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Gravitational Wave Decay: Implications for cosmological scalar-tensor theories

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Abstract

The recent discovery that gravitational waves and light travel with the same speed, with an error below 10^{-15}, has greatly constrained the parameter space of infrared modifications of gravity. In this thesis we study the phenomenology of gravitational-wave propagation in modifications of gravity relevant for dark energy with an additional scalar degree of freedom. Of particular interest are Horndeski and Beyond Horndeski models surviving after the event GW170817. Here the dark energy field is responsible for the spontaneous breaking of Lorentz invariance on cosmological scales. This implies that gravitons γ can experience new dispersion phenomena and in particular they can decay into dark energy fluctuations π.

First, we study the perturbative decay channels $\gamma \to \pi\pi$ and $\gamma \to \gamma\pi$ in Beyond Horndeski models. The first process is found to be large and thus incompatible with recent gravitational-wave observations. This provides a very stringent constraint for the particular coefficient $\tilde{m}_4^2$ of the Effective Field Theory of Dark Energy or, in the covariant language, on quartic Beyond Horndeski operators. We then study how the same coupling affects at loop level the propagation of gravitons. It is found that the new contribution modifies the dispersion relation in a way that is incompatible with current observations, giving bounds of the same magnitude as the decay.

Next, we improve our analysis of the decay by taking into account the large occupation number of gravitons and dark energy fluctuations in realistic situations. When the operators m_3^3 (cubic Horndeski) and $\tilde{m}_4^2$ are present, we show that the gravitational wave acts as a classical background for π and affects its dynamics, with π growing exponentially. In the regime of small gravitational-wave amplitude, we compute analytically the produced π and the change in the gravitational wave. For the operator m_3^3, π self-interactions are of the same order as the resonance and affect the growth in a way that cannot be described analytically. For the operator $\tilde{m}_4^2$, in some regimes self-interactions remain under control and our analysis improves the bounds from the perturbative decay, ruling out quartic Beyond Horndeski operators from having any relevance for cosmological applications.

Finally, we show that in the regime of large amplitude for the gravitational wave π becomes unstable. If m_3^3 takes values relevant for cosmological applications, we conclude that dark energy fluctuations feature ghost and gradient instabilities in presence of gravitational waves of typical binary systems. Taking into account the populations of binary systems, we find that the instability is triggered in the whole Universe. The fate of the instability and the subsequent

time-evolution of the system depends on the UV completion, so that the theory may end up in a state very different from the original one. In conclusion, the only dark-energy theories with sizeable cosmological effects that avoid these problems are k-essence models, with a possible conformal coupling with matter.

Publications

The main material of this thesis has appeared previously in the following publications:

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1 | Introduction

Scientific efforts of the last century in the field of cosmology led to a radical change in our understanding of the universe. Remarkably, we are currently in the era of *precision cosmology*, where several cosmological parameters can be measured by different experiments with unprecedented accuracy.

Probes at different cosmological scales have all confirmed that our universe is with good accuracy *homogeneous*, *isotropic* and has negligible spatial curvature [4–6]. Moreover, all the observations so far are consistent with *General Relativity* (GR). These and many more features are very well explained in our current cosmological model (Λ-cold dark matter or ΛCDM in short) by a handful of parameters, describing the abundance of different energy components (the ingredients of the universe). Additionally, we are reaching the level of precision of the few percent, making it possible to rule out competing theoretical models.

Despite all these recent observational and theoretical progress, a fundamental understanding for most of the energy components in ΛCDM is however still lacking. Undoubtedly the most puzzling aspect in this picture remains *dark energy* (DE). Among the most impressive discoveries in the era of modern cosmology, with crucial implications for fundamental physics, is the notion that we are currently undergoing a phase of *accelerated* expansion [7–9], or in other words the universe is DE-dominated. This discovery was further confirmed by a multitude of other probes, such as Cosmic Microwave Background (CMB) radiation [10–15], and Large-Scale Structure [16–23]. As the name suggests, within ΛCDM, DE is modelled simply by a cosmological constant Λ measured to be around $\Lambda \sim M_{\rm Pl}^2 H_0^2 \sim (10^{-3}\,{\rm eV})^4$ and comprises $\sim 70\,\%$ of the energy budget of the universe today.

The smallness of Λ remains hard to reconcile with the gargantuan contributions to the vacuum energy expected from Standard Model fields, which would predict a value at least as large as the weak scale $\sim (1\,{\rm TeV})^4$. Although this could be dismissed as being a *fine-tuning* problem, the theoretical challenges for circumventing such issue make the explanation for the value of the cosmological constant one of the hardest unresolved problems in modern physics [24, 25]. From such a perspective, this problem seems to put into question some basic assumptions of cosmology and particle physics: no dynamical solution to this problem can be found in GR [25]. Given the lack of concrete solutions to the smallness of the cosmological constant, it is then worthwhile to explore and then test possible deviations from Λ, in the hope to obtain evidence for new physics. Many of the investigated routes involve the introduction of new degrees of freedom, extra dimensions or *anthropic* arguments [26–30]. The first two possibilities are usually described in terms of new light degrees of freedom, able to affect the dynamics at horizon scales for both the background evolution and perturbations, thus providing an *infrared* (IR) modification of gravity.

In this scenario, DE could be regarded as a fluid evolving over cosmological scales. Its equation of state parameter $w_{\rm DE}$ is constrained by CMB, Type Ia supernovae and Baryon acoustic oscillations (BAO) to be close to that of a cosmological constant ($w_{\rm DE} \simeq -1$, with a few-percent accuracy) [31], making it a very different fluid from the ones we are accustomed to. Up to now, there is no solid evidence that DE differs from Λ at the level of cosmological background evolution. However, the common feature of these models is to provide deviations at the level of perturbations, where the new degrees of freedom associated with this fluid should appear.

A regime of great phenomenological interest at the level of perturbations is the *quasi-static* regime, when Fourier modes are well within the horizon and deviations from ΛCDM could potentially appear. This limit mimics the Newtonian limit of GR, and time derivatives can usually be discarded, making modelling easier. The typical equations for the Newtonian potentials Φ and Ψ, in Fourier space, are

$$-\boldsymbol{k}^2\Phi = 4\pi\mu G_{\rm N}\,\delta\rho\,, \qquad \Phi/\Psi = \gamma\,, \tag{1.1}$$

where $G_{\rm N}$ is Newton's constant ($M_{\rm Pl}^2 = (8\pi G_{\rm N})^{-1}$), $\delta\rho$ is the perturbed energy density of matter and μ (modified Newton constant) and γ (gravitational slip) parametrize deviations from GR. These last two parameters are typically functions of both time and momentum $\boldsymbol{k}$; GR is recovered when they are both equal to one. This parametrization is often used to quote bounds in cosmological and astrophysical tests of GR.

Cosmological probes

These parameters modify, at a cosmological level, the growth of structures and affect a multitude of observables. For example, they leave an imprint in the low-ℓ multipoles of the CMB through the integrated Sachs-Wolfe effect, and also affect the lensing of photons. At present, CMB and LSS probes such as Planck, BOSS, KIDS and DES give bounds on μ and γ of order $\sim \mathcal{O}(1)$, still leaving room for large departures from GR [15, 32, 33]. This might seem a rather unpleasant result given that solar system tests of gravity are much more constraining, reaching levels of $\sim \mathcal{O}(10^{-5})$ precision [34]. Nonetheless, it is impressive how these cosmological bounds can be obtained in the first place, given that they require extrapolating gravity to extremely large scales.

The power of the bounds on gravity at short scales suggests a general property that any credible IR modification of gravity should possess: it needs to be *screened* on short scales, or in presence of large densities, in order to pass solar system tests. On the other hand, large effects can appear in large-scale observables, where bounds on gravity are weaker. In other terms, even strong bounds at short scales are not a showstopper for modified gravity, at least so far.

It is probable that in the next decades the situation will dramatically improve, thanks to a multitude of high-precision cosmological probes ready to launch. For instance experiments such as Euclid, DESI, LSST, SKA and the Simons observatory promise to collect an unprecedented number of cosmological data and improve the current constraints down to $\sim \mathcal{O}(10^{-2})$ [35–39]. At that stage, a deviation from ΛCDM will potentially be a clear sign of new physics.

Gravitational wave probes

Remarkably, on top of these incredible advancements in precision cosmology, a new observational window on the universe is opening thanks to gravitational waves (GWs). Indeed, in September 2015 the quest for the direct detection of GWs has finally succeeded with the first observation of a black hole-black hole (BH-BH) merger by the two LIGO interferometers at Hanford and Livingston [40]. Since then LIGO has completed two observing runs, collecting a total of ten BH-BH and one neutron star-neutron star (NS-NS) merger detections [41]. Along the way Virgo, a third interferometer based in Italy, started collecting data and detecting events, giving more statistical power to the LIGO detections. After a period of upgrades to the detectors, a third observing run is now in progress and promises to yield many more GWs events.

By design these ground-based interferometers work in the frequency range spanning from few Hz to around 1 kHz, making them especially sensitive to the merger phase of stellar-mass BHs and to the inspiralling phase of NS-NS systems. In the particular case of BH-BH binaries, the electromagnetic emissions are usually quite low, and indeed no such systems where known before the first LIGO detection. This sole fact shows how GWs are opening a new important window in astrophysics and cosmology, that will help to answer many puzzles regarding the formation and evolution of binary systems [42].

Moreover, GWs can test GR in the strong-gravity regime (probing the close-horizon geometry of BHs and observing relativistic systems), becoming complementary to solar system tests of gravity. Perhaps not surprisingly, the waveforms observed so far are found to be in agreement with the GR predictions [43, 44].

In this regard, the NS-NS event GW170817 detected by the LIGO/Virgo detectors has been of exceptional relevance. Indeed, together with this event was associated a γ-ray burst (GRB), measured by Fermi-GBM and INTEGRAL [45–47], that allowed a direct comparison between gravitational and electromagnetic radiation properties. Remarkably the subsequent transient electromagnetic counterpart was observed by more than fifty telescopes and tracked for several days afterwards.

Between the many discoveries made possible by this event, the most relevant for testing GR was the almost simultaneous detection of the GW and GRB signals [47]. The event is measured at a distance of around 40 Mpc whilst the time delay between gravitational and electromagnetic radiation is of the order of seconds (more details are given in Sec. 3.2). This puts an incredibly strong bound on the speed of propagation of GWs c_T, whose difference with the speed of light c is bound to be $|c_T - c|/c \lesssim 10^{-15}$ [47]. Given that the signals travelled over cosmological distances, this translates in a powerful constrain for IR modifications of gravity [48–51], several orders of magnitude better than what can be achieved by cosmological probes.

In low energy modifications of gravity, DE often behaves as a fluid and thus provides a preferred reference frame, spontaneously breaking Lorentz invariance. Therefore, the dynamics of GWs and DE fluctuations is in general Lorentz violating and dispersion phenomena can take place. This is analogous to what happens in condensed matter systems, where light acquires a different speed when passing through a material and can also be absorbed. In this case, by studying the propagation of light one can infer the properties of the material itself. This idea can be applied to GWs, and in the context of DE theories the constraint on c_T already tells us important properties of the DE "fluid". Motivated by this analogy, in this thesis we want to study more deeply the dispersion properties of GWs in DE models to extend the current constraints on IR modifications of gravity.

In chapter 2 we are going to introduce the DE theories we are going to focus on, with particular emphasis on the Effective Field Theory formulation of DE. Of great interest for our purposes are *Galileon theories*, that we discuss in Sec. 2.5. A detailed analysis of the constraints on these theories coming from $c_T = c$ is then given in chapter 3.

Due to the spontaneous breaking of Lorentz invariance, we find that GWs can decay into DE fluctuations, as we discuss in chapter 4. In this chapter, we perform a perturbative analysis of this decay channel and we also study the dispersion phenomena associated with the absorption of gravitons. Depending on the model, gravitons might not survive over cosmological scales: given that GW experiments do not report significant deviations from GR, we are then able to place new bounds on DE theories.

In chapter 5 we extend the perturbative analysis of the decay of GWs by considering the more realistic situation in which the wave is classical, and resonances play an important role. Indeed, we show that the decay rate is greatly enhanced, improving the previous bounds coming from the perturbative decay. Our calculations are reliable in the particular regime of *narrow resonance*, where the classical enhancement can be treated analytically. Moreover, we also need to keep DE non-linearities under control.

Obtaining bounds outside of these regimes is much harder since the system becomes fully non-linear. However, in the particular limit in which the amplitude of the GW becomes large, we are able to study the propagation of GWs. We study this limit in chapter 6 and we show that DE fluctuations can become unstable for large enough GW amplitudes. Strictly speaking, this does not imply a tension between theory and data, but rather it signifies a loss of predictability of the theory of DE. On the other hand, GR perfectly describes the propagation of GWs, and therefore modifications of gravity featuring instabilities appear as highly disfavoured. In light of current and future GW observations, we then elaborate on the implications of our constraints for DE theories in Sec. 6.6.

The future of GW detections is unmistakably bright given the multitude of new ground-based interferometers about to join the existing network of detectors, with KAGRA and LIGO-India being the first in line [52]. Moreover, another major milestone will be the Laser Interferometer Space Antenna (LISA), a space-based interferometer operating in the range of frequencies of the mHz, that will be operational in the 2030s [53]. This spur of new GW data will certainly greatly improve our current understanding of astrophysics, gravitational dynamics and fundamental physics.

2 | Dark Energy theories

The results in this thesis are in great part obtained within the framework of the Effective Field Theory of Dark Energy (EFT of DE). Given its relevance, this latter subject is briefly reviewed in this chapter, together with a discussion of its covariant formulation. Most of the material follows [54] and [55].

2.1 The EFT principle

It is predicted that in the next decades there will be an unprecedented collection of data, from cosmological and GW probes, as already outlined in chapter 1. Thanks to this effort, there is hope for a better understanding of the nature of the cosmological constant and potentially it will enable us to discriminate between GR and modified gravity. Given the complexity involved in this challenging task, it is paramount to have a unified theoretical framework able to encompass a sufficiently large number of competing models, ready to be compared against ΛCDM.

Regarding DE, many theoretical models are described by a single scalar degree of freedom (on top of the metric and matter fields), at least in the regimes relevant to cosmology. The situation is indeed very similar to the case of inflation which, for most of the models, is described solely in terms of the inflaton. This field ultimately defines when inflation ends, and thus acts as a "clock" (providing a privileged foliation of space-time). Put in a different and perhaps more general way, inflation is a theory where time-diffeomorphisms are spontaneously broken or better, non-linearly realized.

This point of view hints at the almost inevitable presence of a scalar field in a theory with early-times acceleration. Of course, this reasoning does not immediately generalize to DE since, as far as the actual observations can tell, the current accelerated expansion can continue indefinitely in the future, as in ΛCDM. Rather, inflation suggests that deviations from a simple cosmological constant are possible, with an additional scalar being an obvious candidate to explain the current acceleration. In what follows we will take this point of view.

As it is well known in particle physics, the structure of the action for the light degrees of freedom is fixed in terms of the symmetry-breaking pattern [56, 57]. The remarkable generality of this statement makes the low-energy theory essentially unique, irrespectively of the detailed mechanism responsible for the symmetry breaking.[1]

Moreover, the effects of ultraviolet (UV) physics in the low-energy effective theory are typically encapsulated by non-renormalizable operators (in d-dimensions these are operators with mass dimension larger than d) and are therefore suppressed in the IR. This is indeed intuitively expected: physics at short scales should not have a significant impact at large scales. The expansion in terms of non-renormalizable operators is, in spirit, very similar to the multipole expansion in electrodynamics. At large distances even very complicated sources are well described by their lower moments (starting form monopole and dipole), and only with a good experimental accuracy one can probe larger multipoles. In other words, sources appear simple not because they generally are, but because we observe them from far away. Similarly, for a given experimental precision, low-energy EFTs are described by a small set of parameters (and thus appear simple) even though the description in terms of UV degrees of freedom can be very complicated and even strongly coupled.

Contrary to what was previously believed, it is nowadays well understood that the non-renormalizability of EFTs does not limit their predictive power and they are as predictive as renormalizable theories [58]. Their regime of validity is however limited, meaning that they are valid up to some energy cutoff (in the case of the multipole expansion in electrodynamics, up to distances around the size of the source), beyond which a new effective description is needed to make predictions.

The generality of the aforementioned features makes EFTs ubiquitous, as demonstrated by the multitude of applications they find in different branches of physics, from condensed matter to particle physics. In the contest of gravitation and cosmology, these techniques have been applied for example to GWs [59], inflation [55, 60, 61], LSS [62] DE [54, 63, 64] and Black Holes [65, 66].

In the case of inflation, these constructions go under the name of EFT of Inflation [55, 60, 61]. As far as current observations go, it is still unclear what is the role of the inflaton ϕ in relation to other fundamental fields and in most of the inflationary models ϕ has the sole purpose of being the source for the accelerated expansion. To capture this latter feature in full generality, without committing to a particular model, the EFT approach appears very convenient. The main philosophy, in this case, is then to be agnostic about the mechanism leading to the accelerated expansion at the level of the background evolution, and to only deal with the inflaton perturbations. One can therefore take this perspective and, as in the examples known from particle physics, the action for fluctuations is then fixed in terms of symmetries. Keeping

[1] One can alternatively take the point of view of a low-energy observer (at energies below the symmetry breaking scale). In this case, one constructs the action by fixing the degrees of freedom and the linearly-realized (or unbroken) symmetries. The presence of non-linearly realized symmetries then imposes particular relations between the *Wilson coefficients* of the EFT.

on with the analogy, the inflaton perturbations act then as *Goldstone modes* for non-linearly realized time-translations.

This construction has many useful features. For instance, the relevant degrees of freedom are identified as the inflaton and metric perturbations around the Friedmann-Robertson-Walker (FRW) background metric. Such perturbations are directly responsible for the CMB anisotropies and for the structures we observe today in the universe, hence they have a clear and direct connection with physical observables. Furthermore, on large scales, these perturbations remain small, even at late times. Therefore, the action can be simply organized as an expansion in number of perturbations rather than in fundamental fields. In this way, the dynamics is dominated by the quadratic action, with higher orders becoming more important for instance at shorter scales (where indeed non-linearities become stronger).

2.2 EFT action in unitary gauge

The construction of the EFT action follows very straightforwardly from the field content and symmetries of the system we are trying to model. In our case, the DE phase is characterized by an accelerated cosmic expansion. In other words, the metric at the background level is a FRW metric

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\,\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\ , \tag{2.1}$$

where $a(t)$ is the scale factor and $i, j = 1, 2, 3$ are spatial indices (here and in the rest of the thesis we work with mostly-plus spacetime metrics). The requirement of having an accelerated expansion then reads $\ddot{a}/a = H^2 + \dot{H} > 0$, where H is the Hubble function and the dot stands for a time derivative (the limiting case of pure de Sitter corresponds of course to $\dot{H} = 0$).

We are interested in the case in which DE is due to a dynamical scalar field $\phi = \phi_0(t)+\delta\phi(x)$, where $\phi_0(t)$ is the time-dependent background value and $\delta\phi(x)$ are its fluctuations. As already mentioned, this field acts as a clock since it provides a preferred time-slicing of spacetime. Because of this fact, in analogy with particle physics, we then say that time-translations are spontaneously broken by the vacuum expectation value of ϕ. The fluctuations $\delta\phi(x)$ then represent the Nambu-Goldstone boson associated to the broken symmetry $t \to t(x)$.

It is worth to point out that the symmetry-breaking pattern we described is not the only possibility. Indeed, as in the case of solid inflation, different realizations for a quasi-dS phase are possible [67, 68]. A possible drawback in these alternatives is the presence of more fields, that make such alternatives somewhat less minimal.

More practically, given a symmetry-breaking pattern it is possible to construct, in a geometrical way, the action for the metric and DE fluctuations. In our case in particular, it is always possible to choose a particular gauge, called *unitary gauge*, where $\delta\phi(x) = 0$ and ϕ corresponds to our time coordinate. Equivalently, we are foliating spacetime with space-like hypersurfaces of constant ϕ, hence there is a preferred time-like vector n^μ and a spatial metric

$h_{\mu\nu}$ that we define as

$$n_\mu \equiv -\frac{\partial_\mu \phi}{\sqrt{-\partial_\alpha \phi \partial^\alpha \phi}} \ , \qquad h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu \ , \tag{2.2}$$

where n_μ is normalized so that $n_\mu n^\mu = -1$. Given this normalization, then $h_{\mu\nu} n^\nu = 0$. Additionally, in unitary gauge we have $n_\mu = -\delta^0_\mu (-g^{00})^{-1/2}$, where g^{00} is the time-time component of the inverse metric $g^{\mu\nu}$.

2.2.1 General form of the unitary gauge action

In unitary gauge, all the degrees of freedom (in our case a scalar and a massless graviton) live "inside" the metric $g_{\mu\nu}$. This means that in this gauge the action must be geometrical since it can depend only on the different components of the metric. As opposed to more familiar cases, since we have partially fixed the gauge, the action can depend on combinations of the metric that are not manifestly invariant under 4-dimensional diffeomorphisms. For sure the action must be invariant under the residual (unfixed) gauge transformations. Specifically, one is still allowed to perform time-dependent spatial diffs., that infinitesimally act as

$$t \to t \ , \qquad x^i \to x^i + \xi^i(x) \ , \tag{2.3}$$

and are linearly realized. Physically, these transformations correspond to the freedom we have in reparameterizing the coordinates of the equal ϕ-hypersurfaces. Thus, the action is allowed to depend on geometrical invariant quantities describing the spatial hypersurfaces and their embedding in the 4-dimensional geometry. The extrinsic and intrinsic geometry of the hypersurfaces are conveniently described respectively by the extrinsic curvature K^μ_ν, defined as

$$K^\mu_\nu \equiv h^{\mu\rho} \nabla_\rho n_\nu \ , \tag{2.4}$$

and by the 3-dimensional Riemann tensor ${}^{(3)}R_{\mu\nu\rho\sigma}$, constructed out of the spatial metric $h_{\mu\nu}$.[2] From these definitions it follows $K^\mu_\nu n^\nu = 0 = K^\mu_\nu n_\mu$, so that the indices of K^μ_ν are spatial. Similarly, also the spatial curvature has only spatial indices.

[2]The spatial curvature is defined in term of the commutator between spatial derivatives $\mathcal{D}_\alpha$ acting on a generic vector v^γ projected on the hypersurface

$$[\mathcal{D}_\alpha, \mathcal{D}_\beta] v^\gamma = {}^{(3)}R^\gamma_{\ \delta\alpha\beta} v^\delta \ . \tag{2.5}$$

This covariant derivative $\mathcal{D}_\alpha$ acts on generic tensors on the hypersurface as

$$\mathcal{D}_\alpha T^{\beta_1 \dots \beta_n}_{\gamma_1 \dots \gamma_m} \equiv h^\sigma_\alpha h^{\nu_1}_{\gamma_1} \dots h^{\nu_m}_{\gamma_m} h^{\beta_1}_{\mu_1} \dots h^{\beta_n}_{\mu_n} \nabla_\sigma T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} \ . \tag{2.6}$$

Additionally, the 3d Riemann satisfies the Gauss-Codazzi relation

$${}^{(3)}R_{\alpha\beta\gamma\delta} = h^\mu_\alpha h^\nu_\beta h^\sigma_\gamma h^\rho_\delta R_{\mu\nu\sigma\rho} - K_{\alpha\gamma} K_{\beta\delta} + K_{\beta\gamma} K_{\alpha\delta} \ . \tag{2.7}$$

Of course, on top of these quantities, the action can still contain fully covariant objects such as the various contractions of the 4-dimensional Riemann $R_{\mu\nu\rho\sigma}$, and can also involve covariant derivatives ∇_μ. Furthermore, since time-translations are broken, explicit dependencies on time t are allowed (in other words the field $\phi_0(t)$ can also appear). Finally, because of the preferred time direction, we can also allow for an explicit dependence on g^{00} that can indeed be obtained by contracting $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ in unitary gauge.

Overall the gravitational part of the action in unitary gauge can be written, in complete generality, as [55]

$$S = \int \mathscr{L}\left[R_{\mu\nu\rho\sigma}, K_{\mu\nu}, g^{00}, \nabla_\mu, t\right]\sqrt{-g}\,\mathrm{d}^4x \; . \tag{2.8}$$

The form above for practical applications is still too general. Therefore, following the principles of EFTs of Sec. 2.1, we would like to expand (2.8) as a series in the number of perturbations and derivatives. Before doing so, it is however useful to introduce the ADM decomposition of the metric.

2.2.2 ADM decomposition

In most applications, it is very useful to perform an Arnowitt-Deser-Misner (ADM) decomposition of the metric [69]. As will be seen in later chapters, the different variables introduced in this splitting have a clear connection with the degrees of freedom we will be studying. Furthermore, in ADM it is somewhat easier to obtain the perturbative couplings between different fields.

We start by defining the *lapse function* $N \equiv (-g^{00})^{-1/2}$. Let us consider the vector $\partial_t = t^\mu\partial_\mu$, then it is easy to show that $Nn^\mu t_\mu = 1$. Despite this latter result, in general the vectors t^μ and Nn^μ differ. This is because the spatial coordinates, that we call x^i, are not always orthogonal to the spatial hypersurfaces Σ_t of constant t, as can be pictured in fig. 2.1. The difference is measured by the *shift vector* N^μ, which is defined through the relation

$$t^\mu \equiv Nn^\mu + N^\mu \; . \tag{2.9}$$

From this definition and the aforementioned property of N, it follows that $t^\mu N_\mu = 0$ and $n^\mu N_\mu = 0$, so N^μ has only spatial indices.

The lapse N and shift N^i are very useful for writing the different components of the metric:

$$g_{00} = t^\mu t_\mu = -N^2 + N_iN^i \; , \qquad g_{0i} = t^\mu h_{i\mu} = N_i \; , \qquad g_{ij} = h_{ij} \; , \tag{2.10}$$

where the spatial indices are contracted with the metric h_{ij}. In a more compact form, we have

$$\mathrm{d}s^2 = -N^2\,\mathrm{d}t^2 + h_{ij}(\,\mathrm{d}x^i + N^i\,\mathrm{d}t)(\,\mathrm{d}x^j + N^j\,\mathrm{d}t) \; . \tag{2.11}$$

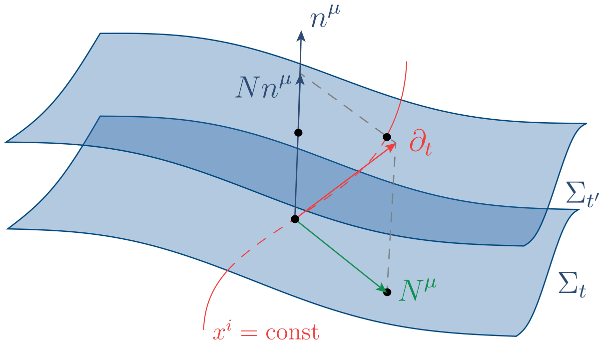


Fig. 2.1 The curves x^i = const. intersect the spatial slices Σ_t and so define the shift vector N^μ and the lapse function N.

Moreover, the inverse components are

$$g^{00} = -\frac{1}{N^2}\,, \qquad g^{0i} = \frac{N^i}{N^2}\,, \qquad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}\,, \tag{2.12}$$

where h^{ij} is the metric inverse of h_{ij}. Another useful property is that the determinant of the metric is decomposed as $\sqrt{-g} = N\sqrt{h}$, where h here stands for the determinant of the spatial metric. Finally, the extrinsic curvature (2.4) decomposes to

$$K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - \mathcal{D}_i N_j - \mathcal{D}_j N_i\right)\,, \tag{2.13}$$

where the dot stands for the time derivative.

2.2.3 Expansion of the action

At this point we are ready to expand the action (2.8) in perturbations around a given FRW background. We define $\delta g^{00} \equiv g^{00} + 1$ and $\delta K^\mu_\nu \equiv K^\mu_\nu - H\delta^\mu_\nu$. These two operators vanish when evaluated in the FRW background metric (2.1), so that they start at linear order in perturbations. Also, they are tensors under the time-dependent spatial diffs. (2.3). This is hardly a trivial property and follows from the high degree of symmetry of the background.[3] A similar definition for perturbations of the Riemann can be done as well, but it will not be of particular interest for us. The 3d Riemann instead starts already at linear order in perturbations since the spatial curvature is assumed to vanish.

[3]For less symmetric backgrounds, similar decompositions are possible at the price of losing the residual gauge invariance [66]. The relation to other gauges is then less obvious.

Schematically, the expansion can be written as

$$
\begin{aligned}
S_{\rm EFT} = \int \Big[& \frac{M_\star^2}{2} f(t) R - \Lambda(t) - c(t) g^{00} \\
& + \frac{1}{2} m_2^4(t) (\delta g^{00})^2 + \frac{1}{3} M_3^4(t) (\delta g^{00})^3 \\
& - \frac{1}{2} m_3^3(t) \delta K \delta g^{00} - \bar{M}_2^2(t) \delta K^2 - \bar{M}_3^2(t) \delta K^\mu_{\ \nu} \delta K^\nu_{\ \mu} + \dots \Big] \sqrt{-g}\, \mathrm{d}^4 x \,,
\end{aligned}
\tag{2.14}
$$

where the dots stand for operators at higher order in perturbations or derivatives. This expansion is not the most general but allows us to outline the main feature of the EFT.

First, each term contains a time-dependent coefficient that, as explained above, is allowed when the time translations are broken.

It is possible to notice that there are only three operators (those in the first line) contributing at the background and linear level. This is general since other possible linear terms can be re-written, using integrations by parts, in terms of these operators and higher-order contributions [55].

Among the operators contributing at linear level, the first represents the usual Einstein-Hilbert term of GR, but with a time-dependent Planck mass, characterized by $f(t)$. Of course, one is always allowed to make a field redefinition of the metric in such a way to remove this time dependence, without changing the physics. This corresponds to moving from the Jordan to the Einstein frame. To do so, however, we need first to specify the coupling between the metric and matter. In our case, we prefer to limit ourselves with a minimal coupling in the Jordan frame, or in other words, we assume the weak equivalence principle (WEP). The matter action for a generic matter field ψ_m is then of the form $S_m[g_{\mu\nu}, \psi_m]$. Notice that in Jordan frame the stress-energy tensor for matter $T^{(m)}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$ is covariantly conserved, while in other frames this would not be the case.

Also, the background equations can be obtained by requiring that the tadpoles in (2.14) vanish. This requirement then fixes the functions $c(t)$ and $\Lambda(t)$. By assuming matter is in the form of a perfect fluid, $T^{\mu\,(m)}_{\ \nu} = \mathrm{diag}(-\rho_m, p_m, p_m, p_m)$, then the tadpoles vanish for

$$
c(t) = \frac{1}{2} \left[M_\star^2 \left(-\ddot{f} + \dot{f} H - 2 f \dot{H} \right) - \rho_m - p_m \right] \,, \tag{2.15}
$$

$$
\Lambda(t) = \frac{1}{2} \left[M_\star^2 \left(\ddot{f} + 5 \dot{f} H + 2 f \dot{H} + 6 f H^2 \right) - \rho_m + p_m \right] \,. \tag{2.16}
$$

Since $T^{(m)}_{\mu\nu}$ is covariantly conserved ($\nabla_\mu T^{\mu\nu\,(m)} = 0$), then we also have $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$.[4] This shows that indeed the parameters $c(t)$ and $\Lambda(t)$ are fixed in terms of the desired unperturbed

[4] The equivalence between the cancellation of the tadpoles and the usual Friedmann equations can be made more transparent. We can vary $S_{\rm EFT}$ with respect to $g^{\mu\nu}$ and define a fictitious dark energy stress-energy tensor $T^{(\rm DE)}_{\mu\nu}$ as follows

$$
\frac{2}{\sqrt{-g}} \frac{\delta S_{\rm EFT}}{\delta g^{\mu\nu}} \equiv M_\star^2 G_{\mu\nu} - T^{(\rm DE)}_{\mu\nu} \,, \tag{2.17}
$$

FRW history. On the other hand, the remaining time-dependent couplings in (2.14) are in principle left generic and independent on the background. Of course, their time dependence can be related to background quantities in some specific models.

To make the overall constriction more concrete, we can consider how (2.14) arises from a specific covariant model of a scalar ϕ. Let us take the Lagrangian $\mathscr{L} = X^2/\tilde{\Lambda}^4$, where $X \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ and $\tilde{\Lambda}$ is some energy scale. This model does not have a well-defined Minkowski limit, meaning that the fluctuations around the trivial vacuum expectation value $\phi_0 = 0$ are strongly coupled (this theory does not have a standard kinetic term for ϕ). However, $\mathscr{L}$ is perfectly fine around a time-dependent background $\phi_0(t)$ (for $\tilde{\Lambda}^2 > 0$), and indeed it describes a fluid of radiation ($\rho_\phi = p_\phi/3$) as it can be checked by computing its stress-energy tensor. This is perfectly consistent: we cannot require the fluid to make sense around $\phi_0 = 0$, because in this limit the fluid is not even there.

By adding to $\mathscr{L}$ the standard Einstein-Hilbert Lagrangian for gravity, this model can be relevant for DE or inflation, albeit with very different values for $\tilde{\Lambda}$. It is useful to go to the unitary gauge by setting $\phi = \phi_0(t)$. In doing so we see that

$$\mathscr{L} = \frac{\dot{\phi}_0^4}{\tilde{\Lambda}^4}(g^{00})^2 = -\frac{\dot{\phi}_0^4}{\tilde{\Lambda}^4} - 2\frac{\dot{\phi}_0^4}{\tilde{\Lambda}^4}g^{00} + \frac{\dot{\phi}_0^4}{\tilde{\Lambda}^4}(\delta g^{00})^2 \, . \tag{2.19}$$

Therefore, in terms of the operators in eq. (2.14) this model reduces to

$$\Lambda(t) = \frac{\dot{\phi}_0^4}{\tilde{\Lambda}^4} \, , \qquad c(t) = m_2^4(t) = 2\Lambda(t) \, . \tag{2.20}$$

Thanks to this simple example, we see how general the EFT action really is. Indeed, it contains even cases one would not naively consider starting from a covariant approach [70].

2.3 Action for the scalar mode and Stueckelberg procedure

The action S_{EFT}, as it stands, is defined in a specific gauge. In many applications it is however preferable to move to other gauges, where for instance the dynamics of the scalar mode is easier to treat, at least at high energies. Given that gauge symmetries are never truly broken, it is possible to perform a “broken” transformation without changing the physics and move to away from the unitary gauge. Moreover, full gauge invariance can be reintroduced by promoting the gauge parameter to a field transforming non-linearly under the broken group.

This procedure, that goes under the name of *Stueckelberg procedure* (or trick), is analogous to what happens for massive, interacting, gauge bosons. In these cases, at high enough energies,

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor. Then, varying the full action $S_{\text{EFT}} + S_m$ we obtain the Einstein equations $M_\star^2 G_{\mu\nu} = T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(\text{DE})}$ that, at the background level, give the Friedmann equations

$$3M_\star^2 H^2 = \rho_m + \rho_{\text{DE}} \, , \qquad 6M_\star^2(\dot{H} + H^2) = \rho_m + 3p_m + \rho_{\text{DE}} + 3p_{\text{DE}} \, , \tag{2.18}$$

where ρ_{DE} and p_{DE} represent the energy density and pressure of DE, and are given in terms of $c(t)$ and $\Lambda(t)$.

the longitudinal mode decouples from the transverse modes thanks to the equivalence theorem for massive gauge bosons. To illustrate this example in a minimal way, we can consider a massive Proca field with a quartic self-interaction in flat space

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu - \frac{g^2}{4}(A_\mu A^\mu)^2 \,, \tag{2.21}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength, m is the mass and g is a dimensionless coupling that we assume smaller than unity. Here both the mass term and the quartic self-interaction do break the U(1) gauge symmetry so that A_μ propagates two transverse and one longitudinal modes. However, we can always restore gauge invariance by mean of the Stueckelberg procedure. To do so, we perform a broken transformation $A_\mu \to A_\mu + \partial_\mu \xi$ and then promote ξ to a field $\xi \to -\pi$. At this point gauge invariance is recovered if π transforms as $\pi \to \pi - \xi$. Clearly, the gauge $\xi = -\pi$ corresponds to the unitary gauge. In any case, once π is reintroduced the Lagrangian becomes

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}(D_\mu \pi)^2 - \frac{g^2}{4}(D_\mu \pi D^\mu \pi)^2 \,, \tag{2.22}$$

where $D_\mu \pi \equiv \partial_\mu \pi - A_\mu$ is the covariant derivative for π; now the Lagrangian is manifestly invariant under U(1).

The field π can be canonically normalized as $\pi_c = m\,\pi$. We can also notice that there is a quadratic kinetic mixing between A_μ and π of the form $m^2 \partial_\mu \pi A^\mu = m\,\partial_\mu \pi_c A^\mu$. This means that at energies larger than m we can neglect the mixing so that the longitudinal and transverse modes are effectively decoupled.

Furthermore, π becomes strongly-coupled at energies of the order m/g, which are parametrically larger than m for $g \ll 1$. The advantage of this formulation is now clear, since for energies E in the range $m \ll E \ll m/g$ we can study the longitudinal sector focusing only on π, and this mode will also dominate in the scattering amplitudes. Notice that we have not taken into account factors of 4π in the estimate for the scale of strong coupling. For simplicity we will neglect them also in future estimates.

The same procedure can be applied similarly to the action (2.14). Let us consider for simplicity a particular operator $\int c(t) g^{00} \sqrt{-g}\, \mathrm{d}^4 x$. Under a broken diff. $t \to \tilde{t} = t + \xi^0(x)$, $x^i \to \tilde{x}^i = x^i$ this operator, after changing integration variable from x to $\tilde{x}$, becomes

$$\int c(t - \xi^0(x)) \partial_\mu (t - \xi^0) \partial_\nu (t - \xi^0) g^{\mu\nu}(x) \sqrt{-g}\, \mathrm{d}^4 x \,, \tag{2.23}$$

As in the case of the vector boson (2.21), we introduce π by making the substitution $\xi^0 \to -\pi$. In doing so, the action becomes manifestly invariant under the full 4d diffs., with π transforming non-linearly as

$$\pi \to \pi - \xi^0 \,. \tag{2.24}$$

By going to sufficiently high energies we can simplify the mixing between the metric $g_{\mu\nu}$ and π by retaining only the leading-order contributions. The energies of this expansion depend on the operators we include in (2.14), but typically they are of order H. Therefore, for studying physics at short scales this is often a reliable expansion.

From the example of this operator it is easy to understand how to systematically reintroduce π for a generic operator of eq. (2.14). Indeed, for a generic function of time $f(t)$ we substitute $f(t) \to f(t+\pi)$, and for the components of the inverse metric we make the replacements

$$g^{00} \to (1+\dot{\pi})^2 g^{00} + 2(1+\dot{\pi})\partial_i \pi g^{0i} + \partial_i \pi \partial_j \pi g^{ij} \, , \tag{2.25}$$

$$g^{0i} \to (1+\dot{\pi}) g^{0i} + \partial_k \pi g^{ki} \, , \tag{2.26}$$

$$g^{ij} \to g^{ij} \, , \tag{2.27}$$

which follow from the transformation rules for $g^{\mu\nu}$ under time diffs. and are exact to all orders in π. To obtain the Stueckelberg transformation for $g_{\mu\nu}$ instead, we can first obtain the Jacobian and its inverse, for the broken transformations. They read

$$\left(\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\right) = \begin{pmatrix} 1+\dot{\pi} & \partial_i \pi \\ \partial_i \pi & \delta_{ij} \end{pmatrix} \, , \qquad \left(\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\right) = \frac{1}{D} \begin{pmatrix} 1 & \partial_i \pi \\ \partial_i \pi & D\delta_{ij} + \partial_i \pi \partial_j \pi \end{pmatrix} \, , \tag{2.28}$$

where $D \equiv 1 + \dot{\pi} - \delta_{ij}\partial_i \pi \partial_j \pi$. From the latter matrix it is straightforward to obtain the exact replacements for $g_{\mu\nu}$. It is however more useful for our applications to use transformations up to a given order in π or number of perturbations. In terms of the ADM variables (2.11), we have

$$N \to N(1-\dot{\pi}) + \mathcal{O}(2) \, , \tag{2.29}$$

$$N^i \to N^i + N^2 h^{ik} \partial_k \pi + \mathcal{O}(2) \, , \tag{2.30}$$

$$h_{ij} \to h_{ij} + \mathcal{O}(2) \, , \tag{2.31}$$

where $\mathcal{O}(2)$ stands for terms at least quadratic in perturbations.

Finally, from these expressions and the definitions of the extrinsic curvature (2.13) and the 3d Riemann we have

$$\delta K^i_j \to \delta K^i_j - \dot{H}\pi \delta^i_j - N h^{ik} \partial_k \partial_j \pi + \mathcal{O}(2) \, , \tag{2.32}$$

$$R \to R + \frac{4}{a^2} \nabla^2 \pi + \mathcal{O}(2) \, , \tag{2.33}$$

where we define $\nabla^2 \equiv \delta_{ij}\partial_i \partial_j$. When needed, we will consider also higher orders in these formulas as we will explain later on.

In complete analogy with the U(1) case, the Stueckelberg procedure allows us to also investigate the mixing between the metric and π. To illustrate this point, we can consider the operator characterized by $c(t)$, letting the operators in the second and third line of eq. (2.14) to zero with also $f(t) = 1$ and no matter ($\rho_m = 0 = p_m$). In this case one has a mixing between $\dot{\pi}$ and δg^{00} of the form $M_\star^2 \dot{H}\, \dot{\pi} \delta g^{00}$. The canonical normalization for π and the metric can be also read from eqs. (2.14) and (2.25) and are around $\pi_c \sim M_\star \dot{H}^{1/2} \pi$, $\delta g_c^{00} \sim M_\star \delta g^{00}$. After rewriting the mixing in terms of the canonical fields, we see that π_c demixes from δg_c^{00} at energies above $\sim \dot{H}^{1/2}$. When considering additional operators in eq. (2.14) this conclusion might change and a more detailed analysis of the mixing is necessary.

2.4 Most general EFT of Dark Energy

The construction outlined in the previous sections shows how the EFT of DE can be constructed and how it is connected to some exemplifying models. Most of the time, and especially for phenomenological considerations, it is best to employ a systematic expansion of the action. In general, from an EFT perspective, we can always do a truncation in the number of perturbations and derivatives so to have a reasonable perturbative expansion.

It is also very convenient to impose an additional requirement and to further limit the operators we consider. A reasonable requirement for instance is for the theory to be trustworthy even in regimes where classical non-linearities become important (the interaction terms for cosmological perturbations become as important as the quadratic ones). In particular, if the equations of motion involve more than two (time) derivatives per field typically the system becomes unstable due to the appearance of so-called *Ostrogradski ghosts* [71].

As a prototypical example of this kind of instability, let us consider the following Lagrangian for a scalar with higher derivatives [72]

$$\mathscr{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2M^2}(\Box \phi)^2 - V(\phi) \,, \tag{2.34}$$

where M is some energy scale and $\Box \phi \equiv \eta^{\mu\nu}\partial_\mu \partial_\nu \phi$. This system needs more than two initial conditions for setting up an initial-value problem (also $\ddot{\phi}$ and $\dddot{\phi}$ need to be specified as initial conditions), suggesting the presence of an additional degree of freedom on top of ϕ. The new field, that we call σ, can be made manifest by considering the Lagrangian

$$\mathscr{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - \partial_\mu \phi \partial^\mu \sigma - \frac{M^2}{2}\sigma^2 - V(\phi) \,. \tag{2.35}$$

Indeed, by integrating-out σ one exactly recovers eq. (2.34). The Lagrangian (2.35) contains a kinetic mixing between the fields in eq. (2.35), that we can diagonalize by replacing $\phi = \tilde{\phi} - \sigma$. We then get

$$\mathscr{L} = -\frac{1}{2}(\partial_\mu \tilde{\phi})^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{M^2}{2}\sigma^2 - V(\tilde{\phi}, \sigma) \,. \tag{2.36}$$

The wrong sign of the kinetic term for σ alone does not give any problem at the level of the equations of motion (as long as $M^2 > 0$). Once interactions with other fields such as $\tilde{\phi}$ are considered however, the system becomes unstable since the energy is unbounded from below. Considering quantum mechanics, the situation worsen: the presence of the ghost implies that the vacuum is unstable due to the spontaneous creation of both positive and negative energy states. Moreover, because of boost invariance, final configurations with different momenta contribute the same amount to the decay rate. The rate of instability thus formally diverges (independently on the specific form of the interaction), since the possible final states are infinitely degenerate.

However, this scenario is not as bleak as it first appears. By looking at eq. (2.36) we see that, as long as we stay in regimes where the energy is below M, we do not excite the ghost. As a confirmation of this, the operator in eq. (2.34) with higher-derivatives is generically expected to appear once quantum corrections are considered (it is generated for example by integrating-out massive fields around M), meaning that in this setting the ghost is an artefact of the perturbative expansion and cannot be trusted as being a legitimate additional state.

In more general situations, Lagrangians also contain non-linear higher-derivative terms. In these cases, the theory will break down (the ghost will be excited) once classical non-linearities become large, even when the energies are far below the typical scale suppressing higher-derivative operators. From an EFT point of view, we can then say that theories with ghosts generically have a smaller regime of validity (smaller ϕ at which the EFT breaks down) than naively expected but are not inherently problematic from the get-go.

For cosmological applications, we are often facing non-linear regimes for the fields, and therefore we prefer to consider theories that feature no Ostrogradski ghosts in the regimes of our interest. Although this is not, strictly speaking, a direct implication it would be rather strange if DE were to change description at short (astrophysical) scales.

Given these motivations we can then focus on theories with at most second-order equations of motion. It has been shown that the most general EFT of DE action with this property at the non-linear level has the following form [73, 74]

$$
\begin{aligned}
S_{\rm EFT} = \int \Bigg[& \frac{M_\star^2}{2} f(t) R - \Lambda(t) - c(t) g^{00} \\
& + \frac{m_2^4(t)}{2} (\delta g^{00})^2 - \frac{m_3^3(t)}{2} \delta K \delta g^{00} - m_4^2(t) \delta \mathcal{K}_2 + \frac{\tilde{m}_4^2(t)}{2} \delta g^{00\,(3)}R \\
& - \frac{m_5^2(t)}{2} \delta g^{00} \delta \mathcal{K}_2 - \frac{m_6(t)}{3} \delta \mathcal{K}_3 - \tilde{m}_6(t) \delta g^{00} \delta \mathcal{G}_2 - \frac{m_7(t)}{3} \delta g^{00} \delta \mathcal{K}_3 \Bigg] \sqrt{-g}\, \mathrm{d}^4 x \, .
\end{aligned}
\tag{2.37}
$$

Here we define

$$
\begin{aligned}
\delta \mathcal{K}_2 &\equiv \delta K^2 - \delta K^\nu_{\ \mu} \delta K^\mu_{\ \nu} \, , \\
\delta \mathcal{G}_2 &\equiv \delta K^\nu_{\ \mu}\, {}^{(3)}R^\mu_{\ \nu} - \delta K\, {}^{(3)}R/2 \, , \\
\delta \mathcal{K}_3 &\equiv \delta K^3 - 3\, \delta K \delta K^\nu_{\ \mu} \delta K^\mu_{\ \nu} + 2\, \delta K^\nu_{\ \mu} \delta K^\mu_{\ \rho} \delta K^\rho_{\ \nu} \, .
\end{aligned}
\tag{2.38}
$$

This action is a generalization of eq. (2.14), where particular higher-order operators have been included. Additionally, eq. (2.37) is obtained by retaining the leading operators in the number of spatial derivatives, that are the most relevant for the non-linear regimes of structure formation and screening. Subleading operators in spatial derivatives have typically more powers of δg^{00}, and are often negligible even for GW physics. We will consider them separately when needed.

Notably, eq. (2.37) has been shown to be equivalent to *Horndeski theories* [75, 76] which are indeed the most general covariant scalar-tensor theories with at most second order-equations of motion. Furthermore, the case $m_4^2 \neq \tilde{m}_4^2$, $m_6 \neq \tilde{m}_6$ corresponds to an extension, called *Beyond Horndeski* or Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theories [77–79]. As noted in [73] at the quadratic level, for $m_4^2 \neq \tilde{m}_4^2$ the equations of motion for the metric and the scalar can, in some gauges, appear to be higher-order. However, when re-casted in terms of the true propagating degrees of freedom they reduce to second order, thus avoiding Ostrogradski ghosts. This extends to cubic level for $m_6 \neq \tilde{m}_6$.

Other types of operators, belonging to the so-called Degenerate Higher-Order Scalar-Tensor (DHOST) class, are discussed in App. A. These models avoid the presence of Ostrogradski ghosts by imposing degeneracy conditions in the action and so propagate a single scalar mode [80, 81].

Although equivalent, this covariant formulation is more suitable for certain types of computations (especially for computing quantum corrections) and for this reason we include it in the following sections.

2.5 Weakly Broken Galileons

The requirement of having second-order equations of motion, as already remarked in the previous section, does not seem to stem from some deep physical principle. On the contrary, theories are usually understood and constructed in terms of symmetries. Following the Wilsonian approach, the reason behind this is simply that quantum corrections do generate all possible operators compatible with a given set of symmetries (unless the symmetry is anomalous). Consequently, operators leading to higher-order equations are generically present unless they are incompatible with the symmetries. The case of Horndeski theories is no exception, and loops generically spoil the classical tuning needed to avoid Ostrogradski ghosts, unless certain (approximate) symmetries are enforced.

It is therefore advisable to build theories starting from symmetry arguments, and in the case at hand for DE we will be dealing with internal symmetries for the scalar field, that turn out to be approximate once gravity is taken into account. Such breaking can completely spoil the structure of the action unless some peculiar non-minimal couplings with gravity are present.

The role of quantum corrections in the EFT of DE is extremely important for the phenomenology of GW observations since in these situations one starts probing the irrelevant operators generated at loop level.

2.5.1 Galileons in flat space

Concerning DE, the scalar field ϕ is required to have a mass small enough so to have significant effects on cosmological scales. This feature can be re-stated in terms of symmetries by requiring ϕ to be shift-invariant, at least approximately. We are going to consider instead a generalized form of shift symmetry that turns out to have remarkable quantum features. Such transformation is called *Galileon transformation* and acts as

$$\phi \to \phi + c + b_\mu x^\mu \ , \tag{2.39}$$

where b_μ and c are constants and x^μ are the flat coordinates in Minkowski space. The name for the transformation comes of course from the analogous Galilean transformations in Newtonian mechanics, where the velocity of a particle $\dot{x}^i$ shifts upon changing reference frame $\dot{x}^i \to \dot{x}^i + v^i$.

Theories invariant under (2.39) in the absence of gravity go under the name of *Galileons*. Clearly, a standard kinetic term for ϕ is invariant, up to a boundary term, as can be realized after integrating by parts. Another invariant operator is the tadpole ϕ, but we will not consider its role in what follows. Other trivially invariant operators are constructed with two or more derivatives acting on ϕ, such as $(\Box\phi)^2$. Terms containing less than two derivatives per field are at first sight forbidden. This is actually not correct, and remarkably there is a finite set of operators invariant under (2.39) up to boundary terms.[5] Overall, in four dimensions the most general Lagrangian non-trivially invariant under Galileon transformation takes the form

$$\mathscr{L}^{\,\mathrm{Gal}} = -\frac{1}{2}(\partial_\mu\phi)^2 + \sum_{i=3}^{5} \frac{c_i}{\Lambda_3^{3(i-2)}} \mathscr{L}_i^{\,\mathrm{Gal}} \ , \tag{2.40}$$

where

$$\begin{aligned} \mathscr{L}_3^{\,\mathrm{Gal}} &= (\partial_\mu\phi)^2\,[\Phi] \ , \\ \mathscr{L}_4^{\,\mathrm{Gal}} &= (\partial_\mu\phi)^2\left([\Phi]^2 - [\Phi^2]\right) \ , \\ \mathscr{L}_5^{\,\mathrm{Gal}} &= (\partial_\mu\phi)^2\left([\Phi]^3 - 3\,[\Phi][\Phi^2] + 2\,[\Phi^3]\right) \ . \end{aligned} \tag{2.41}$$

Here we define $\Phi_{\mu\nu} \equiv \partial_\mu\partial_\nu\phi$ and square brackets stand for traces of this matrix and of its powers (for example $[\Phi] = \Box\phi$). The c_is are generic coefficients and the energy scale Λ_3 is up to now arbitrary. The latter scale represents the scale of strong coupling for Galileons, hence one expects heavy degrees of freedom to enter at most at Λ_3. Moreover, the operators $\mathscr{L}_3^{\,\mathrm{Gal}}$, $\mathscr{L}_4^{\,\mathrm{Gal}}$ and $\mathscr{L}_5^{\,\mathrm{Gal}}$ are often referred respectively as cubic, quartic and quintic Galileons.[6]

The cubic Galileon interaction was first discovered in the context of the Dvali-Gabadadze-Porrati (DGP) model [83]. This is a higher-dimensional model where Standard Model fields are

[5] It is argued in [82] that this can be understood as the fact that these operators are Wess-Zumino terms for broken spacetime symmetries.

[6] In d spacetime dimension there are $d-1$ non-trivial Galileons (not counting the tadpole and the kinetic term). For instance, for $d=2$ only the cubic Galileon remains, while the other reduce to total derivatives.

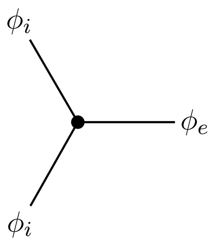
confined on a four-dimensional brane embedded in a 5d bulk spacetime. The main motivation for such a construction is that it allows for self-accelerating cosmological solutions (solutions where the late-time acceleration is not due to a cosmological constant).[7] In DGP, the brane-bending mode is described by a scalar ϕ with cubic Galileon self-interactions [84].

Given the peculiar symmetry of ϕ, the cubic Galileon Lagrangian was extended to its general form (2.40) in [90]. Remarkably, Galileon interactions also appear in the context of de Rham-Gabadadze-Tolley (dRGT) massive gravity, where ϕ describes the longitudinal mode of the graviton [91]. In the context of cosmology, Galileons have found applications most notably for cosmic acceleration [92, 93], inflation [94–97], alternatives to inflation and violations of the null energy condition [98–100]. Finally, in more formal aspects of Quantum Field Theory (QFT) particular versions of the Galileons play a role for instance in the proof of the *a-theorem* [101, 102]. Given this vast range of applications and the good properties we are going to discuss, Galileons have been considered almost as synonymous for well-behaved higher-derivative theories.

The Lagrangian (2.40) contains on average more than one derivative per field, but the equations of motion are still second order. As advertised the Galileon symmetry enforces this property, provided one can neglect trivially invariant operators with more derivatives. One can argue that this is the case thanks to the peculiar renormalization properties of (2.40). Note however that Galileon invariance alone *does not* guarantee second-order equations of motion.

In [84] it was first noted that the cubic Galileon does not get renormalized at any loop order. On the other hand, operators with more derivatives are generated via quantum corrections. Such property extends to the full $\mathscr{L}^{\text{Gal}}$ in (2.40) and goes under the name of *non-renormalization theorem* for Galileons [103]. The proof follows essentially from integrations by parts, as we can see as an example for $\mathscr{L}_3^{\text{Gal}}$.

