# Regularity of the free boundary for a two phase Bernoulli problem

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# The Bernoulli Free Boundary Problem

Let $\lambda_0, \lambda_+, \lambda_- \geq 0$ be given and for $D \subset \mathbb{R}^d$ let us consider

$$J(u, D) = \int_D |\nabla u|^2 + \lambda_+ |\{u > 0\}| + \lambda_- |\{u < 0\}| + \lambda_0 |\{u = 0\}|.$$

and the minimization problem

(TPBP) $$\min_{u|_{\partial D} = g} J(u, D).$$

where $g$ is a given function.

# The Bernoulli Free Boundary Problem: some remarks

A few simple properties.

- Minimizers are easily seen to exist.
- Uniqueness in general fails.
- A minimizers would like to be harmonic where it is $\neq 0$, but the functional might penalize to be always non zero and/or might impose a "balance" between the negative and positive phase

# The Bernoulli Free Boundary Problem: some remarks

When $\lambda_0, \lambda_- = 0$ *and* $g \geq 0$, the problem reduces to the *one phase free boundary problem*:

(OPBP)
$$\min_{u=g,\, u\geq 0} \widehat{J}(u, D)$$
$$\widehat{J}(u, D) := \int_D |\nabla u|^2 + \lambda_+ |\{u > 0\}|$$

# Motivations

- These problems have been introduced in the 80's by Alt-Caffarelli (**??**) and by Alt-Caffarelli-Friedmann (**??**) motivated by some problems in flows with jets and cavities.

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  *Since then they have been the model problems for a huge class of free boundary problems.*

- More recently these types of problems turned out to have applications in the study of shape optimization problems.

# Shape Optimization Problems

Let us consider the following minimization problem:

$$\min_{U \subset D} \mathrm{Cap}(U, \Omega) - \lambda |U|$$

where

$$\mathrm{Cap}(U, D) = \min \Big\{ \int_D |\nabla u|^2 \quad u \in W_0^{1,2}(D), u = 1 \text{ on } U \Big\}$$

is the Newtonian capacity of $U$ relative to $D$.

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*$u = 1 - v$ solves a one phase problem.*

# Shape Optimization Problems

Let us consider the following *minimal partition problem*:

$$\min\Big\{\sum_i \lambda(D_i) + m_i|D_i| \quad D_i \subset D, \quad D_i \cap D_j = \emptyset \text{ if } i \neq j\Big\}.$$

Here $\lambda(D_i)$ is the first eigenvalue of the Dirichlet Laplacian on $D_i$.

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Solutions can be obtained by exploiting the techniques introduced by Buttazzo and Dal Maso.

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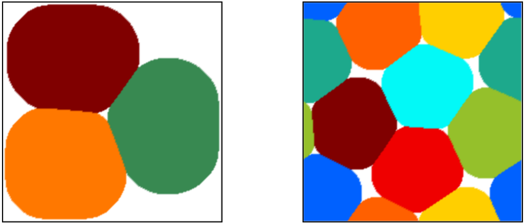


One can show (Spolaor-Trey-Velichkov):

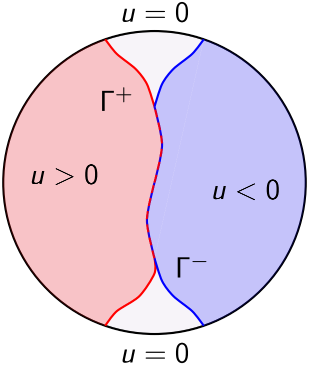
- There are no *triple points* $\partial D_i \cap \partial D_j \cap \partial D_k = \emptyset$.
- If $u_i, u_j$ are the first (positive) eigenfunctions of $D_i, D_j$ then $v = u_i - u_j$ is a (local) minimizer of

$$\int |\nabla v|^2 + m_i |\{v > 0\}| + m_j |\{v < 0\}| + \text{H.O.T.}$$

# Back to the Bernoulli free boundary problem

We are interested in the regularity of $u$ and of the free boundary:

$$\Gamma = \Gamma^+ \cup \Gamma_-$$
$$\Gamma^+ = \partial\{u > 0\} \qquad \Gamma^- = \partial\{u < 0\}.$$



