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Quantum principal bundles, Gauge groupoids and Coherent Hopf 2-algebras

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QUANTUM PRINCIPAL BUNDLES, GAUGE GROUPOIDS AND COHERENT HOPF 2-ALGEBRAS

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Preface

This thesis is based on the work during my Ph.D studies at SISSA supervised by Prof. Dr. Giovanni Landi. The main reference are the following:

[1] M. Dubois-Violette, X. Han, G. Landi,
Principal fibrations over non-commutative spheres,
Reviews in Mathematical Physics Vol. 30, No. 10,1850020 (2018).

[2] X. Han, G. Landi,
On the gauge group of Galois objects,
arXiv:2002.06097.

[3] X. Han,
On the coherent Hopf 2-algebras,
arXiv:2005.11207.

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Contents

1. Introduction

The duality between spaces and algebras of functions on spaces is the basis of noncommutative geometry. One gives up the commutativity of the algebras of functions while replacing them by appropriate classes of noncommutative associative algebras which are considered as "algebras of functions" on (virtual) "noncommutative spaces".

For instance, one may consider noncommutative associative algebras generated by coordinate functions that satisfy relations other than the commutation between them, thus generalizing the polynomial algebras and thereby defining noncommutative vector spaces.

There is also a duality between groups and Hopf algebras, where structures on the algebra of the quantum space represent the corresponding group structures. More precisely, the coproduct, counit and antipode of a quantum group will represent the product, unit and inverse operation in a usual group.

This duality can be extended to groupoids and corresponding Hopf algebroids.

Similarly, the quantisation of a space with a group action leads to the notion of a coaction of a Hopf algebra. In order to quantise a principal bundle one needs additional structures on the corresponding algebras and this leads to the notion of Hopf-Galois extension.

Finally, motivated by the idea of quantisation of groups, one can also consider the duality between 2-groups and quantum 2-groups. We know that a 2-group is a monoidal category, such that all morphisms are invertible and all objects are weakly invertible. So there is a corresponding quantisation on the 2-structure, which we call a Hopf 2-algebra.

This thesis is divided into three parts: quantum principal bundles, gauge groups of Galois objects and coherent Hopf 2-algebras.

1.1. Principal fibrations over noncommutative spheres.

In the papers [13] and [14] noncommutative finite-dimensional Euclidean spaces and noncommutative products of them were defined. These "spaces" were given in the general framework of the theory of regular algebras, which are a natural noncommutative generalization of the algebras of polynomials. Noncommutative spheres and noncommutative product of spheres were also defined. These are examples of noncommutative spherical manifolds related to the vanishing of suitable Connes–Chern classes of projections or unitaries in the sense of the work [11] and [10].

In this part we go one step further and consider actions of (classical) groups on noncommutative spheres and quotients thereof. In particular we present examples of noncommutative four-spheres that are base spaces of SU(2)-principal bundles with noncommutative seven-spheres as total spaces. This means that the noncommutative algebra of coordinate functions on the four-sphere is identified as a subalgebra of invariant elements for an action of the classical group SU(2) on the noncommutative algebra of coordinate functions on the seven-sphere. Conditions for these to qualify as noncommutative principal bundles are satisfied. The four-sphere algebra is generated by the entries of a projection which yields a noncommutative vector bundle over the sphere. Under suitable conditions the components of the Connes–Chern character of the projection vanish except for the second (the top) one. The latter is then a nonzero Hochschild cycle that plays the role of the volume form for the noncommutative four-sphere.

This part is organized as follows. In Section 8 we recall from [13] and [14] some results on the quadratic algebras which we will need and some of the solutions for noncommutative spheres $\mathbb{S}^7_R$ that we will use later on (here R is a matrix of deformation parameters). In Section 9, out of the functions on the seven-sphere $\mathbb{S}^7_R$ we construct a projection in a matrix algebra over these functions, whose entries are invariant for a right action of SU(2) and thus generate a subalgebra that we identify as the coordinate algebra of a four-sphere $\mathbb{S}^4_R$. We also show that this algebra inclusion is a noncommutative principal bundle with classical structure group SU(2). Sections 9.3 and 9.4 relate Connes–Chern characters of idempotents and unitaries to Hochschild cycles and noncommutative volume forms on a four-sphere $\mathbb{S}^4_R$. In Section 9.5 the $*$-structure on the algebra of functions of $\mathbb{S}^4_R$ is related to the vanishing of a component of the Connes–Chern character of the projection (and of a related unitary), a fact that puts some restrictions on the possible deformation matrices R, but makes the spheres examples of noncommutative spherical manifolds [11], [10]. We also exhibit explicit families of noncommutative four-sphere algebras.

1.2. **On the gauge group of Galois objects.** The study of groupoids on one hand and gauge theories on the other hand is important in different areas of mathematics and physics. In particular these subjects meet in the notion of the gauge groupoid associated to a principal bundle. In view of the considerable amount of recent work on noncommutative principal bundles it is desiderable to come up with a noncommutative version of groupoids and their relations to noncommutative principal bundles, for which one needs to have a better understanding of bialgebroids.

In this part, we will consider the Ehresmann–Schauenburg bialgebroid associated with a noncommutative principal bundle as a quantization of the classical gauge groupoid. Classically, bisections of the gauge groupoid are closely related to gauge transformations. Here we show that under some conditions on the base space algebra of the noncommutative principal bundle, the gauge group of the principal bundle is group isomorphic to the group of bisections of the corresponding Ehresmann–Schauenburg bialgebroid.

For a bialgebroid there is a notion of coproduct and counit but in general not of an antipode. An antipode can be defined for the Ehresmann–Schauenburg bialgebroid of a Galois object which (the bialgebroid that is) is then a Hopf algebra. Now, with H a Hopf algebra, a H-Galois object is a noncommutative principal bundle over a point in a sense: a H-Hopf–Galois extension of the ground field $\mathbb{C}$. In contrast to the classical situation where a bundle over a point is trivial, for the isomorphism classes of noncommutative principal bundles over a point this need not be the case. Notable examples are the group Hopf algebras $\mathbb{C}[G]$, whose corresponding principal bundles are $\mathbb{C}[G]$-graded algebras and are classified by the cohomology group $H^2(G, \mathbb{C}^\times)$, and Taft algebras T_N, the equivalence classes of T_N-Galois objects are in bijective correspondence with the abelian group $\mathbb{C}$.

Thus the central part is dedicated to the Ehresmann–Schauenburg bialgebroid of a Galois object and to the study of the corresponding groups of bisections, be they algebra maps from the bialgebroid to the ground field (and thus characters) or more general transformations. These are in bijective correspondence with the group of gauge transformations of the Galois object. We study in particular the case of Galois objects for H a general cocommutative Hopf algebra and in particular a group Hopf algebra. A nice class of examples comes from Galois objects and corresponding Ehresmann–Schauenburg bialgebroids for the Taft algebras T_N, an example that we work out in full detail.

Automorphisms of a (usual) groupoid with natural transformations form a strict 2-group or, equivalently, a crossed module. The crossed module involves the product of bisections and the composition of automorphisms, together with the action of automorphisms on bisections by conjugation. Bisections are the 2-arrows from the identity morphisms to automorphisms, and the composition of bisections can be viewed as the horizontal composition of 2-arrows. This construction is extended to the Ehresmann–Schauenburg bialgebroid of a Hopf–Galois extension by constructing a crossed module for the bisections and the automorphisms of the bialgebroid.

This part is organised as follows: In Section 11 we recall all the relevant concepts and notation on gauge groups that we need. In Section 12, we first have Ehresmann–Schauenburg bialgebroids and the group of their bisections, then we show that when the base algebra belongs to the centre of the total space algebra, the gauge group of a noncommutative principal bundle is group isomorphic to the group of bisections of the corresponding Ehresmann–Schauenburg bialgebroid. In Section 13 we consider Galois objects, which can be viewed as noncommutative principal bundles over a point. Several examples are studied here, such as Galois objects for a cocommutative Hopf algebra, in particular group Hopf algebras, regular Galois objects (Hopf algebras as self-Galois objects) and Galois objects of Taft algebras. In Section 14, we study the crossed module structure in terms of the bisection and the automorphism groups of an Ehresmann–Schauenburg bialgebroid. When restricting to a Hopf algebra, we show that characters and automorphisms also form a crossed module structure, and this can generate the representation theory of 2-groups (or crossed modules) on Hopf algebras. We work out in detail this construction for the Taft algebras.

1.3. **On Coherent Hopf 2-algebras.** The study of higher group theory and quantum group theory is becoming more and more important in various branches of mathematics and physics, such as topological field theory and quantum gravity. In this part, the main idea of constructing a coherent quantum 2-group (or coherent Hopf 2-algebra) is to make a quantization on the 2-arrows corresponding to a 2-category.

In [25] and [15], the researchers constructed a strict quantum 2-group, while here we construct a generalization of them. The quantum groups corresponding to the objects and morphisms are not necessarily coassociative, and thus there will be a corresponding coherence condition. For noncoassociative quantum groups, Hopf coquasigroups [20] usually offer some new properties and interesting examples, such as functions on Cayley algebras, hence we are motivated to build the coherent Hopf 2-algebra by Hopf coquasigroups. Moreover, the nonassociativity of Cayley algebras are controlled by a 3-cocycle coboundary corresponding to a 2-cochain. This fact will play an important role in satisfying the coherence condition of a coherent Hopf 2-algebra. We choose a special quantization for the 2-structure, such that by definition a coherent Hopf 2-algebra will be composed of two Hopf coquasigroups, which correspond to the "quantum" object and morphisms.

For a coherent 2-group, since all the morphisms are invertible, there is a groupoid structure based on of the composition of morphisms. Therefore, a quantum groupoid or Hopf algebroid will naturally exist in the coherent Hopf 2-algebra. These facts result in two different structures for the quantum 2-arrows. On one hand, it is a Hopf coquasigroup, which corresponds to the 'horizontal' coproduct (or tensor coproduct); on the other hand it is also a Hopf algebroid, which corresponds to the 'vertical' coproduct (or morphism cocomposition). These two kinds of coproducts also satisfy the interchange law. Moreover, the antipode of the Hopf coquasigroup preserves the coproduct of the Hopf algebroid while the antipode of the Hopf algebroid is a coalgebra map, which contradicts the usual property that the antipode is an anti-coalgebra map. The coherence condition will be described by a coassociator, which satisfies the "3-cocycle" condition. When we consider Hopf algebras instead of Hopf coquasigroups with trivial coassociator, we will get a strict Hopf 2-algebra.

For a strict 2-group, there is an equivalent definition, called the crossed module of group. In the "quantum case" [15], the researchers construct a crossed comodule of Hopf algebra as a strict quantum 2-group. Here we show that under some conditions, a crossed comodule of Hopf algebra is a strict Hopf 2-algebra with trivial coassociator. Several examples are also shown here, which can be characterised by the corresponding bialgebra morphisms.

We also produce a generalisation for crossed comodules of Hopf algebras, i.e. crossed comodules of Hopf coquasigroups, by replacing the pair of Hopf algebras with a special pair of Hopf coquasigroups. We show that if a Hopf coquasigroup is quasi-coassociative, one can construct a special crossed comodule of a Hopf coquasigroup and furthermore a coassociator, using which one can construct a coherent Hopf 2-algebra. An example is a Hopf coquasigroup which consists of functions on unit Cayley algebra basis. This Hopf coquasigroup is quasi coassociative and we will give all the structure maps precisely. Finally, we show that the coassociator is indeed controlled by a 3-coboundary cocycle corresponding to a 2-cochain.

In Section 15 coherent Hopf 2-algebras are defined and several of their properties are studied. Section 16 is devoted to a generalisation of crossed comodules of Hopf algebras, which is shown to be a strict Hopf 2-algebra under some conditions. In Section 17, we will first give the definition of quasi-coassociative Hopf coquasigroup, and then construct a crossed comodule of a Hopf coquasigroup and futhermore a coherent Hopf 2-algebra. In Section 18, the finite dimensional coherent Hopf 2-algebra is discussed, and through an investigation into the dual pairing, we make clear why quasi-coassociative coquasigroups are the quantization of quasiassociative quasigroups; we also consider an example built by functions on a Cayley algebra basis.

Before we talk about quantum principal bundles, gauge groupoids and coherent 2-Hopf algebras, it will be necessary to introduce principal bundles, groupoids and coherent 2-groups, which we will do from Section 2 to 4. From Section 5 to 7 we will then recall all the relevant concepts and notation on Hopf algebras, Hopf coquasigroups, noncommutative principal bundles (Hopf–Galois extensions) and Hopf algebroids that we need.

2. Classical principal bundles and examples

We start with the definition of principal fiber bundle associated to a group G in the category of topological spaces.

Recall that a fiber bundle over a topological space M is a triple (P, F, M), which consists of three topological spaces P, F and M, together with a surjective map $\pi : P \to M$, such that for all $x \in M$, the fiber $\pi^{-1}(x)$ is homeomorphic to F. Moreover, the bundle map is locally trivial, i.e. for all $x \in M$, there is an open neighborhood U of x and a homeomorphic map $f : U \times F \to \pi^{-1}(U)$ with $p_1 = \pi \circ f$, where p_1 is the projection on the first factor U.

Definition 2.1. A *principal G-bundle* over M is a fiber bundle (P, M, G, π), which consists of two topological spaces P and M, and a topological group G, together with a bundle map $\pi : P \to M$ and a continuous right G-action $\rho : P \times G \to P$ over M, such that:

- The canonical map
$$\chi : P \times G \to P \times_M P, \qquad (p, g) \mapsto (p, pg), \tag{2.1}$$
is a homeomorphism, where $P \times_M P$ is the subspace of $P \times P$:
$$P \times_M P = \{(p, q) \in P \times P \quad | \quad \pi(p) = \pi(q)\}.$$

A principal G-bundle (P, G, M, π) is *trivial*, if there is a G-equivariant diffeomorphism between P and $M \times G$.

Let (P, M, G, π) and (P', M, G, π') be two principal G-bundles over M, then a morphism between them is a G-equivariant continuous map from P to P'.

It is known that there is a bijective correspondence between the isomorphism class of principal G-bundles over M and the homotopy classes of continuous maps from X to BG, where BG is the classifying space of the Lie group G. As a result, every principal G-bundle over a point is isomorphic to each other.

Example 2.2. Recall that there are only four kinds of finite dimensional division algebras over $\mathbb{R}$, that is: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$ with dimensions 1, 2, 4 and 8 respectively. We know the real and complex numbers are commutative and associative. The quaternions are associative but not commutative. The octonions are not commutative or associative. Now we can construct four different fiber bundles in terms of the division algebras, which we can also call Hopf fibrations:

$$S^0 \hookrightarrow S^1 \to S^1,$$
$$S^1 \hookrightarrow S^3 \to S^2$$
$$S^3 \hookrightarrow S^7 \to S^4$$
$$S^7 \hookrightarrow S^{15} \to S^8.$$

The simplest one is $S^0 \hookrightarrow S^1 \to S^1$. Considering S^1 as the unit complex numbers and S^0 as the unit real numbers (which are ± 1), the bundle map $\pi : S^1 \to S^1$ is given by $\pi(z) := z^2$ for any $z \in S^1$. The right action of S^0 is the multiplication of ± 1 on S^1.

The rest of the fibrations can be constructed in the same way, here we only consider the fibration $S^1 \hookrightarrow S^3 \to S^2$:

The sphere S^3 can be views as the set of elements $(x, y) \in \mathbb{C}^2$, such that $\|x\|^2 + \|y\|^2 = 1$. When considering S^1 as the unit complex numbers, there is a right action of S^1 on S^3 by multiplication on the right, i.e. $\rho : ((x, y), w) \mapsto (xw, yw)$, where $(x, y) \in S^3$ and $w \in S^1$. The bundle map $\pi : S^3 \to S^2$ is given by

$$\pi(x, y) := (2xy^*, \|x\|^2 - \|y\|^2) \in S^4 = \{(z, r) \in \mathbb{H} \times \mathbb{R} | zz^* + r^2 = 1\}. \tag{2.2}$$

First we can see that the projection is invariant under the group action, and

$$(2xy^*)(2xy^*)^* + (\|x\|^2 - \|y\|^2)^2 = \|x\|^4 + \|y\|^4 + 2\|x\|^2\|y\|^2 = 1.$$

Thus we get a well defined principal $U(1)$-bundle. Similarly, $S^3 \hookrightarrow S^7 \to S^4$ is a principal $SU(2)$-bundle, when we replace complex numbers by quaternions. However, the Hopf fibration $S^7 \hookrightarrow S^{15} \to S^8$ is only a fiber bundle, but not a principal bundle, since by the non-associativity of octonions, the diagonal action of S^7 on S^{15} (i.e. $(x, y) \triangleleft g = (xg, yg)$) can not preserve the fiber over S^8.

But in this case we can still construct an action of S^7 on S^{15} by charts:

$$(x, y) \triangleleft w := ((xy^{-1})(yw), yw), \tag{2.3}$$

for any $(x, y) \in S^{15}$ with $y \neq 0$ and $w \in S^7$, where S^7 can be viewed as unit octonions. Similarly, we can define

$$(x, y) \triangleleft w := (xw, (yx^{-1})(xw)), \tag{2.4}$$

for any $(x, y) \in S^{15}$ with $x \neq 0$ and $w \in S^7$. We can see that this action is invariant under the projection $\pi : S^{15} \to S^8$. Indeed,

$$((xy^{-1})(yw))((xy^{-1})(yw))^* = xx^*, \qquad (yw)(yw)^* = yy^*$$

and

$$((xy^{-1})(yw))(yw)^* = xy^*,$$

since the subalgebra of octonions generated by any two elements is associative. However, this action is not associative, i.e. $((x, y) \triangleleft w) \triangleleft z \neq (x, y) \triangleleft (wz)$.

3. Groupoids

In this section we will first recall the definition of groupoids and then talk about the gauge or Ehresmann groupoid as an example.

A groupoid is a small category with all the morphisms invertible. More precisely, we have the following definition:

Definition 3.1. A *groupoid* (G^0, G^1, s, t, id) consists of two sets G^0 (objects) and G^1 (morphisms), and three maps $s, t : G^1 \to G^0$ (called respectively the source and target map) and $id : G^0 \to G^1$ (called the identity map). Moreover, for any two morphisms f, g with $s(f) = t(g)$, there is a binary operation (called the composition of morphisms) $\circ : G^1{}_s\times_t G^1 \to G^1$, where $G^1{}_s\times_t G^1$ is the pullback of the morphisms s and t, i.e.

$$G^1{}_s\times_t G^1 := \{(f, g) \in G^1 \times G^1 \quad | \quad s(f) = t(g)\},$$

such that the following conditions are satisfied:

(1)
$$s(g \circ h) = s(h), \qquad t(g \circ h) = t(g).$$
for any $(g, h) \in G^1{}_s\times_t G^1$.

(2) $(g \circ h) \circ k = g \circ (h \circ k)$, for any $(g, h), (h, k) \in {G^1}_s \times_t G^1$.
(3) $s(id_a) = t(id_a) = a$, for any $a \in G^0$, where we denote the image of id by id_a.
(4) For any $g \in G^1$, we have

$$g \circ id_{s(g)} = g = id_{t(g)} \circ g.$$

(5) For each $g \in G^1$, there is an inverse $g^* \in G^1$, such that

$$s(g) = t(g^*), \qquad t(g) = s(g^*).$$

Moreover, $g \circ g^* = id_{t(g)}$ and $g^* \circ g = id_{s(g)}$.

It is well known that a group is a groupoid, which consists of only one object.

Example 3.2. Here we give as an example the gauge groupoid associated to a principal bundle and examine the corresponding group of bisections; we shall mostly follow the book [27]. Let $\pi : P \to M$ be a principal bundle over the manifold M with (Lie) structure group G. Consider the diagonal action of G on $P \times P$ given by $(u, v)g := (ug, vg)$; denote by $[u, v]$ the orbit of (u, v) and by $\Omega = P \times_G P$ the collection of orbits. Then (M, Ω, s, t, id) is a groupoid (called the *gauge* or *Ehresmann groupoid* of the principal bundle), with Ω being the morphisms and M being the objects, and the source and target projections given by

$$s([u, v]) := \pi(v), \qquad t([u, v]) := \pi(u). \tag{3.1}$$

The identity map $id : M \to P \times_G P$ is given by

$$m \mapsto \mathrm{id}_m := [u, u] \tag{3.2}$$

for $m \in M$ and u any element in $\pi^{-1}(m)$. And the partial multiplication $[u, v'] \cdot [v, w]$, defined when $\pi(v') = \pi(v)$ is given by

$$[u, v] \cdot [v', w] = [u, wg], \tag{3.3}$$

for the unique $g \in G$ such that $v = v'g$. [1] One can always choose representatives such that $v = v'$ and the multiplication is then simply $[u, v] \cdot [v, w] = [u, w]$. The inverse is

$$[u, v]^{-1} = [v, u]. \tag{3.4}$$

A *bisection* of the gauge groupoid Ω is a map $\sigma : M \to \Omega$, which is right-inverse to the source projection, $s \circ \sigma = \mathrm{id}_M$, and is such that $t \circ \sigma : M \to M$ is a diffeomorphism. The collection of bisections, denoted $\mathcal{B}(\Omega)$, form a group: given two bisections σ_1 and σ_2, their multiplication is defined by

$$\sigma_1 * \sigma_2(m) := \sigma_1\big((t \circ \sigma_2)(m)\big)\sigma_2(m), \qquad \text{for} \quad m \in M. \tag{3.5}$$

The identity is the object inclusion $m \mapsto \mathrm{id}_m$, simply denoted id, with inverse given by

$$\sigma^{-1}(m) = \Big(\sigma\big((t \circ \sigma)^{-1}(m)\big)\Big)^{-1}; \tag{3.6}$$

here $(t \circ \sigma)^{-1}$ is a diffeomorphism of M, while the outer inversion is the one in (3.4).

The subset $\mathcal{B}_{P/G}(\Omega)$ of 'vertical' bisections, that is those bisections that are right-inverse to the target projection as well, $t \circ \sigma = \mathrm{id}_M$, form a subgroup of $\mathcal{B}(\Omega)$.

It is a classical result [27] that there is a group isomorphism between $\mathcal{B}(\Omega)$ and the group of principal (G-equivariant) bundle automorphisms of the principal bundle,

$$\mathrm{Aut}_G(P) := \{\varphi : P \to P\, ;\ \varphi(pg) = \varphi(p)g\}\,, \tag{3.7}$$

[1] Here one is really using the classical translation map $t : P \times_M P \to G$, $(ug, u) \mapsto g$.

while $\mathcal{B}_{P/G}(\Omega)$ is group isomorphic to the group of gauge transformations, that is the subgroup of principal bundle automorphisms which are vertical (project to the identity on the base space):

$$\mathrm{Aut}_{P/G}(P) := \{\varphi : P \to P;\ \varphi(pg) = \varphi(p)g\ ,\ \pi(\varphi(p)) = \pi(p)\}. \tag{3.8}$$

4. Quasigroups and 2-groups

In this section we will first talk about quasigroups, then we will introduce (coherent) 2-groups.

4.1. **Quasigroups.**

Definition 4.1. A *quasigroup* is a set G with a product and identity, for each element g there is a inverse $g^{-1} \in G$, such that $g^{-1}(gh) = h$ and $(hg^{-1})g = h$ for any $h \in G$.

For a quasigroup, the *multiplicative associator* $\beta : G^3 \to G$ is defined by

$$g(hk) = \beta(g, h, k)(gh)k, \tag{4.1}$$

for any $g, h, k \in G$.

The group of associative elements $N(G)$ is given by

$$N(G) = \{a \in G|(ag)h = a(gh), \quad g(ah) = (ga)h, \quad (gh)a = g(ha), \quad \forall g, h \in G\},$$

which sometimes called the 'nucleus' [20]. A quasigroup is called *quasiassociative*, if β has its image in $N(G)$ and $uN(G)u^{-1} \subseteq N(G)$.

It is clear that any element in $N(G)$ can pass through the brackets of a product. For example, $(g(hx))k = (gh)(xk)$ for any $g, h, k \in G$ and $x \in N(G)$. Therefore, we have the following proposition, and hereafter we use $m^n_I : G \times G \times \cdots \times G \to G$ to denote an n-th iterated product on G (with index I to distinguish different kinds of iterated products).

Proposition 4.2. *Let $N(G)$ be the associative elements of a quasigroup G, and let $m^n_I, m^n_J : G \times G \times \cdots \times G \to G$ be both n-th iterated products on G. If*

$$m^n_I(g_1, g_2, \cdots, 1, \cdots, g_n) = m^n_J(g_1, g_2, \cdots, 1, \cdots, g_n), \tag{4.2}$$

for any $g_1, g_2, \cdots g_n \in G$ with 1 is inserted into the m-th position ($1 \le m \le n+1$). Then

$$m^n_I(g_1, g_2, \cdots, x, \cdots, g_n) = m^n_J(g_1, g_2, \cdots, x, \cdots, g_n),$$

for any $x \in N(G)$ inserted into the m-th position.

Proof. We can prove this proposition inductively. For $n = 2$, this is obvious by the definition of $N(G)$. Now we consider the case for $n \ge 3$. We can see both sides of equation (4.2) are equal to $m^{n-1}_K(g_1, g_2, \cdots, g_n)$ for an $(n-1)$-th iterated product m^{n-1}_K. Let $m^{n-1}_K(g_1, g_2, \cdots, g_n) = m^p_{K'}(g_1, \cdots, g_p)m^q_{K''}(g_{p+1}, \cdots, g_n)$ for some iterated coproducts $m^p_{K'}$ and $m^q_{K''}$ with $p + q = n - 2$. Assume this proposition is correct for $n = N - 1$. We have two cases for the n-th iterated products m^n_I, m^n_J:

The first case is $m^n_I(g_1, g_2, \cdots, h, \cdots, g_n) = m^{i'}_{I'}(g_1, \cdots, h, \cdots, g_{i'+1})m^{i''}_{I''}(g_{i'+2}, \cdots, g_n)$ and $m^n_J(g_1, g_2, \cdots, h, \cdots, g_n) = m^{j'}_{J'}(g_1, \cdots, h, \cdots, g_{j'+1})m^{j''}_{J''}(g_{j'+2}, \cdots, g_n)$, with $i' + i'' = j' + j'' = n - 1$ (h is inserted into the m-th position and $m + 1 \le i', j'$). Here i' has to be equal to j', otherwise we can't get equation (4.2). Thus, $m^{i''}_{I''}(g_{i'+2}, \cdots, g_n) = m^{j''}_{J''}(g_{j'+2}, \cdots, g_n)$. Moreover, $m^{i''}_{I''} = m^{j''}_{J''} = m^q_{K''}$. Then we can use the hypotheses for $m^{i'}_{I'}$ and $m^{j'}_{J'}$ with $i' = j' \le n$. It is similar to consider the case $m^n_{I/J}(g_1, g_2, \cdots, h, \cdots, g_n) =$

$m^{i'}_{I'/J'}(g_1,\cdots g_{i'+1})m^{i''}_{I''/J''}(g_{i'+2},\cdots,h,\cdots,g_n)$ (h is inserted into the m-th position and $m+1>i',j'$).

The second case is $m^n_I(g_1,g_2,\cdots,h,\cdots,g_n)=m^{i'}_{I'}(g_1,\cdots,h,\cdots,g_{i'+1})m^{i''}_{I''}(g_{i'+2},\cdots,g_n)$, and $m^n_J(g_1,g_2,\cdots,h,\cdots,g_n)=m^{j'}_{J'}(g_1,\cdots g_{j'+1})m^{j''}_{J''}(g_{j'+2},\cdots,h,\cdots,g_n)$. Since (4.2) is equal to $m^{n-1}_K(g_1,g_2,\cdots,g_n)$, we have $i'+1=m=p+1$, i.e. $m^n_I(g_1,g_2,\cdots,g_{i'},h,g_{i'+1}\cdots,g_n)=m^{i'}_{I'}(g_1,\cdots,h)m^{i''}_{I''}(g_{i'+1},\cdots,g_n)=m^{i'}_{I'}(g_1,\cdots,g_p,h)m^{i''}_{K''}(g_{p+1},\cdots,g_n)$. Similarly, $m^n_J=m^n_J(g_1,g_2,\cdots,h,\cdots,g_n)=m^p_{K'}(g_1,\cdots g_p)m^{q+1}_{J''}(h,\cdots,g_n)$. Define two iterated products $m^{p+1}_G:=m\circ(m^p_{K'}\times id_G)$ and $m^{q+1}_H:=m\circ(id_G\times m^q_{K''})$. By the hypotheses we have $m^{p+1}_G(g_1\cdots,g_p,x)=m^{p+1}_{I'}(g_1\cdots,g_p,x)$ and $m^{q+1}_H(x,g_1,\cdots,g_q)=m^{q+1}_{J''}(x,g_1,\cdots,g_q)$. Therefore, we have

$$\begin{aligned}m^n_I(g_1,g_2,\cdots,x,\cdots,g_n)=&m^{p+1}_{I'}(g_1\cdots,g_p,x)m^{i''}_{K''}(g_{p+1},\cdots,g_n)\\=&m^{p+1}_G(g_1\cdots,g_p,x)m^{i''}_{K''}(g_{p+1},\cdots,g_n)\\=&m^p_{K'}(g_1\cdots,g_p)m^{q+1}_H(x,g_{p+1},\cdots,g_n)\\=&m^p_{K'}(g_1\cdots,g_p)m^{q+1}_{J''}(x,g_{p+1},\cdots,g_n)\\=&m^n_J(g_1,g_2,\cdots,x,\cdots,g_n),\end{aligned}$$

for any $x\in N(G)$, where the 3rd step uses $x\in N(G)$. □

Proposition 4.2 will be useful in the proof of Theorem 4.8. In [20] we have the following:

Lemma 4.3. *Let G be a quasigroup. If it is quasiassociative, then we have the following 3-cocycle condition:*

$$(g\beta(h,k,l)g^{-1})\beta(g,hk,l)\beta(g,h,k)=\beta(g,h,kl)\beta(gh,k,l) \tag{4.3}$$

for any $g,h,k,l\in G$.