Let us consider a generic one-particle irreducible (1PI) Feynman diagram. We can focus on a generic external leg with associated external field ϕ_e, while the two internal legs are indicated as ϕ_i. We want to show there are at least two derivatives acting on ϕ_e, so that $\mathscr{L}_3^{\text{Gal}}$ does not get radiative corrections. The only worrisome part of the diagram is then



$$\supset 2\,\partial_\mu\phi_e\partial^\mu\phi_i\Box\phi_i = 2\,\partial_\mu\phi_e\partial_\nu\left[\partial^\mu\phi_i\partial^\nu\phi_i - \frac{1}{2}\eta^{\mu\nu}(\partial_\sigma\phi_i)^2\right]\,. \tag{2.42}$$

[7] This self-accelerating branch is however plagued by ghost-like instabilities [84–89].

Because of momentum conservation at the vertex we can move the derivative ∂_ν to ϕ_e, thus showing that only the innocuous term $\sim \partial^2\phi_e$ is generated.

As expected, trivially-invariant operators are generated at the scale Λ_3, hence Ostrogradski ghosts do not appear as long as we stay within the Galileon EFT. Furthermore, the structure of each $\mathscr{L}_i^{\rm Gal}$ in (2.40) is preserved at the quantum level, avoiding tuning in the coefficients of the different terms.

One might still worry about the fate of the Galileon symmetry in the presence of small symmetry-breaking operators. As a particular example, we can consider the effect of operators preserving the shift symmetry for ϕ but only having one derivative per field. For instance, let us focus on the simplest case

$$\mathscr{L} = -\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{\Lambda_3^3}(\partial_\mu\phi)^2\Box\phi + \frac{1}{\Lambda_2^4}(\partial_\mu\phi)^4 \,, \tag{2.43}$$

where we assume $\Lambda_2 \gg \Lambda_3$ so that the new operator is a slight correction to the cubic Galileon Lagrangian. One can worry in this situation that eq. (2.43) is not radiatively stable anymore, since operators of the form $(\partial_\mu\phi)^{2n}$ are expected to appear suppressed by the scale Λ_2. However, because of the property (2.42) one cannot obtain such terms by using the cubic Galileon vertex and only quartic vertices can be used to obtain such operators. If loops are cut-off at the scale Λ_3, then the corrections $(\partial_\mu\phi)^{2n}$ will be suppressed by an additional ratio $(\Lambda_3/\Lambda_2)^4$, making such terms negligible in comparison with the tree-level action.

2.5.2 Galileons in curved spacetime

The Galileon symmetry of eq. (2.39), defined in terms of the flat coordinates x^μ, cannot be implemented in an arbitrary spacetime. A quick way to see this is by noting that in spacetimes that are not flat in general there are no covariantly conserved vectors b_μ. Therefore, as soon as gravity is made dynamical the symmetry is lost. Motivated by the previous discussion on the weak breaking of the symmetry, and the fact that gravitational interactions are highly suppressed by $M_{\rm Pl}$, one can rightfully expect the breaking to be small even in this case.

However, the situation is not completely trivial, and if Galileons are minimally coupled with the metric this property does not hold, with the symmetry-breaking operators generated relatively close to Λ_3. In this sense, the hierarchy between Galileon and symmetry-breaking terms would be greatly lost. Remarkably, there exists a peculiar non-minimal coupling of ϕ to $g_{\mu\nu}$ such that loop corrections do preserve the Galileon structure of the action. The price to pay is that operators of the form $(\partial_\mu\phi)^{2n}$, for n generic, are still generated (even in their absence in the classical action), but fortunately they are suppressed by a scale parametrically larger than Λ_3, that lies in between Λ_3 and $M_{\rm Pl}$. Since the dominant interactions remain the Galileon ones, and symmetry-breaking operators remain confined at higher scales, these theories are dubbed *Weakly Broken Galileon* (WBG) theories [104]. This non-trivial result is the remnant

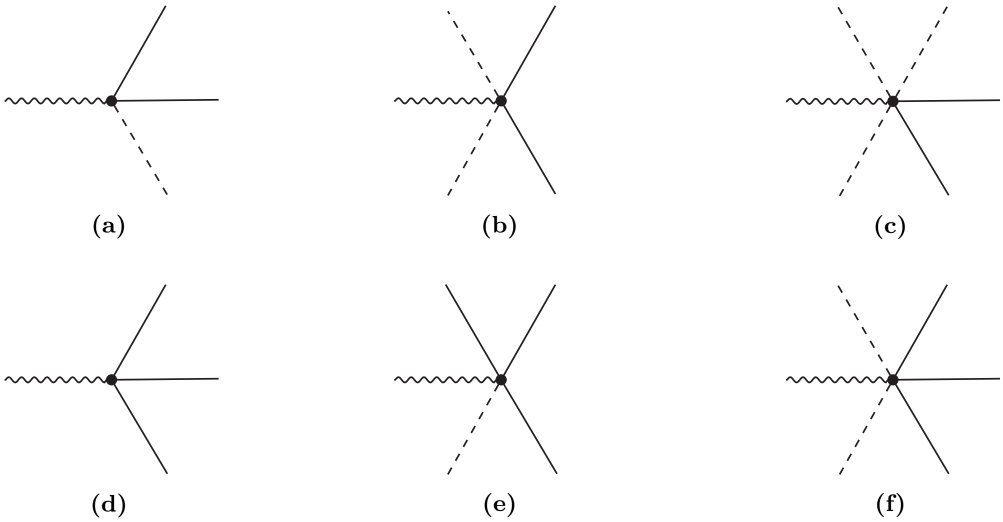


Fig. 2.2 Vertices of (2.40) minimally coupled to the gravity. Gravitons are indicated by wavy lines, while solid lines represents $\partial\phi$ and dashed lines $\partial^2\phi$.

of the non-renormalization theorem in flat space, but it also crucially depends on the couplings between the metric and the scalar.

We can better motivate the conclusions above by sketching the proof given in [104]. Let us first consider $\mathscr{L}^{\,\mathrm{Gal}}$ minimally coupled to gravity. The coupling with the metric is then simply obtained by replacing partial derivatives with covariant ones, and $\eta_{\mu\nu}$ with $g_{\mu\nu}$. We also expand the metric around Minkowski space $g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}$, where the fluctuations $\gamma_{\mu\nu}$ can be canonically normalized as $\gamma^c_{\mu\nu} = M_{\mathrm{Pl}}\gamma_{\mu\nu}$. By expanding the covariant derivatives acting on ϕ, one can pick up interactions containing one derivative on $\gamma^c_{\mu\nu}$ from the Christoffel symbols schematically as $\nabla^2\phi \supset M_{\mathrm{Pl}}^{-1}\partial\gamma^c\partial\phi$. The corresponding operators are the leading ones breaking Galileon invariance (subleading ones contain more powers of $\gamma^c_{\mu\nu}$ and so are further suppressed by M_{Pl}).

A quick inspection at the structure of eq. (2.40) naively suggests there are various terms of this type, with up to three $(\partial\phi)$-legs, that schematically are of the form given in fig. 2.2. The first column (fig. 2.2a and fig. 2.2d) is obtained from the operators of $\mathscr{L}_3^{\mathrm{Gal}}$, the second (fig. 2.2b and fig. 2.2e) in $\mathscr{L}_4^{\mathrm{Gal}}$ and the third (fig. 2.2c and fig. 2.2f) in $\mathscr{L}_5^{\mathrm{Gal}}$.

It is easy to realize that the most worrisome vertices for the fate of the Galileon symmetry are those in the second line of fig. 2.2. This because they contain more $(\partial\phi)$-legs for fixed number of $\gamma^c_{\mu\nu}$'s (and thus M_{Pl}), so we should first focus on them.

Operators of the form

$$\frac{(\partial_\mu\phi)^{2n}}{\Lambda^{4(n-1)}}\,, \tag{2.44}$$

are generated by the set of 1PI diagrams whose external legs are attached to the vertices of fig. 2.2. We see that by inserting enough times the troublesome vertex with three $(\partial\phi)$-legs we can generate the terms (2.44) suppressed by the very low scale $\Lambda = \left(\Lambda_3^5 M_{\rm Pl}\right)^{1/6}$. On the other hand, the cubic Galileon does not actually give any diagram with three $(\partial\phi)$s and one graviton, because one can always move the derivative acting on $\gamma^c_{\mu\nu}$ to one of the other fields so that the vertex fig. 2.2d can be written as fig. 2.2a. Symmetry breaking operators in the case of $\mathscr{L}_3^{\rm Gal}$ then appear at the much larger scale $\Lambda = (M_{\rm Pl}/\Lambda_3)\left(\Lambda_3^3 M_{\rm Pl}\right)^{1/4}$.

Except for the cubic Galileon, the Galileon symmetry is, in the case of minimal coupling, greatly spoiled. This can be realized by noticing for instance that, in cosmological applications, one often needs at least mild Galileon non-linearities ($\partial^2\phi_0 \sim \Lambda_3^3$) and at the same time ϕ should dominate the present-time evolution ($H_0^2 M_{\rm Pl}^2 \sim (\partial\phi_0)^2$). From these requirements we see that, in this regime, operators of the form (2.44) generated by $\mathscr{L}_4^{\rm Gal}$ and $\mathscr{L}_5^{\rm Gal}$ would dominate over the classical contributions and the theory could not be considered in any sense a weakly broken Galileon. In the case of $\mathscr{L}_3^{\rm Gal}$ instead, the scale is high enough so that the effect of the operators (2.44) remains mild.

Given this result, a natural question arises: is it possible to find a peculiar coupling with gravity such that, in analogy with $\mathscr{L}_3^{\rm Gal}$, the remaining vertices in the second line of fig. 2.2 vanish? The question has been addressed in [104] by performing a detailed inspection of the vertices and has a positive answer. The optimal non-minimal coupling that preserves Galileon symmetry as much as possible is of the form

$$\begin{aligned}
\mathscr{L}_3^{\rm cov} &= \sqrt{-g}\,\mathscr{L}_3^{\rm Gal}\,, \\
\mathscr{L}_4^{\rm cov} &= \sqrt{-g}\left[(\partial_\mu\phi)^4 R - 4\,\mathscr{L}_4^{\rm Gal}\right]\,, \\
\mathscr{L}_5^{\rm cov} &= \sqrt{-g}\left[(\partial_\mu\phi)^4\, G^{\mu\nu}\Phi_{\mu\nu} + \frac{2}{3}\mathscr{L}_5^{\rm Gal}\right]\,,
\end{aligned} \tag{2.45}$$

where from now on $\Phi_{\mu\nu}$ is constructed with covariant derivative, and the overall Lagrangian is a linear combination of these terms with the same energy scales as in (2.40). The theory above actually coincides with the *Covariant Galileon* [105], a covariantization of the flat-space Galileon that preserves the second-order equations for both ϕ and the metric.[8] This apparently coincidental result can be partially understood by noticing that the dangerous terms in fig. 2.2 have a derivative acting on $\gamma^c_{\mu\nu}$. Hence, upon variation w.r.t. the metric, the equations of motion would contain higher-order terms such as $\partial^3\phi$. In the absence of such vertices on the other hand the equations remain second order (the particular Galileon structure is also essential for this to work).

[8]Covariant Galileons can also be obtained through different considerations, for instance as 4d EFTs arising from higher-dimensional constructions, where ϕ describes the brane-bending mode of a probe-brane in 5d [106]. By taking the non-relativistic limit for the brane motion one can indeed recover eq. (2.45).

For our purposes we can focus on values for Λ_3 relevant for DE. Therefore, we define, without loss of generality, its value to be

$$\Lambda_3 \equiv (H_0^2 M_{\rm Pl})^{1/3} \,, \tag{2.46}$$

where H_0 is the Hubble constant. This scale corresponds roughly with $\sim 10^{-13}$ eV or, in terms of a length scale, with $\sim (1000\text{ km})^{-1}$. Then, assuming the dimensionless coefficients in eq. (2.40) to be of order one, the operators (2.44) are generated from (2.45) at scales above or equal to

$$\Lambda_2 \equiv (H_0 M_{\rm Pl})^{1/2} \,, \tag{2.47}$$

which is parametrically larger than Λ_3 ($\Lambda_3/\Lambda_2 \sim 10^{-10}$).

2.5.3 Weakly broken Galileons

It is now also clear that eq. (2.45) can also be complemented with symmetry-breaking operators without altering its renormalizations properties, provided these have a suppression of Λ_2^4 for each $(\partial\phi)^2$. Such operators, even if they re-dress higher-derivative terms already present in the classical Lagrangian, do not alter the counting of two $(\partial\phi)$-legs per $M_{\rm Pl}$.

A particular structure compatible with this requirement, and obtained in [104], is the following WBG Lagrangian

$$\mathscr{L}^{\rm WBG} = \sum_{i=2}^{5} \mathscr{L}_i^{\rm WBG} \sqrt{-g} \,, \tag{2.48}$$

with

$$\begin{aligned}
\mathscr{L}_2^{\rm WBG} &= \Lambda_2^4\, G_2(X) \,, \\
\mathscr{L}_3^{\rm WBG} &= \frac{\Lambda_2^4}{\Lambda_3^3}\, G_3(X)[\Phi] \,, \\
\mathscr{L}_4^{\rm WBG} &= \frac{\Lambda_2^8}{\Lambda_3^6}\, G_4(X)R + 2\frac{\Lambda_2^4}{\Lambda_3^6}\, G_{4,X}(X)\left([\Phi]^2 - [\Phi^2]\right) \,, \\
\mathscr{L}_5^{\rm WBG} &= \frac{\Lambda_2^8}{\Lambda_3^9}\, G_5(X)G^{\mu\nu}\Phi_{\mu\nu} - \frac{\Lambda_2^4}{3\Lambda_3^9}\, G_{5,X}(X)\left([\Phi]^3 - 3[\Phi][\Phi^2] + 2[\Phi^3]\right) \,.
\end{aligned} \tag{2.49}$$

Here the $G_i(X)$ are arbitrary functions of the variable $X = -(\partial_\mu\phi)^2/\Lambda_2^4$. The subscript "$X$" stands for derivatives with respect to X. The extra factors of Λ_2 appearing in (2.49) are placed in such a way as to recover the Covariant Galileon once we set $G_i(X) = X$ for all i's. This also gives the correct scaling for the non-Galileon operators.

From the EFT perspective, the functions $G_i(X)$ should be expressible as a Taylor expansion in X, with order-one coefficients. Operators such as $(\partial_\mu\phi)^4\Box\phi$ in eq. (2.48) appear then with the correct scale $(\Lambda_3^3\Lambda_2^4)^{1/7}$. Thanks to the arguments outlined in this section they, together with the Galileon invariant operators in (2.48), get negligible corrections at loop level with a suppression of the order of $(\Lambda_3/\Lambda_2)^4$. In this sense (2.48) can be considered as invariant under Galileon symmetry as the covariant Galileon (2.45). Notice that, even when coupling with gravity is turned off, the former differs from the standard Galileon Lagrangian (2.40) (to recover the Galileons from a WBG theory, one needs to send both $M_{\rm Pl}$ and Λ_2 to infinity while keeping Λ_3 fixed).

To summarize, there are three different types of operators with different scaling:

- operators of the form
$$\frac{(\nabla\phi)^{2n}}{\Lambda_2^{4(n-1)}}\frac{(\nabla^2\phi)^m}{\Lambda_3^{3m}}\,, \tag{2.50}$$
with second derivatives having the Galileon structure, are present in the classical Lagrangian (2.48) and receive quantum corrections suppressed by $(\Lambda_3/\Lambda_2)^4$;

- operators of the form
$$\frac{(\nabla\phi)^{2n}}{\Lambda_2^{4n}}\frac{(\nabla^2\phi)^m}{\Lambda_3^{3m-4}}\,, \tag{2.51}$$
with second derivatives with arbitrary contractions, are not explicitly written in (2.48) but are generated quantum mechanically. They are subleading in comparison with the previous ones;

- operators of the form
$$\frac{\nabla^m(\nabla^2\phi)^n}{\Lambda_3^{3n+m-4}}\,, \tag{2.52}$$
with second derivatives with arbitrary contractions, are generated at the scale Λ_3 even in the standard Galileon Lagrangian.

The WBG Lagrangian (2.48), not surprisingly belongs to a subclass of Horndeski theories [75, 76]. The latter theories are more general, in the sense that they allow for arbitrary dependencies on both X and ϕ in the functions G_i, without committing to a peculiar power-counting for such terms. As already mentioned in Sec. 2.4, the defining property of Horndeski theories is that of having second-order equations of motion.

2.5.4 Beyond Horndeski

Interestingly enough, eq. (2.48) is not the most general Lagrangian with the properties above. Indeed, the derivation outlined so far relies on the absence of the vertices with too many $(\partial\phi)$s per $M_{\rm Pl}$. Although sufficient, this is not a necessary requirement in order to satisfy the prescribed renormalization properties of WBG theories. Cancellations can in principle also occur at the loop level, and if this happens the corresponding new classes of operators can be consistently added to $\mathscr{L}^{\rm WBG}$.

This logical possibility is seen to happen for Beyond Horndeski operators [107].[9] These new Beyond Horndeski operators with the explicit scaling are

$$\frac{1}{\Lambda_3^6} F_4(X)\, \epsilon^{\mu\nu\rho}_{\sigma} \epsilon^{\mu'\nu'\rho'\sigma} \nabla_\mu \phi \nabla_{\mu'} \phi\ \Phi_{\nu\nu'} \Phi_{\rho\rho'}\ , \tag{2.53}$$

and

$$\frac{1}{\Lambda_3^9} F_5(X)\, \epsilon^{\mu\nu\rho\sigma} \epsilon^{\mu'\nu'\rho'\sigma'} \nabla_\mu \phi \nabla_{\mu'} \phi\ \Phi_{\nu\nu'} \Phi_{\rho\rho'} \Phi_{\sigma\sigma'}\ , \tag{2.54}$$

where $\epsilon_{\mu\nu\rho\sigma}$ denotes the totally antisymmetric Levi-Civita tensor.

Properly speaking, in the context of Horndeski and Beyond Horndeski, the functions G_i and the new F_4 and F_5 also depend on ϕ. Additionally, for the sake of being consistent with the literature, we now prefer to rescale these functions so to incorporate the dimensionfull constants present in eqs. (2.49), (2.53) and (2.54). In doing so, we also redefine X such that $X \equiv (\partial_\mu \phi)^2$ so that all scales are implicit; when needed the explicit scaling will be emphasized. With this notation, the full Beyond Horndeski (or GLPV) action takes the form

$$S = \int \sum_{i=2}^{5} \mathscr{L}_i \sqrt{-g}\, \mathrm{d}^4 x\ , \tag{2.55}$$

where

$$\begin{aligned}
\mathscr{L}_2 &= G_2(\phi, X)\ ,\\
\mathscr{L}_3 &= G_3(\phi, X)[\Phi]\ ,\\
\mathscr{L}_4 &= G_4(\phi, X) R - 2G_{4,X}(\phi, X)\left([\Phi]^2 - [\Phi^2]\right)\\
&\quad - F_4(\phi, X)\, \epsilon^{\mu\nu\rho}_{\sigma} \epsilon^{\mu'\nu'\rho'\sigma} \nabla_\mu \phi \nabla_{\mu'} \phi\, \Phi_{\nu\nu'} \Phi_{\rho\rho'}\ ,\\
\mathscr{L}_5 &= G_5(\phi, X) G^{\mu\nu} \Phi_{\mu\nu} + \frac{1}{3} G_{5,X}(\phi, X)\left([\Phi]^3 - 3[\Phi][\Phi^2] + 2[\Phi^3]\right)\\
&\quad - F_5(\phi, X)\, \epsilon^{\mu\nu\rho\sigma} \epsilon^{\mu'\nu'\rho'\sigma'} \nabla_\mu \phi \nabla_{\mu'} \phi\ \Phi_{\nu\nu'} \Phi_{\rho\rho'} \Phi_{\sigma\sigma'}\ .
\end{aligned} \tag{2.56}$$

Furthermore, to avoid Ostrogradski ghosts these functions need to satisfy a degeneracy condition [80, 78]:

$$X G_{5,X} F_4 = 3F_5\left[G_4 - 2X G_{4,X} - (X/2) G_{5,\phi}\right]\ . \tag{2.57}$$

Because of this we see that only one between the two Beyond Horndeski functions can be chosen arbitrarily.

Alternatively, Beyond Horndeski can be seen as arising from Horndeski theories upon making the following disformal redefinition of the metric

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = g_{\mu\nu} + D(X, \phi) \nabla_\mu \phi \nabla_\nu \phi\ , \tag{2.58}$$

[9] More precisely [107] explicitly shows that Beyond Horndeski terms quadratic in $\nabla^2\phi$, with the right scalings, belong to $\mathscr{L}^{\mathrm{WBG}}$. Even if not proven, the remaining Beyond Horndeski terms, that are cubic in the second derivatives, are nonetheless expected to behave in a similar way since they can be obtained through a field redefinition of the Horndeski Lagrangian [79, 78].

then the degeneracy condition (2.57) arises by requiring that both F_4 and F_5 are generated by the same function $D(X, \phi)$ [78]. Of course the field redefinition (2.58), if not singular, relates two physically equivalent Lagrangians. Assuming the WEP, different theories are then obtained once we choose which metric is minimally coupled to matter. We thus refer to Beyond Horndeski as theories where, in the Jordan frame, at least one between F_4 and F_5 is present.

2.6 Phenomenological aspects of WBG theories

The WBG class of theories appears as a promising candidate for DE, being a legitimate modification of gravity on large scales. Other theories such *Chameleons* [108, 109] (which are essentially given by the function $G_2(\phi)$ in Horndeski) seem in to be in tension with the requirements of producing self-accelerating solutions and, at the same time, being able to hide at short scales [110–112]. In other words, acceleration must be, in a way or another, given by a cosmological constant so that one cannot really regard such theories as modifications of gravity. These no-go theorems of course do not rule these theories out, but they limit their theoretical appeal.

Of course, WBG are built around the Galileon structure, but it is worthwhile to point out that they contain, as a special subclass, $P(X)$ theories. The latter correspond to having only the function $G_2(X)$ turned on, with eventually also a dependence on ϕ. They were first studied in the context of inflation [113, 114] and then for DE, where they are referred to as k-essence models. A particularly interesting case of $P(X)$ is the Dirac-Born-Infield (DBI in short) Lagrangian, which arises in string theory as the effective action for D-branes [115–117]. Another popular model is the ghost-condensate [61, 118] which, differently from $P(X)$, crucially relies on higher-order spatial derivatives and can lead for instance to violations of the null energy condition [60]. On top of its cosmological applications, $P(X)$ finds ample use in the study of condensed-matter systems such as fluids [119].

A necessary requirement for any serious modification of gravity is the presence of some screening mechanism, hiding fifth forces in regimes where GR is well established. To achieve this, $P(X)$ relies on the so-called *kinetic screening* or k-mouflage [120]. In this case, if ϕ is coupled universally with matter, then close to a matter source classical non-linearities become large (meaning $\partial\phi \gg \Lambda_2^2$) so that ϕ deviates from the usual $1/r$ radial profile. In turn, this suppresses the fifth force mediated by ϕ so that its effects remain out of current experimental reach.

This mechanism comes with its features and drawbacks. For one, it turns out that even when classical non-linearities dominate, the theory remains generically under radiative control so that loop corrections never compete with the classical terms even for $X/\Lambda_2^4 \gg 1$ (the same also applies to higher derivative corrections) [121]. On the other hand, if the parameters of the model allow for screening, then scalar perturbations around the radial profile feature superluminalities (the speed of sound exceeds the speed of light along some direction). As an example, for $G_2(X) = -X + \epsilon X^2/\Lambda_2^4$, $\epsilon = -1$ allows for screening but leads to $c_s > 1$. This worrying (or exciting) feature seems to suggest that it is not possible to obtain such models as IR limits of Lorentz invariant UV theories [122].

2.6.1 Vainshtein screening

Regarding models where Galileon-like operators dominate over the single-derivative interactions, screening works with a similar principle as for kinetic screening, with the difference that non-linearities become large when $\partial^2\phi \gg \Lambda_3^3$. This type of mechanism goes under the name of *Vainshtein screening* [123] and was first applied to massive gravity where it also lifts the van Dam-Veltman-Zakharov (vDVZ) discontinuity [124] (see for instance [28] for a review on the subject).

The essential features of the Vainshtein mechanism are already captured by the cubic Galileon conformally coupled to matter with gravitational strength[10]

$$\mathscr{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{\Lambda_3^3}(\partial_\mu\phi)^2\Box\phi + \frac{g}{M_{\rm Pl}}\phi T^\mu_{\ \mu} \,, \tag{2.59}$$

with the dimensionless coefficient g taken to be of order unity and $T^\mu_{\ \mu}$ being the trace of the stress-energy tensor of matter. The equation of motion for ϕ, in the absence of gravity, is

$$\Box\phi + \frac{2}{\Lambda_3^3}\left[(\Box\phi)^2 - (\partial_\mu\partial_\nu\phi)^2\right] = -\frac{g}{M_{\rm Pl}}T^\mu_{\ \mu} \,. \tag{2.60}$$

The case of most interest is of a static, spherically symmetric distribution of matter (modelling for instance the Sun or a galaxy) at distances larger than the radius of the object. We can then approximate $T^\mu_{\ \mu} = -M\delta^{(3)}(\boldsymbol{x})$, where M is the total mass of the classical object and $\boldsymbol{x}$ is the spatial vector with components x^i. Due to spherical symmetry, we can also write $\phi = \hat{\phi}(r)$. Notice moreover that, because of the shift symmetry of the cubic Galileon, the left-hand side (LHS) of eq. (2.60) is a total divergence. Hence, we can integrate both sides over a spherical region around the origin and obtain an algebraic equation for $\hat{\phi}' \equiv \partial_r\hat{\phi}$

$$\hat{\phi}' + \frac{4(\hat{\phi}')^2}{\Lambda_3^3 r} = \frac{gM}{4\pi r^2 M_{\rm Pl}} \,. \tag{2.61}$$

There are two solutions of this equation, and we focus on the one that vanishes at infinity. (We will see in later chapters the other solution features ghost-like instabilities.) Explicitly, it is given by

$$\hat{\phi}' = \frac{\Lambda_3^3 r}{8}\left(1 - \sqrt{1 + \frac{4}{\pi}\left(\frac{r_{\rm V}}{r}\right)^3}\right) \,, \tag{2.62}$$

[10] Here we are following [29] and [85].

where the Vainshtein radius $r_{\rm v}$ is defined as

$$r_{\rm v} \equiv \frac{1}{\Lambda_3}\left(\frac{gM}{M_{\rm Pl}}\right)^{1/3} . \tag{2.63}$$

We see that $r_{\rm v}$ represents the distance at which classical non-linearities kick in. Phenomenologically, $r_{\rm v}$ is required to be much larger than the typical radius of the object in question, and this is guaranteed by the fact that $M \gg M_{\rm Pl}$ and that the scale $\Lambda_3 \sim (1000\ {\rm km})^{-1}$ is around the typical size of a compact object.

Before analyzing the properties of the solution, we should comment about the role of classical and quantum non-linearities. Classical non-linearities are characterized by the size of the non-linear terms in the equation of motion in comparison with the linear ones. On the other hand, the leading quantum corrections for the Galileons depend on the typical size of the second derivatives over Λ_3^3 (as can be seen from the structure of eq. (2.52)). In this case one needs to be careful: in presence of a ϕ background the relevant scale might not be simply Λ_3 but could be re-dressed by the background itself. We define then two different expansion parameters, measuring respectively the size of classical and quantum non-linearities

$$\alpha_{\rm cl} \equiv \frac{\partial^2\phi}{\Lambda_3^3} , \qquad \alpha_{\rm q} \equiv \frac{\partial^2}{\Lambda_3^2} . \tag{2.64}$$

Of course, a large $\alpha_{\rm cl}$ does not imply the loss of control over quantum corrections pretty much like in GR. For the latter case indeed, one can trust the highly non-linear solution close to the horizon of a Black Hole (with large enough mass compared to $M_{\rm Pl}$) without worrying about quantum corrections (that are important much closer to the singularity).

Going back to the Vainshtein solution, far outside the Vainshtein radius ($r \gg r_{\rm v}$) $\hat{\phi}'$ falls off as $1/r^2$ and distant objects feel a fifth force with gravitational strength. Moreover, in this region both parameters in eq. (2.64) remain small

$$\alpha_{\rm cl} \sim \left(\frac{r_{\rm v}}{r}\right)^{1/3} \ll 1 , \quad \alpha_{\rm q} \sim (r\Lambda_3)^{-2} \ll 1 , \tag{2.65}$$

and the theory is linear.

Near the source ($r \ll r_{\rm v}$), on the other hand, $\hat{\phi}'$ changes behaviour and grows as slowly as $r^{-1/2}$. This dependence is such that the fifth force is greatly suppressed with respect to gravity

$$\frac{F_\phi}{F_{\rm grav}} \sim \left(\frac{r}{r_{\rm v}}\right)^{3/2} \ll 1 , \tag{2.66}$$

so that screening is effective and Galileons can pass solar-system tests of gravity.[11] As expected, in this regime $\alpha_{\rm cl} \gg 1$ but quantum corrections don't seem small at shorter distances since

[11] Note that the precision of solar-system tests such as LLR is so high that fifth forces are nowadays not far from the experimental sensitivity [125].

$\alpha_{\rm q} \sim (r\Lambda_3)^{-2}$. From this expression, it would seem that below the scale Λ_3 the theory loses predictability and there would be no hope to explain gravitational experiments on Earth within the Galileon EFT. Fortunately, this is not entirely correct as we can see by inspecting the action for the fluctuations for ϕ.

If we expand the field as $\phi = \hat{\phi} + \varphi$ and add a perturbation $\delta T_{\mu\nu}$ to $T_{\mu\nu}$, then the Lagrangian for φ is

$$\mathscr{L} = Z^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{\Lambda_3^3}(\partial_\mu\varphi)^2\Box\varphi + \frac{g}{M_{\rm Pl}}\varphi\,\delta T^\mu_{\ \mu} \,, \tag{2.67}$$

where the kinetic matrix $Z^{\mu\nu}$ depends on the background configuration and is given by

$$Z^{\mu\nu} = -\frac{1}{2}\eta^{\mu\nu}\frac{2}{\Lambda_3^3}\left(\partial^\mu\partial^\nu\hat{\phi} - \eta^{\mu\nu}\Box\hat{\phi}\right) \,, \tag{2.68}$$

By direct inspection, $Z^{\mu\nu}$ evaluated on (2.62) is well-behaved for all values of r, even inside the Vainshtein radius.[12]

For $\alpha_{\rm cl} \ll 1$ this matrix reduces to the standard kinetic matrix for ϕ. However, deep inside the Vainshtein radius the contributions from non-linearities dominate. We can understand what this entails by characterizing $Z^{\mu\nu}$ by its typical eigenvalues, for simplicity. Very roughly, we can write $Z^{\mu\nu} \sim ZA^{\mu\nu}$ with Z being a (positive) typical eigenvalue and $A^{\mu\nu}$ a constant matrix with $\mathcal{O}(1)$ entries. Then the fluctuations are canonically normalized as $\varphi_c \equiv \sqrt{Z}\varphi$. In turn this rescaling changes the strong-coupling scale and also the coupling with matter. Specifically, Λ_3 and g get rescaled to $\tilde{\Lambda}_3 \equiv \sqrt{Z}\Lambda_3$ ad $\tilde{g} \equiv g/\sqrt{Z}$.

In the Vainshtein regime $Z \sim \alpha_{\rm cl} \sim (r_{\rm v}/r)^{3/2} \gg 1$. As one would expect, the resulting large kinetic term for φ is such that φ becomes weakly coupled ($\tilde{\Lambda}_3 \gg \Lambda_3$, $\tilde{g} \ll g$). This is, of course, another sign that screening is operative, and the Galileon decouples from matter. Additionally, we see that at the scale Λ_3 the EFT does not break down: the correct parameter controlling quantum corrections in this regime is $\alpha_{\rm q} = \partial^2/\tilde{\Lambda}_3^2$.

To assess whether $\tilde{\Lambda}_3$ is large enough, so to trust this description for table-top experiments of gravity, we can look at its typical value on Earth. If M is the mass of the Earth and $g = 1$, we find on the Earth's surface that

$$r_{\rm v} \simeq 10^{14}\ {\rm km}\,, \quad Z \sim (r_{\rm v}/r)^{3/2} \simeq 10^{15}\,, \quad \tilde{\Lambda}_3 \simeq (1\ {\rm cm})^{-1}\,. \tag{2.69}$$

Gravity has been tested so far up to $\sim \mu$m scales, therefore $\tilde{\Lambda}_3$, although greatly reduced, seems still quite large. This might not be as tragic as it seems given that gravity remains weakly coupled at these scales, contrary to ϕ.[13] Moreover, since at the onset of the breakdown

[12] This matrix features however superluminalities, which are a generic property of Galileons and derivatively-coupled theories [90, 126].

[13] Strictly speaking this holds for cubic Galileon-type interactions. For quartic and quintic this is not the case, as we discuss in Sec. 4.4.2.

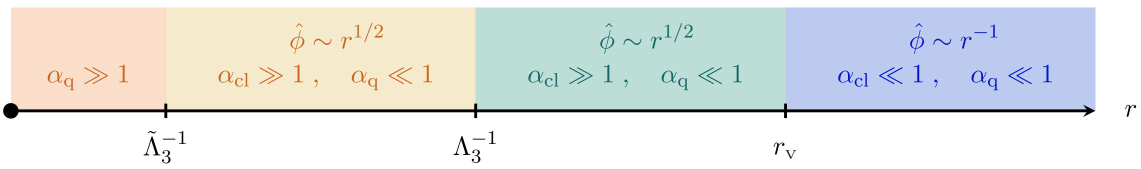


Fig. 2.3 Different regions around a screened source. Above r_V (blue) the solution behaves as in the free theory with small classical and quantum non-linearities. Inside the Vainshtein radius (green and yellow regions) the solution changes less rapidly due to classical non-linearities. For $\tilde{\Lambda}_3^{-1} < r < \Lambda_3^{-1}$ quantum corrections are still under control provided one correctly identifies the strong-coupling scale $\tilde{\Lambda}_3$. At shorter scales (orange) the EFT breaks down.

of the EFT ϕ does not source large gravitational backreactions, it is then at least reasonable to expect no large deviations at slightly shorter scales [85]. The situation is summarized in fig. 2.3.

We conclude by mentioning another potential problem in the Vainshtein regime (or in any other regime with $\alpha_{cl} \gg 1$). Although above $\tilde{\Lambda}_3$ the quantum-corrections parameter α_q remains small, the series of operators (2.52) is nonetheless formally divergent. To see this we can focus on operators with $m = 2$ and large n: such terms are suppressed by one power of α_q but at the same time are enhanced by n powers of α_{cl}.

Such operators are expected to arise from a generic UV completion but, since they are associated with power-law divergences, it is technically natural to tune them to zero in the EFT. The robust quantum corrections that cannot be tuned to zero (the logarithmic divergences) are associated instead to operators with at least three derivatives per field [85]. As long as the background is such that these higher derivatives are sufficiently suppressed in comparison to $\tilde{\Lambda}_3$, then the problem can in principle be shifted into a requirement for the UV completion.

2.7 Connection with the covariant formulation

After introducing the covariant theory for DE, we are now in a position to relate it to the EFT of DE formulation. In Sec. 2.2.3 we already saw, for a particular example, how to write a covariant Lagrangian in terms of the unitary gauge operators of eq. (2.37). The procedure is essentially the same also in the case of the full WBG (or Beyond Horndeski) Lagrangian.

Recalling the definitions for n_μ (2.2) and $K_{\mu\nu}$ (2.4), it is not surprising that the higher-derivatives, characteristic of the Galileons, in unitary gauge yield terms containing powers of $\delta K_{\mu\nu}$. Moreover, the generic functions $G_i(\phi, X)$, $F_i(\phi, X)$ will become functions of g^{00} and t; by expanding them in powers of δg^{00} we see that we generate a tower of operators. These are, as already explained, for the most part innocuous as they are generally subleading in numbers of spatial derivatives. Therefore, the great freedom in the functions of two variables G_i and F_i collapses, at the level of the EFT of DE, to the freedom in few time-dependent coefficients. (Of course, this is not true once the theory enters particular highly non-linear regimes, where all the powers of δg^{00} should be resummed.)

These functions, evaluated on the background for ϕ, are then mapped into the coefficients of the EFT of DE as follows

$$
\begin{aligned}
M^2 &\equiv M_\star^2 f + 2m_4^2 = 2G_4 - 4XG_{4,X} - X\left(G_{5,\phi} + 2H\dot\phi G_{5,X}\right) + 2X^2F_4 - 6H\dot\phi X^2 F_5 \ , \\
m_4^2 &= \tilde m_4^2 + X^2F_4 - 3H\dot\phi X^2 F_5 \ , \\
\tilde m_4^2 &= -\left[2XG_{4,X} + XG_{5,\phi} + (H\dot\phi - \ddot\phi)XG_{5,X}\right] \ , \\
m_5^2 &= X\left[2G_{4,X} + 4XG_{4,XX} + H\dot\phi\left(3G_{5,X} + 2XG_{5,XX}\right) + G_{5,\phi}\right. \ , \\
&\qquad \left. + XG_{5,X\phi} - 4XF_4 - 2X^2F_{4,X} + H\dot\phi X\left(15F_5 + 6XF_{5,X}\right)\right] \ , \\
m_6 &= \tilde m_6 - 3\dot\phi X^2 F_5 \ , \\
\tilde m_6 &= -\dot\phi X G_{5,X} \ , \\
m_7 &= \frac{1}{2}\dot\phi X\left(3G_{5,X} + 2XG_{5,XX} + 15XF_5 + 6X^2F_{5,X}\right) \ ,
\end{aligned}
\tag{2.70}
$$

where the time-dependence is left implicit. For the moment we do not write explicitly the correspondence with the coefficients m_2 and m_3, since their lengthy expressions will not be of much use for now. Their expressions are found for instance in [74, 73].

By inspecting eq. (2.70) we notice a hierarchical structure in the contributions to the m_i's. Indeed, m_6, $\tilde m_6$ and m_7 only depend on the quintic terms from $\mathscr{L}_5$, while m_4, $\tilde m_4$ and m_5 depend on both quartic and quintic terms. For m_3 the contributions can be seen to come from all quintic, quartic and cubic terms.

The different m_i coefficients give in general peculiar effects already at the level of linear perturbations. As it will become clear, however, linear cosmological observables only depend on certain linear combinations of them. It is therefore convenient to also provide such basis of coefficients since they have a more intuitive meaning. Following [127, 128] we define

$$
\begin{aligned}
&\alpha_{\rm K} \equiv \frac{2c + 4m_2^4}{M^2H^2} \ , \quad \alpha_{\rm B} \equiv \frac{M_\star^2\dot f - m_3^3}{2M^2H} \ , \quad \alpha_{\rm M} \equiv \frac{M_\star^2\dot f + 2(m_4^2)^{\cdot}}{M^2H} \ , \\
&\alpha_{\rm T} \equiv -\frac{2m_4^2}{M^2} \ , \qquad \alpha_{\rm H} \equiv \frac{2\left(\tilde m_4^2 - m_4^2\right)}{M^2} \ , \qquad \alpha \equiv \alpha_{\rm K} + 6\alpha_{\rm B}^2 \ .
\end{aligned}
\tag{2.71}
$$

Moreover, we can sum-up their role for DE and their physical interpretation

- *kineticity* $\alpha_{\rm K}$: it normalizes the kinetic term for scalar fluctuations. Moreover it also controls the speed of sound c_s for the scalar mode, with larger values suppressing c_s. In the covariant formulation this coefficient arises in k-essence models [113, 129, 113], and receives contributions from all the functions G_i and F_i. In the EFT of DE it gets contributions from the operator $(\delta g^{00})^2$;

- *braiding* $\alpha_{\rm B}$: kinetic mixing between the scalar and the metric [130]. Because of this mixing, it also enters in the canonical normalization of the scalar through together with $\alpha_{\rm K}$ in the combinations α. It is responsible for the clustering of DE below a certain scale [127] and contributes to the variation of the Newton's constant and the gravitational slip (1.1). In the covariant theory, it gets contributions from all operators, except G_2. The braiding arises from the operators $\delta g^{00}\delta K$ and from the time-dependent Planck mass in the EFT language;

- *running of the Planck mass* $\alpha_{\rm M}$: is the rate of variation of the Planck mass. It measures the non-minimal coupling in theories such as Brans-Dicke [131] and also produces anisotropic stresses, contributing to the gravitational slip. Horndeski models source this coefficient through G_4 and G_5. In the EFT it arises from $f(t)$ and the coefficient m_4;

- *tensor speed excess* $\alpha_{\rm T}$: modification of the speed of propagation for gravitons in comparison with the speed of light. Receives contributions from G_4, G_5, F_4 and F_5. It is related to the operator $\delta\mathcal{K}_2$ of the EFT;

- *deviation from Horndeski* $\alpha_{\rm H}$: characterizes the size of Beyond Horndeski terms for linear perturbations. It is also referred to as *kinetic matter mixing* since it leads to a kinetic mixing between matter and the scalar mode. It modifies the Poisson equation for the Newtonian potentials, adding a dependence on the matter velocity potential [132]. Of course, it vanishes in the absence of both F_4 and F_5.

These are all dimensionless coefficients whose size tells us how large the modification of gravity is. They offer a convenient basis where to quote phenomenological constrains, given that they are tightly related to physical quantities. Among these coefficients, $\alpha_{\rm K}$ is poorly constrained by data since it does not enter in the equations of motion in the quasi-static regime and thus has little impact on observations [133].

The linear equations for cosmological perturbations, in both the covariant and EFT approaches, have also been implemented numerically [134–137]. This has allowed exploring the parameter space of (Beyond) Horndeski compatible with available cosmological data (see e.g. [138–140]).

3 | Dark Energy after GW170817

After discussing the main theoretical features of DE models based on Galileon interactions in chapter 2, in this chapter we are going to switch to the phenomenology of these theories in the context of GWs. Of particular interest is the NS-NS event GW170817, which provided incredibly strong constraints on the propagation of GWs, with large repercussions for modified gravity.

After briefly reviewing the production of GWs in GR, we are going to study the implications of this event for the operators of the EFT of DE, following [48, 141].

3.1 Gravitational waves at interferometers

Up to now, thanks to the LIGO/Virgo collaboration, we have been able to detect GW events from compact binary systems where the mass of each object is around the mass of the Sun $M_\odot$. In particular, current experiments are mostly sensitive to the GWs emitted just before the merger. This phase, called *inspiral phase*, is characterized by an adiabatic GW emission while the two compact objects move in an elliptic orbit. The release of radiation in GWs tends to reduce the eccentricity of the orbit in such a way that, closer to the merger, the orbit has circularized with great accuracy.

During the inspiral, the two objects are still greatly separated, and their circular velocities are still far from being relativistic. In this regime their motion is still well described by Newtonian mechanics but, due to the slow loss of energy in GWs, the orbit over long time scales shrinks, bringing the two bodies closer and closer (as in the Hulse-Taylor binary pulsar [142]). At shorter distances, the velocities in turn increase, making the GW emission stronger. Such evolution results in the typical *chirping* profile in the GW signal, where the amplitude and frequency of the wave increase up to the point of merger.

Much before merger the problem can be treated analytically, at least at the leading orders, thank to the fact that the GW represent only a tiny modification of the background geometry. In this regime, in GR, the waveform for the GW $\gamma_{\mu\nu}$ can be obtained with the famous quadrupole formula. If we consider two objects with masses m_1 and m_2 in circular orbit, then the two polarizations $(+, \times)$ of the wave at the detector are given by [143]

$$\begin{aligned} \gamma_{+}(t) &= \frac{4}{r}\left(G_{\rm N} M_c\right)^{5/3}\left(\pi f(t)\right)^{2/3}\left(\frac{1+\cos^2\iota}{2}\right)\cos\left[\varphi(t)\right] \ , \\ \gamma_{\times}(t) &= \frac{4}{r}\left(G_{\rm N} M_c\right)^{5/3}\left(\pi f(t)\right)^{2/3}\cos\iota\cos\left[\varphi(t)\right] \ . \end{aligned} \tag{3.1}$$

Here we defined

$$M_c \equiv \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} \tag{3.2}$$

as the *chirp mass*, ι as the angle between the normal to the orbit and the line-of-sight and r as the distance from the binary. Moreover, $f(t)$ is the time-dependent frequency of the GW (which is twice the orbital frequency). Because of the inspiral, it evolves as

$$f(t) = \frac{1}{\pi}\left[\frac{5}{256(t_{\rm c}-t)}\right]^{3/8}\left(G_{\rm N} M_c\right)^{-5/8} \ , \tag{3.3}$$

where $t_{\rm c}$ is the time of coalescence. The phase $\varphi(t) \equiv \int_{t_0}^{t} 2\pi f(t')\,{\rm d}t'$ is then given by

$$\varphi(t) = -2\left(\frac{t_{\rm c}-t}{5 G_{\rm N} M_c}\right)^{5/8} + \varphi_0 \ . \tag{3.4}$$

with φ_0 being its value at $t_{\rm c}$.

The amplitude in eq. (3.1) diverges at $t_{\rm c}$. However, it is clear that before that point the approximations used do not hold any longer. For instance, as the objects approach each other one cannot reliably model them as point-masses. The leading modification is then given by the fact that the objects eventually get in contact with each other so that the chirp profile is cut-off at some maximum frequency. A good approximation for this f is obtained by noticing that in the Schwarzschild geometry there is a maximum radius below which circular orbits become unstable. This is referred to as the *Innermost Stable Circular Orbit* (ISCO) and the associated radius is $r_{\rm ISCO} = 6G_{\rm N}(m_1+m_2)$. Using Kepler's law, the GW frequency at the ISCO is

$$f_{\rm ISCO} = \frac{1}{6^{3/2}\pi}\frac{1}{G_{\rm N}(m_1+m_2)} \ . \tag{3.5}$$

For equal mass objects with few times the mass of the Sun, $f_{\rm ISCO} \sim 400$ Hz while for more massive BH binaries m_1, $m_2 \sim 10\ M_\odot$ the maximum frequency is lower, at around $f_{\rm ISCO} \sim 100$ Hz. LIGO/Virgo are of course most sensitive in this frequency range, making these the perfect candidate events for their interferometers. On the other hand supermassive Black Holes with masses $\sim 10^6\ M_\odot$ or more, merge at mHz frequencies (this being the range of LISA).

Since GWs typically are observed after they travel over cosmological distances, the expansion of the Universe gives a contribution in eq. (3.1). Clearly, the frequency gets redshifted as $f = (1+z)f^{(\rm obs)}$, where $f^{(\rm obs)}$ is the observed frequency at the detector and z is the redshift

$(z+1 \equiv a(t_{\rm obs})/a(t))$. Moreover, the correct formula for the waveform is obtained by replacing in (3.1) the distance r by the *luminosity distance* $d_L \equiv (1+z)a(t_{\rm obs})r$, and the chirp mass by $\mathcal{M}_c \equiv (1+z)M_c$.

3.2 Properties of GW170817 and GRB170817 A

Arguably the most surprising and rich GW event measured so far has been GW170817 [46], with its associated electromagnetic (EM) counterpart GRB170817 A. Aside from the important astrophysical information learned thanks to this event, the detection of both GW and EM signals allowed to greatly reduce the available parameter space in modifications of gravity, as we will see.

On August 17, 2017, the two LIGO detectors in Hanford and Livingston observed a GW signal entering in their sensitivity band at around 30 Hz and then sweeping up in frequency, lasting more than one minute. At the time, this was the loudest signal observed, with a combined signal-to-noise ratio of around 30. On the other side of the Ocean, the third GW interferometer Virgo did not measure a strong signal, pointing to the fact that the sky location of the event was in a blind spot of this detector. Despite the lack of a convincing third signal, this information helped with the sky-localization of the GW source, which has been established with good accuracy.

The long time observation of the signal allowed for a very precise measurement of the chirp mass (as the latter sets the frequency in eq. (3.1)). The relatively low value of M_c, and the low value estimated for the masses (that can be inferred including higher Post-Newtonian (PN) corrections in the waveform), suggested that the event consisted of the merger of two NSs. The properties of the EM counterpart also strongly support this conclusion. Moreover, the luminosity distance was estimated to be around 40 Mpc, making the event the closest one detected yet.

Finally, other valuable information was obtained regarding the tidal deformability parameter, relevant for the equation of state for NSs. The bound in this case is loose, but points against less compact equations of state. The parameters of this event are summarized in table 3.1.

m_1	m_2	M_c	d_L	z
$(1.36-1.60)\,M_\odot$	$(1.17-1.36)\,M_\odot$	$1.188^{+0.004}_{-0.002}\,M_\odot$	40^{+8}_{-14} Mpc	$0.008^{+0.002}_{-0.003}$

Table 3.1 Main parameters obtained from the analysis of GW170817 [46]. The redshift is obtained assuming a ΛCDM cosmology with cosmological parameters from Planck.

At the same time of this detection, a short GRB was identified by Fermi-GRB and INTEGRAL [45, 47] in the same region of the sky. Thanks to the great sky localization obtained by

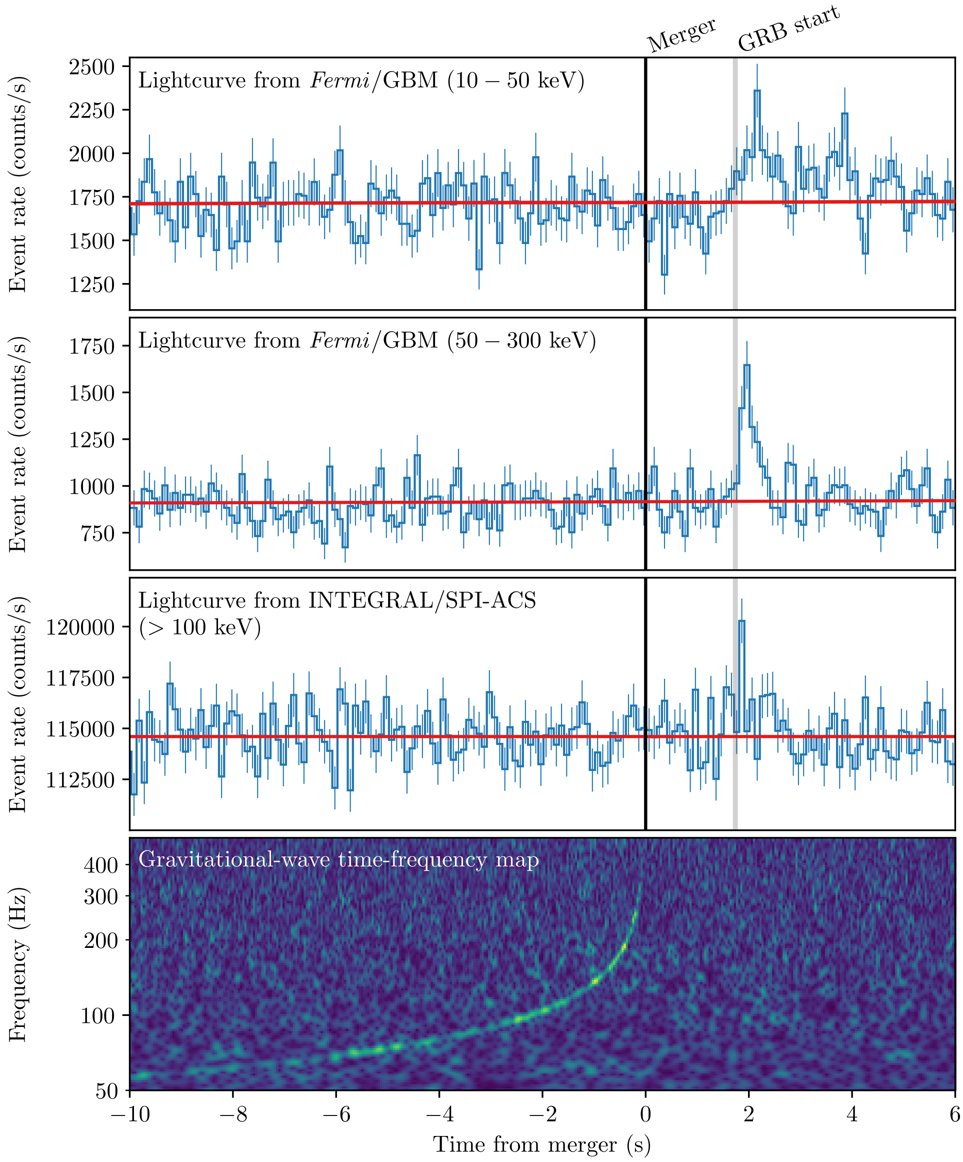


Fig. 3.1 Comparison between the arrival times of GW170817 and GRB170817 A, taken from [47]. Lower panel: observed frequency of the GW as a function of time obtained combining the two LIGO signals. Color-map indicated the strain amplitude of the signal, which increases towards the merger. Upper panels: GRB detected by Fermi/GRB and INTEGRAL. The main peaks are delayed in comparison with the merger.

LIGO/Virgo, it was possible to establish that the origin of the GRB was indeed related to the merger, making it the first multimessenger-astronomy event recorded.

Crucially, the GRB event was detected with an arrival time delayed by

$$\Delta t = 1.734 \pm 0.054 \text{ s} \tag{3.6}$$

with respect to the merger time obtained from GW data. The gravitational and EM signals are shown in fig. 3.1. This time difference is moreover consistent with models for the GRB emission, in which the γ-ray is expected to be emitted after a few seconds from the merger.

Together with the value of the distance, this measurement allows for a very stringent constraint on the difference between the speed of propagation for GWs c_T and the speed of light c. Because of the low redshift of the event, the bound can be obtained as if in flat space using the distance d. Then the bound comes as follows. Let us say the merger happened at $t = 0$ and that the arrival time of the GW signal is t_T (corresponding to the peak in the GW amplitude). Also, assume that the EM signal is emitted at a slightly different time $t = \delta t$, and arrives at a time t_E so that $\Delta t = t_E - t_T$ (from (3.6), $\Delta t > 0$). If we define $\Delta c \equiv c_T - c$, then at leading order we have that

$$\frac{\Delta c}{c} = c\,\frac{\Delta t - \delta t}{d}\ . \tag{3.7}$$

Astrophysical considerations point to a positive δt. Using this prior we see that, as expected, for $\Delta c > 0$ the strongest bound comes from setting $\delta t = 0$ so assigning the whole delay to Δt. On the other hand, if $\Delta c < 0$, δt can weaken the bound. In this case one can for instance be conservative and use a very broad prior $\delta t < 10$ s and $d \gtrsim 37$ Mpc. Combining both sides of the constraint one then has [42]

$$-2\times 10^{-15} \lesssim \frac{\Delta c}{c} \lesssim 4\times 10^{-16}\ , \tag{3.8}$$

Because of the large distance compared to a total delay of few seconds, this bound is incredibly strong on both sides.

Notice that previous bounds on c_T from GWs were of the order $\Delta c/c \lesssim \mathcal{O}(1)$ since they relied on the travel time of the GW between the two LIGO detectors. On the other hand, indirect constraints on c_T were also available before the NS-NS event and are based on the emission of gravitational Cherenkov radiation by cosmic rays [144]. For kinematic reasons, particles moving faster than c_T can lose energy by emitting gravitons. Since however cosmic rays are observed, this gives a lower bound similar to the one of eq. (3.8), but it is based on physical at a totally different energy scale. GWs give also additional bounds on fundamental physics, but so far (3.8) is arguably the one with the strongest implications.