# Known results

- $u$ is Lipschitz, Alt-Caffarelli (one phase), Alt-Caffarelli-Friedmann (two-phase).
- If $u$ is a solution of the *one-phase* problem, then $\Gamma^+$ is smooth outside a (relatively) closed set $\Sigma_+$ with $\dim_{\mathcal{H}} \le d - 5$ (Alt-Caffarelli, Weiss, Jerison-Savin, a recent new proof from De Silva).
- There is a minimizer in dimension $d = 7$ with a point singularity (De Silva-Jerison).
- If $u$ is a solution of the *two phase* problem and $\lambda_0 \ge \min\{\lambda_+, \lambda_-\}$, then $\Gamma^+ = \Gamma^- = \Gamma$ is smooth. (Alt-Caffarelli-Friedmann, Caffarelli, De Silva-Ferrari-Salsa).

# The case $\lambda_0 \geq \min\{\lambda_+, \lambda_-\}$

If $\lambda_- \leq \lambda_0$, let $v$ be the harmonic function which is equal to $u^-$ on $\partial(D \setminus \{u > 0\})$. Then

$$w = u^+ - v$$

satisfies

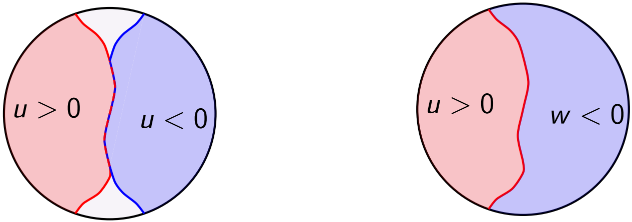
$$J(w, D) \leq J(u, D).$$

since

$$\lambda_- |\{w < 0\}| \leq \lambda_- |\{u < 0\}| + \lambda_0 |\{u = 0\}|$$

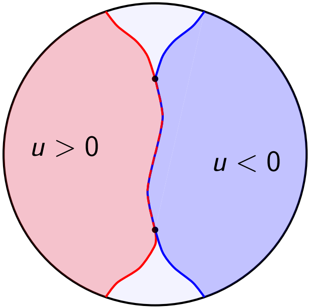
and

$$\int |\nabla v|^2 \leq \int |\nabla u^-|^2$$



# The case $\lambda_0 < \min\{\lambda_+, \lambda_-\}$

When $\lambda_0 < \min\{\lambda_+, \lambda_-\}$ the three phases may co-exist and *branch points* might appear.



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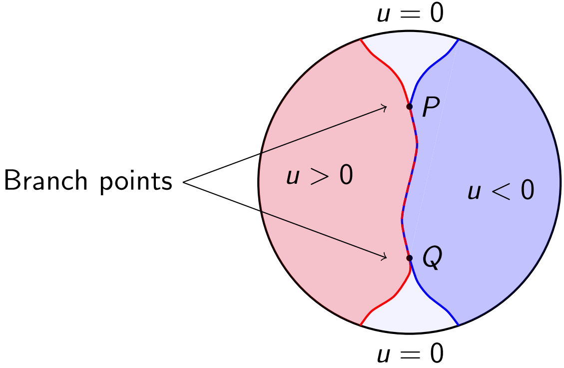


# Main result

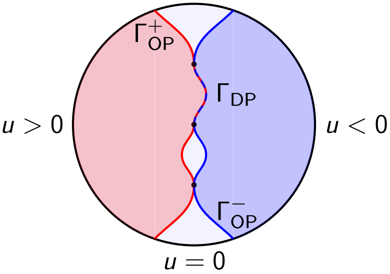
**Theorem D.-Spolaor-Velichkov '19 (Spolaor-Velichkov'16 for $d = 2$)**

