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THESIS

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MAGISTER PHILOSOPHIAE

"LYAPOUNOV CHARACTERISTIC EXPONENTS AND ATTENUATIONS IN RANDOMLY DISTORTED OPTICAL WAVEGUIDES"

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A K N O W L E D G E M E N T S

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INTRODUCTION

In this thesis, we discuss the influence of random geometrical microdeformations on the attenuation of light signals travelling inside an optical fiber.

By geometrical microdeformations we mean microbendings of the axis of the fiber and slight variations of the diameter of the fiber.

Very little is known, experimentally, on such a subject (*). In particular, no statistic on the microdeformations of actual fibers is available.

The only results in the literature are numerical ones, such as those in [20]. (see §3.4 for a discussion of these results).

Here, we discuss a particular, disctretized model of a randomly microdeformed fiber, in which we require that the "allowed" geometrical deformation exibit some kind of continuity. This requirement creates correlation between the stochastic variables defining the model.

Most of the works on disordered systems available in the literature are about the asymptotic behaviour fo the solutions of a discretized Schroedinger equation with random potentials indipendentely distributed at each lattice site. This problem has been introduced by P.W. Anderson [2], and it is usually referred to as Anderson's localization; it is connected with the diffusion properties of an electron in a solid presenting random impurities.

Two viewpoints are possible in studying localization: one static, the other dynamic: in the first one studies the spectrum of the Hamiltonian $H(V)$ and tries to predict under which conditions, for almost all V, it is purely point or continuous; in the second, one studies the time-evolution of a solution,

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(*) In fact, the amplitude of the deformations that can cause the losses measured on actual fibers are too small to be detected by optical means, the only available.

and in particualr the probability of finding the associated particle in a finite region of the space. It is well known [26], that these two approaches are equivalent, the continuous spectrum corresponding to solutions that diffuse at infinity, the pure point spectrum to solutions that remain essentially confined in a finite region of the space during the evolution. In the literature, these are referred to as extended or localized states.

In the picture of the localization problem associated to Anderson's model, the following results have been proved (for almost all potentials V):
In dimension 1:
a) the spectrum of the Hamiltonian H(V) does not contains the absolutely continuous component (see [23]);
b) the spectrum is purely point and the relative eigenfunctions exponentially localized (see [13], [7], [15], [8]);
c) the rates of exponential decrease is at least equal to the Lyapounov Characteristic Exponent (see §2.1 for a definition) (see [14], [7]);
d) in presence of an electric field, arbitrairily small, the spectrum becames absolutely continuous (see [4]).
In higher dimensions:
e) in dimension 3, as the disorder is varied, a transition between localized and extended states has been numerically observed (see [24]);
f) in two and three dimensions, for large disorder or high energy, all eigenstates are exponentially localized and the spectrum is purely point.(see [9], [10]);
g) in a two-dimensional strip it has been shown (see [16]) that all the states are exponentially localized; the same behaviour seems to be exibited by the whole bidimensional system (see [1]).

Also our problem is a localization one, though different from the one discussed above. We have tackled such a problem through the study of the Lyapounov Characteristic exponents of the product of the random matrices associated with our model. This technique is the same one which has been used to obtain the results a),c),e),g).

In particular, we will show how it is possible to evaluate numerically the LCE's of our model. This is a first step toward an understanding of the behaviour of our disordered system. Also, it could give an hint to which kind af a statistic actual fiber could obey to.

The exposition is organized as follows:
In chapter 1 we describe our discretized model of a randomly distorted fiber and define the associated transfer matrices.
Chapter 2 is devoted to the exposition of the theory of Lyapounov Characteristic Exponents, together with their main properties and the description of the method which allows the computation of all of them.
In Chapter 3 we give the detailed construction of the random transfer matrices connected with our discretized equation in the special case of a 2-dimensional model and indicate how explicit numerical calculations of all the LCE's of the family of such random matrices can be carried out. A brief discussion of the numerical results obtained by Marcuse is also presented.

We want to point out that this thesis is just a first report on a ongoing joint research with Prof. F. Bentosela. Our aim, after the numerical calculations will be carried out, is to see whether results analogous to those proved for Anderson's model can be proved in this different situation.

Chapter 1

DISCRETIZED MODEL OF A DISTRORTED FIBER.

In fig. 1 is showed the transversal section of a coaxial step-index optical fiber. Such an optical waveguide is employed for long distance communications and may tipically be a few kilometers in length. It consists of a core of a dielectric material with refractive index n_1, sorrounded by a cladding of another dieletric material whose refractive index n_2 is less than n_1.

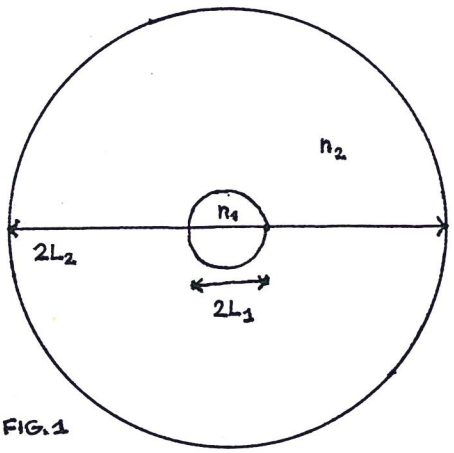

FIG.1

All dieletric waveguides support, in addition to the infinite number of cladding modes, a finite number of guided modes confined in the core region and transmitted by means of total internal reflections occurring at the core/cladding separation surface.

The diameter of the fiber core can usually vary between few micrometers and few dozens of micrometers, the external diamater of the cladding between one and two hundreds of micrometers. In most optical fibers these parameters are chosen so as to allow only one guided mode (single-mode fibers), to avoid pulse distorsions caused by the different group delays of the modes exhibited by multi-modes fibers. The guided mode propagates without losses along the axis of the fiber, provided that the waveguide structure is free of imperfections. But actual waveguides are never perfect, there are always index inhomogeneities or geometrical defects as microbends of the fiber axis or slight random changes of the core width. These deformation of the fiber structure determine the coupling between the different modes of the waveguide. This coupling causes a power transfer from the guided mode to the cladding modes and, since the power carried awayfrom the core region into the cladding is dissipated by the lossy coating sorrounding the fiber, results in waveguide losses.

In what follows we shall be concerned in the study of attenuations dues to geometrical imperfections of the fiber structure as random microbends of the

geometrical imperfections of the fiber structure as microbends of the of the axis or random core diameter variations.

Experiments performed on single-mode optical fibers seem in fact to indicate that are the random fluctuations in the geometry of the fiber which are mainly responsible for attenuations.

The losses of single-mode step-index fibers experimentally measured are of the order of few decibel per kilometer. The microscopical size of microbends that can cause such losses (on the order of nanometers) results in the difficulty to obtain experimental informations about the geometrical defects of actual fibers.

For a general discussion of optical fibers, see [19].

§ 1.1 Distorted waveguides.

In fig. 2 is shown a coaxial (cylindrical) step-index fiber with an undistorted refractive index profile. In the case of such a perfectly cylindrical fiber, the refractive index function n(x,y,z) does not depend on the x-coordinate, here assumed to be the axis of the fiber. It is of the form:

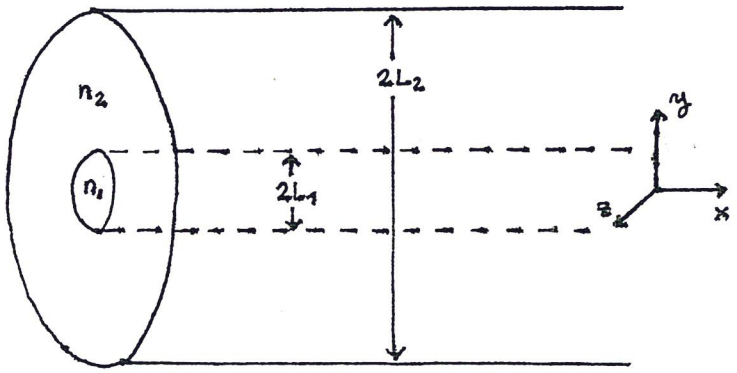

FIG.2

$$(1.2.1)\qquad n(y,z) = \begin{cases} n_1 & \text{if } y^2+z^2 \le L_1 \\ & \forall x \\ n_2 & \text{if } L_1 < y^2+z^2 \le L_2 \end{cases}$$

where n_1 and n_2 are the refractive indices of the core and cladding materials, and L_1, L_2 their radii.

As we have already said in the introduction, we will be interested in studying attenuations due to geometrical imperfections only. We can essentially distinguish between two kinds of geometrical defects: microbendings of the fiber axis and variation of the core and cladding diameters. We will treat

separately each of these cases.

To treat the microbending case, we shall allow the center of the fiber to perform slight random variations of its position with respect to the x-axis, while the diameters L_1 and L_2 will be kept fixed. The model correspons to a refractive index function defined as:

$$(1.1.2) \qquad n(y-y_0(x), z-z_0(x)) = \begin{cases} n_1 & \text{if } (y-y_0(x))^2 + (z-z_0(x))^2 \leq L_1 \\ n_2 & \text{if } L_1 < (y-y_0(x))^2 + (z-z_0(x))^2 \leq L_2 \end{cases}$$

where $y_0(x)$ and $z_0(x)$ are the y and z-coordinates of the center of the fiberat some poin x.

In the second case we shall fix the position of the center of the fiber to be fixed on the x-axis, while the diameters L_1 and L_2 will vary randomly with x. We shall assume for simplicity that such random width variations do not affect the rotational symmetry of the fiber with respect to the x-axis. The function n(x,y,z) will then be:

$$(1.1.3) \qquad n(x) = \begin{cases} n_1 & \text{if } (y^2+z^2) \leq L_1(x) \\ n_2 & \text{if } L_1(x) < (y^2+z^2) \leq L_2(x) \end{cases}$$

As we have mentioned above, these two kinds of geometrical random irregularities seem to be the main cause of waveguide losses.

In either cases we shall assume some kind of probability distributions for the random fiber defects, and we will study the behaviour of the fiber attenuations as the "disorder" is varied.

As it is usually the case for any kind of disordered system, we can model the distorted fiber with an ensemble Ω consitued by an infinite number of sample systems ω, each one corresponding to a particular choice of the fiber geometrical parameters (position of the centre of the fiber or values of the fiber diameter) at each point, together with the recourrence probability of each sample.

Once a probability measure has assigned on the space of all allowed geometrical shapes of the distorted fiber (all possible configurations ω of the system), one is interested in getting results on the behaviour of a "typical" fiber, i.e. results holding for almost all configurations ω with respect to the assigned probability measure.

§ 1.2. Discretized equation and random transfer matrices.

The light signal travelling inside the waveguide is a solution of the wave equation:

(1.2.1) $$\frac{\partial^2}{\partial t^2}\tilde{\Psi}(x,y,z,t) - v^2(x,y,z)\,\Delta\tilde{\Psi}(x,y,z,t) = 0$$

is which the propagation velocity is $v = c/n$, where c is the the light propagation velocity in vacuum and $n(x,y,z)$ is one of the refractive index functions defined in the previous section (see (1.1.2) and (1.1.3)) for $y^2+z^2 \leq L_2(x)$. We will assume $\tilde{\Psi}(x,y,z)$ to be zero for $y^2+z^2 \geq L_2(x)$; i.e. we will assume Dirichlet boundary conditions on the boundary of the region $\{(x,y,z) \in \mathbb{R}^3 \mid y^2+z^2 \leq L_2(x)\}$. One way, particularly suitable for numerical calculations, to study such an equation is to discretize it, i.e. to consider it defined not on $\mathbb{R}^3$, but on the three dimensional lattice (of lattice side $a \in \mathbb{R}$):

(1.2.2) $$\mathbb{Z}^3_a \equiv \{\underline{x} \in \mathbb{R}^3 \mid \underline{x} = (m_1 a, m_2 a, m_3 a);\ m_1, m_2, m_3 \in \mathbb{Z}\}$$

One then has to deal with the following discrtied operator, acting on the Hilbert space $\ell^2(\mathbb{Z}^3_a)$:

(1.2.3) $$H\Psi = -\frac{c^2}{n^2}\,\Delta_d\Psi$$

(1.2.4) $$\Psi \in \ell^2(\mathbb{Z}^3_a) = \{\Psi(\underline{m}),\ \underline{m} \in \mathbb{Z}^3_a \mid \sum_{\underline{m}\in\mathbb{Z}^3_a} |\Psi(\underline{m})|^2 < \infty\}$$

with scalar product $(\Phi|\Psi) = \sum_{\underline{m}\in\mathbb{Z}^3_a} \overline{\Phi}(\underline{m})\Psi(\underline{m})$, where Δ_d is the discretized Laplacian operator on $\ell^2(\mathbb{Z}^3_a)$:

(1.2.5) $$(\Delta_d\Psi)(\underline{m}) = -\frac{6}{a^2}\Psi(\underline{m}) + \frac{1}{a^2}\sum_{\underline{m}'\,n.n.\,\underline{m}} \Psi(\underline{m}')$$

the sum running over all $\underline{m}'$ nearest neighbours of $\underline{m}$ $\mathbb{Z}^3_a$. We want to look for solutions of the eigenvalues equation:

(1.2.6) $$H\Psi(m_1,m_2,m_3) = -\frac{c^2}{n^2(m_1,m_2,m_3)}\,\Delta_d\Psi(m_1,m_2,m_3) = E\Psi(m_1,m_2,m_3)$$

and we will choose Dirichlet boundary conditions on the external surface of the fiber's cladding.

we first rewrite the equation (1.2.6) esplicitely as:

$$(1.2.7)\quad \frac{1}{a^2}\Big[6\Psi(m_1,m_2,m_3) - \Psi(m_1+1,m_2,m_3) - \Psi(m_1-1,m_2,m_3) + \\ - \Psi(m_1,m_2+1,m_3) - \Psi(m_1,m_2-1,m_3) - \Psi(m_1,m_2,m_3+1) + \\ - \Psi(m_1,m_2,m_3-1)\Big] = \frac{n^2(m_1,m_2,m_3)}{c^2} E\, \Psi(m_1,m_2,m_3)$$

or

$$(1.2.8)\quad \frac{1}{a^2}\Big(2\Psi(m_1,m_2,m_3) - \Psi(m_1+1,m_2,m_3) - \Psi(m_1-1,m_2,m_3) + \\ + \Big[\Big(\frac{4}{a^2} - \frac{n^2(m_1,m_2,m_3)\,E}{c^2}\Big)\Psi(m_1,m_2,m_3) - \frac{1}{a^2}\big(\Psi(m_1,m_2+1,m_3) + \\ + \Psi(m_1,m_2-1,m_3) + \Psi(m_1,m_2,m_3+1) + \Psi(m_1,m_2,m_3-1)\big)\Big] = 0$$

The term in square braket in equation (1.2.8) corresponds (for each fixed m_1) to the operator:

$$(1.2.9)\quad H^{(m_1)} = -\Delta_d^{(1)} - E\frac{n^2}{c^2}$$

where $\Delta_d^{(1)}$ is the discretized Laplacian operator in the two dimensional lattice $\mathbb{Z}_a^2$, and $H^{(m_1)}$ is a two-dimensional Schroedinger operator, where the potential has the shape of an infinitely deep (this because of Dirichlet boundary conditions) cylindrical hole presenting a small cylindrical well at his bottom as showed in fig. 3.

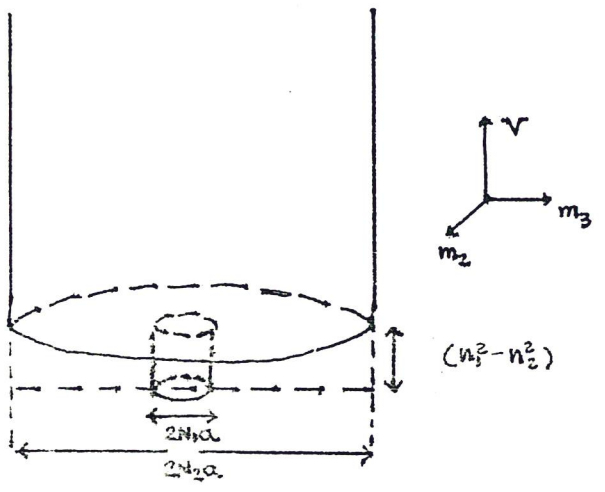

FIG. 3

Once we have obtained the solutions of the eigenvalue equation

(1.2.10) $$H^{(m_1)}\phi_j^{(m_1)}(m_2,m_3,E) = \lambda_j^{(m_1)}(E)\,\phi_j^{(m_1)}(m_2,m_3,E)$$

we can decompose all vectors $\Psi(m_1,m_2,m_3)$, m_1 being fixed, on the eigenvectors $\phi_j^{(m_1)}$ of $H^{(m_1)}$:

(1.2.11) $$\Psi(m_1,m_2,m_3) = \sum_j a_j(m_1)\,\phi_j^{(m_1)}(m_2,m_3)$$

where the sum runs over all the eigenvectors $\phi_j^{(m_1)}$ and

(1.2.12) $$a_j(m_1) = (\phi_j^{(m_1)}(m_2,m_3) \mid \Psi(m_1,m_2,m_3))$$

$(\ \mid\ )$ being the scalar product in $\ell^2(\mathbb{Z}_a^2)$ (we have omitted the E dependence to semplify the notation). Making use of the developement (1.2.11), the equation (1.2.8) becames:

(1.2.13) $$\frac{1}{a^2}\Big[2\sum_j a_j(m_1)\,\phi_j^{(m_1)}(m_2,m_3) - \sum_j a_j(m_1+1)\,\phi_j^{(m_1+1)}(m_2,m_3) + \\ + \sum_j a_j(m_1-1)\,\phi_j^{(m_1-1)}(m_2,m_3)\Big] + \sum_j a_j(m_1)\,\lambda_j^{(m_1)}\phi_j^{(m_1)}(m_2,m_3) = 0$$

Taking the scalar product of (1.2.13) with $\phi_k^{(m_1)}(m_2,m_3)$, one gets:

(1.2.14) $$\frac{2}{a^2}a_k(m_1) - \frac{1}{a^2}\sum_j a_j(m_1+1)\,K_{kj}(m_1,m_1+1) - \frac{1}{a^2}\sum_j a_j(m_1-1)\,K_{kj}(m_1,m_1-1) + \\ + a_k(m_1)\,\lambda_k^{(m_1)} = 0$$

where we have posed:

(1.2.15) $$K_{kj}(m_1,m_1\pm 1) = (\phi_k^{(m_1)} \mid \phi_j^{(m_1\pm 1)})$$

the K_{ij}'s depending on the energy E and onthe shape configuration ω of the fiber. We have thus obtained the following system of coupled equations:

(1.2.16) $$\Big[\frac{2}{a^2} + \lambda_k^{(m_1)}\Big]a_k(m_1) - \frac{1}{a^2}\sum_j\Big[K_{kj}(m_1,m_1+1)\,a_j(m_1+1) + \\ + K_{kj}(m_1,m_1-1)\,a_j(m_1-1)\Big] = 0$$

which can be rewritten into a matricial form as:

(1.2.17) $$D^{(m_1)}A(m_1) - K(m_1,m_1+1)A(m_1+1) - K(m_1,m_1-1)A(m_1-1) = 0$$

where $K = [K_{ij}]$, $A = [a_j]$, $D^{(m_1)} = [(a^2\lambda_j + 2)\delta_{ij}]$

Remark. 1.2.1. In the case of a perfect (non distorted) fiber, $K_{ij}(m_1, m_1+1) = \delta_{ij}$, $\forall m_1$, and the equations in (1.2.16) are no more coupled. The system (1.2.16) becames in this case:

$$\left[\frac{2}{a^2} + \lambda_j(E)\right] a_j(m_1) - \frac{1}{a^2} a_j(m_1+1) - \frac{1}{a^2} a_j(m_1-1) = 0$$

the solutions then being

$$a_j(m_1) = \exp(i k_j m_1 a)$$

the corresponding eigenvalues (not depending on m_1) being

$$\lambda_j(E) = -\frac{2}{a^2}(1 - \cos k_j a)$$

For $\lambda_j(E) > 0$, all solutions $a_j(m_1)$ are exponentially decreasing/increasing and the corresponding modes do not propagate. In fact the eigenstates of (1.2.6) are given by

$$\Psi_j(m_1, m_2, m_3, E) = \phi_j^{(m_1)}(m_2, m_3, E) \exp(i k_j m_1 a)$$

This implies that, when studying propagation (and attenuation) along the m_1 axis, one can neglet the components of the solution along the $\Phi_j(E)$ such that $\lambda_j(E) > 0$.

Remark 1.2.2. In analogy with the simple case of a perfect fiber, to perform numerical calculations we shall make the assumption that all the modes of a distorted fiber corresponding to eigenvalues $\lambda_j^{(m_1)}(E)$ greater than some constant (to be precised) can be negleted. In such a way we will be allowed to retain only a smaller number of coupled equations from system (1.2.16)

Remark 1.2.3. We remark that the qualitative results one can obtain in the study of a disordered system will not depend on the continuum or discrete character of the equation describing it. In particular, since the eigenvalues of the bottom of the spectrum of $H^{(m_1)}$, if the lattice shift a is taken to be sufficiently small, are independent from a, the number of modes to be taken into account in ourt computation (i.e. the number of eigenvectors $\phi_j^{(m_1)}(E)$ of $H^{(m_1)}$ such that their corresponding eigenvalues $\lambda_j^{(m_1)}(E)$ are smaller then some constant) will not depend on a, too.

Remark 1.2.4. In the case of microbends of the axis of the fiber two eigenvalue equations (1.2.10) corresponding to two neighbouring sections, and the two sets of corresponding eigenfunctions are just translated one with respect to the other, while the eigenvalues $\lambda_j(E,m_1)$ are independent from m_1. This fact implies that the diagonal matrix $D(m_1)$ does not depent on m_1, while $K(m_1,m_1+1)$ will depend only on the relative positions of the fiber at the sections m_1 and m_1+1. We will suppose that the variations of the position of the center of the fiber occur only in the m_2-direction, and, denoting with $\gamma(m_1)$ the value of its m_2-coordinate at the m_1-section we will also assume that $\gamma(m_1+1)$ can take (with certain probabilities) only the values $\gamma(m_1)$, $\gamma(m_1)+1$ and $\gamma(m_1)-1$. Since each matrix element K_{kj} involves a scalar product between the k-th and the j-th eigenvector of $H^{(m_1)}$, taken at neighbours sections of the waveguide, each of them can take only three different values:

$$K_{kj}(m_1,m_1+1) = \begin{cases} \delta_{kj} & \text{if } \gamma(m_1+1)=\gamma(m_1) \\ K^+_{kj} & \text{if } \gamma(m_1+1)=\gamma(m_1)+1 \\ K^-_{kj} & \text{if } \gamma(m_1+1)=\gamma(m_1)-1 \end{cases}$$

$K^{\pm}_{kj}$ not depending on m_1. The matrices $K^{\pm}$ will be described explicitely, together with their properties, in the third chapter, when we shall treat a two dimensional case.

On the other hand, in the random width's case, both $D(m_1)$ and the sets of eigenvectors of $H^{(m_1)}$ will depend on the choosen section m_1.

Denoting with $w(m_1)$ the waveguide diameter at m_1, we will still have

$$K_{kj}(m_1,m_1+1) = \begin{cases} \delta_{kj} & \text{if } w(m_1+1)=w(m_1) \\ K^+_{kj} & \text{if } w(m_1+1)=w(m_1)+\delta w \\ K^-_{kj} & \text{if } w(m_1+1)=w(m_1)-\delta w \end{cases}$$

but now $K^{\pm}_{kj} = K^{\pm}_{kj}(-W(m_1))$. This means that, in this second case, there will be as many different matrices K as twice the number of the possible different widths of the diameter, plus the identity matrix.

Equation (1.2.17) provides a recurrence formula for calculating $A(m_1+1)$ once $A(m_1)$ and $A(m_1-1)$ are known:

(1.2.18)
$$\begin{pmatrix} A(m_1+1) \\ A(m_1) \end{pmatrix} = S_{m_1+1} \begin{pmatrix} A(m_1) \\ A(m_1-1) \end{pmatrix} =$$
$$= \left(\begin{array}{c|c} K^{-1}(m_1, m_1+1)\, D(m_1) & -K^{-1}(m_1, m_1+1)\, K(m_1, m_1-1) \\ \hline \mathbb{1} & 0 \end{array}\right) \begin{pmatrix} A(m_1) \\ A(m_1-1) \end{pmatrix}$$

Starting with an initial condition $A(0)$, $A(-1)$, the solution at the point m_1+1, $A(m_1+1)$ is then obtained by iterated applications of the transfer random matrices S_n, $n=1,\ldots,m_1+1$ $(S_n = S_n(E,\omega))$:

(1.2.19)
$$\begin{pmatrix} A(m_1+1) \\ A(m_1) \end{pmatrix} = S_{m_1+1} S_{m_1} S_{m_1-1} \cdots S_1 \begin{pmatrix} A(0) \\ A(-1) \end{pmatrix}$$

Denoting as $\mathcal{S}_N$ the N-th product of the matrices S_m:

(1.2.20)
$$\mathcal{S}_N(\omega) = \prod_{m=1}^{N} S_m$$

the asymptotic behaviour of our solution is given by the behaviour of

(1.2.21)
$$\| \mathcal{S}_N(\omega)\, A^{(0)} \| \qquad \text{as } N \to \infty$$

$A^{(0)}$ denoting the initial condition $\begin{pmatrix} A(0) \\ A(-1) \end{pmatrix}$.