3.3 Implications for Dark Energy models

The bound on the speed of propagation coming from GW170817 can be applied to the DE models of chapter 2. This is possible mainly for two reasons: the relatively low-energy of the observed GW and the large distance from the event.

To understand the importance of the first, one can focus on the scaling of the operators of the WBG theory (2.48). In the presence of Galileon operators, the value for the strong-coupling scale is around Λ_3. As already emphasized, this scale is in the ballpark of 10^{-13} eV, corresponding to frequencies of ~ 100 Hz. Notice that for this reason the bounds on c_T from Cherenkov radiation cannot be applied since this effect is relevant for much higher energies. The EFT of DE can then possibly be applied to the propagation of GWs in the LIGO/Virgo range. Notice however that GW frequencies are still quite close to Λ_3. As in the case of the scattering of massive gauge bosons in the Standard Model, new physics (in this case the Higgs) is required to appear below or at the strong-coupling scale of the low-energy EFT. Therefore, the UV completion can show-up even below Λ_3, as explained in [145]. However, since concrete examples of (partial) UV completions of Galileon theories are at present not known, it is difficult to make precise claims about the true scale at which this is supposed to happen for DE. For now, we will assume the low-energy EFT can describe the propagation of waves at LIGO/Virgo. We leave additional remarks on this point for later chapters.

Moreover, the propagation of GWs happens over Mpc scales. Even if part of this path resides within a region where the Vainshtein screening is active, the expectation is that at least some parts of it explore unscreened regions (40 Mpc is around the linear scale for large scale structure). Therefore, one should be allowed to use the EFT description evaluated around the cosmological background, so to constrain the parameters relevant for cosmology.

Note also that the EFT coefficients, together with c_T, are time dependent and that the event GW170817 is observed at a very low redshift. This implied that the constraint is actually for the value of c_T today and in principle, one could still have $c_T \neq 1$ at other times. This possibility seems to imply a large degree of tuning however (one would have to explain why this happens for $z = 0$), and will not be discussed in the following.

The specific constraints for Horndeski and Beyond Horndeski have been obtained in [48–51]. First, we are going to discuss such bounds in the context of the EFT of DE and next, we will generalize the derivation, taking into account a possible loophole. The latter part overlaps with the results obtained independently in [141].

Given this premise, we would like to obtain the quadratic action for gravitons to see what parameters in the EFT action (2.37) contribute to the speed of propagation c_T. First, we consider the tensor perturbations γ_{ij} in the FRW metric as $\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)(e^\gamma)_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$. Moreover, we require γ_{ij} to be transverse ($\partial_i\gamma_{ij} = 0$) and traceless ($\delta_{ij}\gamma_{ij} = 0$), so that it describes the two graviton polarizations. In order to compare the speed of gravitons and photons, we work in a “frame” where photons are minimally coupled to the metric.

At quadratic order in γ_{ij} there are only two operators of eq. (2.37) that contribute: the usual Einstein-Hilbert term and the operator $m_4^2\delta\mathcal{K}_2$ defined in (2.38). The first yields the usual relativistic kinetic term for γ_{ij}, so it doesn’t spoil the luminal speed. The second however, as we can see from the expression for K_{ij} (2.13), modifies both the normalization and the speed of γ_{ij}. After some straightforward steps, and after noticing that $\delta K = 0$ and $h = a^3(t)$, one has the following quadratic action for gravitons

$$S_\gamma^{(2)} = \int \frac{M_\star^2 f}{8} \left[\left(1 + \frac{2m_4^2}{M_\star^2 f}\right) \dot{\gamma}_{ij}^2 - a^{-2}(\partial_k \gamma_{ij})^2 \right] a^3 \, \mathrm{d}^4 x \; . \tag{3.9}$$

By rescaling γ_{ij} so to have a canonically normalized field we obtain that the speed c_T is given by

$$c_T^2 = 1 - \frac{2m_4^2}{M^2} \; , \tag{3.10}$$

where M^2 is defined in the first line of eq. (2.70). Notice that m_4^2 can take either sign, despite being written as a square. Also, the tensor speed excess parameter α_T in eq. (2.71) is equal to $c_T^2 - 1$.

Then the bound (3.8) translates to a bound on m_4^2/M^2 or on α_T (notice that we switched again to $c = 1$ units). The parameter α_T represents the size of a particular deviation from GR at cosmological scales. In the foreseeable future, we will only be able to test cosmology with $\mathcal{O}(10^{-2})$ precision. On the other hand, the bound in consideration is so strong, of the order of 10^{-15}, that for all practical purposes it implies $\alpha_\mathrm{T} = 0$, or $m_4^2 = 0$, for any phenomenological application.

As it stands, this bound has been applied to the unperturbed cosmological history. To avoid tuning, we would also like to require that c_T remains luminal even for slightly different histories, which for instance can be obtained by varying the energy density and pressure of dark matter by, say, few percents. In the EFT of DE, for instance, this variation is equivalent to an independent homogeneous and isotropic variation of the metric, *i.e.* of $g^{00}(t)$ and of the Hubble function as a function of time. Once we perform this variation, the scalar operators δg^{00} and δK present in the non-linear terms of (2.37) shift the value of m_4^2. For now, we consider δg_b^{00} and δH_b as the independent variations of the background, while

$$\delta K_b = 3\delta H_b - \frac{3}{2} H \delta g_b^{00} \; . \tag{3.11}$$

Only cubic and higher operators in (2.37), starting from $\tilde{m}_4^2$, can give contributions under the change of background.

Let us focus on linear changes in m_4^2 first. The contributions from $\tilde{m}_4^2$ and m_5^2 change respectively the spatial and the time kinetic terms for γ_{ij} in a way proportional to δg_b^{00}. Overall, at linear order in δg_b^{00} they give

$$\delta m_4^2 = \frac{1}{2}(\tilde{m}_4^2 - m_5^2)\delta g_b^{00} \; , \tag{3.12}$$

where δm_4^2 is the variation of m_4^2 under the change of the background.

Moving on to $m_6 \delta\mathcal{K}_3$, we see that we can obtain quadratic contributions by placing one of its $\delta K_{\mu\nu}$s on the background. In doing so, we have $\delta\mathcal{K}_3 = \delta K_b \delta\mathcal{K}_2$ and, using (3.11), we obtain the corresponding shift

$$\delta m_4^2 = m_6 \delta H_b - \frac{1}{2} m_6 H \delta g_b^{00} \, . \tag{3.13}$$

The last contribution at linear order comes from the operator $\tilde{m}_6 \delta g^{00} \delta \mathcal{G}_2$, which is slightly more involved. The only place where we can place the variation of the background is on δg^{00}. Then, we can rewrite $\delta \mathcal{G}_2$, thanks to eq. (8) of [73], only in terms of ${}^{(3)}\!R$ and its time derivative plus terms that give total derivatives in the Lagrangian. This operator thus changes the spatial kinetic term for gravitons, shifting c_T as well. Effectively this is equivalent to the following change in m_4^2

$$\delta m_4^2 = \frac{1}{2} \left[\tilde{m}_6 H \delta g_b^{00} + (\tilde{m}_6 \delta g_b^{00})\dot{} \right] \, . \tag{3.14}$$

At linear order the overall contribution, which is the sum of (3.12), (3.13) and (3.14) should be set to zero. Since δH_b can be varied independently and it only appears in (3.13), we are forced to set $m_6 = 0$. Moreover, (3.14) is the only instance where $\delta \dot{g}_b^{00}$ enters, thus requiring $\tilde{m}_6 = 0$ as well. Finally, (3.12) allows for a tuning $\tilde{m}_4^2 = m_5^2$, making the contribution vanish.

Similarly, at second order we have a contribution from $m_7 \delta g^{00} \delta \mathcal{K}_3$ when δg^{00} and one $\delta K_{\mu\nu}$ are on the background. Since this is the only possible term containing δH_b, it needs to vanish as well.

Notice that there are other higher-order operators we didn't write explicitly in (2.37) that contain more δg^{00} powers with respect to their lower-order counterparts. For instance, these are operators such as $(\delta g^{00})^2\, {}^{(3)}\!R$ and $(\delta g^{00})^2\, \delta \mathcal{K}_2$. We see that a tuning, similar to the one between $\tilde{m}_4^2$ and m_5^2, can be made also for these two terms (and also for higher-orders terms).

Overall, the constraint applied around the perturbed history is remarkably strong and in summary we have

$$m_4 = 0 \, , \qquad \tilde{m}_4^2 = m_5^2 \, , \qquad m_6 = \tilde{m}_6 = m_7 = 0 \, . \tag{3.15}$$

It is instructive to see how these constraints can be obtained starting from the covariant Beyond Horndeski theory. In that case one requires m_4^2 (given in terms of the Beyond Horndeski functions (2.70)) to vanish for any background, meaning that this must hold for any value of $\ddot{\phi}$, $\dot{\phi}$ and H. This translates into the conditions

$$G_{5,X} = 0 \, , \qquad 2G_{4,X} - XF_4 + G_{5,\phi} = 0 \, , \qquad F_5 = 0 \, , \tag{3.16}$$

that must hold generally, and not only on a specific background for ϕ.

One can easily see that imposing (3.16) in (2.70) gives exactly the conditions (3.15), obtained from the EFT of DE.

Additionally, we notice that the Beyond Horndeski term F_5 is excluded while G_5 can be only a function of ϕ. The remaining Beyond Horndeski term F_4 remains unconstrained, but is tuned to be related to G_4 and G_5 (in the EFT language this corresponds to the tuning $\tilde{m}_4^2 = m_5^2$).

The surviving Lagrangian after the measurement of $c_T = 1$ is then given by

$$\mathscr{L}_{c_T=1} = G_2(\phi, X) + G_3(\phi, X)\Box\phi + f(\phi, X)R - \frac{4}{X} f_{,X}(\phi, X)\left(\nabla^\mu\phi\nabla^\nu\phi\,\Phi_{\mu\nu}[\Phi] - \nabla^\mu\phi\nabla_\lambda\phi\,\Phi_{\mu\nu}\Phi^{\lambda\nu}\right) , \tag{3.17}$$

where $f(\phi, X) \equiv G_4(\phi, X) + XG_{5,\phi}/2$. This Lagrangian can be obtained by performing some integrations by parts in (2.56). One can also explicitly check that indeed the last line in (3.17) does not contribute the graviton's speed-excess.[1] Notice also that conformal redefinitions of the metric do not alter the light-cone of gravitons. Hence, other surviving models are obtained by performing a redefinition $g_{\mu\nu} \to \Omega(\phi, X)g_{\mu\nu}$ in (3.17) (while leaving the matter sector unaffected). This brings the Lagrangian in the form of a DHOST theory (see [48] for an explicit expression).

There is another question that is important to address at this stage: the radiative stability of eq. (3.17). Given the extreme precision of the bound for c_T, it is in principle conceivable that quantum corrections can spoil this choice. In the context of Horndeski for instance, from the discussion in Sec. 2.5.3 we expect the operator $G_5(X)$ to be generated. However, because of the renormalization properties of WBG theories, the contributions to c_T from these corrections scale as $(\Lambda_3/\Lambda_2)^4 \sim 10^{-40}$ and are thus negligible (see eq. (2.50)). The same conclusion also applies to Beyond Horndeski terms [107]. Furthermore, the quantum-generated terms with more derivatives of eq. (2.52) do not give troublesome contributions as well. In conclusion, one can indeed establish that the choice $c_T = 1$ is technically natural [48]. It would be interesting to explore whether this choice can be related to some symmetry argument.

3.4 Possible loopholes

In deriving $\mathscr{L}_{c_T=1}$ we claimed that the variations of $\ddot{\phi}$, $\dot{\phi}$ and H are independent of each other, so that one is forced to set their coefficients to zero in m_4^2. As we are going to show in more detail, this is not always the case and their values, depending on the model, can be related by the equation of motion for the scalar.

In terms of δg_b^{00}, $\delta \dot{g}_b^{00}$ and δH_b, the equation for ϕ corresponds to an equation for δg_b^{00} and in some cases, through this equation, $\delta \dot{g}_b^{00}$ is fixed by the other variations. Inspired by the possible loophole found in [146], we would like to perform a detailed calculation showing how in principle alternatives to (3.17) can be obtained in this fashion. Ultimately, however, we will

[1] To see this one can use the Gauss-Codazzi relation

$$R = {}^{(3)}R + K_{\mu\nu}K^{\mu\nu} - K^2 + 2\nabla_\mu\left(Kn^\nu - n^\nu\nabla_\nu n^\mu\right) \tag{3.18}$$

to replace R in (3.17). After integrating by parts, $\mathscr{L}_{c_T=1}$ can be recast as

$$\mathscr{L}_{c_T=1} = P(\phi, X) + Q(\phi, X)\Box\phi + f(\phi, X)\left({}^{(3)}R + K_{\mu\nu}K^{\mu\nu} - K^2\right) , \tag{3.19}$$

where the free functions P and Q are related respectively to G_2 and G_3, but also depend on f.

also show that these additional cases are unattainable once we include spatial inhomogeneities, closing the possible loophole in the argument of the previous section.

3.4.1 Homogeneous variation of the background history

First we are going to work up to linear order in δm_4^2 thus neglecting operators such as m_7 and we also keep considering homogeneous perturbations of the background history $\delta g_b^{00}(t)$, $\delta H_b(t)$.

The equations of motion in unitary gauge for these latter variables are obtained by varying the action with respect to the metric. As already outlined in (2.15) and (2.16), in the presence of a matter stress-energy tensor $T_{\mu\nu}^{(m)} = \mathrm{diag}(\rho_m, p_m, p_m, p_m)$ the unperturbed background Einstein's equations give a set of equations for $c(t)$ and $\Lambda(t)$ (the dependence on t will from now on remain implicit). By combining these two, one obtains the equivalent relations

$$3M_\star^2\left(fH^2+\dot{f}H\right)=\rho_m+c+\Lambda\ , \tag{3.20}$$

$$3M_\star^2\dot{f}\dot{H}+6M_\star^2H^2\dot{f}=\dot{\Lambda}+\dot{c}+6cH\ , \tag{3.21}$$

with the former corresponding to a generalized Friedmann equation and the latter being the conservation of the fictitious DE stress-energy tensor (defined in eq. (2.17)).

We can obtain the equation of motion for δg_b^{00} by including, in this latter equation, linear perturbations of the background history. Notice that the only operators of eq. (2.14) contributing at this order are m_2^4 and m_3^3. After some work we obtain

$$\lambda_{\dot{g}}\,\delta\dot{g}_b^{00}=\lambda_g\delta g_b^{00}+\lambda_H\delta H_b+\lambda_{\dot{H}}\delta\dot{H}_b\ , \tag{3.22}$$

where the λ coefficients are defined as

$$\begin{aligned}
\lambda_{\dot{g}}&\equiv -c-2m_2^4+\frac{3}{2}H(M_\star^2\dot{f}-m_3^3)\ ,\\
\lambda_g&\equiv \dot{c}+2(m_2^4)^{\cdot}+3\dot{H}(m_3^3-M_\star^2\dot{f})+\frac{3}{2}H\left[4m_2^4+(m_3^3)^{\cdot}+3Hm_3^3-4M_\star^2H\dot{f}+4c\right]\ ,\\
\lambda_H&\equiv -3\left[2c+(m_3^3)^{\cdot}+3m_3^3H-4H\dot{f}M_\star^2\right]\ ,\\
\lambda_{\dot{H}}&\equiv 3(\dot{f}M_\star^2-m_3^3)\ .
\end{aligned} \tag{3.23}$$

Here we have simplified the expressions using eqs. (3.20) and (3.21). In the covariant language, (3.22) corresponds to the equation of motion for ϕ, as already mentioned.

In light of this equation, which constrains the possible variations we are allowed to perform, we can now revisit the variation of δm_4^2, that overall reads

$$\delta m_4^2=\frac{1}{2}[\tilde{m}_4^2-m_5^2+\dot{\tilde{m}}_6+H(\tilde{m}_6-m_6)]\delta g_b^{00}+m_6\delta H_b+\frac{1}{2}\tilde{m}_6\delta\dot{g}_b^{00}\ . \tag{3.24}$$

As already stressed, there are three quantities we can vary independently and for convenience our basis of choice in the previous section has been δg_b^{00}, $\delta\dot{g}_b^{00}$ and δH_b (equivalently we could vary $\delta\rho_m$, δp_m and δg^{00} for instance). Provided that the coefficient $\lambda_{\dot{H}}$ in eq. (3.22) is different from zero, one can also move to the basis δg_b^{00}, δH_b and $\delta\dot{H}_b$. Indeed, through that equation a variation in $\delta\dot{g}_b^{00}$ can be traded for a variation of $\delta\dot{H}_b$.

Looking at (3.24), there are no terms involving $\delta\dot{H}_b$, but instead we find $\delta\dot{g}_b^{00}$. Therefore, as long as $\lambda_{\dot{H}} \neq 0$ we can use the same procedure of the previous section and obtain the constraint (3.15). However, this leaves open the possibility of having $\lambda_{\dot{H}} = 0$, where $\delta\dot{g}_b^{00}$ is completely fixed by δg_b^{00} and δH_b.

Physically this situation corresponds to $m_3^3 = M_\star^2 \dot{f}$, or in other words no kinetic braiding ($\alpha_{\rm B} = 0$). Indeed, from its very definition, braiding relates $\dot{H}$ with $\ddot{\phi}$ (see for instance eq. (3.8) of [127]). Our condition then appears as very reasonable, since we are looking for a way to avoid the relation between the two quantities.

In going forward, let us focus on this possibility. In this case we now need to replace $\delta\dot{g}_b^{00}$ in eq. (3.24) with its expression in eq. (3.22). After doing so, we are left with the equation

$$\delta m_4^2 = \frac{1}{2}\left(\tilde{m}_4^2 - m_5^2 - m_6 H + \dot{\tilde{m}}_6 + \tilde{m}_6 H + \frac{\lambda_g}{\lambda_{\dot{g}}}\tilde{m}_6\right)\delta g_b^{00} + \left(m_6 + \frac{\lambda_H}{2\lambda_{\dot{g}}}\tilde{m}_6^2\right)\delta H_b \, . \tag{3.25}$$

Imposing the constraint $c_T = 1$ then requires the coefficients of δg_b^{00} and δH_b in the above equation to vanish. Explicitly the overall constraints are

$$2m_6(c + 2m_2^4) + 3\tilde{m}_6\left[2c + M_\star^2(\ddot{f} - H\dot{f})\right] = 0 \, , \tag{3.26}$$

$$\begin{aligned}&(c + 2m_2^4)[m_5^2 - \tilde{m}_4^2 + \dot{\tilde{m}}_6 + H(\tilde{m}_6 - m_6)] \\ &\qquad - \tilde{m}_6\left[(c + 2m_2^4)^{\cdot} + 6H(c + m_2^4) + \frac{3}{2}HM_\star^2(\ddot{f} - H\dot{f})\right] = 0 \, ,\end{aligned} \tag{3.27}$$

and overall we also have

$$m_3^3 - M_\star^2 \dot{f} = 0 \, , \tag{3.28}$$

that we used to simplify (3.26) and (3.27). These equations can be solved to find a new class of models compatible with the requirement that GWs travel at the speed of light. A possible solution for these equations is of course given by (3.15), but there are additional solutions. To make this more evident we can combine our conditions to obtain the simpler requirements

$$m_6 = -\frac{3}{2}\tilde{m}_6\frac{2c + M_\star^2(\ddot{f} - H\dot{f})}{(c + 2m_2^4)} \, , \quad \tilde{m}_6\frac{\mathrm{d}}{\mathrm{d}t}\log(c + 2m_2^4) = \dot{\tilde{m}}_6 - 2H\tilde{m}_6 + m_5^2 - \tilde{m}_4^2 \, . \tag{3.29}$$

We can notice that these models allow for Beyond Horndeski terms, since in general $\tilde{m}_4^2 \neq m_4^2$ and $\tilde{m}_6 \neq m_6$. Moreover, due to the appearance of c, the various coefficients need to be related, in a non-trivial way, with the background solution.

From the point of view of the EFT of DE, the solution for the m_i coefficients is enough to define a particular set of theories. On the other hand, in the covariant formulation, the requirement that the theory belongs to Beyond Horndeski specifies a unique model satisfying these constraints, which was identified in [146]. It is actually possible to derive this theory starting from the EFT parameters together with (3.29) (the main idea of this being that all the EFT coefficients m_i should, in a covariant theory, be related to background quantities). However showing this will not be extremely useful, so here we just verify that the particular model of [146], once translated to the EFT language, satisfies our constraints.

3.4.2 Loophole in the covariant theory

The authors of [146] argued for the existence of a Horndeski model, prematurely excluded by [48], where gravitons propagate luminally thanks to a dynamical mechanism.[2] It is indeed possible to have a tensor speed excess $\alpha_{\rm T}$, or m_4^2, different from zero but vanishing once the background equation of motion for the scalar field is imposed. The particular structure they uncover, in terms of the Horndeski functions, is the following

$$\begin{aligned} G_2 &= -3\mu W'''(\phi)X\sqrt{-X} + \sigma - \frac{\nu e^{W(\phi)}}{X} \, , \qquad G_3 = -6\mu W''(\phi)\sqrt{-X} \, , \\ G_4 &= \kappa_G + \frac{3}{2}\mu W'(\phi)\sqrt{-X} \, , \qquad G_5 = -\frac{6\mu}{\sqrt{-X}} \, , \end{aligned} \tag{3.30}$$

where κ_G, μ, ν and σ are dimensionful constants (in the notation of [146] Λ is used instead of σ) and the functions F_4, F_5 are set to zero. Also, $W(\phi)$ is a function of ϕ and primes denotes its derivatives.

The background equations for the scale-factor and for the scalar read respectively

$$6\kappa_G H^2 - 12\mu H^3 + \sigma - 3\frac{\nu e^W}{X} = \rho_m \, , \tag{3.31}$$

$$\mathcal{E}_\phi \equiv \frac{6\nu}{X^2}\left(\ddot{\phi} - H\dot{\phi} - \frac{1}{2}W'\dot{\phi}^2\right) = 0 \, . \tag{3.32}$$

Using the mapping (2.70) it is easy to obtain the coefficient m_4^2 in this theory, and it is given by

$$m_4^2 = \frac{3\mu}{\sqrt{-X}}\left(\ddot{\phi} - H\dot{\phi} - \frac{1}{2}W'\dot{\phi}^2\right) = \frac{\mu}{2\nu}(-X)^{3/2}\mathcal{E}_\phi \, . \tag{3.33}$$

By construction, this coefficient is proportional to the background equation of motion for ϕ. As promised then, this theory dynamically evades the constraints from the $c_T = 1$ measurement.

We can moreover explicitly check that the Horndeski functions (3.30) give EFT coefficients satisfying eqs. (3.26), (3.27) and (3.28). The coefficients f, $\tilde{m}_4$, m_5, m_6 and $\tilde{m}_6$ are directly

[2] For another example of dynamical mechanism leading to $c_T = 1$ see [147]. In this case the speed approaches unity only at late times.

obtained from the mapping (2.70) and (3.30). Instead, the parameter c can be obtained by comparing the background Friedmann equations (3.31) and (3.20) (the dependence on Λ can be removed by using eq. (3.21)). Finally, the parameters m_2^4 and m_3^3 are again given in terms of the Horndeski functions (see for example [127] for explicit expressions). After applying this mapping we obtain

$$\begin{aligned} &M_\star^2 f = 2\kappa_G - 6H\mu\,, \quad c = -3\mu(H\dot{H} - \ddot{H}) + \frac{\nu e^W \left(W'X + 2\ddot{\phi}\right)}{2HX\sqrt{-X}}\,, \\ &m_2^4 = -\frac{c}{2} - \frac{3\nu}{2X}e^W\,, \quad m_3^3 = -6\dot{H}\mu\,, \\ &\tilde{m}_4^2 = m_4^2\,, \quad m_5 = 0, \quad m_6 = \tilde{m}_6 = -3\mu\,. \end{aligned} \tag{3.34}$$

One can check that, by using these relations together the equation of motion $\mathcal{E}_\phi$, indeed this model solves the constraints (3.26), (3.27) and (3.28). Thus, this is a valid realization of the wider mechanism at play in the EFT of DE.

In this derivation we focused on Beyond Horndeski operators. Within the larger class of DHOST theories one could then hope to obtain new loopholes in the constraint $c_T = 1$. However, this is not possible, as we show in more detail in App. B. This negative result can be understood intuitively from the fact that the additional DHOST operators introduce a sort of kinetic braiding, mixing $\delta\dot{g}_b^{00}$ and $\delta\dot{H}_b$ at the level of the perturbed background equations [148]. As in the (Beyond) Horndeski case, the loophole relies on having no braiding, and so one is forced to set the DHOST operators to zero. Once again, one then goes back to the less exotic set of constraints (3.15).

3.4.3 Closing the loophole: the role of spatial curvature

Thus far we have considered homogeneous variations of the background quantities, and we have found that non-trivial models satisfying $c_T = 1$ can be obtained. However, as shown in [146], the presence of inhomogeneities prevents the model from properly working, since c_T does not remain unity when GWs propagate through inhomogeneities. Instead of studying the full case of $\boldsymbol{x}$-dependent variations it seems plausible that, to the end of showing that the loophole is not viable, we can approximate inhomogeneities by adding spatial curvature to the universe. As a matter of fact, [146] shows that, as far as their model is concerned, this is enough to rule out the loophole.

To reproduce this result within the EFT we just need to look at the modifications in the equation of motion for δg_b^{00} (3.22) in the presence of curvature. Indeed, it is easy to realize that there are no additional operators in eq. (2.37) contributing to δm_4^2 at linear order when turning on spatial curvature.

Let us consider a spatially curved FRW metric in Cartesian coordinates

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a(t)^2 \left(\delta_{ij} + \frac{\kappa\, x^i x^j}{1-\kappa|\boldsymbol{x}^2|}\right) \mathrm{d}x^i\,\mathrm{d}x^j \,, \tag{3.35}$$

where κ is the spatial curvature, and focus on how this affects the equation for δg_b^{00}. On the one hand operators containing only $\delta K^\nu_{\ \mu}$s are not affected by spatial curvature, in the sense that $\delta K^\nu_{\ \mu}$ starts already at the perturbation level for $\kappa = 0$. On the other hand, clearly, the background value of the spatial curvature becomes different from zero: ${}^{(3)}R^\nu_{\ \mu} = \frac{2\kappa}{a^2}\delta_\nu^{\ \mu}$. This implies that the operators $\tilde m_4^2$ and $\tilde m_6$ in (2.37) now start respectively at linear and quadratic order.

The term $\tilde m_4^2$ now contributes directly to the background equations of motion (3.20) and (3.21).[3] For what concerns the equation at linear order instead, its contribution does not involve the term $\delta\dot H$, as it is easy to realize. As we are going to motivate, this is why $\tilde m_4^2$ is not essential for understanding the failure of the loophole once $\kappa \neq 0$.

The most important operator in this discussion turns out to be $\tilde m_6$. Since this operator starts at quadratic order now, it enters in the equation of motion (3.22). Moreover, its contribution can be easily obtained by noticing that the only way it can contribute at linear order is when ${}^{(3)}R^\nu_{\ \mu}$ is evaluated on the background (otherwise we would get higher-order terms). By doing so and by using the formula for ${}^{(3)}R^\nu_{\ \mu}$ we obtain

$$-\,\tilde m_6\,\delta g^{00}\left(\delta K^\nu_{\ \mu}\,{}^{(3)}R^\mu_{\ \nu} - \frac{1}{2}\delta K\,{}^{(3)}R\right) \supset \tilde m_6\,\frac{\kappa}{a^2}\delta g^{00}\delta K \,. \tag{3.38}$$

We clearly see that the effect of $\tilde m_6$, in presence of curvature, is to shift the coefficient m_3^3. According to eq. (2.37) we have $m_3^3 \to m_3^3 - \frac{2\kappa}{a^2}\tilde m_6$.[4] This means that this operator also gives contributions in eq. (3.22) that contain $\delta\dot H$. The equation now becomes

$$\lambda_{\dot g}^{(\kappa)}\,\delta\dot g_b^{00} = \lambda_g^{(\kappa)}\delta g_b^{00} + \lambda_H^{(\kappa)}\delta H_b + \lambda_{\dot H}^{(\kappa)}\delta\dot H_b \,, \tag{3.39}$$

where the coefficients $\lambda^{(\kappa)}$ are given by

$$\begin{aligned} \lambda_{\dot g}^{(\kappa)} &\equiv \lambda_{\dot g} + \frac{3\kappa}{a^2}\tilde m_4^2\,, \quad & \lambda_g^{(\kappa)} &\equiv \lambda_g - \frac{6\kappa}{a^2}H\tilde m_4^2 - \frac{3\kappa}{a^2}(\tilde m_4^2)^{\cdot}\,, \\ \lambda_H^{(\kappa)} &\equiv \lambda_H + \frac{6\kappa}{a^2}\tilde m_4^2\,, \quad & \lambda_{\dot H}^{(\kappa)} &\equiv \lambda_{\dot H} - \frac{6\kappa}{a^2}\tilde m_6 \,. \end{aligned} \tag{3.40}$$

[3] For reference, these equations in the presence of curvature read

$$3M_\star^2\left(fH^2 + \dot f H\right) + \frac{3\kappa}{a^2}\left(M_\star^2 f + 2\tilde m_4^2\right) = \rho_m + c + \Lambda \,, \tag{3.36}$$

$$3M_\star^2\dot f\dot H + 6M_\star^2H^2\dot f + \frac{3\kappa}{a^2}\left(M_*^2\dot f + 2(\tilde m_4^2)^{\cdot} + 2H\tilde m_4^2\right) = \dot\Lambda + \dot c + 6\,cH \,. \tag{3.37}$$

[4] In order to have sizable modification of gravity these coefficients need to scale as $m_3^3 \sim HM_{\rm Pl}^2$ and $\tilde m_6 \sim M_{\rm Pl}^2H^{-1}$. On the other hand $\kappa a^{-2} \sim H^2$, therefore m_3^3 and $\tilde m_6\kappa a^{-2}$ have the same scaling.

Since there are no new contributions to δm_4^2 once curvature is turned on, we see that $\tilde{m}_6$ modifies the relation (3.28), crucial for the loophole to work. This condition then becomes

$$m_3^3 - M_\star^2 \dot{f} - \frac{2\kappa}{a^2}\tilde{m}_6 = 0 \, . \tag{3.41}$$

At this point we should demand this relation to hold for any value of κ, meaning that it should hold even for universes with a slightly different spatial curvature. The same requirement arises also because κ can be interpreted as the effect of inhomogeneities with very small momenta, and so we need c_T not to be spoiled by inhomogeneities along the path of the GW. This requirement then forces $\tilde{m}_6$ to vanish or, in the covariant language, $G_{5,X} = 0$. This is consistent with the conclusions of [146], and the model (3.30) reduces to a subclass of the models already available from [48]. Also in the more general case of the EFT the same happens, and all the conditions (3.29) collapse to (3.15). As done in [141], this conclusion can be further motivated by a more detailed analysis of inhomogeneities.

4 | Gravitational Wave Decay into Dark Energy

The observation of the gravitational-wave event GW170817 and its EM counterpart, presented in the previous chapter, started the detailed study of the propagation of GWs. At variance with a cosmological constant—the simplest explanation of the present acceleration—models of DE act as a sort of "medium", through which GWs travel. In the same way one uses the propagation of EM waves to study a material, GWs propagating through DE can be used to test these theories. Like a normal material, DE defines a preferred frame and thus spontaneously breaks Lorentz invariance. This implies that in general the speed of GWs may be different from the speed of light. The recent observations put severe bounds on this possibility and therefore strong constraints on some DE models, as seen in Sec. 3.3.

In this chapter, following the results in [1], we want to study another phenomenon that is possible due to the breaking of Lorentz invariance: the decay of GW into DE fluctuations. In a Lorentz invariant theory, a massless particle can only decay into two or more massless particles with all momenta exactly aligned. Measurable quantities must be summed over these collinear emissions to get rid of spurious IR divergences [149, 58]. Once Lorentz invariance is broken, the excitations of DE will in general move at a speed different from the one of gravitons and the decay is allowed. For other works studying the damping of gravitational waves see e.g. [150–155].

We will study the decay of gravitons in the framework of the EFT of DE. We specify it to the subset of theories with GWs travelling at the speed of light since the others are not compatible with the recent data in Sec. 4.1 (assuming that the regime of validity of the EFT of DE encompasses the LIGO/Virgo scales [145]). In App. C we discuss the invariance of the results under a disformal transformation.

In Sec. 4.2 we derive the cubic coupling $\gamma\pi\pi$ (where π describes the DE fluctuations, as in Sec. 2.3) and compute the decay rate of the process $\gamma \to \pi\pi$. The process turns out to be very large and thus incompatible with observations, for a particular operator of the EFT of DE: $\frac{1}{2}\tilde{m}_4^2(t)\,\delta g^{00}\left({}^{(3)}R + \delta K_\mu^{\ \nu}\delta K_\nu^{\ \mu} - \delta K^2\right)$. This conclusion holds if $\tilde{m}_4$ is large enough to play any role in modifying gravity and potentially affecting large-scale structure measurements. In the framework of GLPV theories, setting $\tilde{m}_4 = 0$ (or equivalently $\alpha_{\rm H} = 0$, where $\alpha_{\rm H}$ is defined in eq. (2.71)) and requiring GWs to travel at the speed of light corresponds in the covariant language to restricting to Horndeski up to the cubic Lagrangian (in particular no beyond Horndeski terms survive). First the calculations are performed in Newtonian gauge, while in App. D they are done in the spatially flat gauge. Afterwards, in Sec. 4.3, we provide the derivation of the coupling $\gamma\gamma\pi$ and the computation of the decay rate of $\gamma \to \gamma\pi$, and this turns out to be subdominant.

The coupling $\gamma\pi\pi$ can be used to make a loop with external γ legs: in other words, as we study in Sec. 4.4, at one-loop the graviton propagator is corrected. The calculable, i.e. log-divergent, corrections give a sizeable dispersion of gravitational waves: a higher-dimension operator that is quadratic in the graviton and violates Lorentz-invariance is generated. Also, this effect can be used to rule out the operator proportional to $\tilde{m}_4$. We study in general the radiative generation of higher dimension operators that can correct the graviton propagation. This allows one to rule out the operator $\tilde{m}_4$ even when the speed of π is larger or equal to the speed of GWs and the decay $\gamma \to \pi\pi$ is impossible. The remaining theories, corresponding to Horndeski up to cubic order, do not generate sizeable higher derivative corrections.

4.1 Free theory

To be compatible with the constraints from the GW170817 event, in the following we will assume that gravitational waves propagate at the speed of light, $c_T = 1$. Moreover, the function f in (2.37) can be set to be constant by a conformal redefinition of the metric, which does not change the speeds of propagation. In general, the conformal transformation changes the couplings between matter and the DE field but the interactions between gravitons and DE do not depend on these matter couplings. Therefore, there is no loss of generality in choosing this frame. We will discuss the constraints in a more generic frame in App. C.

With the above assumptions and this last simplification, the unitary-gauge action becomes

$$\begin{aligned} S = \int \Big[& \frac{M_{\rm Pl}^2}{2} R - \Lambda - c\, g^{00} + \frac{m_2^4}{2} (\delta g^{00})^2 - \frac{m_3^3}{2}\, \delta K \delta g^{00} \\ & + \frac{\tilde{m}_4^2}{2}\, \delta g^{00} \left({}^{(3)}\!R + \delta K_\mu^{\ \nu} \delta K_\nu^{\ \mu} - \delta K^2 \right) \Big] \sqrt{-g}\, \mathrm{d}^4 x \,, \end{aligned} \tag{4.1}$$

where the (time-independent) Planck mass squared is $M_{\rm Pl}^2 = M_\star^2 f$. For simplicity, in the following we will assume that the mass scales m_3^3 and $\tilde{m}_4^2$ are time independent but taking into account their slow time dependence is straightforward. Recall that to have sizeable effects for structure formation one typically needs $m_2^4 \sim M_{\rm Pl}^2 H_0^2$, $m_3^3 \sim M_{\rm Pl}^2 H_0$ and $\tilde{m}_4 \sim M_{\rm Pl}$.

We will first expand the action at quadratic order and then derive the graviton-scalar interactions in Sec. 4.2. For later convenience, it is useful to use the standard ADM metric decomposition, where the metric line element is given by eq. (2.11). Moreover, from the extrinsic curvature one can pull-out the dependence on the lapse N and define $K_{ij} = \frac{1}{N} E_{ij}$.

In this section we will work in Newtonian gauge, defined by

$$N^2 = 1 + 2\Phi \,, \qquad N_i = 0 \,, \qquad h_{ij} = a^2(t)(1 - 2\Psi)(e^\gamma)_{ij} \,, \tag{4.2}$$

with $\partial_i \gamma_{ij} = \gamma_{ii} = 0$. The derivation of the quadratic action and the graviton-scalar interactions in spatially flat gauge is left to App. D.

The time-diffeomorphism invariance of the action can be restored by the usual Stueckelberg trick, introduced in Sec. 2.3. By indicating the Goldstone bosons of broken time diffeomorphisms again by π, under a time coordinate change $t \to t + \pi(x)$ we recall that the relevant operators of the EFT action reintroduce π as given in eqs. (2.25), (2.32) and (2.33).

Varying the action with respect to Φ and focussing on the sub-Hubble limit by keeping only the leading terms in spatial derivatives, one obtains

$$2M_{\rm Pl}^2 \nabla^2 \Psi + m_3^3 \nabla^2 \pi + 4\tilde{m}_4^2 \nabla^2 (\Psi + H\pi) = 0 \,, \tag{4.3}$$

which can be solved for Ψ in terms of π,

$$\Psi = -\frac{m_3^3 + 4\tilde{m}_4^2 H}{2(M_{\rm Pl}^2 + 2\tilde{m}_4^2)} \pi \,. \tag{4.4}$$

Variation with respect to Ψ in the same limit yields

$$M_{\rm Pl}^2 \nabla^2 (\Phi - \Psi) + 2\tilde{m}_4^2 \nabla^2 (\Phi - \dot{\pi}) = 0 \,. \tag{4.5}$$

Since the frequencies involved in the gravitational wave experiments that concern us here are much higher than the Hubble rate, one can focus on the highest number of time derivatives per field (i.e. we assume $H\pi \ll \dot{\pi}$) and express Φ in terms of $\dot{\pi}$,

$$\Phi = \frac{2\tilde{m}_4^2}{M_{\rm Pl}^2 + 2\tilde{m}_4^2} \dot{\pi} \,. \tag{4.6}$$

Plugging these solutions back into the action one obtains (see also [73])

$$S_\pi^{(2)} = \int {\rm d}^4 x M_{\rm Pl}^2 \frac{3m_3^6 + 4M_{\rm Pl}^2 (c + 2m_2^4)}{4(M_{\rm Pl}^2 + 2\tilde{m}_4^2)^2} \left[\dot{\pi}^2 - c_s^2 (\partial_i \pi)^2 \right] \,, \tag{4.7}$$

where c_s^2 is the speed of sound squared, which is given by

$$c_s^2 = \frac{4\left(M_{\rm Pl}^2 + 2\tilde{m}_4^2\right)^2 c - M_{\rm Pl}^2 \left(m_3^3 - 2M_{\rm Pl}^2 H\right)\left(m_3^3 + 4\tilde{m}_4^2 H\right) + 8M_{\rm Pl}^2 \tilde{m}_4^2 \left(M_{\rm Pl}^2 + 2\tilde{m}_4^2\right) \dot{H}}{M_{\rm Pl}^2 \left[3m_3^6 + 4M_{\rm Pl}^2 (c + 2m_2^4)\right]} \,. \tag{4.8}$$

Once again, we consider frequencies much higher than H: we can assume that we are in Minkowski spacetime and set $a = 1$, as we did for eq. (4.7).

A comment is in order here. It is a known peculiar feature of Beyond Horndeski theories that the dynamics of π is affected by the mixing with matter fluctuations [77, 132]. However, this mixing is neglected in eq. (4.7) by neglecting matter fluctuations in its derivation, in eqs. (4.4) and (4.6). This is justified by the fact that the mixing would depend on the local environment and on scales of order 1000 km one cannot rely on small perturbations around the cosmological average value. Since the mixing depends on the position, in the following the coefficients of the π action, in particular the speed of sound, should be considered as weakly position-dependent. This approximation does not change our conclusions, however. (Neglecting matter fluctuations becomes exact in the limit in which DE dominates in the Friedmann equations.)

Getting back to eq. (4.7), we define the canonically normalized field π_c as

$$\pi_c \equiv \frac{M_{\rm Pl}\big[3m_3^6 + 4M_{\rm Pl}^2(c + 2m_2^4)\big]^{\frac{1}{2}}}{\sqrt{2}(M_{\rm Pl}^2 + 2\tilde{m}_4^2)}\pi \ . \tag{4.9}$$

The quadratic action for the graviton can be found by expanding the Einstein-Hilbert term. One gets

$$S_\gamma^{(2)} = \int {\rm d}^4x \frac{M_{\rm Pl}^2}{8}\left[\dot{\gamma}_{ij}^2 - (\partial_k\gamma_{ij})^2\right] \ . \tag{4.10}$$

Defining the Fourier decomposition of γ_{ij} as

$$\gamma_{ij}(t,\boldsymbol{x}) = \int \frac{{\rm d}^3\boldsymbol{k}}{(2\pi)^3} \sum_{\sigma=\pm} \epsilon_{ij}^\sigma(\boldsymbol{k})\gamma_{\boldsymbol{k}}^\sigma(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \ , \tag{4.11}$$

where $+$ and $-$ are the two polarizations of the graviton, with

$$\epsilon_{ij}^\sigma(\boldsymbol{k})\delta^{ij} = k^i\epsilon_{ij}^\sigma(\boldsymbol{k}) = 0 \ , \qquad \epsilon_{ij}^\sigma(\boldsymbol{k})\epsilon_{ij}^{\star\sigma'}(\boldsymbol{k}) = 2\delta_{\sigma\sigma'} \ , \tag{4.12}$$

the canonical normalized Fourier modes of the graviton are

$$\gamma_{ij}^c \equiv \frac{M_{\rm Pl}}{\sqrt{2}}\gamma_{ij} \ . \tag{4.13}$$

For later convenience, we note that the tensor product of two polarizations has to be transverse in each of its indices and traceless in two couples of indices. It is thus given by

$$\sum_{\sigma=\pm} \epsilon_{ij}^\sigma(\boldsymbol{k})\epsilon_{mn}^{\star\sigma}(\boldsymbol{k}) = \lambda_{im}\lambda_{jn} + \lambda_{in}\lambda_{jm} - \lambda_{ij}\lambda_{mn} \ , \qquad \lambda_{ij} \equiv \delta_{ij} - \frac{k_ik_j}{\boldsymbol{k}^2} \ . \tag{4.14}$$

4.2 Graviton decay into $\pi\pi$

As mentioned at the beginning of the chapter, the dominant decay channel is the decay of gravitational waves into two scalar fluctuations. In this section, we compute the interaction vertex and the rate associated with this decay.

4.2.1 Interaction vertex $\gamma\pi\pi$

Let us compute the cubic vertex of the interaction $\gamma\pi\pi$ in the gauge specified in (4.2). To obtain cubic vertices using the ADM splitting we loosely follow [156]: at this order we only need the solution to the constraint equations for Φ and Ψ at linear order. Although in this gauge Φ and Ψ are not Lagrange multipliers, the procedure remains unaffected but, as a check, we will also perform the calculation in another gauge in App. D.

We first inspect the Einstein-Hilbert term in the action (4.1), to see if it can generate such a coupling. Since the 4d Ricci scalar is 4d diffeomorphism invariant, we do not need to perform the Stueckelberg trick on it. We can decompose it in the $3+1$ quantities using the Gauss-Codazzi relation, i.e.

$$S_{\rm EH} = \frac{M_{\rm Pl}^2}{2}\int \left[N\, {}^{(3)}\!R + N^{-1}\left(E_{ij}E^{ij} - E^2\right)\right]\sqrt{h}\,{\rm d}^4x\,. \tag{4.15}$$

One can verify that the 3d scalar quantities $\sqrt{h}$, E and $E_{ij}E^{ij}$ do not yield any contribution linear in γ_{ij}. While ${}^{(3)}\!R$ gives a term linear in γ_{ij}, this contains fewer derivatives than the terms discussed below. Therefore, we disregard $S_{\rm EH}$.

Discarding the operators proportional to Λ, c, m_2^4 and m_3^3, which do not contain linear terms in γ_{ij}, we focus on the operator proportional to $\tilde{m}_4^2$, whose contribution to the action is

$$S_{\tilde{m}_4} = \frac{\tilde{m}_4^2}{2}\int \delta g^{00}\left[{}^{(3)}\!R + \delta K_{ij}\delta K^{ij} - \delta K^2\right] N\sqrt{h}\,{\rm d}^4x\,. \tag{4.16}$$

For the 3d Ricci, eq. (2.33) is not enough, since we need to perform the Stueckelberg trick at linear order in both π and in γ. Starting from the linear expression ${}^{(3)}\!R = \partial_i\partial_j h_{ij} - \nabla^2 h$ and using the following transformations under a time-diffeomorphism,

$$h_{ij} \to h_{ij} - N_i\partial_j\pi - N_j\partial_i\pi + \mathcal{O}(\pi^2)\,, \tag{4.17}$$

$$\partial_i \to \partial_i - \partial_i\pi\partial_0 + \mathcal{O}(\pi^2)\,, \tag{4.18}$$

one gets, neglecting the expansion of the universe,

$${}^{(3)}\!R \to {}^{(3)}\!R - 2\partial_i\partial_j(N_i\partial_j\pi) + 2\nabla^2(N_i\partial_i\pi) - \partial_i\pi\partial_j\dot{h}_{ij} - \partial_i(\partial_j\pi\dot{h}_{ij}) + \partial_i(\partial_i\pi\dot{h}) + \partial_i\pi\partial_i\dot{h}\,, \tag{4.19}$$

which in our gauge becomes

$$ {}^{(3)}R \to {}^{(3)}R - \dot{\gamma}_{ij}\partial_i\partial_j\pi \ . \tag{4.20} $$

By multiplying by δg^{00} after the Stueckelberg trick (see eq.(2.25)) this term generates the following contribution to the action

$$ -\frac{\tilde{m}_4^2}{2}\int (2\Phi - 2\dot{\pi})\dot{\gamma}_{ij}\partial_i\partial_j\pi \ \mathrm{d}^4x \ , \tag{4.21} $$

where we have retained only terms with the highest number of time derivatives.

For the terms quadratic in the extrinsic curvature in the bracket of eq. (4.16), it is enough to use the linear Stueckelberg trick, eq. (2.32). While δK^2 does not generate terms linear in γ_{ij} unsuppressed by H, $\delta K_{ij}\delta K^{ij}$ generates $-\dot{\gamma}_{ij}\partial_i\partial_j\pi$. Multiplying by δg^{00}, this gives an identical contribution as eq. (4.21). Replacing Φ using eq. (4.6) and integrating by parts, we finally obtain

$$ S_{\gamma\pi\pi} = \frac{M_{\rm Pl}^2\tilde{m}_4^2}{M_{\rm Pl}^2 + 2\tilde{m}_4^2}\int \ddot{\gamma}_{ij}\partial_i\pi\partial_j\pi \ \mathrm{d}^4x \ . \tag{4.22} $$

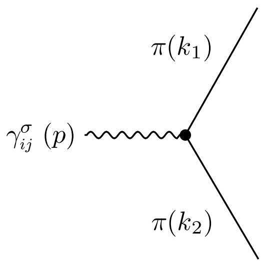
Using the canonically normalized fields defined in eqs. (4.9) and (4.13), the interaction vertex (4.22) becomes

$$ \mathscr{L}_{\gamma\pi\pi} = \frac{1}{\Lambda_\star^3}\ddot{\gamma}^c_{ij}\partial_i\pi_c\partial_j\pi_c \ , \tag{4.23} $$

with

$$ \Lambda_\star^3 \equiv M_{\rm Pl}\frac{3m_3^6 + 4M_{\rm Pl}^2(c + 2m_2^4)}{2\sqrt{2}\,\tilde{m}_4^2(M_{\rm Pl}^2 + 2\tilde{m}_4^2)} \ . \tag{4.24} $$

In the following we denote by p^μ, k_1^μ and k_2^μ, respectively the 4-momentum of the decaying graviton and of the two π fields in the final state. Therefore, in diagrammatic form in Fourier space, for a given polarization σ the interaction vertex reads



$$ = 2 \times \frac{1}{\Lambda_\star^3}p^2\, k_{1m}k_{2n}\left[\frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) - \frac{1}{3}\delta_{ij}\delta_{mn}\right] \ , \tag{4.25} $$

where the factor of 2 comes from the two possibilities of associating k_1 and k_2.

4.2.2 Decay rate

Let us define the matrix element $i\mathscr{A}$ for a given polarization state σ as

$$\langle\{p,\sigma\};\mathrm{in}|k_1,k_2;\mathrm{out}\rangle \equiv (2\pi)^4\delta^{(4)}(p^\mu - k_1^\mu - k_2^\mu)\, i\mathscr{A} \ . \tag{4.26}$$

The decay rate reads

$$\Gamma_{\gamma\to\pi\pi} = \frac{1}{2}\times\frac{1}{2E_p}\int\frac{\mathrm{d}^3\boldsymbol{k}_1}{(2\pi)^3 2E_{k_1}}\frac{\mathrm{d}^3\boldsymbol{k}_2}{(2\pi)^3 2E_{k_2}}(2\pi)^4\delta^{(4)}(p^\mu - k_1^\mu - k_2^\mu)\,\langle|i\mathscr{A}|^2\rangle \ , \tag{4.27}$$

(the factor 1/2 in front of the integral comes from considering identical final particles) where, for any 4-vector q^μ, E_q denotes its time component and $\langle|i\mathscr{A}|^2\rangle$ is the square of the matrix element $i\mathscr{A}$ averaged over all possible initial polarizations for the in-state. Before evaluating this explicitly, we can simplify the integral.

Integrating over $\mathrm{d}^3\boldsymbol{k}_2$ removes $\delta^{(3)}(\boldsymbol{p}-\boldsymbol{k}_1-\boldsymbol{k}_2)$. Then, let us define $p\equiv|\boldsymbol{p}|$, $k_1\equiv|\boldsymbol{k}_1|$ and $k_2\equiv|\boldsymbol{k}_2|$. Integrating over $\mathrm{d}k_1$ using the on-shell conditions (we neglect the mass of π assuming that it is much smaller than the typical frequency under consideration)

$$E_p = p \ , \qquad E_{k_1} = c_s k_1 \ , \qquad E_{k_2} = c_s k_2 \ , \tag{4.28}$$

removes $\delta(E_p - E_{k_1} - E_{k_2})$. For this last step, it is convenient to define $\Omega\equiv\boldsymbol{k}_1\cdot\boldsymbol{p}/(k_1 p)$ and express k_2 in terms of k_1 as

$$k_2 = \sqrt{k_1^2 + p^2 - 2pk_1\Omega} \ . \tag{4.29}$$

Finally, assuming $0 < c_s < 1$ and expressing k_1 in terms of p and Ω using

$$k_1 = \frac{p(1-c_s^2)}{2c_s(1-c_s\Omega)} \ , \tag{4.30}$$

one obtains

$$\Gamma_{\gamma\to\pi\pi} = \frac{1}{4p}\frac{1}{16\pi c_s^3}\int_{-1}^{1}\frac{1-c_s^2}{(1-c_s\Omega)^2}\,\langle|i\mathscr{A}|^2\rangle\ \mathrm{d}\Omega \ . \tag{4.31}$$

We can now compute $\langle|i\mathscr{A}|^2\rangle$. The matrix element of the vertex (4.25) is

$$i\mathscr{A} = \frac{2i}{\Lambda_\star^3}p^2\, k_{1i}k_{2j}\,\epsilon_{ij}^{\star\sigma}(\boldsymbol{p}) \ . \tag{4.32}$$

Averaging over all possible initial polarizations for the in-state, using energy-momentum conservation, $k_2^\mu = p^\mu - k_1^\mu$, the transversality of the polarization tensor, $p^i\epsilon_{ij}^\sigma(\boldsymbol{p}) = 0$, and eq. (4.14), we find

$$\langle|i\mathscr{A}|^2\rangle \equiv \frac{1}{2}\sum_{\sigma=\pm}|i\mathscr{A}|^2 = \frac{2p^4}{\Lambda_\star^6}k_1^4(1-\Omega^2)^2 \ . \tag{4.33}$$

Finally, replacing this expression inside the integral and integrating over $\mathrm{d}\Omega$ using eq. (4.30) we obtain

$$\Gamma_{\gamma\to\pi\pi} = \frac{p^7(1-c_s^2)^2}{480\pi c_s^7 \Lambda_\star^6} \,. \tag{4.34}$$

At this point we can consider how this result applies for GWs at interferometers. Up to now there are no indications from data of deviations from the expected signals from GR, therefore any deviation of the waveform below, say, few percent are to be considered ruled-out. For LIGO/Virgo observations we have $p \sim \Lambda_3$. Requiring that the GWs are stable over cosmological distances $\sim H_0^{-1}$, one gets

$$\frac{\Lambda_3}{H_0}\left(\frac{\Lambda_3}{\Lambda_\star}\right)^6 \frac{(1-c_s^2)^2}{480\pi c_s^7} \sim 10^{20}\left(\frac{\Lambda_3}{\Lambda_\star}\right)^6 \frac{(1-c_s^2)^2}{480\pi c_s^7} \lesssim 1 \,, \tag{4.35}$$

which implies that $\Lambda_\star \gg \Lambda_3$. To compare with large-scale structure constraints, we can write the scale $\Lambda_\star$ in terms of quantities constrained by observations. In particular, using the definitions of the α coefficients in (2.71) one finds

$$\left(\frac{\Lambda_3}{\Lambda_\star}\right)^3 = \frac{\alpha_{\rm H}(1+\alpha_{\rm H})}{\sqrt{2}\,\alpha} \,, \tag{4.36}$$

so that, from eq. (4.35), $\alpha_{\rm H}$—and thus $\tilde m_4^2$—must vanish for any practical purpose. Notice that one cannot avoid this conclusion taking α very large: this limit corresponds to $c_s \ll 1$ and further enhances the decay rate eq. (4.34). Moreover, in the same way one cannot take $\alpha_{\rm H}$ close to -1. Indeed, in this case the speed of sound squared becomes negative, as one can see from eq. (4.8), and the system is unstable.

For interesting values of $\tilde m_4^2$ the decay rate of the GWs is so large that no wave will reach the detector. For this reason it is not worthwhile to look at the precise effects on the luminosity distance as a function of the frequency. Concerning the produced scalar modes, these will not form a possibly detectable burst, but they will be emitted in different directions and spread in space. Notice also that our perturbative calculation does not take into account the presence of a large number of quanta giving rise to a classical wave: coherent effects will further enhance the loss of energy into scalar waves. This effect will be discussed in the next chapter.

