ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Thesis submitted for the degree of "Magister Philosophiae"

ASYMPTOTIC ANALYSIS
OF
NON EQUICOERCIVE MINIMUM PROBLEMS

Candidate:
Valeria CHIADO' PIAT

Supervisor:
Prof. Gianni DAL MASO

Academic Year 1987/88

TRIESTE

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CONTENTS

0. Introduction

In this thesis work we deal with a typical problem in the asymptotic calculus of variations, that is the limit behaviour of a sequence of minimization problems. The study of this topic, together with other questions of variational convergence, that have considerably developed in the last twenty years, has a relevant interest both from the theoretical point of view and from the aspect of the applications to different branches of mathematics and to other sciences.

Roughly speaking, many physical phenomena find their mathematical formulation in minimization problems that sometimes cannot be treated with numerical techniques for the heaviness of computations. The alternative approach of approximation gives rise to mathematical problems of the type studied in this thesis work. A tipical example is given by the behaviour of non homogeneous media with a fine periodic structure that at present can be suitably approached with the different techniques of the homogenization theory.

Among the mathematical tools generally known as variational convergences, the theory of Γ-convergence (for which we refer to [17] and [2]) turns out to be the most appropriate for the study of sequences of minimum problems. Following Attouch (see[2]), the idea is that, given a sequence of functions, the notion of Γ-convergence "may be regarded as the 'weakest' notion which allows to approach the limit in the corresponding minimization problems".

This fact has a precise formulation in the following theorem (Corollary 2.4 in [17]): if a sequence $(F_h)_h$ of extended real valued functions on a topological space X is equicoercive, i.e. for every $t\in \mathbf{R}$ $\bigcup\{F_h \geq t\}$ is relatively compact in X, and $(F_h)_h$ Γ-converges to a function F, then the sequence of minimum values of $(F_h)_h$ tends to the minimum value of F. Additional hypotheses imply also the convergence of the corresponding minimizers.

The case of non equicoercive sequences has been extensively studied but complete results have been proved only for some classes of problems, like the ones of homogenization type (see [4], [26]), or for particular situations, as Neumann problems in domains with holes (see for instance [23], [31]) where the lack of coercivity is "concentrated" on some special sets. This thesis concerns a situation of this second type.

More precisely, we deal with the limit behaviour of a sequence of non-equicoercive integral functionals of the form $F_h^{\varphi}: L^p(\Omega) \to [0,+\infty]$

$$(0.1) \qquad F_h^{\varphi}(u) = \begin{cases} \displaystyle\int_\Omega f_h(x,Du)dx & \text{if } u-\varphi \in W_0^{1,p}(\Omega) \\ +\infty & \text{otherwise}, \end{cases}$$

where Ω is a bounded open subset of $\mathbf{R}^N$, $N \geq 2$, $p > 1$, and $\varphi \in W^{1,p}(\Omega)$. We assume that $(f_h)_h$

are non-negative Borel functions defined on $\Omega \times \mathbf{R}^N$, convex on $\mathbf{R}^N$ for a.e. $x \in \Omega$, satisfying the following inequalities

(0.2) $\qquad 0 \le f_h(x,\xi) \le c_2 |\xi|^p \qquad\qquad$ in $B_h \times \mathbf{R}^N$

(0.3) $\qquad c_1 |\xi|^p \le f_h(x,\xi) \le c_2 |\xi|^p \qquad\qquad$ in $\Omega_h \times \mathbf{R}^N$,

with $0 < c_1 \le c_2$, $\Omega_h \cup B_h = \Omega$ and $\Omega_h \cap B_h = \emptyset$. For the domain of equicoerciveness Ω_h we shall consider two different situations.

First of all, we shall deal with the case in which the sets B_h are fragmented into an increasing number of connected homothetic components, i.e.

$$B_h = \bigcup_{i \in I_h} B_h^i \ ,$$

where $B_h^i = x_h^i + r_h^i B$, $x_h^i \in \Omega$, $r_h^i > 0$ and B is a given closed subset of $\mathbf{R}^N$. Under suitable assumptions on such sets B_h, we study the asymptotic behaviour, as h goes to infinity, of the minimum problems with Dirichlet boundary conditions

(0.4) $\qquad m_h = \inf_{u \in L^p(\Omega)} F_h^\varphi(u)$,

and of their solutions, when F_h^φ satisfies (0.1) - (0.3).

Problems similar to (0.4) have been extensively studied by many authors. Most of the papers cited in the References study the associated Euler equations, instead of the minimum problems themselves. Some of them (see, for instance, [10], [14]) deal with Neumann BVP's for the Laplace operator on perforated domains and require a periodic distribution of the holes. Eigenvalue problems under similar assumptions are considered in [32], [33], [30]. Homogenization results for non-coercive functionals when u is a vector valued function can be found in [1]. The non-periodic case is studied in [23], [31] and, with Dirichlet boundary conditions, in [9] and [11], [12] which regard also the related obstacle problems.

Our approach to problem (0.4) is in the framework of Γ-convergence theory and follows the ideas of Mortola and Profeti (see [26]) who studied the corresponding periodic quadratic case.

The first step (see Section 1) consists in individuating a limiting minimum problem

(0.5) $\qquad m = \inf_{u \in L^p(\Omega)} F^\varphi(u)$

defined by the Γ-limit F^φ of the given sequence (0.1) and in proving that F^φ is an integral and coercive functional.

Then, by applying a general result in Γ-convergence theory (see Theorem 1.1), our problem reduces to find a sequence of minimum points u_h of F_h^{φ}, which is strongly convergent in $L^p(\Omega)$. This in fact implies the convergence of the corresponding minimum values m_h (see Corollary 2.6).

The main difficulty, at this point, is clearly the lack of a priori bounds in $W^{1,p}(\Omega)$ for the minimizers of F_h^{φ}, that usually come from equicoerciveness assumptions. Nevertheless, extending the ideas of [26] to our non-quadratic, non-periodic case, we are actually able to prove the strong compactness of a sequence of minimum points, taking the weak maximum principle into account in the estimates of the L^p-norm on each B_h^i. The geometric assumptions on such sets play here a crucial role.

In the remaining part of Section 2 and in Section 3 we replace (0.2) by

$$(0.6) \qquad f_h(x,\xi) = 0 \qquad \text{in } B_h \times \mathbf{R}^N$$

and weaken the hypothesis on B_h, dropping the assumption of fragmentation into homothetic components. More generally, we require $(\Omega_h)_h$ to satisfy a suitable extension property for functions in the Sobolev class $W^{1,p}(\Omega_h)$. A condition of this type was introduced by Hruslov in [23], where an analogous problem for the Laplace operator is studied. In this setting problem (0.4) takes the simpler form

$$(0.7) \qquad m_h = \inf \{ \int_{\Omega_h} f_h(x,Du)dx \ : u \in W^{1,p}(\Omega_h),\ u = \varphi \text{ on } \partial\Omega \},$$

and all the results of the first part, concerning Γ-convergence and convergence of minima and minimizers, still hold, with slight modifications in the proofs.

In Section 3, under the same assumptions, we restrict ourselves to the particular case in which f_h is a quadratic form in ξ, i.e

$$f_h(x,\xi) = a_{ij}^h(x)\xi_i\xi_j ,$$

that vanishes on $B_h \times \mathbf{R}^N$. The main purpose is then to study the asymptotic behaviour of the weak solutions, the eigenvalues, and the eigenspaces related to the following mixed BVP:

$$(0.8) \qquad \begin{cases} -D_i(a_{ij}^h D_j u) + \lambda u = g & \text{in } \Omega_h \\ \dfrac{\partial u}{\partial \upsilon_h} = 0 & \text{on } \partial B_h \\ u = 0 & \text{on } \partial\Omega , \end{cases}$$

where $g \in L^2(\Omega)$, $\lambda>0$ and $\partial/\partial \upsilon_h$ denotes the conormal derivative operator at ∂B_h. To this aim, according to the usual variational formulation, we introduce the functional

$$\Phi_h(u) = \int_{\Omega_h} f_h(x,Du)\, dx \;+\; \int_{\Omega_h} (\lambda u^2 - 2gu)\, dx$$

obtained perturbing F_h^0 (see (0.1) with $\varphi=0$) by the addition of continuous terms, and consider the related minimum problem on $L^2(\Omega)$.

First of all, we prove that $(\Phi_h)_h$ Γ-converges to the functional

$$\Phi(u) = \int_{\Omega} a_{ij} D_j u D_i u\, dx \;+\; \int_{\Omega} (\lambda u^2 - 2gu) b\, dx \ ,$$

where b denotes the $L^\infty(\Omega)$-weak* limit of the characteristic functions of $(\Omega_h)_h$.

Then, provided $(\Omega_h)_h$ satisfies the extension property mentioned above, we obtain the strong compactness of a sequence $(u_h)_h$ of minimizers of Φ_h, that are clearly weak solutions to (0.8). The limit u of such a sequence turns out to minimize Φ on $L^2(\Omega)$, that is to solve the following limit equation

$$(0.9) \qquad \begin{cases} -D_i(a_{ij}\, D_j u) + \lambda b u^2 = bg & \text{in } \Omega \\ u \in H^1_o(\Omega) \ . \end{cases}$$

Finally, we consider the spectral behaviour of the operators defining equation (0.8). The same problem has been studied by Boccardo and Marcellini in [3], in the case of uniformly elliptic operators and by Vanninathan and Shamaev (see [32], [33], [30]), for the homogenization of elliptic eigenvalue problems with periodic holes.

In our case, even if we do not have the uniform convergence of the resolvent operators, that are simply pointwise convergent, under the above assumptions, we obtain (see Theorem 3.6) the convergence of the eigenvalues to the ones of the limit operator defining equation (0.9), and the Mosco-convergence of corresponding spaces of eigenvectors.

A detailed proof of this theorem is provided in Appendix A, where some basic results on the spectral properties of compact operators are also recalled, with a special regard to the variational characterization of their eigenvalues.

In Appendix B the problem of determining some analytic formula for the Γ-limit is briefly mentioned through two simple cases. The former concerns a sequence of quadratic integral functionals whose coefficients a^h_{ij} tend in measure to a matrix of functions b_{ij} that turn out to be the coefficients of the integrand of the Γ-limit. The latter is the case of homogenization of non linear, non equicoercive integral functionals. For both cases we just state the results and send back to the original papers ([15] and [4], respectively) for the proofs and more references.

The results of Sections 1, 2, 3 will be published in the paper [8].

1. Γ-convergence results

In this section we consider sequences of non-coercive integral functionals. We are interested in their behaviour with respect to Γ-convergence and in (possibly) integral representation formulas for the Γ-limits. Therefore we remind, first of all, some basic definitions.

Let Ω be a bounded open subset of $\mathbf{R}^N$, $N \geq 2$, and $p \in \mathbf{R}$, $p > 1$. Given a sequence of functionals $F_h\colon L^p(\Omega) \to \mathbf{R}$, we say that

$$F' = \Gamma\text{-}\liminf_{h \to \infty} F_h$$

if for every $u \in L^p(\Omega)$ the following conditions are satisfied:

1. $\forall\, u_h \to u$ strongly in $L^p(\Omega)$ $\quad F'(u) \leq \liminf_{h \to \infty} F_h(u_h)$
2. $\exists\, u_h \to u$ strongly in $L^p(\Omega)$ $\quad F'(u) \geq \liminf_{h \to \infty} F_h(u_h)$.

Moreover, we say that

$$F'' = \Gamma\text{-}\limsup_{h \to \infty} F_h$$

if 1. and 2. are verified with F' replaced by F'' and liminf replaced by limsup, respectively. Finally, we say that $(F_h)_h$ Γ-converges to F and write

$$F = \Gamma\text{-}\lim_{h\to\infty} F_h \qquad \text{or} \qquad F_h \xrightarrow{\Gamma} F \quad ,$$

if $F' = F'' = F$ or, equivalently, if for every $u \in L^p(\Omega)$ one has

$$3.\ \forall\, u_h \to u \quad \text{strongly in } L^p(\Omega) \qquad F(u) \le \liminf_{h\to\infty} F_h(u_h)$$

$$4.\ \exists\, u_h \to u \quad \text{strongly in } L^p(\Omega) \qquad F(u) \ge \limsup_{h\to\infty} F_h(u_h).$$

The next Theorem 1.1, which is nothing but a particular case of a basic theorem in Γ-convergence theory that suitably applies to many problems in calculus of variations, is due to De Giorgi-Franzoni ([17], Corollary 2.4). It will be needed in Section 2 in order to deduce convergence for minimizers and minima of our functionals.

THEOREM 1.1

Let $(F_h)_h$ *be a sequence of functions from* $L^p(\Omega)$ *to the extended reals and* $(u_h)_h$ *a sequence in* $L^p(\Omega)$, *such that for every* h

$$F_h(u_h) = \min_{v \in L^p(\Omega)} F_h(v).$$

Then, if $(F_h)_h$ Γ*-converges to* F *and* $(u_h)_h$ *converges to* u *in* $L^p(\Omega)$, *it follows that*

$$F_h(u_h) \to \min_{v \in L^p(\Omega)} F(v) \qquad \text{and} \qquad F(u) = \min_{v \in L^p(\Omega)} F(v)\ .$$

In order to deal with integral functionals of the type

$$F_h(u) = \int_\Omega f_h(x,Du)\, dx\ ,$$

we consider now a sequence $(f_h)_{h\in\mathbf{N}}$, $f_h: \Omega\times\mathbf{R}^N \to [0,+\infty[$, of non negative Borel functions which are defined on $\Omega\times\mathbf{R}^N$ and are convex on $\mathbf{R}^N$ for almost every x in Ω. We assume that there exist two positive constants c_1, c_2, with $0 < c_1 \le c_2$, such that

$$(1.1) \qquad 0 \le f_h(x,\xi) \le c_2|\xi|^p \qquad \text{in } \Omega \times \mathbf{R}^N$$

$$(1.2) \qquad c_1|\xi|^p \le f_h(x,\xi) \qquad \text{in } \Omega_h \times \mathbf{R}^N$$

where $\Omega_h = \Omega - B_h$ and B_h is the union of a finite family $\{B_h^i : i \in I_h\}$ of closed sets of the form $B_h^i = x_h^i + r_h^i B$, with $x_h^i \in \Omega$ and $r_h^i > 0$. We suppose that $B \subseteq \mathbf{R}^N$ is a given closed set with non empty interior and regular boundary (say Lipschitz-continuous). Moreover, we assume that

$0 \in B$, $\operatorname{diam} B \leq 1$ and $B \subset\subset D$, where D is a fixed open subset of $\mathbf{R}^N$. From now on we shall use the following notation:

$$(1.3)\qquad \begin{cases} D_h^i = x_h^i + r_h^i D \\ C = D - B \\ C_h^i = D_h^i - B_h^i \\ B_h = \bigcup\limits_{i\in I_h} B_h^i \ , \quad D_h = \bigcup\limits_{i\in I_h} D_h^i \ , \quad C_h = \bigcup\limits_{i\in I_h} C_h^i \ . \end{cases}$$

Let us suppose that

$$(1.4)\qquad D_h^i \cap D_h^j = \emptyset \qquad \forall\ i,j \in I_h,\ i\neq j, \forall\ h \in \mathbf{N}$$

$$(1.5)\qquad D_h^i \subseteq \Omega \qquad \forall\ i \in I_h,\ \forall\ h \in \mathbf{N}$$

$$(1.6)\qquad r_h = \max_{i\in I_h} r_h^i \to 0 \qquad \text{as} \quad h \to \infty.$$

Let us denote by $\mathcal{A}=\mathcal{A}(\Omega)$ the family of the open subsets of Ω and define the sequence of functionals $(F_h)_{h\in \mathbf{N}}$, $F_h\colon L^p(\Omega) \times \mathcal{A} \to [0,+\infty]$ as follows:

$$(1.7)\qquad F_h(u,A) = \begin{cases} \displaystyle\int_A f_h(x,Du)\,dx & \text{if } u \in W^{1,p}_{loc}(A), \\ +\infty & \text{otherwise.} \end{cases}$$

The following theorem gives, up to a subsequence, an integral representation formula for the Γ-limit of $(F_h)_h$ in the space $W^{1,p}$.

