcal{N} = 2$ super QED. As we will see in the next section, the highly nontrivial statement of Seiberg-Witten theory is that there are special points of the Coulomb branch in which additional BPS hypermultiplets become massless and enter the description. These typically have nonzero magnetic charge, so that they cannot be captured by a weakly-coupled Lagrangian description.

The reason why we focus on the Coulomb branch is that on the so-called "Higgs branch", where the gauge group is completely broken, the low-energy theory is just a NLSM, whose metric is Hyperkähler because of $\mathcal{N} = 2$ supersymmetry and is classically exact.

Consider an $\mathcal{N} = 2$ nonabelian gauge theory. The Coulomb branch in this case is the space of vacua parametrized by vevs of the vector mulitplet scalar ϕ along the Cartan direction of the gauge group. This is because its potential has the form

$$V(\phi) \propto [\phi, \bar{\phi}]^2, \tag{1.7}$$

which is zero when ϕ is in the Cartan of the gauge algebra. Of course, this is not a gauge invariant statement and it is true only up to conjugation. For $SU(N)$, we can write this in matrix form as

$$\langle\phi\rangle \sim \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{pmatrix} \equiv \boldsymbol{a}, \qquad\qquad \sum_{\alpha=1}^{N} a_\alpha = 0, \tag{1.8}$$

where $\sim$ means "in the same conjugacy class".

Based on all our considerations above, the low-energy effective Lagrangian of an $\mathcal{N}=2$ gauge theory coupled to massive matter, in the Coulomb branch, is

$$\mathcal{L}_{eff} = Im\left[\tau_{IJ}(\phi)\left(\partial_\mu\bar{\phi}^I\partial^\mu\phi^J + \frac{1}{2}F^I_{\mu\nu}F^{J\mu\nu}\right)\right] + \text{fermions}. \tag{1.9}$$

In fact, for a general $\mathcal{N}=2$ theory, any two-derivative action is defined by a single holomorphic (possibly multivalued) function called the prepotential, and the Lagrangian is the integration of this function over chiral $\mathcal{N}=2$ superspace:

$$\mathcal{L}_{eff} = \int d^4\theta F(\Phi). \tag{1.10}$$

The complexified gauge coupling τ_{IJ} in (1.9) is a nontrivial function of all the coordinates of the Coulomb branch, given in terms of the prepotential by the formula

$$\tau_{IJ} = \frac{\partial^2 F}{\partial\phi^I\partial\phi^J}. \tag{1.11}$$

An important property of $\mathcal{N}=2$ low-energy physics is electro-magnetic duality. The BPS spectrum consists in general of dyons in addition to magnetic and electric monopoles. Their electric and magnetic charges, that we will denote by q, p respectively, are specified by two integer numbers n_m, n_e, and given by

$$p = \frac{4\pi}{e}n_m, \qquad q = n_e e - \frac{\theta e}{2\pi}n_m. \tag{1.12}$$

Here e is the $U(1)$ coupling constant and θ the theta-angle. Because of the theta-term, n_m contributes to the electric charge, so that a "magnetic monopole" is actually a dyon: this is known as the Witten effect. Nontheless, we will denote in the following by electric/magnetic monopole a state with numbers respectively $(n_e, 0)$ or $(0, n_m)$, and by dyon a state with (n_e, n_m). For BPS states, the central charge has the following form

$$Z = \boldsymbol{a}_{\mathrm{SW}}\cdot\boldsymbol{n_e} + \boldsymbol{a}^D_{\mathrm{SW}}\cdot\boldsymbol{n_m} + \frac{1}{\sqrt{2}}\sum_{i=1}^{N_f} m_i S_i. \tag{1.13}$$

In the formula above, $\boldsymbol{a}_{SW}$ (the subscript SW stands for Seiberg-Witten, and the reason for this will become clear in the following section) and are functions on the moduli space that at tree-level coincide with the Cartan vev, and $\boldsymbol{a_D}$ are defined by

$$\frac{\partial F}{\partial a}. \tag{1.14}$$

m_i are the masses of eventual N_f hypermultiplets, and S_i are the charges for the corresponding $U(1)_f$ flavor symmetries. The statement of electric-magnetic duality is that there is an $Sp(2n,\mathbb{Z})$ acting on $(\boldsymbol{n_e},\boldsymbol{n_m})$ as

$$\begin{pmatrix} \boldsymbol{n_e} \\ \boldsymbol{n_m} \end{pmatrix} \to \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \boldsymbol{n_e} \\ \boldsymbol{n_m} \end{pmatrix}, \tag{1.15}$$

and on the $U(1)^r$ coupling constants $\boldsymbol{\tau} = (\tau_{IJ})$ as

$$\boldsymbol{\tau} \to (A\boldsymbol{\tau} + B)(C\boldsymbol{\tau} + D)^{-1}. \tag{1.16}$$

In general a duality transformation is not a symmetry of the theory, but rather sends one theory to a physically equivalent one, described in a different language. This observation has a very important consequence: consider performing a loop in the moduli space of the theory. Because the initial and final point are the same, the physical system does not change. However this is only true up to duality, so that we have the possibility that after a non-trivial loop γ we encounter a nontrivial $Sp(2n,\mathbb{Z})$ monodromy of the form (1.16). This is possible only if the loop is non-contractible, which means that it has to encircle one (or more) singular points of the moduli space, which are known to correspond physically to points where some states in the spectrum become massless. This concept of monodromy, which will be a red thread in different forms in the course of this thesis, is the starting point for the exact Seiberg-Witten solution of the low-energy dynamics for $\mathcal{N} = 2$ theories.

1.3 The Seiberg-Witten Solution

Given the considerations above, it is possible to rephrase the problem of the low-energy BPS spectrum and prepotential for $\mathcal{N} = 2$ theories as that of studying periods of an abelian differential over an appropriate Riemann Surface. We will first take a longer, purely gauge-theoretical, route to study this problem, using the example of pure $SU(2)$ SYM. We will mostly follow the original paper [7], while in the next section we will show how all the concepts introduced can be rephrased in terms of an integrable system from classical mechanics.

First let us introduce introduce the general Seiberg-Witten construction as a kind of "recipe" that outputs the low-energy physics of the gauge theory. We will then motivate this recipe by considering the explicit example of pure $SU(2)$ SYM. The main object is the so-called Seiberg-Witten curve. This is a Riemann Surface described as a branched covering of the Coulomb branch, by a polynomial equation in $\mathbb{C}^2$

$$\Sigma : P(y, z; \Lambda, u, m) = 0 \tag{1.17}$$

where λ, z are coordinates of $\mathbb{C}^2$, while the coefficients of the polynomial parametrize the low-energy theory we are interested in. The curve comes equipped with a meromorphic differential

$$dS = ydz, \tag{1.18}$$

which in this context is called the Seiberg-Witten differential. It has the property that

$$\frac{\partial}{\partial a_i} dS = \omega_i, \tag{1.19}$$

where ω_i are holomorphic differentials on the curve Σ, canonically normalized with respect to a basis $\{A_1, \ldots, A_g, B_1, \ldots B_g\}$ of A- and B-cycles of the curve, so that

$$\oint_{A_i} \omega_j = \delta_{ij}, \qquad \oint_{B_i} \omega_j = \tau_{ij}. \tag{1.20}$$

τ_{ij}, the matrix of gauge couplings, is then obtained as the period matrix of the Seiberg-Witten curve. This last property can also be written as

$$a_i = \oint_{A_i} dS, \qquad a_i^D = \oint_{B_i} dS, \tag{1.21}$$

so that the BPS spectrum is computed by the A- and B-cycle integrals of the Seiberg-Witten differential. From the above relations, it also follows that the gauge coupling matrix can be obtained as

$$\frac{\partial a_i^D}{\partial a_j} = T_{ij}. \tag{1.22}$$

From these ingredients, one can then reconstruct the full prepotential. Let us see how this description comes to be in the simplest example of $SU(2)$ super Yang-Mills theory.

1.3.1 Example: Pure Super Yang-Mills

In this case the Coulomb branch is parametrized by a singule gauge-invariant coordinate

$$u = \frac{1}{2} \text{tr} \langle \phi^2 \rangle = a^2 + \text{quantum corrections}. \tag{1.23}$$

The quantum corrections will determine a_{SW} as a function of u, allowing us to read from (1.13) the low-energy BPS spectrum. On the other hand, finding the prepotential F will allow us to read the low-energy Lagrangian and gauge couplings. Consider first the semiclassical regime $u \to \infty$. In this region of the moduli space the theory is weakly coupled, so that the prepotential is well-approximated by the perturbative (tree-level and one-loop) result

$$F = \tau a^2 + \frac{a^2}{2\pi i} \log \left(\frac{a^2}{\Lambda^2} \right). \tag{1.24}$$

The dual variable a_D is given by

$$a_D = \frac{\partial F}{\partial a} = a(\tau + \frac{1}{i\pi}) - ia \log \frac{a^2}{\Lambda^2}. \tag{1.25}$$

To probe whether the Coulomb branch has singular points, we can perform a monodromy around a large circle in this perturbative region, sending

$$u \to e^{2\pi i} u. \tag{1.26}$$

Using the fact that $u \simeq a^2$ (the result being exact in the strict limit), we have

$$a \to -a, \qquad a_D \to -a_D + 2a, \tag{1.27}$$

which yields the monodromy matrix

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}. \tag{1.28}$$

Because u is given by the vev of a complex scalar field, in the semiclassical regime $u \in \mathbb{P}^1$ (we have included the possibility that the vev be infinity). Assuming this is not changed in the fully quantum regime, we must have that this monodromy around infinity is compensated by monodromies around singularities in the interior of the Coulomb branch, and these must satisfy the monodromy relation of a Riemann sphere:

$$M_\infty = M_1 \dots M_k. \tag{1.29}$$

The number of other singular points is yet undetermined, but we can put some constraints using the symmetries of the gauge theory.

First note that due to the chiral anomaly, the $U(1)_R$ symmetry is broken to a $\mathbb{Z}_8$ by quantum corrections. Further, the order parameter u is acted non-trivially upon by a $\mathbb{Z}_2 \subset \mathbb{Z}_8$ as

$$\mathbb{Z}_2 : u \to -u, \tag{1.30}$$

so that the symmetry breaking is $U(1)_R \to \mathbb{Z}_2$. Because of this, the singular points of the Coulomb branch should be either $0, \infty$, which are fixed points under the $\mathbb{Z}_2$ symmetry, or they should come in pairs conjugated under it.

We can already exclude the case in which there are only two singular points $0, \infty$: if this was the case, we would have $M_0 = M_\infty$, and $u = a^2$ would be a good coordinate on the whole Coulomb branch. Then it would be globally true that

$$\tau = \tau(a^2) = -\frac{8}{2\pi i} \log \frac{a}{\Lambda} + f(a^2), \tag{1.31}$$

with f meromorphic with singularities only at $0, \infty$. This would imply that $\operatorname{Im} \tau \sim 1/g^2$ is a harmonic function on the cut plane $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$, so that it should be either constant (which we know not to be true, due to one-loop corrections), or unbounded from below (leading to nonunitary physics). We will now proceed assuming we have the minimal number of singularities allowed by the above considerations: the weakly-coupled point at ∞, and two finite-distance singularities $\pm u_0$, conjugated by the $\mathbb{Z}_2$ symmetry[2]. Further, we will set $u_0 = \Lambda^2$, so that as $\Lambda \to 0$ (classical limit) we reobtain the tree-level description of only two singular points at $u = 0, \infty$. The solution is actually very simple in this case: we determined that

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \tag{1.32}$$

from the semiclassical dynamics. Further, we can use the physical input that singularities in the moduli space correspond to a state becoming massless. Such a state must be invariant, by definition, under the monodromy corresponding to the point where it becomes massless. One can then see that if (n_m, n_e) are the magnetic and electric charges of such a state, the monodromy matrix must have the form

$$M(n_m, n_e) = \begin{pmatrix} 1 + 2n_m n_e & 2n_e^2 \\ -2n_m^2 & 1 - 2n_m n_e \end{pmatrix}. \tag{1.33}$$

It is possible to show that the only solution, modulo $SL(2, \mathbb{Z})$ conjugation, of the equation $M_\infty = M_{\Lambda^2} M_{-\Lambda^2}$ subject to this constraint is given by the following:

$$M_{\Lambda^2} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad M_{-\Lambda^2} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \tag{1.34}$$

which tells us that the state becoming massless at $u = \Lambda^2$ has $(n_m, n_e) = (1, 0)$, so that Λ^2 is known as the *monopole point*, while the state becoming massless at $u = -\Lambda^2$ has $(n_m, n_e) = (1, -1)$, so that $-\Lambda^2$ is known as the *dyon point*. We actually already know enough to solve the low-energy theory without recurring to the full Seiberg-Witten construction. *Solving* here means: find multivalued holomorphic functions $a(u)$, $a_D(u)$ on the three-punctured sphere $\mathbb{P}^1 \setminus \{\Lambda^2, -\Lambda^2, \infty\}$ with the monodromy representation described above, with the asymptotic behavior as $u \to \infty$ given by

$$a \sim \sqrt{u}, \qquad a_D \sim -\frac{8\sqrt{u}}{2\pi i} \log \frac{\sqrt{u}}{\Lambda}. \tag{1.35}$$

[2]Historically, the Seiberg-Witten solution for Super Yang-Mills was derived by assuming the existence of only the minimal number of singular points consistent with the above considerations: the weakly-coupled point at ∞, and two singularities at finite $\pm u_0$. This assumption has later been verified in various ways:

1. From a purely field-theoretical point of view, it is possible to obtain the Seiberg-Witten prepotential by a nonperturbative instanton computation (see Chapter 3), which verifies the consequences of this assumption;

2. It is possible to obtain the $\mathcal{N} = 2$ theories we will be considering as a low-energy limit of string theory on a brane configuration. The Seiberg-Witten curve arises then naturally from the geometry of the brane system, and one can see "directly" the number of singular points of the Coulomb branch. This construction has an M-theory uplift and generalization, known as Class S construction, that we will review in Chapter 2.

Such problem is solved by Gauss' hypergeometric function ${}_2F_1$:

$$a(u) = \sqrt{2(\Lambda^2 + u)}\,{}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1; \frac{2}{1 + u/\Lambda^2}\right), \tag{1.36}$$

$$a_D(u) = i\frac{\Lambda + u/\Lambda}{2}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 2; \frac{1 - u/\Lambda^2}{2}\right). \tag{1.37}$$

From this, one can obtain the prepotential and the gauge coupling τ.

For illustration purposes, we can recover this result from the Seiberg-Witten curve. Observe that these integral monodromies introduce a covering structure on the complex u-plane. Such structure can be encoded in a Riemann Surface, which in this context is known as the Seiberg-Witten curve. For this particular case, the curve is

$$\lambda^2 - \Lambda^2 \mathrm{Ch}\, z = u, \tag{1.38}$$

while the differential is

$$dS = \lambda dz. \tag{1.39}$$

Using $\lambda = \sqrt{u + \Lambda^2 \mathrm{Ch}\, z}$, one finds that

$$a = \oint_A \sqrt{u + \Lambda^2 \mathrm{Ch}\, z} dz. \tag{1.40}$$

This is related to the previous parametrization by the change of variables [9]

$$dS = \sqrt{u + \Lambda^2 \mathrm{Ch}\, z} dz = \frac{w - u}{y(w)} dw, \tag{1.41}$$

where $y(w)$ is the Seiberg-Witten curve in the elliptic parametrization

$$y^2 = (w^2 - \Lambda^4)(w - u). \tag{1.42}$$

In this variables, the integral is easily seen to give rise to the integral representation of the hypergeometric function above:

$$a(u) = \int_{-\Lambda^2}^{\Lambda^2} dw \sqrt{\frac{w - u}{(w + \Lambda^2)(w - \Lambda^2)}}. \tag{1.43}$$

1.4 Intermezzo: a brief overview of Integrable Systems

In the next section we will rephrase everything we just said more efficiently in the language of finite dimensional, classical integrable systems. Before doing that, however, we have to introduce some basic objects that will appear again and again in this thesis, as well as some fundamental terminology, being concerned only with the system itself and not to its connection to high-energy physics.

High-energy physicists are used to think of nonlinearities to be associated with interactions: the study of such sysyems usually involves splitting the Lagrangian or Hamiltonian into a "free" part, associated with linear evolution equations, and an "interacting part", which adds nonlinearities to the equations, that are treated perturbatively. Of course, this splitting is rather artificial (and in fact this "interaction picture" is mathematically inconsistent in Quantum Field Theory because of Haag's theorem [34]), but physically well-motivated: because we deal with local interaction, we can envision an asymptotic region where the states, which are the particles we observe, are indeed those of the free Hamiltonian, and the interactions are negligible. We can say that the systems we usually deal with are "kinematically trivial" (the kinematics is given by the free Hamiltonian), but dynamically nontrivial (there are interactions).

Integrable systems are a very special case where the perspective gets completely reversed: there is a special set of variables in terms of which the evolution is free. However, these are unphysical variables that live in an unphysical space: the Jacobian (or rather the Prym) of a so-called "spectral curve". The crucial thing is, that all the nonlinearities of the time evolution are encoded in this change of variables, so that we might say that the system is "kinematically nonlinear", but "dynamically trivial", as there is a change of variables that gives us no interaction at all. Having given this very broad and vague distinction, let us be a bit more precise.

The reader might be familiar with the usual "elementary" definition of integrability: an integrable system is a dynamical system with a phase space of dimension $2n$, with n conserved Hamiltonians in involution. While this is a universally accepted definition in most cases, it does not extend well to integrable field theories, nor to the quantum level. In fact, we will see later that some much less trivial systems (so-called isomonodromic systems) have non-conserved Hamiltonians, but still give rise to integrable equations. We will thus give a different definition of what is an Integrable System, that works in all the cases above:

> An integrable system is given by the compatibility conditions of an overdetermined system of linear problems.

This definition is much less intuitive, but indeed correct in all the cases we mentioned above. The most common instance of compatibility condition is given by the so-called Lax equations, that are defined by two linear problems of the form

$$\begin{cases} L\psi = 0, \\ \frac{d\psi}{dt} = M\psi, \end{cases} \qquad \Longrightarrow \qquad \frac{dL}{dt} = [M, L]. \tag{1.44}$$

If we are dealing with a system whose equations of motion are in this form, (L, M) is known as a Lax pair for the system. The above formula is very general, and depending on the nature of the linear operators one deals with different dynamical systems (for example, L might be a linear differential operator on some L^2 space). Since we will be considering finite-dimensional integrable systems, we will always consider $L, M \in GL(N, \mathbb{C})$ or $SL(N, \mathbb{C})$, where N is a measure of the number of degrees of freedom of the system: in this case L is just referred to as the Lax matrix. Further, they will depend not only on the parameters defining the system, but meromorphically on a complex variable z, and the equations will be required to be valid for every value of z. These systems are then defined at the intersection between complex analysis and linear algebra, so that one should not be surprised if the most natural language to describe them is that of Algebraic Geometry. Note that (1.44) implies that the spectrum of L is conserved, and in fact these equations make manifest that we have (in the generic case where all the eigenvalues of L are distinct), N conserved quantities, making contact with good old Liouville definition of integrability.

From elementary Linear Algebra, we know that the eigenvalues of a matrix are the zeros of its characteristic polynomial:

$$P(z, \lambda) = \det(\lambda - L(z)) = 0. \tag{1.45}$$

Since the Lax matrix depends on a complex parameter z, we see that the characteristic polynomial defines an algebraic (or rather rational) curve, which is called the spectral curve of the system. This is a polynomial in λ, and a rational function in z. Its moduli are the integrals of motion, parametrized by the coefficients of the characteristic polynomial:

$$u_k = e_k(L). \tag{1.46}$$

e_k are elementary symmetric polynomials, and for $SL(N)$ they are simply

$$u_k = \frac{1}{k} \operatorname{tr} L^k. \tag{1.47}$$

As we mentioned above, an important ingredient in the solution of an integrable system is the change to action/angle variables. Here this change of variables is simply the Abel map, from the spectral curve to its Jacobian:

$$\begin{aligned} \mathcal{A} : \Sigma &\longrightarrow Jac(\Sigma) \sim \mathbb{C}^g / \Lambda_T \\ P &\longrightarrow \int_{P_0}^{P} \boldsymbol{\omega}, \end{aligned} \tag{1.48}$$

where Λ_τ is the lattice defined by the period matrix T_{ij} of the Riemann surface, and

$$\boldsymbol{\omega} \equiv \left(\omega^1, \dots, \omega^g\right) \tag{1.49}$$

is a basis of holomorphic differentials on Σ, canonically normalized so that

$$\oint_{A_i} \omega^j = \delta_{ij}, \qquad \oint_{B_i} \omega^j = \tau_{ij}. \tag{1.50}$$

For a given Integrable System, we can define a generating differential

$$dS = \sum_i p_i dq_i, \tag{1.51}$$

where p_i, q_i are canonically conjugated coordinates of the system. It is only defined up to an exact form. The important property of this differential is that the action variables can be obtained from it by contour integrating:

$$a_j = \oint_{A_j} dS. \tag{1.52}$$

An important property of this canonical differential is in fact that its derivative with respect to the action variables gives the holomorphic differentials ω_i (in cohomology): since by definition

$$\delta a_j = \oint_{A_j} \frac{\partial dS}{\partial a_i} \delta a_i = \delta_j{}^i \delta a_i, \tag{1.53}$$

we have

$$\frac{\delta dS}{\delta a_i} = \omega_i + d\xi, \tag{1.54}$$

where ξ is some exact form. It turns out to be convenient to define also the dual action variables

$$a_j^D = \oint_{B_j} dS. \tag{1.55}$$

Of course, since both of these are action variables for the same system, they cannot be independent. The relation between them is easily found to be

$$\frac{\partial a_j^D}{\partial a_i} = \oint_{B_j} \frac{\partial dS}{\partial a_i} = T_{ij}. \tag{1.56}$$

1.5 Seiberg-Witten Theory as an Integrable System

We have now all the ingredients that we need to make a precise correspondence between Seiberg-Witten theory and Integrable Systems. First of all it should come to no surprise that the spectral curve of the Integrable Sysyem, the characteristic polynomial of L is nothing else than the spectral curve. In fact, the original starting point in the correspondence between gauge theory and intgerable systems was exactly to find such a system described by a spectral curve equal to the Seiberg-Witten curve under consideration [9] (see [35] for a review from this point of view).

Further, it turns out that the action differential $dS = pdq$ is in fact the Seiberg-Witten differential. Given these two ingredients, the rest of the correspondence follows, which we can pack in the following table

Seiberg-Witten Theory	Integrable Systems
SW Curve	Spectral Curve
Matrix of gauge coupings	Period Matrix of the Spectral Curve
Quantum Coulomb branch coordinates u_k	Hamiltonians of the IS
SW Differential	Generating differential pdq
SW periods a, a_D	Action coordinates

In particular, pure $SU(2)$ super Yang-Mills is described in the above sense by an integrable system with Hamiltonian

$$H = p^2 + \Lambda^2 \mathrm{Ch}\, q = u, \tag{1.57}$$

which is a complexified version of the Hamiltonian of the simple pendulum (they coincide if $q \in i\mathbb{R}$), as promised in the introduction. We will see how to systematically obtain the correct integrable system that describes the low-energy theory for a given (large class of) $\mathcal{N} = 2$ quantum field theories in Chapter 2.

Chapter 2

String Theory Embeddings

In this chapter we will review several string theory constructions of four-dimensional theories with extended supersymmetry. We will start by reviewing the type IIA construction of linear quiver gauge theory from [36] and its M-theory uplift. We then review how this construction is generalized to build a large class of theories, non necessarily Lagrangian, called Class S theories [4]. These constructions give us the chance to introduce a remarkable family of integrable systems, the Hitchin system, that arises quite naturally from the string theory construction, giving a better physical understanding of the correspondence between Seiberg-Witten theory and integrable systems. We will then briefly describe a different construction of the same four-dimensional gauge theories, called their geometric engineering realization, given by type IIA compactified on a Calabi-Yau threefold, which in our case will be a (non-compact) toric Calabi-Yau manifold. The M-theoretic uplift then describes a five-dimensional $\mathcal{N} = 1$ theory on $\mathbb{R}^4 \times S^1$, which has an equivalent T-dual description of type IIB on an NS5-D5 (p, q) brane web. We will conclude by mentioning the relation of this with Topological String theory on the same Calabi-Yau threefold, a point of view allows an explicit computation of the partition function and prepotential of the theory, as a formal power series in the Kähler parameters of the toric Calabi-Yau.

2.1 Witten's construction of linear quiver theories

The idea of this construction is to consider a setup of semi-infinite D4-branes stretched between NS5-branes. They are extended in the directions as specified in Table 2.1, that is they have the following configuration: the NS5 branes are extended in the directions $x^0, \ldots, x^5$, and the i-th NS5 brane is localized at $x^7 = x^8 = x^0 = 0$, $x^6 = v_i$; the D4-branes share the directions $x^0, \ldots, x^3$ with the NS5 branes (these are the directions where the four-dimensional theory will live), while they are finite in the direction x^6, ending on two different NS5s. There are $n + 1$ NS5-branes labeled by $j = 0, \ldots, n$, such that between the brane j and $j - 1$ stretch N_j D4-branes, as depicted in Figure 2.1 for the case $n = 1$, $N_1 = 2$, that gives a low-energy $\mathcal{N} = 2$ pure $SU(2)$ super Yang-Mills theory on the D4-branes. For the analysis it turns out to be useful to use the complex coordinate $z = x^4 + ix^5$. The D4-branes have definite value of z, that we denote by a^j_α, $\alpha = 1, \ldots, N_j$, $j = 1, \ldots, n$. These are the vevs of the scalar in the vector multiplet living on the D4 brane: on the N_j D4

	0	1	2	3	4	5	6	7	8
NS5	x	x	x	x	x	x			
D4	x	x	x	x			x		

TABLE 2.1

branes between the NS5 labeled by $j-1$ and j there lives a $U(1)^{N_j}$ gauge theory, enhanced to $U(N_j)$ in the D4 coinciding limit. The resulting theory is a linear quiver gauge theory with gauge group $U(N_1) \times \cdots \times U(N_n)$, which can be reduced to a $SU(N_1) \times \cdots \times SU(N_n)$ gauge theory by enforcing the condition

$$\sum_{\alpha} a^i_{\alpha} - \sum_{\beta} a^{i+1}_{\beta} = q_i. \tag{2.1}$$

This condition can be seen to arise from a semiclassical stability analysis of the brane configuration.

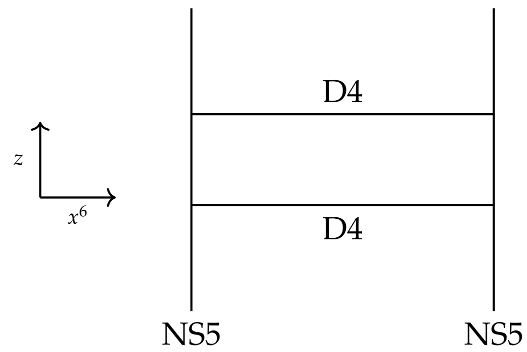


FIGURE 2.1: Brane construction for $SU(2)$ pure SYM

For $n > 1$, the quiver gauge theory has bifundamental hypermultiplets coupled to the gauge group $SU(N_j)$ and $SU(N_{j+1})$, given by strings stretching between different sets of D4-branes. Their mass is given by the average separation of the branes:

$$m_j = \sum_{\alpha} \left(\frac{a^j_{\alpha}}{N_{\alpha}} - \frac{a^{j+1}_{\alpha}}{N_{j+1}} \right). \tag{2.2}$$

The gauge coupling is determined by the separation along the x^6-direction:

$$\frac{1}{g_j^2} = \frac{x^6_j - x^6_{j-1}}{g_s}, \tag{2.3}$$

where g_s is the string coupling constant. One last point is that of brane balancing: for the configuration to be "balanced" in the sense that there are no net forces on the NS5 branes from the D4 branes, which translates to the four-dimensional field theory being conformal (as opposed to free) in the UV, we actually need to add semi-infinite D4 branes on the left and right side of the brane diagram, so that each NS5 brane has the same number of D4's ending on it from the left and from the right. The result of this operation would be, for example, to turn Figure 2.1 into Figure 2.2, engineering $SU(2)$ SYM with $N_f = 4$[1].

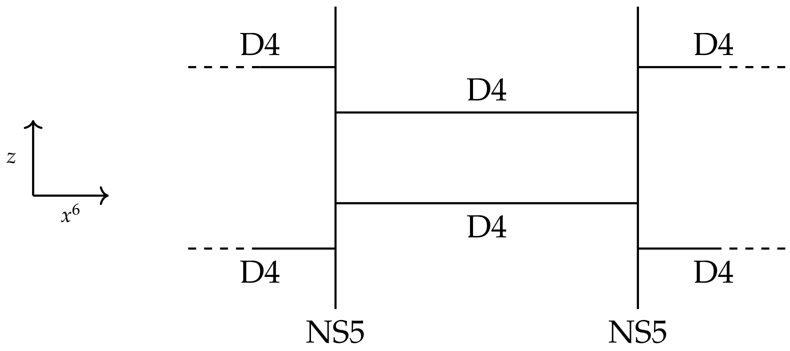


FIGURE 2.2: Brane construction for $SU(2)$ SYM with four fundamental flavors

We can uplift this picture to M-theory, by adding a tenth compact direction x^9. This has the nice feature of resolving the endings of the D4 branes on NS5 branes, which are not well understood, because all the IIA branes are now regions of a single M5-brane in M-theory (think of thickening all the lines of Figure 2.1, 2.2 into cylinders). The world-volume of this M5-brane is $\mathbb{R}^4 \times \Sigma$, where Σ is the Seiberg-Witten curve of the theory. We thus see that the geometry of the Seiberg-Witten curve, which from the field theoretical point of view was some obscure auxiliary object, appears naturally in M-theory.

The low-energy physics can be checked to be the correct one expected from Seiberg-Witten theory: on the world-volume of the M5-brane there is a 2-form β whose field strength T is self-dual. If the world-volume is $\mathbb{R}^4 \times \Sigma$, the zero-modes of β give rise to g abelian gauge fields on $\mathbb{R}^4$, where g is the genus of Σ. This is because we can write a self-dual 3-form as

$$T = F \wedge \Lambda + *F \wedge *\Lambda, \tag{2.4}$$

where F is a 2-form and Λ a 1-form. The Bianchi identity $dT = 0$ yields

$$\begin{cases} dF = d*F = 0, \\ d\Lambda = d*\Lambda = 0. \end{cases} \tag{2.5}$$

The equations for F are just Maxwell's equations, while those for Λ tell us that Λ is a harmonic 1-form on Σ, or in other terms a holomorphic differential. The space of holomorphic differentials on a Riemann surface is g-dimensional, and each solution corresponds to a different embedding of

[1]A more detailed description of the unbalanced case, giving asymptotically free theories, would involve dealing with the brane bending phenomenon [37], which in the terminology of class S theories, still to be introduced, yields irregular singularities. Since this is not really relevant for this thesis, we will not delve in such details.

Maxwell's equations into the 3-form equations, so that we get g abelian gauge fields. The complexified gauge couplings of these gauge fields describe the Jacobian of Σ. Note that in this construction the Seiberg-Witten curve will always be hyperelliptic, i.e. it will be be given by a covering of the Riemann sphere (the z-plane of the NS5 branes, which are the various sheets connected by the D4 branes, which assume the role of "branch points" connecting the sheets). Let us see how this discussion is generalized by Gaiotto's "Class S" construction.

2.2 Class S theories

The above construction for linear quiver gauge theories can be generalized to a much wider class of theories, most of which nonlagrangian, known as class S [4, 37]. The general idea is that the construction of the previous section realizes gauge theories on their Coulomb branch. The UV physics of the gauge theory is realized when the NS5 branes of the IIA setup coincide: the M-theory uplift of this configuration is a stack of N coinciding M5-branes. As we mentioned in section 2.1, the massless degree of freedom living on an NS5-brane is an self-dual antisymmetric tensor, which in the limit of coinciding branes gives not an enhanced $U(N)$ gauge symmetry like in the case of D-branes, but rather a nonlagrangian theory of tensionless strings [38]. The theory on the world-volume of this configuration is then a nonlagrangian six-dimensional superconformal $(2,0)$ theory. Such theories have an ADE classification, and the case of N coincident M5-branses is the A_{N-1} theory. We will denote such six-dimensional theories by its corresponding alebra $\mathfrak{g}$ in the ADE classification, $\mathcal{S}(\mathfrak{g})$.

We want to study the case where the world-volume of these M5-branes is $\mathbb{R}^{1,3} \times C_{g,n}$, where $C_{g,n}$ is a Riemann Surface with marked points (these marked points correspond to the inclusion of a pointlike defect in the six-dimensional QFT). Let us note immediately that the configuration of the previous section, describing the IR theory on the Coulomb branch, is simply achieved by separating these M5-branes. The resulting configuration is N copies of an M5 brane wrapping $C_{g,n}$, connected by tubes, which is the same as a single M5 brane wrapping an N-covering of $C_{g,n}$. This covering is the Seiberg-Witten curve Σ of the four-dimensional theory. As we already mentioned, the construction of the previous section corresponds to the case where C is of genus zero, while this is the more general case.

Compactifying directly the six-dimensional SCFT on an arbitrary Riemann surface would break four-dimensional supersymmetry. To preserve four-dimensional $\mathcal{N} = 2$ we have to perform a partial topological twist: the breaking pattern induced by the compactification, for the bosonic part of the symmetry superalgebra, is

$$\mathfrak{osp}(6,2|4) \supset \mathfrak{so}(5,1) \oplus \mathfrak{so}(5) \to \mathfrak{so}(3,1) \oplus \mathfrak{so}(2)_C \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2)_R. \quad (2.6)$$

Without the topological twist, the supersymmetric charges $Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}$ would lie in the representation of the unbroken symmetries given by

$$\left(\left((2,1)_{\frac{1}{2}} \oplus (1,2)_{-\frac{1}{2}}\right) \otimes \left(2_{\frac{1}{2}} \oplus 2_{-\frac{1}{2}}\right)\right). \tag{2.7}$$

To twist, we identify the diagonal subalgebra of $\mathfrak{so}(2)_C \oplus \mathfrak{so}(2)_R$ with the holonomy algebra of the Riemann Surface $C_{g,n}$. After doing this, the transformation of the supercharges under the remaining $\mathfrak{so}(3,1) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2)'_C$ is

$$(2,1;2)_1 \oplus (2,1;2)_0 \oplus (1,2;2)_0 \oplus (1,2;2)_{-1}. \tag{2.8}$$

This means that eight of the supercharges $Q^A_\alpha, \bar{Q}_{\dot{\alpha}A}$ are scalars on the Riemann surface and span a four-dimenensional $\mathcal{N} = 2$ supersymmetry algebra, while the other eight become one-forms on $C_{g,n}$, that we will denote with $Q^{\alpha A}_z, \bar{Q}^{\dot{\alpha} A}_{\bar{z}}$. Since the anticommutator of the supercharges $\overline{Q}$ is

$$\left\{\bar{Q}^{\dot{\alpha}A}, \bar{Q}^{\dot{\beta}B}_{\bar{z}}\right\} \propto \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{AB}\partial_{\bar{z}}, \tag{2.9}$$

any chiral operator must satisfy

$$\partial_{\bar{z}}\langle\mathcal{O}\rangle = 0, \tag{2.10}$$

i.e. chiral operators are holomorphic objects on the Riemann Surface. More precisely, if a chiral operators $\mathcal{O}$ has $\mathfrak{so}(2)_R$ charge d_k, it is a section of the holomorphic bundle K^{d_k}. The Coulomb branch will be parametrized by all the possible gauge invariant chiral operators, so that it will be given by

$$\mathcal{B} = \bigoplus_{k=1}^{r} H^0(C_{g,n}, K^{\otimes d_k}). \tag{2.11}$$

The curve $C_{g,n}$ that the branes are wrapping is in fact a holomorphic curve inside a hyperkähler manifold Q of complex dimension two, that we can consider locally around $C_{g,n}$ as the cotangent bundle $T^*C_{g,n}$. The Seiberg-Witten curve Σ, wrapped by the branes when we consider them to be separated, is a different curve inside the same hyperkähler manifold Q. This means that we can express it by choosing Darboux coordinates (x,z) for T^*C as

$$x^K + \sum_{k=2}^{K} u_k(z)x^{K-k} = 0. \tag{2.12}$$

The collection of coefficients $u_k \in H^0(C, K^{\otimes k})$ coordinatize a point in the Coulomb branch $\mathcal{B}$. The symplectic form of T^*M is given by

$$\Omega = \frac{l^3}{2\pi^2} dx \wedge dz, \tag{2.13}$$

so that the canonical one-form

$$dS = xdz \tag{2.14}$$

is the Seiberg-Witten differential.

2.3 Intermezzo: Hitchin Systems

We now take again a short break from the high-energy theory discussion, in order to introduce some further concepts from the world of integrability, namely the Hitchin System [39]. This is an algebraic (i.e. complex) Integrable System, whose construction is purely geometrical. We will see in the next section how this geometry arises naturally in the context of Class S theories, but in fact the relation between Seiberg-Witten theory and Hitchin systems was argued more than fifteen years before the Class S construction by Donagi and Witten [10].

The starting point of the construction is a holomorphic vector bundle $\mathcal{E}$ with gauge group G, connection A, and fiber V an N-dimensional vector space[2], over a Riemann surface C_g of genus g. It is possible to endow this bundle with an End $\mathcal{E}$-valued differential Φ, called in this context the "Higgs field" (no relation whatsoever with the *actual* Higgs field from the Standard Model). This is a holomorphic section of $\mathcal{E}$, which means that it satisfies

$$\overline{D}_A\Phi = \overline{\partial}\Phi + [\bar{A}, \Phi] = 0. \tag{2.15}$$

The pair $(\mathcal{E}, \Phi)$ is called a Higgs bundle. It was shown by Hitchin that the cotangent space to the space of connection and Higgs fields carries a (pre)symplectic structure

$$\omega_H = \int_{C_g} \langle \delta\Phi \wedge \delta A \rangle, \tag{2.16}$$

where $\langle , \rangle$ is the Cartan-Killing form for the gauge group G. After quotienting by the gauge group, the space of holomorphic connections and Higgs fields turns into the moduli space of holomorphic Higgs bundles $\mathcal{M}_H$. This quotient turns $T^*\mathcal{M}_H$ into a finite-dimensional symplectic manifold, whose symplectic form is the gauge-fixed version of ω_H. This is the phase space of a finite-dimensional integrable system, for which Φ is the Lax matrix. The spectral curve is then

$$\Sigma : \det(\lambda - \Phi(z)) = 0, \tag{2.17}$$

which is an N-fold covering of C_g if $N = \dim V$. The action generating differential, which will be the Seiberg-Witten differential, is

$$dS = \lambda dz. \tag{2.18}$$

[2]For all practical applications, we will always consider the group to be $GL(N)$ or $SL(N)$, acting on N-vectors. Then Φ is an $N \times N$ matrix. However, we will try to stay general when possible and not specify G when it is not needed.

The dimension of the Hitchin system is

$$\dim T^*\mathcal{M}_H = 2\dim G(g-1), \tag{2.19}$$

so that there is no interesting Hitchin system for $g \leq 1$. In order to have interesting system for $g = 0, 1$, we need to generalize the discussion to Riemann Surfaces $C_{g,n}$ with marked points. The extension to such object of Hitchin's construction was carried out by Donagi and Markman [40, 41], and the holomorphic condition for Φ turns into a meromorphic one, which adds sources at the marked points, so that now

$$\bar{\partial}\Phi + [\bar{A}, \Phi] = \sum_k \Phi_k \delta^2(z - z_k). \tag{2.20}$$

The dimension of the phase space is now

$$\dim T^*\mathcal{M}_H = \begin{cases} 2\dim G(n-3), & g = 0, \\ 2\dim G(n-1), & g = 1, \\ 2\dim G(g-1+n), & g \geq 2. \end{cases} \tag{2.21}$$

2.4 Hitchin moduli space and Seiberg-Witten theory

We observed that if we compactify the six-dimensional SCFT $S(\mathfrak{g})$ on the punctured Riemann surface $C_{g,n}$ with partial topological twist, the resulting four-dimensional theory is an $\mathcal{N} = 2$ theory, that we will denote by $\mathcal{S}(\mathfrak{g}, C_{g,n})$. An upside of this framework is that the identification of Seiberg-Witten theory with the Hitchin system can be seen quite nicely by further compactifying to three dimensions.

To understand why this is the case, consider the IR limit of the compactification of the four-dimensional $\mathcal{N} = 2$ theory on a circle S^1_R . Let the S^1 direction be the time direction, and decompose the gauge field as $A = (A_0, A_i)$. The low-energy degrees of freedom in 3d, starting from regular points of the Coulomb branch in 4d, are the following:

- r abelian gauge fields A_i^α, which in 3d are dual to r periodic real scalars;
- r periodic real scalars given by the holonomies of the gauge field along the S^1 (characterized by A_0);
- r complex scalars which parametrize the four-dimensional Coulomb branch.

At regular points, the 3d target space is then $T^{2r} \times \mathcal{B}_{reg}$. To have the global description we must include singular points of $\mathcal{B}$, over which some cycle of T^{2r} degenerates. The target space is then a complete holomorphic symplectic manifold fibered over the whole Coulomb branch $\mathcal{B}$, $\mathcal{M}_{SW} : T^{2r} \to \mathcal{B}$: the Coulomb branch $\mathcal{B}$ is parametrized by the gauge-invariant coordinates

u_k, while the torus fibration essentially associates the Seiberg-Witten differential dS to every point of the Coulomb branch. We wish to identify this fibration with the Hitchin fibration of the previous section. We will do this by inverting the order of the two compactifications. In general two different compactification limits need not commute, but since we will focus only on the BPS sector of these theories, no phase transitions occurs due to the partial topological twist implemented during the compactification on $C_{g,n}$.

If we compactify $S(\mathfrak{g})$ on S^1_R, the low-energy theory will be given by an $\mathcal{N} = 2$ maximally supersymmetric 5d super Yang-Mills with gauge group G and coupling $g_{5d} = R^2$. The bosonic part of the action is five-dimensional Yang-Mills coupled to the vector multiplet scalars Y^I, $I = 1, \ldots, 5$:

$$S_{bos} = \frac{R}{8\pi^2} \int_{\mathbb{R}^{1,2} \times C_{g,n}} \mathrm{tr} \left(\frac{1}{R^2} F \wedge *F + DY^I \wedge *DY^I \right). \tag{2.22}$$

When we compactify on $C_{g,n}$, the IR theory is given by a sigma model into the moduli space of the BPS equations for 5d SYM on this geometry. The resulting equations are a special case of the self-dual Yang-Mills equations [42, 43]. These have the form

$$\begin{cases} F + R^2[\Phi, \bar{\Phi}] = \sum_k A_k \delta^2(z - z_k), \\ \bar{D}_A \Phi = \sum_k \Phi_k \delta^2(z - z_k) \\ \bar{D}_A \bar{\Phi} = \sum_k \bar{\Phi}_k \delta^2(z - z_k). \end{cases} \tag{2.23}$$

Thus, from the first compactification we know that the resulting 3d theory is a nonlinear sigma-model with target space the Seiberg-Witten moduli space; from the second argument however, we know that the same three-dimensional theory is a nonlinear sigma model with target space the moduli space of solutions of (2.23). We want to show that this is the same as Hitchin's moduli space. From 3d $\mathcal{N} = 4$ supersymmetry, we know that the target space of these equations must be hyperkähler. This means that it admits a $\mathbb{CP}^1$ worth of complex structures, that we will parametrize by $\zeta \in \mathbb{CP}^1$. The BPS equations above can then be rephrased in an invariant form as the flatness conditions for the $G_{\mathbb{C}}$-connection

$$\mathcal{A} = A + \frac{R}{\zeta}\Phi + R\zeta\bar{\Phi}, \tag{2.24}$$

which give

$$\begin{aligned} \sum_k \mathcal{A}_k \delta^2(z - z_k) &= \bar{\partial}\mathcal{A} + \partial\bar{\mathcal{A}} + [\mathcal{A}, \bar{\mathcal{A}}] \\ &= F + R^2[\Phi, \bar{\Phi}] + \frac{R}{\zeta}\overline{D}_A\Phi + R\zeta D_A\bar{\Phi}, \end{aligned} \tag{2.25}$$

where we defined

$$\mathcal{A}_k = A_k + \frac{R}{\zeta}\Phi_k + R\zeta\bar{\Phi}_k. \tag{2.26}$$

The BPS equation then follow from the requirement that this equation be satisfied for arbitrary ζ. In particular, if we take $\zeta \to 0$, the equation becomes simply the Hitchin equations of the previous section,

$$\overline{D}_A \Phi = \sum_k \Phi_k \delta^2(z - z_k). \tag{2.27}$$

We thus get the sought-for result that the Seiberg-Witten moduli space, read in the complex structure $\zeta \to 0$, is the same as the Hitchin moduli space. As we argued in the previous section, the Hitchin moduli space has the structure of an algebraic integrable system. This provides a general geometric framework for the observations we made in Chapter 1, relating Seiberg-Witten theory to integrable systems.

A final remark: the equations above define the punctures by specifying sources in the BPS equations of the compactification, at marked points of the Riemann Surface. These are implementing singular boundary conditions for the fields: to see that, consider for simplicity $A = g^{-1}\partial g$, so that it can be gauged away. Then Hitchin equation becomes

$$\bar{\partial}\Phi = \sum_k \Phi_k \delta^2(z - z_k), \tag{2.28}$$

which has solution

$$\Phi(z) = \sum_k \frac{\Phi_k}{z - z_k}. \tag{2.29}$$

However, the solution is only defined modulo gauge transformation, so that the relevant data specifying the puncture z_k is not Φ_k *per se*, but only the coadjoint orbit

$$\mathcal{O}_k \equiv \left\{ \Phi_k \in SL(N, \mathbb{C}) : \Phi_k = C_k \boldsymbol{\theta}_k C_k^{-1} \right\}, \quad \boldsymbol{\theta}_k = \mathrm{diag}(\theta_1, \ldots, \theta_N). \tag{2.30}$$

A coadjoint orbit is a symplectic space, over which one can define the Kirillov-Konstant Poisson bracket

$$\left\{ \Phi_k^a, \Phi_k^b \right\} = C^{ab}{}_c \Phi_k^c, \tag{2.31}$$

where $C^{ab}{}_c$ are the structure constant of the relative Lie algebra, in this case $\mathfrak{sl}_N$. The residues Φ_k encode (at least part) of the degrees of freedom of the Hitchin System. Physically, the data at the coadjoint orbit specifies a flavor group: in the $SU(2)$ case only one choice is possible, for the coadjoint orbit to have conjugacy class $\boldsymbol{\theta}_k = \theta_k \sigma_3$, which gives an $SU(2)$ flavor group with Cartan parameters specified by θ_k. For $SU(N)$ instead, many choices are possible, corresponding to different types of punctures. Of particular importance to us will be *full punctures*, specified by a generic $\boldsymbol{\theta}_k = \mathrm{diag}(\theta_1, \ldots, \theta_N)$, and *minimal punctures*, specified by $\boldsymbol{\theta}_k = \theta_k \boldsymbol{\omega}_1$, where $\boldsymbol{\omega}_1$ is the first fundamental weight of $\mathfrak{sl}_N$:

$$\boldsymbol{\omega}_1 = \left(\tfrac{N-1}{N}, \ -\tfrac{1}{N}, \ -\tfrac{1}{N}, \ \ldots, \ -\tfrac{1}{N} \right). \tag{2.32}$$

In general, one can consider as data also coadjoint orbits for different algebras than $\mathfrak{sl}_N$, but we will not discuss this in this thesis.

Let us conclude the discussion of class S theories with the following observation: we saw that the world-volume theory of M5-branes wrapping a Riemann surface with punctures is a four-dimensional $\mathcal{N} = 2$ QFT. What is this theory? A partial answer is given by Witten's construction, that we studied in Section 2.1: if the Riemann surface is $C_{0,n}$, genus zero with an arbitrary number of simple punctures, then the resulting four-dimensional theory is a linear quiver gauge theory. In the general case, the resulting theory need not be even a Lagrangian theory, but it is determined by the observation that a $\mathfrak{sl}_N$ coadjoint orbit at a puncture associates an $SU(N)$ flavor group to the corresponding four-dimensional theory.

We could discuss here how to identify what type of four-dimensional theory one ends up with just by looking at the Riemann Surface and the data at the punctures. However, we will postpone this discussion to the next Chapter, where we will see that a useful way to understanding these theories goes through conformal blocks of an appropriate conformal symmetry algebra living on the Riemann Surface $C_{g,n}$.

2.5 Geometric Engineering

There is another way of obtaining $\mathcal{N} = 2$ Quantum Field Theories in four dimensions, known as geometric engineering [6, 44]. This consists in compactifying type IIA string theory on a singular CY_3. To obtain an ADE gauge group, one needs the Calabi-Yau manifold to be an ADE singularity[3] fibered over a Riemann Surface. The field content is determined by the compactification: a Riemann Surface with genus g yields a theory with g adjoint hypermultiplets, and fundamental matter can be obtained by blowing up points on the ADE singular curve. The Calabi-Yau space must be singular in order to have enhanced gauge symmetry: the W-boson states will be given by D2 branes wrapping vanishing 2-cycles of the singularity [45–47].

The prototypical example of this is pure $SU(2)$ gauge theory, which is obtained according to the rules above by compactifying type IIA on a nontrivial fibration of an A_1 singularity over $\mathbb{P}^1$. In addition to the $\mathbb{P}^1$ constituting the base of the fibration, of area Q_b, from the Dynkin diagram of A_1 one gets a $\mathbb{P}^1$ vanishing cycle in the fiber, of area Q_f, and because of this the space is known as local $\mathbb{P}^1 \times \mathbb{P}^1$[4]. The areas of the base and fiber $\mathbb{P}^1$'s are related to the Kähler parameters t_b, t_f of the Calabi-Yau, and to the gauge theory

[3] An ADE singularity is an orbifold that locally looks like $\mathbb{C}^2//\Gamma$, where Γ is a finite subgroup of $SU(2)$ given by the ADE classification, and the coordinates of $\mathbb{C}^2$ transform in the fundamental representation of $SU(2)$. Its blowup is a union of $\mathbb{P}^1$'s having one point in common with each other, forming a "blown-up version" of an ADE Dynkin diagram of Γ.

[4] This is the local Hirzebruch surface known as local $\mathbb{F}_0$.

quantities by

$$Q_b = e^{-t_b} = \left(\frac{R\Lambda}{2}\right)^4, \qquad Q_f = e^{-t_f} = e^{-2Ra}, \tag{2.33}$$

where $\Lambda \sim e^{-1/g^2}$ is the dynamically generated scale of the gauge theory and a is the Cartan vev of the scalar in the vector multiplet. From this identification it's clear that the large volume limit $Q_b \to \infty$ corresponds to the weakly-coupled gauge theory phase $g \to 0$.

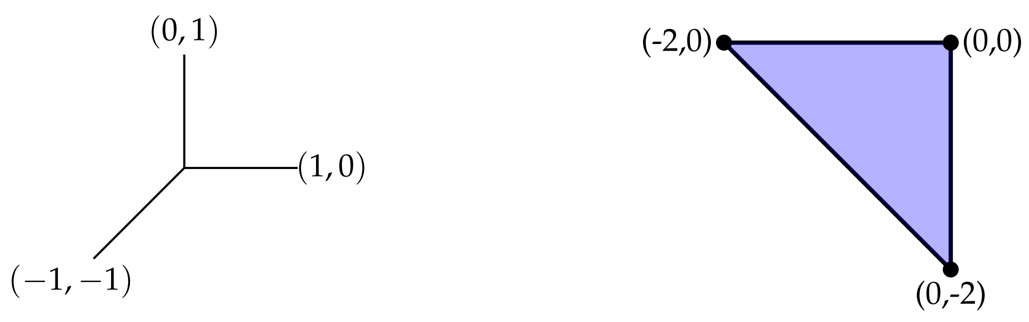
Local $\mathbb{F}_0$ is an example of what is called a local toric Calabi-Yau threefold, which are non-compact Calabi-Yau manifolds with a toric action that degenerates over its fixed points. For such singular manifolds, there is a dual way of describing the QFTs obtained in the IR, as the theories living on a (p, q) brane web in type IIB string theory [48–50]. A (p, q) brane in type IIB is a bound state of p D5-branes and q NS5-branes. The tension of such an object is given in terms of the tension T_{D5} of a D5-brane by

$$T_{p,q} = |p + \tau q| T_{D5}, \tag{2.34}$$

where τ is the axio-dilaton of type IIB. A (p, q) web is a configuration of intersecting (p, q) branes with four-common directions (where the physical spacetime will be located), so that all the nontrivial geometry can be represented by a two-dimensional diagram depicting the two directions not shared by the branes. Such a configuration is stable provided the (p, q) charge is conserved:

$$\sum_i p_i = \sum_i q_i = 0. \tag{2.35}$$

Because of this, for example, an NS5-brane and a D5-brane cannot simply end on each other as in the IIA construction of Section 2.1, but they must form a (-1,-1) bound state, as in Figure 2.3a. Further, to preserve five-dimensional



(A) Basic (p,q) brane vertex

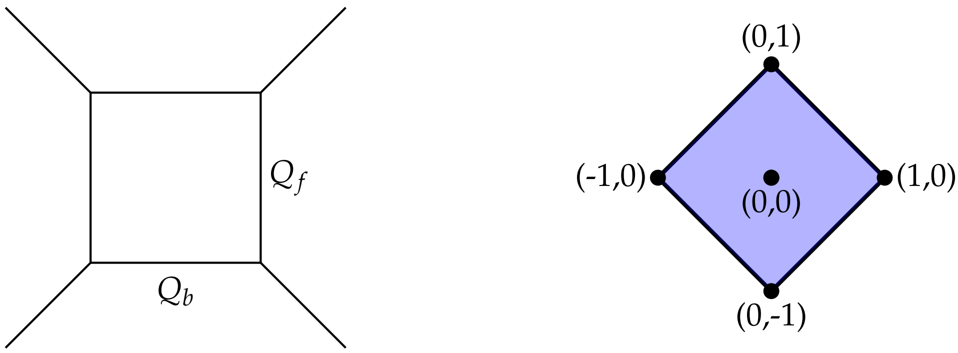
(B) Newton polygon of $\mathbb{C}^3$

FIGURE 2.3

$\mathcal{N} = 1$ supersymmetry (and make the configuration truly stable because of the BPS condition), we must require that the slope of a configuration in the (x, y) plane must divide $p + \tau q$. There is a way to associate to any toric

Calabi-Yau compactification in type IIA the corresponding type IIB (p,q) brane web, and it goes as follows.

One can associate to any toric CY_3 its Newton polygon (we will not delve in details of toric geometry that are needed to understand this fact, see [51] for a concise, physically oriented overview), encoding all the relevant geometrical data. The (p,q) brane web is then obtained from the Newton polygon simply as the dual graph: to every face we associate a vertex, and to every external edge an orthogonal one. This is illustrated for the case of the basic vertex, which corresponds to the toric geometry of $\mathbb{C}^3$ (viewed as a $T^2 \times \mathbb{R}$ fibration over $\mathbb{R}^3$) of Figure 2.3, and in the example of local $\mathbb{F}_0$ in Figure 2.4, where we show the Newton polygon and the brane web side by side. The above correspondence between toric CY_3 and (p,q) brane webs can be



(A) (p,q) web for local $\mathbb{F}_0$ (B) Newton polygon for local $\mathbb{F}_0$

FIGURE 2.4

derived by using local mirror symmetry [52–54]. The Kähler moduli of the CY_3 are mapped to deformations that preserve the outgoing branes of the web, i.e. the sizes of internal compact directions in the diagram.

The appealing feature about this correspondence is that toric CY_3 can be constructed by gluing local patches isomorphic to $\mathbb{C}^3$ with nontrivial transition functions. This translates in the brane picture to the gluing of trivalent vertices like that in Figure 2.3a. In fact, one can compute the full partition function of the geometrically engineered theory by using the fact that the above setup, of M-theory on a toric $CY_3 \times S^1$, is equivalent to that of Topological String theory on the same CY_3. The gluing picture of the (p,q) web then gets a precise computational meaning, that we will briefly review in Section 3.3.2, after we introduce the concept of nonperturbative gauge theory partition function.

Chapter 3

Instanton Counting and AGT Correspondence

Another important ingredient of our story is given by the exact computation, through the methods of equivariant localization, of nonperturbative partition functions and prepotentials of $\mathcal{N} = 2$ supersymmetric gauge theories [14], and their identification with conformal blocks of two-dimensional Conformal Field Theories. This latter correspondence is called the AGT correspondence, from the authors of the seminal paper [55]. In this chapter we will first recall some essential facts about instanton counting and nonperturbative partition functions, and then review the necessary notions from the AGT correspondence, for the gauge theory with and without the insertion of extended line and surface operators.

3.1 The instanton moduli space

Let us first briefly review the role of instantons in a general four-dimensional nonabelian gauge theory. The starting point is the usual Yang-Mills action, written in terms of the curvature two-form $F = D_A A$:

$$\begin{aligned} S_{YM} &= \frac{1}{2g^2} \int \mathrm{tr}\, F \wedge *F \\ &= \frac{1}{4g^2} \int \mathrm{tr}\, (F \pm *F) \wedge *(F \pm *F) \mp \frac{1}{g^2} \int \mathrm{tr}\, F \wedge F \\ &\geq \mp \frac{1}{g^2} \int \mathrm{tr}\, F \wedge F. \end{aligned} \tag{3.1}$$

The last term is proportional to the second Chern class of the bundle defined by the connection A:

$$-\frac{1}{16\pi^2} \int \mathrm{tr}\, F \wedge F = \int c_2(F) \equiv n. \tag{3.2}$$

n is called the instanton number. Putting the two equations above together, we find

$$S_{YM} \geq \frac{4\pi}{g^2} |n|, \tag{3.3}$$

the bound being saturated by (anti-) self-dual configuration, i.e. configuration satisfying the Bogomol'nyi-Prasad-Sommerfeld (BPS), or (anti-) self-dual Yang-Mills equations

$$F = \pm * F. \tag{3.4}$$

A configuration satisfying the BPS bound with instanton number n (respectively $-n$) are called n-instanton (respectively n anti-instanton) configurations, and they are all inequivalent extrema of the Yang-Mills action. Note that these are very special solutions to the Yang-Mills equations: these latter are second-order nonlinear ODEs, while the BPS equations are first-order.

The solutions of the BPS equations come in families depending on a number of parameters. These parameters constitute the so-called n-instanton moduli space $\mathcal{M}_{G,n}$. There is an algebraic way of describing this moduli space, due to Atiyah, Drinfeld, Hitchin, Manin [56, 57], from which it takes the name ADHM construction. This is important for the actual computation of the partition function through equivariant localization, but we will not enter in such details, as this would make us stray too far away from our main discussion.

At the quantum level, the path-integral decomposes into a sum over inequivalent saddles, each of which with its own perturbative series. Setting $A = A^{inst} + \delta A$, we have

$$Z = \int [dA] e^{-S_{YM}} = \sum_n e^{-8\pi^2|n|} g \int [d^{\dim \mathcal{M}_{G,n}} X] e^{-S_{QM}[X(\tau)]} \int [\delta A] (\dots). \tag{3.5}$$

In particular, we have a series of instanton contributions determined by a quantum mechanical path-integral in terms of the moduli of the instanton configuration (this is known as moduli space approximation), and over each of these configuration we have a corresponding perturbative series.

In the $\mathcal{N} = 2$ theories that we consider here, the (anti-) instanton configurations preserve half the supersymmetry so that, due to non-renormalization theorems, the perturbative series consists only of the tree-level (Z_{cl}) and 1-loop (Z_{pert}) contributions. Further, the moduli space approximation is exact, because the superymmetric quantum mechanical action localizes, so that the partition function takes the form

$$Z = Z_{cl} Z_{pert} Z_{inst}, \tag{3.6}$$

where the instanton contribution is given by a series

$$Z_{inst} = \sum_{n=0}^{\infty} q^n Z_n. \tag{3.7}$$

Here we defined the instanton counting parameter

$$q = e^{2\pi i \tau}, \qquad \tau = \frac{2\pi i}{g^2} + \frac{\theta}{2\pi}. \tag{3.8}$$

In conformal theories, τ is just the UV gauge coupling, while for non-conformal theories the coupling runs, so that q is proportional to a power of the dynamically generated energy scale $\Lambda \sim e^{1/g^2(E)}$, E being the renormalization scale.

What we have said up to now is essentially correct, but we have cheated a little bit: there is an important technical complication, which is the fact that the instanton moduli space $\mathcal{M}_{G,n}$ is a singular space. It has UV singularities due to pointlike instantons, because the Yang-Mills action is classically scale-invariant, so that we can shrink the instantons to points, and IR singularities, due to the fact that $\mathbb{R}^4$ is noncompact, so that we can consider instantons infinitely far apart from each other. To deal with these problems, one needs to regularize the moduli space in some way: it turns out that the UV singularities are resolved by considering a noncommutative deformation of spacetime, so that pointlike instantons do not exist anymore because of the uncertainty principle on the spacetime coordinats, while the IR singularities are dealt with by deforming the theory with the so-called Ω-deformation. The original theory is recovered by sending the deformation parameters to zero [14, 22].

We will not delve into the details of the noncommutative deformation, as they will be not relevant for us, being only a technical point in the derivation of the explicit expressions for Z_{inst}. We will need, however, to recall some basic details on the Ω-deformation, that plays an important role in our story.

3.2 Ω-background and instanton partition functions

To compute the instanton partition functon through equivariant localization[1], one introduces the Omega-background. This can be seen either as a two-parameter deformation of the theory, or of spacetime: both interpretations are correct, since putting a theory on a fixed curved background is the same as deforming it in an appropriate way.

In order to define the Omega-background, fiber the spacetime $\mathbb{R}^4 \approx \mathbb{C}^2$ with coordinates (z, w) over a circle of radius S^1_R, $R = \beta/2\pi$, in such a way that

$$(z, w, 0) \sim \left(e^{i\beta\epsilon_1} z, e^{i\beta\epsilon_2} w, \beta\right). \tag{3.9}$$

The theory is then defined over a five-dimensional space that is the total space of a $\mathbb{C}^2$ fibration over S^1_R. This fibration can also be seen as a five-dimensional supergravity background (the nontrivial bundle is then interpreted as a nontrivial background for the graviphoton in the spin-2 multiplet). In particular, the ϵ_1, ϵ_2 parameters define a $U(1)^2$ isometry of this space, that induces a $U(1)^2$ symmetry acting on our fields. Further, we require finite-energy boundary conditions: the fields must tend to a vacuum configuration at infinity of $\mathbb{R}^4$. We will choose a point in the Coulomb branch, so that this is achieved by setting the gauge field boundary conditions to be

[1] See [58, 59] for detailed, but old fashioned, reviews, or [60] for a more recent review focused on instanton counting. See [61] for a mathematical introduction to the topic. The original computations of four-dimensional instanton partition functions were performed in [14, 19].

pure gauge, while the scalar of the vector multiplet satisfy the boundary conditions

$$\phi \to \mathrm{diag}(a_1, \dots, a_N) \equiv \boldsymbol{a}. \tag{3.10}$$

This asymptotically breaks the gauge group to $U(1)^r$, where $r = \mathrm{rk}\, G$, and gives a further $U(1)^r$ symmetry parametrized by $\boldsymbol{a}$. Finally, if we have N_f matter hypermultiplets, we have also a $U(1)^{N_F}$ subgroup of the flavor symmetry, parametrized by the masses $m_1, \dots m_{N_f}$.

It turns out that after the noncommutative and Omega-deformation, the instanton partition function is an equivariant characteristic class for this $U(1)^{r+N_f+2}$ action. Because of this, by an infinite-dimensional generalization of the Atiyah-Bott theorem, it localizes at the fixed points of this toric action. It was shown in [22] that in the case $G = SU(N)$ or $U(N)$, which are the cases that will be relevant to us, these are labeled by colored partitions, i.e. by N-tuples of Young diagrams

$$\mathbf{Y} \equiv (Y_1, \dots, Y_N), \tag{3.11}$$

of total length equal to the instanton number

$$\sum_i |Y_i| = n. \tag{3.12}$$

We will not reproduce the computation of the instanton partition function, but rather write the final result, for which we need to introduce some notations.

A Young diagram Y can be specified by providing the height λ_i of its columns:

$$Y = (\lambda_1, \dots, \lambda_N). \tag{3.13}$$

The transposed diagram Y^T is specified in the same way by the lengths λ'_i of the rows of Y,

$$Y^T = (\lambda'_1, \dots, \lambda'_N). \tag{3.14}$$

Let $s = (i, j)$ be a box in the Young diagram specified by the row i and the column j. Two important functions that we will use are the arm length $A_Y(s)$ and the leg length $L_Y(s)$, defined by

$$A_Y(s) = \lambda_i - j, \qquad L_Y(s) = \lambda'_j - i. \tag{3.15}$$

In terms of these, the instanton partition function

$$Z_{inst} = \sum_n q^n Z_n \tag{3.16}$$

for gauge group $SU(N)$ or $U(N)$, with N_f hypermultiplets in the fundamental representation and N_A hypermultiplets in the adjoint representation, can be written in such a way that the n-instanton cuntributions are a product of elementary building blocks:

$$Z_n = \sum_{\mathbf{Y},|\mathbf{Y}|=n} Z_{vect}(a,\mathbf{Y}) \prod_{i=1}^{N_f} Z_{fund}(\boldsymbol{a},\mathbf{Y},m_i) \prod_{j=1}^{N_A} Z_{adj}(\boldsymbol{a},\mathbf{Y},\mu_j). \tag{3.17}$$

These factors have the following form:

$$\begin{aligned} Z_{vect}(\boldsymbol{a},\mathbf{Y}) &= \frac{1}{\prod_{i,j=1}^{N} \prod_{s\in Y_i} E(a_{ij},Y_i,Y_j,s) \prod_{t\in Y_j} \left[\epsilon_+ - E(a_{ji},Y_j,Y_i,t)\right]}, \\ Z_{adj}(\boldsymbol{a},\mathbf{Y},m) &= \prod_{i,j=1}^{N} \prod_{s\in Y_i} \left[E(a_{ij},Y_i,Y_j,s) - m\right] \prod_{t\in Y_j} \left[\epsilon_+ - E(a_{ji},Y_j,Y_i,t) - m\right], \\ Z_{fund}(\boldsymbol{a},\mathbf{Y},m) &= \prod_{i=1}^{N} \prod_{s\in Y_i} \left[\phi(a_i,s) - m + \epsilon_+\right], \end{aligned} \tag{3.18}$$

where we defined

$$E(\boldsymbol{a},Y_1,Y_2,s) = a - \epsilon_1 L_{Y_2}(s) + \epsilon_2 \left(A_{Y_1}(s) + 1\right), \tag{3.19}$$

$$a_{ij} = a_i - a_j, \qquad\qquad \epsilon_+ = \epsilon_1 + \epsilon_2, \tag{3.20}$$

$$\phi(a,s) = a + \epsilon_1(i-1) + \epsilon_2(j-1). \tag{3.21}$$

In fact, all these factors, as well as that of an antifundamental hypermultiplet, can be obtained as a special case of a single one, that of the bifundamental hypermultiplet. This is given by

$$\begin{aligned} Z_{bifund}(\boldsymbol{a},\mathbf{Y},\boldsymbol{b},\mathbf{W};m) = \prod_{i=1}^{N}\prod_{j=1}^{M}\prod_{s\in Y_i} &\left[E(a_i - b_j, Y_i, W_j, s) - m\right] \\ &\times \left[\epsilon_+ - E(b_j - a_i, W_j, Y_i, t) - m\right], \end{aligned} \tag{3.22}$$

and the other contributions can be written in terms of Z_{bifund} as

$$Z_{adj}(\boldsymbol{a},\mathbf{Y},m) = Z_{bifund}(\boldsymbol{a},\mathbf{Y},\boldsymbol{a},\mathbf{Y},m), \qquad Z_{vect} = \frac{1}{Z_{adj}(\boldsymbol{a},\mathbf{Y},0)}, \tag{3.23}$$

$$Z_{fund}(\boldsymbol{a},\mathbf{Y},m) = Z_{bifund}(\boldsymbol{a},\mathbf{Y},0,\emptyset,m), \tag{3.24}$$

$$Z_{antifund}(\boldsymbol{a},\mathbf{Y},m) = Z_{bifund}(0,\emptyset,\boldsymbol{a},\mathbf{Y},m). \tag{3.25}$$

3.2.1 Five-dimensional partition functions

Recall that our starting point for the four-dimensional partition functions was actually a five-dimensional setup. The five-dimensional expressions, with finite R are the partition functions for the five-dimensional theories engineered by (p,q) brane webs, and we will write them down in this section.