In particular, as we shall discuss in the following chapter, the relevant quantities to be investigated are the so called Lyapounov Characteristic

Exponents (LCE's) of the product of the matrices $S_m(\omega)$, defined as:

$$(1.2.22)\qquad \chi(\omega, A^{(0)}) \equiv \lim_{N\to\infty} \frac{1}{N} \log \langle A^{(0)} | \mathcal{S}_N^* \mathcal{S}_N A^{(0)} \rangle$$

their existence, for μ-almost all $\omega \in \Omega$, being guaranteed by Oseledec's non commutative ergodic theorem. The next chapter is therefore devoted to this subject.

Chapter 2

LYAPOUNOV CHARACTERISTIC EXPONENTS.

As we have seen in the preceding chapter, we are interested in the behaviour of infinite products of $n \times n$ random matrices, depending on $\omega \in \Omega$, Ω a probability space.

A very useful tool in studying this kind of objects is the theory of Lyapounov Characteristic Exponents (LCE), first introduced by Lyapounov [18] for dynamical systems and later extended, essentially through Oseledec's theorem [22] to ergodic systems. This chapter will be devoted to expose this theory.

In §2.1 we will define LCE's for infinite products of matrices and state their most important properties. §2.2 will be devoted to Oseledec's theorem, which we will state in the improved version given by Raghunathatan [25]; we will also discuss how LCE's can be used to obtain results on the behaviour of disordered systems. In the last section we will state a theorem by Benettin et al. [3], on which our numerical calculations of the LCE's will be based.

§ 2.1. Lyapounov Caracteristic Exponents: definition and basic properties.

Let $M(n;\mathbb{K}) = \{(a_{ij})_{1 \le i,j \le n} : a_{ij} \in \mathbb{K}\}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let

$$S : \mathbb{Z} \to M(n;\mathbb{K}) \qquad i \mapsto S_i \in M(n;\mathbb{K})$$

and set:

$$\mathcal{S}_N = \prod_{i=1}^{N} S_i \tag{2.1.1}$$

We are interested in the behaviour of $\mathcal{S}_N$ for $N \to \infty$. If

$$\limsup_{n \to \infty} \frac{1}{n} \log \| S_n \| < \infty \tag{2.1.2}$$

then, $\forall v \in \mathbb{K}^n$, $v \neq 0$, we can define:

$$(2.1.3) \quad \chi(S,v) = \limsup_{N\to\infty} \frac{1}{N} \log \|S_N v\| < \infty$$

(the existence of vectors $v \neq 0$ such that $\chi(S,v) = -\infty$ being not excluded). The number $\chi(S,v)$ is called the Lyapounov Caracteristic Exponent (LCE) of the vector v with respect to the family of matrices $\{S_j\}_{j\in\mathbb{Z}}$. The definition (2.1.3) is extended to the zero vector, $0 \in \mathbb{K}^n$, setting $\chi(S,0) = -\infty$.

From definition (2.1.3) easily follows:

$$(2.1.4) \quad \chi(S, v+w) \leq \max\{\chi(S,v), \chi(S,w)\}$$

and

$$(2.1.5) \quad \chi(S, \alpha v) = \chi(S,v)$$

By property (2.1.5), we can consider $\chi(S,v)$ as a function defined on the set of the one-dimensional subsets of $\mathbb{K}^n$. From properties (2.1.4),(2.1.5) it follows that, $\forall \alpha \in \mathbb{K}$, the set

$$\{v \in \mathbb{K}^n \mid \chi(S,v) \leq \alpha\}$$

is a vector subspace of $\mathbb{K}^n$.

The functions $\{\chi(S,v)\}_{v\in\mathbb{K}^n}$ can take at most n different values (see [18]) - including $-\infty$- say:

$$(2.1.6) \quad \chi_1 > \cdots > \chi_m \qquad 1 \leq m \leq n$$

Let

$$(2.1.7) \quad L_i = \{v \in \mathbb{K}^n \mid \chi(S,v) \leq \chi_i\}$$

($L_i = L_i(S)$). By (2.1.6), one has:

$$(2.1.8) \quad \{0\} \equiv L_{m+1} \subset L_m \subset \cdots \subset L_1 = \mathbb{K}^n$$

$$(2.1.9) \quad L_{i+1} \neq L_i \qquad 1 \leq i \leq m$$

and

(2.1.10) $\chi(S,v) = \chi_i$ iff $v \in L_i \setminus L_{i+1}$

The numbers χ_i, $1 \le i \le m$, are called LCE's of the family $\{S_j\}_{j\in\mathbb{Z}}$.

We will now introduce the notion of LCE's of higher order.

Let e^p be a p-dimensional subspace of $\mathbb{K}^n$, $1 \le p \le n$, and U a bounded open subset of e^p.

One then defines the LCE, of order p, of the subspace e^p, with respect to the family $\{S_j\}_{j\in\mathbb{Z}}$, to be the quantity:

(2.1.11) $$\chi(S,e^p) = \limsup_{N\to\infty} \frac{1}{N} \log \mathrm{Vol}^p(\mathcal{S}_N U)$$

where Vol^p is the p-dimensional euclidean volume. One can easily see that the definition (2.1.11) does not depend on the particular choice of $U \subset e^p$. In particular, let us choose U to be the open parallelepiped generated by p linearly independent vectors $\{e_i^p\}_{1\le i\le p}$ belonging to e^p. By Hadamard's inequality [28], the euclidean volume of U is less or equal than the product of the lenghts of its sides. We then get:

$$\begin{aligned}\chi(S,e^p) &\le \limsup_{N\to\infty} \frac{1}{N} \log \prod_{i=1}^{p} \|\mathcal{S}_N e_i^p\| \\ &= \limsup_{N\to\infty} \frac{1}{N} \sum_{i=1}^{p} \log \|\mathcal{S}_N e_i^p\| \\ &\le \sum_{i=1}^{p} \chi(S,e_i^p) < \infty \qquad (2.1.12)\end{aligned}$$

In the case one can replace, in the definition of the LCE's of any order $\limsup_{N\to\infty}$ by $\lim_{N\to\infty}$, then the LCE's are called the Exact LCE's of the family $\{S_j\}_{j\in\mathbb{Z}}$.

§ 2.2. Oseledec's non commutative ergodic theorem.

Let now (Ω, Σ, μ) be a probability space. Suppose we have a measurable map:

$$S : \mathbb{Z} \times \Omega \to M(n; \mathbb{K})$$
$$(i, \omega) \mapsto S_i(\omega)$$

$\forall \omega \in \Omega$, we can define (see §2.1) the LCE's of the family of random matrices $\{S_j(\omega)\}_{j \in \mathbb{Z}}$, that we shall denote as .

Oseledec's theorem assures the existence, for μ-almost every $\omega \in \Omega$, of the exact LCE's of $\{S_j(\omega)\}_{j \in \mathbb{Z}}$. Raghunatan's [25] version of Oseledec's theorem is the following:

Theorem 2.2.1. (Noncommutative Ergodic Theorem).
Let $\theta : \Omega \to \Omega$ be a measure preserving transormation (i.e. $(\theta\mu)(B) = \mu(\theta^{-1}B) = \mu(B)$, $\forall B$ measurable subset of Ω) and let $\mathcal{S} : \mathbb{Z} \times \Omega \to M(n; \mathbb{K})$ be a meausurable map s.t. $\forall p, q$, $q \geq 0$

$$\mathcal{S}(p, \theta^q \omega)\mathcal{S}(q, \omega) = \mathcal{S}(p+q, \omega) \tag{2.2.1}$$

If

$$\int \log^+ \|\mathcal{S}(1, \omega)\| \, d\mu(\omega) < \infty \tag{2.2.2}$$

then there exists a measurable subset $\Omega' \subset \Omega$, with $\mu(\Omega') = 1$, such that, $\forall \omega \in \Omega'$, the limits

$$\lim_{N \to \infty} \frac{1}{N} \log \|\mathcal{S}(N, \omega) v\|$$

exist $\forall v \in \mathbb{K}^n$.

Remark 2.2.2. In its original form, as stated in [22], Oseledec's theorem is proved under the additional hypothesis: 1°) that the map θ and the matrices $\mathcal{S}(1, \omega)$, $(\forall \omega \in \Omega)$, are inverible, and 2°) $(\mathcal{S}(1, \omega))^{\pm 1} \in L_1(\Omega, \mu)$.
The improved result obtained in [25] uses a weaker result by [11].

We will now clarify the relationship between Oseledec's theorem and the exact LCE's.

Let

$$\mathcal{S}(p,\omega) = \mathcal{S}_p(\omega) = S_p(\omega)\cdots S_1(\omega)$$

with this choice for $\mathcal{S}$, Oseledec's theorem proves the existence of the exact LCE's under the following hypothesis:

(2.2.3) $\exists\,\theta : \Omega \to \Omega$ meausure preserving transformation

(2.2.4) $S_{p+q}(\omega) = S_p(\theta^q\omega) \qquad \forall p \in \mathbb{Z}\,,\ \forall q \geq 0$

(2.2.5) $\int \log^+ \| S_1(\omega) \|\, d\mu(\omega) < \infty$

In this case, the exact LCE's are θ-invariant. If the measure μ is θ - ergodic (*), then these limits are constants for μ-almost all $\omega \in \Omega$. [17]

Let us now consider a discretized disordered system. With this we mean that our probability space is the set of all possible configurations of the system under consideration. We will take $\omega \in \Omega$ to be a function

$$\omega : \mathbb{Z}_a^d \to \mathbb{R}^n$$

defining the configuration $\omega \equiv \{\omega(\underline{m})\}_{\underline{m}\in\mathbb{Z}_a^d}$. For example, $\omega(\underline{m})$ could be the value of some random potential in the lattice site $\underline{m}$ (this is the case when one is studying Schroedinger equation with random potentials); in our case, $\omega(\underline{m}) = \omega(m_1)$ will be the position of the fiber's centre $\gamma(m_1)$ and/or the diameter of the fiber's core $w(m_1)$ in the plane $\underline{m}\cdot\underline{e}_1 = m_1$ assumed to be the plane perpendicular to the fiber axis.

It is now necessary to define a probability measure μ on Ω. We will require that such a measure makes the functions $\Omega \ni \omega \mapsto \omega_i(\underline{m}) \in \mathbb{R}$ μ-measurable (i.e. that $\omega \mapsto \omega_i(\underline{m})$ are stochastic variables). The standard way to define such a measure is to assign finite dimensional probability

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(*) A probability measure μ, invariant under a transformation θ, is said to be θ-ergodic iff the only sets which are θ-invariant are sets of μ-measure 0 or 1, or, equivalentely, if the only functions which are θ-invariant are the constants.

distributions to the values $\omega_i(\underline{m})$. It is then a standard procedure (see for example [29]) the construction of the probability space (Ω, Σ, μ). Suppose the probability measure μ has been assigned on Ω. Then the probability distribution of a stochastic variable ξ (i.e. $\xi : \Omega \to \mathbb{R}$ is measurable), is:

$$(2.2.6) \qquad \mu_\xi(B) = \mu\{\omega \in \Omega \mid \xi(\omega) \in B\}$$

where B is some Borel subset of $\mathbb{R}$.

Let ξ_1, ξ_2 be stochastic variables. Let

$$(2.2.7) \qquad \mu_{\xi_1\xi_2}(B_1 \times B_2) = \mu\{\omega \in \Omega \mid \xi_1(\omega) \in B_1 \text{ and } \xi_2(\omega) \in B_2\}$$

be their joint distribution. If

$$(2.2.8) \qquad \mu_{\xi_1\xi_2}(B_1 \times B_2) = \mu_{\xi_1}(B_1)\mu_{\xi_2}(B_2)$$

$\forall B_1, B_2$ Borel subsets of $\mathbb{R}$, then the two random variables ξ_1, ξ_2 are said to be independent.

Besides the configuration space Ω , our discretized disordered system will also consists of a transfer equation, of the kind of (1.2.18), involving products of random transfer matrices $S_i(\omega)$, $i \in \mathbb{Z}$, $\omega \in \Omega$.