Before concluding the section, let us briefly discuss the case $c_s \geq 1$. For $c_s^2 = 1$, energy-momentum conservation implies that the πs are collinear with γ, i.e. $\Omega = 1$. In this configuration the decay is forbidden by the conservation of the angular momentum: the graviton with helicity 2 cannot decay collinearly into scalar particles. (Indeed, in this limit the interaction (4.32) vanishes by the transversality of the graviton polarization.) Instead, the case $c_s > 1$ is kinematically forbidden by energy-momentum conservation. We will discuss in Sec. 4.4 that also in the case $c_s \geq 1$ the operator proportional to $\tilde m_4^2$ must be negligibly small.

4.3 Graviton decay into $\gamma\pi$

4.3.1 Interaction $\gamma\gamma\pi$

To compute the cubic vertex of the interaction $\gamma\gamma\pi$ in (4.1), we proceed analogously to what we did for $\gamma\pi\pi$. Let us start once more from the Einstein-Hilbert term, eq. (4.15). Focussing on the terms containing γ_{ij}, it is easy to verify that

$$ {}^{(3)}R \supset -\frac{1}{4}\left(\partial_k\gamma_{ij}\right)^2 + \mathcal{O}(\gamma^3) , \qquad E_{ij}E^{ij} \supset \frac{1}{4}\left(\dot\gamma_{ij}\right)^2 + \mathcal{O}(\gamma^3) , \tag{4.37} $$

while $E^2 \supset \mathcal{O}(\gamma^3)$. The Einstein-Hilbert term is covariant, so we don't need to apply the Stueckelberg procedure to it. Therefore, using $N = 1 - 2\Phi + \mathcal{O}(\Phi^2)$, it only contributes with

$$ -\frac{M_{\rm Pl}^2}{8}\int \Phi\left[\left(\partial_k\gamma_{ij}\right)^2 + \left(\dot\gamma_{ij}\right)^2\right] \mathrm{d}^4x \; . \tag{4.38} $$

Analogously we can compute the contribution from the operator $\tilde m_4^2$, eq. (4.16). We find the cubic operator

$$ \frac{\tilde m_4^2}{8}\int (2\Phi - 2\dot\pi)\left[-\left(\partial_k\gamma_{ij}\right)^2 + \left(\dot\gamma_{ij}\right)^2\right] \mathrm{d}^4x \; . \tag{4.39} $$

Combining these two contributions and replacing Φ by using eq. (4.6), we obtain

$$ S_{\gamma\gamma\pi} = -\frac{M_{\rm Pl}^2\tilde m_4^2}{2(M_{\rm Pl}^2 + 2\tilde m_4^2)}\int \dot\pi\dot\gamma_{ij}^2 \; \mathrm{d}^4x \; . \tag{4.40} $$

Note that, despite appearances, this vertex does not change the speed of propagation of gravitons, even in the presence of a background of $\dot\pi$. Indeed, this vertex comes from the contribution in eq. (4.39), which just modifies the normalization of γ, and from the contribution of the Einstein-Hilbert term, eq. (4.38). The latter expresses the coupling between the kinetic terms of the graviton and the scalar metric Φ, which deforms the graviton-cone. But the same coupling and deformation are also experienced by minimally (or conformally) coupled photons and matter so that at the end gravitons travel on the light-cone.

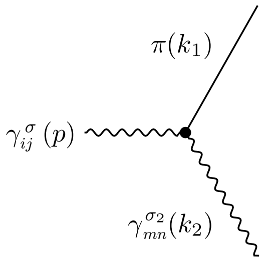
In terms of canonically normalized fields, the interaction vertex becomes

$$ \mathscr{L}_{\gamma\gamma\pi} = -\frac{1}{\Lambda_{\gamma\gamma\pi}^2}\dot\pi_c(\dot\gamma_{ij}^c)^2 \; , \tag{4.41} $$

with

$$ \Lambda_{\gamma\gamma\pi}^2 \equiv \frac{M_{\rm Pl}}{\sqrt{2}\,\tilde m_4^2}\left[3m_3^6 + 4M_{\rm Pl}^2(c + 2m_2^4)\right]^{\frac{1}{2}} \; . \tag{4.42} $$

Denoting by p^μ, k_1^μ and k_2^μ respectively the 4-momentum of the decaying graviton, of the π field and of the graviton in the final state, in diagrammatic form the vertex reads



$$= -\frac{2i^4}{\Lambda^2_{\gamma\gamma\pi}} E_p E_{k_1} E_{k_2} \left[\frac{1}{2}\left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right) - \frac{1}{3}\delta_{ij}\delta_{mn}\right] . \tag{4.43}$$

Note that this vertex has fewer derivatives than the vertex for $\gamma\pi\pi$, see eq. (4.25), and the scale $\Lambda_{\gamma\gamma\pi}$ is much larger than $\Lambda_\star$ defined in eq. (4.24), i.e. $\Lambda_{\gamma\gamma\pi} \sim \Lambda_2 \gg \Lambda_\star \sim \Lambda_3$. Thus, we expect a smaller decay rate than the one from $\gamma\pi\pi$ and a weaker constraint on $\tilde{m}_4^2$. We will come back to this point at the end of the section.

4.3.2 Decay rate

The decay rate reads

$$\Gamma_{\gamma\to\gamma\pi} = \frac{1}{2E_p} \int \frac{\mathrm{d}^3\boldsymbol{k}_1}{(2\pi)^3 2E_{k_1}} \frac{\mathrm{d}^3\boldsymbol{k}_2}{(2\pi)^3 2E_{k_2}} (2\pi)^4 \delta^{(4)}(p^\mu - k_1^\mu - k_2^\mu) \left\langle |i\mathscr{A}|^2 \right\rangle , \tag{4.44}$$

where $\langle |i\mathscr{A}|^2\rangle$ is the matrix element squared and averaged over the polarizations of the initial and final states. As done in Sec. 4.2.2, we can remove $\delta^{(3)}(\boldsymbol{p} - \boldsymbol{k}_1 - \boldsymbol{k}_2)$ by integrating over $\mathrm{d}^3\boldsymbol{k}_2$. Moreover, integrating over $\mathrm{d}k_1$ using the on-shell conditions

$$E_p = p , \qquad E_{k_1} = c_s k_1 , \qquad E_{k_2} = k_2 , \tag{4.45}$$

we can remove $\delta(E_p - E_{k_1} - E_{k_2})$. To do that, we express k_2 in terms of k_1 and $\Omega = \boldsymbol{p}\cdot\boldsymbol{k}_1/(pk_1)$ using eq. (4.29). In the following we assume $0 < c_s \leq \Omega$; the case $c_s > \Omega$, and thus $c_s > 1$, is kinematically forbidden. Replacing k_1 using

$$k_1 = \frac{2p(\Omega - c_s)}{1 - c_s^2} , \tag{4.46}$$

we obtain

$$\Gamma_{\gamma\to\gamma\pi} = \frac{1}{2p} \frac{1}{4\pi c_s (1 - c_s^2)} \int_{c_s}^{1} \left\langle |i\mathscr{A}|^2 \right\rangle \mathrm{d}\Omega . \tag{4.47}$$

Let us now compute $\langle |i\mathscr{A}|^2\rangle$. This is given by

$$\left\langle |i\mathscr{A}|^2 \right\rangle = \frac{1}{2} \sum_{\sigma=\pm} \sum_{\sigma_2=\pm} |i\mathscr{A}|^2 , \tag{4.48}$$

where the tree-level amplitude reads

$$i\mathscr{A} = -\frac{2}{\Lambda^2_{\gamma\gamma\pi}} E_p E_{k_1} E_{k_2}\, \epsilon^{\star\sigma}_{ij}(\boldsymbol{p})\epsilon^{\sigma_2}_{ij}(\boldsymbol{k}_2) \ . \tag{4.49}$$

Using this expression, eq. (4.14) and the transversality condition, after some straightforward algebra we find

$$\langle |i\mathscr{A}|^2\rangle = \frac{2}{\Lambda^4_{\gamma\gamma\pi}}(c_s\, p\, k_1 k_2)^2 \left[3 + 6\frac{(\boldsymbol{k}_2\cdot\boldsymbol{p})^2}{\boldsymbol{k}_2^2\,\boldsymbol{p}^2} + \frac{(\boldsymbol{k}_2\cdot\boldsymbol{p})^4}{\boldsymbol{k}_2^4\,\boldsymbol{p}^4}\right] \ . \tag{4.50}$$

After we replace this result in (4.47) and perform the integral over $\mathrm{d}\Omega$ we find

$$\Gamma_{\gamma\to\gamma\pi} = \frac{p^5}{32\pi\Lambda^4_{\gamma\gamma\pi}}\mathcal{F}(c_s) \ , \tag{4.51}$$

where

$$\begin{aligned}\mathcal{F}(c_s) \equiv & -\frac{1}{c_s^{10}}(1-c_s^2)^3(5-c_s^2)(\tanh^{-1}c_s+1) + \frac{1}{c_s^{10}(1+c_s)^5}\Bigg[5+30c_s+59c_s^2+\frac{17c_s^3}{3} \\ & -\frac{416c_s^4}{3} - \frac{509c_s^5}{3} + 22c_s^6 + \frac{6458c_s^7}{35} + \frac{2329c_s^8}{21} - \frac{10496c_s^9}{315} - \frac{3791c_s^{10}}{63} \\ & + \frac{3743c_s^{11}}{105} - \frac{3782c_s^{12}}{63} + \frac{3263c_s^{13}}{63}\Bigg] \ .\end{aligned} \tag{4.52}$$

The function $\mathcal{F}(c_s)$ vanishes for $c_s = 0$ and reaches its maximum value $\mathcal{F}(c_s^{\max}) \approx 3.50$ at $c_s^{\max} \approx 0.19$. Additionally, $\mathcal{F}(1) = 4/3$.

Applying this result to LIGO/Virgo energies, $p \sim \Lambda_3$, and requiring that the decay is slower than a Hubble time one gets

$$\left(\frac{\Lambda_3}{\Lambda_2}\right)^2\left(\frac{\Lambda_2}{\Lambda_{\gamma\gamma\pi}}\right)^4\frac{\mathcal{F}(c_s)}{32\pi} \sim 10^{-20}\left(\frac{\Lambda_2}{\Lambda_{\gamma\gamma\pi}}\right)^4 \lesssim 1 \ . \tag{4.53}$$

Since

$$\left(\frac{\Lambda_2}{\Lambda_{\gamma\gamma\pi}}\right)^4 = \frac{\alpha_{\mathrm{H}}^2}{4\,\alpha} \ , \tag{4.54}$$

the constraint on α_{H}, and thus on $\tilde{m}_4^2$, is rather weak. Given this suppression, in the following we will focus on the coupling $\gamma\pi\pi$ instead.

4.4 Loop corrections and dispersion

We now move to study the loop corrections to the graviton propagator induced by the coupling $\gamma\pi\pi$, since is the dominant decay channel. As argued in [48] and explained in chapter 3, setting

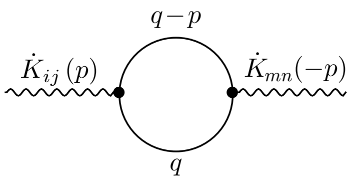
$c_T = 1$ is stable under quantum corrections. However, since Lorentz invariance is spontaneously broken, loop corrections could modify the dispersion relation of gravitons (i.e. provide an energy-dependent phase-velocity) at a level in principle detectable by current gravitational waves experiments [157, 158]. The bounds on a possible non-trivial dispersion are even tighter than the ones on c_T since they rely on the comparison among different frequencies and are not limited by the astrophysical uncertainty on the emission time. Moreover, the result for the decay rate obtained in the previous section suggests that these dispersion effects are of a conspicuous size. Indeed, it is well known that absorption is often accompanied by dispersion effects of the same magnitude. In this section we want to investigate these effects by computing loop-corrections to the graviton propagator and look at possible higher-derivative corrections.

4.4.1 Graviton self-energy

As already done for the decay rate, we focus on the interaction vertex (4.25), which turns out to be the dominant coupling at the energy scales relevant for gravitational wave experiments. The corresponding term in the action can be cast in a manifestly 3-dimensional covariant form as $\sim \nabla^0 K_{\mu\nu}\partial^\mu\pi\partial^\nu\pi$. Therefore, operators generated at loop-level from this interaction do preserve diffeomorphism invariance. To better motivate this let us open a brief parenthesis. Radiative corrections will generate terms that are manifestly invariant under time-dependent spatial diffs. In the vertex $\delta g^{00}\delta K_{ij}\delta K^{ij}$ one has an external δK_{ij} leg, which is explicitly covariant under time-dependent spacial diffs. One can integrate by parts and move derivatives that act on the internal πs on the external leg: this shows that the operator δK_{ij}^2 is not renormalized in compliance with the non-renormalization theorem of Galileons discussed in Sec. 2.5. Since we are interested in the effect on the propagation of gravitational waves, we disregard spatial derivatives acting on K_{ij}: these will contribute to operators that depend on $\partial_i K_{ij}$, and these cannot affect gravitational waves, since they are transverse. The external leg can thus be taken of the form $\partial_0 K_{ij}$. (Invariance under time-dependent spatial diffs at all orders implies one gets a structure $\nabla^0 K_{ij}$; here we disregard higher-order terms.)

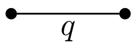
Things are less transparent for the interaction $\delta g^{00\,(3)}R$. In the calculation one has to take out of ${}^{(3)}R$ a gravitational wave and a scalar so that one cannot keep objects that are explicitly covariant under time-dependent spatial diffs. To check the invariance it is useful to look at the terms linear in π that originate from ${}^{(3)}R$, eq. (4.19). One can explicitly check the invariance of eq. (4.19) under time-dependent spatial diffs: $h_{ij} \to h_{ij} + \partial_i\xi_j + \partial_j\xi_i$, $N^i \to N^i + \partial_0\xi^i$. In particular, since we are interested only in the effect on gravitational waves, one can disregard terms that vanish for transverse, traceless perturbations and focus on the two terms: $\partial_i\partial_j\pi(-\dot h_{ij} + 2\partial_j N_i) = -\frac{1}{2}\partial_i\partial_j\pi K^{ij}$. The generated terms relevant for gravitational waves have the same structure as in the case $\delta g^{00}\delta K_{ij}\delta K^{ij}$.

Given the above and in order to keep covariance manifest we choose to express $\dot\gamma_{ij}$ as $2K_{ij}$. In the following we adopt dimensional regularization in $d \equiv 4-\varepsilon$ dimensions and we work at lowest order in the coupling $\tilde{m}_4^2$. Then, at 1-loop, the only diagram contributing to the graviton propagator we need to evaluate is



$$= i\Pi_{ijmn}(p)\frac{M_{\rm Pl}^2}{2}\dot{K}_{ij}(p)\dot{K}_{mn}(-p)\ . \tag{4.55}$$

Indeed, tadpole diagrams with virtual massless fields vanish in dimensional regularization since they do not contribute to logarithmic divergences. To maintain the correct dimensions, the scale $\Lambda_\star$ is replaced by $\Lambda_{\star d} = \Lambda_\star\, \mu^{-\varepsilon/6}$, where μ is an arbitrary energy scale.[1] Additionally, the propagator for π is



$$= \frac{-i}{-q_0^2 + c_s^2\boldsymbol{q}^2 - i\epsilon} = \frac{-i}{\bar{q}^2 - i\epsilon}\ , \tag{4.56}$$

where in the last equality we have defined

$$\bar{q}^\mu \equiv (q_0, c_s\boldsymbol{q})\ . \tag{4.57}$$

At this point we are ready to evaluate the amplitude of the diagram (4.55) as:

$$\begin{aligned} i\Pi_{ijmn}(p) = \frac{1}{2}\times\left(\frac{-4i}{\Lambda_{\star d}^3}\right)^2 \int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{-i}{\bar{q}^2 - i\epsilon}\frac{-i}{(\bar{q}-\bar{p})^2 - i\epsilon}\frac{1}{4}\left[q_i(q-p)_j + q_j(q-p)_i\right]\times \\ \times\left[q_m(q-p)_n + q_n(q-p)_m\right] \end{aligned} \tag{4.58}$$

One can now insert a Feynman parameter x and change the variable of integration to $k \equiv \bar{q} - \bar{p}x$. Notice that terms with powers of $\boldsymbol{p}$ in the numerator are not relevant for gravitational waves and can be disregarded as they would generate operators containing $\partial_i K_{ij}$ that vanish for transverse-traceless perturbations. The same also holds for terms proportional to the trace of the extrinsic curvature. Using the suffix $^{(\mathrm{TT})}$ to denote that we restrict to these terms, one gets

$$i\Pi_{ijmn}^{(\mathrm{TT})}(p) = \frac{8\mu^\varepsilon}{\Lambda_\star^6 c_s^{d+3}}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{k_i k_j k_m k_n}{[k^2 + \bar{p}^2 x(1-x) - i\epsilon]^2}\ . \tag{4.59}$$

Due to the rotational symmetry of the integral over k in (4.59), we can use

$$k_i k_j k_m k_n = \frac{\boldsymbol{k}^4}{d^2-1}\left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} + \delta_{ij}\delta_{mn}\right)\ , \tag{4.60}$$

where the last term in the parenthesis can be dropped, since it yields a term proportional to the trace of the extrinsic curvature.

[1] Note that in d spacetime dimensions γ_{ij}^c and π_c have dimension $d/2-1$.

After these steps, we define $\Delta \equiv \bar{p}^2 x(1-x) - i\epsilon$ and we compute the integral in eq. (4.59) using that $\boldsymbol{k}^4 = (k^2 - k_0^2)^2$ and

$$\begin{aligned} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(k^2-k_0^2)^2}{(k^2+\Delta)^2} &= \mathcal{I}\left[1 + \frac{\Omega_{d-1}}{\Omega_d}\int_0^\pi (\cos^4\phi - 2\cos^2\phi)\sin^{d-2}\phi \, \mathrm{d}\phi\right] \\ &= \mathcal{I}\,\frac{d^2-1}{d(d+2)} \,, \end{aligned} \tag{4.61}$$

where $\mathcal{I}$ and Ω_d, the area of the $(d-1)$-sphere, are given by

$$\mathcal{I} \equiv \frac{i}{(4\pi)^{d/2}} \frac{d(d+2)}{4} \frac{\Gamma(-d/2)}{\Delta^{-d/2}} \,, \qquad \Omega_d \equiv \frac{2\pi^{d/2}}{\Gamma(d/2)} \,. \tag{4.62}$$

Thus, we get

$$i\Pi^{(\mathrm{TT})}_{ijmn}(p) = \frac{2i\mu^\varepsilon(\bar{p}^2 - i\epsilon)^{d/2}}{\Lambda_\star^6 c_s^{d+3}} \frac{\Gamma(-d/2)}{(4\pi)^{d/2}} \frac{\Gamma(1+d/2)^2}{\Gamma(2+d)} \left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right) . \tag{4.63}$$

From this, by taking the limit $d \to 4 - \varepsilon$ and expanding at leading order in ε, we obtain the divergent contribution to the effective action in momentum space,

$$S_{\text{eff}} = \frac{M_{\text{Pl}}^2}{480\pi^2\Lambda_\star^6 c_s^7} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \bar{p}^4 \dot{K}_{ij}(p)\dot{K}_{ij}(-p) \left[\frac{1}{\varepsilon} + \frac{23}{15} - \frac{\gamma_{\mathrm{E}}}{2} - \frac{1}{2}\log\left(\frac{\bar{p}^2}{4\pi c_s^2 \mu^2} - i\epsilon\right)\right] \,, \tag{4.64}$$

where γ_{E} is the Euler-Mascheroni constant.

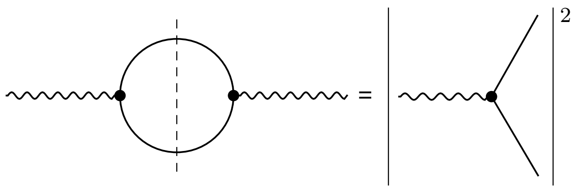
After introducing the suitable counterterm to remove the divergent part in the limit $\varepsilon \to 0$, one is left with a dispersion relation for the gravitational waves of the form

$$\omega^2 = \boldsymbol{k}^2 - \frac{\boldsymbol{k}^8(1-c_s^2)^2}{480\pi^2\Lambda_\star^6 c_s^7} \log\left(-(1-c_s^2)\frac{\boldsymbol{k}^2}{\mu_0^2} - i\epsilon\right) \,. \tag{4.65}$$

Here μ_0 is an unknown constant that must be fixed by experiments. This dispersion relation is not Lorentz-invariant and since the momenta relevant for observations are of order Λ_3, it is not compatible with the recent GW results (see [157, 158] for experimental constraints on GW modified dispersion relations) unless $\tilde{m}_4$ is very small or c_s is very close to unity. Notice that the higher derivative correction cannot be set to zero since it runs logarithmically with the scale $\boldsymbol{k}^2$. Notice also that the correction to the propagation of GWs is there even when $c_s > 1$ and the decay of the GW cannot take place. This is indeed consistent with the fact that the loop in (4.55) involves the propagation of off-shell πs, hence there is no kinematic restriction to the calculation. If one starts with an action with $c_T = 1$, this condition is stable under radiative corrections. However, eq. (4.65) shows that higher-derivative non-Lorentz invariant operators are generated.

In the calculation, we did not take into account loops of the Fadeev-Popov ghost fields. The ghosts appear in any theory with gauge redundancy as a way to express the determinant of the variation of the gauge condition with respect to the gauge parameters. In general, this determinant is field dependent and needs to be included in the action via the Fadeev-Popov procedure. This happens for the two gauges we are using in this analysis, the Newtonian gauge and, in App. D, the spatially flat one (in particular both gauges feature residual gauge freedom at zero momentum and this non-physical modes must be cancelled by the ghosts). The ghost action only depends on the chosen gauge condition and, in particular, it does not depend on the operators that describe the dynamics of the fluctuations around the FRW background. The loop of ghosts with two external graviton lines will therefore be independent of $\tilde{m}_4$ and as such not suppressed by the low scale Λ_3.

We also observe that unitarity of the S-matrix, in the form of the optical theorem, provides a non-trivial check of our results thus far. Indeed, this theorem sets an equality between the imaginary part of the graviton self-energy (evaluated on-shell) and the decay rate times the energy we computed in (4.34). Furthermore, this relation can be expressed diagrammatically as



(4.66)

and remarkably does not depend on the renormalization procedure one employs.

We can readily show this equality by evaluating the imaginary part of (4.65) and comparing the result with $\Gamma\omega$. Of course the only term contributing to the imaginary part is the logarithm. For $c_s^2 < 1$ the argument of the logarithm is negative, so we have

$$\operatorname{Im}\omega^2 = -\frac{\boldsymbol{k}^8(1-c_s^2)^2}{480\pi^2\Lambda_\star^6 c_s^7}\operatorname{Im}\log(-\boldsymbol{k}^2(1-c_s^2)-i\epsilon) = \frac{\boldsymbol{k}^8(1-c_s^2)^2}{480\pi\Lambda_\star^6 c_s^7} = \Gamma\omega. \tag{4.67}$$

Conversely for $c_s^2 > 1$ the argument of the logarithm is positive and as expected we find no imaginary part, in agreement with (4.34) where in this case the result is zero.

The experimental constraints on the imaginary and real part of the dispersion relations are similar

$$\frac{\operatorname{Im}\,\omega^2}{\omega^2} \lesssim \frac{1}{\omega d_S}\,, \qquad \frac{\operatorname{Re}\,(\omega^2-k^2)}{\omega^2} \lesssim \frac{1}{\omega d_S}\,, \tag{4.68}$$

where d_S is the distance of the source. Indeed, neither the amplitude nor the phase of a given Fourier mode can have an order one modification travelling from the source to the detector.[2]

[2]For the real part the constraints come from comparing different frequencies, i.e. looking at the distortion of the expected signal. In the case of a quadratic dispersion relation with $c_T \neq 1$ the signal is not distorted and one has to rely on an optical counterpart, with somewhat looser bounds.

The bound reads

$$\frac{1}{\omega d_S} \sim 10^{-18} \times \frac{2\pi \times 100\,\mathrm{Hz}}{\omega} \frac{40\,\mathrm{Mpc}}{d_S} . \tag{4.69}$$

4.4.2 Higher-derivative corrections

The calculation above shows that radiative corrections generate operators suppressed by powers of ∂/Λ_3. Even if we concentrated on the logarithmic divergences, which do not depend on the UV physics, one expects that power divergences will be generated as well. These operators are in general not Lorentz invariant and affect the propagation of tensor modes: since ∂/Λ_3 is not very small in the recent observations of GWs, this setup is ruled out by observations. (We expect this conclusion to hold also in the case $c_s = 1$, even though the calculable corrections of the previous section vanish.) One has both a large decay rate of gravitational waves and a sizeable distortion of the signal. It is however important to point out two possible ways out.

First, these conclusions do not apply to the operator $\delta g^{00}\delta K$ or, in the covariant language, to the cubic Galileon/Horndeski. Even though the strong coupling scale is Λ_3, it is easy to realise that the coupling with gravitational waves is very suppressed. Indeed, if one considers this operator with a size that is relevant for modifications of gravity on cosmological scales (corresponding to a cutoff of order Λ_3), one can easily read the coupling with gravity

$$H M_{\rm Pl}^2 \delta g^{00} \delta K \sim H M_{\rm Pl}^2 \dot\pi \partial_i \partial_j \pi \gamma_{ij} \,. \tag{4.70}$$

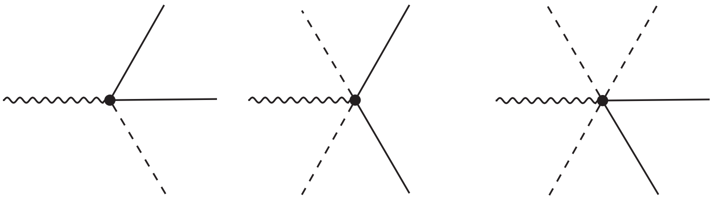
We obtain a coupling that, if compared with the one of the previous sections, is suppressed by a much larger scale, around $\Lambda_2 = (M_{\rm Pl} H_0)^{1/2}$ (defined in eq. (2.47)). This strongly suppresses both the decay rate of gravitational waves and the loop corrections to the propagation.

Since this point is quite important, it is worthwhile repeating it in the covariant language. Making again explicit the dependence on the scales Λ_2 and Λ_3, as in Sec. 2.5, one starts with a Horndeski action of the form

$$\mathscr{L}_2 = \Lambda_2^4 G_2\left(\frac{(\partial\phi)^2}{\Lambda_2^4}\right) , \qquad \mathscr{L}_3 = \Lambda_2^4 G_3\left(\frac{(\partial\phi)^2}{\Lambda_2^4}\right)\frac{\Box\phi}{\Lambda_3^3} , \qquad \mathscr{L}_{4,5} = \dots , \tag{4.71}$$

where the explicit form of the $\mathscr{L}_4$ and $\mathscr{L}_5$ operators is given in (2.49). As already discussed, the action is characterised by the two scales $\Lambda_2 \gg \Lambda_3$. This form of the action is stable under radiative corrections: the functions G_i receive corrections that are parametrically suppressed by $(\Lambda_3/\Lambda_2)^4 \ll 1$ compared to original action. (In particular this implies the stability of the condition that GWs travel luminally, at the 2-derivative level, $c_T = 1$.) This result is based on the non-renormalization theorem of Galileons and on the small breaking induced by gravity, discussed in Sec. 2.5.3. We recall that the gravitational interactions present in the covariant

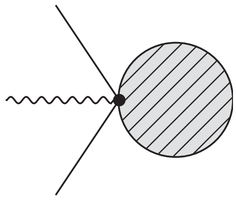
Galileons [105] are of the form



(4.72)

where again solid lines represent a single derivative acting on the scalar, while dashed lines more than one derivative; the wavy line is a graviton. The crucial point is that one has one graviton and thus one power of $1/M_{\rm Pl}$ for each $(\partial\phi)^2$.

Let us now consider the renormalization of operators with external graviton lines (we take the graviton canonically normalized, i.e. a dimension one field γ^c):



(4.73)

The scaling still works in the same way: one power of $1/M_{\rm Pl}$ for each $(\partial\phi)^2$ (one can have more powers of $(\partial\phi)^2$ as external legs, but each carries its $1/M_{\rm Pl}$ since the functions G_i above are characterised by Λ_2 and not Λ_3 in the action). This implies the generation of operators of the schematic form

$$\Lambda_3^4\,\mathcal{F}\left(\frac{(\partial\phi)^2}{\Lambda_2^4},\frac{\partial^2\phi}{\Lambda_3^3},\frac{\gamma^c}{\Lambda_3},\frac{\partial}{\Lambda_3}\right)\,. \tag{4.74}$$

Since on the background solution one has $\dot\phi^2\sim\Lambda_2^4$ one sees that the action for tensors is characterised by the only scale Λ_3: in particular one has sizeable corrections to the propagation if the frequency is not well below Λ_3. Notice however that the conclusion does not apply to the cubic Horndeski, i.e. to the first interaction of (4.72). Since graphs must be 1PI one is forced to have one leg with a single derivative inside the loop: this changes the scaling and suppresses the final result by $(\Lambda_3/\Lambda_2)^4$. Therefore, a theory with only cubic Galileon/Horndeski is viable since it does not affect the graviton propagation at the scale Λ_3 (or at least it is technically natural to make this assumption). Notice that this setup is consistent since the non-renormalization theorem guarantees that if quartic and quintic terms are zero at the beginning, they will be generated only with a very suppressed coefficient. One can generalise the argument to Beyond Horndeski theories following [107].

The second caveat, already mentioned in the previous chapter, is that the theory that describes cosmological perturbations may break down at energies parametrically lower than Λ_3 [145]. Of course nothing forbids that a theory changes before reaching its unitarity cutoff. In this case the EFT of DE cannot be used to describe the recent observations of propagation of gravitational waves. Since we do not know of any explicit UV completion of the DE theories we are studying, it is difficult to reach general conclusions. Naively, one expects the speed of gravitons to approach the speed of light as $c_T - 1 \sim M^2/\omega^2$, where M is the typical mass of the new degrees of freedom. Therefore, to satisfy the experimental bounds one needs $M \lesssim (10^{11}\,\mathrm{km})^{-1}$. Notice that this scenario seems to be at odds with the requirement of recovering GR at short scales. Indeed, if the unitarity cutoff is so low, in the Vainshtein regime discussed in Sec. 2.6.1 the redressed cutoff would be far from the μm scale, and a UV completion would be needed in order to explain physics at the solar-system scale.

An interesting open direction would be to study the constraints imposed by causality and analyticity on a scenario in which gravitational waves have a different speed at different frequencies. In the analogous problem of light propagating in a material, one can derive general conclusions on the absorption of light given its frequency-dependent speed. Indeed, the real and imaginary part of the index of refraction are related by the Kramers-Kronig relations. It is worthwhile studying whether similar techniques can be applied to the propagation of gravitational waves.

4.5 Discussion and outlooks

The observation of gravitational waves has opened a new way of constraining dark energy and modified gravity. This is made possible by the fact that the cutoff of the scalar-tensor EFT describing dark energy, $\Lambda_3 = (M_{\rm Pl} H_0^2)^{1/3}$, lays within the LIGO/Virgo band. At these energy scales, interactions involving gravitons and dark energy fluctuations become large. In the presence of spontaneous breaking of Lorentz invariance, this makes gravitons decay at a catastrophically large rate.

To be compatible with the GW170817 measurements we have restricted our study to theories where gravitons propagate at the speed of light. In the covariant language these are described by the Lagrangian in eq. (3.17). Moreover, we have focused on cubic interactions. In this case, two channels of decay are possible: $\gamma \to \pi\pi$ from the coupling $\alpha_{\rm H} M_{\rm Pl}^2 \ddot{\gamma}_{ij} \partial_i \pi \partial_j \pi$ and $\gamma \to \gamma\pi$ from the coupling $\alpha_{\rm H} M_{\rm Pl}^2 \dot{\pi} \dot{\gamma}_{ij}^2$. Here $\alpha_{\rm H}$ is a dimensionless time-dependent function measuring the Beyond Horndeski character of the theory. It is defined in eq. (2.71) and for the theory with $c_T = 1$ (3.17) is given by $\alpha_{\rm H} = -2X f_{,X}/f$.

We have studied the decay rate for $\gamma \to \pi\pi$ in Sec. 4.2, finding that it is roughly given by $\Gamma_{\gamma\to\pi\pi} \sim \alpha_{\rm H}^2 \omega^7/\Lambda_3^6$ (the full expression can be found in eq. (4.34) with (4.24)). The coupling $\gamma\gamma\pi$ contains one less derivative than $\gamma\pi\pi$. Therefore, the decay rate for $\gamma \to \gamma\pi$ is much smaller: $\Gamma_{\gamma\to\gamma\pi} \sim \alpha_{\rm H}^2 \omega^5/\Lambda_2^4$, where $\Lambda_2 = (M_{\rm Pl} H_0)^{1/2}$. This decay gives constraints on $\alpha_{\rm H}$ that are much looser than the other channel and it is studied in Sec. 4.3.

The absence of this effect at LIGO/Virgo frequencies $\omega \sim \Lambda_3$ implies that $\alpha_{\rm H}$ is practically zero. Thus, the surviving theory is

$$\mathscr{L}_{c_T=1,\text{ no decay}} = f(\phi)R + G_2(\phi, X) + G_3(\phi, X)\Box\phi \,. \tag{4.75}$$

It is interesting to formulate this theory in the context of the more general DHOST theories, discussed in App. A. These theories can be obtained starting from Beyond Horndeski and performing an invertible conformal transformation that depends on X, i.e. $g_{\mu\nu} \to C(\phi, X)g_{\mu\nu}$ [79, 159, 160] (we assume C is not linear in X so that the transformation is invertible). Since this does not change the light-cone, we can do the same with the theory above, obtaining

$$\mathscr{L}_{c_T=1,\text{ no decay}} = G_2(\phi, X)+G_3(\phi, X)\Box\phi+C(\phi, X)R+\frac{6C_{,X}(\phi, X)^2}{C(\phi, X)}\nabla^\mu\phi\nabla_\lambda\phi\,\Phi_{\mu\nu}\Phi^{\lambda\nu} \,. \tag{4.76}$$

Here we have redefined the free functions G_2 and G_3 after the transformation and reabsorbed the dependence on $f(\phi)$ in C. This is the most general degenerate theory compatible with $c_T^2 = 1$ and with the absence of graviton decay.[3]

The closeness of the cutoff to the LIGO/Virgo band is also responsible for modifying the dispersion relation of the gravitational waves, see Sec. 4.4. For $c_s < 1$, this is expected because the dispersion is related to the decay by the optical theorem, as we explain in Sec. 4.4.1. For $c_s > 1$, even if the decay of gravitons is kinematically forbidden, the loop corrections to the graviton propagation are still present and give practically the same bound. In the case $c_s = 1$, the decay rate and the calculable part of the loop corrections that we studied vanish. On the other hand, power-law divergent terms are expected and would provide similar constraints to those obtained for $c_s \neq 1$. We conclude that the absence of $\tilde{m}_4^2$ holds for any value of the scalar speed c_s. Interestingly, as explained in Sec. 4.4.2 radiative corrections in the surviving theory eq. (4.75) (and its degenerate version (4.76)) do not generate measurable effects in the graviton dispersion relation even at the LIGO/Virgo scales.

As already mentioned in the current and previous chapters, our conclusions do not hold if the theories at hand break down at a scale parametrically smaller than Λ_3 [145]. It would be interesting to investigate further whether an example of such a proposal can be constructed that successfully reproduces GR on short scales. Moreover, in this chapter we have studied the perturbative decay of gravitational waves, neglecting possible coherent effects. Given the very high occupation number of gravitons in the observed waves, these effects are indeed important and their absence can be used to rule out another corner of the parameter space of these theories. This will be investigated in the next two chapters.

[3]In terms of the dimensionless coefficients defined in eq. (2.71) and (A.6) (see [148] for more details), for DHOST theories we find that neither $\alpha_{\rm H}$ nor β_1, the coefficient parameterizing the presence of higher-order operators, vanish. However, in the absence of decay these coefficients are not independent but are related by $\alpha_{\rm H} = -2\beta_1$. This implies that the screening mechanism based on quartic terms studied in [161–163] is absent. We thank M. Crisostomi and K. Koyama for pointing this out.

5 | Resonant Decay of Gravitational Waves into Dark Energy

In the previous chapter it was pointed out that in modified-gravity theories gravitational waves (GWs) can decay into dark energy fluctuations as a consequence of the spontaneous breaking of Lorentz invariance. Moreover, since the relative speed between GWs and light is now constrained with 10^{-15} accuracy, only models where gravitons travel at the same speed as light were considered.

In particular, we showed that some operators of the EFT of DE display a cubic $\gamma\pi\pi$ interaction, that can mediate the decay of gravitons. Of course, this can happen only if both energy and momentum can remain conserved during the process, which is when scalar fluctuations propagate subluminally. The same vertex is also responsible for an anomalous GW dispersion, for speeds of scalar propagation different from that of light. Depending on the energy scale suppressing this interaction, these two effects can be important at frequencies observed by LIGO/Virgo and can constrain these theories.

The bound derived in [1] is based on a perturbative calculation, in which individual gravitons are assumed to decay independently of each other. But a classical GW is a collection of many particles with very large occupation number and particle production must be treated as a collective process in which many gravitons decay simultaneously. We are going to argue that the classical GW acts as a background for the propagation of scalar fluctuations. In this chapter, following the results in [2], we study this process in the limit where the GW background acts as a small periodic perturbation (narrow resonance). Larger amplitudes of the GW can induce tachyon or ghost instabilities: we will study this possibility in the next chapter.

As in the previous chapter, we work within the framework of the EFT of DE, expanded around a flat FRW background. Moreover, we still focus on theories where gravitons travel luminally and, for later convenience, we split the EFT of DE action in the sum of three actions,

$$S = S_0 + S_{m_3} + S_{\tilde{m}_4} \ , \tag{5.1}$$

where

$$S_0 = \int \left[\frac{M_{\rm Pl}^2}{2} {}^{(4)}R - \lambda(t) - c(t) g^{00} + \frac{m_2^4(t)}{2} (\delta g^{00})^2 \right] \sqrt{-g}\, {\rm d}^4 x \;, \tag{5.2}$$

$$S_{m_3} = - \int \frac{m_3^3(t)}{2} \delta K \delta g^{00} \sqrt{-g}\, {\rm d}^4 x \;, \tag{5.3}$$

$$S_{\tilde{m}_4} = \int \frac{\tilde{m}_4^2(t)}{2} \delta g^{00} \left({}^{(3)}R + \delta K_\mu^\nu \delta K_\nu^\mu - \delta K^2 \right) \sqrt{-g}\, {\rm d}^4 x \;. \tag{5.4}$$

The first action, S_0, contains the Einstein-Hilbert term and the minimal scalar field Lagrangian, which describe the dynamics of the background. Notice that we have again removed any time-dependence in front of the Einstein-Hilbert term by a conformal transformation, which leaves the graviton speed unaffected. We use the notation $\lambda(t)$, and not $\Lambda(t)$ as usual, to avoid confusion with the energy scale Λ suppressing the cubic vertices that we are going to introduce. The operator m_2^4 does not change the background but affects the speed of propagation of scalar fluctuations, c_s^2. We recall once again that its typical value is $\sim M_{\rm Pl}^2 H_0^2$. This action was studied in details in [63] and in the covariant language it describes quintessence [164] or, more generally, a dark energy with scalar field Lagrangian $P(\phi, X)$.

The operator in the second action, S_{m_3}, introduces the kinetic mixing between the scalar field and gravity discussed in Sec. 2.7. In the covariant language it corresponds to the cubic Horndeski Lagrangian, of the form $G_3(\phi, X)\Box\phi$. In the regime that leads to sizeable modifications of gravity (i.e. for $m_3^3 \sim M_{\rm Pl}^2 H_0$), the operator contained in S_{m_3} displays a $\gamma\pi\pi$ interaction suppressed by an energy scale of order $\Lambda_2 = (M_{\rm Pl} H_0)^{1/2} \sim 10^{-3}\,$eV. This energy scale is much greater than the typical LIGO/Virgo frequency. For this reason, in [1] the parameter m_3^3 remains unconstrained by the graviton decay computed in perturbation theory.

Finally, the operator in the third action, $S_{\tilde{m}_4}$ has been extensively studied in the previous chapter in the context of GW propagation. This operator displays a $\gamma\pi\pi$ interaction and is constrained with the perturbative decay because the vertex is suppressed by an energy scale close to LIGO/Virgo frequencies.

In Sec. 5.1, after expanding the action $S_0 + S_{m_3}$ in perturbations in the Newtonian gauge (the same calculation is repeated in the spatially-flat gauge in Sec. D.3), we study the effect of a classical GW background on the π dynamics, for the operator m_3^3 (Sec. 5.1.1) and $\tilde{m}_4^2$ (Sec. 5.1.2). The regime of small GWs can be studied analytically and leads to the so-called *narrow* parametric resonance, which is the subject of Sec. 5.2. There we compute the energy density of π produced by the parametric instability due to the oscillating GWs (in Sec. 5.2.2) and we re-interpret the π production in the narrow-resonance regime as an effect of Bose enhancement of the perturbative decay in App. E. The back-reaction on the GW signal is computed in Sec. 5.2.3 for a linearly polarized wave, while the case of elliptical polarization is discussed in Sec. 5.2.4. In Sec. 5.2.5 we check that energy is conserved in this process, as expected (the details of the calculations are given in App. F).

The treatment in Sec. 5.2 neglects scalar-field nonlinearities, which are studied in Sec. 5.3. The operator m_3^3 contains cubic self-interactions (typical of cubic-Galileon models) suppressed by the scale $\Lambda_3 = (M_{\rm Pl} H_0^2)^{1/3} \sim 10^{-13}\,$eV, which is much smaller than the one appearing in the vertex $\gamma\pi\pi$. Thus, these become relevant and probably halt the parametric resonance well before the GWs are affected by the back-reaction (Sec. 5.3.1). This makes the results of Sec. 5.2 applied to this operator inconclusive. The situation is different for the operator $\tilde{m}_4^2$: in this case the scale that suppresses non-linearities is the same that appears in the coupling $\gamma\pi\pi$. The leading non-linearities are quartic in the regime of interest and are suppressed with respect to a naive estimate due to the particular structure of Galileon interactions. At least in some region of parameters non-linearities do not halt the parametric instability due to the oscillating GWs. In Sec. 5.4.1 we therefore study in which range of parameters one expects a modification of the GW signal. Moreover, in Sec. 5.4.2 we discuss *precursors*, higher harmonics induced in the GW signal by the produced π, that enter the observational band earlier than the main signal. We conclude discussing the main results of the article and possible future directions in Sec. 5.5.

5.1 Graviton-scalar-scalar vertices

Let us derive the interaction $\gamma\pi\pi$ from the action (5.1), using the Newtonian gauge. For the time being, we neglect the self-interactions of the π field; they will be discussed later, in Sec. 5.3. We initially focus on the operator m_3^3; as a check, in Sec. D.3 we perform the same calculation in spatially-flat gauge.

5.1.1 m_3^3-operator

Let us consider the action $S_0 + S_{m_3}$. One can restore the π dependence in a generic gauge with the Stueckelberg procedure. For our purposes we can use the results given in Sec. 2.3 but for δK it will be useful to keep also quadratic contributions in π [54]

$$\delta K \to \delta K - h^{ij}\partial_i\partial_j\pi + \frac{2}{a^2}\partial_i\pi\partial_i\dot{\pi} + \dots \, . \tag{5.5}$$

In Newtonian gauge, the ADM components of the metric are as in eq. (4.2). Contrary to Sec. 4.1, in the absence of $\tilde{m}_4$ the relation between Φ and π does not involve time derivative. This can be verified by solving the constraint equations at linear order. Varying the above action with respect to Φ and Ψ and focussing on the sub-Hubble limit by keeping only the leading terms in spatial derivatives, one obtains

$$2M_{\rm Pl}^2\nabla^2\Psi + m_3^3\nabla^2\pi = 0\,, \qquad M_{\rm Pl}^2\nabla^2(\Phi - \Psi) = 0\,. \tag{5.6}$$

From now on we will always consider the Minkowski limit, i.e. that time and spatial derivatives are much larger than Hubble. These equations can be solved in terms of π,

$$\Phi = \Psi = -\frac{m_3^3}{2M_{\rm Pl}^2}\pi \ . \tag{5.7}$$

Using these relations, one can find the kinetic term of the π field. After a straightforward calculation, the canonically normalized scalar and tensor perturbations are then given by

$$\pi_c \equiv \sqrt{\alpha} M_{\rm Pl} H \pi \ , \qquad \gamma_{ij}^c \equiv \frac{M_{\rm Pl}}{\sqrt{2}}\gamma_{ij} \ , \tag{5.8}$$

where the parameter α, controlling the kinetic term of π (and required to be positive to avoid ghost instabilities), is defined in eq. (2.71). In terms of these, the interaction term, after integrating by parts, reads

$$-\frac{1}{2}\sqrt{-g}\ m_3^3 \delta K \delta g^{00} \supset \frac{1}{\Lambda^2}\dot\gamma_{ij}^c \partial_i \pi_c \partial_j \pi_c \ , \tag{5.9}$$

where

$$\Lambda^2 \equiv \frac{4M_{\rm Pl}^2(c+2m_2^4)+3m_3^6}{\sqrt{2}m_3^3 M_{\rm Pl}} = -\frac{\alpha}{\sqrt{2}\alpha_{\rm B}}\frac{H}{H_0}\Lambda_2^2 \ , \tag{5.10}$$

and in the right-hand side we have defined the dimensionless quantity $\alpha_{\rm B}$, that in this frame is given by[1]

$$\alpha_{\rm B} = -\frac{m_3^3}{2M_{\rm Pl}^2 H} \ . \tag{5.12}$$

We drop the symbol of canonical normalization: γ and π will indicate for the rest of the chapter the canonically normalized fields. The total action we are interested in is then

$$S_0 + S_{m_3} = \int {\rm d}^4 x \left[\frac{1}{4}(\dot\gamma_{ij})^2 - \frac{1}{4}(\partial_k \gamma_{ij})^2 + \frac{1}{2}\dot\pi^2 - \frac{c_s^2}{2}(\partial_i \pi)^2 + \frac{1}{\Lambda^2}\dot\gamma_{ij}\partial_i\pi\partial_j\pi\right] \ , \tag{5.13}$$

and c_s is the same as in eq. (4.8), but with $\tilde m_4^2$ set to zero

$$c_s^2 = \frac{4M_{\rm Pl}^2 c - m_3^3(m_3^3 - 2M_{\rm Pl}^2 H)}{4M_{\rm Pl}^2(c+2m_2^4)+3m_3^6} = \frac{2}{\alpha}\left(\frac{c}{M_{\rm Pl}^2 H^2} - \alpha_{\rm B} - \alpha_{\rm B}^2\right) \ . \tag{5.14}$$

[1] To write the action (5.1), we used a conformal transformation (possibly dependent on $X = (\partial_\mu \phi)^2$) to set to constant the effective Planck mass and to zero higher-order operators of DHOST theories. Using the notation of [148] and the transformation formulae contained therein, one can check that in a general frame the only relevant parameter is

$$\alpha_{\rm B} - \frac{\alpha_{\rm M}}{2}(1-\beta_1) + \beta_1 - \dot\beta_1/H \ . \tag{5.11}$$

The perturbative decay rate of the graviton can be computed following exactly the same procedure followed for $\tilde{m}_4^2$ in Sec. 4.2. It gives

$$\Gamma_{\gamma\to\pi\pi} = \frac{p^5(1-c_s^2)^2}{480\pi c_s^7 \Lambda^4} \,, \tag{5.15}$$

where p is the momentum of the decaying graviton. One can check that this is negligible for frequencies relevant for GW observations, since by eq. (5.10) Λ is of order Λ_2.

The equation of motion of π from the Lagrangian (5.13) reads

$$\ddot{\pi} - c_s^2 \nabla^2 \pi + \frac{2}{\Lambda^2} \dot{\gamma}_{ij} \partial_i \partial_j \pi = 0 \,. \tag{5.16}$$

Let us use the classical background solution of the GW travelling in the $\hat{z}$ direction[2] with a linear polarization. Without loss of generality we take the + polarization (the × one can be obtained by a 45° rotation of the axes)

$$\gamma_{ij} = M_{\rm Pl} h^+ \epsilon_{ij}^+ \,, \qquad h^+(t,z) \equiv h_0^+ \sin(\omega(t-z)) \,, \tag{5.17}$$

where h^+ is the dimensionless strain of the GW and the polarisation tensor is $\epsilon_{ij}^+ = \mathrm{diag}(1,-1,0)$. In this chapter we will always be away from the source generating the GW, i.e. in the weak field regime $h^+ \ll 1$. Substituting the solution (5.17) into eq. (5.16), one gets (modulo an irrelevant phase)

$$\ddot{\pi} - c_s^2 \nabla^2 \pi + c_s^2 \beta \cos[\omega(t-z)](\partial_x^2 - \partial_y^2)\pi = 0 \,, \tag{5.18}$$

where the parameter β is defined as

$$\beta \equiv \frac{2\omega M_{\rm Pl} h_0^+}{c_s^2 |\Lambda^2|} = \frac{2\sqrt{2}|\alpha_{\rm B}|}{\alpha c_s^2} \frac{\omega}{H} h_0^+ \,, \qquad \text{for} \quad m_3^3 \neq 0 \,, \; \tilde{m}_4^2 = 0 \qquad (\alpha_{\rm B} \neq 0 \,, \; \alpha_{\rm H} = 0) \,. \tag{5.19}$$

In the following, to evaluate the right-hand side of the above definition we will use $\omega = \Lambda_3$ and $H = H_0$, in which case $\omega/H \sim 10^{20}$. Moreover, note that α and c_s^2 in the second equality appear in the combination αc_s^2. Therefore, thanks to eqs. (5.14), β defined above depends only on $\alpha_{\rm B}$ (or m_3) and we can tune this parameter to make β small.

5.1.2 $\tilde{m}_4^2$-operator

We now consider the operator $\tilde{m}_4^2$. To avoid large π nonlinearities, discussed in Sec. 5.3, we focus on the case $m_3 = 0$ and consider the action $S_0 + S_{\tilde{m}_4}$. The operator $\tilde{m}_4^2$ has been studied in chapter 4 and details on the calculations can be found there.

[2] Here we use Cartesian spatial coordinates (x, y, z).

In this case, the action for the canonically normalized fields reads

$$S_0 + S_{\tilde{m}_4} = \int \mathrm{d}^4x \left[\frac{1}{4}(\dot{\gamma}_{ij})^2 - \frac{1}{4}(\partial_k \gamma_{ij})^2 + \frac{1}{2}\dot{\pi}^2 - \frac{c_s^2}{2}(\partial_i \pi)^2 + \frac{1}{\Lambda_\star^3}\ddot{\gamma}_{ij}\partial_i\pi\partial_j\pi \right] , \tag{5.20}$$

where the sound speed of scalar fluctuations is now

$$c_s^2 = \frac{2}{\alpha}\left[(1+\alpha_{\rm H})^2 \frac{c}{M_{\rm Pl}^2 H^2} + \alpha_{\rm H} + \alpha_{\rm H}(1+\alpha_{\rm H})\frac{\dot{H}}{H^2}\right] , \tag{5.21}$$

and we assumed $\alpha_{\rm H} \ll 1$ since this will be the regime of interest. This operator, as shown in Sec. 4.2.1, contains an interaction $\gamma\pi\pi$ suppressed by the scale $\Lambda_\star^3$ (defined in eq. (4.24)).

The evolution equation for π then reads

$$\ddot{\pi} - c_s^2\nabla^2\pi + \frac{2}{\Lambda_\star^3}\ddot{\gamma}_{ij}\partial_i\partial_j\pi = 0 \, . \tag{5.22}$$

Substituting the solution (5.17) into this equation gives

$$\ddot{\pi} - c_s^2\nabla^2\pi - \frac{2\omega^2 M_{\rm Pl} h_0^+}{\Lambda_\star^3}\sin[\omega(t-z)](\partial_x^2 - \partial_y^2)\pi = 0 \, . \tag{5.23}$$

One sees that the evolution equation for π for the $\tilde{m}_4^2$ operator is very similar to the case of the m_3^3 operator, with the replacement $\Lambda^2 \to \Lambda_\star^3\omega^{-1}$. We can thus apply eq. (5.18), with β now defined as

$$\beta \equiv \frac{2\omega^2 M_{\rm Pl} h_0^+}{c_s^2|\Lambda_\star^3|} = \frac{\sqrt{2}|\alpha_{\rm H}|}{\alpha c_s^2}\left(\frac{\omega}{H}\right)^2 h_0^+ \, , \qquad \text{for} \quad m_3^3 = 0 \, , \ \tilde{m}_4^2 \neq 0 \qquad (\alpha_{\rm B} = 0 \, , \ \alpha_{\rm H} \neq 0) \, . \tag{5.24}$$

Analogously to the m_3 case, because of eqs. (5.21), β defined above depends only on $\alpha_{\rm H}$ (or $\tilde{m}_4$) and also in this case we can tune this parameter to make β small.

5.2 Narrow resonance

Equation (5.18) describes a harmonic oscillator with periodic time-dependent frequency, which can lead to parametric resonance. As explained at the beginning of this chapter, we are going to focus on the *narrow-resonance* regime, which corresponds to $\beta \ll 1$. In this case, the solution of the equation of motion of π can be treated analytically and features an exponential growth of scalar fluctuations.

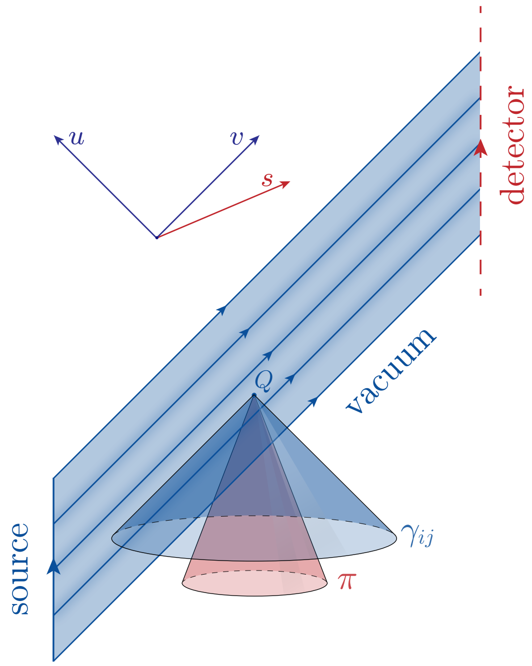


Fig. 5.1 Space-time diagram in (t, z)-coordinates indicating the path taken by the gravitational-wave wave packet (blue region). The π light-cone is narrower than the gravitational wave one.

5.2.1 Parametric resonance

The GW is emitted at $t = 0$ in the z direction and is detected at some later time, see fig. 5.1. Using the light-cone coordinates,

$$u \equiv t - z\,, \qquad v \equiv t + z\,, \tag{5.25}$$

its solution for $t > 0$ can be written as

$$\gamma_{ij}(u) = M_{\rm Pl} h_0^+ \sin(\omega u)\, \epsilon_{ij}^+\,, \tag{5.26}$$

where h_0^+ can be taken as constant, since it varies slowly compared to the GW frequency.