The instanton partition function for $\mathcal{N}=1$ 5d $SU(N)$ SYM with N_f fundamental flavors and Chern-Simons level $k=0,1,\dots,N-1$ is given by a sum over N-tuples if partitions $\boldsymbol{\lambda}(\lambda_1,\dots,\lambda_N)$ with counting parameter t:

$$Z_{inst}=\sum_{\boldsymbol{\lambda}} t^{|\boldsymbol{\lambda}|} Z^{CS}_{\boldsymbol{\lambda}} Z^{fund}_{\boldsymbol{\lambda}} Z^{gauge}_{\boldsymbol{\lambda}}. \tag{3.26}$$

The Chern-Simons factor given by

$$Z^{CS}_{\boldsymbol{\lambda}}=\prod_{i=1}^{N} T_{\lambda_i}(u_i;q_1,q_2)^k, \qquad T_\lambda(u;q_1,q_2)=\prod_{(i,j)\in\lambda} u q_1^{i-1} q_2^{j-1}, \tag{3.27}$$

while the matter and gauge contributions can all be written in terms of the single building block

$$N_{\lambda,\mu}(u,q_1,q_2)=\prod_{s\in\lambda}\left(1-uq_2^{-a_\mu(s)-1}q_1^{l_\lambda(s)}\right)\prod_{s\in\mu}\left(1-uq_2^{a_\lambda(s)}q_1^{-l_\mu(s)-1}\right), \tag{3.28}$$

in the following way:

$$Z^{fund}_{\boldsymbol{\lambda}}=\prod_{i=1}^{N_f}\prod_{\alpha=1}^{N} N_{\lambda,\varnothing}(Q_i u_\alpha), \tag{3.29}$$

$$Z^{gauge}_{\boldsymbol{\lambda}}=\prod_{i,j=1}^{N}\frac{1}{N_{\lambda_i,\lambda_j}(u_i/u_j;q_1,q_2)}. \tag{3.30}$$

The perturbative contribution is given by the following:

$$Z_{cl}=e^{-\log t\frac{\sum_{i=1}^{N}(\log u_i)^2}{2\log q_1\log q_2}-k\frac{\sum_{i=1}^{N}(\log u_i)^3}{6\log q_1\log q_2}}, \tag{3.31}$$

$$Z_{\text{1-loop}}=\frac{\prod_{1\le\alpha\neq\beta\le N}\left(u_\alpha/u_\beta;q_1,q_2\right)_\infty}{\prod_{i=1}^{N_f}\prod_{\alpha=1}^{N}\left(Q_i u_\alpha;q_1,q_2\right)_\infty}. \tag{3.32}$$

Here $\left(u_i/u_j;q_1,q_2\right)_\infty$ is the multiple q-Pochhammer symbol, defined by

$$\begin{aligned}(z;q_1,\dots,q_n)_\infty &\equiv \prod_{i_1,\dots,i_N=0}^{\infty}\left(1-z\prod_{k=1}^{n} q_k^{i_k}\right)\\ &=\exp\left(-\sum_{m=1}^{\infty}\frac{z^m}{m}\prod_{k=1}^{N}\frac{1}{1-q_k^m}\right).\end{aligned} \tag{3.33}$$

In all the formulae above the following notations are used, in terms of the four-dimensional gauge theory parameters:

$$u_\alpha = e^{\beta a_\alpha}, \qquad Q_i = e^{-\beta m_i}, \qquad q_1 = e^{\beta\epsilon_1}, \qquad q_2 = e^{\beta\epsilon_2}. \tag{3.34}$$

An important property of the double Pochhammer symbol, that we will use repeatedly in Chapter 8, is the following:

$$\frac{(zq;q,q^{-1})_\infty}{(z;q,q^{-1})_\infty} = (zq;q)_\infty, \qquad \frac{(zq^{-1};q,q^{-1})_\infty}{(z;q,q^{-1})_\infty} = \frac{1}{(z;q)_\infty}, \tag{3.35}$$

$$\frac{(zq;q,)_\infty}{(z;q)_\infty} = \frac{1}{1-z}. \tag{3.36}$$

The full partition function $Z(u;t)$ is given by $Z = Z_{cl} Z_{1-\mathrm{loop}} Z_{inst}$.

3.3 The AGT correspondence

Roughly speaking, the AGT correspondence [55] is the identification of instanton partition functions of four-dimensional $\mathcal{N}=2$ theories in the Omega-background with conformal blocks of two-dimensional CFTs. This identification can be obtained by rephrasing the results of section 3.2 on instanton partition functions in the language of Vertex Operator Algebras: introduce the Hilbert space

$$\mathbf{V}_{G,\boldsymbol{a}} = \bigoplus_{n=0}^{\infty} \mathbf{V}_{G,\boldsymbol{a},n}, \qquad \mathbf{V}_{G,\boldsymbol{a},n} \equiv \bigoplus_{p} \mathbb{C}|p\rangle. \tag{3.37}$$

The second sum runs over all the fixed points p of the $U(1)^{r+2}$ action on the instanton moduli space parametrized by $\boldsymbol{a}, \epsilon_1, \epsilon_2$. The collection of the states defined by these fixed points define a basis of the Hilbert space. As we just said, for $U(N)$ and $SU(N)$ gauge groups such fixed points are N-tuples of Young diagrams $\mathbf{Y}$, so that we can write (omitting the ϵ_1, ϵ_2 dependence)

$$\mathbf{V}_{G,\boldsymbol{a},n} \equiv \bigoplus_{\mathbf{Y},|\mathbf{Y}|=n} \mathbb{C}|\mathbf{Y};a\rangle. \tag{3.38}$$

The inner product between these states is defined to be equal to the contribution of the vector multiplet:

$$\langle \mathbf{Y}'; \boldsymbol{a}' | \mathbf{Y}; \boldsymbol{a}\rangle = \delta_{\mathbf{Y},\mathbf{Y}'} \delta_{\boldsymbol{a},\boldsymbol{a}'} \frac{1}{Z_{vect}(\boldsymbol{a})}. \tag{3.39}$$

In this framework, a bifundamental hypermultiplet under the gauge groups G_1, G_2 can be viewed as an intertwining (vertex) operator

$$V_{\boldsymbol{a}',m,\boldsymbol{a}} : \mathbf{V}_{G_1,\boldsymbol{a}} \to \mathbf{V}_{G_2,\boldsymbol{a}'}, \tag{3.40}$$

with matrix elements

$$\langle \mathbf{Y}; \boldsymbol{a} | V_{\boldsymbol{a},m,\boldsymbol{a}'} | \mathbf{Y}'; \boldsymbol{a}' \rangle \equiv Z_{bifund}(\mathbf{Y}, \boldsymbol{a}; \mathbf{Y}', \boldsymbol{a}'). \tag{3.41}$$

The vector space $\mathbf{V}_{G,\boldsymbol{a},n}$ is naturally related to the equivariant cohomology of the instanton moduli space, as follows

$$\mathbf{V}_{g,\boldsymbol{a},n} = H^*_{G\times U(1)^2}(\mathcal{M}_{G,n}) \otimes \mathcal{S}_G. \tag{3.42}$$

The factor $\mathcal{S}_G$ is the so-called quotient field of the equivariant cohomology of a point $H^*_{G\times U(1)^2}(pt)$, but let us ignore it to simplify the following discussion.

We will be concerned only with $G = U(N)$: in this case it was proven [62, 63] that there is a natural W_N algebra action on $\mathbf{V}_{G,\boldsymbol{a}}$[2]. In this case, $\mathbf{V}_{G,\boldsymbol{a}}$ gets identified with a module of $W_N \times \mathfrak{F}$, where $\mathfrak{F}$ is the Fock space of a free boson[3]:

$$\mathbf{V}_{G,\boldsymbol{a}} = \mathcal{V}_{\boldsymbol{a}'} \otimes \mathfrak{F}_k, \tag{3.43}$$

where

$$k = \sum_{i=1}^{N} a_i, \qquad \mathbf{a'} = \mathbf{a} - k\left(\frac{1}{N}, \dots, \frac{1}{N}\right). \tag{3.44}$$

The number operator $\mathbf{N}$ acting on $\mathbf{V}_{G,a}$, that counts the instanton number, is identified with the zero-mode L_0 of the Virasoro subalgebra of W_N, with central charge

$$c = N - 1 + N(N^2-1)\left(\frac{\epsilon_1+\epsilon_2}{\epsilon_1\epsilon_2}\right)^2. \tag{3.45}$$

The creation operators L_{-m} are instead raising operators for the instanton number, and create descendants by acting on the highest state $|\boldsymbol{a}\rangle$ of $\mathcal{V}_a$, with weight

$$\Delta_a = \mathbf{a} \cdot (\mathbf{Q} - a). \tag{3.46}$$

$\mathbf{Q}$ is the background charge in the free-field realization of the W_N algebra, and is given by

$$\mathbf{Q} = \left(b + \frac{1}{b}\right)\left(\frac{N}{2}, \frac{N}{2} - 1, \dots, -\frac{N}{2}, -\frac{N}{2}\right), \qquad b^2 = \frac{\epsilon_1}{\epsilon_2}. \tag{3.47}$$

The vertex operators $V_{a',m,a}$ are primary fields of W_N. In general, instanton partition functions are identified with conformal blocks of W_N, up to an extra $U(1)$ factor due to the presence of the Fock space $\mathfrak{F}_m$; in fact, it is possible to identify the full partition function with the conformal block, by properly normalizing the states and the vertex operators, so as to include the perturbative

[2] W_N algebras, first introduced by Zamolodchikov [64], are infinite-dimensional algebras with generators up to spin N [65–67]. They are a higher-spin generalization of the Virasoro algebra, which is the particular case W_2, generated by the energy-momentum tensor $T(z)$ of spin 2. See below for more techinical details in the case of Virasoro central charge c=N-1.

[3] This identification has been proven rigorously only in the case of $SU(2)$ [68, 69], while for the general case of $SU(N)$ it has not been proven, but only thoroughly checked [70–72].

contribution to the partition function.

The identification of the W_N conformal block with the gauge theory partition function holds only for W_N conformal blocks where the vertex operators are so-called "semi-degenerate" fields, which are vertex operators with charge defined as

$$\boldsymbol{\theta} = \theta \boldsymbol{h}_1, \tag{3.48}$$

where $\boldsymbol{h}_1$ is the highest weight of the first fundamental representation of $\mathfrak{sl}_N$.

In order not to delve into technicalities we do not need, we will here describe these operators in the case of Virasoro central charge

$$c = N - 1, \tag{3.49}$$

corresponding to the self-dual Omega-background in terms of gauge theory. In the following we will use fundamental weights plus zero vector given by

$$\begin{aligned}
\boldsymbol{\omega}_0 &= (0, 0, 0, \dots, 0)\,, \\
\boldsymbol{\omega}_1 &= (\frac{N-1}{N}, \frac{-1}{N}, \frac{-1}{N}, \dots, \frac{-1}{N})\,, \\
\boldsymbol{\omega}_2 &= (\frac{N-2}{N}, \frac{N-2}{N}, \frac{-2}{N}, \dots, \frac{-2}{N})\,, \qquad (3.50)\\
&\vdots \\
\boldsymbol{\omega}_{N-1} &= (\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \dots, \frac{1-N}{N})\,,
\end{aligned} \tag{3.51}$$

to be distinguished from the weights of the first fundamental representation of $\mathfrak{sl}_N$:

$$\begin{aligned}
\boldsymbol{h}_1 &= (\frac{N-1}{N}, \frac{-1}{N}, \frac{-1}{N}, \dots, \frac{-1}{N})\,, \\
\boldsymbol{h}_2 &= (\frac{-1}{N}, \frac{N-1}{N}, \frac{-1}{N}, \dots, \frac{-1}{N})\,, \\
&\vdots \\
\boldsymbol{h}_N &= (\frac{-1}{N}, \frac{-1}{N}, \frac{-1}{N}, \dots, \frac{N-1}{N})\,.
\end{aligned} \tag{3.52}$$

A W_N algebra can be embedded in a $\widehat{\mathfrak{sl}}_N$ algebra, and in fact a W_N CFT can be represented as a constrained WZNW model through the so-called quantum Drinfeld-Sokolov reduction [73, 74]. In particular, for $c = N - 1$ there is a realization of the W_N algebra in terms of free bosons φ_k, subject to the relation

$$\sum_{k=1}^{N} \varphi_k = 0. \tag{3.53}$$

The W_N algebra generators are defined in terms of the $U(1)$ currents generated by these free bosons:

$$J_k = i\partial\varphi_k, \qquad W^{(j)} = \sum_{1\leq i_1 \leq N} :J_{i_1}\dots J_{i_j}:, \tag{3.54}$$

where $j = 2,\dots N$. In particular, note that $W^{(2)}$ is the Sugawara energy-momentum tensor associated to the current algebra.

Analogously to the case of Virasoro, where we can find a basis of the Verma module $\mathcal{V}_{\boldsymbol{\theta}}$ labeled by partitions, in the W_N case we can find a basis labeled by $N-1$-tuples of partitions $\lambda^{(j)} = (\lambda_1^{(j)},\dots,\lambda_k^{(j)})$, given by

$$|\boldsymbol{\lambda},\boldsymbol{\theta}\rangle \equiv W^{(N)}_{-\lambda^{(N)}}\dots W^{(2)}_{-\lambda^{(2)}}|\boldsymbol{\theta}\rangle \equiv W_{\boldsymbol{\lambda}}|\boldsymbol{\theta}\rangle, \tag{3.55}$$

where $W^{(j)}_{-\lambda}$ represents the product of W_N generators

$$W^{(j)}_{-\lambda} = W^{(j)}_{-\lambda_1}\dots W^{(j)}_{-\lambda_k}, \tag{3.56}$$

where $k = |\lambda|$, the length of the partition. However, differently from the $N = 2$ case, a generic matrix element of descendants operators cannot be written solely in terms of primary matrix elements by using the conformal Ward identities. One class of fields for which this is possible is that of quasi-degenerate fields, for which the conformal weight $\boldsymbol{\theta}$ is proportional to the weight of the first fundamental representation of $\mathfrak{sl}_N$: $\boldsymbol{\theta} = \nu\boldsymbol{\omega}_1$.

The Verma module defined by this highest weight state has $N-2$ null-state decoupling equations, that allow the matrix elements of $V_{\nu\boldsymbol{\omega}_1}$ and its descendants to be expressed in terms of its primary matrix elements

$$\langle\boldsymbol{\theta}'|V_{\nu\boldsymbol{\omega}_1}|\boldsymbol{\theta}\rangle \equiv \mathcal{N}(\boldsymbol{\theta},\nu\boldsymbol{\omega}_1,\boldsymbol{\theta})z^{\Delta_{\boldsymbol{\theta}'}-\Delta_{\nu\omega_1}-\Delta_{\boldsymbol{\theta}}}, \tag{3.57}$$

where

$$\Delta_{\boldsymbol{\theta}} = -e_2(\boldsymbol{\theta}) = \frac{\boldsymbol{\theta}^2}{2}, \tag{3.58}$$

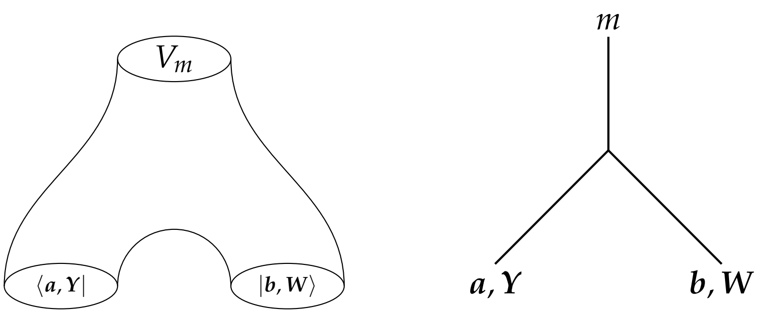
e_2 being the second elementary symmetric polynomial in $\theta_1,\dots,\theta_N$. Differently from what happened in the case of the Virasoro algebra, where the gauge theory partition function was given by conformal blocks of Virasoro primary fields, here the partition function is given by conformal blocks including two W_N primaries with arbitrary weights, while the rest are semi-primary fields.

3.3.1 Pants decompositions and conformal blocks

We now illustrate a convenient technique to relate partition function of class S theories to conformal blocks, based pants decompositions of a Riemann surface. Consider the most basic Virasoro conformal block, a three-point function of a vertex operator with primary fields or descendants. By the AGT correspondence, this is the bifundamental contribution of the instanton parition function:

$$Z_{bifund}(\boldsymbol{a}, \boldsymbol{Y}, \boldsymbol{b}, \boldsymbol{W}, m) = \langle \boldsymbol{a}, \boldsymbol{Y} | V_m(1) | \boldsymbol{b}, W \rangle. \tag{3.59}$$

By the usual identification of zero and infinity with the in-out states in radial quantization, we can associate this object with a three-punctured sphere $C_{0,3}$. The punctures are the insertion of the states at $0, 1, \infty$, which we have fixed using global conformal symmetry. By shrinking this Riemann Surface onto its maximal degeneration diagram, we get the usual "stick figure" representation of conformal blocks: we reproduce $C_{0,3}$ and its associated diagram in Figure 3.1.



(A) Trinion

(B) Three-point sphere conformal block

FIGURE 3.1

It is a well-known property of any CFT that the computation any correlation function can be reduced to sum of products of three-point functions by repeated use of the OPE. In the operator formalism we are using, this statement means simply we can insert the identity

$$\mathbb{I} = \sum_{\mathbf{Y}} q^{L_0} |\boldsymbol{a}, \boldsymbol{Y}\rangle \langle \boldsymbol{a}, \boldsymbol{Y}| = \sum_{\mathbf{Y}} q^{(a+|\mathbf{Y}|)^2} |\boldsymbol{a}, \boldsymbol{Y}\rangle \langle \boldsymbol{a}, \boldsymbol{Y}| \tag{3.60}$$

in a conformal block as many times as we want. The meaning of q is the following: the above formula corresponds to attaching two punctures Q_1, Q_2 with insertion of states $|\boldsymbol{a}, \boldsymbol{Y}\rangle$, and to do this, one uses the so-called "gluing construction" [75–77], that we now briefly review.

Let $q \in \mathbb{C}$, with $|q| < 1$, and choose two local coordinates z_1, z_2 such that $z_i(Q_i) = 0$, and pick a neighborhood

$$D_i = \left\{ P_i : |z_i(P_i)| < |q|^{-1/2} \right\}. \tag{3.61}$$

We want to glue together the annuli

$$A_i = \left\{ P_i : |q|^{1/2} < |z_i(P_i)| < |q^{-1/2}| \right\}. \tag{3.62}$$

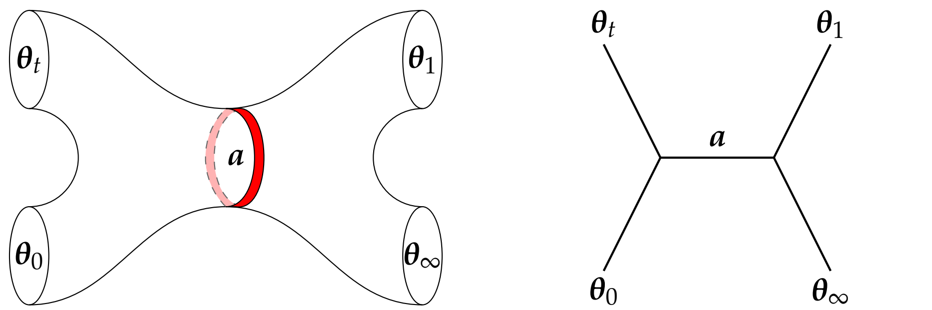
To do this, one identifies two points P_1, P_2, lying on A_1, A_2 respectively, if they satisfy

$$z_1(P_1)z_2(P_2) = q. \tag{3.63}$$

Let us see how this works in practice with an example. Consider the four-point conformal block of primaries on the sphere, this reads as

$$\langle \boldsymbol{\theta}_\infty | V_{\boldsymbol{\theta}_1}(1) V_{\boldsymbol{\theta}_t}(t) | \boldsymbol{\theta}_0 \rangle = \sum_{\mathbf{Y}} q^{(a+|\mathbf{Y}|^2)} \langle \boldsymbol{\theta}_\infty | V_{\boldsymbol{\theta}_1}(1) | \boldsymbol{a}, \boldsymbol{Y} \rangle \langle \boldsymbol{a}, \boldsymbol{Y} | V_{\boldsymbol{\theta}_t}(t) | \boldsymbol{\theta}_0 \rangle, \tag{3.64}$$

where $q = t(1-t)$, and we have assumed $0 < |t| < 1$. Basically we have to sum over all the possible descendants in the intermediate channel between t and 1. We can represent this geometrically following the ideas we outlined above for the three-point function. The sum over the intermediate channels represents the gluing of two trinions along a puncture, as depicted in Figure 3.2.



(A) Pants decomposition of four-punctured sphere

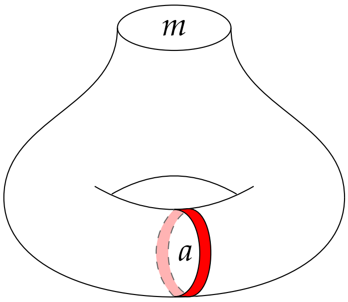
(B) Four-point sphere conformal block (s-channel)

FIGURE 3.2

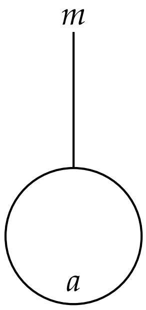
What is the meaning of this in the four-dimensional field theory? In the class S construction [4], the operation of gluing three-point spheres has the effect of gauging the flavor symmetries at the punctures that are being glued. This geometric construction gives us a convenient "recipe" to read off the four-dimensional gauge theory from the six-dimensional A_1 compactification: a particular pants decomposition corresponds to a duality frame in which there are some weakly coupled gauge groups given by the internal legs. The gluing parameter is then related to the gauge coupling in the usual way:

$$q = e^{2\pi i \tau}, \tag{3.65}$$

while the data $\boldsymbol{a}$ at the punctures that are being glued (that must lie in the same coadjoint orbit) will give the vev of the scalar in the vector multiplet. The external legs are instead the matter hypermultiplets. Thus, for example, the sphere with four punctures corresponds to $\mathcal{N} = 2$ $SU(2)$ super Yang-Mills with $N_f = 4$ flavors. Another example that will be very important for us is the torus with one puncture. It can be obtained by gluing two external legs of the same three-punctured sphere, as in Figure 3.3. In this case,



(A) Pants decomposition of one-punctured torus



(B) One-point torus Conformal Block

FIGURE 3.3

the gluing prescription takes the usual form of the prescription for the torus correlators as an operator trace:

$$\langle V_m(0)\rangle = \mathrm{tr}\, q^{L_0} V_m(0) = \sum_{\mathbf{Y}} q^{(a+|\mathbf{Y}|)^2} \langle \boldsymbol{a}, \mathbf{Y}| V_m(0) |\boldsymbol{a}, \mathbf{Y}\rangle. \tag{3.66}$$

The corresponding gauge theory is $SU(2)$ super Yang-Mills with an adjoint hypermultiplet, also called $\mathcal{N} = 2^*$, that will be our main object of study in Chapter 5.

The discussion above generalizes to linear and circular quiver gauge theories: we already argued in Chapter 2, studying Witten's brane construction, that the A_1 six-dimensional theory compactified on a Riemann sphere with puncture produces a linear quiver gauge theory. This appears evident from the above prescription on pants decomposition: for example, the compactification on the Riemann Surface $C_{0,5}$, in the pants decomposition of Figure 3.4, produces a linear quiver gauge theory with two $SU(2)$ groups, a bifundamental hypermultiplet coupled to both, and two fundamental hypermultiplets coupled to each.

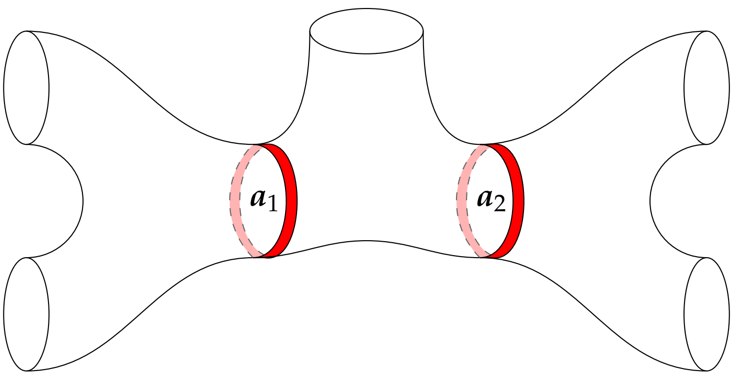


FIGURE 3.4: A pants decomposition for the five-punctured sphere

Extending further this new perspective, we can infer easily what will be the field theory given any Riemann Surface, something which was not immediately obvious from the original brane construction. For example, the Riemann Surface $C_{1,2}$, with pants decomposition shown in Figure 3.5, will produce a four-dimensional theory with two $SU(2)$ gauge groups, and two bifundamental hypermultiplets under both of them. Theories of this type are called circular quiver gauge theories, and they will be our main topic in Chapter 6.

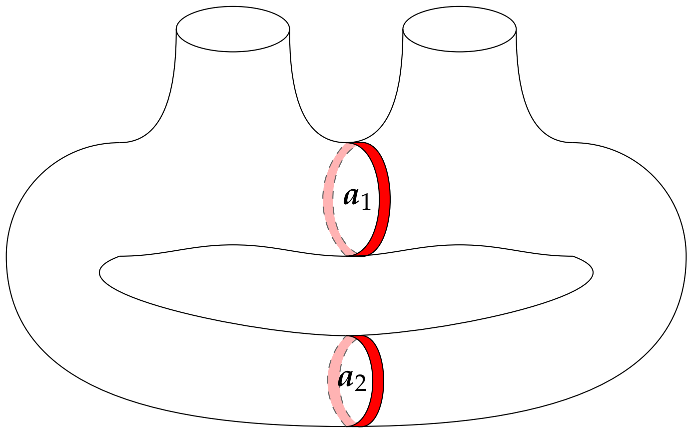


FIGURE 3.5: A pants decomposition for the two-punctured torus

More generally, one can build complicated $SU(2)$ quiver gauge theories by compactifying the six-dimensional A_1 SCFT on a punctured Riemann Surface $C_{g,n}$. It has been shown through geometrical arguments [77] that the partition function of these theories in the Omega background is given by Virasoro conformal blocks on the higher genus punctured Riemann Surface: however, in this case we do not know a general expression for the partition function, or equivalently for the conformal block, in terms of Nekrasov factors. This is because in the pants decomposition we necessarily have trinions with all legs internal to the diagram (an example of this, in the case of a compact genus 2 Riemann Surface, is given in Figure 3.6. The factor for an internal trinion would be given in the CFT by summing over matrix elements with all descendant states, for which we do not have an expression. It is however possible to compute these conformal blocks by using recurrence relations [78, 79] or, in certain cases, some group-theoretic arguments on Nekrasov partition functions [80].

Until now, our discussion on gluing and gauging has dealt only with $SU(2)$ theories, whose partition functions are given by Virasoro conformal blocks, and for which the AGT correspondence has been proven in the general case. As we already mentioned, the higher rank $SU(N)$ case, however, is much less understood. The discussion regarding the pants decomposition is of course the same, apart from the fact that now the trinions can have many types of data attached. The building block which is understood, and that gives Nekrasov partition functions when composed under gluing, is now the trinion with semi-degenerate data $\boldsymbol{\theta} = \theta \boldsymbol{h}_1$ at the punctures.

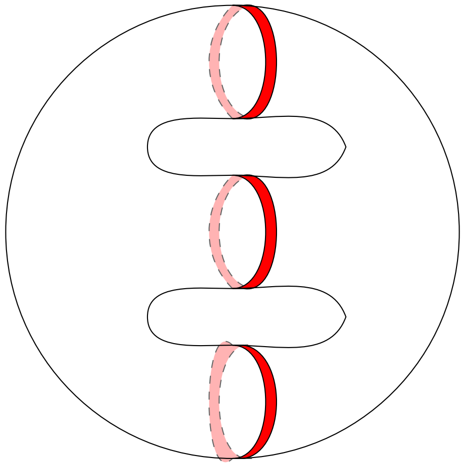


FIGURE 3.6: A pants decomposition for the genus two curve

3.3.2 Topological Vertex, five-dimensional gauge theory and $q - W_N$ conformal blocks

As we saw in Section 2.5 the gauge theories we are studying can be engineered by type IIB string theory on (p, q) brane webs, or equivalently by Topological String Theory on the toric Calabi-Yau geometry mirror to such a configuration. This setup allows to extract more quantitative information, through the formalism of the (refined) Topological Vertex [24, 81]. This in turn connects five-dimensional Nekrasov partition functions, of the $\mathcal{N} = 1$ gauge theory on Omega-deformed $\mathbb{R}^4 \times S^1$ [22, 23, 82], to conformal blocks of q-deformed conformal algebras [83–87]. We will outline these aspects rather briefly, since most of this thesis is devoted to the study of four-dimensional aspects.

Recall that the (p, q) web diagram of the brane configuration engineering a five-dimensional gauge theory can be constructed by gluing together trivalent vertices. In the Topological Vertex approach one associates to each leg a Young diagram, and regards the internal legs as propagators. The partition function of (refined) Topological String Theory on this geometry can be computed in a similar fashion to the Feynman diagram method for scattering amplitudes: multiply by an appropriate factor $V(Y_1, Y_2, Y_3)$ for every vertex in the diagram and multiply by an appropriate propagator factor $\Delta(Y)$ involving a sum over all Young Diagrams for the internal edges. The explicit expressions involving the topological vertex can become quite involved, and since we will not need them directly we will avoid writing them down. Instead, we note here that the partition function computed by the Topological Vertex is a formal power series in all its parameters. In certain regions of the moduli space it can be resummed, and in the weakly-coupled region for the engineered gauge theory it can be seen to coincide with the Nekrasov partition function of the five-dimensional gauge theory that we wrote down in Section 3.2.1. Physically, this is because the logarithm of Nekrasov partition function is the prepotential of the gauge theory in the supergravity graviphoton background, which however also coincide with the free energy of refined Topological Strings [88–92].

The Topological Vertex setup can be formalized in the language of VOA as we have done in the previous section for four-dimensional Nekrasov partition functions. This generalizes to this case the AGT correspondence in the following way, consider the basis $|Y\rangle$ of a Hilbert space $\tilde{V}_1$, labeled by Young diagrams. The inner product is the Topological Vertex propagator

$$\langle Y|Y'\rangle = \Delta(Y)\delta_{Y,Y'}. \tag{3.67}$$

This space has a natural action of a Ding-Yohara-Mihi (DIM) algebra, for which the topological vertex is an intertwiner [93]. When computing the partition functions associated with $SU(N)$ gauge theories, one deals with configurations of N parallel D4-branes, leading to the space

$$\tilde{V}_1^{\otimes N} \simeq \tilde{V}_{U(N)}, \tag{3.68}$$

that has a q-W_N action in the same way as the space $V_{G,\boldsymbol{a},n}$ had a W_N action in the four-dimensional case. The five-dimensional version of the AGT correspondence is then the identification of five-dimensional gauge theory partition functions with q-deformed W_N conformal blocks.

3.4 Surface Operators

A surface operator in a QFT is an extended codimension 2 defect. To define such an operator, consider a $U(N)$ gauge field on $\mathbb{R}^4 \simeq \mathbb{C}^2$, put the surface operator at $z_2 = 0$ and write $z_2 = re^{i\theta}$. The effect of the surface operator is to create a singularity at $r \simeq 0$ so that

$$A \sim \mathrm{diag}(\alpha_1, \ldots, \alpha_N) i d\theta. \tag{3.69}$$

We call a *full surface operator* one such that the α_i are all distinct, i.e. s.t. the subgroup commuting with the divergent part is $U(1)^N$.

For consistency, the unboken gauge group at $z_2 = 0$ must be $U(1)^N$. For every $U(1)$ gauge field, defining a line bundle of the surface operator, we can associate the Chern classes

$$l_i = \frac{1}{2\pi}\int_{z_2=0} F_i, \tag{3.70}$$

or in vector notation, $\boldsymbol{l} \in \mathbb{Z}^N$

$$\boldsymbol{l} = \frac{1}{2\pi}\int_{z_2=0} \mathbf{F}. \tag{3.71}$$

Because of the presence of nontrivial line bundles, the definition of instanton number is now modified to

$$\frac{1}{8\pi^2}\int_{\mathbb{C}^2\setminus\{z_2=0\}} \mathrm{tr}\, F_A \wedge F_A = k + \frac{1}{2}\boldsymbol{\alpha}\cdot\boldsymbol{l}. \tag{3.72}$$

Instead of parametrizing the moduli space of instantons in the presence of a surface operator by $k, \boldsymbol{l}$ separately, it turns out to be convenient to recur to the vector

$$\mathbf{k} = (k_1, \ldots, k_N) = (k, k+l_1, k+l_1+l_2, \ldots, k+l_1+\cdots+l_{N-1}) \tag{3.73}$$

If we think of the theory $\mathcal{S}(C_{g,n}, D)$ on a four-manifold X_4 as realized from a stack of $M5$-branes wrapping the Riemann surface $C_{g,n}$, the UV theory can produce extended objects for the IR field theory in various ways. Specifically, surface operators in these theories have two different M-theory realizations [94, 95]:

1. Intersection with another set of M5 branes, wrapping $C_{g,n}$;
2. M2-branes with 2-dimensional support on X_4 and pointlike support z on C;

In general this different (UV) M-theoretic origin of the surface operators will translate into different (IR) gauge-theoretic properties. However, in the case of $SU(2)$ gauge theory, the two UV construction lead to the same gauge theory defect [95]. More generally, surface operators given by M5 branes yield full surface defects in the four-dimensional theory, while those realized by M2 branes give minimal defects, which are the same for $SU(2)$ since any element of the Cartan has the form $\boldsymbol{\theta} = \theta\sigma_3$.

Because the M2-branes intersect the Riemann Surface $C_{g,n}$ at one point z, the CFT representation for the minimal surface defects is given by an insertion of a local (degenerate) field; on the other hand, because the M5 branes are wrapping the Riemann surface, the full surface defects modify the two-dimensional theory itself, and the four-dimensional gauge theory partition function in the presence of such defects is given by changing the two-dimensional symmetry algebra from W_N to $\widehat{\mathfrak{sl}}_N$. The infrared duality between the two kinds of defects corresponds to an isomonorphism between Virasoro and Kac-Moody conformal blocks, first discovered in [96, 97].A detailed description of instanton counting in the presence of such defects is beyond our scope, and is developed in [98–101]. We will briefly outline the CFT description of these partition functions.

3.4.1 M5-brane defects and Kac-Moody Conformal Blocks

The partition function $\widehat{Z}$ in the presence of a codimension two surface defect can be expressed in such a way as to make apparent the connection with 2d CFT. To study the gauge theory partition function in the presence of such a defect, one can proceed similarly as what we did for the "vanilla" partition function: we introduce a space $\hat{\mathcal{H}}_{\boldsymbol{a}}$ with inner product

$$\langle \boldsymbol{a}, \boldsymbol{\lambda} | \boldsymbol{a}, \boldsymbol{\lambda}' \rangle = \frac{\delta_{\lambda\lambda'}}{\hat{z}_{vect}(\boldsymbol{a}, \boldsymbol{\lambda})}, \tag{3.74}$$

and analogously for the intertwining operators. However, now we have N counting operators because of the additional relevant Chern classes:

$$\mathbf{K}_i|\boldsymbol{\lambda}, \boldsymbol{a}\rangle = k_i(\lambda)|\boldsymbol{\lambda}, \boldsymbol{a}\rangle. \tag{3.75}$$

It has been proven [98, 99] that the space $\widehat{\mathcal{H}}$ arries an action of the Kac-Moody algebra $\widehat{\mathfrak{sl}}(N)_k$, with level determined by the Omega-background parameters as

$$k = -N - \frac{\epsilon_2}{\epsilon_1}. \tag{3.76}$$

Again, the space decomposes into

$$\hat{\mathcal{H}}_{\boldsymbol{a}} = \mathbb{V}_{\frac{a}{\epsilon_1} - \rho} \otimes \mathfrak{F}, \tag{3.77}$$

where $\mathbb{V}_j$ is the Kac Moody Verma module of highest weight j. $\sum k_i$ is the Virasoro number operator $\mathbf{N}$, and k_i measures the weight in the i-th fundamental representation. Because of this, the partition function in the presence of a full surface defect is related to a Kac-Moody, rather than W_N, conformal block.

3.4.2 M2-brane defects, loop operators and degenerate fields

The partition function in the presence of a codimension-4 surface operator, given in M-theory by an M2-brane with world-volume in our spacetime four-manifold, localized at a point $z \in C$, is given by a conformal block with a degenerate field insertion.

We will first explain the situation in the case of an $SU(2)$ gauge theory, so that the conformal blocks are Virasoro conformal blocks. The field whose insertion gives the partition function in the presence of the defect is the degenerate field $\phi_{2,1}$ at level two, defined by the null-vector equation

$$\left(L_{-1}^2 + b^2 L_{-2}\right) \phi_{2,1} = 0, \tag{3.78}$$

where

$$b^2 = \frac{\epsilon_1}{\epsilon_2} \tag{3.79}$$

as before. Note that while this null-vector decouples from correlation functions of minimal models, this is not true for the conformal blocks, which are just representation-theoretic objects. In particular, the degenerate field is a Virasoro primary with Liouville charge $-b/2$. If the partition function of the four-dimensional gauge theory is

$$Z_{4d} = \langle \mathcal{O} \rangle \tag{3.80}$$

for an appropriate string $\mathcal{O}$ of vertex operators, the partition function in the presence of the defect will be simply given by

$$\widehat{Z}_{4d} = \langle \phi_{2,1}(z) \mathcal{O} \rangle. \tag{3.81}$$

In the case of W_N algebra, the picture is essentially the same, but now one has to use W_N *fully degenerate* fields [102]. Because these will be important in our discussions, let us recall now their definition.

3.5 Fusion, braiding and loop operators

An important property of degenerate fields is their fusion rule with Virasoro Verma modules [103]. As we did before, we will discuss in detail the $SU(2)$ case, mentioning at the end the higher rank generalization.

Denote by $[\phi_{2,1}]$ the Verma module generated by $\phi_{2,1}$, and by $[V_a]$ the Verma module of highest weight Δ_a, whose primary field we represent by a vertex operator V_a. Then the fusion rule is

$$[\phi_{2,1}] \times [V_a] = [V_{a-b/2}] + [V_{a+b/2}]. \tag{3.82}$$

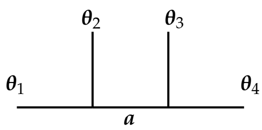
In fact, this property allows the CFT computation of another gauge theory quantity, namely expectation values of supersymmetric loop operators, which can be electric, magnetic or dyonic, in terms of CFT objects called Verlinde loop operators.

3.5.1 Fusion algebra of degenerate fields

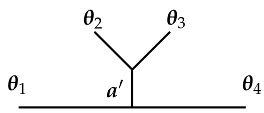
To understand this, we have to note first that, even though a CFT correlator is a single-valued object, a conformal block is generically multivalued: the chiral (holomorphic) and antichiral (antiholomorphic) conformal blocks have to be combined in an appropriate way so that the correlation function is single-valued. The multivaluedness can be understood in the following way: consider a conformal block defined through radial quantization on a sphere. Each vertex operator insertion insertion at a point z_k introduces a branch cut on a circle $|z| = |z_k|$. Then, if we have an operator insertion $\mathcal{O}(z)$ at some point z we can study how the conformal block is analytically continued from the region $|z| < |z_k|$ to the region $|z| > |z_k|$. This operation of analytic continuation corresponds to finding the relation between conformal blocks with different time ordering, or in terms of operators betweeen the matrix elements

$$\langle 0 | \ldots \mathcal{O}(z) V(z_k) \ldots | 0 \rangle, \qquad \langle 0 | \ldots V(z_k) \mathcal{O}(z) \ldots | 0 \rangle, \tag{3.83}$$

where ... denotes possible other operator insertions. It was shown in [104] (see also [94, 105, 106] for more gauge-theory oriented reviews) that all such operations can be encoded in elementary local "moves" on conformal blocks. These are local in the sense that, given a pants decomposition as in Section 3.3.1, we need only to focus on a subgraph of the conformal block given by a four- or three-punctured sphere [4]. The first transformation, called fusion, is the linear transformation between the sphere 4-point conformal block in s-channel and t-channel:

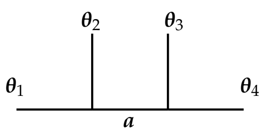


$$= \sum_a F_{aa'} \begin{bmatrix} \theta_2 & \theta_3 \\ \theta_1 & \theta_4 \end{bmatrix}$$

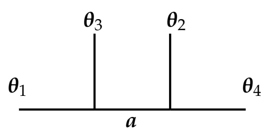


. (3.84)

It is given by a matrix $F_{aa'}$, depending on all the weights of the vertex operator insertion, as well as on the intermediate charges. These latter determine also the number of entries of the matrix, equal to the possible fields that can appear in the t-channel leg. The second transformation is the braiding of two legs in the four-point conformal block:

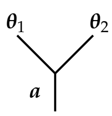


$$= \sum_a B^{(\epsilon)}_{aa'} \begin{bmatrix} \theta_2 & \theta_3 \\ \theta_1 & \theta_4 \end{bmatrix}$$

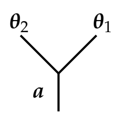


. (3.85)

This depends on another parameter ϵ, which amounts to choosing whether we are braiding clockwise or counterclockwise. When we omit such dependence, the counterclockwise ($\epsilon = +$) orientation is always implied. Finally, we have the "flip" operation, that can be seen as a particular case of the braiding move, when one of the external legs is the identity:



$$= \Omega(\epsilon)_a^{\theta_1,\theta_2}$$



. (3.86)

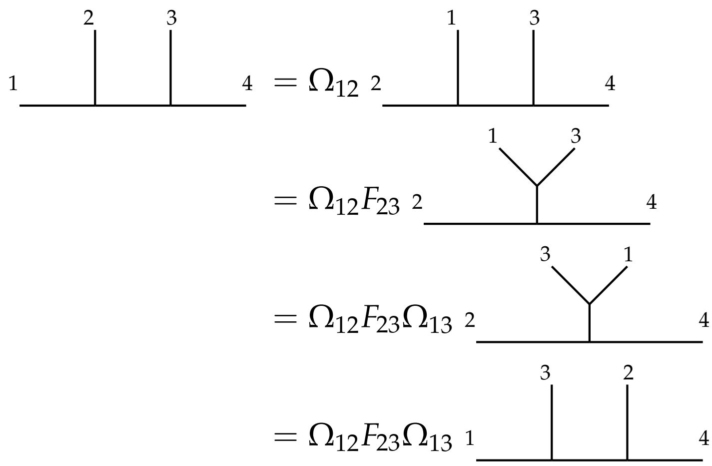
For generic values of the conformal weights, among these only the flip factor Ω is known, which is

$$\Omega(\epsilon)_a^{\theta_1,\theta_2} = e^{i\pi\epsilon\left(\Delta_{\theta_1}+\Delta_{\theta_2}-\Delta_a\right)}. \tag{3.87}$$

These three transformations are not independent, because we can obtain the fusion move as composition of the braiding move and flip moves (or vice

[4] In fact, the Seiberg-Moore groupoid, consisting of all such transformation of conformal blocks, is generated by the transformations F, Ω above, together with the S-duality transformation between the torus blocks with modular parameter τ and $1/\tau$ respectively. We will not use these latter.

versa). This happens as follows (we omit the indices to avoid excessive cluttering):

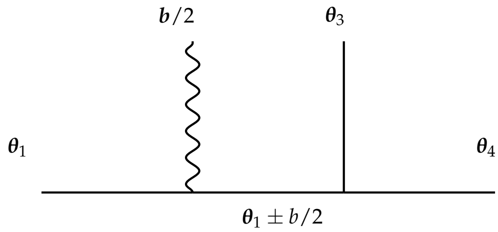


(3.88)

which means that

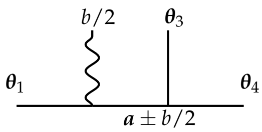
$$F_{aa'}\begin{bmatrix}\theta_2 & \theta_3\\ \theta_1 & \theta_4\end{bmatrix} = e^{-i\pi\epsilon\left(\Delta_{\theta_1}+\Delta_{\theta_3}-\Delta_a-\Delta_{a'}\right)}B^{(\epsilon)}_{aa'}\begin{bmatrix}\theta_2 & \theta_4\\ \theta_1 & \theta_3\end{bmatrix}. \tag{3.89}$$

Although, as we said, the braiding/fusion matrices are not known for generic weights of the external legs, we can write explicitly the braiding matrix B when one of the external fields is a degenerate field $\phi_{2,1}$. In this case, because of equation (3.82), the intermediate weight of the four-point conformal block can be only shifts of $\pm b/2$. It is then convenient to introduce the



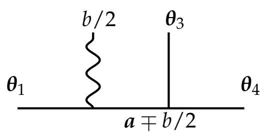
following notation:

$\langle 1|\phi_{\pm}(z)V_3|4\rangle =$



(3.90)

$\langle 1|\tilde{\phi}_{\pm}(z)V_3|4\rangle =$



. (3.91)

The insertion of ϕ_s, $s = \pm$ in a conformal block then is a 2-vector with elements given by the two fusion channels of $\phi_{(2,1)}$. This vector is in the fundamental representation for the $SL(2,\mathbb{C})$ action given by the Seiberg-Moore groupoid. Finally, and importantly, this action is explicit, since in this case the braiding matrix is known to be

$$B(a',m,a) \equiv B^{(+)} \begin{bmatrix} \theta & b/2 \\ a' & a \end{bmatrix} = \begin{pmatrix} \frac{\cos\pi(\theta+a+a')}{\sin 2\pi a} & \frac{\cos\pi(\theta+a'-a)}{\sin 2\pi a} \\ -\frac{\cos\pi(\theta+a-a')}{\sin 2\pi a} & -\frac{\cos\pi(\theta-a-a')}{\sin 2\pi a} \end{pmatrix}. \quad (3.92)$$

Another important ingredient in the next sections will be the OPE of the degenerate fields $\phi, \tilde{\phi}$. This is given by

$$\tilde{\phi}_s(w)\phi_{s'}(z) \sim \frac{\delta_{s,s'}}{(w-z)^{1/2}} + J_{ss'}(z) + (w-z)\left[\delta_{ss'}T(z) + \partial_z J_{ss'}(z)\right]. \quad (3.93)$$

Here $J_{ss'}(z)$ is the level-1 descendant, while $T(z)$ is the energy-momentum tensor that generates the Virasoro algebra through its OPE.

The higher rank generalization goes along the same lines, but studies analytic continuation of fully degenerate fields, for which $\boldsymbol{\theta} = \boldsymbol{h}_1 = \boldsymbol{\omega}_1$ (first fundamental representation of $\mathfrak{sl}_N$) or $\boldsymbol{\theta} = -\boldsymbol{h}_N = \boldsymbol{\omega}_{N-1}$ (last fundamental representation of $\mathfrak{sl}_N$). In this case there are additional null states that imply further constraints in order for the $\mathcal{N}$'s to be nonvanishing. The fusion of a completely degenerate field with a primary state is

$$V_{\boldsymbol{h}_1}|\boldsymbol{\theta}\rangle = \sum_{s=1}^{N} \mathcal{N}(\boldsymbol{\theta}+\boldsymbol{h}_s, \boldsymbol{h}_1, \boldsymbol{\theta}) z^{\Delta_{\boldsymbol{\theta}+\boldsymbol{h}_s} - \Delta_{\boldsymbol{h}_1} - \Delta_{\boldsymbol{\theta}}} |\boldsymbol{\theta}+\boldsymbol{h}_s\rangle. \quad (3.94)$$

It turns out to be convenient to restrict the completely degenerate field to a specific fusion channel by using projectors $\mathcal{P}_{\boldsymbol{\theta}}$:

$$\phi_{s,\boldsymbol{\theta}} \equiv \mathcal{P}_{\boldsymbol{\theta}+\boldsymbol{h}_s} V_{\boldsymbol{h}_1} \mathcal{P}_{\boldsymbol{\theta}}, \qquad \bar{\phi}_{s,\boldsymbol{\theta}} \equiv \mathcal{P}_{\boldsymbol{\theta}-\boldsymbol{h}_s} V_{-\boldsymbol{h}_N} \mathcal{P}_{\boldsymbol{\theta}}, \quad (3.95)$$

where $s = 1, \dots N$. These "reduced" fields have just one fusion channel:

$$\phi_{s,\boldsymbol{\theta}}|\boldsymbol{\theta}\rangle = \mathcal{N}(\boldsymbol{\theta}+\boldsymbol{h}_s, \boldsymbol{h}_1, \boldsymbol{\theta}) y^{\Delta_{\boldsymbol{\theta}+\boldsymbol{h}_s} - \Delta_{\boldsymbol{h}_1} - \Delta_{\boldsymbol{\theta}}} |\boldsymbol{\theta}+\boldsymbol{h}_s\rangle, \quad (3.96)$$

$$\bar{\phi}_{s,\boldsymbol{\theta}}|\boldsymbol{\theta}\rangle = \mathcal{N}(\boldsymbol{\theta}-\boldsymbol{h}_s, -\boldsymbol{h}_N, \boldsymbol{\theta}) y^{\Delta_{\boldsymbol{\theta}-\boldsymbol{h}_s} - \Delta_{\boldsymbol{h}_N} - \Delta_{\boldsymbol{\theta}}} |\boldsymbol{\theta}-\boldsymbol{h}_s\rangle, \quad (3.97)$$

and OPEs

$$\phi_s(z)\bar{\phi}_{s'}(w) \sim \frac{\delta_{s,s'}}{(z-w)^{(N-1)/N}}, \quad (3.98)$$

$$\phi_s(z)\phi_{s'}(w) \sim 0, \qquad \bar{\phi}_s(z)\bar{\phi}_{s'}(w) \sim 0. \quad (3.99)$$

As it happened for the case of "plain" partition functions, only the case with semi-degenerate insertions is under control, and the braiding matrix of a fully degenerate and semi-degenerate fields takes the form

$$\begin{aligned}
\mathsf{B}_{lj}(\boldsymbol{\sigma}',\nu,\boldsymbol{\sigma}) &= e^{-i\pi((N-1)/N+\sigma_l-\sigma_j')}e^{i\pi N((\boldsymbol{\sigma}+\boldsymbol{h}_l,\boldsymbol{h}_l)-(\boldsymbol{\sigma}_j',\boldsymbol{h}_l))} \\
&\quad\times\prod_{k\neq l}\frac{\sin\pi((\nu+1)/N+\sigma_j'-\sigma_k)}{\sin\pi(\sigma_k-\sigma_l)}, \\
&= e^{\pi i(\nu+1/N)}\prod_{k\neq l}\frac{1-e^{-2\pi i((\nu+1)/N+\sigma_j'-\sigma_k)}}{1-e^{-2\pi i(\sigma_l-\sigma_k)}}.
\end{aligned} \tag{3.100}$$

3.5.2 Gauge theory loop operators from Verlinde loop operators

Whithin the AGT correspondence, the meaning of Verlinde loop operators is to represent the algebra of supersymmetric loop operators of the corresponding gauge theory. Recall that a pants decomposition corresponds to a weak coupling limit of the corresponding gauge theory. Further, it provides us automatically with a canonical basis of A- and B-cycles, which are dual under the symplectic pairing given by the intersection form of cycles. In view of the electric-magnetic splitting, A-cycles are "electric", while B-cycles are "magnetic": more precisely, the Verlinde loop operator representing analytic continuation of a degenerate field insertion $\phi_{2,1}$ along an A-cycle of the underlying punctured Riemann surface translate in gauge theory terms to a Wilson loop in the fundamental representation of the gauge group, while the Verlinde loop operator along the B-cycle represents a 't Hooft loop operator in the fundamental representation. This correspondence was used to classify loop operators of theories in Class $\mathcal{S}$ in [107]. The line operators described above are supported on the surface operator, since they are given by analytic continuation of the degenerate field insertion. To obtain a loop operator supported in the bulk, with no reference to a surface operator, one uses the OPE (3.93): since the identity Verma module is in the fusion product of two degenerate fields, one first creates two surface defects by inserting nearby degenerate fields in the conformal block, then analytically continues one of them to form a closed loop on the Riemann Surface, and finally fuses them back to have an identity insertion (we can see this as a brane-antibrane annihilation from the six-dimensional viewpoint).

Part II

Isomonodromic Deformations and Class S Theories

Chapter 4

Isomonodromic deformations

Having discussed the physical setting of $\mathcal{N} = 2$ class S theories and their relation to Hitchin systems, we will in this chapter introduce their relation with isomonodromic deformation equations. We will start by showing the prototypical example of such equations, given by Painlevé equations. After this discussion, we will turn to a more general discussion of isomonodromic deformations starting from linear systems of ODEs. This discussion will provide an explanation for the meaning of this name, since these are the deformations preserving the monodromies of the solution to a linear system of ODEs in the complex domain. These deformations are Hamiltonian flows, and the generating function for the Hamiltonians is called isomonodromic tau function. We will then review the CFT construction for the solution of these linear system [28, 29, 108], together with the Painlevé/gauge theory correspondence [15, 16] relating the isomonodromic tau function to dual gauge theory partition functions.

4.1 Painlevé equations

We cannot resist to the temptation of starting this chapter with a seemingly unrelated problem: the study of nonlinear special functions (see [13] for a detailed account). To understand this, let us first make the observation that all the special functions which are used in a physicist or mathematician's everyday life can be defined by the (linear) differential equation which they solve. For example, one can define the exponential as the solution of

$$\frac{du}{dx} = u, \tag{4.1}$$

or the hypergeometric function ${}_2F_1(z; a, b, c)$ as the solution of the hypergeometric equation

$$z(1-z)\frac{d^2u}{dz^2} + [c - (a+b+1)z]\frac{du}{dz} - abu = 0. \tag{4.2}$$

Not all differential equations one can come up with, however, lead to well-defined functions (i.e. globally defined objects over the domain we are considering) as their solutions. Such a requirement of global existence (as opposed to the local existence, which is guaranteed by Cauchy existence theorem) is a stringent one and is one of the hallmarks of integrability.

Painlevé, together with his collaborators Fuchs and Gambier, identified what is the obstacle for the existence of a global solution: it is the existence of movable critical point. A critical point is a branch point (or more generally a point of multivaluedness) for the solution of the ODE, and is a singularity of the corresponding differential equation. It is movable if it depends on the initial conditions. When a function does not possess movable critical points it is said to have the Painlevé property. All linear ODEs possess this property, but this is a very nontrivial requirement for nonlinear ODEs.

The aim of Painlevé and his collaborators was to extend the results known for linear ODEs to nonlinear ones: they managed to classify all the nonlinear equations of order one and two that give rise to globally defined solutions. Up to Möbius transformation, there are six inequivalent first-order nonlinear ODEs with this property, and their general solutions are all expressible in terms of linear special functions. On the other hand, there are 50 inequivalent nonlinear ODEs of order two with this property, and the solutions of six of these equations are not expressible in terms of previously known special functions. These are the six Painlevé equations, and their solutions define bona fide *nonlinear* special functions. The Painlevé equations are the following (in order from Painlevé I to VI):

$$u'' = 6u^2 + x, \tag{4.3}$$

$$u'' = 2u^3 + xu + \alpha, \tag{4.4}$$

$$u'' = \frac{u'^2}{u} - \frac{u'}{x} + \frac{\alpha u^2 + \beta}{x} + \gamma u^3 + \frac{\delta}{u}, \tag{4.5}$$

$$u'' = \frac{u'^2}{2u} + \frac{3}{2}u^3 + 4xu^2 + 2(x^2 - \alpha)u + \frac{\beta}{u}, \tag{4.6}$$

$$u'' = \left[\frac{1}{2u} + \frac{1}{u-1}\right] u'^2 - \frac{u'}{x} + \frac{(u-1)^2}{x^2}\left[\alpha u + \frac{\beta}{u}\right] + \gamma\frac{u}{x} + \delta\frac{u(u+1)}{u-1}, \tag{4.7}$$

$$\begin{aligned} u'' = &\frac{1}{2}\left[\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right] u'^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right] u' \\ &+ \frac{u(u-1)(u-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{u^2} + \gamma\frac{x-1}{(u-1)^2} + \delta\frac{x(x-1)}{(u-x)^2}\right]. \end{aligned} \tag{4.8}$$

Since these are the most general nonlinear ODEs defining nonlinear special functions, it should come to no surprise that they have an impressively wide range of applications: from Statistical physics [109–111] to Topological QFT [112], 2d Quantum Gravity [113], and many other fields of physics. Most recently, they proved to play a role in four and five-dimensional $\mathcal{N} = 2$ supersymmetric field theories and Topological String/M-theory on local Calabi-Yau threefolds [16, 114, 115].

4.2 Linear systems of ODEs and isomonodromy

The way that Painlevé equations enter into our discussions of two-dimensional CFT and four-dimensional supersymmetric gauge theories, is through their connection with finite-dimensional integrable systems, or rather with their *isomonodromic* deformations. Recall that in Section 1.4 we characterized integrability by the existence of a Lax pair: this means that the equations of motion can be written as the compatibility equations for the system

$$\begin{cases} L(z;\{a_i\})Y = 0, \\ \frac{dY}{dt} = M(z;\{a_i\})Y. \end{cases} \tag{4.9}$$

The Lax matrix L depends on a spectral parameter z, together with a set of other parameters, that we denote by $\{a_i\}$. The time evolution becomes linear on the spectral curve

$$\det(\lambda - L(z)) = 0, \tag{4.10}$$

which is a rational curve in z, algebraic in λ. Now suppose we want to include the spectral parameter z in the set of times. The first equation above then gets deformed to

$$\partial_z Y = L(z;\{a_i\})Y. \tag{4.11}$$

This is a linear system of N first order ODEs, that will have critical points at the singularities of L, which we take to be rational. The solution Y will develop multivaluedness around these critical points, which are determined by the so-called *monodromy data* of the system, that we denote by $\{\alpha_i\} \subset \{a_i\}$. In particular, the Let us see how we can complete this linear system to a consistent Lax pair. Assume there is a time ξ that allows for a consistent Lax pair:

$$\begin{cases} \partial_z Y = LY, \\ \partial_\xi Y = M_\xi Y. \end{cases} \tag{4.12}$$

When we analytically continue Y around a singular point z_k of the linear system $\partial_z Y = LY$, because this is a critical point, it will be multiplied by a monodromy matrix (an $SL(N)$ representation of the fundamental group for the corresponding punctured Riemann Surface)[1]

$$Y(\gamma_{z_k} z) = Y(z)M_k. \tag{4.13}$$

[1] We will here surrender to the traditional notation, that denotes both monodromy matrices and matrices in the Lax pair by M. Hopefully it will be clear from the context to what M-matrix we are referring to.

This monodromy does not depend on the spectral parameter z. Then, the only way for the system (4.12) to make sense, is for the monodromy matrix to satisfy

$$\partial_\xi M_{z_k} = 0. \tag{4.14}$$

Such time ξ is then called an *isomonodromic time*, since the monodromies of the linear system are independent from it. In particular, the parameters $\{a_i\}$ are split into monodromy data $\{\alpha_i\}$ and isomonodromic times $\{t_i\}$, and the equations for isomonodromic deformations are the compatibility conditions for the system

$$\begin{cases} \partial_z Y = LY, \\ \partial_{t_i} Y = M_i Y, \end{cases} \tag{4.15}$$

which then take the form

$$\partial_z M_i + \partial_{t_i} L + [M_i, L] = 0. \tag{4.16}$$

4.2.1 Painlevé VI from isomonodromic deformations

The discussion above was fairly general, but the connection with Painlevé equations is still not clear. In this section we illustrate how the sixth Painlevé equation can be obtained from the isomonodromic deformation conditions of a linear system of ODEs with four simple poles on the Riemann sphere, following the discussion in [116].

Consider the following Lax pair on the Riemann sphere:

$$L = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}, \qquad M_t = -\frac{A_t}{z-t}. \tag{4.17}$$

Here we fixed by an $SL(2,\mathbb{C})$ transformation three of the simple poles of $L(z)$ at $0, 1, \infty$, so that we are left with one free pole t. The matrices A_0, A_1, A_t parametrized the residues at the punctures situated at finite z. Further, there is an A_∞ that parametrizes the residue at infinity. Due to Cauchy theorem they satisfy the constraint

$$A_0 + A_1 + A_t + A_\infty = 0. \tag{4.18}$$

We assume them to be diagonalizable, and set

$$A_k = G_k \boldsymbol{\theta}_k G_k^{-1}, \qquad \boldsymbol{\theta}_k = \theta_k \sigma_3. \tag{4.19}$$

By an overall (constant) conjugation of L, which is a symmetry of the linear system $d_z Y = LY$, we can set one of the residues to be diagonal, so let us fix

$$A_\infty = \theta_k \sigma_3. \tag{4.20}$$

The Lax equations

$$\partial_t L + \partial_z M + [M, L] = 0 \tag{4.21}$$

talke the form

$$\dot{A}_0 = -\frac{1}{t}[A_0, A_t], \qquad \dot{A}_1 = \frac{1}{1-t}[A_1, A_t], \tag{4.22}$$

$$\dot{A}_t = \frac{1}{t}[A_0, A_t] - \frac{1}{1-t}[A_1, A_t]. \tag{4.23}$$

Using the constraint (4.18) we can eliminate A_t and write

$$\dot{A}_0 = \frac{1}{t}[A_0, A_1 + A_\infty], \qquad \dot{A}_1 = -\frac{1}{1-t}[A_1, A_0 + A_\infty]. \tag{4.24}$$

This is convenient, because we already fixed A_∞ to be constant and diagonal. The monodromy data in this case consists of the parameters $\{\theta_k\}$. To check that the time evolution preserves them, i.e. it is isomonodromic, we have just to check that $\det A_k$ is time-independent for $k = 0, 1, t$, since the residues are 2×2 traceless matrices. Let us see how this works, for example, for A_0:

$$\frac{\partial}{\partial t} \log \det A_0 = \operatorname{tr} A_0^{-1} \dot{A}_0 = \frac{1}{t} \operatorname{tr} \left(A_0^{-1} [A_0, A_1 + A_\infty] \right) = 0, \tag{4.25}$$

and the same can be done for A_1, A_t. To reduce the system of equations above to a single, scalar ODE, we have to parametrize in an appropriate way the matrices A_0, A_1, in a way that implements the constraints

$$\operatorname{tr} A_i = 0, \qquad \det A_i = \frac{\theta_i^2}{4}. \tag{4.26}$$

One such parametrization is the following:

$$A_0 = \frac{1}{2} \begin{pmatrix} z_0(t) & u_0(t)\,(\theta_0 - z_0(t)) \\ \frac{\theta_0 + z_0(t)}{u_0(t)} & -z_0(t) \end{pmatrix}, \tag{4.27}$$

$$A_1 = \frac{1}{2} \begin{pmatrix} z_1(t) & u_1(t)\,(\theta_1 - z_1(t)) \\ \frac{\theta_1 + z_1(t)}{u_1(t)} & -z_1(t) \end{pmatrix}, \tag{4.28}$$

so that everything is parametrized in terms of four qunatities z_0, z_1, u_0, u_1. In fact, among these we will use only z_0, z_1, instead choosing as the other two parameters k, y, defined by

$$L(z)_{12} \equiv \frac{k(z-y)}{2z(z-1)(z-t)}. \tag{4.29}$$

In terms of these four variables, the equations of motion take the form

$$\dot{z}_0 = -\frac{Z}{2t}, \qquad \dot{z}_1 = -\frac{Z}{2(1-t)}, \tag{4.30}$$

$$\dot{y} = \frac{(1-\theta_\infty)y(1-y) - \zeta}{t(1-t)}, \qquad \dot{k} = k(1-\theta_\infty)\frac{y-t}{t(1-t)} \tag{4.31}$$

$$\xi = z_0 + z_1, \qquad \zeta = t(1-y)z_0 + (1-t)yz_1, \tag{4.32}$$

$$Z = -\left[\frac{1}{t(1-y)} + \frac{1}{(1-t)y}\right]\frac{\zeta^2}{y-t} + \frac{2\zeta\xi}{y-t} - \frac{t(1-y)}{(1-t)y}\theta_0^2 + \frac{(1-t)y}{t(1-y)}\theta_1^2. \tag{4.33}$$

It requires some nontrivial algebra to write an equation only in one variable from here. The idea is to once again differentiate the equation for y with respect to the time, and use the other equations to eliminate z_0, z_1, k. One arrives then to the following equation

$$\begin{aligned}\ddot{y} = &\frac{1}{2}\left[\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right]\dot{y}^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right]\dot{y} \\ &+ \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right],\end{aligned} \tag{4.34}$$

with

$$\alpha = \frac{(1-\theta_\infty)^2}{2}, \qquad \beta = -\frac{\theta_0^2}{2}, \qquad \gamma = \frac{\theta_1^2}{2}, \qquad \delta = \frac{1-\theta_t^2}{2}. \tag{4.35}$$

This is nothing but the Painlevé VI equation (4.8), with the parameters expressed in terms of the monodromy data of the corresponding linear system.