*Let $u$ be a local minimizer of $J$. Let us define*

$$\Gamma^{\pm} = \partial\{\pm u > 0\} \qquad \Gamma_{DP} = \Gamma^{+} \cap \Gamma^{-} \qquad \Gamma^{\pm}_{OP} = \Gamma^{\pm} \setminus \Gamma_{DP},$$

*Then*

- $\Gamma^{\pm}$ *are* $C^{1,\alpha}$ *manifolds outside relatively closed set* $\Sigma^{\pm}$ *with* $\dim_{\mathcal{H}}(\Sigma^{\pm}) \leq d - 5$.
- $\Gamma_{DP} \cap \Sigma^{\pm} = \emptyset$. *In particular* $\Gamma_{DP}$ *is a closed subset of a* $C^{1,\alpha}$ *graph.*



# Steps in the proof

As it is customary in Geometric Measure Theory, the above result is based on two steps:

- Blow up analysis.
- $\varepsilon$-regularity theorem.

# Optimality conditions

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What are the optimality conditions on the free boundary?

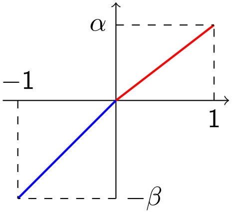
They can be formally obtained by performing inner variations

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} J(u_\varepsilon) = 0 \qquad u_\varepsilon(x) = u(x + \varepsilon X(x)) \quad X \in C_c(D; \mathbb{R}^d)$$

# Optimality condition

Let us assume that $u$ is one dimensional:

$$u = \alpha x_+ - \beta x_-$$



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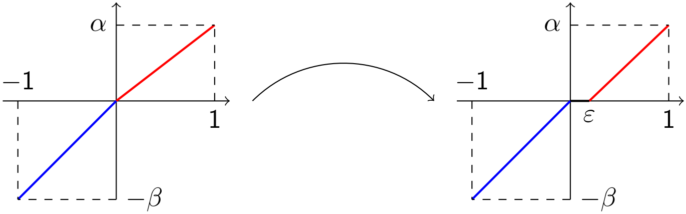

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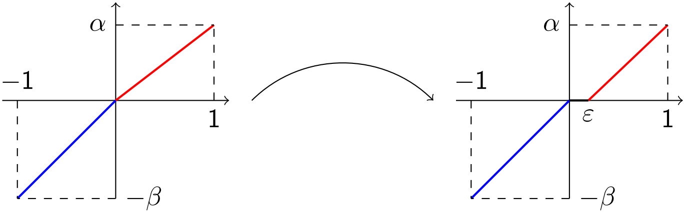
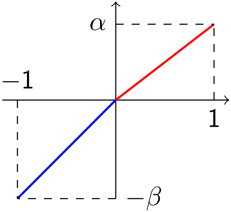



$$0 \le J(u_\varepsilon) - J(u) = \frac{\alpha^2}{(1-\varepsilon)} - \alpha^2 - (\lambda_+ - \lambda_0)\varepsilon$$
$$= \alpha^2 \varepsilon - (\lambda_+ - \lambda_0)\varepsilon + o(\varepsilon)$$

# Optimality condition

Moreover

$$u = \alpha x_+ - \beta x_-$$



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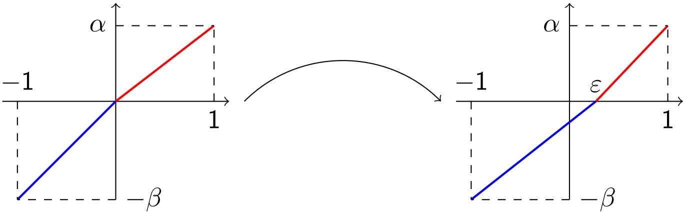
Moreover

$$u = \alpha x_+ - \beta x_- \qquad u_\varepsilon = \frac{\alpha}{1-\varepsilon}(x-\varepsilon)_+ - \frac{\beta}{1+\varepsilon}(x-\varepsilon)_-$$




$$0 \leq J(u_\varepsilon) - J(u) = (\alpha^2 - \beta^2)\varepsilon - (\lambda_+ - \lambda_-)\varepsilon + o(\varepsilon)$$