Eq. (5.18) takes the form

$$\ddot{\pi} - c_s^2 \nabla^2 \pi + c_s^2 \beta \cos(\omega u)\, (\partial_x^2 - \partial_y^2)\pi = 0\,. \tag{5.27}$$

For $c_s < 1$, which we assume in the following, u is a time-like variable for the π metric: hypersurfaces of constant u are space-like to the π-cone. It is then convenient to define the variable s,

$$s \equiv -t + c_s^{-2} z\,, \tag{5.28}$$

which is orthogonal to u and is thus space-like with the π metric, see fig. 5.1, and use the coordinates $\tilde{\boldsymbol{x}} = (x, y, s)$ to describe the spatial foliations. Since the π-lightcone is narrower than the one of GWs, the solution for the scalar will only be sensitive to a finite region of the GW background. This is the reason why one can approximate the GW as a plane wave with constant amplitude and disregard the process of emission of the GW from the astrophysical source. In particular, for this to be a good approximation we need to require that the past light-cone of π overlaps only with a region of the GW with constant amplitude. From fig. 5.1 one sees that this is a very weak requirement. It is enough that $1 - c_s$ is larger than the ratio between the duration of the GW signal (of order seconds) and the scale of variation of the amplitude (of order Mpc). Plugging the numbers one gets $1 - c_s \gtrsim 10^{-14}$.

Since for π there is translational invariance in $\tilde{\boldsymbol{x}}$, it is useful to decompose π in Fourier modes as

$$\pi(u, \tilde{\boldsymbol{x}}) = \int \frac{\mathrm{d}^3\tilde{\boldsymbol{p}}}{(2\pi)^3} e^{i\tilde{\boldsymbol{p}}\cdot\tilde{\boldsymbol{x}}} \pi_{\tilde{\boldsymbol{p}}}(u) \,, \tag{5.29}$$

where $\tilde{\boldsymbol{p}} = (p_x, p_y, p_s)$ is conjugate to $\tilde{\boldsymbol{x}}$. In the absence of the interaction with the gravitational wave, one can relate p_s to the momentum written in the original coordinates,

$$p_s = \frac{c_s^2}{1 - c_s^2}(p_z - c_s|\boldsymbol{p}|) \,. \tag{5.30}$$

In the following we are going to use this change of variable also when $\beta \neq 0$, although plane waves in the original coordinates (t, x, y, z) are not solution of eq. (5.27).

Then we can quantize π straightforwardly. More specifically, we decompose $\pi_{\tilde{\boldsymbol{p}}}$ as

$$\pi_{\tilde{\boldsymbol{p}}}(u) = \frac{1}{c_s\sqrt{2p_u}} \left[f_{\tilde{p}}(u)\hat{a}_{\tilde{\boldsymbol{p}}} + f^\star_{\tilde{p}}(u)\hat{a}^\dagger_{-\tilde{\boldsymbol{p}}} \right] \,, \tag{5.31}$$

where

$$p_u \equiv \frac{c_s}{1 - c_s^2}(|\boldsymbol{p}| - c_s p_z) \,, \tag{5.32}$$

and $\hat{a}_{\tilde{\boldsymbol{p}}}$ and $\hat{a}^\dagger_{-\tilde{\boldsymbol{p}}}$ are the usual creation and annihilation operators satisfying the commutation relations, $[\hat{a}_{\tilde{\boldsymbol{p}}}, \hat{a}^\dagger_{\tilde{\boldsymbol{p}}'}] = (2\pi)^3\delta^{(3)}(\tilde{\boldsymbol{p}} - \tilde{\boldsymbol{p}}')$. The normalization is chosen for convenience. Indeed, for $\beta = 0$ the evolution equation for π satisfies a free wave equation and each Fourier mode can be described as an independent quantum harmonic oscillator. We assume that in this case π is in the standard Minkowski vacuum, given by[3]

$$f_{\tilde{p}}(u) = e^{-ip_u u} \,, \qquad (\beta = 0) \,. \tag{5.34}$$

[3] It is straightforward to verify that eq. (5.34) is equivalent to the standard Minkowski vacuum, i.e.

$$\pi(x) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2c_s|\boldsymbol{p}|}} \left(e^{-ip\cdot x}\hat{a}_{\boldsymbol{p}} + e^{ip\cdot x}\hat{a}^\dagger_{-\boldsymbol{p}} \right) \,, \tag{5.33}$$

upon use of $\mathrm{d}^3\tilde{\boldsymbol{p}}/\mathrm{d}^3\boldsymbol{p} = c_s p_u/|\boldsymbol{p}|$ and, consequently, of $\hat{a}_{\tilde{\boldsymbol{p}}} = [|\boldsymbol{p}|/(c_s p_u)]^{1/2}\,\hat{a}_{\boldsymbol{p}}$.

To study the parametric resonance, we will now show that eq. (5.27) can be written as a Mathieu equation [165]. First, in terms of the new coordinates, eq. (5.27) becomes

$$[(1-c_s^2)\partial_u^2 - c_s^{-2}(1-c_s^2)\partial_s^2 - c_s^2(\partial_x^2+\partial_y^2)]\pi + c_s^2\beta\cos(\omega u)(\partial_x^2-\partial_y^2)\pi = 0 \; . \tag{5.35}$$

For convenience we can also define the dimensionless time variable τ,

$$\tau \equiv \frac{\omega u}{2} \; . \tag{5.36}$$

For each Fourier mode, f satisfies

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\tau^2} + [A - 2q\cos(2\tau)]f = 0 \; , \tag{5.37}$$

with

$$A = 4\frac{c_s^2\,\boldsymbol{p}^2}{\omega^2}\frac{(1-c_s\Omega)^2}{(1-c_s^2)^2} = \frac{4p_u^2}{\omega^2} \; , \tag{5.38}$$

$$q = 2\beta\frac{c_s^2\,\boldsymbol{p}^2}{\omega^2}\frac{(1-\Omega^2)\cos(2\varphi)}{1-c_s^2} \; . \tag{5.39}$$

To write A and q we have decomposed the vector $\boldsymbol{p}$ in polar coordinates

$$\boldsymbol{p} = |\boldsymbol{p}|(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \; , \tag{5.40}$$

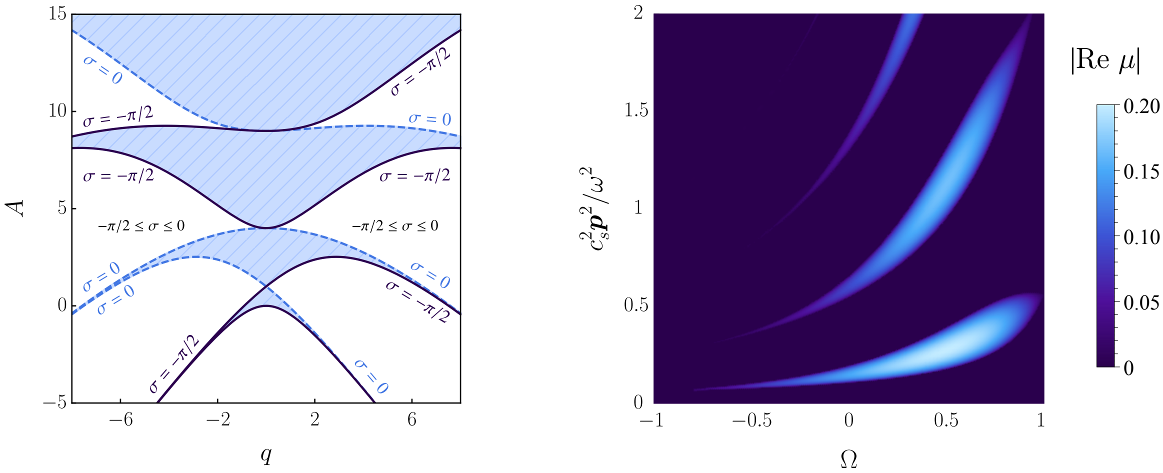
and we have defined $\Omega \equiv p_z/|\boldsymbol{p}| = \cos\theta$.

The general solution of the Mathieu equation is of the form $e^{\pm\mu\tau}P(\tau)$, where $P(\tau+\pi) = P(\tau)$ [165]. If the characteristic exponent μ has a real part, the solution of the Mathieu equation is unstable for generic initial conditions. Since μ is a function of A and q, the instability region can be represented on the (q, A)-plane, see fig. 5.2a. The unstable regions are also shown in fig. 5.2b on the plane $(\Omega, c_s^2\,\boldsymbol{p}^2/\omega^2)$, for specific values of the other parameters (in the example in the figure we take $\beta = 0.8$ and $c_s = 1/2$). Notice that the maximal exponential growth is reached when the ratio q/A has its maximum at $\Omega = c_s$ and $\varphi = 0$.

5.2.2 Energy density of π

In this subsection we compute the energy density of π produced by the coupling with the GW. This is given by (see eq. (5.78) and explanation below)

$$\rho_\pi = \frac{1}{2}\langle 0|\left[\dot\pi^2 + c_s^2(\partial_i\pi)^2\right]|0\rangle \; . \tag{5.41}$$



(a) **(b)**

Fig. 5.2 Left panel (fig. 5.2a): Instability chart for the Mathieu equation (5.37). Coloured regions are stable while empty regions are unstable. Each instability band is spanned by $\sigma \in \left(-\frac{\pi}{2}, 0\right)$, see eq. (5.61) and below for details. Right panel (fig. 5.2b): Instability bands as functions of Ω and $c_s^2 \boldsymbol{p}^2/\omega^2$ assuming $\varphi = 0$. Light blue regions are unstable: the colour grading indicates the coefficient of instability $|\mathrm{Re}\,\mu|$. The plot is obtained using $\beta = 0.8$ and $c_s = 1/2$. (We choose a large value of β for this figure because the instability bands can be easily located, otherwise the bands would be too narrow to be seen.)

Decomposing in Fourier modes and in the $\hat{a}$ and $\hat{a}^\dagger$ coefficients using respectively eqs. (5.29) and (5.31), the energy density can be rewritten as

$$\begin{aligned}\rho_\pi &= \int \frac{\mathrm{d}^3\tilde{\boldsymbol{p}}}{(2\pi)^3} \frac{1}{4c_s^2 p_u} \left[|f'_{\tilde{p}} - ip_s f_{\tilde{p}}|^2 + c_s^2 |f'_{\tilde{p}} - ic_s^{-2} p_s f_{\tilde{p}}|^2 + 2ip_s (f'_{\tilde{p}} f^\star_{\tilde{p}} - f'^\star_{\tilde{p}} f_{\tilde{p}}) + c_s^2 (p_x^2 + p_y^2) |f_{\tilde{p}}|^2 \right] \\ &= \int \frac{\mathrm{d}^3\tilde{\boldsymbol{p}}}{(2\pi)^3} \frac{1}{4c_s^2 p_u} \left\{ (1 + c_s^2) |\partial_u f_{\tilde{p}}|^2 + |f_{\tilde{p}}|^2 \left[p_s^2 (1 + c_s^{-2}) + c_s^2 (p_x^2 + p_y^2) \right] + 4 p_s p_u \right\} , \end{aligned} \tag{5.42}$$

where we have simplified the expression on the right-hand side using that the Wronskian is time-independent, $\mathscr{W}[f_{\tilde{p}}(u), f^\star_{\tilde{p}}(u)] = -2ip_u$.[4] One can verify that in the limit $\beta = 0$ the above expression reduces to the energy density of the vacuum, i.e. $\rho_\pi^0 = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\omega_p}{2}$.

We can simplify the right-hand side further by making some approximations. Since we are not interested in following the oscillatory behaviour of ρ_π, we can perform an average in τ over many periods of oscillation. Since the amplitude of the periodic part of $f_{\tilde{\boldsymbol{p}}}$ is bounded to be less

[4] In general, we can write f as a linear combination of Mathieu-sine and cosine functions, respectively $\mathcal{S}$ and $\mathcal{C}$ [165]. We fix the boundary conditions of the solution such that π is in the vacuum, i.e. the function f satisfies eq. (5.34), at $u = 0$. This gives

$$f(\tau) = \frac{-i2p_u}{\omega} \mathcal{S}(A, q; \tau) + \mathcal{C}(A, q; \tau) \ . \tag{5.43}$$

Using this expression and the properties of these functions, we can check that the Wronskian $\mathscr{W}[f_{\tilde{p}}(u), f^\star_{\tilde{p}}(u)]$ is constant and with the above normalization is given by $-2ip_u$. As a consequence, the commutation relation in the "interacting" region are satisfied at all times.

than unity, it is reasonable to take $\langle |f_{\tilde{p}}|^2\rangle_T \simeq \langle |\partial_\tau f_{\tilde{p}}|^2\rangle_T \simeq e^{2\mu\tau}/2$ in eq. (5.42). This educated guess will be confirmed in Sec. 5.2.3. Changing integration variables, from $\tilde{\boldsymbol{p}}$ to $\xi \equiv c_s^2 \boldsymbol{p}^2/\omega^2$ and the angular variables Ω and φ, we find

$$\rho_\pi \simeq \frac{\omega^4}{(2\pi)^3 16 c_s^3} \int \left[\frac{1+c_s^2}{4} + \xi \frac{(c_s - \Omega)^2(1+c_s^2)}{(1-c_s^2)^2} + \xi(1-\Omega^2)\right] e^{2\mu(\xi,\Omega,\varphi)\tau} \mathrm{d}\Omega\, \mathrm{d}\varphi\, \mathrm{d}\xi\ . \tag{5.44}$$

We now want to solve the integral on the right-hand side. Since it is dominated by the unstable modes, we restrict the domain of integration to the bands of instability. Actually, we are going to focus on the first instability band for two reasons. The instability rate of the higher bands goes as $\mu_m \sim q^m/(m!)^2 \sim \beta^m$ [165, 166], which implies that for small β the first band is the most unstable. Second, the π produced in this band will modify the GW signal with the original angular frequency ω. (With some contributions at higher frequencies that will be discussed in Section 5.4.2.) We are going to work in the regime $\beta \ll 1$ corresponding to $q \ll 1$, the regime of narrow resonance.

In this situation we can restrict the integral to the first unstable band, which is defined by [165]

$$A_- \le A \le A_+\ , \qquad A_\pm = 1 \pm |q|\ . \tag{5.45}$$

Within this region, the value of the exponent μ is

$$\mu \simeq \frac{1}{2}\sqrt{(A_+ - A)(A - A_-)}\,. \tag{5.46}$$

Using the definitions (5.38) and (5.39) in eq. (5.45), the boundary region above can be rewritten in terms of ξ,

$$\xi_- \le \xi \le \xi_+\ , \qquad \xi_\pm = \frac{(1-c_s^2)^2}{4(1-c_s\Omega)^2}\left[1 \pm \beta|\cos(2\varphi)|\frac{(1-\Omega^2)(1-c_s^2)}{2(1-c_s\Omega)^2}\right]\ , \tag{5.47}$$

which fixes the domain of integration in eq. (5.44).

The integral in (5.44) can be then solved with the saddle-point approximation. In general, an integral of the form

$$\mathcal{I}(\tau) = \int g(\boldsymbol{X}) e^{h(\boldsymbol{X})\tau}\, \mathrm{d}^3\boldsymbol{X}\ , \tag{5.48}$$

can be approximated for large τ by

$$\mathcal{I}(\tau) \approx \frac{1}{\sqrt{\det \mathrm{H}_{ij}}}\left(\frac{2\pi}{\tau}\right)^{3/2} g(\boldsymbol{X}_0) e^{h(\boldsymbol{X}_0)\tau}\ , \tag{5.49}$$

where $\boldsymbol{X}_0$ is the maximum of the exponent, i.e. the solution of $\partial_i h(\boldsymbol{X}) = 0$ with the Hessian $\mathrm{H}_{ij} \equiv -\partial_i\partial_j h(\boldsymbol{X}_0)$ positive definite. (If there are more than one maximum one should sum over them.)

In our case we have to maximise μ of eq. (5.46), by requiring that $\partial\mu/\partial\xi$, $\partial\mu/\partial\Omega$ and $\partial\mu/\partial\varphi$ vanish. This happens for

$$\xi = \frac{1}{4-\beta^2}\,, \qquad \Omega = c_s\,, \qquad \varphi = n\pi/4\,, \qquad (n = 0,1,2,\ldots)\,, \tag{5.50}$$

but we discard the solutions with n odd, because they make q (and the integration region) vanish, see eq. (5.39). There are four relevant saddle points ($\varphi = 0, \pi/2, \pi, 3\pi/2$) giving the same value to the integral; without loss of generality we choose $\varphi = 0$ and then multiply the final result by 4. The expression of the Hessian matrix at the saddle point is

$$\mathrm{H}_{ij} = 4\begin{pmatrix} \frac{\left(4-\beta^2\right)^{3/2}}{\beta} & -\frac{2c_s\sqrt{4-\beta^2}}{\beta(1-c_s^2)} & 0 \\ -\frac{2c_s\sqrt{4-\beta^2}}{\beta(1-c_s^2)} & \frac{4c_s^2\left(4-\beta^2\right)+2\beta^2}{(1-c_s^2)^2\beta(4-\beta^2)^{3/2}} & 0 \\ 0 & 0 & \frac{4\beta}{(4-\beta^2)^{3/2}} \end{pmatrix}. \tag{5.51}$$

Working at leading order in β and using the saddle-point approximation with these values in eq. (5.44), the energy density of π as a function of $u = t - z$ reads

$$\rho_\pi(u) \approx \frac{\omega^{5/2}(1-c_s^2)}{c_s(8\pi u)^{3/2}\sqrt{\beta}} \exp\left(\frac{\beta}{4}\omega u\right)\,. \tag{5.52}$$

We can compare ρ_π with the energy density of the gravitational wave, which is roughly constant,

$$\rho_\gamma \simeq (M_{\rm Pl}\omega h_0^+)^2\,. \tag{5.53}$$

For instance, for $\beta = 0.1$ and $c_s = 1/2$, we get $\rho_\pi \simeq \rho_\gamma$ after $\tau/\pi \simeq 750$ cycles.

The exponential growth studied in this section can also be seen as a consequence of Bose enhancement, see App. E.

5.2.3 Modification of the gravitational waveform

The parametric production of π suggests that its back-reaction will modify the background gravitational wave. In this subsection we estimate this effect remaining in the narrow-resonance limit, i.e. $|q| \ll 1$ ($\beta \ll 1$). Here we focus again on the case of the operator m_3^3.

To compute the back-reaction on γ_{ij}, we start from the action (5.13). The equation of motion for γ_{ij} is

$$\ddot{\gamma}_{ij} - \nabla^2\gamma_{ij} + \frac{2}{\Lambda^2}\Lambda_{ij,kl}\partial_t\left(\partial_k\pi\partial_l\pi\right) = 0\,, \tag{5.54}$$

where, given a direction of propagation of the wave $\boldsymbol{n}$, $\Lambda_{ij,kl}(\boldsymbol{n})$ is the projector into the traceless-transverse gauge, defined by

$$\Lambda_{ij,kl}(\boldsymbol{n}) \equiv (\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l) - \frac{1}{2}(\delta_{ij} - n_i n_j)(\delta_{kl} - n_k n_l) \ . \tag{5.55}$$

We focus again on a wave traveling in the $\hat{\boldsymbol{z}}$ direction. Using light-cone variables the equation above becomes

$$\begin{aligned} &\partial_u \partial_v \gamma_{ij} + \frac{1}{4\Lambda^2} \partial_u J_{ij}(u) = 0 \ , \\ &J_{ij}(u) \equiv \Lambda_{ij,kl} \partial_k \pi \partial_l \pi = \Lambda_{ij,kl} \int \frac{\mathrm{d}^3 \tilde{\boldsymbol{p}}}{(2\pi)^3} \frac{2 p_k p_l}{c_s^2 p_u} |f_{\tilde{p}}(u)|^2 \ . \end{aligned} \tag{5.56}$$

We can then split the solution into a homogeneous and a forced one,

$$\gamma_{ij} \equiv \bar{\gamma}_{ij} + \Delta\gamma_{ij} \ . \tag{5.57}$$

The former reads

$$\bar{\gamma}_{ij}(u,v) = \bar{\gamma}_{ij}(u,0) + \bar{\gamma}_{ij}(0,v) - \bar{\gamma}_{ij}(0,0) \ , \tag{5.58}$$

while for the latter, which represents the back-reaction due to π, we find

$$\Delta\gamma_{ij}(u,v) = -\frac{1}{4\Lambda^2} \left[J_{ij}(u) - J_{ij}(0) \right] v \ . \tag{5.59}$$

Because it is transverse and traceless, the source J_{ij} can be projected into a plus and cross polarization. We will focus on the plus polarization. In this case, we can proceed as in the previous subsection and rewrite the integral in terms of Ω, φ and ξ,

$$J_{ij}(u) = \frac{\omega^4}{4(2\pi)^3 c_s^5} \epsilon_{ij}^{+} \int \xi \, (1-\Omega^2) \cos(2\varphi) |f_{\tilde{p}}(u)|^2 \, \mathrm{d}\Omega \; \mathrm{d}\varphi \; \mathrm{d}\xi \ . \tag{5.60}$$

Then, we can evaluate this integral in the narrow-resonance regime with the saddle-point approximation. However, we will now use an approximation for the solution $f_{\tilde{p}}(u)$ that takes into account its oscillatory behaviour. In particular, we will split the integral in two regions: $q > 0$ and $q < 0$. Since the sign of q is controlled by the angle φ through $\cos 2\varphi$, this corresponds to splitting the integration over φ.

For $q > 0$, the solution of the Mathieu equation in the first instability band can be approximated by (see pag. 72 of [165])

$$f(\tau) \simeq c_+ \, e^{\mu\tau} \sin(\tau - \sigma) + c_- \, e^{-\mu\tau} \sin(\tau + \sigma) \ , \qquad (q > 0) \ , \tag{5.61}$$

where $\mu > 0$ and $\sigma \in (-\frac{\pi}{2}, 0)$ is a parameter which depends on A and q. It is real inside the instability bands, as shown in the instability chart for the Mathieu equation in fig. 5.2a. More specifically, in the first instability band one has

$$A = 1 - q\cos(2\sigma) + \mathcal{O}(q^2)\ , \tag{5.62}$$

$$\mu = -\frac{1}{2}q\sin(2\sigma) + \mathcal{O}(q^2)\ . \tag{5.63}$$

The coefficients c_+ and c_- can be fixed by demanding that at $\tau = 0$ we recover the vacuum solution.

The case $q < 0$ can be obtained by noting that when q changes sign, μ does it as well while A remains the same. The only way to implement this is to consider the simultaneous change $\sigma \to \sigma' = -\sigma - \frac{\pi}{2}$. By performing these two transformations for q and σ on $f(\tau)$ one obtains

$$f(\tau) \simeq c'_- e^{-\mu\tau}\cos(\tau+\sigma) - c'_+ e^{\mu\tau}\cos(\tau-\sigma)\ , \qquad (q<0)\ . \tag{5.64}$$

We can now integrate the right-hand side of (5.60) starting from the interval $\varphi \in (-\frac{\pi}{4}, \frac{\pi}{4})$. Replacing the growing mode solution of eq. (5.61) in the integrand, we obtain

$$\frac{\omega^4}{4(2\pi)^3 c_s^5}\int_{-\pi/4}^{\pi/4}\mathrm{d}\varphi\int_{-1}^{1}\mathrm{d}\Omega\int\mathrm{d}\xi\,\xi\,(1-\Omega^2)\cos(2\varphi)|c_1|^2\sin^2(\tau-\sigma)e^{2\mu\tau}\ . \tag{5.65}$$

The τ dependence in $\sin(\tau-\sigma)$ seems to change the saddle point computed by maximizing μ. However, by rewriting the sine as exponential functions one can check that its effect is simply to add a constant to the exponent so that the saddle point remains the same as the one we computed in Sec. 5.2.2, see eq. (5.50). At the saddle point, eqs. (5.62) and (5.63) give $\tan(2\sigma) \simeq 2/\beta$ while c_+ and c'_+ can be computed by requiring vacuum initial conditions, eq. (5.34), at $\tau = 0$. Working for small β, this gives

$$\sigma \simeq -\frac{\pi+\beta}{4}\ , \qquad c_+ \simeq c'_+ \simeq \frac{1-i}{\sqrt{2}}\ . \tag{5.66}$$

Then, applying eq. (5.49) to the integral in (5.65), the latter can be solved, giving

$$\frac{\omega^4(1-c_s^2)^2}{2c_s^5(16\pi\tau)^{3/2}\sqrt{\beta}}|c_+|^2\sin^2(\tau-\sigma)e^{\beta\tau/2}\ , \tag{5.67}$$

where for c_+ and σ we must use the saddle-point values, eq. (5.66). We can now repeat this exercise for each part of the integral, using the growing mode of either eqs. (5.61) or (5.64) depending on the sign of q. Summing them together, we obtain

$$J_{ij} \simeq \frac{\omega^4(1-c_s^2)^2}{c_s^5(16\pi\tau)^{3/2}\sqrt{\beta}}\left[|c_+|^2\sin^2(\tau-\sigma) - |c'_+|^2\cos^2(\tau-\sigma)\right]e^{\beta\tau/2}\epsilon^+_{ij}\ . \tag{5.68}$$

Replacing in the solution for $\Delta\gamma_{ij}$, eq. (5.59), the expression of $J_{ij}(u)$ of eq. (5.68), with σ, c_+ and c'_+ fixed by the saddle-point approximation from eq. (5.66), the back-reaction on the

GWs due to π reads

$$\Delta\gamma_{ij}(u,v) \simeq -\frac{v}{4\Lambda^2}\frac{(1-c_s^2)^2}{c_s^5\sqrt{\beta}}\frac{\omega^{5/2}}{(8u\pi)^{3/2}}\sin\left(\omega u+\frac{\beta}{2}\right)\exp\left(\frac{\beta}{4}\omega u\right)\epsilon_{ij}^{+} , \tag{5.69}$$

where we have dropped $J_{ij}(0)$ which is negligible at late time. Thus, the back-reaction grows exponentially in u, as expected from the growth of the energy of π, eq. (5.52), and linearly in v. Note that the right-hand side of the above equation diverges for $\beta \to 0$. This is because it has been obtained using the saddle-point approximation, which assumes that $\beta\omega u$ is large. In App. E we check that this result reduces to the perturbative calculation when the occupation number is small enough.

Notice that there is no production of cross polarization. Indeed, the integrand of the source term in eq. (5.56) now contains $2p_xp_y$ instead of $p_x^2-p_y^2$. Since $p_xp_y \propto \sin(2\varphi)=0$ and the saddle points are such that $\sin(2\varphi)=0$, the cross-polarized waves are not generated by the back-reaction of dark energy fluctuations produced by plus-polarized waves and eq. (5.69) represents the full result.

5.2.4 Generic polarization

So we far we have been discussing linearly polarized waves. Since the resonant effect is non-linear, one cannot simply superimpose the result for linear polarization in order to get a general polarization. In this subsection we are going to consider a more generic polarization state, that we parametrise as follows

$$\gamma_{ij} = M_{\rm Pl}h_0\left[\cos\alpha\sin(\omega u)\epsilon_{ij}^{+}+\sin\alpha\cos(\omega u)\epsilon_{ij}^{\times}\right] , \tag{5.70}$$

where $0\le\alpha<2\pi$ is an angle characterizing the GW polarization. Note that the state of polarization for generic α is *elliptical*, like the one coming from binary systems [143].

Following the same procedure as in Sec. 5.2.1, the Mathieu (5.37)becomes

$$\frac{\mathrm{d}^2f}{\mathrm{d}\tau^2}+[A-2q\cos(2\tau+\hat{\theta})]f=0 , \qquad \tan\hat{\theta}\equiv\tan\alpha\tan(2\varphi) , \tag{5.71}$$

with

$$q=\sqrt{2}\beta\frac{c_s^2\boldsymbol{p}^2}{\omega^2}\frac{(1-\Omega^2)}{1-c_s^2}\sqrt{1+\cos(2\alpha)\cos(4\varphi)} , \tag{5.72}$$

while A remains the same as before. One needs to shift $\tau\to\tau+\hat{\theta}/2$ in order to use the same form for the Mathieu solution. Given this change in q, one can easily obtain the modified saddle points which are given by

$$\xi=\frac{1}{4-\frac{\beta^2}{2}[1+(-1)^n\cos(2\alpha)]} , \qquad \Omega=c_s , \qquad \varphi=n\pi/4 , \qquad (n=0,1,2,\dots) . \tag{5.73}$$

Thus, the saddle points for Ω and φ are unaffected by the angle α. From the saddle point found above one can see that choosing n to be even selects the $+$ polarization, whereas n odd corresponds to $\times$ polarization. The exponent μ of eq. (5.46) can be evaluated on these saddle points and, at leading order in β, is given by

$$\mu \simeq \frac{\beta}{4\sqrt{2}}\sqrt{1+(-1)^n\cos(2\alpha)}\,. \tag{5.74}$$

Here one can define μ^+ and $\mu^\times$ corresponding to $+$ and $\times$ polarizations respectively as

$$\mu^+ \equiv \frac{\beta}{4\sqrt{2}}\sqrt{1+\cos(2\alpha)}\,, \qquad \mu^\times \equiv \frac{\beta}{4\sqrt{2}}\sqrt{1-\cos(2\alpha)}\,. \tag{5.75}$$

Several comments are in order at this stage. First, for $0<\alpha<\pi/4$ the $+$ contribution in the initial wave is larger. In this case one has $\mu^+>\mu^\times$ which means that the $+$ polarization dominates also in the back-reaction for $\Delta\gamma_{ij}$. For $\pi/4<\alpha<\pi/2$ the $\times$ mode dominates. For $\alpha=\pi/4$ both polarization states grow with the same rate, meaning that a circularly polarized wave remains circular. Moreover, by setting $\alpha=0$ ($\alpha=\pi/2$) we recover the results of the previous sections for the case of $+$ ($\times$) polarization.

5.2.5 Conservation of energy

To check our results and to get a better understanding of the system, it is useful to verify that energy is conserved in the production of the π field and the corresponding modification of the GW, $\Delta\gamma$. From the Lagrangian (5.13), one can derive the Noether stress-energy tensor, which is conserved on-shell as a consequence of translational invariance

$$\partial_\mu T^\mu_{\ \nu}=0\,. \tag{5.76}$$

(Notice that this $T^\mu_{\ \nu}$ will be different from the pseudo stress-energy tensor of GR.) Let us consider the region represented in fig. 5.3. Given the symmetries of the system, it is useful to take the left and right boundaries as null, instead of time-like, surfaces.

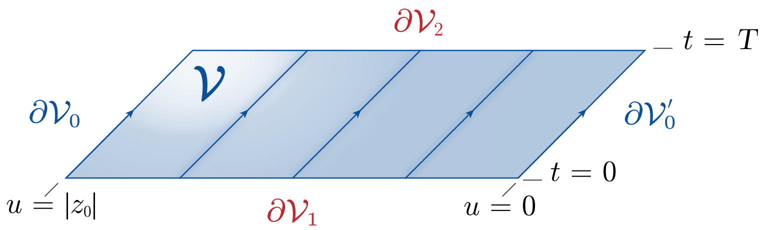


Fig. 5.3 Region $\mathcal{V}$ over which we are checking the conservation of the stress-energy tensor. The boundaries $\partial\mathcal{V}_0$ and $\partial\mathcal{V}_0'$ are null hypersurfaces at $u=const$.

At first sight, it is somewhat puzzling that while both the original GW and the induced π are only displaced as time proceeds (see fig. 5.1 and fig. 5.3), so that their contribution to the total energy does not depend on time, $\Delta\gamma$ grows on later time slices since it is proportional to v. What we are going to verify is that the variation of the energy between $\partial\mathcal{V}_1$ and $\partial\mathcal{V}_2$ due to the change in $\Delta\gamma$ is compensated by a flux of energy across the null boundary $\partial\mathcal{V}_0$.[5] (There is no flux across the right null boundary $\partial\mathcal{V}_0'$ since π is in the vacuum for $u = 0$.)

The Noether stress-energy tensor of the action (5.13) is given by

$$T^0_{\ 0} = (T^0_{\ 0})_\gamma + (T^0_{\ 0})_\pi \ , \tag{5.77}$$

$$(T^0_{\ 0})_\gamma \equiv -\frac{1}{4}\left[\dot\gamma_{ij}^2 + (\partial_k\gamma_{ij})^2\right] \ , \quad (T^0_{\ 0})_\pi \equiv -\frac{1}{2}\left[\dot\pi^2 + c_s^2(\partial_i\pi)^2\right] \ , \tag{5.78}$$

$$T^0_{\ i} = -\frac{1}{2}\dot\gamma_{kl}\partial_i\gamma_{kl} - \dot\pi\partial_i\pi - \frac{1}{\Lambda^2}\partial_i\gamma_{kl}\partial_k\pi\partial_l\pi \ , \tag{5.79}$$

$$T^i_{\ 0} = \frac{1}{2}\dot\gamma_{kl}\partial_i\gamma_{kl} + c_s^2\dot\pi\partial_i\pi - \frac{2}{\Lambda^2}\dot\gamma_{ij}\dot\pi\partial_j\pi \ , \tag{5.80}$$

$$\begin{aligned} T^i_{\ j} = {} & \frac{1}{2}\partial_i\gamma_{kl}\partial_j\gamma_{kl} + c_s^2\partial_i\pi\partial_j\pi - \frac{2}{\Lambda^2}\dot\gamma_{ik}\partial_j\pi\partial_k\pi \\ & + \frac{1}{2}\delta^i_j\left[\frac{1}{2}\dot\gamma_{kl}^2 - \frac{1}{2}(\partial_m\gamma_{kl})^2 + \dot\pi^2 - c_s^2(\partial_l\pi)^2 + \frac{2}{\Lambda^2}\dot\gamma_{kl}\partial_k\pi\partial_l\pi\right] \ . \end{aligned} \tag{5.81}$$

Notice that the total energy of the system is simply the sum of the kinetic energy of γ and π without a contribution due to interactions. (This is a consequence of the interaction term being linear in $\dot\gamma$.) This means that the production of π must be compensated by a decrease of the γ kinetic energy. Using the splitting in eq. (5.57) and defining $\bar\rho_\gamma \equiv \frac{1}{4}(\dot{\bar\gamma}_{ij})^2 + \frac{1}{4}(\partial_k\bar\gamma_{ij})^2$, $\rho_\pi \equiv \frac{1}{2}\dot\pi^2 + \frac{c_s^2}{2}(\partial_i\pi)^2$, the components (5.78)-(5.81) can be written up to second order in perturbation. For instance

$$T^0_{\ 0} = -(\bar\rho_\gamma + \rho_\pi + \frac{1}{2}\dot{\bar\gamma}_{ij}\Delta\dot\gamma_{ij} + \frac{1}{2}\partial_k\bar\gamma_{ij}\partial_k\Delta\gamma_{ij}) + \mathcal{O}(\Delta\gamma^2) \tag{5.82}$$

$$T^i_{\ 0} = \frac{1}{2}\dot{\bar\gamma}_{kl}\partial_i\bar\gamma_{kl} + \frac{1}{2}\Delta\dot\gamma_{kl}\partial_i\bar\gamma_{kl} + \frac{1}{2}\dot{\bar\gamma}_{kl}\partial_i\Delta\gamma_{kl} + c_s^2\dot\pi\partial_i\pi - \frac{2}{\Lambda^2}\dot{\bar\gamma}_{ij}\dot\pi\partial_j\pi + \mathcal{O}(\Delta\gamma^2) \ . \tag{5.83}$$

Gauss theorem works also when the region has null boundaries (see for instance [167]):

$$\int_\mathcal{V}\partial_\mu T^{\mu 0}\mathrm{d}^4x = \oint_{\partial\mathcal{V}} T^{\mu 0} n_\mu \,\mathrm{d}S \ . \tag{5.84}$$

The only subtlety in the case of null boundaries is that one does not know how to choose the normalization of the null vector n^μ orthogonal to the surface (of course when the boundary is null this vector also lies on the surface). A related ambiguity is that there is no natural volume form on the boundary to perform the integration, since the induced metric on a null surface is degenerate. The two ambiguities compensate each other. If one chooses a 3-form $\tilde\varepsilon$ as a volume

[5]Notice that, while the original GW is described by a wave packet localised in a certain interval of u, π waves are present at arbitrary large u and thus will always contribute to the flux.

form on the null surface one needs the covector n_μ to satisfy

$$n \wedge \tilde{\varepsilon} = \varepsilon \ , \tag{5.85}$$

with ε the volume 4-form, the one used to perform the integration in $\mathcal{V}$ in equation (5.84). This equation generalises the concept of orthonormal vector in the Gauss theorem and one can show that it implies eq. (5.84), see [167]. In our case, if one chooses to perform the integral over the null boundary as $\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y$, which corresponds to a 3-form $\tilde{\varepsilon}_{\alpha\beta\gamma}$ which is completely antisymmetric in the variables t, x and y, one has to normalise the orthogonal vector n^μ such that

$$\frac{1}{4}\varepsilon_{\alpha\beta\gamma\delta} = n_{[\alpha}\tilde{\varepsilon}_{\beta\gamma\delta]} \ . \tag{5.86}$$

This is satisfied by the vector $n^\mu = (1, 0, 0, 1)$ in the Minkowski coordinates (t, x, y, z).

We now apply eq. (5.84) to the region depicted in fig. 5.3 and use eq. (5.76). There is no dependence on x and y, so we can factor out the surface $\mathrm{d}x\,\mathrm{d}y$:

$$\int_{\partial\mathcal{V}_2} T^{00}\,\mathrm{d}z - \int_{\partial\mathcal{V}_1} T^{00}\,\mathrm{d}z = -\int_{\partial\mathcal{V}_0} \left(T^{00} - T^{z0}\right)\mathrm{d}t \ . \tag{5.87}$$

We dropped the contribution of the surface $\partial\mathcal{V}_0'$ since all fields vanish on this surface. The LHS is the difference in energy between $t = T$ and $t = 0$, while the RHS gives the flux of energy across $\partial\mathcal{V}_0$. As shown in App. F, neither $\bar{\gamma}$ nor $\Delta\gamma$ contribute to the energy flux across $\partial\mathcal{V}_0$ (intuitively GWs move parallel to this surface). Conversely, since π only depends on u, it does not contribute to the difference in energy. Since the flux of energy is only due to π, which is constant on $\partial\mathcal{V}_0$, the flux is proportional to T. We see that the dependence $\Delta\gamma \propto T$ in the LHS is necessary for the cancellation. Notice that the sign of $\Delta\gamma_{ij}$ in (5.69) is the correct one: it implies that the amplitude of GW is *decreasing*. Indeed, since the total energy is just the sum of the kinetic energy of γ and π, γ must decrease in amplitude as π grows. In App. F we check these statements and verify eq. (5.87).

5.3 Nonlinearities

We now want to look at the effects of π non-linearities (non-linearities of γ are suppressed by further powers of M_{Pl}). In particular, we are going to study the effect of non-linearities on the exponential amplification of dark energy fluctuations that occurs in the narrow-resonance regime.

5.3.1 m_3^3-operator

We start from the operator (5.3). In this case, the cubic self-interaction in the Lagrangian is[6]

$$\frac{1}{\Lambda_{\rm B}^3}\Box\pi(\partial_i\pi)^2 \, , \tag{5.88}$$

where $\Box = -\partial_t^2 + \nabla^2$ and

$$\Lambda_{\rm B}^3 \equiv -\frac{[3m_3^6 + 4M_{\rm Pl}^2(c+2m_2^4)]^{3/2}}{\sqrt{2}m_3^3 M_{\rm Pl}^3} = \frac{\alpha^{3/2}}{\alpha_{\rm B}}\left(\frac{H}{H_0}\right)^2\Lambda_3^3 \, . \tag{5.89}$$

At early times, when π is in the vacuum state, it is safe to neglect these terms. However, as π grows their importance increases and they become comparable to the resonance term

$$\frac{1}{\Lambda^2}\dot{\gamma}_{ij}\partial_i\pi\partial_j\pi \, . \tag{5.90}$$

Since $\Lambda_{\rm B} \ll \Lambda$, we expect this to happen rather quickly. Comparing the above operators, this takes place when $\Box\pi \sim (\Lambda_{\rm B}^3/\Lambda^2)\dot{\gamma}_{ij} \sim \sqrt{\alpha}H\dot{\gamma}_{ij}$, which can be written as

$$(\partial_i\pi)^2 \sim \alpha c_s^2 (h_0^+)^2 \Lambda_2^4 \, , \tag{5.91}$$

by using $\dot{\gamma}_{ij} \sim \omega M_{\rm Pl} h_0^+$ and $\Box\pi \sim (\omega/c_s)\partial_i\pi$. When this happens, both the energy density of π and the modification of the GW, see eq. (5.59), are small. Indeed, using the above equation one finds

$$\frac{\rho_\pi}{\rho_\gamma} \sim \frac{(\partial_i\pi)^2}{(M_{\rm Pl}\omega h_0^+)^2} \sim \alpha\left(\frac{c_s H_0}{\omega}\right)^2 \ll 1 \, , \qquad \frac{\Delta\gamma}{\bar{\gamma}} \sim \frac{v(\partial_i\pi)^2}{\Lambda^2 M_{\rm Pl} h_0^+} \sim vH_0 h_0^+ \alpha_{\rm B} c_s^2 \ll 1 \, . \tag{5.92}$$

After this point one can no longer trust the Mathieu equation and the solutions for π used earlier, eq. (5.61): non-linear terms change the fundamental frequency of the oscillator in eq. (5.37), so that originally unstable modes are driven out of their instability bands and a more sophisticated analysis is required. The same conclusion can be obtained from the Boltzmann analysis of App. E. When the resonance term is modified by $\sim \mathcal{O}(1)$ corrections, particles are produced also outside the thin-shell of momenta Δk. This dispersion in momentum space implies that the Bose-enhancement factor, and in turn the growth index μ_k, are affected.

The above estimate is also in agreement with numerical results in the preheating literature (see e.g. [168, 169]). These simulations suggest that even small self-interactions of the produced fields are enough to qualitatively change the development of the resonance. In these works it was also shown that attractive potentials for the reheated particles modify, and eventually shut

[6] Strictly speaking, to obtain the cubic Galileon self-interaction from the EFT of DE one has to include also a quartic operator $\tilde{m}_3^3(t)\left(\delta g^{00}\right)^2\delta K$. In this chapter this distinction will not play a role. More details on the cubic interactions are given in the next chapter.

down, the exponential growth found at linear level. This conclusion is not surprising, since in these cases large field expectation values contribute positively to their effective mass, making the decay kinetically disfavoured as soon as large field values are reached. This result cannot be applied to our case because the derivative self-interactions in eq. (5.88) do not enter with a definite sign in the action. To reach a definitive answer, in the narrow resonance regime, one would need a full numerical analysis.

5.3.2 $\tilde{m}_4^2$-operator

We will now take $m_3^3 = 0$ and focus on the self-interaction of π generated by the operator of eq. (5.4). This choice results technically natural since m_3^3 corresponds, in the covariant theory, to cubic Horndeski operators that feature a weakly-broken Galilean invariance (recall that the same is true for the choice $\tilde{m}_4^2 = m_5^2$ when imposing $c_T = 1$).

Clearly, on sub-Hubble scales the most important nonlinearities are due to operators containing two derivatives per field. Since we are interested in the regime $\alpha_{\rm H} \ll 1$, in this case the most relevant non-linearities are not cubic but quartic. Indeed, cubic non-linearities are suppressed by $\alpha_{\rm H}/\alpha^{3/2} \cdot \partial^2\pi/\Lambda_3^3$, while the quartic ones by $\alpha_{\rm H}/\alpha^2 \cdot (\partial^2\pi/\Lambda_3^3)^2$. Thus, for $\alpha_{\rm H} \ll 1$, quartic non-linearities become relevant for a smaller value of $\partial^2\pi/\Lambda_3^3$ (unless α is huge, we will not be interested in this regime below). Notice that the cut-off of the theory is thus of order $\alpha^{1/3}\alpha_{\rm H}^{-1/6}\Lambda_3$. This scale is much larger than ω for the values of $\alpha_{\rm H}$ we are interested in, so that the GW experimental results are well within the regime of validity of the theory. See fig. 5.4 for a comparison of the scales in the problem.

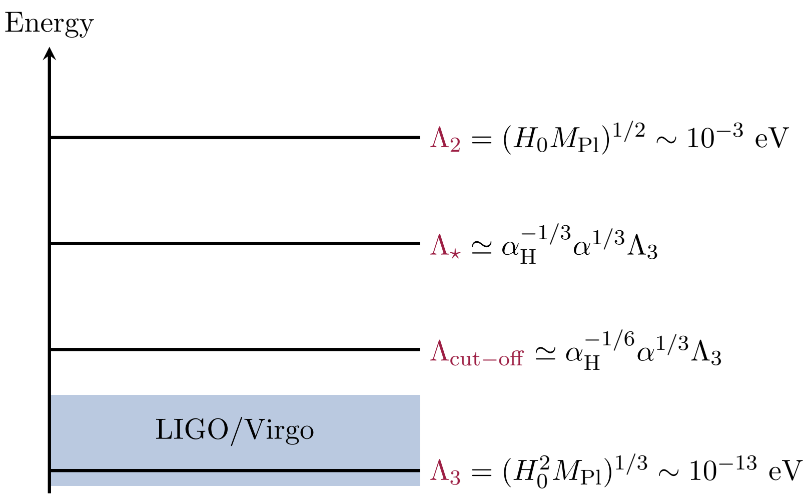


Fig. 5.4 The various energy scales in the limit $\alpha_{\rm H} \ll 1$. We assume $\alpha \gg \alpha_{\rm H}^{1/2}$ so that $\Lambda_{\rm cut-off} \gg \Lambda_3$. Notice that we do not consider the regime of parametrically small c_s, which may modify the estimates above.

Let us start with a naive estimate for $c_s \sim 1$. The resonance will be affected by non-linearities when

$$\frac{\alpha_{\rm H}}{\alpha^2}\left(\frac{\partial^2\pi}{\Lambda_3^3}\right)^2 \sim \beta \, . \tag{5.93}$$

Going through the same calculations as in the case of the m_3^3-operator one gets the constraint

$$\frac{\Delta\gamma}{\bar{\gamma}} \sim \frac{v\,\omega(\partial_i\pi)^2}{\Lambda_\star^3 M_{\rm Pl} h_0^+} \lesssim \beta\alpha(vH_0)\frac{H_0}{\omega h_0^+} \, . \tag{5.94}$$

For typical values of the parameters the RHS of this inequality is at most of order unity. This would mean that non-linearities are important when the GW signal is substantially modified.

Actually, it turns out that the estimate above is not quite correct, as a consequence of the detailed structure of the quartic Galileon interaction. We want to evaluate the importance of the interactions on the modes that grow fastest, i.e. on the saddle point. Using (u,s,x,y) coordinates, one has $p_s = 0$ on the saddle, see eqs. (5.30) and (5.50). This can be understood from the equation of motion, eq. (5.35): for a given frequency of a π wave, i.e. for a given momentum $|\boldsymbol{p}|$, one maximises the forcing term if $p_s = 0$. Therefore, in these coordinates the derivative with respect to s vanishes. However, since the coordinate u is null, the inverse metric satisfies $g^{uu} = 0$. This means that also the derivative with respect to u will not appear in the interaction (there are only cross terms $\propto \partial_u\partial_s$). Therefore, we are left only with the two coordinates x and y. However, the structure of the quartic Galileon is such that it vanishes if one has less than three dimensions. We conclude that the quartic π self-interaction vanishes on the saddle point. To get a correct estimate one is forced to look at deviations from the saddle, i.e. one has to estimate what is the typical range of p_s around $p_s = 0$ that contributes to the integrals like eq. (5.65).

The rms value of the various variables can be read from the inverse of the Hessian matrix eq. (5.51). The momentum p_s can be written in terms of the integration variables (ξ,Ω,φ) as

$$p_s = \frac{c_s\omega}{1-c_s^2}\sqrt{\xi}(\Omega - c_s) \, . \tag{5.95}$$

One can write

$$(\Delta p_s)^2 = \frac{\partial p_s}{\partial X^i}\frac{\partial p_s}{\partial X^j}\Delta X^i \Delta X^j = \frac{\partial p_s}{\partial X^i}\frac{\partial p_s}{\partial X^j}({\rm H}^{-1})^{ij} \, . \tag{5.96}$$

Evaluated at the saddle, the matrix of the first derivatives of p_s is zero, except for the entry (Ω,Ω) which is given by

$$\left(\frac{\partial p_s}{\partial\Omega}\right)^2 = \frac{c_s^2\omega^2}{(1-c_s^2)^2(4-\beta^2)} \, . \tag{5.97}$$

The entry (Ω,Ω) of the inverse of the Hessian reads $\left({\rm H}^{-1}\right)^{\Omega\Omega} = \frac{(1-c_s^2)^2}{4\beta\tau}(4-\beta^2)^{3/2}$. Therefore, we obtain

$$\Delta p_s = \frac{c_s\omega(4-\beta^2)^{1/4}}{2\sqrt{\beta\tau}} \simeq \frac{c_s\omega}{\sqrt{2\beta\tau}} \, . \tag{5.98}$$

Using this estimate one can revise the bound above as

$$\frac{\Delta\gamma}{\bar{\gamma}} \sim \frac{v\,\omega(\partial_i\pi)^2}{\Lambda_\star^3 M_{\rm Pl} h_0^+} \lesssim \beta c_s^3 \alpha(vH_0)\frac{H_0}{\omega h_0^+}\sqrt{\beta\tau}\ . \tag{5.99}$$

It is important to stress that the coefficient of the quartic self-interaction of π is tied by symmetry with the one of the operator $\ddot{\gamma}_{ij}\partial_i\pi\partial_j\pi$, so that the effect of self-interactions cannot be suppressed. This statement holds even considering models with $c_T \neq 1$: since we are considering a regime of very small $\alpha_{\rm H}$, comparable values for $c_T - 1$ are not ruled out experimentally. If one does not impose the constraints on the speed of GWs, instead of the single operator of eq. (5.4) one has two independent coefficients

$$\frac{\tilde{m}_4^2}{2}\delta g^{00}\,{}^{(3)}\!R + \frac{m_5^2}{2}\delta g^{00}\left(\delta K^\nu_\mu \delta K^\mu_\nu - \delta K^2\right)\ . \tag{5.100}$$

In terms of these parameters, the coefficient of the quartic Galileon was calculated in [170] and it reads

$$\frac{2}{M_{\rm Pl}^2}(\tilde{m}_4^2 + m_5^2)\ . \tag{5.101}$$

On the other hand the coefficient of the operator $\ddot{\gamma}_{ij}\partial_i\pi\partial_j\pi$ reads (see eq. (C.11) and the related discussion of App. C)

$$\frac{1}{M_{\rm Pl}^2}(\tilde{m}_4^2 + m_5^2 c_T^2)\ . \tag{5.102}$$

The two coincide, modulo a factor of 2, up to relative corrections suppressed by $c_T^2 - 1$.

5.4 Observational signatures for $\tilde{m}_4^2$

5.4.1 Fundamental frequency

In the narrow resonance regime, $\beta \ll 1$, with the replacement $\Lambda^2 \to \Lambda_\star^3\omega^{-1}$, from eq. (5.69) one gets a relative modification of the GW given by

$$\frac{\Delta\gamma(u,v)}{\bar{\gamma}} \simeq -\frac{v}{4\Lambda_\star^3 M_{\rm Pl} h_0^+}\frac{(1-c_s^2)^2}{c_s^5\sqrt{\beta}}\frac{\omega^{7/2}}{(8u\pi)^{3/2}}\exp\left(\frac{\beta}{4}\omega u\right)\ . \tag{5.103}$$

Here we are interested in the phenomenological consequences of the (possibly large) backreaction on GWs. In particular, since the effect we pointed out relies on having many oscillations of the GW, it is interesting to look at effects for waves originated from binary systems. In such scenario, the amplitude of GWs in the plus polarization, as can be read from (3.1), is given by

$$h_0^+ \sim \frac{4}{\sqrt{2}r}(G_{\rm N}M_c)^{5/3}(\pi f)^{2/3}\ , \tag{5.104}$$

where r is the distance from the source, M_c is the chirp mass of the binary system and the factor of $\sqrt{2}$ at the denominator comes from our convention for the normalization of the graviton (see eq. (5.8)). The number of cycles of the gravitational wave is given by

$$N_{\rm cyc} = \frac{\omega u}{2\pi} = f u \; , \tag{5.105}$$

where we have defined the gravitational wave frequency $f \equiv \omega/2\pi$. In our calculations we took a time-independent frequency, while in reality f increases with time, during the binary inspiralling. The frequency can be taken as roughly constant for the number of cycles $N_{\rm cyc}$ required for f to double in size. In particular, the number of cycles between f and $2f$ can be obtained by using the time evolution of the phase in eq. (3.4) and inverting the relation between f and t in (3.3). In the end is given by

$$\bar{N}_{\rm cyc} = \frac{4-2^{1/3}}{128\pi}(\pi G M_c f)^{-5/3} \; . \tag{5.106}$$

Inverting these relations we can express the chirp mass as a function of the number of cycles and the frequency. Using this in eq. (5.104) we obtain

$$h_0^+ \sim \frac{0.006}{f \bar{N}_{\rm cyc} r} \; . \tag{5.107}$$

In the calculation we approximate the GW amplitude as a constant, therefore we have to limit $v \lesssim r$.