Proof. Multiply both sides by $((gh)k)l$, we have on one hand

$$\begin{aligned}&(g\beta(h,k,l)g^{-1})\beta(g,hk,l)\beta(g,h,k)(((gh)k)l)\\=&(g\beta(h,k,l)g^{-1})\beta(g,hk,l)\Big(\big(\beta(g,h,k)((gh)k)\big)l\Big)=(g\beta(h,k,l)g^{-1})\beta(g,hk,l)((g(hk))l)\\=&(g\beta(h,k,l)g^{-1})(g((hk)l))=g\big(\beta(h,k,l)((hk)l)\big)=g(h(kl))\end{aligned}$$

On the other hand,

$$\beta(g,h,kl)\beta(gh,k,l)(((gh)k)l)=\beta(g,h,kl)((gh)(kl))=g(h(kl)).$$

□

4.2. **Coherent 2-groups.** We know from [37] that a coherent 2-group is a monoidal category, in which every object is weakly invertible and every morphism is invertible. More precisely, we have the following definition:

Definition 4.4. A *coherent 2-group* is a monoidal category $(G,\otimes,I,\alpha,r,l)$, where $\otimes: G\otimes G\to G$ is the multiplication functor, I is the unit, $\alpha:\otimes\circ(\otimes\times id_G)\implies\otimes\circ(id_G\times\otimes)$ is the associator, $r_g:g\otimes I\to g$ and $l_g:I\otimes g\to g$ are the right and left unitor (α, r and l are natural equivalence), such that the following diagrams commute.:

(1)

$$\begin{tikzcd}[column sep=small]
& (g\otimes h)\otimes(k\otimes l) \arrow[dr, "\alpha_{g,h,k\otimes l}"] & \\
((g\otimes h)\otimes k)\otimes l \arrow[ur, "\alpha_{g\otimes h,k,l}"] \arrow[d, "\alpha_{g,h,k}\otimes \mathrm{id}_l"'] & & g\otimes(h\otimes(k\otimes l)) \\
(g\otimes(h\otimes k))\otimes l \arrow[rr, "\alpha_{g,h\otimes k,l}"'] & & g\otimes((h\otimes k)\otimes l) \arrow[u, "\mathrm{id}_g\otimes \alpha_{h,k,l}"']
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
(g\otimes I)\otimes h \arrow[rr, "\alpha_{g,I,h}"] \arrow[dr, "r_g\otimes \mathrm{id}_h"'] & & g\otimes(I\otimes h) \arrow[dl, "\mathrm{id}_g\otimes l_h"] \\
& g\otimes h &
\end{tikzcd}$$

(3) Moreover, there is an additional functor $\iota : G \to G$, and there are two natural equivalences $i_g : g \otimes \iota(g) \to I$ and $e_g : \iota(g) \otimes g \to I$, such that the following diagram commutes:

$$\begin{tikzcd}
(g\otimes\iota(g))\otimes g \arrow[r, "i_g\otimes \mathrm{id}_g"] \arrow[d, "\alpha_{g,\iota(g),g}"'] & I\otimes g \arrow[r, "l_g"] & g \arrow[d, "\mathrm{id}_g"] \\
g\otimes(\iota(g)\otimes g) \arrow[r, "\mathrm{id}_g\otimes e_g"] & g\otimes I \arrow[r, "r_g"] & g
\end{tikzcd}$$

A *strict 2-group* is a coherent 2-group, such that all the natural transformations α, l, r, i and e are identity.

There are several equivalent definitions of strict 2-group, one is called the "crossed module":

Definition 4.5. A *crossed module* (M, N, ϕ, γ) consists of two groups M, N together with a group morphism $\phi : M \to N$ and a group morphism $\gamma : N \to \mathrm{Aut}(M)$ such that, denoting $\gamma_n : M \to M$ for every $n \in N$, the following conditions are satisfied:

(1) $\phi(\gamma_n(m)) = n\phi(m)n^{-1}$, for any $n \in N$ and $m \in M$;

(2) $\gamma_{\phi(m)}(m') = mm'm^{-1}$, for any $m, m' \in M$.

Theorem 4.6. *There is a bijective correspondence between a strict 2-group and a crossed module.*

Proof. Given a strict 2-group $(G, \otimes, I)$, we know both the set of objects G^0 and morphism G^1 are groups with the tensor product as group multiplications, and with I and id_I as units. Let $s, t : G^1 \to G^0$ be the source and target of morphisms in G. Let $H := ker(s) \subseteq G^1$, then H is a subgroup of G^1. Define $\gamma : G^0 \to \mathrm{Aut}(H)$ by

$$\gamma_g(\psi) := id_g \otimes \psi \otimes id_{g^{-1}}, \tag{4.4}$$

where $g \in G^0$ and $\psi \in H$ with $s(\psi) = I$, $t(\psi) = h \in G^0$. We can see $\gamma_g(\psi) : I \to ghg^{-1}$ is a well defined morphism with its source as I, where we use gg' to denote the tensor product $g \otimes g'$ of any objects g and g'. Therefore, $\gamma : G^0 \to \mathrm{Aut}(H)$ is a well defined group action. With the target map $t : G^1 \to G^0$, we can check (H, G^0, t, γ) is a crossed module. First we can see

$$t(\gamma_g(\psi)) = ghg^{-1} = gt(\psi)g^{-1}.$$

Let $\psi : I \to h$ and $\psi' : I \to h'$ be two morphisms, we have

$$\begin{aligned}\gamma_{t(\psi)}(\psi') =& id_h \otimes \psi' \otimes id_{h^{-1}} = (id_h \otimes \psi' \otimes id_{h^{-1}}) \circ (\psi \otimes id_I \otimes \psi^{-1})\\ =&(id_h \circ \psi) \otimes (\psi' \circ id_I) \otimes (id_{h^{-1}} \circ \psi^{-1}) = \psi \otimes \psi' \otimes \psi^{-1},\end{aligned}$$

where the third step uses the interchange law between the tensor product and morphism composition. So we get a crossed module (H, G^0, t, γ).

Once we have a crossed module (M, N, ϕ, γ), we can construct a group $M \ltimes N$, with multiplication:

$$(m, n) \bullet (m', n') := (m\gamma_n(m'), nn'),$$

for any $m, m' \in M$ and $n, n' \in N$. We can see that the multiplication is associative with unit $(1_M, 1_N)$. The inverse is given by

$$(m, n)^{-1} = (\gamma_{n^{-1}}(m^{-1}), n^{-1}).$$

Define two group morphisms $s, t : M \ltimes N \to N$ by

$$s(m, n) := n, \qquad t(m, n) := \phi(m)n. \tag{4.5}$$

Define the identity morphism by $id : N \to M \ltimes N$

$$id(n) := (1_M, n). \tag{4.6}$$

There is a composition product $\circ : M \ltimes N_s \times_t M \ltimes N \to M \ltimes N$:

$$(m', \phi(m)n) \circ (m, n) := (m'm, n).$$

The inverse of the composition product $(m, n)^*$ is given by

$$(m, n)^* := (m^{-1}, \phi(m)n).$$

Clearly, $(N, M \ltimes N, s, t, id)$ is a groupoid. The interchange law of products can be checked as follows: On one hand we have

$$\begin{aligned}&((m_1', \phi(m_1)n_1) \circ (m_1, n_1)) \bullet ((m_2', \phi(m_2)n_2) \circ (m_2, n_2)) \qquad (4.7)\\ =&(m_1'm_1, n_1) \bullet (m_2'm_2, n_2) = (m_1'm_1\gamma_{n_1}(m_2'm_2), n_1n_2). \qquad (4.8)\end{aligned}$$

On the other hand, by using the axioms of crossed module we have

$$\begin{aligned}&((m_1', \phi(m_1)n_1) \bullet (m_2', \phi(m_2)n_2)) \circ ((m_1, n_1) \bullet (m_2, n_2))\\ =&(m_1'\gamma_{\phi(m_1)n_1}(m_2'), \phi(m_1)n_1\phi(m_2)n_2) \circ (m_1\gamma_{n_1}(m_2), n_1n_2)\\ =&(m_1'\gamma_{\phi(m_1)n_1}(m_2')m_1\gamma_{n_1}(m_2), n_1n_2) = (m_1'\gamma_{\phi(m_1)}(\gamma_{n_1}(m_2'))m_1\gamma_{n_1}(m_2), n_1n_2)\\ =&(m_1'm_1(\gamma_{n_1}(m_2'))m_1^{-1}m_1\gamma_{n_1}(m_2), n_1n_2) = (m_1'm_1\gamma_{n_1}(m_2')\gamma_{n_1}(m_2), n_1n_2)\\ =&(m_1'm_1\gamma_{n_1}(m_2'm_2), n_1n_2),\end{aligned}$$

where the second step is well defined since

$$\begin{aligned}\phi(m_1\gamma_{n_1}(m_2))n_1n_2 =&\phi(m_1)\phi(\gamma_{n_1}(m_2))n_1n_2 = \phi(m_1)n_1\phi(m_2)n_1^{-1}n_1n_2\\ =&\phi(m_1)n_1\phi(m_2)n_2.\end{aligned}$$

As a result, we have strict 2-group, with objects N (with the group multiplication as the tensor product on objects), morphisms $M \ltimes N$, tensor product $\bullet$ (for morphisms) and morphism composition $\circ$. Now we briefly check the correspondence is bijective.

On one hand, given a strict 2-group, we can see $H \ltimes G^0$ is isomorphic to G^1. Indeed, for any morphism $\psi : g \to g'$, $(\psi \otimes id_{g^{-1}}, g)$ belongs to $H \ltimes G^0$. For any two morphisms

$\psi_1 : g_1 \to g_1'$ and $\psi_2 : g_2 \to g_2'$ we know $\psi_1 \otimes \psi_2$ is a morphism from $g_1 \otimes g_2$ to $g_1' \otimes g_2'$. We can see

$$\begin{aligned}(\psi_1 \otimes id_{g_1^{-1}}, g_1) \bullet (\psi_2 \otimes id_{g_2^{-1}}, g_2) =&(\psi_1 \otimes id_{g_1^{-1}} \otimes id_{g_1} \otimes \psi_2 \otimes id_{g_2^{-1}} \otimes id_{g_1^{-1}}, g_1 \otimes g_2)\\ =&(\psi_1 \otimes \psi_2 \otimes id_{g_2^{-1}} \otimes id_{g_1^{-1}}, g_1 \otimes g_2).\end{aligned}$$

We can also see the correspondence $\psi \mapsto (\psi \otimes id_{g^{-1}}, g)$ is bijective: Assume $(\psi \otimes id_{g^{-1}}, g) = (id_I, I)$, then the source of ψ is I, so $\psi = \psi \otimes id_I = id_I$, and any $(\psi, g) \in H \ltimes G^0$ has a preimage $\psi \otimes id_g \in G^1$.

On the other hand, given a crossed module (M, N, ϕ, γ), since $s(m, n) = n$, we get $\ker(s) \subseteq M \ltimes N$ is isomorphic to M. □

In general the objects of a coherent 2-group can be any unital set with a binary operation. However, in this paper we are interested in a more restricted case where the objects of the corresponding monoidal category form a quasigroup, such that l, r, i, e are identity natural transformations. We can see that if we make more requirements on the associator α, then this kind of coherent 2-group satisfies the following property:

Proposition 4.7. *Let $(G, \otimes, I, \alpha, r, l, \iota, i, e)$ be a coherent 2-group. Assume G^0 is a quasigroup (with the tensor product as the group multiplication), and that l, r, i, e are identity natural transformations, and assume in addition that the associator α satisfies the following:*

$$\alpha_{1,g,h} = \alpha_{g,1,h} = \alpha_{g,h,1} = id_{gh} \tag{4.9}$$

$$\alpha_{g,g^{-1},h} = \alpha_{g^{-1},g,h} = id_h = \alpha_{h,g,g^{-1}} = \alpha_{h,g^{-1},g} \tag{4.10}$$

for any $g, h \in G^0$, then we have the following properties:

(i) *The morphisms and their compositions form a groupoid.*
(ii) *The identity map from the objects to the morphisms preserves the tensor product, i.e. $id_g \otimes id_h = id_{g\otimes h}$, for any objects g, h.*
(iii) *The source and target maps of morphisms also preserve the tensor product.*
(iv) *The composition and tensor product of morphisms satisfy the interchange rule:*

$$(\phi_1 \otimes \phi_2) \circ (\psi_1 \otimes \psi_2) = (\phi_1 \circ \psi_1) \otimes (\phi_2 \circ \psi_2), \tag{4.11}$$

for any composable pair of morphisms ϕ_1, ψ_1 and ϕ_2, ψ_2.
(v) *The morphisms and their tensor product form a quasigroup.*

Proof. The first four of properties are direct results of the definition of a coherent 2-group.

For (v), let $\phi : g \to h$ and $\psi : k \to l$ be two morphisms. We know the inverse of ϕ (in the sense of the tensor product) is $\phi^{-1} := \iota(\phi) : g^{-1} \to h^{-1}$. By the naturality of α we have

$$\psi = \alpha_{h,h^{-1},l} \circ ((\phi \otimes \phi^{-1}) \otimes \psi) = (\phi \otimes (\phi^{-1} \otimes \psi)) \circ \alpha_{g,g^{-1},k} = \phi \otimes (\phi^{-1} \otimes \psi),$$

where in the first and last steps we use (4.10) and the natural transformation i being trivial. Similarly, we also have $\psi = (\psi \otimes \phi) \otimes \phi^{-1}$. By the same method, we can check $(id_1 \otimes \phi) \otimes \psi = \phi \otimes \psi = \phi \otimes (\psi \otimes id_1)$. Indeed,

$$(id_1 \otimes \phi) \otimes \psi = \alpha_{1,h,l} \circ ((id_1 \otimes \phi) \otimes \psi) = (id_1 \otimes (\phi \otimes \psi)) \circ \alpha_{1,g,k} = \phi \otimes \psi,$$

where we also use that the natural transformations l, r are trivial. Thus the morphisms with their tensor product form a quasigroup.

□

Theorem 4.8. *If G is a quasiassociative quasigroup, then we can construct a coherent 2-group.*

Proof. Define $H := N(G) \ltimes G$, then H is a quasigroup with the product given by:

$$(n, g) \otimes (m, h) := (n(gmg^{-1}), gh).$$

The inverse corresponding to this product is given by

$$(n, g)^{-1} := (g^{-1}n^{-1}g, g^{-1}),$$

which is well defined since G is quasiassociative. Before we check if H is a quasigroup, we can see that the adjoint action $Ad : G \to \mathrm{Aut}(N(G))$ given by $Ad_g(m) := gmg^{-1}$ is well defined, since on one hand

$$((gh)m(gh)^{-1})g = ((gh)m(gh)^{-1})((gh)h^{-1}) = (((gh)m(gh)^{-1})(gh))h^{-1} = ((gh)m)h^{-1},$$

and on the other hand

$$\begin{aligned}&(g(hmh^{-1})g^{-1})g = g(hmh^{-1}) = ((gh)h^{-1})(hmh^{-1}) = (gh)(h^{-1}(hmh^{-1}))\\ =&(gh)(mh^{-1}) = ((gh)m)h^{-1},\end{aligned}$$

therefore we have

$$Ad_{gh}(m) = (gh)m(gh)^{-1} = g(hmh^{-1})g^{-1} = Ad_g \circ Ad_h(m),$$

for any $g, h \in G$ and $m \in N(G)$. Now we can see H is a quasigroup:

$$\begin{aligned}&((n, g) \otimes (m, h)) \otimes (m, h)^{-1} = (n(gmg^{-1}), gh) \otimes (h^{-1}m^{-1}h, h^{-1})\\ =&((n(gmg^{-1}))(Ad_{gh}(Ad_{h^{-1}}(m^{-1}))), (gh)h^{-1}) = ((n(gmg^{-1}))(gm^{-1}g^{-1}), g)\\ =&(n, g),\end{aligned}$$

for any (n, g), $(m, h) \in H$. Similarly, we have $(n, g)^{-1} \otimes ((n, g) \otimes (m, h)) = (m, h)$.

To construct a coherent 2-group, we also need the objects to be G and $\otimes : G \times G \to G$ to be the multiplication of G. The source and target maps $s, t : H \to G$ are given by

$$s(n, g) := g \qquad t(n, g) := ng.$$

The identity morphism $id : G \to H$ is given by

$$id(g) := (1, g).$$

The composition $\circ : H_s \times_t H \to H$, is given by

$$(n, mg) \circ (m, g) := (nm, g).$$

The composition inverse is given by

$$(n, g)^* := (n^{-1}, ng).$$

Thus we get a groupoid with the corresponding composition, inverse, source and target maps. Moreover, the maps s, t and id preserve the tensor product. Indeed, by using that G is quasiassociative we have

$$\begin{aligned}&t((n, g) \otimes (m, h)) = t(n(gmg^{-1}), gh) = (n(gmg^{-1}))gh = n((gmg^{-1})(gh))\\ =&n(((gmg^{-1})g)h) = n((gm)h) = n(g(mh)) = (ng)(mh) = t(n, g) \otimes t(m, h).\end{aligned}$$

The interchange law of products can be also proved as in Theorem 4.6.

The associator $\alpha : G \times G \times G \to H$ is given by

$$\alpha_{g,h,k} := (\beta(g, h, k), (gh)k),$$

this is well defined since the image of β belongs to $N(G)$, and the source and target of $\alpha_{g,h,k}$ are $(gh)k$ and $g(hk)$. Because G is a quasigroup, we have $\alpha_{g,g^{-1},h} = \alpha_{h,g^{-1},g} = (1,h) = id_h$ and $\alpha_{1,g,h} = \alpha_{g,1,h} = \alpha_{g,h,1} = id_{gh}$ for any $g, h \in G$. We can also see that α is a natural transformation. Indeed, let (l,g), (m,h) and (n,k) belong to H, we can see on one hand

$$\begin{aligned}
&\alpha_{lg,mh,nk} \circ (((l,g) \otimes (m,h)) \otimes (n,k)) \\
=&(\beta(lg,mh,nk), ((lg)(mh))(nk)) \circ ((l(Ad_g(m)))(Ad_{gh}(n)), (gh)k) \\
=&(\beta(lg,mh,nk)\Big((l(Ad_g(m)))(Ad_{gh}(n))\Big), (gh)k)
\end{aligned}$$

and on the other hand

$$\begin{aligned}
&((l,g) \otimes ((m,h) \otimes (n,k))) \circ \alpha_{g,h,k} \\
=&(lAd_g\big(mAd_h(n)\big), g(hk)) \circ (\beta(g,h,k), (gh)k) \\
=&(\Big(lAd_g\big(mAd_h(n)\big)\Big)\beta(g,h,k), (gh)k).
\end{aligned}$$

Since

$$\begin{aligned}
&\beta(lg,mh,nk)\Big((l(Ad_g(m)))(Ad_{gh}(n))\Big)(gh)k \\
=&(lg)((mh)(nk)) \\
=&\Big(lAd_g\big(mAd_h(n)\big)\Big)\beta(g,h,k)(gh)k
\end{aligned}$$

we get

$$\beta(lg,mh,nk)\Big((l(Ad_g(m)))(Ad_{gh}(n))\Big) = \Big(lAd_g\big(mAd_h(n)\big)\Big)\beta(g,h,k),$$

therefore,

$$\alpha_{lg,mh,nk} \circ (((l,g) \otimes (m,h)) \otimes (n,k)) = ((l,g) \otimes ((m,h) \otimes (n,k))) \circ \alpha_{g,h,k}.$$

The pentagon for being a coherent 2-group is satisfied by Lemma 4.3. Indeed, let $g, h, k, l \in G$, on one hand we have

$$\begin{aligned}
&\alpha_{g,h,kl} \circ \alpha_{gh,k,l} = (\beta(g,h,kl), (gh)(kl)) \circ (\beta(gh,k,l), ((gh)k)l) \\
=&(\beta(g,h,kl)\beta(gh,k,l), ((gh)k)l)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(id_g \otimes \alpha_{h,k,l}) \circ \alpha_{g,hk,l} \circ (\alpha_{g,h,k} \otimes id_l) \\
=&(g\beta(h,k,l)g^{-1}, g((hk)l)) \circ \big(\beta(g,hk,l), (g(hk))l\big) \circ (\beta(g,h,k), ((gh)k)l) \\
=&((g\beta(h,k,l)g^{-1})\beta(g,hk,l)\beta(g,h,k), ((gh)k)l),
\end{aligned}$$

by using Lemma 4.3 we get the pentagon. Thus we get a coherent 2-group, where the natural transformations l, r, e, i are identity. □

5. Algebras, coalgebras and all that

First we recall some material about Hopf algebras, Hopf coquasigroups and corresponding modules and comodules. We also review the more general notions of rings and corings over an algebra as well as the associated notion of a Hopf algebroid. We then move onto Hopf–Galois extensions, as noncommutative principal bundles.

To be definite we work over the field $\mathbb{C}$ of complex numbers but in the following this could be substituted by any field k. Algebras are assumed to be unital and associative with morphisms of algebras taken to be unital.

Definition 5.1. A *bialgebra* is an algebra H with two algebra maps $\Delta_H : H \to H \otimes H$ (called the coproduct) and $\epsilon : H \to \mathbb{C}$ (called the counit), such that

$$(id_H \otimes \Delta_H) \circ \Delta_H = (\Delta_H \otimes id_H) \circ \Delta, \quad (\mathrm{id}_H \otimes \epsilon_H) \circ \Delta_H = \mathrm{id}_H = (\epsilon_H \otimes id_H) \circ \Delta_H, \tag{5.1}$$

where the algebra multiplication on $H \otimes H$ is given by $(h \otimes g) \cdot (h' \otimes g') := (hh' \otimes gg')$ for any $h \otimes g, h' \otimes g' \in H \otimes H$. Moreover, if there is a linear map $S : H \to H \otimes H$ (called the antipode), such that

$$m_H \circ (S \otimes id_H) \circ \Delta_H = 1_H \epsilon_H = m_H \circ (id_H \otimes S) \circ \Delta_H, \tag{5.2}$$

where m_H is the product on H, then H is called a *Hopf algebra*. Let H and G be two Hopf algebras, a Hopf algebra morphism from H to G is an algebra map $\psi : H \to G$, such that

$$(\psi \otimes \psi) \circ \Delta_H = \Delta_G \circ \psi, \qquad \epsilon_G \circ \psi = \epsilon_H. \tag{5.3}$$

In the following context, we will always use Sweedler index notation to denote the image of a coproduct, i.e.

$$\Delta_H(h) = h_{(1)} \otimes h_{(2)}. \tag{5.4}$$

For the coproduct of a bialgebra $\Delta : H \to H \otimes H$ we use the Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ (sum understood), and its iterations: $\Delta^n = (\mathrm{id}_H \otimes \Delta_H) \circ \Delta_H^{n-1} : h \mapsto h_{(1)} \otimes h_{(2)} \otimes \cdots \otimes h_{(n+1)}$. However, this n-th iterated coproduct is not unique, if the coproduct is not coassociative.

In order to give a coherent Hopf 2-algebra in the last part, which is a weaker version of a Hopf 2-algebra, we need a more general algebra structure which is a Hopf coquasigroup [20].

Definition 5.2. A *Hopf coquasigroup* H is an unital associative algebra, equiped with counital algebra homomorphisms $\Delta : H \to H \otimes H$, $\epsilon : H \to \mathbb{C}$, and a linear map $S_H : H \to H$ such that

$$(m_H \otimes id_H)(S_H \otimes id_H \otimes id_H)(id_H \otimes \Delta)\Delta = 1 \otimes id_H = (m_H \otimes id_H)(id_H \otimes S_H \otimes id_H)(id_H \otimes \Delta)\Delta, \tag{5.5}$$

$$(id_H \otimes m_H)(id_H \otimes S_H \otimes id_H)(\Delta \otimes id_H)\Delta = id_H \otimes 1 = (id_H \otimes m_H)(id_H \otimes id_H \otimes S_H)(\Delta \otimes id_H)\Delta. \tag{5.6}$$

A morphism between two Hopf coquasigroups is an algebra map $f : H \to G$, such that for any $h \in H$, $f(h)_{(1)} \otimes f(h)_{(2)} = f(h_{(1)}) \otimes f(h_{(2)})$ and $\epsilon_G(f(h)) = \epsilon_H(h)$.

Remark 5.3. Hopf coquasigroup are a generalisation of Hopf algebras, for which the coproduct is not necessarily coassociative. As a result, we cannot use Sweedler index notion in the same way as for Hopf algebras (but we still use $h_{(1)} \otimes h_{(2)}$ as the image of a coproduct), so in general we don't have: $h_{(1)(1)} \otimes h_{(1)(2)} \otimes h_{(2)} = h_{(1)} \otimes h_{(2)} \otimes h_{(3)} = h_{(1)} \otimes h_{(2)(1)} \otimes h_{(2)(2)}$. It is given in [20] that the linear map S_H (which we likewise call the 'antipode') of H has similar property as the antipode of a Hopf algebra. That is:

- $h_{(1)}S_H(h_{(2)}) = \epsilon(h) = S_H(h_{(1)})h_{(2)}$,
- $S_H(hh') = S_H(h')S_H(h)$,
- $S_H(h)_{(1)} \otimes S_H(h)_{(2)} = S_H(h_{(2)}) \otimes S_H(h_{(1)})$,

for any $h, h' \in H$.

Given a Hopf coquasigroup H, define a linear map $\beta : H \to H \otimes H \otimes H$ by

$$\beta(h) := h_{(1)(1)}S_H(h_{(2)})_{(1)(1)} \otimes h_{(1)(2)(1)}S_H(h_{(2)})_{(1)(2)} \otimes h_{(1)(2)(2)}S_H(h_{(2)})_{(2)} \tag{5.7}$$

for any $h \in H$. We can see that

$$\beta * ((\Delta \otimes id_H) \circ \Delta) = (id_H \otimes \Delta) \circ \Delta, \tag{5.8}$$

where we denote by $*$ the convolution product in the dual vector space $H' := \mathrm{Hom}(H, \mathbb{C})$, $(f * g)(h) := f(h_{(1)})g(h_{(2)})$. More precisely, (5.8) can be written as

$$h_{(1)(1)(1)}S_H(h_{(1)(2)})_{(1)(1)}h_{(2)(1)(1)} \otimes h_{(1)(1)(2)(1)}S_H(h_{(1)(2)})_{(1)(2)}h_{(2)(1)(2)} \otimes h_{(1)(1)(2)(2)}S_H(h_{(1)(2)})_{(2)}h_{(2)(2)}$$
$$= h_{(1)} \otimes h_{(2)(1)} \otimes h_{(2)(2)},$$

which can be derived from the definition of a Hopf coquasigroup. For any $h \in H$ we will always denote the image of β by

$$\beta(h) = h^{\hat{1}} \otimes h^{\hat{2}} \otimes h^{\hat{3}}.$$

A Hopf coquasigroup H is coassociative if and only if $\beta(h) = \epsilon(h)1_H \otimes 1_H \otimes 1_H$ if and only if H is a Hopf algebra.

Given a Hopf coquasigroup H, a *left H-comodule* is a vector space V carrying a left H-coaction, that is a $\mathbb{C}$-linear map $\delta^V : V \to H \otimes V$ such that

$$(id_H \otimes \delta^V) \circ \delta^V = (\Delta \otimes id_V) \circ \delta^V \ , \quad (\epsilon \otimes id_V) \circ \delta^V = id_V \ . \tag{5.9}$$

In Sweedler notation, $v \mapsto \delta^V(v) = v^{(-1)} \otimes v^{(0)}$, and the left H-comodule properties read

$$v^{(-1)}{}_{(1)} \otimes v^{(-1)}{}_{(2)} \otimes v^{(0)} = v^{(-1)} \otimes v^{(0)(-1)} \otimes v^{(0)(0)} \ ,$$
$$\epsilon(v^{(-1)})\, v^{(0)} = v \ ,$$

for all $v \in V$. The $\mathbb{C}$-vector space tensor product $V \otimes W$ of two H-comodules is a H-comodule with the left tensor product H-coaction

$$\delta^{V\otimes W} : V \otimes W \longrightarrow H \otimes V \otimes W, \quad v \otimes w \longmapsto v^{(-1)}w^{(-1)} \otimes v^{(0)} \otimes w^{(0)}. \tag{5.10}$$

A H-comodule map $\psi : V \to W$ between two H-comodules is a $\mathbb{C}$-linear map $\psi : V \to W$ which is H-equivariant (or H-colinear), that is, $\delta^W \circ \psi = (id_H \otimes \psi) \circ \delta^V$.

In particular, a *left H-comodule algebra* is an algebra A, which is a left H-comodule such that the multiplication and unit of A are morphisms of H-comodules. This is equivalent to requiring the coaction $\delta : A \to H \otimes A$ to be a morphism of unital algebras (where $H \otimes A$ has the usual tensor product algebra structure). Corresponding morphisms are H-comodule maps which are also algebra maps.

In the same way, a left H-comodule coalgebra is a coalgebra C, which is a left H-comodule and such that the coproduct and the counit of C are morphisms of H-comodules. Explicitly, this means that, for each $c \in C$,

$$c^{(-1)} \otimes c^{(0)}{}_{(1)} \otimes c^{(0)}{}_{(2)} = c_{(1)}{}^{(-1)}c_{(2)}{}^{(-1)} \otimes c_{(1)}{}^{(0)} \otimes c_{(2)}{}^{(0)} \ ,$$
$$\epsilon_C(c) = c^{(-1)}\epsilon_C(c^{(0)}) \ .$$

Corresponding morphisms are H-comodule maps which are also coalgebra maps.

Clearly, there are right H-comodule versions of the above ones. Here we use lower Sweedler notation to denote the image of a right coaction $\delta_V : V \to V \otimes H$, i.e. $\delta_V(v) = v_{(0)} \otimes v_{(1)}$. More precisely, with a Hopf coquasigroup H, a *right H-comodule* is a vector space V carrying a right H-coaction, that is, a $\mathbb{C}$-linear map $\delta_V : V \to V \otimes H$ such that

$$(id_V \otimes \Delta) \circ \delta_V = (\delta_V \otimes id_H) \circ \delta_V \ , \quad (id_V \otimes \epsilon) \circ \delta_V = id_V \ . \tag{5.11}$$

In Sweedler notation, $v \mapsto \delta_V(v) = v_{(0)} \otimes v_{(1)}$, and the right H-comodule properties read,

$$\begin{aligned} v_{(0)} \otimes (v_{(1)})_{(1)} \otimes (v_{(1)})_{(2)} &= (v_{(0)})_{(0)} \otimes (v_{(0)})_{(1)} \otimes v_{(1)} =: v_{(0)} \otimes v_{(1)} \otimes v_{(2)} \ , \\ v_{(0)}\, \epsilon(v_{(1)}) &= v \ , \end{aligned}$$

for all $v \in V$. The $\mathbb{C}$-vector space tensor product $V \otimes W$ of two H-comodules is a H-comodule with the right tensor product H-coaction

$$\delta_{V \otimes W} : V \otimes W \longrightarrow V \otimes W \otimes H \ , \quad v \otimes w \longmapsto v_{(0)} \otimes w_{(0)} \otimes v_{(1)} w_{(1)} \ . \tag{5.12}$$

A H-comodule map $\psi : V \to W$ between two H-comodules is a $\mathbb{C}$-linear map $\psi : V \to W$ which is H-equivariant (or H-colinear), that is, $\delta_W \circ \psi = (\psi \otimes id_H) \circ \delta_V$.