# Optimality conditions

We get the following problem

$$\begin{cases} \Delta u = 0 & \text{on } \{u \neq 0\} \\ |\nabla u^{\pm}|^2 = \lambda_{\pm} - \lambda_0 & \text{on } \Gamma^{\pm}_{\mathrm{OP}} \\ |\nabla u^{+}|^2 - |\nabla u^{-}|^2 = \lambda_{+} - \lambda_{-} & \text{on } \Gamma_{\mathrm{DP}} \\ |\nabla u^{\pm}|^2 \geq \lambda_{\pm} - \lambda_0 & \text{on } \Gamma^{\pm} \end{cases}$$

# Blow up analysis

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Let $x_0 \in \Gamma$ and $r > 0$. Let

$$u_{x_0,r}(x) = \frac{u(x_0 + rx)}{r} \qquad (u(x_0) = 0).$$

Then $\{u_{x_0,r}\}_{r>0}$ is pre-compact in $C^0$ and every limit point is *one-homogeneous* (Weiss Monotonicity Formula).

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Then $\{u_{x_0,r}\}_{r>0}$ is pre-compact in $C^0$ and every limit point is *one-homogeneous* (Weiss Monotonicity Formula).

If $x_0 \in \Gamma$ is regular it is easy to see that there is a unique limit $v_{x_0}$ and

$$v_{x_0} = \begin{cases} \pm\sqrt{\lambda_\pm - \lambda_0}(x \cdot \boldsymbol{e}_{x_0})_\pm & \text{if } x_0 \in \Gamma_{\mathrm{OP}}^{\pm} \\ \alpha_+(x \cdot \boldsymbol{e}_{x_0})_+ - \alpha_-(x \cdot \boldsymbol{e}_{x_0})_- & \text{if } x_0 \in \Gamma_{\mathrm{DP}} \\ \qquad \alpha_\pm \geq \sqrt{\lambda_\pm - \lambda_0}, \quad \alpha_+^2 - \alpha_-^2 = \lambda_+ - \lambda_- & \end{cases}$$

where $\boldsymbol{e}_{x_0}$ is the normal to $\Gamma$ at $x_0$.

# Regular points

We are going to call a point *regular* if $\{u_{x_0,r}\}$ admits *one* limit point of the above form (for some $\boldsymbol{e}$).

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4. The new step is to understand what happens at *branch points* and to put everything together.

# The $\varepsilon$-regularity theorem at one phase point

Let us show De Silva's proof at *one-phase* points ($\lambda_+ = 1, \lambda_-, \lambda_0 = 0$, $\boldsymbol{e} = \boldsymbol{e}_1$).
Assume that in $B_1$

$$u^+ \approx (x_1)_+ \qquad u^+ = x_1 + \varepsilon v_\varepsilon \quad \text{on } \{u > 0\} \qquad \varepsilon := \|u^+ - x_1\|_{L^\infty(\{u>0\}\cap B_1)}$$

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What are the equation satisfied by $v_\varepsilon$?

$$\Delta v_\varepsilon = 0 \qquad \text{on } \{u > 0\} \approx B_1^+.$$

moreover

$$1 = |\nabla u^+|^2 = 1 + \varepsilon \partial_1 v_\varepsilon + o(\varepsilon) \qquad \text{on } \partial\{u > 0\}$$

i.e.

$$\partial_1 v_\varepsilon \approx 0 \qquad \text{on } \partial\{u > 0\} \approx \{x_1 = 0\}$$