Expressing β in terms of $\alpha_{\rm H}$, replacing h_0^+ from the above relation and using $v = r$, eq. (5.103) becomes

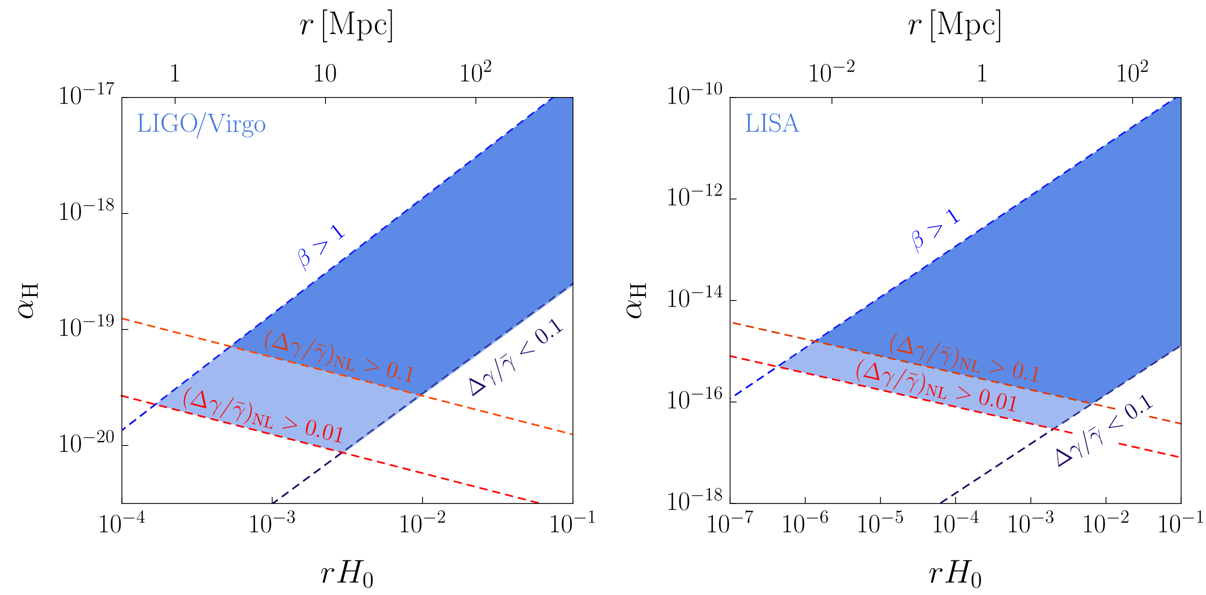
$$\begin{aligned} \frac{\Delta\gamma}{\bar{\gamma}} \sim -6\cdot 10^{-3}\frac{(1-c_s^2)^2}{c_s^3}\left(\frac{\alpha_{\rm H}}{\alpha c_s^2}\right)^{\frac{1}{2}}\left(\frac{H_0}{M_{\rm Pl}}\right)^{\frac{1}{6}}\left(\frac{2\pi f}{\Lambda_3}\right)^{\frac{11}{2}}(rH_0)^{\frac{5}{2}}\times \\ \times \exp\left[\frac{0.12}{rH_0}\left(\frac{M_{\rm Pl}}{H_0}\right)^{\frac{1}{3}}\frac{\alpha_{\rm H}}{\alpha c_s^2}\frac{2\pi f}{\Lambda_3}\right] \; . \end{aligned} \tag{5.108}$$

Note that this expression is independent of $\bar{N}_{\rm cyc}$. Sizeable effects in the GW waveform can be obtained when the argument of the exponential is $\sim \mathcal{O}(10^2)$, which translates into

$$\alpha_{\rm H} \gtrsim 10^{-17}\cdot rH_0 \cdot \frac{\Lambda_3}{2\pi f}\alpha c_s^2 \; . \tag{5.109}$$

We have also to impose the constraint of eq. (5.99), which using eq. (5.107) reads

$$\frac{\Delta\gamma}{\bar{\gamma}} \lesssim 18(\beta \bar{N}_{\rm cyc})^{3/2} c_s^3 \alpha (rH_0)^2 \equiv \left(\frac{\Delta\gamma}{\bar{\gamma}}\right)_{\rm NL} \; , \tag{5.110}$$



(a) **(b)**

Fig. 5.5 The blue regions indicate where our approximations apply and one has a sizeable modification of the GW signal. Above the red dashed lines the effect of π non-linearities is small. The upper blue dashed lines indicate the line $\beta = 1$: our analytical approximation holds for $\beta \ll 1$ and that we extrapolate to $\beta \lesssim 1$. One has a sizeable effect on the GW above the lower dashed blue lines. This mainly depends on the exponential of $\beta \bar{N}_{\rm cyc}$ and since $\bar{N}_{\rm cyc}$ is fixed once M_c and f are given, this constraint is to a good approximation a lower bound on β. Left panel (fig. 5.5a): LIGO/Virgo case: $f = 30$ Hz, $M_c = 1.188\ \mathrm{M}_\odot$ as for GW170817. Right panel (fig. 5.5b): LISA case: $f = 10^{-2}$ Hz, $M_c = 30\ \mathrm{M}_\odot$ as for GW150914.

where the term $\beta \bar{N}_{\rm cyc}$ roughly coincides with the argument of the exponential in eqs. (5.103) and (5.108). Finally, a further constraint to impose is the narrow resonance condition $\beta \ll 1$.

In fig. 5.5a and fig. 5.5b we plot these three constraints as a function of $\alpha_{\rm H}$ and the distance between the source and the resonant decay of the GW into π (therefore this is not the distance between the source and the detector). The first plot is done for a ground-based interferometer with LIGO/Virgo-like sensitivity; in order to maximise the number of oscillation we choose a neutron-star event similar to GW170817 [46]. The second plot is for a space-based interferometer with LISA-like sensitivity [171] and a binary black hole event similar to GW150914 [40]. The blue region corresponds to a sizeable modification of the GW signal, calculable within our approximation. The neutron-star merger GW170817 is at a distance of approximately 40 Mpc. The absence of sizeable effects ($\Delta\gamma > 0.1\gamma$; of course future measurements will improve this sensitivity) in the observed event puts constraints on a resonant effect that takes place at less than 40 Mpc from the source and rules out the interval: $3 \times 10^{-20} \lesssim \alpha_{\rm H} \lesssim 10^{-18}$. Future measurements by LISA of events similar to GW150914 can, on the other hand, constrain the range $10^{-16} \lesssim \alpha_{\rm H} \lesssim 10^{-10}$.

It may be useful to summarize here the set of assumptions for the validity of the constraints in these figures:

- Narrow resonance regime, i.e. $\beta \sim \alpha_{\rm H}(\omega/H)^2 h_0^+ \lesssim 1$. This is diplayed by the blue-dashed line in the figures.
- The modification of the GW is sufficiently sizable to be observed, i.e. $|\Delta\gamma| > 0.1\gamma$. This is diplayed by the purple-dashed line in the figures. The modification takes place outside the Vainshtein radius, which for the values of $\alpha_{\rm H}$ that we are considering corresponds to distances from the source smaller than those shown on the horizontal axis. Moreover, to be relevant the modification must occur before the detection.
- Nonlinearities in π are small so that their effect on the Mathieu equation is negligible, see discussion in Sec. 5.3.2. This is displayed by the red-dashed lines in the figures.
- In the figures we assume $c_s = 1/2$ and the constraints would change for different values of c_s. From eq. (5.69) the effect is proportional to $(1-c_s^2)^2$ so that it is suppressed for values of c_s close to 1. Moreover, as we discussed in Sec. 5.2.1, for $c_s - 1 \lesssim 10^{-14}$ the π lightcone is too wide and our calculations do not apply. (Notice also that in our estimates of the various scales we are assuming that c_s is not parametrically small.)

For comparison, the bound coming from the perturbative decay of the graviton, eq. (4.35), reads $\alpha_{\rm H} \lesssim 10^{-10}$. Such values of $\alpha_{\rm H}$ are outside the narrow resonance regime studied in this chapter. One expects that the effect of large occupation number and nonlinearities of π are also important for larger values of $\alpha_{\rm H}$ so that this perturbative bound should be revised. Moreover, notice that when the GW is closer to the source, its amplitude is larger and β will exceed unity. At a certain point one enters the Vainshtein regime and the coupling $\gamma\pi\pi$ is reduced (since we are considering very small values of the coupling, the Vainshtein radius will be correspondingly small). We will study all these aspects in the next chapter, where we will show that nonetheless the bound on $\alpha_{\rm H}$ in the regime $\beta > 1$ can still be applied.

5.4.2 Higher harmonics and precursors

Since for the operator $\tilde{m}_4^2$ one can trust the parametric growth and the change in the GW signal $\Delta\gamma$, we want to study this correction in more detail. At leading order in q the change $\Delta\gamma$ has the same frequency as the original wave. However, corrections to the leading solution (5.61) introduce higher harmonics in the GW signal. Higher harmonics appear in two quantitatively different ways. First, we can consider higher instability bands, see fig. 5.2a. However, in the narrow resonance approximation the instability coefficient in the m-th band scales as $\mu_m \sim q^m/(m!)^2 \sim \beta^m$ [165, 166]. Therefore, in the long τ limit they are exponentially suppressed compared to the fundamental band.

Second, we can stay in the first band and look at the modification of the π solution at higher orders in q. This will give an effect which is only suppressed by positive powers of q. If we include the first two corrections to eq. (5.61), we have[7]

$$\begin{aligned} f(\tau) &\simeq c_+ e^{(\mu+\delta\mu)\tau} F(\tau;\sigma) + c_- e^{(-\mu-\delta\mu)\tau} F(\tau;-\sigma) \\ F(\tau;\sigma) &= \sin(\tau-\sigma) + s_3 \sin(3\tau-\sigma) + c_3 \cos(3\tau-\sigma) + s_5 \sin(5\tau-\sigma) \\ s_3 &\equiv -\frac{1}{8}q + \frac{1}{64}q^2 \cos(2\sigma)\ , \quad c_3 \equiv \frac{3}{64}q^2 \sin(2\sigma)\ , \quad s_5 \equiv \frac{1}{192}q^2\ . \end{aligned} \tag{5.111}$$

Clearly, in this case the higher frequency component grows as fast as the original frequency. Therefore, it is not necessary to specify the correction $\delta\mu$. The correction to $|f_{\tilde{p}}|^2$ at order q^2 is thus

$$\begin{aligned} |f_{\tilde{p}}|^2 \simeq |c_+|^2 e^{2\mu\tau} \Big[& \sin^2(\tau-\sigma) + 2s_3 \sin(\tau-\sigma)\sin(3\tau-\sigma) + s_3^2 \sin^2(3\tau-\sigma) + \\ & 2c_3 \sin(\tau-\sigma)\cos(3\tau-\sigma) + 2s_5 \sin(\tau-\sigma)\sin(5\tau-\sigma)\Big]\ . \end{aligned} \tag{5.112}$$

On the other hand, when q becomes negative the solution can be obtained by performing the transformation $\tau \to \tau + \pi/2$, or equivalently $q \to -q$ together with $\sigma \to -\sigma - \pi/2$. By using the latter transformation and by recalling that by flipping the sign of q we send the decreasing mode into the growing one, we get the contribution for negative q:

$$\begin{aligned} |f_{\tilde{p}}|^2 \simeq |c'_+|^2 e^{2\mu\tau} \Big[& \cos^2(\tau-\sigma) - 2s_3 \cos(\tau-\sigma)\cos(3\tau-\sigma) + s_3^2 \cos^2(3\tau-\sigma) + \\ & 2c_3 \cos(\tau-\sigma)\sin(3\tau-\sigma) + 2s_5 \cos(\tau-\sigma)\cos(5\tau-\sigma)\Big]\ . \end{aligned} \tag{5.113}$$

By direct calculation we can also show that, as in (5.66), $|c_+| = |c'_+| = 1$ at lowest order in q. We also note that, since all the frequencies in the expressions above are multiples of ω, the position of the saddle point is not affected and it is simply fixed by the exponential function. Hence, the expression for $J(u)$ can be obtained by using the saddle-point procedure as in (5.68). We obtain

$$\begin{aligned} \Delta\gamma_{ij}(u,v) \simeq -\frac{v}{4\Lambda^2}\frac{(1-c_s^2)^2}{c_s^5\sqrt{\beta}}\frac{\omega^{5/2}}{(8u\pi)^{3/2}} \exp\left(\frac{\beta}{4}\omega u\right)\epsilon^+_{ij} \Bigg\{ & \sin\left(\omega u + \frac{\beta}{2}\right) - \frac{\beta}{8}\cos(\omega u) \\ + \frac{\beta^2}{768}\left[12\sin\left(\omega u - \frac{\beta}{2}\right) + 6\sin\left(\omega u + \frac{\beta}{2}\right) + \sin\left(3\omega u + \frac{\beta}{2}\right)\right] & \Bigg\}\ . \end{aligned} \tag{5.114}$$

(The result for the case of $\tilde{m}_4^2$, which is the focus of this section, can be obtained by the replacement $\Lambda^2 \to \Lambda_\star^3\omega^{-1}$.) We are interested in the τ-dependent oscillatory part of (5.114). Compared to (5.68) we get a correction at the fundamental frequency linear in β (with no

[7]In this section we write the formulas for the case of m_3^3, so that they can be easily compared with Sec. 5.2.3. Notice that, since in this case π is coupled with $\dot{\gamma}$, while in the case $\tilde{m}_4^2$ it is coupled with $\ddot{\gamma}$, there is an overall shift of $\pi/2$ between the two cases.

phase-shift). Moreover, to find the first higher harmonics we have to go to order β^2, where indeed we encounter a term with three-times the frequency ω. Note that by going to the second band instead, we would find higher harmonics of frequency 2ω.

Higher harmonics enter the bandwidth of the detector at earlier time compared to the main signal. In some sense they are *precursors* of the main signal. To assess their potential observability, one has to consider that also the post-Newtonian expansion generates terms with a frequency which is a multiple of the original one. It would be interesting to explore these effects further, with a detailed analysis.

5.5 Discussion and outlooks

In this chapter we have studied the effects of a classical GW on scalar field fluctuations, in the context of dark energy models parametrised by the EFT of DE. The GW acts as a classical background and modifies the dynamics of dark energy perturbations, leading to parametric resonant production of π fluctuations. This regime is described by a Mathieu equation, where modes in the unstable bands grow exponentially. In the regime of narrow resonance, corresponding to a small amplitude of the GW, the instability can be studied analytically in detail. One can also include the back-reaction of the produced π on the original GW, which leads to a change in the signal. The resonant growth is however very sensitive to the non-linearities of the produced fields: when non-linearities of π are sizeable the resonance is damped. This regime cannot be captured analytically: numerical simulations are needed to confirm this effect. This happens for the operator m_3^3, while a sizeable modification of the GW signal is possible for the operator $\tilde{m}_4^2$, at least in some range of parameters, see fig. 5.5a and fig. 5.5b.

Many questions remain still open. For larger amplitudes of the GW, one exits the regime of narrow resonance: the GW may induce instabilities, which we study in the next chapter. This will also allow to revise the robustness of the perturbative calculation of [1], discussed in chapter 4. In the next chapter we will also mention how these effects are changed in the presence of Vainshtein screening. Simulations are probably required if one wants to study the regime of parametric resonance in the presence of sizeable π self-interactions. Similar effects are at play in preheating after inflation and are also addressed numerically, see e.g. [168, 169]. It would also be interesting to try to envisage methods to look for the small changes in the GW signal. For instance, the generation of higher harmonics that enter the detection bandwidth before the main signal is a striking possible effect if it can be disentangled from post-Newtonian corrections. The effects that we studied look quite generic to all theories in which gravity is modified with a quite low cut-off. Therefore, it would be nice to explore other setups, starting from the DGP model [83]. This model features the same non-linear interaction of π as the ones of the m_3^3 operator; however the extra-dimensional origin changes the dynamics of GWs so our analysis cannot be applied directly. All these points are worthwhile considering in the future.

6 | Dark-Energy Instabilities induced by Gravitational Waves

In chapter 4 we studied the decay induced by the Beyond Horndeski term $\tilde{m}_4^2$ perturbatively, i.e. when individual gravitons decay independently of each other, and we showed that the absence of perturbative decay implies that $\Lambda_\star \gtrsim 10^3\Lambda_3$, setting a tight bound on the parameter space of the model. In particular, the absence of decay sets a bound on this operator: $|\tilde{m}_4^2| \lesssim 10^{-10}M_{\rm Pl}^2$. Equivalently, in terms of the dimensionless parameter $\alpha_{\rm H}$ this translates in the bound $|\alpha_{\rm H}| \lesssim 10^{-10}$. For more general theories Beyond Horndeski, such as DHOST theories, the constraint becomes $\alpha_{\rm H} + 2\beta_1 \lesssim 10^{-10}$.[1] This rules out the possibility of observing the effects of these theories in the large-scale structure.

In chapter 5 instead, we extended this study to consider coherent effects due to the large occupation number of the GW, acting as a classical background for π. In this case, a better description of the system is that of parametric resonance, where π fluctuations are described by a Mathieu equation and are exponentially produced by parametric instability. We focused on the regime of narrow resonance, obtained when the GW induces a small perturbation on the π equation. This regime can be used to probe only very small values of $\alpha_{\rm H}$. In particular, within the validity of our approximations the resonant decay takes place in the range $10^{-20} \lesssim |\alpha_{\rm H}| \lesssim 10^{-17}$ for frequencies of interest for LIGO/Virgo and $10^{-16} \lesssim |\alpha_{\rm H}| \lesssim 10^{-10}$ for LISA, as shown in fig. 5.5a and fig. 5.5b.

Another operator leading to the decay is m_3^3. In this case, the scale Λ that suppresses this cubic interaction is typically much higher than Λ_3, i.e. $\Lambda \sim \Lambda_2 = (H_0 M_{\rm Pl})^{1/2}$ and the perturbative decay is negligible. Moreover, non-linearities in the dark energy field become sizeable much before the effect of narrow resonance is relevant, possibly quenching the coherent instability. Therefore, the study of the perturbative and resonant decay for this operator remains inconclusive.

In this chapter we study the effect of a classical GW on π in the regime where the amplitude of the wave is large, i.e. far from the narrow resonance, focussing on the stability of π perturbations. We initially concentrate on the operator m_3^3, while deviations from this case are studied in

[1] The consequences of this constraint on the Vainshtein mechanism in these theories have been studied in [172, 170].

App. G. Inspired by the analysis of [85] reviewed in Sec. 6.2.1, in the rest of Sec. 6.2 we compute the non-linear classical solution of π generated by the GW and we study the stability of π fluctuations, outlining the differences with the analysis of [85]. We consider two different regimes: subluminal and luminal speed of π fluctuations, respectively examined in Sec. 6.2.2 and 6.2.3. Both cases display instabilities and qualitatively agree. This means that, as soon as the parameter β (defined in eq. (5.19) and (5.24)) becomes larger than unity, the fluctuations of π become ghost or gradient unstable. A more refined estimate of when this happens is given in the next sections.

After this happens, the evolution of the system and its endpoint cannot be described without the knowledge of the UV completion of the theory. We discuss this issue in Sec. 6.3 with an example that displays similar instabilities and whose UV completion is known. Anyway, the theory must change qualitatively in the regions where the instability develops. In Sec. 6.4 we study whether the populations of binary systems and their production of GWs is enough to trigger the instability in the whole Universe. Stellar and massive BHs are able to globally induce the instability in the regime where one has a sizeable effect on structure formation ($|\alpha_{\rm B}| \gtrsim 10^{-2}$). The instability is triggered by GWs as long as 10^{10} km, so that our conclusions are robust unless the theory is modified on even longer scales. In Sec. 6.5, we discuss the application of our study to the operator $\tilde{m}_4^2$ as well, and we derive strong bounds of order $|\alpha_{\rm H}| \lesssim 10^{-20}$. Finally, we discuss our conclusions and future prospects in Sec. 6.6.

6.1 The action

In this section we are going to summarize what are the couplings we have to take into account if we want to study the dynamics of π and γ_{ij}. Since self-interactions of π are important to consider, we introduce an additional operator to the EFT of DE action (2.37) in such a way to recover, in the covariant formulation, the typical terms appearing in cubic covariant Galileon models. Therefore, we consider the following action in unitary gauge,

$$S = \int \Big[\frac{M_{\rm Pl}^2}{2}R - \lambda(t) - c(t)g^{00} + \frac{m_2^4(t)}{2}(\delta g^{00})^2 - \frac{m_3^3(t)}{2}\delta g^{00}\delta K - \frac{\tilde{m}_3^3(t)}{8}(\delta g^{00})^2\delta K\Big]\sqrt{-g}\,{\rm d}^4x\,, \tag{6.1}$$

and focus in particular on the cubic Galileon, i.e. $\tilde{m}_3^3 = -m_3^3$ [54]. Generalizations of this case are discussed in App. G. As usual, we take a constant $M_{\rm Pl}$ with a proper choice of frame, while the general case is discussed at the end of the chapter. The operator proportional to m_2^4 introduces self-interactions but in the cosmological setting these are suppressed by $\Lambda_2 \gg \Lambda_3$ and can be dropped for this discussion, because they are irrelevant for the stability. For the same reason, we can ignore higher powers of δg^{00}.

We follow again the same procedure as in the previous chapter, working in Newtonian gauge, with the solution of the constraints given by eq. (5.7). After applying the usual Stueckelberg procedure, we obtain quadratic and cubic terms in the action for π and γ_{ij}. In the following we will use canonically normalized fields, given in eq. (5.8), and we will drop the symbol of canonical normalizations. Neglecting the expansion of the Universe, the action for π reads $S_\pi = \int \mathscr{L}\, \mathrm{d}^4x$, where the π Lagrangian is

$$\mathscr{L} = \frac{1}{2}\left[\dot{\pi}^2 - c_s^2(\partial_i\pi)^2\right] - \frac{1}{\Lambda_{\mathrm{B}}^3}\Box\pi(\partial\pi)^2 + \frac{1}{\Lambda^2}\dot{\gamma}_{ij}\partial_i\pi\partial_j\pi + \frac{m_3^3}{2\sqrt{\alpha}M_{\mathrm{Pl}}^3 H}\pi\dot{\gamma}_{ij}^2 \,, \tag{6.2}$$

where, as usual $\Box\pi = \eta^{\mu\nu}\partial_\mu\partial_\nu\pi$ and $(\partial\pi)^2 = \eta^{\mu\nu}\partial_\mu\pi\partial_\nu\pi$. The parameters c_s, Λ and Λ_{B} are given in eq. (5.14), (5.10) and (5.89). Let us comment on the last operator, containing the cubic coupling $\gamma\gamma\pi$. Such vertex is not directly obtained from the operator $m_3^3\delta g^{00}\delta K$. Instead, it comes from the Einstein-Hilbert term of the action (6.1) when replacing the potentials Φ and Ψ with π *via* eq. (5.7).[2]

For the GW, we will use again the classical background solution travelling in the $\hat{z}$ direction with linear polarization $+$, used in the previous chapter,

$$\gamma_{ij} = M_{\mathrm{Pl}} h_0^+ \sin\left[\omega(t-z)\right]\epsilon_{ij}^+ \,, \tag{6.3}$$

where h_0^+ is the dimensionless strain amplitude. For later convenience, we also recall that the parameter controlling the parametric resonance is

$$\beta = \frac{2\omega M_{\mathrm{Pl}} h_0^+}{c_s^2|\Lambda^2|} = \frac{2\sqrt{2}|\alpha_{\mathrm{B}}|}{\alpha c_s^2}\frac{\omega}{H}h_0^+ \,. \tag{6.4}$$

6.2 Classical solutions and stability of perturbations

In any non-linear theory one can investigate the stability of a given solution by looking at the kinetic term of small perturbations around it. This was done for the cubic Galileon (equivalent to the decoupling limit of the Dvali-Gabadadze-Porrati (DGP) model) in [85], in the absence of GWs. It was proven that solutions that are stable at spatial infinity are stable everywhere, provided the sources are non-relativistic. The analysis was later extended to higher Galileons in [173], where such strong statement does not hold and one expects that general non-linear solutions feature instabilities. Here we want to extend the analysis of [85] including GWs[3] and

[2]Since we know that a tensor perturbation γ_{ij} couples with the metric in the same way as a scalar field does, one can easily obtain this interaction by considering the Lagrangian of minimally coupled scalar field and replacing the scalar field by γ_{ij}.

[3]As discussed in [174], the DGP model is not a local theory of a scalar field and thus is not included in the ordinary EFT of DE action. However, the structure of the non-linear terms is analogous and the arguments used in [85] can be applied straightforwardly to the EFT of DE. On the other hand, the brane-bending mode in the DGP model is not a scalar under 4d diffs [84], so that the coupling with GWs will be different from the one discussed here.

considering a generic speed of propagation c_s. (In order to compare with the result of [85], in the main text we stick to the non-linearity of the cubic Galileon, i.e. $\tilde{m}_3^3 = -m_3^3$. In App. G we consider the more general case $\tilde{m}_3^3 \neq -m_3^3$.)

For convenience, we define $\bar{\eta}_{\mu\nu} \equiv \mathrm{diag}(-1, c_s^2, c_s^2, c_s^2)$ and $\bar{\Box}\pi \equiv \bar{\eta}^{\mu\nu}\partial_\mu\partial_\nu\pi = -\ddot{\pi} + c_s^2\partial_k^2\pi$. In this section, indices are raised and lowered with the usual Minkowski metric. Moreover, we define

$$\Gamma_{\mu\nu} \equiv \frac{\dot{\gamma}_{\mu\nu}}{\Lambda^2} \, . \tag{6.5}$$

Using the above definitions, the action for π, eq. (6.2), becomes

$$\mathscr{L} = -\frac{1}{2}\bar{\eta}^{\mu\nu}\partial_\mu\pi\partial_\nu\pi - \frac{1}{\Lambda_{\rm B}^3}\Box\pi(\partial\pi)^2 + \Gamma_{\mu\nu}\partial^\mu\pi\partial^\nu\pi - \frac{\Lambda_{\rm B}^3}{2}\pi\Gamma_{\mu\nu}^2 \, . \tag{6.6}$$

In the following we will use that $\partial^\mu\Gamma_{\mu\nu} = \eta^{\mu\nu}\Gamma_{\mu\nu} = \bar{\eta}^{\mu\nu}\Gamma_{\mu\nu} = 0$. For our GW solution (6.3) we have

$$\Gamma_{00} = \Gamma_{0i} = 0 \, , \qquad \Gamma_{ij} = \frac{\beta c_s^2}{2}\cos\left[\omega(t-z)\right]\epsilon_{ij}^+ \, , \tag{6.7}$$

where we have used the definition of β, eq. (6.4). The third term of eq. (6.6) suggests that for $\beta > 1$ the scalar π features a gradient instability, because $\Gamma_{\mu\nu}$ changes sign in time. This conclusion, although substantially correct, is premature, since the GW also sources a background for π and this affects through non-linearities the behaviour of perturbations.

Let us split the field in a classical background part plus fluctuations, i.e. $\pi = \hat{\pi}(t, \boldsymbol{x}) + \delta\pi(t, \boldsymbol{x})$. In general $\hat{\pi}$ will be sourced by the term $\dot{\gamma}_{\mu\nu}^2$ of (6.2), corresponding to the last term of eq. (6.6), and also by astrophysical matter sources (that here we have not specified). Let us first study the background solution $\hat{\pi}$ in the presence of a classical background for $\Gamma_{\mu\nu}$. Its equation of motion reads

$$\bar{\Box}\hat{\pi} - \frac{2}{\Lambda_{\rm B}^3}\left[(\partial_\mu\partial_\nu\hat{\pi})^2 - \Box\hat{\pi}^2\right] - 2\Gamma_{\mu\nu}\partial^\mu\partial^\nu\hat{\pi} - \frac{\Lambda_{\rm B}^3}{2}\Gamma_{\mu\nu}^2 = 0 \, . \tag{6.8}$$

Following [85], we define the matrix

$$\mathcal{K}_{\mu\nu} \equiv -\frac{1}{\Lambda_{\rm B}^3}\partial_\mu\partial_\nu\hat{\pi} \, , \tag{6.9}$$

and rewrite the above equation as

$$\mathcal{K}^{\mu\nu}\bar{\eta}_{\mu\nu} + 2\left(\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} - \mathcal{K}^2\right) - 2\Gamma_{\mu\nu}\mathcal{K}^{\mu\nu} + \frac{1}{2}\Gamma_{\mu\nu}^2 = 0 \, . \tag{6.10}$$

Due to the Galileon symmetry and the fact the equations of motion are second order, eq. (6.8) reduces to an algebraic equation for the second derivatives of $\hat{\pi}$. Eq. (6.10) can be rewritten solely in terms of $\tilde{\mathcal{K}}_{\mu\nu} \equiv \mathcal{K}_{\mu\nu} - \frac{1}{2}\Gamma_{\mu\nu}$, and becomes

$$\tilde{\mathcal{K}}^{\mu\nu}\bar{\eta}_{\mu\nu} + 2\left(\tilde{\mathcal{K}}_{\mu\nu}\tilde{\mathcal{K}}^{\mu\nu} - \tilde{\mathcal{K}}^2\right) = 0 \, . \tag{6.11}$$

The stability of a generic solution of eq. (6.10) can be assessed by studying the quadratic Lagrangian for the perturbations $\delta\pi$. These are assumed to be of a wavelength much shorter than the typical variation of $\hat{\pi}$. Expanding the action (6.6) at quadratic order in $\delta\pi$, after some integrations by parts we obtain

$$\mathscr{L}_{(2)} = Z^{\mu\nu}(x)\,\partial_\mu\delta\pi\partial_\nu\delta\pi \,, \qquad Z^{\mu\nu} \equiv -\frac{1}{2}\bar{\eta}^{\mu\nu} - 2\left(\tilde{\mathcal{K}}^{\mu\nu} - \eta^{\mu\nu}\tilde{\mathcal{K}}\right) \,, \tag{6.12}$$

where the indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$. In general, for time-dependent kinetic terms like (6.12) there is no clear definition of stability. However, in the limit that we consider here where $Z^{\mu\nu}(x)$ changes much slower than the fluctuations $\delta\pi$, the requirement of stability simply translates in the absence of ghost or gradient instabilities for the perturbations, i.e. $Z^{00} > 0$ and that $Z^{0i}Z^{0j} - Z^{ij}Z^{00}$ is a positive-definite matrix at each point [85, 119]. As explained in [119], a theory can be stable even when $Z^{00} < 0$, provided it features superluminal excitations and one can boost to a frame in which $Z^{00} > 0$.[4] However, in our problem we have a privileged frame, the cosmological one, where the Cauchy problem must be well-defined. Therefore, stability must be manifest in this specific frame.

It is important to stress that the Newtonian gauge is very special for our analysis. In this chapter we are interested in a fully non-linear analysis, but we solved for Ψ and Φ linearly, see eq. (5.7). This is justified in Newtonian gauge because, even when the equation of motion of π becomes non-linear, Φ and Ψ remain small and higher-order terms can be neglected. (This is analogous to what happens for non-linearities in the Large-Scale Structure: perturbation theory for the density contrast δ breaks down on short scales, but Φ and Ψ remain small and perturbative.) This does not happen in other gauges. For instance, in spatially-flat gauge one can take the solution for the shift function, see eq. (D.16) in App. D. In the regime of interest $\alpha_\psi \sim 1$ and Galileon non-linearities are relevant for $\partial^2\pi_c \sim H^2 M_{\rm Pl}$, which implies $\partial^2\pi \sim H$. Therefore, the perturbation in the extrinsic curvature is $\delta K \sim \nabla^2\psi \sim H$, which is of the same order as the background value. Thus, higher-order corrections in the constraint equations become relevant, since the Einstein-Hilbert action contains terms quadratic in the extrinsic curvature that cannot be neglected. A similar behaviour occurs in comoving gauge. The analysis in these gauges is therefore much more complicated. As a partial check of our calculation, in Sec. D.3 we verify that our Newtonian action matches the action in spatially flat gauge, but we do this only at the perturbative level, at cubic order.

It is important to stress that, although our analysis is done in a particular gauge, the matrix $\tilde{\mathcal{K}}_{\mu\nu}$ is a covariant tensor:

$$\tilde{\mathcal{K}}_{\mu\nu} = \mathcal{K}_{\mu\nu} - \frac{1}{2}\Gamma_{\mu\nu} = -\frac{\alpha_{\rm B}}{\alpha H}\cdot\nabla_\mu\nabla_\nu\phi \,, \tag{6.13}$$

[4] In the absence of superluminality the sign of Z^{00} is invariant under Lorentz transformations. The positive definiteness of the matrix $Z^{0i}Z^{0j} - Z^{ij}Z^{00}$ is always invariant.

where $\phi \equiv t + \pi$ is the complete DE scalar field (not in canonical normalization) and it is a scalar quantity under all diffs. (The second equality works only if we neglect non-linear terms involving π and Christoffel's symbols: one can check that these are subdominant with respect to the terms we kept.) Therefore, the matrix $Z^{\mu\nu}$ is a covariant tensor[5] and the conditions for stability are gauge independent.

The matrix $Z_{\mu\nu}$ is characterized by the classical non-linear solution of eq. (6.10). To better see the connection between stability and background evolution, it is useful to invert the second relation in (6.12) and express $\tilde{\mathcal{K}}_{\mu\nu}$. We obtain

$$\tilde{\mathcal{K}}_{\mu\nu} = -\frac{1}{2}\left(Z_{\mu\nu} - \frac{1}{3}Z\eta_{\mu\nu}\right) - \frac{1}{4}\bar{\eta}_{\mu\nu} + \frac{1}{12}(1+3c_s^2)\eta_{\mu\nu} \,. \tag{6.14}$$

Using this expression to replace $\tilde{\mathcal{K}}_{\mu\nu}$, the equation for the background, eq. (6.11), becomes an equation containing only quadratic terms in $Z_{\mu\nu}$, i.e.,

$$\frac{1}{3}Z^2 - (Z_{\mu\nu})^2 = \frac{3c_s^2 - 1}{6} \,. \tag{6.15}$$

Remarkably, the terms containing $\Gamma_{\mu\nu}$ have cancelled out: we obtain the same equation as the one derived in [85] without GWs, although here we have neglected the presence of matter sources, and we have considered a generic c_s^2. This is to be expected since $\dot{\gamma}_{\mu\nu}$ can be set to zero *locally* by a proper change of coordinates; thus, its value cannot affect eq. (6.15). On the other hand, the solution for $\hat{\pi}$ requires a global knowledge of the GWs. We can now use this equation to discuss the stability of the solution.

6.2.1 Stability in the absence of GWs

To warm up, we will first review the argument for the case $\Gamma_{\mu\nu} = 0$ and $c_s^2 = 1$, analogous to the DGP case discussed in [85]. A configuration that turns off at spatial infinity, i.e. for which $\mathcal{K}_{\mu\nu} = 0$ and $Z_{\mu\nu} = -\eta_{\mu\nu}/2$, is stable in this limit. One can show that such a solution cannot become unstable at any other point $\boldsymbol{x}$. The proof is made by further assuming that the matrix $Z_{\mu\nu}(x)$ is diagonalizable by means of a Lorentz boost, in such a way that it can be taken to the form $Z^{\mu}_{\ \nu} = \mathrm{diag}(z_0, z_1, z_2, z_3)$.

Using this form, eq. (6.15) reduces, for $c_s = 1$, to

$$-\frac{2}{3}\left[(z_0^2 + \ldots + z_3^2) - (z_0 z_1 + z_0 z_2 + \ldots + z_2 z_3)\right] = \frac{1}{3} \,. \tag{6.16}$$

In this frame, stability requires that $z_\mu < 0$, for all $\mu = \{0, 1, 2, 3\}$. Marginally stable solutions, on the other hand, lie on the hyper-planes defined by $z_\mu = 0$, for some μ. A stable solution can become unstable if and only if the solution crosses one of these critical hyper-planes at

[5] Actually, $\bar{\eta}^{\mu\nu}$ depends on π perturbations but again the terms that we are neglecting are subdominant with respect to the ones we kept.

some intermediate point in the evolution. For this to happen, these critical hyper-planes should intersect the space of solutions. Without loss of generality we can consider the plane $z_0 = 0$, $z_i \neq 0$ in (6.16), that now reduces to

$$-\frac{1}{3}\left[(z_1 - z_2)^2 + (z_1 - z_3)^2 + (z_2 - z_3)^2\right] = \frac{1}{3} \, . \tag{6.17}$$

This equation does not admit any solution because the two sides have different signs: a stable solution at infinity remains stable everywhere. Notice that the right-hand side of the equation above is replaced by $(3c_s^2 - 1)/6$ for a general c_s, see eq. (6.15). Therefore, the stability of the system, even in the absence of GWs, is not guaranteed for $c_s < 1/\sqrt{3}$.

In the next two sections we are going to explicitly show the presence of instabilities around a GW background, respectively for $c_s < 1$ and $c_s = 1$. For $c_s > 1/\sqrt{3}$, this is somewhat surprising, since eq. (6.15) is qualitatively the same as in the absence of GWs. The catch is that the matrix $Z_{\mu\nu}$ will not be diagonalizable. Indeed, diagonalizability can be proven in the case of non-relativistic sources, but it does not hold for a GW background, which is clearly relativistic.

6.2.2 The effect of GWs, $c_s < 1$

To study the case $c_s < 1$ we start from eq. (6.8). For large GW amplitudes ($\beta > 1$), the $\gamma\pi\pi$ interaction leads to a wrong sign of the spatial kinetic term for $\delta\pi$. However, to confirm this assessment we need to take into account the effect of the tadpole $\gamma\gamma\pi$ and of the self-interactions of π. The tadpole will generate a background for $\hat{\pi}$ that, in turn, modifies the action for fluctuations through eq. (6.12).

The setup we are considering here is the same as in chapter 5 (see fig. 5.1). It is convenient to introduce again the null coordinates

$$u = t - z \, , \qquad v = t + z \, , \tag{6.18}$$

which implies $\partial_t = \partial_u + \partial_v$ and $\partial_z = \partial_v - \partial_u$. Moreover, the following relations will be useful below,

$$\partial_t^2 = \partial_u^2 + \partial_v^2 + 2\partial_u\partial_v \, , \qquad \partial_z^2 = \partial_u^2 + \partial_v^2 - 2\partial_u\partial_v \, , \qquad \partial_t\partial_z = \partial_v^2 - \partial_u^2 \, . \tag{6.19}$$

In the presence of a background for the GW, of the form $\gamma_{ij}(u)$, we can solve the equation (6.8) for $\hat{\pi}$. In this case, since $c_s < 1$, there is no intersection between the region where the source is active and the past light-cone of π is finite. As explained in the previous chapter this is reasonable even for values of c_s very close to one. Therefore, we have translational invariance along v (at least as long as we are considering points far away from the emission of γ). For this reason we will look for solutions of the form $\hat{\pi}(u)$. Notice that the non-linear interaction arising from the cubic Galileon vanishes when π depends solely on the variable u. Indeed, eq. (6.19) implies

$$\partial_t^2\hat{\pi} = \partial_u^2\hat{\pi}\,, \qquad \partial_z^2\hat{\pi} = \partial_u^2\hat{\pi}\,, \qquad \partial_t\partial_z\hat{\pi} = -\partial_u^2\hat{\pi}\,, \tag{6.20}$$

and thus that

$$\bar{\Box}\hat{\pi} = -(1-c_s^2)\partial_u^2\hat{\pi}\,, \qquad (\partial_\mu\partial_\nu\hat{\pi})^2 - \Box\hat{\pi}^2 = 0\,. \tag{6.21}$$

Therefore, defining

$$\varphi \equiv \frac{\hat{\pi}}{\Lambda_{\rm B}^3}\ , \tag{6.22}$$

where $\Lambda_{\rm B}^3$ is defined in eq. (5.89), eq. (6.8) gives

$$\varphi''(u) = -\frac{\Gamma_{\mu\nu}^2}{2(1-c_s^2)} = -\frac{\beta^2 c_s^4}{4(1-c_s^2)}\cos^2(\omega u)\ \ , \tag{6.23}$$

where we used eq. (6.7). The solution implies that $\varphi''(u) \le 0$. Note that, as one expects, the limit $c_s \to 1$ is singular: the past light-cone of π becomes sensitive to the details of the emission of the GW.

We can now use this solution to compute the kinetic matrix for the π fluctuations, eq. (6.12). Its non vanishing elements are given by

$$\begin{aligned}
Z^{00} &= \frac{1}{2} + 2\varphi''(u) = \frac{1}{2}\left[1 - \frac{\beta^2 c_s^4}{1-c_s^2}\cos^2(\omega u)\right]\ , \\
Z^{11} &= -\frac{1}{2}c_s^2 + \Gamma^{11} = -\frac{c_s^2}{2}\left[1 - \beta\cos(\omega u)\right]\ , \\
Z^{22} &= -\frac{1}{2}c_s^2 + \Gamma^{22} = -\frac{c_s^2}{2}\left[1 + \beta\cos(\omega u)\right]\ , \\
Z^{33} &= -\frac{1}{2}c_s^2 + 2\varphi''(u) = -\frac{c_s^2}{2}\left[1 + \frac{\beta^2 c_s^2}{1-c_s^2}\cos^2(\omega u)\right]\ , \\
Z^{03} &= Z^{30} = 2\varphi''(u) = -\frac{\beta^2 c_s^4}{2(1-c_s^2)}\cos^2(\omega u)\ .
\end{aligned} \tag{6.24}$$

The background $\hat{\pi}$ does not affect the entries Z^{11} and Z^{22} (it contributes only through $\Box\hat{\pi}$ which vanishes since $\hat{\pi} = \hat{\pi}(u)$): they feature a gradient instability for $\beta > 1$. On the other hand, one can easily verify that the condition $(Z^{03})^2 - Z^{00}Z^{33} > 0$ is satisfied, i.e. the gradient instability does not appear in this direction. One has a ghost instability, $Z^{00} < 0$, for

$$\frac{\beta^2 c_s^4}{1-c_s^2} > 1\ . \tag{6.25}$$

These results seem to contradict what we discussed in the previous section, where we stated that the stability is guaranteed provided $c_s^2 > 1/3$. However, in order to prove stability one has to assume that the matrix $Z^{\mu\nu}$ is diagonalizable via a boost at each point. This is possible only when $|Z^{03}| < \frac{1}{2}|Z^{00} + Z^{33}|$ [119]. In our case this condition gives

$$\frac{\beta^2 c_s^4}{1-c_s^2} < \frac{1-c_s^2}{4} \ . \tag{6.26}$$

Comparing this inequality with eq. (6.25), one sees that the $Z^{\mu\nu}$ is not diagonalizable when there is a ghost instability, so there is no contradiction with the result of Sec. 6.2.1. The inequality (6.26) can be written as

$$c_s^2 < \frac{1}{1+2\beta} \ . \tag{6.27}$$

In the presence of a gradient instability, $\beta > 1$, the right-hand side is smaller than $1/3$. Therefore, the matrix $Z^{\mu\nu}$ can be diagonalized only if $c_s^2 < 1/3$. Again there is no contradiction with what we discussed above since for $c_s^2 < 1/3$ there is no guarantee of stability.

6.2.3 The effect of GWs, $c_s = 1$

Let us now turn to $c_s = 1$. This case is qualitatively different since the light-cone of π is as wide as the one of the GWs, see fig. 5.1, so that we should not expect the same kind of instabilities. As before, we take a GW of the form $\gamma_{ij}(u)$, but now one cannot assume that $\hat{\pi}$ only depends on u; in general it will also depend on v and it will be sensitive to the source of GWs. However, for simplicity, we stick to the equations in the absence of sources.

To find $\hat{\pi}$ we write (6.8) in terms of the null coordinates u and v and using eq. (6.19) we find

$$\Box\hat{\pi} = -4\partial_u\partial_v\hat{\pi} \ , \tag{6.28}$$

$$(\partial_\mu\partial_\nu\hat{\pi})^2 = (\partial_t^2\hat{\pi})^2 - 2(\partial_t\partial_z\hat{\pi})^2 + (\partial_z^2\hat{\pi})^2 = 8\left[\partial_u^2\hat{\pi}\partial_v^2\hat{\pi} + (\partial_u\partial_v\hat{\pi})^2\right] \ . \tag{6.29}$$

Equation (6.8), written in terms of $\varphi = \hat{\pi}\Lambda_{\rm B}^{-3}$, becomes

$$\partial_u\partial_v\varphi + 4\left[\partial_u^2\varphi\,\partial_v^2\varphi - (\partial_u\partial_v\varphi)^2\right] = -\frac{\Gamma_{\mu\nu}^2}{8} \ . \tag{6.30}$$

(Notice that the coupling $\gamma\pi\pi$ does not contribute.)

It is not clear how to determine the most general solution of the above non-linear equation. However, two solutions can be easily obtained by considering the separation of variables $\varphi(u,v) = U(u)V(v)$. Then, one can make the LHS of (6.30) independent of v by taking $V(v) = v$. In this case, the equation takes a very simple form in terms of $U(u)$,

$$U'(u)^2 - \frac{1}{4}U'(u) - \frac{\Gamma_{\mu\nu}^2}{32} = 0 \ , \tag{6.31}$$

with solutions

$$U'_\pm(u) = \frac{1}{8}\left(1 \pm \sqrt{1+2\Gamma_{\mu\nu}^2}\right) \ , \tag{6.32}$$

where the solution that recovers the linear one at small couplings is $U'_-(u)$. (The conclusions about stability are not altered by considering the other branch.)

We can now check whether the solution (6.32) is stable or not. The kinetic matrix eq. (6.12) is given by

$$\begin{aligned} Z^{00} &= \frac{1}{2} + 2\left[vU''(u) - 2U'(u)\right] \ , \\ Z^{11} &= -\frac{1}{2} + \Gamma^{11} + 8U'(u) \ , \\ Z^{22} &= -\frac{1}{2} + \Gamma^{22} + 8U'(u) \ , \\ Z^{33} &= -\frac{1}{2} + 2\left[vU''(u) + 2U'(u)\right] \ , \\ Z^{03} &= Z^{30} = 2vU''(u) \ . \end{aligned} \tag{6.33}$$

First, we focus on possible gradient instabilities. One can easily check, using (6.32), that the components Z^{11} and Z^{22} are negative, so there is no gradient instability in these directions. The matrix is non-diagonal in the block t-z and stability requires $(Z^{03})^2 - Z^{00}Z^{33} > 0$. In our case

$$(Z^{03})^2 - Z^{00}Z^{33} = \left(\frac{1}{2} - 4U'(u)\right)^2 \geq 0 \ ; \tag{6.34}$$

the matrix $Z^{\mu\nu}$ is thus free from gradient instabilities.

Let us turn now to ghost instabilities. As already pointed out at the beginning of Sec. 6.2, ghosts are present whenever Z^{00} becomes negative. From eq. (6.33) we see that this is possible: the term linear in v can be negative and larger than the other positive contributions. To see this more explicitly, we can replace $U(u)$ in Z^{00} with the solution (6.32). We get

$$Z^{00} = \frac{1 + 2\Gamma^2_{\mu\nu} - v\Gamma_{\mu\nu}\partial_u\Gamma^{\mu\nu}}{2(1+2\Gamma^2_{\mu\nu})^{1/2}} \simeq \frac{1 - \Gamma^2_{\mu\nu}\omega v}{2(1+2\Gamma^2_{\mu\nu})^{1/2}} \ , \tag{6.35}$$

where in the last equality we used that $\omega v \gg 1$ and we approximated $\partial_u\Gamma_{\mu\nu} = \omega\Gamma_{\mu\nu}\tan(\omega u) \simeq \omega\Gamma_{\mu\nu}$ (valid for a plane wave). Using $\Gamma^2_{\mu\nu} \simeq \beta^2/2$, the condition to avoid a ghost becomes

$$\beta^2 \lesssim \frac{2}{\omega v} \ . \tag{6.36}$$

Since after a few oscillations $\omega v \gg 1$, we conclude that also for $c_s = 1$ the system becomes unstable. Notice that also in this case the matrix $Z^{\mu\nu}$ is not diagonalizable (the condition $|Z^{03}| < \frac{1}{2}|Z^{00} + Z^{33}|$ is not satisfied by eq. (6.33), but the two sides are actually equal) so that there is no contradiction with the result of section 6.2.1.

Even if the solution we studied is not unique and does not take into account the effect on $\hat{\pi}$ of sources, we conclude that the system is generically unstable and avoids the stability argument of [85], because the GW background is relativistic and gives a non-diagonalizable $Z^{\mu\nu}$.[6] For the same reason, one expects the system to be unstable also for $c_s > 1$ (although it is not clear whether a theory of this kind allows a standard UV completion). In this case, it is not clear how to find a simple ansatz for the solution $\varphi(u, v)$, so that a dedicated study would be needed.

6.2.4 Vainshtein effect on the instability

So far we assumed that GWs are the only source of $\hat{\pi}$. However, astrophysical objects also source $\hat{\pi}$ and a corresponding matrix $Z^{\mu\nu}$. As discussed above, in the presence of non-relativistic sources this matrix is healthy and it gives rise to the Vainshtein effect. From the discussion in Sec. 2.6.1, the large $Z^{\mu\nu}$ of eq. (2.68) gives a more weakly coupled theory, in which the effect of π is suppressed. The Vainshtein effect will also suppress the instability we are studying: the astrophysical background makes the kinetic term large and healthy, while the dangerous vertex $\gamma\pi\pi$ is *not* enhanced (one does not have a term $\partial^2\hat{\pi}\,\dot{\gamma}\partial\pi\partial\pi$, since it would have too many derivatives). Therefore, in regions with large $Z^{\mu\nu}$ the parameter β is effectively suppressed and the instabilities can thus be stopped. However, the condition of large $Z^{\mu\nu}$ cannot be maintained over cosmological scales. Both analytical arguments [74] and simulations [176] indicate that Vainshtein screening is negligible over sufficiently large scales, say larger than 1 Mpc. We are going to discuss more in detail the consequences of this in Sec. 6.4. This means that averaged over these large scales the effect of astrophysical sources is negligible.[7] Since the GWs we observe travel over cosmological distances, one expects that on average the effect of Vainshtein screening is small and that over most of their travel the gradient instability is active. We will come back to this point below, in Sec. 6.4.

6.3 Fate of the instability

In order to understand the implications of the instability we discussed, one would like to know the fate of it. In this Section we want to argue that the dynamics of the instability and its endpoint are UV sensitive and cannot be studied without knowing the UV theory. First of all, notice that it is not possible to follow the development of the instability looking at what happens at the matrix $Z^{\mu\nu}$ in the presence of the growing perturbations. Since the most unstable modes

[6]Even in the absence of GWs, perturbations around a plane wave $\hat{\pi}(u)$, with $c_s = 1$, are unstable [175], as it is easy to check. This suggests that the instability is generic in a relativistic setting.

[7]Since astrophysical sources are with good approximation non-relativistic, the entries Z^{0i} of the matrix $Z^{\mu\nu}$ are negligible (see discussion at the beginning of Sec. 6.2). In order to stabilize the gradient instability one should have that all the eigenvalues of the spatial part of the matrix Z^{ij} are large, much larger than the standard kinetic term, i.e. parametrically larger than unity (in absolute value). This means that also the trace should be parametrically larger than unity. To avoid the instability these conditions should be maintained over all the trajectory of the GW, i.e. over cosmological distances. This however cannot happen. If the trace were large over large regions, it would imply that the trace of the average of $Z^{\mu\nu}$ over a large region is sizeable. This is in contradiction with the statement that linear perturbation theory is recovered over sufficiently large scales.

are the shortest, the instability generates a configuration of $\hat{\pi}$ with very large gradients and the analysis of the previous sections is only useful to understand the behaviour of modes with wavelength much shorter than the variation of the background.

Since we do not know of any UV completion of the theories we are discussing, to gain some intuition on the possible outcome of the instability we now discuss a toy model that features gradient and ghost-like instabilities, and whose UV completion is known. Consider a U(1)-symmetric theory for a complex scalar h, with a quartic Mexican-hat potential, in the absence of gravity:[8]

$$\mathscr{L}_{\rm UV} = -|\partial_\mu h|^2 - V(|h|) \;, \qquad V(|h|) = \lambda\big(|h|^2 - v^2\big)^2 \;. \tag{6.37}$$

In the broken phase with vacuum-expectation value $\langle h\rangle = v$ for h we have a massless degree of freedom (the Goldstone boson), and a heavy one (the "Higgs"). It makes sense to integrate out the latter and write down a low-energy effective theory for the former.

For small λ, one can integrate out the Higgs at tree level. Let us define $h = h_0\, e^{i\phi}$. If we are interested in terms with the minimum number of derivatives acting on ϕ, one can solve the classical equation of motion for a constant h_0 in a constant X field, $X \equiv (\partial\phi)^2$, and plug the result back into the action. One gets

$$h_0^2 = -\frac{1}{2\lambda}X + v^2 = \frac{1}{4\lambda}(\mu^2 - 2X) \;, \tag{6.38}$$

where $\mu^2 \equiv 4\lambda v^2$ is the mass of the radial direction. Plugging this back into the action, one gets the Lagrangian

$$P(X) \simeq -\frac{1}{4\lambda}\, X\,(\mu^2 - X) \;. \tag{6.39}$$

Remarkably, the tree-level effective action stops at quadratic order in X, that is at fourth order in ϕ. The function $P(X)$ will receive corrections suppressed by λ at loop level. Notice that the validity of this action is not limited to small X, provided *derivatives* of X are small: operators with derivatives acting on X are suppressed by powers of ∂/μ.

Consider a background $\hat{\phi}$ with $\partial_\mu\hat{\phi} \equiv C_\mu$ and small perturbations about it, $\hat{\phi} + \delta\phi$. The matrix $Z_{\mu\nu}$ in this case, see eq. (6.12), is given by

$$Z^{\mu\nu} = 2\hat{P}'' C^\mu C^\nu + \hat{P}'\eta^{\mu\nu} \;. \tag{6.40}$$

If C^μ is time-like, that is if $\hat{X} < 0$, we can choose a frame such that $C^0 = \pm(-\hat{X})^{1/2}$, $C_i = 0$. In this frame we have

$$\mathscr{L}_2 = -(2\hat{P}''\hat{X} + \hat{P}')\delta\dot{\phi}^2 + \hat{P}'(\nabla\delta\phi)^2 \;, \qquad \hat{X} < 0 \;. \tag{6.41}$$

[8] This analysis is based on unpublished work by P. Creminelli and A. Nicolis. See also [177].

For stability we thus want

$$2\hat{P}''\hat{X}+\hat{P}'<0\,,\qquad \hat{P}'<0\,. \tag{6.42}$$

If instead C^μ is space-like, $\hat{X}>0$, we can go to a frame where $C^0=0$, $C_i^2=\hat{X}$, where we get

$$\mathscr{L}_2=-\hat{P}'\delta\dot{\phi}^2+\hat{P}'(\nabla_\perp\delta\phi)^2+(2\hat{P}''\hat{X}+\hat{P}')(\nabla_\parallel\delta\phi)^2\,,\qquad \hat{X}>0\,. \tag{6.43}$$

The parallel and normal directions are of course relative to $\vec{C}$. Thus, we see that in this case as well the conditions for stability are those given in eq. eq. (6.42).

For the case we are studying, eq. (6.39), one has

$$2\hat{P}''\hat{X}+\hat{P}'=\frac{1}{4\lambda}(6\hat{X}-\mu^2)\,,\qquad \hat{P}'=\frac{1}{4\lambda}(2\hat{X}-\mu^2)\,. \tag{6.44}$$

The system is stable for

$$\hat{X}<\frac{1}{6}\mu^2\,. \tag{6.45}$$

It is interesting that for such values of $\hat{X}$, the propagation speed is always subluminal—a non-trivial check about the consistency of the effective theory. In the range

$$\frac{1}{6}\mu^2<\hat{X}<\frac{1}{2}\mu^2\,, \tag{6.46}$$

the $(\nabla_\parallel\delta\phi)^2$ in eq. (6.43) has the wrong sign, thus signalling a tachyon-like instability which, unlike a real tachyon instability, is dominated by the UV. That is, we have exponentially growing modes $\sim e^{k_\parallel t}$. The shorter the wavelength, the faster the growing rate. Finally, for

$$\hat{X}>\frac{1}{2}\mu^2 \tag{6.47}$$

all terms in eq. (6.43) have wrong signs. This in the low-energy effective theory looks like a ghost-like instability.

It is interesting to understand these pathologies in terms of the UV theory (6.37). There, the kinetic energy is positive definite. There is no room for ghost-instabilities, and the only instabilities present in certain regions of field space are real tachyons, with a decay rate of order μ. Let us therefore consider small fluctuations of the radial mode h_0 and of ϕ in the UV theory, about a background configuration with constant $\hat{X}$ and $\hat{h}_0$, related by eq. (6.38),

$$h_0\to\hat{h}_0+\delta h\,,\qquad \phi\to\hat{\phi}+\delta\phi\,. \tag{6.48}$$

Expanding the Lagrangian (6.37) at quadratic order we get

$$\mathscr{L}_{\rm UV} \to \dot{\delta h}^2 + \dot{\delta\tilde{\phi}}^2 - \begin{pmatrix} \delta h \\ \delta\tilde{\phi} \end{pmatrix} \cdot \begin{pmatrix} -\nabla^2 + (-2\hat{X} + \mu^2) & 2\sqrt{\hat{X}}\,\nabla_\parallel \\ -2\sqrt{\hat{X}}\,\nabla_\parallel & -\nabla^2 \end{pmatrix} \cdot \begin{pmatrix} \delta h \\ \delta\tilde{\phi} \end{pmatrix}, \tag{6.49}$$

where we canonically normalized the angular fluctuations by defining $\delta\tilde{\phi} = \hat{h}_0\,\delta\phi$, and we specialized to the positive-$\hat{X}$ case (spacelike C^μ), given that this is the region where the pathologies discussed above show up.