Similarly, a *right H-comodule algebra* is an algebra A which is a right H-comodule such that the comodule map $\delta_V : V \to V \otimes H$ is an algebra map. Corresponding morphisms are H-comodule maps which are also algebra maps.

A *right H-comodule coalgebra* is a coalgebra C which is a right H-comodule such that the coproduct and the counit of C are morphisms of H-comodules. Explicitly, this means that, for each $c \in C$,

$$\begin{aligned} (c_{(1)})_{(0)} \otimes (c_{(2)})_{(0)} \otimes (c_{(1)})_{(1)} (c_{(2)})_{(1)} &= (c_{(0)})_{(1)} \otimes (c_{(0)})_{(2)} \otimes c_{(1)} \ , \\ \epsilon(c_{(0)}) c_{(1)} &= \epsilon(c) 1_H \ . \end{aligned}$$

Corresponding morphisms are H-comodule maps which are also coalgebra maps. Clearly, the comodule and the comodule algebra of a Hopf algebra can be given in the same way.

Next, let H be a bialgebra and let A be a right H-comodule algebra. An (A, H)-relative Hopf module V is a right H-comodule with a compatible left A-module structure. That is the left action $\triangleright_V : A \otimes V \to V$ is a morphism of H-comodules such that $\delta^V \circ \triangleright_V = (\triangleright_V \otimes id_H) \circ \delta^{A \otimes V}$. Explicitly, for all $a \in A$ and $v \in V$,

$$(a \triangleright_V v)_{(0)} \otimes (a \triangleright_V v)_{(1)} = a_{(0)} \triangleright_V v_{(0)} \otimes a_{(1)} v_{(1)} \ . \tag{5.13}$$

A morphism of (A, H)-relative Hopf modules is a morphism of right H-comodules which is also a morphism of left A-modules. In a similar way one can consider the case for the algebra A to be acting on the right, or with left and right A-actions.

Definition 5.4. A *coassociative pair* (A, B, ϕ) consists of a Hopf coquasigroup B and a Hopf algebra A, together with a Hopf coquasigroup morphism $\phi : B \to A$, such that

$$\begin{cases} \phi(b_{(1)(1)}) \otimes b_{(1)(2)} \otimes b_{(2)} = & \phi(b_{(1)}) \otimes b_{(2)(1)} \otimes b_{(2)(2)} \\ b_{(1)(1)} \otimes \phi(b_{(1)(2)}) \otimes b_{(2)} = & b_{(1)} \otimes \phi(b_{(2)(1)}) \otimes b_{(2)(2)} \\ b_{(1)(1)} \otimes b_{(1)(2)} \otimes \phi(b_{(2)}) = & b_{(1)} \otimes b_{(2)(1)} \otimes \phi(b_{(2)(2)}). \end{cases} \tag{5.14}$$

Clearly, for any morphism ϕ between two Hopf algebras B and A, (A, B, ϕ) is a coassociative pair.

Remark 5.5. A coassociative pair can be viewed as a quantisation of a group and quasigroup, such that there is a quasigroup morphism which maps the group into the associative elements of the quasigroup. More precisely, let H be a group, G be a quasigroup, and $\phi : H \to G$ be a morphism of quasigroup, such that the $\phi(H) \subseteq N(G)$. So we have

$$\begin{cases} (\phi(h)g)g' = \phi(h)(gg') \\ (g\phi(h))g' = g(\phi(h)g') \\ (\phi(h)g)g' = \phi(h)(gg'). \end{cases}$$

due to relations (5.14). Since the image of ϕ belongs to the associative elements of G, $\phi(h)$ can pass through brackets. In other words, let m_I and m_J be two iterated products, such that $m_I(G \otimes G \cdots \otimes G \otimes 1 \otimes G \otimes \cdots \otimes G) = m_J(G \otimes G \cdots \otimes G \otimes 1 \otimes G \otimes \cdots \otimes G)$ (where 1 is inserted into the m-th position), then $m_I(G \otimes G \cdots \otimes G \otimes \phi(h) \otimes G \otimes \cdots \otimes G) = m_J(G \otimes G \cdots \otimes G \otimes \phi(h) \otimes G \otimes \cdots \otimes G)$ (where 1 is replaced by $\phi(h)$). For example, assume the 4th iterated products $m_I, m_J : G \otimes G \otimes G \otimes G \to G$ given by $m_I(g \otimes g' \otimes g'' \otimes g''') := (g(g'g''))g'''$ and $m_I(g \otimes g' \otimes g'' \otimes g''') := (gg')(g''g''')$, then we have $(g(g'\phi(h)))g'' = (gg')(\phi(h)g'')$ for any $g, g', g'' \in G$ and $h \in H$.

We know for a Hopf coquasigroup, the n-th iterated coproducts Δ^n are not always equal when $n \geq 2$ (n can be equal to 0, in which case Δ^0 is identity map). However, given a coassociative pair (A, B, ϕ), there is an interesting property due to Remark 5.5 and Proposition 4.2:

Proposition 5.6. *Let (A, B, ϕ) be a coassociative pair of a Hopf algebra A and a Hopf coquasigroup B, and let Δ^n_I, Δ^n_J both n-th iterated coproducts on B. Let $\Delta^n_I(b) = b_{I_1} \otimes b_{I_2} \otimes \cdots \otimes b_{I_{n+1}}$ and $\Delta^n_J(b) = b_{J_1} \otimes b_{J_2} \otimes \cdots \otimes b_{J_{n+1}}$. If*

$$b_{I_1} \otimes b_{I_2} \otimes \cdots \otimes \epsilon_B(b_{I_m}) \otimes \cdots \otimes b_{I_{n+1}} = b_{J_1} \otimes b_{J_2} \otimes \cdots \otimes \epsilon_B(b_{J_m}) \otimes \cdots \otimes b_{J_{n+1}} \tag{5.15}$$

for $1 \leq m \leq n+1$, then we have

$$b_{I_1} \otimes b_{I_2} \otimes \cdots \otimes \phi(b_{I_m}) \otimes \cdots \otimes b_{I_{n+1}} = b_{J_1} \otimes b_{J_2} \otimes \cdots \otimes \phi(b_{J_m}) \otimes \cdots \otimes b_{J_{n+1}}.$$

Proof. We can prove this proposition inductively. For $n = 2$, this is obvious by the definition of a coassociative pair. Now we consider the case for $n \geq 3$. We can see both sides of equation (5.15) are equal to the image of an $(n-1)$-th iterated coproduct Δ^{n-1}_K, which can be written as $\Delta^{n-1}_K = (\Delta^p_{K'} \otimes \Delta^q_{K''}) \circ \Delta$ for some iterated coproducts $\Delta^p_{K'}$, $\Delta^q_{K''}$ with $p + q = n - 2$. Assume this proposition is correct for $n = N - 1$. We have two cases for the index of I_m and J_m:

The first case is that the first index of I_m and J_m are the same (where the first index means the first Sweedler index on the left, for example the first index of $b_{(2)(2)(1)}$ is 2). When $m \leq (p+2)$, the first indices of I_m and J_m have to be 1. In this case, we can see $\Delta^n_I = (\Delta^{p+1}_{I_1} \otimes \Delta^q_{K''}) \circ \Delta$ and $\Delta^n_J = (\Delta^{p+1}_{J_1} \otimes \Delta^q_{K''}) \circ \Delta$, for some $(p+1)$-th iterated coproducts $\Delta^{p+1}_{I_1}$ and $\Delta^{p+1}_{J_1}$. Then we can apply the hypotheses for the terms, whose first indices are 1. For $m \geq p+3$, the situation is the similar.

The second case is that the first indices of I_m and J_m are different. Assume the first index of I_m is 1 and J_m is 2. In this case m has to be equal to $p+2$, and $\Delta^n_I = (\Delta^{p+1}_E \otimes \Delta^q_{K''}) \circ \Delta$ and $\Delta^n_J = (\Delta^p_{K'} \otimes \Delta^{q+1}_F) \circ \Delta$ for some iterated $(p+1)$-th coproduct Δ^{p+1}_E with $(id_B^{\otimes p} \otimes \epsilon_B) \circ \Delta^{p+1}_E = \Delta^p_{K'}$ and iterated $(q+1)$-th coproduct Δ^{q+1}_F with $(\epsilon_B \otimes id_B^{\otimes q}) \circ \Delta^{q+1}_F = \Delta^q_{K''}$. Define $\Delta^{p+1}_G := (\Delta^p_{K'} \otimes id_B) \circ \Delta$ and $\Delta^{q+1}_H := (id_B \otimes \Delta^q_{K''}) \circ \Delta$ (notice that Δ^{p+1}_E is not necessarily equal to Δ_G^{p+1} and Δ_F^{p+1} is not necessarily equal to Δ_H^{p+1}), then we can see

$$\begin{aligned}
&b_{I_1} \otimes b_{I_2} \otimes \cdots \otimes \phi(b_{I_m}) \otimes b_{I_{m+1}} \otimes \cdots \otimes b_{I_{n+1}} \\
=&b_{(1)E_1} \otimes b_{(1)E_2} \otimes \cdots \otimes \phi(b_{(1)E_{p+2}}) \otimes b_{(2)K''_1} \otimes \cdots \otimes b_{(2)K''_{q+1}} \\
=&b_{(1)G_1} \otimes b_{(1)G_2} \otimes \cdots \otimes \phi(b_{(1)G_{p+2}}) \otimes b_{(2)K''_1} \otimes \cdots \otimes b_{(2)K''_{q+1}} \\
=&b_{(1)(1)K'_1} \otimes b_{(1)(1)K'_2} \otimes \cdots \otimes b_{(1)(1)K'_{p+1}} \otimes \phi(b_{(1)(2)}) \otimes b_{(2)K''_1} \otimes \cdots \otimes b_{(2)K''_{q+1}} \\
=&b_{(1)K'_1} \otimes b_{(1)K'_2} \otimes \cdots \otimes b_{(1)K'_{p+1}} \otimes \phi(b_{(2)(1)}) \otimes b_{(2)(2)K''_1} \otimes \cdots \otimes b_{(2)(2)K''_{q+1}} \\
=&b_{(1)K'_1} \otimes b_{(1)K'_2} \otimes \cdots \otimes b_{(1)K'_{p+1}} \otimes \phi(b_{(2)H_1}) \otimes b_{(2)H_2} \otimes \cdots \otimes b_{(2)H_{q+2}} \\
=&b_{(1)K'_1} \otimes b_{(1)K'_2} \otimes \cdots \otimes b_{(1)K'_{p+1}} \otimes \phi(b_{(2)F_1}) \otimes b_{(2)F_2} \otimes \cdots \otimes b_{(2)F_{q+2}} \\
=&b_{J_1} \otimes b_{J_2} \otimes \cdots \otimes \phi(b_{J_m}) \otimes b_{J_{m+1}} \otimes \cdots \otimes b_{J_{n+1}},
\end{aligned}$$

where $b_{(1)E_1} \otimes b_{(1)E_2} \otimes \cdots \otimes \phi(b_{(1)E_{p+2}}) := \Delta_E^{p+1}(b_{(1)})$ and similar for the rest. The 2nd and 6th steps use the hypotheses for $n \leq N-1$, and the 4th step uses the definition of a coassociate pair.

□

From this proposition, we can make a generalisation by using the proposition twice: If

$$\begin{aligned}
&b_{I_1} \otimes b_{I_2} \otimes \cdots \otimes \epsilon_B(b_{I_m}) \otimes \cdots \otimes \epsilon_B(b_{I_{m'}}) \otimes \cdots \otimes b_{I_{n+1}} \\
=&b_{J_1} \otimes b_{J_2} \otimes \cdots \otimes \epsilon_B(b_{J_m}) \otimes \cdots \otimes \epsilon_B(b_{J_{m'}}) \otimes \cdots \otimes b_{J_{n+1}}
\end{aligned}$$

for $1 \leq m < m' \leq n+1$. Then we have

$$\begin{aligned}
&b_{I_1} \otimes b_{I_2} \otimes \cdots \otimes \phi(b_{I_m}) \otimes \cdots \otimes \phi(b_{I_{m'}}) \otimes \cdots \otimes b_{I_{n+1}} \\
=&b_{J_1} \otimes b_{J_2} \otimes \cdots \otimes \phi(b_{J_m}) \otimes \cdots \otimes \phi(b_{J_{m'}}) \otimes \cdots \otimes b_{J_{n+1}}.
\end{aligned}$$

There is a dual version of the Hopf coquasigroup, which is the Hopf quasigroup [20]:

Definition 5.7. A *Hopf quasigroup* A is a coassociative coalgebra with a coproduct $\Delta : A \to A \otimes A$ and counit $\epsilon : A \to k$, together with a unital and possibly nonassociative algebra structure, such that the coproduct and counit are algebra maps. Moreover, there is an linear map (the antipode) $S_A : A \to A$ such that:

$$m(id_A \otimes m)(S_A \otimes id_A \otimes id_A)(\Delta \otimes id_A) = \epsilon \otimes id_A = m(id_A \otimes m)(id_A \otimes S_A \otimes id_A)(\Delta \otimes id_A) \tag{5.16}$$

$$m(m \otimes id_A)(id_A \otimes S_A \otimes id_A)(id_A \otimes \Delta) = id_A \otimes \epsilon = m(m \otimes id_A)(id_A \otimes id_A \otimes S_A)(id_A \otimes \Delta). \tag{5.17}$$

A Hopf quasigroup is a Hopf algebra if and only if it is associative.

6. Hopf algebroids

In the following, we will give an introduction to Hopf algebroids. (There are different kinds of Hopf algebroids, here we mainly follow the definition in [8]).

For an algebra B a *B-ring* is a triple (A, μ, η). Here A is a B-bimodule with B-bimodule maps $\mu : A \otimes_B A \to A$ and $\eta : B \to A$, satisfying the following associativity condition:

$$\mu \circ (\mu \otimes_B id_A) = \mu \circ (id_A \otimes_B \mu) \tag{6.1}$$

and unit condition,

$$\mu \circ (\eta \otimes_B id_A) = A = \mu \circ (id_A \otimes_B \eta). \tag{6.2}$$

A morphism of B-rings $f : (A, \mu, \eta) \to (A', \mu', \eta')$ is a B-bimodule map $f : A \to A'$, such that $f \circ \mu = \mu' \circ (f \otimes_B f)$ and $f \circ \eta = \eta'$. Here for any B-bimodule M, the balanced tensor product $M \otimes_B M$ is given by

$$M \otimes_B M := M \otimes M / \langle m \otimes bm' - mb \otimes m' \rangle_{m,m' \in M,\, b \in B}.$$

From [8, Lemma 2.2] there is a bijective correspondence between B-rings (A, μ, η) and algebra morphisms $\eta : B \to A$. Starting with a B-ring (A, μ, η), one obtains a multiplication map $A \otimes A \to A$ by composing the canonical surjection $A \otimes A \to A \otimes_B A$ with the map μ. Conversely, starting with an algebra map $\eta : B \to A$, a B-bilinear associative multiplication $\mu : A \otimes_B A \to A$ is obtained from the universality of the coequaliser $A \otimes A \to A \otimes_B A$ which identifies an element $ar \otimes a'$ with $a \otimes ra'$.

Dually, for an algebra B a *B-coring* is a triple (C, Δ, ϵ). Here C is a B-bimodule with B-bimodule maps $\Delta : C \to C \otimes_B C$ and $\epsilon : C \to B$, satisfying the following coassociativity and counit conditions,

$$(\Delta \otimes_B id_C) \circ \Delta = (id_C \otimes_B \Delta) \circ \Delta, \quad (\epsilon \otimes_B id_C) \circ \Delta = id_C = (id_C \otimes_B \epsilon) \circ \Delta. \tag{6.3}$$

A morphism of B-corings $f : (C, \Delta, \epsilon) \to (C', \Delta', \epsilon')$ is a B-bimodule map $f : C \to C'$, such that $\Delta' \circ f = (f \otimes_B f) \circ \Delta$ and $\epsilon' \circ f = \epsilon$.

Definition 6.1. Given an algebra B, a *left B-bialgebroid* $\mathcal{L}$ consists of an $(B \otimes B^{op})$-ring together with a B-coring structures on the same vector space $\mathcal{L}$ with mutual compatibility conditions. From what said above, a $(B \otimes B^{op})$-ring $\mathcal{L}$ is the same as an algebra map $\eta : B \otimes B^{op} \to \mathcal{L}$. Equivalently, one may consider the restrictions

$$s := \eta(\,\cdot\, \otimes_B 1_B) : B \to \mathcal{L} \quad \text{and} \quad t := \eta(1_B \otimes_B \,\cdot\,) : B^{op} \to \mathcal{L}$$

which are algebra maps with commuting ranges in $\mathcal{L}$, called the *source* and the *target* map of the $(B \otimes B^{op})$-ring $\mathcal{L}$. Thus a $(B \otimes B^{op})$-ring is the same as a triple $(\mathcal{L}, s, t)$ with $\mathcal{L}$ an algebra and $s : B \to \mathcal{L}$ and $t : B^{op} \to \mathcal{L}$ both algebra maps with commuting range.

Thus, for a left B-bialgebroid $\mathcal{L}$ the compatibility conditions are required to be

(i) The bimodule structures in the B-coring $(\mathcal{L}, \Delta, \epsilon)$ are related to those of the $B \otimes B^{op}$-ring $(\mathcal{L}, s, t)$ via

$$b \triangleright a \triangleleft b' := s(b)t(b')a \qquad \text{for } b, b' \in B,\ a \in \mathcal{L}. \tag{6.4}$$

(ii) Considering $\mathcal{L}$ as a B-bimodule as in (6.4), the coproduct Δ corestricts to an algebra map from $\mathcal{L}$ to

$$\mathcal{L} \times_B \mathcal{L} := \left\{ \sum\nolimits_j a_j \otimes_B a'_j \mid \sum\nolimits_j a_j t(b) \otimes_B a'_j = \sum\nolimits_j a_j \otimes_B a'_j s(b), \text{ for all } b \in B \right\}, \tag{6.5}$$

where $\mathcal{L} \times_B \mathcal{L}$ is an algebra via component-wise multiplication.

(iii) The counit $\epsilon : \mathcal{L} \to B$ is a left character on the B-ring $(\mathcal{L}, s, t)$, that is it satisfies the properties, for $b \in B$ and $a, a' \in \mathcal{L}$,
  (1) $\epsilon(1_{\mathcal{L}}) = 1_B$, (unitality)
  (2) $\epsilon(s(b)a) = b\epsilon(a)$, (left B-linearity)
  (3) $\epsilon(as(\epsilon(a'))) = \epsilon(aa') = \epsilon(at(\epsilon(a')))$, (associativity) .

Similarly, we have the definition of right bialgebroid:

Definition 6.2. Given an algebra B, a *right B-bialgebroid* $\mathcal{R}$ consists of an $(B \otimes B^{op})$-ring together with a B-coring structures on the same vector space $\mathcal{R}$ with mutual compatibility conditions. From what is said above, a $(B \otimes B^{op})$-ring $\mathcal{R}$ is the same as an algebra map $\eta : B \otimes B^{op} \to \mathcal{R}$. Equivalently, one may consider the restrictions

$$s := \eta(\,\cdot\, \otimes_B 1_B) : B \to \mathcal{R} \quad \text{and} \quad t := \eta(1_B \otimes_B \,\cdot\,) : B^{op} \to \mathcal{R}$$

which are algebra maps with commuting ranges in $\mathcal{R}$, called the *source* and the *target* map of the $(B \otimes B^{op})$-ring $\mathcal{R}$. Thus a $(B \otimes B^{op})$-ring is the same as a triple $(\mathcal{R}, s, t)$ with $\mathcal{R}$ an algebra and $s : B \to \mathcal{R}$ and $t : B^{op} \to \mathcal{R}$ both algebra maps with commuting ranges.

Thus, for a right B-bialgebroid $\mathcal{R}$ the compatibility conditions are required to be

(i) The bimodule structures in the B-coring $(\mathcal{R}, \Delta, \epsilon)$ are related to those of the $B \otimes B^{op}$-ring $(\mathcal{R}, s, t)$ via

$$b \triangleright a \triangleleft b' := as(b')t(b) \qquad \text{for } b, b' \in B,\ a \in \mathcal{R}. \tag{6.6}$$

(ii) Considering $\mathcal{R}$ as a B-bimodule as in (6.6), the coproduct Δ corestricts to an algebra map from $\mathcal{R}$ to

$$\mathcal{R} \times^B \mathcal{R} := \left\{ \sum\nolimits_j a_j \otimes_B a'_j \mid \sum\nolimits_j s(b)a_j \otimes_B a'_j = \sum\nolimits_j a_j \otimes_B t(b)a'_j, \text{ for all } b \in B \right\}, \tag{6.7}$$

where $\mathcal{R} \times^B \mathcal{R}$ is an algebra via component-wise multiplication.

(iii) The counit $\epsilon : \mathcal{R} \to B$ is a right character on the B-ring $(\mathcal{R}, s, t)$, that is it satisfies the properties, for $b \in B$ and $a, a' \in \mathcal{R}$,

(1) $\epsilon(1_{\mathcal{R}}) = 1_B$, (unitality)

(2) $\epsilon(as(b)) = \epsilon(a)b$, (right B-linearity)

(3) $\epsilon(s(\epsilon(a))a') = \epsilon(aa') = \epsilon(t(\epsilon(a))a')$, (associativity) .

Remark 6.3. Consider a left B-bialgebroid $\mathcal{L}$ with s_L, t_L the corresponding source and target maps. If the images of s_L and t_L belong to the centre of H (which implies B is a commutative algebra, since the source map is injective), we can construct a right bialgebroid with the same underlying k-algebra $\mathcal{L}$ and B-coring structure on $\mathcal{L}$, but a new source and target map $s_R := t_L$, $t_R := s_L$. Indeed, they have the same bimodule structure on $\mathcal{L}$, since $r \triangleright b \triangleleft r' = s_L(r)t_L(r')b = bt_R(r)s_R(r')$. With the same bimodule structure and the same coproduct and counit, one can get the same B-coring. By using the assumption that the images of s_L and t_L belong to the centre of H, we can find that all the conditions for being a right bialgebroid can be satisfied. Similarly, under the same assumption, a right bialgebroid can induce a left bialgebroid. Since the image of B belongs to the center of H, we can also see ϵ is an algebra map, indeed, $\epsilon(bb') = \epsilon(bs(\epsilon(b'))) = \epsilon(s(\epsilon(b'))b) = \epsilon(b)\epsilon(b')$.

To make a proper definition of 'quantum' groupoid, a left or right bialgebroid is not sufficient, since we still need to have the antipode, which plays the role of the inverse of 'quantum groupoid'. Its inclusion allows us to define a Hopf algebroid [8] like so:

Definition 6.4. Given two algebras B and C, a *Hopf algebroid* $(\mathcal{H}_L, \mathcal{H}_R, S)$ consists of a left B-bialgebroid $(\mathcal{H}_L, s_L, t_L, \Delta_L, \epsilon_L)$ and a right C-bialgebroid $(\mathcal{H}_R, s_R, t_R, \Delta_R, \epsilon_R)$, such that their underlying algebra H is the same. The *antipode* $S : H \to H$ is a linear map. Let $\mu_L : H \otimes_{s_L} H \to H$ be the B-ring (H, s_L) product induced by s_L (where the tensor product $\otimes_{s_L}$ means: $hs_L(b) \otimes_{s_L} h' = h \otimes_{s_L} s_L(b)h$), and $\mu_R : H \otimes_{s_R} H \to H$ be the C-ring (H, s_R) product induced by s_R, such that all the structures above satisfy the following axioms:

(i) $s_L \circ \epsilon_L \circ t_R = t_R, \quad t_L \circ \epsilon_L \circ s_R = s_R, \quad s_R \circ \epsilon_R \circ t_L = t_L, \quad t_R \circ \epsilon_R \circ s_L = s_L.$

(ii) $(\Delta_L \otimes_C id_H) \circ \Delta_R = (id_H \otimes_B \Delta_R) \circ \Delta_L$ and $(\Delta_R \otimes_B id_H) \circ \Delta_L = (id_H \otimes_C \Delta_L) \circ \Delta_R$.

(iii) For $b \in B$, $c \in C$ and $h \in H$, $S(t_L(b)ht_R(c)) = s_R(c)S(h)s_L(b)$.

(iv) $\mu_L \circ (S \otimes_B id_H) \circ \Delta_L = s_R \circ \epsilon_R$ and $\mu_R \circ (id_H \otimes_C S) \circ \Delta_R = s_L \circ \epsilon_L$.

Remark 6.5. We can see axiom (i) makes the coproduct Δ_L (Δ_R resp.) into a C-bimodule map (B-bimodule map resp.), so that (ii) is well defined. The axiom (ii) makes H both a $\mathcal{H}_L$-$\mathcal{H}_R$ bicomodule and a $\mathcal{H}_R$-$\mathcal{H}_L$ bicomodule, since the regular coactions Δ_L and Δ_R commute.

In order to make sure axiom (iv) is well defined, we need axiom (iii), where $S \otimes_B H : H \otimes_B H \to H \otimes_{s_L} H$ maps the tensor product $\otimes_B$ into a different tensor product $\otimes_{s_L}$, so that μ_L makes sense.

Remark 6.6. In particular, given a Hopf algebroid as above $(\mathcal{H}_L, \mathcal{H}_R, S)$ such that

(1) $B = C$;
(2) $s_L = r_R$ and $t_L = s_R$, with their images belong to the center of H;
(3) the coproduct and counit of $\mathcal{H}_L$ coincide with the coproduct and counit of $\mathcal{H}_R$;

with the help of Remark 6.3, we know that the left and right bialgebroid structures are compatible with each other. In other words, the right bialgebroid $\mathcal{H}_R$ is constructed from $\mathcal{H}_L$ as in Remark 6.3. Therefore, axioms (i) and (ii) are satisfied automatically. The axiom (iii) asserts that $S \circ s_L = t_L$, and $S \circ t_L = s_L$. We use Δ and ϵ to denote the coproduct and counit for both $\mathcal{H}_L$ and $\mathcal{H}_R$, s to denote $s_L = t_R$, and t to denote $t_L = s_R$. So (iv) can be written as:

$$\mu_L \circ (S \otimes_B id_H) \circ \Delta = t \circ \epsilon, \tag{6.8}$$

and

$$\mu_R \circ (id_H \otimes_B S) \circ \Delta = s \circ \epsilon. \tag{6.9}$$

A Hopf algebroid of this kind is denoted by $(H, s, t, \Delta, \epsilon, S)$.

From now on we will only consider the left bialgebroids, Hopf algebroids whose underlying algebra B is commutative and the images of whose source and target maps belongs to the center. With the help of Remark 6.6 we have the definition of central Hopf algebroids, which is a simplification of the Hopf algebroid of Definition 6.4.

Definition 6.7. A *central Hopf algebroid* is a left bialgebroid $(H, s, t, \Delta, \epsilon)$ over an algebra B, the images of whose source and target maps belong to the center of H, together with a linear map $S : H \to H$, such that:

(1) For any $h \in H$ and $b, b' \in B$,
$$S(t(b)hs(b')) = t(b')S(h)s(b). \tag{6.10}$$
(2) $\mu_L \circ (S \otimes_B id_H) \circ \Delta = t \circ \epsilon$ and $\mu_R \circ (id_H \otimes_B S) \circ \Delta = s \circ \epsilon$,

where $\mu_L : H \otimes_s H \to H$ is the B-ring (H, s) product induced by s, and $\mu_R : H \otimes_t H \to H$ is the B-ring (H, t) product induced by t.

Let C be a B-coring. We denote by $\star : {}_B\mathrm{Hom}_B(C, B) \times {}_B\mathrm{Hom}_B(C, B) \to A$ the convolution product $(f \star g)(c) := f(c^{(1)})g(c^{(2)})$, where $c^{(1)} \otimes_B c^{(2)}$ is the image of the coproduct of the coring, and ${}_B\mathrm{Hom}_B(C, B)$ is the vector space of B-bimodule maps. In [6] we know ${}_B\mathrm{Hom}_B(C, B)$ is a B-ring with unit $\epsilon : C \to B$.

In the following we will use lower Sweedler notation for the coproduct of a Hopf coquasigroup (include Hopf algebra) and upper notation for a Hopf algebroid. Whenever we say Hopf algebroid, we mean central Hopf algebroid.

We finish this part with an additional notion that we shall use in Section 14.1.

Definition 6.8. Let $(\mathcal{L}, \Delta, \epsilon, s, t)$ be a left bialgebroid over the algebra B. An automorphism of the bialgebroid $\mathcal{L}$ is a pair (Φ, φ) of algebra automorphisms, $\Phi : \mathcal{L} \to \mathcal{L}$, $\varphi : B \to B$ such that:

(i) $\Phi \circ s = s \circ \varphi$;
(ii) $\Phi \circ t = t \circ \varphi$;
(iii) $(\Phi \otimes_B \Phi) \circ \Delta = \Delta \circ \Phi$;
(iv) $\epsilon \circ \Phi = \varphi \circ \epsilon$.

In fact, the map φ is uniquely determined by Φ via $\varphi = \epsilon \circ \Phi \circ s$ and one can just say that Φ is a bialgebroid automorphism. Automorphisms of a bialgebroid $\mathcal{L}$ form a group by composition that we simply denote $\mathrm{Aut}(\mathcal{L})$. The automorphisms for a right bialgebroid are given similarly.

Remark 6.9. Here the pair of algebra maps (Φ, φ) can be viewed as a bialgebroid map (cf. [34], §4.1) between two copies of $\mathcal{L}$ with different source and target map (and so B-bimodule structure). If s, t are the source and target maps on $\mathcal{L}$, one defines new source and target maps on $\mathcal{L}$ by $s' := s \circ \varphi$ and $t' := t \circ \varphi$ with the new bimodule structure given by $b \triangleright_\varphi c \triangleleft_\varphi \tilde{b} := s'(b)t'(\tilde{b})a$, for any $b, \tilde{b} \in B$ and $a \in \mathcal{L}$ (see (6.4)). Therefore we get a new left bialgebroid with product, unit, coproduct and counit unchanged.

Clearly, from conditions (i) and (ii) Φ is a B-bimodule map: $\Phi(b \triangleright c \triangleleft \tilde{b}) = b \triangleright_\varphi \Phi(c) \triangleleft_\varphi \tilde{b}$. Condition (iii) is well defined once conditions (i) and (ii) are satisfied (the balanced tensor product in (iii) is induced by s' and t'). Conditions (iii) and (iv) imply Φ is a coring map, therefore (Φ, φ) is an isomorphism between the original and the new bialgebroids.