# The $\varepsilon$-regularity theorem at one phase point

In other words $v_\varepsilon$ is almost a solution of a Neumann problem

(NP)
$$\begin{cases} \Delta v = 0 & \text{on } B_1^+ \\ \partial_1 v = 0 & \text{on } \{x_1 = 0\} \cap B_1 \end{cases}$$

The $C^2$ regularity theory for the (**??**) allows to show the existence of

$$\mathbb{S}^{d-1} \ni \boldsymbol{e} = \boldsymbol{e}_1 + \varepsilon \nabla v(0) + O(\varepsilon^2) \qquad \big(\boldsymbol{e}_1 \perp \nabla v(0)\big)$$

such that for $\rho, \delta \ll 1$

$$\|u^+ - (x \cdot \boldsymbol{e})_+\|_{L^\infty(\{u>0\}\cap B_\rho)} \leq \rho^{2-\delta} \|u^+ - (x_1)_+\|_{L^\infty(\{u>0\}\cap B_1)}.$$

# What happens at branch points?

Assume $\lambda_\pm = 1$, $\lambda_0 = 0$. At branch points

$$u \approx (x_1)_+ - (x_1)_- + \varepsilon v_\varepsilon^+ + \varepsilon v_\varepsilon^-$$

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The functions $v_\varepsilon^\pm$ are almost solutions of a thin two membrane problem (this was first observed by Andersson-Shahgholian-Weiss).

$$\begin{cases} \Delta u = 0 & \text{on } \{u \neq 0\} \\ |\nabla u^\pm|^2 = 1 & \text{on } \Gamma_{\mathrm{OP}}^\pm \\ |\nabla u^+|^2 = |\nabla u^-|^2 & \text{on } \Gamma_{\mathrm{DP}} \\ |\nabla u^\pm|^2 \geq 1 & \text{on } \Gamma^\pm \end{cases} \Rightarrow \begin{cases} \Delta v^\pm = 0 & \text{on } B_1^\pm \\ \partial_1 v^\pm = 0 & \text{on } \{v^+ \neq v^-\} \cap \{x_1 = 0\} \\ \partial_1 v^+ = \partial_1 v^- & \text{on } \{v^+ = v^-\} \cap \{x_1 = 0\} \\ \partial_1 v^\pm \geq 0 & \text{on } \{x_1 = 0\} \end{cases}$$

$C^{1,\frac{1}{2}}$ regularity for the two membrane problem would to conclude (same caveat).

# The $\varepsilon$-regularity theorem at one phase point: compactness

The key point to make the above proofs rigorous is *compactness* of $v_\varepsilon^\pm$.

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A good topology is $C^0$ (solutions will be intended in the viscosity sense) which is the topology where the sequences are bounded. Some a-priori regularity theory is needed (De Silva: adapt Savin's "Half Harnack inequality").

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Furthermore the functions are defined on varying domains ($\Gamma$-convergence).

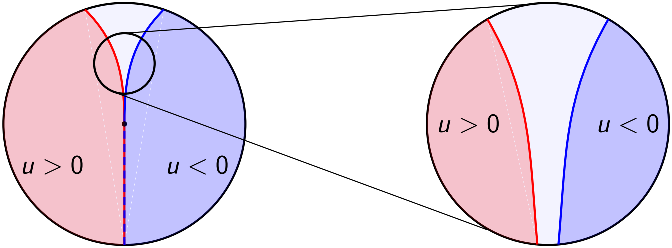
# Compactness at branch points

In order to prove compactness one does not only to deal with the case where

$$u \approx (x_1)_+ - (x_1)_- \qquad \text{but also} \qquad u \approx (x_1 + \delta_1)_+ - (x_1 + \delta_1)_-$$

with $\delta_1, \delta_2 \ll 1$. This is the behavior close to branch points.

Indeed this is the local picture close a branch point:



THANK YOU
FOR YOUR ATTENTION!

Thank you
for your attention!

Happy Birthday Gianni!