The other Painlevé equations can be obtained in a similar way from different linear systems, which now have to include higher order poles, that can be obtained by starting from PVI and colliding singularities. This leads to the so-called Painlevé confluence diagram in Figure 4.3. We will come back to this point, and its gauge theory significance, in Section 4.4. Before doing that, however, let us show how the theory of isomonodromic deformations comes into play for the four-dimensional gauge theories and two-dimensional CFTs of the previous chapters.

4.3 Isomonodromic deformations and CFT on the Riemann Sphere

We saw how the sixth Painlevè equation can be obtained from the isomonodromy condition of a linear system with four simple poles on the Riemann sphere. This condition can be naturally generalized to linear systems with an arbitrary number of simple poles, for which the isomonodromic deformation equations take the form of the so-called Schlesinger system. In this section we will show how two-dimensional Conformal Field Theory can be used to study this problem, a fact that can be rephrased naturally in gauge theory terms by using the AGT correspondence of Chapter 3.

Our starting point is now a system of linear ODEs

$$\begin{cases} \partial_z Y(z) = L(z)Y(z), \\ Y(z_0) = \mathbb{I}_N. \end{cases} \tag{4.36}$$

in the complex domain, where both Y and L are $N \times N$ matrices, and $L(z)$ is a meromorphic matrix with simple poles at points $z_1, \dots, z_n$:

$$L(z) = \sum_{k=1}^{n} \frac{A_k}{z - z_k}, \tag{4.37}$$

and A_k are constant matrices. The problem of finding the solution $Y(z)$ of this linear system can be rephrased as the following Riemann-Hilbert Problem (RHP): find a matrix-valued, holomorphic and multivalued function on $\mathbb{P}^1 \setminus \{z_1, \dots, z_n\}$, with the following properties:

$$\begin{cases} Y(z \sim z_k) = G_k(z)(z - z_k)^{\boldsymbol{\theta}_k} C_k, & \\ \det Y(z) \neq 0, & z \neq z_k, \\ Y(z_0) = \mathbb{I}_N, & \end{cases} \tag{4.38}$$

where $\boldsymbol{\theta}_k \equiv \mathrm{diag}(\theta_k^1, \dots, \theta_k^N)$ is the diagonal matrix of eigenvalues[2] of A_k, that is diagonalized at z_k by holomorphic matrix-valued functions $G_k(z)$:

$$\boldsymbol{\theta}_k = G_k^{-1}(z_k) A_k G_k(z_k). \tag{4.39}$$

The monodromies acquired by $Y(z)$ upon analytic continuation around z_k are given by

$$M_k = C_k^{-1} e^{2\pi i \boldsymbol{\theta}_k} C_k. \tag{4.40}$$

Because of this, the $\boldsymbol{\theta}_k$'s are called local monodromy exponents. The problem of finding a solution to the linear system (4.36) is completely equivalent to that of finding a solution to the Riemann-Hilbert Problem (4.38), a particular instance of the so-called Riemann-Hilbert correspondence (see e.g. [12, 13]).

The isomonodromic flows correspond to moving the positions z_k of the punctures, three of which can be fixed by a homographic transformation. They are given by the compatibility conditions of the Lax system

$$\begin{cases} \partial_z Y(z) = L(z)Y(z), \\ \partial_{z_k} Y(z) = M(z)Y(z), \end{cases} \tag{4.41}$$

with

$$M_{z_k} = -\frac{A_k}{z - z_k}. \tag{4.42}$$

[2]Subject to the non-resonance condition $\theta_k - \theta_l \notin \mathbb{Z}\ \forall k \neq l$, $\theta_k \notin \mathbb{Z}\ \forall k$ [111]. We always assume our matrices are diagonalizable.

The compatibility conditions

$$\partial_z M_{z_k}(z) + \partial_{z_k} L(z) + [L(z), M(z)] = 0 \tag{4.43}$$

take the form of the Schlesinger system

$$\frac{\partial A_j}{\partial z_i} = \frac{[A_j, A_i]}{z_j - z_i}, \qquad i \neq j, \qquad \frac{\partial A_j}{\partial z_j} = -\sum_{i \neq j} \frac{[A_j, A_i]}{z_j - z_i}. \tag{4.44}$$

The time evolutions with times z_k are given by Hamiltonian flows with Hamiltonians

$$H_k = \frac{1}{2}\mathrm{Res}_{\,z_k} \mathrm{tr}\, L^2(z) = \frac{1}{2}\sum_{j \neq i} \frac{\mathrm{tr}\,(A_j A_i)}{z_j - z_i}. \tag{4.45}$$

It follows from the compatibility of the time evolutions

$$\partial_{z_i} H_j = \partial_{z_j} H_i, \tag{4.46}$$

that all these Hamiltonians are generated by a single function $\mathcal{T}$, called the isomonodromic tau function $\mathcal{T}$, first defined in [111, 117] and then generalized in [118, 119]

$$\partial_{t_k} \log \mathcal{T} = H_k. \tag{4.47}$$

We will see in Chapter 7 how in general, for Fuchsian isomonodromic problems on higher genus Riemann surfaces, the number of isomonodromic times is equal to the (complex) dimension of the moduli space of genus g curves with n punctures $\dim \mathcal{M}_{g,n} = 3g - 3 + n$, and the formula above gets generalized in terms of overlap of the quadratic differential $\mathrm{tr}\, L^2$ with the relevant Beltrami differentials.

It was already noted in the late '70s [120–124] that two-dimensional Quantum Field Theory provides a useful framework to solve this class of problems. In fact, the results of those papers have been recently extended and simplified, by using the much more powerful tools developed in the past few decades in the context of two-dimensional Conformal Field Theory [28, 29, 108, 125].

In order to find a solution to the RHP (4.38), one has to find a matrix function $Y(z)$ with prescribed monodromies, singular behavior and normalization. It is possible to engineer such an object by constructing a W_N conformal block with the desired properties[126], but to make the essential points clearer in this section we will consider the 2×2 case, that is solved by the more familiar conformal theories with Virasoro symmetry [28]. More precisely, our aim is to show that it is possible to construct the monodromy representation of (4.38) by using the fusion algebra of degenerate fields we introduced in Section 3.5. For the rest of the section, it will be implicitly assumed everywhere that the matrices are 2×2 and the conformal blocks are usual Virasoro blocks. In the next section, we will reformulate the solution in terms of free fermion conformal blocks, and we will discuss the solution to the $N \times N$ system within that framework.

Consider a $c = 1$ Virasoro conformal block with vertex operator insertions

V_k at the location z_k of the punctures. The vertex operator have Liouville charge θ_k, which amounts to a conformal weight

$$\Delta_k = \theta_k^2. \tag{4.48}$$

We normalize the vertex operators so that their matrix elements between primary states coincide with the 1-loop term in the corresponding trinion factor for the quiver gauge theory dual to the conformal block in AGT correspondence:

$$\langle \sigma | V_k | \sigma' \rangle \equiv N(\sigma, \theta_k, \sigma') \tag{4.49}$$
$$= \frac{G(1+\sigma-\sigma'+\theta_k)G(1+\sigma'-\sigma+\theta_k)G(1-\theta_k-\sigma-\sigma')G(1+\sigma+\sigma'-\theta_k)}{G(1+2\theta_k)G(1+2\sigma)G(1-2\sigma')}.$$

G is the Barnes' G-function, which can be viewed as a "higher" version of the Gamma function, since they are defined by

$$\Gamma(z+1) = z\Gamma(z), \qquad G(z+1) = \Gamma(z)G(z). \tag{4.50}$$

Denote by $\phi_s, \tilde{\phi}_s$, with $s = \pm$, the degenerate field insertions as defined by equations (3.90) and (3.91), and consider the following conformal block, whose diagram is given in Figure 4.1:

$$\mathcal{Y}_{rs}(z; z_0) = (z - z_0)^{1/2} \frac{\langle V_1(z_1) \ldots V_n(z_n) \tilde{\phi}_r(z_0) \phi_s(z) \rangle}{\langle V_1(z_1) \ldots V_n(z_n) \rangle}. \tag{4.51}$$

To avoid cluttering of indices we will also use the matrix notation

$$\mathcal{Y}(z; z_0) = (z - z_0)^{1/2} \frac{\langle V_1(z_1) \ldots V_n(z_n) \tilde{\phi}(z_0) \otimes \phi(z) \rangle}{\langle V_1(z_1) \ldots V_n(z_n) \rangle}. \tag{4.52}$$

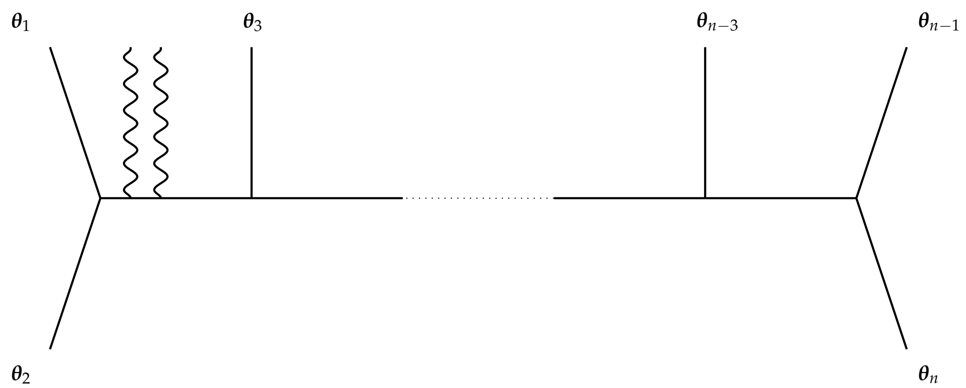


FIGURE 4.1: n point sphere conformal block with degenerate field insertions

This object has *almost* the desired properties. First of all, the correct behavior at the points $z_0, z_1, \ldots, z_n$ is given by the OPE of degenerate fields with vertex operators, while the nontrivial thing to check is that we have the correct monodromy representation. This is given by analytically continuing the variable z in the conformal block along closed loops on the punctured sphere, which can be achieved by using the Verlinde loop operators we introduces in Section 3.5.

The chiral block (4.52), however, does not yet have nice monodromy transformations. Indeed, because of the fusion rules of degenerate fields with primaries (3.82), the internal weights of the conformal block get shifted when performing a fusion operation. For the construction of the solution to the RHP from Virasoro conformal block in [28] it was crucial to show that even though a single fusion operation shifts the weights by half-integers, to perform a complete cycle on the sphere one has to perform each fusion twice, so that monodromy operations in the CFT at genus zero yield in general integer shifts of internal weights. Because of this, if we take a discrete Fourier series of the conformal block over all the internal weights, we have

$$\begin{aligned} Y(z;z_0) &= (z-z_0)^{1/2} \sum_{\mathbf{n}\in\mathbb{Z}^{n-3}} e^{i\mathbf{n}\cdot\boldsymbol{\eta}} \mathcal{Y}(z,z_0;\{\theta_k\},\{\sigma_k+n_k\}) \\ &\equiv (z-z_0)^{1/2}\mathcal{Y}^D(z,z_0). \end{aligned} \tag{4.53}$$

The subscript D stands for "dual", and in general we will denote by this the Fourier series of Virasoro/W_N conformal blocks. The monodromies of Y around the punctures are computable by means of the Moore-Seiberg formalism and are constructed from the CFT's fusion and braiding matrices following the procedures we outlined in Section 3.5.2: the expressions are rather involved, especially in the case of many punctures, and we refer to the original papers for their explicit form [28, 125]. Finally, one can obtain the tau function from Y by expanding the above expression for $z \sim z_0$: by using

$$L(z) = \partial_z Y(z) Y^{-1}(z), \qquad \operatorname{tr} L(z) = 0 \tag{4.54}$$

we can write

$$\frac{1}{z-z_0}\operatorname{tr} Y^{-1}(z_0)Y(z) = \frac{2}{z-z_0} + \frac{z-z_0}{2}\operatorname{tr} L^2(z) + \dots. \tag{4.55}$$

On the CFT side, the expansion can be done by using the OPE of $\phi, \tilde{\phi}$ in (4.52), and equating order by order in $z-z_0$ we get

$$\begin{aligned} \frac{1}{2}\operatorname{tr} L^2(z) &= \frac{\langle V_1(z_1)\dots V_n(z_n)T(z)\rangle_D}{\langle V_1(z_1)\dots V_n(z_n)\rangle_D} \\ &= \sum_{k=1}^{n}\left[\frac{\Delta_k}{(z-z_k)^2} + \frac{1}{z-z_k}\partial_{z_k}\log\langle V_1(z_1)\dots V_n(z_n)\rangle_D\right]. \end{aligned} \tag{4.56}$$

By using (4.47) and taking the residues in z_k of the above expression, we get that the tau function is simply the Fourier transformed chiral conformal block of primary fields

$$\mathcal{T} = \langle V_1(z_1)\dots V_n(z_n)\rangle_D. \tag{4.57}$$

It is also possible, and as we will see in Chapter 5 advisable, to construct a solution of the linear system (4.36) by using (twisted) free fermions [28, 29] instead of degenerate fields. Let us recall how they are defined and how the above construction carries through in this case.

4.3.1 Free fermion reformulation

It was realized already in [28] that all the construction outlined above can be conveniently reformulated using N-component free fermions instead of degenerate fields. This was then extended to the case of $SL(N)$ systems in [125, 126]. The reason why there are these two representations of Y and $\mathcal{T}$ lies in the following fermionization formulae: out of Virasoro/W_N degenerate fields one can construct N-component free fermions by the addition of a $U(1)$ boson φ satisfying the OPE

$$\varphi(z)\varphi(w) \sim -\frac{1}{2}\log(w-z), \tag{4.58}$$

so that the fields

$$\psi_s(z) = e^{i\varphi(z)}\phi_s(z), \qquad \bar{\psi}_s(z) = e^{-i\varphi(z)}\bar{\phi}_s(z) \tag{4.59}$$

satisfy the fermion VOA

$$\bar{\psi}_s(z)\psi_{s'}(w) \sim \frac{\delta_{ss'}}{z-w}. \tag{4.60}$$

We now introduce these free fermions in more detail following [29], without starting from degenerate fields, since we will see that this is the language one is forced to use when trying to generalize the construction of the previous section to genus $g > 0$. We define N-component free complex fermions, collecting them in two vectors $\psi, \bar{\psi}$, by their Fourier expansion in cylindrical coordinates:

$$\psi(z) = \sum_{r\in\mathbb{Z}+1/2} \psi_r e^{2\pi i(r+\boldsymbol{a}+\frac{1}{2})z}, \qquad \bar{\psi}(z) = \sum_{r\in\mathbb{Z}+1/2} \bar{\psi}_r e^{2\pi i(r-\boldsymbol{a}-\frac{1}{2})}, \tag{4.61}$$

or in components

$$\psi_\alpha(z) = \sum_{r\in\mathbb{Z}+1/2} \psi_{\alpha,r} e^{2\pi i(r+a_\alpha+\frac{1}{2})z}, \quad \bar{\psi}_\alpha(z) = \sum_{r\in\mathbb{Z}+1/2} \bar{\psi}_{\alpha,r} e^{2\pi i(r-a_\alpha-\frac{1}{2})} \tag{4.62}$$

Here $\boldsymbol{a}$ lies in the Cartan of $\mathfrak{sl}_N$, and the Fourier modes of the components $\psi_\alpha(z)$, $\psi_\beta(z)$ satisfy the usual canonical anticommutation relations

$$\{\psi_{\alpha,r}, \psi_{\beta,s}\} = \{\bar{\psi}_{\alpha,r}, \bar{\psi}_{\beta,s}\} = 0, \qquad \{\bar{\psi}_{\alpha,r}, \psi_{\beta,s}\} = \delta_{\alpha,\beta}\delta_{r,-s}, \tag{4.63}$$

$$r,s \in \mathbb{Z}+1/2, \qquad \alpha,\beta = 1,\dots,N. \tag{4.64}$$

The fermionic bilinear operators

$$J_{\alpha\beta}(z) \equiv: \bar{\psi}_\alpha(z)\psi_\beta(z) : \tag{4.65}$$

generate a twisted $\widehat{\mathfrak{gl}}(N)_1$ algebra, whose Cartan subalgebra can be used to define a $W_N \otimes \mathfrak{F}$ subalgebra. Its generators are given as elementary symmetric polynomials of the Cartan currents:

$$W_n(z) \equiv \sum_{\alpha_1<\cdots<\alpha_n} : J_{\alpha_1}\dots J_{\alpha_n} : \tag{4.66}$$

where $n = 1, \dots, N$, and

$$J_\alpha(z) = J_{\alpha\alpha}(z). \tag{4.67}$$

These generators can be split into W_N and $\mathfrak{F}_a$ generators by the replacement

$$J_\alpha(z) \to J_\alpha(z) + j(z), \tag{4.68}$$

where $j(z)$ is identified with the $U(1)$ current of $\mathfrak{F}$, while after the replacement $\sum J_\alpha = 0$. We will however for convenience consider directly the original $\widehat{\mathfrak{gl}}(N)_1$ currents.

As a consequence of what we just said, the fermionic Hilbert space $\mathcal{H}$ can be decomposed in sectors with definite $\widehat{\mathfrak{gl}}(N)_1$ charge given by a vector $\boldsymbol{n} \in \mathbb{Z}^N$:

$$\mathcal{H} = \bigoplus_{\boldsymbol{n}\in\mathbb{Z}^N} \mathcal{H}_{\boldsymbol{n}}. \tag{4.69}$$

From the free fermions we can also define vertex operators in an axiomatic way by their braiding relations involving free fermions, i.e. as intertwiners (for more details, see [29]): if one analytically extends a matrix element involving $\psi(z)$ along a contour γ that interchanges its time-ordering with a vertex operator $V_{\boldsymbol{\theta}}$ going counterclockwise above the insertion of the vertex operator, then

$$\bar{\psi}(\gamma \cdot z)V_{\boldsymbol{\theta}}(0) = V_{\boldsymbol{\theta}}(0)B^{-1}\bar{\psi}(z), \qquad \psi(\gamma \cdot z)V_{\boldsymbol{\theta}}(0) = V_{\boldsymbol{\theta}}(0)\psi(z)B. \tag{4.70}$$

Although our discussion will be fully general, the explicit form of B is known, for the N-component system, only in the semi-degenerate case. Let us denote by $\tilde{B}$ the braiding matrix defined by

$$V_{\boldsymbol{\theta}}(0)\bar{\psi}(\tilde{\gamma} \cdot z) = \tilde{B}^{-1}\bar{\psi}(z)V_{\boldsymbol{\theta}}(0), \qquad V_{\boldsymbol{\theta}}(0)\psi(\tilde{\gamma} \cdot z) = \psi(z)\tilde{B}V_{\boldsymbol{\theta}}(0), \tag{4.71}$$

where $\tilde{\gamma}$ follows the same orientation as γ, but goes below the insertion of the vertex operator: see the second and third step in Figure 6.2. In the notation of (3.85), we have $B = B^{(+)}$, $\tilde{B} = B^{(-)}$. We can compute the monodromies around any punctures by iterating these two moves, noting that $\tilde{\gamma} \circ \gamma$ represents a noncontractible contour around the point of insertion of the vertex:

$$\begin{aligned}\langle\sigma|\dots V_{\boldsymbol{\theta}}(z_k)\psi(z)\dots|\sigma'\rangle &\to \langle\sigma|\dots\psi(z)V_{\boldsymbol{\theta}}(z_k)\dots|\sigma'\rangle\\ &= \langle\sigma|\dots V_{\boldsymbol{\theta}}(z_k)\psi(z)\dots|\sigma'\rangle B_k \\ &\to \langle\sigma|\dots V_{\boldsymbol{\theta}}(0)\psi(z)\dots|\sigma'\rangle \tilde{B}_k B_k.\end{aligned} \tag{4.72}$$

The monodromy as composition of braiding operation is represented pictori-

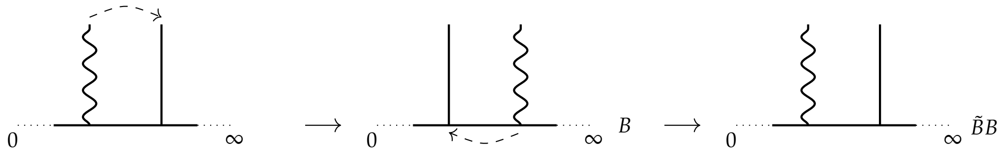
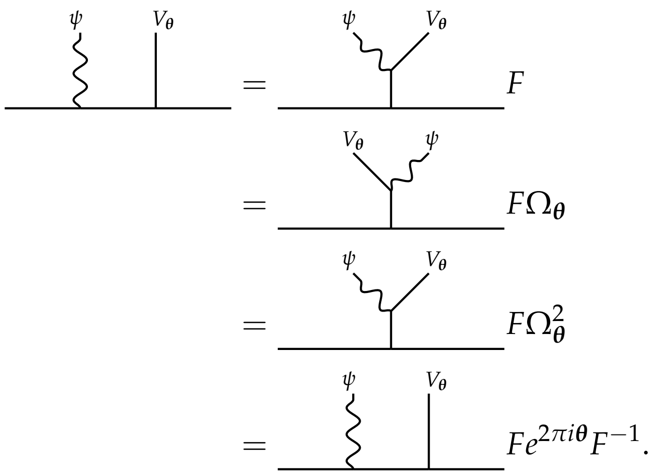


FIGURE 4.2: Braiding of a fermion with a vertex operator. The wavy line represents the insertion of a free fermion operator, while the solid line represents the insertion of a vertex.

ally in Figure 4.2. This monodromy is in the correct conjugacy class, because if we represent the above operation using fusion matrices, we find, schematically,



(4.73)

where we used $\Omega_{\boldsymbol{\theta}}^2 = e^{2\pi i\boldsymbol{\theta}}$. More generally, if we consider the n-point sphere conformal block, the monodromy around a puncture will be obtained by first applying the braiding move until the fermion is close to the vertex operator inserted at the puncture, then applying the above local monodromy operation, and finally by bringing the fermion back with the inverse braiding move, so that the monodromy around the puncture z_k is given by

$$M_k = B_2 \dots B_{n-1} F_k e^{2\pi i \boldsymbol{\theta}_k} F_k^{-1} B_{n-1}^{-1} \dots B_2. \tag{4.74}$$

To be able to compute all the monodromies, we also need a further ingredient: when the fermion is inserted near zero, its monodromy is diagonal, and given by [3]

$$\psi(\gamma_0 \cdot z)|\boldsymbol{a}\rangle = \psi(z)|\boldsymbol{a}\rangle e^{2\pi i \boldsymbol{a}}. \tag{4.75}$$

In fact, this is not only true for the primary state $|\boldsymbol{a}\rangle$ but also for all descendants

$$|\mathbf{M}, \boldsymbol{a}\rangle \equiv \psi_{\alpha_1,-p_1} \dots \psi_{\alpha_l,-p_l} \bar{\psi}_{\beta_1,-q_1} \dots \bar{\psi}_{\beta_l,-q_l} |\boldsymbol{a}\rangle, \tag{4.76}$$

labeled by the coloured Maya diagram

$$\mathbf{M} = \{((\alpha_1, -p_1), \dots, (\alpha_l, -p_l)), ((\beta_1, -q_1), \dots, (\beta_l, -q_l))\}. \tag{4.77}$$

Analogous statements hold if the fermion is inserted instead near infinity. These last points follows from the solution of the problem on the three punctured sphere: by repeated insertions of the identity

$$\begin{aligned} \langle \sigma | \dots \psi(z) V_{\boldsymbol{\theta}}(z_k) \dots |\sigma'\rangle = \sum_{\mathbf{M},\mathbf{M}'} & \langle \sigma | \dots |\mathbf{M}, \boldsymbol{a}\rangle \langle \mathbf{M}, \boldsymbol{a} | \psi(z) V_{\boldsymbol{\theta}}(z_k) | \mathbf{M}', \boldsymbol{a}'\rangle \\ & \times \langle \mathbf{M}', \boldsymbol{a}' | \dots |\sigma'\rangle \end{aligned} \tag{4.78}$$

we can reduce the problem of computing monodromies around arbitrary punctures to a repeated use of the rules described above. Geometrically we are considering a pants decomposition, and using it to write the solution of the complicated problem with n punctures in terms of the solvable 3-point problem.

From all the considerations above, the normalized solution $Y(z; z_0)$ for the $N \times N$ linear system is written in terms of free fermionic conformal blocks as:

$$Y(z; z_0) = (z - z_0) \frac{\langle V_1(z_1) \dots V_n(z_n) \bar{\psi}(z_0) \otimes \psi(z) \rangle}{\langle V_1(z_1) \dots V_n(z_n) \rangle}. \tag{4.79}$$

In this case, the normalization factor is $z - z_0$, as it must compensate the simple pole of the fermion propagator. It turns out to be better, in view of future generalizations, to write this in a slightly different way: note that the normalized solution $Y(z; z_0)$ can be written in terms of a solution with an arbitrary normalization $Y(z)$ in the following way:

$$Y(z; z_0) = Y^{-1}(z_0) Y(z_0). \tag{4.80}$$

Then the free fermionic correlator is simply the following kernel constructed in terms of these objects:

$$K(z; z_0) \equiv \frac{\langle V_1(z_1) \dots V_n(z_n) \bar{\psi}(z_0) \otimes \psi(z) \rangle}{\langle V_1(z_1) \dots V_n(z_n) \rangle} = \frac{Y^{-1}(z_0) Y(z)}{z - z_0}. \tag{4.81}$$

Note already that the description using free fermions seems the most natural, since the Fourier transform in this case simply comes from the decomposition of the free fermionic Hilbert space. The tau function has again the expression:

[3] Recall that in our notations $e^{2\pi i \boldsymbol{a}} = \mathrm{diag}(e^{2\pi i a_i}, \dots, e^{2\pi i a_N})$

$$\mathcal{T} = \langle V_1 \dots V_n \rangle. \tag{4.82}$$

Note that because the tau function can be written as a correlator of primary fields in a free fermionic CFT, it coincides, up to a factor coming from the normalization of the vertex operators, with Nekrasov-Okounkov dual partition function [22] for a linear quiver gauge-theory, with quiver diagram given by the conformal block as in Figure 4.1, but without the wiggly lines. If one considers instead the purely Liouville representation using degenerate fields, in order to reach the same conclusion one must use the AGT correspondence [55].

4.4 The Painlevé-Gauge theory correspondence

The identification of the isomonodromic tau function with a dual gauge theory partition function has been studied in greater detail in the case of four punctures corresponding to the sixth Painlevé equation and to the $N_f = 4$ theory. We already anticipated in Section 4.1 that all other Painlevé equations can be obtained from the sixth by a degeneration procedure. To understand better this fact from a gauge theory perspective, let us first write what are the parameters of Painlevé VI from a gauge theory point of view. As we have seen, the monodromy exponents $\theta_0, \theta_t, \theta_1, \theta_\infty$ are the external charges of the conformal blocks. In terms of these, the physical hypermultiplet masses are ass follows:

$$m_1 = \theta_0 + \theta_t, \quad m_2 = \theta_0 - \theta_t, \quad m_3 = \theta_1 + \theta_\infty, \quad m_4 = \theta_1 - \theta_\infty. \tag{4.83}$$

The degeneration from Painlevé VI to Painlevé V is achieved by setting

$$\theta_1 = \frac{\Lambda + \theta_*}{2}, \qquad \theta_\infty = \frac{\Lambda - \theta_*}{2}, \tag{4.84}$$

and sending $\Lambda \to \infty$. The tau function is given by

$$\tau_V(t) = \lim_{\Lambda\to\infty} \left(\frac{t}{\Lambda}\right)^{\theta_0^2+\theta_t^2} \tau_{VI}(t/\Lambda). \tag{4.85}$$

We see in particular that we are sending the mass $m_4 \to \infty$, and appropriately rescaling the instanton counting parameter. This is in fact exactly the prescription for the holomorphic decoupling of the hypermultiplet of mass m_4, and the tau function of Painlevé V will be given by the dual partition function of the $N_f = 3$ theory. One can go on with this procedure, following the Painlevé confluence diagram in Figure 4.3, showing that the correspondence with isomonodromic problems is more general than the case with only regular punctures on the Riemann Surface: all the Painlevé equations in the upper part of the confluence diagram are obtained from the previous ones by holomorphic decoupling, and correspond to $SU(2)$ super Yang-Mills with different number of hypermultiplets in the fundamental representation

[15]. The Painlevé equations on the lower part of the diagram have solutions

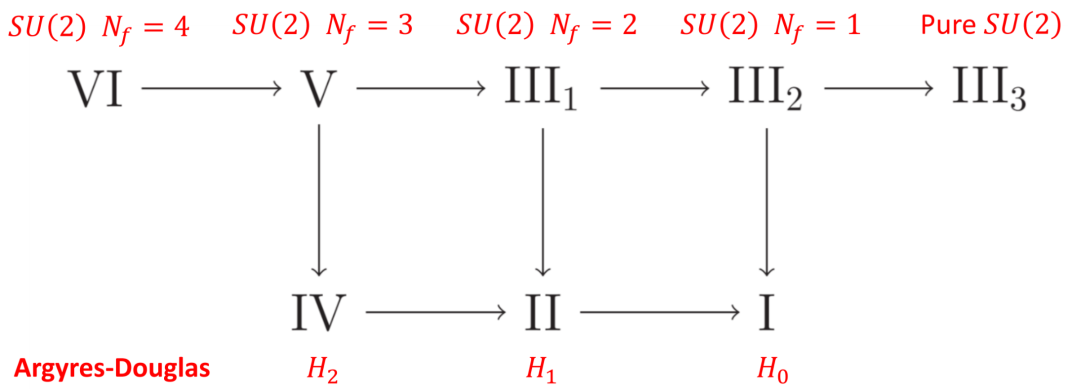


FIGURE 4.3: Confluence Diagram of Painlevé equations and corresponding QFTs

with more intricate analytic behavior, one feature of which is that they do not admit an expansion around $t = 0$. Recalling that $t = e^{2\pi i\tau}$, this means that the corresponding field theory does not admit a weakly coupled expansion. This is due to the fact that the limit that brings from the upper to the lower part of the confluence diagram is the limit that brings to the special point of the Coulomb branch given by Argyres Douglas theories [17, 18], which are isolated fixed points describing nonlagrangian SCFTs. However, an expansion around the point $t = \infty$ is possible, and by studying this case a strongly coupled expansion for the prepotential of these theories was derived [16].

What is the meaning of these equations from a gauge theory perspective? For asymptotically conformal field theories, like $SU(2)$ with four massive flavors, the gauge coupling is an exactly marginal deformation. The differential equation in t describes then an exactly marginal deformation of the theory, traced along the conformal manifold. For the asymptotically free cases, as well as for Argyres-Douglas theories, the gauge coupling describes a relevant deformation of the theory, so that we are considering RG equations along a relevant direction.

Having said this, we will from now on focus solely on isomonodromic deformations in the case of regular punctures, corresponding to asymptotically conformal gauge theories. Our aim in the next three Chapters will be to extend the results we reviewed in this one to more general class S theories, involving the compactification of the A_{N-1} theory on a punctured Riemann surface.

Chapter 5

One-punctured torus, $\mathcal{N} = 2^*$ theory

In this chapter we discuss the first original result of this thesis, from the paper [1]. We will generalize the connection between isomonodromic deformations, two-dimensional conformal field theory, and four-dimensional supersymmetric gauge theories that we described in the previous chapter, to the case of Riemann Surfaces of genus $g \geq 1$, with regular punctures. We will see that the generalization is not completely straightforward, as there are additional complications due to the presence of several nonequivalent flat bundles on Riemann Surfaces with genus $g > 0$. Here we study in detail the first nontrivial higher genus case, the $SL(2)$ isomonodromic system on the torus with one regular puncture. From the class $\mathcal{S}$ theory point of view, the six-dimensional A_1 theory compactified on a torus with one regular puncture gives the $\mathcal{N} = 2^*$ theory, which is $\mathcal{N} = 2$ $SU(2)$ super Yang-Mills with one adjoint massive hypermultiplet, that can be regarded as a mass-deformation of $\mathcal{N} = 4$ SYM and is UV finite: the partition function of this theory turns out to be closely related to the isomonodromic tau function for this problem, albeit in a more complicated way than what happened for linear quiver gauge theories.

The only isomonodromic time is here the modular parameter of the torus τ, which is the (complexified) gauge coupling of the $\mathcal{N} = 2^*$ theory: by using the flow in this variable we will be able to give an exact relation between the IR and UV gauge couplings for $\mathcal{N} = 2^*$ in terms of theta functions.

5.1 Isomonodromic deformations and torus conformal blocks

5.1.1 Linear systems on the torus

To generalize the Lax matrix (4.37) to the case of the torus, we must take into account the Riemann-Roch theorem. Because of it, there is no function with only one simple pole on the torus, and in general a Lax matrix $L(z)$ required to have simple poles at given points will transform nontrivially along the A

and B cycles[1]:

$$L(z+1) = T_A L(z) T_A^{-1}, \qquad L(z+\tau) = T_B L(z) T_B^{-1}. \tag{5.1}$$

Here the twists T_A, T_B satisfy

$$T_A T_B^{-1} T_A^{-1} T_B = \zeta, \tag{5.2}$$

where

$$\zeta = e^{2\pi i c_1/N}, \tag{5.3}$$

and $c_1 = 0, \ldots, N-1$ is the first Chern class of the bundle with the centre of $SL(N)$ as structure group, which classifies the inequivalent flat bundles on the torus [127]. One can go from one bundle to the other by means of singular gauge transformations, known as Hecke transformations [128], so that without loss of generality we will deal with the case $c_1 = 0$, corresponding to the Lax matrix of the N-particles elliptic Calogero-Moser system [2]. This is known to describe isomonodromic deformations on the one-punctured torus, with isomonodromic time τ [129, 130]. It is also the Lax matrix of the integrable system describing the Seiberg-Witten theory of four-dimensional $SU(N)$ super Yang-Mills with one hypermultiplet in the adjoint representation of the gauge group, or $\mathcal{N} = 2^*$ theory [131–133].

The $SL(2,\mathbb{C})$ linear system (4.36) with one simple pole at $z = 0$ on the torus is then

$$\begin{cases} \partial_z Y(z|\tau) = L(z|\tau) Y(z|\tau), \\ Y(z_0|\tau) = \mathbb{I}_2, \end{cases} \qquad L(z|\tau) = \begin{pmatrix} p & m x(2Q,z) \\ m x(-2Q,z) & -p \end{pmatrix}, \tag{5.5}$$

$$T_A = \mathbb{I}_2 \qquad T_B = e^{2\pi i Q}, \qquad \zeta = 1 \tag{5.6}$$

where

$$x(u,z) = \frac{\theta_1(z-u|\tau)\theta_1'(\tau)}{\theta_1(z|\tau)\theta_1(u|\tau)}, \tag{5.7}$$

[1] In general, L transforms as a connection, so in the transition functions T_A, T_B there could be also a nonhomogeneous term. However, these matrices can be chosen so that they are z-independent up to a scalar multiple [127].

[2] TYhis is the $SL(N)$ Hitchin system for the one-punctured torus. It describes a one-dimensional integrable system of particles on the torus with repulsive potential, and its Hamiltonian in the $SL(2)$ case is

$$H_{CM} = \frac{p^2}{2} + g^2 \wp(2Q|\tau). \tag{5.4}$$

and we also used the notation [3]

$$e^{2\pi i \mathbf{Q}} = e^{2\pi i Q\sigma_3}. \tag{5.8}$$

As a consequence of (5.1), the solution Y will have, besides the usual monodromies acting on the right, also twists acting on the left:

$$\begin{aligned} Y(\gamma_A \cdot z|\tau) &= Y(z)M_A, \\ Y(\gamma_B \cdot z|\tau) &= e^{2\pi i \mathbf{Q}} Y(z)M_B, \\ Y(\gamma_k \cdot z|\tau) &= Y(z)M_k, \end{aligned} \tag{5.9}$$

where γ_A, γ_B denote analytic continuation along the A- and B-cycle, while γ_0 denotes analytic continuation along a small loop around the puncture at $z = 0$. Differently from the monodromies, the twists are not constant along the isomonodromic flows. In this case with one puncture, we can easily use identity (A.14) and the defining equation to compute the isomonodromic Hamiltonian

$$\begin{aligned} H_\tau &= \frac{1}{2}\oint_A \operatorname{tr} L^2(z)dz = \int_{0+ih}^{1+ih} dz \left[p^2 - m^2\left(\wp(2Q|\tau) - \wp(z|\tau)\right)\right] \\ &= p^2 - m^2\wp(2Q|\tau) - 2m^2\eta_1(\tau), \end{aligned} \tag{5.10}$$

associated to the time $2\pi i\tau$. The last term comes from

$$\int_{0+ih}^{1+ih} dz\wp(z|\tau) = -\int_{0+ih}^{1+ih} dz\zeta'(z|\tau) = \zeta(0) - \zeta(1) = -2\eta_1(\tau), \tag{5.11}$$

where we offset the integration contour by a small imaginary value ih to avoid the singularity in the integrand along the real axis. Since this last term is a function of τ only, it does not contribute to the isomonodromy deformation equations, which are the Hamilton equations for this Hamiltonian[4] and

---

[3]Here and below we use the standard Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Bold letters always mean either vectors, or diagonal matrices.

[4]This equation is the isomonodromy deformation equation because it is equivalent to the Lax pair equation

$$2\pi i\partial_\tau L + \partial_z M + [M, L] = 0, \tag{5.12}$$

which can be shown, by using equation (A.15), to be the zero-curvature compatibility condition of the system

$$\begin{cases} \partial_z Y = LY, \\ 2\pi i\partial_\tau Y = -MY, \end{cases} \tag{5.13}$$

where M is the other matrix of the Lax pair of L:

$$M = m\begin{pmatrix} \wp(2Q) & y(2Q, z) \\ y(-2Q, z) & \wp(2Q) \end{pmatrix}, \tag{5.14}$$

and $y(u, z) = \partial_u x(u, z)$.

take the form of a special case of Painlevé VI for Q [129, 130]:

$$(2\pi i)^2 \frac{d^2 Q}{d\tau^2} = m^2 \wp'(2Q). \tag{5.15}$$

We see that the twist is the Painlevé transcendent, and as such is a function of τ that in general cannot be expressed in terms of usual special functions.

5.1.2 Monodromies of torus conformal blocks

We will now derive the CFT expression for the solution of the linear system (5.5) and for the tau function of the isomonodromic problem. To this end, consider first the chiral block with degenerate field insertions

$$\begin{aligned}\mathcal{Y}(z, z_0|\tau, a, m) &= \langle V_m(0)\tilde{\phi}(z_0) \otimes \phi(z)\rangle \\ &= \frac{1}{Z(\tau)} \mathrm{tr}\,_{\mathcal{V}_a} \left(q^{L_0} V_m(0)\tilde{\phi}(z_0) \otimes \phi(z)\right)\end{aligned} \tag{5.16}$$

for the Virasoro algebra at $c = 1$, where $\phi_i, \tilde{\phi}_i$ are degenerate fields as in Section 3.5.1, V_m is a primary field with Liouville charge m, and weight $\Delta_m = m^2$, and $Z(\tau)$ is the torus partition function of the CFT:

$$Z(\tau) = \mathrm{tr}\,_{\mathcal{V}_a} \left(q^{L_0} \right). \tag{5.17}$$

Because of the CFT solution to the RHP on the sphere, that we reviewed in Section 4.3, the object

$$\mathcal{Y}^D(z, z_0) = \sum_n e^{in\eta} \mathcal{Y}(z, z_0; \tau, m, a+n) \tag{5.18}$$

has monodromies on the one-punctured torus in prescribed conjugacy classes [5]

$$M_A \sim ie^{2\pi i a}, \qquad M_1 \sim e^{2\pi i m}. \tag{5.19}$$

Differently from what happens in the spherical case, in the case of the torus we do not always have just integer shifts of the internal charges when transporting z around a closed loop: when we move the degenerate field around the B-cycle of the torus, we perform fusion with the primary V_m only once, so that because of the fusion rules (3.82) the internal weights get shifted by half-integers, as is shown in Figure 5.1. As a consequence, the Fourier transform $\mathcal{Y}^D$, that includes only integer shifts, will not transform into itself under B-cycle monodromy. Let us make this observation more precise by computing how $\mathcal{Y}$ transforms when the degenerate field goes in a loop around the B-cycle.

[5] The factor of i in M_A comes from the Jacobian of the transformation from the plane to the cylinder for a field of dimension $\frac{1}{4}$.

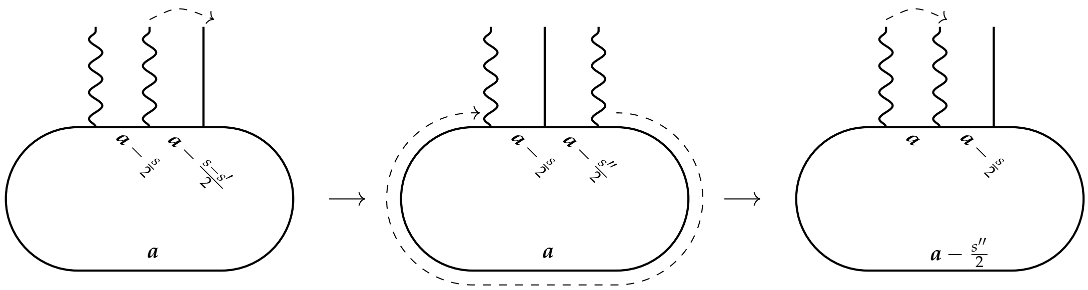


FIGURE 5.1: B-cycle monodromy for the one-punctured torus

The transformation is given in Figure 5.1, which means that

$$\mathcal{Y}_{ss'}(\gamma_B\cdot z)=i\sum_{s''=\pm}\mathcal{Y}_{ss''}(z)e^{-\frac{s''}{2}\overleftarrow{\partial}_a}e^{i\pi s''(a-s/2)}F_{s''s'}(a-s/2,m,a)e^{-i\pi s'a} \tag{5.20}$$

where we denoted by $e^{-\frac{s''}{2}\overleftarrow{\partial}_a}$ the operator that acts on the left by shifting $a\to a-s''/2$ (operators must act from the right because such is the action of the monodromies). Now we use the property

$$F_{s''s'}(\boldsymbol{a}-s/2,m,a)=ss''F_{s''s'}(a,m-1/2,a), \tag{5.21}$$

of the fusion matrix, which can be easily checked given its explicit expression (3.82), together with

$$e^{i\pi s''(a-s/2)}=-ss''ie^{i\pi s''a} \tag{5.22}$$

to rewrite the above monodromy action as a matrix action (operator valued, because of the shifts):

$$\mathcal{Y}(\gamma_B\cdot z)=\mathcal{Y}e^{-\frac{1}{2}\overleftarrow{\partial}_a\sigma_3}e^{i\pi\boldsymbol{a}}F(a,m-1/2,a)e^{-i\pi\boldsymbol{a}}. \tag{5.23}$$

At first glance, it would seem that due to the half-integer shifts in the intermediate channel, in the case of the torus one should consider the Fourier transform

$$\sum_n e^{\frac{in\eta}{2}}\mathcal{Y}(z,z_0;\tau,m,a+n/2), \tag{5.24}$$

as it diagonalizes the shift operator $e^{-\frac{1}{2}\overleftarrow{\partial}_a}$. However, doing so the monodromy action along the A-cycle is spoiled, since

$$\mathcal{Y}(\gamma_A\cdot z,a+n)=i\mathcal{Y}(z,a+n)e^{-2\pi i\boldsymbol{a}}, \tag{5.25}$$

$$\mathcal{Y}(\gamma_A\cdot z,a+n+1/2)=-i\mathcal{Y}(z,a+n+1/2)e^{-2\pi i\boldsymbol{a}}. \tag{5.26}$$

This can be remedied by using instead the free fermion formulation that we reviewed in Section 4.3.1. Recall that this amounts to the addition of an extra $U(1)$ boson, leading to the following fermionization formulae for degenerate fields:

$$\psi(z) \equiv e^{i\varphi(z)}\phi(z), \qquad \bar{\psi}(z) \equiv e^{-i\varphi(z)}\tilde{\phi}(z). \tag{5.27}$$

The presence of the $U(1)$ boson shifts all monodromy exponents by the eigenvalue σ of the zero-mode of $\partial\varphi$: we might as well set $\sigma = 0$, however as we will see, it will turn out to be meaningful to keep this factor for a while. The effect of adding the $U(1)$ boson is that the additional sign along the A-cycle cancels out with that of the degenerate fields, so that the free fermion conformal block

$$K(z,z_0) \equiv \sum_{n,k} e^{\frac{in\eta}{2}} e^{4\pi i(\rho+1/2)(n/2+k+1/2)} \mathrm{tr}\,_{\mathcal{V}_{a+\frac{n}{2}}\otimes\mathfrak{F}_{a+\frac{n}{2}+k+\frac{1}{2}}} \left(q^{L_0} V_m(0)\bar{\psi}(z_0)\otimes\psi(z)\right)$$
$$\equiv \langle V_m(0)\bar{\psi}(z_0)\otimes\psi(z)\rangle$$

has numerical monodromies along all noncontractible cycles. In this definition the shifts of the Virasoro highest weight $a \to a + n/2$ and of the Heisenberg charge ($\sigma + 1/2 \to \sigma + 1/2 + n/2 + k$) correspond to the shift of the fermionic charge(s) of two-component fermions by $(n+k,k)$. By using the above considerations, together with expression (5.23) for the B-cycle monodromy, and the action

$$K(z,z_0;\tau,m,a,\sigma)e^{-\frac{1}{2}\overleftarrow{\partial_a}\sigma_3}e^{-\frac{1}{2}\overleftarrow{\partial_\sigma}} = K(z,z_0;\tau,m,a,\sigma)e^{i\frac{\eta}{2}\sigma_3+2\pi i\rho}, \tag{5.28}$$

we see that the monodromies around the A- and B-cycles of the torus take the form

$$\hat{M}_A = e^{-2\pi i \boldsymbol{a} - 2\pi i\sigma} \equiv e^{-2\pi i\sigma}M_A, \tag{5.29}$$

$$\hat{M}_B = e^{i\frac{\eta}{2}\sigma_3+2\pi i\rho}e^{i\pi\boldsymbol{a}}F(a,m-1/2,a)e^{-i\pi\boldsymbol{a}} \equiv e^{2\pi i\rho}M_B. \tag{5.30}$$

Above, M_A, M_B are the part of the conformal block monodromies $\hat{M}_A, \hat{M}_B$ that are independent from the additional $U(1)$ charges σ,ρ, and they have unit determinant. As we will see in the next section, they are the monodromy matrices of the linear system solution Y. [6]

Let us make some final remarks on this computation. The A-cycle monodromy is encoded in the mode expansion of the complex fermions:

$$\psi(z) = \sum_{p\in\mathbb{Z}+\frac{1}{2}} \psi_p e^{2\pi i(p-\boldsymbol{a}-(\sigma+1/2))z} \tag{5.31}$$

which comes from the change of coordinate from the plane to the torus $w = e^{-2\pi iz}$. We see that the 1/2 shift of σ must be added in order to cancel antiperiodicity of the natural mode expansion. In the same way, in the computation of the B-cycle monodromy we shifted by 1/2 the parameter ρ in order to

[6] Recall that $\det Y(z) = 1$, so that the extra $U(1)$ factors $e^{2\pi i\rho}$ and $e^{-2\pi i\sigma}$ from the point of view of the linear system are introduced artificially. In fact, they are arbitrary and we can set them to any value, but it turns out to be convenient to keep them arbitrary throughout the computations.

cancel the (-1) factor coming from fermions re-ordering [7].

The signs of the shifts by n are defined by the expression for L_0, twisted by two fermionic charges $\mathbf{H}_1$ and $\mathbf{H}_2$:

$$L_0 = const + L_0^{(0)} + \mathbf{H}_1(\sigma + a) + \mathbf{H}_2(\sigma - a), \tag{5.32}$$

where

$$[L_0^{(0)}, \psi_{i,p}] = -n\psi_{i,p}, \qquad [\mathbf{H}_i, \psi_{j,p}] = \delta_{ij}\psi_{j,p}. \tag{5.33}$$

5.1.3 CFT solution to the Riemann-Hilbert problem and tau function

To relate the free fermion correlator (5.28) to the solution Y of the linear system (5.5), we need to take care of two things: the fact that the correlator has an additional simple pole in $z = z_0$ with residue one that Y does not have, and the fact that Y has, in addition to the monodromies acting from the right, twists acting from the left, as in (5.1.1), while the free fermion correlator only has monodromies. The generalization of the sphere kernel (4.81) to a kernel on the torus is then:

$$Y^{-1}(z_0)\Xi(z - z_0)Y(z) = \frac{\langle V_m(0)\bar{\psi}(z_0) \otimes \psi(z)\rangle}{\langle V_m(0)\rangle}, \tag{5.34}$$

where we defined the matrix

$$\Xi(z) \equiv \mathrm{diag}\left(x(\sigma\tau + \rho - Q, z), x(\sigma\tau + \rho + Q, z)\right) e^{-2i\pi\sigma z} \tag{5.35}$$

whose transformations along the two cycles of the torus are such that it cancels the twists of the solution Y, while also giving the additional shifts due to the $U(1)$ boson charge σ. The two sides of the equations have the same singularities with same singular behavior, and same monodromies both in z and z_0. Because of this, they coincide. It is also useful to introduce new notations

$$\widetilde{Q}_1 = -\sigma\tau - \rho + Q, \quad \widetilde{Q}_2 = -\sigma\tau - \rho - Q, \quad \widetilde{\mathbf{Q}} = \mathrm{diag}(\widetilde{Q}_1, \widetilde{Q}_2). \tag{5.36}$$

As in the case of the sphere, we now expand both sides of the equation to obtain the tau function. In particular, we need only to expand the trace of (5.34)

$$\sum_\alpha \frac{\langle \psi_\alpha(z)\bar{\psi}_\alpha(z_0)V_m(0)\rangle}{\langle V_m(0)\rangle} = -\mathrm{tr}\left[Y(z)Y^{-1}(z_0)\Xi(z - z_0)\right]. \tag{5.37}$$

[7] These two shifts mean that we are fermionizing the degenerate fields into fermions which are periodic along both cycles on the torus (in the sense that no additional signs are involved in the computation of monodromies). The shift in σ amounts to the periodicity condition on the cylinder, while that in ρ is implemented in the operator formalism we are using by an insertion of $(-)^F$ in all the traces over the whole fermionic Hilbert space.

The expansion of the first two factors is:

$$Y(z)Y^{-1}(z_0) = \left(\mathbb{I} + (z-z_0)L(z_c) + \frac{(z-z_0)^2}{2}L^2(z_c)\right), \tag{5.38}$$

where $z_c = \frac{z+z_0}{2}$. Then, we expand the theta functions in the diagonal matrix:

$$\frac{\theta_1'(\tau)}{\theta_1(z-z_0|\tau)} = \frac{1}{z-z_0} - \frac{z-z_0}{6}\frac{\theta_1'''(\tau)}{\theta_1'(\tau)} + O((z-z_0)^3), \tag{5.39}$$

$$\begin{aligned} e^{-2\pi i\sigma(z-z_0)}\frac{\theta_1(z-z_0+\widetilde{\mathbf{Q}}|\tau)}{\theta_1(\widetilde{\mathbf{Q}}|\tau)} &= 1 + (z-z_0)\left(\frac{\theta_1'(\widetilde{\mathbf{Q}}|\tau)}{\theta_1(\widetilde{\mathbf{Q}}|\tau)} - 2\pi i\sigma\right) + \\ + \frac{(z-z_0)^2}{2}&\left(\frac{\theta_1''(\widetilde{\mathbf{Q}}|\tau)}{\theta_1(\widetilde{\mathbf{Q}}|\tau)} - \frac{\theta_1'(\widetilde{\mathbf{Q}}|\tau)}{\theta_1(\widetilde{\mathbf{Q}}|\tau)}4\pi i\sigma + (2\pi i\sigma)^2\right) + O((z-z_0)^3) \end{aligned} \tag{5.40}$$

Putting everything together, we find that the $O(z-z_0)$ term in the expansion above is

$$\begin{aligned} &\sum_\alpha \frac{\langle \frac{1}{2} : \partial\bar{\psi}_\alpha(z_c)\psi_\alpha(z_c) + \partial\psi_\alpha(z_c)\bar{\psi}_\alpha(z_c) : V_m(0)\rangle}{\langle V_m(0)\rangle} = \frac{\langle T(z_c)V_m(0)\rangle}{\langle V_m(0)\rangle} \\ &= \frac{1}{2}\mathrm{tr}\left(L^2(z_c) + \frac{\theta_1''(\widetilde{\mathbf{Q}}|\tau)}{\theta_1(\widetilde{\mathbf{Q}}|\tau)} + 2(L(z_c)-2\pi i\sigma)\frac{\theta_1'(\widetilde{\mathbf{Q}}|\tau)}{\theta(\widetilde{\mathbf{Q}}|\tau)} + (2\pi i\sigma)^2 - \frac{1}{3}\frac{\theta_1'''(\tau)}{\theta_1'(\tau)}\right). \end{aligned} \tag{5.41}$$

There are some additional terms with respect to what we found for the sphere. However they can be rearranged in a more convenient form. We use the fact that the diagonal part of the Lax matrix (5.5) consists of the momenta of the Hamiltonian system [127] to write

$$\begin{aligned} &\mathrm{tr}\left(\frac{\theta_1''(\widetilde{\mathbf{Q}}|\tau)}{\theta_1(\widetilde{\mathbf{Q}}|\tau)} + 2(L(z_c)-2\pi i\sigma)\frac{\theta_1'(\widetilde{\mathbf{Q}}|\tau)}{\theta(\widetilde{\mathbf{Q}}|\tau)} + (2\pi i\sigma)^2 - \frac{1}{3}\frac{\theta_1'''(\tau)}{\theta_1'(\tau)}\right) = \\ &= \sum_{i=1}^{2}\left(\frac{\theta_1''(\widetilde{Q}_i)}{\theta_1(\widetilde{Q}_i|\tau)} + 2(p_i - 2\pi i\sigma)\frac{\theta_1'(\widetilde{Q}_i|\tau)}{\theta_1(\widetilde{Q}_i|\tau)} + (2\pi i\sigma)^2 - \frac{1}{3}\frac{\theta_1'''(\tau)}{\theta_1(\tau)}\right). \end{aligned} \tag{5.42}$$

We now use the heat equation for θ_1, as well as the relation for the coordinates and momenta of the nonautonomous system

$$\theta_1'' = 4\pi i\partial_\tau\theta_1, \qquad p_i = 2\pi i\partial_\tau Q_i, \tag{5.43}$$

to write the above expression as

$$4\pi i\sum_{i=1}^{2}\left[\frac{\theta'_{1,\tau}(\widetilde{Q}_i|\tau)}{\theta_1(\widetilde{Q}_i|\tau)}+\partial_\tau(Q_i-\sigma\tau-\rho)\frac{\theta'_1(\widetilde{Q}_i|\tau)}{\theta_1(\widetilde{Q}_i|\tau)}+(2\pi i\sigma)^2-\frac{1}{3}\frac{\partial_\tau\theta'(\tau)}{\theta_1(\tau)}\right]=$$
$$=4\pi i\sum_{i=1}^{2}\partial_\tau\log e^{\pi i\tau\sigma^2}\frac{\theta_1(\widetilde{Q}_i|\tau)}{\theta'_1(\tau)^{1/3}}=4\pi i\partial_\tau\log\left(e^{2\pi i\tau\sigma^2}\frac{\theta_1(\widetilde{Q}_1|\tau)}{\eta(\tau)}\frac{\theta_1(\widetilde{Q}_2|\tau)}{\eta(\tau)}\right)=$$
$$=4\pi i\partial_\tau\log\left(Z_{twist}(Q,\rho,\sigma,\tau)\right),\tag{5.44}$$

where in the last equality we noted that the argument of the logarithm is the partition function of two free complex twisted fermions, with twists defined by the Lax connection (5.5) with extra $U(1)$ shift. Plugging the last formula into (5.41) we find

$$\frac{1}{2}\mathrm{tr}\,L^2(z_c)+2\pi i\partial_\tau\log\left(Z_{twist}(Q,\rho,\sigma,\tau)\right)=\frac{\langle V_m(0)T(z_c)\rangle}{\langle V_m\rangle}.\tag{5.45}$$

We can use the Virasoro Ward identity for an energy-momentum tensor insertion on the torus, which takes the form

$$\frac{\langle T(z)V_m(0)\rangle}{\langle V_m\rangle}=\langle T\rangle+m^2\left[\wp(z|\tau)+2\eta_1(\tau)\right]+2\pi i\partial_\tau\log\langle V_m\rangle.\tag{5.46}$$

In order to identify the tau function from this equation, we will use a general result from [127], in which the authors study isomonodromic deformation problems on elliptic curves. The relation between tr L^2 and the isomonodromic Hamiltonian, specialized to the case of one puncture, is

$$\frac{1}{2}\mathrm{tr}\,L^2(z)=H_\tau+m^2(\wp(z)+2\eta_1).\tag{5.47}$$

m^2 is the Casimir of the orbit around the puncture, and the isomonodromic Hamiltonian turns out to be

$$H_\tau=2\pi i\partial_\tau\log\mathcal{T}=2\pi i\partial_\tau\log\left(\frac{\langle V_m\rangle}{Z_{twist}(\tau)}\right),\tag{5.48}$$

so that the correlation function is

$$Z^D(\tau)=\langle V_m\rangle=\sum_{n,k\in\mathbb{Z}}e^{4\pi i(\rho+1/2)(n/2+k+1/2)}e^{\frac{in\eta}{2}}\mathrm{tr}_{\mathcal{V}_{a+n/2}\otimes\mathcal{F}_{\sigma+1/2+n/2+k}}(q^{L_0}V_m(0))$$
$$=Z_{twist}(\tau)\mathcal{T}(\tau)$$
$$=e^{2\pi i\tau\sigma^2}\eta(\tau)^{-2}\theta_1(\sigma\tau+\rho+Q(\tau))\theta_1(\sigma\tau+\rho-Q(\tau))\mathcal{T}(\tau).\tag{5.49}$$

The tau function is then the correlator of primary fields in a free fermionic chiral CFT, as in the case of the sphere, but instead of being normalized by the partition function of the CFT itself, it is normalized with the partition function of twisted complex fermions, with twists given by the isomonodromic problem under consideration. This result is more general, and holds for the case of the n-punctured torus, and for more general $N \times N$ linear systems, solved by W_N free fermionic Conformal Field Theories. This is the result of the paper [2], which we will discuss in the next chapter.

In fact, note that this is nothing but the free fermion expression of the dual gauge theory partition function Z^D that appears in the original paper by Nekrasov and Okounkov [22], so that

$$\mathcal{T}(\eta, a, m, \tau) = \frac{G(1+m)^2}{G(1+2m)} \frac{Z^D(\eta, a, m, \rho, \sigma, \tau)}{Z_{twist}(\eta, a, m, \rho, \sigma, \tau)}. \tag{5.50}$$

Where the additional m-dependent factor comes due to our normalization of the vertices, as we show below. In fact, since we are defining the isomonodromic tau function by the property

$$2\pi i \partial_\tau \log \mathcal{T} = H_\tau, \tag{5.51}$$

constant multiplicative factors are irrelevant, and we may as well consider the tau function to be

$$\mathcal{T}_{gauge}(\tau) \equiv \frac{Z^D(\tau)}{Z_{twist}(\tau)}. \tag{5.52}$$

At this point we exploit the arbitrary $U(1)$ charges to establish relations between the Fourier series of Virasoro conformal blocks, which we will call Virasoro dual partition functions, and the full dual partition function, computed over the free fermionic Hilbert space. First we expand this latter over its pure Virasoro and Heisenberg contributions:

$$\begin{aligned} Z^D(\tau) &= \eta(\tau)^{-1} \sum_{n,k\in\mathbb{Z}} e^{i\eta n/2} e^{4\pi i(\rho+1/2)(k+n/2+1/2)} q^{(\sigma+1/2+k+n/2)^2} \mathrm{tr}_{\mathcal{V}_{a+n/2}} \left(q^{L_0} V_m(0) \right) \\ &= Z_0^D(\tau)\eta(\tau)^{-1} e^{2\pi i\tau\sigma^2} \sum_{k\in\mathbb{Z}} e^{2\pi i\tau(k+1/2)^2} e^{4\pi i(k+1/2)(\sigma\tau+\rho+1/2)} \\ &+ Z_{1/2}^D(\tau)\eta(\tau)^{-1} e^{2\pi i\tau\sigma^2} \sum_{k\in\mathbb{Z}} e^{2\pi i\tau k^2} e^{4\pi i k(\sigma\tau+\rho+1/2)} \\ &= -Z_0^D(\tau) e^{2\pi i\tau\sigma^2} \eta(\tau)^{-1} \theta_2(2\sigma\tau+2\rho|2\tau) \\ &+ Z_{1/2}^D(\tau) e^{2\pi i\tau\sigma^2} \eta(\tau)^{-1} \theta_3(2\sigma\tau+2\rho|2\tau). \end{aligned} \tag{5.53}$$

The above equation defines Z_0^D and $Z_{1/2}^D$, which are Fourier transforms of the one-point torus conformal block containing respectively only integer or half-integer shifts:

$$Z_{\epsilon/2}^D(\eta, a, m, \tau) = \sum_{n\in\mathbb{Z}+\frac{\epsilon}{2}} e^{in\eta} \mathrm{tr}_{\mathcal{V}_{a+n}} (q^{L_0} V_m(0)). \tag{5.54}$$

The trace over the Fock space has been resummed and yields the theta function and Dedekind eta factors. Notice further that the variables σ and ρ enter the above formula only through the two theta functions. Now we use the addition formula for theta functions

$$\theta_1(x-y|\tau)\theta_1(x+y|\tau) = \theta_3(2x|2\tau)\theta_2(2y|2\tau) - \theta_2(2x|2\tau)\theta_3(2y|2\tau) \quad (5.55)$$

and rewrite the relation between dual partition function and isomonodromic tau function (5.49) in the form

$$\begin{aligned} Z^D(\tau) = \mathcal{T}(\tau)\frac{e^{2\pi i\sigma^2\tau}}{\eta(\tau)^2} \\ \times\left(-\theta_2(2\sigma\tau+2\rho|2\tau)\theta_3(2Q|2\tau)+\theta_3(2\sigma\tau+2\rho|2\tau)\theta_2(2Q|2\tau)\right). \end{aligned} \quad (5.56)$$

Comparing the two formulas we find two relations, free from σ and ρ:

$$\begin{aligned} Z_0^D(\tau) &= \eta(\tau)^{-1}\theta_3(2Q|2\tau)\mathcal{T}(\tau)\,, \\ Z_{1/2}^D(\tau) &= \eta(\tau)^{-1}\theta_2(2Q|2\tau)\mathcal{T}(\tau)\,. \end{aligned} \quad (5.57)$$

One consequence of these formulas is the following:

$$\frac{\theta_3(2Q|2\tau)}{\theta_2(2Q|2\tau)} = \frac{Z_0^D(\tau)}{Z_{1/2}^D(\tau)}. \quad (5.58)$$

This allows us to express the solution Q of the isomonodromic system in terms of CFT/gauge theory objects. We will show in the next section that in fact one can recover the conformal blocks directly from the equation for Q (5.15). Another possibility is to use a "minimal choice" for the extra charge of the $U(1)$ boson, setting $\sigma=\rho=0$. Then the above expression becomes

$$\mathcal{T}(\eta,a,m,\tau) = \frac{\eta(\tau)^2}{\theta_1(Q|\tau)^2}Z^D(\eta,a,m,0,0,\tau). \quad (5.59)$$

Let us see explicitly which objects of the free fermionic CFT yield the classical, perturbative and instanton part of the dual partition function. Expanding the trace in the basis of descendants $|a,\mathbf{Y}\rangle$ one recovers the instanton expansion: the factor q^{L_0} yields the classical partition function and the instanton counting parameter, the normalization of the vertex operator (4.49) gives the perturbative contribution

$$\begin{aligned} N(a,m,a) &= \frac{G(1+m)^2G(1-m-2a)G(1-m+2a)}{G(1+2m)G(1-2a)G(1+2a)} \\ &= \frac{G(1+m)^2}{G(1+2m)}Z_{pert}(a,m) \end{aligned} \quad (5.60)$$

together with the extra m-dependent factor in (5.50). Finally, the expansion of the conformal block itself in the basis of descendants labeled by partitions

$|a, \mathbf{Y}\rangle$ yields as usual the instanton contributions to the partition function [8].

5.2 Tau function and Nekrasov functions

In this section we perform some more detailed computations regarding the tau function, providing further support for our formulae (5.57) and (5.59). We also clarify the relation between the Painlevé equation (5.15) and the $\mathcal{N} = 2^*$ gauge theory, providing the precise dictionary between the initial conditions of the equation and the Coulomb branch coordinates of the gauge theory.