7. Hopf–Galois extensions

In this section we will briefly recall Hopf–Galois extensions, as noncommutative principal bundles. These are H-comodule algebras A with a canonically defined map $\chi : A \otimes_B A \to A \otimes H$ which is required to be invertible.

Definition 7.1. Let H be a Hopf algebra and let A be a H-comodule algebra with coaction δ^A. Consider the subalgebra $B := A^{coH} = \{b \in A \mid \delta^A(b) = b \otimes 1_H\} \subseteq A$ of coinvariant elements and the corresponding balanced tensor product $A \otimes_B A$. The extension $B \subseteq A$ is called a *H-Hopf–Galois extension* if the *canonical Galois map*

$$\chi := (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes_B \delta^A) : A \otimes_B A \longrightarrow A \otimes H \ , \quad a' \otimes_B a \mapsto a' a_{\,(0)} \otimes a_{\,(1)} \tag{7.1}$$

is bijective.

We can see that this bijective canonical map is dual to the canonical map of a principal bundle (see Definition 2.1), which is why we call Hopf Galois extensions noncommutative principal bundles.

Remark 7.2. In the following, we shall always implicitly assume that for the Hopf Galois extension $B \subseteq A$, the algebra A is *faithfully flat* as a left B-module. This means that taking the tensor product $\otimes_B A$ with a sequence of right B-modules produces an exact sequence if and only if the original sequence is exact. Finite-rank, free or projective modules are examples of faithfully flat modules.

The canonical map χ is a morphism of relative Hopf modules for A-bimodules and right H-comodules (cf. [1]). Both $A \otimes_B A$ and $A \otimes H$ are A-bimodules. The left A-module structures are the left multiplication on the first factors while the right A-actions are:

$$(a \otimes_B a')a'' := a \otimes_B a'a'' \quad \text{and} \quad (a \otimes h)a' := aa'_{\,(0)} \otimes ha'_{\,(1)} \ .$$

As for the H-comodule structure, the natural right tensor product H-coaction as in (5.12):

$$\delta^{A\otimes A} : A\otimes A \to A\otimes A\otimes H, \quad a\otimes a' \mapsto a_{(0)}\otimes a'_{(0)}\otimes a_{(1)}a'_{(1)} \tag{7.2}$$

for all $a, a' \in A$, descends to the quotient $A\otimes_B A$ because $B \subseteq A$ is the subalgebra of H-coinvariants. Similarly, $A\otimes H$ is endowed with the tensor product coaction, where one regards the Hopf algebra H as a right H-comodule with the right adjoint H-coaction

$$\mathrm{Ad} : h \longmapsto h_{(2)} \otimes S(h_{(1)})\, h_{(3)} \ . \tag{7.3}$$

The right H-coaction on $A\otimes H$ is then given, for all $a\in A,\ h\in H$ by

$$\delta^{A\otimes H}(a\otimes h) = a_{(0)}\otimes h_{(2)}\otimes a_{(1)}\, S(h_{(1)})\, h_{(3)} \in A\otimes H\otimes H \ . \tag{7.4}$$

Since the canonical Galois map χ is left A-linear, its inverse is determined by the restriction $\tau := \chi^{-1}_{|_{1_A\otimes H}}$, named the *translation map*,

$$\tau = \chi^{-1}_{|_{1_A\otimes H}} : H \to A\otimes_B A\ , \quad h\mapsto h^{<1>}\otimes_B h^{<2>} \ .$$

The translation map enjoys a number of properties that we list here for later use. Firstly, it was shown in [5, Prop. 3.6] that,

$$(\mathrm{id}\otimes_B \delta^A)\circ\tau = (\tau\otimes\mathrm{id})\circ\Delta\ , \quad (\tau\otimes S)\circ\mathrm{flip}\circ\Delta = (\mathrm{id}\otimes\mathrm{flip})\circ(\delta^A\otimes_B\mathrm{id})\circ\tau \ .$$

On an element $h\in H$ these respectively read

$$h^{<1>}\otimes_B {h^{<2>}}_{(0)}\otimes {h^{<2>}}_{(1)} = {h_{(1)}}^{<1>}\otimes_B {h_{(1)}}^{<2>}\otimes h_{(2)}\ , \tag{7.5}$$

$${h^{<1>}}_{(0)}\otimes_B h^{<2>}\otimes {h^{<1>}}_{(1)} = {h_{(2)}}^{<1>}\otimes_B {h_{(2)}}^{<2>}\otimes S(h_{(1)})\ . \tag{7.6}$$

Furthermore, from [6, Lemma 34.4], for any $a\in A$ and $h,k\in H$, we have the following:

$$h^{<1>}{h^{<2>}}_{(0)}\otimes {h^{<2>}}_{(1)} = 1_A\otimes h\ , \tag{7.7}$$

$$h^{<1>}h^{<2>} = \epsilon(h)1_A\ , \tag{7.8}$$

$$(hk)^{<1>}\otimes_B (hk)^{<2>} = k^{<1>}h^{<1>}\otimes_B h^{<2>}k^{<2>}\ , \tag{7.9}$$

$${h_{(1)}}^{<1>}\otimes_B {h_{(1)}}^{<2>}{h_{(2)}}^{<1>}\otimes_B {h_{(2)}}^{<2>} = h^{<1>}\otimes_B 1_A\otimes_B h^{<2>}\ , \tag{7.10}$$

$$a_{(0)}{a_{(1)}}^{<1>}\otimes_B {a_{(1)}}^{<2>} = 1_A\otimes_B a\ , \tag{7.11}$$

for any $h,k\in H$ and $a\in A$. Here we also give a proof:

Proof. For (7.7), applying $\chi\circ\tau$ on the right hand side we can get the result directly. Applying χ on both sides of (7.11), we get

$$\begin{aligned}\chi(a_{(0)}{a_{(1)}}^{<1>}\otimes_B {a_{(1)}}^{<2>}) &= a_{(0)}{a_{(1)}}^{<1>}{{a_{(1)}}^{<2>}}_{(0)}\otimes {{a_{(1)}}^{<2>}}_{(1)}\\ &= a_{(0)}\otimes a_{(1)} = \chi(1\otimes_B a),\end{aligned}$$

where the second step uses (7.7). Since χ is bijective, we get (7.11). By applying $id_A\otimes\varepsilon_H$ on (7.7) we can get (7.8). By applying $\chi\otimes id_H$ on the left hand side of (7.5) we get

$$\begin{aligned}\chi(h^{<1>}\otimes_B {h^{<2>}}_{(0)})\otimes {h^{<2>}}_{(1)} &= h^{<1>}{h^{<2>}}_{(0)}\otimes {h^{<2>}}_{(1)}\otimes {h^{<2>}}_{(2)}\\ &= 1\otimes h_{(1)}\otimes h_{(2)}\\ &= {h_{(1)}}^{<1>}{{h_{(1)}}^{<2>}}_{(0)}\otimes {{h_{(1)}}^{<2>}}_{(1)}\otimes h_{(2)}\\ &= \chi({h_{(1)}}^{<1>}\otimes_B {h_{(1)}}^{<2>})\otimes h_{(2)},\end{aligned}$$

where the 1st and 3rd steps use (7.7), therefore we get (7.5). By applying $id_A\otimes_B\chi^{-1}$ on both sides of (7.5) we get

$$h^{<1>}\otimes_B {h^{<2>}}_{(0)}{{h^{<2>}}_{(1)}}^{<1>}\otimes_B {{h^{<2>}}_{(1)}}^{<2>} = {h_{(1)}}^{<1>}\otimes_B {h_{(1)}}^{<2>}{h_{(2)}}^{<1>}\otimes_B {h_{(2)}}^{<2>}.$$

Then by using (7.11) on the left hand side we can get (7.10). Applying the canonical map to both sides of (7.9), the left hand side is equal to $1 \otimes gh$, while the right hand side is equal to

$$h^{<1>} g^{<1>} g^{<2>}{}_{(0)} h^{<2>}{}_{(0)} \otimes g^{<2>}{}_{(1)} h^{<2>}{}_{(1)} = 1 \otimes gh,$$

where (7.7) is used here. There are other H-comodule structures on $A \otimes_B A$ and $A \otimes H$, which are: for $A \otimes_B A$ the coaction is given by $\delta'(a \otimes_B a') := a_{(0)} \otimes a' \otimes a_{(1)}$, for any $a \otimes_B a' \in A \otimes_B A$. For $A \otimes H$ the corresponding H-comodule coaction is given by $\delta''(a \otimes h) := a_{(0)} \otimes h_{(2)} \otimes a_{(1)} S(h_{(1)})$, for any $a \otimes h \in A \otimes H$. It was given in [2] that the canonical map χ is a right H-comodule map (so is χ^{-1}) for this comodule structure. So for (7.6), we have

$$\begin{aligned} h^{<1>}{}_{(0)} \otimes_B h^{<2>} \otimes h^{<1>}{}_{(1)} &= \delta' \circ \chi^{-1}(1 \otimes h) \\ &= (\chi^{-1} \otimes id_H) \circ \delta''(1 \otimes h) \\ &= \chi^{-1}(1 \otimes h_{(2)}) \otimes S(h_{(1)}) \\ &= h_{(2)}{}^{<1>} \otimes_B h_{(2)}{}^{<2>} \otimes S(h_{(1)}). \end{aligned}$$

□

Two H-Hopf–Galois extensions A, A' of a fixed algebra B are isomorphic provided there exists an isomorphism of H-comodule algebras $A \to A'$. This is the algebraic counterpart for noncommutative principal bundles of the geometric notion of isomorphisms of principal G-bundles with a fixed base space. As in the geometric case this notion is relevant in the classification of noncommutative principal bundles, (cf. [19]).

For structure Hopf algebras H which are cosemisimple and have bijective antipodes, Theorem I of [33] grants additional nice properties. In particular, the surjectivity of the canonical map implies its bijectivity. Moreover, in order to prove the surjectivity of χ, it is enough to prove that for any generator h of H, the element $1 \otimes h$ is in the image of the canonical map. Indeed, if $\chi(g_k \otimes_B g'_k) = 1 \otimes g$ and $\chi(h_l \otimes_B h'_l) = 1 \otimes h$ for $g, h \in H$, then $\chi(g_k h_l \otimes_B h'_l g'_k) = g_k h_l \chi(1 \otimes_B h'_l g'_k) = 1 \otimes hg$, using the fact that the canonical map restricted to $1 \otimes_B A$ is a homomorphism. Extension to all of $A \otimes_B A$ then follows from left A-linearity of χ. It is also easy to write down an explicit expression for the inverse of the canonical map. Indeed, one has $\chi^{-1}(1 \otimes hg) = g_k h_l \otimes_B h'_l g'_k$ in the above notation so that the general form of the inverse follows again from left A-linearity.

In the following we are also interested in *Galois objects*: given a Hopf algebra H, a H-Galois object is a H-Hopf–Galois extension of $\mathbb{C}$. These can be thought of a noncommutative principal bundle over a point. It is well known (cf. [19]) that the set $\mathrm{Gal}_H(\mathbb{C})$ of isomorphic classes of H-Galois objects need not be trivial. This is in contrast to the fact that any (usual) fibre bundle over a point is trivial.

Part 1. Principal fibrations over noncommutative spheres

In this part we use Einstein convention of summing over repeated up-down indices. An algebra is always an associative algebra and a graded algebra is meant to be $\mathbb{N}$-graded.

8. A family of quadratic algebras

8.1. **General definitions and properties.** In [13] and [14] there were considered complex algebras $\mathcal{A}_R$ generated by two sets of hermitian elements $x = (x_1, x_2) = (x_1^\lambda, x_2^\alpha)$, with $\lambda \in \{1, \dots, N_1\}$ and $\alpha \in \{1, \dots, N_2\}$, subject to relations

$$\begin{aligned} x_1^\lambda x_1^\mu &= x_1^\mu x_1^\lambda \ , \qquad x_2^\alpha x_2^\beta = x_2^\beta x_2^\alpha, \\ x_1^\lambda x_2^\alpha &= R^{\lambda\alpha}_{\beta\mu}\, x_2^\beta x_1^\mu \ , \qquad x_2^\alpha x_1^\lambda = \overline{R}^{\lambda\alpha}_{\beta\mu}\, x_1^\mu x_2^\beta \end{aligned} \tag{8.1}$$

for a 'matrix' $(R^{\lambda\alpha}_{\beta\mu})$. Here $\overline{R}^{\lambda\alpha}_{\beta\mu} \in \mathbb{C}$ is the complex conjugates of the $R^{\lambda\alpha}_{\beta\mu} \in \mathbb{C}$. The class of relevant matrices R was defined by a series of conditions that we recall momentarily.

The quadratic complex algebra $\mathcal{A}_R$ is a graded algebra $\mathcal{A}_R = \oplus_{n\in\mathbb{N}} (\mathcal{A}_R)_n$ which is connected, that is $(\mathcal{A}_R)_0 = \mathbb{C}\mathbb{1}$. Moreover, the quadratic relations (8.1) of $\mathcal{A}_R$ imply that there is a unique structure of $*$-algebra on $\mathcal{A}_R$ for which the x_1^λ ($\lambda \in \{1, \dots, N_1\}$) and the x_2^α ($\alpha \in \{1, \dots, N_2\}$) are hermitian, $x_1^\lambda = (x_1^\lambda)^*$ and $x_2^\alpha = (x_2^\alpha)^*$. This structure is graded in the sense that one has $f^* \in (\mathcal{A}_R)_n \Leftrightarrow f \in (\mathcal{A}_R)_n$ and $\mathcal{A}_R$ is the quadratic $*$-algebra generated by the hermitian elements x_1^λ and x_2^α with the relations (8.1).

The $x_1^\lambda x_1^\mu$ for $\lambda \leq \mu$ and the $x_2^\alpha x_2^\beta$ for $\alpha \leq \beta$ are linearly independent in $(\mathcal{A}_R)_2$ and generate $(\mathcal{A}_R)_2$ together with the $x_1^\lambda x_2^\alpha$. It is also natural to assume that the $x_2^\alpha x_1^\lambda$ are independent which implies the equations

$$\overline{R}^{\lambda\alpha}_{\beta\mu} R^{\mu\beta}_{\gamma\nu} = \delta^\lambda_\nu \delta^\alpha_\gamma \tag{8.2}$$

which in turn imply that the $x_1^\lambda x_2^\alpha$ are also independent. Finally this implies in particular that the $x_1^\lambda x_1^a$ with $\lambda \leq \mu$, the $x_2^\alpha x_2^\beta$ with $\alpha \leq \beta$ and the $x_1^\nu x_2^\gamma$ define a basis of $(\mathcal{A}_R)_2$ while by definition the elements x_1^λ and the x_2^α form a basis of $(\mathcal{A}_R)_1$.

The classical (commutative) solution is given by

$$(R_0)^{\lambda\alpha}_{\beta\mu} = \delta^\lambda_\mu \delta^\alpha_\beta$$

and $\mathcal{A}_{R_0}$ is the coordinate algebra over the product $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. Thus, the algebra $\mathcal{A}_R$ is though to define by duality the noncommutative product of $\mathbb{R}^{N_1} \times_R \mathbb{R}^{N_2}$, that is $\mathcal{A}_R$ is the algebra of coordinate functions on the noncommutative vector space $\mathbb{R}^{N_1} \times_R \mathbb{R}^{N_2}$.

If we collect together the coordinates, defining the x^a for $a \in \{1, 2, \dots, N_1 + N_2\}$ by $x^\lambda = x_1^\lambda$ and $x^{\alpha+N_1} = x_2^\alpha$, the relations (8.1) with (8.2) can be written in the form

$$x^a x^b = \mathcal{R}^{a\,b}_{c\,d}\, x^c x^d \ . \tag{8.3}$$

The $\mathcal{R}^{ab}_{cd}$ are the matrix elements of an endomorphism $\mathcal{R}$ of $(\mathcal{A}_R)_1 \otimes (\mathcal{A}_R)_1$. It follows from (8.2) that the $\mathcal{R}$ matrix is involutive, that is

$$\mathcal{R}^2 = I \otimes I \tag{8.4}$$

where I is the identity mapping of $(\mathcal{A}_R)_1$ onto itself. One next imposes that the matrix $\mathcal{R}$ satisfies the Yang-Baxter equation

$$(\mathcal{R} \otimes I)(I \otimes \mathcal{R})(\mathcal{R} \otimes I) = (I \otimes \mathcal{R})(\mathcal{R} \otimes I)(I \otimes \mathcal{R}), \tag{8.5}$$

which then breaks in a series of conditions on the starting matrix $R^{\lambda\alpha}_{\beta\mu}$ in (8.1). By [13] we have the following property:

Proposition 8.1. *The Yang-Baxter equation* (8.5) *for $\mathcal{R}$ is equivalent to the following*

$$\begin{cases} R^{\lambda\alpha}_{\gamma\rho} R^{\rho\beta}_{\delta\mu} = R^{\lambda\beta}_{\delta\rho} R^{\rho\alpha}_{\gamma\mu} & \textit{for indices} \quad (a,b,c) = (\lambda\alpha\beta) \\ \overline{R}^{\lambda\alpha}_{\gamma\rho} \overline{R}^{\rho\beta}_{\delta\mu} = \overline{R}^{\lambda\beta}_{\delta\rho} \overline{R}^{\rho\alpha}_{\gamma\mu} & \textit{for indices} \quad (a,b,c) = (\alpha\beta\lambda) \\ \overline{R}^{\lambda\alpha}_{\gamma\rho} R^{\rho\beta}_{\delta\mu} = R^{\lambda\beta}_{\delta\rho} \overline{R}^{\rho\alpha}_{\gamma\mu} & \textit{for indices} \quad (a,b,c) = (\alpha\lambda\beta) \end{cases} \tag{8.6}$$

and

$$\begin{cases} R^{\lambda\alpha}_{\gamma\nu} R^{\mu\gamma}_{\beta\rho} = R^{\mu\alpha}_{\gamma\rho} R^{\lambda\gamma}_{\beta\nu} & \textit{for indices} \quad (a,b,c) = (\lambda\mu\alpha) \\ \overline{R}^{\lambda\alpha}_{\gamma\nu} \overline{R}^{\mu\gamma}_{\beta\rho} = \overline{R}^{\mu\alpha}_{\gamma\rho} \overline{R}^{\lambda\gamma}_{\beta\nu} & \textit{for indices} \quad (a,b,c) = (\alpha\lambda\mu) \\ R^{\lambda\alpha}_{\gamma\nu} \overline{R}^{\mu\gamma}_{\beta\rho} = \overline{R}^{\mu\alpha}_{\gamma\rho} R^{\lambda\gamma}_{\beta\nu} & \textit{for indices} \quad (a,b,c) = (\lambda\alpha\mu) \end{cases} \tag{8.7}$$

for the matrices $R^{\lambda\alpha}_{\beta\mu}$ and $\overline{R}^{\lambda\alpha}_{\beta\mu}$.

Finally, additional conditions on the matrix $R^{\lambda\alpha}_{\beta\mu}$ comes by requiring that both quadratic elements $(x_1)^2 = \sum_{\lambda=0}^{N_1} (x_1^\lambda)^2$ and $(x_2)^2 = \sum_{\alpha=0}^{N_2} (x_2^\alpha)^2$ of $\mathcal{A}_R$ be central.

Lemma 8.2. *If $\mathcal{R}$ satisfies the reality condition* (8.2)*, together with the requirement that the quadratic elements $(x_1)^2$ and $(x_2)^2$ be central, then we have the symmetry conditions:*

$$R^{\lambda\beta}_{\alpha\mu} = R^{\mu\alpha}_{\beta\lambda} = \overline{R}^{\mu\beta}_{\alpha\lambda} = (R^{-1})^{\beta\mu}_{\lambda\alpha}. \tag{8.8}$$

Moreover, the quadratic conditions

$$R^{\lambda\beta}_{\alpha\rho} R^{\rho\delta}_{\gamma\mu} = R^{\lambda\delta}_{\gamma\rho} R^{\rho\beta}_{\alpha\mu} \qquad \textit{and} \qquad R^{\lambda\beta}_{\gamma\nu} R^{\mu\gamma}_{\alpha\rho} = R^{\mu\beta}_{\gamma\rho} R^{\lambda\gamma}_{\alpha\nu} \tag{8.9}$$

are equivalent (under (8.8)*) to the cubic relations of the Yang–Baxter equations.*

Proof. Assume $(x_1)^2 = \sum_{\lambda=0}^{N_1} (x_1^\lambda)^2$ belongs to the center, then we have

$$\sum_{\lambda=0}^{N_1} (x_1^\lambda)^2 x_2^\gamma = \sum_{\lambda=0}^{N_1} x_1^\lambda x_1^\lambda x_2^\gamma = \sum_{\lambda=0}^{N_1} x_1^\lambda R^{\lambda\gamma}_{\beta\nu} x_2^\beta x_1^\nu = \sum_{\lambda=0}^{N_1} R^{\lambda\gamma}_{\beta\nu} R^{\lambda\beta}_{\alpha\mu} x_2^\alpha x_1^\mu x_1^\nu,$$

thus we can conclude

$$\sum_{\lambda=0}^{N_1} R^{\lambda\gamma}_{\beta\nu} R^{\lambda\beta}_{\alpha\mu} = \delta^\gamma_\alpha \delta_{\mu\nu}. \tag{8.10}$$

Similarly, by assuming $(x_2)^2$ belongs to the center we also have

$$\sum_{\alpha=0}^{N_2} R^{\lambda\alpha}_{\beta\rho} R^{\rho\alpha}_{\gamma\mu} = \delta^\lambda_\mu \delta_{\beta\gamma}. \tag{8.11}$$

From the reality condition (8.2) we know $\overline{R}^{\mu\beta}_{\alpha\lambda} = (R^{-1})^{\beta\mu}_{\lambda\alpha}$, by (8.10) and (8.11) we get $R^{\lambda\beta}_{\alpha\mu} = R^{\mu\alpha}_{\beta\lambda} = (R^{-1})^{\beta\mu}_{\lambda\alpha}$ by considering the 'transposition' in the indices λ, μ (and α, β resp.). Moreover, with the help of (8.8) we can clearly replace (8.6) and (8.7) by (8.9). □

The general solution of these equations was given in [14] as follows. By setting

$$\widehat{R}^{\lambda\alpha}_{\mu\beta} = R^{\lambda\alpha}_{\beta\mu}$$

for the endomorphism $\hat{R} = (\widehat{R}^{\lambda\alpha}_{\mu\beta})$ of $\mathbb{R}^{N_1} \otimes \mathbb{R}^{N_2}$ one has the representation

$$\widehat{R} = \sum_r A_r \otimes B_r + \mathrm{i} \sum_a C_a \otimes D_a \tag{8.12}$$

with the A_r real symmetric $N_1 \times N_1$ matrices and the B_r real symmetric $N_2 \times N_2$ matrices, both set taken to be linearly independent; and the C_a real anti-symmetric $N_1 \times N_1$ matrices and the D_a real anti-symmetric $N_2 \times N_2$ matrices (again both set taken to be linearly independent). Furthermore, they are such that

$$[A_r, A_s] = 0, \quad [A_r, C_a] = 0, \quad [C_a, C_b] = 0 \tag{8.13}$$

$$[B_r, B_s] = 0, \quad [B_r, D_a] = 0, \quad [D_a, D_b] = 0 \tag{8.14}$$

for $r, s \in \{1, \ldots, p\}$ and $a, b \in \{1, \ldots, q\}$, with normalization condition

$$\sum_{r,s} A_r A_s \otimes B_r B_s + \sum_{a,b} C_a C_b \otimes D_a D_b = \mathbb{1}_{N_1} \otimes \mathbb{1}_{N_2}, \tag{8.15}$$

a translation of the condition in (8.4)

With the quadratic elements $(x_1)^2 = \sum_{\lambda=1}^{N_1} (x_1^\lambda)^2$ and $(x_2)^2 = \sum_{\alpha=1}^{N_2} (x_2^\alpha)^2$ of $\mathcal{A}_R$ being central, one may consider the quotient algebra

$$\mathcal{A}_R / \left((x_1)^2 - \mathbb{1}, (x_2)^2 - \mathbb{1}\right)$$

which defines by duality the noncommutative product $\mathbb{S}^{N_1-1} \times_R \mathbb{S}^{N_2-1}$ of the classical spheres $\mathbb{S}^{N_1-1}$ and $\mathbb{S}^{N_2-1}$. Indeed, for $R = R_0$, the above quotient is the restriction to $\mathbb{S}^{N_1-1} \times \mathbb{S}^{N_2-1}$ of the polynomial functions on $\mathbb{R}^{N_1+N_2}$.

Furthermore, with the central quadratic element $x^2 = \sum_{a=1}^{N_1+N_2} (x^a)^2 = (x_1)^2 + (x_2)^2$, one may also consider the quotient of $\mathcal{A}_R$

$$\mathcal{A}_R / \left(x^2 - \mathbb{1}\right).$$

This defines (by duality) the noncommutative $(N_1 + N_2 - 1)$-sphere $\mathbb{S}_R^{N_1+N_2-1}$ shown in [14] to be a noncommutative spherical manifold in the sense of [11] and [10].

8.2. **Some quaternionic geometry.** When $N_1 = N_2 = 4$, explicit solutions for the matrix $R^{\lambda\alpha}_{\beta\mu}$ were given in [13] and [14] by using results on the geometry of quaternions.

The space of quaternions $\mathbb{H}$ is identified with $\mathbb{R}^4$ in the usual way:

$$\mathbb{H} \ni q = x^0 e_0 + x^1 e_1 + x^2 e_2 + x^3 e_3 \quad \longmapsto \quad x = (x^\mu) = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4. \tag{8.16}$$

Here $e_0 = 1$ and the imaginary units e_a obey the multiplication rule of the algebra $\mathbb{H}$:

$$e_a e_b = -\delta_{ab} + \sum_{c=1}^{3} \varepsilon_{abc} e_c.$$

From this it follows an identification of the unit quaternions $\mathrm{U}_1(\mathbb{H}) = \{q \in \mathbb{H} \,|\, q\bar{q} = 1\}$ with the euclidean three-sphere $\mathbb{S}^3 = \{x \in \mathbb{R}^4 \,;\, ||x||^2 = \sum_\mu (x^\mu)^2 = 1\}$.

With the identification (8.16), left and right multiplication of quaternions are represented by matrices acting on $\mathbb{R}^4$:

$$L_{q'} q := q'q \quad \to \quad E^+_{q'}(x) \qquad \text{and} \qquad R_{q'} q := qq' \quad \to \quad E^-_{q'}(x).$$

For q a unit quaternion, both E^+_q and E^-_q are orthogonal matrices. In fact the unit quaternions form a subgroup of the multiplicative group $\mathbb{H}^*$ of non vanishing quaternions. When restricting to these, one has then the identification

$$\mathrm{U}_1(\mathbb{H}) \simeq \mathrm{SU}(2),$$

that is E_q^+ and E_q^-, for $q \in \mathrm{U}_1(\mathbb{H})$, are commuting $\mathrm{SU}(2)$ actions (each in the 'defining representation') on $\mathbb{R}^4$, or together an action of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ on $\mathbb{R}^4$, with L/R denoting left/right action. This action is the adjoint one, an action of $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R/\mathbb{Z}_2$.

Let us denote $E_a^\pm = E_{e_a}^\pm$ for the imaginary units. By definition one has that

$$E_a^+ E_b^- = E_b^- E_a^+, \qquad E_a^\pm E_b^\pm = -\delta_{ab}\mathbb{1} \pm \sum_{c=1}^{3} \varepsilon_{abc} E_c^\pm.$$

In the following, it will turn out to be more convenient to change a sign to the 'right' matrices: we shall rather use matrices $J_a^+ := E_a^+$ and $J_a^- := -E_a^-$. For these one has

$$J_a^+ J_b^- = J_b^- J_a^+, \qquad J_a^\pm J_b^\pm = -\delta_{ab}\mathbb{1} + \sum_{c=1}^{3} \varepsilon_{abc} J_c^\pm,$$

that is the matrices $J_a^\pm$ are two copies of the quaternionic imaginary units. Indeed for these 4×4 real matrices $J_a^\pm$ one can explicitly compute

$$(J_a^\pm)_{\mu\nu} = \mp(\delta_{0\mu}\delta_{a\nu} - \delta_{a\mu}\delta_{0\nu}) + \sum_{b,c=1}^{3} \varepsilon_{abc}\delta_{b\mu}\delta_{c\nu}. \tag{8.17}$$

With the identification $\mathrm{U}_1(\mathbb{H}) \simeq \mathrm{SU}(2)$, when acting on $\mathbb{R}^4$, the matrices $J_a^\pm$ are a representation of the Lie algebra $\mathrm{su}(2)$ of $\mathrm{SU}(2)$, or taken together a representation of $\mathrm{su}(2)_L \oplus \mathrm{su}(2)_R$. For the standard positive definite scalar product on $\mathbb{R}^4$, the six matrices $J_a^\pm$ are readily checked to be antisymmetric, ${}^tJ_a^\pm = -J_a^\pm$, and one finds in addition that

$$-\tfrac{1}{4}\,\mathrm{tr}(J_a^\pm J_b^\pm) = \delta_{ab}.$$

Then, the matrices $(J_1^\pm, J_2^\pm, J_3^\pm)$ are canonically an orthonormal basis of $\Lambda^2_\pm\mathbb{R}^{4*} \simeq \mathbb{R}^3$ considered as an oriented three-dimensional euclidean space with the orientation of this basis; mapping $J_a^\pm \to J_a^\mp$ amounts to exchange the orientation. On the other hand, the nine matrices $J_a^+ J_b^-$ are an orthonormal basis for the space of symmetric trace-less 4×4 matrices.