First, notice that for $\hat{X} = \frac{1}{2}\mu^2$ the mass term for the radial fluctuation δh goes to zero. This means that at this particular point in field space we cannot get a local low-energy effective theory for the $\tilde{\phi}$ by integrating δh out. Also, at the same point the radial background $\hat{h}_0$ goes to zero—see eq. (6.38)—and it remains zero for even larger values of $\hat{X}$. We thus see that the ghost instability we encounter in the low-energy theory for the angular mode starting from $\hat{X} = \frac{1}{2}\mu^2$, is a sign that at those values of $\hat{X}$ the low-energy theory just makes no sense—the derivative expansion breaks down at zero energy.

Then, we see from the structure of eq. (6.49) that the background configuration is stable if and only if the gradient/mass matrix has positive eigenvalues. For plane-waves with momentum k_i parallel to $C_i = \partial_i\hat{\phi}$, the determinant of such matrix is

$$k_\parallel^2\left[k_\parallel^2 + (\mu^2 - 6\hat{X})\right]. \tag{6.50}$$

We thus see that for $\hat{X} > \frac{1}{6}\mu^2$, the gradient/mass matrix develops a negative eigenvalue in a finite range of momenta, $0 < k_\parallel^2 < (6\hat{X} - \mu^2)$. This signals an instability with a rate of order μ. Indeed, from the low-energy viewpoint the instability was UV-dominated, and we see that in the UV theory it is saturated at $k_\parallel \sim \mu$. At higher energies the UV theory makes perfect sense.

What can we learn from this example about the instability induced by GWs?

- Instabilities can arise from a perfectly sensible theory when one goes in a certain region of field space and from the EFT perspective one can only conclude that the instability exists in the regime of validity of the EFT itself: the theory may be completely healthy in the UV.

- The example we discussed has a well-defined Hamiltonian bounded from below, hence at most the instability can convert this finite amount of energy into the unstable modes. Therefore one can only conclude that an energy of order $\Lambda_{\rm UV}^4$, with $\Lambda_{\rm UV}$ the cut-off of the theory, is damped into the unstable modes; all further developments depend on the UV completion. Since in our case $\Lambda_{\rm UV}^4$ is parametrically smaller than the energy density of the GWs (which accidentally is of order Λ_2^4 for the typical amplitudes and frequencies detected by LIGO/Virgo) one cannot conclude that the GW signal will be affected.

- The appearance of the instability may signal that the EFT breaks down. This happens in the example above in the case of the ghost instability: the range of applicability of the EFT shrinks to zero. The regime of validity of the EFT is not only determined by the requirement that frequencies are sufficiently small, but it can be modified in the presence of a sizeable background. Therefore it may be that the instability we studied is simply telling us that the EFT of DE breaks down. This means that we are unable to describe the propagation of GWs unless we know the UV completion of the theory.

Notice that both in the case in which the instability can be described within the EFT and in the case in which the EFT breaks down at the instability, in order to continue the time evolution of the system one needs the UV completion.

6.4 Phenomenological consequences

Let us explore the phenomenological consequences of the instability we studied. First of all, as it is clear from the toy model we described in the previous section, without a UV completion one cannot conclude that a sizeable amount of energy goes into π. The instability may be saturated at the cut-off scale Λ_3 or even at a lower scale. This means that it is not guaranteed that the instability leads to a backreaction on the GW signal that can be seen at the interferometers.[9] In the following, we will concentrate on the question whether a generic point in the Universe is affected by the instability. For this we do not need to focus on the particular events observed by LIGO/Virgo (or eventually LISA and pulsar timing array, see e.g. [178]) but one has to consider the effect of all GW emissions.

Let us neglect momentarily the Vainshtein effect. The Universe is populated by binary systems and these trigger the instability in points that are close enough to the source to have $\beta > 1$. Let us divide the Universe in spheres of 10 Mpc radius and ask whether the instability is triggered in these regions. Since in first approximation the Universe is homogeneous on scales of 10 Mpc, one expects that all regions behave approximately in the same way. If within a region and in a time comparable to the age of the Universe, there is at least one binary event that gives $\beta > 1$ at a distance of 10 Mpc, one can conclude that this event will trigger the instability over the whole sphere (and thus in the whole Universe). In the following we are also going to explore regions of 1 Mpc. In this case, since the Universe is inhomogeneous on this scale, using the same criteria as before one can only conclude that sufficiently dense regions reached the instability. Indeed, the events will be mostly localized in overdensities and may not be able to trigger the instability in underdense regions.

The parameters needed to characterize the instabilities discussed in Sec. 6.2.2 are the amplitude h_0^+ and the frequency f. Long before the merger, the amplitude h_0^+ can be written as in eq. (5.104). Recall that this is a reasonable approximation until the orbit reaches

[9] In fact, using eq. (5.92) one can straightforwardly show that $\Delta\gamma/\bar{\gamma} \sim \Lambda_3^4/(\Lambda_2^4 h_0^+) \ll 1$.

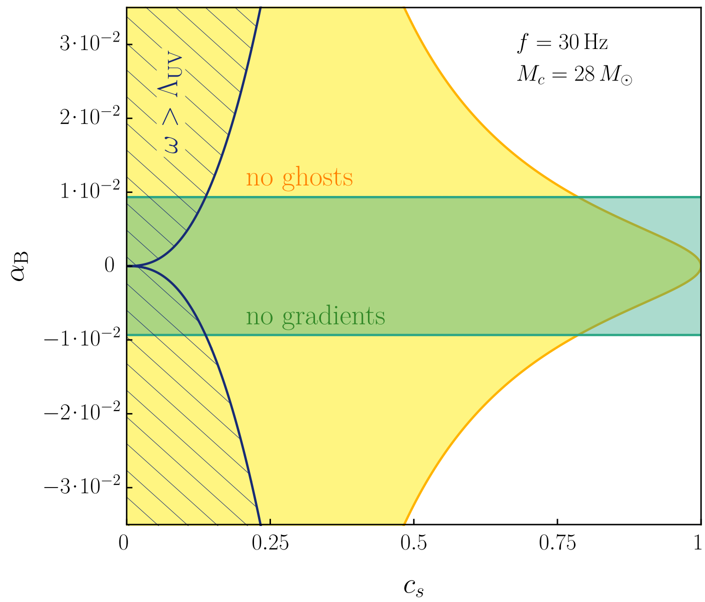


Fig. 6.1 Stability regions on the plane $(c_s, \alpha_{\mathrm{B}})$. The yellow region indicates where ghost instabilities of eq. (6.25) are absent in which $\beta < c_s^{-2}\sqrt{1-c_s^2}$. The green region indicates where gradient instabilities are absent, i.e. $\beta < 1$. The fact that this curve is independent of c_s follows from the choice $\alpha = 1/(2c_s^2)$, see eqs. (5.10) and (6.4). In the region with the blue diagonal grid, the frequency of the GW is above the perturbative unitarity bound, $\omega > \Lambda_{\mathrm{UV}}$ (see footnote 10), and our analysis cannot be applied. In the plot we have used $M_c = 28 M_\odot$ and $f = 30\,\mathrm{Hz}$.

the ISCO. Assuming equal the binary is made out of equal mass objects, eq. (3.5) becomes $f_{\mathrm{ISCO}} \simeq 0.034/(\pi G_{\mathrm{N}} M_c)$.

Figure 6.1 focusses on stellar mass BHs; for concreteness we chose $M_c = 28\,M_\odot$ as for GW150914 and $f = 30\,\mathrm{Hz}$. We take the distance to be $1\,\mathrm{Mpc}$. Taking a distance of $10\,\mathrm{Mpc}$ would require, in order to keep the same h_0^+, to consider times closer to the coalescence. However, this corresponds to larger frequencies and one goes in a regime that cannot be trusted, since the frequency is higher than the unitarity cut-off.[10] In fig. 6.1 we plot the gradient and ghost instabilities in the plane $(c_s, \alpha_{\mathrm{B}})$ together with the unitarity cut-off. Models with $\alpha_{\mathrm{B}} \gtrsim 10^{-2}$ are affected by one or both instabilities, but the cut-off is quite close.

On the other hand, if one considers massive BHs, frequencies are many orders of magnitude smaller than the unitarity cut-off. In fig. 6.2 we plot the threshold $\beta = 1$ as a function of the chirp mass of the binary for $1\,\mathrm{Mpc}$ and $10\,\mathrm{Mpc}$ distances. Independently of the chirp mass,

[10] The cut-off can be obtained as the energy scale at which perturbative unitarity is lost. In order to explicitly get such scale for m_3^3 we focus on the leading term in (6.2): the dominant interaction in the small-c_s limit is $\sim -\nabla^2\pi(\partial_i\pi)^2/\Lambda_{\mathrm{B}}^3$. Following [179, 180] we find that for such interaction perturbative unitarity in the $\pi\pi \to \pi\pi$ scattering is lost when

$$\frac{\omega^6}{\Lambda_{\mathrm{B}}^6 c_s^{11}} < \frac{3\pi}{4}\,, \tag{6.51}$$

where here ω is the energy of π.

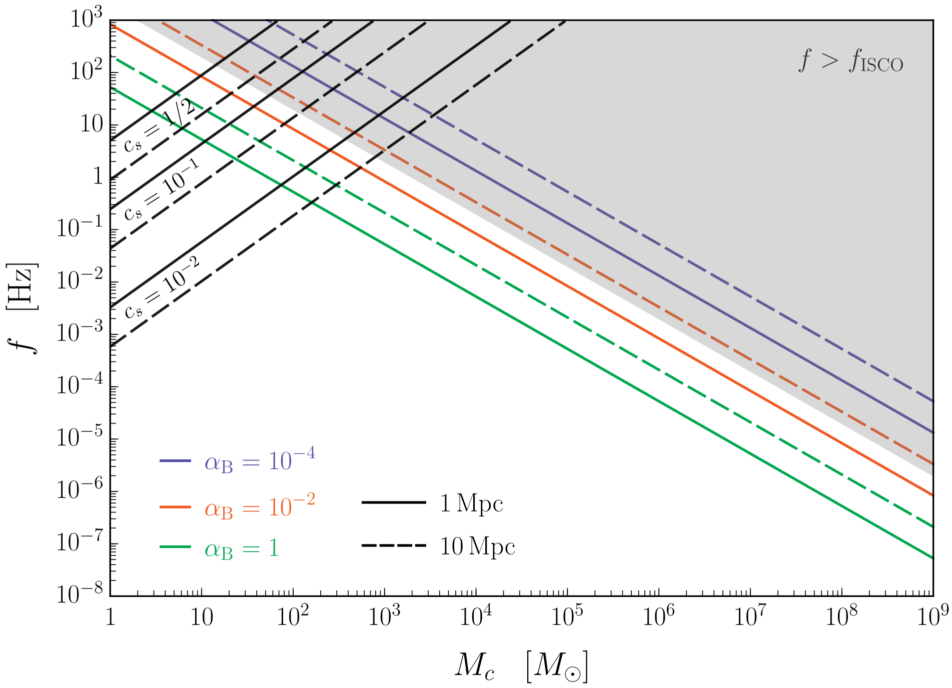


Fig. 6.2 Gradient-instability lines, $\beta = 1$, for different value of α_B as a function of the chirp mass of the binary system, evaluated at a distance of 1 Mpc (solid lines) and 10 Mpc (dashed lines). The grey region cannot be trusted because it would correspond to extrapolating the orbit beyond the ISCO. Regions above the black lines have frequencies larger than the unitarity cut-off $\omega > \Lambda_{\rm UV}$ (the three lines correspond to different values of c_s). At fixed $\beta = 1$, we expressed the cut-off frequency as a function of M_c using (5.104). All lines are evaluated with the choice $\alpha = 1/(2c_s^2)$.

the instability is triggered close to the ISCO for values of α_B that are of interest for future LSS experiments, i.e. $\alpha_B \gtrsim 10^{-2}$. Although there is some degree of uncertainty on the rate of massive BH mergers, one can be quite sure that in a region of 10 Mpc many mergers of halos, and therefore binary mergers of massive BHs, took place in the last Hubble time. To be more quantitative, in the range $10^7\, M_\odot < M_c < 10^8\, M_\odot$ one estimates between 5 and 50 events in a volume of 10 Mpc radius between $z = 1$ and $z = 0$ [181]. Rates are larger, but considerably more uncertain, for smaller masses [182].

Let us now discuss the role of screening. As we discussed above, in regions with large field non-linearities the threshold of instability can be lifted by the Vainshtein mechanism. If the typical radius at which the screening is effective is of order 10 Mpc or smaller, then our conclusions do not qualitatively change. There may be very non-linear regions where the instability did not occur, but in most of the Universe the instability takes place. Following [74], one can estimate the scale at which the Vainshtein mechanism is relevant assuming a power-law Universe with matter power spectrum $P(k) \propto k^n$, where the relevant value near the non-linear scale for the real Universe is $n \simeq -2$. In our case one finds $\lambda_{\rm V} \sim [\alpha_B/(c_s^2\alpha)]^{\frac{4}{3+n}} \lambda_{\rm NL}$, which shows that for small α_B the Vainshtein scale $\lambda_{\rm V}$ is in general much shorter than 10 Mpc, which roughly corresponds to the non-linear scale for structure formation $\lambda_{\rm NL}$ (see also [176] for an estimate of the Vainshtein scale in numerical N-body simulations, confirming these estimates).

What can we conclude if a model lies in the unstable region? As we discussed, the endpoint of the instability is unknown and requires knowledge of the UV. Naively one can imagine that a certain amount of πs with energy close to the cut-off is generated until their backreaction stops the instability. It looks difficult to argue that the theory around this new state will resemble the original one and give similar predictions: the πs produced by the instability must qualitatively change the theory to make it stable, so that one expects that also the other predictions of the theory will be affected. One cannot make any firm prediction without understanding the fate of the instability and this requires a UV completion.

Another possibility is that the EFT breaks down at the instability, so that the instability itself cannot be trusted. Notice however that the frequencies involved may be as low as 10^{10} km. In this case one has to declare the impossibility to say anything about any process that has to do with GWs. Moreover, all the successes of GR on shorter scales cannot be explained. Analyticity arguments can be used to argue that a theory with an approximate Galilean symmetry must break down at a very large scale, of order 10^7 km in the range of parameters we are discussing [183].[11] Although it is not straightforward to apply these arguments in a cosmological context, where Lorentz invariance is spontaneously broken, it is an independent indication that the theories at hand must break down at extremely large scales.

6.5 Instabilities in Beyond Horndeski

The analysis of the previous sections focused on the stability of cubic Horndeski theories. Here we want to consider another quadratic operator, $\tilde{m}_4^2$, that we already discussed in the perturbative and resonant decay. The operator, give by the action (5.4), is highly constrained by the analysis of the previous chapters. However, as we noticed in Sec. 5.4, the regime of perturbative decay corresponds, in astrophysical situations, to $\beta > 1$ where π non-linearities can become important. Building on the previous analysis for m_3^3, here we want to study this regime, and assess whether the bound should be revised or not. The stability analysis will show that π non-linearities are not sufficient to prevent the system from becoming unstable.

The Lagrangian of π in the presence of all the relevant non-linearities schematically reads

$$\begin{aligned}\mathscr{L}_\pi = -\frac{1}{2}\bar{\eta}^{\mu\nu}\partial_\mu\pi\partial_\nu\pi + \frac{1}{\Lambda_\star^3}\ddot{\gamma}_{ij}\partial_i\pi\partial_j\pi - \frac{\partial^2\pi}{\Lambda_\star^3}(\partial\pi)^2 \\ + \frac{(\partial\pi)^2}{\Lambda_c^6}[(\Box\pi)^2 - (\partial_\mu\partial_\nu\pi)^2] - \frac{\alpha_{\rm H}}{2\sqrt{\alpha}HM_{\rm Pl}}\dot{\pi}\dot{\gamma}_{ij}^2\,,\end{aligned} \tag{6.52}$$

[11] See however [184, 185] for some recent developments.

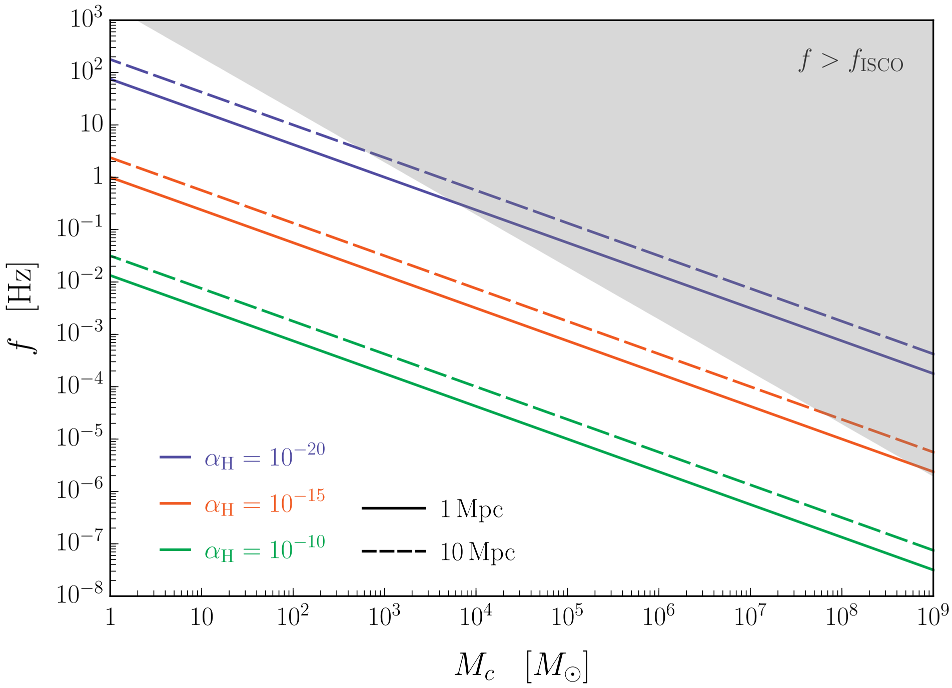


Fig. 6.3 Gradient-instability lines $\beta = 1$ for different value of $\alpha_{\rm H}$ as a function of the chirp mass of the binary system. The grey region cannot be trusted because it would correspond to extrapolating the orbit beyond the ISCO.

where $\Lambda_\star \simeq {\alpha_{\rm H}}^{-1/3}\alpha^{1/3}\Lambda_3$ and $\Lambda_c \simeq {\alpha_{\rm H}}^{-1/6}\alpha^{1/3}\Lambda_3$ (see the discussion in Sec. 5.3.2). The second term gives an instability similar to the one discussed for m_3^3 and is characterized by the parameter β of eq. (5.24).

It turns out that the analysis for this case is simpler, since to assess the stability of the system it is enough to look at the $\gamma\pi\pi$ interaction, while all additional non-linearities are negligible: the system is unstable for $\beta > 1$. We are going to verify this statement below. In fig. 6.3 we plot the instability region as a function of the chirp mass and frequency. Notice that in this case the unitarity cut-off of the theory does not appear in the figure, since it is much higher that the frequencies of interest. The absence of instability is a constraint much tighter than the perturbative bound of eq. (4.35). On the other hand, the narrow-resonance regime gives even better constraints for $\alpha_{\rm H}$, of the order of $\beta \lesssim 10^{-2}$ (see fig. 5.5).

Let us now verify that the other non-linear terms in the Lagrangian of eq. (6.52) can be neglected. (For simplicity in the following we take $\alpha \sim 1$ and $c_s \sim 1$.) First, let us estimate the size of the induced background $\hat{\pi}$ sourced by $\gamma\gamma\pi$. Neglecting π non-linearities and using the Lagrangian (6.52), one can estimate

$$\hat{\pi} \sim \frac{M_{\rm Pl}\alpha_{\rm H}\omega(h_0^+)^2}{H} \, . \tag{6.53}$$

In order for this estimate to be correct we must check that the cubic and quartic self-interactions of (6.52) are negligible. At the level of the equations of motion, the contributions of the cubic and quartic terms are schematically given by $\mathcal{E}_{(3)} \sim (\partial^2\hat{\pi})^2\,\Lambda_\star^{-3}$ and $\mathcal{E}_{(4)} \sim (\partial^2\hat{\pi})^3\,\Lambda_c^{-6}$. These have to be compared with $\mathcal{E}_{(2)} \sim \partial^2\hat{\pi}$. Using eq. (6.53) we have

$$\frac{\mathcal{E}_{(3)}}{\mathcal{E}_{(2)}} \sim \alpha_{\rm H}^2\left(\frac{\omega}{H}\right)^3 (h_0^+)^2 \sim \beta^2\frac{H}{\omega}\,, \qquad \frac{\mathcal{E}_{(4)}}{\mathcal{E}_{(2)}} \sim \alpha_{\rm H}^3\left(\frac{\omega}{H}\right)^6 (h_0^+)^4 \sim \beta^3 h_0^+\,. \tag{6.54}$$

For $\beta \gtrsim 1$ both these ratios are very small and (6.53) is valid. (The approximation may not be correct for $\beta \gg 1$, but in this case one can reduce to $\beta \gtrsim 1$ considering weaker—i.e. farther—GW sources.) The background $\hat{\pi}$ will affect the kinetic term of perturbations $Z_{\mu\nu}$: we have to compare its contribution with the one of GWs, $\partial_u\Gamma_{\mu\nu}$ (note that for $\tilde{m}_4^2$ the relevant parameter is $\ddot{\gamma}_{ij}$ rather than $\dot{\gamma}_{ij}$). For the cubic self-interaction in (6.52) one gets

$$\frac{\partial^2\hat{\pi}}{\ddot{\gamma}} \sim \alpha_{\rm H}\frac{\omega}{H}h_0^+ \sim \beta\frac{H}{\omega} \ll 1\,. \tag{6.55}$$

For the quartic self-interaction in (6.52) one has

$$\frac{(\partial^2\hat{\pi})^2\Lambda_\star^3}{\ddot{\gamma}\Lambda_c^6} \sim \alpha_{\rm H}^2\left(\frac{\omega}{H}\right)^4 (h_0^+)^3 \sim \beta^2 h_0^+ \ll 1\,. \tag{6.56}$$

We conclude that one can trust the bound plotted in fig. 6.3.

As already mentioned, the stability properties of the cubic Galileon interactions (in the absence of GWs) do not hold for the quartic and quintic Galileon [173]. This means that these theories are in general unstable in the Vainshtein regime, even before considering GWs. However, the two instabilities are quite different: one is only present in the non-linear regime of π, while the instability we discuss in this chapter holds outside the Vainshtein regime and extend to the whole Universe.

6.6 Discussion and outlooks

In this chapter we have studied the effect of a large GW background on the stability of the EFT of DE. We have discussed two operators where this effect is relevant: (5.3), associated to the dimensionless function $\alpha_{\rm B}$, and (5.4), associated to $\alpha_{\rm H}$. We have first focused the analysis on the former, because this operator remains unconstrained by the perturbative decay of gravitons, since the scale suppressing the coupling $\gamma\pi\pi$ is typically too high (see discussion in Sec. 4.4.2). Moreover, the resonant decay is (probably) quenched by the non-linear self-couplings of π, so that also in this regime there are no conclusive bound on this operator from the decay of GWs (see Sec. 5.3.1).

The stability of perturbations for this operator is studied in Sec. 6.2. For $c_s < 1$, perturbations of π become necessarily unstable in the presence of a GW background with

$$\beta \sim \frac{|\alpha_{\rm B}|}{\alpha c_s^2} \frac{\omega}{H_0} h_0^+ > 1 \,, \tag{6.57}$$

since the kinetic matrix $Z^{\mu\nu}$ presents either ghost or gradient instabilities. These conclusions are not at variance with the well-known theorem that ensures stability for the DGP model [85] in the absence of GWs. In this case, for $c_s^2 > 1/3$ the theorem can be extended to the m_3^3 operator, but its assumptions break down when a GW background is present. The case $c_s = 1$ is also discussed, assuming that the π background is linear in the light-cone coordinate $v = t + r$. In this case we find that ghost instabilities are generic. Our conclusions do not extend directly to the DGP model, because the coupling of the scalar bending mode to tensor modes is different. It would be interesting to verify the stability of the DGP model in the presence of a GW background.

Although the instabilities we have discussed appear in the presence of GWs, the expectation is that they generically arise whenever the assumptions of the stability theorem of [85] are violated (for instance when dealing with relativistic systems instabilities can arise). Examples of this behaviour, in the absence of gravity, have been studied in [175, 186].

The physical implication of these instabilities is unclear, since the most unstable modes are the closest to the cut-off. Sensible conclusions can only be drawn with the knowledge of the UV completion of the theory. We discussed this in Sec. 6.3 by an example unrelated to our theory: a U(1)-symmetric theory for a complex scalar with a Mexican-hat potential. In the broken phase, at low energy this system can be described by an effective $P(X)$-theory for the angular mode ϕ. Even if the effective theory presents ghost or gradient instabilities for certain values of P' and P'', the UV completion remains perfectly healthy. From this example one can argue that it is not possible to continue the time evolution of the system without knowing the complete theory.

In Sec. 6.4 we explore for which values of the dimensionless function $\alpha_{\rm B}$ the EFT becomes unstable everywhere in the Universe, losing predictability. The bounds are shown fig. 6.1 and 6.2 for different GW sources and roughly correspond to $|\alpha_{\rm B}| \gtrsim 10^{-2}$. They are thus very close to the forecasted limits on this parameter reachable with future large-scale structure observations (see e.g. [187, 138–140]). For this reason, it would be interesting to improve our analysis considering a more refined estimate of the abundance of massive BH binaries [181, 182]. Indeed, the limits obtained from these events are the most interesting, as they correspond to frequencies well below the cut-off of the theory. Of course, a logical possibility is that the EFT breaks down at extremely large scales and it cannot be used to study any GW event. In this scenario also all the classical tests of GR are outside the EFT and one cannot rely on screening mechanisms to explain the success of GR at short scales.

Parameters

EFT of DE operator	(4.1)	$\tilde{m}_4^2$	m_3^3
GLPV covariant Lagrangian	(3.17)	$-\frac{2Xf_{,X}}{f}$	$\frac{2Xf_{,X}}{f} + \frac{\dot{\phi}XG_{3,X}}{2Hf}$
Dimensionless function α_i	(2.71)	$\alpha_{\rm H}$	$\alpha_{\rm B}$
After conformal transformation	(A.6)	$\alpha_{\rm H} + 2\beta_1$	$\alpha_{\rm B} - \frac{\alpha_{\rm M}}{2}(1-\beta_1) + \beta_1 - \frac{\dot{\beta}_1}{H}$

Constraints

Perturbative decay	(4.35)	$\lvert\alpha_{\rm H}\rvert \gtrsim 10^{-10}$	Irrelevant ($\lvert\alpha_{\rm B}\rvert \gtrsim 10^{10}$)
Narrow resonance	(5.108)	$3 \times 10^{-20} \lesssim \lvert\alpha_{\rm H}\rvert \lesssim 10^{-17}$ (fig. 5.5a) $10^{-16} \lesssim \lvert\alpha_{\rm H}\rvert \lesssim 10^{-10}$ (fig. 5.5b)	Not applicable (large non-linearities)
Instability	(6.25)	$\lvert\alpha_{\rm H}\rvert \gtrsim 10^{-20}$ (fig. 6.3)	$\lvert\alpha_{\rm B}\rvert \gtrsim 10^{-2}$ (fig. 6.2)

Table 6.1 Summary of the results of the constraints outlined in this thesis, and obtained in [1–3] (we assume $c_T = 1$). The top table summarizes the relations between covariant operators and the EFT of DE. The bottom table summarizes the constraints, for the corresponding operators, of the previous chapters.

In Sec. 6.5 we discuss the instability of the operator $\tilde{m}_4^2$, triggered by a GW background with

$$\beta \sim \frac{|\alpha_{\rm H}|}{\alpha c_s^2}\left(\frac{\omega}{H_0}\right)^2 h_0^+ \, . \tag{6.58}$$

This is easier to study because one can neglect non-linearities of π. The bounds on $\alpha_{\rm H}$ based on $c_T = 1$ are shown in fig. 6.3: the effective theory becomes unstable for $|\alpha_{\rm H}| \gtrsim 10^{-20}$. These are much smaller values than those constrained by the perturbative decay. If $c_T = 1$ is relaxed, the combination constrained by our analysis is $(\tilde{m}_4^2 + m_5^2 c_T^2)/M_{\rm Pl}^2$, instead of $\tilde{m}_4^2/M_{\rm Pl}^2$ (see eq. (5.102)).

Throughout our discussion, we considered the effects of $\alpha_{\rm B}$ and $\alpha_{\rm H}$ independently but we do not expect that the combination of the two operators can provide better stability properties for π. Indeed, the lack of a general theorem for stability in the presence of GWs suggests that our conclusions hold in a more general theory, where both operators are turned on. In particular, we expect $\alpha_{\rm H}$ to also contribute to the operator $\dot{\gamma}_{ij}\partial_i\pi\partial_j\pi$: it may be possible then to tune $\alpha_{\rm H}$ and $\alpha_{\rm B}$ to set the operator to zero. However, the dominant operator $\ddot{\gamma}_{ij}\partial_i\pi\partial_j\pi$, which has more derivatives, would then lead to instability since it cannot be removed by tuning other parameters.

We summarize these results and those of chapter 4 and 5 in table 6.1, using different notations. For simplicity, in eqs. (6.1) and (5.4) we have assumed that our starting theory has a constant effective Planck mass and no higher-derivative operators such as those appearing in DHOST theories. However, our results also apply after a conformal transformation with conformal factor depending on ϕ and $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, of the form $g_{\mu\nu} \to C(\phi, X)g_{\mu\nu}$. In the fourth line of the table we provide the corresponding parameters to which our analysis applies.

We conclude that for what concerns large-scale structure surveys, the surviving single-field theory that avoids the aforementioned issues is a k-essence theory [129, 188], modulo the above conformal transformation. In the covariant language, its action reads

$$\mathscr{L} = G_2(\phi, X) + C(\phi, X)R + \frac{6C_{,X}(\phi, X)^2}{C(\phi, X)}\nabla^\mu\phi\nabla_\nu\phi\,\Phi_{\mu\nu}\Phi^{\nu\lambda}\,, \tag{6.59}$$

Note that there is no Vainshtein screening in these theories [170]: some other mechanism (see e.g. [29, 189] and references therein) is required to screen the fifth force on astrophysical scales. It would be interesting to investigate this further in the future.

Conclusions

The discovery of GWs and their direct measurement has opened, in the last five years, an incredibly rich observational window for cosmology and astrophysics, with far-reaching implications for fundamental physics. Indeed, waves at LIGO/Virgo can probe a very interesting range of scales, that fit right in between those probed by solar system tests of gravity and the much larger scales of cosmological surveys. Among the various GW events detected so far, GW170817 as been the one with the most valuable information for DE theories. The corresponding measurement that GWs travel at the speed of light has dramatically changed the landscape of viable DE models, pointing to less exotic explanations for the cosmological constant.

In the wake of this measurement, in this thesis we have further studied the propagation of GWs in DE theories, with the intent of placing additional constraints on the surviving models. After reviewing the previous constraints coming from GW170817, in chapter 3 we have re-evaluated the implications of $c_T = 1$ in more details, uncovering new models (previously considered ruled-out) that satisfy this constraint thanks to a dynamical mechanism. In particular, working within the EFT of DE, we have found that these new theories allow for quintic Beyond Horndeski operators to be present. Ultimately, however, they are not viable since they spoil $c_T = 1$ in the presence of inhomogeneities. This derivation has allowed us to close a possible loophole in the argument of [48–51] and is in agreement with the findings of [141].

Another interesting aspect discussed in this thesis is the presence of dispersion effects in the propagation of GWs. In theories relevant for cosmology, Lorentz invariance is spontaneously broken due to the time-dependent vev of the DE field. In such a situation, the DE fluctuations π can travel with different speed compared to GWs, thus allowing gravitons γ to decay [1]. This possibility is studied in chapter 4, where we have shown that the quartic Beyond Horndeski operator in the EFT of DE (surviving after GW170817) leads to a very large decay rate, incompatible with observations. The decay is also tied to dispersion effects thanks to unitarity, in the form of the optical theorem. Interestingly, to obtain constraints from the decay and dispersion, no EM counterpart is required, thus making our results very robust. Moreover, we have analysed in detail the radiative corrections in theories with a covariant Galileon structure, evaluated around the cosmological background. We have concluded that loop diagrams generically lead to large dispersion effects, suppressed by the low scale Λ_3. Since

GW observations have not detected significant deviations from GR, these theories cannot lead to observable new features on large scales. As an exception, the only viable subclass not leading to appreciable dispersion is cubic Horndeski.

In the analysis of chapter 4 we have focused on a *perturbative* analysis, where GWs are treated as an independent collection of free gravitons. Realistic waves are however classical in nature, and therefore coherent effects can lead to sizeable enhancements in the decay rate of gravitons. To produce more realistic constraints, in chapter 5 we have discussed the resonant effects associated with the large occupation number of γ and π. The classical equation describing the evolution of DE fluctuations in the presence of a GW background can be recast in the form of a Mathieu equation, and in the *narrow-resonance* regime we have been able to treat the problem analytically. The field π in this regime can grow exponentially in time, reaching very large occupation numbers. In turn, this can lead to large backreaction effects on the original wave, leading to very peculiar signatures such as *precursor signals*.

For what concerns quartic Beyond Horndeski operators, we have shown that π self-interactions are not able to halt the resonance, and therefore the constraints on these theories are much tighter than what initially estimated from the perturbative analysis. Again, cubic Horndeski theories are shown to remain safe even in the narrow-resonance regime. In this case, π non-linearities quickly become of the same order of the resonance in the equations of motion, making our analysis inconclusive. A possible way forward in this regime could be to study the system numerically and to take into account all the relevant non-linearities. However, we expect that self-interaction modify the instability bands in the Mathieu equation; the instability-rate should then be considerably lower, producing weak constraints.

The regime of large amplitude for the GW is instead much more significant for placing constraints on cubic Horndeski models. In chapter 6, inspired the analysis of [85], we have studied the stability properties of π in the presence of such a GW background. Contrary to the case where gravity is turned off and the dynamics is non-relativistic, we have shown that π can generically feature both ghost and gradient instabilities. Remarkably, this can happen for GWs of astrophysical origin that can be detected by LIGO/Virgo and, in the future, by LISA. Moreover, this type of backgrounds is ubiquitous in the current universe and the instability is not just localized around astrophysical sources, hence the theory cannot be applied to cosmology and structure formation. As in the case of the narrow-resonance regime, we do not expect to have a large backreaction on γ even in the presence of these catastrophic instabilities. Rather, the effective theory describing π and γ breaks down, and a UV completion should then take its place. This however does not alleviate the situation, as it would imply the impossibility of predicting gravity on short scales within the limits of validity of the effective theory. Indeed, at present no concrete UV completion for Galileon theories is known.

The constraints we presented in this thesis greatly limit the number of viable models for DE, with Galileon theories highly disfavoured. Nonetheless, the study of the implications of GW observations is still far from over. For instance, an interesting avenue for future investigation is the phenomenology of GWs in higher-dimensional theories of gravity, such as the DGP model. In this latter case, instabilities similar to the ones discussed in chapter 6 are expected, but a full analysis is still missing. It would then be interesting to explore this possibility further.

A | DHOST Theories

Discussing Beyond Horndeski theories in Sec. 2.5.4 we encountered a situation where, upon making a disformal transformation [190], we recover the Horndeski class. This might appear strange at first, since redefinitions of the metric should not affect physics, but in the original frame the equations look higher-order while in the second the equations are second-order. Upon closer inspection though, the Beyond Horndeski theory can be recast as a second-order system thanks to its degeneracy condition (2.57), thus avoiding the Ostrogradski theorem. This feature has been extensively studied and finally extended in [80, 81], where new classes of theories, generalizing the situation of Beyond Horndeski, have been introduced. Here only one scalar mode propagates despite the fact that the Euler-Lagrange equations are higher-order. Such models are referred to as Degenerate Higher-Order Scalar-Tensor (DHOST) theories and, as their name suggests, once written as second-order systems (by adding new variables), possess a degenerate kinetic-term for the fields.

All DHOST Lagrangian quadratic and cubic in the second derivatives of ϕ have been identified, respectively in [80] and [191]. The covariant action for DHOST is

$$\begin{aligned} S_{\rm DHOST} = \int \Big[& G_2(X,\phi) + G_3(X,\phi)\Box\phi + G_4(X,\phi)R + C^{\mu\nu\rho\sigma}_{(2)}\Phi_{\mu\nu}\Phi_{\rho\sigma} \\ & + G_5(X,\phi)G^{\mu\nu}\Phi_{\mu\nu} + C^{\mu\nu\rho\sigma\gamma\delta}_{(3)}\Phi_{\mu\nu}\Phi_{\rho\sigma}\Phi_{\gamma\delta}\Big]\sqrt{-g}\,\mathrm{d}^4x\ . \end{aligned} \tag{A.1}$$

Here the tensors $C^{\mu\nu\rho\sigma}_{(2)}$ and $C^{\mu\nu\rho\sigma\gamma\delta}_{(3)}$ are the most general tensors constructed out of $\nabla_\mu\phi$ and $g_{\mu\nu}$. At quadratic order $C^{\mu\nu\rho\sigma}_{(2)}$ can be written as

$$C^{\mu\nu\rho\sigma}_{(2)}\Phi_{\mu\nu}\Phi_{\rho\sigma} = \sum_{I=1}^{5} a_I(X,\phi)\mathscr{L}_I^{(2)}\ , \tag{A.2}$$

and the $\mathscr{L}_I^{(2)}$'s are

$$\begin{aligned} &\mathscr{L}_1^{(2)} = [\Phi^2]\ , && \mathscr{L}_2^{(2)} = [\Phi]^2, && \mathscr{L}_3^{(2)} = \nabla^\mu\phi\nabla^\nu\phi\,\Phi_{\mu\nu}[\Phi]\ , \\ &\mathscr{L}_4^{(2)} = \nabla^\mu\phi\nabla_\nu\phi\,\Phi_{\mu\rho}\Phi^{\rho\nu}\ , && \mathscr{L}_5^{(2)} = (\nabla^\mu\phi\nabla^\nu\phi\,\Phi_{\mu\nu})^2\ . \end{aligned} \tag{A.3}$$

For what concerns $C_{(3)}^{\mu\nu\rho\sigma\gamma\delta}$ one similarly has

$$C_{(3)}^{\mu\nu\rho\sigma\gamma\delta}\Phi_{\mu\nu}\Phi_{\rho\sigma}\Phi_{\gamma\delta} = \sum_{I=1}^{10} b_I(X,\phi)\mathscr{L}_I^{(3)} \ , \tag{A.4}$$

with

$$\begin{aligned}
&L_1^{(3)} = [\Phi]^3 \ , && L_2^{(3)} = [\Phi][\Phi^2] \ , && L_3^{(3)} = [\Phi^3] \ , \\
&L_4^{(3)} = \nabla^\mu\phi\nabla^\nu\phi\,\Phi_{\mu\nu}[\Phi]^2 \ , && L_5^{(3)} = \nabla^\mu\phi\nabla_\nu\phi\,\Phi_{\mu\rho}\Phi^{\rho\nu}[\Phi] \ , && L_6^{(3)} = \nabla^\mu\phi\nabla^\nu\phi\,\Phi_{\mu\nu}[\Phi^2] \ , \\
&L_7^{(3)} = \nabla_\mu\phi\nabla_\sigma\phi\,\Phi^{\mu\nu}\Phi_{\nu\rho}\Phi^{\rho\sigma} \ , && L_8^{(3)} = \nabla_\mu\phi\nabla^\rho\phi\nabla_\sigma\phi\nabla_\lambda\phi\,\Phi^{\mu\nu}\Phi_{\nu\rho}\Phi^{\sigma\lambda} \ , && \\
&L_9^{(3)} = [\Phi]\left(\nabla^\mu\phi\nabla^\nu\phi\,\Phi_{\mu\nu}\right)^2 \ , && L_{10}^{(3)} = \left(\nabla^\mu\phi\nabla^\nu\phi\,\Phi_{\mu\nu}\right)^3 \ . &&
\end{aligned} \tag{A.5}$$

The degeneracy conditions are the imposed as relations between the functions a_I, b_I [191].

From the point of view of the EFT of DE, DHOST were studied in [148]. In this context, and at the quadratic order in perturbations, DHOST provide new operators in addition to the ones found in eq. (2.37) and are of the form

$$S_{\rm DHOST} = \int M^2 \left[-\frac{\alpha_{\rm L}}{3}\delta K^2 + 2\beta_1\,\delta K\,V + \frac{\beta_2}{2}V^2 + \frac{\beta_3}{2}(\partial_i\delta g^{00})^2 \right] \sqrt{-g}\,{\rm d}^4x \ , \tag{A.6}$$

where $V \equiv -\frac{1}{2}(\dot g^{00} - N^i\partial_i g^{00})/g^{00}$ and M^2 is defined in (2.70). The functions of time $\alpha_{\rm L}$, β_1, β_2 and β_3 are the analogue of the α-coefficients of [127] that we discussed in Sec. 2.7. More specifically, $\alpha_{\rm L}$ provides a detuning in the time-kinetic term for gravitons (which usually has the form $K_{\mu\nu}^2 - K^2$). This coupling appears for instance in Horava-gravity [192] and its generalizations. Additionally, β_1 and β_2 resemble respectively the kinetic braiding $\alpha_{\rm B}$ and the kineticity $\alpha_{\rm K}$, while β_3 is associated with the gradient energy for δg^{00}.

The degeneracy conditions of the covariant theory translate into conditions among the coefficients just introduced (and also the usual αs). By studying the Newtonian limit of (A.6) however, one finds that theories with $\alpha_{\rm L} \neq 0$ give an infinite value for the Newton's constant $G_{\rm N}$ [148]. In what appears to be the phenomenologically relevant case for DE of $\alpha_{\rm L} = 0$, the degeneracy conditions are

$$\beta_1 = -\frac{1+\alpha_{\rm H}}{1+\alpha_{\rm T}} \ , \quad \beta_2 = -6\beta_1^2 \ , \quad \beta_3 = 2\frac{(1+\alpha_{\rm H})^2}{1+\alpha_{\rm T}} \ . \tag{A.7}$$

Remarkably, this set of DHOST theories is exactly the one that can be obtained from Beyond Horndeski through an X-dependent conformal redefinition for the metric

$$g_{\mu\nu} \to C(\phi, X) g_{\mu\nu} \ . \tag{A.8}$$

B | No new loophole in DHOST theories

This appendix is dedicated to the possible loopholes arising in the constraint $c_T = 1$, that we discussed in chapter 3. Specifically we want to investigate whether we can find new classes of theories evading the constraints coming from $c_T = 1$ and, in particular, we focus on DHOST theories. As explained in the main text, the new operators arising in these theories can be obtained by performing a conformal redefinition of the metric and therefore they do not change the lightcone of gravitons. Because of this, they are not constrained from the GW170817 measurement. In the derivation of Sec. 3.4 however they were not included, so we might wonder whether they change our conclusions. Thus, we are going to repeat the same procedure, but in the presence of the additional DHOST operators.

From the explanation of App. A, the EFT action now contains the four additional operators of eq. (A.6). In practice, however, we are going to use the degeneracy condition (A.7), so that $\alpha_{\rm L} = 0$ and $\beta_2 = -6\beta_1^2$. An additional simplification comes from the fact that the operator β_3 contains spatial derivatives, thus it does not play any role when considering homogeneous and isotropic perturbations of the background history.

The two remaining operators start at quadratic order, therefore they affect the equation of motion for δg_b^{00} (3.22). Of course this equation of motion now depends on higher derivatives of δg^{00}, and indeed we find terms up to $\partial_t^3 \delta g^{00}$, making the analysis less straightforward. Indeed, it is far less clear at this point what are the quantities we are allowed to vary independently. Nonetheless, because of the degeneracy conditions the dynamical system can still be formulated in terms of the usual variables (in other words we can recast the equations of motion in terms of $\delta \dot{g}^{00}$, δH and lower derivatives). Given the rapid increase in the complexity of the equations we are going to deal with, we are going to just outline how to recast the system in this form.

To obtain such formulation we follow the similar procedure used in [193]. We start with the Einstein equations, that we obtain by varying the overall action (2.37) plus the DHOST terms (A.6) and matter. We write schematically these equations as $\mathcal{E}_{\mu\nu} = 0$. For our purposes we can then focus on two of these equations, that we expand up to linear order in δg_b^{00} and δH_b:

$$\mathcal{E}_1 \equiv \mathcal{E}_{00} = \mathcal{E}_1^{(0)} + \mathcal{E}_1^{(1)} + \ldots = 0 \,, \tag{B.1}$$

$$\mathcal{E}_2 \equiv g^{\mu\nu}\mathcal{E}_{\mu\nu} = \mathcal{E}_2^{(0)} + \mathcal{E}_2^{(1)} + \ldots = 0 \,, \tag{B.2}$$

where the suffix stands for the order of the terms. One can explicitly check that the dependence of these functions of the following form

$$\mathcal{E}_1^{(1)} = \mathcal{E}_1^{(1)}(\delta g_b^{00}, \delta \dot{g}_b^{00}, \delta \ddot{g}_b^{00}, \delta H_b, \delta\rho_m) \,, \tag{B.3}$$

$$\mathcal{E}_2^{(1)} = \mathcal{E}_2^{(1)}(\delta g_b^{00}, \delta \dot{g}_b^{00}, \delta \ddot{g}_b^{00}, \delta H_b, \delta \dot{H}_b, \delta\rho_m, \delta p_m) \,. \tag{B.4}$$

To reduce the order in which δg_b^{00} appear, we first take a linear combination of $\mathcal{E}_1^{(1)}$ and $\mathcal{E}_2^{(1)}$ in such a way to remove $\delta \ddot{g}_b^{00}$. It turns out the combination we seek is

$$\mathcal{E}_3^{(1)} \equiv \mathcal{E}_1^{(1)} + \frac{\beta_1}{\beta_1 - 1}\mathcal{E}_2^{(1)} = 0 \,. \tag{B.5}$$

As a bonus, this new equation does not depend on $\delta \dot{H}_b$ as well. From this "master equation" we can express $\delta \dot{g}_b^{00}$ in terms of δg_b^{00} and matter, thus removing the dependence on δH_b. To do so, we take a time-derivative of eqs. (B.5) and use eq. (B.3) and (B.4) to remove respectively the terms $\delta \ddot{g}_b^{00}$ and $\delta \dot{H}_b$. Finally, we use eq. (B.5) to replace δH_b in this new equation and so we obtain an equation for $\delta \dot{g}_b^{00}$ of the form

$$\delta \dot{g}_b^{00} = \mathcal{B}_1(\delta g_b^{00}, \delta\rho_m, \delta\dot{\rho}_m, \delta p_m, \delta \dot{p}_m) \,, \tag{B.6}$$

for some function $\mathcal{B}_1$ that, to avoid cluttering, we don't to write down explicitly. This equation is analogous to eq. (4.33) of [193], where contrary to our case an equation of state for matter is also assumed.

Moreover, we can also get an equation for $\delta \dot{H}_b$. First we use eq. (B.6) in (B.5) so to get

$$\delta H_b = \delta H_b(\delta g_b^{00}, \delta\rho_m, \delta\dot{\rho}_m, \delta p_m, \delta \dot{p}_m) \,. \tag{B.7}$$

By taking a time-derivative of this last expression and using once again (B.5) we obtain

$$\delta \dot{H} = \mathcal{B}_2(\delta g^{00}, \delta\rho_m, \delta\dot{\rho}_m, \delta\ddot{\rho}_m, \delta p_m, \delta \dot{p}_m, \delta \ddot{p}_m) \,. \tag{B.8}$$

Equations (B.6) and (B.8) show that the dynamical system is kept up to second order in time for both $\phi(t)$ and $a(t)$. We can do a further step and remove also higher derivatives of matter quantities. This can be done following [148] around eq. (4.6). Indeed, by considering the combination

$$\delta H_b - \frac{\beta_1}{2}\delta \dot{g}_b^{00} = \tilde{\lambda}_g \delta g_b^{00} + \tilde{\lambda}_\rho \delta\rho_m + \tilde{\lambda}_p \delta p_m \tag{B.9}$$

we remove derivatives in the matter sector. The relevant coefficients in this equation are

$$\tilde{\lambda}_\rho \equiv \frac{1}{6M^2\left[(\alpha_{\rm B}+1)H-\dot{\beta}_1\right]}\,, \quad \tilde{\lambda}_p \equiv -\frac{\beta_1}{2M^2\left[(\alpha_{\rm B}+1)H-\dot{\beta}_1\right]}\,, \tag{B.10}$$

while we do not write explicitly the coefficient $\tilde{\lambda}_g$ given its lengthy expression.

As a consistency check one finds that, by taking $\beta_1 = 0$ the dependences on $\delta\dot{g}_b^{00}$ and δp_m drop out from eq. (B.9). This is indeed the usual case where δH_b is fixed by the matter and ϕ energy densities.

For what concerns the loophole in the constraint from c_T, we need to understand whether the variation $\delta\dot{g}_b^{00}$ can, for some values of the parameters, depend only on δg^{00} and δH_b. In such a case we might have a chance to find a non-trivial model with $c_T = 1$. The independent variations we were considering as our basis in eq. Sec. 3.4 where δg_b^{00}, δH_b and $\delta\dot{H}_b$ (or $\delta\dot{g}_b^{00}$ instead of the latter). The reason why this was possible in Horndeski is that, by varying $\delta\rho_m$ and δp_m one can vary independently δH_b and $\delta\dot{H}_b$. With DHOST this is far less transparent given that $\delta\rho_m$ and δp_m (and their derivatives) enter in the expressions for both $\delta\dot{g}_b^{00}$ and δH_b. It is thus more convenient for us to switch basis, and work with matter variations instead of variations of Hubble.

At this point we can replace the term δH_b in δm_4^2 (3.24) with its value given by eq. (B.9).

$$\begin{aligned}\delta m_4^2 = \frac{1}{2}(\tilde{m}_6 + m_6\beta_1)\delta\dot{g}_b^{00} + m_6(\tilde{\lambda}_\rho\delta\rho_m + \tilde{\lambda}_p\delta p_m)\\ + \frac{1}{2}[\tilde{m}_4^2 - m_5^2 + \dot{\tilde{m}}_6 - H(m_6 - \tilde{m}_6) + 2m_6\tilde{\lambda}_g]\delta g_b^{00}\,.\end{aligned} \tag{B.11}$$

In this expression the only dependence on $\delta\dot{\rho}_m$ and $\delta\dot{p}_m$ comes from $\delta\dot{g}_b^{00}$, and cannot be removed by tuning the parameters in eq. (B.6) while keeping $\beta_1 \neq 0$ (this comes from an explicit formula for $\mathcal{B}_1$). Since we can vary the time-derivatives of the matter quantities independently of $\delta\rho_m$ and δp_m, and since δm_4^2 needs to be robustly set to zero, we are forced to choose β_1 so to remove $\delta\dot{g}_b^{00}$. This means that $\beta_1 m_6 = -\tilde{m}_6$.

However, we also need the coefficients of $\delta\rho_m$ and δp_m to vanish in eq. (B.11). If we insist in keeping $\beta_1 \neq 0$ (so to have a DHOST model), then from the explicit expressions for the coefficients (B.10) we see there is only one options: setting $m_6 = 0$. Combined with the previous condition this means also $\tilde{m}_6 = 0$. Furthermore, at this point the only surviving term in eq. (B.11) is δg_b^{00}, and to make its coefficient vanish we require $\tilde{m}_4^2 = m_5^2$. Clearly enough, overall this is just the constraint (3.15). In conclusion, in the presence of DHOST we cannot avoid the requirements of [48], thus showing no other loophole is possible.

C | Generic disformal frame for $\gamma\pi\pi$

In Sec. 4.2.2 we have seen that the parameter $\tilde{m}_4^2$ has to vanish to suppress the gravitational wave decay. We have made the calculation in a frame where gravitons travel at a speed $c_T = 1$, so that several of the EFT parameters are absent from the beginning. Combining the new constraint $\tilde{m}_4^2 = 0$ with those coming from the speed of gravitons, eq. (3.15), one finds that the EFT simplifies considerably:

$$\dot{f} = m_4^2 = \tilde{m}_4^2 = m_5^2 = m_6 = \tilde{m}_6 = m_7 = 0 \,, \tag{C.1}$$

where the time independence of f can be set by a conformal transformation. Here we want to see the consequences of the absence of gravitational wave decay in a generic disformal frame and show that our results can be written in a frame independent way.

Exceptionally in this appendix we will use the following notation: we will denote by a hat quantities in the special frame where $\hat{c}_T = 1$, while quantities without a hat are in a generic frame. Starting from the generic action (2.37), it is possible to show that the cubic interaction $\gamma\pi\pi$ is controlled by the scale

$$\Lambda_\star^3 = \frac{2\sqrt{2} M H^2 c_T^2\, \alpha}{\left[1 + \alpha_{\rm H} - c_T^2(1 + \alpha_{\rm V})\right](1 + \alpha_{\rm H})} \,, \tag{C.2}$$

where the dimensionless quantity

$$\alpha = \frac{2c + 4m_2^4}{M^2 H^2} + \frac{3}{2}\left(\frac{M_*^2 \dot{f} - m_3^3}{M^2 H}\right)^2 \,, \tag{C.3}$$

already defined in eq. (2.71), sets the normalization of the scalar fluctuations and we have also defined the dimensionless quantity (see e.g. [163])

$$\alpha_{\rm V} \equiv -\frac{2m_5^2}{M^2} \,. \tag{C.4}$$

The other quantities are hopefully already familiar.

Generalizing the calculations of Sec. 4.2.2 in the frame where $c_T \neq 1$ (in this frame photons and gravitons move at the same speed, as required by experiments, but not equal to unity) we can derive the decay rate in a generic frame. This reads

$$\Gamma_{\gamma\to\pi\pi} = \frac{E_p^7\,(1-c_s^2/c_T^2)^2}{480\pi c_s^7 \Lambda_\star^6}\,, \tag{C.5}$$

where $E_p = c_T\, p$ and $\Lambda_\star^6$ is obtained from squaring eq. (C.2) above. This expression generalizes the one in eq. (4.34) to a generic frame.