THEOREM 1.2

Under the assumptions (1.1) - (1.7), *there exist a subsequence* $(F_{\sigma(h)})_h$ *of the sequence* (1.7) *and a functional* $F\colon L^p(\Omega) \times \mathcal{A} \to [0,+\infty]$ *such that*

$$(1.8)\qquad F_{\sigma(h)}(\,\cdot\,,A) \quad \Gamma\text{-converges to} \quad F(\,\cdot\,,A)$$

for every $A \in \mathcal{A}$. *Moreover, there exists a Borel function* $f\colon \Omega \times \mathbf{R}^N \to [0,+\infty[$, *convex on* $\mathbf{R}^N$ *for almost every* $x\in \Omega$, *for which*

$$(1.9) \qquad F(u,A) = \begin{cases} \int_A f(x,Du)dx & \text{if } u \in W^{1,p}(A) \\ +\infty & \text{otherwise} \end{cases}$$

and

$$(1.10) \qquad c_0 |\xi|^p \leq f(x,\xi) \leq c_2 |\xi|^p \qquad \text{a.e. } x \in \Omega, \forall \xi \in \mathbf{R}^N,$$

for a suitable constant c_0, *with* $0 < c_0 \leq c_2$.

We present here some technical lemmas, which are needed for the proof of Theorem 1.2. They will also be exploited in Section 2, to prove some compactness result. We begin with the following extension lemma, proved for instance in [13] (see Lemma 3).

LEMMA 1.3

Given $r > 0$ *and* $x_0 \in \mathbf{R}^N$, *we denote by* B_r, C_r, D_r *the sets* $B_r = x_0 + rB$, $D_r = x_0 + rD$, $C_r = D_r - B_r$. *Then for every* $u \in W^{1,p}(C_r)$, *there exists an extension* $\tilde{u} \in W^{1,p}(D_r)$ *such that*

$$(1.11) \qquad \tilde{u} = u \qquad \text{on } C_r$$

$$(1.12) \qquad \int_{D_r} |D\tilde{u}|^p dx \leq c_3 \int_{C_r} |Du|^p dx ,$$

where the constant c_3 *depends on* N, p, B *and* D, *but not on* r *and* u. ∎

Let us denote by 1_{Ω_h} the characteristic function of the set Ω_h:

$$(1.13) \qquad 1_{\Omega_h}(x) = \begin{cases} 1 & x \in \Omega_h \\ 0 & x \in B_h . \end{cases}$$

By the Banach-Alaoglu theorem it is not restrictive to assume that $(1_{\Omega_h})_h$ converges to a non negative function $b \in L^\infty(\Omega)$, in the weak* topology of $L^\infty(\Omega)$:

$$(1.14) \qquad 1_{\Omega_h} \rightarrow b \qquad L^\infty(\Omega)\text{-weak*}.$$

In the following lemma we prove that b is strictly positive.

LEMMA 1.4

If the sequence (B_h) *satisfies the assumptions* (1.4), (1.5), (1.6) *then*

(1.15) $$b(x) \geq \beta > 0 \qquad a.e.\ x \in \Omega,$$

where $\beta = |C|/|D|$.

PROOF:

By the Lebesgue derivation theorem, it is enough to prove that

(1.16) $$\int_{B(x_o,\rho)} b\,dx \geq \beta\,|B(x_o,\rho)|\ ,$$

for every ball $B(x_0,\rho)$ of centre x_0 and radius ρ, contained in Ω. Since b is the weak*-limit of $(1_{\Omega_h})_h$, (1.16) is equivalent to

(1.17) $$\lim_{h\to\infty} |B_h \cap B(x_0,\rho)| \leq \frac{|B|}{|D|}\,|B(x_0,\rho)|\,.$$

To prove (1.17) we observe that, since for h large enough $B_h^i \cap B(x_0,\rho) \neq \varnothing$ implies $D_h^i \subseteq B(x_0,\rho + r_h \mathrm{diam}D)$, we have

$$|D|/|B|\,|B_h \cap B(x_0,\rho)| \leq |D_h \cap B(x_0,\rho + r_h\mathrm{diam}D)| \leq |B(x_0,\rho + r_h\mathrm{diam}D)|$$

and then

$$\frac{|D|}{|B|}\lim_{h\to\infty} |B_h \cap B(x_0,\rho)| \leq \lim_{h\to\infty} |B(x_0,\rho + r_h\,\mathrm{diam}D)| = |B(x_0,\rho)|\ ,$$

which concludes the proof. ∎

PROOF OF THEOREM 1.2:

Theorem 4.3 and Proposition 2.4 in [5] prove the existence of a subsequence $(F_{\sigma(h)})_h$ and a Borel function $f: \Omega \times \mathbf{R}^N \to [0,+\infty[$, convex on $\mathbf{R}^N$ for almost every $x \in \Omega$, such that, for every $A \in \mathcal{A}$, the Γ-limit of $(F_{\sigma(h)}(\,.\,,A))_h$ exists on $W^{1,p}(A)$ and is given by

$$\int_A f(x,Du)dx,$$

for each $u \in W^{1,p}(A)$. For simplicity, along this proof we denote such a subsequence by $(F_h)_h$ and we set

$$F'(u,A) = (\Gamma\text{-}\liminf_{h \to \infty} F_h)(u,A).$$

To complete the proof of (1.8), (1.9) we now show that

$$F'(u,A) = +\infty, \tag{1.18}$$

for every $A \in \mathcal{A}$ and $u \in L^p(A)-W^{1,p}(A)$. Actually, $F'(u,A) < +\infty$ implies $u \in W^{1,p}(A)$. In fact, let us fix $A \in \mathcal{A}$ and $u \in L^p(A)$ such that $F'(u,A) < +\infty$. From the definition of Γ-liminf and from (1.2), it follows the existence of a sequence $(u_h)_h$ in $L^p(A)$ that tends to a function u in the norm topology, such that up to a subsequence

$$c_1 \lim_{h \to \infty} \int_{\Omega_h \cap A} |Du_h|^p dx \le F'(u,A).$$

Our aim is to show that this implies

$$\int_A |Du|^p dx \le cF'(u,A), \tag{1.19}$$

which clearly proves that $u \in W^{1,p}(A)$.

For technical reasons we choose an arbitrary open subset $A' \subset\subset A$ and we denote by J_h the subset of I_h defined as follows

$$J_h = \{i \in I_h : B_h^i \cap A' \ne \varnothing\}.$$

Note that, for h large enough, $i \in J_h \Longrightarrow D_h^i \subseteq A$. Using Lemma 1.3, we can find suitable functions

$$\tilde{u}_h \in W^{1,p}(A' \cup \bigcup_{i \in J_h} D_h^i)$$

such that

$$\tilde{u}_h = u_h \qquad \text{on } (A' \cup \bigcup_{i \in J_h} D_h^i) \cap \Omega_h$$

$$\int_{D_h^i} |D\ddot{u}_h|^p dx \le c \int_{C_h^i} |Du_h|^p dx,$$

for every $h \in \mathbf{N}$ and $i \in J_h$. Since the constant c in the preceding inequality does not depend on i and h and the sets $(D_h^i)_i$ do not intersect each others for h fixed, we obtain that

$$\text{(1.20)} \qquad \limsup_{h \to \infty} \int_{A'} |D\ddot{u}_h|^p dx \le cF'(u,A).$$

Let us now consider for every $t \in \mathbf{R}^+$, the function $\tilde{u}_h^t = (\ddot{u}_h \vee t) \wedge (-t)$. It is clear that $(\tilde{u}_h^t)$ has at least a subsequence which is bounded in $W^{1,p}(A')$, independently of h. Then, passing if necessary to a further subsequence, $(\tilde{u}_h^t)$ tends for every t weakly in $W^{1,p}(A')$ and strongly in $L^p(A')$ to a function $v_t \in W^{1,p}(A')$, depending on t. It can be proved that $v_t = u^t = (u \vee t) \wedge (-t)$ on A'; in fact, since $\tilde{u}_h^t = u_h^t$ on $\Omega_h \cap A'$,

$$\int_{A' \cap \Omega_h} |u^t - v_t|^p dx \le c \int_{A' \cap \Omega_h} |u^t - u_h^t|^p dx + c \int_{A' \cap \Omega_h} |\tilde{u}_h^t - v_t|^p dx ,$$

where the right hand side tends to zero as h goes to infinity, while

$$\int_{A' \cap \Omega_h} |u^t - v_t|^p dx \longrightarrow \int_{A'} b\, |u^t - v_t|^p dx.$$

This yields

$$\int_{A'} b\, |u^t - v_t|^p dx = 0$$

and by (1.15) $u^t = v_t$ on A'. As a consequence $u^t \in W^{1,p}(A')$ for every $t \in \mathbf{R}^+$ and moreover, by (1.20), we have

$$\int_{A'} |Du^t|^p dx \le \liminf_{h \to \infty} \int_{A'} |D\tilde{u}_h^t|^p dx \le cF'(u,A).$$

Taking into account that $u \in L^p(A')$ and that

$$\|u^t\|_{L^p(A')} \le \|u\|_{L^p(A')}$$

it follows that $(u^t)_t$ is bounded in $W^{1,p}(A')$ and hence, up to a subsequence, tends to u weakly in $W^{1,p}(A')$. This immediatly yields (1.19).

In order to complete the proof of the theorem, we now show that the two inequalities in (1.10) hold. Taking $u_h = u$ in condition 3. of the definition of Γ-limit, from (1.1) we obtain

$$\int_A f(x,Du)dx \leq c_2 \liminf_{h \to \infty} \int_A |Du_h|^p dx.$$

for every $A \in \mathcal{A}$ and $u \in W^{1,p}(A)$. Taking now $u(x) = \xi \cdot x$, with $\xi \in \mathbf{Q}^N$ and using the Lebesgue derivation theorem we first have

$$f(x,\xi) \leq c_2 |\xi|^p \qquad \forall\, x \in \Omega_0, \forall\, \xi \in \mathbf{Q}^N,$$

where Ω_0 is a suitable subset of Ω, whose complement in Ω has Lebesgue measure zero. Since $f(x,.)$ is convex for a.e. $x \in \Omega$, it finally follows

$$(1.21) \qquad f(x,\xi) \leq c_2 |\xi|^p \qquad \forall x \in \Omega_0 \quad , \forall \xi \in \mathbf{R}^N.$$

In order to prove the first inequality in (1.10), we arbitrarily choose $A \in \mathcal{A}$ and $u \in W^{1,p}(A)$; from the definition of Γ-limit and from (1.2), there exists a sequence $(u_h)_h$ in $L^p(A) \cap W^{1,p}_{loc}(A)$ which tends to u in the L^p-norm and such that

$$\int_A f(x,Du)dx \geq c_1 \liminf_{h \to \infty} \int_{\Omega_h \cap A} |Du_h|^p dx.$$

Hence, up to a subsequence,

$$c_1 \lim_{h \to \infty} \int_{\Omega_h \cap A} |Du_h|^p dx \leq \int_A f(x,Du)dx < +\infty$$

Applying the same argument as in the proof of (1.19), we obtain that

$$\frac{c_1}{c_3} \int_A |Du|^p dx \leq \int_A f(x,Du)dx,$$

where c_3 is the constant defined in Lemma 1.3. From this inequality, taking $u(x) = \xi \cdot x$ and using again the Lebesgue derivation theorem and the convexity of $f(x,.)$, we finally have

$$f(x,\xi) \geq c_0 |\xi|^p \qquad \text{for a.e. } x \in \Omega, \forall \xi \in \mathbf{R}^N,$$

where $c_0 = c_1/c_3$. ∎

From now on, we shall always suppose that

(1.22) $$F_h(\,.\,,A) \xrightarrow{\Gamma} F(\,.\,,A)$$

for every $A \in \mathcal{A}$, where F satisfies (1.9) and (1.10). To study the asymptotic behaviour of minimum problems for the functionals F_h with Dirichlet boundary conditions, given $\varphi \in W^{1,p}(\Omega)$, we consider the functionals F_h^φ, F^φ: $L^p(\Omega) \to [0,+\infty]$, defined as follows

(1.23) $$F_h^\varphi(u) = \begin{cases} F_h(u,\Omega) & \text{if } u - \varphi \in W_0^{1,p}(\Omega) \\ +\infty & \text{otherwise} \end{cases},$$

(1.24) $$F^\varphi(u) = \begin{cases} F(u,\Omega) & \text{if } u - \varphi \in W_0^{1,p}(\Omega) \\ +\infty & \text{otherwise} \end{cases}.$$

Taking into account the preceding theorems and lemmas, we can immediately prove a result about Γ-convergence for the sequence $(F_h^\varphi)_h$ and integral representation of its Γ-limit.

THEOREM 1.5

In addition to the assumptions (1.1) - (1.7), *we assume that* F_h *satisfies* (1.22). *Then the sequence* $(F_h^\varphi)_h$ *Γ-converges in the norm topology of* $L^p(\Omega)$ *to the functional* F^φ.

PROOF:

We have to check that for every $u \in L^p(\Omega)$

(a) $$F^\varphi(u) \leq (\Gamma\text{-}\liminf_{h\to\infty} F_h^\varphi)(u)$$

(b) $$F^\varphi(u) \geq (\Gamma\text{-}\limsup_{h\to\infty} F_h^\varphi)(u).$$

To prove (a) we can assume that u makes the right hand side finite. Then there exist an increasing sequence of integers $(h_k)_k$ and a sequence $(u_k)_k$, with $(u_k - \varphi) \in W_0^{1,p}(\Omega)$, which tends to u in the L^p-norm, such that

$$(\Gamma\text{-}\liminf_{h\to\infty} F_h^{\varphi})(u) = \lim_{k\to\infty} F_{h_k}^{\varphi}(u_k) = \lim_{k\to\infty} F_{h_k}(u_k,\Omega) < +\infty.$$

Then, the following inequality

$$\int_{\Omega_{h_k}} |Du_k|^p dx \le \text{const}$$

holds for k large enough. Taking into account Lemma 1.3, one can find a sequence $(\tilde{u}_k)_k$, $\tilde{u}_k - \varphi \in W_0^{1,p}(\Omega)$, which converges weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ to some function $v \in W_0^{1,p}(\Omega) + \varphi$, such that

$$\tilde{u}_k|_{\Omega_{h_k}} = u_k|_{\Omega_{h_k}} .$$

Using the same argument as in the proof of Theorem 1.2 (where we prove $v_t = u^t$ on A'), one obtains that $u = v$ in Ω and hence $u - \varphi \in W_0^{1,p}(\Omega)$. Since $F_h(\,.\,,\Omega) \le F_h^{\varphi}(\,.\,)$, we then have:

$$F^{\varphi}(u) = \int_{\Omega} f(x,Du)dx = (\Gamma\text{-}\lim_{h\to\infty} F_h)(u,\Omega) \le (\Gamma\text{-}\liminf_{h\to\infty} F_h^{\varphi})(u),$$

that is (a).