8.3. **Noncommutative quaternionic tori and spheres.** Referring to the above, we have explicit solutions for the deformation matrix in (8.12). Firstly, with any vector $\mathbf{u} = (u^1, u^2, u^3) \in \mathbb{R}^3$ we get antisymmetric matrices

$$J_{\mathbf{u}}^+ := u^1 J_1^+ + u^2 J_2^+ + u^3 J_3^+ \qquad \text{or} \qquad J_{\mathbf{u}}^- := u^1 J_1^- + u^2 J_2^- + u^3 J_3^-.$$

With this notation, consider the matrix

$$R_{\beta\mu}^{\lambda\alpha} = u^0\,\delta_\mu^\lambda\delta_\beta^\alpha + \mathrm{i}\,(J_{\mathbf{v}}^+)_\mu^\lambda\,(J_{\mathbf{u}}^+)_\beta^\alpha. \tag{8.18}$$

Clearly, all the commutation relation (8.13) and (8.14) are satisfied. Thus, with this matrix, we define $\mathcal{A}_R$ as the $*$-algebra generated by the hermitian elements x_1^λ and x_2^α, $\lambda, \alpha \in \{0,1,2,3\}$, with relations

$$x_1^\lambda x_1^\mu = x_1^\mu x_1^\lambda, \qquad x_2^\alpha x_2^\beta = x_2^\beta x_2^\alpha, \qquad x_1^\lambda x_2^\alpha = R_{\beta\mu}^{\lambda\alpha} x_2^\beta x_1^\mu \tag{8.19}$$

But using the action of $\mathrm{SO}(3)$ one can always rotate $\mathbf{v}$ to a fixed direction $\widehat{\mathbf{u}}$, and in this case the resulting matrix R has parameters $u^0 \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^3$ constrained to

$$(u^0)^2 + \mathbf{u}^2 = 1,$$

that is they make up a three-dimensional sphere $\mathbb{S}^3$. There is in fact a residual 'gauge' freedom in that one can use a rotation around the direction $\widehat{\mathbf{u}}$ to remove one component of the vector $\mathbf{u}$. Thus if $\widehat{\mathbf{u}}_1$ and $\widehat{\mathbf{u}}_2$ are two orthogonal unit vectors (say in the canonical basis), we get families of noncommutative spaces determined by the matrices

$$R^{\lambda\alpha}_{\beta\mu} = u^0 \delta^\lambda_\mu \delta^\alpha_\beta + \mathrm{i}\,(J^+_1)^\lambda_\mu\,(u^1 J^+_1 + u^2 J^+_2)^\alpha_\beta, \tag{8.20}$$

and parameters constrained by a two-dimensional sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{S}^3/\mathbb{S}^1 = \mathbb{S}^2$ being

$$(u^0)^2 + (u^1)^2 + (u^2)^2 = 1.$$

These constructions lead to natural quaternionic generalisations of the toric four-dimensional noncommutative spaces described in [11] for which the space of deformation parameter is $\mathbb{P}^1(\mathbb{R}) = \mathbb{S}^1/\mathbb{Z}_2 = \mathbb{S}^1$.

Indeed, in parallel to the complex case were there is an action of the classical torus $\mathbb{T}^2$, there is now an action of the classical quaternionic torus $T^2_{\mathbb{H}} = \mathrm{U}_1(\mathbb{H}) \times \mathrm{U}_1(\mathbb{H}) = \mathbb{S}^3 \times \mathbb{S}^3 = \mathrm{SU}(2) \times \mathrm{SU}(2)$ by $*$-automorphisms of the algebra $\mathcal{A}_R$ given as follows.

In view of the commutations of the J^-_a with the J^+_b for $a, b \in \{1, 2, 3\}$, the mappings $x_1 \mapsto J^-_a x_1$, $x_2 \mapsto J^-_b x_2$ for $a, b \in \{1, 2, 3\}$ leave the relations (8.19) of $\mathcal{A}_R$ invariant and thus define $*$-automorphisms of the $*$-algebra $\mathcal{A}_R$. By setting $q = q^0 + q^a e_a \in \mathbb{H}^* \mapsto q^0 \mathbb{1} + q^a J^-_a$ with obvious conventions, one has from last section (right quaternionic) actions $x_1 \mapsto (q^0_1 \mathbb{1} + q^a_1 J^-_a) x_1$ and $x_2 \mapsto (q^0_2 \mathbb{1} + q^a_2 J^-_a) x_2$ of the multiplicative group $\mathbb{H}^* \times \mathbb{H}^*$ as automorphisms of the $*$-algebra $\mathcal{A}_R$,

This induces an action of $\mathrm{U}_1(\mathbb{H}) \times \mathrm{U}_1(\mathbb{H})$ on $\mathcal{A}_R$ by restriction to the $q \in \mathrm{U}_1(\mathbb{H})$ which passes to the quotient by the ideal generated by the two central elements $(x_1)^2 = \sum_\lambda (x^\lambda_1)^2$, $(x_2)^2 = \sum_\alpha (x^\alpha_2)^2$ and defines an action of the classical quaternionic torus $\mathrm{U}_1(\mathbb{H}) \times \mathrm{U}_1(\mathbb{H})$ by $*$-automorphisms of the coordinate algebra

$$\mathcal{A}((T^2_{\mathbb{H}})_R) = \mathcal{A}_R / \big((x_1)^2 - \mathbb{1}, (x_2)^2 - \mathbb{1}\big)$$

of the "noncommutative" quaternionic torus $(T^2_{\mathbb{H}})_R$. The action also passes to the quotient by the ideal generated by the central element $(x_1)^2 + (x_2)^2$ and defines an action of the classical quaternionic torus $\mathrm{U}_1(\mathbb{H}) \times \mathrm{U}_1(\mathbb{H})$ by $*$-automorphisms of the coordinate algebra

$$\mathcal{A}(\mathbb{S}^7_R) = \mathcal{A}_R / \big((x_1)^2 + (x_2)^2 - \mathbb{1}\big)$$

of a noncommutative seven-sphere $\mathbb{S}^7_R$. As we shall see in what follows, when restricting to the diagonal action of $\mathrm{U}_1(\mathbb{H}) \subset \mathrm{U}_1(\mathbb{H}) \times \mathrm{U}_1(\mathbb{H})$ on $\mathcal{A}(\mathbb{S}^7_R)$ will result into a SU(2)-principal bundles $\mathbb{S}^7_R \to \mathbb{S}^4_R$ on a noncommutative four-sphere.

9. Principal fibrations

We are going to define natural SU(2)-principal bundles $\mathbb{S}^7_R \to \mathbb{S}^4_R$ in the 'dual' sense of a coordinate algebra $\mathcal{A}(\mathbb{S}^4_R)$ on a four-sphere that is identified as the invariant subalgebra of the coordinate algebra $\mathcal{A}(\mathbb{S}^7_R)$ on a seven-sphere, for an action of the group SU(2).

9.1. **A canonical projection.**

In parallel with (8.16) consider the two quaternions

$$x_1 = x^\mu_1 e_\mu, \quad x_2 = x^\alpha_2 e_\alpha,$$

with commutation relations among their components governed by a matrix $R^{\lambda\alpha}_{\beta\mu}$ as in (8.1). Then, when restricting to the sphere $\mathbb{S}^7_R$ the vector-valued function

$$|\psi\rangle = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \tag{9.1}$$

has norm

$$\langle \psi, \psi \rangle = ||x_1||^2 + ||x_2||^2 = \mathbb{1}$$

and thus we get a projection

$$p = |\psi\rangle\langle\psi| = \begin{pmatrix} x_2x_2^* & x_2x_1^* \\ x_1x_2^* & x_1x_1^* \end{pmatrix}, \tag{9.2}$$

that is $p = p^* = p^2$. Define coordinate functions $Y = Y^0e_0 + Y^ke_k$ and Y^4 by

$$Y^4 = ||x_2||^2 - ||x_1||^2 \qquad \text{and} \qquad \tfrac{1}{2}Y = x_2x_1^* \tag{9.3}$$

so that the projection (9.2) is written as

$$p = |\psi\rangle\langle\psi| = \tfrac{1}{2}\begin{pmatrix} 1+Y^4 & Y \\ Y^* & 1-Y^4 \end{pmatrix}. \tag{9.4}$$

The condition $p^2 = p$ leads to

$$YY^* + (Y^4)^2 = 1 \quad \text{and} \quad Y^*Y + (Y^4)^2 = 1 \tag{9.5}$$

$$YY^4 = Y^4Y \quad \text{and} \quad Y^*Y^4 = Y^4Y^*. \tag{9.6}$$

Thus the coordinate function Y^4 is central while comparing the first two conditions requires $YY^* = Y^*Y$ and that this is a (central) multiple of the identity. A direct computation translates these to the conditions

$$-(Y^{0*}Y^k - Y^{k*}Y^0) + \varepsilon_{kmn}Y^{m*}Y^n = 0, \tag{9.7}$$

$$Y^0Y^{k*} - Y^kY^{0*} + \varepsilon_{kmn}Y^mY^{n*} = 0 \tag{9.8}$$

for $k, r, m = 1, 2, 3$ and totally antisymmetric tensor ε_{krm}, together with

$$\sum_{\mu=0}^{3}(Y^{\mu*}Y^\mu - Y^\mu Y^{\mu*}) = 0. \tag{9.9}$$

Then condition (9.5) reduces to a four-sphere relation

$$\sum_{\mu=0}^{3} Y^{\mu*}Y^\mu + (Y^4)^2 = 1 = \sum_{\mu=0}^{3} Y^\mu Y^{\mu*} + (Y^4)^2. \tag{9.10}$$

Being Y^4 central, these relations also give that both $\sum_{\mu=0}^{3} Y^{\mu*}Y^\mu$ and $\sum_{\mu=0}^{3} Y^\mu Y^{\mu*}$ are central as well. In view of the relations (9.10), the elements Y^μ generate the $*$-algebra $\mathcal{A}(\mathbb{S}^4_R)$ of a four-sphere $\mathbb{S}^4_R$. This four-sphere $\mathbb{S}^4_R$ is the suspension (by the central element Y^4), of a three-sphere $\mathbb{S}^3_R$ obtained by reducing (9.10) to

$$\sum_{\mu=0}^{3} Y^{\mu*}Y^\mu = 1 = \sum_{\mu=0}^{3} Y^\mu Y^{\mu*}. \tag{9.11}$$

Remark 9.1. Up to the change $Y^0 \mapsto -Y^0$, the relations (9.7)-(9.8) are the same as the relations (2.4)-(2.6) of [10].

Clearly, the coordinate function Y^4 is hermitian. On the other hand, as we shall see, the coordinate functions $Y^{\mu*}$, $\mu = 0, 1, 2, 3$, while not hermitian, are not independent from the Y^μ's with the explicit dependence determined by the matrix $R^{\lambda\alpha}_{\beta\mu}$. In fact the commutation relations (9.7)-(9.8) are not additional relations but they are determined by the $R^{\lambda\alpha}_{\beta\mu}$ which gives the commutation relations among the starting x's.

Lemma 9.2. *With the definitions in (9.3) it holds that*

$$\tfrac{1}{2}Y^0 = \sum_{\mu=0}^{3} x_2^\mu x_1^\mu, \qquad \tfrac{1}{2}Y^k = x_2^k x_1^0 - x_2^0 x_1^k - \varepsilon_{knm}x_2^n x_1^m \tag{9.12}$$

and

$$\tfrac{1}{2}Y^{0*} = \sum_{\mu=0}^{3} x_1^\mu x_2^\mu, \qquad \tfrac{1}{2}Y^{k*} = x_1^0 x_2^k - x_1^k x_2^0 + \varepsilon_{knm} x_1^n x_2^m\,. \tag{9.13}$$

Proof. A direct computation. □

9.2. **Noncommutative** $\mathrm{SU}(2)$**-principal bundles.** As mentioned, due to the relations (9.10), the elements Y^μ generate the $*$-algebra $\mathcal{A}(\mathbb{S}^4_R)$ of a four-sphere $\mathbb{S}^4_R$. The algebra inclusion $\mathcal{A}(\mathbb{S}^4_R) \hookrightarrow \mathcal{A}(\mathbb{S}^7_R)$ is a principal $\mathrm{SU}(2)$ bundle in the following sense.

With $|\psi\rangle$ the vector-valued function in (9.1), let the action of a unit quaternion $w \in \mathrm{U}_1(\mathbb{H}) \simeq \mathrm{SU}(2)$ on $\mathbb{S}^7_R$ be obtained from the following action on the generators:

$$\alpha_w(|\psi\rangle) = |\psi\rangle\, w = \begin{pmatrix} x_2 w \\ x_1 w \end{pmatrix}. \tag{9.14}$$

Clearly, the projection p and then the algebra $\mathcal{A}(\mathbb{S}^4_R)$ are invariant for this action.

On the other hand, in general the action (9.14) does not preserve the commutation relations of the $\mathbb{S}^7_R$ and thus not results into an action by $*$-automorphisms of the coordinate algebra $\mathcal{A}(\mathbb{S}^7_R)$. Let us assume this is the case, that is the action preserves the commutation relations and postpone to later on the study of deformations that meet this condition.

Dually we can also apply the general theory of Section 7 to construct a noncommutative principal bundle $\mathcal{A}(\mathbb{S}^4_R) \subseteq \mathcal{A}(\mathbb{S}^7_R)$ with Hopf algebra $\mathcal{A}(SU(2))$. In order to make the coaction clear, we will use complex coordinate to construct the Hopf Galois extension.

We know the Hopf algebra $\mathcal{A}(SU(2))$ is an unital complex $*$-algebra generated by $\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2$ subject to the relation $\omega_1\overline{\omega}_1 + \omega_2\overline{\omega}_2 = 1$. The coproduct, counit and antipode is given by:

$$\Delta : \begin{pmatrix} \omega_1 & \omega_2 \\ -\overline{\omega}_2 & \overline{\omega}_1 \end{pmatrix} \mapsto \begin{pmatrix} \omega_1 & \omega_2 \\ -\overline{\omega}_2 & \overline{\omega}_1 \end{pmatrix} \otimes \begin{pmatrix} \omega_1 & \omega_2 \\ -\overline{\omega}_2 & \overline{\omega}_1 \end{pmatrix},$$

with counit $\epsilon(\omega_1) = \epsilon(\overline{\omega}_1) = 1$, $\epsilon(\omega_2) = \epsilon(\overline{\omega}_2) = 0$ and antipode $S(\omega_1) = \overline{\omega}_1$, $S(\omega_2) = -\omega_2$.

For the coordinate x_1^μ and x_2^μ, we define

$$z_1 := x_1^0 + x_1^1 e_1, \quad z_2 := x_1^2 + x_1^3 e_1, \quad z_3 := x_2^0 + x_2^1 e_1, \quad z_4 := x_2^2 + x_2^3 e_1. \tag{9.15}$$

Or equivalently, $x_1 = z_1 + z_2 e_2$ and $x_2 = z_3 + z_4 e_2$, where e_1, e_2 and e_3 are Quaternion basis, with $e_1 e_2 = e_3$.

The coaction $\delta : \mathcal{A}(\mathbb{S}^7_R) \to \mathcal{A}(\mathbb{S}^7_R) \otimes \mathcal{A}(SU(2))$ is given by:

$$\delta : (z_1, z_2, z_3, z_4) \mapsto (z_1, z_2, z_3, z_4) \otimes \begin{pmatrix} \omega_1 & \omega_2 & 0 & 0 \\ -\overline{\omega}_2 & \overline{\omega}_1 & 0 & 0 \\ 0 & 0 & \omega_1 & \omega_2 \\ 0 & 0 & -\overline{\omega}_2 & \overline{\omega}_1 \end{pmatrix}. \tag{9.16}$$

By exchanging the coordinate we can see that the algebra generated by $\mathcal{A}(\mathbb{S}^4_R)$ can also be written as

$$\mathcal{A}(\mathbb{S}^4_R) = \mathbb{C}[1, z_1\overline{z_3} + z_2\overline{z_4}, -z_1 z_4 + z_2 z_3, z_1\overline{z_1} + z_2\overline{z_2}], \tag{9.17}$$

i.e. $\mathcal{A}(\mathbb{S}^4_R)$ is generated by $\alpha := 2(z_1\overline{z_3} + z_2\overline{z_4})$, $\beta := 2(-z_1 z_4 + z_2 z_3)$ and $\gamma := z_1\overline{z_1} + z_2\overline{z_2} - z_3\overline{z_3} + z_4\overline{z_4}$. By direct computation we can see $\mathcal{A}(\mathbb{S}^4_R)$ is the coinvariant subalgebra of the coaction δ. Moreover, we have

$$\alpha = Y^{0*} - Y^{1*} e_1, \qquad \beta = -Y^{2*} - Y^{3*} e_1, \qquad \gamma = -Y^4. \tag{9.18}$$

Proposition 9.1. *Let the action of* $\mathrm{SU}(2)$ *in* (9.14) *be by* $*$*-automorphisms of the coordinate algebra* $\mathcal{A}(\mathbb{S}^7_R)$ *and let* $H = \mathcal{A}(\mathrm{SU}(2))$*. Then the canonical map*

$$\chi : \mathcal{A}(\mathbb{S}^7_R) \otimes_{\mathcal{A}(\mathbb{S}^4_R)} \mathcal{A}(\mathbb{S}^7_R) \to \mathcal{A}(\mathbb{S}^7_R) \otimes H, \qquad \chi(p' \otimes p) = p'\delta(p)$$

is bijective.

Proof. With $|\psi\rangle$ as in (9.1), one has that

$$\chi\big(\, \langle\psi| \otimes_{\mathcal{A}(\mathbb{S}^4_R)} |\psi\rangle \,\big) = \langle\psi|\, \delta(|\psi\rangle) = \langle\psi,\psi\rangle \otimes w = \mathbb{1} \otimes w.$$

Since $\mathcal{A}(\mathrm{SU}(2))$ is cosemisimple, surjectivity is equivalent to bijectivity (as described in the Section 7). More precisely, we can check on the generators that

$$\chi(\overline{z_1} \otimes_{\mathcal{A}(\mathbb{S}^4_R)} z_1 + z_2 \otimes_{\mathcal{A}(\mathbb{S}^4_R)} \overline{z_2} + \overline{z_3} \otimes_{\mathcal{A}(\mathbb{S}^4_R)} z_3 + z_4 \otimes_{\mathcal{A}(\mathbb{S}^4_R)} \overline{z_4}) = 1 \otimes \omega_1.$$
$$\chi(\overline{z_1} \otimes_{\mathcal{A}(\mathbb{S}^4_R)} z_2 - z_2 \otimes_{\mathcal{A}(\mathbb{S}^4_R)} \overline{z_1} + \overline{z_3} \otimes_{\mathcal{A}(\mathbb{S}^4_R)} z_4 - z_4 \otimes_{\mathcal{A}(\mathbb{S}^4_R)} \overline{z_3}) = 1 \otimes \omega_2.$$
$$\chi(\overline{z_2} \otimes_{\mathcal{A}(\mathbb{S}^4_R)} z_1 - z_1 \otimes_{\mathcal{A}(\mathbb{S}^4_R)} \overline{z_2} + \overline{z_4} \otimes_{\mathcal{A}(\mathbb{S}^4_R)} z_3 - z_3 \otimes_{\mathcal{A}(\mathbb{S}^4_R)} \overline{z_4}) = -1 \otimes \overline{\omega}_2.$$
$$\chi(\overline{z_2} \otimes_{\mathcal{A}(\mathbb{S}^4_R)} z_2 + z_1 \otimes_{\mathcal{A}(\mathbb{S}^4_R)} \overline{z_1} + \overline{z_4} \otimes_{\mathcal{A}(\mathbb{S}^4_R)} z_4 + z_3 \otimes_{\mathcal{A}(\mathbb{S}^4_R)} \overline{z_3}) = 1 \otimes \overline{\omega}_1.$$

□

Notice that the Hopf Galois extension we construct above is associated with a $*$-structure. In other words, both the Hopf algebra $H = \mathcal{A}(SU(2))$ and the comodule algebra $A = \mathcal{A}(\mathbb{S}^7_R)$ are $*$-algebras, such that the coproduct, counit and coaction preserve the $*$-structure, i.e. $(* \otimes *) \circ \Delta = \Delta \circ *$, $\epsilon(h^*) = \overline{\epsilon(h)}$ for any $h \in H$, and $(* \otimes *) \circ \delta = \delta \circ *$.

9.3. **Connes–Chern characters.** Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$ and let $\widetilde{\mathcal{A}} = \mathcal{A}/\mathbb{C}\mathbb{1}$ be the quotient of $\mathcal{A}$ by the scalar multiples of the unit $\mathbb{1}$. Given an idempotent,

$$e = (e^i_j) \in \mathrm{Mat}_r(\mathcal{A}) \qquad e^2 = e,$$

the component $\mathrm{ch}_k(e)$ of the (reduced) Chern character of e is the element of $\mathcal{A} \otimes (\widetilde{\mathcal{A}})^{\otimes 2k}$, given explicitly by the formula

$$\begin{aligned} \mathrm{ch}_k(e) &= \lambda_k \left\langle (e - \tfrac{1}{2}\,\mathbb{1}) \otimes e^{\otimes 2k} \right\rangle \\ &= \lambda_k \left(e^{i_0}_{i_1} - \tfrac{1}{2}\,\delta^{i_0}_{i_1} \right) \otimes e^{i_1}_{i_2} \otimes e^{i_2}_{i_3} \cdots \otimes e^{i_{2k}}_{i_0}. \end{aligned} \tag{9.19}$$

Here δ_{ij} is the usual Kronecker symbol and the λ_k normalization constants.

Similarly, for a unitary

$$U = (U^i_j) \in \mathrm{Mat}_r(\mathcal{A}) \qquad UU^* = U^*U,$$

the component $\mathrm{ch}_{k+\frac{1}{2}}(U)$ of the Chern character of U is the element of $\mathcal{A} \otimes (\widetilde{\mathcal{A}})^{\otimes(2k+1)}$ given explicitly by the formula

$$\begin{aligned} \mathrm{ch}_{k+\frac{1}{2}}(U) &= \Big\langle \underbrace{U \otimes U^* \otimes U \otimes U^* \otimes \cdots U \otimes U^*}_{2(k+1)} - \underbrace{U^* \otimes U \otimes U^* \otimes U \cdots \otimes U^* \otimes U}_{2(k+1)} \Big\rangle \\ &= \lambda_k \left(U^{i_0}_{i_1} \otimes U^{*i_1}_{\ \ i_2} \otimes U^{i_2}_{i_3} \otimes \cdots \otimes U^{*i_{2k+1}}_{\ \ i_0} - U^{*i_0}_{\ \ i_1} \otimes \cdots \otimes U^{i_{2k+1}}_{i_0} \right) \end{aligned} \tag{9.20}$$

with λ_k again normalization constants.

The crucial property of the components $\mathrm{ch}_k(e)$ or $\mathrm{ch}_{k+\frac{1}{2}}(U)$ is that they define a *cycle* in the (b, B) bicomplex of cyclic homology [9], [23], that is

$$B\,\mathrm{ch}_k(e) = b\,\mathrm{ch}_{k+1}(e) \qquad \text{or} \qquad B\,\mathrm{ch}_{k+\frac{1}{2}}(U) = b\,\mathrm{ch}_{k+\frac{3}{2}}(U) \tag{9.21}$$

where b is the Hochschild boundary operator and B is the Connes boundary operator.

For a noncommutative spherical manifold [11], [10], one asks that the components of the character vanish but a top one that, due to (9.21) is then a (non zero) Hochschild cycle and plays the role of the volume form for the noncommutative manifold. Specifically, in even dimensions, for $n = 2m$ one asks

$$\mathrm{ch}_k(e) = 0, \qquad \text{for all} \quad k = 0, 1, \dots m-1, \tag{9.22}$$

and $\mathrm{ch}_m(e)$ (with $b\,\mathrm{ch}_m(e) = 0$ from (9.21)) is the volume form. Similarly, in odd dimensions, for $n = 2m+1$ the vanishing condition becomes

$$\mathrm{ch}_{k+\frac{1}{2}}(U) = 0, \qquad \text{for all} \quad k = 0, 1, \dots m-1, \tag{9.23}$$

and $\mathrm{ch}_{m+\frac{1}{2}}(U)$ (with $b\,\mathrm{ch}_{m+\frac{1}{2}}(U) = 0$ from (9.21)) is the volume form.

9.4. **Volume forms.** We have already observed that the unit radius conditions in (9.5) requires that the 'quaternion' $Y = Y^0 e_0 + Y^k e_k$ be such that $YY^* = Y^*Y \in \mathbb{1}_2 \otimes \mathcal{A}(\mathbb{S}^4_R)$ (in fact be in the centre of $\mathcal{A}(\mathbb{S}^4_R)$. An important role is played by the components of the Connes–Chern character in cyclic homology of Y,

$$\mathrm{ch}_{\frac{1}{2}}(Y) = \langle Y \otimes Y^* - Y^* \otimes Y \rangle \tag{9.24}$$

and

$$\mathrm{ch}_{\frac{3}{2}}(Y) = \langle Y \otimes Y^* \otimes Y \otimes Y^* - Y^* \otimes Y \otimes Y^* \otimes Y \rangle\,. \tag{9.25}$$

Here $\langle \,\cdot\, \rangle$ indicates the partial matrix trace over $\mathbb{M}_2(\mathbb{C})$, thinking of $\mathbb{H}$ as a subset of $\mathbb{M}_2(\mathbb{C})$. We know that when $\mathrm{ch}_{\frac{1}{2}}(Y) = 0$ the element $\mathrm{ch}_{\frac{3}{2}}(Y)$ is a Hochschild cycle which gives a volume element for the three-sphere $\mathbb{S}^3_R$ obtained by the unitarity conditions $YY^* = Y^*Y = \mathbb{1}_2$.

On the other hand, for the projection p in (9.4) one has at once that

$$\mathrm{ch}_0(p) = \big\langle (p - \tfrac{1}{2}\,\mathbb{1}) \big\rangle = 0.$$

Moreover, being the four-sphere $\mathbb{S}^4_R$ the suspension by the central element Y^4 of the three-sphere $\mathbb{S}^3_R$, the vanishing $\mathrm{ch}_{\frac{1}{2}}(Y) = 0$ would also imply the vanishing of the component $\mathrm{ch}_1(p)$ (cf. [10, Theorem 2]), where

$$\mathrm{ch}_1(p) = \big\langle (p - \tfrac{1}{2}\,\mathbb{1}) \otimes p^{\otimes 2} \big\rangle\,.$$

Then, similarly to before, the element $\mathrm{ch}_2(p) = \big\langle (p - \tfrac{1}{2}\,\mathbb{1}) \otimes p^{\otimes 4} \big\rangle$ is a Hochschild cycle which gives a volume element for the four-sphere $\mathbb{S}^4_R$.

We see that the vanishing $\mathrm{ch}_{\frac{1}{2}}(Y) = 0$ makes both the sphere $\mathbb{S}^3_R$ as well as its suspension sphere $\mathbb{S}^4_R$, noncommutative spherical manifolds in the sense of [11], [10].

9.5. **An analysis of the $*$-structure.** Due to relation (9.9), one expect the elements Y^{a*}, $a = 0, 1, 2, 3$, to be expressed in terms of the elements Y^a, $a = 0, 1, 2, 3$. This fact requires conditions on the possible deformation matrix R, while giving nicer properties for both spheres $\mathbb{S}^4_R$ and $\mathbb{S}^3_R$. Indeed, they becomes spherical manifolds as mentioned at the end of previous section. A direct computation shows that the vanishing $\mathrm{ch}_{\frac{1}{2}}(Y) = 0$ is equivalent to the condition

$$\sum_{\mu=0}^{3} (Y^{\mu *} \otimes Y^{\mu} - Y^{\mu} \otimes Y^{\mu *}) = 0. \tag{9.26}$$

One has then the following

Lemma 9.3. [10, Lemma 2] *The condition* (9.26) *is satisfied, that is* $\mathrm{ch}_{\frac{1}{2}}(Y) = 0$*, if and only if there is a symmetric unitary matrix* $\Lambda \in \mathbb{M}_4(\mathbb{C})$ *such that*

$$Y^{\mu *} = \Lambda^{\mu}{}_{\nu} Y^{\nu} \qquad \mu, \nu = 0, 1, 2, 3\,. \tag{9.27}$$

In turn, the condition $\mathrm{ch}_{\frac{1}{2}}(Y) = 0$ is left unchanged by a linear change in generators as

$$Y^{\mu} \mapsto u\, S^{\mu}{}_{\nu}\, Y^{\nu} \tag{9.28}$$

with $u \in \mathrm{U}(1)$ and $S \in \mathrm{SO}(4)$ a real rotation. Under this transformation, the symmetric unitary matrix Λ in (9.27) transforms as

$$\Lambda \mapsto u^2\, S^t\, \Lambda S. \tag{9.29}$$

Then it can be diagonalized by a real rotation S and with a further normalization (by a factor $u \in \mathrm{U}(1)$) it can alway be put in the form

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\mathrm{i}\theta_1} & 0 & 0 \\ 0 & 0 & e^{\mathrm{i}\theta_2} & 0 \\ 0 & 0 & 0 & e^{\mathrm{i}\theta_3} \end{pmatrix}, \tag{9.30}$$

for suitable angles $\theta_1, \theta_2, \theta_3$ (see [10, §2]).

10. The quaternionic family of four-spheres

Let us now consider the quaternionic deformations mentioned in section 8.3 governed by the deformation matrix in (8.20) and in particular the noncommutative seven-sphere. As we have seen, there is a compatible action of $\mathrm{U}_1(\mathbb{H}) \times \mathrm{U}_1(\mathbb{H})$ by $*$-automorphisms of the corresponding coordinate algebra. Then for the action in (9.14) we may take the diagonal action by $w = w^0 + w^a e_a \in \mathrm{U}_1(\mathbb{H}) \mapsto w^0 \mathbb{1} + w^a J_a^- \in \mathrm{SU}(2)$, written explicitly on generators as $x_1 \mapsto (w^0\mathbb{1} + w^a J_a^-)x_1$ and $x_2 \mapsto (w^0\mathbb{1} + w^a J_a^-)x_2$.