If $S_i(\omega) = S(\omega(i\underline{e}_1))$, (which is, for example, true if $\omega_i(\underline{m})$ is independent from $\omega_j(\underline{m}')$, for $\underline{m} \neq \underline{m}'$), then condition (2.2.4) implies:

$$(2.2.9) \qquad S(\omega((p+q)\underline{e}_1)) = S((\theta^q\omega)(p\underline{e}_1))$$

Defining, $\forall q \in \mathbb{Z}$, the transformation $\theta : \Omega \to \Omega$, requested by Oseledec's theorem, to be the lattice translation along the $\underline{e}_1$ axis, i.e.

$$(2.2.10) \qquad \theta\underline{m} = \underline{m} - \underline{e}_1 \qquad \underline{m} \in \mathbb{Z}_a^d$$

acting on $\omega \in \Omega$ as:

$$(2.2.11) \qquad \theta\omega(\underline{m}) = \omega(\underline{m} + \underline{e}_1)$$

then (2.2.9) is automatically satisfied, and Oseledec's theorem will hold under the hypothesis that the measure μ is invariant under translations along the $\underline{e}_1$ axis and (2.2.5). In the case in which the $\omega_j(\underline{m})$ are mutually independent

random variables, with a common distribution law, it is immediate to see that the probability measure μ is invariant under translations and ergodic. Hypothesis (2.2.5) is then satisfied taking a bounded probability distribution for the transfer matrices $S_j(\omega)$(probability distribution that will derive from the probability distribution of the random variables $\omega_j(\underline{m})$). Actually, the invariance property of the measure (2.2.3) requested by Oseledec's theorem and hypothesis (2.2.5) are much less restrictive then the situation of independence, and are fulfilled by more general classes of variables, for example in the case in which they behave as an ergodic Markov chain. (This, in particular, will be our case in chapter 3).

The proof of theorem 2.2.1 consists then in showing that the matrices $(\mathcal{S}_N^*(\omega)\,\mathcal{S}_N(\omega))^{1/2N}$ converge, as $N \to \infty$, to an asymptotic matrix Λ(Oseledec's matrix), for each ω belonging to a subset $\Omega' \subset \Omega$, with $\mu(\Omega') = 1$:

$$(2.2.12) \qquad \Lambda = \lim_{N\to\infty} (\mathcal{S}_N^*\,\mathcal{S}_N)^{1/2N}$$

Let us denote by v_j the eigenvector of Λ , corresponding to the eigenvalue λ_j. Then $\lambda_j = \exp(\chi(\omega, v_j))$, $\chi(\omega, v_j)$ being the exact LCE of the vector v_j , with respect to the family fo random matrices $\{S_i(\omega)\}$, $i \in \mathbb{Z}$. (see 2.1.3). In fact, $\Lambda v_j = \lambda_j v_j$ implies:

$$\lambda_j = \frac{\langle v_j | \Lambda v_j \rangle}{\langle v_j | v_j \rangle} = \lim_{N\to\infty} \frac{\langle v_j | \mathcal{S}_N^* \mathcal{S}_N v_j \rangle^{1/2N}}{\langle v_j | v_j \rangle} = \lim_{N\to\infty} \frac{\| \mathcal{S}_N v_j \|^{1/N}}{\| v_j \|^2}$$

and then:

$$(2.2.13) \qquad \log \lambda_j = \lim_{N\to\infty} \frac{1}{N} \log \| \mathcal{S}_N v_j \| \equiv \chi(\omega, v_j)$$

THe spaces L_i, defined in (2.1.7) are then the eigenspaces of the Oseledec's matrix corresponding to the eigenvalues smaller than the i-th eigenvalue λ_i. If $V_1, \ldots, V_m$ are the eigenspaces of Λ corresponding to the eigenvalues $\lambda_1 = \exp(\chi_1), \ldots, \lambda_m = \exp(\chi_m)$, then $\forall\, v_j \in V_j$, the exact LCE's $(\chi_j)_{1 \le j \le m}$ of the family $\{S_i(\omega)\}_{i \in \mathbb{Z}}$, are given by:

(2.2.14) $\chi_j = \chi(\omega, v_j)$

For a detailed discussion of LCE's and Oseledec's theorem see also [17].

Remark 2.2.4 We want to point out that Oseledec's theorem appears as a generalization to (non-commutative) random matrices of the Birkhoff Ergodic theorem. In fact, let us suppose that θ is a measure preserving transformation on a probability space (Ω, Σ, μ), and let $f \in L^1(\Omega, \mu)$, $f(\omega) \in \mathbb{C}$. Define $\lambda_i \in L^1(\Omega, \mu)$ as $\lambda_i(\omega) = e^{f(\theta^i \omega)}$. Consider the family $\{Q_i(\omega)\}_{i \in \mathbb{Z}}$ of diagonal, unimodular, commuting matrices, where:

$$Q_i(\omega) = \begin{pmatrix} \lambda_i(\omega) & 0 \\ 0 & \lambda_i^{-1}(\omega) \end{pmatrix}$$

Such a family satisfies the hypothesis of Oseledec's theorem (more precisely, (2.2.3),(2.2.4),(2.2.5), the last one being nothing but the requirement $f \in L^1$). Then we know that exists, for μ-almost every $\omega \in \Omega$

$$\Lambda(\omega) = \lim_{N \to \infty} \left[\left(\prod_{i=1}^{N} Q_i \right)^* \left(\prod_{i=1}^{N} Q_i \right) \right]^{1/2N} = \lim_{N \to \infty} \left[\prod_{i=1}^{N} \begin{pmatrix} |\lambda_i|^2 & 0 \\ 0 & |\lambda_i|^{-2} \end{pmatrix} \right]^{1/2N}$$

which will be a matrix of the form:

$$\Lambda(\omega) = \begin{pmatrix} \exp(\chi(\omega)) & 0 \\ 0 & \exp(-\chi(\omega)) \end{pmatrix}$$

with

$$\begin{aligned} \chi(\omega) &= \lim_{N \to \infty} \frac{1}{N} \log \left\| \prod_{i=1}^{N} \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \\ &= \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \log |\lambda_i(\omega)| \\ &= \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} |f(\theta^i \omega)| \end{aligned} \tag{2.2.15}$$

We have already remarked that such a $\chi(\omega)$ is θ-invariant. If the measure is θ-ergodic, then χ is a (for μ-almost every $\omega \in \Omega$) a constant, precisely:

$$\chi(\omega) = \chi = \int_{\Omega} |f(\omega)| \, d\mu(\omega) \tag{2.2.16}$$

(this can be seen integrating the (2.2.15), knowing that $\chi(\omega)$ is a constant).

Then we have obtained:

Theorem (Birkhoff) (*). Let Θ be a measure preserving transformation on a probability space (Ω, Σ, μ). Then, $\forall f \in L^1(\Omega, \mu)$, the limit

$$\hat{f}(\omega) = \lim_{N\to\infty} \frac{1}{N} \sum_{m=0}^{N-1} f(\Theta^m \omega)$$

exists, for μ-almost every $\omega \in \Omega$, and it is Θ-invariant, i.e.:

$$\hat{f}(\omega) = \hat{f}(\Theta\omega)$$

Moreover, if Θ is ergodic, $\hat{f}(\omega)$ is equal to the (constant) mean value of f:

$$\hat{f}(\omega) = \int_\Omega f(\omega)\, d\mu(\omega)$$

This example shows that Oseledec's theorem introduces in a natural way a mean value through a Large Number Law; and, moreover, that by multiplying the N-th matrix $\mathcal{S}_N$ by its adjoint, it eliminates the phases of the eigenvalues of $\mathcal{S}_\infty$ which do not obey to such a law. Finally, Oseledec proves that $\mathcal{S}_N^* \mathcal{S}_N$ behaves, for N large, as Λ^{2N} and differs from its mean taken over all possible configurations.

The non commutative ergodic theorem 2.2.1 can also be seen as a generalization of the spectral theory, which allows the analysis of the N-th iterates of some matrix Q, the LCE's in this (non disordered) case corresponding to the logarithm of the moduli of the eigenvalues of Q.

Finally, let us briefly discuss the relation between Lyapunov Exponents, describing the rates of growth of the norms $\|\mathcal{S}_N v\|$ as $N\to\infty$ for some $v \in \mathbb{R}^n$, and the behaviour of the (ℓ^2) solutions of equations (1.2.6).
First of all we recall that any solution of (1.2.6) which behaves as $e^{-\ell|m_1|}$, $\ell > 0$, for m_1 large, is said to be (exponentially) localized, with localization length $1/\ell$.
Borland [6] conjectured that the LCE's are closely related to the localization length of an eigenstate (if it exists at such an energy), in the sense that

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(*) see, for example, [5]

$\chi_i > 0$ should imply the existence of a solution with localization length $1/\chi_i$. For one dimensinal Schroedinger operators with random potentials, Matsuda and Ishii [21], [14] prove, under additional hypothesis that the LCE's (which in this case are only two $\chi_1 = -\chi_2$) are different from zero using a theorem of Furstenberg [12], and from this fact Pastur [23] deduces the absence of absolutely continuous spectrum (with probability one), thus proving a weaker form of Borland's conjecture.

Following this interpretation of the LCE χ_i as the inverse of the localization length of a possible eigenstate, one is interested in the evaluation of all LCE's, and in particular of the smaller one, that should correspond to the maximal localization length.

Recalling the definition of the attenuation ß:

$$ß = -10 \log_{10} \frac{I}{I_o}$$

I and I_o being the intensities of the output and input signals, respectevely, it is clear that the LCE's are proportional to the expected attenuation of a "typical" fiber belonging to the statistical ensemble (Ω, Σ, μ).

§ 2.3. A method for computing all LCE's.

From what we have been discussing at the end of §2.2, it is clear that, to determine the asymptotic behaviour of the solutions of our random transfer equation (1.2.18), we have to compute all the LCE's of the family $\{\mathcal{S}_j(\omega)\}_{j\in\mathbb{Z}}$. In particular, one would like to get estimations on the smaller LCE, which should correspond to the maximal localization length and, in our picture, to the smaller attenuation of a typical fiber belonging to the ensemble $\mathcal{L}$.

In principle, if one could know a priori the spaces L_i (defined in (2.1.7)), and make exact computations, it should be possible to estimate all LCE's using the (2.2.13). In fact, it would be sufficient in that case to take any vector $v_i \in L_i \setminus L_{i+1}$ to obtain

$$\chi_i = \lim_{N\to\infty} \frac{1}{N} \log \| \mathcal{S}_N v_i \| \qquad 1 \le i \le m \tag{2.3.1}$$

On the other hand, since all subspaces $L_2,\dots,L_m$ have positive codimension, and thus zero Lebesgue measure, it is clear that any random choice of a $v \in \mathbb{R}^n$ should take to the computation of the maximal LCE, χ_1.

In particular, if one wants to compute the LCE's numerically, even the exact knowledge of the spaces L_i is not enough to allow the computation of the LCE's different from the maximal one.

In fact, the unavoidable numerical errors reproduce a situation equivalent to that of a random choice of the initial vector, leading in any case to the evaluation of the maximal LCE. (for an example of such a pathology, see [3 II part]. Nevertheless one can show that, in the same way that the random choice of a one dimensional subspace in $\mathbb{R}^N$ leads to the computation of χ_1, a random choice of a p-dimensianal subspace of $\mathbb{R}^N$ gives the maximal LCE of order p, $1 \le p \le m$. This fact is at the basis of the method developed by Benettin, Galgani, Giorgilli and Strelcyn [3], for the computation of all LCE's.

Let us introduce the following

Definition 2.3.1. A basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$ is said to be a normal

(b) $\forall$ p-dimentional subspace $e^p \subset \mathbb{R}^n$ one has:

$$\chi(\omega, e^p) = \sum_{k=1}^{p} \alpha_{i_k} \tag{2.3.5}$$

with a suitable sequence $1 \leq i_1 \leq i_2 \leq \ldots \leq i_p \leq m$,

(c) $\forall$ p-dimentional subspace $e^p \subset \mathbb{R}^n$:

$$\chi(\omega, e^p) = \min \sum_{i=1}^{p} \chi(\omega, e'_i) \tag{2.3.6}$$

where the minimum is taken over all bases $\{e'_1 \ldots e'_p\}$ of $e^p \subset \mathbb{R}^n$.

We remark that, for $p = n$, point (b) of Theorem 2.3.3 reads:

$$\chi(\omega, \mathbb{R}^n) = \sum_{i=1}^{n} \alpha_i$$

and is an immediate consequence of the regularity of $\{S_j(\omega)\}_{j \in \mathbb{Z}}$ since, being

$$|\det S_N| = (\mathrm{Vol}^n(S_N U)) / (\mathrm{Vol}^n U)$$

$\forall U$ bounded subset of $\mathbb{R}^n$,

$$\chi(\omega, \mathbb{R}^n) = \lim_{N \to \infty} \frac{1}{N} \log |\det S_N| \tag{2.3.7}$$

Point (a) is an extension of Oseledec's Theorem as stated in [22] , points (b) and (c), under a slightly different form, are stated in [25] and also in [27].