5.2.1 Periodicity of the tau functions

Equation (5.15) is invariant under the transformation

$$Q \mapsto Q + \frac{n}{2} + \frac{\tau k}{2}, \qquad k, n \in \mathbb{Z}. \tag{5.61}$$

We denote this transformation by $\delta_{\frac{n}{2},\frac{k}{2}}$. One might ask the following question: what are the transformation properties of the tau function and dual partition functions under $\delta_{\frac{n}{2},\frac{k}{2}}$? The tau function after the transformation is defined by

$$\begin{aligned} 2\pi i \partial_\tau \log(\delta_{\frac{n}{2},\frac{k}{2}} \mathcal{T}) = (2\pi i)^2 (\partial_\tau Q + k/2)^2 - m^2 \wp(2Q) - 2m^2 \eta_1(\tau) = \\ = 2\pi i \partial_\tau \mathcal{T} + (2\pi i)^2 (k^2/4 + k \partial_\tau Q). \end{aligned} \tag{5.62}$$

Therefore

$$\delta_{\frac{n}{2},\frac{k}{2}} \mathcal{T} = C_{\frac{n}{2},\frac{k}{2}} \cdot q^{k^2/4} e^{2\pi i k Q} \mathcal{T} \tag{5.63}$$

Using now the relation (5.57) between the tau function and the dual partition functions we compute transformations of Z_0^D, $Z_{1/2}^D$:

$$\begin{aligned} \delta_{\frac{n}{2},\frac{k}{2}} Z_0^D = C_{\frac{n}{2},\frac{k}{2}} \eta(\tau)^{-1} \theta_3(2Q + n + k\tau | 2\tau) q^{k^2/4} e^{2\pi i k Q} \mathcal{T} = \\ = \left[\begin{array}{l} k \in 2\mathbb{Z} : C_{\frac{n}{2},\frac{k}{2}} \eta(\tau)^{-1} \theta_3(2Q\tau | 2\tau) \mathcal{T} = C_{\frac{n}{2},\frac{k}{2}} Z_0^D \\ k \in 1 + 2\mathbb{Z} : C_{\frac{n}{2},\frac{k}{2}} \eta(\tau)^{-1} \theta_2(2Q\tau | 2\tau) \mathcal{T} = C_{\frac{n}{2},\frac{k}{2}} Z_{1/2}^D \end{array} \right. \end{aligned} \tag{5.64}$$

In this way we see that the dual partition functions have much better behaviour than the tau function $\mathcal{T}$, under the shifts (5.61). As we will show in the following section, such shifts of parameters correspond to simple shifts of the initial data:

$$\delta_{\frac{n}{2},\frac{k}{2}}(\eta, a) = (\eta + 2\pi n, a + \frac{k}{2}) \tag{5.65}$$

[8]The precise statement is that to get Nekrasov factors one has to make $\mathrm{Res}\,_0 L(z)dz$ of rank 1 by an appropriate $U(1)$ shift. It is the standard AGT trick: see also discussion in the end of Section 5.3.

Therefore the transformations of the dual partition functions look as follows:

$$\delta_{\frac{n}{2},\frac{k}{2}} Z^D_{1/2} = e^{-i\eta\frac{k}{2}} \cdot \left[\begin{array}{l} k \in 2\mathbb{Z} : Z^D_{1/2} \\ k \in 1+2\mathbb{Z} : Z^D_0 \end{array} \right. , \tag{5.66}$$

and we can conclude that

$$C_{\frac{n}{2},\frac{k}{2}} = e^{-i\eta\frac{k}{2}}, \qquad \delta_{\frac{n}{2},\frac{k}{2}} \mathcal{T} = q^{k^2/4} e^{ik(2\pi Q - \frac{\eta}{2})} \mathcal{T}. \tag{5.67}$$

5.2.2 Asymptotic calculation of the tau function

We start from the following ansatz for the solution of the non-autonomous Calogero equation:

$$Q = \alpha\tau + \beta + \frac{1}{2\pi i} \sum_{n=0}^{\infty} \sum_{k=-n}^{\infty} c_{n,k} q^n e^{4\pi i k(\alpha\tau+\beta)} = \alpha\tau + \beta + \frac{1}{2\pi i} X \tag{5.68}$$

The series expansion of $\wp(z|\tau) + 2\eta_1(\tau) = -\partial_z^2 \log\theta_1(z|\tau)$ looks as follows:

$$\frac{1}{(2\pi i)^2}(-\partial_z^2 \log\theta_1(z|\tau)) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} k q^{kn} e^{2\pi i k z} + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k q^{kn} e^{-2\pi i k z} \tag{5.69}$$

Let us introduce the notation

$$s \equiv e^{2\pi i(\alpha\tau+\beta)}. \tag{5.70}$$

We also introduce a formal parameter of expansion ϵ in the following way: $q \mapsto q \cdot \epsilon^2$, $s \mapsto s \cdot \sqrt{\epsilon}$. Then we can rewrite equation (5.15) as

$$\sum_{n=0}^{\infty} \sum_{k=-n}^{\infty} (n+2\alpha k)^2 c_{n,k} q^n s^{2k} \epsilon^{2n+k} =$$
$$= m^2 \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} k^2 q^{kn} s^{2k} e^{kX(q,s)} \epsilon^{k(2n+1)} - m^2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k^2 q^{kn} s^{-2k} e^{-kX(q,s)} \epsilon^{k(2n-1)} \tag{5.71}$$

One can see that powers of ϵ in the r.h.s. are at least one, therefore higher-order coefficients $c_{n,k}$ become functions of the lower-order ones, and thus equation can be solved order-by-order starting from $c_{0,0} = 0$ [9].

[9] This algorithm is not optimal for the computation of non-trivial coefficients: as we will see later, it gives a lot of zero terms in the dual partition functions.

After this is done, we need to compute the logarithm of the isomonodromic tau function:

$$\log \mathcal{T} = \alpha^2 \log q + \sum_{n,k} q^n s^k b_{n,k} = \alpha^2 \log q + Y(q,s)$$

$$\sum_{n,k} (n+2\alpha k) q^n s^k b_{n,k} = \left(\sum_{n,k} (n+2\alpha k) q^n s^{2k} c_{n,k} \right)^2 - \tag{5.72}$$

$$-m^2 \left(\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} k q^{kn} s^{2k} e^{kX(q,s)} + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k q^{kn} s^{-2k} e^{-kX(q,s)} \right)$$

Then we compute the two dual partition functions:

$$Z_0^D = q^{\alpha^2 - 1/24} \sum_{n=1}^{\infty} p(n) q^n \cdot \sum_{n=-\infty}^{\infty} q^{n^2} s^{2n} e^{2nX(q,s)} \cdot e^{Y(q,s)+2\alpha X(q,s)} = \sum_{n,k} z_{n,k} q^{n+\alpha^2-1/24} s^{2k},$$

$$Z_{1/2}^D = q^{\alpha^2 - 1/24} \sum_{n=1}^{\infty} p(n) q^n \cdot \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} s^{2n+1} e^{(2n+1)X(q,s)} \cdot e^{Y(q,s)+2\alpha X(q,s)} =$$

$$= \sum_{n,k} z'_{n,k} q^{n+\alpha^2+5/24} s^{2k-1}. \tag{5.73}$$

Results

We solved the equation asymptotically up to ϵ^6.

All the information about the function $f(q,s) = \sum f_{n,k} q^n s^{2k}$ is encoded in the list of non-zero coefficients $f_{n,k}$: we will denote such coefficients by points (k,n) in the integer plane (sometimes it will be shifted by some fractional numbers which we neglect). We call the set of such points *support* of the function f.

First we show the support of the solution $X(q,s)$ in Fig. 5.2. In this picture the gray region contains all coefficients that were computed in the asymptotic expansion. Dotted lines show monomials with the same order of ϵ. The full support is bounded from below by the lines $n = 0$ and $n = -k$.

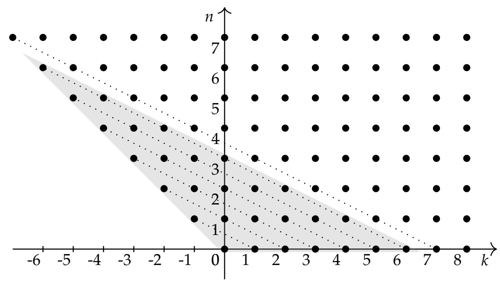


FIGURE 5.2: Support of $X(q,s)$

The first few terms of the expansion look as follows:

$$\begin{aligned} c_{0,1} &= \frac{m^2}{4\alpha^2}, \quad c_{1,-1} = -\frac{m^2}{(2\alpha-1)^2}, \quad c_{0,2} = \frac{m^2\left(8\alpha^2+m^2\right)}{32\alpha^4}, \\ c_{1,0} &= -\frac{(4\alpha-1)m^4}{2\alpha^2(2\alpha-1)^2}, \quad c_{2,-2} = -\frac{m^2\left(8\alpha^2-8\alpha+m^2+2\right)}{2(2\alpha-1)^4}, \quad \ldots \end{aligned} \tag{5.74}$$

The support of the dual partition functions is shown in Fig. 5.3.

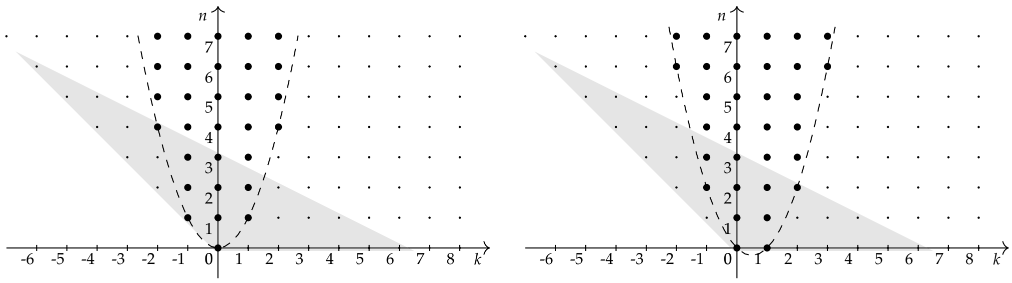


FIGURE 5.3: Support of Z_0^D (left) and of $Z_{1/2}^D$ (right).

We see that some non-trivial cancellation happened and a lot of coefficients that naively might be non-zero (denoted by small dots) actually vanish.

Values of the first non-trivial coefficients are given by

$$\begin{gathered} z_{0,0} = 1, \quad z'_{0,0} = 1, \quad z'_{0,1} = 1 - \frac{m^2}{4\alpha^2}, \quad z_{1,-1} = \frac{(m^2-(1-2\alpha))^2}{(1-2\alpha)^2}, \\ z_{1,1} = \frac{(m^2-4\alpha^2)^2(m^2-(2\alpha+1)^2)}{16\alpha^4(2\alpha+1)^2}, \quad \ldots \end{gathered} \tag{5.75}$$

We also found experimentally that normalized values of all other non-trivial coefficients can be given in terms of a single function, which will be identified with the toric conformal block:

$$\mathcal{B}(a,m,q)=1+q\left(\frac{(m^2-1)m^2}{2a^2}+1\right)+q^2\left(\frac{3(m^2-4)(m^2-1)^2m^2}{16(a^2-\frac{1}{4})^2}-\right.$$
$$\left.-\frac{(m^2-3)(m^2-1)(m^2+2)m^2}{4(a^2-\frac{1}{4})}+\frac{(m^2-1)(m^4-m^2+2)m^2}{4a^2}+2\right)+$$
$$+q^3\left(\frac{(m^2-9)(m^2-4)^2(m^2-1)^2m^2}{36(a^2-1)^2}+\frac{(m^2-4)(m^2-1)^2(2m^4-2m^2+9)m^2}{48(a^2-\frac{1}{4})^2}-\right.$$
$$-\frac{(m^2-4)(m^2-1)(11m^6-106m^4+131m^2+108)m^2}{216(a^2-1)}+$$
$$+\frac{(m^2-1)(3m^8-22m^6+65m^4-46m^2+24)m^2}{24a^2}-$$
$$\left.-\frac{(m^2-1)(8m^8-24m^6+15m^4+m^2-162)m^2}{108(a^2-\frac{1}{4})}+3\right)+O(q^3)=\sum_{n=0}^{\infty}\mathcal{B}_n(a,m)q^n \tag{5.76}$$

Namely, the ratios of coefficients are

$$z_{n,0}/z_{0,0}=\mathcal{B}_n(\alpha,m),\quad n=0,1,2,3,\quad z_{2,1}/z_{1,1}=\mathcal{B}_1(\alpha+1,m),\quad n=0,1,2,$$
$$\frac{z_{n+1,-1}}{z_{1,-1}}=\mathcal{B}_n(\alpha-1,m),\quad \frac{z_{3,1}}{z_{1,1}}=\mathcal{B}_2(\alpha-1,m),\quad \frac{z'_{-1,3}}{z'_{-1,2}}=\mathcal{B}_1(a-\frac{3}{2},m)$$
$$\frac{z'_{n,0}}{z'_{0,0}}=\mathcal{B}_n(\alpha-\frac{1}{2},m),\quad n=0,1,2,3,\quad \frac{z'_{n,1}}{z'_{0,1}}=\mathcal{B}_n(\alpha+\frac{1}{2},m),\quad n=0,1,2. \tag{5.77}$$

We see that the latter formula is in complete agreement with (5.57) if $\mathcal{B}$ is a conformal block. In fact, we can check that it actually coincides with the AGT formula

$$\mathcal{B}(a,m,q)=\prod_{n=1}^{\infty}(1-q^n)^{1-2m^2}\sum_{Y_+,Y_-}q^{|Y_+|+|Y_-|}\prod_{\epsilon,\epsilon'=\pm}\frac{N_{Y_\epsilon,Y_{\epsilon'}}(m+(\epsilon-\epsilon')a)}{N_{Y_\epsilon,Y_{\epsilon'}}((\epsilon-\epsilon')a)} \tag{5.78}$$

where Nekrasov factors are given by

$$N_{\lambda,\mu}(x)=\prod_{s\in\lambda}(x+a_\lambda(s)+l_\mu(s)+1)\prod_{t\in\mu}(x-l_\lambda(t)-a_\mu(t)-1). \tag{5.79}$$

5.2.3 Asymptotic computation of monodromies

At the moment we have two in principle different parameterizations: one in terms of (α,β), which are the initial data of the Painlevé equation, and another in terms of the monodromy data (a,η), which are the Coulomb branch coordinates, or equivalently the conformal block parameters corresponding to the pants decomposition. We need to know precisely how they are related.

To compute this identification we use the fact that the evolution is isomonodromic, so monodromies can be computed in the limit $\tau \to +i\infty$.

$$\theta_1(z|\tau) = 2q^{1/8}\left(\sin \pi z - q \sin 3\pi z + \ldots\right). \tag{5.80}$$

One can take just the first term of expansion until it becomes smaller than the first correction. This occurs when $\sin \pi z \approx e^{2\pi i \tau} \sin 3\pi z$, so for $z = \pm\tau$. We will choose two copies of the A-cycle with $\mathrm{Im}(z) = \pm\mathrm{Im}(\tau/2)$ and work in the region between them. So for all computations to be consistent we need to have $-1/4 < \mathrm{Re}(\alpha) < 1/4$ (one can easily overcome this constraint taking more terms in the expansion). Also convergence of the series (5.68) requires $\alpha > 0$. So for simplicity we just take α to have a sufficiently small positive real part.

Our approximation for x is then

$$x(u,z) \approx \frac{\pi \sin \pi(z-u)}{\sin \pi z \sin \pi u} = \frac{2\pi i}{e^{2\pi i u}-1} - \frac{2\pi i}{e^{2\pi i z}-1}. \tag{5.81}$$

The first terms of the expansion of Q look as

$$Q(\tau) = \alpha\tau + \beta + \frac{m^2}{8\pi i \alpha^2} e^{4\pi i(\alpha\tau+\beta)} + \ldots \tag{5.82}$$

The leading behavior of the connection matrix is

$$L(z|\tau) = 2\pi i \begin{pmatrix} \alpha + \frac{m^2}{2\alpha} e^{4\pi i(\alpha\tau+\beta)} & \frac{m}{e^{4\pi i(\alpha\tau+\beta)}-1} - \frac{m}{e^{2\pi i z}-1} \\ \frac{m}{e^{-4\pi i(\alpha\tau+\beta)}-1} - \frac{m}{e^{2\pi i z}-1} & -\alpha - \frac{m^2}{2\alpha} e^{4\pi i(\alpha\tau+\beta)} \end{pmatrix}. \tag{5.83}$$

Further expanding up to first order in $e^{4\pi i \alpha\tau}$ we get

$$L(z|\tau) = 2\pi i \begin{pmatrix} \alpha & -\frac{m e^{2\pi i z}}{e^{2\pi i z}-1} \\ -\frac{m}{e^{2\pi i z}-1} & -\alpha \end{pmatrix} + 2\pi i e^{4\pi i(\alpha\tau+\beta)} \begin{pmatrix} \frac{m^2}{2\alpha} & -m \\ m & -\frac{m^2}{2\alpha} \end{pmatrix}. \tag{5.84}$$

One may notice that there is an equality

$$L(z|\tau) = U_0 L_0(z|\tau) U_0^{-1} + o\left(e^{4\pi i(\alpha\tau+\beta)}\right), \tag{5.85}$$

where

$$U_0 = 1 + \frac{m}{2\alpha} e^{4\pi i(\alpha\tau+\beta)} \sigma_1, \quad L_0(z|\tau) = 2\pi i \begin{pmatrix} \alpha & -\frac{m e^{2\pi i z}}{e^{2\pi i z}-1} \\ -\frac{m}{e^{2\pi i z}-1} & -\alpha \end{pmatrix}. \tag{5.86}$$

This equality is reminiscent of the isomonodromic deformation equation (5.13). The solution of the linear system in the region $z\sim\frac{\tau}{2}$ is given by

$$\begin{gathered}Y(z)=(1-e^{2\pi iz})^m U_0\times\\ \times\begin{pmatrix} {}_2F_1(m,m+2\alpha,2\alpha,e^{2\pi iz}) & \frac{me^{2\pi iz}}{1-2\alpha}{}_2F_1(1+m,1+m-2\alpha,2-2\alpha,e^{2\pi iz}) \\ \frac{m}{2\alpha}{}_2F_1(1+m,m+2\alpha,1+2\alpha,e^{2\pi iz}) & {}_2F_1(m,1+m-2\alpha,1-2\alpha,e^{2\pi iz})\end{pmatrix}\times\\ \times\mathrm{diag}((-e^{2\pi iz})^{\alpha},(-e^{2\pi iz})^{-\alpha})\end{gathered} \tag{5.87}$$

Now we compute the analytic continuation of this solution to the region $z\sim -\frac{\tau}{2}$ along the imaginary line $\mathrm{Re}(z)=1/2$ [10] :

$$Y(z)=\sigma_1 Y(-z)\sigma_1\widetilde{M_B},\qquad \widetilde{M_B}=\begin{pmatrix}\frac{\Gamma(2\alpha)^2}{\Gamma(2\alpha+m)\Gamma(2\alpha-m)} & \frac{\Gamma(2-2\alpha)\Gamma(-1+2\alpha)}{\Gamma(m)\Gamma(1-m)}\\ \frac{\Gamma(1-2\alpha)\Gamma(2\alpha)}{\Gamma(1-m)\Gamma(m)} & \frac{\Gamma(1-2\alpha)^2}{\Gamma(1-m-2\alpha)\Gamma(1+m-2\alpha)}\end{pmatrix}. \tag{5.89}$$

We wish to compute also the B-cycle monodromy. Its defining relation is

$$Y(\frac{\tau}{2}+x)=e^{2\pi i\mathbf{Q}}Y(-\frac{\tau}{2}+x)M_B. \tag{5.90}$$

Using (5.89) we get

$$Y(\frac{\tau}{2}+x)=e^{2\pi i\mathbf{Q}}\sigma_1 Y(\frac{\tau}{2}-x)\sigma_1\widetilde{M_B}M_B. \tag{5.91}$$

We write this expression down keeping only the first orders:

$$\begin{gathered}(1+\frac{m}{2\alpha}e^{4\pi i(\alpha\tau+\beta)}\sigma_1)\begin{pmatrix}1 & 0\\ \frac{m}{2\alpha} & 1\end{pmatrix}\times\mathrm{diag}(e^{\pi i\alpha\tau},e^{-\pi i\alpha\tau})\times\mathrm{diag}((-e^{2\pi ix})^{\alpha},(-e^{2\pi ix})^{-\alpha})=\\ =\mathrm{diag}\left(e^{2\pi i(\alpha\tau+\beta)},e^{-2\pi i(\alpha\tau+\beta)}+\frac{m^2}{4\alpha^2}e^{2\pi i(\alpha\tau+\beta)}\right)(1+\frac{m}{2\alpha}e^{4\pi i(\alpha\tau+\beta)}\sigma_1)\begin{pmatrix}1 & \frac{m}{2\alpha}\\ 0 & 1\end{pmatrix}\times\\ \times\mathrm{diag}(e^{-\pi i\alpha\tau},e^{\pi i\alpha\tau})\times\mathrm{diag}((-e^{2\pi ix})^{\alpha},(-e^{2\pi ix})^{-\alpha})\times\widetilde{M_B}M_B.\end{gathered} \tag{5.92}$$

By using also the relation

$$\sigma_1\widetilde{M_B}\sigma_1=\widetilde{M_B}^{-1} \tag{5.93}$$

we can then express M_B as

$$M_B=\sigma_1\widetilde{M_B}\sigma_1\times\mathrm{diag}(e^{-2\pi i\beta},e^{2\pi i\beta})+O(e^{4\pi i\alpha\tau}). \tag{5.94}$$

[10]To do this we use connection formula for hypergeometric function

$$\begin{gathered}{}_2F_1(a,b,c,z)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1(a,a-c+1,a-b+1,z^{-1})+\\ +\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1(b,b-c+1,b-a+1,z^{-1})\end{gathered} \tag{5.88}$$

We see that to our precision M_B is actually constant when $\tau \to i\infty$, its value is given by

$$M_B = \begin{pmatrix} \frac{\Gamma(1-2\alpha)^2}{\Gamma(1-m-2\alpha)\Gamma(1+m-2\alpha)} e^{-2\pi i\beta} & \frac{\sin \pi m}{\sin \pi\alpha} e^{2\pi i\beta} \\ -\frac{\sin \pi m}{\sin \pi\alpha} e^{-2\pi i\beta} & \frac{\Gamma(2\alpha)^2}{\Gamma(2\alpha+m)\Gamma(2\alpha-m)} e^{2\pi i\beta} \end{pmatrix}. \tag{5.95}$$

To compare this with the monodromy matrix (5.29) computed by braiding we now change normalization

$$e^{2\pi i\beta} = e^{2\pi i\beta'}/r, \qquad M'_B = \mathrm{diag}(r^{1/2}, r^{-1/2}) M_B \mathrm{diag}(r^{-1/2}, r^{1/2}), \tag{5.96}$$

where

$$r = \frac{\Gamma(2\alpha)\Gamma(1-2\alpha-m)}{\Gamma(1-2\alpha)\Gamma(2\alpha-m)}. \tag{5.97}$$

The new monodromy is

$$M'_B = \begin{pmatrix} \frac{\sin \pi(2\alpha-m)}{\sin 2\pi\alpha} e^{-2\pi i\beta'} & \frac{\sin \pi m}{\sin 2\pi\alpha} e^{2\pi i\beta'} \\ -\frac{\sin \pi m}{\sin 2\pi\alpha} e^{-2\pi i\beta'} & \frac{\sin \pi(2\alpha+m)}{\sin 2\pi\alpha} e^{2\pi i\beta'} \end{pmatrix}. \tag{5.98}$$

Corresponding A-cycle monodromy is clearly given by the formula

$$M'_A = M_A = \begin{pmatrix} e^{2\pi i a} & 0 \\ 0 & e^{-2\pi i a} \end{pmatrix}. \tag{5.99}$$

These monodromies are related to those computed from CFT (5.29) by a conjugation with the matrix

$$C = \begin{pmatrix} 0 & -ie^{\pi i a} \\ ie^{-\pi i a} & \end{pmatrix}. \tag{5.100}$$

We can in fact check explicitly that

$$\begin{gathered} M'_A = C M_A^{(CFT)} C^{-1} = \begin{pmatrix} e^{2\pi i a} & 0 \\ 0 & e^{-2\pi i a} \end{pmatrix}, \\ M'_B = C M_B^{(CFT)} C^{-1} = \begin{pmatrix} \frac{\sin \pi(2a-m)}{\sin 2\pi a} e^{-i\eta/2} & \frac{\sin \pi m}{\sin 2\pi a} e^{i\eta/2} \\ -\frac{\sin \pi m}{\sin 2\pi a} e^{-i\eta/2} & \frac{\sin \pi(2a+m)}{\sin 2\pi a} e^{i\eta/2} \end{pmatrix}, \end{gathered} \tag{5.101}$$

after we made the identification

$$\alpha = a, \qquad \eta = 4\pi\beta'. \tag{5.102}$$

5.2.4 Initial conditions of Painlevé and Coulomb branch coordinates

To further check that the identification (5.102) is indeed correct, one may check that the two asymptotic expansions have the following forms:

$$\begin{aligned} \eta(\tau)^{-1}\theta_3(2Q|2\tau)\mathcal{T}(\tau) &= q^{-1/24}\sum_{n\in\mathbb{Z}} C_n e^{4\pi i n\beta} q^{(\alpha+n)^2}\mathcal{B}(\alpha+n,m,q) \\ \eta(\tau)^{-1}\theta_2(2Q|2\tau)\mathcal{T}(\tau) &= q^{-1/24}\sum_{n\in\mathbb{Z}+\frac{1}{2}} C_n e^{4\pi i n\beta} q^{(\alpha+n)^2}\mathcal{B}(\alpha+n,m,q) \end{aligned} \tag{5.103}$$

Where the structure constants are given by explicit formula

$$\begin{aligned} C_n &= \frac{G(1-m+2(\alpha+n))G(1-m-2(\alpha+n))}{G(1+2(\alpha+n))G(1-2(\alpha+n))} \\ &\times \frac{G(1+2\alpha)G(1-2\alpha)}{G(1-m+2\alpha)G(1-m-2\alpha)}\left(\frac{\Gamma(2\alpha)\Gamma(1-m-2\alpha)}{\Gamma(1-2\alpha)\Gamma(2\alpha-m)}\right)^{2n}. \end{aligned} \tag{5.104}$$

We can in particular verify that $C_0 = C_{-\frac{1}{2}} = 1$, which is consistent with the "experimental" results from the previous section. We also see that after the redefinition[11]

$$e^{2i\beta'} = e^{2i\beta}\frac{\Gamma(2\alpha)\Gamma(1-m-2\alpha)}{\Gamma(1-2\alpha)\Gamma(2\alpha-m)}, \tag{5.105}$$

the tau functions may be rewritten as

$$\begin{aligned} \eta(\tau)^{-1}\theta_3(2Q|\tau)\mathcal{T}(\tau) &= \frac{q^{-1/24}}{C(\alpha)}\sum_{n\in\mathbb{Z}} C(\alpha+n) e^{4\pi i n\beta'} q^{(\alpha+n)^2}\mathcal{B}(\alpha+n,m,q) \\ \eta(\tau)^{-1}\theta_2(2Q|\tau)\mathcal{T}(\tau) &= \frac{q^{-1/24}}{C(\alpha)}\sum_{n\in\mathbb{Z}+\frac{1}{2}} C(\alpha+n) e^{4\pi i n\beta'} q^{(\alpha+n)^2}\mathcal{B}(\alpha+n,m,q) \end{aligned} \tag{5.106}$$

where

$$C(\alpha) = \frac{G(1-m+2\alpha)G(1-m-2\alpha)}{G(1+2\alpha)G(1-2\alpha)}. \tag{5.107}$$

Again, comparing with expressions (5.57) we find $a = \alpha$, $\eta = 4\pi\beta'$, consistently with what we found in the asymptotic computation of the monodromy matrices.

5.2.5 A self-consistency check

In the final part of this section we compute the derivative with respect to τ of the conformal block, using the generalized Wick's theorem [134], and provide a consistency check for the relation 5.34 between free fermion CFT

[11] Note this is the same redefinition of (5.127), if $\eta = 4\pi\beta'$, which we will see is the case.

and the linear system 5.5. Namely, we compute

$$\begin{aligned} F(w,y)_{\alpha\beta} &= 2\pi i\partial_\tau \left(Y(w)^{-1}\Xi(y-w)Y(y)\right)_{\alpha\beta} = 2\pi i\partial_\tau \frac{\langle\bar\psi_\alpha(w)\psi_\beta(y)V_m(0)\rangle}{\langle V_m(0)\rangle} \\ &= 2\pi i\frac{\partial_\tau\langle\bar\psi_\alpha(w)\psi_\beta(y)V_m(0)\rangle}{\langle V_m(0)\rangle} - 2\pi i\frac{\langle\bar\psi_\alpha(w)\psi_\beta(y)V_m(0)\rangle}{\langle V_m(0)\rangle}\frac{\partial_\tau\langle V_m(0)\rangle}{\langle V_m(0)\rangle}. \end{aligned} \tag{5.108}$$

We then note that the derivative of any correlator with respect to the modular parameter τ can be realized by an integral over the A-cycle of an insertion of the energy-momentum tensor $T(z)$:

$$2\pi i\partial_\tau\langle\mathcal{O}\rangle = \oint_A \langle\mathcal{O}T(z)\rangle dz. \tag{5.109}$$

Further, we can use the explicit expression of the free fermion energy-momentum tensor

$$T(z) = \frac{1}{2}\sum_\gamma (:\partial\bar\psi_\gamma(z)\psi_\gamma(z): + :\partial\psi_\gamma(z)\bar\psi_\gamma(z):) \tag{5.110}$$

where : : denotes regular part of the OPE. Then,

$$\begin{aligned} F(w,y)_{\alpha\beta} = \sum_\gamma \oint_A dz \frac{\langle:\partial\bar\psi_\gamma(z)\psi_\gamma(z):\bar\psi_\alpha(w)\psi_\beta(y)V_m(0)\rangle}{\langle V_m(0)\rangle} - \\ - \frac{\langle:\partial\bar\psi_\gamma(z)\psi_\gamma(z):V_m(0)\rangle\langle\bar\psi_\alpha(w)\psi_\beta(y)V_m(0)\rangle}{\langle V_m(0)\rangle^2}. \end{aligned} \tag{5.111}$$

Here we added a total derivative to $T(z)$ since it does not change correlator. Now we can compute this expression using the generalized Wick theorem [134]. The second term cancels the one containing pairing between $\bar\psi_\alpha(w)$ and $\psi_\beta(y)$. So finally we get only

$$\begin{aligned} F(w,y)_{\alpha\beta} &= \sum_\gamma \oint_A dz \frac{\langle\partial\bar\psi_\gamma(z)\psi_\beta(y)V_m(0)\rangle\langle\psi_\gamma(z)\bar\psi_\alpha(w)V_m(0)\rangle}{\langle V_m(0)\rangle^2} \\ &= -\sum_\gamma \oint_A dz \left(Y(w)^{-1}\Xi(z-w)Y(z)\right)_{\alpha\gamma} \partial_z \left(Y(z)^{-1}\Xi(y-z)Y(y)\right)_{\gamma\beta} \\ &= \sum_\gamma \oint_A dz\partial_z \left(Y(w)^{-1}\Xi(z-w)Y(z)\right)_{\alpha\gamma} \left(Y(z)^{-1}\Xi(y-z)Y(y)\right)_{\gamma\beta} \\ &= \oint_A dz \left(Y(w)^{-1}\left(\Xi(z-w)L(z) + \partial_z\Xi(z-w)\right)\Xi(y-z)Y(y)\right)_{\alpha\beta}. \end{aligned} \tag{5.112}$$

Comparing (5.108) with (5.112) and using (5.13) we get the identity

$$\begin{aligned} M(w)\Xi(y-w) - \Xi(y-w)M(y) + 2\pi i\partial_\tau\Xi(y-w) = \\ = \oint_A dz \left(\Xi(z-w)L(z) + \partial_z\Xi(z-w)\right)\Xi(y-z). \end{aligned} \tag{5.113}$$

We can plug in the explicit expression for L, M and see that the relation will hold iff the following relations are satisfied:

$$\begin{aligned} 2\pi i \partial_\tau x(Q, w-y) &= p\partial_Q x(Q, w-y) - \partial_w \partial_Q x(Q, w-y) \\ &= \oint_A dz \left[p x(Q, w-z) x(Q, z-y) - \partial_w x(Q, w-z) x(Q, z-y) \right], \end{aligned} \tag{5.114}$$

$$\begin{aligned} &y(2Q, w) x(Q, y-w) - y(2Q, y) x(-Q, y-w) \\ &= \oint_A dz x(2Q, z) x(-Q, z-w) x(Q, y-z). \end{aligned} \tag{5.115}$$

To find the r.h.s. we need to compute two integrals:

$$\begin{aligned} I_1(w, y) &= \oint_A dz\, x(Q, w-z) x(Q, z-y), \\ I_2(w, y) &= \oint_A dz\, x(q_1, z-w) x(q_2 - q_1, z) x(q_2, y-z). \end{aligned} \tag{5.116}$$

I_1 and $\partial_w I_1$ contribute to diagonal elements, whereas I_2 defines off-diagonal ones. First we consider the integral over the boundary of the cut torus:

$$I_1^\epsilon = \oint_{\partial \mathbb{T}} dz\, x(Q, w-z) x(Q+\epsilon, z-y). \tag{5.117}$$

The function inside the integral is not periodic under $z \to z + \tau$, but rather acquires a phase $e^{-2\pi i \epsilon}$. If we take the combination of two contour integrals over A-cycles shifted by τ in the opposite directions, they enter with opposite signs times the quasi-periodicity:

$$I_1^\epsilon = (1 - e^{-2\pi i \epsilon}) \oint_A dz\, x(Q, w-z) x(Q+\epsilon, z-y). \tag{5.118}$$

On the other hand, this integral can be computed by residues:

$$I_1^\epsilon = 2\pi i \left[x(Q+\epsilon, w-y) - x(Q, w-y) \right]. \tag{5.119}$$

Now we take a limit $\epsilon = 0$:

$$I_1 = 2\pi i \lim_{\epsilon \to 0} \frac{x(Q+\epsilon, w-y) - x(Q, w-y)}{1 - e^{-2\pi i \epsilon}} = \partial_Q x(Q, w-y). \tag{5.120}$$

By plugging this result in (5.114) we see that it is satisfied. The second integral can be computed in the same manner, noting that the integrand now has quasi-periodicity $e^{2\pi i \epsilon}$:

$$\begin{aligned} I_2^\epsilon &= \oint_{\partial \mathbb{T}} dz\, x(q_1, z-w) x(q_2 - q_1 + \epsilon, z) x(q_2, y-z) = \\ &= 2\pi i \left(x(q_1, y-w) x(q_2 - q_1 + \epsilon, y) - x(q_1, -w) x(q_2, y) \right. \\ &\qquad \left. - x(q_2 - q_1 + \epsilon, w) x(q_2, y-w) \right) \end{aligned} \tag{5.121}$$

The expansion of this integral up to the first order yields I_2:

$$I_2(w,z) = y(2Q,w)x(Q,y-w) - y(2Q,y)x(-Q,y-w) \tag{5.122}$$

because of which (5.115) is satisfied.

5.3 Gauge theory, topological strings and Fredholm determinants

We have shown how the free fermion formulation is necessary, in the case of genus one, in order to construct the correct monodromy representation. In [29] it was shown that, in the case of the four-punctured sphere, one can sum up the expression for the tau function into a single Fredholm determinant by means of the generalized Wick theorem for free fermions. This determinant for the generic tau function on the sphere with n punctures was also constructed in [135] in a mathematically rigorous way. This was then shown to satisfy the Jimbo-Miwa-Ueno definition of the tau function and to reproduce the expansion of the relevant Nekrasov partition function. More recent understanding of such determinant formulas in the sphere case, together with a simplified proof, can be found in [136].

In [135] the RHP is solved by decomposing the n-punctured sphere in trinions or pairs of pants, which are three-punctured spheres, thus reducing the RHP on the sphere with n punctures to the problem of properly gluing solutions to RHPs on three-punctured spheres, with punctures at $0, 1, \infty$. These are given, normalized by their asymptotics at zero, by

$$Y_0(w) = (1-w)^{(m-\gamma)} \times \\ \times \begin{pmatrix} {}_2F_1(m, m+2a, 2a, w) & \frac{-mw}{2a-1} {}_2F_1(1+m, 1+m-2a, 2-2a, w) \\ \frac{m}{2a} {}_2F_1(1+m, m+2a, 1+2a, w) & {}_2F_1(m, 1+m-2a, 1-2a, w) \end{pmatrix}, \tag{5.123}$$

where ${}_2F_1$ are hypergeometric functions, and γ is a $U(1)$ shift about which we will comment later. Note that the solution above is essentially the same as what we saw in the previous secion, equation (5.87), after the change of variables from spherical to cylindrical $w = e^{2\pi i z}$. There, it appeared in the study of the asymptotic behavior of the solution Y on the torus. The solution above is well-defined as a series in w, convergent for $|w| < 1$, so we also define another normalized solution of the same problem, well-defined as a series in w^{-1}:

$$Y_\infty(w) = \sigma_1 Y_0(1/w) \sigma_1. \tag{5.124}$$

In this subsection, motivated by [29] and [135], we compute the relevant Fredholm determinant by pants decomposition of the one-punctured torus as in Fig. 5.4. Namely, we expand the trace in (5.49) by inserting the identity operator on the free-fermion states, compute the matrix elements of V_m by using the generalized Wick theorem, and finally arrive at the Fredholm determinant expression given below.

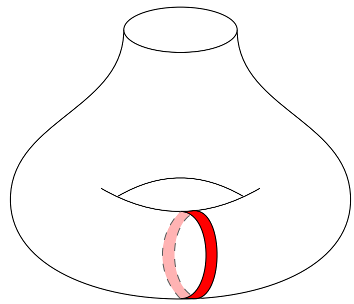


FIGURE 5.4: Pants decomposition of one-punctured torus

The one-punctured torus is obtained from the three-punctured sphere by gluing two legs of the trinion as in Figure 5.4. We will take these to be the legs corresponding to the punctures at $0, \infty$ in spherical coordinates, or $\pm i\infty$ in cylindrical coordinates. We thus have to show how this translates to an operation on the solution of the three-punctured RHP.

To glue the solutions defined at zero and infinity, we define the following integral kernels:

$$\mathsf{a}(w,w') = D\frac{Y_0(qw)^{-1}Y_0(w') - \mathbb{I}}{qw - w'}, \qquad \mathsf{b}(w,w') = -D\frac{Y_0(qw)^{-1}Y_\infty(w')}{qw - w'},$$
$$\mathsf{c}(w,w') = D^{-1}\frac{Y_\infty(w/q)^{-1}Y_0(w')}{w/q - w'}, \qquad \mathsf{d}(w,w') = D^{-1}\frac{\mathbb{I} - Y_\infty(w/q)^{-1}Y_\infty(w')}{w/q - w'}, \tag{5.125}$$

where the diagonal matrix D is given by the formula

$$D = -q^{(1+\sigma)}e^{2\pi i\rho}\mathrm{diag}(q^{-a}e^{-2\pi i\beta}, q^a e^{2\pi i\beta}), \tag{5.126}$$

and $e^{2\pi i\beta}$ is given by

$$e^{2\pi i\beta} = e^{i\eta/2}\frac{\Gamma(1-2a)\Gamma(2a-m)}{\Gamma(2a)\Gamma(1-2a-m)}. \tag{5.127}$$

Let us now specify the Hilbert spaces on which the above operators act. There are at least two equivalent choices: the first one is to consider them as operators, acting on the space of (row) vector-valued functions

$$\mathcal{H} = L^2(S^1) \otimes \mathbb{C}^2 \tag{5.128}$$

on the circle $S^1 = \{w, |w| = R\}$, where $|q| < R < 1$. In this realisation, the action of an operator A on a function f is defined by the integral

$$(\mathsf{A}f)(w) = \oint \frac{dw'}{2\pi i} f(w')\mathsf{A}(w', w). \tag{5.129}$$

We will instead employ a different description of this Hilbert space, which is better suited for computational purposes and that makes contact naturally with the free fermion description: we will use the Fourier expansions of the kernels

$$\mathsf{a}(w,w') = \sum_{p,q\in\mathbb{Z}'_-} \frac{\mathsf{a}_{pq}}{w^{p+\frac{1}{2}} w'^{q+\frac{1}{2}}}, \qquad \mathsf{b}(w,w') = \sum_{p\in\mathbb{Z}'_-,q\in\mathbb{Z}'_+} \frac{\mathsf{b}_{pq}}{w^{p+\frac{1}{2}} w'^{q+\frac{1}{2}}}$$
$$\mathsf{c}(w,w') = \sum_{p\in\mathbb{Z}'_+,q\in\mathbb{Z}'_-} \frac{\mathsf{c}_{pq}}{w^{p+\frac{1}{2}} w'^{q+\frac{1}{2}}}, \qquad \mathsf{d}(w,w') = \sum_{p,q\in\mathbb{Z}'_+} \frac{\mathsf{d}_{pq}}{w^{p+\frac{1}{2}} w'^{q+\frac{1}{2}}}, \tag{5.130}$$

where $\mathbb{Z}'_+ = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{3}, \ldots\}$, $\mathbb{Z}'_- = \{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots\}$ are positive and negative half-integer subsets respectively, and $\mathbb{Z}' = \mathbb{Z}'_- \sqcup \mathbb{Z}'_+$. In terms of Fourier modes we can describe our Hilbert space as $\mathcal{H} = \mathbb{C}^{\mathbb{Z}'} \otimes \mathbb{C}^2$, whose basis vectors are labelled by a pair (p, α) of one half-integer number $p \in \mathbb{Z}'$ and one matrix index $\alpha \in \{1, 2\}$. One can also define two subspaces of $\mathcal{H}$: $\mathcal{H}_+$, corresponding to $\mathbb{Z}'_+$ — the subspace of non-negative Fourier modes, and $\mathcal{H}_-$, corresponding to $\mathbb{Z}'_-$ — the subspace of negative Fourier modes. We can easily see from (5.130) that the operators $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ act non-trivially only between the following sub-spaces:

$$\mathsf{a} : \mathcal{H}_+ \to \mathcal{H}_-, \quad \mathsf{b} : \mathcal{H}_- \to \mathcal{H}_-, \quad \mathsf{c} : \mathcal{H}_+ \to \mathcal{H}_+, \quad \mathsf{d} : \mathcal{H}_- \to \mathcal{H}_+. \tag{5.131}$$

Using the above definitions, together with the results of [135], one can write down the following expression for the dual partition function $Z^D(\tau)$:

$$Z^D(\tau) = N(a, m, a) q^{a^2+(\sigma+1/2)^2-1/12} e^{2\pi i(\rho+1/2)} \prod_{n=1}^{\infty} (1-q^n)^{-2\gamma^2} \det(\mathbb{I} + K), \tag{5.132}$$

where the operator K can be written as a block matrix with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$:

$$K = \begin{pmatrix} \mathsf{c} & \mathsf{d} \\ \mathsf{a} & \mathsf{b} \end{pmatrix}. \tag{5.133}$$

Alternatively, one can sum up the whole Fourier modes and define a single matrix integral kernel as

$$K(w,w') = \mathsf{a}(w,w') + \mathsf{b}(w,w') + \mathsf{c}(w,w') + \mathsf{d}(w,w'), \tag{5.134}$$

acting on $L^2(S^1) \otimes \mathbb{C}^2$. Let us now make some comments on (5.132). The factor $N(a, m, a)$, that we defined in (5.60), accounts both for the vertex normalization in CFT and the one-loop factor of the Nekrasov partition function, while q^{a^2} is the classical contribution to the partition function. The Fredholm determinant $\det(1 + K)$ is then identified with the instanton part of Nekrasov-Okounkov partition function for $\mathcal{N} = 2^*$ gauge theory, up to a free fermion normalization depending on the background charges (γ, σ, ρ), which are arbitrary and can be set to any value. Let us see how interesting results can be obtained by specializing these $U(1)$ charges to prescribed values.

First, note that the r.h.s. of (5.132) does not depend on γ since the $U(1)$ factor $\eta(q)^{-2\gamma^2}$ cancels the same contribution from the determinant. The advantage of having this extra shift γ is that one can consider the two cases $\gamma = 0$ and $\gamma = m$ [12]. If $\gamma = 0$, the summation over principal minors of K gives Virasoro conformal blocks as in (5.78), but in this case such minors have a complicated form. If one puts instead $\gamma = m$, the minors of K turn into factorized Nekrasov expressions: technically the $U(1)$ contribution in front of the determinant, together with the $\eta(\tau)^{-1}$ from (5.53), cancels the $U(1)$ factor in (5.78). For the explicit computations of minors see [135].

Let us consider now ρ, σ. Note that dependence of $Z^D(\eta, a, m, \rho, \sigma, \tau)$ on these parameters, given by (5.53), is quite simple, as the only meaningful combination is $\sigma\tau + \rho$. Let us then put $\sigma = 0$ and study the ρ-dependence. From (5.49) we can see that zeroes of $Z^D(\eta, a, m, \rho, \tau)$ in ρ define the solution $Q(\tau)$ of the Painlevé VI equation (5.15):

$$Z^D(\eta, a, m, \pm Q(\tau) + k + l\tau, \tau) = 0, \tag{5.135}$$

so that Q may be found as zero of the Fredholm determinant (5.132). This formula is definitely the deautonomization of Krichever's formula [137] which gives coordinates of N particles $Q_i(t)$ in the elliptic Calogero-Moser system, describing the low-energy Seiberg-Witten theory of $\mathcal{N} = 2^*$, as zeroes of the theta function: $\Theta(\vec{U}Q_i(t) + \vec{V}t + \vec{W}) = 0$. Indeed, in the next section, by studying explicitly the isospectral/autonomous limit, we find Krichever's formula, as well as the exact relation between the UV and IR coupling of the gauge theory. The generalization of (5.135) to $N > 2$ will be given in [2].

To explain the origin of (5.135) in the spirit of [135] we notice the following: if one substitutes $\rho = Q(\tau)$, then the first row of $Y(w)$ has the following periodicity properties:

$$Y_{1i}(w+1) = Y_{1i}(w), \quad Y_{1i}(w+\tau) = e^{2\pi i Q} Y_{1i}(w) M_B = Y_{1i}(w)\hat{M}_B. \tag{5.136}$$

We thus see that the two functions $Y_{11}(w)$ and $Y_{12}(w)$ are globally defined functions on the torus with prescribed monodromies $\hat{M}_A$ and $\hat{M}_B$, so it is possible to restrict them to the boundaries of the red strip in Fig. 5.4 and therefore they belong to the space of functions that have analytic continuations with prescribed monodromies inside and outside the red strip simultaneously. This breaks the decomposition of the space of all functions into the space of functions analytic inside and outside the red strip, which holds in the generic position, and this is indicated by the vanishing of the determinant. Before passing on, let us report some other determinantal identities one can find by playing around with the $U(1)$ charges in (5.132): notice that

[12] The first case corresponds to $\mathrm{Res}_{w=1} L(w)dw \sim \mathrm{diag}(m, -m)$, whereas the second one corresponds to $\mathrm{Res}_{w=1} L(w)dw \sim \mathrm{diag}(2m, 0)$. The second normalization was used in [22].

$K|_{\rho \mapsto \rho+1/2} = -K$. Combining this observation with (5.53) and with periodicity properties of theta functions we find

$$\begin{aligned} Z^D_{1/2}(\tau)\eta(\tau)^{-1}\theta_3(2\rho|2\tau) &= \frac{1}{2}N(a,m,a)q^{a^2-1/3}e^{2\pi i\rho}\left(\det(\mathbb{I}-K)-\det(\mathbb{I}+K)\right), \\ Z^D_{0}(\tau)\eta(\tau)^{-1}\theta_2(2\rho|2\tau) &= \frac{1}{2}N(a,m,a)q^{a^2-1/3}e^{2\pi i\rho}\left(\det(\mathbb{I}-K)+\det(\mathbb{I}+K)\right), \end{aligned} \tag{5.137}$$

where we put $\gamma = 0$ to simplify the formulas. For arbitrary γ everything is the same. Another option is to substitute $\rho = \frac{1}{4}$ and $\rho = \frac{1}{4} + \frac{\tau}{2}$ into (5.53) in order to cancel each of the two theta functions:

$$\begin{aligned} Z^D_{1/2}(\tau)\eta(\tau)^{-1}\theta_4(2\tau) &= -iN(a,m,a)q^{a^2-1/3}\det\left(\mathbb{I}+K|_{\rho=\frac{1}{4}}\right), \\ Z^D_{0}(\tau)\eta(\tau)^{-1}\theta_4(2\tau) &= N(a,m,a)q^{a^2+5/12}\det\left(\mathbb{I}+K|_{\rho=\frac{1}{4}+\frac{\tau}{2}}\right). \end{aligned} \tag{5.138}$$

5.3.1 Line operators and B-branes

Let us comment at this point on the gauge/string theoretical interpretation of the Riemann-Hilbert kernel. Recall first what we reviewed in section 3.4: theories of class S have a canonical surface operator [138], given in the M-theory construction by an M2 brane embedded in spacetime $\mathbb{R}^4$, localized at a point z of the Riemann Surface. In the AGT correspondence, this is described by an insertion of a degenerate field of weight $\pm b/2$ at z. Wilson and 't Hooft loops living on the surface operator are computed by means of Verlinde loop operators using braiding and fusion of degenerate fields [94, 105], in the same way as the monodromies of the fundamental solution of the linear system are computed in the CFT approach to isomonodromy equations. When the Verlinde loop operator goes around an A-cycle of the Riemann surface, the monodromy operation acts multiplicatively on the conformal block. The corresponding line operator in gauge theory is a Wilson loop. When we go around a B-cycle, there is a shift of the internal Liouville momentum. In this case the line operator carries magnetic charge and thus it is a 't Hooft loop. A choice of A-cycle and B-cycles on the Riemann surface corresponds to a choice of S-duality frame in the gauge theory, and the modular group of the Riemann surface is the S-duality group.

The Verlinde loop operators represent the braiding algebra of Wilson and 't Hooft loops at an operatorial level. The Fourier basis that is adopted in the isomonodromic setting is necessary in order to have an object that transforms linearly onto itself by monodromy so that we can identify the correlator with the fundamental solution of the linear system (4.36) on the sphere. On this basis of the Hilbert space both 't Hooft loops and Wilson loops act by multiplication, which is possible because they are commuting operators. This basis can therefore be regarded as an S-duality complete basis of the Coulomb branch for loop operators, since Wilson and 't Hooft loops are treated on the same footing.

The situation is different in the case of the one-punctured torus, which was analyzed in this chapter. In this case, the Wilson and 't Hooft loop operators for $SU(2)$ anticommute, and by using only degenerate fields one is not able to construct a S-duality complete basis. Indeed, on the torus the addition of an extra $U(1)$ boson in the CFT is required, leading to a free fermionic CFT. The dictionary between monodromy and gauge theory data is the following: the monodromy at the puncture parametrises the mass of the adjoint hypermultiplet

$$\mathrm{Tr} M_1 = 2\cos(2\pi m)\,, \tag{5.139}$$

the monodromy along the A-cycle parametrises the v.e.v. of the Wilson loop in the fundamental representation

$$\mathrm{Tr} M_A = 2\cos(2\pi a) \tag{5.140}$$

in terms of the v.e.v. of the scalar field of the $\mathcal{N} = 2$ vector multiplet, and finally the combined monodromy around the A and the B cycles

$$\mathrm{Tr} M_A M_B = \frac{1}{\sin(2\pi a)}\left[\sin\pi(2a - m)e^{-i(\eta/2 - 2\pi a)} + \sin\pi(2a + m)e^{i(\eta/2-2\pi a)}\right] \tag{5.141}$$

parametrises the v.e.v. of the minimal dyonic 't Hooft loop operator as computed in [13] [105]. From the view point of the Painlevé transcendent, (a, η) are related to the initial conditions of $Q(\tau)$ as we showed in Section 5.2.2, while from the Hitchin system perspective formulas (5.140) and (5.141) together with $\mathrm{Tr} M_B$ are the Darboux coordinates on the moduli space of $SL(2, \mathbb{C})$ flat connections on the one punctured torus [139].

This CFT has a more natural interpretation in the topological string setting, where loop operators are computed in terms of brane amplitudes. Our proposal is that the isomonodromy deformation of the integrable system associated to the classical Seiberg-Witten curve is described in terms of topological B-brane amplitudes as discussed in [140]. There – see sect. 4.7 and further elaborated in [141, 142] – it is also suggested that for higher genus Riemann surfaces the most natural framework is given by considering the free-fermion grand-canonical partition function leading precisely to the Fourier basis and thus to the Nekrasov-Okounkov partition function. Indeed this latter can be regarded as a character of $\mathcal{W}_{1+\infty}$-algebra as in the topological B-brane setting of [140].

5.3.2 The autonomous/Seiberg-Witten limit

In this subsection we analyse the autonomous limit which gives the isospectral integrable system describing the Seiberg-Witten geometry of $\mathcal{N} = 2^*$, namely elliptic Calogero-Moser. We proceed by finding the explicit solution

[13] To compare with [105], put $b = i$, $a \to ia$ remembering that our conformal weights are related to the Liouville charge by $\Delta_a = a^2$, while in [105] the usual convention $\delta_a = -a^2$ is used.

of the equation

$$H_\tau = (2\pi i\partial_\tau Q)^2 - m^2(\wp(2Q|\tau) + 2\eta_1(\tau)) \tag{5.142}$$

in the scaling limit $H_\tau = \hbar^{-2}(u + O(\hbar))$, $m = \hbar^{-1}\mu$, for small variations of τ, namely $\tau = \tau_0 + \hbar t$, where $t \ll \hbar^{-1}$ in the limit $\hbar \to 0$ [14]. In this limit the scaled Hamiltonian is the Coulomb branch parameter of the gauge theory. Indeed in the above limit we have

$$\begin{aligned} 0 = \det(L - \lambda \mathbb{I}_2) &= H_\tau + m^2(\wp(2z) + 2\eta_1(\tau)) + \lambda^2 \\ &\simeq \frac{1}{\hbar^2}\left[u + \mu^2\left(\wp(2z) + 2\eta_1(\tau)\right) + \tilde{\lambda}^2\right], \end{aligned} \tag{5.143}$$

which is the Seiberg-Witten curve for the $\mathcal{N} = 2^*$ theory, so that the energy parameter is identified with the Coulomb branch modulus u[15]. In this limit, equation (5.142) takes the form of the energy conservation law

$$u = (2\pi i\partial_t Q)^2 - \mu^2(\wp(2Q|\tau_0) + 2\eta_1(\tau_0)). \tag{5.144}$$

As for any one-dimensional Hamiltonian system, we can integrate it by quadratures:

$$t - t_0 = \int^Q \frac{2\pi i dQ}{\sqrt{u + 2\mu^2\eta_1(\tau) + \mu^2\wp(2Q|\tau_0)}}, \tag{5.145}$$

however, to explicitly compute this integral we have to perform a couple of changes of variables. First we introduce the new variable

$$y = \frac{\theta_2(2Q|2\tau_0)}{\theta_3(2Q|2\tau_0)}, \tag{5.146}$$

that satisfies

$$(2\pi i\partial_t y)^2 = 4\pi^2\theta_4(2\tau_0)^4\frac{\theta_1(2Q|2\tau_0)^2\theta_4(2Q|2\tau_0)^2}{\theta_3(2Q|2\tau_0)^4}\left(u + 2\mu^2\eta_1 + \mu^2\wp(2Q|\tau_0)\right), \tag{5.147}$$

Where (A.16) has been used. Now substitute

$$\begin{aligned} \wp(2Q|\tau_0) + 2\eta_1(\tau_0) = &-4\pi i\partial_{\tau_0}\log\theta_2(\tau_0) \\ &+ \left(\pi\theta_4(2\tau_0)^2\frac{\theta_2(2Q|2\tau_0)\theta_3(2Q|2\tau_0)}{\theta_1(2Q|2\tau_0)\theta_4(2Q|2\tau_0)}\right)^2. \end{aligned} \tag{5.148}$$

Introducing

$$\tilde{u} = u - 4\pi i\mu^2\partial_{\tau_0}\log\theta_2(\tau_0) \tag{5.149}$$

[14] such small variations preserve the integrals of motion. There is also another part of the problem: to find slow evolution of the integrals of motion at the time scale $t \sim \hbar^{-1}$. The general approach to this problem, which gives rise to Whitham equations, is given in [143]. Relation of this approach to our general solution of the non-autonomous problem still has to be uncovered.

[15] Note that all the quantities in the isomonodromic setting are dimensionless, being measured in Omega-background units.

we rewrite (5.144) as

$$\begin{aligned}(i\partial_t y)^2 = \tilde{u}\theta_4(2\tau_0)^4 \frac{\theta_1(2Q|2\tau_0)^2\theta_4(2Q|2\tau_0)^2}{\theta_3(2Q|2\tau_0)^4} \\ + \pi^2\mu^2\theta_4(2\tau_0)^8 \frac{\theta_2(2Q|\tau_0)^2}{\theta_3(2Q|\tau_0)^2}.\end{aligned} \tag{5.150}$$

By using (A.17) we finally rewrite (5.144) as

$$\begin{aligned}(i\partial_t y)^2 = \tilde{u}\theta_2(2\tau_0)^2\theta_3(2\tau_0)^2 \left(1 - \frac{\theta_3(2\tau_0)^2}{\theta_2(2\tau_0)^2} y^2\right)\left(1 - \frac{\theta_2(2\tau_0)^2}{\theta_3(2\tau_0)^2} y^2\right) \\ + \pi^2\mu^2\theta_4(2\tau_0)^8 y^2.\end{aligned} \tag{5.151}$$

We see that the problem is reduced to the computation of an elliptic integral. In order to do this, we introduce the new variable ϕ through

$$y = \frac{\theta_2(2\phi|2\tau_{SW})}{\theta_3(2\phi|2\tau_{SW})}, \tag{5.152}$$

where τ_{SW} is the complex modulus of the covering curve (5.143), that is the infrared gauge coupling of the $\mathcal{N} = 2^*$ gauge theory. This is given by the polynomial in the r.h.s., and we get the expression

$$\begin{aligned}&(2\pi i\partial_t\phi)^2 \cdot \theta_2(2\tau_{SW})^2\theta_3(2\tau_{SW})^2 \left(1 - \frac{\theta_3(2\tau_{SW})^2}{\theta_2(2\tau_{SW})^2} y^2\right)\left(1 - \frac{\theta_2(2\tau_{SW})^2}{\theta_3(2\tau_{SW})^2} y^2\right) \\ &= \tilde{u}\theta_2(2\tau_0)^2\theta_3(2\tau_0)^2 \left(1 - \frac{\theta_3(2\tau_0)^2}{\theta_2(2\tau_0)^2} y^2\right)\left(1 - \frac{\theta_2(2\tau_0)^2}{\theta_3(2\tau_0)^2} y^2\right) + \pi^2\mu^2\theta_4(2\tau_0)^8 y^2\end{aligned} \tag{5.153}$$

To linearize the equation on ϕ we wish to cancel two bi-quadratic polynomials by solving this explicit equation on $2\tau_{SW}$:

$$\frac{\theta_2(2\tau_{SW})^2}{\theta_3(2\tau_{SW})^2} + \frac{\theta_3(2\tau_{SW})^2}{\theta_2(2\tau_{SW})^2} = \frac{\theta_2(2\tau_0)^2}{\theta_3(2\tau_0)^2} + \frac{\theta_3(2\tau_0)^2}{\theta_2(2\tau_0)^2} - \frac{\mu^2}{\tilde{u}} \frac{\pi^2\theta_4(2\tau_0)^8}{\theta_2(2\tau_0)^2\theta_3(2\tau_0)^2} \tag{5.154}$$

The solution for ϕ is then given by the formula

$$\phi = \frac{\sqrt{\tilde{u}}}{2\pi i} \frac{\theta_2(2\tau_0)\theta_3(2\tau_0)}{\theta_2(2\tau_{SW})\theta_3(2\tau_{SW})} t + \phi_0/2 = \omega t + \phi_0/2. \tag{5.155}$$

Collecting together the two changes of variables we find that the coordinate $Q(t)$ should be found from the solution of the equation

$$\frac{\theta_2(2Q(t)|2\tau_0)}{\theta_3(2Q(t)|2\tau_0)} = \frac{\theta_2(2\omega t + \phi_0)|2\tau_{SW})}{\theta_3(2\omega t + \phi_0)|2\tau_{SW})}. \tag{5.156}$$

This result has to be compared with (5.58): we see that in the isospectral limit dual partition functions can be effectively replaced by theta-functions. In fact, this formula coincides with the one in [144], expressing the exact solution of the elliptic Calogero-Moser model. As a byproduct, we found the explicit relation (5.154) between the UV coupling τ_0 and the IR coupling $\tau_{SW}(\tau_0, \mu^2/u)$ for the $\mathcal{N} = 2^*$ theory.

In other terms, equation (5.58) is the explicit solution of the renormalisation group flow of $\mathcal{N} = 2^*$ theory in a self-dual Ω-background, while eq. (5.154) is the corresponding Seiberg-Witten limit. The results above could be compared with the small mass expansion in terms of modular forms as found in [145]. Actually, our finding suggests that modular anomaly equations are explicitly solved in terms of the corresponding isomonodromy problem, while their SW limit in terms of the corresponding Calogero-Moser system.

Chapter 6

Torus with many punctures, circular quiver gauge theories

In this chapter, based on [2], we will analyse how the identification between gauge theory partition function and the τ-function of a suitable isomonodromy deformation problem arises for A_{N-1} class $\mathcal{S}$ theories on the torus, a typical example of which is a circular quiver $\mathcal{N} = 2$ $d = 4$ $SU(N)$ SUSY gauge theory, depicted in Figure 6.1, in a self-dual Ω-background and which are the integrable systems involved, generalizing the result of the previous chapter to this more general case.

We show that the expression (5.49) is generalized to this case in the following way:

$$\mathcal{T} = Z^D \prod_i \frac{\eta(\tau)}{\theta_1(Q_i(\{z_k\},\tau) - \sigma\tau - \rho)}, \tag{6.1}$$

where Q_i are again the dynamical variables of the isomonodromic system. These solve a system of coupled nonlinear differential equations corresponding to an elliptic version of the Schlesinger system, in which the times are the punctures' positions $z_1, \ldots, z_n$ and the elliptic modulus τ [127], and Z^D is again a free fermionic conformal block on the torus, given by

$$Z^D = \mathrm{tr}_{\mathcal{H}} \left(q^{L_0} (-)^F e^{2\pi i \boldsymbol{\eta} \cdot \boldsymbol{J}_0} V_1 \ldots V_n \right). \tag{6.2}$$

$\boldsymbol{J}_0$ are charges under the Cartan of a twisted $\widehat{\mathfrak{gl}(N)}_1$ algebra and $\boldsymbol{\eta}$ their fugacities. σ, ρ are the $U(1)$ charge and fugacity of this $\widehat{\mathfrak{gl}(N)}_1$. When the vertex operators $V_1 \ldots V_n$ are semi-degenerate fields of W_N, through the AGT correspondence Z^D is identified with the dual partition function of a circular quiver gauge theory, while for more general values of their W-charges the derivation, while formally holding at the level of CFT, does not have a known gauge theory counterpart, and thus an explicit combinatorial expression in terms of Nekrasov functions [14, 22, 146]. Note that while the representation (6.2) corresponds to the dual partition function of a circular quiver gauge theory, by applying fusion transformations on the vertex operators it is possible to obtain the other class $\mathcal{S}$ theories corresponding to the same number of punctures on the torus. The corresponding tau functions will differ by connection constants determined by the fusion kernels, as it happens in the case of the sphere [20, 21, 147, 148]. The construction will

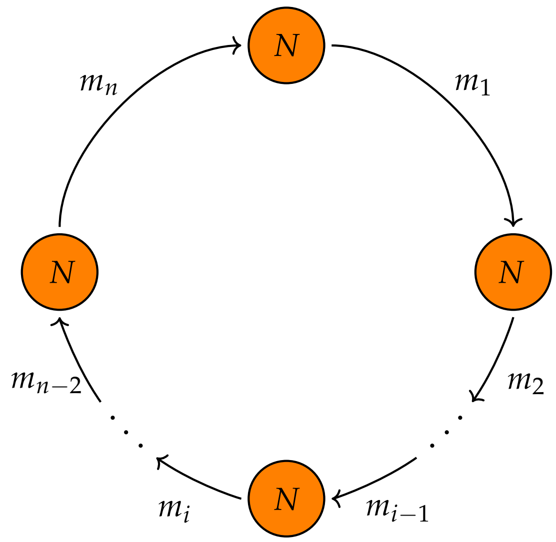


FIGURE 6.1: Circular quiver gauge theory corresponding to the torus with n punctures: for every puncture z_i we have a hypermultiplet of mass m_i sitting in the bifundamental representation of two different $SU(N)$ gauge groups. The case $n = 1$ is special, as the hypermultiplet is in the bifundamental representation for the same $SU(N)$ gauge group, so that it is an adjoint hypermultiplet and the theory is the $\mathcal{N} = 2^*$ theory.

again contain additional $U(1)$ parameters over which the tau function does not depend. We show however that, like in the $SL(2)$ case, the zeroes of the dual partition function in these additional variables are exactly the solutions Q_i of the nonautonomous system. The condition $Z^D = 0$ is therefore shown to be the nonautonomous generalization of the algebro-geometric solution of the Calogero-Moser model [144]. Moreover, the fact that the tau function does not depend on σ, ρ can be made explicit by decomposing the trace in (6.2) into different $\mathfrak{sl}_N$ sectors, labeled by $j = 1, \ldots, N$. This will allow us to write a generalization of equation (5.53), where now we have N different representations of the same tau function $\mathcal{T}$ (instead of two):

$$Z_j^D = \frac{\Theta_j(\mathbf{Q})}{\eta(\tau)^{N-1}} \mathcal{T}, \tag{6.3}$$

where now Z_j^D, $j = 0, \ldots, N-1$ are N different dual partition functions for the $SU(N)$ quiver theory, with different shifts in the Fourier series over the Coulomb branch parameters, and Θ_j are Riemann theta functions given by equation (6.119). First of all, however, let us construct the isomonodromic problem of interest.

6.1 General Fuchsian system on the torus

We are going to study monodromy preserving deformations of linear systems on the torus of the form

$$\partial_z Y(z|\tau) = L(z|\tau)Y(z|\tau), \tag{6.4}$$

where L, Y are $N \times N$ matrices and L, the Lax matrix, has n simple poles located at $\{z_1, \dots, z_n\}$. As it happened in the case of one puncture, $L(z)dz$ is not a single-valued matrix differential, but rather has the following twist properties along the torus A- and B-cycles [127, 149–151]:

$$L(z+1) = T_A L(z) T_A^{-1}, \qquad L(z+\tau) = T_B L(z) T_B^{-1}. \tag{6.5}$$

As can be seen from (6.4), these twists will act on the solution Y of the linear system on the left, in addition to the usual right-action by monodromies. Now the isomonodromic times are $\tau, z_1, \dots, z_n$, and they preserve the monodromies but not the twists. In fact, as was already discussed in Chapter 5, the twists are essentially parametrized by the dynamical variables of the isomonodromic system. The analytic continuation of Y along the generators $\gamma_1, \dots, \gamma_n, \gamma_A, \gamma_B$ of $\pi_1(\Sigma_{1,n})$ is then

$$\begin{cases} Y(\gamma_k \cdot z|\tau) = Y(z|\tau)M_k, \\ Y(z+1|\tau) = T_A(\{z_i\}, \tau)Y(z|\tau)M_A, \\ Y(z+\tau|\tau) = T_B(\{z_i\}, \tau)Y(z|\tau)M_B. \end{cases} \tag{6.6}$$

Together with the singular behavior of Y around $z_1, \dots, z_n$, which are its branch points, these conditions fix completely $Y(z|\tau)$.