We now prove (b) adapting an argument used in [5] (see the proof of Theorem 3.1) to obtain the subadditivity of the Γ-limit. We can assume $F^{\varphi}(u)$ finite, from which $F^{\varphi}(u) = F(u,\Omega)$ and $u - \varphi \in W_0^{1,p}(\Omega)$. From (1.22) there exists $u_h \to u$ in $L^p(\Omega)$, such that

$$F(u,\Omega) = \lim_{h\to\infty} F_h(u_h,\Omega).$$

Setting $\Phi(x) = \text{dist}(x,\partial\Omega)$, we then have $\Phi \in W_0^{1,p}(\Omega)$ and $|D\Phi| \le 1$ a.e. in Ω. Let us now define

$$v_h = \begin{cases} u - \Phi & u_h < u - \Phi \\ u_h & |u_h - u| \le \Phi \\ u + \Phi & u_h > u + \Phi \end{cases},$$

that is $v_h = [(u - \Phi) \vee u_h] \wedge (u + \Phi)$. Since $|v_h - u| \le \Phi$ with $\Phi \in W_0^{1,p}(\Omega)$, then $v_h - u \in W_0^{1,p}(\Omega)$, from which it follows that

$$F_h^{\varphi}(v_h) = F_h(v_h,\Omega).$$

Moreover $|v_h - u| \leq |u_h - u|$ a.e. in Ω and hence $v_h \to u$ in $L^p(\Omega)$. This implies that

$$(\Gamma\text{-}\limsup_{h\to\infty} F_h^{\varphi})(u) \leq \limsup_{h\to\infty} F_h^{\varphi}(v_h) = \limsup_{h\to\infty} F_h(v_h,\Omega).$$

We finally get the conclusion if we prove that

$$\limsup_{h\to\infty} F_h(v_h,\Omega) \leq \lim_{h\to\infty} F_h(u_h,\Omega).$$

But this is an easy consequence of the estimate

$$F_h(v_h,\Omega) \leq F_h(u_h,\Omega) + c_2 \int_{\{|u_h-u|>\Phi\}} (|Du|^p + 1)dx$$

and of the fact that $|\{|u_h - u| > \Phi\}| \to 0$ as h goes to infinity. ∎

2. Compactness of minima and minimizers

In addition to the assumptions (1.1) - (1.7), in this section we shall always suppose that for every $A \in \mathcal{A}$ the sequence $(F_h(\,.\,,A))_h$ Γ-converges to a functional $F(\,.\,,A)$ satisfying (1.9) and (1.10). Our interest is now to investigate the asymptotic behaviour of the sequence of Dirichlet minimum problems of the form

(2.1) $$m_h = \inf_{u\in L^p(\Omega)} F_h^{\varphi}(u).$$

More precisely we want to prove that

(2.2) $$m_h \to m \qquad \text{as} \quad h \to \infty,$$

where m denotes the minimum value of the Γ-limit F^{φ}:

(2.3) $$m = \min_{u\in L^p(\Omega)} F^{\varphi}(u).$$

To this aim, given h, we introduce the relaxed functional (or L^p-lower semicontinuous envelope) $sc^- F_h^{\varphi}$ defined as follows

$$\text{(2.4)} \qquad sc^-F_h^\varphi(u) = \inf \{\liminf_{n\to\infty} F_h^\varphi(u_n) : u_n \in L^p(\Omega),\ u_n \to u \text{ strongly in } L^p(\Omega)\}$$

for every $u \in L^p(\Omega)$. It is well known that $sc^-F_h^\varphi$ can be characterized as the greatest L^p-l.s.c. functional which is less then or equal to F_h^φ on $L^p(\Omega)$ and that

$$\inf_{u\in L^p(\Omega)} F_h^\varphi(u) = \inf_{u\in L^p(\Omega)} sc^-F_h^\varphi(u).$$

In the following theorem we briefly consider the particular case in which f_h is identically zero on $B_h \times \mathbf{R}^N$ and show that the corresponding relaxed functional $sc^-F_h^\varphi$ takes a very simple integral form.

THEOREM 2.1

In addition to assumptions (1.1) - (1.5), *suppose that*

$$\text{(2.5)} \qquad f_h(x,\xi) = 0 \qquad \text{on } B_h \times \mathbf{R}^N$$

and consider F_h^φ *as given by* (1.23). *Then the relaxed functional* $sc^-F_h^\varphi$ *defined by* (2.4) *has the following structure*

$$\text{(2.6)} \qquad sc^-F_h^\varphi(u) = \begin{cases} \int_{\Omega_h} f_h(x,Du)dx & \text{if } u|_{\Omega_h} \in W^{1,p}(\Omega_h),\ u = \varphi \text{ on } \partial\Omega \\ +\infty & \text{otherwise in } L^p(\Omega), \end{cases}$$

for every $h \in \mathbf{N}$.

PROOF:

First of all, we note that the equality $u=\varphi$ on $\partial\Omega$ in (2.6) means simply that if $\psi \in C_0^\infty(\Omega)$ such that $\psi=1$ a.e. in a neighbourhood of B_h, then $(1-\psi)(u-\varphi) \in W_0^{1,p}(\Omega)$. In particular, if $\partial\Omega$ is regular enough (say Lipschitz-continuous) then $u=\varphi$ on$\partial\Omega$ in the sence of the trace operator from $W^{1,p}(\Omega_h)$ to $L^p(\partial\Omega)$.

In order to prove (2.6), let us set

$$\overline{F}_h^{\varphi}(u) = \begin{cases} \displaystyle\int_{\Omega_h} f_h(x,Du)dx & \text{if } u_{|\Omega_h} \in W^{1,p}(\Omega_h),\ u=\varphi \text{ on } \partial\Omega \\ +\infty & \text{otherwise in } L^p(\Omega), \end{cases}$$

with h fixed in **N**. We want to prove that

i) $\overline{F}^{\varphi}_h$ is l.s.c. on $L^p(\Omega)$

ii) $\overline{F}^{\varphi}_h \le F^{\varphi}_h$ on $L^p(\Omega)$

iii) if $G: L^p(\Omega) \to [0,+\infty]$ is l.s.c. and $G \le F^{\varphi}_h$, then $G \le \overline{F}^{\varphi}_h$ on $L^p(\Omega)$.

Let us prove i). Given $u \in L^p(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$ such that

$$\liminf_{h\to\infty} \overline{F}_h^{\varphi}(u_n) < +\infty$$

there exists a subsequence $(u_{\sigma(n)})_n$ for which

$$\int_{\Omega_h} |Du_{\sigma(n)}|^p dx \le c.$$

Then, passing, if necessary, to a further subsequence, $u_{\sigma(h)}$ tends to u weakly in $W^{1,p}(\Omega_h)$, as n goes to infinity. Since $\overline{F}^{\varphi}_h$ is finite on $u_{\sigma(h)}$ and u, and the functional

$$\int_{\Omega_h} f_h(x,Dv)\,dx$$

is weakly-lower semicontinuous on $W^{1,p}(\Omega_h)$, it follows that

$$\overline{F}_h^{\varphi}(u) \le \liminf_{n\to\infty} \overline{F}_h^{\varphi}(u_{\sigma(n)}),$$

which proves i).

The proof of ii) turns out right from the definitions.

To show that iii) holds, let us consider a functional $G: L^p(\Omega) \to [0,+\infty]$, l.s.c. on $L^p(\Omega)$, $G \le F^{\varphi}_h$. It is enough to prove that for every $u \in L^p(\Omega)$ such that $u_{|\Omega_h} \in W^{1,p}(\Omega_h)$ and $u=\varphi$ on $\partial\Omega$ it follows

$$G(u) \leq \int_{\Omega_h} f_h(x,Du)dx \, .$$

Given such a function u, let us consider a sequence $(\varphi_n)_n$ in $C_0^\infty(\Omega)$ such that supp $\varphi_n \subset\subset B_h$ for every n and $\varphi_n \to (u - E_h u)$ in $L^p(\Omega)$, as n tends to infinity. If we set $u_n = \varphi_n + E_h(u_{|\Omega_h})$, then $(u_n - \varphi) \in W_0^{1,p}(\Omega)$, $u_n = u$ on Ω_h and $u_n \to u$ in $L^p(\Omega)$. Since $G \leq F^\varphi_h$, this implies

$$G(u) \leq \liminf_{n\to\infty} G(u_n) \leq \liminf_{n\to\infty} F_h^\varphi(u_n) = \liminf_{n\to\infty} \int_{\Omega_h} f_h(x,Du_n)dx = \int_{\Omega_h} f_h(x,Du)dx \, ,$$

that concludes the proof. ∎

Going back to the general assumptions (1.1) - (1.7), we remark that the problem of proving an integral representation formula for $sc^-F_h{}^\varphi$ on the space $X(F_h{}^\varphi)=\{u\in L^p(\Omega):\ sc^-F_h{}^\varphi(u) < +\infty\}$ is still an open question. Anyway, many results are already known for functionals of this type on suitable subspaces of $X(F_h{}^\varphi)$. For more details see, for instance, [7], that concerns a representation formula on the space of the Lipschitz-continuous functions, [19], for the L^2-lower semicontinuous envelope of quadratic functionals on $H^1(\Omega)$, [6], regarding integral representation on the Sobolev space $W^{1,p}$, and the wide bibliography therein.

In order to prove (2.2) we study existence and properties of the minimizers of $sc^-F_h{}^\varphi$ and we individuate a set M_h^* of minimum points with the following property: if $u_h \in M_h^*$ for every $h \in N$, then $(u_h)_h$ has at least one subsequence which converges to some $u\in L^p(\Omega)$ in the L^p-norm. As a consequence of Theorem 1.1, one obtains that u solves problem (2.3) and that (2.2) is then fulfilled. Moreover in the case F^φ is strictly convex, one has also that the whole sequence $(u_h)_h$ tends to u.

The existence of minimizers for $sc^-F_h{}^\varphi$, which in the general case is not trivial, is proved in Theorem 2.3. The proof goes through an approximation argument that characterizes the elements of the class M_h^*. Some technical lemmas, concerning extension properties (see Lemma 1.3) and local L^p-estimates (Lemma 2.2) for the minimum points u_h, are also needed. They allow also to show the compactness of sequences of the form $(u_h)_h$, $u_h \in M_h^*$. We start now by establishing Lemma 2.2.

LEMMA 2.2

Let B_r, C_r, D_r, N, p *be defined as in Lemma* 1.3 *and let* $u\in W^{1,p}(D_r)$ *be a solution of the* *following minimum problem*

(2.7) $\qquad G(u) \le G(v) \qquad \forall\, v \in W^{1,p}(D_r) \,/\, v-u \in W_0^{1,p}(D_r)$

with $G\colon W^{1,p}(D_r) \to [0, +\infty]$ *defined by*

$$G(v) = \int_{D_r} g(x,Dv)\,dx.$$

Let us assume that $g\colon D_r \times \mathbf{R}^N \to [0, +\infty[$ *is a Borel function,* $g(x, .\,)$ *is convex for a.e.* $x \in D$ *and there exist three real constants* $k_0, k_1, k_2,\ 0 < k_0 < k_1 \le k_0$, *such that*

(2.8) $\qquad k_0 \,|\xi|^p \le g(x,\xi) \le k_2 \,|\xi|^p \qquad$ *in* $B_r \times \mathbf{R}^N$

(2.9) $\qquad k_1 \,|\xi|^p \le g(x,\xi) \le k_2 \,|\xi|^p \qquad$ *in* $C_r \times \mathbf{R}^N$.

Then the function u *satisfies the following estimates:*

(2.10) $\qquad \sup_{B_r} |u|^p \le c_4 ⨍_{C_r} |u|^p dx$

(2.11) $\qquad \int_{B_r} |u|^p dx \le c_5 \int_{C_r} |u|^p dx$

where $⨍$ *denotes the average and the constants* c_4, c_5 *depend on* k_1, k_2, p, N, B, *and* D, *but are independent of* k_0 *and* r.

PROOF:

It is enough to prove (2.10) since it immediately implies (2.11). Let us assume that $u \in W^{1,p}(D_r)$ is a solution of the minimum problem (2.7); then, for every open set $A \subseteq D_r$, u satisfies the weak maximum principle

(2.12) $\qquad \sup_{A} u = \sup_{\partial A} u$

(see, for instance, [21], Theorem 4.3) and the local estimate

(2.13) $\qquad \sup_{B(y,R/2)} |u| \le c_6 \left(⨍_{B(y,R)} |u|^p dx\right)^{1/p}$

for every y, R such that $B(y,R) = \{x \in \mathbf{R}^N: |x-y| < R\}$ is contained in C_r (see [20], Theorem 2.1). Here the constant c_6 depends only on N, p and k_2/k_1.

Now (2.10) follows easily from (2.12) and (2.13); in fact, since $B \subset\subset D$, there exist an open set A and a positive number ε, such that $B \subset\subset A \subset\subset D$ and $\{x \in \mathbf{R}^N: d(x,A) < 2\varepsilon\} \subset C$. Then, setting $A_r = x_0 + rA$ we have

$$\sup_{B_r} u \le \sup_{\partial A_r} u \le \sup_{y \in \partial A_r} \sup_{B(y,\varepsilon r)} |u| \le$$

$$c_6 \sup_{y \in \partial A_r} \left(\fint_{B(y,\varepsilon r)} |u|^p dx\right)^{1/p} \le \frac{c_6}{|B(0,\varepsilon r)|^{1/p}} \sup_{y \in \partial A_r} \left(\int_{B(y,\varepsilon r)} |u|^p dx\right)^{1/p} \le$$

$$\frac{c_6}{|B(0,\varepsilon r)|^{1/p}} \left(\int_{C_r} |u|^p dx\right)^{1/p}.$$

Taking the power p, it follows that

$$\sup_{B_r} |u|^p \le |\sup_{B_r} u|^p \le (\varepsilon r)^{-N} c_6 \int_{C_r} |u|^p dx = c_4 \fint_{C_r} |u|^p dx,$$

where c_4 depends on N, p, k_2/k_1, B and D, but not on k_0 and r. ∎

In the proof of the next existence theorem, we need to approximate $sc^- F_h^\varphi$ with a sequence of coercive functionals $(G^\varphi_{h,s})$. Therefore we first introduce the functionals Ψ_p, $G_{h,s}$, $G^\varphi_{h,s}$: $L^p(\Omega) \times \mathcal{A} \to [0,+\infty]$ defined as follows:

(2.14)
$$\Psi_p(u,A) = \begin{cases} \int_A |Du|^p dx & \text{if } u \in W^{1,p}(A) \\ +\infty & \text{otherwise} \end{cases}$$

(2.15)
$$G_{h,s}(u,A) = (F_h + \frac{1}{s}\Psi_p)(u,A)$$

$$(2.16) \qquad G^{\varphi}_{h,s}(u) = \begin{cases} G_{h,s}(u,\Omega) & \text{if } u-\varphi \in W^{1,p}_0(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

THEOREM 2.3

If we assume (1.1) - (1.5), *then for every* $h \in \mathbf{N}$ *there exists* $u_h \in L^p(\Omega)$ *such that*

$$m_h = sc^- F_h{}^{\varphi}(u_h).$$

PROOF:

In order to prove the theorem, we first show that $(G^{\varphi}_{h,s})_s$ has a sequence of minimizers which is bounded in $L^p(\Omega)$.