By Theorem 2.3.3, it is easy to see that, given a p-dimensional subspace $e^p \subset \mathbb{R}^n$, the necessary and sufficient condition to be satisfied to have

$$\chi(\omega, e^p) = \alpha_1 + \ldots + \alpha_p \tag{2.3.8}$$

is the following "random condition":

Condition 2.3.4. $\forall j$, $2 \leq j \leq m$ one has:

$$\dim (e^p \cap L_j) = \max \left(0, p - \sum_{i=1}^{j-1} k_i\right) \tag{2.3.9}$$

k_i being the number of repetitions of α_i , i.e., $\forall 1 \leq i \leq m$:

$$k_i = \dim L_i - \dim L_{i+1} \tag{2.3.10}$$

Let us illustrate this in the simple case in which $m = n$, i.e. all multiplicities of the χ_i's are equal to one. From point (b) of Theorem 2.3.3 follows

basis, with respect to the family $\{S_j(\omega)\}_{j\in\mathbb{Z}}$, if, for any other basis $\{e'_1,\dots,e'_n\}$ of $\mathbb{R}^n$:

$$(2.3.2)\qquad \sum_{i=1}^{n}\chi(\omega,e_i)\le\sum_{i=1}^{n}\chi(\omega,e'_i)$$

We remark that, even if a normal basis (when it exists) is never unique, nevertheless the numbers $\{\chi(\omega,e_i)\}_{1\le i\le n}$ do not depend on the particular choice of the normal basis.

Recalling the definition of LCE's, in fact we will have:

$$\chi(\omega,e_i)\equiv\alpha_i=\chi_{k_i}$$

where $1\le i\le n$, $1\le k_i\le m$, and we will order the basis $\{e_1,\dots,e_n\}$ in such a way that $\chi_1=\alpha_1\le\alpha_2\le\dots\le\alpha_n=\chi_m$

Definition 2.3.2. The family $\{S_j(\omega)\}_{j\in\mathbb{Z}}$ is called regular if, $\forall j\in\mathbb{Z}$:

1°) $S_j(\omega)$ is invertible,

2°) $\lim_{N\to\infty}\frac{1}{N}\log|\det S_N|$ exists and it is finite,

3°) there exists a basis $\{e_1\dots e_n\}$ of $\mathbb{R}^n$ such that

$$(2.3.3)\qquad \lim_{N\to\infty}\frac{1}{N}\log|\det S_N|=\sum_{i=1}^{n}\chi_i(\omega,e_i)$$

Note that any basis such that (2.3.3) holds is a normal one, since, by Hadamard's inequality recalled in section 2.2, one has, for any basis $\{e'_1\dots e'_n\}$ of $\mathbb{R}^n$, that

$$(2.3.4)\qquad \lim_{N\to\infty}\frac{1}{N}\log|\det S_N|\le\sum_{i=1}^{n}\chi_i(\omega,e'_i)$$

It is easy to prove the following:

Theorem 2.3.2. In the hypothesis of Oseledec's Theorem (Thm 2.2.1), if $\forall j\in\mathbb{Z}$ and $\forall\omega\in\Omega$, $S_j(\omega)$ is invertible, then $\exists\,\Omega'\subset\Omega$, $\mu(\Omega')=1$, such that $\forall\omega\in\Omega'$ the family $\{S_j(\omega)\}_{j\in\mathbb{Z}}$ is a regular family.

In [3], Benettin et al. prove:

Theorem 2.3.3. Let $\{S_j(\omega)\}_{j\in\mathbb{Z}}$ be a regular family. Then:

(a) the exact LCEs of any order exist,

(2.3.11) $\chi(\omega, e^p) \leq \alpha_1 + \ldots + \alpha_p$

In order to have $\chi(\omega, e^p) < \alpha_1 + \ldots + \alpha_p$, it is clearly necessary that $e^p \supset L_{p+1}$ which is surely not the case if e^p is spanned by p-linearly independent vectors, randomly choosen in $\mathbb{R}^n$. (In the same way in which a plane in $\mathbb{R}^3$, chosen at random between those planes passing through the origin, does not contain a given straight line through the origin).

Thus one obtains the following:

Theorem 2.3.5. Let $\{S_j(\omega)\}_{j\in\mathbb{Z}}$ be a regular family and e^p a p-dimensional subspace of $\mathbb{R}^n$ ($1 \leq p \leq n$), satisfying Condition 2.3.4 w.r. to the family $\{S_j(\omega)\}_{j\in\mathbb{Z}}$. Then :

(2.3.12) $$\chi(\omega, e^p) = \sum_{i=1}^{p} \alpha_i$$

Condition 2.3.4 being obviously satisfied within a numerical procedure, theorem 2.3.5 furnishes a constructive method for the computation of all LCE's of the family $\{S_j(\omega)\}_{j\in\mathbb{Z}}$, method which will be specialized to our case in chapter 3.

Chapter 3

COMPUTATION OF THE LCE's

In this chapter, we want to discuss the strategy for the numerical computation of all LCE's of the family of random transfer matrices in the case of a two-dimensional distorted fiber's model.

We start in §3.1 by transforming the matrices $S_j(\omega)$ in such a way that the new transfer matrices $T_j(\omega)$ are symplectic so that we will be in the position to use the symmetry property of the LCE's, holding in the symplectic case, as a numerical test for our computations.

The two-dimensional problem of an optical fiber presenting random micro-bends of its axis is described in details in §3.2, together with the explicit construction of the transfer matrices.

§3.3 is devoted to the description of the main steps through which the numerical computations of all LCE's can be carried out, while in §3.4 we present some numerical results on optical fibers losses obtained, through different methods, by D. Marcuse. [20]

§ 3.1. A transformation of the transfer matrices.

As we have seen in chapter 1, we are interested in the study of the asymptotic behaviour of the solutions of the following transfer equation:

$$(3.1.1)\quad \begin{pmatrix} A(m_1+1) \\ A(m_1) \end{pmatrix} = S_{m_1+1} \begin{pmatrix} A(m_1) \\ A(m_1-1) \end{pmatrix} = \left(\begin{array}{c|c} K^{-1}(m_1, m_1+1)\, D(m_1) & -K^{-1}(m_1, m_1+1)\, K(m_1, m_1-1) \\ \hline \mathbb{1} & 0 \end{array}\right) \begin{pmatrix} A(m_1) \\ A(m_1-1) \end{pmatrix}$$

where

$$(3.1.2)\quad K_{ij}(m_1, m_1 \pm 1) \equiv \left(\phi_i^{(m_1)}(m_2, m_3, E) \mid \phi_j^{(m_1)}(m_2, m_3, E)\right)$$

$$(3.1.3)\quad D_{ij}(m_1) \equiv \left(2 + a^2 \lambda_i^{(m_1)}(E)\right) \delta_{ij}$$

and $\phi_i^{(m_1)}(m_2, m_3, E)$ are the solutions, $\lambda_i^{(m_1)}(E)$ being the corresponding eigenvalues, of the the two-dimensional problem:

$$(3.1.4)\quad (H^{(m_1)}\phi_i^{(m_1)})(m_2,m_3,E) = \frac{4}{a^2}\phi_i^{(m_1)}(m_2,m_3,E) - \frac{1}{a^2}\sum_{\substack{(m_2',m_3') \\ \text{n.n. } (m_2,m_3)}} \phi_i^{(m_1)}(m_2',m_3',E) + \\ - \frac{E}{c^2}n^2(m_2,m_3)\,\phi_i^{(m_1)}(m_2,m_3) = \lambda_i^{(m_1)}(E)\,\phi_i^{(m_1)}(m_2,m_3,E)$$

the sum running over all lattice sites (m_2',m_3') nearest neighbours of the site (m_2,m_3).

First of all, we remark that in the case the transfer matrices $S_j(\omega)$ are symplectic ones, the following symmetry property for the 2n-LCE (note that, if our state vector are n-dimensional, the transfer matrices are 2n-dimensional) of the family $\{S_j(\omega)\}_{j\in\mathbb{Z}}$,

$$(3.1.5)\quad \alpha_1 \geq \ldots \geq \alpha_{2n}$$

(not all necessarely different), hold:

$$(3.1.6)\quad \alpha_{2n-j+1} = -\alpha_j \qquad 1 \leq j \leq 2n$$

Such a property is a direct consequence of the symplecticity of the Oseledec's matrix Λ. In fact if Λ is symplectic

$$\Lambda^* J \Lambda = J \quad \text{or} \quad J\Lambda J^{-1} = (\Lambda^*)^{-1} = (\Lambda^{-1})^* \quad \text{where} \quad J = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right)$$

and, since $\det(\Lambda)=1$, one has:

$$\begin{aligned} \det(\Lambda - \lambda \mathbb{I}) &= \det(J(\Lambda - \lambda\mathbb{1})J^{-1}) = \det(\Lambda^{-1} - \lambda\mathbb{1}) \\ &= \det(\Lambda^{-1}(\mathbb{1} - \lambda\Lambda)) = \det(\mathbb{I} - \lambda\Lambda) \\ &= \det(\lambda(\lambda^{-1}\mathbb{1} - \Lambda)) = \lambda^{2n}\det(\lambda^{-1}\mathbb{1} - \Lambda) \end{aligned}$$

and then, ($\lambda = 0$ being not an eigenvalue of Λ),

$$\det(\Lambda - \lambda\mathbb{1}) = 0 \quad \text{iff} \quad \det(\Lambda - \lambda^{-1}\mathbb{1}) = 0$$

This means that if $\exp(\alpha_j)$ is an eigenvalue of Λ, so is $\exp(-\alpha_j)$, and then, ordering the α_j's as in (3.1.5), the (3.1.6) holds.

Our transfer matrices defined in (3.1.1) are not symplectic (relative to

the form J given above):

$$S^{*}_{m_1+1}\, J\, S_{m_1+1} = \left(\begin{array}{c|c} \left[K^{-1}(m_1,m_1+1)^{*} - K^{-1}(m_1,m_1+1)\right] D(m_1) & K^{-1}(m_1,m_1+1)\, K(m_1,m_1-1) \\ \hline -K^{*}(m_1,m_1-1)\, K^{-1\,*}(m_1,m_1+1) & 0 \end{array}\right) \neq J$$

On the other hand, we would like to use the symmetry property (3.1.6) of LCE's as a useful numerical test for the numerically calculated values of the LCE's.

For this reason, we perform the following transformation: we define the sequence of matrices $J(m_1)$ as:

(3.1.7)
$$\begin{aligned} J(0) &= \mathbb{1} \\ J(1) &= K^{-1}(0,1) \\ J(2) &= K^{-1}(1,2)\, K(1,0) \\ &\vdots \\ J(m_1-1) &= K^{-1}(m_1-2,m_1-1)\, K(m_1-2,m_1-3)\, K^{-1}(m_1-4,m_1-3)\, K(m_1-4,m_1-5)\ldots \\ J(m_1) &= K^{-1}(m_1-1,m_1)\, K(m_1-1,m_1-2)\, K^{-1}(m_1-3,m_1-2)\, K(m_1-3,m_1-4)\ldots \\ J(m_1+1) &= K^{-1}(m_1,m_1+1)\, K(m_1,m_1-1)\, K^{-1}(m_1-2,m_1-1)\, K(m_1-2,m_1-3)\ldots \\ &\vdots \end{aligned}$$

constructed in such a way that:

(3.1.8) $$K(m_1,m_1+1)\, J(m_1+1) = K(m_1,m_1-1)\, J(m_1-1)$$

Rewriting equation (1.3.17)

(3.1.9) $$K(m_1,m_1+1)\, A(m_1+1) + K(m_1,m_1-1)\, A(m_1-1) = D(m_1)\, A(m_1)$$

as

(3.1.10) $$\begin{aligned} K(m_1,m_1+1)\, J(m_1+1)\, J^{-1}(m_1+1)\, A(m_1+1) + K(m_1,m_1-1)\, J(m_1-1)\, J^{-1}(m_1-1)\, A(m_1-1) = \\ = D(m_1)\, J(m_1)\, J^{-1}(m_1)\, A(m_1) \end{aligned}$$

and defining

(3.1.11) $$\textcircled{H}(m_1) = J^{-1}(m_1)\, A(m_1)$$

From property (3.1.8) follows:

$$(3.1.12)\quad K(m_1, m_1+1)\, J(m_1+1)\left[\Theta(m_1+1) + \Theta(m_1-1)\right] = D(m_1)\, J(m_1)\, \Theta(m_1)$$

or

$$(3.1.13)\quad \begin{pmatrix} \Theta(m_1+1) \\ \hline \Theta(m_1) \end{pmatrix} = T_{m_1+1} \begin{pmatrix} \Theta(m_1) \\ \hline \Theta(m_1-1) \end{pmatrix} =$$

$$= \left(\begin{array}{c|c} J^{-1}(m_1+1)\, K^{-1}(m_1, m_1+1)\, D(m_1)\, J(m_1) & -\mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right) \begin{pmatrix} \Theta(m_1) \\ \hline \Theta(m_1-1) \end{pmatrix}$$

Denoting as

$$(3.1.14)\quad G_{m_1+1} = \left(\begin{array}{c|c} J^{-1}(m_1+1) & 0 \\ \hline 0 & J^{-1}(m_1) \end{array}\right)$$

we have that:

$$(3.1.15)\quad T_{m_1+1} = G_{m_1+1}\, S_{m_1+1}\, G_{m_1}^{-1}$$

Let us check that the new tranfer matrices T_{m_1+1}, defined in (3.1.13) are in fact symplectic.