Proposition 10.1. *Given the commutation relations for the x's for the matrix* (8.20)*, one has* $Y^{\mu *} = \Lambda^{\mu}{}_{\nu} Y^{\nu}$ *for* $\Lambda \in \mathbb{M}_4(\mathbb{C})$ *a symmetric unitary matrix given explicitly by:*

$$\begin{pmatrix} Y^{0*} \\ Y^{3*} \end{pmatrix} = \begin{pmatrix} u^0 + \mathrm{i}u^1 & \mathrm{i}u^2 \\ \mathrm{i}u^2 & u^0 - \mathrm{i}u^1 \end{pmatrix} \begin{pmatrix} Y^0 \\ Y^3 \end{pmatrix} \tag{10.1}$$

and

$$\begin{pmatrix} Y^{1*} \\ Y^{2*} \end{pmatrix} = \begin{pmatrix} u^0 + \mathrm{i}u^1 & \mathrm{i}u^2 \\ \mathrm{i}u^2 & u^0 - \mathrm{i}u^1 \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}. \tag{10.2}$$

Proof. A comparison with the matrices (8.17) shows that $\frac{1}{2}Y^a = (J_+^a)_{\alpha\lambda}\, x_2^{\alpha} x_1^{\lambda}$ for $a = 0, 1, 2, 3$, with $J_+^0 = \mathbb{1}$. This allows one to write $\frac{1}{2}Y^{a*} = (J_+^a)_{\alpha\lambda}\, x_1^{\lambda} x_2^{\alpha} = (J_+^a)_{\alpha\lambda}\, R^{\lambda\alpha}_{\beta\mu}\, x_2^{\beta} x_1^{\mu}$. Then, with the R matrix in (8.20), a direct computation of (9.13) yields

$$\begin{aligned} \tfrac{1}{2}Y^{a*} &= (J_a^+)_{\alpha\lambda}\, R^{\lambda\alpha}_{\beta\mu}\, x_2^{\beta} x_1^{\mu} \\ &= \left(u^0 J_a^+ - \mathrm{i}\, u^1 J_1^+ J_a^+ J_1^+ - \mathrm{i}\, u^2 J_2^+ J_a^+ J_1^+\right)_{\beta\mu} x_2^{\beta} x_1^{\mu} \\ &= \tfrac{1}{2}\left(u^0 Y^a - \mathrm{i}\, u^1 J_1^+ Y^a J_1^+ - \mathrm{i}\, u^2 J_2^+ Y^a J_1^+\right), \end{aligned}$$

from which one gets the explicit expressions in (10.1) and (10.2). ∎

With the $*$-structure of the previous proposition one shows that none of the generators is normal, that is $Y^{\mu *}Y^{\mu} \neq Y^{\mu}Y^{\mu *}$ while the condition (9.9) is automatically satisfied. On the other hand, the commutation relations (9.7) and (9.8) can be written as:

$$\begin{aligned}
(u^0 + \mathrm{i}u^1)(Y^1Y^0 - Y^0Y^1) + \mathrm{i}u^2(Y^1Y^3 - Y^0Y^2) &= 0\\
(u^0 - \mathrm{i}u^1)(Y^3Y^2 - Y^2Y^3) + \mathrm{i}u^2(Y^3Y^1 - Y^2Y^0) &= 0\\
u^0(Y^2Y^0 - Y^0Y^2) - \mathrm{i}u^1(Y^1Y^3 + Y^3Y^1) + \mathrm{i}u^2(Y^1Y^0 - Y^3Y^2) &= 0\\
u^0(Y^3Y^1 - Y^1Y^3) - \mathrm{i}u^1(Y^0Y^2 + Y^2Y^0) + \mathrm{i}u^2(Y^0Y^1 - Y^2Y^3) &= 0\\
u^0(Y^3Y^0 - Y^0Y^3) + \mathrm{i}u^1(Y^1Y^2 + Y^2Y^1) + \mathrm{i}u^2\big((Y^2)^2 - (Y^1)^2\big) &= 0\\
u^0(Y^2Y^1 - Y^1Y^2) + \mathrm{i}u^1(Y^0Y^3 + Y^3Y^0) + \mathrm{i}u^2\big((Y^3)^2 - (Y^0)^2\big) &= 0.
\end{aligned}$$

For the structure Λ in (10.1) and (10.2), the matrix

$$\Lambda' = \begin{pmatrix} u^0 + \mathrm{i}u^1 & \mathrm{i}u^2 \\ \mathrm{i}u^2 & u^0 - \mathrm{i}u^1 \end{pmatrix} \tag{10.3}$$

being symmetric and unitary, can be diagonalized by a real rotation S: one finds eigenvalues $\lambda_{\pm} = u^0 \pm \mathrm{i}\sqrt{(u^1)^2 + (u^2)^2} = u^0 \pm \mathrm{i}\sqrt{1 - (u^0)^2}$. With a further normalization by the factor $u^0 - \mathrm{i}\sqrt{(u^1)^2 + (u^2)^2} \in \mathrm{U}(1)$, the matrix Λ' can be put in the form

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}\theta} \end{pmatrix}, \tag{10.4}$$

and a direct computation gives:

$$e^{\mathrm{i}\theta} = \frac{u^0 + \mathrm{i}\sqrt{(u^1)^2 + (u^2)^2}}{u^0 - \mathrm{i}\sqrt{(u^1)^2 + (u^2)^2}} = \big(u^0 + \mathrm{i}\sqrt{(u^1)^2 + (u^2)^2}\,\big)^2. \tag{10.5}$$

The sphere $\mathbb{S}^4_R = \mathbb{S}^4_\theta$ is then (isomorphic to) a θ-deformation, as the one introduced in [11]

Part 2. On the Gauge group of Galois objects

11. The gauge groups

In [5] gauge transformations for a noncommutative principal bundles were defined to be invertible and unital comodule maps, with no additional requirement. In particular they were not asked to be algebra morphisms. A drawback of this approach is that the resulting gauge group might be very big, even in the classical case; for example the gauge group of the a G-bundle over a point would be much bigger than the structure group G. On the other hand, in [2] gauge transformations were required to be algebra homomorphisms. This implies in particular that they are invertible.

In the line of the paper [2] we are lead to the following definition.

Definition 11.1. Given a Hopf–Galois extension $B = A^{coH} \subseteq A$. Consider the collection

$$\mathrm{Aut}_H(A) := \{F \in \mathrm{Hom}_{\mathcal{A}^H}(A, A) \mid F|_B \in \mathrm{Aut}(B)\}, \tag{11.1}$$

of right H-comodule unital algebra morphisms of A which restrict to automorphisms of B, and the sub-collection

$$\mathrm{Aut}_{ver}(A) := \{F \in \mathrm{Aut}_H(A) \mid F|_B = \mathrm{id}_B\}. \tag{11.2}$$

of 'vertical' ones, that is that in addition are left B-module morphisms.

Thus elements $F \in \mathrm{Aut}_H(A)$ preserve the (co)-action of the structure quantum group since they are such that $\delta^A \circ F = (F \otimes \mathrm{id})\delta^A$ (or $F(a)_{(0)} \otimes F(a)_{(1)} = F(a_{(0)}) \otimes a_{(1)}$). And if in $\mathrm{Aut}_{ver}(A)$ they also preserve the base space algebra B. These will be called the gauge group and the vertical gauge group respectively: in parallel with [2, Prop. 3.6], $\mathrm{Aut}_H(A)$ and $\mathrm{Aut}_{ver}(A)$ are groups when B is restricted to be in the centre of A by the following proposition:

Proposition 11.2. *Let $B = A^{coH} \subseteq A$ be a H-Hopf–Galois extension with B in the centre of A. Then $\mathrm{Aut}_H(A)$ is a group with respect to the composition of maps*

$$F \cdot G := G \circ F$$

for all $F, G \in \mathrm{Aut}_H(A)$. For $F \in \mathrm{Aut}_H(A)$ its inverse $F^{-1} \in \mathrm{Aut}_H(A)$ is given by

$$F^{-1} := m \circ ((F|_B)^{-1} \otimes \mathrm{id}) \circ (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes F \otimes_B \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ \delta^A \tag{11.3}$$

where τ is the translation map, that is for all $a \in A$,

$$F^{-1}(a) := (F|_B)^{-1}\big(a_{(0)} F(a_{(1)}{}^{<1>})\big)\, a_{(1)}{}^{<2>}\,. \tag{11.4}$$

In particular the vertical homomorphisms $\mathrm{Aut}_{ver}(A)$ form a subgroup of $\mathrm{Aut}_H(A)$.

Proof. The group multiplication is clearly well defined with unit the identity map on A. Next, we compute that somewhat 'implicitly', $a_{(0)} F(a_{(1)}{}^{<1>}) \otimes_B a_{(1)}{}^{<2>} \in B \otimes_B A$. Indeed

$$\begin{aligned}
(\delta^A \otimes_B \mathrm{id}_A)(a_{(0)} F(a_{(1)}{}^{<1>}) \otimes_B a_{(1)}{}^{<2>}) &= a_{(0)(0)} F(a_{(1)}{}^{<1>})_{(0)} \otimes a_{(0)(1)} F(a_{(1)}{}^{<1>})_{(1)} \otimes_B a_{(1)}{}^{<2>} \\
&= a_{(0)} F(a_{(2)}{}^{<1>}{}_{(0)}) \otimes a_{(1)} a_{(2)}{}^{<1>}{}_{(1)} \otimes_B a_{(2)}{}^{<2>} \\
&= a_{(0)} F(a_{(2)(2)}{}^{<1>}) \otimes a_{(1)} S(a_{(2)(1)}) \otimes_B a_{(2)(2)}{}^{<2>} \\
&= a_{(0)} F(a_{(1)}{}^{<1>}) \otimes 1_H \otimes_B a_{(1)}{}^{<2>},
\end{aligned}$$

where the 2nd step uses that F is H-equivalent map, the 3rd step uses (7.6); since δ^A is right B–linear, everything is well defined. Now, being B the coinvariant subalgebra of A for the coaction, we have the exact sequence,

$$0 \longrightarrow B \xrightarrow{i} A \xrightarrow{\delta^A - \mathrm{id}_A \otimes \mathrm{id}_H} A \otimes H \longrightarrow 0$$

And, since A is faithful flat as left B-module, we also have exactness of the sequence,

$$0 \longrightarrow B \otimes_B A \overset{i \otimes_B \mathrm{id}_A}{\longrightarrow} A \otimes_B A \overset{(\delta^A - \mathrm{id}_A \otimes \mathrm{id}_H) \otimes_B \mathrm{id}_A}{\longrightarrow} A \otimes H \otimes_B A \longrightarrow 0\,.$$

Thus, the equality $(\delta^A \otimes_B \mathrm{id}_A)(a_{(0)} F(a_{(1)}{}^{<1>}) \otimes_B a_{(1)}{}^{<2>}) = a_{(0)} F(a_{(1)}{}^{<1>}) \otimes 1_H \otimes_B a_{(1)}{}^{<2>}$ shows that $a_{(0)} F(a_{(1)}{}^{<1>}) \otimes_B a_{(1)}{}^{<2>} \in B \otimes_B A$ and thus F^{-1} in (11.4) is well defined. Let us check that F^{-1} is an algebra map:

$$\begin{aligned} F^{-1}(aa') &= (F|_B)^{-1}\big((aa')_{(0)} F((aa')_{(1)}{}^{<1>})\big)\,(aa')_{(1)}{}^{<2>} \\ &= (F|_B)^{-1}\big(a_{(0)} a'_{(0)} F(a'_{(1)}{}^{<1>}) F(a_{(1)}{}^{<1>})\big)\, a_{(1)}{}^{<2>} a'_{(1)}{}^{<2>} \\ &= F^{-1}(a) F^{-1}(a'), \end{aligned}$$

where the 2nd step uses (7.9), and the last step uses the fact that $a'_{(0)} F(a'_{(1)}{}^{<1>}) \otimes_B a'_{(1)}{}^{<2>} \in B \otimes_B A$ and B belongs to the centre of A, thus F^{-1} is an algebra map. Also for any $b \in B$, $F^{-1}(b) = (F|_B)^{-1}(b)$, so $F^{-1}|_B \in \mathrm{Aut}(B)$. Then, for any $a \in A$

$$\begin{aligned} F^{-1}(F(a)) &= (F|_B)^{-1}\big(F(a)_{(0)} F(F(a)_{(1)}{}^{<1>})\big)\, F(a)_{(1)}{}^{<2>} \\ &= (F|_B)^{-1}\big(F(a_{(0)}) F(a_{(1)}{}^{<1>})\big)\, a_{(1)}{}^{<2>} \\ &= (F|_B)^{-1}\big(F(a_{(0)} a_{(1)}{}^{<1>})\big)\, a_{(1)}{}^{<2>} \\ &= a, \end{aligned}$$

where the 2nd step uses the H-equivariance of F, and the last step uses (7.11). Finally,

$$\begin{aligned} F(F^{-1}(a)) &= F\big((F|_B)^{-1}(a_{(0)} F(a_{(1)}{}^{<1>})) a_{(1)}{}^{<2>}\big) \\ &= a_{(0)} F(a_{(1)}{}^{<1>}) F(a_{(1)}{}^{<2>}) \\ &= a. \end{aligned}$$

Thus F^{-1} is the inverse map of $F \in \mathrm{Aut}_H(A)$. The map F^{-1} is H-equivariant as well so it belongs to $\mathrm{Aut}_H(A)$. Indeed, for any $a \in A$ we have

$$a_{(0)} \otimes a_{(1)} = F(F^{-1}(a))_{(0)} \otimes F(F^{-1}(a))_{(1)} = F(F^{-1}(a)_{(0)}) \otimes F^{-1}(a)_{(1)},$$

where the last step uses the H-equivariance of F. Applying $F^{-1} \otimes \mathrm{id}_H$ on both sides of the last equation we get

$$F^{-1}(a_{(0)}) \otimes a_{(1)} = F^{-1}(a)_{(0)} \otimes F^{-1}(a)_{(1)}$$

so F^{-1} is H-equivariant. We conclude that $\mathrm{Aut}_H(A)$ is a group.
As for the vertical automorphisms, clearly $\mathrm{Aut}_{ver}(A)$ is closed for map compositions and one sees that $F^{-1} \in \mathrm{Aut}_{ver}(A)$ when $F \in \mathrm{Aut}_{ver}(A)$. Thus $\mathrm{Aut}_{ver}(A)$ is also a group. ☐

Remark 11.3. A similar proposition was first given in [2], for a Hopf algebra H which is a coquasitriangular Hopf algebra, and A is a quasi-commutative H-comodule algebra. As a consequence, B belongs to the centre of A. In the present paper, we only require B to belongs to the centre of A without assuming H to be a coquasitriangular Hopf algebra.

For the sake of the present paper, where we are concerned mainly with Galois objects, and seek to study their gauge groups with relations to bisections of suitable groupoids, there is no restriction in assuming that the base space algebra B be in the centre.

12. Ehresmann–Schauenburg bialgebroids

To any Hopf–Galois extension $B = A^{co\,H} \subseteq A$ one associates a B-coring [6, §34.13] and a bialgebroid [6, §34.14]. These can be viewed as a quantization of the gauge or Ehresmann groupoid that is associated to a principal fibre bundle (cf. [27]).

12.1. **Ehresmann corings.** The coring can be given in few equivalent ways. Let $B = A^{co\,H} \subseteq A$ be a Hopf–Galois extension with right coaction $\delta^A : A \to A \otimes H$. Recall the diagonal coaction (7.2), given for all $a, a' \in A$ by

$$\delta^{A\otimes A} : A \otimes A \to A \otimes A \otimes H, \quad a \otimes a' \mapsto a_{(0)} \otimes a'_{(0)} \otimes a_{(1)}a'_{(1)}\,,$$

with corresponding B-bimodule of coinvariant elements,

$$(A \otimes A)^{coH} = \{a \otimes \tilde{a} \in A \otimes A\,;\; a_{(0)} \otimes \tilde{a}_{(0)} \otimes a_{(1)}\tilde{a}_{(1)} = a \otimes \tilde{a} \otimes 1_H\}. \tag{12.1}$$

Lemma 12.1. *Let τ be the translation map of the Hopf–Galois extension. Then the B-bimodule of coinvariant elements in (12.1) is the same as the B-bimodule,*

$$\mathcal{C} := \{a \otimes \tilde{a} \in A \otimes A :\; a_{(0)} \otimes \tau(a_{(1)})\tilde{a} = a \otimes \tilde{a} \otimes_B 1_A\}. \tag{12.2}$$

Proof. Let $a \otimes \tilde{a} \in (A \otimes A)^{coH}$. By applying $(\mathrm{id}_A \otimes \chi)$ on $a_{(0)} \otimes a_{(1)}{}^{<1>} \otimes_B a_{(1)}{}^{<2>}\tilde{a}$, we get

$$\begin{aligned} a_{(0)} \otimes a_{(1)}{}^{<1>} a_{(1)}{}^{<2>}{}_{(0)}\tilde{a}_{(0)} \otimes a_{(1)}{}^{<2>}{}_{(1)}\tilde{a}_{(1)} &= a_{(0)} \otimes \tilde{a}_{(0)} \otimes a_{(1)}\tilde{a}_{(1)} \\ &= a \otimes \tilde{a} \otimes 1_H = a \otimes \chi(\tilde{a} \otimes_B 1_A) \\ &= (\mathrm{id}_A \otimes \chi)(a \otimes \tilde{a} \otimes_B 1_A), \end{aligned}$$

where the first step uses (7.7). This shows that $(A \otimes A)^{coH} \subseteq \mathcal{C}$.
Conversely, let $a \otimes \tilde{a} \in \mathcal{C}$. By applying $(\mathrm{id}_A \otimes \chi^{-1})$ on $a_{(0)} \otimes \tilde{a}_{(0)} \otimes a_{(1)}\tilde{a}_{(1)}$ and using the fact that χ^{-1} is left A-linear and (7.9), we get

$$\begin{aligned} a_{(0)} \otimes \tilde{a}_{(0)}\tilde{a}_{(1)}{}^{<1>} a_{(1)}{}^{<1>} \otimes_B a_{(1)}{}^{<2>}\tilde{a}_{(1)}{}^{<2>} &= a_{(0)} \otimes a_{(1)}{}^{<1>} \otimes_B a_{(1)}{}^{<2>}\tilde{a} \\ &= a \otimes \tilde{a} \otimes_B 1_A \\ &= (\mathrm{id}_A \otimes \chi^{-1})(a \otimes \tilde{a} \otimes 1_H), \end{aligned}$$

where in the first step (7.11) is used. This shows that $\mathcal{C} \subseteq (A \otimes A)^{coH}$. □

We have then the following definition [6, §34.13].

Definition 12.2. Let $B = A^{co\,H} \subseteq A$ be a Hopf–Galois extension with translation map τ. If A is faithful flat as a left B-module, then the B-bimodule $\mathcal{C}$ in (12.2) is a B-coring with coring coproduct

$$\Delta(a \otimes \tilde{a}) = a_{(0)} \otimes \tau(a_{(1)}) \otimes \tilde{a} = a_{(0)} \otimes a_{(1)}{}^{<1>} \otimes_B a_{(1)}{}^{<2>} \otimes \tilde{a}, \tag{12.3}$$

and counit

$$\epsilon(a \otimes \tilde{a}) = a\tilde{a}. \tag{12.4}$$

By applying the map $m_A \otimes \mathrm{id}_H$ to elements of (12.1), it is clear that $a\tilde{a} \in B$. The above B-coring is called the *Ehresmann* or *gauge coring*; we denote it $\mathcal{C}(A, H)$.

Whenever the structure Hopf algebra H has an invertible antipode, the Ehresmann coring can also be given as an equaliser (see [18]). Indeed, let H be a Hopf algebra with invertible antipode. And let $B = A^{co\,H} \subseteq A$ be a H-Hopf–Galois extension, with right coaction $\delta^A : A \to A \otimes H$, $a \mapsto \delta^A(a) = a_{(0)} \otimes a_{(1)}$. Via the inverse of S one has also a left H coaction ${}^A\delta : A \to H \otimes A$, ${}^A\delta(a) := S^{-1}(a_{(1)}) \otimes a_{(0)}$. One also shows that $B := {}^{coH}A = \big\{b \in A \mid {}^A\delta(b) = 1_H \otimes b\big\}$. Using the left B-linearity of δ^A and the right B-linearity of ${}^A\delta$ one has a B-bimodule,

$$\begin{aligned} A^H \,\square\, {}^H\!A &= \ker(\delta^A \otimes \mathrm{id}_A - \mathrm{id}_A \otimes {}^A\delta) \\ &= \big\{a \otimes \tilde{a} \in A \otimes A :\; a_{(0)} \otimes a_{(1)} \otimes \tilde{a} = a \otimes S^{-1}(\tilde{a}_{(1)}) \otimes \tilde{a}_{(0)}\big\} \end{aligned} \tag{12.5}$$

Lemma 12.3. *The bimodule $A^H \,\square\, {}^H\!A$ is the same as the bimodules $\mathcal{C}$ and $(A \otimes A)^{coH}$.*

Proof. Let $a\otimes\tilde a\in\mathcal{C}$. Then, by applying $(\mathrm{id}_A\otimes\mathrm{id}_H\otimes m_A)\circ(\mathrm{id}_A\otimes{}^A\delta\otimes\mathrm{id}_A)$ on $a_{(0)}\otimes {a_{(1)}}^{<1>}\otimes_B {a_{(1)}}^{<2>}\tilde a=a\otimes\tilde a\otimes_B 1_A$ we get, for the left hand side,

$$\begin{aligned}a_{(0)}\otimes S^{-1}({a_{(1)}}^{<1>}{}_{(1)})\otimes {a_{(1)}}^{<1>}{}_{(0)}{a_{(1)}}^{<2>}\tilde a &= a_{(0)}\otimes S^{-1}(S(a_{(1)(1)}))\otimes {a_{(1)(2)}}^{<1>}{a_{(1)(2)}}^{<2>}\tilde a\\ &= a_{(0)}\otimes a_{(1)}\otimes\tilde a,\end{aligned}$$

using (7.6) in the first step and (7.8) in the second one. As for the right hand side, we get $a\otimes S^{-1}(\tilde a_{(1)})\otimes\tilde a_{(0)}$. Thus $a_{(0)}\otimes a_{(1)}\otimes\tilde a=a\otimes S^{-1}(\tilde a_{(1)})\otimes\tilde a_{(0)}$, and $a\otimes\tilde a\in A^H\,\square\,{}^HA$.

Conversely, assume $a\otimes\tilde a\in A^H\,\square\,{}^HA$. By applying $(\mathrm{id}_A\otimes\mathrm{id}_A\otimes m_A)\circ(\mathrm{id}_A\otimes\tau\otimes\mathrm{id}_A)$ on $a_{(0)}\otimes a_{(1)}\otimes\tilde a=a\otimes S^{-1}(\tilde a_{(1)})\otimes\tilde a_{(0)}$, we get

$$a_{(0)}\otimes {a_{(1)}}^{<1>}\otimes_B {a_{(1)}}^{<2>}\tilde a=a\otimes S^{-1}(\tilde a_{(1)})^{<1>}\otimes_B S^{-1}(\tilde a_{(1)})^{<2>}\tilde a_{(0)}. \tag{12.6}$$

Now, using (7.7) in the second step, we have

$$\begin{aligned}\chi(S^{-1}(\tilde a_{(1)})^{<1>}\otimes_B S^{-1}(\tilde a_{(1)})^{<2>}\tilde a_{(0)}) &= S^{-1}(\tilde a_{(1)})^{<1>}S^{-1}(\tilde a_{(1)})^{<2>}{}_{(0)}\tilde a_{(0)(0)}\otimes S^{-1}(\tilde a_{(1)})^{<2>}{}_{(1)}\tilde a_{(0)(1)}\\ &= \tilde a_{(0)}\otimes S^{-1}(\tilde a_{(2)})\tilde a_{(1)}=\tilde a\otimes 1=\chi(\tilde a\otimes_B 1_A),\end{aligned}$$

From this $S^{-1}(\tilde a_{(1)})^{<1>}\otimes_B S^{-1}(\tilde a_{(1)})^{<2>}\tilde a_{(0)}=\tilde a\otimes_B 1$ which, when substituting in the right hand side of (12.6) yields $a_{(0)}\otimes {a_{(1)}}^{<1>}\otimes_B {a_{(1)}}^{<2>}\tilde a=a\otimes\tilde a\otimes_B 1_A$. Thus $a\otimes\tilde a\in\mathcal{C}$. ☐

Finally the coproduct (12.3) translates to the coproduct on $A^H\,\square\,{}^HA$ written as,

$$\Delta(a\otimes\tilde a)=a\otimes\tau(S^{-1}(\tilde a_{(1)}))\otimes\tilde a_{(0)}, \tag{12.7}$$

The Ehresmann coring of a Hopf–Galois extension is in fact a bialgebroid, called the *Ehresmann–Schauenburg bialgebroid* (cf. [6, 34.14]). One see that $\mathcal{C}(A,H)=(A\otimes A)^{coH}$ is a subalgebra of $A\otimes A^{op}$; indeed, given $a\otimes\tilde a, a'\otimes\tilde a'\in(A\otimes A)^{coH}$, one computes $\delta^{A\otimes A}(aa'\otimes\tilde a'\tilde a)=a_{(0)}a'_{(0)}\otimes\tilde a'_{(0)}\tilde a_{(0)}\otimes a_{(1)}a'_{(1)}\tilde a'_{(1)}\tilde a_{(1)}=a_{(0)}a'\otimes\tilde a'\tilde a_{(0)}\otimes a_{(1)}\tilde a_{(1)}=aa'\otimes\tilde a'\tilde a\otimes 1_H$.

Definition 12.4. Let $\mathcal{C}(A,H)$ be the coring of a Hopf–Galois extension $B=A^{co\,H}\subseteq A$, with A faithful flat as a left B-module. Then $\mathcal{C}(A,H)$ is a (left) B-bialgebroid with product

$$(a\otimes\tilde a)\bullet_{\mathcal{C}(A,H)}(a'\otimes\tilde a')=aa'\otimes\tilde a'\tilde a, \tag{12.8}$$

for all $a\otimes\tilde a$, $a'\otimes\tilde a'\in\mathcal{C}(A,H)$ (and unit $1\otimes 1\in A\otimes A$). The target and the source maps are given by

$$t(b)=1\otimes b,\quad\text{ and }\quad s(b)=b\otimes 1. \tag{12.9}$$

We refer to [6, 34.14] for the checking that all defining properties are satisfied.

12.2. **The groups of bisections.** The bialgebroid of a Hopf–Galois extension can be view as a quantization (of the dualization) of the classical gauge groupoid (see Example 3.2), of a (classical) principal bundle. Dually to the notion of a bisection on the classical gauge groupoid there is the notion of a bisection on the Ehresmann–Schauenburg bialgebroid. And in particular there are vertical bisections. These bisections correspond to automorphisms and vertical automorphisms (gauge transformations) respectively.

Definition 12.5. Let $\mathcal{C}(A,H)$ be the left Ehresmann–Schauenburg bialgebroid associate to a Hopf–Galois extension $B=A^{coH}\subseteq A$. A bisection of $\mathcal{C}(A,H)$ is a unital algebra map $\sigma:\mathcal{C}(A,H)\to B$, such that $\sigma\circ t=\mathrm{id}_B$ and $\sigma\circ s\in\mathrm{Aut}(B)$.

In general the collections of all bisections do not have additional structure. As a particular case that parallels Proposition 11.2 we have the following.

Proposition 12.6. *Consider the left Ehresmann–Schauenburg bialgebroid $\mathcal{C}(A,H)$ associate to a Hopf–Galois extension $B = A^{coH} \subseteq A$. If B belong to the centre of A, then the set of all bisections of $\mathcal{C}(A,H)$ is a group, denoted $\mathcal{B}(\mathcal{C}(A,H))$, with product defined by*

$$\begin{aligned}\sigma_1 * \sigma_2(a \otimes \tilde{a}) &:= (\sigma_2 \circ s)\big(\sigma_1(a_{(0)} \otimes a_{(1)}{}^{<1>})\big)\, \sigma_2(a_{(1)}{}^{<2>} \otimes \tilde{a}), \\ &= \sigma_2\big(\sigma_1(a_{(0)} \otimes a_{(1)}{}^{<1>})\, a_{(1)}{}^{<2>} \otimes \tilde{a}\big) \\ &= \sigma_2\big(\sigma_1((a \otimes \tilde{a})_{(1)})\, (a \otimes \tilde{a})_{(2)}\big) \end{aligned} \tag{12.10}$$

for any bisections σ_1, σ_2 and any element $a \otimes \tilde{a} \in \mathcal{C}(A,H)$. The unit of this group is the counit of the bialgebroid. And for any bisection σ, its inverse is given by

$$\sigma^{-1}(a \otimes \tilde{a}) = (\sigma \circ s)^{-1}\big(a\sigma(\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>})\, \tilde{a}_{(1)}{}^{<2>}\big). \tag{12.11}$$

Here $(\sigma \circ s)^{-1}$ is the inverse of $\sigma \circ s \in \mathrm{Aut}(B)$.

Proof. The second equality in (12.10) follows from the fact that bisections are taken to be algebra maps. The expressions on the right hand side of (12.10) and (12.11) are well defined. For any bisection σ and any $b \in B$, $a \in A$ the condition $\sigma \circ t = \mathrm{id}_B$ yields:

$$\sigma(a_{(0)} \otimes a_{(1)}{}^{<1>} b)\, a_{(1)}{}^{<2>} = \sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})\, b\, a_{(1)}{}^{<2>}\,. \tag{12.12}$$

As for the multiplication in (12.10): for bisections σ_1, σ_2 and any $b \in B$, we have

$$\sigma_1 * \sigma_2(s(b)) = \sigma_1 * \sigma_2(b \otimes 1) = \sigma_2(s(\sigma_1(b \otimes 1))) = (\sigma_2 \circ s) \circ (\sigma_1 \circ s)(b).$$

Being both $\sigma_1 \circ s$ and $\sigma_2 \circ s$ automorphisms of B, we have $(\sigma_1 * \sigma_2) \circ s \in \mathrm{Aut}(B)$. Similarly one shows that $\sigma_1 * \sigma_2(t(b)) = b$ for $b \in B$, that is $(\sigma_1 * \sigma_2) \circ t = \mathrm{id}_B$. Also, the multiplication is associative: let σ_1, σ_2, σ_3 be bisections, and let $a \otimes \tilde{a} \in \mathcal{C}(A,H)$. From

$$((\Delta \otimes_B \mathrm{id}_{\mathcal{C}(A,H)}) \circ \Delta)(a \otimes \tilde{a}) = a_{(0)} \otimes a_{(1)}{}^{<1>} \otimes_B a_{(1)}{}^{<2>} \otimes a_{(2)}{}^{<1>} \otimes_B a_{(2)}{}^{<2>} \otimes \tilde{a},$$

we have (using in the second step that $\sigma_3 \circ s$ is an algebra map):

$$\begin{aligned} &((\sigma_1 * \sigma_2) * \sigma_3)(a \otimes \tilde{a}) \\ &\quad= (\sigma_3 \circ s)\Big((\sigma_2 \circ s)\big(\sigma_1(a_{(0)} \otimes a_{(1)}{}^{<1>})\big)\, \sigma_2(a_{(1)}{}^{<2>} \otimes a_{(2)}{}^{<1>})\Big)\, \sigma_3(a_{(2)}{}^{<2>} \otimes \tilde{a}) \\ &\quad= (\sigma_3 \circ s)\Big((\sigma_2 \circ s)\big(\sigma_1(a_{(0)} \otimes a_{(1)}{}^{<1>})\big)\Big)\, (\sigma_3 \circ s)\big(\sigma_2(a_{(1)}{}^{<2>} \otimes a_{(2)}{}^{<1>})\big)\, \sigma_3(a_{(2)}{}^{<2>} \otimes \tilde{a}) \\ &\quad= ((\sigma_2 * \sigma_3) \circ s)\big(\sigma_1(a_{(0)} \otimes a_{(1)}{}^{<1>})\big)\, (\sigma_2 * \sigma_3)(a_{(1)}{}^{<2>} \otimes \tilde{a}) \\ &\quad= (\sigma_1 * (\sigma_2 * \sigma_3))(a \otimes \tilde{a})\,. \end{aligned}$$

The assumption that B belongs to the centre of A implies that the product $\sigma_1 * \sigma_2$ is an algebra map. Indeed, for $a \otimes \tilde{a}$ and $a' \otimes \tilde{a}' \in \mathcal{C}$ one has,

$$\begin{aligned} &\sigma_1 * \sigma_2(aa' \otimes \tilde{a}'\tilde{a}) \\ &= (\sigma_2 \circ t)(\sigma_1((aa)'_{(0)} \otimes (aa')_{(1)}{}^{<1>}))\, (\sigma_2((aa')_{(1)}{}^{<2>} \otimes \tilde{a}'\tilde{a})) \\ &= (\sigma_2 \circ t)(\sigma_1(a_{(0)}a'_{(0)} \otimes a'_{(1)}{}^{<1>} a_{(1)}{}^{<1>}))\, (\sigma_2(a_{(1)}{}^{<2>} a'_{(1)}{}^{<2>} \otimes \tilde{a}'\tilde{a})) \\ &= (\sigma_2 \circ t)(\sigma_1(a_{(0)} \otimes a_{(1)}{}^{<1>}))\, (\sigma_2 \circ t)(\sigma_1(a'_{(0)} \otimes a'_{(1)}{}^{<1>}))\, (\sigma_2(a'_{(1)}{}^{<2>} \otimes \tilde{a}'))(\sigma_2(a_{(1)}{}^{<2>} \otimes \tilde{a})) \\ &= (\sigma_1 * \sigma_2)(a \otimes \tilde{a})\, (\sigma_1 * \sigma_2)(a' \otimes \tilde{a}') \end{aligned}$$

The 2nd step uses (7.9), the 3rd step uses the fact that σ_1 and σ_2 are both algebra maps, the last step uses that B belongs to the centre.