We can now check that this result can be obtained from the decay rate in the frame with $\hat{c}_T = 1$, i.e. (see eq. (4.34))

$$\hat{\Gamma}_{\gamma\to\pi\pi} = \frac{\hat{p}^7\,(1-\hat{c}_s^2)^2}{480\pi \hat{c}_s^7 \hat{\Lambda}_\star^6}\,, \tag{C.6}$$

with (see eq. (4.24))

$$\hat{\Lambda}_\star^3 = \frac{\sqrt{2}\hat{M}_{\rm Pl}\hat{H}^2\hat{\alpha}}{\hat{\alpha}_{\rm H}(1+\hat{\alpha}_{\rm H})}\,. \tag{C.7}$$

(Notice that in the $\hat{c}_T = 1$ frame $\hat{\alpha}_{\rm V} = -\hat{\alpha}_{\rm H}$ and one recovers this equation from eq. (C.2).) When moving from the $\hat{c}_T = 1$ to the $c_T \neq 1$ frame, momenta do not change (i.e. $\hat{p} = p$) but the scale $\hat{\Lambda}_\star^6$ gets rescaled. Indeed, using the effects of a disformal transformation studied in [194, 132, 148] one can show that

$$\hat{M}_{\rm Pl} = c_T^{1/2} M\,, \qquad \hat{H} = H/c_T\,, \tag{C.8}$$

and

$$\hat{\alpha} = \frac{4\alpha c_T^4}{\left[1+\alpha_{\rm H}+c_T^2(1+\alpha_{\rm V})\right]^2}\,, \qquad \hat{\alpha}_{\rm H} = \frac{1+\alpha_{\rm H}-c_T^2(1+\alpha_{\rm V})}{1+\alpha_{\rm H}+c_T^2(1+\alpha_{\rm V})}\,. \tag{C.9}$$

Confronting eqs. (C.2) and (C.7) using these expressions shows that $\hat{\Lambda}_\star^3 = c_T^{1/2}\Lambda_\star^3$. Using this result and $\hat{c}_s = c_s/c_T$ in eq. (C.6) one sees that the dimensionless decay rate Γ/H is invariant,

$$\frac{\hat{\Gamma}_{\gamma\to\pi\pi}}{\hat{H}} = \frac{\Gamma_{\gamma\to\pi\pi}}{H}\,, \tag{C.10}$$

as expected.

To conclude, eq. (C.2) shows that the frame-invariant combination of parameters that is constrained by the absence of decay is

$$\frac{1+\alpha_{\rm H}-c_T^2(1+\alpha_{\rm V})}{2} = \frac{1}{M^2}\left[\tilde{m}_4^2 + m_5^2\left(1-\frac{2m_4^2}{M^2}\right)\right] = 0\,. \tag{C.11}$$

From eq. (2.70), for a quartic GLPV theory this constraint reads

$$2G_{4,X}^2 - XG_{4,X}F_4 + 2G_4G_{4,XX} - 2G_4F_4 - XF_{4,X}G_4 = 0\,. \tag{C.12}$$

As expected, eq. (C.11) cannot be put to zero by a disformal transformation $g_{\mu\nu} \to g_{\mu\nu} + D(\phi, X)\nabla_\mu\phi\nabla_\nu\phi$, as one can check using that

$$\alpha_{\rm H} \to \frac{1+\alpha_{\rm H}}{1+\alpha_{\rm X}} - 1 \,, \qquad \alpha_{\rm V} \to \frac{1+\alpha_{\rm V}}{(1+\alpha_{\rm D})(1+\alpha_{\rm X})} - 1 \,, \qquad c_T^2 \to c_T^2(1+\alpha_{\rm D}) \,, \tag{C.13}$$

where $\alpha_{\rm D} \equiv -XD/(1+XD)$ and $\alpha_{\rm X} \equiv -X^2 D_{,X}$.

D | Interactions in spatially-flat gauge

D.1 Gauge transformation

To write the metric in Newtonian gauge, we start with the general decomposition

$$\mathrm{d}s^2 = -(1+2\Phi)\,\mathrm{d}t^2 + a(t)^2\,(1-2\Psi)\,(e^{\gamma})_{ij}\,(\,\mathrm{d}x^i + N^i\,\mathrm{d}t)(\,\mathrm{d}x^j + N^j\,\mathrm{d}t)\;, \tag{D.1}$$

where $\delta^{ij}\gamma_{ii} = 0$. We can further decompose the vector part N^i into a scalar and a transverse vector as $N_i = \partial_i\psi + \hat{N}_i$ where $\partial_i\hat{N}_i = 0$ (in this section, indices are raised and lowered using the unperturbed metric $\bar{g}_{00} = -1$ and $\bar{g}_{ij} = a(t)^2\delta_{ij}$, we use $\nabla^2 = \partial_i\partial_i$, and hatted quantities are divergenceless). To go to Newtonian gauge, we use three diffeomorphisms to make the tensor transverse, $\partial_j\gamma_{ij} = 0$, and one diffeomorphism to make the vector transverse, $\psi = 0$. In this gauge, we also have the Goldstone mode $\pi(x)$ which appears explicitly in the action (i.e. after the Stueckelberg trick).

Another common gauge choice is the spatially-flat gauge (see e.g. [195]), where the metric is written in the general decomposition

$$\mathrm{d}s^2 = -(1+\delta N)^2\mathrm{d}\tilde{t}^2 + a(\tilde{t})^2(e^{\tilde{\gamma}})_{ij}(\mathrm{d}\tilde{x}^i + \tilde{N}^i\mathrm{d}\tilde{t})(\mathrm{d}\tilde{x}^j + \tilde{N}^j\mathrm{d}\tilde{t})\;. \tag{D.2}$$

The four gauge conditions in this case are that the tensor is transverse and traceless, $\tilde{\partial}_j\tilde{\gamma}_{ij} = 0$ and $\delta^{ij}\tilde{\gamma}_{ij} = 0$. Thus, in the decomposition of the vector, $\tilde{N}_i = \tilde{\partial}_i\tilde{\psi} + \hat{\tilde{N}}_i$, the scalar $\tilde{\psi}$ is still present (here $\tilde{\partial}_\mu \equiv \partial/\partial\tilde{x}^\mu$). The Goldstone field in this gauge is denoted by $\tilde{\pi}(\tilde{x})$.

Now, we wish to find the gauge transformation that connects the two above gauges to linear order. Under the gauge transformation $x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu$, the metric changes as usual

$$\tilde{g}_{\mu\nu}(\tilde{x}(x)) = g_{\rho\sigma}(x)\frac{\partial x^\rho}{\partial\tilde{x}^\mu}\frac{\partial x^\sigma}{\partial\tilde{x}^\nu}\;. \tag{D.3}$$

Infinitesimally, this gives

$$\Delta g_{\mu\nu}(x) \equiv \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = -\xi^\sigma\partial_\sigma g_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu \tag{D.4}$$

where on the right-hand side, and in the rest of this section, all derivatives without a tilde are taken with respect to the x coordinates, and all fields are evaluated at the point x. Expanding the metric around a time-dependent background $g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(x)$, this gives the following relation between the fluctuations

$$\delta\tilde{g}_{\mu\nu}(x) = \delta g_{\mu\nu}(x) + \Delta g_{\mu\nu}(x) \ . \tag{D.5}$$

Additionally, the transformation of the Goldstone field is dictated by the fact that it non-linearly realizes time-diffemorphisms: $\tilde{\pi}(\tilde{x}(x)) = \pi(x) - \xi^0(x)$, or infinitesimally as

$$\Delta\pi(x) \equiv \tilde{\pi}(x) - \pi(x) = -\xi^\sigma\partial_\sigma\pi - \xi^0 \ . \tag{D.6}$$

This gives the following relationships among the fields

$$\delta N = \Phi + \partial_0\xi_0 \tag{D.7}$$
$$a^2\tilde{\gamma}_{ij} = a^2\left(\gamma_{ij} - 2\Psi\delta_{ij} - 2H\xi^0\delta_{ij}\right) - \partial_i\xi_j - \partial_j\xi_i \tag{D.8}$$
$$\tilde{N}_i = N_i - \partial_0\xi_i - \partial_i\xi_0 \tag{D.9}$$
$$\tilde{\pi} = \pi - \xi^0 \ . \tag{D.10}$$

It is also convenient to parametrize the spatial part of the diffeomorphism into a scalar and a transverse vector: $\xi_i = \partial_i\xi + \hat{\xi}_i$ where $\partial_i\hat{\xi}^i = 0$. Requiring that both γ_{ij} and $\tilde{\gamma}_{ij}$ be transverse and traceless gives

$$\xi^0 = -\frac{\Psi}{H} \ , \quad \text{and} \quad \nabla^2\xi = 0 \ . \tag{D.11}$$

The remaining tensor part of eq. (D.8) gives

$$\partial_i\hat{\xi}_j + \partial_j\hat{\xi}_i = a^2(\gamma_{ij} - \tilde{\gamma}_{ij}) \ , \tag{D.12}$$

while the scalar and vector parts of eq. (D.9) give

$$\nabla^2\xi_0 = -\nabla^2\tilde{\psi} \ , \quad \text{and} \quad \partial_0\hat{\xi}_i = \hat{N}_i - \hat{\tilde{N}}_i \ . \tag{D.13}$$

D.2 Vertices in spatially-flat gauge for $\tilde{m}_4^2$

In this subsection we redo the computations of Sec. 4.2 in spatially-flat gauge. As we will see, because we are in a different gauge, the relevant vertices emerge from different terms in the action. In the spatially-flat gauge, δN and $\tilde{N}^i$ are Lagrange multipliers, and for the cubic action, we only need their expressions to first order [156]. Variation of the action (4.1) with respect to $\tilde{\psi}$ gives the constraint equation for δN, and variation with respect to δN gives the constraint equation for $\tilde{\psi}$:

$$\begin{aligned}\frac{\delta S}{\delta\tilde{\psi}} &= a^3\,\nabla^2\left[-2M_{\rm Pl}^2(\dot{H}\tilde{\pi}+H\delta N)+m_3^3(\delta N-\dot{\tilde{\pi}})\right]\\ \frac{\delta S}{\delta\,\delta N} &= 2a^3\left[2m_2^4+3Hm_3^3-M_{\rm Pl}^2(3H^2+\dot{H})\right]\delta N+a^3(2M_{\rm Pl}^2\dot{H}-4m_2^4-3Hm_3^3)\dot{\tilde{\pi}}\\ &\quad+3a^3\dot{H}(m_3^3-2HM_{\rm Pl}^2)\tilde{\pi}+a\,\nabla^2\left[(m_3^3+4H\tilde{m}_4^2)\tilde{\pi}+(m_3^3-2HM_{\rm Pl}^2)\tilde{\psi}\right]\,.\end{aligned} \tag{D.14}$$

Setting the above to zero gives the solutions (see [196] for the case $\tilde{m}_4^2=0$),

$$\delta N = \frac{m_3^3}{m_3^3-2M_{\rm Pl}^2H}\dot{\tilde{\pi}}+\frac{2M_{\rm Pl}^2\dot{H}}{m_3^3-2M_{\rm Pl}^2H}\tilde{\pi}\equiv\alpha_N\dot{\tilde{\pi}}+\tilde{\alpha}_N\tilde{\pi} \tag{D.15}$$

and

$$\tilde{\psi} = -\frac{m_3^3+4H\tilde{m}_4^2}{m_3^3-2HM_{\rm Pl}^2}\tilde{\pi}-\frac{3m_3^6H-4HM_{\rm Pl}^2(\dot{H}M_{\rm Pl}^2-2m_2^4)}{(m_3^3-2M_{\rm Pl}^2H)^2}\frac{a^2}{\tilde{\nabla}^2}\dot{\tilde{\pi}}\equiv\alpha_\psi\tilde{\pi}+\tilde{\alpha}_\psi\frac{a^2}{\tilde{\nabla}^2}\dot{\tilde{\pi}}\,, \tag{D.16}$$

where $\tilde{\nabla}^2\equiv\tilde{\partial}_i\tilde{\partial}_i$. For the purposes of this section we are going to keep the leading terms in derivatives α_N and α_ψ in eq. (D.15) and (D.16) (the subleading terms $\tilde{\alpha}_N$ and $\tilde{\alpha}_\psi$ contribute to $\gamma\pi\pi$ but with fewer derivatives). Notice that one can also obtain the same results by directly using the equations for the gauge transformation in Sec. D.1. These solutions can then be plugged back into the action so that it is simply a functional of $\tilde{\gamma}_{ij}$ and $\tilde{\pi}$.

As before, one can then look at the quadratic Lagrangian to find the canonical normalization of the fields and the speed of sound for $\tilde{\pi}$. Because we have not changed the tensor part of the metric, the normalization for $\tilde{\gamma}_{ij}$ is the same as in eq. (4.13). For $\tilde{\pi}$, we can use the results of the last section to immediately see the answer. Using eqs. (D.10), (D.11), (5.7), (4.6) and (4.9), we find

$$\tilde{\pi}=\frac{2HM_{\rm Pl}^2-m_3^3}{4H\tilde{m}_4^2+2HM_{\rm Pl}^2}\pi=\frac{2HM_{\rm Pl}^2-m_3^3}{\sqrt{2}HM_{\rm Pl}(3m_3^6+4M_{\rm Pl}^2(c+2m_2^4))^{\frac{1}{2}}}\pi_c\,. \tag{D.17}$$

Because only the normalization of π changes between the gauges, the speed of sound c_s^2 is the same as in (4.8).

D.2.1 Vertex $\gamma\pi\pi$

Our main focus is to show that the S-matrix elements in the two gauges are equal. At this level then, we are allowed to simplify our expressions by using the quadratic equations of motion for the fields. Indeed, this corresponds to a perturbative field-redefinition that doesn't affect matrix elements. Given this premise, now we move on to the non-linear $\gamma\pi\pi$ vertex. This vertex receives contributions both from the Einstein-Hilbert term $S_{\rm EH}$ in (4.15) and the dark-energy term $S_{\tilde{m}_4}$ in (4.16). There are two different contributions from $N^{-1}\left(E_{ij}E^{ij}-E^2\right)$ in the Einstein-Hilbert term (4.15): the first has the form $\delta N\dot{\tilde{\gamma}}_{ij}\partial_i\partial_j\tilde{\psi}$ and comes from the

$E_{ij}E^{ij}$ term, and the second has the form $\partial_i\tilde{\psi}\,\partial_j\tilde{\psi}\,\nabla^2\tilde{\gamma}_{ij}$ and comes from both $E_{ij}E^{ij}$ and E^2. For the last term mentioned, we can use the linear equation of motion $\nabla^2\gamma_{ij} = a^2\ddot{\gamma}_{ij}$ (the term $3H\dot{\gamma}_{ij}$ in the equation can be neglected, since it has fewer derivatives) so that the vertex has two time and two spatial derivatives, which is the form in (4.22). More specifically, we have

$$\begin{aligned} S_{\rm EH} &\supset \frac{1}{2}M_{\rm Pl}^2 \int a\left(\alpha_N\alpha_\psi\,\dot{\tilde{\pi}}\dot{\tilde{\gamma}}_{ij}\partial_i\partial_j\tilde{\pi} + \frac{1}{2a^2}\alpha_\psi^2\partial_i\tilde{\pi}\partial_j\tilde{\pi}\nabla^2\tilde{\gamma}_{ij}\right)\mathrm{d}^4x \\ &= \frac{2H\tilde{m}_4^2M_{\rm Pl}^2(m_3^3+4H\tilde{m}_4^2)}{(m_3^3-2HM_{\rm Pl}^2)^2}\int a\,\dot{\tilde{\pi}}\dot{\tilde{\gamma}}_{ij}\partial_i\partial_j\tilde{\pi}\,\mathrm{d}^4x \end{aligned} \tag{D.18}$$

where, as always for the case of $\tilde{m}_4$, we are in the high energy limit. The contribution from $S_{\tilde{m}_4}$ comes both from the Stueckelberg discussed after (4.16), and from $\delta K_{ij}\delta K^{ij}$ in the same manner as just discussed for the Einstein-Hilbert term. More specifically, we have

$$\begin{aligned} S_{\tilde{m}_4} &\supset \tilde{m}_4^2(1-\alpha_N)(2+\alpha_\psi)\int a\,\dot{\tilde{\pi}}\dot{\tilde{\gamma}}_{ij}\partial_i\partial_j\tilde{\pi}\,\mathrm{d}^4x \\ &= \frac{2H\tilde{m}_4^2M_{\rm Pl}^2[-m_3^3+4H(\tilde{m}_4^2+M_{\rm Pl}^2)]}{(m_3^3-2HM_{\rm Pl}^2)^2}\int a\,\dot{\tilde{\pi}}\dot{\tilde{\gamma}}_{ij}\partial_i\partial_j\tilde{\pi}\,\mathrm{d}^4x\,. \end{aligned} \tag{D.19}$$

In total, then, we have

$$S_{\tilde{\gamma}\tilde{\pi}\tilde{\pi}} = \frac{8H^2M_{\rm Pl}^2\tilde{m}_4^2(2\tilde{m}_4^2+M_{\rm Pl}^2)}{(m_3^3-2HM_{\rm Pl}^2)^2}\int a\,\dot{\tilde{\pi}}\dot{\tilde{\gamma}}_{ij}\partial_i\partial_j\tilde{\pi}\,\mathrm{d}^4x\,. \tag{D.20}$$

Indeed, one can check that this is the same result that one would obtain by starting with the vertex in Newtonian gauge (4.22) and using (D.17) to write it in the spatially flat gauge.

Since the coupling $\gamma\gamma\pi$ in the presence of $\tilde{m}_4$ turns out to be irrelevant for the constraints, here we avoid providing the check in spatially-flat gauge.

D.3 Vertices in spatially-flat gauge for m_3^3

In this section we are interested in obtaining, in spatially-flat gauge, the couplings between π and γ_{ij} in the case of $\tilde{m}_4^2 = 0$, $m_3^3 \neq 0$, that are present in the Lagrangian (6.2).

D.3.1 Vertex $\gamma\pi\pi$

First, we are going to check the calculation of Sec. 5.1.1 for $\gamma\pi\pi$ in spatially-flat gauge. For what concerns the quadratic action, there is no difference from the case of $\tilde{m}_4$ of Sec. D.2 (of course in the coefficients given, $\tilde{m}_4$ has to be set to zero), so we can immediately discuss interactions. Here we keep a distinction between the canonical field π_c and the non-canonical field $\tilde{\pi}$, unlike in the main text. We are going to investigate the non-linear $\gamma\pi\pi$ vertex which contains three or more derivatives (we will drop all the rest). Starting from the action (5.1) this vertex can get contributions through the Einstein-Hilbert and the S_{m_3} terms. Let us first consider the contribution from S_{m_3} term. As usual, the extrinsic curvature is defined by

$$\tilde{K}_{ij} = \frac{1}{2N}(\dot{\tilde{h}}_{ij} - \tilde{\mathcal{D}}_i\tilde{N}_j - \tilde{\mathcal{D}}_i\tilde{N}_j) \ , \tag{D.21}$$

where $\tilde{\mathcal{D}}_i$ denotes the 3d covariant derivative with respect to the induced metric $\tilde{h}_{ij}$. It is straightforward to show that a variation of $\tilde{K}$ contains $a^{-2}\tilde{\gamma}_{ij}\tilde{\partial}_i\tilde{\partial}_i\tilde{\psi}$. After performing the Stueckelberg trick of eq. (2.25) and (5.5), the contribution from S_{m_3} which has three derivatives reads

$$S_{m_3} \supset \frac{2m_3^3 M_{\rm Pl}^4 H^2}{(m_3^3 - 2M_{\rm Pl}^2 H)^2}\int a\, \dot{\tilde{\gamma}}_{ij}\tilde{\partial}_i\tilde{\pi}\tilde{\partial}_j\tilde{\pi}\ {\rm d}^4\tilde{x} \ , \tag{D.22}$$

where we have taken the term $-2(1-\alpha_N)\dot{\tilde{\pi}}$ from $\delta\tilde{g}^{00}$ and used eqs. (D.15), (D.16).

Using the canonical normalizations both for $\tilde{\pi}$ (D.17) and $\tilde{\gamma}_{ij}$ one obtains the same interaction that we have obtained in the Newtonian gauge (5.9). Thus, we expect the contribution arising from $S_{\rm EH}$ to cancel out. To show this, notice that the contribution from the Einstein-Hilbert term comes from $N(\tilde{K}_{ij}\tilde{K}^{ij} - \tilde{K}^2)$. More specifically, we have

$$S_{\rm EH} \supset \frac{M_{\rm Pl}^2}{2}\int a\ \left(4H\delta N\tilde{\gamma}_{ij}\tilde{\partial}_i\tilde{\partial}_j\tilde{\psi} + \delta N\dot{\tilde{\gamma}}_{ij}\tilde{\partial}_i\tilde{\partial}_j\tilde{\psi} + \frac{1}{2a^2}\tilde{\nabla}^2\tilde{\gamma}_{ij}\tilde{\partial}_i\tilde{\psi}\tilde{\partial}_j\tilde{\psi}\right){\rm d}^4\tilde{x} \ , \tag{D.23}$$

where we have performed a few integrations by parts. Using (D.15) and (D.16) in $S_{\rm EH}$ one obtains

$$\begin{aligned} S_{\rm EH} \supset \frac{M_{\rm Pl}^2}{2}\int a\ \Big\{&\alpha_\psi(\alpha_N+\alpha_\psi)\dot{\tilde{\pi}}\dot{\tilde{\gamma}}_{ij}\tilde{\partial}_i\tilde{\partial}_j\tilde{\pi} - a^2\tilde{\alpha}_\psi(\alpha_N+\alpha_\psi)\tilde{\pi}\ddot{\tilde{\gamma}}_{ij}\tilde{\partial}_i\tilde{\partial}_j\frac{1}{\tilde{\nabla}^2}\dot{\tilde{\pi}} \\ &+\Big[H\alpha_\psi(2\alpha_N+\alpha_\psi) - \alpha_\psi(\tilde{\alpha}_N+\dot{\alpha}_\psi) + \alpha_N\tilde{\alpha}_\psi c_s^2\Big]\dot{\tilde{\gamma}}_{ij}\tilde{\partial}_i\tilde{\pi}\tilde{\partial}_j\tilde{\pi}\Big\}\,{\rm d}^4\tilde{x} \ , \end{aligned} \tag{D.24}$$

where we have used the linear equations of motion for $\tilde{\gamma}_{ij}$, $\ddot{\tilde{\gamma}}_{ij} + 3H\dot{\tilde{\gamma}}_{ij} - \frac{1}{a^2}\tilde{\nabla}^2\tilde{\gamma}_{ij} = 0$, and for $\tilde{\pi}$, $\ddot{\tilde{\pi}} + 3H\dot{\tilde{\pi}} - \frac{c_s^2}{a^2}\tilde{\nabla}^2\tilde{\pi} = 0$.

Notice that the first two terms on RHS vanish due to the fact that $\alpha_N + \alpha_\psi = 0$. Let us consider the prefactor of the last term. Using the expression of c_s^2 (5.14) and the definitions of the α-parameters in eqs. (D.15) and (D.16), the prefactor can be rewritten as

$$H\alpha_\psi(2\alpha_N+\alpha_\psi) - \alpha_\psi(\tilde{\alpha}_N+\dot{\alpha}_\psi) + \alpha_N\tilde{\alpha}_\psi c_s^2 = H\alpha_\psi(\alpha_N+\alpha_\psi) + \frac{H\tilde{\alpha}_N^2}{M_{\rm Pl}^2\dot{H}^2}(M_{\rm Pl}^2\dot{H}\alpha_\psi - c\alpha_\psi) \ . \tag{D.25}$$

In our case the coupling with matter has been neglected ($\rho_m = p_m = 0$), therefore the parameter c is equal to $-M_{\rm Pl}^2\dot{H}$ (see e.g. [55, 54]). The prefactor is then given by

$$H\alpha_\psi(2\alpha_N+\alpha_\psi) - \alpha_\psi(\tilde{\alpha}_N+\dot{\alpha}_\psi) + \alpha_N\tilde{\alpha}_\psi c_s^2 = (\alpha_\psi+\alpha_N)(H\alpha_\psi + \frac{H\tilde{\alpha}_N^2}{\dot{H}}) \tag{D.26}$$

which is again zero since $\alpha_N + \alpha_\psi = 0$. Therefore, $S_{\rm EH}$ gives no contribution to our $\gamma\pi\pi$ vertex and S_{m_3}, on the other hand, gives the same result that we have obtained in the Newtonian gauge (5.9).

D.3.2 Vertex $\gamma\gamma\pi$

In this last part we are going to complete the check of the various interactions used in this thesis. The last vertex to consider is the cubic term $\gamma\gamma\pi$ in the Lagrangian (6.2), that was computed in Newtonian gauge. As already done, we are going to limit our check to the case in which matter is negligible ($\rho_m = p_m = 0$); in this case one has $c = -M_{\rm Pl}^2\dot{H}$.

It is straightforward to realize that the vertex $\gamma\gamma\pi$ is not generated by the term $m_3^3\delta\tilde{g}^{00}\delta\tilde{K}$ in the action (it is not possible to get γ^2 out of either δg^{00} or δK, at cubic order). On the other hand, the sought out vertex is generated by the Einstein-Hilbert term. In order to simplify the derivation we are going to exploit the fact that, as in Newtonian gauge, tensor perturbations $\tilde{\gamma}_{ij}$ couple to the metric as a minimally-coupled scalar does (this statement can be verified explicitly by expanding the Einstein-Hilbert term up to cubic order). Therefore, the quadratic Lagrangian for $\tilde{\gamma}_{ij}$ is

$$\begin{aligned}\mathscr{L} &= -\frac{M_{\rm Pl}^2}{8}\tilde{g}^{\mu\nu}\tilde{\partial}_\mu\tilde{\gamma}_{ij}\tilde{\partial}_\nu\tilde{\gamma}_{ij}\sqrt{-\tilde{g}} \\ &= -a^3\frac{M_{\rm Pl}^2}{8}\left[-\dot{\tilde{\gamma}}_{ij}^2 + a^{-2}(\tilde{\partial}_k\tilde{\gamma}_{ij})^2 + \delta N\left(\dot{\tilde{\gamma}}_{ij}^2 + a^{-2}(\tilde{\partial}_k\tilde{\gamma}_{ij})^2\right) + \frac{2}{a^2}\tilde{\partial}_k\tilde{\psi}\tilde{\partial}_k\tilde{\gamma}_{ij}\dot{\tilde{\gamma}}_{ij}\right] . \end{aligned} \tag{D.27}$$

The first two terms in the last equation are the standard kinetic term for the graviton, and the remaining terms contribute to our cubic vertex.

Let us focus on these relevant terms. By replacing δN and $\tilde{\psi}$ by the constraints (D.15) and (D.16) we obtain, after several integrations by part and after dropping terms with less than two derivatives,

$$\begin{aligned}\mathscr{L}_{\gamma\gamma\pi} = &- a^3\frac{M_{\rm Pl}^2}{8}\Bigg[(\alpha_N+\alpha_\psi)\left(\dot{\tilde{\gamma}}_{ij}^2 + a^{-2}(\tilde{\partial}_k\tilde{\gamma}_{ij})^2\right)\dot{\tilde{\pi}} + (\alpha_\psi H + \dot{\alpha}_\psi + \tilde{\alpha}_N + c_s^2\tilde{\alpha}_\psi)\frac{1}{a^2}(\tilde{\partial}_k\tilde{\gamma}_{ij})^2\tilde{\pi} \\ &- (3H\alpha_\psi - \dot{\alpha}_\psi - \tilde{\alpha}_N - c_s^2\tilde{\alpha}_\psi)\dot{\tilde{\gamma}}_{ij}^2\,\tilde{\pi}\Bigg] ,\end{aligned} \tag{D.28}$$

where we have used the linear equations of motion for $\tilde{\gamma}_{ij}$, $\ddot{\tilde{\gamma}}_{ij} + 3H\dot{\tilde{\gamma}}_{ij} - \frac{1}{a^2}\tilde{\nabla}^2\tilde{\gamma}_{ij} = 0$ and for $\tilde{\pi}$, $\ddot{\tilde{\pi}} + 3H\dot{\tilde{\pi}} - \frac{c_s^2}{a^2}\tilde{\nabla}^2\tilde{\pi} = 0$. The first term in the above equation vanishes since $\alpha_N + \alpha_\psi = 0$. Then, using the expression for c_s^2 in eq. (5.14), also the second term of (D.28) vanishes (notice that our expression for c_s^2 assumes $\dot{m}_3 = 0$, but the cancellation works also in the more general case [78]). Therefore, one is left only with the term in the last line, which simplifies to

$$\mathscr{L}_{\gamma\gamma\pi} = \frac{a^3}{2}\frac{m_3^3M_{\rm Pl}^2H}{2M_{\rm Pl}^2H - m_3^3}\dot{\tilde{\gamma}}_{ij}^2\,\tilde{\pi} . \tag{D.29}$$

After using (D.17) and (5.8) to go to canonical normalization for $\tilde{\pi}$ and $\tilde{\gamma}_{ij}$, equation (D.29) matches exactly with the vertex in Newtonian gauge (6.2).

E | Parametric resonance as Bose enhancement

In this Appendix we want to reinterpret the exponential growth due to parametric resonance as the Bose enhancement of the perturbative decay $\gamma \to \pi\pi$. To see this, we study the Boltzmann equation for the number density of dark energy fluctuations. We denote by $n^{\pi}_{\boldsymbol{k}}$ and $n^{\gamma}_{\boldsymbol{k}}$ the occupation numbers, respectively, of π and γ. Moreover, the number density for the particle species π is defined as

$$n_\pi \equiv \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} n^{\pi}_{\boldsymbol{k}} \; , \tag{E.1}$$

and an analogous definition holds for γ.

Let us consider a collection of gravitons with frequency ω, each of them decaying into two π-particles. For concreteness, we will focus on the case $c_s^2 \ll 1$, for which the two momenta of π, $\boldsymbol{k}$ and $-\boldsymbol{k}$, have opposite directions and equal magnitudes $k = \omega/(2c_s)$. Following [197] and denoting by $\Gamma_{\gamma\to\pi\pi}$ the tree-level decay rate (see e.g. (5.15)), the rate of change of n_π in a given volume V is

$$\frac{\mathrm{d}n_\pi}{\mathrm{d}u} \simeq 2 \times \frac{\Gamma_{\gamma\to\pi\pi}}{V}\left[(n^{\pi}_{\boldsymbol{k}}+1)(n^{\pi}_{-\boldsymbol{k}}+1)n^{\gamma}_{\omega} - n^{\pi}_{\boldsymbol{k}} n^{\pi}_{-\boldsymbol{k}}(n^{\gamma}_{\omega}+1)\right] , \tag{E.2}$$

where the factor of 2 accounts for two identical particles in the final state. As explained earlier in Sec. 5.2.1, for small c_s we can use u as time. On the right-hand side, we have neglected integration over the angle φ which would appear when considering only one of the GW polarizations. For $n^{\pi}_{\boldsymbol{k}} = n^{\pi}_{-\boldsymbol{k}} = n^{\pi}_{k}$ and $n^{\gamma}_{\omega} \gg \{n^{\pi}_{\boldsymbol{k}}, 1\}$ we find

$$\frac{\mathrm{d}n_\pi}{\mathrm{d}u} \simeq 2\Gamma_{\gamma\to\pi\pi} n_\gamma \left(1 + 2n^{\pi}_{k}\right) \; , \tag{E.3}$$

where we have introduced the number density of gravitons, here given by $n_\gamma = n^{\gamma}_{\omega}/V$.

The produced π-particles end up populating the spherical shell $k = k_0 \pm \Delta k/2$ of radius $k_0 \simeq \omega/(2c_s)$ and thickness Δk, so that their occupation number is related to their number density by

$$n^{\pi}_{k} \simeq \frac{n_\pi}{4\pi k_0^2 \Delta k/(2\pi)^3} \; . \tag{E.4}$$

The thickness is given by comparing the time-independent part of the equation of motion for π, eq. (5.27), with the amplitude of its periodic part. Using $\Omega = c_s \ll 1$ valid for small c_s, we obtain

$$\Delta k = \beta k_0 \ll k_0 \, . \tag{E.5}$$

Plugging the expressions of k_0 and Δk into (E.4) and using $n_\gamma \simeq \omega (M_{\rm Pl} h_0^+)^2$, the occupation number can be written as

$$n_k^\pi = 4\beta c_s^7 \pi^2 \left(\frac{\Lambda}{\omega}\right)^4 \frac{n_\pi}{n_\gamma} \, , \tag{E.6}$$

where we have focused on the case of the operator m_3^3 (which can be straightforwardly extended to the case of the operator $\tilde{m}_4^2$ by the replacement $\Lambda^2 \to \Lambda_\star^3 \omega^{-1}$, see discussion in Sec. 5.1.2) and used the definition of β, eq. (5.19), to rewrite $(M_{\rm Pl} h_0^+)^2$. The Bose condensation effect becomes important for $n_k^\pi \gg 1$ or, using the above equation, for

$$n_\pi \gg \frac{n_\gamma}{4\beta c_s^7 \pi^2} \left(\frac{\omega}{\Lambda}\right)^4 \, . \tag{E.7}$$

In this case, we can solve eq. (E.3) with the decay rate given by eq. (5.15). This gives

$$n_\pi \propto \exp\left(\frac{\pi\beta}{30} \omega u\right) \, , \tag{E.8}$$

which displays an instability similar to that encountered above in eq. (5.52), but with a different exponent. Notice that the approach of this subsection is approximate and does not reproduce the correct numerical factors in the timescale of the instability. Of course, a more precise calculation would give the same answer.

It is useful to check that our formula for the modification of the GW, eq. (5.69), smoothly interpolates with the perturbative decay result, eq. (5.15), when the occupation number becomes small. In the regime in which Bose enhancement is negligible $n_\pi/n_\gamma \sim \Gamma u$, see fig. 5.1. Using this in eq. (E.6) we get $n_k^\pi \sim \beta \omega u$. Not surprisingly this is the parameter that enters the exponential growth of the instability. Our saddle-point treatment is valid for $\beta\omega u \gg 1$, but we expect that, when $\beta \omega u \sim 1$, it gives a result of the same order as the perturbative decay of gravitons. Indeed, if we plug this equality in eq. (5.69) one gets (for $c_s \ll 1$)

$$\Delta\gamma \simeq \frac{v}{\Lambda^2} \frac{1}{c_s^5} \beta \omega^4 \simeq M_{\rm Pl} h_0^+ \Gamma v \, . \tag{E.9}$$

This is indeed the perturbative result: the original GW, $M_{\rm Pl} h_0^+$, changes with a rate Γ for a time of order v.

F | Details on the conservation of energy

F.1 m_3^3-operator

In this Appendix we check the conservation of energy discussed in Sec. 5.2.5. First, let us verify that GWs do not contribute to the flux of energy across $\partial\mathcal{V}_0$ (see fig. 5.3). This is clearly true for $\bar{\gamma}_{ij}$ but it holds at order $\Delta\gamma$ too. Indeed we have, using $\dot{\bar{\gamma}}_{ij} = -\partial_z\bar{\gamma}_{ij}$ and $\partial_k\bar{\gamma}_{ij} = \delta_{kz}\partial_z\bar{\gamma}_{ij}$,

$$\begin{aligned}\left(T^{00} - T^{z0}\right)\big|_{u=|z_0|} \supset\ & \frac{1}{4}\left[(\dot{\bar{\gamma}}_{ij})^2 + (\partial_k\bar{\gamma}_{ij})^2\right] + \frac{1}{2}\dot{\bar{\gamma}}_{ij}\partial_z\bar{\gamma}_{ij} \\ & + \frac{1}{2}\dot{\bar{\gamma}}_{ij}\Delta\dot{\gamma}_{ij} + \frac{1}{2}\partial_k\bar{\gamma}_{ij}\partial_k\Delta\gamma_{ij} + \frac{1}{2}\dot{\bar{\gamma}}_{ij}\partial_z\Delta\gamma_{ij} + \frac{1}{2}\Delta\dot{\gamma}_{ij}\partial_z\bar{\gamma}_{ij} \\ & = 0 + \frac{1}{2}\dot{\bar{\gamma}}_{ij}(\Delta\dot{\gamma}_{ij} - \partial_z\Delta\gamma_{ij}) + \frac{1}{2}\dot{\bar{\gamma}}_{ij}(\partial_z\Delta\gamma_{ij} - \Delta\dot{\gamma}_{ij}) = 0\ .\end{aligned} \tag{F.1}$$

Let us now calculate the LHS of eq. (5.87), i.e. the variation of the total energy in the region. This is only due to $\Delta\gamma_{ij}$, since π and $\bar{\gamma}_{ij}$ depend on u only. One has

$$\begin{aligned}T^{00} \supset\ & \frac{1}{2}\dot{\bar{\gamma}}_{ij}\Delta\dot{\gamma}_{ij} + \frac{1}{2}\partial_k\bar{\gamma}_{ij}\partial_k\Delta\gamma_{ij} = \frac{1}{2}\dot{\bar{\gamma}}_{ij}\left(\Delta\dot{\gamma}_{ij} - \partial_z\Delta\gamma_{ij}\right) = \frac{1}{2}\dot{\bar{\gamma}}_{ij}2\partial_u\Delta\gamma_{ij} \\ & = -\dot{\bar{\gamma}}_{ij}\left(\frac{v}{4\Lambda^2}\partial_u J_{ij}(u)\right) = -\dot{\bar{\gamma}}_{ij}\epsilon^+_{ij}\left(\frac{v}{4\Lambda^2}\partial_u\left\langle(\partial_x\pi)^2 - (\partial_y\pi)^2\right\rangle\right) \equiv v\mathcal{F}(u)\ .\end{aligned} \tag{F.2}$$

The integral over $\partial\mathcal{V}_2 \cup \partial\mathcal{V}_1$ reduces to

$$\begin{aligned}\int_{\partial\mathcal{V}_2} T^{00}\,\mathrm{d}z - \int_{\partial\mathcal{V}_1} T^{00}\,\mathrm{d}z &= \int_{T-|z_0|}^{T} v\mathcal{F}(u)\big|_{t=T}\,\mathrm{d}z - \int_{-|z_0|}^{0} v\mathcal{F}(u)\big|_{t=0}\,\mathrm{d}z \\ &= \int_{T-|z_0|}^{T}(T+z)\mathcal{F}(T-z)\,\mathrm{d}z - \int_{-|z_0|}^{0} z\mathcal{F}(-z)\,\mathrm{d}z \\ &= \int_{-|z_0|}^{0}(2T+\tilde{z})\mathcal{F}(-\tilde{z})\,\mathrm{d}\tilde{z} - \int_{-|z_0|}^{0} z\mathcal{F}(-z)\,\mathrm{d}z \\ &= 2T\int_{-|z_0|}^{0}\mathcal{F}(-z)\,\mathrm{d}z\,.\end{aligned} \tag{F.3}$$

The RHS of eq. (5.87) gets contribution only from π:

$$\int_{\partial\mathcal{V}_0}\left(T^{00}-T^{z0}\right)\mathrm{d}t=\int_0^T\left(T^{00}_\pi\ -T^{z0}_\pi\right)|_{u=|z_0|}\mathrm{d}t=T\left(T^{00}_\pi-T^{z0}_\pi\right)|_{u=|z_0|}\,. \tag{F.4}$$

The equation for energy conservation, eq. (5.87), becomes

$$\begin{aligned}0&=T\left(T^{00}_\pi-T^{z0}_\pi\right)|_{u=|z_0|}+2T\int_{-|z_0|}^0\mathcal{F}(-z)\,\mathrm{d}z\\&=T\left[\left(T^{00}_\pi-T^{z0}_\pi\right)|_{u=|z_0|}-\frac{1}{2\Lambda^2}\int_0^{|z_0|}\dot{\bar{\gamma}}_{ij}(u)\epsilon^+_{ij}\partial_u\left\langle(\partial_x\pi)^2-(\partial_y\pi)^2\right\rangle\,\mathrm{d}u\right]\,.\end{aligned} \tag{F.5}$$

Note that the linear v dependence of $\Delta\gamma$ is essential: if it was not the case then the two terms in (F.5) would have a different T dependence, with no chance of adding up to zero. To explicitly check energy conservation one should integrate (F.5). A faster way to check this is to take a derivative with respect to $|z_0|$ (or equivalently u). The resulting equation can be shown to be satisfied by the solution for π, (5.37). After simplifying T and taking the derivative with respect to z_0, (F.5) becomes

$$\partial_u\left[\langle T^{00}_\pi(u)\rangle-\langle T^{z0}_\pi(u)\rangle\right]-\frac{1}{2\Lambda^2}\dot{\bar{\gamma}}_{ij}(u)\epsilon^+_{ij}\partial_u\left\langle(\partial_x\pi)^2-(\partial_y\pi)^2\right\rangle=0\,. \tag{F.6}$$

The expression of $\langle T^{00}_\pi\rangle$ is given by eq. (5.42) while

$$\langle T^{z0}_\pi(u)\rangle=-c_s^2\left\langle\dot{\pi}\partial_z\pi\right\rangle=-\int\frac{\mathrm{d}^3\tilde{\boldsymbol{p}}}{(2\pi)^3}\frac{1}{4c_s^2p_u}\left[-2c_s^2|\partial_uf_{\tilde{p}}|^2-2p_s^2|f_{\tilde{p}}|^2+\text{const}\right]\,. \tag{F.7}$$

Therefore we can write

$$\begin{aligned}&\partial_u\left[\langle T^{00}_\pi(u)\rangle-\langle T^{z0}_\pi(u)\rangle\right]=\\&=\int\frac{\mathrm{d}^3\tilde{\boldsymbol{p}}}{(2\pi)^3}\frac{1}{4c_s^2p_u}\left[\partial_uf^\star_{\tilde{p}}\left((1-c_s^2)\partial_u^2f_{\tilde{p}}+f_{\tilde{p}}\left((1-c_s^2)c_s^{-2}p_s^2+c_s^2(p_x^2+p_y^2)\right)\right)+\text{h.c.}\right].\end{aligned} \tag{F.8}$$

The remaining term contains

$$\begin{aligned}\partial_u\left\langle(\partial_x\pi)^2-(\partial_y\pi)^2\right\rangle&=\partial_u\int\frac{\mathrm{d}^3\tilde{\boldsymbol{p}}}{(2\pi)^3}\frac{1}{4c_s^2p_u}2(p_x^2-p_y^2)|f_{\tilde{p}}|^2\\&=\int\frac{\mathrm{d}^3\tilde{\boldsymbol{p}}}{(2\pi)^3}\frac{1}{4c_s^2p_u}2(p_x^2-p_y^2)f^{\star\prime}_{\tilde{p}}f_{\tilde{p}}+\text{h.c.}\end{aligned} \tag{F.9}$$

At this point by adding up these contributions we can collect the terms with $\partial_uf^\star_{\tilde{p}}$. They are

$$\partial_uf^\star_{\tilde{p}}\left[(1-c_s^2)\partial_u^2f_{\tilde{p}}+f_{\tilde{p}}\left((1-c_s^2)c_s^{-2}p_s^2+c_s^2(p_x^2+p_y^2)-\frac{1}{\Lambda^2}\dot{\gamma}^b_{ij}(u)\epsilon^+_{ij}(p_x^2-p_y^2)\right)\right]+\text{h.c.} \tag{F.10}$$

This equation is solved by $f_{\tilde{p}}$: this is easily seen by comparing with equation (5.37).

F.2 $\tilde{m}_4^2$-operator

In this section we are going to use the same logic as the one in the previous section to verify the conservation of energy for $\tilde{m}_4^2$-operator. According to the Lagrangian (5.20), the components of the energy-momentum tensor are given by

$$T^0_{\ 0} = -\left[\frac{1}{4}\dot{\gamma}_{ij}^2 + \frac{1}{4}(\partial_k\gamma_{ij})^2 + \frac{1}{2}\dot{\pi}^2 + \frac{1}{2}c_s^2(\partial_i\pi)^2 - \frac{2}{\Lambda_\star^3}\dot{\gamma}_{ij}\partial_i\dot{\pi}\partial_j\pi\right] , \tag{F.11}$$

$$T^0_{\ i} = -\frac{1}{2}\dot{\gamma}_{kl}\partial_i\gamma_{kl} - \dot{\pi}\partial_i\pi + \frac{2}{\Lambda_\star^3}\partial_i\gamma_{kl}\partial_k\dot{\pi}\partial_l\pi - \frac{1}{\Lambda_\star^3}\partial_i\dot{\gamma}_{kl}\partial_k\pi\partial_l\pi , \tag{F.12}$$

$$T^i_{\ 0} = \frac{1}{2}\dot{\gamma}_{kl}\partial_i\gamma_{kl} + c_s^2\dot{\pi}\partial_i\pi - \frac{2}{\Lambda_\star^3}\ddot{\gamma}_{ij}\dot{\pi}\partial_j\pi , \tag{F.13}$$

$$\begin{aligned} T^i_{\ j} = {} & \frac{1}{2}\partial_i\gamma_{kl}\partial_j\gamma_{kl} + c_s^2\partial_i\pi\partial_j\pi - \frac{2}{\Lambda_\star^3}\ddot{\gamma}_{ik}\partial_j\pi\partial_k\pi \\ & + \frac{1}{2}\delta^i_j\left[\frac{1}{2}\dot{\gamma}_{kl}^2 - \frac{1}{2}(\partial_m\gamma_{kl})^2 + \dot{\pi}^2 - c_s^2(\partial_l\pi)^2 + \frac{2}{\Lambda_\star^3}\ddot{\gamma}_{kl}\partial_k\pi\partial_l\pi\right] . \end{aligned} \tag{F.14}$$

Notice that there is an extra term in $T^0_{\ 0}$ due to the interaction $\gamma\pi\pi$, unlike the case of m_3^3-operator. Since this new piece is second order in π, it can be approximated as $-\frac{2}{\Lambda_\star^3}\dot{\bar{\gamma}}_{ij}\partial_i\dot{\pi}\partial_j\pi$.

Let us first consider the RHS of (5.87), taking into account that $\dot{\bar{\gamma}}_{ij} = -\partial_z\bar{\gamma}_{ij}$ and $\partial_k\bar{\gamma}_{ij} = \delta_{kz}\partial_z\bar{\gamma}_{ij}$,

$$\begin{aligned} \int_{\partial\mathcal{V}_0}\left(T^{00} - T^{z0}\right)|_{u=|z_0|} &= T\left(\frac{1}{2}\dot{\pi}^2 + \frac{c_s^2}{2}(\partial_k\pi)^2 + c_s^2\dot{\pi}\partial_z\pi - \frac{2}{\Lambda_\star^3}\dot{\bar{\gamma}}_{ij}\partial_i\dot{\pi}\partial_j\pi\right)_{u=|z_0|} \\ &\equiv T\left(T_\pi^{00} - T_\pi^{z0} - \frac{2}{\Lambda_\star^3}\dot{\bar{\gamma}}_{ij}\partial_i\dot{\pi}\partial_j\pi\right)_{u=|z_0|} , \end{aligned} \tag{F.15}$$

where all terms involving only GWs added up to zero because of eq. (F.1). The LHS of (5.87) gets contributions only from $\Delta\gamma_{ij}$, as for the m_3^3-operator case. We have

$$T^{00} \supset \dot{\bar{\gamma}}_{ij}\left(\frac{v}{4\Lambda_\star^3}\partial_u^2 J_{ij}(u)\right) = \dot{\bar{\gamma}}_{ij}\epsilon^+_{ij}\left(\frac{v}{4\Lambda_\star^3}\partial_u^2\left\langle(\partial_x\pi)^2 - (\partial_y\pi)^2\right\rangle\right) \equiv v\tilde{\mathcal{F}}(u) . \tag{F.16}$$

Integrating T^{00} over $\partial\mathcal{V}_2 \cup \partial\mathcal{V}_1$ gives

$$\int_{\partial\mathcal{V}_2} T^{00}\,\mathrm{d}z - \int_{\partial\mathcal{V}_1} T^{00}\,\mathrm{d}z = 2T\int_{-|z_0|}^{0}\tilde{\mathcal{F}}(-z)\,\mathrm{d}z , \tag{F.17}$$

similarly to eq. (F.3). Using (F.15) and (F.17), eq. (5.87) becomes

$$\begin{aligned} 0 &= T\left(T^{00} - T^{z0}\right)|_{u=|z_0|} + 2T\int_{-|z_0|}^{0} \tilde{\mathcal{F}}(-z)\,\mathrm{d}z \\ &= T\left[\left(T^{00}_\pi - T^{z0}_\pi - \frac{2}{\Lambda_\star^3}\dot{\bar{\gamma}}_{ij}\partial_i\dot{\pi}\partial_j\pi\right)_{u=|z_0|} + \frac{1}{2\Lambda_\star^3}\int_0^{|z_0|}\dot{\bar{\gamma}}_{ij}(u)\epsilon^+_{ij}\partial_u^2\left\langle(\partial_x\pi)^2 - (\partial_y\pi)^2\right\rangle\,\mathrm{d}u\right] . \end{aligned} \tag{F.18}$$

Like in the previous section, one can verify this equation by taking a derivative with respect to $|z_0|$ (or equivalently u):

$$\partial_u\left[\langle T^{00}_\pi(u)\rangle - \langle T^{z0}_\pi(u)\rangle\right] - \frac{1}{\Lambda_\star^3}\ddot{\bar{\gamma}}_{ij}(u)\epsilon^+_{ij}\partial_u\left\langle(\partial_x\pi)^2 - (\partial_y\pi)^2\right\rangle = 0\ . \tag{F.19}$$

As we have shown before, the first term of LHS can be expressed as

$$\begin{aligned} &\partial_u\left[\langle T^{00}_\pi(u)\rangle - \langle T^{z0}_\pi(u)\rangle\right] = \\ &= \int\frac{\mathrm{d}^3\tilde{\boldsymbol{p}}}{(2\pi)^3}\frac{1}{4c_s^2p_u}\left[\partial_u f^\star_{\tilde{p}}\left((1-c_s^2)\partial_u^2 f_{\tilde{p}} + f_{\tilde{p}}\left((1-c_s^2)c_s^{-2}p_s^2 + c_s^2(p_x^2+p_y^2)\right)\right) + \text{h.c.}\right] . \end{aligned} \tag{F.20}$$

Similarly, the second term can be rewritten as

$$-\frac{1}{\Lambda_\star^3}\ddot{\bar{\gamma}}_{ij}(u)\epsilon^+_{ij}\partial_u\left\langle(\partial_x\pi)^2 - (\partial_y\pi)^2\right\rangle = -\frac{1}{\Lambda_\star^3}\ddot{\bar{\gamma}}_{ij}(u)\epsilon^+_{ij}\int\frac{\mathrm{d}^3\tilde{\boldsymbol{p}}}{(2\pi)^3}\frac{1}{4c_s^2p_u}2(p_x^2-p_y^2)f^{\star\prime}_{\tilde{p}}f_{\tilde{p}} + \text{h.c.} \tag{F.21}$$

Adding up (F.20) and (F.21) together one therefore obtains

$$\partial_u f^\star_{\tilde{p}}\left[(1-c_s^2)\partial_u^2 f_{\tilde{p}} + f_{\tilde{p}}\left((1-c_s^2)c_s^{-2}p_s^2 + c_s^2(p_x^2+p_y^2) - \frac{1}{\Lambda_\star^3}\ddot{\gamma}^b_{ij}(u)\epsilon^+_{ij}(p_x^2-p_y^2)\right)\right] + \text{h.c.} \tag{F.22}$$

This coincides with the equation of motion for $f_{\tilde{p}}$, which can be obtained by expanding eq. (5.23) in Fourier modes.

G | Deviation from cubic Galileon in the stability analysis

The discussion of Sec. 6.2 assumes that the relevant cubic non-linearities are of the form $\tilde{m}_3^3 = -m_3^3$ (as is the case for cubic Galileon interactions). However, one could wonder whether a different choice of operators can make the theory stable around GW backgrounds. To address this possibility, in this appendix we are going to study the stability properties of theories that deviate from the cubic Galileon for the case $c_s < 1$. For concreteness, we focus on the case $\tilde{m}_3^3 = -m_3^3(1+\eta)$, with $\eta \neq 0$ parametrizing such deviations.

The leading non-linear interactions of π arising from this coupling are again cubic. The Lagrangian takes then the form

$$\mathscr{L} = -\frac{1}{2}\bar{\eta}_{\mu\nu}\partial^\mu\pi\partial^\nu\pi - \frac{1}{\Lambda_{\rm B}^3}\Box\pi(\partial\pi)^2 + \frac{\eta}{\Lambda_{\rm B}^3}\ddot{\pi}(\partial_i\pi)^2 + \Gamma_{\mu\nu}\partial^\mu\pi\partial^\nu\pi - \frac{\Lambda_{\rm B}^3}{2}\pi\Gamma_{\mu\nu}\Gamma^{\mu\nu} \; . \tag{G.1}$$

Notice that the terms proportional to η do not change the couplings with $\Gamma_{\mu\nu}$ and $\Gamma_{\mu\nu}\Gamma^{\mu\nu}$: the operator $(\delta g^{00})^2\delta K$ yields interactions between γ_{ij} and π that only start at quartic order. Straightforwardly, the equation of motion reads

$$\bar{\Box}\pi - \frac{2}{\Lambda_{\rm B}^3}\left[(\partial_\mu\partial_\nu\pi)^2 - \Box\pi^2\right] + \frac{2\eta}{\Lambda_{\rm B}^3}\left[(\partial_i\dot{\pi})^2 - \ddot{\pi}\nabla^2\pi\right] - 2\,\Gamma_{\mu\nu}\partial^\mu\partial^\nu\pi - \frac{\Lambda_{\rm B}^3}{2}\Gamma_{\mu\nu}\Gamma^{\mu\nu} = 0 \; . \tag{G.2}$$

Following the discussion of Sec. 6.2.2, we have that for $c_s < 1$ the solution is a function of u only: $\varphi = \varphi(u)$. In this case one can check that there are no contributions proportional to η, hence the above equation reduces to (6.23).

At this stage we can compute the kinetic matrix $Z^{\mu\nu}$ for perturbations $\delta\pi$. By expanding (G.1) at quadratic order we obtain

$$Z^{\mu\nu} = -\frac{1}{2}\bar{\eta}^{\mu\nu} + \Gamma^{\mu\nu} + \frac{2}{\Lambda_{\rm B}^3}\left[\partial^\mu\partial^\nu\hat{\pi} - \eta^{\mu\nu}\Box\hat{\pi}\right] + \eta\mathcal{R}^{\mu\nu} \; , \tag{G.3}$$

where the matrix $\mathcal{R}^{\mu\nu}$ is defined as

$$\mathcal{R}^{\mu\nu} \equiv \frac{1}{\Lambda_{\rm B}^3} \begin{pmatrix} \nabla^2\hat{\pi} & -\partial_j\dot{\hat{\pi}} \\ \\ -\partial_i\dot{\hat{\pi}} & \ddot{\hat{\pi}}\,\delta_{ij} \end{pmatrix} . \tag{G.4}$$

This expression for $Z^{\mu\nu}$ should be compared with the case $\eta = 0$ of eq. (6.12).

Using the u-dependent solution (6.23) for $\varphi(u)$ and the change of variables of eq. (6.19), one finds the non-vanishing components of $Z^{\mu\nu}$, that are given by

$$\begin{aligned} Z^{00} &= \frac{1}{2} + (2+\eta)\varphi''(u) \ , \\ Z^{11} &= -\frac{1}{2}c_s^2 + \Gamma^{11} + \eta\varphi''(u) \ , \\ Z^{22} &= -\frac{1}{2}c_s^2 + \Gamma^{22} + \eta\varphi''(u) \ , \\ Z^{33} &= -\frac{1}{2}c_s^2 + (2+\eta)\varphi''(u) \ , \\ Z^{03} &= Z^{30} = (2+\eta)\varphi''(u) \ . \end{aligned} \tag{G.5}$$

Now we can see that with this choice of solution the contributions arising from η-term are the same in all the entries: $\eta\varphi''$. To avoid gradient instabilities along x, one requires $\eta > 0$ and sufficiently large. However, with this choice one clearly encounters ghosts. Hence, for any value of η the system remains unstable.

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