Since $G^{\varphi}_{h,s}$ is l.s.c. and coercive in $L^p(\Omega)$, for every $h,s \in \mathbf{N}$ there exists a solution $u_{h,s} \in L^p(\Omega)$ of the minimum problem

$$(2.17) \qquad m_{h,s} = \min_{u \in L^p(\Omega)} G^{\varphi}_{h,s}(u).$$

From (1.2) one first obtains that

$$(2.18) \qquad \int_{\Omega_h} |Du_{h,s}|^p dx \le \text{const.}$$

Let us prove that

$$(2.19) \qquad \int_{\Omega_h} |u_{h,s}|^p dx \le \text{const.}$$

By Lemma 1.3 and assumption (1.4), there exists $\tilde{u}_{h,s} \in W^{1,p}_0(\Omega) + \varphi$ such that

$$(2.20) \qquad \int_{\Omega} |D(\tilde{u}_{h,s} - \varphi)|^p dx \le c_3 \int_{\Omega_h} |D(u_{h,s} - \varphi)|^p dx$$

and from Poincaré inequality

$$(2.21) \qquad \int_{\Omega_h} |u_{h,s} - \varphi|^p dx \le \int_{\Omega} |\tilde{u}_{h,s} - \varphi|^p dx \le c_\Omega \int_{\Omega} |D(\tilde{u}_{h,s} - \varphi)|^p dx.$$

From (2.21), (2.20) and (2.18), we then have (2.19). Moreover we can show that

$$(2.22) \qquad \int_{B_h} |u_{h,s}|^p dx \le \text{const}$$

uniformly with respect to h and s; in fact from (2.11) it follows that

$$\int_{B_h^i} |u_{h,s}|^p dx \le c_5 \int_{C_h^i} |u_{h,s}|^p dx$$

from which, taking the sum over i and using (1.4) we obtain

$$(2.23) \qquad \int_{B_h} |u_{h,s}|^p dx \le c_5 \int_{\Omega_h} |u_{h,s}|^p dx.$$

This last inequality, together with (2.19), gives (2.22). Hence we have finally

$$(2.24) \qquad \int_{\Omega} |u_{h,s}|^p dx \le \text{const}$$

for every $s,h \in \mathbf{N}$.

Now, up to a subsequence, $u_{h,s}$ coverges weakly in $L^p(\Omega)$ as $s \to \infty$, to some function $u_h \in L^p(\Omega)$, which, as we are going to show, minimizes $sc^- F^\varphi{}_h$. In fact, since $(G_{h,s})_s$ is monotonically decreasing and pointwise convergent to F_h as $s \to \infty$, we have

$$sc^- F_h^\varphi(u_h) \le \liminf_{s \to \infty} F_h^\varphi(u_{h,s}) \le \liminf_{s \to \infty} G_{h,s}^\varphi(u_{h,s}) .$$

But the fact that $u_{h,s}$ minimizes $G^\varphi{}_{h,s}$ on $L^p(\Omega)$ yields

$$G^\varphi{}_{h,s}(u_{h,s}) \le G^\varphi{}_{h,s}(v)$$

for every $v \in L^p(\Omega)$ and hence

$$sc^{-}F_h^{\varphi}(u_h) \leq \lim_{s\to\infty} G_{h,s}^{\varphi}(v) = F_h^{\varphi}(v)$$

for every $v \in L^p(\Omega)$. Then we have

$$sc^{-}F_h^{\varphi}(u_h) \leq \inf_{v\in L^p(\Omega)} F_h^{\varphi}(v) = \inf_{v\in L^p(\Omega)} sc^{-}F_h^{\varphi}(v) ,$$

that concludes the proof. ∎

REMARK 2.4

The limiting function $u_h \in L^p(\Omega)$, which has been determined in the preceding theorem, has the following properties:

(2.25) $$u_h|_{\Omega_h} \in W^{1,p}(\Omega_h)$$

(2.26) $$u_{h,s}|_{\Omega_h} \to u_h|_{\Omega} \quad \text{weakly in } W^{1,p}(\Omega_h), \text{ as } s \to \infty$$

(2.27) $$u_h|_{\partial\Omega} = \varphi|_{\partial\Omega}$$

(2.28) $$\int_{\Omega_h} | u_{h,s} - u_h |^p dx \to 0 \qquad \text{as } s \to \infty.$$

In fact (2.25) and (2.26) follow from (2.18) and (2.19); taking into account (2.20) and Rellich's Theorem one gets also (2.28). Finally (2.27) can be obtained noticing that, for every $s \in \mathbf{N}$, $(u_{h,s}-\varphi)$ belongs to the set of v's in $W^{1,p}(\Omega_h)$ such that $v|_{\partial\Omega}= 0$, which is a weakly closed subspace of $W^{1,p}(\Omega_h)$. ∎

In order to state the main compactness theorem, from now on, we shall indicate by M_h the class of the solutions of the minimum problem (2.1) and by M_h^* the subset formed by the functions $u_h \in L^p(\Omega)$ which are weak-L^p-limits of sequences $(u_{h,s})_s$ of minimizers of $G^{\varphi}_{h,s}$:

(2.29) $M_h = \{u \in L^p(\Omega): sc^{-}F_h^{\varphi}(u) = m_h\}$

(2.30) $M_h^* = \{u \in L^p(\Omega): \exists\ \sigma(s)\uparrow\infty, \exists\ u_s \to u \text{ weakly in } L^p(\Omega), G^{\varphi}_{h,\sigma(s)}(u_s) = m_{h,\sigma(s)}\}$.

THEOREM 2.5

Let us assume that (1.1) - (1.7) *hold and let* $(u_h)_h$ *be such that* $u_h \in M_h^*$*, for every* h. *Then there exist an increasing sequence of integers* $(\tau(h))_h$ *and a function* $u \in W^{1,p}(\Omega)$*, with* $u-\varphi \in W_0^{1,p}(\Omega)$*, such that* $u_{\tau(h)} \to u$ *in the norm topology of* $L^p(\Omega)$.

PROOF:

Since u_h is a minimum point for $sc^-F_h^\varphi$ and (1.1) - (1.3) hold, it follows that

(2.31) $$\int_{\Omega_h} |Du_h|^p dx \leq \text{const.}$$

By Lemma 1.3, there exists a sequence of functions $(\tilde{u}_h)_h$ in $W_0^{1,p}(\Omega) + \varphi$ such that

(2.32) $$\tilde{u}_h|_{\Omega_h} = u_h|_{\Omega_h}$$

(2.33) $$\int_{\Omega} |D\tilde{u}_h|^p dx \leq c \int_{\Omega_h} |Du_h|^p dx.$$

But (2.31), (2.32), (2.33) imply that $(\tilde{u}_h)_h$ is bounded in $W^{1,p}(\Omega)$ and hence it has a subsequence, still denoted by $(\tilde{u}_h)_h$, such that

(2.34) $$\tilde{u}_h \to u \qquad \text{weakly in } W^{1,p}(\Omega) \text{ and strongly in } L^p(\Omega)$$

where $u \in W_0^{1,p}(\Omega) + \varphi$.

Our aim is now to prove that also $(u_h)_h$ tends to the same u in the L^p-norm. From (2.18) we have

$$\int_{\Omega} |u_h - u|^p dx = \int_{\Omega - D_h} |\tilde{u}_h - u|^p + \sum_{i \in I_h} \int_{D_h^i} |u_h - u|^p dx$$

where the first term tends to zero because of (2.34). Setting

$$v_h^i = \fint_{D_h^i} u$$

the second one can be estimated as follows:

$$(2.35) \qquad \sum_{i \in I_h} \int_{D_h^i} |u_h - u|^p dx \le c \sum_{i \in I_h} \int_{D_h^i} |u_h - v_h^i|^p dx + c \sum_{i \in I_h} \int_{D_h^i} |v_h^i - u|^p dx.$$

Using Poincaré inequality and assumption (1.6), we obtain for the second term in the right hand side of (2.35) that

$$(2.36) \qquad \sum_{i \in I_h} \int_{D_h^i} |v_h^i - u|^p dx \le c r_h^p \int_{\Omega} |Du|^p dx \to 0.$$

The estimate of the first term is more delicate. We can actually prove the next inequality:

$$(2.37) \qquad \sum_{i \in I_h} \int_{D_h^i} |u_h - v_h^i|^p dx \le c \sum_{i \in I_h} \int_{C_h^i} |u_h - v_h^i|^p dx.$$

In fact, since $u_h \in M_h^*$, u_h is the weak-L^p-limit of a sequence $(u_{h,s})_s$ of minimizers of $G^{\varphi}{}_{h,s}$. In particular for each $h \in \mathbf{N}$ and $i \in I_h$

$$u_{h,s} - v_h^i|_{D_h^i} \to u_h - v_h^i|_{D_h^i} \qquad \text{as } s \to \infty, \text{ weakly in } L^p(D_h^i).$$

Hence we have

$$\sum_{i \in I_h} \int_{D_h^i} |u_h - v_h^i|^p dx \le \sum_{i \in I_h} \liminf_{s \to \infty} \int_{D_h^i} |u_{h,s} - v_h^i|^p dx.$$

Moreover $(u_{h,s} - v_h^i)$, restricted to D_h^i, is a local minimizer of $G_{h,s}$ (defined by (2.13)) on D_h^i i.e.

$$G_{h,s}(u_{h,s} - v_h^i, D_h^i) \le G_{h,s}(w, D_h^i)$$

for every $w \in W^{1,p}(D_h^i)$ such that $w - (u_{h,s} - v_h^i) \in W_0^{1,p}(D_h^i)$. Hence, by Lemma 2.2, $(u_{h,s} - v_h^i)$ satisfies an estimate of the type (2.11) on D_h^i, which implies

$$\sum_{i \in I_h} \liminf_{s \to \infty} \int_{D_h^i} |u_{h,s} - v_h^i|^p dx \le c_5 \sum_{i \in I_h} \liminf_{s \to \infty} \int_{C_h^i} |u_{h,s} - v_h^i|^p dx.$$

Since the convergence of $(u_{h,s} - v_h^i)_s$ is strong in C_h^i,(see (2.28)), we finally obtain

$$c_5 \sum_{i \in I_h} \liminf_{s \to \infty} \int_{C_h^i} | u_{h,s} - v_h^i |^p dx \le c_5 \sum_{i \in I_h} \int_{D_h^i} | u_h - v_h^i |^p dx,$$

which concludes the proof of (2.37). Now, to complete the proof of the theorem, it is enough to show that the right hand side of (2.37) tends to zero as h goes to infinity. To this aim we prove that

$$\sum_{i \in I_h} \int_{C_h^i} | u_h - v_h^i |^p \, dx \to 0, \tag{2.38}$$

as h goes to infinity. Since $\tilde{u}_h = u_h$ in Ω_h, then

$$\sum_{i \in I_h} \int_{C_h^i} | u_h - v_h^i |^p dx \le \sum_{i \in I_h} \int_{C_h^i} | \tilde{u}_h - u |^p dx + \sum_{i \in I_h} \int_{C_h^i} | u - v_h^i |^p dx,$$

from which, taking (3.34) and (3.36) into account, we obtain (2.38). Hence we have finally proved that

$$u_h \to u \qquad \text{strongly in } L^p(\Omega), \tag{2.39}$$

as h goes to infinity. ∎

COROLLARY 2.6

In the hypothesis of Theorem 2.5 *it follows that* $m_h \to m$, *with* m_h, m *defined by* (2.1), (2.3) *respectively.*

PROOF:

Trivial, using Theorems 1.1, 1.5, 2.5. ∎

REMARK 2.7

If the limiting functional F^φ is strictly convex, then the minimum point u of (2.3) is unique. This implies that the whole sequence $(u_h)_h$ converges to u, strongly in $L^p(\Omega)$. ∎

The assumptions (1.1) - (1.6) stated in Section 1 contain as a particular case the situation in which the integrands f_h are identically zero on the holes B_h, i.e.

$$f_h(x,\xi) = 0 \qquad \text{in } B_h \times \mathbf{R}^N.$$

In this case the asymptotic behaviour of problem (2.1) can be investigated in a different way, weakening the hypothesis concerning the structure of the holes and allowing more general geometric configurations.

To this aim we assume the following:

(2.40) Ω is a bounded subset of $\mathbf{R}^N$ and for every $h \in \mathbf{N}$ B_h is a compact subset of Ω that lies locally on one side of its boundary; ∂B_h is supposed to be Lipschitz-continuous. The complement of B_h in Ω will be denoted by Ω_h.

(2.41) The sequence of characteristic functions $(1_{\Omega_h})_h$ is supposed to $L^\infty(\Omega)$-weak* converge to a limiting function $b \in L^\infty(\Omega)$ such that

$$b \geq \beta > 0 \qquad \text{a.e. in } \Omega,$$

where β is a real constant.

(2.42) For every $h \in \mathbf{N}$, $f_h: \Omega \times \mathbf{R}^N \to [0,+\infty[$ is a Borel function, convex on $\mathbf{R}^N$ for almost every $x \in \Omega$ and satisfying

(a) $c_1 |\xi|^p \leq f_h(x,\xi) \leq c_2 |\xi|^p \qquad \text{in } \Omega_h \times \mathbf{R}^N$

(b) $f_h(x,\xi) = 0 \qquad \text{in } B_h \times \mathbf{R}^N,$

where c_1 and c_2 are independent of h and $0 < c_1 \leq c_2$.

(2.43) The sequence $(\Omega_h)_h$, has the Uniform Local Extension Property (ULEP) according to the following definition.

DEFINITION 2.8

The sequence of sets $(\Omega_h)_h$ *has the* Uniform Local Extension Property (ULEP) *with constant* $c > 0$, *if for every pair* A', A *of open subsets of* $\mathbf{R}^N$ *such that* $A' \subset\subset A$ *and for every* $h \in \mathbf{N}$ *there exist a linear and continuous extension operator* $E_h: W^{1,p}(A \cap \Omega_h) \to W^{1,p}(A' \cap \Omega)$ *and a constant* $c_h > 0$ *such that for every* $u \in W^{1,p}(A \cap \Omega_h)$

i) $E_h u = u \qquad \text{a.e. in } A' \cap \Omega_h$

ii) $$\int_{A' \cap \Omega_h} |D(E_h u)|^p dx \leq c_h \int_{A \cap \Omega_h} |Du|^p dx$$

iii) $\limsup_{h \to \infty} c_h \leq c < +\infty.$

REMARK 2.9

Note that the sets Ω, Ω_h, B_h defined in Section 1 are a particular case of (2.40) and by Lemma 1.4 satisfy (2.41). Moreover, an extension property of the type (2.43) holds, by Lemma 1.3. ∎

REMARK 2.10

Note that the bound iii) in the previous definition is independent of the sets A, A'. If the conditions of Definition 2.8 are fulfilled just for $A' = \Omega$ and $A = \mathbf{R}^N$, then the sequence $(\Omega_h)_h$ is said to be satisfying the Strong Connectivity Condition (SCC) with constant c (see [23]). It is an easy matter to prove that SCC implies Ω_h to be connected for h large enough. ∎

Under assumptions (2.40) - (2.43) most of the results of Sections 1 and 2 concerning Γ-convergence, integral representation of Γ-limits, existence and compactness of minimizers and minima for problem (2.1), are still valid, while some proofs need suitable modifications. What in fact loses its meaning in this new context is Lemma 2.1, whose application (see proof of Theorem 2.5) is based on the homothetic structure of the holes considered before. In order to avoid confusion and misunderstandings, we now consider one by one the true results, giving the proof if necessary.