As discussed in [127], for the group $SL(N, \mathbb{C})$ there are N inequivalent Lax matrices of this kind characterized by the commutation relation of the twists:

$$T_A T_B^{-1} T_A^{-1} T_B = e^{2\pi i c_1/N}, \tag{6.7}$$

where $c_1 = 0, \dots, N-1$ is the first Chern class of the flat bundle with connection Ldz having the centre of $SL(N, \mathbb{C})$ as structure group. It is possible to relate Lax matrices characterizing inequivalent bundles by means of singular gauge transformations, called Hecke modifications of the bundle [128]. Another possible approach, as in [143], is to consider instead a single-valued Lax matrix with additional simple poles at the so-called 'Tyurin points'. We will discuss the CFT solution to the problem defined by this latter Lax matrix, and its relation to our approach, in Section 6.4.

Because of (6.6), it is possible to define the following kernel:

$$K(z', z) \equiv Y^{-1}(z')\Xi(z', z)Y(z), \tag{6.8}$$

where Ξ is defined so that it has one simple pole at $z = z'$, and transforms as

$$\Xi(z'+1, z) = T_A \Xi(z', z), \qquad \Xi(z', z+1) = \Xi(z', z)T_A^{-1}, \tag{6.9}$$

$$\Xi(z'+\tau,z) = T_B M_B^{U(1)} \Xi(z',z), \quad \Xi(z',z+\tau) = \Xi(z',z)\left(M_B^{U(1)}\right)^{-1} T_B^{-1}, \tag{6.10}$$

in such a way that its transformation cancels the twists of Y. We also included the possibility for Ξ to introduce further $U(1)$ factors, which will be useful to compare with the free fermion description. Because of this, along a closed cycle γ, K transforms as follows

$$K(\gamma\cdot z',z) = \hat{M}_\gamma^{-1} K(z',z), \qquad K(z',\gamma\cdot z) = K(z',z)\hat{M}_\gamma, \tag{6.11}$$

where

$$\hat{M}_\gamma = M_\gamma M_\gamma^{U(1)} \tag{6.12}$$

is the $GL(N)$ representative of γ in the monodromy group, while M_γ is its $SL(N)$ representative (the monodromy of the solution Y).

Keeping in mind the aforementioned fact that we can straightforwardly change from one bundle to another by means of a (singular) gauge transformation, from now on we consider the case $c_1 = 0$ of a topologically trivial bundle, for which the Lax matrix has the form

$$L(z|\tau) = \mathbf{p} + \sum_{k=1}^{n} L^{(k)}, \tag{6.13}$$

where

$$\mathbf{p} = \mathrm{diag}(p_1,\dots,p_N) \tag{6.14}$$

and

$$L_{ij}^{(k)} = \delta_{ij}\frac{\theta_1'(z-z_k)}{\theta_1(z-z_k)} S_{ii}^{(k)} + (1-\delta_{ij})\frac{\theta_1'(0)\theta(z-z_k-Q_i+Q_j)}{\theta_1(z-z_k)\theta_1(-Q_i+Q_j)} S_{ji}^{(k)}, \tag{6.15}$$

where the parameters $S_{ii}^{(k)}$ are subject to the constraint

$$\sum_k S_{ii}^{(k)} = 0, \tag{6.16}$$

so that we have the correct quasi-periodicity properties (6.5). The monodromy preserving deformations of (6.4) involve moving the singular points $z_1,\dots,z_k$ (one of which can be fixed using the automorphisms of the torus), and the modular parameter τ. These flows are generated by the Hamiltonians, given by the trace of the Lax matrix squared

$$\frac{1}{2}\mathrm{tr}\, L^2(z) = H_\tau + \sum_{k=1}^{n} H_k E_1(z-a_k) + C_2^k E_2(z-a_k), \tag{6.17}$$

where E_1, E_2 are the Eisenstein functions (see Appendix A for their definition), C_2^k is the Casimir at the orbit of z_k, while H_k, H_τ generate the flows with times z_k and $2\pi i\tau$ respectively, and can be computed by performing contour integrals:

$$H_k = \oint_{\gamma_k} \frac{dz}{2\pi i} \frac{1}{2} \mathrm{tr}\, L^2(z), \qquad H_\tau = \oint_A dz \frac{1}{2} \mathrm{tr}\, L^2(z). \tag{6.18}$$

These Hamiltonians can all be obtained as usual from the logarithmic derivative of a single tau function [127, 149–151]:

$$\partial_{z_k} \log \mathcal{T} = H_k, \qquad 2\pi i \partial_\tau \log \mathcal{T} = H_\tau. \tag{6.19}$$

6.2 Kernel and tau function from free fermions

We now show that the kernel (6.8) has the following expression in terms of free fermion conformal blocks:

$$\begin{aligned} K(z', z) &= Y^{-1}(z') \Xi(z - z', \mathbf{Q}) Y(z) \\ &= \frac{\langle V_{\theta_1}(z_1) \dots V_{\theta_n}(z_n) \bar{\psi}(z') \otimes \psi(z) \rangle}{\langle V_{\theta_1}(z_1) \dots V_{\theta_n}(z_n) \rangle}, \end{aligned} \tag{6.20}$$

where

$$\Xi(z - z', \mathbf{Q}) = \mathrm{diag}\left(x(\sigma\tau + \rho - Q_1, z), \dots, x(\sigma\tau + \rho - Q_n, z) \right], \tag{6.21}$$

x being the Lamé function defined in Appendix A. The notation $\langle \dots \rangle$ stands for

$$\langle \mathcal{O} \rangle = \mathrm{tr}_{\mathcal{H}} \left(q^{L_0} (-)^F e^{2\pi i \boldsymbol{\eta} \cdot J_0} \mathcal{O} \right), \tag{6.22}$$

where $\mathcal{H}$ is our free fermionic Hilbert space (4.69), J_0^i are the $\widehat{\mathfrak{gl}(N)}_1$ Cartan charges and η_i their fugacities. The insertion of $(-)^F$ shifts the periodicity condition of our fermions around the B-cycle of the torus, and will be relevant in the computation of the B-cycle monodromy. As discussed in Section 6.1, we included the $U(1)$ charge and fugacity in the definition of Ξ, that we denoted by

$$\sigma = \frac{1}{N} \sum_{i=1}^{N} \sigma_i, \qquad \rho = \frac{1}{N} \sum_{i=1}^{N} \eta_i. \tag{6.23}$$

It will be also useful to introduce $\mathfrak{sl}_n$ projections of the charge vectors

$$\tilde{\sigma}_i = \sigma_i - \sigma, \qquad \tilde{\eta}_i = \eta_i - \rho. \tag{6.24}$$

As in the 2×2 case, the matrix Ξ gives the LHS of the equation a simple pole, that in the RHS is due to the OPE of the free fermions, while also producing the $U(1)$ part of the monodromies, absent in Y but present by construction in the CFT. Further and most importantly, it cancels both the twists of the solution Y, so that the kernel K has monodromies acting from both left and right as in equation (6.11). Our goal will be to show that the vertex operators can be defined in such a way that the RHS has given monodromies acting in exactly such a way with prescribed conjugacy class, which together with the identical singular behavior around $z, z' \sim z_k$, $z \sim z'$ coming from the OPE of the free fermions with the vertex operators shows that the two objects coincide.

In this section we compute the monodromies following the method explained in Section 4.3.1: the vertex operators are defined through their action on free fermions, so it is possible to realize a monodromy with prescribed conjugacy class at every puncture. Operationally, if one wants to compute the monodromy around the cycle γ_n, for example, the operation is the following (we are summing over repeated indices)[1]:

$$\begin{aligned}\langle V_{\theta_1}(z_1)\dots V_{\theta_n}(z_n)\bar{\psi}_i(z')\psi_j(z)\rangle &\to -\langle V_{\theta_1}(z_1)\dots V_{\theta_n}(z_n)\psi_j(z)\bar{\psi}_i(z')\rangle \\ &\to -\langle V_{\theta_1}(z_1)\dots \psi_k(z)V_{\theta_n}(z_n)\bar{\psi}_i(z')\rangle (B_n)^k{}_j \\ &\to -\langle V_{\theta_1}(z_1)\dots V_{\theta_n}(z_n)\psi_k(z)\bar{\psi}_i(z')\rangle (\tilde{B}_n B_n)^k{}_j \\ &\to \langle V_{\theta_1}(z_1)\dots V_{\theta_n}(z_n)\bar{\psi}_i(z')\psi_k(z)\rangle (\tilde{B}_n B_n)^k{}_j,\end{aligned} \tag{6.25}$$

so that the monodromy around z_n is

$$M_n = \tilde{B}_n B_n = F_n^{-1} e^{2\pi i \theta_n} F_n \sim e^{2\pi i \theta_n}. \tag{6.26}$$

Following the same idea, one can compute the monodromy around an arbitrary puncture z_α: one perform a braiding around every puncture from z_n to $z_{\alpha+1}$, then twice around z_α, then again around z_α to z_n in the opposite direction as before: since the braiding and fusion operations are local, the discussion is completely analogous to that in Section 4.3.1. The operation is represented graphically in Figure 6.2 for the puncture z_1 in the two-punctured torus. The result is that the monodromy around an arbitrary puncture z_α is given by

$$\begin{aligned}M_\alpha &= B_n^{-1}\dots B_{\alpha+1}^{-1}\tilde{B}_\alpha B_\alpha B_{\alpha+1}\dots B_n \\ &= (F_\alpha B_{\alpha+1}\dots B_n)^{-1} e^{2\pi i \theta_\alpha}(F_\alpha B_{\alpha+1}\dots B_n) \sim e^{2\pi i \theta_\alpha}.\end{aligned} \tag{6.27}$$

The monodromy around the A-cycle is fixed by our choice of gluing: it is given by

$$M_A = e^{2\pi i \boldsymbol{a}}. \tag{6.28}$$

Finally, the monodromy around the B-cycle can be computed in the following way. First we go once around every z_k:

$$\begin{aligned}\langle V_{\theta_1}(z_1)\dots V_{\theta_n}(z_n)\bar{\psi}_i(z')\psi_j(z)\rangle &\to -\langle V_{\theta_1}(z_1)\dots V_{\theta_n}(z_n)\psi_j(z)\bar{\psi}_i(z')\rangle \\ &\to \dots \to -\langle \psi_k(z) V_{\theta_1}(z_1)\dots V_{\theta_n}(z_n)\bar{\psi}_i(z')\rangle (B_1\dots B_n)^k{}_j.\end{aligned} \tag{6.29}$$

Now, to go around the B-cycle we have to bring the fermion back to the original position without crossing again the other operators. This is done by

[1] Recall that $B = B^{(+)}$, $\tilde{B} = B^{(-)}$ in the notations of Chapter 3.

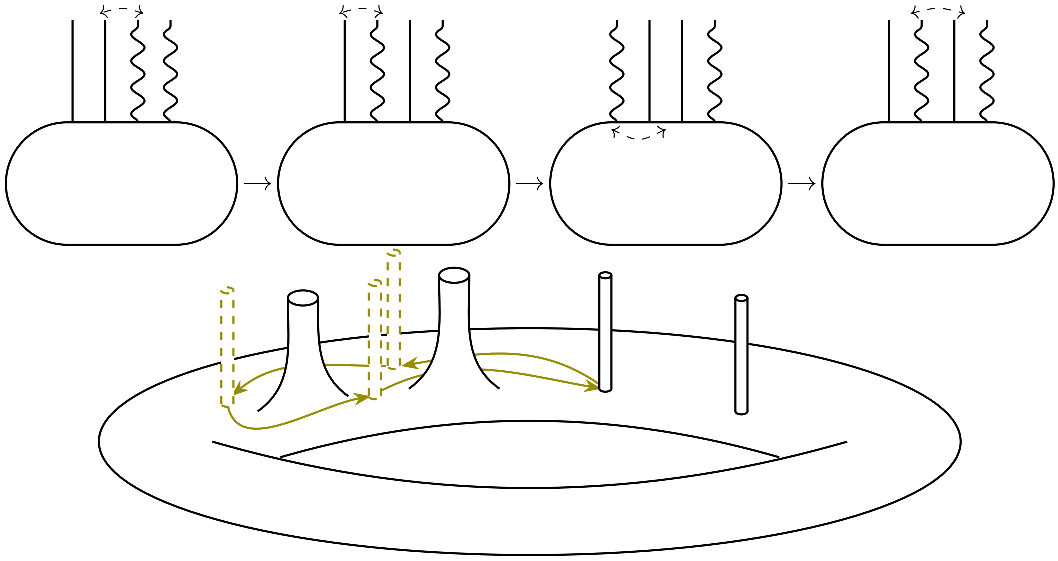


FIGURE 6.2: Monodromy of a fermion around a puncture through braiding on the two-punctured torus. On the upper side, the steps that compose the monodromy operation are represented in terms of conformal block diagrams. On the lower side, the meaning of the conformal block diagram is drawn on the torus: the thin cylinders represent the fermions, while the larger tubes represent the vertex operators. The intermediate steps are drawn in olive green.

using the cyclicity of the trace, but in fact in doing so we also have to take into account the insertion of $(-)^F e^{2\pi i\eta\cdot J_0}$:

$$\begin{aligned}
&-\langle \psi_k(z)V_{\theta_1}(z_1)\dots V_{\theta_n}(z_n)\bar\psi_i(z')\rangle (B_1\dots B_n)^k{}_j\\
&= -\mathrm{tr}_{\mathcal{H}}\left(q^{L_0}(-)^F e^{2\pi i\eta\cdot J_0}\psi_k(z)V_{\theta_1}(z_1)\dots V_{\theta_n}(z_n)\bar\psi_i(z')\right)(B_1\dots B_n)^k{}_j\\
&\to \mathrm{tr}_{\mathcal{H}}\left(\psi_k(z)q^{L_0}(-)^F e^{2\pi i\eta\cdot J_0}V_{\theta_1}(z_1)\dots V_{\theta_n}(z_n)\bar\psi_i(z')\right)(e^{2\pi i\eta}B_1\dots B_n)^k{}_j\\
&= \langle V_{\theta_1}(z_1)\dots V_{\theta_n}(z_n)\bar\psi_i(z')\psi_k(z)\rangle\cdot e^{2\pi i\rho}(e^{2\pi i\tilde\eta}B_1\dots B_n)^k{}_j
\end{aligned} \tag{6.30}$$

so that

$$M_B = e^{2\pi i\rho}e^{2\pi i\tilde\eta}B_1\dots B_n. \tag{6.31}$$

The two sides of equation (6.20) have prescribed monodromies and singular behavior, and so they coincide. To compute the tau function we have to expand the trace of equation (6.20) for $z\sim z'$. The computation is analogous to the one of the previous section, with some important technical differences.

By expanding the LHS, we get a term involving the Lax matrix

$$Y(z+t/2)Y^{-1}(z-t/2) = \left(\mathbb{I} + tL(z) + \frac{t^2}{2}L^2(z)\right), \tag{6.32}$$

and two terms from the expansion of the matrix Ξ:

$$\frac{\theta_1'(0)}{\theta_1(t)} = \frac{1}{t} - \frac{t}{6}\frac{\theta_1'''}{\theta_1'} + O(t^3), \tag{6.33}$$

$$\frac{\theta_1(t-\tilde{Q}_i)}{\theta_1(-\tilde{Q}_i)} = 1 + t\frac{\theta_1'(-\tilde{Q}_i)}{\theta_1(-\tilde{Q}_i)} + \frac{t^2}{2}\frac{\theta_1''(-\tilde{Q}_i)}{\theta_1(-\tilde{Q}_i)}. \tag{6.34}$$

Here we introduced

$$\tilde{Q}_i = Q_i - \sigma\tau - \rho. \tag{6.35}$$

On the RHS, the expansion consists of the OPE for the fermions, yielding

$$\operatorname{tr}\psi(z+t/2)\otimes\bar{\psi}(z-t/2) = \frac{N}{t} + Nj(z) + \frac{t}{2}T(z) + O(t^2). \tag{6.36}$$

The $O(t)$ term relates the expectation value of the energy-momentum tensor to the trace squared of the Lax matrix:

$$\begin{aligned}\frac{\langle T(z)V_1\dots V_n\rangle}{\langle V_1\dots V_n\rangle} &= \frac{1}{2}\operatorname{tr}L^2(z) + \operatorname{tr}L(z)\frac{\theta_1'(\tilde{\mathbf{Q}})}{\theta_1(\tilde{\mathbf{Q}})} + \frac{1}{2}\operatorname{tr}\frac{\theta_1''(\tilde{\mathbf{Q}})}{\theta_1(\tilde{\mathbf{Q}})} - \frac{N}{6}\frac{\theta_1'''(0)}{\theta_1'(0)} \\ &\equiv \frac{1}{2}\operatorname{tr}L^2(z) + t(z)\end{aligned} \tag{6.37}$$

We see that, as happened in the previous chapter, in the genus one case there is a correction to the relation that one has in genus zero (Chapter 4), encoded in $t(z)$. We wish now to determine the expression for the tau function by computing contour integrals of (6.37) and comparing with (6.18) and (6.19). From (6.37) we see that we can split the tau function in two parts:

$$\mathcal{T} = \mathcal{T}_0\mathcal{T}_1, \tag{6.38}$$

which are defined by the following equations:

$$\begin{aligned}\partial_{z_k}\log\mathcal{T}_0 &= \oint_{\gamma_k}\frac{dz}{2\pi i}\langle T(z)V_1\dots V_n\rangle, \\ 2\pi i\partial_\tau\log\mathcal{T}_0 &= \oint_A dz\frac{1}{2}\langle T(z)V_1\dots V_n\rangle,\end{aligned} \tag{6.39}$$

$$\partial_{z_k}\log\mathcal{T}_1 = -\oint_{\gamma_k}\frac{dz}{2\pi i}t(z), \qquad 2\pi i\partial_\tau\log\mathcal{T}_1 = -\oint_A dz t(z). \tag{6.40}$$

The first term would be there also in the genus zero case, while the second term is a new feature appearing in higher genus. $\mathcal{T}_0$ is computed by applying the Virasoro Ward identity:

$$\langle T(z)V_1\dots V_n\rangle = \langle T\rangle + \sum_{k=1}^n E_1(z-z_k)\partial_k\log\langle V_1\dots V_n\rangle + \sum_{k=1}^n \boldsymbol{\theta}_k^2 E_2(z-a_k), \tag{6.41}$$

yielding

$$\mathcal{T}_0 = \langle V_1 \dots V_n \rangle. \tag{6.42}$$

We now turn to computing the contour integrals of $t(z)$: since we have

$$\sum_i S_{ii}^{(k)} = 0, \qquad \int_0^1 dz \frac{\theta_1'(z-z_k)}{\theta_1(z-z_k)} = \pi i \tag{6.43}$$

when z_k lies in the fundamental domain. Then, the only contribution to the τ-derivative of $\mathcal{T}_1$ will be

$$\begin{aligned} -2\pi i \partial_\tau \log \mathcal{T}_1 &= \mathrm{tr}\, \boldsymbol{p} \frac{\theta_1'(\tilde{\boldsymbol{Q}})}{\theta_1(\tilde{\boldsymbol{Q}})} + \frac{1}{2} \mathrm{tr} \frac{\theta_1''(\tilde{\boldsymbol{Q}})}{\theta_1(\tilde{\boldsymbol{Q}})} - \frac{N}{6} \frac{\theta_1'''(0)}{\theta_1'(0)} \\ &= 2\pi i \mathrm{tr}\, \partial_\tau \tilde{\boldsymbol{Q}} \frac{\theta_1'(\tilde{\boldsymbol{Q}})}{\theta_1(\tilde{\boldsymbol{Q}})} + 2\pi i \mathrm{tr} \frac{\partial_\tau \theta_1(\tilde{\boldsymbol{Q}})}{\theta_1(\tilde{\boldsymbol{Q}})} - 2\pi i \frac{N}{3} \frac{\partial_\tau \theta'(0)}{\theta_1'(0)} \\ &= 2\pi i \partial_\tau \left(\mathrm{tr} \log \theta_1(\tilde{\boldsymbol{Q}}) - N \log \eta(\tau) \right) \end{aligned} \tag{6.44}$$

Therefore

$$\mathcal{T}_1 = f(\{z_k\}) \frac{\eta(\tau)^N}{\prod_i \theta_1(\tilde{Q}_i(\{z_k\}, \tau))} = \frac{f(\{z_k\})}{Z_{twist}(\tilde{\boldsymbol{Q}}(\{z_k\}, \tau))}, \tag{6.45}$$

where $f(\{z_k\})$ is an arbitrary function of the punctures' positions, left undetermined by the integration. In fact, let us show that $f(\{z_k\}) = 1$: computing the residues of $t(z)$ yields

$$-\partial_{z_k} \log \mathcal{T}_1 = \sum_i S_{ii}^{(k)} \frac{\theta_1'(\tilde{Q}_i)}{\theta_1(\tilde{Q}_i)}. \tag{6.46}$$

At first sight, the RHS doesn't look like a total z_k-derivative. However, let us consider the $\boldsymbol{p}$-dependent part of the corresponding Hamiltonian:

$$H_k = \frac{1}{2} \mathrm{Res}_{\,z_k} \mathrm{tr}\, \mathcal{A}(z)^2 = \sum_i S_{ii}^{(k)} p_i + \dots \tag{6.47}$$

from which it follows that $\partial_{z_k} Q_i = S_{ii}^{(k)}$. Therefore

$$\partial_{z_k} \log \mathcal{T}_1 = \sum_i \frac{\theta_1'(\tilde{Q}_i)}{\theta_1(\tilde{Q}_i)} \partial_{z_k} \tilde{Q}_i = \partial_{z_k} \log \prod_i \theta_1(\tilde{Q}_i) \tag{6.48}$$

Therefore in (6.45) $f(\{z_k\}) = const$, and we can put without loss of generality $f(\{z_k\}) = 1$, as promised. The isomonodromic tau function is

$$\mathcal{T}(\{z_k\}, \tau) = \frac{1}{Z_{twist}(\tilde{\boldsymbol{Q}}(\tau))} \langle V_1(z_1) \dots V_n(z_n) \rangle. \tag{6.49}$$

Let us remark that the CFT arguments used above are valid for general vertex insertions. However, in order to have explicit calculable expressions one needs to consider the insertion of (semi-) degenerate fields. In this case, the fermionic correlator is identified with the dual partition function of a circular quiver gauge theory with gauge group $U(N)^n$ and n hypermultiplets in bifundamental representations of the gauge groups, as encoded in the conformal block diagram. Therefore, the above equality can be rewritten as

$$\mathcal{T} = \frac{Z^D(\tau, \{z_k\}|\{\boldsymbol{a}_k\}, \{\boldsymbol{\eta}_k\}, \{\boldsymbol{\theta}_k\})}{Z_{twist}(\tilde{\mathbf{Q}}(\tau, \{z_k\}))}, \tag{6.50}$$

where we made explicit the dependence on all the intermediate channel charges $\boldsymbol{a}_k$, $k = 1, \dots n$, together with their duals entering in the Fourier transform $\boldsymbol{\eta}_k$, and set $\boldsymbol{a} \equiv \boldsymbol{a}_1$.

6.3 Torus monodromies with Verlinde loop operators

In this section we show an alternative proof of formulas (6.20) and (6.49) for the kernel and tau function respectively, using Verlinde loop operators acting on (semi-) degenerate representations of W_N algebras, along the lines of [126]. The necessary definitions about degenerate fields and W_N algebras were provided in Section 3.5.1. The advantage of this proof is that one has to keep track of everything explicitly (in particular, all the internal charges) throughout the computation; this is also its disadvantage, however, because it is much more notationally heavy and can become quite technical.

6.3.1 General setup

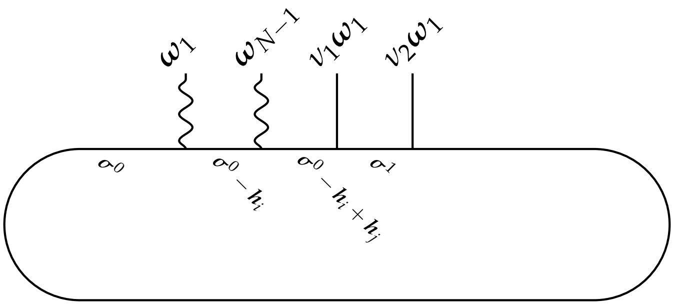


FIGURE 6.3: Toric conformal blocks with $n = 2$ semi-degenerate and 2 degenerate fields.

We wish to study the monodromy properties of the torus conformal block with insertions of two W_N completely degenerate fields, ϕ and $\bar{\phi}$, and n semi-degenerate W-primaries V:

$$\mathcal{Y}_{ij}(\boldsymbol{\sigma}^0;\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^{n-1}|z,z_0) = \\ = \mathrm{tr}_{\mathcal{H}_{\sigma^0}}\left(q^{L_0}\phi_i(z)\bar{\phi}_j(z_0)V_{\nu_1}(z_1)\mathcal{P}_{\sigma^1}V_{\nu_2}(z_2)\ldots V_{\nu_{n-1}}(z_{n-1})\mathcal{P}_{\sigma^{n-1}}V_{\nu_n}(z_n)\right). \tag{6.51}$$

In this formula the operators V_{ν_k} are semi-degenerate W-primaries with W-charges given by $\boldsymbol{\theta}_k = \nu_k\boldsymbol{\omega}_1$, where $\boldsymbol{\omega}_1$ is the first fundamental weight of A_{N-1}. Operators ϕ_i and $\bar{\phi}_j$ are completely degenerate fields with W-charges given by $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_{N-1}$, respectively. The indices i and j label fusion channels.

The normalization of V_{ν_k} is given by:

$$\langle\boldsymbol{\sigma}'|V_\nu|\boldsymbol{\sigma}\rangle \equiv \mathcal{N}^+(\boldsymbol{\sigma}',\nu\boldsymbol{\omega}_1,\boldsymbol{\sigma}), \tag{6.52}$$

where

$$\mathcal{N}^\pm(\boldsymbol{\sigma}',\nu\boldsymbol{\omega}_1,\boldsymbol{\sigma}) = \frac{\prod_{lj}G(1\mp\nu/N\pm\sigma_l\mp\sigma'_j)}{\prod_{k<m}G(1+\sigma_k-\sigma_m)G(1-\sigma'_k+\sigma'_m)}. \tag{6.53}$$

We also fix normalization of the completely degenerate field by[2]

$$\langle\boldsymbol{\sigma}|\phi_i(1)|\boldsymbol{\sigma}-\boldsymbol{h}_i\rangle = e^{i\pi N(\boldsymbol{\sigma},\boldsymbol{h}_i)}\mathcal{N}^-(\boldsymbol{\sigma},\boldsymbol{\omega}_1,\boldsymbol{\sigma}-\boldsymbol{h}_i). \tag{6.54}$$

As in equation (3.95), $\mathcal{P}_{\sigma^k}$ is the projection operator onto the W-algebra representation with charge $\boldsymbol{\sigma}^k$, expressing the fact that the conformal block has fixed intermediate charges. It is useful to expand the trace of (6.51) as a sum of diagonal matrix elements:

$$\mathcal{Y}(\boldsymbol{\sigma}^0;\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^{n-1}|z,z_0) = \sum_{\boldsymbol{Y}} q^{|\boldsymbol{Y}|}\mathcal{Y}^{(\boldsymbol{Y})}(\boldsymbol{\sigma}^0;\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^{n-1}|z,z_0) \tag{6.55}$$

where vector of Young diagrams $\boldsymbol{Y}$ labels W-algebra descendants, and we defined the matrix element between descendants

$$\mathcal{Y}^{(\boldsymbol{Y})} = \langle\boldsymbol{\sigma}^0,\boldsymbol{Y}|\phi(z)\otimes\bar{\phi}(z_0)V_{\nu_1}\ldots V_{\nu_n}|\boldsymbol{\sigma}^0,\boldsymbol{Y}\rangle. \tag{6.56}$$

We remind one of the the main results of [126, Theorem 5.1]: the Fourier transformation of $\mathcal{Y}^{(\boldsymbol{Y})}$ over all internal W-charges has number-valued (not operator-valued as generically happens) monodromies around $0,\infty$ and the insertion points $z_1,\ldots,z_n$, as a function of z and z_0, independent from $\boldsymbol{Y}$. The Fourier transform is defined by

$$\mathcal{Y}^{(\boldsymbol{Y})D}(\boldsymbol{\sigma}^0;\boldsymbol{\sigma}^1,\boldsymbol{\eta}^1,\ldots,\boldsymbol{\sigma}^{n-1},\boldsymbol{\eta}^{n-1}|z,z_0) = \\ = \sum_{\boldsymbol{w}^i\in Q_{A_{N-1}}} e^{2\pi i\sum_{i=1}^{n-1}(\boldsymbol{\eta}^i,\boldsymbol{w}^i)}\mathcal{Y}^{(\boldsymbol{Y})}(\boldsymbol{\sigma}^0;\boldsymbol{\sigma}^1+\boldsymbol{w}^1,\ldots,\boldsymbol{\sigma}^{n-1}+\boldsymbol{w}^{n-1}|z,z_0), \tag{6.57}$$

[2] This parameterization differs from one in [126] by the factor $e^{i\pi(1-N)(\boldsymbol{\sigma},\boldsymbol{h}_i)}$.

where $Q_{A_{N-1}}$ is the $\mathfrak{sl}_N$ root lattice. Moreover, for the case $\boldsymbol{Y} = \emptyset$ the function $\mathcal{Y}^{0;D}$ gives the solution of the $n+2$ point Fuchsian system on the sphere. So using the results of [126] we get automatically the following statement: the function $\mathcal{Y}^D$, given by the formula[3]

$$\mathcal{Y}^D = \sum_{w^i \in Q_{A_{N-1}}} \sum_{\boldsymbol{Y}} e^{2\pi i \sum_{i=1}^{n-1} (\eta^i, w^i)} \mathcal{Y}(\sigma^0; \sigma^1 + w^1, \ldots, \sigma^{n-1} + w^{n-1} | z, z_0) \tag{6.58}$$

has number-valued monodromies M_k around all z_k, and also number-valued A-cycle monodromy $M_A = e^{2\pi i \sigma^0}$, since after taking trace we identify A-cycle with the loop around 0 or ∞ on the initial sphere. The problem now is to find a linear combination of $\mathcal{Y}^D$ that has number-valued monodromy around the B-cycle.

6.3.2 B-cycle monodromy operator

The main ingredient in the computation, as in the case of free fermions, is the braiding move exchanging two insertions in a four-point conformal block, as in Figure 6.4, where we see how the free fermion braiding can be expressed in terms of the W_N braiding matrix B, given below in equation (6.59).

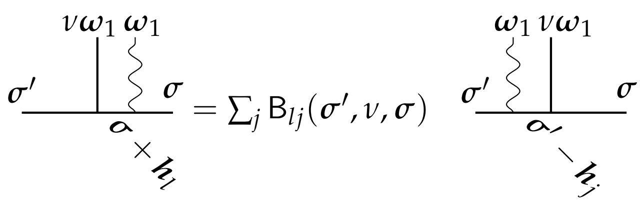


FIGURE 6.4: Braiding transformation of conformal blocks.

It is a local transformation of conformal blocks, and maps a conformal block to a linear combination of other conformal blocks with different intermediate dimensions. Since it is local, it can be studied for conformal blocks with one degenerate, one semi-degenerate and two arbitrary fields: in this case the conformal block is given by a generalized hypergeometric function ${}_N F_{N-1}$, so the computation of the fusion matrix F is equivalent to re-expansion of hypergeometric function around zero in the vicinity of infinity, see [126] and references therein. The analytic continuation between these two region is performed around a semidegenerate field insertion in the counter-clockwise direction. These conformal blocks can be obtained directly from geometric engineering in topological string theory, as in [152, 153]. We perform the sequence of braiding transformations that correspond to the B-cycle monodromy pictorially, exemplified in the case of two punctures, in Fig. 6.5.

From the figure we can see that after analytic continuation along the B-cycle, the intermediate charges are shifted: in other words, we have an

[3] In all formulas letter "D" stands for "dual".

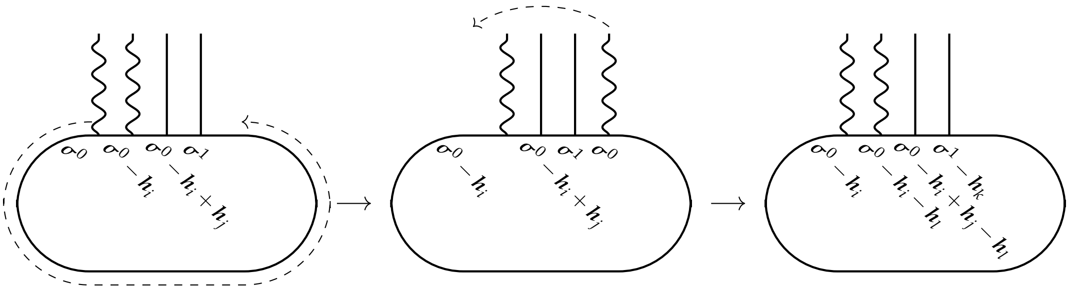


FIGURE 6.5: Monodromy of degenerate field.

operator-valued monodromy matrix $\hat{M}_B$, containing shift operators. The main problem, as in the previous chapter, will be to turn this matrix into number-valued matrix M_B. Before going through the whole computation let us make the following observation: while in the spherical case all monodromies led to shifts in the A_{N-1} root lattice (generated by $\boldsymbol{h}_i - \boldsymbol{h}_j$), in the toric case the single B-cycle monodromy also simultaneously shifts all the charges by a single $\boldsymbol{h}_i$. Therefore the arbitrary shift vector, which appears here and will have to appear in the Fourier transform, has the form $(\boldsymbol{w}^0 + \boldsymbol{\omega}_k, \boldsymbol{w}^1 + \boldsymbol{\omega}_k, \ldots, \boldsymbol{w}^{n-1} + \boldsymbol{\omega}_k)$, where $\boldsymbol{w}^l \in Q_{A_{N-1}}$ are the elements of A_{N-1} root lattice. To get the proper kernel for the Riemann-Hilbert problem it will be necessary to sum over this set: the essential difference from the naive expectation is the presence of the extra shift by the fundamental weight $\boldsymbol{\omega}_k$.

Now we perform the precise computations along the lines of [126]. The explicit formula for the fusion kernel is given by (3.100), that we write here for convenience:

$$\mathsf{B}_{lj}(\boldsymbol{\sigma}', \nu, \boldsymbol{\sigma}) = e^{\pi i(\nu+1/N)} \prod_{k\neq l} \frac{1 - e^{-2\pi i((\nu+1)/N + \sigma'_j - \sigma_k)}}{1 - e^{-2\pi i(\sigma_l - \sigma_k)}} . \tag{6.59}$$

The main advantage of the normalization (6.54) is that the braiding matrix is periodic under $\sigma_i \mapsto \sigma_i + 1$ or $\sigma'_i \mapsto \sigma'_i + 1$. In matrix notation, the braiding of Figure 6.4 takes the form

$$\mathcal{P}_{\sigma'} V_\nu(z) \vec{\phi}(\gamma \cdot y) \mathcal{P}_\sigma = \mathsf{B}(\boldsymbol{\sigma}', \nu, \boldsymbol{\sigma}) \cdot \mathcal{P}_{\sigma'} \vec{\phi}(y) \Phi(z) \mathcal{P}_\sigma , \tag{6.60}$$

Another basic operation is the permutation of a degenerate field and a projector:

$$\vec{\phi}(z) \mathcal{P}_\sigma = \nabla_\sigma \mathcal{P}_\sigma \vec{\phi}(z) . \tag{6.61}$$

Here ∇_σ is a diagonal matrix with entries given by the shift operators:

$$(\nabla_\sigma)_{ii} \mathcal{P}_\sigma = \mathcal{P}_{\sigma + \boldsymbol{h}_i} . \tag{6.62}$$

The appearance of such operators makes monodromy matrices operator- valued. The transformation of the conformal block (6.51) when we analytically continue in z along the B-cycle is expressed as a sequence of these operations:

in order to write it down, it is convenient to introduce the column vectors

$$\mathcal{Y}_j(z) = \left(\mathcal{Y}_{1,j}, \ldots, \mathcal{Y}_{N,j}\right)^T , \tag{6.63}$$

constructed from the lines of $\mathcal{Y}$. In terms of these, we can write the monodromy transformation as

$$\mathcal{Y}_j(\gamma_B \cdot z) = \hat{M}_B^T \mathcal{Y}_j(z) , \tag{6.64}$$

where

$$\begin{aligned} \hat{M}_B^T =& \nabla_{\sigma^0}^{-1} \mathsf{B}(\boldsymbol{\sigma}^{n-1}, \nu_n, \boldsymbol{\sigma}^0) \nabla_{\sigma^{n-1}}^{-1} \mathsf{B}(\boldsymbol{\sigma}^{n-2}, \nu_{n-1}, \boldsymbol{\sigma}^{n-1}) \\ & \ldots \nabla_{\sigma^1}^{-1} \mathsf{B}(\boldsymbol{\sigma}^0 + \boldsymbol{h}_j, \nu_1, \boldsymbol{\sigma}^1) e^{\pi i (1-N)/N}. \end{aligned} \tag{6.65}$$

To compute braiding of two degenerate fields we used the simple identity

$$\mathsf{B}(\boldsymbol{\sigma}^0, -1, \boldsymbol{\sigma}^0 + \boldsymbol{h}_j - \boldsymbol{h}_l)_{lk} = e^{\pi i (1-N)/N}. \tag{6.66}$$

To further simplify the form of the monodromy matrix $\hat{M}_B$ we do some manipulations in order to make all shift operators act only on the conformal blocks, but not on the other matrices. We will denote a shift operator that acts only on the conformal block by $\tilde{\nabla}$. This can be done with the help of the following identities:

$$\begin{aligned} \mathsf{B}(\boldsymbol{\sigma}', \nu, \boldsymbol{\sigma} \pm \boldsymbol{h}_m) &= -\mathsf{B}(\boldsymbol{\sigma}', \nu \pm 1, \boldsymbol{\sigma}) , \\ \mathsf{B}(\boldsymbol{\sigma}' \pm \boldsymbol{h}_m, \nu, \boldsymbol{\sigma}) &= -\mathsf{B}(\boldsymbol{\sigma}', \nu \mp 1, \boldsymbol{\sigma}) \end{aligned} \tag{6.67}$$

and their obvious consequence:

$$\nabla_{\sigma}^{-1} \mathsf{B}(\boldsymbol{\sigma}', \nu, \boldsymbol{\sigma}) = -\tilde{\nabla}_{\sigma}^{-1} \mathsf{B}(\boldsymbol{\sigma}', \nu + 1, \boldsymbol{\sigma}). \tag{6.68}$$

Naively one might think that $\hat{M}_B^T$ acts differently on different rows of Ψ, but due to (6.67) this dependence disappears. Simplified form of the monodromy matrix is

$$\begin{aligned} \hat{M}_B^T = (-1)^n e^{\pi i (1-N)/N} \tilde{\nabla}_{\sigma^0}^{-1} \mathsf{B}(\boldsymbol{\sigma}^{n-1}, \nu_n - 1, \boldsymbol{\sigma}^0) \tilde{\nabla}_{\sigma^{n-1}}^{-1} \mathsf{B}(\boldsymbol{\sigma}^{n-2}, \nu_{n-1} - 1, \boldsymbol{\sigma}^{n-1}) \ldots \\ \ldots \tilde{\nabla}_{\sigma^1}^{-1} \mathsf{B}(\boldsymbol{\sigma}^1, \nu_2 - 1, \boldsymbol{\sigma}^2) \tilde{\nabla}_{\sigma^1}^{-1} \mathsf{B}(\boldsymbol{\sigma}^0, \nu_1 - 1, \boldsymbol{\sigma}^1). \end{aligned} \tag{6.69}$$

6.3.3 Fourier transformation

One can easily verify using (6.67) that

$$\begin{aligned} \nabla_{\sigma^i} \otimes \nabla_{\sigma^i}^{-1} \otimes \hat{M}_B &= \tilde{\nabla}_{\sigma^i} \otimes \tilde{\nabla}_{\sigma^i}^{-1} \otimes \hat{M}_B , \\ \nabla_{\sigma^0} \otimes \nabla_{\sigma^1} \otimes \cdots \otimes \nabla_{\sigma^{n-1}} \otimes \hat{M}_B &= \tilde{\nabla}_{\sigma^0} \otimes \tilde{\nabla}_{\sigma^1} \otimes \cdots \otimes \tilde{\nabla}_{\sigma^{n-1}} \otimes \hat{M}_B . \end{aligned} \tag{6.70}$$

this means that the matrix $\hat{M}_B$ is periodic with respect to shifts by the vectors $(\boldsymbol{\omega}_j, \ldots, \boldsymbol{\omega}_j) + (\boldsymbol{w}^0, \ldots, \boldsymbol{w}^{n-1})$, where $\boldsymbol{w}^i \in Q_{A_{N-1}}$. We can thus construct a Fourier transformation of the fundamental solution in order to (almost) diagonalize all shift operators simultaneously:

$$\mathcal{Y}_k^D \equiv \sum_{w^i \in Q_{A_{N-1}}} e^{2\pi i \sum_{i=0}^{n-1}(\eta^i, w^i+\omega_k)} \mathcal{Y}\left(\{\sigma^i + w^i + \omega_k\}\right) . \tag{6.71}$$

The shift operators act on this expression as follows:

$$\nabla_{\sigma^0}^{-1} \otimes \nabla_{\sigma^1}^{-1} \otimes \cdots \otimes \nabla_{\sigma^{n-1}}^{-1} \mathcal{Y}k^D = e^{2\pi i \tilde{\eta}^0} \otimes e^{2\pi i \tilde{\eta}^1} \otimes \cdots \otimes e^{2\pi i \tilde{\eta}^{n-1}} \mathcal{Y}_{k-1}^D \tag{6.72}$$

This means that one can replace

$$\nabla_{\sigma^0}^{-1} \otimes \nabla_{\sigma^1}^{-1} \otimes \cdots \otimes \nabla_{\sigma^{n-1}}^{-1} \to e^{2\pi i \tilde{\eta}^0} \otimes e^{2\pi i \tilde{\eta}^1} \otimes \cdots \otimes e^{2\pi i \tilde{\eta}^{n-1}} T^{-1} , \tag{6.73}$$

where the operator T shifts the index $k \in \mathbb{Z}/N\mathbb{Z}$:

$$T : \mathcal{Y}_k^D \mapsto \mathcal{Y}_{k-1}^D. \tag{6.74}$$

Thanks to this, the B-cycle monodromy matrix of $\mathcal{Y}^D$ is given by

$$\begin{aligned}\hat{M}_B^T = (-1)^n e^{\pi i(1-N)/N} e^{2\tilde{\pi} i \eta^0} \mathsf{B}(\sigma^{n-1}, \nu_n - 1, \sigma^0) e^{2\pi i \tilde{\eta}^{n-1}} \mathsf{B}(\sigma^{n-2}, \nu_{n-1} - 1, \sigma^{n-1}) \ldots \\ \ldots e^{2\pi i \tilde{\eta}^2} \mathsf{B}(\sigma^1, \nu_2 - 1, \sigma^2) e^{2\pi i \tilde{\eta}^1} \mathsf{B}(\sigma^0, \nu_1 - 1, \sigma^1).\end{aligned} \tag{6.75}$$

The A-cycle monodromy can be computed in the obvious way, but it is different in the sectors with different shifts ω_k:

$$M_{A,k} = e^{2\pi i(\tilde{\sigma}^0 - \omega_k)} = e^{2\pi i k/N} e^{2\pi i \tilde{\sigma}^0} . \tag{6.76}$$

To fix this issue it is necessary to introduce an extra $U(1)$ boson $\varphi(z)$ with the OPE

$$\varphi(z)\varphi(w) \sim -\frac{1}{N} \log(z-w) \tag{6.77}$$

Using this boson we turn W-degenerate fields into N–component fermions:

$$\begin{aligned}\psi_i(z) &= \phi_i(z) \otimes e^{i\varphi(z)} , \\ \bar{\psi}_i(z) &= \bar{\phi}_i(z) \otimes e^{-i\varphi(z)} .\end{aligned} \tag{6.78}$$

After analogous, but quite simpler considerations w.r.t. the ones reported above, we arrive at the result that, for the $U(1)$ factor, the B–cycle monodromy is just the charge-shifting operator for $U(1)$ charge, and the A-cycle monodromy is just some number, different in the different sectors:

$$\begin{aligned}\hat{M}_B^{U(1)} &= e^{2\pi i(\rho + \frac{N-1}{2N})} \left(T^{U(1)}\right)^{-1} , \\ M_{A,k}^{U(1)} &= e^{-2\pi i k/N} e^{2\pi i \sigma^{U(1)}} ,\end{aligned} \tag{6.79}$$

where the $U(1)$ shift operator is defined as

$$T^{U(1)}f(\sigma) = f\left(\sigma + \frac{1}{N}\right) \tag{6.80}$$

We are finally able to construct the following object, which is invariant under the action of $T \cdot T^{U(1)}$:

$$K^{U(N)}(z, z_0) = \sum_{k=0}^{N-1} \mathcal{Y}_k^D(z, z_0)\mathcal{Y}_k^{U(1)}(z, z_0), \tag{6.81}$$

that has number-valued monodromies:

$$\begin{gathered} M_B^T = (-1)^n e^{2\pi i \eta^0} \mathsf{B}(\boldsymbol{\sigma}^{n-1}, \nu_n - 1, \boldsymbol{\sigma}^0) e^{2\pi i \eta^{n-1}} \mathsf{B}(\boldsymbol{\sigma}^{n-2}, \nu_{n-1} - 1, \boldsymbol{\sigma}^{n-1}) \dots \\ \dots e^{2\pi i \eta^2} \mathsf{B}(\boldsymbol{\sigma}^1, \nu_2 - 1, \boldsymbol{\sigma}^2) e^{2\pi i \eta^1} \mathsf{B}(\boldsymbol{\sigma}^0, \nu_1 - 1, \boldsymbol{\sigma}^1), \\ M_A^{U(N)} = e^{2\pi i \sigma^0}, \end{gathered} \tag{6.82}$$

giving a solution to the Riemann-Hilbert problem.

Finally, let us note that from this we can read the explicit form of the fermion braiding matrix B_k used in the previous section:

$$B_k = -\mathsf{B}^T(\boldsymbol{\sigma}^{k-2}, \nu_{k-1} - 1, \boldsymbol{\sigma}^{k-1}) e^{2\pi i \eta^{k-1}}. \tag{6.83}$$

6.4 Relation to Krichever's connection

We wish now to connect the solution we found in the previous sections to the solution of the linear system defined by the Lax matrix

$$\begin{gathered} L_{ii}(z|\tau) = p_i + \sum_k L_{ii}^{(k)} \left[\zeta(z - z_k) - \zeta(z - Q_i) - \zeta(Q_i - z_m)\right], \qquad (6.84) \\ \sum_m L_m^{ii} = -1, \\ L_{ij}(z|\tau) = \sum_k L_{ij}^{(k)} \left[\zeta(z - z_k) - \zeta(z - Q_j) - \zeta(Q_i - z_k) + \zeta(Q_i - Q_j)\right], \quad i \neq j, \end{gathered}$$

obtained following Krichever's construction [143, 154], which is a different approach to the construction of Lax matrices on elliptic curves, that also extends to algebraic curves of higher genus.

Recall that Riemann-Roch theorem forced us, in the $g > 0$ case, to introduce the twist factors that we discussed in Section 6.1. More specifically, a Lax matrix is a meromorphic matrix-valued differential with poles specified by a divisor on the Riemann surface. The space of $r \times r$ matrix functions with degree d divisor of poles has dimension $r^2(d - g + 1)$. Besides the Lax pair matrices L, M, the Lax equation involves also their commutator: if n, m are the degrees of the divisors of L, M respectively, the degree of their commutator is $n + m$. We thus have $r^2(n + m - g + 1)$ equations, but only $r^2(n+m-2g+1)$ unknown functions modulo gauge equivalence. Unless $g=0$, this results in an overdetermined system of equations. One way of dealing with this is tensoring with some other bundle, which is technically what we do when we introduce twists: our Lax matrix was not a meromorphic differential but rather a section of some other bundle, so we cannot straightforwardly apply Riemann-Roch theorem as above.

There exists another way to handle this problem, which is to consider the linear system as defining a vector bundle of degree rg, instead of a degree zero bundle as in the construction with twists. Then the determinant bundle will vanish at rg points, and one can show that the Lax matrix $L(z)$ for such a linear system will have additional simple poles at extra points, the so-called Tyurin points. These simple poles have residue one, so that from the point of view of the linear system they are apparent singularities around which the solution of the linear system

$$\partial_z Y^{Kr}(z|\tau) = L^{Kr}(z|\tau)Y^{Kr}(z|\tau) \tag{6.85}$$

will have no monodromies. The Riemann-Hilbert problem for Y^{Kr} is modified as following: instead of having (6.6), we have

$$\begin{cases} Y(\gamma_k \cdot z|\tau) = Y(z|\tau)M_k, & k=1,\dots,n \\ Y(z+1|\tau) = Y(z|\tau)M_A, & \\ Y(z+\tau|\tau) = Y(z|\tau)M_B, & \\ \det Y(Q_i|\tau) = 0, & i=1,\dots,r, \end{cases} \tag{6.86}$$

with Lax matrix given by (6.84). To make contact with our fermionic construction, first recall that in Section 6.1 it was mentioned that starting from the original Lax matrix (6.13) it is possible to go to a description involving a different one by means of a singular gauge transformation, so that to find the CFT description of this approach we should find a $g(z|\tau)$ such that

$$L^{Kr} = gLg^{-1} + \partial_z g g^{-1}. \tag{6.87}$$

For the one-punctured torus with a single pole at zero, the Lax matrices are

$$L^{Kr}_{ij} = m\frac{\theta_1(z+Q_i-Q_j)\theta_1(z-Q_i)\theta_1(Q_j)}{\theta_1(z)\theta_1(z-Q_j)\theta_1(Q_i-Q_j)\theta_1(Q_i)}, \tag{6.88}$$

$$L^{Kr}_{ii} = p_i + E_1(z-Q_i) - E_1(z) + E_1(Q_i), \tag{6.89}$$

$$L^{CM}_{ij} = mx(Q_i-Q_j,z), \qquad L^{CM}_{ii} = p_i, \tag{6.90}$$

so that the gauge transformation is relatively easy to find, and is given by

$$g(z) = \operatorname{diag}\left[\frac{\theta_1'(0)\theta_1(z+Q_i)}{\theta_1(z)\theta_1(Q_i)}\right]. \tag{6.91}$$

To generalize this to the case of many punctures, it is convenient instead to consider the Riemann-Hilbert Problem (6.86). Such solution, that we will from now on denote by Y^{Kr}, can be constructed from $Y(z)$ in the following way:

$$Y^{Kr}(z) = \mathrm{diag}\frac{\theta_1'(0)\theta_1(z-z_1+Q_i)}{\theta_1(z-z_1)\theta_1(Q_i-z_1)} \times Y(z) \equiv g(z)Y(z). \tag{6.92}$$

We see that $\det Y^{Kr}(z_1-Q_i) = 0$ in all points Q_i, and also all its singular exponents in the point z_1 are shifted. One way to obtain this solution is the following: consider first the kernel (6.20):

$$K(z,z_0) = Y(z_0)^{-1}\frac{\theta_1'(0)\theta_1(z-z_0+\boldsymbol{Q})}{\theta_1(z-z_0)\theta_1(\boldsymbol{Q})}Y(z), \tag{6.93}$$

and then send $z_0 \to z_1$:

$$Y^{Kr}(z) = \lim_{z_0\to z_1} Y(z_0)K(z,z_0). \tag{6.94}$$

This formula has clear CFT interpretation: near z_1 the behavior of the solution is

$$Y(z_0) = G_1(z_0-z_1)(z_0-z_1)^{\boldsymbol{\theta}_1}C_1, \tag{6.95}$$

where $G_1(z)$ is holomorphic and invertible around $z=0$. Therefore

$$K(z,z_0) = C_1^{-1}(z_0-z_1)^{-\boldsymbol{\theta}_1}G_1(z_0-z_1)^{-1}Y^{Kr}(z). \tag{6.96}$$

Because of the limit $z_0 \to z_1$, in the CFT we have to consider the OPE of the fermion $\bar{\psi}_\alpha(z_0)$ with the primary field $V_{\boldsymbol{\theta}_1}(z_1)$:

$$\bar{\psi}_\alpha(z_0)V_{\boldsymbol{\theta}_1}(z_1) = \sum_\beta \left(C_1^{-1}\right)_{\alpha\beta}(z_0-z_1)^{-\theta_{1,\beta}}\delta_\beta V_{\boldsymbol{\theta}_1}(z_1) + \ldots, \tag{6.97}$$

where $\delta_\beta V_{\boldsymbol{\theta}_1}$ is a field with shifted W-charge $\boldsymbol{\theta}_1 \mapsto \boldsymbol{\theta}_1 - \boldsymbol{h}_\beta$ [4]. Now comparing (6.97) with (6.94) we can identify

$$Y^{Kr}(z)_{\alpha\beta} = \sum_\gamma G_1(0)_{\alpha\gamma}\frac{\langle\psi_\alpha(z)\delta_\gamma V_{\boldsymbol{\theta}_1}(z_1)V_{\boldsymbol{\theta}_2}(z_2)\ldots V_{\boldsymbol{\theta}_n}(z_n)\rangle}{\langle V_{\boldsymbol{\theta}_1}(z_1)\ldots V_{\boldsymbol{\theta}_n}(z_n)\rangle}. \tag{6.98}$$

We see that up to normalization (which is not actually fixed) Krichever's solution has the nice CFT interpretation of the expectation value of a single fermion in the presence of all the vertex operators. The expression of the two-fermionic correlator in terms of Krichever's solution can be obtained by

[4] Notice that in the general W_N case fields $\delta_\beta V$ are rather problematic. The only well-understood fields are the ones with $\boldsymbol{\theta}_1 = \nu_1\boldsymbol{\omega}_1$, but fields with charge $\boldsymbol{\theta}_1 - \boldsymbol{h}_\beta$ generally do not lie in this class.

applying the gauge transformation (6.92):

$$K(z,z_0) = Y(z_0)^{-1}\frac{\theta_1'(0)\theta_1(z-z_0+\boldsymbol{Q})}{\theta_1(z-z_0)\theta_1(\boldsymbol{Q})}Y(z) =$$
$$= Y^{Kr}(z_0)^{-1}\frac{\theta_1(z_0-a_1+\boldsymbol{Q})}{\theta_1(z_0-a_1)}\frac{\theta_1'(0)\theta_1(z-z_0+\boldsymbol{Q})}{\theta_1(z-z_0)\theta_1(\boldsymbol{Q})}\frac{\theta_1(z-a_1)}{\theta_1(z-a_1+\boldsymbol{Q})}Y^{Kr}(z). \tag{6.99}$$

We thus see that Krichever's solution becomes less natural than $Y(z)$ if we wish to express the two-fermionic correlator, because it contains a more involved diagonal matrix between $Y^{Kr}(z_0)^{-1}$ and $Y^{Kr}(z)$. On the other hand, contrary to what happens in the twisted formulation, the solution itself can be obtained from the CFT, not only the kernel.

6.5 Solution of the elliptic Schlesinger system

As a further application of our results we will now show how, starting from equations (6.49), (6.50), one can obtain a formula for the solution of the Calogero-like variables Q_i of the elliptic Schlesinger system. This formula generalizes the algebro-geometric solution of the elliptic Calogero-Moser model found in [144] to the nonautonomous case with many punctures, and suggests a double role of the dual partition function from the point of view of integrable systems: on the one hand, being proportional to the tau function, its vanishing locus includes the Malgrange divisor, where the Riemann-Hilbert problem is no longer solvable [118, 119, 155]. On the other hand, we have an extra vanishing locus, which generalizes the Riemann theta divisor of the Krichever/Seiberg-Witten curve, whose points are the solution to the equations of motion of the isomonodromic system. Note that this is essentially a consequence of our choice of twists, or analogously of the choice of Calogero-like dynamical variables. As a byproduct, we will also obtain a direct link between the isomonodromic tau function and the $SU(n)$, rather than $U(n)$, gauge theory.

However, note that in the case of more than one puncture, the Calogero-like variables Q_i do not specify the whole system: there are additional spin variables satisfying the Kirillov-Kostant Poisson bracket for $\mathfrak{sl}(N)$ [151] that will not enter in the following discussion: while it may be that there is some further connection between Z^D and these remaining dynamical variables, this does not seem evident at the moment.

In order to obtain the aforementioned result, we first split Z^D in various components having different types of $\mathfrak{gl}_N$ shifts:

$$Z^D = \mathrm{tr}_{\mathcal{H}}(-)^F e^{2\pi i \boldsymbol{\eta}\cdot J_0} q^{L_0} V = \sum_{\mathbf{n}\in\mathbb{Z}^N} \mathrm{tr}_{\mathcal{H}_{\mathbf{n}}}(-)^F e^{2\pi i \boldsymbol{\eta}\cdot J_0} q^{L_0} V, \tag{6.100}$$

where we denoted by V the whole string of vertex operators. To perform the splitting, it is convenient to decompose $\boldsymbol{\eta}$ as

$$\boldsymbol{\eta} = \eta_1\boldsymbol{\omega}_1 + \cdots + \eta_{N-1}\boldsymbol{\omega}_{N-1} + N\rho\mathbf{e} \equiv \tilde{\boldsymbol{\eta}} + N\rho\mathbf{e}, \tag{6.101}$$

where

$$\mathbf{e} \equiv \frac{1}{N}\left(1, \ldots, 1\right), \tag{6.102}$$

and $\boldsymbol{\omega}_k$ are the fundamental weights of $\mathfrak{sl}_N$, normalized as

$$\boldsymbol{\omega}_k \cdot \boldsymbol{\omega}_k = k\frac{N-k}{N}. \tag{6.103}$$

We also decompose $\mathbf{n}$ as

$$\mathbf{n} = (n_1, \ldots, n_N) \equiv \tilde{\mathbf{n}} + N\left(k + \frac{j}{N}\right)\mathbf{e}, \tag{6.104}$$

where we separated the traceless part from the $U(1)$ factor

$$J_0^{U(1)} = \mathbf{n}\cdot\mathbf{e} = \frac{n}{N} \equiv k + \frac{j}{N}, \tag{6.105}$$

with $j = 0, \ldots, N-1$. The space $\mathcal{H}_{\mathbf{n}}$ analogously decomposes into a W_N highest weight module plus a Fock space, with $U(1)$ charge given by

$$\mathcal{H}_{\mathbf{n}} = \mathcal{W}_{\mathbf{a}+\tilde{\mathbf{n}}} \oplus \mathfrak{F}_{\sigma+1/2+k+j/N}, \tag{6.106}$$

where we shifted the $U(1)$ charge σ by $1/2$ to get consistent signs in the monodromy, as in the 2×2 case. Then,

$$Z^D = \sum_{j=0}^{N-1}\sum_{k,\tilde{\mathbf{n}}} \mathrm{tr}_{\,\mathcal{W}_{\mathbf{a}+\tilde{\mathbf{n}}}}\left(e^{2\pi i\boldsymbol{\eta}\cdot\tilde{\mathbf{n}}}q^{L_0}V\right)\mathrm{tr}_{\,\mathfrak{F}_{\sigma+k+j/N+1/2}}\left(e^{2\pi i N(\rho+1/2)(k+j/N)}q^{L_0}\right), \tag{6.107}$$

where we encoded the fermion number operator into a shift of ρ by $1/2$. However, we must note that j is not independent of the W_N charge shift. In fact, if we parametrize

$$\mathbf{n} = (n_1+k, n_2+k, \ldots, n_{N-1}+k, k), \tag{6.108}$$

the $U(1)$ charge is indeed

$$J_0^{U(1)} = k + \frac{n_1+\cdots+n_{N-1}}{N} \equiv k + \frac{j}{N}, \tag{6.109}$$

but we also have

$$\frac{n_1+\ldots n_{N-1}}{N} = \mathbf{n}\cdot\boldsymbol{\omega}_{N-1}, \tag{6.110}$$

so that j/N is the shift in the W_N weight along the $\boldsymbol{\omega}_{N-1}$ direction. Then,

$$\begin{aligned} Z^D &= \frac{1}{\eta(\tau)} \sum_{j=0}^{N-1} Z_j^D \sum_k e^{2\pi i N(\rho+1/2)(k+j/N)} e^{N\pi i\tau(\sigma+k+j/N+1/2)^2} \\ &= \frac{q^{\sigma^2} q^{N/8} q^{N(\sigma\tau+1/2)/2}}{\eta(\tau)} \sum_{j=0}^{N-1} \theta_{N\tau} \begin{bmatrix} j/N \\ 0 \end{bmatrix} (N(\rho+1/2+(\sigma+1/2)\tau)) Z_j^D, \end{aligned} \tag{6.111}$$

where we defined

$$Z_j^D \equiv \sum_{\tilde{\boldsymbol{n}} \in Q_{A_{N-1}}, \tilde{\boldsymbol{n}} \cdot \boldsymbol{\omega}_{N-1} = j/N} \operatorname{tr}_{\mathcal{W}_{\boldsymbol{a}+\tilde{\boldsymbol{n}}}} e^{2\pi i \boldsymbol{\eta}\cdot\tilde{\boldsymbol{n}}} q^{L_0} V. \tag{6.112}$$

We should now compare the equation above with

$$Z^D = \mathcal{T} \prod_i \frac{\theta_1(Q_i - \sigma\tau - \rho)}{\eta(\tau)}. \tag{6.113}$$

First of all, from this expression we see that $\sigma\tau + \rho = Q_i$ are zeros of Z_D. In other words, the solutions of the nonautonomous system are given by

$$Z^D|_{\sigma\tau+\rho=Q_i} = 0, \qquad i = 1, \dots, N. \tag{6.114}$$

This is a generalization to the nonautonomous case of the condition $\theta(Q) = 0$, expressing the solution of the autonomous integrable system as the vanishing theta divisor of the Seiberg-Witten curve, which is the autonomous limit of our description. Further, the decomposition (6.111) is a deformation of the one expressing the Riemann theta function associated to the Seiberg-Witten curve as a sum over $N-1$ Jacobi theta functions with characteristics shifted by j [144].

We can further write the isomonodromic tau function in a way that is manifestly independent from the $U(1)$ charges. By writing all the theta functions in their q-series representation, we have

$$\begin{aligned} Z^D &= \frac{q^{\sigma^2}}{\eta(\tau)^N} (i)^N \mathcal{T} \sum_{n_1,\dots,n_N} (-)^{n_1+\dots n_N} e^{2\pi i\tau\left[(n_1+1/2)^2+\dots+(n_N+1/2)^2\right]/2} \\ &\quad \times e^{2\pi i[(n_1+1/2)(-Q_1+\sigma\tau+\rho)+\dots+(n_N+1/2)(-Q_N+\sigma\tau+\rho)]} \\ &= \frac{q^{\sigma^2}}{\eta(\tau)^N} (i)^N \mathcal{T} \sum_{\mathbf{n}\in\mathbb{Z}^N} e^{2\pi i \mathbf{n}\cdot(-\mathbf{Q}+(\sigma\tau+\rho+1/2)\mathbf{e})} e^{2\pi i(\mathbf{n}+\mathbf{e}/2)^2\tau/2}. \end{aligned} \tag{6.115}$$

We decompose, similarly as before,

$$\mathbf{n} = \tilde{\mathbf{n}} + N(n + j/N)\mathbf{e}, \tag{6.116}$$

and find

$$\begin{aligned} Z^D &= i^N \mathcal{T} \frac{q^{\sigma^2}}{\eta(\tau)^N} (i)^N e^{i\pi N(\sigma\tau+\rho)} \\ &\times \sum_{j=0}^{N-1} \sum_{\tilde{\mathbf{n}},k} \left(e^{2\pi i \tilde{\mathbf{n}}\cdot\mathbf{Q}} e^{i\pi\tilde{\mathbf{n}}^2\tau} \right) \left(e^{i\pi N(k+j/N)} e^{i\pi N\tau(k+j/N+1/2)^2} e^{2\pi i N(\sigma\tau+\rho)(k+j/N)} \right) \\ &= \mathcal{T} \frac{q^{\sigma^2}}{\eta(\tau)^N} e^{i\pi N(\sigma\tau+\rho+1/2)} \sum_{j=0}^{N-1} \Theta_j(\mathbf{Q}) \sum_k e^{i\pi N\tau(k+j/N)^2} e^{2\pi i N((\sigma+1/2)\tau+\rho+1/2)} \\ &= \mathcal{T} \frac{q^{\sigma^2}}{\eta(\tau)^N} e^{i\pi N(\sigma\tau+\rho+1/2)} \sum_{j=0}^{N-1} \Theta_j(\mathbf{Q}) \theta_{N\tau} \left[\begin{array}{c} j/N \\ 0 \end{array} \right] (N((\sigma+1/2)\tau+\rho+1/2))\,. \end{aligned} \tag{6.117}$$

Comparing the two expressions, we see that

$$Z_j^D = \frac{e^{i\pi N\rho}}{\eta(\tau)^{N-1}} \Theta_j(\mathbf{Q}) \mathcal{T}, \tag{6.118}$$

where

$$\Theta_j(\mathbf{Q}) = \sum_{\mathbf{n}\in Q_{A_{N-1}}:\ \mathbf{n}\cdot\boldsymbol{\omega}_{N-1}=j/N} e^{2\pi i\mathbf{n}\cdot\mathbf{Q}} e^{i\pi\mathbf{n}^2\tau}. \tag{6.119}$$

Chapter 7

Higher genus Class S Theories and isomonodromy

In the previous two chapters we illustrated the results of [1, 2], where we extended the connection between supersymmetric gauge theories, 2d CFT and isomonodromic deformations to the case of circular quiver gauge theories with an arbitrary number of $SU(N)$ nodes, corresponding to CFT and isomonodromic deformations on the torus.

In this chapter we extend further these results to the case of a general Riemann surface with genus g and arbitrary number n of simple punctures. We will first recall the general Hamiltonian construction for isomonodromic deformation on Riemann Surfaces of [127, 129], while also defining our choice for the canonical coordinates of the isomonodromic Hamiltonian system. We will then show how isomonodromic deformations take the form of the flatness conditions for a connection over the total space $T_{g,n} \times C_{g,n}$, where $C_{g,n}$ is the punctured Riemann surface and $T_{g,n}$ the Teichmüller space of surfaces with genus g and n punctures. We also discuss the relation to the Hitchin system and the class S construction. After this, we will show how the relation between tau function and free fermion conformal blocks can be extended to this case: this automatically extends the correspondence also for the gauge theory when the AGT correspondence has been shown to hold (for instance, in the $SL(2)$ case [77]).