Thus $\sigma_1 * \sigma_2$ is a well defined algebra map. Next, we check ϵ is the unit of this multiplication. Firstly, since B is taken to belong to the centre of A the counit ϵ is an algebra map. Indeed, for any $a \otimes \tilde{a} \in \mathcal{C}(A, H)$,

$$\epsilon(aa' \otimes \tilde{a}'\tilde{a}) = aa'\tilde{a}'\tilde{a} = a'\tilde{a}'a\tilde{a} = \epsilon(a \otimes \tilde{a})\epsilon(a' \otimes \tilde{a}'),$$

the 2nd step using that B belongs to the centre. Then,

$$\begin{aligned}\sigma * \epsilon\,(a \otimes \tilde{a}) &= (\epsilon \circ t)\big(\sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})\big)\,\epsilon(a_{(1)}{}^{<2>} \otimes \tilde{a}) = (\epsilon \circ t)\big(\sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})\big)\,a_{(1)}{}^{<2>}\tilde{a} \\ &= (\epsilon \circ t)\big(\sigma(a \otimes \tilde{a})\big) \\ &= \sigma(a \otimes \tilde{a}),\end{aligned}$$

where the 3rd step uses the definition of $\mathcal{C}$. Similarly, for any $a \otimes \tilde{a} \in \mathcal{C}(A, H)$:

$$\begin{aligned}\epsilon * \sigma(a \otimes \tilde{a}) &= (\sigma \circ t)\big(\epsilon(a_{(0)} \otimes a_{(1)}{}^{<1>})\big)\sigma(a_{(1)}{}^{<2>} \otimes \tilde{a}) \\ &= (\sigma \circ t)\big(a_{(0)}a_{(1)}{}^{<1>}\big)\,\sigma(a_{(1)}{}^{<2>} \otimes \tilde{a}) \\ &= \sigma(a \otimes \tilde{a}),\end{aligned}$$

and for the last equality we use $a_{(0)}a_{(1)}{}^{<1>} \otimes_B a_{(1)}{}^{<2>} = 1 \otimes_B a$. Thus ϵ is the unit.

Next, let us check that the inverse of a bisection σ as given in (12.11), is well defined. The quantity $a\sigma(\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>})\tilde{a}_{(1)}{}^{<2>}$, the argument of $(\sigma \circ s)^{-1}$ in (12.11), belongs to B. Indeed, with δ^A the coaction as in 7.1, one has

$$\begin{aligned}\delta^A(a\sigma(\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>})\tilde{a}_{(1)}{}^{<2>}) &= a_{(0)}\sigma(\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>})(\tilde{a}_{(1)}{}^{<2>})_{(0)} \otimes a_{(1)}(\tilde{a}_{(1)}{}^{<2>})_{(1)} \\ &= a_{(0)}\sigma(\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>})\tilde{a}_{(1)}{}^{<2>} \otimes a_{(1)}\tilde{a}_{(2)} \\ &= a_{(0)}\sigma(\tilde{a}_{(0)(0)} \otimes \tilde{a}_{(0)(1)}{}^{<1>})\tilde{a}_{(0)(1)}{}^{<2>} \otimes a_{(1)}\tilde{a}_{(1)} \\ &= a\sigma(\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>})\tilde{a}_{(1)}{}^{<2>} \otimes 1_H,\end{aligned}$$

where the 1st step uses that σ is valued in B, the 2nd use (7.5), the last one uses (12.1).

And for any $b \in B$, $\sigma^{-1}(s(b)) = (\sigma \circ s)^{-1}(b)$, so $\sigma^{-1} \circ s = (\sigma \circ s)^{-1} \in \mathrm{Aut}(B)$; also $\sigma^{-1}(t(b)) = (\sigma \circ s)^{-1}(\sigma(b \otimes 1)) = (\sigma \circ s)^{-1}((\sigma \circ s)(b))) = b$, so $\sigma^{-1} \circ t = \mathrm{id}_B$.
Next, let us show σ^{-1} is indeed the inverse of σ. For $a \otimes \tilde{a} \in \mathcal{C}$, we have

$$\begin{aligned}&(\sigma^{-1} * \sigma)(a \otimes \tilde{a}) \\ &\quad= (\sigma \circ s)(\sigma^{-1}(a_{(0)} \otimes a_{(1)}{}^{<1>}))\,\sigma(a_{(1)}{}^{<2>} \otimes \tilde{a}) \\ &\quad= (\sigma \circ s)\Big((\sigma \circ s)^{-1}\big(a_{(0)}\sigma(a_{(1)}{}^{<1>}{}_{(0)} \otimes a_{(1)}{}^{<1>}{}_{(1)}{}^{<1>})a_{(1)}{}^{<1>}{}_{(1)}{}^{<2>}\big)\Big)\,\sigma(a_{(1)}{}^{<2>} \otimes \tilde{a}) \\ &\quad= a_{(0)}\sigma(a_{(1)}{}^{<1>}{}_{(0)} \otimes a_{(1)}{}^{<1>}{}_{(1)}{}^{<1>})a_{(1)}{}^{<1>}{}_{(1)}{}^{<2>}\,\sigma(a_{(1)}{}^{<2>} \otimes \tilde{a}) \\ &\quad= a_{(0)}\sigma\big(a_{(2)}{}^{<1>} \otimes S(a_{(1)})^{<1>})S(a_{(1)})^{<2>}\,\sigma(a_{(2)}{}^{<2>} \otimes \tilde{a}) \\ &\quad= a_{(0)}\sigma\big(a_{(2)}{}^{<1>}a_{(2)}{}^{<2>} \otimes \tilde{a}S(a_{(1)})^{<1>}\big)\,S(a_{(1)})^{<2>} \\ &\quad= a_{(0)}\tilde{a}\,S(a_{(1)})^{<1>}S(a_{(1)})^{<2>} \\ &\quad= a\tilde{a} \\ &\quad= \epsilon(a \otimes \tilde{a}),\end{aligned}$$

where the 4th step uses (7.6), the 5th step uses that B belongs to the centre of A, the 6th and 7th steps use (7.8). On the other hand,

$$\begin{aligned}
(\sigma * \sigma^{-1})(a \otimes \tilde{a}) &= (\sigma^{-1} \circ s)\big(\sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})\big)\, \sigma^{-1}(a_{(1)}{}^{<2>} \otimes \tilde{a}) \\
&= (\sigma \circ s)^{-1}\big(\sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})\big)\, (\sigma \circ s)^{-1}(a_{(1)}{}^{<2>} \sigma(\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>})\tilde{a}_{(1)}{}^{<2>}) \\
&= (\sigma \circ s)^{-1}\big(\sigma(a_{(0)}\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>} a_{(1)}{}^{<1>}\big)\, a_{(1)}{}^{<2>}\tilde{a}_{(1)}{}^{<2>}) \\
&= (\sigma \circ s)^{-1}\big(\sigma((a\tilde{a})_{(0)} \otimes (a\tilde{a})_{(1)}{}^{<1>}\big)\, (a\tilde{a})_{(1)}{}^{<2>}) \\
&= (\sigma \circ s)^{-1}(\sigma(a\tilde{a} \otimes 1)) \\
&= (\sigma \circ s)^{-1}\big((\sigma \circ s)(a\tilde{a})\big) \\
&= a\tilde{a} \\
&= \epsilon(a \otimes \tilde{a}),
\end{aligned}$$

where the second step uses $\sigma^{-1}(s(b)) = (\sigma \circ s)^{-1}(b)$, the 3rd step uses that B belongs to the centre of A, the 4th step uses (7.9), and the 5th step uses that $a\tilde{a} \in B$.

Finally, the map σ^{-1} is an algebra map:

$$\begin{aligned}
\sigma^{-1}(aa' \otimes \tilde{a}'\tilde{a}) &= (\sigma \circ s)^{-1}\Big(aa'\sigma\big((\tilde{a}'\tilde{a})\big)_{(0)} \otimes (\tilde{a}'\tilde{a})_{(1)}{}^{<1>}\Big)(\tilde{a}'\tilde{a})_{(1)}{}^{<2>} \\
&= (\sigma \circ s)^{-1}\big(aa'\sigma(\tilde{a}'{}_{(0)}\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>}\tilde{a}'_{(1)}{}^{<1>})\tilde{a}'_{(1)}{}^{<2>}\tilde{a}_{(1)}{}^{<2>}\big) \\
&= (\sigma \circ s)^{-1}\big(aa'\sigma(\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>})\sigma(\tilde{a}'_{(0)} \otimes \tilde{a}'_{(1)}{}^{<1>})\tilde{a}'_{(1)}{}^{<2>}\tilde{a}_{(1)}{}^{<2>}\big) \\
&= (\sigma \circ s)^{-1}\big(aa'\sigma(\tilde{a}'{}_{(0)} \otimes \tilde{a}'_{(1)}{}^{<1>})\tilde{a}'_{(1)}{}^{<2>}\sigma(\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>})\tilde{a}_{(1)}{}^{<2>}\big) \\
&= (\sigma \circ s)^{-1}\big(a'\sigma(\tilde{a}'{}_{(0)} \otimes \tilde{a}'_{(1)}{}^{<1>})\tilde{a}'_{(1)}{}^{<2>}a\sigma(\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>})\tilde{a}_{(1)}{}^{<2>}\big) \\
&= (\sigma \circ s)^{-1}\big(a'\sigma(\tilde{a}'{}_{(0)} \otimes \tilde{a}'_{(1)}{}^{<1>})\tilde{a}'_{(1)}{}^{<2>}\big)(\sigma \circ s)^{-1}\big(a\sigma(\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>})\tilde{a}_{(1)}{}^{<2>}\big) \\
&= \sigma^{-1}(a \otimes \tilde{a})\, \sigma^{-1}(a' \otimes \tilde{a}');
\end{aligned}$$

the second step uses (7.9), the 3rd step uses σ is an algebra map, the 5th one uses that the image of σ and $a'\sigma(\tilde{a}'{}_{(0)} \otimes \tilde{a}'{}_{(1)}{}^{<1>})\tilde{a}'{}_{(1)}{}^{<2>}$ are in B, which is in the centre of A. □

Remark 12.7. Having asked that bisections are algebra maps, they are B-linear in the sense of the coring bimodule structure in (6.6). That is, for any bisection σ and $b \in B$,

$$\sigma\big((a \otimes \tilde{a}) \triangleleft b\big) = \sigma\big(t(b) \bullet_{\mathcal{C}} a \otimes \tilde{a}\big) = \sigma(a \otimes \tilde{a})\, \sigma(t(b)) = \sigma(a \otimes \tilde{a})\, b$$

and

$$\sigma\big(b \triangleright (a \otimes \tilde{a})\big) = \sigma\big(s(b) \bullet_{\mathcal{C}} a \otimes \tilde{a}\big) = \sigma(a \otimes \tilde{a})\, \sigma(s(b)) = \sigma(a \otimes \tilde{a})\, (\sigma \circ s)(b)\,.$$

Among all bisections an important role is played by the vertical ones.

Definition 12.8. Let $\mathcal{C}(A, H)$ be the left Ehresmann–Schauenburg bialgebroid associate to a Hopf–Galois extension $B = A^{co\,H} \subseteq A$. A *vertical bisection* is a bisection of $\mathcal{C}$ which is also a left inverse for the target map s, that is $\sigma \circ s = \mathrm{id}_B$.

Then the following statement is immediate.

Corollary 12.9. *Consider the left Ehresmann–Schauenburg bialgebroid $\mathcal{C}(A, H)$ associate to a Hopf–Galois extension $B = A^{co\,H} \subseteq A$. If B belong to the centre of A, then the set $\mathcal{B}_{ver}(\mathcal{C}(A, H))$ of all vertical bisections of $\mathcal{C}$ is a group, a subgroup of the group of all bisections $\mathcal{B}(\mathcal{C}(A, H))$, with the restricted product given by*

$$\sigma_1 * \sigma_2(a \otimes \tilde{a}) := \sigma_1(a_{(0)} \otimes a_{(1)}{}^{<1>})\, \sigma_2(a_{(1)}{}^{<2>} \otimes \tilde{a}) \tag{12.13}$$

for any vertical bisections σ_1, σ_2. Moreover, the inverse of a vertical bisection is given by

$$\sigma^{-1}(a\otimes\tilde a) = a\sigma(\tilde a_{(0)}\otimes\tilde a_{(1)}{}^{<1>})\,\tilde a_{(1)}{}^{<2>} = \sigma(\tilde a_{(0)}\otimes\tilde a_{(1)}{}^{<1>})\,a\,\tilde a_{(1)}{}^{<2>}\,. \tag{12.14}$$

Proof. The right hand side of both (12.13) and (12.14) is seen to be a vertical bisection. □

Remark 12.10. We notice that the product (12.13) on vertical bisections is just the convolution product due to the second expression for the coproduct in (12.3),

$$\begin{aligned}\sigma_1 * \sigma_2(a\otimes\tilde a) &= (\sigma_1\otimes_B\sigma_2)\circ\Delta(a\otimes\tilde a)\\ &= \sigma_1(a_{(0)}\otimes a_{(1)}{}^{<1>})\,\sigma_2(a_{(1)}{}^{<2>}\otimes\tilde a).\end{aligned} \tag{12.15}$$

We show directly that the product in (12.15) is well defined, Indeed, for $b\in B$ we have

$$\begin{aligned}\sigma_1(a_{(0)}\otimes a_{(1)}{}^{<1>}\,b)\,\sigma_2(a_{(1)}{}^{<2>}\otimes\tilde a) &= \sigma_1\big((a_{(0)}\otimes a_{(1)}{}^{<1>})\bullet_{\mathcal{C}}(1\otimes b)\big)\,\sigma_2(a_{(1)}{}^{<2>}\otimes\tilde a)\\ &= \sigma_1(a_{(0)}\otimes a_{(1)}{}^{<1>})\,\sigma_1(1\otimes b)\,\sigma_2(a_{(1)}{}^{<2>}\otimes\tilde a)\\ &= \sigma_1(a_{(0)}\otimes a_{(1)}{}^{<1>})\,b\,\sigma_2(a_{(1)}{}^{<2>}\otimes\tilde a)\\ &= \sigma_1(a_{(0)}\otimes a_{(1)}{}^{<1>})\,\sigma_2(b\otimes 1)\,\sigma_2(a_{(1)}{}^{<2>}\otimes\tilde a)\\ &= \sigma_1(a_{(0)}\otimes a_{(1)}{}^{<1>})\,\sigma_2(b\,a_{(1)}{}^{<2>}\otimes\tilde a)\end{aligned}$$

with the 4th step coming from σ_2 being vertical.

12.3. **Bisections and gauge groups.** Recall the Definition 11.1 and the Proposition 11.2 concerning the gauge group of a Hopf–Galois extension. We have the following results.

Proposition 12.11. *Let $B = A^{coH}\subseteq A$ be a Hopf–Galois extension, and let $\mathcal{C}(A,H)$ be the corresponding left Ehresmann–Schauenburg bialgebroid. If B is in the centre of A, then there is a group isomorphism $\alpha : \mathrm{Aut}_H(A)\to\mathcal{B}(\mathcal{C}(A,H))$. The isomorphism α restricts to an isomorphism between vertical subgroups $\alpha : \mathrm{Aut}_{ver}(A)\to\mathcal{B}_{ver}(\mathcal{C}(A,H))$.*

Proof. Let $F\in\mathrm{Aut}_H(A)$ and define $\sigma_F\in\mathcal{B}(\mathcal{C}(A,H))$ by

$$\sigma_F(a\otimes\tilde a) := F(a)\tilde a, \tag{12.16}$$

for any $a\otimes\tilde a\in\mathcal{C}(A,H)$. This is well defined since

$$\begin{aligned}\delta^A(F(a)\tilde a) &= (F(a)\tilde a)_{(0)}\otimes(F(a)\tilde a)_{(1)}\\ &= F(a)_{(0)}\tilde a_{(0)}\otimes F(a)_{(1)}\tilde a_{(1)} = F(a_{(0)})\tilde a_{(0)}\otimes a_{(1)}\tilde a_{(1)}\\ &= F(a)\tilde a\otimes 1_H,\end{aligned}$$

where the last equality use (12.1), thus $F(a)\tilde a\in B$. And σ_F is an algebra map, since

$$\begin{aligned}\sigma_F((a'\otimes\tilde a')\bullet_{\mathcal{C}(A,H)}(a\otimes\tilde a)) &= \sigma_F(a'a\otimes\tilde a\tilde a') = F(a'a)\tilde a\tilde a'\\ &= F(a')F(a)\tilde a\tilde a' = F(a')(F(a)\tilde a)\tilde a' = F(a')\tilde a'\sigma_F(a\otimes\tilde a)\\ &= \sigma_F(a'\otimes\tilde a')\sigma_F(a\otimes\tilde a),\end{aligned}$$

where the 5th equality uses that B is in the centre of A. It is clear that $\sigma_F\circ t = \mathrm{id}_B$ and $\sigma_F\circ s = F|_B\in\mathrm{Aut}(B)$. Thus σ_F is a well defined bisection. By the definition (12.16),

$$\sigma_{\mathrm{id}_A}(a\otimes\tilde a) = a\tilde a = \epsilon(a\otimes\tilde a)$$

and for any $a \otimes \tilde{a} \in \mathcal{C}(A, H)$ we have

$$\begin{aligned}\sigma_G * \sigma_F(a \otimes \tilde{a}) &= (\sigma_F \circ s)\big(\sigma_G(a_{(0)} \otimes a_{(1)}{}^{<1>})\big)\, \sigma_F(a_{(1)}{}^{<2>} \otimes \tilde{a}) \\ &= \sigma_F\big(G(a_{(0)})a_{(1)}{}^{<1>} \otimes 1)\big)\, \sigma_F(a_{(1)}{}^{<2>} \otimes \tilde{a}) \\ &= F(G(a_{(0)})a_{(1)}{}^{<1>})F(a_{(1)}{}^{<2>})\tilde{a} \\ &= F\big(G(a_{(0)})a_{(1)}{}^{<1>}a_{(1)}{}^{<2>}\big)\, \tilde{a} \\ &= F(G(a))\, \tilde{a} \\ &= \sigma_{(G\cdot F)}(a \otimes \tilde{a}),\end{aligned}$$

where the 5th step uses (7.8).

Conversely, given a bisection σ, one can define an algebra map $F_\sigma : A \to A$ by

$$F_\sigma(a) := \sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})a_{(1)}{}^{<2>}\, . \tag{12.17}$$

We have already seen (cf. (12.12)) that the right hand side of (12.17) is well defined. Clearly $F_\sigma(b) = (\sigma \circ s)(b)$ for any $b \in B$, so $F_\sigma|_B \in \mathrm{Aut}(B)$. Moreover,

$$\begin{aligned}F_\sigma(aa') &= \sigma((aa')_{(0)} \otimes (aa')_{(1)}{}^{<1>})(aa')_{(1)}{}^{<2>} = \sigma(a_{(0)}a'_{(0)} \otimes (a_{(1)}a'_{(1)})^{<1>})(a_{(1)}a'_{(1)})^{<2>} \\ &= \sigma(a_{(0)}a'_{(0)} \otimes a'_{(1)}{}^{<1>}a_{(1)}{}^{<1>})a_{(1)}{}^{<2>}a'_{(1)}{}^{<2>} \\ &= \sigma(a'_{(0)} \otimes a'_{(1)}{}^{<1>})\sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})a_{(1)}{}^{<2>}a'_{(1)}{}^{<2>} \\ &= F_\sigma(a)F_\sigma(a'),\end{aligned}$$

where in the third step we use (7.9). Also, F_σ is H-equivalent:

$$\begin{aligned}F_\sigma(a)_{(0)} \otimes F_\sigma(a)_{(1)} &= \sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})a_{(1)}{}^{<2>}{}_{(0)} \otimes a_{(1)}{}^{<2>}{}_{(1)} \\ &= \sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})a_{(1)}{}^{<2>} \otimes a_{(2)} \\ &= F_\sigma(a_{(0)}) \otimes a_{(1)},\end{aligned}$$

where the 2nd step uses (7.5), thus $F_\sigma \in \mathrm{Aut}_H(A)$.

The map α is an isomorphism. Indeed for any $a \otimes \tilde{a} \in \mathcal{C}(A, H)$ and any $\sigma \in \mathcal{B}(\mathcal{C}(A, H))$:

$$\sigma_{F_\sigma}(a \otimes \tilde{a}) = F_\sigma(a)\tilde{a} = \sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})a_{(1)}{}^{<2>}\tilde{a} = \sigma(a \otimes \tilde{a}),$$

where the last step uses (12.2). On the other hand, for any $a \in A$ and any $F \in \mathrm{Aut}_H(A)$:

$$F_{\sigma_F}(a) = \sigma_F(a_{(0)} \otimes a_{(1)}{}^{<1>})a_{(1)}{}^{<2>} = F(a_{(0)})a_{(1)}{}^{<1>}a_{(1)}{}^{<2>} = F(a).$$

Finally, for a vertical automorphism $F \in \mathrm{Aut}_{ver}(A)$, it is clear that the corresponding $\sigma_F \in \mathcal{B}_{ver}(\mathcal{C}(A, H))$, and conversely from $\sigma \in \mathcal{B}_{ver}(\mathcal{C}(A, H))$ we have $F_\sigma \in \mathrm{Aut}_{ver}(A)$. □

12.4. **Extended bisections and gauge groups.** We have already mentioned that gauge transformations for a noncommutative principal bundles could be defined without asking them to be algebra homomorphisms [5]. Mainly for the sake of completeness we record here a version of them via bialgebroid and bisections. To distinguish them from the analogous concepts introduced in the previous section, and for lack of a better name, we call the extended gauge transformation and extended bisections.

In the same vein of [5] we have the following definition.

Definition 12.12. Given a Hopf–Galois extension $B = A^{coH} \subseteq A$. Its *extended gauge group* $\mathrm{Aut}^{ext}_H(A)$ consists of invertible H-comodule unital maps $F : A \to A$ such that their restrictions $F|_B \in \mathrm{Aut}(B)$ and such that $F(ba) = F(b)F(a)$ for any $b \in B$ and $a \in A$. The *extended vertical gauge group* $\mathrm{Aut}^{ext}_{ver}(A)$ is made of elements $F \in \mathrm{Aut}^{ext}_H(A)$ whose restrictions $F|_B = \mathrm{id}_B$. The group structure is map composition.

In parallel with this we have then the following.

Definition 12.13. Let $\mathcal{C}(A,H)$ be the left Ehresmann–Schauenburg bialgebroid of the Hopf–Galois extension $B = A^{co\,H} \subseteq A$. An *extended bisection* is an unital convolution invertible (in the sense of (12.10)) map $\sigma : \mathcal{C}(A,H) \to B$, such that $\sigma \circ t = \mathrm{id}_B$ and $\sigma \circ s \in \mathrm{Aut}(B)$, which in addition is B-linear in the sense of the B-coring structure on $\mathcal{C}$ (cf. Remark 12.7). That is, $\sigma\big((a \otimes \tilde{a}) \triangleleft b\big) = \sigma(a \otimes \tilde{a})\, b$, and $\sigma\big(b \triangleright (a \otimes \tilde{a})\big) = \sigma(a \otimes \tilde{a})\,(\sigma \circ s)(b)$. The set of all extended bisections which are invertible for the product (12.10):

$$(\sigma_1 * \sigma_2)(a \otimes \tilde{a}) := (\sigma_2 \circ s)\big(\sigma_1(a_{(0)} \otimes a_{(1)}{}^{<1>})\big)\, \sigma_2(a_{(1)}{}^{<2>} \otimes \tilde{a}). \tag{12.18}$$

will be denote by $\mathcal{B}^{ext}(\mathcal{C}(A,H))$, while $\mathcal{B}^{ext}_{ver}(\mathcal{C}(A,H))$ will denote those which are invertible and vertical, that is such that $\sigma \circ s = \mathrm{id}_B$ as well.

Lemma 12.14. *The product (12.18) is well defined.*

Proof. We need to check that $\sigma_1 * \sigma_2$ is B-linear in the sense of the definition. Now, for any $a \otimes a' \in \mathcal{C}$ and $b \in B$ we have

$$\begin{aligned}
(\sigma_1 * \sigma_2)((a \otimes \tilde{a}) \triangleleft b) &= (\sigma_1 * \sigma_2)((a \otimes \tilde{a}b) \\
&= (\sigma_2 \circ s)\big(\sigma_1(a_{(0)} \otimes a_{(1)}{}^{<1>})\big)\, (\sigma_2(a_{(1)}{}^{<2>} \otimes \tilde{a}b)) \\
&= (\sigma_2 \circ s)\big(\sigma_1(a_{(0)} \otimes a_{(1)}{}^{<1>})\big)\, \sigma_2\big((a_{(1)}{}^{<2>} \otimes \tilde{a}) \triangleleft b)\big) \\
&= (\sigma_1 * \sigma_2)(a \otimes a')\, b.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\sigma_1 * \sigma_2)(b \triangleright (a \otimes \tilde{a})) &= (\sigma_1 * \sigma_2)(ba \otimes \tilde{a}) \\
&= (\sigma_2 \circ s)\big(\sigma_1(ba_{(0)} \otimes a_{(1)}{}^{<1>})\big)\, (\sigma_2(a_{(1)}{}^{<2>} \otimes \tilde{a})) \\
&= (\sigma_2 \circ s)\big(\sigma_1(b \triangleright (a_{(0)} \otimes a_{(1)}{}^{<1>}))\big)\, (\sigma_2(a_{(1)}{}^{<2>} \otimes \tilde{a})) \\
&= (\sigma_2 \circ s)\big(\sigma_1(a_{(0)} \otimes a_{(1)}{}^{<1>})\, (\sigma_1 \circ s)(b)\big)\, (\sigma_2(a_{(1)}{}^{<2>} \otimes \tilde{a})) \\
&= (\sigma_2 \circ s)(\sigma_1 \circ s)(b)\, (\sigma_2 \circ s)\big(\sigma_1(a_{(0)} \otimes a_{(1)}{}^{<1>})\big)\, \sigma_2(a_{(1)}{}^{<2>} \otimes \tilde{a}) \\
&= (\sigma_1 * \sigma_2)(a \otimes \tilde{a})\, \big((\sigma_1 * \sigma_2) \circ s\big)(b).
\end{aligned}$$

Were the last step uses the identity $\big((\sigma_1 * \sigma_2) \circ s\big)(b) = (\sigma_2 \circ s)(\sigma_1 \circ s)(b)$, and the last but one one the fact that $\sigma \circ s \in \mathrm{Aut}(B)$ and that B is in the centre. ☐

Remark 12.15. We remark that (12.11) is now not the inverse for the product in (12.18) since, in contrast to Proposition 12.6 we are not asking the bisections be algebra maps.

Finally, in analogy with Proposition 12.11 we have the following.

Proposition 12.16. *Let $B = A^{coH} \subseteq A$ be a Hopf–Galois extension, and let $\mathcal{C}(A,H)$ be the corresponding left Ehresmann–Schauenburg bialgebroid. If B belongs to the centre of A, there is a group isomomorphism $\widehat{\alpha} : \mathrm{Aut}^{ext}_H(A) \to \mathcal{B}^{ext}(\mathcal{C}(A,H))$. The isomorphism restricts to an isomorphism $\widehat{\alpha}_v : \mathrm{Aut}^{ext}_{ver}(A) \to \mathcal{B}^{ext}_{ver}(\mathcal{C}(A,H))$ between vertical subgroups.*

Proof. This uses the same methods as Proposition 12.11. Given $F \in \mathrm{Aut}^{ext}_H(A)$, define its image as in (12.16): $\sigma_F(a \otimes \tilde{a}) = F(a)\tilde{a}$. Being B in the centre, for all $b \in B$ we have,

$$\begin{aligned}
\sigma_F(a \otimes \tilde{a}b) &= F(a)\tilde{a}\, b = \sigma_F(a \otimes \tilde{a})\, b\,, \\
\sigma_F(ba \otimes \tilde{a}) &= F(ba)\tilde{a} = F(b)F(a)\tilde{a} = \sigma_F(a \otimes \tilde{a})\, (\sigma_F \circ s)(b),
\end{aligned}$$

that is, $\sigma_F\big((a \otimes \tilde{a}) \triangleleft b\big) = \sigma_F(a \otimes \tilde{a})\, b$, and $\sigma_F\big(b \triangleright (a \otimes \tilde{a})\big) = \sigma_F(a \otimes \tilde{a})\,(\sigma_F \circ s)(b)$. Conversely, for $\sigma \in \mathcal{B}^{ext}(\mathcal{C}(A,H))$, define its image as in (12.17): $F_\sigma(a) = \sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})a_{(1)}{}^{<2>}$. Then $F_\sigma(ba) = \sigma(ba_{(0)} \otimes a_{(1)}{}^{<1>})a_{(1)}{}^{<2>} = (\sigma \circ s)(b)F_\sigma(a) = F_\sigma(b)F_\sigma(a)$, due to B in the centre of A. The rest of the proof goes as that of Proposition 12.11. (minus the algebra map parts). ☐

13. Bisections and gauge groups of Galois objects

From now on we shall concentrate on *Galois objects* of a Hopf algebra H. These are noncommutative principal bundles over a point. In contrast to the classical result that any fibre bundle over a point is trivial, the set $\mathrm{Gal}_H(\mathbb{C})$ of isomorphic classes of H-Galois objects need not be trivial (cf. [4], [19]). We shall illustrate later on this non-triviality with examples coming from group algebras and Taft algebras.