Theorems 1.2 and 1.5 are still valid when assumptions (1.1) - (1.7) are replaced by (2.40) - (2.43); their proofs require simply to use the ULEP in place of Lemma 1.3. Moreover Theorem 2.3 can be replaced by the following one.

THEOREM 2.11

Under assumptions (2.40) - (2.43), *for every* $h \in \mathbf{N}$ *there exists* $u_h \in L^p(\Omega)$ *such that*

$$m_h = sc^- F_h^{\varphi}(u_h) \quad ,$$

where m_h *and* $sc^- F^{\varphi}_h$ *are given by* (2.1) *and* (2.6), *respectively.*

PROOF:

Let $(u_n)_n$ be a minimizing sequence, that is

$$sc^- F_h^{\varphi}(u_n) \longrightarrow m_h$$

as n tends to infinity. Then $sc^-F^{\varphi}{}_h(u_n)$ is bounded for n large enough and hence from (2.6) $u_n \in W^{1,p}(\Omega_h)$, $u_n = \varphi$ on $\partial\Omega$. But from (2.42-a) it follows that

$$\int_{\Omega_h} |Du_n|^p dx \leq c$$

independently of n. Taking into account the ULEP, it follows that $E_h u_n$ is bounded in $W^{1,p}(\Omega)$ and hence weakly converges to some function u_h, with $u_h - \varphi \in W_0^{1,p}(\Omega)$. By the lower semicontinuity of $sc^-F^{\varphi}{}_h$ we obtain

$$sc^-F_h^{\varphi}(u_h) \leq \liminf_{n \to \infty} sc^-F_h^{\varphi}(E_h u_n) = \liminf_{n \to \infty} \int_{\Omega_h} f_h(x, Du_n)dx = m_h$$

that concludes the proof. ∎

COROLLARY 2.12

In the hypothesis of Theorem 2.11 *it follows that* $m_h \to m$, *with* m_h, m *defined by* (2.1), (2.3) *respectively.*

PROOF:

From the Γ-convergence results (Theorems 1.2, 1.5) and the existence of minimum points for (2.1) (Theorems 2.11) it follows directly the conclusion using Theorem 1.1. In fact if u_h is a solution to problem (2.1), then

$$\int_{\Omega_h} |Du_h|^p dx \leq c$$

independently of h. By the ULEP one can find a sequence of functions $E_h u_h \in W_0^{1,p}(\Omega) + \varphi$, such that $E_h u_h = u_h$ on Ω_h, which are bounded in $W^{1,p}(\Omega)$, are still solutions to (2.1) because of the particular structure (2.6) of $sc^-F^{\varphi}{}_h$, and finally have a subsequence which is strongly convergent in $L^p(\Omega)$ to some function u. ∎

3. The quadratic case with holes: resolvent and spectrum convergence

In this section we consider a case in which f_h is a quadratic form

(3.1) $$f_h(x,\xi) = a_{ij}^h(x)\xi_i\xi_j$$

where (a_{ij}^h) is a N×N symmetric matrix of functions in $L^\infty(\Omega)$, that satisfy the following conditions

(3.2) $$c_1 |\xi|^2 \leq a_{ij}^h(x)\xi_i\xi_j \leq c_2 |\xi|^2 \qquad \text{in } \Omega_h \times \mathbf{R}^N$$

(3.3) $$a_{ij}^h(x) = 0 \qquad \text{in } B_h,$$

with $0 < c_1 \leq c_2$. We assume that Ω, Ω_h, B_h satisfy (2.40). Moreover we suppose that $(\Omega_h)_h$ verifies (2.41) and the SCC in the sense of the following definition (see [23]).

DEFINITION 3.1

The sequence of sets $(\Omega_h)_h$ *satisfies the* Strong Connectivity Condition (SCC) *with constant* $c > 0$ *if for every* $h \in \mathbf{N}$ *there exists a linear and continuous extension operator* $E_h: H^1(\Omega_h) \to H^1(\Omega)$ *such that*

i) $E_h u = u$ a.e. in Ω_h

ii) $$\int_\Omega |D(E_h u)|^2 dx \leq c \int_{\Omega_h} |Du|^2 dx$$

for every $u \in H^1(\Omega_h)$. ∎

We assume that there exists a functional F satisfying (1.9) such that the sequence of functionals $F_h^0: L^2(\Omega) \to [0,+\infty]$ Γ-converges to the functional F^0, where F_h^0 and F^0 are given by (1.23) and (1.24), respectively, with $\varphi = 0$ and $p = 2$. Under our assumptions it can be proved that the integrand $f: \Omega \times \mathbf{R}^N \to [0,+\infty[$ corresponding to F^0 in (1.9) is a quadratic form of the type

(3.4) $$f(x,\xi) = a_{ij}(x)\xi_i\xi_j$$

and a_{ij} is a N×N symmetric matrix in $L^\infty(\Omega)$ such that

(3.5) $$c_0 |\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq c_2 |\xi|^2 \qquad \text{a.e. } x \in \Omega, \forall \xi \in \mathbf{R}^N$$

for a suitable constant c_0, $0 < c_0 \leq c_2$ (see [28], [29]) for the proof of (3.4) and [15] for (3.5)).

Under these assumptions we consider the following boundary value problem

$$
(3.6)\qquad \begin{cases} -D_i(a^h_{ij}\,D_ju)+\lambda u=g & \text{in } \Omega_h \\ \dfrac{\partial u}{\partial \upsilon_h}=0 & \text{on } \partial B_h \\ u=0 & \text{on } \partial\Omega\ , \end{cases}
$$

where $\partial/\partial\upsilon_h$ denotes the conormal derivative operator at ∂B_h

$$
\frac{\partial u}{\partial \upsilon_h}=a^h_{ij}\,D_ju\,n^i_h
$$

and n_h is the outer normal at ∂B_h. The main purpose of this section is to study the behaviour of the weak solutions, the eigenvalues, and the eigenspaces related to problem (3.6) as h goes to infinity.

First of all, in order to briefly deal with existence of the weak solutions, we introduce the function space Y_h, defined by

$$
Y_h=\{v\in H^1(\Omega_h):\ v=0\ \text{ on } \partial\Omega\}.
$$

As an immediate consequence of Lax-Milgram Lemma, for every h there exists one and only one weak solution $u_h\in Y_h$ to problem (3.6), according to the usual variational formulation. Note that, since (3.6) defines its solution only on Ω_h, every extension of u_h to a function $\tilde{u}_h\in L^2(\Omega)$ minimizes the functional

$$
F^0_h(v)+\int_{\Omega_h}(\lambda v^2-2gv)dx
$$

in the class $L^2(\Omega)$. That is true in particular when $\tilde{u}_h=E_hu_h$, with E_h given by Definition 3.1. In order to make an asymptotic analysis of problem (3.6) we need a further Γ-convergence result.

THEOREM 3.2

If $(g_h)_h$ *is a sequence in* $L^2(\Omega)$ *which strongly converges to some* $g\in L^2(\Omega)$, *then*

$$
(3.7)\qquad F^0_h(v)+\int_{\Omega_h}(\lambda v^2-2g_hv)dx \longrightarrow F^0(v)+\int_{\Omega}(\lambda v^2-2gv)b\,dx,
$$

where $b \in L^\infty(\Omega)$ *is defined by* (2.41).

PROOF:

In order to semplify notation, let us set

$$(3.8) \qquad G_h(v) = \int_{\Omega_h} (\lambda v^2 - 2g_h v)dx \quad , \quad G(v) = \int_{\Omega} (\lambda v^2 - 2gv)b\,dx \,.$$

The proof of (3.7) is an easy consequence of the fact that, for every sequence $(v_h)_h$ in $L^2(\Omega)$ which is strongly convergent in $L^2(\Omega)$ to some v, we have

$$(3.9) \qquad G_h(v_h) \to G(v) \qquad \text{as } h \to +\infty$$ ∎

We now introduce the operator A_h: $Y_h \to Y_h'$, defined as follows

$$(3.10) \qquad A_h u = g \quad \Longleftrightarrow \quad \int_{\Omega_h} a_{ij}^h D_j u\, D_i v\, dx = \langle g,v\rangle \qquad \forall\, v \in Y_h \,,$$

where $\langle \cdot\,,\cdot \rangle$ denotes the duality pairing between the spaces Y_h and Y_h'. According to (3.10), problem (3.6) takes the form

$$(3.11) \qquad (A_h + \lambda I)u = g \,,$$

where I: $Y_h \to Y_h'$ denotes the canonical immersion defined by

$$\langle Iu,v\rangle = \int_{\Omega} uv\, dx \qquad \forall \; v \in Y_h .$$

Moreover we define two operators which are going to express the limiting problem. Let A: $H_0^1(\Omega) \to H^{-1}(\Omega)$ be the operator defined by

$$(3.12) \qquad Au = g \quad \Longleftrightarrow \quad \begin{cases} -D_i(a_{ij}\, D_j u) = g & \text{in } \Omega, \\ u \in H_o^1(\Omega) \;, \end{cases}$$

and denote by B: $L^2(\Omega) \to L^2(\Omega)$ the operator of pointwise multiplication by b, i.e.

$$(Bu)(x) = b(x)\, u(x) \qquad \text{a.e. in } \Omega.$$

With this notation we prove the following theorem.

THEOREM 3.3

If $(g_h)_h$ *is a sequence in* $L^2(\Omega)$, *which strongly converges to* $g \in L^2(\Omega)$, λ *is non negative and* $u_h^\lambda \in Y_h$ *is such that*

(3.13) $$(A_h + \lambda I)u_h^\lambda = g_h|_{\Omega_h},$$

then the sequence $(E_h u_h^\lambda)_h$, *where* E_h *is given by Definition* 3.1, *converges weakly in* $H_0^1(\Omega)$ *and strongly in* $L^2(\Omega)$ *to the unique solution* $u^\lambda \in H_0^1(\Omega)$ *of*

(3.14) $$(A + \lambda B)\, u^\lambda = Bg.$$

PROOF:

After recalling that $E_h u_h^\lambda$ is a minimizer of the functional $F_h^0 + G_h$, (G_h is defined by (3.8)), in the class $L^2(\Omega)$ and that

$$F_h^0 + G_h \xrightarrow{\ \Gamma\ } F^0 + G,$$

(see Theorem 3.2), it remains to prove that $(E_h u_h^\lambda)_h$ is bounded in $H_0^1(\Omega)$. This is almost immediate, since (3.2) implies

(3.15) $$c_1 \int_{\Omega_h} |Du_h^\lambda|^2 dx \le \text{const}$$

and from ii) in Definition 1.3 one obtains

(3.16) $$\int_{\Omega} |D(E_h u_h^\lambda)|^2 dx \le \text{const}.$$

Hence, there exists at least one convergent subsequence of $(E_h u_h^\lambda)_h$, whose limit u^λ has to be minimizer of the limit functional $F^0 + G$ (see Theorem 1.1). This is equivalent to say that u^λ solves equation (3.14). From the uniqueness of u^λ, which is due to the strong convexity of the functional $F^0 + G$ (see (3.5)), we can conclude that the whole sequence converges to u^λ

(3.17) $$E_h u_h^\lambda \to u^\lambda \quad \text{weakly in } H_0^1(\Omega), \text{ strongly in } L^2(\Omega).$$ ∎

REMARK 3.4

The result stated in the preceding theorem, considered in the case $g_h = g$ for every h, can be interpreted as the strong convergence of the resolvent operators $(A_h + \lambda I)^{-1}$, up to the composition with restriction and extension operators. If we denote by R_h the restriction operator from $L^2(\Omega)$ to $L^2(\Omega_h)$, we actually have

(3.18) $$(E_h(A_h + \lambda I)^{-1}R_h)g \to ((A + \lambda B)^{-1}B)g \ ,$$

strongly in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$, for every $g \in L^2(\Omega)$ and every $\lambda \geq 0$. ∎

The remaining part of Section 3 concerns the spectral behaviour of the sequence $(A_h)_h$ as $h \to \infty$. From classical spectral theory for compact operators, one can easily prove that, for every h, there exists a sequence $(\lambda_h^n)_n$ of positive real numbers and a sequence of functions $(u_h^n)_n$ in Y_h such that

(3.19) $$\forall\, n \in \mathbf{N} \qquad A_h u_h^n = \lambda_h^n u_h^n \ ,$$

(3.20) $$0 < \lambda_h^1 \leq \lambda_h^2 \leq \ldots\ldots \leq \lambda_h^n \leq \ldots\ldots \to +\infty \ ,$$

(3.21) the sequence $(u_h^n)_n$ is an orthonormal basis for $L^2(\Omega_h)$.

Moreover, the following variational characterization of λ_h^n holds:

(3.22) $$\lambda_h^n = \min_{\substack{u \in Y_h \\ (u,u_h^j)_h = 0 \\ j = 1,\ldots,n-1}} \frac{\langle A_h u,u \rangle_{Y_h',Y_h}}{\|u\|^2_{L^2(\Omega_h)}} = \min_{\substack{V \subseteq Y_h \\ \dim V = n}} \max_{v \in V} \frac{\langle A_h u,u \rangle_{Y_h',Y_h}}{\|u\|^2_{L^2(\Omega_h)}} \ ,$$

where $(u,v)_h$ denotes the scalar product in $L^2(\Omega_h)$.

An analogous result holds for the eigenvalue problem $Au = \lambda Bu$. More precisely, there exist a sequence $(\lambda^n)_n$ of positive real numbers and a sequence of functions $(u^n)_n$ in $H_0^1(\Omega)$ for which

(3.23) $$\forall\, n \in \mathbf{N} \qquad Au^n = \lambda^n B u^n \ ,$$

(3.24) $$0 < \lambda^1 \leq \lambda^2 \leq \ldots\ldots \leq \lambda^n \leq \ldots\ldots \to +\infty \ ,$$

(3.25) the sequence $(u^n)_n$ is an orthonormal basis in $L^2(\Omega)$, with respect to the scalar product $(. , .)_b$ defined by

(3.26) $$(u,v)_b = \int_\Omega uvb\,dx \qquad \forall\, u, v \in L^2(\Omega)\ .$$

Moreover, the following variational characterization of λ^n holds:

(3.27) $$\lambda^n = \min_{\substack{u \in H_0^1(\Omega) \\ (u,u^i)_b = 0 \\ i = 1,\dots,n-1}} \frac{\langle Au,u\rangle_{H^{-1},H_0^1}}{\| u \|_b^2} = \min_{\substack{V \subseteq H_0^1(\Omega) \\ \dim V = n}} \max_{u \in V} \frac{\langle Au,u\rangle_{H^{-1},H_0^1}}{\| u \|_b^2}\ .$$

Finally, we say that $\lambda \in (\lambda^n)^n$ has multiplicity $m(\lambda)$, if $m(\lambda) = \mathrm{card}\{n : \lambda = \lambda^n\}$.

Before stating the main result about spectrum convergence, we introduce the following definition (see [27]).