7.1 Hamiltonian approach to isomonodromic deformations

The general idea is that a flat connection determines a representation of the monodromy group of a Riemann surface. Isomonodromic flows (for regular singularities, at least) can be seen as deformations of the flat connection where the times are local coordinates in the Teichmüller space of the surface, which leave invariant the conjugacy classes of the monodromies. We will work in this section with an arbitrary (complex) Lie algebra $\mathfrak{g}$.

Let $\mathcal{A}_{g,n}$ be the space of affine connections on a Riemann surface $C_{g,n}$ of genus g with n marked points z_k, $k = 1, \dots n$. The complex structure of the Riemann surface induces a polarization of the connection a, i.e. a splitting into $(1,0)$ and $(0,1)$ parts

$$d + a \equiv (\partial_z + A_z)dz + (\bar{\partial}_{\bar{z}} + \bar{A}_{\bar{z}})d\bar{z} \equiv (dz\partial_z + A) + (d\bar{z}\partial_{\bar{z}} + \bar{A}). \tag{7.1}$$

The puctures are described by boundary conditions at the marked points $\{z_k\}$, in terms of coadjoint orbits, similarly to the case of the Hitchin system that we described in Section 2.4:

$$\mathcal{O}_k = \{S_k = g^{-1} S_k^0 g, g \in G, S_k^0 \in \mathfrak{g}^*\}. \tag{7.2}$$

The affine space of connections $\mathcal{A}(C_{g,n})$ [1] is

$$\mathcal{A}(C_{g,n}) = \left\{ (A, \bar{A}) : \begin{cases} \bar{A}|_{U_k} = 0, \\ A|_{U_k} = \frac{S_k}{z - z_k} + O(1), \end{cases} \quad S_k \in \mathcal{O}_k \right\}. \tag{7.3}$$

This space carries a symplectic structure, given by two separate parts:

$$\omega = \omega_0 + \omega_{KK}. \tag{7.4}$$

Here ω_0 is the Atiyah-Bott symplectic form [156]

$$\omega_0 = \int_{C_{g,n}} \langle \delta A, \delta \bar{A} \rangle, \tag{7.5}$$

with $\langle , \rangle$ Killing form of $\mathfrak{g}$, and ω_{KK} is the Kirillov-Konstant symplectic structure on the coadjoint orbits:

$$\omega_{KK} = \sum_k \langle S_k, g^{-1}\delta g \wedge g^{-1} \delta g \rangle. \tag{7.6}$$

We can impose the flatness condition on this space as a moment map constraint:

$$\mu_A = F_A - \sum_{k=1}^{n} S_k \delta^2(z - z_k) : \mathcal{A}(C_{g,n}) \to \mathbb{C}, \tag{7.7}$$

where F_A is the curvature

$$(F_A)_{z\bar{z}} = \partial_z A_{\bar{z}} + \partial_{\bar{z}} A_z + [A_{\bar{z}}, A_z]. \tag{7.8}$$

The moduli space $\mathcal{A}_{flat}$ of flat connections is obtained by imposing the moment map condition, and the moduli space Bun_{flat} of flat bundles is obtained by quotienting by the gauge group $\mathcal{G}$:

$$\mathcal{A}_{flat}(C_{g,n}) = \mu^{-1}(0), \quad \mathrm{Bun}_{flat}(C_{g,n}) = \mu^{-1}(0)/\mathcal{G} = \mathcal{A}(C_{g,n})//\mathcal{G}, \tag{7.9}$$

where we adopted the usual notation $//$ for the symplectic quotient. In particular, after imposing the flatness condition and gauge-fixing, the resulting

[1] In general, the boundary condition for the connection will include simple poles for $\bar{A}$ around the marked points z_k. However, we can obtain the boundary conditions below by gauging away $\bar{A}$ in a small neighborhood U_k of the point z_k (see e.g. [127, 129]).

space $\mathrm{Bun}_{flat}(C_{g,n})$ is finite dimensional:

$$\dim \mathrm{Bun}_{flat}(C_{g,n}) = 2\dim\mathfrak{g}(g-1) + \sum_{k=1}^{n} \dim \mathcal{O}_k. \tag{7.10}$$

We want to study motions in this space induced by changes in the $3g-3+n$ Riemann surface moduli. These are the punctures' positions z_k, $k = 1, \dots n$, together with the $3g-3$ moduli m_s, $s = 1, \dots 3g-3$ of the corresponding compact surface C_g: we will denote them in the following collectively as t_i, $i = 1, \dots 3g-3+n$. The deformations are induced by a chiral diffeomorphism

$$w = z - \varepsilon(z,\bar{z}), \qquad \bar{w} = \bar{z}. \tag{7.11}$$

Under which partial derivatives and differentials transform as

$$\partial_w = \partial_z, \qquad \partial_{\bar{w}} = \partial_{\bar{z}} + \mu(z,\bar{z})\partial_z, \tag{7.12}$$

$$dw = dz - \mu(z,\bar{z})d\bar{z}, \qquad d\bar{w} = d\bar{z}. \tag{7.13}$$

Here we introduced the Beltrami differential

$$\mu(z,\bar{z}) = \frac{\partial_{\bar{z}}\varepsilon(z,\bar{z})}{1-\partial_z\varepsilon(z,\bar{z})} = \partial_{\bar{z}}\varepsilon(z,\bar{z}) + O(\varepsilon^2) \in \Omega^{(-1,1)}(C_{g,n}). \tag{7.14}$$

Note that

$$\mu(w(z,\bar{z}),\bar{w}(z,\bar{z})) = \partial_{\bar{w}}\varepsilon(w,\bar{w}) + O(\varepsilon^2). \tag{7.15}$$

The moduli space $\mathcal{M}_{g,n}$ of punctured Riemann surfaces is identified with the space of Beltrami differentials modulo diffeomorphisms, and its tangent space is the Teichmüller space $T_{g,n}$. This allows us to write any Beltrami differential as

$$\mu(z,\bar{z}) = \sum_{i=1}^{3g-3+n} t_l \mu_l(z,\bar{z}), \tag{7.16}$$

where t_l are local coordinates of $T_{g,n}$, that will be identified with the space of isomonodromic times. The Beltrami differentials related to the moving points z_k are simply

$$\mu_k(z,\bar{z}) = \partial_{\bar{z}}\chi_k(z,\bar{z}), \tag{7.17}$$

where χ_k is the characteristic function of an infinitesimal neighborhood U_k of z_k. From the transformation of the differentials and partial derivatives we can obtain the relation between the polarization of the connection in the two set of coordinates:

$$A_z = A_w, \qquad A_{\bar{z}} = A_{\bar{w}} - \mu(z,\bar{z})A_w. \tag{7.18}$$

To make the Hamiltonian properties of $\mathrm{Bun}_{flat}(C_{g,n})$ under the flows $\{m_\alpha, z_k\} \in T_{g,n}$ manifest, we go to the extended Hamilton-Jacobi phase space.

This means considering, together with variations of the usual phase space coordinates, also those of the times, which will be "canonically conjugated" to the Hamiltonians in the sense that the symplectic form becomes

$$\omega = \omega_0 + \omega_{KK} - \sum_l \delta H_l \delta t^l. \tag{7.19}$$

By performing the transformation (7.11) on the symplectic form, we obtain explicitly

$$\omega = \omega_0 + \omega_{KK} - \int_{C_{g,n}} \langle \delta A, A \rangle \delta\mu = \omega - \frac{1}{2}\sum_{l=1}^{3g-3+n} \delta t_l \delta \int_{C_{g,n}} \langle A, A \rangle \mu_l. \tag{7.20}$$

This shows that the Hamiltonians inducing the flows are

$$H_l = \frac{1}{2}\int_{C_{g,n}} \langle A, A \rangle \mu_l. \tag{7.21}$$

The equations of motion following from (7.20) are free equations on the affine space $\mathcal{A}(C_{g,n})$

$$\partial_{m_l} A = 0, \qquad \partial_{m_l} \bar{A} = A\mu_l. \tag{7.22}$$

As we will now show, these become nontrivial, and identified with isomonodromic deformation equations, after going to a gauge where A is meromorphic, imposing the flatness condition (7.7). Let us comment on these points: while on the sphere it is possible to completely gauge away the antiholomorphic part of the connection, on a higher genus Riemann surface this is not possible, since there can exist topologically nontrivial connections. Let us now define

$$\partial \equiv dw\partial_w, \qquad \bar{\partial} \equiv d\bar{w}\partial_{\bar{w}} = d\bar{w}\left(\partial_{\bar{z}} + \mu(z,\bar{z})\partial_z\right) \tag{7.23}$$

By means of a regular gauge transformation

$$A \to f(\partial + A)f^{-1}, \qquad \bar{A} \to f(\bar{\partial} + \bar{A})f^{-1} \tag{7.24}$$

it is at most possible to fix

$$\partial\bar{A}(w,\bar{w}) = 0. \tag{7.25}$$

The flatness condition in this gauge is

$$\bar{\partial}A + [\bar{A}, A] = \sum_k S_k \delta^2(w - z_k). \tag{7.26}$$

These two equations tell us that in this gauge $\bar{A}$ is an anti-holomorphic differential, but, differently from what happens for the Schlesinger system on the Riemann sphere, in general A will not be meromorphic. To make contact with isomonodromy deformations, where we want our Lax matrix to be meromorphic, we thus have to further gauge-fix $\bar{A}$ to zero. In order to do this, we have to find a function f such that

$$\bar{A} = f^{-1}\bar{\partial} f. \tag{7.27}$$

After this operation, the flatness condition is simply

$$\bar{\partial} L = \sum_k S_k \delta^2(w - z_k), \tag{7.28}$$

where we denoted by L the $(1,0)$ component of the connection in this gauge, because it will be the Lax matrix of our isomonodromic system, and we see that it is meromorphic. However, it is no longer a well-defined object on $C_{g,n}$, since the gauge transformation (7.27) is not single valued: because $\bar{A}$ is anti-holomorphic, f carries a representation of the monodromy group of the compact Riemann surface $C_{g,0}$,:

$$f(\gamma_{A_j} \cdot P) = T_{A_j} f(P), \qquad f(\gamma_{B_j} \cdot P) = T_{B_j} f(P), \tag{7.29}$$

$$\prod_j T_{A_j} T_{B_j} T_{A_j}^{-1} T_{B_j}^{-1} = id. \tag{7.30}$$

Here $(\gamma_{A_j} \cdot)$ and $(\gamma_{B_j} \cdot)$ denote analytic continuation along an A_j- (B_j-) cycle. Because of this, L is a twisted meromorphic differential on the Riemann Surface $C_{g,n}$, with twists given by the matrices T_{A_j}, T_{B_j} above, which means that

$$\gamma'_{A_j}(z) L(\gamma_{A_j} z) = T_{A_j} L(z) T_{A_j}^{-1}, \qquad \gamma'_{B_j}(z) L(\gamma_{B_j} z) = T_{B_j} L(z) T_{B_j}^{-1}. \tag{7.31}$$

We wish now to see what is the form of the equations of motion (7.22) after gauge-fixing. By writing

$$A = f^{-1} L f + f^{-1} \partial f, \quad \bar{A} = f^{-1} \bar{\partial} f, \quad M_s \equiv (\partial_{t_s} f) f^{-1} \equiv (\partial_s f) f^{-1}, \tag{7.32}$$

we have that the first equation of (7.22) becomes

$$0 = \partial_s A = \partial_s \left(f^{-1} L f + f^{-1} \partial f \right) = f^{-1} \left(\partial_s L + \partial M_s + [L, M_s] \right) f. \tag{7.33}$$

The matrix M_s is defined by the second equation of motion: we have

$$\partial_s \bar{A} = f^{-1} \bar{\partial} M_s f + \mu_s f^{-1} \partial f = \mu_s A = \mu_s f^{-1} L f + \mu_s f^{-1} \partial f. \tag{7.34}$$

In terms of the Lax pairs (L, M_s), $s = 1, \dots, 3g - 3 + n$, the two equations of motion, together with the moment map equation, are then

$$\begin{cases} \partial_s L + \partial M_s + [L, M_s] = 0, \\ \bar{\partial} M_s = \mu_s L, \\ \bar{\partial} L = 0, \end{cases} \tag{7.35}$$

which are the consistency condition for the linear systems

$$\begin{cases} \partial Y = LY, \\ \partial_s Y = -M_s Y, \\ \bar{\partial} Y = 0. \end{cases} \tag{7.36}$$

The last equation tells us that Y is holomorphic in the transformed coordinates, while the first two equations mean that it is multivalued along the nontrivial cycles, and that the evolution with respect to the moduli is isomonodromic for the linear system $dY = LY$. In this gauge, L will have the following form:

$$L(z) = dz\Pi^b_j t_a \omega^j(z)^a{}_b + dz \sum_{k=1}^{n} S^b_k t_a \Theta(z, z_k)^a{}_b, \tag{7.37}$$

where $\omega^j(z)^a{}_b$ is a basis of twisted holomorphic differentials, and $\Theta(z, z_k)$ are twisted meromorphic one-forms with one simple pole at $z = z_k$, and we are using Einstein's convention for summation of repeated indices. In terms of the Lax matrix L, the Hamiltonians take the form of Hitchin Hamiltonians

$$H_s = \int_{C_{g,n}} \frac{1}{2} \mathrm{tr}\, L^2(z) \mu_s(z, \bar{z}). \tag{7.38}$$

The above formulas are completely general; however, we are able to provide an explicit parametrization of the Lax matrix (7.37) only for bundles admitting a Schottky parametrization, for which $T_{A_i} = id$. In this case, the twisted differentials can be expressed as twisted Poincaré series, and we give their expressions in Appendix B. Of course, differently from what happened in genus one, the space of flat bundles admitting such coordinates is only an open subspace of Bun_{flat}. A description covering the whole space of flat bundles should resort to some other set of coordinates, e.g. to Fock-Goncharov coordinates for the flat bundle, which cover the whole space with transition functions given by mutations [157], but we are unaware of explicit expressions for the twisted differentials in that case. Sadly, this means that in Section 7.1.3 we will be able to define canonical variables on our reduced space only for Schottky bundles. The description in terms of Schottky bundles has however the following nice features:

- The space of Schottky bundles appears naturally in the context of two-dimensional conformal field theory and in particular in KZB equations [158–161], which are a quantization of isomonodromic deformation equations [162, 163] so that it will allow us to relate the isomonodromic tau function generating the Hamiltonians (7.38) to free fermion conformal blocks;

- Related to this point is the fact that Schottky bundles naturally arise when studying a pants decomposition of a Riemann Surface [164], so that one could think of using this description in the future to generalize the construction of [135, 136] to higher genus Riemann Surfaces;

It is also our hope that, as it happened in the case of genus one [2, 128], it turns out to be possible to find an explicit Hecke modification to bring the Schottky bundle to the one defined by the Tyurin parametrization with apparent singularities [143], thus ensuring the generality of our approach.

7.1.1 Relation to class S theories

The above construction also clarifies the appearance of isomonodromic deformations in the study of Class S theories. Recall that the BPS equations for the compactification of 5d $\mathcal{N} = 2$ super Yang-Mills on the Riemann Surface $C_{g,n}$ took the form of a flatness condition for the connection

$$\mathcal{A} = A + \frac{R}{\zeta}\Phi + R\zeta\bar{\Phi}, \tag{7.39}$$

where A was the restriction of the five-dimensional gauge field to the Riemann surface, while Φ was the scalar of the five-dimensional vector multiplet. The boundary conditions at the marked points for $\mathcal{A}$ were of the type (in a local coordinate around z_k)

$$\mathcal{A}(z \sim z_k) =\sim \left(\frac{R\rho}{2\zeta} + \frac{\alpha}{2i}\right)\frac{dz}{z - z_k} + \left(\frac{R\zeta\bar{\rho}}{2} - \frac{\alpha}{2i}\right)\frac{d\bar{z}}{\bar{z} - \bar{z}_k}, \tag{7.40}$$

where α, ρ are the boundary condition at the puncture for A, Φ respectively. The twistorial parameter ζ kept track of the $\mathbb{P}^1$ worth of complex structures for the hyperkähler moduli space. For a generic $\zeta \in \mathbb{C}^\star$, the space is identified exactly with the moduli space of flat complex connections $\mathcal{M}_{flat}$. The gauge-fixing $\partial\bar{A} = 0$ gauges away the antiholomorphic polar part, leaving us with a flat connection a satisfying

$$a(z \sim z_k) = S_k^0 \frac{dz}{z - z_k} + O(1), \tag{7.41}$$

where S_k will be a function of the original data R, ζ, ρ, α determined by the gauge transformation. We see that one arrives naturally at the starting point of Section 7.1. The isomonodromic flows describe the change of the compactified theory under deformations of the moduli of the Riemann Surface, while keeping constant the (monodromy) data defining the compactification itself. The much less trivial statement, for which we have only indirect proofs from two-dimensional CFT, is that the Hamiltonians inducing these flows are generated by the dual partition function of the four-dimensional QFT.

7.1.2 Isomonodromic deformations from flatness of a universal connection

We will now show how the isomonodromic deformation equations derived above are just the flatness conditions of a unique universal connection over

$C_{g,n} \times T_{g,n}$. In order to treat in a symmetric manner the "Horizontal" directions $C_{g,n}$ and the "longitudinal" directions $T_{g,n}$we will introduce also antiholomorphic coordinates of the Teichmüller space $\bar{t}^s$, which amounts to consider a non-chiral diffeomorphism

$$\begin{cases} w = z - \varepsilon(z,\bar{z}), \\ \bar{w} = \bar{z} - \bar{\varepsilon}(z,\bar{z}). \end{cases} \tag{7.42}$$

Because of this, we will have not only a Beltrami differential $\mu = \bar{\partial}\varepsilon$, but also its "conjugate" $\bar{\mu} = \partial\bar{\varepsilon}$. The flat connection that we introduce is

$$\nabla \equiv d + s + a + M, \tag{7.43}$$

where

$$a = Adz + \bar{A}d\bar{z}, \qquad M = \delta t^s M_s + \delta\bar{t}^s M_{\bar{s}}. \tag{7.44}$$

The de Rham differential along the $C_{g,n}$ directions are defined as follows:

$$d = \partial + \bar{\partial} = dw\left(\partial_z + \bar{\mu}\partial_{\bar{z}}\right) + d\bar{w}\left(\partial_{\bar{z}} + \mu\partial_z\right), \tag{7.45}$$

and the action of the longitudinal differential s acting on the $\mathcal{M}_{g,n}$ direction is:

$$sA = \delta t^s\partial_s A + \delta\bar{t}^s\left(\partial_{\bar{s}}A - \bar{\mu}_s\bar{A}\right), \quad s\bar{A} = \delta t^s(\partial_s\bar{A} - \mu_s A) + \delta\bar{t}^s\partial_{\bar{s}}\bar{A}, \tag{7.46}$$

$$sM = \delta t^r \wedge \delta t^s\partial_s M_r + \delta t^r \wedge \delta\bar{t}^s\left(\partial_{\bar{s}}M_r - \partial_r\bar{M}_s\right) + \delta\bar{t}^r \wedge \delta\bar{t}^s\partial_{\bar{s}}\bar{M}_r. \tag{7.47}$$

Then the flatness condition for ∇ is

$$0 = \nabla^2 = (da + a \wedge a) + (sa + dM + a \wedge M) + (sM + M \wedge M). \tag{7.48}$$

The first term is a two-form on the Riemann surface, the last term is a two-form on moduli space, while the second term is mixed. If we expand the equations in components, we have that the vanishing of the various terms give the system of equations

$$\begin{cases} \bar{\partial}A + \partial\bar{A} + [A,\bar{A}] = 0, \\ \partial_s A + \partial M_s + [A, M_s] = 0, \\ \partial_s\bar{A} + \bar{\partial}M_s + [\bar{A}, M_s] = \mu_s A, \\ \partial_{\bar{s}}A + \partial\bar{M}_s + [A,\bar{M}_s] = \bar{\mu}_s\bar{A}, \\ \partial_{\bar{s}}\bar{A} + \bar{\partial}\bar{M}_s + [\bar{A},\bar{M}_s] = 0, \\ sM + M \wedge M = 0. \end{cases} \tag{7.49}$$

The first and last equations state that $d + a$ and $s + M$ are separately flat connections, respectively on the Riemann surface $C_{g,n}$ and on Teichmüller space. In particular, the last equation tells us that locally $M_s = (\partial_s f)f^{-1}$.

The second and third equations are the isomonodromic deformation equation and the definition of the Lax pairs, for a generic choice of gauge in which $\bar{A} \neq 0$, while the fourth and fifth equations are their specular version with respect to the anti-holomorphic coordinates of the moduli space.

7.1.3 Poisson brackets and canonical variables

We finally turn to the study of the symplectic structure (7.20) in the multivalued gauge-fixing (7.32). From now on, we will cease to distinguish between $(z, \bar{z})$ and $(w, \bar{w})$ coordinates, since we will always work in the transformed coordinates. Note that because of equation (7.15), we can also directly consider the Beltrami differential as depending on the same transformed coordinates as the Lax pairs. By construction, the residues at the punctures satisfy the Kirillov-Konstant Poisson bracket

$$\{S_k^a, S_m^b\} = \delta^{km} C^{ab}{}_c S_k^c, \tag{7.50}$$

where $C^{ab}{}_c$ are the structure constants of our Lie algebra $\mathfrak{g}$. We have to determine the Poisson brackets for the additional $\dim G(g-1)$ moduli characterizing the flat bundle. One way to do this would be to study the Dirac bracket induced by the flatness condition on the Poisson bracket for flat connections on a Riemann surface,

$$\{A(z), \bar{A}(z')\} = \delta^2(z - z'). \tag{7.51}$$

Doing so, however, one encounters technical difficulties related to the presence of zero-modes for the Dolbeault differentials in the space of one-forms [149]. We will circumvent this analysis by studying instead the quasi-periodicities of the linear system (7.36) in the Schottky parametrization, that we define in Appendix B. The drawback of this approach is that our canonical variables are defined only on the Schottky subset of $\mathcal{M}_{flat}$. Consider the flows induced by the motions of the punctures' positions, so that

$$M_{z_k}(z) = -\partial_{z_k} Y(z) Y(z)^{-1}. \tag{7.52}$$

Equation (7.52) tells us that M_{z_k} is a 1-form in z_k, and not quite an automorphic 0-form in z, since

$$M_{z_k}(\gamma_{B_j} \cdot z) = T_{B_j} M_{z_k} T_{B_j}^{-1} - \partial_{z_k} T_{B_j} T_{B_j}^{-1}. \tag{7.53}$$

These properties are partly expressed by the equation

$$\bar{\partial} M_{z_k}(z) = \mu_{z_k}(z, \bar{z}) L(z), \tag{7.54}$$

with $\mu_{z_k} = \chi_k(z, \bar{z})$ the characteristic function of an infinitesimal neighborhood of the puncture z_k. This last equationd can't see, however, the affine part of the transformation, which is independent of z. These considerations tell us that

$$M_{z_k}(z) = \Theta(z_k, z)_{ab} t^b S_k^a - \sum_{j=1}^{g} \omega^j(z_k)^a{}_c R_j(z)^b{}_a t_b B_j^c. \tag{7.55}$$

where $R_j(z)^a{}_b$ are twisted holomorphic 0-forms in z, a[2]. By using the properties of the twisted 1-form Θ (B.6), we find that along a B-cycle the M-matrix transforms as

$$\begin{aligned} M_{z_k}(\gamma_{B_j}(z)) &= (AdT_j^{-1})^{cb} t_b \left[\Theta(z_k, z)^a{}_c (S_k)_a - \sum_{j=1}^{g} \omega^j(z_k)^a{}_d B_j^d R_j(z)_{ca} \right] \\ &\quad - \Theta(z_k, \gamma_{B_j}^{-1}(w_0))^a{}_c (AdT_{B_j}^{-1})^{cb} t_b (S_k)_a \\ &= T_{B_j} M_{z_k} T_{B_j}^{-1} - \omega^j(z_k)^a{}_c (AdT_{B_j}^{-1})^{cb} t_b (S_k)_a. \end{aligned} \tag{7.56}$$

Now write

$$\begin{aligned} \partial_{z_k} T_{B_j} T_{B_j}^{-1} = T_{B_j} T_{B_j}^{-1} \partial_{z_k} T_{B_j} T_{B_j}^{-1} &= (AdT_{B_j}^{-1})^a{}_b (T_{B_j}^{-1} \partial_{z_k} T_{B_j})_a t^b \\ &\equiv (AdT_{B_j}^{-1})_{ab} \partial_{z_k} Q_j^a t^b. \end{aligned} \tag{7.57}$$

This equations defines the variables Q_j^a. By comparing equations (7.53) and (7.56), we see that

$$\partial_{z_k} Q_a^j = \omega^j(z_k)^b{}_a (S_k)_b. \tag{7.58}$$

On the other hand, we also have

$$H_{z_k} = \frac{1}{2} res_{z_k} \mathrm{tr}\, L^2(z) = \sum_{m \neq k} \Theta(z_m, z_k)_{ab} S_k^a S_m^b + \sum_{j=1}^{g} \omega^j(z_k)_{ab} \Pi_j^b S_k^a. \tag{7.59}$$

Imposing that H_{z_k} is the generator of time evolution in z_k, i.e.

$$\partial_{z_k} Q_c^j = \{H_{z_k}, Q_c^j\}, \tag{7.60}$$

we find the canonical Poisson bracket

$$\{Q_c^j, \Pi_k^b\} = \delta_k^j \delta_c^b. \tag{7.61}$$

7.2 Linear systems and free fermions on Riemann surfaces

We now turn to the free fermion solution of the isomonodromic problem. By this we mean that we want to study the Fuchsian linear system of ODEs on a Riemann surface $C_{g,n}$ of genus $g \geq 2$ arising from the Hamiltonian reduction

[2] We will not specify their form, as we will not need it.

procedure of the previous section:

$$d_z Y(z) = L(z)Y(z), \tag{7.62}$$

L is the Lax matrix resulting from meromorphic gauge fixing of a Schottky bundle, so that it is a matrix-valued twisted meromorphic differential with simple poles at $z_1, \ldots z_n$ and quasi-periodicity properties

$$L(\gamma_{A_j} z) = L(z), \qquad L(\gamma_{B_j} z) = T_{B_j} L(z) T_{B_j}^{-1} \tag{7.63}$$

along a given homology basis $\{A_i, B_i\}$ of C_g. The Lax matrix is given by equation (7.37) that we recall here for convenience:

$$L(z) = dz \Pi_j^b t_a \omega^j(z)^a{}_b + dz \sum_{k=1}^{n} S_k^b t_a \Theta(z, z_k)^a{}_b, \tag{7.64}$$

where the twisted differentials $\Theta(z, z_k)^a{}_b, \omega^j(z)^a{}_b$ are given in terms of twisted Poincaré series by equations (B.6), (B.12). The residues S_k^a at the punctures have prescribed conjugacy classes

$$\boldsymbol{S}_k \equiv S_k^a t_a \sim \boldsymbol{\theta}_k \equiv \mathrm{diag}(\theta_1, \ldots, \theta_N). \tag{7.65}$$

The solution Y of the linear system is instead a matrix-valued 0-form, with quasi-periodicities

$$Y(\gamma_{A_i} z) = T_{A_i} Y(z) M_{A_i}, \quad Y(\gamma_{B_i} z) = T_{B_i} Y(z) M_{B_i}, \quad Y(\gamma_{z_i} z) = Y(z) M_i, \tag{7.66}$$

where the matrices $M_i, M_{A_i} M_{B_i}$ are the monodromies of the linear system, and as such they satisfy

$$\prod_i M_{A_i} M_{B_i} M_{A_i}^{-1} M_{B_i}^{-1} = M_1 \ldots M_n. \tag{7.67}$$

The $g \dim G$ parameters Π_i^a's are not all independent, since they represent the $\dim G(g-1)$ momenta of the isomonodromic Hamiltonian system. In our expressions this follows from the fact that the twisted differentials (B.12) have a simple pole at a base-point w_0: in order for this not to be a pole of the Lax matrix, we have to require that the residue vanishes: by using (B.7) and (B.13) we find the condition

$$\sum_{j=1}^{g} \left(g(B_j)^a{}_b - \delta^a{}_b \right) \Pi_j^b = \sum_k S_k^a, \tag{7.68}$$

which are $\dim G$ linear equations constraining the Π_j^a. The above problem can be linked to free fermion conformal blocks by considering our usual kernel:

$$K(w, z) \equiv \frac{\langle V_1(z_1) \ldots V_n(z_n) \bar{\psi}(w) \otimes \psi(z) \rangle}{\langle V_1 \ldots V_n \rangle}. \tag{7.69}$$

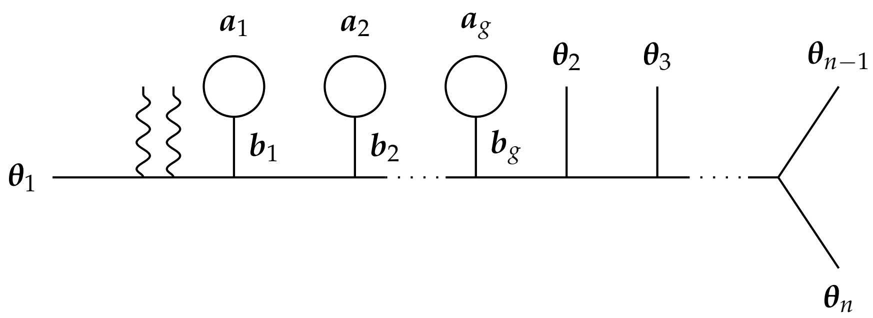


FIGURE 7.1: Higher genus conformal block

We will show in this section that such a two-point function of free fermions with vertex operator insertions is related to the solution of the linear system (7.62) in the same way as in the torus case:

$$K(w,z) = Y^{-1}(w)\Xi(w,z)Y(z). \tag{7.70}$$

However, now $\Xi(w,z)$ is a twisted 1/2-differential in both w and z, transforming in the (anti-) fundamental representation under $SL(N)$

$$\begin{aligned}(\gamma'_{A_j,B_j})^{1/2}\Xi(w,\gamma_{A_j,B_j}z) &= \Xi(w,z)T^{-1}_{A_j,B_j},\\ (\gamma'_{A_j,B_j})^{1/2}\Xi(\gamma_{A_j,B_j}w,z) &= T_{A_j,B_j}\Xi(w,z)\end{aligned} \tag{7.71}$$

in such a way as to cancel the twists of Y. Ξ can be given a physical interpretation as the Green's function of an auxiliary system of N complex free fermions twisted in the same as the Lax matrix along the A- and B-cycles. The way we are going to prove this statement is to show, as done for lower genera in the previous sections, that K has prescribed monodromies in given conjugacy classes, together with prescribed singular behavior at $z = z_1, \dots z_n, w$, and thus solves a higher genus version of a Riemann-Hilbert problem. To show this, we are going to use a pants decomposition, resulting in a conformal block diagram as in Figure 7.1, and by using the Moore-Seiberg construction [104] the braiding transformations of free fermions and vertex operators will yield the monodromy matrices. The choice of pants decomposition in the figure is particularly convenient, because it allows us to express almost all the monodromy matrices in terms of those already computed in lower genera. We will not delve into details of the monodromy computations, since there are no essential differences with what we discussed already: namely, all the monodromy matrices are obtained by braiding and fusion operation of a free fermion with the vertex operators, which are local. In general such operations shift the fermionic charges in the internal charges, so that they are not simple matrices but rather they are operator-valued: this problem is taken care of by the prescription that all the chiral correlators include a sum over all the internal fermionic charges. Let us quickly show what are the monodromy matrices in terms of braiding and fusion matrices: clearly, the monodromy around the vertex operator V_1 is simply

$$M_1 = e^{2\pi i \theta_1}. \tag{7.72}$$

To perform a monodromy around a puncture z_k, $k > 1$, we have to braid the fermion clockwise through the legs $b_1, \ldots, b_g, \theta_2, \ldots, \theta_{k-1}$, perform braiding clockwise twice around θ_k, and then braid back in anti-clockwise direction to the starting position. The resulting monodromy matrix is

$$M_k^{(g)} = B_{b_1} \ldots B_{b_g} M_k^{(0)} (B_{b_1} \ldots B_{b_g})^{-1} \equiv C_k M_k^{(0)} C_k^{-1}, \tag{7.73}$$

where we recognized that, up to conjugation, the monodromy around the puncture is that around an $n-1$-punctured sphere, that we denoted by $M_k^{(0)}$. To perform a monodromy around an A-cycle A_i, we have to first braid the fermion through the vertical lines $b_1, \ldots b_{i-1}$, and then fuse it twice, so that the fermion leg lies on the circle a_i. Finally, one has to perform all these moves backwards after a local monodromy operation $e^{2\pi i \mathbf{a}_i}$, so that

$$M_{A_i}^{(g)} = B_{b_1} \ldots B_{b_{i-1}} F_{b_i} F_{a_i} e^{2\pi i \mathbf{a}_i} (B_{b_1} \ldots B_{b_{i-1}} F_{b_i} F_{a_i})^{-1} \equiv C_{a_i} M_{A_i}^{(1)} C_{a_i}^{-1}. \tag{7.74}$$

Essentially the same moves must be performed for the B-cycle monodromy, but now instead of doing a local monodromy around the cycle A_i the fermion has to go around the B-cycle B_i and braid clockwise through the leg b_i before coming back:

$$M_{B_i}^{(g)} = B_{b_1} \ldots B_{b_{i-1}} F_{b_i} F_{a_i} B_{b_i} (B_{b_1} \ldots B_{b_{i-1}} F_{b_i} F_{a_i})^{-1} \equiv C_{a_i} M_{B_i}^{(1)} C_{a_i}^{-1}. \tag{7.75}$$

These statements do not really require more discussion, as they are direct consequence of the results already obtained in lower genera. Note also that the same free fermion conformal block is related to solutions of linear problems given by different Lax matrices by properly choosing the contraction kernel. A bit more care is required to extract from this observation an expression for the tau function. In order to do this, we will use the fact we already noted, that the kernel Ξ described above has the properties of the two-point function of an auxiliary N-component complex fermion system, so that we can write:

$$\Xi(w,z) \equiv \langle \bar{\Psi}(w) \otimes \Psi(z) \rangle. \tag{7.76}$$

Defining the currents and energy-momentum tensor of this auixiliary system in the usual way from the OPE of the fermions,

$$\begin{aligned} \tilde{J}_{\alpha\beta}(z) &\equiv : \bar{\Psi}_\alpha(z) \Psi_\beta(z) :, \\ \tilde{T}(z) &= \frac{1}{2} \sum_\alpha : \partial \bar{\Psi}_\alpha(z) \Psi_\alpha(z) : - : \bar{\Psi}_\alpha(z) \partial \Psi_\alpha(z) : \end{aligned} \tag{7.77}$$

we can expand (7.69) for z close to w:

$$\begin{aligned}&\frac{1}{2}\mathrm{tr}\,\left[\Xi(z+t/2,z-t/2)Y(z+t/2)Y^{-1}(z-t/2)\right]\\&=\frac{N}{t}+t\left[\frac{1}{2}\mathrm{tr}\,L^2(z)+\langle\tilde{T}(z)\rangle+\mathrm{tr}\,\left(L(z)\langle\tilde{J}(z)\rangle\right)\right]+O(t^2),\end{aligned}\tag{7.78}$$

which gives

$$\frac{\langle V_1\dots V_nT(z)\rangle}{\langle V_1\dots V_n\rangle}=\frac{1}{2}\mathrm{tr}\,L^2(z)+\langle\tilde{T}(z)\rangle+\mathrm{tr}\,\left(L(z)\langle\tilde{J}(z)\rangle\right).\tag{7.79}$$

We wish to use this equation to establish the relation between the isomonodromic tau function and free fermion conformal blocks. To do this, we will contract the expression above with appropriate Beltrami differentials, dual to the moduli of our punctured Riemann surface. Let m^s stand for the $3g-3$ moduli of the compact surface $C_{g,0}$, h^s for the associated quadratic differentials, and μ_s for the Beltrami differential dual to h^s. The integration of the left-hand side gives, by using the Virasoro Ward Identity [159, 165],

$$\int_{C_{g,n}}\mu_s(z)\frac{\langle V_1\dots V_nT(z)\rangle}{\langle V_1\dots V_n\rangle}=\frac{\partial}{\partial m^s}\log\left(Z\langle V_1\dots V_n\rangle\right).\tag{7.80}$$

The first term on the right-hand side gives equation (7.38)

$$\int_{C_{g,n}}\mu_s(z)\frac{1}{2}\mathrm{tr}\,L^2(z)=H_s,\tag{7.81}$$

where H_s is the Hamiltonian generating the isomonodromic flow with time m^s. We can also use the Virasoro Ward Identity for the energy-momentum tensor of the auxiliary twisted fermion system:

$$\int_{C_{g,n}}\mu_s(z)\langle\tilde{T}(z)\rangle=\frac{\partial}{\partial m^s}\log Z_{twist}.\tag{7.82}$$

To deal with the last term, we use the Ward-Identity for the current algebra generated by the twisted free fermions

$$\mathrm{tr}\,(L(z)\langle\tilde{J}(z)\rangle)=\mathrm{tr}\,(L\mathcal{L}\log Z_{twist}),\tag{7.83}$$

where we defined the Lie derivative by

$$\mathcal{L}\equiv t_a\mathcal{L}^a(z)=\sum_{j=1}^{g}t_a\omega^j(z|g)^a{}_b\mathcal{L}^b_j,\tag{7.84}$$

$$\mathcal{L}^v_jF(T_1,\dots,T_g)\equiv 2\pi i\frac{d}{dt}F(T_1,\dots,T_je^{tv},\dots,T_g)|_{t=0}.\tag{7.85}$$

By plugging the Lax matrix (7.37) into the equation, we get

$$\begin{aligned}&\int_{C_{g,n}} \mu_s(z)\mathrm{tr}\left(L\mathcal{L}\log Z_{twist}\right)\\&= \int_{C_{g,n}} \mu_s(z)\left[\frac{1}{2}\sum_{i,j}\omega^i(z)^b{}_c\omega_j(z)_{ba}\Pi_i^c + \sum_{j,k}\omega^j(z)^b{}_a\Theta(z,z_k)_{bc}S_k^c\right]\mathcal{L}_j^a\log Z_{twist}\\&= \sum_j \partial_{m^s}Q_a^j\mathcal{L}_j^a\log Z_{twist},\end{aligned} \tag{7.86}$$

where we observed that the expressions between square brackets is just the time derivative of Q_a^j, obtained from the Hamiltonian H_s using the Poisson bracket (7.61):

$$\partial_{m_s}Q_j^a = \frac{\partial H_s}{\partial \Pi_j^a} = \frac{1}{2}\int_\Sigma \mu_s(z)\left[\sum_{i=1}^g \omega^i(z)^b{}_c\omega_j(z)_{ba}\Pi_i^c + \sum_{k=1}^n \omega^j(z)^b{}_a\Theta(z,z_k)_{bc}S_k^c\right]. \tag{7.87}$$

From all this it follows that (7.79) implies that

$$\begin{aligned}\frac{\partial}{\partial m^s}\log(Z\langle V_1\dots V_n\rangle) &= H_s + \frac{\partial}{\partial m^s}\log Z_{twist} + \sum_j \frac{\partial Q_j^a}{\partial m^s}\mathcal{L}_a^j\log Z_{twist}\\&= H_s + \frac{d}{dm^s}\log Z_{twist}.\end{aligned} \tag{7.88}$$

As a consequence of this relation, contracting equation (7.79) with the Beltrami differential μ_α, we get the following relation between the unnormalized CFT conformal block $Z\langle V\rangle$ and the isomonodromic tau function:

$$\frac{d}{dm^\alpha}\log Z\langle V\rangle = \frac{d}{dm^\alpha}\log(\mathcal{T}Z_{twist}). \tag{7.89}$$

This tells us that

$$Z\langle V\rangle = f(\{z_j\})\mathcal{T}Z_{twist}, \tag{7.90}$$

for some function f, that depends on the times only through the moving singularities $z_1,\dots,z_n$. To show that $f = 1$, one needs only to compute the residue at $z = z_k$ of (7.79) and use (7.58), finding

$$\partial_{z_k}\log(Z\langle V_1\dots V_n\rangle) = \partial_{z_k}\log(\mathcal{T}Z_{twist}). \tag{7.91}$$

Because of this, the relation between the isomonodromic tau function and the non-normalized free fermion correlator:

$$\mathcal{T} = \frac{Z}{Z_{twist}}\langle V_1\dots V_n\rangle. \tag{7.92}$$

We have thus proven the general statement:

> The isomonodromic tau function for a (generic, nonresonant, Schottky) Fuchsian system with n regular singularities on a Riemann surface of genus g is given by the ratio of two objects: a non-normalized free fermion conformal block, with generic charges in the intermediate channels, and a twisted partition function of an auxiliary free fermion system, with twisting along the B-cycle determined by the quasi-periodicities of the Lax matrix defining the isomonodromic system.

The free fermion conformal block should correspond, on the gauge theory side, to the dual partition function [22, 55] of an appropriate class S theory [4]. The AGT correspondence for such cases has however been checked only in the simplest case of a compact genus two Riemann surface, and gauge group $SU(2)$ [80]. Further, differently from the cases of genus zero and one, an explicit series representation in terms of sum over partitions is not available at this time even in this simplest case: this is because there are additional residues in the localization computation, that do not have up to now such a simple description, and in fact correspond to Lagrangian theories only in the case of $SU(2)$ gauge group. A more abstract proof, in arbitrary genus but still for the algebra $\mathfrak{sl}_2$, is given in [77].

From the point of view of CFT, the difficulty in obtaining explicit expressions for the conformal blocks in higher genus is due to the fact that any pants decomposition of a higher genus Riemann surface will involve 3-point functions of descendants. A possible way to obtain explicit series expansions for such conformal blocks would be to represent them using the topological vertex [24, 81], which has recently been used [166] to study the trinion theory T_2, or equivalently the three-point conformal block where all the external legs are descendants, which is the basic building block for the case at hand: in absence of a general proof of the correspondence, one should write the 4d partition function as the limit $q \to 1$ of the (unrefined) Topological String partition function, and check equality of both sides. Another way to obtain explicit expressions for these tau functions, at least in the case of $SL(2,\mathbb{C})$, would be to employ recursion relations for the conformal blocks [78], which have been recently generalized to the case of arbitrary genus [79].

Under the assumption that, as in lower genera, the free fermion conformal block is the dual partition function [22] of the corresponding asymptotically conformal Class $\mathcal{S}$ theory the results of this section imply a generalization of the Painlevé/Gauge theory correspondence to the most general of such theories, with all nodes in the generalized quiver given by the same group $SU(N)$[3]:

[3] In fact, these theories are generically nonlagrangian, so that the name Painlevé/Gauge theory correspondence is a bit of a misnomer.

> The isomonodromic tau function for Fuchsian problem on a punctured Riemann surface $C_{g.n}$ is, up to a factor given by a twisted free fermion partition function, given by the dual partition function of a class $\mathcal{S}$ theory obtained by wrapping N M5-branes on $C_{g,n}$:
>
> $$Z_D = Z_{twist}\mathcal{T}. \tag{7.93}$$

7.3 Conclusions and Outlook

In the last three chapters we gradually extended the correspondence between isomonodromic deformation problems, two-dimensional CFT, and class S asymptotically conformal theories to the general case of a Riemann Surface with an arbitrary number of punctures. One observation it is possible to make is that, as we go to more and more general cases, the expressions tend to be less and less explicit, so that indeed one possible future direction is the detailed study of the analytic properties of the tau functions, or equivalently of 2d conformal blocks or 4d partition functions. Having established this correspondence, one could tackle this problem from any of these directions: we already mentioned in the previous section possible approaches from the CFT or gauge theory side, alternatively one could use the correspondence the other way, and try to generalize the methods of [135, 136] and obtain the gauge theory result by studying the tau function.

Let us observe that renormalisation group equations of the Seiberg-Witten theory can be obtained from blow-up equations as shown in detail in [19] for the $\mathcal{N} = 2$ SYM case. This approach should be generalised to the $\mathcal{N} = 2^*$ theory, the corresponding Painlevé equation arising from the relevant blowup equation.

Another interesting avenue is the inclusion of BPS observables for the gauge theory in the Ω-background. Their couplings can be regarded as additional time variables generating a hierarchy of extra flows [167], which should provide the full character of $\mathcal{W}_{1+\infty}$ algebra.

Finally, an interesting direction for further studies is the relation to surface operators in gauge theory: we already mentioned in chapter 3 the two ways of constructing surface operators [138]: from intersecting another set of M5 branes with the original ones wrapping the Riemann Surface that define the theory (codimension 2 defects) or from M2 branes with endpoints on the original M5s (codimension four defects). The relation between these two types of surface operators, also from the CFT viewpoint, has been discussed in [95]. In the 2d CFT, this amounts to changing the theory itself, and the partition function in the presence of such a surface operator is given by a conformal block of a $\widehat{\mathfrak{sl}}(N)_k$ algebra with an insertion of a certain twist operator $\mathcal{K}$ – see [168], with level k related to the equivariant parameters by requiring that the original Virasoro algebra of the Liouville theory is recovered upon quantum Drinfeld-Sokolov reduction, i.e.

$$k = -N - \frac{\epsilon_2}{\epsilon_1}. \tag{7.94}$$

Because of this, the partition function in the presence of a codimension four surface operator is a solution of KZB equations [158, 159], which are known to be a quantization of isomonodromy deformation equations [162, 163]. For the case of genus zero with four punctures, it has been shown [169, 170] that the classical $k \to \infty$ limit of the partition function with codimension four surface defect reproduces the formula identifying the tau function with the dual gauge theory partition function. For the case of $g > 0$, on the one hand it would be interesting to investigate how the extra factors present in our formulas arise when doing such a procedure on a circular quiver theory, or even more simply in the $\mathcal{N} = 2^*$ theory. On the other hand, it should be noted that we already have a (*twisted*) Kac-Moody algebra in our construction, but with fixed level one. The relation between the appearance of a twisted KM algebra at level one and that of the classical limit of an untwisted KM algebra with the additional insertion of a twist operator $\mathcal{K}$ certainly needs further elucidation. Moreover, it would be interesting to lift our analysis to 5d SUSY gauge theories and group Hitchin systems [171], but on elliptic curves, in which case discrete Painlevé equations should play a central role.

Part III

Discrete Equations and five-dimensional theories

Chapter 8

Five-dimensional SCFTs and q-Painlevé equations

We saw in section 2.5 that QFTs can be embedded in string theory or M-theory via geometric engineering. In this final chapter, based on the results of [3], we relate these constructions, which yield five-dimensional uplifts of the theories we considered in the context of isomonodromic deformations, to q-deformed versions of Painlevé equations. Let us first recall some generalities.

In the geometric engineering framework, five-dimensional QFTs are obtained as the low energy limit of a compactified string theory in a large volume limit, which is needed to decouple its gravitational sector. When they are obtained in this way, the nature of exact dualities gets unveiled through the geometric properties of the string theory background behind it: the string theory on the non-compact Calabi-Yau (CY) background geometry encodes the spectral geometry of integrable systems whose solutions allow to obtain exact results. This is possible because the non-perturbative sector of string theory, described by D-branes, gets transferred through this procedure to the geometrically engineered QFT. The set-up engineered by M-theory compactification on $CY_3 \times S^1$, in the limit of large CY_3 volume and finite S^1 radius, is that of a five dimensional supersymmetric QFT on a circle, whose particles arise from membranes wrapped on the 2-cycles of a suitable non compact CY manifold. As such, the counting of the BPS protected sectors of the theory can be obtained by considering a dual picture given in terms of a topological string on CY_3. The precise dictionary between the two descriptions is obtained by identifying the topological string partition function on the CY_3 with the supersymmetric index of the gauge theory, which is conjectured to capture the exact BPS content of its 5d SCFT completion [172]. More generally, the supersymmetric index of the gauge theory with surface defects is matched with the corresponding D-brane open topological string wave function [24, 141]. The coupling constants and the moduli of the QFT arise from the geometric engineering as CY moduli parameters (Kähler and complex in the A and B-model picture respectively). Therefore, the QFT generated in this way is naturally in a generic phase in which all the coupling constants can be finite. To identify the weakly coupled regimes, one has to consider particular corners in the CY moduli space. In such corners, the topological string theory amplitudes allow for a power expansion in at least one small parameter which is identified with the gauge coupling, while the others are fugacities of global symmetries of the QFT (masses and Coulomb parameters).

The problem we tackle in this chapter, in the general set-up described so far, is that of understanding how to predict the properties of such a supersymmetric index, given the non-compact CY manifold which realizes the five dimensional theory via geometric engineering. We will show that this index satisfies suitable q-difference equations which in the rank one case, namely for Calabi-Yaus whose mirror is a local genus one curve, are well-known in the mathematical physics literature as q-Painlevé equations [27]. These are classified in terms of their symmetry groups as in Fig. 8.1. Remarkably, this classification coincides with the one obtained from string theory considerations in [5]. This allows to describe the grand canonical partition function of topological strings as τ-functions of a discrete dynamical system, whose solutions encode the BPS spectrum of the theory. From this viewpoint the grand canonical partition function is actually vector-valued in the symmetry lattice of the discrete dynamical system at hand. The exact spectrum of the relevant integrable system can be computed from the zeroes of the grand partition function.

The solutions of the discrete dynamical system are naturally parametrised in different ways according to the different BPS chambers of the theory. We will show that the Nekrasov-Okounkov [22] presentation of the supersymmetric index can be recovered in the large volume regions of the Calabi-Yau moduli space which allow the geometric engineering of five-dimensional gauge theories. The expansion parameter is schematically e^{-V}, V being the volume of the relevant cycle corresponding to the instanton counting parameter. Around the conifold point the solution is instead naturally parameterised in terms of a matrix model providing the non-perturbative completion of topological string via topological string/spectral theory correspondence [173]. The case of local $\mathbb{F}_0$ geometry, which engineers pure $SU(2)$ Yang-Mills in five dimensions at zero Chern-Simons level, was discussed in detail in [26]. In this case the matrix model is a q-deformation of the $O(2)$ matrix model describing 2d Ising correlators [174, 175]. The quantum integrable system arising from the quantum Calabi-Yau geometry is two-particle relativistic Toda chain.

We will show that the discrete dynamics is determined from the analysis of the extended automorphism group of the BPS quiver associated to the Calabi-Yau geometry. In this respect let us recall the results [30, 31, 176, 177], where the BPS state spectrum of a class of four-dimensional supersymmetric theories is generated through quiver mutations. The quiver describes the BPS vacua of the supersymmetric theory and encodes the Dirac pairing among the stable BPS particles. The consistency of the Kontsevich-Soibelman formula [178] for the wall crossing among the different stability chambers is encoded in Y- and Q- systems of Zamolodchikov type. While this program have been mostly studied for four-dimensional theories, recently a proposal for BPS quivers for the five-dimensional theory on a circle has been advanced in [179]. The five-dimensional BPS quivers are conjectured to describe the

BPS spectrum of the five-dimensional theory on $\mathbb{R}^4 \times S^1$, and have two extra nodes with respect to the corresponding four-dimensional ones, representing, in properly chosen regimes, the KK tower of states and the five-dimensional instanton monopole which characterise the theory on a circle.

The proposal we make in this chapter is that these very same quivers also encode the q-difference equations satisfied by the SUSY index. These are generated by studying the application of extended quiver symmetries on the relevant cluster algebra variables τ, the latter being identified with a vector-valued topological string grand partition function. The action of the symmetry generators on the cluster algebra coefficients y keeps track of the discrete flows for the τ-functions. As such, once the gauge theory is considered on a self-dual Ω-background, we obtain that its supersymmetric index satisfies a proper set of q-Painlevé equations generated by the extended automorphisms of the quiver. More precisely, we identify different dynamics corresponding to different generators of the extended automorphism group. In a given patch, in which the topological string theory engineers a weakly coupled five dimensional theory, the generator shifting the chosen gauge coupling induces the q-Painlevé dynamics, while the other independent ones act as Bäcklund transformations of the former.

We make a first step towards realizing the above proposal by showing that the discrete flows induced by the extended automorphism group on the BPS quiver generate in a simple way the full BPS spectrum of the 5d SCFT for some examples in the rank one case. At the same time we show that the Nekrasov-Okounkov dual partition function of the 5d gauge theory obtained by relevant deformation of those theories solves the q-Painlevé equations associated to the same discrete flows. This will be accompanied also by the study of the degeneration of the five-dimensional cluster algebra into the four-dimensional one by appropriate decoupling limits. More specifically, we explore the above connection by considering in detail the case of pure $SU(2)$ gauge theory, engineered by local $\mathbb{F}_0$ and local $\mathbb{F}_1$ depending on the value of the Chern-Simons level, as well as the $SU(2)$ gauge theory with two fundamental flavors, or equivalently the one engineered by the local Calabi-Yau threefold over dP_3. This case gives a much richer lattice of bilinear equations than the case of pure gauge, with four independent discrete time evolutions.

It was noticed in [25, 180, 181] that cluster algebras provide a natural framework to describe q-deformed Painlevé equations, together with their higher rank generalizations and quantization (crucial to describe the refined topological string set up). Further, following the results of [26, 182] evidence was provided for the identification of the q-Painlevé tau function with Topological String partition function on toric Calabi-Yau threefolds, or q-deformed conformal blocks. However, while the connection with q-Painlevé equations was derived in many cases, only in the case of pure $SU(N)$ gauge theory (corresponding in the $SU(2)$ case to q-Painlevé III_3) bilinear equations were derived from the cluster algebra. In this chapter we derive from the cluster algebra bilinear equations for the $SU(2)$ theory with two flavors, as well as a bilinear form the the q-Painlevé IV equation from the cluster algebra of the local dP_3 geometry which to our knowledge did not appear in the literature, and we discuss its physical interpretation in terms of the (A_1, D_4) Argyres-Douglas theory.

Notice that, given the geometrical datum of the toric Calabi-Yau, it is possible to obtain its associated quiver from the corresponding dimer model [183, 184], and the A-cluster variables defined from this quiver lead to bilinear equations. In many cases these have been shown to be satisfied by dual partition functions of Topological String theory on this same Calabi-Yau [185], or by q-deformed Virasoro conformal blocks [182, 186–188]. These can also be rephrased in terms of K-theoretic blowup equations [23, 185, 189].

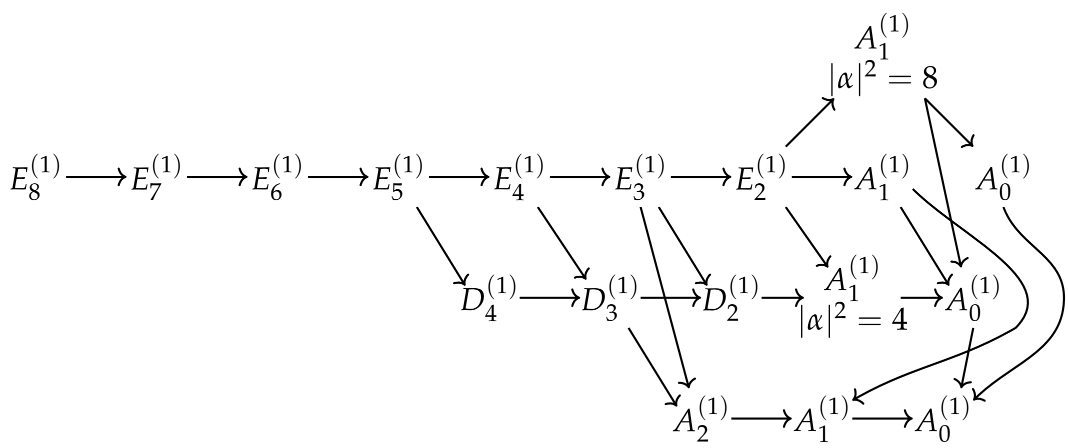


FIGURE 8.1: Sakai's Classification of discrete Painlevé equations by symmetry type

8.1 BPS spectrum of 5d SCFT on S^1 and quiver mutations

The construction that generates the BPS spectrum of a supersymmetric theory through mutations of its BPS quiver is known as mutation algorithm, and was widely employed in the case of four-dimensional $\mathcal{N} = 2$ theories [31, 176, 190]. We recall that, given a quiver with adjacency matrix B_{ij}, the mutation at its k-th node[1] is defined by

$$\mu_k(B_{ij}) = \begin{cases} -B_{ij}, & i = k \text{ or } j = k, \\ B_{ij} + \frac{B_{ik}|B_{kj}| + B_{kj}|B_{ik}|}{2}, & \end{cases} \tag{8.1}$$

[1]This is an example of a cluster algebra structure, that we will introduce more thoroughly in Section 8.2.

The mutations of the BPS charges γ_i are given by

$$\mu_k(\gamma_j) = \begin{cases} -\gamma_j, & j = k, \\ \gamma_j + [B_{kj}]_+ \gamma_k, & \text{otherwise.} \end{cases} \tag{8.2}$$

where we defined $[x]_+ = \max(x, 0)$. In this context, each node of the quiver represents a BPS charge in the upper-half plane, and a mutation μ_k encodes the rotation of a BPS ray vector out of the upper half central charge Z-plane (see [31, 190] for a detailed description) in counterclockwise sense. If the charge is rotated out of the upper-half plane clockwise instead, one has to use a slightly different mutation rule

$$\widetilde{\mu_k}(\gamma_j) = \begin{cases} -\gamma_j, & j = k, \\ \gamma_k + [-B_{kj}]_+ \gamma_k, & \text{otherwise.} \end{cases} \tag{8.3}$$

This construction is most effective when the BPS states lie in a finite chamber, i.e. when the BPS spectrum consists entirely of hypermultiplets. This is not the case for the 5d theories we are considering: due to the intrinsically stringy origin of the UV completion of these theories, in general the BPS spectrum is organised in Regge trajectories of particles with arbitrary higher spin [191, 192]; such chambers of the moduli space are known as wild chambers. In [179] an argument was put forward for the existence of a "tame chamber" of the moduli space. Such a region is characterised by the fact that the higher-spin particles are unstable and decay, and one is left with a hypermultiplet BPS spectrum, giving a situation much similar to the four-dimensional weakly coupled chambers, which are not finite.

8.1.1 Super Yang-Mills, $k = 0$

As an example, Closset and Del Zotto argued that the spectrum for local $\mathbb{F}_0$, engineering pure $SU(2)$ SYM on $\mathbb{R}^4 \times S^1$ with Chern-Simons level $k = 0$, in such a tame chamber is organised as two copies of the weakly coupled chamber of the four-dimensional pure $SU(2)$ gauge theory. The relevant quiver is depicted in Fig. 8.2, and its a adjacency matrix is

$$B = \begin{pmatrix} 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 \end{pmatrix}. \tag{8.4}$$

The spectrum of this theory was originally derived by using the mutation algorithm in [179]. This has been done by using the sequence of mutations

$$\mathbf{m} = \mu_2\mu_4\mu_3\mu_1, \tag{8.5}$$

which represents the wall-crossing arising from clockwise rotations in the upper half-plane of central charges. The n-th iteration of this operator has

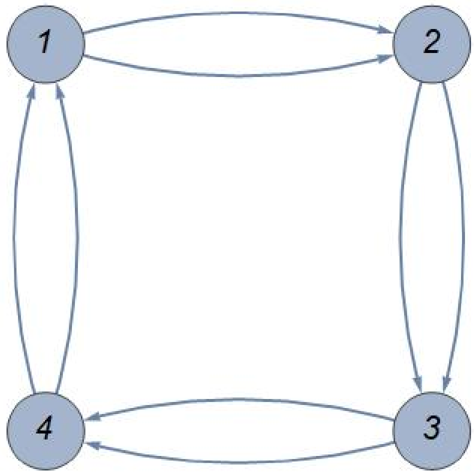


FIGURE 8.2: Quiver associated to local $\mathbb{F}_0$

the following effect on the charges γ_i, $i = 1, \ldots, 4$ of the BPS states:

$$\mathbf{m}^n(\gamma_1) = \gamma_1 + 2n\delta_u, \qquad \mathbf{m}^n(\gamma_2) = \gamma_2 - 2n\delta_u, \tag{8.6}$$
$$\mathbf{m}^n(\gamma_3) = \gamma_3 + 2n\delta_d, \qquad \mathbf{m}^n(\gamma_4) = \gamma_4 - 2n\delta_d, \tag{8.7}$$

with

$$\delta_u = \gamma_1 + \gamma_2, \qquad \delta_d = \gamma_3 + \gamma_4. \tag{8.8}$$

The action of $\mathbf{m}$ corresponds to rotating out of the upper-half plane the BPS charges in the order 1342. The towers of states obtained in this way accumulate on the vector multiplets from one side only. Because of this, the operator $\mathbf{m}$ is not sufficient: in order to construct the full spectrum in this chamber, it is necessary to use also the second operator

$$\hat{\mathbf{m}} = \hat{\mu}_1\hat{\mu}_3\hat{\mu}_2\hat{\mu}_4, \tag{8.9}$$

constructed from right mutations (8.3). The shifts obtained from this operator are

$$\hat{\mathbf{m}}^n(\gamma_1) = \gamma_1 - 2n\delta_u, \qquad \hat{\mathbf{m}}^n(\gamma_2) = \gamma_2 + 2n\delta_u, \tag{8.10}$$
$$\hat{\mathbf{m}}^n(\gamma_3) = \gamma_3 - 2n\delta_d, \qquad \hat{\mathbf{m}}^n(\gamma_4) = \gamma_4 + 2n\delta_d. \tag{8.11}$$

The resulting BPS spectrum consists of two vector multiplets δ_u, δ_d, and two towers of hypermultiplets

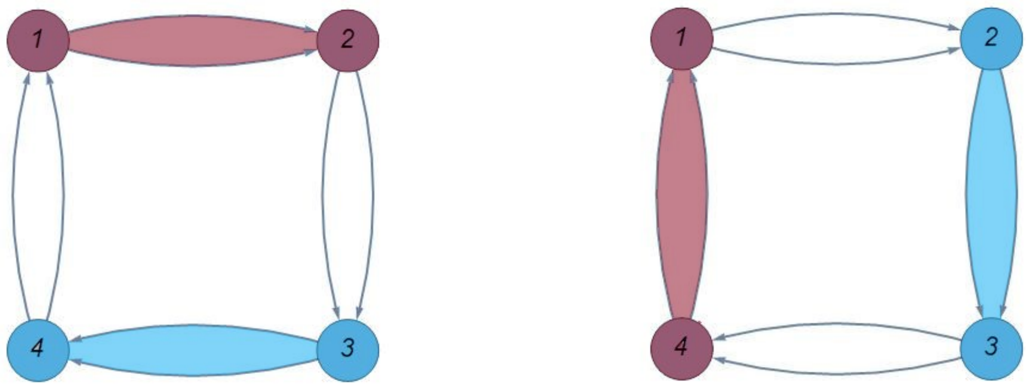
$$\gamma_1 + n\delta_u, \qquad \gamma_2 + n\delta_u, \qquad \gamma_3 + n\delta_d, \qquad \gamma_4 + n\delta_d. \tag{8.12}$$

These are two copies of the weakly coupled spectrum of four-dimensional $\mathcal{N} = 2$ $SU(2)$ pure SYM, which can be thought as being associated to the decomposition of the quiver 8.2 into two four-dimensional Krönecker subquivers, as depicted in Figure 8.3a.

Let us show an alternative derivation of the above result, making use of the group $\mathcal{G}_Q$ of quiver automorphisms. This contains the semidirect product $Dih_4 \ltimes W(A_1^{(1)})$, where Dih_4 is the dihedral group of the square, which consists only of permutations. The automorphisms group is generated by

$$\pi_1 = (1,3)\iota, \qquad \pi_2 = (4,3,2,1), \qquad T_{\mathbb{F}_0} = (1,2)(3,4)\mu_1\mu_3. \tag{8.13}$$

The operator $T_{\mathbb{F}_0}$ is a Weyl translation on the $A_1^{(1)}$ lattice. This operator



(A) Subquiver decomposition for $T_{\mathbb{F}_0}$ (B) Subquiver decomposition for $T'_{\mathbb{F}_0}$

FIGURE 8.3

directly generates the whole BPS spectrum of the theory by acting on the charges. Indeed, by applying the mutation rules (8.2) we obtain

$$T^n_{\mathbb{F}_0}(\gamma_1) = \gamma_1 + n\delta_u, \qquad T^n_{\mathbb{F}_0}(\gamma_2) = \gamma_2 - n\delta_u, \tag{8.14}$$

$$T^n_{\mathbb{F}_0}(\gamma_3) = \gamma_3 + n\delta_d \qquad T^n_{\mathbb{F}_0}(\gamma_4) = \gamma_4 - n\delta_d. \tag{8.15}$$

By computing the Dirac pairing of these states, which is given by the adjacency matrix of the quiver, in particular

$$\langle \delta_u, \delta_d \rangle = \delta_u^T \cdot B \cdot \delta_d = 0, \tag{8.16}$$

we see that $T_{\mathbb{F}_0}$ generates mutually local towers of states

$$\gamma_1 + n\delta_u, \qquad \gamma_4 + n\delta_d, \tag{8.17}$$

and

$$\gamma_2 + n\delta_u, \qquad \gamma_3 + n\delta_d. \tag{8.18}$$

As $n \to \infty$, the towers of states accumulate to the vectors δ_u, δ_d, which are vector multiplets for the four-dimensional quivers that decompose the five-dimensional quiver as in Figure 8.3a. The mutation operators $\mathbf{m}, \hat{\mathbf{m}}$ are related in a simple way to the the time evolution operator:

$$\mathbf{m} = T_{\mathbb{F}_0}^2, \qquad \hat{\mathbf{m}} = T_{\mathbb{F}_0}^{-2}\iota, \tag{8.19}$$

where ι is the inversion. From the perspective of the full automorphism group

$$\widetilde{W}(A_1^{(1)}) \ltimes Dih_4, \tag{8.20}$$

it is natural to consider also another translation operator

$$T'_{\mathbb{F}_0} = (2,3)(1,4)\mu_2\mu_4, \tag{8.21}$$

whose action on the charges is

$$\left(T'_{\mathbb{F}_0}\right)^n(\gamma_1) = \gamma_1 - n(\gamma_1+\gamma_4), \quad \left(T'_{\mathbb{F}_0}\right)^n(\gamma_2) = \gamma_2 + n(\gamma_2+\gamma_3), \tag{8.22}$$

$$\left(T'_{\mathbb{F}_0}\right)^n(\gamma_3) = \gamma_3 - n(\gamma_2+\gamma_3), \quad \left(T'_{\mathbb{F}_0}\right)^n(\gamma_4) = \gamma_4 + n(\gamma_1+\gamma_4). \tag{8.23}$$

This generates different towers of hypermultiplets, which are still organized as two copies of the weakly coupled chamber of four-dimensional super Yang-Mills, with vector multiplets

$$\delta_l = \gamma_1 + \gamma_4, \qquad \delta_r = \gamma_2 + \gamma_3. \tag{8.24}$$

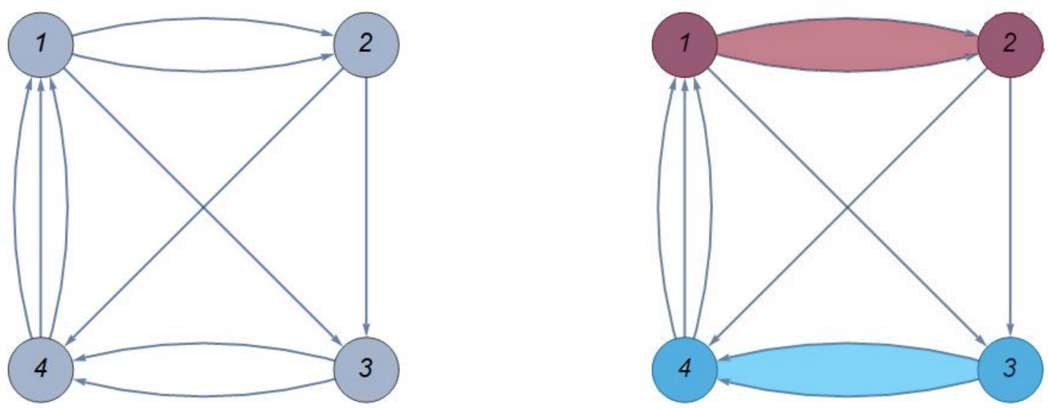
In this way, we find a different infinite chamber, corresponding to the decomposition of the 5d BPS quiver as in Figure 8.3b.