It is sufficient to show that the matrix

$$J^{-1}(m_1+1)\, K^{-1}(m_1, m_1+1)\, D(m_1)\, J(m_1)$$

is equal to its adjoint matrix, or that

$$(3.1.16)\quad J^{-1}(m_1+1)\, K^{-1}(m_1, m_1+1) = J^{*}(m_1)$$

Since:

$$(3.1.17)\quad K_{ij}(m_2, m_1+1) \equiv (\phi_i^{(m_1)} \mid \phi_j^{(m_1+1)}) = \overline{K}_{ji}(m_1+1, m_2)$$

one has:

$$(3.1.18)\quad K(m_1, m_1+1) = K^{*}(m_1+1, m_1)$$

This identity implies (3.1.16) for $m_1 = 1, 2$. In fact one has:

$$J^{*}(1) = \left[K^{-1}(0,1)\right]^{*} = K^{-1}(1,0) = K^{-1}(1,0)\, K(1,2)\, K^{-1}(1,2) \equiv J^{-1}(2)\, K^{-1}(1,2)$$

$$J^*(2) = K^*(1,0)\left[K^{-1}(1,2)\right]^* = K(0,1)K^{-1}(2,1)K(2,3)K^{-1}(2,3) \equiv J^{-1}(3)K^{-1}(2,3)$$

(3.1.10) then follows recoursively $\forall\, m_1$, noting that, if

$$J^*(m_1-1) = J^{-1}(m_1)K^{-1}(m_1-1, m_1)$$

property (3.1.8) implies:

$$J^*(m_1) = J^{-1}(m_1-1)\left[K^{-1}(m_1-1, m_1)\right]^* = J^{-1}(m_1-1)K^{-1}(m_1, m_1-1)$$
$$= J^{-1}(m_1+1)K^{-1}(m_1, m_1+1)$$

As we shall see in detail in the two dimensional case, the matrices $J(m_1)$ are bounded function of the position of the center of the fiber in m_1, $\gamma(m_1)$, in the microbending case and of the core width at the point m_1, $w(m_1)$, in the case of random diameter changes.

If we denote as:

(3.1.19) $$\mathcal{C}_N(\omega) = \prod_{j=1}^{N} T_j(\omega)$$

by (3.1.15) we have:

(3.1.20) $$\lim_{N\to\infty} \frac{1}{N} \log \|\mathcal{C}_N v\| = \lim_{N\to\infty} \frac{1}{N} \log \|G_N \mathcal{S}_N G_0^{-1} v\|$$

Since

(3.1.21) $$\|G_N^{-1}\|^{-1} \|\mathcal{S}_N G_0^{-1} v\| \le \|G_N \mathcal{S}_N G_0^{-1} v\| \le \|G_N\| \|\mathcal{S}_N G_0^{-1} v\|$$

and

(3.1.22) $$0 < \delta_1 \le \|G_N^{-1}\|^{-1} \le \|G_N\| \le \delta_2 < \infty \qquad \forall N$$

it readely follows that the LCE's of the family $\{T_j\}_{j\in\mathbb{Z}}$ are exactly the same as the LCE's of the family $\{S_j\}_{j\in\mathbb{Z}}$.

Let us conclude this section by noting that, by the construction of the matrices $J(m_1)$ (property (3.1.8)), $J^{-1}(m_1+1)K^{-1}(m_1, m_1+1)$ is a constant matrix. More precisely, $\forall\, m_1 \in \mathbb{Z}$:

(3.1.23) $$J^{-1}(m_1+1)K^{-1}(m_1, m_1+1) = \begin{cases} J^{-1}(0)K^{-1}(1,0) \equiv K^{-1}(1,0) & \text{if } m_1+1 \text{ is even} \\ J^{-1}(1)K^{-1}(2,1) \equiv K(0,1)K^{-1}(2,1) & \text{if } m_1+1 \text{ is odd} \end{cases}$$

Then, if we set:

(3.1.24a) $K_{even} = K^{-1}(1,0)D(2n)$

(3.1.24b) $K_{odd} = K(0,1)K^{-1}(2,1)D(2n+1)$

we have to deal with two kinds of transfer matrices T_{m_1+1} :

(3.1.25) $$T_{m_1+1} = \left(\begin{array}{c|c} K_{even}\, J(m_1) & -\mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right) \qquad T_{m_1+1} = \left(\begin{array}{c|c} K_{odd}\, J(m_1) & -\mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right)$$

corresponding respectively to m_1+1 even or odd.

§ 3.2. The two dimensional case.

To fix the ideas, and to semplify the calculations, let us consider the case of random microbends of a two-dimensional single-mode optical waveguide (fig. 1).

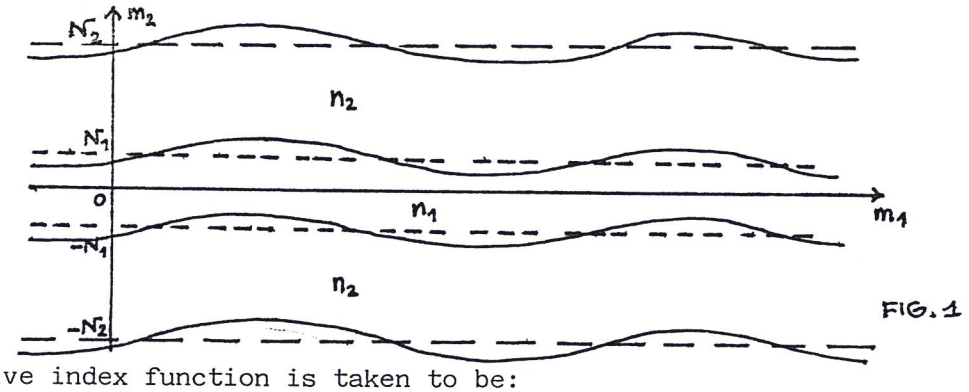


FIG. 1

Its refractive index function is taken to be:

$$(3.2.1)\qquad n(m_2-\gamma(m_1)) = \begin{cases} n_1 \text{ if } |m_2-\gamma(m_1)| \le N_1 \\ n_2 \text{ if } N_1 < |m_2-\gamma(m_1)| \le N_2 \end{cases}$$

$\gamma(m_1)$ being the m_2-coordinate of the center of the fiber at the point m_1. The construction of the transfer matrices T_{m_1} will be carried out in three steps:

step 1: construction of the eigenfunctions $\phi_j(m_2,E)$;

step 2: construction of $K^{\pm}$, their properties ;

step 3: construction of the matrices $J(m_1)$

Step 1. Construction of the eigenfunctions $\phi_j(m_2,E)$.

We have to construct the solutions of the following one-dimensional, discretized eigenvalue equation (m_1 being fixed):

$$(3.2.2)\qquad \frac{1}{a^2}\left[2\phi_j(\tilde m_2)-\phi_j(\tilde m_2+1)-\phi_j(\tilde m_2-1)\right]-E\frac{n^2}{c^2}(\tilde m_2)\,\phi_j(\tilde m_2) = \lambda_j\,\phi_j(\tilde m_2)$$

where $\tilde m_2 \equiv m_2-\gamma(m_1)$, $\phi_j(\tilde m_2) = \phi_j(\tilde m_2,E)$, $\lambda_j = \lambda_j(E)$ and we impose Diriclet boundary condition on the boundary of the cladding:

$$(3.2.3)\qquad \phi_j(\gamma(m_1)\pm N_2) = 0$$

The (ϕ_j, λ_j)'s are then the eigenfuntions and eigenvalues of the problem of a rectangular potential well between two infinite walls (fig. 2), the depth of the well being fixed by the choice of the refractive indexes n_1 and n_2 of the core and cladding's materials and of the wavelength $\tilde{\lambda}$ of the light signal travelling in the waveguide. ($E = \hbar\omega^2 = \hbar c^2 (2\pi)^2 / \tilde{\lambda}^2$). To see how the solutions $\phi_j(\tilde{m}_2)$ are constructed, let us consider, for example, the antisymmetric solutions $\phi_{2j+1}(\tilde{m}_2)$ living above the well, i.e. such that $\lambda_{2j+1} > -n_2^2 E / c^2$.

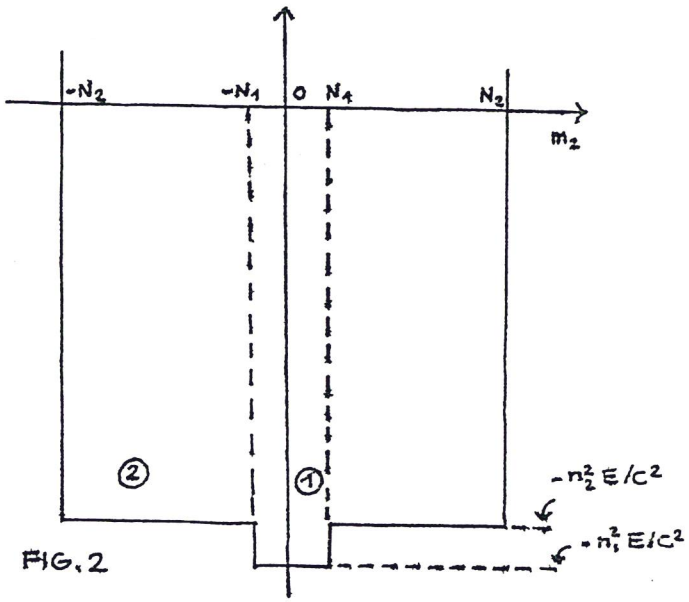


FIG. 2

In the 2 -zone ($-N_2 \leq \tilde{m}_2 < -N_1$) one has (setting i=2j+1)

(3.2.4) $$\phi_i^{(2)}(\tilde{m}_2) = A_2 \sin\left(k_i^{(2)} a\,(\tilde{m}_2 + N_2)\right)$$

the relationship between $k_i^{(2)}$ and λ_i being:

(3.2.5) $$\lambda_i = \frac{2}{a^2}\left(1 - \cos k_i^{(2)} a\right) - E n_2^2 / c^2$$

while in the 1 -zone ($-N_1 \leq \tilde{m}_2 \leq N_1$)

(3.2.6) $$\phi_i^{(1)} = A_1 \sin\left(k_i^{(1)} a\, \tilde{m}_2\right)$$

with

(3.2.7) $$\cos\left(k_i^{(1)} a\right) = \cos\left(k_i^{(2)} a\right) - \frac{a^2}{2}\frac{E}{c^2}\left(n_1^2 - n_2^2\right)$$

Writing equation (3.2.2) for $\tilde{m}_2 = -N_1$ and $\tilde{m}_2 = -N_1 - 1$, one gets the relations between A_1 and A_2:

(3.2.8a) $$A_1 \sin\left(k_i^{(1)} a\,(N_1 + 1)\right) + A_2 \sin\left(k_i^{(2)} a\,(N_2 - N_1 - 1)\right) = 0$$

(3.2.8b) $$A_1 \sin\left(k_i^{(1)} a\, N_1\right) + A_2 \sin\left(k_i^{(2)} a\,(N_2 + N_1)\right) = 0$$

which give the following equation for $k_i^{(1)}$ and $k_i^{(2)}$ ($\forall i=2j+1$):

(3.2.9)
$$F(k_i^{(1)}) = \sin(k_i^{(1)} a (N_1+1)) \cdot \sin(k_i^{(2)} a (N_2-N_1)) + \\ - \sin(k_i^{(2)} a (N_2-N_1) \cdot \sin(k_i^{(1)} a N_1) = 0$$

All (*) $k_i^{(1)}$ can be can be obtained numerically as the zeros of $F(k_{\mathcal{D}_-}^{(1)})$ (for example with Newton's method), the $k_{\mathcal{D}}^{(2)}$'s being given by (3.2.7), and the eigenvalues λ_i being given by (3.2.5).