DEFINITION 3.5

Let $(S_h)_h$, S *be a sequence of convex subsets of a reflexive Banach space* X. *We say that* $(S_h)_h$ Mosco-converges to S, *and we write*

$$S_h \xrightarrow{M} S\ ,$$

if the following relation is fulfilled:

(3.28) $$\text{w-}\limsup_{h \to \infty} S_h = S = \text{s-}\liminf_{h \to \infty} S_h\ .$$

Notation in (3.28) has the following meaning. By

$$\text{w-}\limsup_{h \to \infty} S_h$$

we denote the set of all $x \in X$ for which there exists a sequence $(x_h)_h$ in X that weakly converges to x and such that $x_h \in S_h$ frequently. By

$$\text{s-}\liminf_{h \to \infty} S_h$$

we mean the set of all $x \in X$ for which there exists a sequence $(x_h)_h$ in X that strongly converges to x and such that $x_h \in S_h$ definitely.

We can now state the following theorem:

THEOREM 3.6

According to the notation introduced in (3.19) - (3.21) *and* (3.23) - (3.26), *we have the following:*

(a) $\lambda_h^n \rightarrow \lambda^n$ *as* $h \rightarrow +\infty$, *for every* $n \in \mathbf{N}$

(b) *if* λ^n *has multiplicity* m *and* $\lambda^n = \lambda^{n+1} = ... = \lambda^{n+m-1}$, *and we set*

$$S_h^n = \text{span}\,[E_h u_h^n, ..., E_h u_h^{n+m-1}] \quad , \quad S^n = \text{span}\,[u^n, ..., u^{n+m-1}] \ ,$$

then $(S_h^n)_h$ *Mosco-converges to* S^n *in* $L^2(\Omega)$, *for every* $n \in \mathbf{N}$. ∎

The proof of this result is postponed in Appendix A.

APPENDIX A. Proof of Theorem 3.6.

The present section consists mainly in a detailed proof of Theorem 3.6. Before entering in this topic, we briefly recall some facts that are needed in the following.

First of all, we observe that the Mosco-convergence of convex sets, given by Definition 3.5, has the following property:

(A.1) *Urysohn Property*: Let $(S_h)_h$, S be a sequence of convex subsets of a reflexive Banach space X. Then $(S_h)_h$ Mosco-converges to S if and only if for every subsequence $(S_{\sigma(h)})_h$ there exists a further subsequence $(S_{\sigma(\tau(h))})_h$ that Mosco-converges to S.

For more details see [27], 1.3.(c).

Moreover, in addition to formula (3.22) we recall another useful variational characterization for the eigenvalue λ_h^n of problem (3.19). Let (u_h^n) be the sequence in (3.21) and denote by W_h^n the space $W_h^n = \text{span}[u_h^1, ..., u_h^n]$. Then the following formula holds

(A.2) $$\lambda_h^n = \max_{u \in W_h^n} \frac{\langle A_h u, u \rangle_{Y_h', Y_h}}{\|u\|^2_{L^2(\Omega_h)}}$$

for every h, n∈**N**. A complete proof of (3.22) and (A.2) will be given at the end of this section.

LEMMA A.1

For every n *there exists a positive constant* c=c(n), *independent of* h, *such that*

(A.3) $$\lambda_h^n \leq c(n) \qquad \textit{for every } h \in \mathbf{N}.$$

PROOF:

Let us fix n∈**N** and denote by Λ^n the n^{th} eigenvalue of the Laplace operator in $H_0^1(\Omega)$,

(A.4) $$\Lambda^n = \min_{\substack{W \subseteq H_0^1(\Omega) \\ \dim W = n}} \max_{v \in W} \frac{\int_\Omega |Dv|^2 dx}{\int_\Omega v^2 dx}.$$

Using (3.2), we have that for every $v \in Y_h$

$$\text{(A.5)} \qquad \frac{\langle A_h v, v\rangle_{Y'_h, Y_h}}{\|v\|^2_{L^2(\Omega_h)}} \le c_2 \frac{\int_{\Omega_h} |Dv|^2 dx}{\int_{\Omega_h} v^2 dx} .$$

Moreover, by writing v instead of $E_h v$, the following estimate holds trivially

$$\text{(A.6)} \qquad \frac{\int_{\Omega_h} |Dv|^2 dx}{\int_{\Omega_h} v^2 dx} \le \frac{\int_{\Omega} |Dv|^2 dx}{\int_{\Omega} v^2 dx} \cdot \frac{\int_{\Omega} v^2 dx}{\int_{\Omega_h} v^2 dx}$$

Now, let us denote by W^n the subspace of $H^1_0(\Omega)$ which reaches the minimum in (A.4) and by W^n_h the space of restrictions to Ω_h of functions in W^n, namely $R_h(W^n)$. Since the elements of W^n are eigenvectors of the Laplace operator, and hence analytic, by the unique continuation property it follows that $\dim W^n_h = \dim W^n = n$. Hence, if we can prove that

$$\text{(A.7)} \qquad \frac{\int_{\Omega} v^2 dx}{\int_{\Omega_h} v^2 dx} \le c'(n) \qquad \forall h \in \mathbf{N}, \forall v \in W^n,$$

where c'(n) is independent of h, then the lemma is proved. In fact, using (3.22) and (A.5), we can conclude that

$$\lambda^n_h \le \max_{v \in W^n_h} \frac{\langle A_h v, v\rangle_{Y'_h, Y_h}}{\|v\|^2_{L^2(\Omega_h)}} \le c_2 \max_{v \in W^n_h} \frac{\int_{\Omega_h} |Dv|^2 dx}{\int_{\Omega_h} v^2 dx} =$$

$$= c_2 \max_{v \in W^n} \frac{\int_{\Omega_h} |Dv|^2 dx}{\int_{\Omega_h} v^2 dx} \le c_2\, c'(n) \max_{v \in W^n} \frac{\int_{\Omega} |Dv|^2 dx}{\int_{\Omega} v^2 dx} = c(n)\Lambda^n.$$

In order to prove (A.7) it is enough to show that

$$\text{(A.8)} \qquad \frac{\int_\Omega v^2 dx}{\int_{\Omega_h} v^2 dx} \le c'(n) \qquad \forall h\in N,\ \forall v\in W^n \text{ such that } \int_\Omega v^2 dx = 1,$$

since (A.7) follows then by a simple homogeneity argument. Let us show (A.8). First of all, by (2.41)

$$\text{(A.9)} \qquad \int_{\Omega_h} v^2 dx \to \int_\Omega bv^2 dx$$

for every $v\in W^n$, as h goes to infinity. Now, since the set

$$K = \{v\in W^n : \int_\Omega v^2 dx = 1\}$$

is compact in W^n and the functions in (A.9) are equicontinuous on K, by Ascoli-Arzela' Theorem the convergence in (A.9) is uniform on K. Using the fact that

$$\int_\Omega bv^2 dx \ge \beta \qquad \forall v\in K,$$

we then obtain that

$$\frac{\int_\Omega v^2 dx}{\int_{\Omega_h} v^2 dx} \to \frac{\int_\Omega v^2 dx}{\int_\Omega bv^2 dx},$$

uniformly on K, and this immediately implies (A.8). ∎

PROOF OF THEOREM 3.6:

Let us start with the proof of (a), that will be obtained through four steps.

Step 1. There exist an increasing sequence of integers $\sigma(h)$, a sequence of positive real numbers $\{\overline{\lambda}^n : n\in N\}$ and a sequence of functions $\{\overline{u}^n : n\in N\}\subseteq H^1_0(\Omega)$ such that for every $n\in N$

(A.10) $\lambda^n_{\sigma(h)} \to \bar{\lambda}^n$

(A.11) $E_{\sigma(h)} u^n_{\sigma(h)} \to \bar{u}^n$ weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$

as h goes to infinity. Moreover

(A.12) the sequence $\{\bar{u}^n: n \in \mathbf{N}\}$ is orthonormal in $L^2(\Omega)$ with respect to the scalar product $(\, , \,)_b$ defined by (3.26).

Step 2. If λ^n is given by (3.27) and $\bar{\lambda}^n$ is obtained in step 1, then we have

(A.13) $\{\bar{\lambda}^n: n \in \mathbf{N}\} \subseteq \{\lambda^n: n \in \mathbf{N}\}$,

i.e. every $\bar{\lambda}^n$ is an eigenvalue of problem (3.23).

Step 3.

(A.14) $\{\bar{\lambda}^n: n \in \mathbf{N}\} = \{\lambda^n: n \in \mathbf{N}\}$,

i.e. every eigenvalue of problem (3.23) belongs to the sequence $\{\bar{\lambda}^n: n \in \mathbf{N}\}$.

Step 4. For every $n \in \mathbf{N}$

(A.15) $\bar{\lambda}^n = \lambda^n$,

i.e. $\{\bar{\lambda}^n: n \in \mathbf{N}\}$ coincides with the sequence of the eigenvalues of problem (3.23) repeated according to their multiplicity.

Proof of Step 1. (A.10) follows directly from (A.3). To prove (A.11) it is enough to show that $(E_h u^n_h)_h$ is bounded in $H^1_0(\Omega)$, independently of h. First of all, if we multiply

$$A_h u^n_h = u^n_h \lambda^n_h$$

by u^n_h, we integrate by parts on Ω_h and we take (3.2) into account, then we obtain

$$c_1 \int_{\Omega_h} |Du^n_h|^2 dx \le \lambda^n_h .$$

Hence, from (A.3) and the SCC it follows that

$$\int_\Omega |D(E_h u_h^n)|^2 dx \leq \frac{c_1}{c\, c(n)}$$

for every $h \in \mathbf{N}$, which concludes the proof of (A.11).

Proof of Step 2. Let us consider $(\lambda^n_{\sigma(h)})$, $(u^n_{\sigma(h)})$, $\bar{\lambda}^n$, $\bar{u}^n$ satisfying (A.10), (A.11). We note that $E_{\sigma(h)} u^n_{\sigma(h)}$ solves (3.6) when $\lambda = 0$ and $g = \lambda^n_{\sigma(h)} u^n_{\sigma(h)}$, that is to say it minimizes

$$F^0_{\sigma(h)}(v) - 2 \int_{\Omega_{\sigma(h)}} \lambda^n_{\sigma(h)} u^n_{\sigma(h)} v \, dx$$

on $L^2(\Omega)$. Since

$$\lambda^n_{\sigma(h)} E_{\sigma(h)} u^n_{\sigma(h)} \to \bar{\lambda}^n \bar{u}^n$$

strongly in $L^2(\Omega)$, as h goes to infinity, then by Theorem 3.2 and 1.1 $\bar{u}^n$ minimizes the Γ-limit

$$F^0(v) - 2 \int_\Omega b \bar{\lambda}^n \bar{u}^n v \, dx$$

on $L^2(\Omega)$. This means that $\bar{u}^n$ solves

$$A\bar{u}^n = \bar{\lambda}^n B\bar{u}^n \qquad \text{in } \Omega,$$

i.e. $\bar{\lambda}^n$ is an eigenvalue of problem (3.23) and hence that $\{\bar{\lambda}^n: n \in \mathbf{N}\} \subseteq \{\lambda^n: n \in \mathbf{N}\}$, where $(\lambda^n)_n$, is given by (3.24), (3.27).

Proof of Step 3. First of all we prove that

(A.16) $$0 < \bar{\lambda}^1 \leq \bar{\lambda}^2 \leq \ldots\ldots \to +\infty.$$

From (3.20) and (A.10) we obtain easily that

$$0 \leq \bar{\lambda}^1 \leq \bar{\lambda}^2 \leq \ldots\ldots \leq \bar{\lambda}^n \leq \ldots\ldots \quad .$$

To complete the proof of (A.16) we now show that

(A.17) $\lambda_h^n \geq c c_1 \Lambda^n$ for every $h \in \mathbf{N}$,

where Λ^n is the n^{th} eigenvalue of the Laplace operator in $H_0^1(\Omega)$ given by (A.4) and c is the constant of the SCC (see Definition 3.1). From (A.2), (3.2) and SCC we have in fact

$$\lambda_h^n \geq c\, c_1 \max_{u \in W_h^n} \frac{\int_\Omega |D(E_h u)|^2 dx}{\int_\Omega |E_h u|^2 dx} .$$

Since $\dim E_h(W_h^n) = n$, the right hand side of the preceding inequality is larger than or equal to

$$c\, c_1 \min_{\substack{V \subseteq H_0^1(\Omega) \\ \dim V = n}} \max_{v \in V} \frac{\int_\Omega |Dv|^2 dx}{\int_\Omega v^2 dx} = c\, c_1\, \Lambda^n.$$

The proof of (A.16) follows then from the fact that

$$0 < \Lambda^1 \leq \Lambda^2 \leq \ldots\ldots \to +\infty .$$

We now prove (A.14) arguing by contradiction. We suppose there exists $\lambda \in \{\lambda^n : n \in \mathbf{N}\}$, such that $\lambda \notin \{\overline{\lambda}^n : n \in \mathbf{N}\}$. From (A.16) there exists $m \in \mathbf{N}$ for which

$$\lambda < \overline{\lambda}^{m+1}.$$

Our purpose is now to construct a sequence of functions $w_h \in Y_h$ such that

(A.18) $(w_h, u_h^i)_h = 0$ $i = 1, \ldots\ldots m,$

where $(u,v)_h$ denotes the scalar product in $L^2(\Omega_h)$ and

$$(A.19) \qquad \frac{<A_h w_h, w_h>_{Y_h', Y_h}}{\|w_h\|^2_{L^2(\Omega_h)}} \to \lambda$$

as h goes to infinity. Before proving that this is possible, we show that it brings to the conclusion of the proof. In fact, by (A.18) and by the variational characterization (3.22) of λ_h^{m+1} we get

$$\lambda_h^{m+1} \le \frac{<A_h w_h, w_h>_{Y_h', Y_h}}{\|w_h\|^2_{L^2(\Omega_h)}} ;$$

hence, from the fact that $\lambda_h^{m+1} \to \bar{\lambda}^{m+1}$ and from (A.19) it immediately follows that $\bar{\lambda}^{m+1} \le \lambda$, which clearly contradicts our assumption.

In order to construct w_h we consider a solution u of the eigenvalue problem

$$Au = \lambda Bu ,$$

with $(u,u)_b = 1$. We also consider the solution v_h to

$$\begin{cases} A_h v_h = \lambda u \\ v_h \in Y_h \end{cases}$$

which is uniquely determined and, by Theorem 3.3, satisfies

$$E_h v_h \to u \qquad \text{weakly in } H_0^1(\Omega), \text{ strongly in } L^2(\Omega)$$

as h goes to infinity. If we set

$$w_h = v_h - \sum_{i=1}^{m} (v_h, u_h^i)_h u_h^i$$

with (u_h^i) given by (3.19), then (A.18) is immediately satisfied. Moreover, using the fact that

$$(v_h, u_h^i)_h \rightarrow (u, \bar{u}^i)_b$$

as h goes to infinity and that, since $\lambda \notin \{\bar{\lambda}^n : n \in \mathbf{N}\}$,

$$(u, \bar{u}^i)_b = 0$$

for every i, (A.19) can be easily obtained by direct computation. This concludes the proof of (A.14).

Proof of Step 4. To prove (A.15) it is enough to show that

$$\bar{m}(\lambda) = m(\lambda)$$

for any eigenvalue λ of problem (3.23), where $\bar{m}(\lambda)$ and $m(\lambda)$ denote the multiplicity of λ in the sequence $\{\bar{\lambda}^n : n \in \mathbf{N}\}$ and $\{\lambda^n : n \in \mathbf{N}\}$, respectively, i.e.

$$\bar{m}(\lambda) = \mathrm{card}\{n : \lambda = \bar{\lambda}^n\}$$

and

$$m(\lambda) = \mathrm{card}\{n : \lambda = \lambda^n\}.$$

Given $\lambda \in (\lambda^n)_n$ we claim that

i) $\bar{m}(\lambda) \leq m(\lambda)$
ii) $m(\lambda) \leq \bar{m}(\lambda)$.