We see that considering the natural translation operators associated to the quiver automorphisms builds the correct spectrum for the tame chambers in a simpler way, without the need to consider both left and right mutations. This simplification occurs because we are allowing not just mutations, but also permutations, which are relabelings of the BPS charges. This operation of course has no effect on the resulting spectrum, which is the same as the one emerging from using just the mutation algorithm. However, by using quiver automorphisms, it is possible to construct more elementary dualities of the theory, and the spectrum can be constructed more simply. This plays a crucial rôle in more complicated cases. To illustrate this point we discuss in the following the cases of local $\mathbb{F}_1$ and dP_3.

8.1.2 Super Yang-Mills, $k = 1$

The local $\mathbb{F}_1$ quiver is displayed in Figure 8.4a. This engineers pure $SU(2)$ SYM with 5d Chern-Simons level $k = 1$. The adjacency matrix is

$$B = \begin{pmatrix} 0 & 2 & 1 & -3 \\ -2 & 0 & 1 & 1 \\ -1 & -1 & 0 & 2 \\ 3 & -1 & -2 & 0 \end{pmatrix}. \tag{8.25}$$



(A) Quiver associated to local $\mathbb{F}_1$ (B) 4d subquivers for local $\mathbb{F}_1$

FIGURE 8.4

The traslation operator is given by

$$T_{\mathbb{F}_1} = (1324)\mu_3. \tag{8.26}$$

As far as the spectrum is concerned, this is the same as the local $\mathbb{F}_0$ one. The operators $\mathbf{m}, \tilde{\mathbf{m}}$ are easy to build

$$\mathbf{m} \equiv T_{\mathbb{F}_1}^4 = ((1324)\mu_3)^4, \qquad \hat{\mathbf{m}} \equiv T_{\mathbb{F}_1}^{-4}\iota, \tag{8.27}$$

and associated evolution on the vector of charges γ is

$$T_{\mathbb{F}_1}^{2n}(\gamma) = T_{\mathbb{F}_0}^{n}(\gamma) = \left(\gamma_1 + n\delta_u \quad \gamma_2 - n\delta_u \quad \gamma_3 + n\delta_d \quad \gamma_4 - n\delta_d \right), \tag{8.28}$$

$$T_{\mathbb{F}_1}^{2n-1}(\gamma) = \left(\gamma_3 + n\delta_d, \quad \gamma_4 - n\delta_d, \quad \gamma_2 - n\delta_u, \quad \gamma_1 + n\delta_u. \right) \tag{8.29}$$

We see that even though the introduction of a Chern-Simons level will affect some physical aspects, it does not modify the type of states in the spectrum: again δ_u, δ_d correspond to the vector multiplets of the 4d subquivers depicted in Figure 8.4b. What changes however is the number of tame chambers: because the symmetry group now does not include the Dih_4 factor – as it is clear by inspection of the quiver – there is not the related chamber.

8.1.3 $N_f = 2, k = 0$

When we include matter the situation is much richer, because we encounter the new feature of multiple commuting flows, each characterizing the spectrum in a different chamber of the moduli space. The relevant quiver is the one of dP_3, engineering the $SU(2)$ theory with two flavors, depicted in Figure

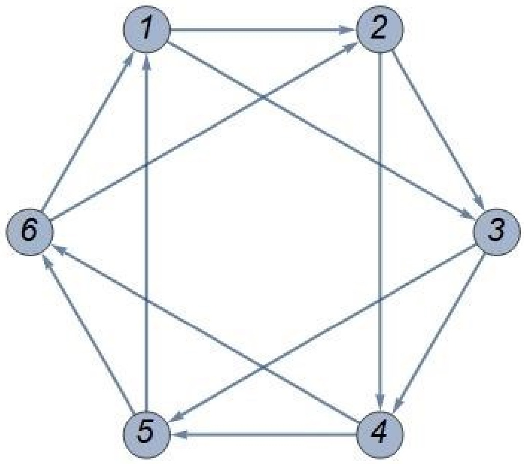


FIGURE 8.5: Quiver for dP_3

8.10. It has adjacency matrix

$$B = \begin{pmatrix} 0 & 1 & 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}. \tag{8.30}$$

The extended Weyl group is $\tilde{W}((A_2 + A_1)^{(1)})$, which is generated by

$$s_0 = (3,6)\mu_6\mu_3 \qquad s_1 = (1,4)\mu_4\mu_1, \qquad s_2 = (2,5)\mu_5\mu_2, \tag{8.31}$$

$$r_0 = (4,6)\mu_2\mu_4\mu_6\mu_2, \qquad r_1 = (3,5)\mu_1\mu_3\mu_5\mu_1, \tag{8.32}$$

$$\pi = (1,2,3,4,5,6), \qquad \sigma = (1,4)(2,3)(5,6)\iota. \tag{8.33}$$

In this case there are four commuting evolution operators, given by Weyl translations of $\tilde{W}((A_2 + A_1)^{(1)})$, acting on the affine root lattice $Q\left((A_2 + A_1)^{(1)}\right)$ [193, 194][2]. One has the three operators

$$T_1 = s_0 s_2 \pi, \qquad T_2 = s_1 s_0 \pi, \qquad T_3 = s_2 s_1 \pi \tag{8.34}$$

satisfying $T_1 T_2 T_3 = 1$, and finally

$$T_4 = r_0 \pi^3. \tag{8.35}$$

[2] For the action on the roots, see Section 8.2.3.

Let us consider the flow T_1 first, given by

$$T_1 = s_0 s_2 \pi = (3,6)\mu_6\mu_3(2,5)\mu_5\mu_2(1,2,3,4,5,6). \tag{8.36}$$

Its action on the BPS charges is the following:

$$T_1^n : \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} \longrightarrow \begin{pmatrix} (n+1)\gamma_1 + n(\gamma_2+\gamma_3), \\ \gamma_2, \\ -(n+1)(\gamma_1+\gamma_2) - n\gamma_3, \\ \gamma_4 + n(\gamma_4+\gamma_5+\gamma_6), \\ \gamma_5, \\ -(n+1)(\gamma_4+\gamma_5) - n\gamma_6 \end{pmatrix} \tag{8.37}$$

We see that T_1 generates infinite towers of hypermultiplets given by

$$\gamma_1 + n(\gamma_1+\gamma_2+\gamma_3), \qquad -(n+1)(\gamma_1+\gamma_2) - n\gamma_3, \tag{8.38}$$
$$\gamma_4 + n(\gamma_4+\gamma_5+\gamma_6), \qquad -(n+1)(\gamma_4+\gamma_5) - n\gamma_6. \tag{8.39}$$

These are the BPS states corresponding to two copies of the weakly-coupled spectrum for the $N_f = 1$ theory in four dimensions, and correspond to the decomposition of the 5d quiver into two 4d subquivers for $N_f = 1$, as in Figure 8.6. One can easily check that the towers of states are mutually local, and as $n \to \infty$ they accumulate on the rays

$$\delta_u^{(1)} = \gamma_1+\gamma_2+\gamma_3, \qquad \delta_d^{(1)} = \gamma_4+\gamma_5+\gamma_6, \tag{8.40}$$

which are indeed the vector multiplets for the 4d $N_f = 1$ subquivers. More precisely, the towers of hypermultiplets above are only half of the towers from $N_f = 1$ theory. To complete the picture here we have to consider, like in the pure gauge case, the states constructed from right mutations: these are generated as before by powers of the inverse of the evolution operator, composed with an inversion ι:

$$T_1^{-n}\iota : \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} \longrightarrow \begin{pmatrix} -n\gamma_1 - (n+1)(\gamma_2+\gamma_3) \\ \gamma_2 \\ n\gamma_3 + (n+1)(\gamma_1+\gamma_2) \\ -n\gamma_4 - (n+1)(\gamma_5+\gamma_6) \\ \gamma_5 \\ (n+1)\gamma_6 + n(\gamma_5+\gamma_4). \end{pmatrix} \tag{8.41}$$

The spectrum of the $N_f = 1$ theory in four dimensions also includes two quarks, that for the subquivers in Figure 8.6 are given by $\gamma_5, \gamma_4+\gamma_6, \gamma_2, \gamma_1 + \gamma_3$. We see that we recover the quarks γ_5, γ_2 as the states that are left invariant by T_1, while the other quarks would be their complementary in the subquiver. We will see below how the remaining quarks can be recovered as the states that are fixed by a different flow.

T_2 is given by

$$T_2 = s_1 s_0 \pi = (1,4)\mu_4\mu_1(3,6)\mu_6\mu_3(1,2,3,4,5,6), \tag{8.42}$$

and acts on the BPS charges as

$$T_2^n : \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} \longrightarrow \begin{pmatrix} -n\gamma_1 - (n+1)(\gamma_5+\gamma_6) \\ (n+1)\gamma_2 + n(\gamma_3+\gamma_4) \\ \gamma_3 \\ -n(\gamma_2+\gamma_3) - n\gamma_4 \\ (n+1)\gamma_5 + n(\gamma_1+\gamma_6) \\ \gamma_6. \end{pmatrix} \tag{8.43}$$

Reasoning as before, we find that the spectrum in this chamber is organized in two copies of the 4d $N_f = 1$ weakly coupled chamber, corresponding to the subquiver decomposition in Figure 8.7, with vector multiplets

$$\delta_2^{(u)} = \gamma_1 + \gamma_5 + \gamma_6, \qquad \delta_2^{(d)} = \gamma_2 + \gamma_3 + \gamma_4. \tag{8.44}$$

Finally, we have

$$T_3 = s_2 s_1 \pi = (2,5)\mu_5\mu_2(1,4)\mu_4\mu_1(1,2,3,4,5,6), \tag{8.45}$$

with action

$$T_3^n : \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} \longrightarrow \begin{pmatrix} \gamma_1 \\ -n\gamma_2 - (n+1)(\gamma_1+\gamma_6) \\ (n+1)\gamma_3 + n(\gamma_4+\gamma_5) \\ \gamma_4 \\ -n\gamma_5 - (n+1)(\gamma_3+\gamma_4) \\ (n+1)\gamma_6 + n(\gamma_1+\gamma_2) \end{pmatrix}, \tag{8.46}$$

which gives another chamber organized as two copies of four-dimensional weakly coupled $N_f = 1$ depicted in Figure 8.8, with vector multiplets

$$\delta_3^{(u)} = \gamma_1 + \gamma_2 + \gamma_6, \qquad \delta_3^{(d)} = \gamma_3 + \gamma_4 + \gamma_5. \tag{8.47}$$

Before considering the evolution T_4, let us make a remark. The picture above suggests that there exists a relation between the different flows in terms of permutations of the nodes of the quiver. Indeed, it is possible to check that we have the relations

$$T_1 = (3,4,5,6,1,2)T_2(5,6,1,2,3,4) = (6,1,2,3,4,5)T_3(2,3,4,5,6,1)\,. \tag{8.48}$$

From the point of view of the BPS spectrum, it is now clear that these three flows will generate the same spectrum up to relabeling of states, i.e. they will differ in what we call electric or magnetic in the field theory.

Another interesting quiver automorphism is given by

$$R_2 = \pi^2 s_1 = (3,5,1)(4,2,6)(1,4)\mu_4\mu_1, \tag{8.49}$$

which is known as half-translation, because it satisfies $R_2^2 = T_2$ (half-translations

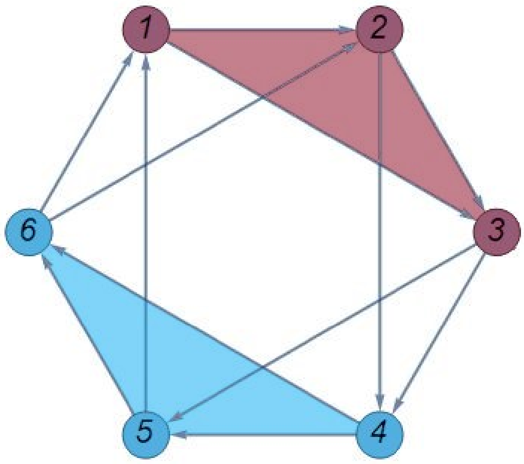


FIGURE 8.6: 4d subquivers for local dP_3, under T_1

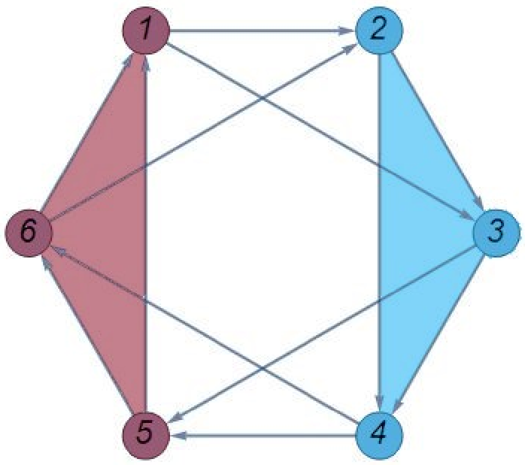


FIGURE 8.7: 4d subquivers for local dP_3, under T_2

for T_1, T_3 can be obtained by using equation (8.48)). Under this quiver automorphism, the BPS charges transform as

$$R_2^{2n+1} : \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} \longrightarrow \begin{pmatrix} -(n+1)\gamma_2 - n(\gamma_3+\gamma_4) \\ n\gamma_1 + (n+1)(\gamma_5+\gamma_6) \\ \gamma_1+\gamma_5 \\ -(n+1)\gamma_5 - n(\gamma_1+\gamma_6) \\ n\gamma_4 + (n+1)(\gamma_2+\gamma_3) \\ \gamma_2+\gamma_4 \end{pmatrix}, \tag{8.50}$$

while of course $R_2^{2n} = T_2^n$. Note that this generates the CPT conjugates of the towers of states as T_2, while the states that are left fixed by the action of R_2 are exactly the missing quarks from our analysis of T_2, so that R_2 generates the full spectrum of the two copies of $N_f = 1$ in the subquivers of Figure 8.7.

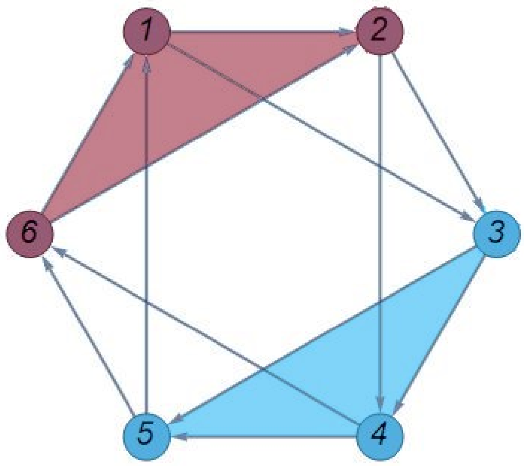


FIGURE 8.8: 4d subquivers for local dP_3, under T_3

Finally, the time evolution T_4 is given by

$$T_4 = r_0\pi^3 = (4,6)\mu_2\mu_4\mu_6\mu_2(4,5,6,1,2,3), \tag{8.51}$$

and acts on the BPS charges as follows:

$$T_4^{3n-2}: \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} \longrightarrow \begin{pmatrix} \gamma_1 + (\gamma_3+\gamma_4) + n\delta \\ \gamma_2 - (\gamma_1+\gamma_2) - n\delta \\ \gamma_3 + (\gamma_5+\gamma_6) + n\delta \\ \gamma_4 - (\gamma_3+\gamma_4) - n\delta \\ \gamma_5 + (\gamma_1+\gamma_2) + n\delta \\ \gamma_6 - (\gamma_5+\gamma_6) - n\delta \end{pmatrix}, \tag{8.52}$$

$$T_4^{3n-1}: \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} \longrightarrow \begin{pmatrix} \gamma_1 + (\gamma_3+\gamma_4+\gamma_5+\gamma_6) + n\delta \\ \gamma_2 - (\gamma_1+\gamma_2+\gamma_3+\gamma_4) - n\delta \\ \gamma_3 + (\gamma_1+\gamma_2+\gamma_5+\gamma_6) + n\delta \\ \gamma_4 - (\gamma_3+\gamma_4+\gamma_5+\gamma_6) - n\delta \\ \gamma_5 + (\gamma_1+\gamma_2+\gamma_3+\gamma_4) + n\delta \\ \gamma_6 - (\gamma_1+\gamma_2+\gamma_5+\gamma_6) - n\delta \end{pmatrix}, \tag{8.53}$$

$$T_4^{3n}: \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} \longrightarrow \begin{pmatrix} \gamma_1 + n\delta \\ \gamma_2 - n\delta \\ \gamma_3 + n\delta \\ \gamma_4 - n\delta \\ \gamma_5 + n\delta \\ \gamma_6 - n\delta \end{pmatrix}. \tag{8.54}$$

We can recognise in this chamber towers of states that accumulate to the same BPS ray $\delta = \gamma_1 + \cdots + \gamma_6$, representing a multiplet with higher spin $s \geq 1$. In this case the four-dimensional interpretation is subtler and more interesting, and we postpone it to Section 8.4.

8.2 Discrete quiver dynamics, cluster algebras and q-Painlevé

The translation operators acting on the BPS quivers we described so far can be regarded as time evolution operators for discrete dynamical systems arising from deautonomization of cluster integrable systems, naturally associated to the geometric engineering of the corresponding five-dimensional gauge theories. This allows to bridge between the BPS quiver description and classical results in the theory of q-Painlevé equations, and actually inspired the reformulation of the BPS quiver analysis that we presented in the previous section. Indeed, the quivers studied in [25] to explore q-Painlevé bilinear equations are exactly the 5d BPS quivers studied in section 8.2. In this respect, the q-Painlevé flows describe wall-crossing of BPS states for the 4d Kaluza-Klein theory obtained by reducing the 5d gauge theory on S^1. We now turn to the study of all the examples we considered up to now from this perspective.

8.2.1 Cluster algebras and quiver mutations

Let us first recall the notion of cluster algebra [195, 196], as well as the two types of cluster variables that will be used throughout the discussion. The ambient field for a cluster algebra $\mathcal{A}$ is a field $\mathcal{F}$ isomorphic to the field of rational functions in $n = \operatorname{rk}\mathcal{A}$ independent variables, with coefficients in $\mathbb{QP}$, where $\mathbb{P}$ is the tropical semifield. The tropical semifield is defined as follows: starting with the free abelian group $(\mathbb{P}, \cdot)$ with usual multiplication, the operation $\oplus$ is defined in terms of a basis[3] $\mathbf{u}$ of $\mathbb{P}$

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)}. \tag{8.55}$$

The cluster algebra $\mathcal{A}$ is determined by the choice of an initial seed. This is a triple $(Q, \boldsymbol{\tau}, \mathbf{y})$, where

- Q is a quiver without loops and 2-cycles, with n vertices;
- $\mathbf{y} = (y_1, \ldots, y_n)$ is an n-tuple of generators of the tropical semifield $(\mathbb{P}, \oplus, \cdot)$ (which in general will not be independent generators, because $\dim \mathbb{P} \leq n$);
- $\boldsymbol{\tau} \equiv (\tau_1, \ldots, \tau_n)$ is an n-tuple of elements of $\mathcal{F}$ forming a free generating set: they are algebraically independent over $\mathbb{QP}$, and $\mathcal{F} = \mathbb{QP}(\tau_1, \ldots, \tau_n)$.

The variables $(\boldsymbol{\tau}, \mathbf{y})$ are called A-cluster variables. We can alternatively define the seed as $(B, \boldsymbol{\tau}, \mathbf{y})$ in terms of the antisymmetric adjacency matrix B of the quiver.

[3]We allow for the free abelian group to have dimension less than n, as it will typically the case for us.

Given these objects, the cluster algebra is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated recursively by applying mutations to the initial seed. A mutation μ_k is an operation defined by its action on a seed:

$$\mu_k(\tau_j) = \begin{cases} \tau_j, & j \neq k, \\ \frac{y_k \prod_{i=1}^n \tau_i^{[B_{ik}]_+} + \prod_{i=1}^{|Q|} \tau_i^{-[B_{ik}]_+}}{\tau_k(1 \oplus y_k)}, & j = k, \end{cases} \tag{8.56}$$

$$\mu_k(y_j) = \begin{cases} y_j^{-1}, & j = k, \\ y_j(1 \oplus y_k^{\operatorname{sgn} B_{jk}})^{B_{jk}}, & j \neq k, \end{cases} \tag{8.57}$$

$$\mu_k(B_{ij}) = \begin{cases} -B_{ij}, & i = k \text{ or } j = k, \\ B_{ij} + \frac{B_{ik}|B_{kj}| + B_{kj}|B_{ik}|}{2}, \end{cases} \tag{8.58}$$

where we defined $[x]_+ = \max(x, 0)$. It is clear from the above expression that the coefficients y_i represent an exponentiated version of the BPS charges γ_i.

An alternative set of variables are the so-called X-cluster variables $\mathbf{x} = (x_1, \ldots, x_n)$, taking values in $\mathcal{F}$. They are defined in terms of the A-variables as

$$x_i = y_i \prod_{j=1}^n \tau_j^{B_{ji}}, \tag{8.59}$$

and their mutation rules are the same as for coefficients, but with ordinary sum instead of semifield sum:

$$\mu_k(x_j) = \begin{cases} x_j^{-1}, & j = k, \\ x_j(1 + x_k^{\operatorname{sgn} B_{jk}})^{B_{jk}}, & j \neq k. \end{cases} \tag{8.60}$$

The X-cluster variables can be considered as coordinates in the so-called X-cluster variety, which is endowed with a degenerate Poisson bracket, with respect to which the X-cluster variables are log-canonically conjugated:

$$\{x_i, x_j\} = B_{ij} x_i x_j. \tag{8.61}$$

Given a convex Newton polygon Δ with area S, it is possible to construct a quiver with $2S$ nodes describing a discrete integrable system in the variables x_i [183, 197]. Due to (8.61), in general the Poisson bracket is degenerate, as there is a space of Casimirs equal to $\ker(B)$. For quivers arising in this way, the quantity

$$q \equiv \prod_i x_i \tag{8.62}$$

is always a Casimir. The system is integrable on the level surface

$$q = 1. \tag{8.63}$$

The number of independent Hamiltonians is the number of internal points of the Newton polygon. The set of discrete time flows of the integrable system is the group $\mathcal{G}_Q$ of quiver automorphisms[4]. We will in fact work with the extended group $\tilde{\mathcal{G}}_Q$, that enlarges $\mathcal{G}_Q$ by the inclusion of the inversion operator ι. This operation reverses all the arrows in the quiver, and acts on the cluster variables as

$$\iota(x_i) = x_i^{-1}, \qquad \iota(y_i) = y_i^{-1}, \tag{8.64}$$

while the variables $\boldsymbol{\tau}$ are invariant, consistently with the relation (8.59).

In [25] it was shown that it is possible to obtain q-Painlevé equations by lifting the constraint $q = 1$, which amounts to the deautonomization of the system. This is no longer integrable in the Liouville sense, since the discrete Hamiltonians are no longer preserved under the discrete flows. The related equations of motion are well-known q-difference integrable equations of mathematical physics, namely q-Painlevé equations: the time evolution describes in this case a foliation, whose slices are different level surfaces of the original integrable system, see e.g. [198] for such a description of q-Painlevé equations. These equations can be obtained geometrically by studying configurations of blowups of eight points on $\mathbb{P}^1 \times \mathbb{P}^1$, or equivalently by configurations of nine blowups on $\mathbb{P}^2$. As in the case of differential Painlevé equations [199], this leads to a classification in terms of the space of their initial conditions, called in this context surface type of the equation, or equivalently by their symmetry groups due to Sakai [27], see Figure 8.1. The former are given by an affine algebra, while the latter turns out to be given by the extended Weyl group of another affine algebra, which is the orthogonal complement of the first one in the group of divisors $Pic(X)$, X being the surface obtained by blowing up points on $\mathbb{P}^1 \times \mathbb{P}^1$.

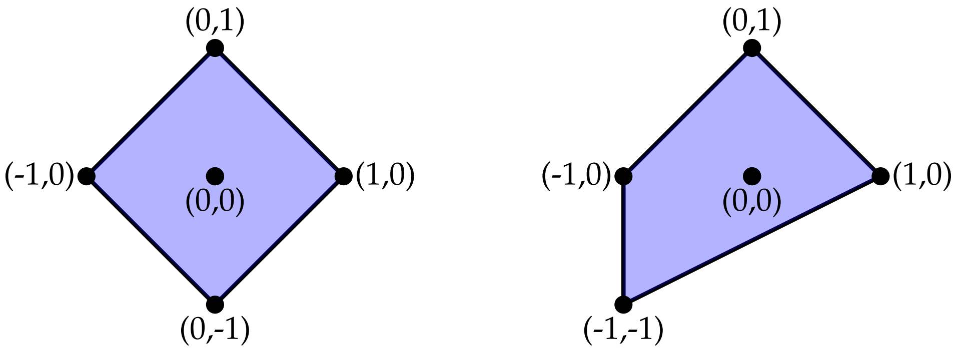
It was further argued in [25] that the time evolution given by the deautonomization of the cluster integrable system, when written in terms of the cluster A-variables $(\boldsymbol{\tau}, \mathbf{y})$ takes the form of bilinear equations, so that we can identify the variables $\boldsymbol{\tau}$ with tau functions for q-Painlevé equations. However, while the q-Painlevé equations in terms of the X-cluster variables were derived for all the Newton polygons with one internal point in [25], their bilinear form was not obtained, except for the Newton polygon of local $\mathbb{F}_0$, corresponding to the q-Painlevé equation of surface type $A_7^{(1)'}$, and local $\mathbb{F}_1$ in [180], corresponding to $A_7^{(1)}$ in Sakai's classification.

In the next section we review these two cases, before turning to the case of dP_3, which corresponds instead to the surface type $A_5^{(1)}$. In fact, this case is much richer, as it admits four commuting discrete flows: we will show that one of these reproduces the bilinear equations considered in [186, 187] for q-Painlevé III_1.

[4]To be more precise, the discrete time flows are given by a subgroup $\mathcal{G}_\Delta \subset \mathcal{G}_Q$, of automorphisms preserving the Hamiltonians. These are given e.g. by spider moves of the associated dimer model, that we are not introducing here. They are given by very specific mutation sequences, that in this chapter we rather view as Weyl translations acting on an affine root lattice. For the cases that we will be concerned with the two groups coincide and we can forget about the distinction.

8.2.2 Pure gauge theory and q-Painlevé III_3

Let us briefly review how q-Painlevé equations are obtained from the quivers associated to local $\mathbb{F}_0$ and local $\mathbb{F}_1$, whose Newton polygons are depicted in Figure (8.9a) and (8.9b). These correspond to the pure $SU(2)$ gauge theory with Chern-Simons level respectively $k = 0, 1$.



(A) Newton polygon for local $\mathbb{F}_0$ (B) Newton polygon for local $\mathbb{F}_1$

FIGURE 8.9

Local $\mathbb{F}_0$: Let us consider first the cluster algebra associated to the quiver in Figure 8.2. This corresponds to local $\mathbb{F}_0$. The group $\mathcal{G}_Q$ of quiver automorphisms contains the symmetry group of the q-Painlevé equation qPIII_3 of surface type $A_7^{(1)'}$, which is the semidirect product $Dih_4 \ltimes W(A_1^{(1)})$. It is generated by

$$\pi_1 = (1,3)\iota, \qquad \pi_2 = (4,3,2,1), \qquad T_{\mathbb{F}_0} = (1,2)(3,4)\mu_1\mu_3. \tag{8.65}$$

The operator $T_{\mathbb{F}_0}$ generates the time evolution of the corresponding q-Painlevé equation, and is a Weyl translation on the underlying $A_1^{(1)}$ lattice. From the adjacency matrix of the quiver

$$B = \begin{pmatrix} 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 \end{pmatrix}, \tag{8.66}$$

we see that the space of Casimirs of the Poisson bracket (8.61) is two-dimensional. We take the two Casimirs to be

$$q = \prod_i x_i = \prod y_i, \qquad t = x_2^{-1}x_4^{-1} = y_2^{-1}y_4^{-1}. \tag{8.67}$$

Therefore, the tropical semifield has two generators, that we take to be the two Casimirs q, t. By fixing the initial conditions for the coefficients, consistently with equation (8.67), to be

$$\mathbf{y} = ((qt)^{1/2}, t^{-1/2}, (qt)^{1/2}, t^{-1/2}), \tag{8.68}$$

one finds that the action of $T_{\mathbb{F}_0}$ on the coefficients yields

$$\bar{q} = q, \qquad \bar{t} = qt, \tag{8.69}$$

while the tau variables evolve as

$$\begin{cases} T_{\mathbb{F}_0}(\tau_1) = \tau_2, \\ T_{\mathbb{F}_0}(\tau_2) = \frac{\tau_2^2+(qt)^{1/2}\tau_4^2}{\tau_1}, \\ T_{\mathbb{F}_0}(\tau_3) = \tau_4, \\ T_{\mathbb{F}_0}(\tau_4) = \frac{\tau_4^2+(qt)^{1/2}\tau_2^2}{\tau_3} \end{cases}, \qquad \begin{cases} T_{\mathbb{F}_0}^{-1}(\tau_1) = \frac{\tau_1^2+t^{1/2}\tau_3^2}{\tau_2}, \\ T_{\mathbb{F}_0}^{-1}(\tau_2) = \tau_1, \\ T_{\mathbb{F}_0}^{-1}(\tau_3) = \frac{\tau_3^2+t^{1/2}\tau_1^2}{\tau_4}, \\ T_{\mathbb{F}_0}^{-1}(\tau_4) = \tau_3, \end{cases} \tag{8.70}$$

leading to the bilinear equations[5]

$$\overline{\tau_1}\underline{\tau_1} = \tau_1^2 + t^{1/2}\tau_3^2, \qquad \overline{\tau_3}\underline{\tau_3} = \tau_3^2 + t^{1/2}\tau_1^2. \tag{8.71}$$

The actual q-Painlevé equation is the equation involving the variables $\mathbf{x}$. It takes the form of a system of two first order q-difference equations, or of a single second-order q-difference equation, in terms of log-canonically conjugated variables

$$F \equiv x_1, \qquad G = x_2^{-1}, \tag{8.72}$$

that satisfy

$$\{F, G\} = 2FG. \tag{8.73}$$

Their time evolution can be studied in a completely analogous way by using the mutation rules (8.60) for X-cluster variables, and leads to the q-Painlevé III_3 equation

$$\overline{G}\underline{G} = \left(\frac{G+t}{G+1}\right)^2. \tag{8.74}$$

Local $\mathbb{F}_1$: We now consider the A-variables associated to the local $\mathbb{F}_1$ quiver of Figure 8.4a, engineering pure $SU(2)$ SYM with 5d Chern-Simons level $k = 1$. The adjacency matrix is

$$B = \begin{pmatrix} 0 & 2 & 1 & -3 \\ -2 & 0 & 1 & 1 \\ -1 & -1 & 0 & 2 \\ 3 & -1 & -2 & 0 \end{pmatrix}. \tag{8.75}$$

[5] We follow the usual convention that one overline denotes a step forward in discrete time, while one underline denotes a step backwards.

The corresponding equation is the q-Painlevé equation of surface type $A_7^{(1)}$, which is a different q-discretization of the differential Painlevé III_3. The time evolution is given by

$$T_{\mathbb{F}_1} = (1324)\mu_3. \tag{8.76}$$

The Casimirs are now

$$q = \prod_i y_i, \qquad t = y_1 y_2^{-1} y_3^2. \tag{8.77}$$

Consistently with this relation, we choose the following initial conditions for the coefficients:

$$\mathbf{y} = (t^{1/2}, t^{1/2}, t^{1/2}, q t^{-3/2}). \tag{8.78}$$

This yields the time evolution of the Casimirs

$$\overline{q} = q, \qquad \overline{t} = q^{1/2} t, \tag{8.79}$$

and of the τ-variables

$$\begin{cases} T_{\mathbb{F}_1}(\tau_1) = \tau_4 \\ T_{\mathbb{F}_1}(\tau_2) = \frac{\tau_4^2 + t^{1/2}\tau_1\tau_2}{\tau_3}, \\ T_{\mathbb{F}_1}(\tau_3) = \tau_1, \\ T_{\mathbb{F}_1}(\tau_4) = \tau_2. \end{cases} \qquad \begin{cases} T_{\mathbb{F}_1}^{-1}(\tau_1) = \tau_3, \\ T_{\mathbb{F}_1}^{-1}(\tau_2) = \tau_4, \\ T_{\mathbb{F}_1}^{-1}(\tau_3) = \frac{t^{1/2}\tau_1^2 + \tau_3\tau_4}{\tau_2}, \\ T_{\mathbb{F}_1}^{-1}(\tau_4) = \tau_1. \end{cases} \tag{8.80}$$

The bilinear equations obtained in this way are

$$\overline{\tau}_4 \underline{\tau_3} = t^{1/2}\tau_1^2 + \tau_3\tau_4, \qquad \overline{\tau}_2 \underline{\tau_1} = \tau_4^2 + t^{1/2}\tau_1\tau_2, \tag{8.81}$$

which are the same as the equations appearing in [180]

$$\tau(qt)\tau(q^{-1}t) = \tau^2 + t^{1/2}\tau(q^{1/2}t)\tau(q^{-1/2}t) \tag{8.82}$$

for the single tau function $\tau_4 \equiv \tau$. The identification is achieved by noting that

$$\begin{cases} \tau_1 = \underline{\tau_4}, \\ \tau_2 = \overline{\tau_1}, \\ \tau_3 = \underline{\underline{\tau_1}}, \end{cases} \tag{8.83}$$

so that the first of our bilinear equations becomes

$$\overline{\overline{\tau}} \underline{\underline{\tau}} = \tau^2 + t^{1/2} \overline{\tau} \underline{\tau}, \tag{8.84}$$

which coincides with (8.82) after using (8.79).

8.2.3 Super Yang-Mills with two flavors and qPIII$_1$

We now turn to consider the quiver associated to dP_3, engineering the $SU(2)$ theory with two flavors, depicted in Figure 8.10. It has adjacency matrix

$$B = \begin{pmatrix} 0 & 1 & 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}, \tag{8.85}$$

and Casimirs

$$a_0 = (y_3 y_6)^{-1/2}, \qquad a_1 = (y_1 y_4)^{-1/2}, \qquad a_2 = (y_2 y_5)^{-1/2}, \tag{8.86}$$

$$b_0 = (y_2 y_4 y_6)^{-1/2}, \qquad b_1 = (y_1 y_3 y_5)^{-1/2}, \tag{8.87}$$

that satisfy

$$a_0 a_1 a_2 = b_0 b_1 = q^{-1/2}, \qquad q = y_1 y_2 y_3 y_4 y_5 y_6. \tag{8.88}$$

As already discussed in section 8.1.3 one has in this case four commuting time evolution operators, given by Weyl translations of $\tilde{W}((A_2 + A_1)^{(1)})$, which act on the affine root lattice $Q\left((A_2 + A_1)^{(1)}\right)$ [193, 194]. These time evolutions are not all associated to the same q-Painlevé equation: the three operators

$$T_1 = s_0 s_2 \pi, \qquad T_2 = s_1 s_0 \pi, \qquad T_3 = s_2 s_1 \pi \tag{8.89}$$

give rise to the q-Painlevé equation qPIII$_1$ in the X-cluster variables, and satisfy $T_1 T_2 T_3 = 1$. On the other hand, the evolution

$$T_4 = r_0 \pi^3 \tag{8.90}$$

yields q-Painlevé IV.

The time evolution of the Casimirs can be obtained easily from the X-cluster variables: it is

$$T_1(a_0, a_1, a_2, b_0, b_1, q) = (q^{1/2} a_0, q^{-1/2} a_1, a_2, b_0, b_1, q), \tag{8.91}$$

$$T_2(a_0, a_1, a_2, b_0, b_1, q) = (a_0, q^{1/2} a_1, q^{-1/2} a_2, b_0, b_1, q) \tag{8.92}$$

$$T_3(a_0, a_1, a_2, b_0, b_1, q) = (q^{-1/2} a_0, a_1, q^{1/2} a_2, b_0, b_1, q), \tag{8.93}$$

$$T_4(a_0, a_1, a_2, b_0, b_1, q) = (a_0, a_1, a_2, q^{-1/2} b_0, q^{1/2} b_1, q). \tag{8.94}$$

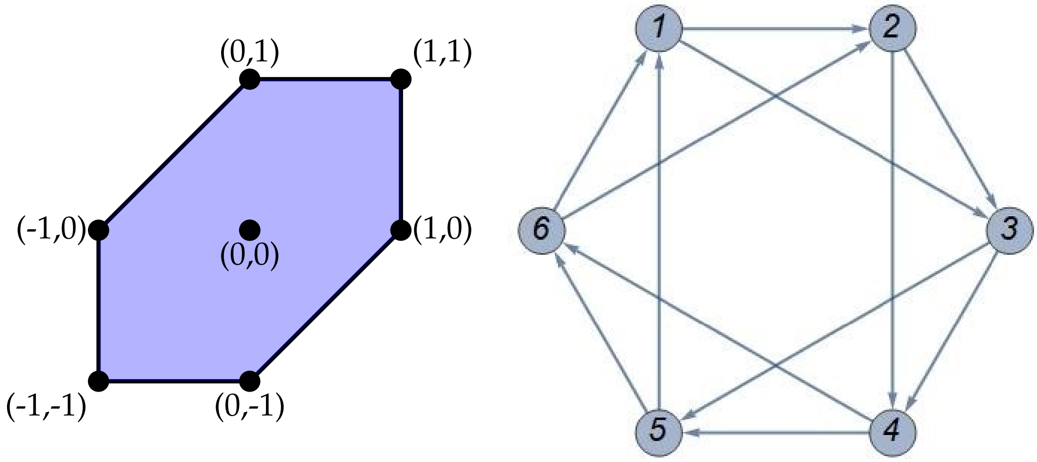


FIGURE 8.10: Newton polygon and quiver for dP_3

This is the counterpart of the fact that T_i are Weyl translations acting on the root lattice $Q((A_2 + A_1)^{(1)})$. If $\alpha_0, \alpha_1, \alpha_2$ are simple roots of $A_2^{(1)}$, and β_0, β_1 are simple roots of $A_1^{(1)}$, the action of T_i as elements of the affine Weyl group is

$$T_1(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\boldsymbol{\alpha}, \boldsymbol{\beta}) + (-1, 1, 0, 0, 0)\delta, \quad T_2(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\boldsymbol{\alpha}, \boldsymbol{\beta}) + (0, -1, 1, 0, 0)\delta, \tag{8.95}$$

$$T_3(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\boldsymbol{\alpha}, \boldsymbol{\beta}) + (1, 0, -1, 0, 0)\delta, \quad T_4(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\boldsymbol{\alpha}, \boldsymbol{\beta}) + (0, 0, 0, 1, -1)\delta, \tag{8.96}$$

where $\delta = \alpha_0 + \alpha_1 + \alpha_2 = \beta_0 + \beta_1$ is the null root of $(A_2 + A_1)^{(1)}$. From each one of these discrete flows we can obtain bilinear equations for the cluster A-variables $\boldsymbol{\tau}$. Once we choose one of the flows as time, the other flows can be regarded as Bäcklund transformations describing symmetries of the time evolution.

Let us define the four tropical semifield generators to be q, t, Q_1, Q_2, and the initial condition on the parameters to be

$$\mathbf{y} = \left(-\frac{1}{Q_2 t^{1/2}}, q^{1/4}t^{1/2}, Q_1 q^{1/4}, -\frac{1}{Q_1 t^{1/2}}, q^{1/4}t^{1/2}, Q_2 q^{1/4}\right) \tag{8.97}$$

which means, in terms of the original parametrization of the Casimirs,

$$a_0^2 = \frac{1}{Q_1 Q_2 q^{1/2}}, \qquad a_1^2 = Q_1 Q_2 t, \qquad a_2^2 = \frac{1}{q^{1/2}t}, \tag{8.98}$$

$$b_0^2 = -q^{-1/2}\frac{Q_1}{Q_2}, \qquad b_1^2 = -q^{-1/2}\frac{Q_2}{Q_1}. \tag{8.99}$$

We now derive bilinear equations for the discrete flows of this geometry: the time evolution for T_1 is

$$\begin{cases} T_1(\tau_1) = \tau_3, \\ T_1(\tau_2) = \frac{\tau_5\tau_6 - Q_2 t^{1/2}\tau_2\tau_3}{\tau_1}, \\ T_1(\tau_4) = \tau_6, \\ T_1(\tau_5) = \frac{\tau_2\tau_3 - Q_1 t^{1/2}\tau_5\tau_6}{\tau_4}, \end{cases} \qquad \begin{cases} T_1^{-1}(\tau_2) = \frac{\tau_4\tau_5 + Q_1 q^{1/4}\tau_1\tau_2}{\tau_3}, \\ T_1^{-1}(\tau_3) = \tau_1, \\ T_1^{-1}(\tau_5) = \frac{\tau_1\tau_2 + Q_2 q^{1/4}\tau_4\tau_5}{\tau_6}, \\ T_1^{-1}(\tau_6) = \tau_4. \end{cases} \tag{8.100}$$

The action on the Casimirs is given by (8.91), that means

$$T_1(Q_1) = q^{-1/2}Q_1, \qquad T_1(Q_2) = q^{-1/2}Q_2, \tag{8.101}$$

We then have

$$\begin{cases} \overline{\tau_2}\underline{\tau_3} = \tau_5\tau_6 - Q_2 t^{1/2}\tau_2\tau_3, \\ \overline{\tau_5}\underline{\tau_6} = \tau_2\tau_3 - Q_1 t^{1/2}\tau_5\tau_6, \\ \underline{\tau_2}\tau_3 = \tau_5\underline{\tau_6} - Q_1 q^{1/2}\tau_2\underline{\tau_3}, \\ \underline{\tau_5}\tau_6 = \tau_2\underline{\tau_3} - Q_2^{1/2} q^{1/2}\tau_5\underline{\tau_6}, \end{cases} \qquad \overline{\tau_i} = \tau_i(q^{-1/2}Q_1, q^{1/2}Q_2). \tag{8.102}$$

The time flow under T_2 for the A-cluster variables is

$$\begin{cases} T_2(\tau_2) = \tau_4, \\ T_2(\tau_3) = \frac{\tau_3\tau_4 + q^{1/4}t^{1/2}\tau_1\tau_6}{\tau_2}, \\ T_2(\tau_5) = \tau_1, \\ T_2(\tau_6) = \frac{\tau_1\tau_6 + q^{1/4}t^{1/2}\tau_3\tau_4}{\tau_5}, \end{cases}, \qquad \begin{cases} T_2^{-1}(\tau_1) = \tau_5, \\ T_2^{-1}(\tau_3) = \frac{-Q_1 t^{1/2}\tau_5\tau_6 + \tau_2\tau_3}{\tau_4}, \\ T_2^{-1}(\tau_4) = \tau_2, \\ T_2^{-1}(\tau_6) = \frac{-Q_2 t^{1/2}\tau_2\tau_3 + \tau_5\tau_6}{\tau_1}, \end{cases} \tag{8.103}$$

where the time evolution is given by

$$T_2(t) = qt, \tag{8.104}$$

leading to the bilinear equations

$$\overline{\tau_3}\tau_2 = q^{1/4}t^{1/2}\overline{\tau_5}\tau_6 + \tau_3\overline{\tau_2}, \qquad \overline{\tau_6}\tau_5 = \overline{\tau_5}\tau_6 + q^{1/4}t^{1/2}\tau_3\overline{\tau_2}, \tag{8.105}$$

$$\overline{\tau_2}\underline{\tau_3} = -Q_1 t^{1/2}\tau_5\tau_6 + \tau_2\tau_3, \qquad \overline{\tau_5}\underline{\tau_6} = -Q_2 t^{1/2}\tau_2\tau_3 + \tau_5\tau_6. \tag{8.106}$$

In particular, from the flow T_2 it is possible to reproduce the bilinear equations of [187], thus obtaining an explicit parametrization of the geometric quantities a_i, b_i coming from the blowup configuration of $\mathbb{P}^1 \times \mathbb{P}^1$ in terms of the Kähler parameters of dP_3.

The discrete flow T_3 is not independent, being simply given by $T_3 = T_1^{-1}T_2^{-1}$, but we write it down for completeness:

$$\begin{cases} T_3(\tau_1) = \frac{\tau_1\tau_2 + Q_2 q^{1/4}\tau_4\tau_5}{\tau_6}, \\ T_3(\tau_3) = \tau_5, \\ T_3(\tau_4) = \frac{\tau_4\tau_5 + Q_1 q^{1/4}\tau_1\tau_2}{\tau_3}, \\ T_3(\tau_6) = \tau_2, \end{cases} \qquad \begin{cases} T_3^{-1}(\tau_1) = \frac{\tau_3\tau_4 + q^{1/4}t^{1/2}\tau_1\tau_6}{,} \\ T_3^{-1}(\tau_2) = \tau_6, \\ T_3^{-1}(\tau_4) = \frac{\tau_1\tau_6 + q^{1/4}t^{1/2}\tau_3\tau_4}{\tau_5}, \\ T_3^{-1}(\tau_5) = \tau_3, \end{cases} \tag{8.107}$$

leading to the bilinear relations

$$\begin{cases} \overline{\tau_1}\underline{\tau_2} = \tau_1\tau_2 + Q_2 q^{1/4}\tau_4\tau_5, \\ \overline{\tau_4}\underline{\tau_5} = \tau_4\tau_5 + Q_1 q^{1/4}\tau_1\tau_2, \\ \underline{\tau_1}\tau_2 = q^{1/4}t^{1/2}\tau_1\underline{\tau_2} + \tau_4\underline{\tau_5}, \\ \underline{\tau_4}\tau_5 = q^{1/4}t^{1/2}\tau_4\underline{\tau_5} + \tau_1\underline{\tau_2}. \end{cases}, \qquad \overline{\tau_i} = \tau_i(q^{1/2}Q_1, q^{1/2}Q_2, q^{-1}t). \tag{8.108}$$

8.2.4 Super Yang-Mills with two flavours, q-Painlevé IV and q-Painlevé II

On top of the previous time evolutions giving rise to $qPIII_1$ equations, there is another the time evolution T_4 from a further automorphism of the dP_3 quiver. This gives rise to the qPIV dynamics and has the following action on the Casimirs, dictated by (8.94):

$$T_4(Q_1) = q^{1/2}Q_1, \qquad T_4(Q_2) = q^{-1/2}Q_2. \tag{8.109}$$

On the tau variables, this amounts to

$$\begin{cases} T_4(\tau_1) = \tau_4, \\ T_4(\tau_2) = \frac{\tau_1\tau_2\tau_6 + \tau_4\tau_5\tau_6 - Q_2 t^{1/2}\tau_2\tau_3\tau_4}{\tau_1\tau_3}, \\ T_4(\tau_3) = \tau_6, \\ T_4(\tau_4) = \frac{\tau_1\tau_2\tau_6 + \tau_4\tau_5\tau_6 + q^{1/4}t^{1/2}\tau_2\tau_3\tau_4}{\tau_3\tau_5}, \\ T_4(\tau_5) = \tau_2, \\ T_4(\tau_6) = \frac{-Q_2 t^{1/2}\tau_1\tau_2\tau_6 + q^{1/4}t^{1/2}\tau_4\tau_5\tau_6 - Q_2 q^{1/4} t\tau_2\tau_3\tau_4}{\tau_1\tau_5}, \end{cases} \tag{8.110}$$

$$\begin{cases} T_4^{-1}(\tau_1) = \frac{\tau_1\tau_2\tau_3 + Q_2 q^{1/4}\tau_3\tau_4\tau_5 + Q_2 q^{1/2}t^{1/2}\tau_1\tau_5\tau_6}{\tau_2\tau_6}, \\ T_4^{-1}(\tau_2) = \tau_5, \\ T_4^{-1}(\tau_3) = \frac{q^{1/4}t^{1/2}\tau_1\tau_2\tau_3 - Q_1 t^{1/2}\tau_3\tau_4\tau_5 - Q_1 q^{1/4} t\tau_1\tau_5\tau_6}{\tau_2\tau_4}, \\ T_4^{-1}(\tau_4) = \tau_1, \\ T_4^{-1}(\tau_5) = \frac{\tau_1\tau_2\tau_3 + Q_2 q^{1/4}\tau_3\tau_4\tau_5 - Q_1 t^{1/2}\tau_1\tau_5\tau_6}{\tau_4\tau_6}, \\ T_4^{-1}(\tau_6) = \tau_3. \end{cases} \tag{8.111}$$

At first sight this seems to lead to cubic equations. However, by following the procedure explained in Appendix C.2, one obtains an equivalent set of bilinear equations

$$\begin{cases} \overline{\tau_6}\underline{\tau_2} - q^{1/4}t^{1/2}\overline{\tau_2}\underline{\tau_6} = -t^{1/2}\left(Q_2 + q^{1/2}Q_1\right)\tau_2\tau_6, \\ Q_+^{1/2}q^{1/4}\overline{\tau_6}\underline{\tau_4} + Q_2 t^{1/2}\overline{\tau_4}\underline{\tau_6} = t^{1/2}\left(Q_2 + q^{1/2}Q_1\right)\tau_4\tau_6, \\ \overline{\tau_4}\underline{\tau_2} - \overline{\tau_2}\underline{\tau_4} = t^{1/2}\left(Q_2 + q^{1/2}Q_1\right)\tau_2\tau_4. \end{cases}, \tag{8.112}$$

$$\overline{\tau_i} = \tau_i(q^{1/2}Q_1, q^{-1/2}Q_2). \tag{8.113}$$

These provide a bilinear form for the qPIV equation, which to our knowledge did not appear in the literature so far.

q-Painlevé II bilinear relations from "half" traslations: Given the root lattice $(A_2 + A_1)^{(1)}$, there exists another time flow that preserves a $(A_1 + A_1')^{(1)}$ sublattice only [200, 201]. It corresponds to the q-Painlevé II equation, and it is given by

$$R_2 = \pi^2 s_1, \tag{8.114}$$

which is still an automorphism of the quiver in Figure 8.10. Because $R_2^2 = T_2$, this flow is also known as half-translation[6]. Its action on the Casimirs is a translational motion (i.e. a good time evolution) only on the locus $a_0 = q^{-1/4}$, i.e. $Q_+ = 1$, on which it acts as

$$R_2(a_1) = q^{1/4}a_1, \qquad R_2(a_2) = q^{-1/4}a_2, \tag{8.115}$$

corresponding to

$$R_2(t) = q^{1/2}t. \tag{8.116}$$

Such restriction constitutes a projective reduction of the system [201]. Its action on the tau-variables reads

$$\begin{cases} R_2(\tau_1) = \frac{\tau_3\tau_4 + q^{1/4}t^{1/2}\tau_1\tau_6}{\tau_2}, \\ R_2(\tau_2) = \tau_6, \\ R_2(\tau_3) = \tau_1, \\ R_2(\tau_4) = \frac{\tau_1\tau_6 + q^{1/4}t^{1/2}\tau_3\tau_4}{\tau_5}, \\ R_2(\tau_5) = \tau_3, \\ R_2(\tau_6) = \tau_4, \end{cases} \qquad \begin{cases} R_2^{-1}(\tau_1) = \tau_3, \\ R_2^{-1}(\tau_2) = \frac{\tau_5\tau_6 - Q_2 t^{1/2}\tau_2\tau_3}{\tau_1}, \\ R_2^{-1}(\tau_3) = \tau_5, \\ R_2^{-1}(\tau_4) = \tau_6, \\ R_2^{-1}(\tau_5) = \frac{\tau_2\tau_3 - Q_1 t^{1/2}\tau_5\tau_6}{\tau_4}, \\ R_2^{-1}(\tau_6) = \tau_2\,. \end{cases} \tag{8.117}$$

By setting $Q_1 = Q_2^{-1} \equiv Q$, we obtain the bilinear equations

$$\begin{cases} \overline{\overline{\tau_3}}\underline{\tau_6} = \tau_3\overline{\tau_6} + q^{1/4}t^{1/2}\overline{\tau_3}\tau_6, \\ \underline{\tau_3}\overline{\overline{\tau_6}} = \overline{\tau_3}\tau_6 + q^{1/4}t^{1/2}\tau_3\overline{\tau_6}, \\ \overline{\tau_3}\underline{\underline{\tau_6}} = \underline{\tau_3}\tau_6 - Q^{-1}t^{1/2}\tau_3\underline{\tau_6}, \\ \underline{\underline{\tau_3}}\overline{\tau_6} = \tau_3\underline{\tau_6} - Qt^{1/2}\underline{\tau_3}\tau_6, \end{cases} \qquad \overline{\tau_i} = \tau_i(q^{1/2}t). \tag{8.118}$$

[6]Other half-traslations can be analogously defined from T_1 and T_3.

We see that in fact these equations are consistent under the further requirement $Q = -1$. This is because the third and fourth equations are obtained by simply applying T_2^{-1} to the first and second one.

According to Sakai's classification (see Figure 8.1) and the analysis in [16] this flow correctly points to the Argyres-Douglas theory of $N_f = 2$ which is, in the four dimensional limit, governed by the differential PII equation.

8.2.5 Summary of dP_3 bilinear equations

To conclude this Section, let us collect here all the flows we found for dP_3 geometry together with the respective bilinear equations:

$$\begin{matrix} T_1,\ \mathrm{qPIII}_1, \\ \overline{Q_+} = q^{-1}Q_+ \end{matrix}, \qquad \begin{cases} \overline{\tau_2}\underline{\tau_3} = \tau_5\tau_6 - Q_2 t^{1/2}\tau_2\tau_3, \\ \overline{\tau_5}\underline{\tau_6} = \tau_2\tau_3 - Q_1 t^{1/2}\tau_5\tau_6, \\ \underline{\tau_2}\tau_3 = \tau_5\underline{\tau_6} - Q_+^{1/2}q^{1/2}\tau_2\underline{\tau_3}, \\ \underline{\tau_5}\tau_6 = \tau_2\underline{\tau_3} - Q_+^{1/2}q^{1/2}\tau_5\underline{\tau_6}, \end{cases} \tag{8.119}$$

$$\begin{matrix} T_2,\ \mathrm{qPIII}_1, \\ \bar{t} = qt, \end{matrix} \qquad \begin{cases} \overline{\tau_2}\underline{\tau_3} = \tau_2\tau_3 - Q_1 t^{1/2}\tau_5\tau_6, \\ \overline{\tau_5}\underline{\tau_6} = \tau_5\tau_6 - Q_2 t^{1/2}\tau_2\tau_3, \\ \tau_2\overline{\tau_3} = \overline{\tau_2}\tau_3 + q^{1/4}t^{1/2}\overline{\tau_5}\tau_6, \\ \overline{\tau_5}\tau_6 = \tau_5\overline{\tau_6} + q^{1/4}t^{1/2}\overline{\tau_2}\tau_3, \end{cases} \tag{8.120}$$

$$\begin{matrix} T_3,\ \mathrm{qPIII}_1, \\ \bar{t} = t/q,\ \overline{Q_+} = qQ_+ \end{matrix} \qquad \begin{cases} \overline{\tau_1}\underline{\tau_2} = \tau_1\tau_2 + Q_+^{1/2}q^{1/4}\tau_4\tau_5, \\ \overline{\tau_4}\underline{\tau_5} = \tau_4\tau_5 + Q_+^{1/2}q^{1/4}\tau_1\tau_2, \\ \underline{\tau_1}\tau_2 = q^{1/4}t^{1/2}\tau_1\underline{\tau_2} + \tau_4\underline{\tau_5}, \\ \underline{\tau_4}\tau_5 = q^{1/4}t^{1/2}\tau_4\underline{\tau_5} + \tau_1\underline{\tau_2}, \end{cases} \tag{8.121}$$

$$\begin{matrix} T_4, \mathrm{qPIV}, \\ \overline{Q_-} = qQ_- \end{matrix} \qquad \begin{cases} \overline{\tau_6}\underline{\tau_2} - q^{1/4}t^{1/2}\overline{\tau_2}\underline{\tau_6} + (tqQ_+)^{1/2}\left(1 + Q_-^{1/2}q^{-1/2}\right)\tau_2\tau_6 = 0, \\ Q_+^{1/2}q^{1/4}\overline{\tau_6}\underline{\tau_4} + q^{1/4}t^{1/2}\overline{\tau_4}\underline{\tau_6} - (tqQ_+)^{1/2}\left(1 + Q_-^{1/2}q^{-1/2}\right)\tau_4\tau_6 = 0, \\ \overline{\tau_4}\underline{\tau_2} - \overline{\tau_2}\underline{\tau_4} - (tqQ_+)^{1/2}\left[1 + Q_-^{1/2}q^{-1/2}\right]\tau_2\tau_4 = 0, \end{cases} \tag{8.122}$$

$$\begin{matrix} R_2,\ \mathrm{qPII}, (Q_1 = Q_2^{-1} = -1) \\ \bar{t} = q^{1/2}t, \end{matrix} \qquad \begin{cases} \overline{\overline{\tau_3}}\underline{\tau_6} = \tau_3\overline{\tau_6} + q^{1/4}t^{1/2}\overline{\tau_3}\tau_6, \\ \underline{\tau_3}\overline{\overline{\tau_6}} = \overline{\tau_3}\tau_6 + q^{1/4}t^{1/2}\tau_3\overline{\tau_6}, \end{cases} \tag{8.123}$$

8.3 Solutions

In this Section we discuss how the solutions of the discrete flow of BPS quivers are naturally encoded in topological string partition functions having as a target space the toric Calabi-Yau varietes associated to the relevant Newton polygons. The corresponding geometries are given by rank two vector bundles over punctured Riemann surfaces. Let us recall that the BPS states of the theory are associated to curves on this geometry that locally minimise the string tension. More specifically, hypermultiplets are associated to open curves ending on the branch points of the covering describing the Riemann surface, while BPS vector multiplets are associated to closed curves[7]. The BPS states are then described in this setting by open topological string amplitudes with boundaries on those curves. The very structure of the discrete flow suggests to expand the τ functions as grand canonical partition functions for the relevant brane amplitudes. Specifically, we propose that

$$\tau_{\{m_i\}}(s_i, Q_i) = \sum_{n_i} s_i^{n_i} Z_{top}(q^{m_i n_i} Q_i) \tag{8.124}$$

where $q = e^{\hbar}$, $\hbar = g_s$ being the topological string coupling, Q_i the Calabi-Yau moduli and s_i the fugacities for the branes amplitudes associated to BPS states with intersection numbers m_i with the cycles associated to the Q_i moduli. These cycles represent a basis associated to the BPS state content of theory in the relevant chamber, the intersection numbers representing the Dirac pairing among them. It is clear from this that the expansion (8.124) for the tau function crucially depends on the BPS chamber. Moreover, distinct flows of the BPS quivers described in the previous sections correspond to bilinear equations in distinct moduli of the Calabi-Yau.

These bilinear equations are in the so called Hirota form and turn out to be equivalent to convenient combinations of blowup equations [23, 202], which consist of many more equations, and suffice to determine recursively the nonperturbative part of the partition function, given the perturbative contribution [203].

In the following we will mainly focus on the expansion of τ functions in the electric weakly coupled frame which is suitable to geometrically engineer five-dimensional gauge theories. In this case the τ function coincides with the Nekrasov-Okounkov partition function.

8.3.1 Local $\mathbb{F}_0$ and qPIII$_3$

We first discuss the pure gauge theory case to gain some perspective. We'll then pass to the richer, and so far less understood, $N_f = 2$ case. Let up point out here that the solution of PIII$_3$ in the strong coupling expansion was worked out in [26] in terms of the relevant Fredholm determinant (or the matrix model in the cumulants expansion). In [26] also the relevant connection problem was solved. Since we are interested in showing the classical expansion of the cluster variables, here we discuss the different asymptotic expansion in the weak coupling $g_s \sim 0$.

[7]Let us notice that in the 5d theories on a circle, one finds in general also "wild chambers" with multiplets of higher spin which can be reached via wall crossing from the "tame" ones. It would be interesting to realise these higher spin multiplets as curves on the spectral geometry.

In our favourite example, the local $\mathbb{F}_0$, the cluster variable $x_2 = G^{-1}$, where G satisfies (see (8.74))

$$G(qt)G(q^{-1}t) = \left(\frac{G(t)+t}{G(t)+1}\right)^2 \tag{8.125}$$

This can be written in terms of Nekrasov-Okounkov partition functions as

$$G = it^{1/4}\frac{\tau_3}{\tau_1}, \tag{8.126}$$

with

$$\tau_1 = \sum_{n\in\mathbb{Z}} s^n Z(uq^n, t), \qquad \tau_3 = \sum_{n\in\mathbb{Z}} s^{n+\frac{1}{2}} Z\left(uq^{n+\frac{1}{2}}, t\right), \tag{8.127}$$

where Z is the full Nekrasov partition function for which we gave explicit formulae in Section 3.2.1. This case corresponds to the $SU(2)$ pure gauge theory with Chern-Simons level $k = 0$, and we set $u_1 = u_2^{-1} = u$, $q_1 = q_2^{-1} = q$. The bilinear equations

$$\overline{\tau_1}\underline{\tau_1} = \tau_1^2 + t^{1/2}\tau_3^2 \tag{8.128}$$

turn into an infinite set of equations for Z:

$$\begin{aligned}\sum_{n,m} s^{n+m} \big[Z(uq^n, qt)\, Z(uq^m, q^{-1}t) - Z(uq^n, t)Z(uq^m, t)& \\ -t^{1/2}Z(uq^{n+1/2}, t)Z(uq^{m-1/2}, t)\big] &= 0,\end{aligned} \tag{8.129}$$

where the coefficient for each power of s must vanish separately. Of course, most of these equations are redundant, but everything is determined by fixing the asymptotics, i.e. the classical contribution for the partition function. Selecting the term $n+m = 1$, for example, we can obtain the following equation for the t^0 coefficient of Z (i.e. the pertirbative contribution):

$$\frac{Z_{\text{1-loop}}(uq^{-1/2})Z_{\text{1-loop}}(uq^{1/2})}{Z^2_{\text{1-loop}}} = \frac{1}{u^2-1}\frac{1}{u^{-2}-1}, \tag{8.130}$$

which is the q-difference equation satisfied by $Z_{\text{1-loop}}$. The term $n+m = 0$ allows us to determine the instanton contribution from the perturbative one in the following way:

$$\begin{aligned}\sum_n Z(uq^n; qt)Z(uq^{-n}; t/q) = \sum_n & Z(uq^n; t)Z(uq^{-n}; t) \\ &- t^{1/2}\sum_n Z(uq^{n+1/2}; t)Z(uq^{-n-1/2}).\end{aligned} \tag{8.131}$$

We can express the above equation in terms of Z_{inst} as

$$\begin{aligned}
0 = &\sum_n t^{2n^2} u^{4n} \prod_{\varepsilon=\pm 1} \left(u^{2\varepsilon} q^{2n\varepsilon}; q, q^{-1}\right)_\infty Z_{inst}(uq^n; qt) Z_{inst}(uq^{-n}; t/q) \\
&- \sum_n t^{2n^2} \prod_{\varepsilon=\pm 1} \left(u^{2\varepsilon} q^{2n\varepsilon}; q, q^{-1}\right)_\infty Z_{inst}(uq^n; t) Z_{inst}(uq^{-n}; t) \\
&+ \sum_n t^{2(n+1/2)^2+1/2} \prod_{\varepsilon=\pm 1} \left(u^{2\varepsilon} q^{2(n+1/2)\varepsilon}; q, q^{-1}\right)_\infty \\
&\times Z_{inst}(uq^{n+1/2}; t) Z_{inst}(uq^{-(n+1/2)}; t).
\end{aligned} \tag{8.132}$$

which gives a recursion relation for the coefficients of the instanton expansion

$$Z_{inst} = \sum_n t^n Z_n \,. \tag{8.133}$$

For example, the one-instanton term is fully determined just by the perturbative contribution:

$$\begin{aligned}
Z_1 &= \frac{2q}{(q-1)^2} \frac{(u^2 q; q, q^{-1})_\infty (u^2/q; q, q^{-1})_\infty (1/u^2 q; q, q^{-1})_\infty (q/u^2; q, q^{-1})_\infty}{(u^2; q, q^{-1})_\infty^2 (1/u^2; q, q^{-1})_\infty^2} \\
&= \frac{2u^2 q}{(q-1)^2 (u^2-1)^2} \,.
\end{aligned} \tag{8.134}$$

which of course correctly reproduces the one-instanton Nekrasov partition function.