Similarly we get the symmetric solutions $\phi_{2j}(\tilde{m}_2)$, $\lambda_{2j} > -n_2^2 E/c^2$, and the (only one) well's solution $\phi_0(\tilde{m}_2)$, $-n_1^2 E/c^2 < \lambda_0 < -n_2^2 E/c^2$.
Setting $k_j^{(1),(2)} a = k_j^{(1),(2)}$ the solutions are:

$$\phi_0(\tilde{m}_2) = \begin{cases} A_0 \cos(k_0^{(1)} N_1) \sinh(k_0^{(2)}(\tilde{m}_2 + N_2)) & -N_2 \le \tilde{m}_2 \le -N_1 \\ A_0 \sinh(k_0^{(2)}(N_2-N_1)) \cos(k_0^{(1)} \tilde{m}_2) & -N_1 \le \tilde{m}_2 \le N_1 \\ A_0 \cos(k_0^{(1)} N_1) \sinh(k_0^{(2)}(N_2 - \tilde{m}_2)) & N_1 \le \tilde{m}_2 \le N_2 \end{cases}$$

$$\phi_{2j+1}(\tilde{m}_2) = \begin{cases} A_{2j+1} \sin(k_{2j+1}^{(1)} N_1) \sin(k_{2j+1}^{(2)}(\tilde{m}_2 + N_2)) & -N_2 \le \tilde{m}_2 \le -N_1 \\ -A_{2j+1} \sin(k_{2j+1}^{(2)}(N_2-N_1)) \sin(k_{2j+1}^{(1)} \tilde{m}_2) & -N_1 \le \tilde{m}_2 \le N_1 \\ -A_{2j+1} \sin(k_{2j+1}^{(1)} N_1) \sin(k_{2j+1}^{(2)}(N_2 - \tilde{m}_2)) & N_1 \le \tilde{m}_2 \le N_2 \end{cases}$$

$$\phi_{2j}(\tilde{m}_2) = \begin{cases} A_{2j} \cos(k_{2j}^{(1)} N_1) \sin(k_{2j}^{(2)}(\tilde{m}_2 + N_2)) & -N_2 \le \tilde{m}_2 \le -N_1 \\ A_{2j} \sin(k_{2j}^{(2)}(N_2-N_1)) \cos(k_{2j}^{(1)} \tilde{m}_2) & -N_1 \le \tilde{m}_2 \le N_1 \\ A_{2j} \cos(k_{2j}^{(1)} N_1) \sin(k_{2j}^{(2)}(N_2 - \tilde{m}_2)) & N_1 \le \tilde{m}_2 \le N_2 \end{cases}$$

(3.2.10)

(*) As already observed in remark 1.2.2, we will actually consider only the eigenfunctions $\phi_j(\tilde{m}_2)$ (and the corresponding k_j) such that $\lambda_j <$ some constant.

where the A_j are (computable) normalization constants.

Step 2. The matrices $K(m_1, m_1 \pm 1)$.

Since we are considering the case in which only displacement of the center of the fiber can occur, the set of eigenfunctions of $H^{(m_1)}$, relative to two consecutive sections m_1, m_1+1, are just translated one with respect to the other by the amount $\delta\gamma = \gamma(m_1+1) - \gamma(m_1)$.

We assume that $\delta\gamma$ can only take the values:

$$\delta\gamma = \begin{cases} +1 \\ -1 \\ 0 \end{cases} \tag{3.2.11}$$

with some probabilities, say p_1, p_2, p_3, $p_1+p_2+p_3=1$. In correspondence with the values taken by $\delta\gamma$, the matrix elements $K_{ij}(m_1, m_1+1)$ will be:

$$K_{ij}(m_1, m_1+1) = \begin{cases} K^+_{ij} & \text{if } \delta\gamma = +1 \\ K^-_{ij} & \text{if } \delta\gamma = -1 \\ \delta_{ij} & \text{if } \delta\gamma = 0 \end{cases} \tag{3.2.12}$$

where $K^{\pm}$ do not depend on m_1. For the matrices $K(m_1, m_1+1)$ the following properties hold:

$$K(m_1, m_1+1) = K^*(m_1+1, m_1) \quad \text{(see (3.1.17))} \tag{3.2.13}$$

$$K^{\pm}(m_1, m_1+1) = K^{\mp}(m_1+1, m_1) \tag{3.2.14}$$

(since $K^+_{ij}(m_1, m_1+1) = \sum_{m_2=-N_2+\gamma(m_1)}^{N_2+\gamma(m_1)} \phi_i(m_2-\gamma(m_1))\,\phi_j(m_2-\gamma(m_1)-1)$

$= \sum_{m_2=-N_2+\gamma(m_1)-1}^{N_2+\gamma(m_1)} \phi_i(m_2-\gamma(m_1)+1)\,\phi_j(m_2-\gamma(m_1)) = K^-(m_1+1, m_1)$

$$K^+ = (K^-)^* \quad \text{(see (3.2.13) and (3.2.14))} \tag{3.2.15}$$

$$K^+K^- \neq K^-K^+ \tag{3.2.16}$$

$$K^+_{ij} = \begin{cases} K^+_{ji} & \text{if } \phi_i \text{ and } \phi_j \text{ have the same symmetry} \\ -K^+_{ji} & \text{if } \phi_i \text{ and } \phi_j \text{ have different symmetry.} \end{cases} \tag{3.2.17}$$

The matrix K^+ (K^- being its adjoint matrix) then looks like:

(3.2.18) $$K^+ = \begin{pmatrix} (\phi_0|\phi_0^+) & (\phi_0|\phi_1^+) & (\phi_0|\phi_2^+) & \cdots \\ -(\phi_0|\phi_1^+) & (\phi_1|\phi_1^+) & (\phi_1|\phi_2^+) & \cdots \\ (\phi_0|\phi_2^+) & -(\phi_1|\phi_2^+) & \ddots & \\ \vdots & & & \end{pmatrix}$$

where $(\phi_i|\phi_j^+) = \sum_{-N_2}^{N_2} \phi_i(\tilde{m}_2)\,\phi_j(\tilde{m}_{2-1})$

To get all matrix elements $K^{\pm}_{ij}$ is then sufficient to compute the elements belonging to the "upper triangle half" of the matrix K^+, i.e. the elements K^+_{ij} with $i \leq j$.

Even if we confine our computation of the eigenvectors $\phi_j(E)$ only to those corresponding, for example, to $\lambda_j(E) < 0$, we will have still to deal with a number of eigenstates of the order of 10^2, and then with the evaluation of about 10^4 scalar products.

To make the numerical procedure quicker, several explicit formulas for the scalar products $(\phi_i|\phi_j^+)$ have been obtained, using the orthonormality of the set $\{\phi_i(E)\}$, the (3.2.9) and the (3.2.7). For example, the scalar product between an antisymmetric eigenfunction ϕ_{2j+1} and the same ϕ_{2j+1} translated is:

(3.2.19) $$(\phi_{2j+1}|\phi^+_{2j+1}) = A^2_{2j+1}\left\{\cos(k^{(2)}_{2j+1}) - C\sin^2((N_2-N_1)k^{(2)}_{2j+1})\cdot\left[N_1 - \frac{\cos((N_1+1)k^{(1)}_{2j+1})\cdot\sin(N_1 k^{(1)}_{2j+1})}{\sin(k^{(1)}_{2j+1})}\right]\right\}$$

$$\left(C = \frac{a^2}{2}\frac{E}{c^2}(n_1^2 - n_2^2)\right)$$

Similar formulas have been obtained for the scalar products involving the other kinds of eigenfunctions, precisely for $(\phi_0|\phi_0^+)$, $(\phi_0|\phi^+_{2j})$, $(\phi_0|\phi^+_{2j+1})$, $(\phi_{2j}|\phi^+_{2j})$, $(\phi_{2j}|\phi^+_{2j'})$, $(\phi_{2j}|\phi^+_{2j+1})$, $(\phi_{2j+1}|\phi^+_{2j'+1})$.

Step 3. The matrices $J(m_1)$.

Our last step toward the construction of the matrices T_{m_1} is the evaluation of the matrices $J(m_1)$ defined in (3.1.7).

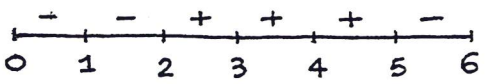
It is easy to convince oneself that $J(m_1)$ depends only on the position of

the center of the fiber at the point m_1 and not on the particular sequence of K's that has brought the center into $\gamma(m_1)$. Let us in fact consider any random path followed by the center of the fiber from zero to a certain m_1. $J(m_1)$ is by definition a product of the type:

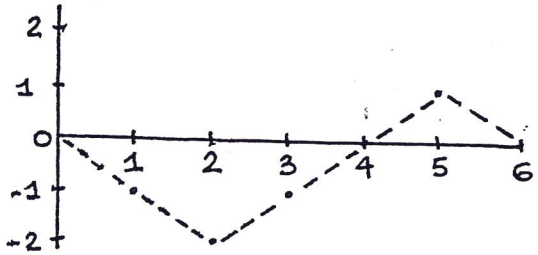
$$J(m_1) = (K^+)^{-1} K^+ (K^-)^{-1} \ldots$$

K^+ and K^- randomly alternated (we are considering only the matrices K^+, K^- since the eventual presence of the identity matrix gives no contribution to $J(m_1)$).

Every time the displacement of the center changes direction, i.e. a K^+ is followed by a K^-, and vice-versa, the K's compensate giving the identity matrix. For example the sequence



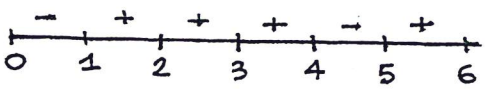
corresponding to the path:



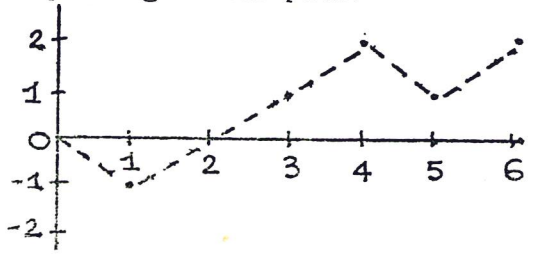
which takes the center of the fiber back to zero after 6 steps, gives:

$$J(6) = K^{-1}(5,6)\,K(5,4)\,K^{-1}(3,4)\,K(3,2)\,K^{-1}(1,2)\,K(1,0)$$
$$= (K^-)^{-1} K^- (K^+)^{-1} K^- (K^-)^{-1} K^+$$
$$= \mathbb{1}$$

On the other hand, the sequence:



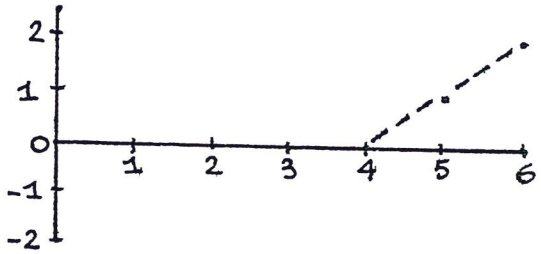
corresponding to the path:



gives:

$$J(6) = (K^+)^{-1} K^+ (K^+)^{-1} K^- (K^+)^{-1} K^+$$
$$= (K^+)^{-1} K^-$$

and is then equivalent (for what concerns J(6)) to any path that brings the center of the fiber into $\gamma(6)=2$, in particular to the path containing only two jumps here described:



From example of this sort we deduce that:

- if $\gamma(m_1)=0$, $J(m_1)$ is the product of an equal number of K^+ and K^-, which compensate giving $J(m_1)=\mathbb{1}$, while
- if $\gamma(m_1)=h\neq 0$, what is left in $J(m_1)$ are the K's corresponding to the "effective" jumps that take the center of the fiber from zero to h.

More precisely, one has: (for $\gamma(m_1)=h$)

$$(3.2.20) \qquad J(m_1) = \begin{cases} \text{for } h\geq 0 & \begin{cases} [K^-(K^+)^{-1}]^{h/2} & \text{if h is even} \\ (K^+)^{-1}[K^-(K^+)^{-1}]^{h/2} & \text{if h is odd} \end{cases} \\ \text{for } h<0 & \begin{cases} [K^+(K^-)^{-1}]^{h/2} & \text{if h is even} \\ (K^-)^{-1}[K^+(K^-)^{-1}]^{h/2} & \text{if h is odd} \end{cases} \end{cases}$$

Remark 3.2.1. Also in the case of random variations of the diameter of the fiber, $w(m_1)$, to each pair of consecutive sections will be associated some matrix $K^{\pm}$ (or the identity matrix), but such matrices will depend on the diameters of the investigated sections. This is a consequence of the fact that the two sets of eigenfunctions, whose scalar products are the elements of the matrices K, will depend on such diameters (as the corresponding eigenvalues). If we assume that the number of eigenstates (corresponding to negative eigenvalues) do not change while the width changes (and this will be the case

for slight variations), the matrices K at different sections will have the same dimension.