If E_λ denotes the eigenspace associated to λ and $\bar{E}_\lambda$ is the space generated by the eigenvectors associated to $\bar{\lambda}^n$, for all the n such that $\bar{\lambda}^n = \lambda$, it is clear that

$$\bar{E}_\lambda \subseteq E_\lambda.$$

Since the sequences $\{\bar{u}^n : n \in \mathbf{N}\}$ and $\{u^n : n \in \mathbf{N}\}$ are orthonormal in $L^2(\Omega)$ with respect to the scalar product $(\ ,\ )_b$, we conclude that

$$\bar{m}(\lambda) = \dim \bar{E}_\lambda$$
$$m(\lambda) = \dim E_\lambda,$$

which implies i).

To prove ii) we argue by contradiction and suppose that

(A.20) $\qquad \overline{m}(\lambda) < m(\lambda).$

From (A.14) and (A.16) there exists $M \in \mathbf{N}$ such that

(A.21) $\qquad \overline{\lambda}^M = \lambda < \overline{\lambda}^{M+1}.$

Let us construct a sequence of real numbers $(\lambda_h)_h$ such that

1) $\lambda_h \to \lambda$ as h goes to infinity,
2) λ_h is not an eigenvalue of A_h,
3) $\lambda_h^M < \lambda_h < \lambda_h^{M+1}$.

Note that 3) can be certainly obtained for h large enough by virtue of (A.21). Let k be the least index such that $\overline{\lambda}^k = \overline{\lambda}^M = \lambda$, so that $\overline{u}^k, \ldots. \overline{u}^M$ be the corresponding eigenvectors defined in Step 1. From (A.20) we can find a solution $u \in H_0^1(\Omega)$ to the eigenvalue problem

$$Au = \lambda Bu$$

such that

$$(u, \overline{u}^i)_b = 0 \qquad i = k, \ldots, M.$$

Moreover we can require also that

$$(u, \overline{u}^i)_b = 0 \qquad i = 1, \ldots, k-1$$

since $\overline{u}^1, \ldots. \overline{u}^{k-1}$ correspond to eigenvalues $\overline{\lambda}^i \neq \lambda$ different from λ.
Let us consider v_h such that

$$\begin{cases} A_h v_h = \lambda_h u \\ v_h \in Y_h \end{cases}$$

and set

$$w_h = v_h - \sum_{i=1}^{M} (v_h, u_h^i)_h u_h^i .$$

Using the fact that for every i = 1,....M

$$(v_h, u_h^i)_h \to (u, \overline{u}^i)_b = 0$$

as h goes to infinity, it is easy to check that

1) $w_h \in Y_h$,
2) $(w_h, u_h^i)_h = 0 \qquad i = 1,....,M$
3) $\dfrac{<A_h w_h, w_h>_{Y_h', Y_h}}{\|w_h\|^2_{L^2(\Omega_h)}} \to \lambda$ as h goes to infinity.

Then, since from the variational characterization (3.22) of λ_h^{M+1} we have

$$\lambda_h^{M+1} \leq \frac{<A_h w_h, w_h>_{Y_h', Y_h}}{\|w_h\|^2_{L^2(\Omega_h)}}$$

and since $\lambda_h^{M+1} \to \overline{\lambda}^{M+1}$, it follows immediately that

$$\overline{\lambda}^{M+1} \leq \lambda$$

which clearly contradicts our assumptions. This concludes the proof of Step 4.

Since by (A.15) the limit in (A.10) does not depend on the particular subsequence, then

$$\lambda_h^n \to \lambda^n,$$

as h goes to infinity and the proof of (a) is then completed.

In order to prove (b), given $n \in \mathbf{N}$, let us consider $(u^n_{\sigma(h)})_h$, $\overline{u}^n$ satisfying (A.11) and set

$$\overline{S}^n = \text{span}[\overline{u}^n, \ldots, \overline{u}^{n+m-1}].$$

According to Definition 3.5 we also set

$$S' = \text{s-}\liminf_{h\to+\infty} S^n_{\sigma(h)}$$

and

$$S'' = \text{w-}\limsup_{h\to+\infty} S^n_{\sigma(h)}.$$

By the Urysohn property (A.1), it is enough to prove that

1) $S'' \subseteq \overline{S}^n$
2) $\overline{S}^n \subseteq S'$
3) $\overline{S}^n = S^n$.

For simplicity in the following we denote $\sigma(h)$ by h. To prove 1) we consider $v \in S''$. By definition there exists a sequence $(v_h)_h$ in $H^1_0(\Omega)$ such that

$$v_h \to v \qquad \text{weakly in } H^1_0(\Omega), \text{ strongly in } L^2(\Omega)$$

as h goes to infinity and $v_h \in S^n_h$ frequently. Then v_h is of the form

$$v_h = \sum_{i=0}^{m-1} c^i_h \, E_h u^{n+i}_h$$

where c^i_h are suitable constants. Since $(v_h)_h$ is bounded in $L^2(\Omega)$ it follows that

$$\sum_{i=0}^{m-1} (c^i_h)^2 = \int_{\Omega_h} v^2_h \, dx \leq \text{const.}$$

Then, up to a subsequence, for every $i = 0,\dots, m-1$, there exists c^i such that

$$c^i_h \to c^i$$

as h goes to infinity. Taking (A.11) into account we obtain that v has the form

$$v = \sum_{i=0}^{m-1} c^i \, \bar{u}^{n+i}$$

and hence it belongs to $\bar{S}^n$.

To prove 2) let v be an element of $\bar{S}^n$. Then v is of the form

$$v = \sum_{i=0}^{m-1} c^i \, \bar{u}^{n+i}.$$

Setting

$$v_h = \sum_{i=0}^{m-1} c^i \, E_h u_h^{n+i},$$

we have that $v_h \in S_h^n$ for every h and that v_h tends to v, strongly in $L^2(\Omega)$, which proves 2).

To prove 3) we observe that $\bar{S}^n$ is a linear subspace of S^n because it is spanned by eigenvectors related to λ^n. Moreover from (3.21) and (A.12) it follows that $\dim\bar{S}^n = n = \dim S^n$ and hence that $\bar{S}^n = S^n$. ∎

In this final part of Appendix A we recall some well known results of classical spectral theory for compact operators. We are going to apply them for proving formulas (3.22) and (A.2).

THEOREM A.2

Let T *be a linear compact operator from a Hilbert space* H *of infinite dimension into itself. If* T *is injective, positive and selfadjoint then its spectrum consists in a non increasing sequence of positive real numbers* (μ_n) *that tends to zero as* n *goes to infinity. Every non-zero eigenvalue* (μ_n) *has a finite number of associated, linearly independent eigenvectors, that is to say:* (μ_n) *has finite multiplicity. There exists an orthonormal basis* $(x_n)_n$ *for* H *consisting of eigenvectors of* T.

Setting $W_k = \text{span}[x_1, \ldots, x_n]$ *and denoting by*

$$R(x) = \frac{(Tx,x)}{\|x\|^2} \qquad \text{(A.22)}$$

the Rayleigh quotient at x, *where* (,) *and* $\|\ \|$ *are the scalar product and the associated norm in* H, *respectively, we have*

(A.23) $$\mu_1 = \max_{x \in H} R(x)$$

$$= \min_{x \in W_1} R(x) = \max_{\substack{V \subseteq H \\ \dim V = 1}} \min_{x \in V} R(x)$$

(A.24) $$\mu_k = \max_{\substack{(x,x_i)=o \\ i=1,\ldots,k-1}} R(x) = \min_{y_1,\ldots,y_{k-1} \in H} \max_{\substack{(x,y_i)=o \\ i=1,\ldots,k-1}} R(x)$$

$$= \min_{x \in W_k} R(x) = \max_{\substack{V \subseteq H \\ \dim V = k}} \min_{x \in V} R(x)$$

for every $k > 1$. ∎

For the proof and more details see for instance [18], chapters VII, X.

Our aim is now to define a suitable compact operator $\hat{T}_h$ to which Theorem A.2 applies and to deduce the variational characterization of its eigenvalues that will turn out to be the inverse of (λ_h^n). The proof of (3.22) and (A.2) will follow easily.

We denote by $T_h : Y'_h \rightarrow Y_h$ the linear operator defined by

(A.25) $$T_h g = u_h \quad \Leftrightarrow \quad \int_{\Omega_h} a_{ij}^h D_j u_h D_i v \, dx = \langle g,v \rangle \qquad \forall v \in Y_h,$$

where $\langle\, , \rangle$ is the duality pairing between Y_h and Y'_h (see also (3.10)).

REMARK A.3

We recall that

$$Y_h = \{ v \in H^1(\Omega_h) : v = 0 \text{ on } \partial\Omega \}$$

is a closed linear subspace of $H^1(\Omega_h)$ and that the norm

$$\|v\|_{Y_h} = \|Dv\|_{L^2(\Omega_h)}$$

is equivalent to the one induced by $H^1(\Omega_h)$. This fact can be easily proved taking into account the SCC (see Definition 3.1). ∎

PROPOSITION A.4

T_h *is a continuous isomorphism from* Y'_h *to* Y_h. *If* A_h *is given by* (3.10), *then* $T_h = A_h^{-1}$.

PROOF:

Taking $v = u_h$ in (A.25) and recalling (3.2), we have

$$c_1 \int_{\Omega_h} |Du_h|^2 \leq \int_{\Omega_h} a_{ij}^h D_j u_h D_i u_h \, dx = \langle g, u_h \rangle_{Y'_h, Y_h} \leq \|g\|_{Y'_h} \|u_h\|_{Y_h} .$$

Hence, from Remark A.3 we conclude that

$$\|u_h\|_{Y_h} \leq c \, \|g\|_{Y'_h}$$

that is to say

$$\|T_h g\|_{Y_h} \leq c \, \|g\|_{Y'_h} ,$$

from which follows the continuity of T_h. The surjectivity and injectivity of T_h turn out right from the definition. The equality $T_h = A_h^{-1}$ follows from (3.10). ∎

We denote, now, by $\hat{T}_h : Y'_h \to Y'_h$ the operator

$$\hat{T}_h = I T_h$$

where I: $Y_h \to Y'_h$ is the canonical immersion defined by

$$\langle Iu, v \rangle_{Y'_h, Y_h} = \int_{\Omega_h} uv \, dx$$

for every u, $v \in Y_h$.

PROPOSITION A.5

$\hat{T}_h$ *is injective, compact, positive and selfadjoint.*

PROOF:

First of all, $\hat{T}_h$ is clearly injective, being the composition of two injective maps. Then let us consider I as the composition of two maps i_1, i_2 defined by

$$i_1 : Y_h \to L^2(\Omega_h)$$

$$i_1(u) = u \qquad \forall u \in Y_h$$

$$i_2 : L^2(\Omega_h) \to Y'_h$$

$$\langle i_2(u), v \rangle_{Y'_h, Y_h} = \int_{\Omega_h} uv\, dx \qquad \forall u, v \in Y_h.$$

By the SCC and Rellich Theorem applied to $H^1_0(\Omega)$ it is easy to prove that i_2 is compact from Y_h into $L^2(\Omega_h)$, while it is clear that i_2 is continuous from $L^2(\Omega_h)$ to Y'_h. Hence I is compact and for the same reason also $\hat{T}_h$.

In order to complete the proof of this proposition we introduce in Y'_h an equivalent norm $\| \, \|_{T_h}$ which is given by the following scalar product

(A.26) $$(f,g)_{T_h} = \langle f, T_h g \rangle_{Y'_h, Y_h} \qquad \forall f, g \in Y'_h.$$

It is easy to verify that $(\, , \,)_{T_h}$ is a scalar product on Y'_h; in fact it is clearly a simmetric bilinear form (note that $a^h_{ij} = a^h_{ji}$). Moreover it is positive definite, that is to say

$$(f,f)_{T_h} \leq 0 \qquad \Rightarrow \qquad f = 0.$$

To prove this fact it is enough to notice that, for every $f \in Y'_h$

$$(f,f)_{T_h} = \langle f, T_h f \rangle_{Y'_h, Y_h} = \int_{\Omega_h} a^h_{ij} D_j u_h D_i u_h\, dx \geq$$

$$c_1 \int_{\Omega_h} |Du_h|^2 dx \geq c\, c_1 \int_{\Omega} |D(E_h u_h)|^2 dx$$

where $u_h = T_h f$. From the preceding inequalities it follows that if $(f,f)_{T_h} \leq 0$, then $E_h u_h = 0$ in Ω and hence $u_h = 0$ in Ω_h. By the injectivity of T_h it follows then that $f = 0$.

The equivalence between $\| \, \|_{T_h}$ and the usual norm in Y'_h can be proved as follows. For any $f \in Y'_h$ we have, as proved above,

$$\|f\|^2_{T_h} = (f,f)_{T_h} \geq c_1 \|Du_h\|^2_{L^2(\Omega_h)}.$$

Finally, since $T_h^{-1} = A_h$ is continuous from Y_h to Y'_h

$$\|u_h\|^2_{Y_h} \geq c'\|A_h u_h\|^2_{Y'_h}$$

and $A_h u_h = f$, we have the inequality

$$\text{(A.27)} \qquad \|f\|^2_{T_h} \geq c''\|f\|^2_{Y'_h}.$$

On the other hand, by the continuity of T_h itself we have for every $f \in Y'_h$

$$\|f\|^2_{T_h} = \langle f, T_h f\rangle_{Y'_h, Y_h} \leq \|f\|_{Y'_h} \|T_h f\|_{Y_h} \leq c\|f\|^2_{Y'_h}$$

that together with (A.27) proves the equivalence between $\| \, \|_{T_h}$ and $\| \, \|_{Y'_h}$.

We can now check without any difficulty that $\hat{T}_h$ is positive and selfadjoint with respect to the new scalar product $(\, , \,)_{T_h}$. For every $f \in Y'_h$, with $T_h f = u_h$, we have in fact

$$(\hat{T}_h f, f)_{T_h} = \langle \hat{T}_h f, T_h f\rangle_{Y'_h, Y_h} = \langle IT_h f, T_h f\rangle_{Y'_h, Y_h} = \int_{\Omega_h} u_h^2 dx \geq 0.$$

Moreover, given $f, g \in Y'_h$ with $u_h = T_h f$, $v_h = T_h g$ we have

$$(\hat{T}_h g, f)_{T_h} = \langle \hat{T}_h g, T_h f\rangle_{Y'_h, Y_h} = \langle Iv_h, u_h\rangle_{Y'_h, Y_h} =$$

$$= \int_{\Omega_h} v_h u_h dx = \langle Iu_h, v_h\rangle_{Y'_h, Y_h} = \langle \hat{T}_h f, T_h g\rangle_{Y'_h, Y_h} = (\hat{T}_h f, g)_{T_h}$$

that concludes the proof. ∎

PROOF OF (3.22), (A.2):

We are now allowed to apply Theorem A.1 to $\hat{T}_h$. It turns out that $\hat{T}_h$ has a decreasing sequence of non negative eigenvectors $(\mu_h^n)_n$ that can be characterized, for instance, by

$$\text{(A.28)} \qquad \mu_h^n = \max_{\substack{(g, g_h^i)_{T_h} = 0 \\ i=1,\dots,n-1}} R_h(x)$$

where $(g_h^n)_n$ is a fixed orthonormal basis consisting of eigenvectors of $\hat{T}_h$ associated to the corresponding eigenvectors $(\mu_h^n)_n$, and

$$(A.29) \qquad R_h(g) = \frac{(\hat{T}_h g, g)_{T_h}}{\|g\|^2_{T_h}}$$

for every $g \in Y'_h$.