8.3.2 q-Painlevé III$_1$ and Nekrasov functions

In the case of the gauge theory with matter, let us first focus on the bilinear equations generated by the translation T_2. As computed in the previous section, these are

$$\overline{\tau_3}\tau_2 = q^{1/4} t^{1/2} \overline{\tau_5}\tau_6 + \tau_3 \overline{\tau_2}, \qquad \overline{\tau_6}\tau_5 = \overline{\tau_5}\tau_6 + q^{1/4} t^{1/2} \tau_3 \overline{\tau_2}, \tag{8.135}$$

$$\overline{\tau_2}\underline{\tau_3} = -Q_1 t^{1/2} \tau_5 \tau_6 + \tau_2 \tau_3, \qquad \overline{\tau_5}\underline{\tau_6} = -Q_2 t^{1/2} \tau_2 \tau_3 + \tau_5 \tau_6. \tag{8.136}$$

Let us crucially note that these coincide with the bilinear equations studied in [187], eqs (4.5-4.8) after relabeling

$$(\tau_2, \tau_3) \to (\tau_1, \tau_2), \qquad (\tau_5, \tau_6) \to (\tau_3, \tau_4), \tag{8.137}$$

and the identification

$$Q_1 = q^{-\theta_1}, \qquad Q_2 = q^{-\theta_2}. \tag{8.138}$$

In the gauge theory, Q_1, Q_2 parametrize the masses of the fundamental hypermultiplets through $\hbar\theta_i = m_i$, where $\hbar$ is the self-dual Ω-background parameter, while t is the instanton counting parameter. In terms of these, the time evolution is

$$T_2(t) = qt, \tag{8.139}$$

so that the discrete time evolution shifts the gauge coupling while the masses stay constant. We can therefore write the bilinear equations as q-difference equations

$$\begin{cases} \tau_2(qt)\tau_3(q^{-1}t) = \tau_2(t)\tau_3(t) - Q_1 t^{1/2}\tau_5(t)\tau_6(t), \\ \tau_5(qt)\tau_6(q^{-1}t) = \tau_5(t)\tau_6(t) - Q_2 t^{1/2}\tau_2(t)\tau_3(t), \\ \tau_2(t)\tau_3(qt) = \tau_2(qt)\tau_3(t) + q^{1/4}t^{1/2}\tau_5(qt)\tau_6(t), \\ \tau_5(qt)\tau_6(t) = \tau_5(t)\tau_6(qt) + q^{1/4}t^{1/2}\tau_2(qt)\tau_3(t). \end{cases} \tag{8.140}$$

It was shown in [187] that the above bilinear equations are solved in terms of the dual partition function for $SU(2)$ SYM with two fundamental flavors. More precisely, in that paper it was shown that if we define

$$Z_0^D \equiv \sum_n s^n Z(Q_1, Q_2, uq^n, t), \tag{8.141}$$

$$Z_{1/2}^D = \sum_n s^n Z(Q_1, Q_2, uq^{n+1/2}, t) = Z_0^D(uq^{1/2}), \tag{8.142}$$

where Z is the Nekrasov partition function for the $N_f = 2$ theory, the τ-functions solving (8.140) can be written as

$$\tau_2 = Z_0^D(Q_1 q^{1/2}, Q_2, tq^{-1/2}), \qquad \tau_3 = Z_0^D(Q_1 q^{-1/2}, Q_2, tq^{1/2}), \tag{8.143}$$

$$\tau_5 = Z_{1/2}^D(Q_1, Q_2 q^{1/2}, tq^{-1/2}), \qquad \tau_6 = Z_{1/2}^D(Q_1, Q_2 q^{-1/2}, tq^{1/2}). \tag{8.144}$$

By using also $\tau_4 = T_2(\tau_2)$, $\tau_1 = T_2(\tau_5)$, we can add to these

$$\tau_1 = Z_{1/2}^D(Q_1, Q_2 q^{1/2}, tq^{1/2}), \qquad \tau_4 = Z_0^D(Q_1 q^{1/2}, Q_2, tq^{1/2}). \tag{8.145}$$

Working in the same way as in Subsection 8.3.1, one can arrive at bilinear equations for Nekrasov functions, but differently from what happened in that simpler case, now one equation does not suffice to determine the non-perturbative contribution from the perturbative one: we have to use both the first and third equations of (8.140). The first equation takes the form

$$\begin{aligned} &\sum_n t^{2n^2} u^{2n} Z_{1\text{-}1} Z_{\text{inst}}(Q_1 q^{1/2}, uq^n; tq^{1/2}) Z_{1\text{-}1} Z_{\text{inst}}(Q_1 q^{-1/2}, uq^{-n}, tq^{-1/2}) \\ &= \sum_n t^{2n^2} u^{-2n} Z_{1\text{-}1} Z_{\text{inst}}(Q_1 q^{1/2}, uq^n; tq^{-1/2}) Z_{1\text{-}1} Z_{\text{inst}}(Q_1 q^{-1/2}, uq^{-n}, tq^{1/2}) \\ &- t^{1/2} Q_1 \sum_{r \in \mathbb{Z}+1/2} t^{2r^2} u^{-2r} Z_{1\text{-}1} Z_{\text{inst}}(Q_2 q^{1/2}, uq^r, tq^{-1/2}) Z_{1\text{-}1} Z_{\text{inst}}(Q_2 q^{-1/2}, uq^{-r}, tq^{1/2}), \end{aligned} \tag{8.146}$$

leading to the following equation on the one-instanton contribution:

$$\begin{aligned}
&q^{-1/2}(1-q)[Z_1(Q_1q^{-1/2})-Z_1(Q_1q^{1/2})]\\
&=\frac{u}{Q_1}\frac{Z_{1\text{-}1}(Q_2q^{-1/2},uq^{1/2})Z_{1\text{-}1}(Q_2q^{1/2},uq^{-1/2})}{Z_{1\text{-}1}(Q_1q^{1/2})Z_{1\text{-}1}(Q_1q^{-1/2})}\\
&+\frac{1}{Q_1u}\frac{Z_{1\text{-}1}(Q_2q^{-1/2},uq^{-1/2})Z_{1\text{-}1}(Q_2q^{1/2},uq^{1/2})}{Z_{1\text{-}1}(Q_1q^{1/2})Z_{1\text{-}1}(Q_1q^{-1/2})}.
\end{aligned} \tag{8.147}$$

The third equation of (8.140) becomes instead

$$\begin{aligned}
&\sum_n t^{2n^2}u^{4n}q^{n^2}Z_{1\text{-}1}Z_{\text{inst}}(Q_1q^{1/2},uq^n;tq^{-1/2})Z_{1\text{-}1}Z_{\text{inst}}(Q_1q^{-1/2},uq^{-n};tq^{3/2})\\
&=\sum_n t^{2n^2}q^{n^2}Z_{1\text{-}1}Z_{\text{inst}}(Q_1q^{1/2},uq^n;tq^{1/2})Z_{1\text{-}1}Z_{\text{inst}}(Q_1q^{-1/2},uq^{-n};tq^{1/2})\\
&+q^{1/4}t^{1/2}\sum_r t^{2r^2}q^{r^2}Z_{1\text{-}1}Z_{\text{inst}}(Q_2q^{1/2},uq^r,tq^{1/2})Z_{1\text{-}1}Z_{\text{inst}}(Q_2q^{-1/2},uq^{-r};tq^{1/2}),
\end{aligned} \tag{8.148}$$

giving an equation for the one-instanton contribution

$$\begin{aligned}
&(1-q)\left[qZ_1(Qq^{1/2})-Z_1(Q_1q^{-1/2})\right]\\
&=q\frac{Z_{1\text{-}1}(uq^{-1/2},Q_2q^{1/2})Z_{1\text{-}1}(uq^{1/2},Q_2q^{-1/2})}{Z_{1\text{-}1}(Q_1q^{-1/2})Z_{1\text{-}1}(Q_1q^{1/2})}\\
&+q\frac{Z_{1\text{-}1}(uq^{-1/2},Q_2q^{-1/2})Z_{1\text{-}1}(uq^{1/2},Q_2q^{1/2})}{Z_{1\text{-}1}(Q_1q^{-1/2})Z_{1\text{-}1}(Q_1q^{1/2})}.
\end{aligned} \tag{8.149}$$

Putting the two equations together, and using the identities (3.35) and (3.36), we obtain the correct one-instanton contribution

$$Z_1=\frac{qu^2}{(1-u^2)^2(1-q)^2}\left[\left(1-\frac{u}{Q_1}\right)\left(1-\frac{u}{Q_2}\right)+\left(1-\frac{1}{uQ_1}\right)\left(1-\frac{1}{uQ_2}\right)\right], \tag{8.150}$$

matching the one computed by instanton counting. One can go on and compute the higher instanton contributions in an analogous way. These two equations are enough to determine the nonperturbative contribution order by order in t, starting from the knowledge of the perturbative contribution, which is the t^0 term.

Let us finally note that all these bilinear equations could be written as lattice equations on $Q((A_2+A_1)^{(1)})$ by noting that all the various tau functions can be obtained starting from a single one, let us say τ_1, since we have

$$\tau_2=T_2^{-1}\left(T_4(\tau_1)\right),\qquad \tau_3=T_1(\tau_1),\qquad \tau_4=T_4(\tau_1), \tag{8.151}$$

$$\tau_5=T_2^{-1}(\tau_1),\qquad \tau_6=T_4^{-1}\left(T_1(\tau_1)\right), \tag{8.152}$$

so that it is possible to introduce, following [194], the tau lattice

$$\begin{aligned} \tau_N^{k,m} &\equiv T_1^k T_2^m T_4^N(\tau_1) \\ &= Z_0^D\left(q^{-\frac{N+k}{2}}Q_1, q^{\frac{N-k+1}{2}}Q_2, q^{\frac{k+N}{2}}u, q^m t\right) . \end{aligned} \tag{8.153}$$

In this notation the original tau-variables can be denoted by

$$\tau_1 \equiv \tau_0^{0,0}, \qquad \tau_2 \equiv \tau_1^{0,-1}, \qquad \tau_3 \equiv \tau_0^{1,0}, \tag{8.154}$$

$$\tau_4 \equiv \tau_1^{0,0}, \qquad \tau_5 \equiv \tau_0^{0,-1}, \qquad \tau_6 \equiv \tau_{-1}^{1,0}, \tag{8.155}$$

and the time flows are integer shifts of the indices of the tau function (8.153). However, not all the flows are compatible with the instanton expansion: from (8.101) and (8.109) we see that the natural expansion parameter for the solution of the T_1 and T_4 flows are respectively Q_1Q_2 and Q_2/Q_1.

In fact, the usual Nekrasov expansion, as defined in Section 3.2.1 by a converging expansion in t, can only solve the equations for T_2, which have t as time parameter: this is because if we try to solve the other equations iteratively by starting with the perturbative contribution as defined in equation (3.32), there is no region in parameter space where all the multiple q-Pocchammer functions with shifted arguments entering the bilinear equations have converging expressions simultaneously. To find a solution one should find an analogue of the perturbative partition function (3.32) which is of order zero not in t, but rather in the appropriate time parameter, solving the order zero of the bilinear equations. This indeed corresponds to an expansion of the topological string partition function (8.124) in the corresponding patch in the moduli space in the Topological Vertex formalism [24].

A preliminary analysis shows that, on top of the evolution in the mass parameters, comparing with the solution in terms of Nekrasov functions, we see that consistency requires also that

$$T_1(u) = q^{1/2}u \quad \text{and} \quad T_4(u) = q^{1/2}u. \tag{8.156}$$

To see why this must hold, one has to consider tau functions related by time evolutions of the flows T_1, T_4. For example the action on the flow T_1 on the solutions τ_1, τ_3 (the same considerations would hold by considering the other tau functions):

$$\tau_1 = Z_0^D(Q_1, Q_2q^{1/2}, uq^{1/2}, tq^{1/2}), \quad \tau_3 = Z_0^D(Q_1q^{-1/2}, Q_2, u, tq^{1/2}), \tag{8.157}$$

$$\tau_3 = T_1(\tau_1), \tag{8.158}$$

Equation (8.156) is consistent with the interpretation of the flows T_1, T_3, T_4 as the Bäcklund transformations of T_2.

8.4 Degeneration of cluster algebras and 4d gauge theory

In the previous sections we saw how the cluster algebra structure yields the q-difference equations satisfied by the partition function of the theory. On the other hand, since it conjecturally encodes the wall-crossing of states for the four-dimensional KK theory, it allows, through a generalized mutation algorithm, to produce the spectrum of the theory in a weak-coupling chamber. Further, it was observed in [179], that the BPS quivers describing the purely four-dimensional theory (with all KK modes decoupled) are contained in the five-dimensional one as subquivers with two fewer nodes: roughly, one of the additional nodes is the five-dimensional instanton monopole, while the other corresponds to the KK tower of states. From the point of view of cluster integrable systems and q-Painlevé equations this was already realized in [25]. Graphically, to go from the 5d theory to the 4d one, one "pops" two nodes of the quiver.

We now show how it is possible to explicitly implement the operation of deleting the two nodes, that brings the five-dimensional quiver to the four-dimensional one, at the level of the full cluster algebra, so that we recover the four-dimensional description of the BPS states. From the gauge theory point of view, the four-dimensional limit $\mathbb{R}^4 \times S^1_R \to \mathbb{R}^4$ is obtained by taking the radius of the five-dimensional circle $R \to 0$. More precisely, one has to scale the Kähler parameters in such a way that the KK modes and instanton particles decouple from the BPS spectrum. This limit is usually achieved by implementing the geometric engineering limit [6, 44], and takes the form

$$t = \left(\frac{R\Lambda}{2}\right)^{4-N_f}, \qquad q = e^{-Rg_s}, \qquad u = e^{-2aR}, \qquad R \to 0. \tag{8.159}$$

We see that this limit amounts to sending

$$q \to 1, \qquad t \to 0, \qquad \frac{t}{(\log q)^{4-N_f}} \text{ finite.} \tag{8.160}$$

Because this limit involves

$$\log q = \log\left(\prod_i x_i\right), \tag{8.161}$$

while the other Casimir is still given by a product of cluster variables, we can already see that it is unlikely for this limit to be able to reproduce cluster algebra transformations. Another way to see this is the case, consider the relation between X- and A-cluster variables for the case of local $\mathbb{F}_0$:

$$x_1 = \left(\frac{\tau_4}{\tau_2}\right)^2 (qt)^{1/2}, \qquad x_2 = \left(\frac{\tau_1}{\tau_3}\right)^2 t^{-1/2}, \tag{8.162}$$

$$x_3 = \left(\frac{\tau_2}{\tau_4}\right)^2 (qt)^{1/2}, \qquad x_4 = \left(\frac{\tau_3}{\tau_1}\right)^2 t^{-1/2}. \tag{8.163}$$

Because of this, if we implement the limit $t \to 0, q \to 1$ by using the geometric engineering prescription, the tau functions, which are given in terms of five-dimensional Nekrasov partition function, will simply go to their four-dimensional limit. Then no X-cluster variable has an interesting limit. In fact, this is instead the continuous limit of the corresponding Painlevé equation, in which the q-discrete equations become differential equations (see e.g. [182] for the explicit implementation of the limit on the 5d Nekrasov functions).

We will now show how to instead implement the limit $q \to 1$, $t \to 0$ for the cases we considered: local $\mathbb{F}_0$, $\mathbb{F}_1$, and dP_3 (respectively 5d pure gauge theory without and with Chern-Simons term, and the theory with $N_f = 2$ hypermultiplets), in such a way that the cluster algebra structure of the quiver is preserved: in particular we will see that:

- The mutations of the five-dimensional quiver degenerate to those of the four-dimensional one in terms of the reduced set of variables;
- The q-Painlevé time flows (or a sub-flow, in the case of $\mathbb{F}_1$), which were given by automorphisms of the five-dimensional quivers, degenerate to appropriate sequences of mutations and permutations which are automorphisms of the four-dimensional ones.

Of course, the recipe taken to implement these limit is quite general, and we have no reason to expect it not to work for the other cases. We will implement the limit on the X-cluster variables, because they carry no ambiguity related to the choice of coefficient/extended adjacency matrix. The four-dimensional cluster A-variables can then be obtained from the X-cluster variables using the adjacency matrix as usual. However, because we are implementing this limit on the X-cluster variables, we do not have an explicit expression in terms of Nekrasov functions for the limiting system.

8.4.1 From local $\mathbb{F}_0$ to the Kronecker quiver

Recall the expression for the Casimirs in terms of the cluster variables:

$$t = x_2^{-1} x_4^{-1} = x_1 x_3 / q, \qquad q = x_1 x_2 x_3 x_4. \tag{8.164}$$

Let us say that we want to decouple the nodes 3,4 on the corresponding quiver, so that we remain with the Kronecker quiver with nodes 1,2 (the red quiver in the Figure 8.3a). We need then to implement the limit

$$t \to 0, \qquad q \to 1, \qquad x_1, x_2 \text{ finite}, \tag{8.165}$$

which, as we argued above, is different from the geometric engineering limit. We then have to take

$$x_3 = qt/x_1 \to 0, \qquad x_4 = \frac{1}{x_2 t} \to \infty, \qquad x_3 x_4 \text{ finite.} \tag{8.166}$$

We are interested in the expressions for the mutations at the remaining nodes after decoupling, as well as that for the q-Painlevé translation. For the mutations this case is very simple: the limit takes the form

$$\begin{aligned}\mu_1(\mathbf{x}) &= \left(x_1^{-1}, \ \tfrac{x_2}{(1+x_1^{-1})^2}, \ x_3, \ (1+x_1)^2 x_4 \right) \\ &\to \left(x_1^{-1}, \ \tfrac{x_2}{(1+x_1^{-1})^2}, \ 0, \ \infty \right),\end{aligned} \tag{8.167}$$

$$\begin{aligned}\mu_2(\mathbf{x}) &= \left(x_1(1+x_2)^2, \ x_2^{-1}, \ \tfrac{x_3}{(1+x_2^{-1})^2}, \ x_4 \right) \\ &\to \left(x_1(1+x_2)^2, \ x_2^{-1}, \ 0, \ \infty \right).\end{aligned} \tag{8.168}$$

We see that the mutations for x_1, x_2 do not involve the variables x_3, x_4, so that no limit is actually necessary, and in fact they are already in the form of mutations for the Kronecker subquiver. Further, these mutations preserve the limiting value of x_3, x_4. In fact, the choice of the subquiver is completely arbitrary: by this limiting procedure we can consider any of the Kronecker subquivers of the quiver 8.2. The limit is less trivial on the q-Painlevé flow:

$$\begin{aligned}T_{\mathbb{F}_0}(\mathbf{x}) &= \left(x_2 \tfrac{(1+x_3)^2}{(1+x_1^{-1})^2}, \ x_1^{-1}, \ x_4 \tfrac{(1+x_1)^2}{(1+x_3^{-1})^2}, \ x_3^{-1} \right) \\ &\to \left(\tfrac{x_2}{(1+x_1)^2}, \ x_1^{-1}, 0, \ \infty \right).\end{aligned} \tag{8.169}$$

Again the limiting value of x_3, x_4 is preserved, while in terms of operations of the Kronecker quiver the q-Painlevé flow becomes

$$T_{\mathbf{F}_0} = (1,2)\mu_1, \tag{8.170}$$

which is an automorphism of the Kronecker quiver.

8.4.2 Local $\mathbb{F}_1$

We can proceed and take the analogous limit for local $\mathbb{F}_1$, for which

$$t = x_1 x_2^{-1} x_3^2, \qquad q = x_1 x_2 x_3 x_4. \tag{8.171}$$

We again focus on the Kronecker subquiver with nodes 1,2, and set

$$x_3 = \left(t x_1^{-1} x_2 \right)^{1/2} \to 0, \qquad x_4 = q t^{-1/2} x_1^{-1/2} x_2^{-3/2} \to \infty. \tag{8.172}$$

The limiting behavior of the mutations is now

$$\begin{aligned}\mu_1(\mathbf{x}) &= \left(x_1^{-1},\ \tfrac{x_2}{(1+x_1^{-1})^2},\ \tfrac{x_3}{1+x_1^{-1}},\ (1+x_1)^3 x_4 \right)\\ &\to \left(x_1^{-1},\ \tfrac{x_2}{(1+x_1^{-1})^2},\ 0,\ \infty \right),\end{aligned} \tag{8.173}$$

$$\begin{aligned}\mu_2(\mathbf{x}) &= \left(x_1(1+x_2)^2,\ x_2^{-1},\ \tfrac{x_3}{1+x_2^{-1}},\ \tfrac{x_4}{1+x_2^{-1}} \right)\\ &\to \left(x_1(1+x_2)^2,\ x_2^{-1},\ 0,\ \infty \right),\end{aligned} \tag{8.174}$$

which again yields the correct limiting behavior. The q-Painlevé flow does not have a good limiting behavior: however its square does, since

$$T_{\mathbb{F}_1}^2 = T_{\mathbb{F}_0} \to (1,2)\mu_1 \tag{8.175}$$

as we saw above.

8.4.3 Local dP$_3$

This case is much more interesting, because we get different decoupling limits, and only one of them is similar to the usual four-dimensional limit, involving $t \to 0$. These are related to the presence of different discrete flows. We consider as usual T_2 first, which we have already seen to be related to the weakly-couplied/instanton counting picture. In analogy to what was done in the previous cases, since

$$T_2(t) = qt, \tag{8.176}$$

we take the limit

$$t \to 0, \qquad\qquad q \to 1, \tag{8.177}$$

by taking a limit on two of the cluster variables. Looking at the quiver for this case, we recognize that the subquiver with vertices 2,3,4,6 (or equivalently 1,3,5,6) gives the BPS quiver of the four-dimensional $N_f = 2$ theory, as in figure 8.11. We will focus on the former case. Because we are "popping out" the nodes 1,5 from the quiver, we want to achieve this by implementing the limit (8.177) directly on the cluster variables. By studying the expressions for the Casimirs (8.86), (8.87) we find

$$x_3x_6 = Q_1Q_2q^{1/2}, \quad x_1x_4 = (Q_1Q_2t)^{-1} \to \infty, \quad x_2x_5 = q^{1/2}t \to 0, \tag{8.178}$$

so that we want to study the limit

$$x_1 = (x_4Q_1Q_2t)^{-1} \to \infty, \qquad\qquad x_5 = q^{1/2}tx_2^{-1} \to 0. \tag{8.179}$$

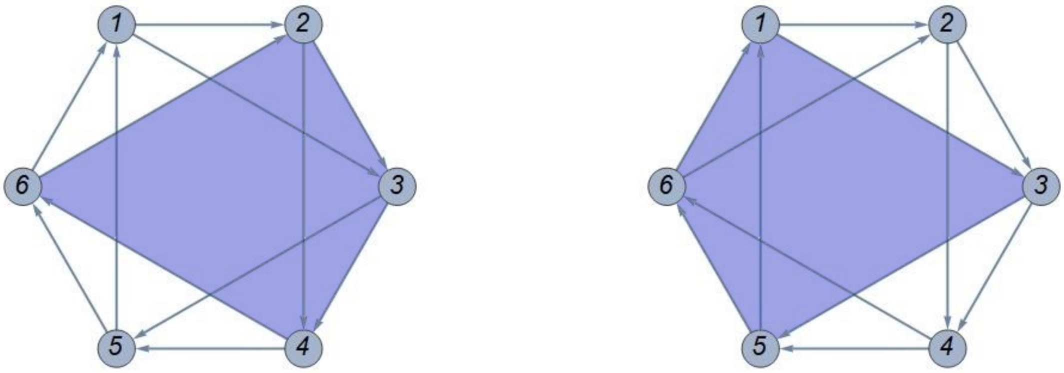


FIGURE 8.11: $N_f = 2$ kite-subquivers for the discrete flow T_2

Taking the limit on the mutations we obtain

$$\begin{aligned} \mu_2(\mathbf{x}) &= \left(x_1(1+x_2),\ x_2^{-1},\ \frac{x_3}{1+x_2^{-1}},\ \frac{x_4}{1+x_2^{-1}},\ x_5,\ (1+x_2)x_6 \right) \\ &\to \left(\infty,\ x_2^{-1},\ \frac{x_3}{1+x_2^{-1}},\ \frac{x_4}{1+x_2^{-1}},\ 0,\ (1+x_2)x_6 \right), \end{aligned} \tag{8.180}$$

and similarly for the other mutations μ_3, μ_4, μ_6, that all degenerate to the mutations of the four-dimensional quiver. The discrete flow also has a "good" limit:

$$\begin{cases} T_2(x_1) = \frac{1+x_5^{-1}}{(1+x_2)x_6}, \\ T_2(x_2) = \frac{x_4(1+x_5)\left(x_6(1+x_2)(1+x_5^{-1})^{-1}\right)}{(1+x_2^{-1})(1+x_3^{-1}(1+x_2^{-1})(1+x_5))}, \\ T_2(x_3) = \frac{1+x_6(1+x_2)(1+x_5^{-1})^{-1}}{x_2\left(1+x_3^{-1}(1+x_2^{-1})(1+x_5)^{-1}\right)}, \\ T_2(x_4) = \frac{1+x_2^{-1}}{x_3(1+x_5)}, \\ T_2(x_5) = \frac{x_1(1+x_2)\left(1+x_3(1+x_5)(1+x_2^{-1})^{-1}\right)}{(1+x_5^{-1})\left(1+x_6^{-1}(1+x_5^{-1})(1+x_2)^{-1}\right)}, \\ T_2(x_6) = \frac{1+x_3(1+x_5)(1+x_2^{-1})^{-1}}{x_5\left(1+x_6^{-1}(1+x_5^{-1})(1+x_2)^{-1}\right)} \end{cases} \to \begin{cases} \infty, \\ \frac{x_4}{(1+x_2^{-1})\left(1+x_3^{-1}(1+x_2^{-1})\right)}, \\ \frac{x_3}{1+x_2(1+x_3)}, \\ \frac{1+x_2^{-1}}{x_3}, \\ 0, \\ x_6\left(1+x_2(1+x_3)\right). \end{cases} \tag{8.181}$$

If we call the variables after taking the limit

$$x_4 \to X_1, \qquad x_2 \to X_2, \qquad x_3 \to X_3, \qquad x_6 \to X_4, \tag{8.182}$$

we have that the limit of the discrete flow is

$$T_2^{(4d)} = (3,2,1)\mu_3\mu_2, \tag{8.183}$$

which is an automorphism of the 4d quiver. We can follow the same logic for the other discrete flows T_1, T_3, T_4: we will from now on discuss only the limits on the discrete time flows, because those on the mutations are rather simple and given by essentially the same computations as above. In the first case the flow is

$$T_1(Q_+) = q^{-1}Q_+, \qquad Q_+ = Q_1Q_2 \tag{8.184}$$

so that the natural guess for the right limit to consider is $Q_+ \to \infty$, in analogy with the previous case. By looking at the Casimirs, we arrive to the conclusion that we can either decouple the nodes 1,3 or 4,6, producing the 4d $N_f = 2$ subquivers in Figure 8.12. The limit we want to implement is then

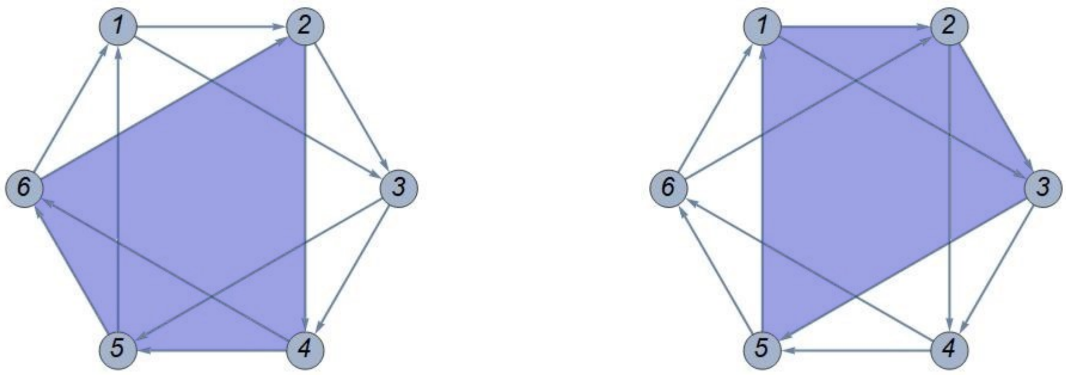


FIGURE 8.12: $N_f = 2$ kite-subquivers for the discrete flow T_1

$$x_4 = (x_1Q_+t)^{-1} \to 0, \qquad x_6 = Q_+q^{1/2}x_3^{-1} \to \infty, \tag{8.185}$$

which gives

$$\begin{cases} T_1(x_1) = \frac{x_3(1+x_4)\left(1+x_5(1+x_1)(1+x_4^{-1})^{-1}\right)}{x_1\left(1+x_2^{-1}(1+x_1^{-1})(1+x_4)^{-1}\right)}, \\ T_1(x_2) = \frac{1+x_5(1+x_1)(1+x_4^{-1})^{-1}}{x_1\left(1+x_2^{-1}(1+x_1^{-1})(1+x_4)^{-1}\right)}, \\ T_1(x_3) = \frac{1+x_1^{-1}}{x_2(1+x_4)}, \\ T_1(x_4) = \frac{x_6(1+x_1)\left(1+x_2(1+x_4)(1+x_1^{-1})^{-1}\right)}{(1+x_4^{-1})\left(1+x_5^{-1}(1+x_4^{-1})(1+x_1)^{-1}\right)}, \\ T_1(x_5) = \frac{1+x_2(1+x_4)(1+x_1^{-1})^{-1}}{x_4\left(1+x_5^{-1}(1+x_4^{-1})(1+x_1)^{-1}\right)}, \\ T_1(x_6) = \frac{1+x_4^{-1}}{x_5(1+x_1)}. \end{cases} \to \begin{cases} \frac{x_3}{(1+x_1^{-1})(1+x_2^{-1}(1+x_1^{-1}))}, \\ \frac{x_2}{1+x_1(1+x_2)}, \\ \frac{1+x_1^{-1}}{x_2}, \\ 0, \\ x_5\left(1+x_1(1+x_2)\right), \\ \infty, \end{cases} \tag{8.186}$$

Which is the same 4d quiver automorphism as for T_2, up to permutations of the nodes. The time evolution T_3 is characterized by

$$T_3(Q_+) = qQ_+, \qquad T_3(t) = q^{-1}t, \tag{8.187}$$

so that the natural limit on the Casimirs is

$$x_3x_6 = Q_+q^{1/2} \to 0, \qquad x_2x_5 = q^{1/2}t \to \infty. \tag{8.188}$$

This can be achieved by decoupling the nodes 3,5 or 2,6, keeping the sub-quivers depicted in Figure 8.13. Choosing the former one for concreteness,

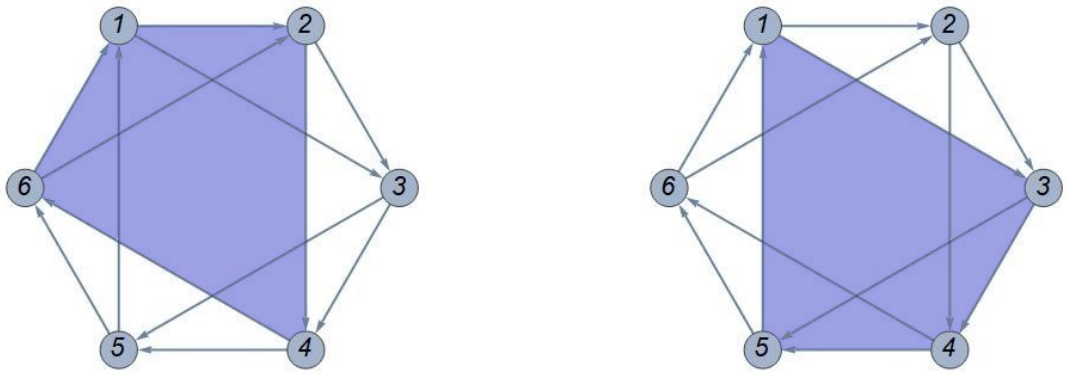


FIGURE 8.13: $N_f = 2$ kite-subquivers for the discrete flow T_3

we want to compute the limit

$$x_3 = Q_+ q^{1/2} x_6^{-1} \to 0, \qquad x_5 = q^{1/2} t x_2^{-1} \to \infty, \tag{8.189}$$

on the discrete evolution T_3. This is given by

$$\begin{cases} T_3(x_1) = \frac{1+x_4(1+x_6)(1+x_3^{-1})^{-1}}{x_6\left(1+x_1^{-1}(1+x_6^{-1})(1+x_3)\right)}, \\ T_3(x_2) = \frac{1+x_6^{-1}}{x_1(1+x_3)}, \\ T_3(x_3) = \frac{x_5(1+x_6)\left(1+x_1(1+x_3)(1+x_6^{-1})^{-1}\right)}{(1+x_3^{-1})\left(1+x_4(1+x_3^{-1})(1+x_6)^{-1}\right)}, \\ T_3(x_4) = \frac{1+x_1(1+x_3)(1+x_6^{-1})^{-1}}{x_3\left(1+x_4^{-1}(1+x_3^{-1})(1+x_6)^{-1}\right)}, \\ T_3(x_5) = \frac{1+x_3^{-1}}{x_4(1+x_6)}, \\ T_3(x_6) = \frac{x_2(1+x_3)\left(1+x_4(1+x_6)(1+x_3^{-1})^{-1}\right)}{(1+x_6^{-1})\left(1+x_1^{-1}(1+x_6^{-1})(1+x_3)^{-1}\right)}, \end{cases} \to \begin{cases} \frac{x_1}{1+x_6(1+x_1)}, \\ \frac{1+x_6^{-1}}{x_1}, \\ 0, \\ x_4(1+x_6(1+x_1)), \\ \infty, \\ \frac{x_2}{(1+x_6^{-1})(1+x_1^{-1}(1+x_6^{-1}))}. \end{cases} \tag{8.190}$$

We see that in all the cases that yielded the time evolution of q-Painlevé III_1, the degeneration of the time flow produces the same automorphism of an appropriate subquiver. It remains to study the flow T_4, which yielded a q-Painlevé IV time evolution, characterized by

$$T_4(Q_-) = qQ_-, \qquad Q_- = \frac{Q_2}{Q_1}. \tag{8.191}$$

In terms of the Casimirs b_0, b_1, this leads to

$$x_2 x_4 x_6 = (qQ_-)^{1/2} \to 0, \qquad x_1 x_3 x_5 = q^{-1} Q_-^{-1/2} \to \infty. \tag{8.192}$$

To achieve this without affecting the Casimirs a_0, a_1, a_2 we have to decouple either the nodes 2,5, or the nodes 3,6, or the nodes 1,4, giving the subquivers in Figure 8.14, and we will consider the first option, given by the limit

$$x_2 = (qQ_-)^{1/2} x_4^{-1} x_6^{-1} \to 0, \qquad x_5 = q^{-1} Q_-^{-1/2} x_1^{-1} x_3^{-1} \to \infty. \tag{8.193}$$

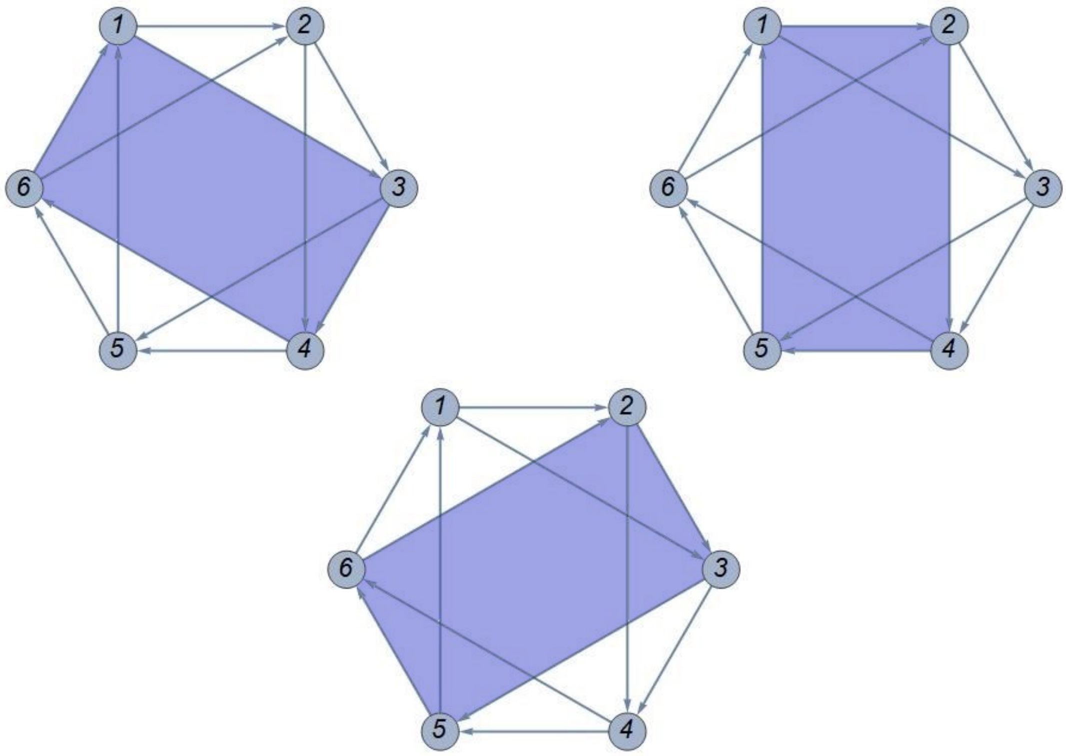


FIGURE 8.14: $N_f = 2$ subquivers for the discrete flow T_4

Here something similar to what happened when we studied the degeneration of the q-Painlevé III_3 associated to local $\mathbb{F}_1$ happens: recall that in that case $T_{\mathbb{F}_1}$ did not have a good degeneration limit as an automorphism of the subquiver, but rather its square did. We observed that this was related to the $\mathbb{Z}_2$-periodicity of the action of $T_{\mathbb{F}_1}$ on the BPS charges. What happens here is that not T_4, but rather T_4^3 has a good action after taking the limit, in particular only for T_4^3 it is true that

$$T_4^3(x_2) \to 0, \qquad T_4^3(x_5) \to \infty, \tag{8.194}$$

consistently with the limit. The resulting sub quiver is the oriented square with arrows of valency one and no diagonals with adjacency matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}. \tag{8.195}$$

The corresponding four dimensional gauge theory has been already classified in [176] as $Q(1,1)$ and shown to correspond to H_3, which is the Argyres-Douglas limit of the $N_f = 3$ with $SU(2)$. All this is consistent with the reductions of the Sakai's table in Figure 8.1. The symmetry type of the five dimensional $SU(2)$ $N_f = 2$ gauge theory is $E_3^{(1)}$. The reduction of the T_1, T_2 and T_3 flows corresponds to the reduction $E_3^{(1)} \to D_2^{(1)}$, the latter being the symmetry type of the four dimensional $SU(2)$ $N_f = 2$ gauge theory. The reduction of the T_4 flow corresponds to the reduction $E_3^{(1)} \to A_2^{(1)}$, the latter being the symmetry type of the H_3 theory. According to Sakai's classification (see Figure 8.1) and the analysis in [16] this flow correctly points to the Argyres-Douglas theory of $N_f = 3$ which is, in the four dimensional limit, governed by the differential PIV equation.

8.5 Conclusions and Outlook

In this final chapter we studied the discrete flows induced by automorphisms of BPS quivers associated to Calabi-Yau geometries engineering 5d quantum field theories. We showed that these flows provide a simple and effective way to determine the BPS spectrum of such theories, producing at the same time a set of bilinear q-difference equations satisfied by the grand canonical partition function of topological string amplitudes. In the rank one case these are known as q-Painlevé equations and admit in a suitable region of the moduli space solutions in terms of Nekrasov-Okounkov, or free fermion, partition functions.

A very attractive feature of this approach is that a simple symmetry principle – the symmetry of the BPS quiver – provides strong constraints on the BPS spectrum and contains a rich and deep set of information which goes well beyond the perturbative approaches to the same theories. Indeed, one can show that the non-perturbative completion of topological string via a spectral determinant presentation, arising in the context of the topological string/ spectral theory correspondence, arises naturally as solution of this system of discrete flow equations [26]. Moreover, some of the flows associated to the BPS quiver directly link to non-perturbative phases of the corresponding gauge theory, as we have seen for a particular flow of the local dP_3 geometry which describe a (A_1, D_4) Argyres-Douglas point, see subsect. 8.4.3. A discussion of the relation between Painlevé equations and Argyres-Douglas points of four-dimensional gauge theories can be found in [16] based on the class $\mathcal{S}$ description of these theories [204].

It is also very interesting that a fully classical construction, the cluster algebra associated to the BPS quiver, contains information about the quantum geometry of the Calabi-Yau. Indeed the zeroes of the τ-functions of the cluster algebra provide the exact spectrum of the associated quantum integrable system, as it was shown in [26] for the local $\mathbb{F}_0$ geometry corresponding to relativistic Toda chain [205], and in [114, 206] for its four-dimensional/non-relativistic limit. The fact the we find that standard topological string – or equivalently 5d gauge theory in the self-dual Ω-background $\epsilon_1 + \epsilon_2 = 0$

rather than in the Nekrasov-Shatashvili [207] background $\epsilon_1 = 0$ – provides a quantisation of the Calabi-Yau geometry is not fully surprising from the view point of equivariant localisation. Indeed the difference between the two cases resides in a different choice of one-parameter subgroup of the full toric action, and therefore contains the same amount of information, although under possibly very non-trivial combinatorial identities. A first instance of this phenomenon was discussed from the mathematical perspective in [61]. For the case at hand, the non-trivial relation between the two approaches is encoded in a suitable limit of blow-up equations [169, 170, 208–210]. There are several directions to further investigate.

Let us notice that, with respect to the framework of [31, 190], to define a BPS chamber one should set the precise order of the arguments of the central charges $Z(\gamma_i)$ for all the charges γ_i in the spectrum. While our method efficiently computes the spectrum, at least in the tame chambers, it doesn't point yet to a precise definition of the corresponding moduli values. This is because we still miss a link with the relevant stability conditions. Our method relies on the existence of patches in the moduli space where the topological string partition function allows finite radius converging expansions[8]. Let us notice that clarifying this point would prepare the skeleton of the demonstration that Kontsevich-Soibelman wall-crossing would be equivalent to the discrete equations (q-Painlevé and higher rank analogues) we obtain. Moreover, the chambers we do compute are "triangular" in the sense of [211]. In this paper it is shown that similar chambers exist for all the class $\mathcal{S}[A_1]$ theories: while multiple affinizations are generically involved, these coincide with the ones computed with our methods, at least in the examples we work out explicitly.

We have seen that when the Calabi-Yau geometry admits several moduli there are various inequivalent flows for the same BPS quiver, but only few of them have a realisation in terms of weakly coupled Lagrangian field theories. In some cases the other flows correspond simply to Bäcklund transformations mapping solutions one into the others. This is the case for example for the fluxes T_1, T_2, T_3 discussed in section 8.2.3. We expect that the full solution to this system of equations will be given in terms of suitable expansions of the topological vertex[9] [24], while its non-pertubative completion should be given by the spectral determinant of the corresponding $N_f = 2$ spectral curve. In other cases the flows are intrinsically non-perturbative, like the flux T_4 discussed in subsect.8.4.3. It would be interesting to characterise the solutions of these flows in terms of supersymmetric indices of four-dimensional gauge theories [214–216].

We expect that the full refined topological string or equivalently the gauge theory in the full Ω background is captured by the *quantum* cluster algebra. The bilinear equations in this case are expected to have a direct relation to the K-theoretic blow-up equations [185].

We have also shown that the X-cluster variables correctly reproduce the ones of the four dimensional BPS quivers under a suitable scaling limit. It would be very interesting to further explore the relation of our results with

[8]See also the comments at the end of section 8.3.2 on this point.

[9]Similar considerations appeared in [212, 213] for the four dimensional case.

the ones on exact WKB methods and TBA equations [37, 209, 217], possibly extending these methods to the q-difference/5d case. For the class S theories an important rôle should be played by the group Hitchin system [171], in the perspective of its quantisation [218, 219].

In the four-dimensional case, the Painlevé/gauge theory correspondence [16] extends also to non-toric cases, corresponding to isomonodromic deformation problems on higher genus Riemann surfaces, see [1, 2] for the genus one case. These have a 5d uplift in terms of q-Virasoro algebra [220] and matrix models [221, 222] whose BPS quiver interpretation would be more than welcome. Also the higher rank extension of BPS quiver flows and the associated tau-functions is to be explored in detail. As a first example, one can consider $SU(N)$ Super Yang-Mills, whose spectral determinant in matrix model presentation was presented in [115]. In the one period phase, this satisfies N-particle Toda chain equations. The corresponding cluster integrable system is discussed in [180]. More in general, our method should extend beyond the rank 1 case and q-Painlevé systems, pointing to more general results about topological string partition functions and discrete dynamical systems.

Part IV

Appendices

Appendix A

Elliptic and theta functions

For elliptic and theta functions we use the notations of [223]. Our torus has periods $(1, \tau)$, and the theta function that we use are

$$\begin{aligned} \theta_1(z|\tau) &\equiv -i \sum_{n\in\mathbb{Z}} (-1)^n q^{(n+\frac{1}{2})^2/2} e^{2\pi i z(n+\frac{1}{2})}, \\ \theta_2(z|\tau) &\equiv \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2/2} e^{2\pi i z(n+\frac{1}{2})}, \\ \theta_3(z|\tau) &\equiv \sum_{n\in\mathbb{Z}} q^{n^2/2} e^{2\pi i z n}, \\ \theta_4(z|\tau) &\equiv \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2} e^{2\pi i z n}, \end{aligned} \tag{A.1}$$

where

$$q = e^{2\pi i \tau}. \tag{A.2}$$

A prime denotes a derivative with respect to z, and when the theta function or its derivatives are evaluated at $z = 0$, we simply denote it by $\theta_\nu(\tau)$ or $\theta_1'(\tau)$, e.t.c. Transformations of θ_1 under elliptic transformations are

$$\theta_1(z+1|\tau) = -\theta_1(z|\tau), \qquad \theta_1(z+\tau|\tau) = -q^{-1} e^{-2\pi i z} \theta_1(z|\tau). \tag{A.3}$$

In the main text we use also Weierstrass $\wp$ and ζ. $\wp$ is a doubly periodic function with a single double pole at $z = 0$, that can be written in terms of θ_1 as

$$\wp(z|\tau) = -\partial_z^2 \log \theta_1(z|\tau) - 2\eta_1(\tau) = -\zeta'(z|\tau), \tag{A.4}$$

where

$$\eta_1(\tau) = -\frac{1}{6}\frac{\theta_1'''(\tau)}{\theta_1'(\tau)}. \tag{A.5}$$

Weierstrass' ζ function is minus the primitive of $\wp$. It has only one simple pole at $z = 0$, and is quasi-elliptic:

$$\zeta(z|\tau) = 2\eta_1(\tau) z + \partial_z \log \theta_1(z|\tau), \tag{A.6}$$

$$\zeta(z+1|\tau) = \zeta(z|\tau) + 2\eta_1(\tau), \quad \zeta(z+\tau|\tau) = \zeta(z|\tau) + 2\tau\eta_1(\tau) - 2\pi i. \tag{A.7}$$

Sometimes it turns out to be convenient to normalize the Weierstrass elliptic functions in a different way, in order to have vanishing A-cycle integral. The functions thus obtained are called Eisenstein functions:

$$E_1(z|\tau) = \partial_z \log\theta(z|\tau) = \zeta(z|\tau) - 2\eta_1(\tau)z, \tag{A.8}$$
$$E_2(z|\tau) = -\partial_z E_1(z|\tau) = \wp(z|\tau) + 2\eta_1(\tau). \tag{A.9}$$

Finally, we use Dedekind's η function, defined as

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{A.10}$$

It is related to the function θ_1 by

$$\eta(\tau) = \left(\frac{\theta_1'(\tau)}{2\pi}\right)^{1/3}. \tag{A.11}$$

Because of the periodicities (A.20), the elliptic transformations of the Lamé function

$$x(u,z) = \frac{\theta_1(z-u|\tau)\theta_1'(\tau)}{\theta_1(z|\tau)\theta_1(u|\tau)} \tag{A.12}$$

are given by

$$x(u,z+1) = x(u,z), \qquad x(u,z+\tau) = e^{2\pi i u}x(u,z). \tag{A.13}$$

The product of Lamé functions satisfies the following identities:

$$x(u,z)x(-u,z) = \wp(z) - \wp(u), \quad x(u,z)y(-u,z) - y(u,z)x(-u,z) = \wp'(u), \tag{A.14}$$

where $y(u,z) = \partial_u x(u,z)$, that are used in computing the isomonodromic Hamiltonian H_τ from the Lax matrix. Further, to show that the zero-curvature equation (5.12) is the compatibility condition for the system (5.13), one has to use the property

$$2\pi i\partial_\tau x(u,z) + \partial_z\partial_u x(u,z) = 0. \tag{A.15}$$

The following theta-function identities are used in the study of the autonomous limit:

$$\partial_z \frac{\theta_2(z|\tau)}{\theta_3(z|\tau)} = -\pi\theta_4^2(\tau)\frac{\theta_1(z|\tau)\theta_4(z|\tau)}{\theta_3(z|\tau)^2}, \tag{A.16}$$

$$\begin{aligned} \frac{\theta_1(2Q|2\tau_0)^2}{\theta_3(2Q|2\tau_0)^2} &= \frac{\theta_2(2\tau_0)^2}{\theta_4(2\tau_0)^2} - \frac{\theta_3(2\tau_0)^2}{\theta_4(2\tau_0)^2}\frac{\theta_2(2Q|2\tau_0)^2}{\theta_3(2Q|2\tau_0)^2}, \\ \frac{\theta_4(2Q|2\tau_0)^2}{\theta_3(2Q|2\tau_0)^2} &= \frac{\theta_3(2\tau_0)^2}{\theta_4(2\tau_0)^2} - \frac{\theta_2(2\tau_0)^2}{\theta_4(2\tau_0)^2}\frac{\theta_2(2Q|2\tau_0)^2}{\theta_3(2Q|2\tau_0)^2}. \end{aligned} \tag{A.17}$$

We also need the Jacobi theta function with characteristics

$$\theta_\tau\begin{bmatrix} a \\ b \end{bmatrix}(z) = \sum_{n\in\mathbb{Z}} e^{i\pi(n+a)^2\tau}e^{2\pi i(z+b)(n+a)} \tag{A.18}$$

Its quasi-periodicity properties of the theta functions are

$$\begin{aligned} \theta_\tau \begin{bmatrix} a \\ b \end{bmatrix} (z+1) &= e^{2\pi i a} \theta_\tau \begin{bmatrix} a \\ b \end{bmatrix} (z), \\ \theta_\tau \begin{bmatrix} a \\ b \end{bmatrix} (z+\tau) &= e^{-i\pi\tau - 2\pi i z - 2\pi i b} \theta_\tau \begin{bmatrix} a \\ b \end{bmatrix} (z), \end{aligned} \tag{A.19}$$

so that

$$\theta_1(z+1|\tau) = -\theta_1(z|\tau), \qquad \theta_1(z+\tau|\tau) = -q^{-1/2} e^{-2\pi i z} \theta_1(z|\tau). \tag{A.20}$$

Appendix B

Schottky uniformization and twisted differentials

In this appendix we review some basic facts about the Schottky parametrization of a Riemann surface, and give Poincaré series representation of the twisted differentials we use in chapter 7.

By the uniformization theorem, we know that any Riemann Surface Σ can be represented as a covering of some base space by a discontinuous Kleinian group Γ. We choose the representation

$$\Sigma = \mathbb{C}/\Gamma, \tag{B.1}$$

where Γ is a Schottky group[1]. The advantage of such a parametrization is that the complex structure of the Teichmüller space has a natural and manifest parametrization.

To specify Γ, we will write a homographic transformation $\gamma \in \mathbb{P}SL(2,\mathbb{C})$ by specifying its fixed points $u_\gamma \neq v_\gamma$ and its multiplier $|q_\gamma| < 1$, through

$$\gamma(z) = \frac{az+b}{cz+d}, \qquad \frac{\gamma(z)-u_\gamma}{\gamma(z)-v_\gamma} = q_\gamma \frac{z-u_\gamma}{z-v_\gamma}. \tag{B.2}$$

The isometric circle of γ, that we will denote with A_γ, is the space on which $|\gamma'(z)| = 1$, i.e. the circle $|cz+d| = 1$. By converse, we will denote the isometric circle of γ^{-1} by A'_γ, which is $|cz+d| = 1$. The Schottky group is the freely generated group

$$\Gamma = \langle \gamma_1, \ldots, \gamma_g \rangle, \tag{B.3}$$

where g is the genus of Σ. The circles A_i are the A-cycles of the curve, and the identification of the circles A_i and $A'_i = -\gamma_i(A_i)$ produces h handles, so that the B-cycles are curves connecting A_i and A'_i. The complex moduli of the Riemann Surface are $3g-3$: $3g$ for the parameters specifying the homographic transformations, -3 because an overall conjugation $M\Gamma M^{-1}$, with $M \in \mathbb{P}SL(2)$ does not change the description. For $g = 1$, we recover the usual construction of the torus as an annulus with identified ends $z \sim qz$.

[1] A Schottky group is a special case of a Kleinian group, that sends the outside of the A_i-cycles onto the inside of the B_i-cycles. For any practical surface, we will only need the explicit construction of the Schottky group outlined in this section.

By using the Schottky parametrization, we define twisted meromorphic differentials. Such objects are defined by their transformation properties under elements of the Schottky group:

$$\gamma'(z)\lambda^a(\gamma(z)) = (\mathrm{Ad}\, g_\gamma)^a{}_b \lambda^b(z). \tag{B.4}$$

We will often write only g_γ instead of Ad g_γ, leaving the representation implicit. One basic such object is the contraction function for Kac-Moody algebra currents with B-cycle twists, given by the twisted Poincaré series

$$\Theta(z,w;w_0)^a{}_b = \sum_{\gamma\in\Gamma}\left[\frac{\gamma'(z)}{\gamma(z)-w} - \frac{\gamma'(z)}{\gamma(z)-w_0}\right](g_\gamma^{-1})^a{}_b. \tag{B.5}$$

It is the nonabelian generalization of $d_z \log E(z,w)$, where E is the prime form. As seen by this expression, our objects depend also on a choice of base-point w_0, at which they have an additional simple pole. We will omit the explicit dependence on the base-point unless it is necessary for the discussion. We now list some of the properties of Θ: first of all it is a twisted automorphic 1-form in z, a, but its transformation laws are not covariant in w, b:

$$\Theta(z,\gamma(w);w_0)^a{}_b = \left[\Theta(z,w)^a{}_c - \Theta(z,\gamma^{-1}(w_0))^a{}_c\right](g_\gamma^{-1})^c{}_b. \tag{B.6}$$

Θ has vanishing A-cycle periods and unit residues:

$$\oint_{A_i}\Theta(z,w)^a{}_b = 0, \quad Res_w\Theta(z,w)^a{}_b = \delta^a{}_b, \quad Res_{w_0}\Theta(z,w;w_0) = -\delta^a{}_b. \tag{B.7}$$

It was noted in [161] that this object is the Green function of a twisted b-c system:

$$\langle b^a(z)c^b(w)\rangle = \Theta(z,w)^{ab}. \tag{B.8}$$

This b-c system defines the currents

$$j^a(z) \equiv f^a{}_{bc} :b^b(z)c^c(z): \tag{B.9}$$

generating a $\widehat{\mathfrak{g}}$ current algebra at level $2h^\vee$. Because of this, one can relate the regularized expression

$$\Theta^{reg}(z)^a{}_b \equiv \lim_{w\to z}\left[\Theta^a{}_b - \frac{\delta^a{}_b}{z-w}\right] \tag{B.10}$$

to the current one-point function

$$f^a{}_{bc}\Theta^{reg}(z)^{bc} = \langle j^a(z)\rangle. \tag{B.11}$$

We can construct a basis of $N^2 g$ twisted differentials from Θ by defining

$$\omega_i(z)^a{}_b \equiv \frac{1}{2\pi i}\Theta(z,\gamma_i^{-1}(w_0)), \tag{B.12}$$

these are nonabelian generalizations of the usual holomorphic differentials on Riemann surfaces, but with an additional pole at w_0, with residue

$$Res_{w_0}\omega_i(z)^a{}_b = g(B_i)^a{}_b - \delta^a{}_b. \tag{B.13}$$

As a consequence of the properties of Θ, they are normalized as to have zero A-periods.

Another important quantity is the derivative of Θ:

$$\partial_w\Theta(z,w)^a{}_b = \sum_{\gamma\in\Gamma}\frac{\gamma'(z)}{(\gamma(z)-w)^2}(g_\gamma^{-1})^a{}_b, \tag{B.14}$$

which is a twisted 1-form with a double pole with zero residue at $z = w$. It is the nonabelian generalization of $d_z d_w \log E(z,w)$. Its regularized version

$$\partial\Theta^{reg}(z)^a{}_b \equiv \lim_{w\to z}\left[\partial_w\Theta(z,w)^a{}_b - \frac{\delta^a{}_b}{z-w}\right] \tag{B.15}$$

gives the one-point function of the b-c system energy-momentum tensor

$$t(z) \equiv -:b^a(z)\partial c_a(z):, \qquad \partial\Theta^{reg}(z)^a{}_a = -\langle t(z)\rangle. \tag{B.16}$$

Appendix C

q-Painlevé III and IV in Tsuda's parametrization

We provide here the choice of parameters for the cluster algebra that reproduces the q-Painlevé III_1 and IV equations of [193]. It turns out for this to be useful to introduce

$$u_1 = a_1 \left(\frac{b_1}{b_0}\right)^{1/3}, \qquad u_2 = a_2 \left(\frac{b_0}{b_1}\right)^{1/3}, \qquad u_3 = a_0 \left(\frac{b_1}{b_0}\right)^{1/3}, \tag{C.1}$$

$$u_4 = a_1 \left(\frac{b_0}{b_1}\right)^{1/3}, \qquad u_5 = a_2 \left(\frac{b_1}{b_0}\right)^{1/3}, \qquad u_6 = a_0 \left(\frac{b_0}{b_1}\right)^{1/3}, \tag{C.2}$$

Note that in [193] the Casimirs have a geometric meaning in terms of points blown-up on $\mathbb{P}^1 \times \mathbb{P}^1$.

C.1 qPIII

Let us consider first the cas of qPIII. We choose as basis for the tropical semifield four independent Casimirs, and parametrize in terms of them the y_i's in such a way that (8.86) and (8.87) are satisfied, together with the correct time evolution (8.91). We choose as independent Casimirs a_1, a_2, q, b_0. To match with [193], we also have to make a different choice for the time evolution parameter q,

$$q = \prod_i y_i^{-1/2}. \tag{C.3}$$

Set

$$y_1 = q^{-1/3} b_0^{2/3} a_1^{-1}, \qquad y_2 = q^{1/3} b_0^{2/3} a_2^{-1}, \qquad y_3 = q^{-4/3} b_0^{2/3} a_1 a_2, \tag{C.4}$$

$$y_4 = q^{1/3} b_0^{-2/3} a_1^{-1}, \qquad y_5 = q^{-1/3} b_0^{2/3} a_2^{-1}, \qquad y_6 = q^{-2/3} b_0^{-2/3} a_1 a_2. \tag{C.5}$$

The time evolution is the following (we only write the relevant τ-variables):

$$\begin{cases} \overline{\tau_2} = \tau_4, \\ \overline{\tau_3} = \frac{a_2 b_0^{2/3}\tau_3\tau_4 + q^{1/3}\tau_1\tau_6}{\tau_2}, \\ \overline{\tau_5} = \tau_1, \\ \overline{\tau_6} = \frac{q^{1/3}a_2\tau_1\tau_6 + \tau_3\tau_4 b_0^{2/3}}{\tau_5}, \end{cases} , \qquad \begin{cases} \underline{\tau_1} = \tau_5, \\ \underline{\tau_3} = \frac{a_1 b_0^{2/3}\tau_5\tau_6 + q^{1/3}\tau_2\tau_3}{\tau_4}, \\ \underline{\tau_4} = \tau_2, \\ \underline{\tau_6} = \frac{q^{1/3}a_1\tau_2\tau_3 + b_0^{2/3}\tau_5\tau_6}{,} \end{cases} \tag{C.6}$$

leading to the bilinear equations

$$\overline{\tau_3}\tau_2 = q^{1/3}(\overline{\tau_5}\tau_6 + u_2\tau_3\overline{\tau_2}), \qquad \overline{\tau_6}\tau_5 = b_0^{2/3}(u_5\overline{\tau_5}\tau_6 + \tau_3\overline{\tau_2}), \tag{C.7}$$

$$\overline{\tau_2}\underline{\tau_3} = q^{1/3}(u_4\tau_5\tau_6 + \tau_2\tau_3), \qquad \overline{\tau_5}\underline{\tau_6} = b_0^{2/3}(u_1\tau_2\tau_3 + \tau_5\tau_6). \tag{C.8}$$

These differ from the bilinear equations of [193] by different overall factors of the RHS, so in principle it would seem that they are different bilinear equations. However they are still equivalent to qPIII. If we define

$$f = \frac{\tau_5\tau_6}{\tau_2\tau_3}, \qquad g = \frac{\tau_1\tau_6}{\tau_3\tau_4}, \tag{C.9}$$

we get the system of first-order q-difference equations

$$\overline{f}f = a_2 g\frac{g + u_5^{-1}}{g + u_2}, \qquad \underline{g}g = \frac{f}{a_1}\frac{f + u_1}{f + u_4^{-1}}, \tag{C.10}$$

which is the qPIII equation appearing in [193].

C.2 qPIV

The action of T_4 on the tau function is the following:

$$\begin{cases} \overline{\tau_1} = \tau_4, \\ \overline{\tau_2} = \frac{q^{4/3}b_0^{2/3}\tau_4\tau_5\tau_6 + a_1 q^{5/3}\tau_2\tau_3\tau_4 + a_1 a_2 b_0^{4/3}\tau_1\tau_2\tau_6}{\tau_1\tau_3}, \\ \overline{\tau_3} = \tau_6, \\ \overline{\tau_4} = \frac{q^{5/3}\tau_4\tau_5\tau_6 + a_1 b_0^{4/3}\tau_2\tau_3\tau_4 + a_1 a_2 q^{1/3} b_0^{2/3}\tau_1\tau_2\tau_6}{\tau_3\tau_5}, \\ \overline{\tau_5} = \tau_2, \\ \overline{\tau_6} = \frac{b_0^{4/3}\tau_4\tau_5\tau_6 + q^{1/3}a_1 b_0^{2/3}\tau_2\tau_3\tau_4 + q^{2/3}a_1 a_2\tau_1\tau_2\tau_6}{\tau_1\tau_5}, \end{cases} \tag{C.11}$$

$$\begin{cases} \underline{\tau_1} = \frac{q^{2/3} b_0^{4/3} \tau_1 \tau_2 \tau_3 + q^{1/3} a_1 \tau_1 \tau_5 \tau_6 + a_1 a_2 b_0^{2/3} \tau_3 \tau_4 \tau_5}{\tau_2 \tau_6}, \\ \underline{\tau_2} = \tau_5, \\ \underline{\tau_3} = \frac{q^{2/3} \tau_1 \tau_2 \tau_3 + q^{1/3} a_1 b_0^{2/3} \tau_1 \tau_5 \tau_6 + a_1 a_2 b_0^{4/3} \tau_3 \tau_4 \tau_5}{\tau_2 \tau_4}, \\ \underline{\tau_4} = \tau_1, \\ \underline{\tau_5} = \frac{q^{2/3} b_0^{2/3} \tau_1 \tau_2 \tau_3 + q^{1/3} a_1 b_0^{4/3} \tau_1 \tau_5 \tau_6 + a_1 a_2 \tau_3 \tau_4 \tau_5}{\tau_4 \tau_6}, \\ \underline{\tau_6} = \tau_3. \end{cases} \tag{C.12}$$

At a first glance these seem trilinear, rather than bilinear, equations. However, take linear combinations of (C.11) in such a way that the first term on the RHS cancels out:

$$\begin{cases} \overline{\tau_6} \tau_1 b_0^{-4/3} - q^{-5/3} \overline{\tau_4} \tau_3 = \frac{\tau_2 \tau_3 \tau_4 a_1 \left(q^{1/3} b_0^{-2/3} - q^{-5/3} b_0^{4/3}\right) + a_1 a_2 \tau_1 \tau_2 \tau_6 \left(q^{2/3} b_0^{-4/3} - q^{-4/3}\right)}{\tau_5}, \\ \overline{\tau_6} \tau_5 b_0^{-2/3} - q^{-4/3} \overline{\tau_2} \tau_3 = \frac{a_1 a_2 \tau_1 \tau_2 \tau_6 \left(q^{2/3} b_0^{-2/3} - q^{-4/3} b_0^{4/3}\right)}{\tau_1}, \\ \overline{\tau_4} \tau_5 q^{-1/3} - b_0^{-2/3} \overline{\tau_2} \tau_1 = \frac{\tau_2 \tau_3 \tau_4 a_1 q^{4/3} \left(q^{-5/3} b_0^{4/3} - q^{1/3} b_0^{-2/3}\right) + a_1 a_2 \tau_1 \tau_2 \tau_6 \left(1 - b_0^{2/3}\right)}{\tau_3}. \end{cases} \tag{C.13}$$

We see that the second equation is now bilinear! We can repeat this procedure to obtain three bilinear equations from (C.11):

$$\overline{\tau_6} \tau_5 b_0^{-2/3} - \overline{\tau_2} \tau_3 q^{-4/3} = a_1 a_2 \left(q^{2/3} b_0^{-2/3} - b_0^{4/3} q^{-4/3}\right) \tau_2 \tau_6, \tag{C.14}$$

$$\overline{\tau_6} \tau_1 q^{-1/3} - \overline{\tau_4} \tau_3 b_0^{-2/3} = \left(b_0^{4/3} q^{-1/3} - q^{5/3} b_0^{-2/3}\right) \tau_4 \tau_6, \tag{C.15}$$

$$\overline{\tau_4} \tau_5 q^{-1/3} - \overline{\tau_2} \tau_1 b_0^{-2/3} = a_1 \left(b_0^{4/3} q^{-1/3} - q^{5/3} b_0^{-2/3}\right) \tau_2 \tau_4. \tag{C.16}$$

(C.17)

We can make these three second-order equations in three variables by using the (C.12):

$$\overline{\tau_6} \underline{\tau_2} b_0^{-2/3} - \overline{\tau_2} \underline{\tau_6} q^{-4/3} = a_1 a_2 \left(q^{2/3} b_0^{-2/3} - b_0^{4/3} q^{-4/3}\right) \tau_2 \tau_6, \tag{C.18}$$

$$\overline{\tau_6} \underline{\tau_4} q^{-1/3} - \overline{\tau_4} \underline{\tau_6} b_0^{-2/3} = \left(b_0^{4/3} q^{-1/3} - q^{5/3} b_0^{-2/3}\right) \tau_4 \tau_6, \tag{C.19}$$

$$\overline{\tau_4} \underline{\tau_2} q^{-1/3} - \overline{\tau_2} \underline{\tau_4} b_0^{-2/3} = a_1 \left(b_0^{4/3} q^{-1/3} - q^{5/3} b_0^{-2/3}\right) \tau_2 \tau_4, \tag{C.20}$$

(C.21)

These are the bilinear equations for qPIV, with the parametrization of [193].

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