It is then easy to show (as for the microbending case) that properties from (3.2.13) to (3.2.17) still hold in this case, each $K^{\pm}$ still having the form (3.2.18), the matrix elements $K^{\pm}_{ij}(m_1,m_1+1)$ now depending on $\tau(m_1)$. In particular a width growth followed by a width decrease (or viceversa) will still produce, for what concerns the matrices $J(m_1)$, a compensation between the relative K's; this will lead to an expression of $J(m_1)$ as a function of $\tau(m_1)$ completely analogous to (3.2.20).

Once we have fixed some h_{max}, that is to say we have fixed the maximal range of the random variations of the center of the fiber to be $[-h_{max},h_{max}]$, the matrix elements of the $2\times h_{max}$ different matrices $J(h)$, $h\neq 0$, can be numerically computed and stocked.

For each $m_1 > 0$, the matrix $K(m_1,m_1+1)$ will be chosen at random (in a way that will be discussed in the next section) to be equal to K^+, K^- or the identity matrix; this choice (clearly equivalent to that of $\gamma(m_1+1)$) will determine the appropriate transfer matrix T_{m_1+1}:

$$(3.2.21)\quad T_{m_1+1} = \begin{cases} \begin{pmatrix} K_{even}\, J(\gamma(m_1+1)) & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} & \text{for } m_1+1 \text{ even} \\ \begin{pmatrix} K_{odd}\, J(\gamma(m_1+1)) & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} & \text{for } m_1+1 \text{ odd} \end{cases}$$

where the matrices (see (3.1.18)):

$$(3.2.22a)\qquad K_{even} \equiv K^{-1}(1,0)\, D$$

$$(3.2.22b)\qquad K_{odd} \equiv K(0,1)\, K^{-1}(2,1)\, D$$

(D being the constant diagonal matrix $D_{ij}=[a^2\lambda_j+2]\delta_{ij}$) are computed at the

beginning of the programme and stocked afterwards.

In the next section we are going to sketch the main steps of the procedure used for the numerical calculation of all the LCE's of the product of the random transfer matrices described so far.

§ 3.3. The numerical procedure for calculating all LCE's.

The aim of this section is to illustrate the computational method that will lead us to the evaluation of all LCE's of the family of random transfer matrices $T_{m_1}(\omega)$.

In the last section we sketched out how the computation of the matrix elements of $T_{m_1}(\omega)$ is carried out.

We have now to specify the way one uses for choosing at random, for each m_1, the jump $\delta\gamma$ which fixes T_{m_1+1}.

Let us still consider, as in the previous section, the case of a two dimensional waveguide whose center is allowed to vary its position with respect to the m_1-axis randomly.

More precisely, we are concerned with the case in which each section of the fiber can be, with respect to the preceding one, either rigidly translated of 1 lattice shift in the positive or negative m_2-direction, or beunshifted; these three possibilities occurring respectively with probabilities p_1, p_2, p_3, where $p_1+p_2+p_3=1$, i. e.

$$(3.3.1)\quad \gamma(m_1+1)\begin{cases} \gamma(m_1)+1 & \text{with probability } p_1 \\ \gamma(m_1)-1 & \text{with probability } p_2 \\ \gamma(m_1) & \text{with probability } p_3 \end{cases}$$

As already mentioned in §3.2, the center of the fiber will be allowed to vary its m_2-coordinate, $\gamma(m_1)$, only within a bounded interval $[-h_{max}, h_{max}]$; so that a "stop" condition has to be imposed on the extreme points $\pm h_{max}$. We will ask that , if $\gamma(m_1) = \pm h_{max}$, then:

$$(3.3.2)\quad \gamma(m_1+1)=\begin{cases} \gamma(m_1) & \text{with probability } p_4 \\ \gamma(m_1)\mp 1 & \text{with probability } p_5 \end{cases}$$

where $p_4+p_5=1$. At any section m_1 one chooses, through a computer's random numbers generator, $\delta\gamma$ to be ± 1 or 0 depending on the values assigned to the transition probabilities p_1, p_2, p_3 (or p_4 and p_5, if the value $\pm h_{max}$ has been

reached in m_1).

For example, a choice of such a probability could be:

(3.3.3) $$p_1 = p_2 = p_3 = \frac{1}{3} \quad \text{and} \quad p_4 = p_5 = \frac{1}{2}$$

The choice of $\delta\gamma$ is the accomplished in this case by letting the computer choose at random a real number $\alpha \in [0,1[$ and associating to it:

(3.3.4) $$\delta\gamma = \begin{cases} +1 & \text{if} \quad \alpha < 1/3 \\ -1 & \text{if} \quad 1/3 \leq \alpha < 2/3 \\ 0 & \text{if} \quad \alpha \geq 2/3 \end{cases}$$

if $|\gamma(m_1)| < h_{max}$, and

(3.3.5) $$\delta\gamma = \begin{cases} -1 \ (\text{resp.}+1) & \text{if} \quad \alpha < 1/2 \\ 0 & \text{if} \quad \alpha \geq 1/2 \end{cases}$$

if $\gamma(m_1) = h_{max}$ (resp. $\gamma(m_1) = -h_{max}$)

The transition probabilities $P(\gamma(m_1)=h, \gamma(m_1+1)=k)$ defined above, give naturally rise to a probability measure on the whole configuration space $\Omega = \{\gamma : \mathbb{Z} \to [-h_{max}, h_{max}]\}$. We have already remarked that, with respect to such a probability measure, $\gamma(m_1)$ is not independent from $\gamma(\tilde{m}_1)$. In fact, it is easy to see that the process $\{\gamma(m_1)\}_{m_1 \in \mathbb{Z}}$ is a Markov process. In spite of that, it is easy to show that the measure is invariant under translation along the m_1-axis, and that the hypothesis of Oseledec's theorem are satisfied.

We want to compute the LCE's $\alpha_1, \ldots, \alpha_{2n}$ of the family of random transfer matrices $T_{m_1}(\omega)$. Given 2n initial vectors $v_1, \ldots, v_{2n} \in \mathbb{R}^{2n}$, choosen numerically at random, so to satisfy the condition 2.3.4, one has to evaluate, according to theorem 2.3.5, the quantities:

(3.3.6) $$\lim_{N \to \infty} \frac{1}{N} \log \text{Vol}^p \left[\mathcal{T}_N(\omega) v_1, \ldots, \mathcal{T}_N(\omega) v_p \right] = \alpha_1 + \ldots + \alpha_p$$

for $p=1,\ldots,n$, where

(3.3.7) $$\mathcal{C}_N(\omega) = \prod_{j=1}^{N} T_j(\omega)$$

$[v_1, \ldots, v_p]$ being the open parallelepiped generated by the vectors $v_1, \ldots, v_p$ of $\mathbb{R}^{2n}$, and Vol^p is the p-dimensional euclidean volume ($\mathrm{Vol}^1[v_1] = \|v_1\|$).
If one applies directly the formula (3.3.6), that is to say, if one computes the products of the matrices $T_j(\omega)$ acting on the vectors v_i, i=1,...p, and calculates the volume spanned by the resulting p vectors, one has to face two kinds of difficulties (see [3, PART II]):

1°) since v is choosen at random, in general one has that $\|\mathcal{C}_N v\|$ grows exponentially with N (this is the case of $\chi_1 > 0$);

2°) since both v_1 and v_2 are choosen at random, the angle between $\mathcal{C}_N v_1$ and $\mathcal{C}_N v_2$ becames in general very small after few steps (this can be viewed as a consequence of theorem 2.3.5 if $\chi_1 > \chi_2$).

This two difficulties cause the computations to exceed rapidly the possibilities of ordinary numerical calculations.

To evaluate the maximal LCE, χ_1, only the first difficulty has to be overcome. This is possible using the following normalization procedure:
one fix an integer M, and defines $\tilde{\mathcal{C}}_k(\omega) = T_{kM}(\omega) T_{kM-1}(\omega) \ldots T_{(k-1)M+1}(\omega)$.
Then one chooses at random a vector $v \in \mathbb{R}^{2n}$, $\|v\| = 1$, and compute recoursively:

$$w_0 = v$$

(3.3.8) $$v_k = \|\tilde{\mathcal{C}}_k w_{k-1}\| \qquad k \geq 1$$

$$w_k = \frac{\tilde{\mathcal{C}}_k w_{k-1}}{v_k}$$

It is easy to see that:

(3.3.9) $$\|\mathcal{C}_{kM} v\| = v_1 \cdots v_k \qquad k \geq 1$$

Then, by (3.3.9), one has:

(3.3.10) $$\alpha_1 = \lim_{k\to\infty} \frac{1}{kM} \log \|\mathcal{C}_{kM} v\|$$
$$= \lim_{k\to\infty} \frac{1}{kM} \sum_{i=1}^{k} \log v_i$$

It is then sufficient to take M to be not too large, in such a way that the v_i ($i \geq 1$) be uniformely bounded to avoid computer's overflow caused by 1°).
In the case in which p-vectors are involved, to avoid 1°) and 2°) it is sufficient to perform the Gram-Schmidt orthonormalization procedure every M steps; that is to say, with the same definition of $\tilde{C}_k$ as before, one sets:

$$
\begin{aligned}
&v_k^{(1)} = \| \tilde{C}_k w_{k-1}^{(1)} \| \\
&w_k^{(1)} = \frac{\tilde{C}_k w_{k-1}^{(1)}}{v_k^{(1)}} \qquad\qquad k \geq 1 \\
&v_k^{(j)} = \| \tilde{C}_k w_{k-1}^{(j)} - \sum_{i=1}^{j-1} \langle w_k^{(i)} \mid \tilde{C}_k w_{k-1}^{(j)} \rangle w_k^{(i)} \| \\
&w_k^{(j)} = \frac{\tilde{C}_k w_{k-1}^{(j)} - \sum_{i=1}^{j-1} \langle w_k^{(i)} \mid \tilde{C}_k w_{k-1}^{(j)} \rangle w_k^{(i)}}{v_k^{(j)}} \qquad j = 2, \dots, p
\end{aligned}
\tag{3.3.11}
$$

One can easily see that:

$$
\begin{aligned}
\mathrm{Vol}^p[\tilde{C}_{kM} v_1, \dots, \tilde{C}_{kM} v_p] &= \\
&= \mathrm{Vol}^p[\tilde{C}_k w_0^{(1)}, \dots, \tilde{C}_k w_0^{(p)}] \times \dots \times \mathrm{Vol}^p[\tilde{C}_k w_{k-1}^{(1)}, \dots, \tilde{C}_k w_{k-1}^{(p)}] \\
&= (v_1^{(1)} \cdots v_k^{(1)}) \cdots (v_1^{(p)} \cdots v_k^{(p)})
\end{aligned}
\tag{3.3.12}
$$

and then, by (3.3.6)

$$
\begin{aligned}
\alpha_1 + \dots + \alpha_p &= \lim_{k\to\infty} \frac{1}{kM} \sum_{i=1}^{k} \log [v_i^{(1)} \cdots v_i^{(p)}] = \lim_{k\to\infty} \frac{1}{kM} \sum_{i=1}^{k} \sum_{j=1}^{p} \log v_i^{(j)} \\
&= \sum_{j=1}^{p} \lim_{k\to\infty} \frac{1}{kM} \sum_{i=1}^{k} \log v_i^{(j)}
\end{aligned}
\tag{3.3.13}
$$

In such a way one also obtain a direct formula for $\alpha_1, \dots, \alpha_{2n}$:

$$
\alpha_p = \lim_{k\to\infty} \frac{1}{kM} \sum_{i=1}^{k} \log v_i^{(p)} \qquad 1 \leq p \leq 2n \tag{3.3.14}
$$

This is the relation we will actually use to compute the LCE's.

§ 3.4. Marcuse's numerical results on scattering losses.

In [20], Marcuse analyses the scattering losses of single-mode optical fibers due to geometrical microdeformations of the fibers, such us random microbends of their axis or random variations of their core width.

He describes the random deformations by means of a deformation function, whose rms variation is fixed, and assumes that the autocorrelation function of the random defect is Gaussian, characterized by the rms variation and the correlation lengthof the deformation function.

Since the random deformation of the geometrical shape of the fiber cause the coupling of the guided mode with the cladding modes, the scattering losses can be studied with the aid of the coupled mode theory (see [19]). The strenght of coupling between the guided and the cladding modes is expressed in terms of the rms variation and of the correlation lenght of thedistortion function.

Then numerical computations are performed to study the dependence of scattering losses on the wavelength of the signal.

The results found by Marcuse are the following:

- step-index fibers with large core radii have higher losses, and the losses of all fibers decrease very sharply with increasing wavelength;
-if the microdeformations have a correlation length of the order of the fiber core radius, the scattering losses are nearly independent from the wavelength; however, if the correlation length is sensibly greater than the core radius, the microdeformations losses increase very sharply with increasing wavelength.
-rms deviations of the order of few micrometers cause very high losses, provided their correlation length is short;
-in general, random fluctuations of the diameter of the fiber core produce higher losses than random microbends with the same rms deformation.

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