Let us see now the relationship existing between the sequences (μ_h^n) and (λ_h^n) given by (A.28) and (3.22) respectively. First of all we can prove that

$$(A.30) \qquad \mu_h^n = (\lambda_h^n)^{-1}$$

for every $n, h \in \mathbf{N}$. Let g_h^n be an eigenvector of $\hat{T}_h$ associated to μ_h^n and set $T_h g_h^n = v_h^n$, or equivalently, $g_h^n = A_h v_h^n$. Then

$$\hat{T}_h g_h^n = \mu_h^n g_h^n,$$

i.e.

$$I v_h^n = \mu_h^n A_h v_h^n,$$

that is to say

$$(\mu_h^n)^{-1} I v_h^n = A_h v_h^n.$$

Comparing the last equation with (3.19) it turns out that $(\mu_h^n)^{-1}$ belongs to the sequence $(\lambda_h^n)_n$ defined by (3.22). Moreover it is easy to prove that

$$(g, g_h^n)_{T_h} = \frac{1}{\mu_h^n} \int_{\Omega_h} v v_h^n dx$$

where $v = T_h g$. This implies that $(v_h^n)_n$, up to a normalization, is an orthonormal basis of $L^2(\Omega_h)$ consisting in eigenvectors of problem (3.19). Hence we can conclude that $(\mu_h^n)^{-1} = \lambda_h^n$, for every $n, h \in \mathbf{N}$.

Now, since for every $g \in Y'_h$, with $T_h g = v$, we have

$$(\hat{T}_h g, g)_{T_h} = \langle \hat{T}_h g, T_h g \rangle_{Y'_h, Y_h} = \langle Iv, v \rangle_{Y'_h, Y_h} = \int_{\Omega_h} v^2 dx$$

and

$$\|g\|^2_{T_h} = \langle g, T_h g\rangle_{Y'_h, Y_h} = \int_{\Omega_h} a^h_{ij} D_j v D_i v dx$$

then the Rayleigh quotient $R_h(g)$ takes the form

(A.31) $$R_h(g) = \frac{\|v\|^2_{L^2(\Omega_h)}}{\langle A_h v, v\rangle}.$$

Then from (A.30), (A.31), (A.23) and (A.24) we deduce (3.22) and (A.2) without any difficulty. ∎

Analogue techniques can be used to study the asymptotic behaviour of the eigenvalue problem

(A.32) $$\begin{cases} -D_i(a^h_{ij} D_j u) = \lambda u & \text{in } \Omega_h \\ \dfrac{\partial u}{\partial \nu_h} = 0 & \text{on } \partial\Omega_h \end{cases},$$

with Neumann conditions on the whole boundary $\partial\Omega_h$, that, according to the usual variational formulation, can be written in the form

(A.33) $$\int_{\Omega_h} a^h_{ij} D_j u D_i v dx = \int_{\Omega_h} \lambda u v dx \qquad \forall v \in H^1(\Omega_h).$$

More precisely, the following theorem holds.

THEOREM A.6

In addition to the assumptions (3.2), (3.3) *let us suppose that* $\partial\Omega$ *is Lipschitz-continuous and that* $(\Omega_h)_h$ *satisfies the* SCC *in the sense of* Definition 3.1 *with the inequality* ii) *replaced by*

(A.34) $$\|Eu\|_{H^1(\Omega)} \le c\, \|u\|_{H^1(\Omega_h)}$$

for every $u \in H^1(\Omega_h)$. *Then there exist a sequence of eigenvalues* (λ_h^n) *and of eigenvectors* (u_h^n) *of problem* (A.32) *that satisfy conditions* (a) *and* (b) *of* Theorem 3.6, *where* λ^n *and* u^n *solve the limit eigenvalue problem*

$$\text{(A.35)} \qquad \begin{cases} -D_i(a_{ij}D_j u) = \lambda\, bu & \text{in } \Omega \\ \dfrac{\partial u}{\partial \upsilon} = 0 & \text{on } \partial\Omega \end{cases},$$

whose variational formulation is given by

$$\text{(A.36)} \qquad \int_\Omega a_{ij} D_j u D_i v dx = \int_\Omega \lambda buv dx \qquad \forall v \in H^1(\Omega).$$

Moreover λ_h^n *has the following variational characterization*

$$\lambda_h^n = \max_{\substack{(u, u_h^i)_h = o \\ i=1,\dots,n-1}} \frac{\int_{\Omega_h} a_{ij}^h D_j u D_i u dx}{\int_{\Omega_h} u^2 dx}.$$

∎

REMARK A.7

Let us denote by T the isomorphism $T: H^{-1}(\Omega) \to H_0^1(\Omega)$ such that $T = A^{-1}$, with A satisfying (3.12), and by I_B the compact immersion of $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ defined by

$$I_B : H_0^1(\Omega) \to H^{-1}(\Omega)$$

$$\langle I_B(u), v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega buv dx$$

for every $u, v \in H_0^1(\Omega)$. Then formula (3.27) can be proved applying Theorem A.1 to the compact operator $\hat{T}: H_0^1(\Omega) \to H_0^1(\Omega)$ given by $\hat{T} = T\, I_B$. ∎

APPENDIX B. Explicit computation of the Γ-limit.

In this section we shall briefly consider two cases in which the Γ-limit of a sequence of functionals can be computed explicitly.

The first one, which is given by the following theorem, is a particular case of a more general situation considered in [15] (see Prop. 4.5). It concerns a sequence of quadratic functionals F_h^0 whose integrands have the coefficients a_{ij}^h that tend in measure on Ω to a matrix of functions b_{ij}: it turns out that b_{ij} are the coefficients of the integrand of the Γ-limit F^0.

THEOREM B.1

In addition to the assumptions (3.1) - (3.3), *with* $(\Omega_h)_h$ *satisfying the* SCC *in the sense of* Definition 3.1, *let us suppose that*

(B.1) $\qquad |B_h| \to 0 \qquad$ *as h goes to infinity,*

(B.2) $\qquad a_{ij}^h \to b_{ij} \qquad$ *in measure on* Ω.

Then $(F_h^0)_h$ Γ*-converges in* $L^2(\Omega)$ *to* F^0, *where* F_h^0 *and* F^0 *are given by* (1.23) *and* (1.24) *respectively, with* $\varphi = 0$ *and* $p = 2$, *and the integrand* (3.4) *of* F^0 *satisfies*

(B.3) $\qquad a_{ij} = b_{ij} \qquad$ *a.e. in* Ω. ∎

The proof of this result and references on similar cases can be found in [15].

Another situation in which the Γ-limit can be computed explicitely and which is also well known is the one of homogenization. Here we shall briefly consider a homogenization problem that will turn out to be a particular case of (1.1) - (1.7) and for which it is possible to prove the so-called homogenization formula for the Γ-limit.

Let $Y =]0,1[^N$ be the unit cube of $\mathbf{R}^N$ and B denote the cube $[\rho,1-\rho]^N$, $0 < \rho < 1/2$.

DEFINITION B.2

We say that a function $f:\mathbf{R}^N \to \overline{\mathbf{R}}$ is *Y-periodic* if $f(x) = f(x + e_i)$ for every $x \in \mathbf{R}^N$ and for every $i = 1,\dots,N$, where $e_1,\dots,e_N$ is the canonical base of $\mathbf{R}^N$.

Let us consider a Borel function $\psi:\mathbf{R}^N \times \mathbf{R}^N \to [0,+\infty[$ with the following properties:

(B.4) $\qquad \psi(\,.\,,\xi)$ is Y-periodic for every $\xi \in \mathbf{R}^N$, (B.5) $\psi(y,.)$ is convex on $\mathbf{R}^N$ for every $y \in \mathbf{R}^N$,

(B.6) $0 \leq \psi(y,\xi) \leq c_2 |\xi|^p$ on $Y \times \mathbf{R}^N$,

(B.7) $c_1 |\xi|^p \leq \psi(y,\xi)$ on $(Y-B) \times \mathbf{R}^N$,

with $0 < c_1 \leq c_2$.

Now we assume that the integrands f_h introduced in Section 1 are of the form

(B.8) $f_h(x,\xi) = \psi(hx,\xi)$

for every $x, \xi \in \mathbf{R}^N$.

Let us set

(B.9) $Y_h^\alpha = \frac{1}{h}(\alpha + Y)$, $B_h^\alpha = \frac{1}{h}(\alpha + B)$,

where $\alpha \in \mathbf{Z}^N$ and suppose that

(B.10) Ω is a parallelepiped whose vertices have integral coordinates.

Then assumptions (1.1) - (1.6) are all satisfied when D is replaced by Y and the set I_h is replaced by the set of multi-indexes $\alpha \in \mathbf{Z}^N$ such that α/h belongs to Ω. Under conditions (B.4)-(B.10) an explicit formula can be given for the Γ-limit of the sequence $(F_h)_h$ defined by (1.7), as the following theorem states.

THEOREM B.3

Suppose that the sequence $(F_h)_h$ *given by* (1.7) *satisfies* (B.4)-(B.10) *and* (1.22). *Then the integrand* f *in* (1.9) *coincides with the function* ψ_0 *given by the following formula*

(B.11) $\psi_0(\xi) = \inf\{ \int_Y \psi(y, Du(y) + \xi)dy : u \in W^{1,p}_{loc}(\mathbf{R}^N),$ u Y-periodic $\}$

for every $\xi \in \mathbf{R}^N$. ∎

The proof of this theorem can be obtained by a direct application of a more general result by Braides (see Theorem 15 in [4]) concerning the homogenization of a wide class of non linear and non equicoercive variational problems. An analogous result for equicoercive sequences of functionals is announced in [25] and proved in [24], where a wide bibliography on this topic can be found.

REFERENCES

[1] ACERBI E. , PERCIVALE D. : Homogenization of noncoercive functionals: periodic materials with soft inclusions. *Appl. Math. Optim.* **17** (1988), 91-102.

[2] ATTOUCH H. : Variational convergence for functions and operators. Pitman, London, 1984.

[3] BOCCARDO L. , MARCELLINI P. : Sulla convergenza delle soluzioni di disequazioni variazionali. *Ann. Mat. Pura Appl. (4)* **110** (1976), 137-159.

[4] BRAIDES A. : Omogeneizzazione di integrali non coercivi. *Ricerche Mat.* **32** (1983), 347-368.

[5] BUTTAZZO G. , DAL MASO G. : Γ-limits of integral functionals. *J. Analyse Math.* **37** (1980), 145-185.

[6] BUTTAZZO G. , DAL MASO G. : Integral representation and relaxation of local functionals. *Nonlinear Anal.* **9** (1985), 515-532.

[7] CARBONE L. , SBORDONE C. : Some properties of Γ-limits of integral functionals. *Ann. Mat. Pura Appl. (4)* **122** (1979), 1-60.

[8] CHIADO' PIAT V. : Convergence of minima for non equicoercive functionals and related problems. Preprint SISSA n.87 (1988).

[9] CIORANESCU D. : Calcul des variations sur des sous-espaces variables. Applications.*C. R. Acad. Sci. Paris Sér. A* **291** (1980), 87-90.

[10] CIORANESCU D. , DONATO P. : Homogénéisation du problème de Neumann non homogène des ouverts perforés. Asymptotic Analysis, to appear.

[11] CIORANESCU D. , MURAT F. : Un terme étrange venu d'ailleurs , I. *Nonlinear partial differential equations and their applications. Collège de France Seminar. Volume II*, 98-138, *Res. Notes in Math., 60, Pitman, London*, 1982.

[12] CIORANESCU D. , MURAT F. : Un terme étrange venu d'ailleurs , II. *Nonlinear partial differential equations and their applications. Collège de France Seminar. Volume III*, 154-178, *Res. Notes in Math., 70, Pitman, London*, 1983.

[13] CIORANESCU D. , SAINT JEAN PAULIN J. : Homogenization in open sets with holes. *J. Math. Anal. Appl.* **71** (1979), 590-607.

[14] CONCA C. , DONATO P. : Non homogeneous Neumann's problems in domains with small holes. Preprint, Univ. Pierre et Marie Curie, Paris, to appear.

[15] DAL MASO G. , GULLIVER R. , MOSCO U. : Paper in preparation.

[16] DAL MASO G. , MODICA L. : Integral functionals determined by their minima. *Rend. Sem. Mat. Univ. Padova* **76** (1986), 255-267.

[17] DE GIORGI E. , FRANZONI T. : Su un tipo di convergenza variazionale. *Rend. Sem. Mat. Brescia* **3** (1979), 63-101.

[18] DUNFORD N. , SCHWARTZ T. J. : Linear operators I, II. Interscience Publishers, New York, 1963.

[19] FUSCO N. , MOSCARIELLO G. : L^2-semicontinuity of functionals of quadratic type. *Ann Mat. Pura Appl. (4)* **129** (1981), 305-326.

[20] GIAQUINTA M. , GIUSTI E. : On the regularity of minima of variational integrals. *Acta Math.* **148** (1982), 31-46.

[21] GIAQUINTA M. , GIUSTI E. : Quasi minima. *Ann. Inst. H. Poincaré. Anal. Non Linéaire* **1** (1984), 79-107.

[22] KESAVAN S. : Homogénéisation et valeurs propres. *C. R. Acad. Sci. Paris Sér. A* **285** (1977), 229-232.

[23] KHRUSLOV E. Ya. : The asymptotic behavior of solutions of the second boundary value problem under fragmentation of the boundary of the domain. *Math. USSR-Sb.* **35** (1979), 266-282.

[24] MARCELLINI P. : Periodic solutions and homogenization of non linear variational problems. *Ann. Mat. Pura Appl. (4)* **117** (1978), 139-152.

[25] MARCELLINI P., SBORDONE C. : Sur quelques questions de G-convergence et d'homogenisation non linéaire. *C. R. Acad. Sci. Paris Sér. A* **284** (1977), 535-537.

[26] MORTOLA S. , PROFETI A. : On the convergence of the minimum points of non equicoercive quadratic functionals. *Comm. Partial Differential Equations* **7 (6)** (1982), 645-673.

[27] MOSCO U. : Convergence of convex sets and solutions of variational inequalities. *Adv. in Math.* **3** (1969), 510-585.

[28] SBORDONE C. : Alcune questioni di convergenza per operatori differenziali del secondo ordine. *Boll. Un. Mat. Ital.* **10** (1974), 672-682.

[29] SBORDONE C. : Sulla G-convergenza di equazioni ellittiche e paraboliche . *Ricerche Mat.* **24** (1975), 76-136.

[30] SHAMAEV A. S. : Averaging of solutions and eigenvalues for boundary value problems for elliptic equations in perforated domains. *Russian Math. Surveys* **37** (1982), 253-254.

[31] SOROKINA A. G. : G-convergence in domains with holes. *Russian Math. Surveys* **38** (1983), 223-224.

[32] VANNINATHAN M. : Homogénéisation des valeurs propres dans les milieux perforés. *C. R. Acad. Sci. Paris Sér. A* **287** (1978), 403-406.

[33] VANNINATHAN M. : Homogeneization of eigenvalue problems in perforated domains. *Proc. Indian Acad. Sci. Math. Sci.* **90** (1981), 239-271.