13.1. Galois objects.

Definition 13.1. Let H be a Hopf algebra, a H-*Galois object* of H is an H-Hopf–Galois extension A of the ground field $\mathbb{C}$.

Thus for a Galois object the coinvariant subalgebra is the ground field $\mathbb{C}$. Now if A is faithfully flat over $\mathbb{C}$, then bijectivity of the canonical Galois map implies $\mathbb{C} = A^{co\,H}$ (cf. [36, Lem. 1.11]) and H is faithfully flat over $\mathbb{C}$ since $A \otimes A$ is faithfully flat over A. Recall from Section 5 that an (A, H)-relative Hopf module M is a right H-comodule with a compatible right A-module structure. That is the action is a morphism of H-comodules such that $\delta^M(ma) = m_{(0)}a_{(0)} \otimes m_{(1)}a_{(1)}$ for all $a \in A$, $m \in M$. We have the following [31]:

Lemma 13.2. *Let M be an (A, H)-relative Hopf module. If A is faithfully flat over $\mathbb{C}$, the multiplication induces an isomorphism*

$$M^{co\,H} \otimes A \to M,$$

whose inverse is $M \ni m \mapsto m_{(0)}{m_{(1)}}^{<1>} \otimes {m_{(1)}}^{<2>} \in M^{co\,H} \otimes A$.

With coaction $\delta^A : A \to A \otimes H$, $\delta^A(a) = a_{(0)} \otimes a_{(1)}$, and translation map $\tau : H \to A \otimes A$, $\tau(h) = h^{<1>} \otimes h^{<2>}$, for the Ehresmann–Schauenburg bialgebroid of a Galois object, being $B = \mathbb{C}$ one has (see also [31, Def. 3.1]):

$$\mathcal{C}(A, H) = \{a \otimes \tilde{a} \in A \otimes A \ : \ a_{(0)} \otimes \tilde{a}_{(0)} \otimes a_{(1)}\tilde{a}_{(1)} = a \otimes \tilde{a} \otimes 1_H\} \tag{13.1}$$

$$= \{a \otimes \tilde{a} \in A \otimes A : \ a_{(0)} \otimes {a_{(1)}}^{<1>} \otimes {a_{(1)}}^{<2>}\tilde{a} = a \otimes \tilde{a} \otimes 1_A\}. \tag{13.2}$$

The coproduct (12.3) and counit (12.4) become $\Delta_{\mathcal{C}}(a \otimes \tilde{a}) = a_{(0)} \otimes {a_{(1)}}^{<1>} \otimes {a_{(1)}}^{<2>} \otimes \tilde{a}$, and $\epsilon_{\mathcal{C}}(a \otimes \tilde{a}) = a\tilde{a} \in \mathbb{C}$ respectively, for any $a \otimes \tilde{a} \in \mathcal{C}(A, H)$. But now there is also an antipode [31, Thm. 3.5] given, for any $a \otimes \tilde{a} \in \mathcal{C}(A, H)$, by

$$S_{\mathcal{C}}(a \otimes \tilde{a}) := \tilde{a}_{(0)} \otimes {\tilde{a}_{(1)}}^{<1>} a {\tilde{a}_{(1)}}^{<2>}. \tag{13.3}$$

Thus the Ehresmann–Schauenburg bialgebroid of a Galois object is a Hopf algebra.

Now, given that $\mathcal{C}(A, H) = (A \otimes A)^{coH}$, Lemma 13.2 yields an isomorphism

$$A \otimes A \simeq \mathcal{C}(A, H) \otimes A, \qquad \widetilde{\chi}(a \otimes \tilde{a}) = a_{(0)} \otimes {a_{(1)}}^{<1>} \otimes {a_{(1)}}^{<2>}\tilde{a}. \tag{13.4}$$

We finally collect some results of [31] (cf. Lemma 3.2 and Lemma 3.3) in the following:

Lemma 13.3. *Let H be a Hopf algebra, and A a (faithfully flat) H-Galois object of H. There is a right H-equivariant algebra map $\delta^{\mathcal{C}} : A \to \mathcal{C}(A, H) \otimes A$ given by*

$$\delta^{\mathcal{C}}(a) = a_{(0)} \otimes {a_{(1)}}^{<1>} \otimes {a_{(1)}}^{<2>}$$

which is universal in the following sense: Given an algebra M and a H-equivariant algebra map $\phi : A \to M \otimes A$, there is a unique algebra map $\Phi : \mathcal{C}(A, H) \to M$ such that $\phi = (\Phi \otimes \mathrm{id}_A) \circ \delta^{\mathcal{C}}$. Explicitly, $\Phi(a \otimes \tilde{a}) \otimes 1_A = \phi(a)\tilde{a}$.

The ground field $\mathbb{C}$ being undoubtedly in the centre, for the bisections of the Ehresmann–Schauenburg bialgebroid $\mathcal{C}(A,H)$ of a Galois object A, we can use all results of previous sections. Clearly, any bisection of $\mathcal{C}(A,H)$ is now vertical as it is vertical any automorphism of the principal bundle A. In fact bisections, being algebra maps, are just characters of the Hopf algebra $\mathcal{C}(A,H)$ with convolution product in (12.13) and inverse in (12.14) that, with the antipode in (13.3) can be written as $\sigma^{-1} = \sigma \circ S_{\mathcal{C}}$, as is the case for characters. From Proposition 12.11 we have then the isomorphism

$$\mathrm{Aut}_H(A) \simeq \mathcal{B}(\mathcal{C}(A,H)) = \mathrm{Char}(\mathcal{C}(A,H)). \tag{13.5}$$

As for extended bisections and automorphisms as in Section 12.4 we have analogously from Proposition 12.16 the isomorphism,

$$\mathrm{Aut}^{ext}_H(A) \simeq \mathcal{B}^{ext}(\mathcal{C}(A,H)) = \mathrm{Char}^{ext}(\mathcal{C}(A,H))\,, \tag{13.6}$$

with $\mathrm{Char}^{ext}(\mathcal{C}(A,H))$ the group of convolution invertible unital maps $\phi : \mathcal{C}(A,H) \to \mathbb{C}$.

13.2. **Hopf algebras as Galois objects.** Any Hopf algebra H is a H-Galois object with coaction given by its coproduct. Then H is isomorphic to the corresponding left bialgebroid $\mathcal{C}(H,H)$.

Let H be a Hopf algebra with coproduct $\Delta(h) = h_{(1)} \otimes h_{(2)}$. For the corresponding coinvariants: $h_{(1)} \otimes h_{(2)} = h \otimes 1$, we have $\epsilon(h_{(1)}) \otimes h_{(2)} = \epsilon(h) \otimes 1$, this imply $h = \epsilon(h) \in \mathbb{C}$ and $H^{co\,H} = \mathbb{C}$. Moreover, the canonical Galois map $\chi : g \otimes h \mapsto gh_{(1)} \otimes h_{(2)}$ is bijective with inverse given by $\chi^{-1}(g \otimes h) := g\, S(h_{(1)}) \otimes h_{(2)}$. Thus H is a H-Galois object.

With $A = H$, the corresponding left bialgebroid becomes

$$\begin{aligned}\mathcal{C}(H,H) &= \{g \otimes h \in H \otimes H :\ g_{(1)} \otimes h_{(1)} \otimes g_{(2)}h_{(2)} = g \otimes h \otimes 1_H\} \\ &= \{g \otimes h \in H \otimes H :\ g_{(1)} \otimes S(g_{(2)}) \otimes g_{(3)}h = g \otimes h \otimes 1_A\}.\end{aligned} \tag{13.7}$$

We have a linear map $\phi : \mathcal{C}(H,H) \to H$ given by $\phi(g \otimes h) := g\,\epsilon(h)$. The map ϕ has inverse $\phi^{-1} : H \to \mathcal{C}(H,H)$, defined by $\phi^{-1}(h) := h_{(1)} \otimes S(h_{(2)})$. This is well defined since

$$\Delta^{H\otimes H}(h_{(1)} \otimes S(h_{(2)})) = h_{(1)} \otimes S(h_{(4)}) \otimes h_{(2)}S(h_{(3)}) = h_{(1)} \otimes S(h_{(2)}) \otimes 1_H,$$

showing that $h_{(1)} \otimes S(h_{(2)}) \in \mathcal{C}(H,H)$. Moreover,

$$\phi(\phi^{-1}(h)) = \phi(h_{(1)} \otimes S(h_{(2)})) = h,$$

and

$$\phi^{-1}(\phi(g \otimes h)) = \epsilon(h)\,\phi^{-1}(g) = \epsilon(h)\, g_{(1)} \otimes S(g_{(2)}) = g \otimes h.$$

Here the last equality is obtained from the condition $g_{(1)} \otimes S(g_{(2)}) \otimes g_{(3)}h = g \otimes h \otimes 1_H$ (for any $g \otimes h \in \mathcal{C}(H,H)$, as in the second line of (13.7)) by applying $\mathrm{id}_H \otimes \mathrm{id}_H \otimes \epsilon$ on both sides and then multiplying the second and third factors:

$$g_{(1)} \otimes S(g_{(2)})\,\epsilon(g_{(3)})\epsilon(h) = g \otimes h\,\epsilon(1_H) \quad \Longrightarrow \quad \epsilon(h)\, g_{(1)} \otimes S(g_{(2)}) = g \otimes h.$$

The map ϕ is an algebra map:

$$\begin{aligned}\phi((g \otimes h) \bullet_{\mathcal{C}} (g' \otimes h')) &= \phi(gg' \otimes h'h) = gg'\epsilon(h')\epsilon(h) \\ &= \phi(g \otimes h) \bullet_{\mathcal{C}} \phi(g' \otimes h').\end{aligned}$$

It is also a coalgebra map:

$$\begin{aligned}(\phi\otimes\phi)(\Delta_{\mathcal{C}}(g\otimes h)) &= (\phi\otimes\phi)(g_{(1)}\otimes g_{(2)}{}^{<1>}\otimes g_{(2)}{}^{<2>}\otimes h)\\ &= (\phi\otimes\phi)(g_{(1)}\otimes S(g_{(2)})\otimes g_{(3)}\otimes h)\\ &= g_{(1)}\otimes g_{(2)}\,\epsilon(h)\\ &= \Delta_H(\phi(g\otimes h));\end{aligned}$$

$$\epsilon_{\mathcal{C}}(g\otimes h) = gh = \epsilon_H(gh) = \epsilon_H(g)\epsilon_H(h) = \epsilon_H(\phi(g\otimes h))\,.$$

13.3. **Cocommutative Hopf algebras.** We start with a class of examples coming from cocommutative Hopf algebras. From [31] (Remark 3.8 and Theorem 3.5.) we have:

Lemma 13.4. *Let H be a cocommutative Hopf algebra, and let A be a H-Galois object. Then the bialgebroid $\mathcal{C}(A,H)$ is isomorphic to H as Hopf algebra.*

Proof. We give a sketch of the proof that uses Lemma 13.3. Start with the coaction $\delta^A : A\to A\otimes H$, $\delta^A(a)=a_{(0)}\otimes a_{(1)}$, and translation map $\tau(h)=h^{<1>}\otimes h^{<2>}$. Firstly, the image of τ is in $\mathcal{C}(A,H)$; indeed, for any $h\in H$, we get

$$\begin{aligned}h^{<1>}{}_{(0)}\otimes h^{<2>}{}_{(0)}\otimes h^{<1>}{}_{(1)}h^{<2>}{}_{(1)} &= h_{(1)}{}^{<1>}{}_{(0)}\otimes h_{(1)}{}^{<2>}\otimes h_{(1)}{}^{<1>}{}_{(1)}h_{(2)}\\ &= h_{(1)(2)}{}^{<1>}\otimes h_{(1)(2)}{}^{<2>}\otimes S(h_{(1)(1)})h_{(2)}\\ &= h^{<1>}\otimes h^{<2>}\otimes 1_H,\end{aligned}$$

where the first step uses (7.5), and the second step uses (7.6). While τ is not an algebra map, being H cocommutative, it is a coalgebra map. Indeed, for any $h\in H$,

$$\begin{aligned}\Delta_{\mathcal{C}}(\tau(h)) &= h^{<1>}{}_{(0)}\otimes\tau(h^{<1>}{}_{(1)})\otimes h^{<2>}\\ &= h_{(2)}{}^{<1>}\otimes\tau(S(h_{(1)}))\otimes h_{(2)}{}^{<2>}\\ &= h_{(2)}{}^{<1>}\otimes h_{(2)}{}^{<2>}h_{(3)}{}^{<1>}\,\tau(S(h_{(1)}))\,h_{(3)}{}^{<2>}h_{(4)}{}^{<1>}\otimes h_{(4)}{}^{<2>}\\ &= h_{(3)}{}^{<1>}\otimes h_{(3)}{}^{<2>}h_{(2)}{}^{<1>}\,\tau(S(h_{(1)}))\,h_{(2)}{}^{<2>}h_{(4)}{}^{<1>}\otimes h_{(4)}{}^{<2>}\\ &= h_{(1)}{}^{<1>}\otimes h_{(1)}{}^{<2>}\ \otimes h_{(2)}{}^{<1>}\otimes h_{(2)}{}^{<2>}\\ &= (\tau\otimes\tau)(\Delta_H(h))\,,\end{aligned}$$

where the 2nd step uses (7.6): $h_{(2)}{}^{<1>}\otimes h_{(2)}{}^{<2>}\otimes S(h_{(1)}) = h^{<1>}{}_{(0)}\otimes h^{<2>}\otimes h^{<1>}{}_{(1)}$, the 3nd step uses twice (7.10): $h_{(1)}{}^{<1>}\otimes h_{(1)}{}^{<2>}h_{(2)}{}^{<1>}\otimes h_{(2)}{}^{<2>} = h^{<1>}\otimes 1_A\otimes h^{<2>}$; the 4rd step uses H is cocommutative: we change the lower indices 2 and 3; and the 5th one uses: $\epsilon(h)\otimes 1_A = \tau(S(h_{(1)})h_{(2)}) = h_{(2)}{}^{<1>}\,\tau(S(h_{(1)}))\,h_{(2)}{}^{<2>}$. Also,

$$\epsilon_{\mathcal{C}}(\tau(h)) = h^{<1>}h^{<2>} = \epsilon(h)1_A.$$

On the other hand, since H is cocommutative, A is also a left H-Galois object with coaction $\delta_L(a)=a_{(1)}\otimes a_{(0)}$ and bijective canonical map $\chi_L(a\otimes\tilde a)=a_{(1)}\otimes a_{(0)}\tilde a$. The corresponding translation map is then $\tau_L=\tau\circ S$ where $S=S^{-1}$ (since H is cocommutative) is the antipode of H. The map τ_L is a coalgebra map being the composition of two such maps (for S this is the case again due to H cocommutative).

From the universality of Lemma 13.3, there is a unique algebra map $\Phi:\mathcal{C}(A,H)\to H$ such that $\delta_L=(\Phi\otimes\mathrm{id}_A)\circ\delta^{\mathcal{C}}$; where $\delta^{\mathcal{C}}: H\to\mathcal{C}(A,H)\otimes H$ as in the lemma. Explicitly, $\Phi(a\otimes\tilde a)\otimes 1_A = \delta_L(a)\tilde a = \chi_{L|_{\mathcal{C}}}(a\otimes\tilde a)$ for $a\otimes\tilde a\in\mathcal{C}(A,H)$. Indeed, with the isomorphism $\widetilde\chi$ in (13.4), the map Φ is such that $\chi_L=(\Phi\otimes\mathrm{id}_A)\circ\widetilde\chi$, thus is an isomorphism since χ_L and $\widetilde\chi$ are such. The map Φ has inverse $\Phi^{-1}=\tau_L$ and thus is a coalgebra map. $\square$

Consequently, the isomorphisms 13.5 and 13.6 for a cocommutative Hopf algebra H are:

$$\mathrm{Aut}_H(A) \simeq \mathcal{B}(\mathcal{C}(A,H)) = \mathrm{Char}(H) \tag{13.8}$$

and

$$\mathrm{Aut}_H^{ext}(A) \simeq \mathcal{B}^{ext}(\mathcal{C}(A,H)) = \mathrm{Char}^{ext}(H)\,, \tag{13.9}$$

with $\mathrm{Char}^{ext}(H)$ the group of convolution invertible unital maps $\phi : H \to \mathbb{C}$ and $\mathrm{Char}(H)$ the subgroup of those which are algebra maps (the characters of H).

13.4. **Group Hopf algebras.** Let G be a group, with neutral element e, and $H = \mathbb{C}[G]$ be its group algebra. Its elements are finite sums $\sum \lambda_g\, g$ with λ_g complex number. We assume that $\{g\,, g \in G\}$ is a vector space basis. The product in $\mathbb{C}[G]$ follows from the group product in G, with algebra unit $1_{\mathbb{C}[G]} = e$. The coproduct, counit, and antipode, making $\mathbb{C}[G]$ a Hopf algebra are defined by $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$, $S(g) = g^{-1}$.

An algebra A is G-graded, that is $A = \oplus_{g\in G}\, A_g$ and $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$, if and only if A is a right $\mathbb{C}[G]$-comodule algebra with coaction $\delta^A : A \to A \otimes \mathbb{C}[G]$, $a \mapsto \sum a_g \otimes g$ for $a = \sum a_g$, $a_g \in A_g$. Moreover, the algebra A is strongly G-graded, that is $A_g A_h = A_{gh}$, if and only if $A_e = A^{co\,\mathbb{C}[G]} \subseteq A$ is Hopf–Galois (see e.g. [29, Thm.8.1.7]). Thus $\mathbb{C}[G]$-Hopf–Galois extensions are the same as G-strongly graded algebras.

In particular, if A is a $\mathbb{C}[G]$-Galois object, that is $A_e = \mathbb{C}$, each component A_g is one-dimensional. If we pick a non-zero element u_g in each A_g, the multiplication of A is determined by the products $u_g u_h$ for each pair g, h of elements of G. We then have

$$u_g u_h = \lambda(g,h)\, u_{gh} \tag{13.10}$$

for a non vanishing $\lambda(g,h) \in \mathbb{C}$. We get then a map $\lambda : G \times G \to \mathbb{C}^\times$ which is in fact a two cocycle $\lambda \in H^2(G, \mathbb{C}^\times)$. Indeed, associativity of the product requires that λ satisfies a 2-cocycle condition, that is for any $g, h \in G$,

$$\lambda(g,h)\lambda(gh,k) = \lambda(h,k)\lambda(g,hk). \tag{13.11}$$

If we choose a different non-zero element $v_g \in A_g$, we shall have $v_g = \mu(g)u_g$, for some non-zero $\mu(g) \in \mathbb{C}$. The multiplication (13.10) will become $v_g v_h = \lambda'(g,h)v_{gh}$ with

$$\lambda'(g,h) = \mu(g)\mu(h)(\mu(gh))^{-1}\lambda(g,h), \tag{13.12}$$

that is the two 2-cocycles λ' and λ are cohomologous. It is easy to check that for any map $\mu(g) : G \to \mathbb{C}^\times$ the assignment $(g,h) \mapsto \mu(g)\mu(h)(\mu(gh))^{-1}$, is a coboundary, that is a 'trivial' 2-cocycle which is cohomologous to $\lambda(g,h) = 1$. Thus the multiplication in A depends only on the second cohomology class of $\lambda \in H^2(G, \mathbb{C}^\times)$, the second cohomology group of G with values in $\mathbb{C}^\times$. We conclude that the equivalence classes of $\mathbb{C}[G]$-Galois objects are in bijective correspondence with the cohomology group $H^2(G, \mathbb{C}^\times)$.

Example 13.5. From [19, Ex. 7.13] we have the following. For any cyclic group G (infinite or not) one has $H^2(G, \mathbb{C}^\times) = 0$. Thus any corresponding $\mathbb{C}[G]$-Galois object is trivial. On the other hand, for the free abelian group of rank $r \geq 2$, one has

$$H^2(\mathbb{Z}^r, \mathbb{C}^\times) = (\mathbb{C}^\times)^{r(r-1)/2}\,.$$

Hence, there are infinitely many isomorphism classes of $\mathbb{C}[\mathbb{Z}^r]$-Galois objects.

Since $H = \mathbb{C}[G]$ is cocommutative, we know from above that the corresponding bialgebroids $\mathcal{C}(A,H)$ are all isomorphic to H as Hopf algebra. It is instructive to show this directly. Clearly, for any $u_g \otimes u_h \in \mathcal{C}(A,H)$ the coinvariance condition $u_g \otimes u_h \otimes gh = u_g \otimes u_h \otimes 1_H$, requires $h = g^{-1}$ so that $\mathcal{C}(A,H)$ is generated as vector space by elements $u_g \otimes u_{g^{-1}}$, $g \in G$, with multiplication

$$(u_g \otimes u_{g^{-1}}) \bullet_{\mathcal{C}} (u_h \otimes u_{h^{-1}}) = \lambda(g,h)\lambda(h^{-1},g^{-1}) u_{gh} \otimes u_{(gh)^{-1}}, \tag{13.13}$$

Lemma 13.6. *The cocycle $\Lambda(g,h) = \lambda(g,h)\lambda(h^{-1},g^{-1})$ is trivial in $H^2(G,\mathbb{C}^\times)$.*

Proof. Firstly, we can always rescale u_e to $\lambda(e,e)\, u_e$ so to have $\lambda(e,e) = 1$. Then the cocycle condition (13.11) yields $\lambda(g,e) = \lambda(e,g) = \lambda(e,e) = 1$, for any $g \in G$. Next, with $u_g u_h = \lambda(g,h)u_{gh}$ and $u_{h^{-1}}u_{g^{-1}} = \lambda(h^{-1},g^{-1})u_{(gh)^{-1}}$, on the one hand we have $u_g u_h u_{h^{-1}} u_{g^{-1}} = \lambda(g,h)\lambda(h^{-1},g^{-1})\lambda(gh,(gh)^{-1})u_e$. On the other hand $u_g u_h u_{h^{-1}} u_{g^{-1}} = \lambda(g,g^{-1})\lambda(h,h^{-1})u_e$. Thus

$$\Lambda(g,h) = \lambda(g,g^{-1})\lambda(h,h^{-1})/\lambda(gh,(gh)^{-1})$$

showing $\Lambda(g,h)$ is trivial since $\Lambda(g,h) = \mu(g)\mu(h)(\mu(gh))^{-1}$ with $\mu(g) = \lambda(g,g^{-1})$. □

Consequently, by rescaling the generators $u_g \to v_g = \lambda(g,g^{-1})^{-\frac{1}{2}}\, u_g$ the multiplication rule (13.10) becomes $v_g v_h = \lambda'(g,h)\, v_{gh}$, with $\lambda'(g,h) = \Lambda(g,h)^{-\frac{1}{2}}\,\lambda(g,h)$ that we rename back to $\lambda(g,h)$ from now on. As for the bialgebroid product in (13.13) one has,

$$(v_g \otimes v_{g^{-1}}) \bullet_{\mathcal{C}} (v_h \otimes v_{h^{-1}}) = v_{gh} \otimes v_{(gh)^{-1}}, \tag{13.14}$$

and the isomorphism $\Phi^{-1} : H \to \mathcal{C}(A,H)$ is simply $\Phi^{-1}(g) = \tau_L(g) = v_g \otimes v_{g^{-1}}$.

As in (13.8), the group of bisections $\mathcal{B}(\mathcal{C}(A,H))$ of $\mathcal{C}(A,H)$, and the gauge group $\mathrm{Aut}_H(A)$ of the Galois object A coincide with the group of characters on $\mathbb{C}[G]$, which is in turn the same as $\mathrm{Hom}(G,\mathbb{C}^\times)$ the group (for point-wise multiplication) of group morphisms from G to $\mathbb{C}^\times$.

Explicitly, since $F \in \mathrm{Aut}_H(A)$ is linear on A, on a basis $\{v_g\}_{g\in G}$ of A, it is of the form

$$F(v_g) = \sum_{h\in G} f_h(g) v_h,$$

for complex numbers, $f_h(g) \in \mathbb{C}$. Then, the H-equivariance of F,

$$F(v_g)_{(0)} \otimes F(v_g)_{(1)} = F(v_{g(0)}) \otimes v_{g(1)} = F(v_g) \otimes g,$$

requires $F(v_g)$ belongs to A_g and we get $f_h(g) = 0$, if $h \neq g$ while $f_g := f_g(g) \in \mathbb{C}^\times$ from the invertibility of F. Finally F is an algebra map:

$$\lambda(g,h) f_{gh} v_{gh} = F(\lambda(g,h)\, v_{gh}) = F(v_g v_h) = F(v_g)F(v_h) = \lambda(g,h) f_g f_h\, v_{gh}\,,$$

implies $f_{gh} = f_g f_h$, for any $g,h \in G$. Thus we re-obtain that $\mathrm{Aut}_H(A) \simeq \mathrm{Hom}(G,\mathbb{C}^\times)$. Note that the requirement $F(v_e = 1_A) = 1 = F_e$ implies that $F_{g^{-1}} = (F_g)^{-1}$.

On the other hand, the group $\mathrm{Aut}_H^{ext}(A)$ and then $\mathcal{B}^{ext}(\mathcal{C}(A,H))$ can be quite big. If $F \in \mathrm{Aut}_H^{ext}(A)$ that is one does not require F to be an algebra map, the corresponding f_g can take any value in $\mathbb{C}^\times$ with the only condition that $f_e = 1$.

13.5. **Taft algebras.** Let $N \geq 2$ be an integer and let q be a primitive N-th roots of unity. The *Taft algebra* T_N, [35], is the N^2-dimensional unital algebra generated by generators x, g subject to relations:

$$x^N = 0\,, \quad g^N = 1\,, \quad xg - q\, gx = 0\,.$$

It is a Hopf algebra with coproduct:

$$\Delta(x) := 1 \otimes x + x \otimes g, \qquad \Delta(g) := g \otimes g\,;$$

counit:

$$\epsilon(x) := 0, \qquad \epsilon(g) := 1\,;$$

and antipode:

$$S(x) := -xg^{-1}, \qquad S(g) := g^{-1}\,.$$

This Hopf algebra is neither commutative nor cocommutative. The four dimensional algebra T_2 is also known as the *Sweedler algebra*.

For any $s \in \mathbb{C}$, let A_s be the unital algebra generated by elements X, G with relations:

$$X^N = s\,, \quad G^N = 1\,, \quad XG - q\,GX = 0\,.$$

The algebra A_s is a right T_N-comodule algebra, with coaction defined by

$$\delta^A(X) := 1 \otimes x + X \otimes g, \qquad \delta^A(G) := G \otimes g. \tag{13.15}$$

Clearly, the corresponding coinvariants are just the ground field $\mathbb{C}$ and so A_s is a T_N-Galois object. It is known (cf. [26], Prop. 2.17, Prop. 2.22, as well as [32]) that any T_N-Galois object is isomorphic to A_s for some $s \in \mathbb{C}$ and that any two such Galois objects A_s and A_t are isomorphic if and only if $s = t$. Thus the equivalence classes of T_N-Galois objects are in bijective correspondence with the abelian group $\mathbb{C}$. For the corresponding Ehresmann–Schauenburg bialgebroid $\mathcal{C}(A_s, T_N) = (A_s \otimes A_s)^{co\,T_N}$.

Lemma 13.7. *The translation map of the coaction* (13.15) *is given on generators by*

$$\tau(g) = G^{-1} \otimes G,$$
$$\tau(x) = 1 \otimes X - XG^{-1} \otimes G.$$

Proof. We just apply the corresponding canonical map to obtain:

$$\chi \circ \tau(g) = G^{-1}G \otimes g = 1 \otimes g,$$
$$\chi \circ \tau(x) = 1 \otimes x + X \otimes g - XG^{-1}G \otimes g = 1 \otimes x.$$

as it should be. ☐

We have then the following:

Proposition 13.8. *For any complex number s there is a Hopf algebra isomorphism*

$$\Phi : \mathcal{C}(A_s, T_N) \simeq T_N.$$

Proof. It is easy to see that the elements

$$\Xi = X \otimes G^{-1} - 1 \otimes XG^{-1}, \qquad \Gamma = G \otimes G^{-1} \tag{13.16}$$

are coinvariants for the right diagonal coaction of T_N on $A_s \otimes A_s$ and that they generate $\mathcal{C}(A_s, T_N) = (A_s \otimes A_s)^{co\,T_N}$ as an algebra. These elements satisfy the relations:

$$\Xi^N = 0, \quad \Gamma^N = 1, \quad \Xi \bullet_{\mathcal{C}} \Gamma = q\, \Xi \bullet_{\mathcal{C}} \Gamma\,. \tag{13.17}$$

Indeed, the last two relations are easy to see. As for the first one, shifting powers of G^{-1} to the right one finds

$$\begin{aligned}\Xi^N &= X^N \otimes G^{-N} + \sum_{r=1}^{N-1} c_r\, X^{N-r} \otimes X^r G^{-N} + (-1)^N \otimes (XG^{-1})^N \\ &= \left[X^N \otimes 1 + \sum_{r=1}^{N-1} c_r\, X^{N-r} \otimes X^r + (-1)^N q^{\frac{n(n-1)}{2}} \otimes X^N\right] G^{-N}\end{aligned}$$

for explicit coefficients c_r depending on q. Then, using the same methods as in [35] one shows that, being q a primitive N-th roots of unity, all coefficients c_r vanish and so $\Xi^N = X^N \otimes G^{-N} + (-1)^N \otimes (XG^{-1})^N$ which then vanishes from $X^N = 0$.

Thus the elements Ξ and Γ generate a copy of the algebra T_N and the isomorphism Φ maps Ξ to x and Γ to g. The map Φ is also a coalgebra map. Indeed,

$$\Delta(\Phi(\Gamma)) = \Delta(g) = g \otimes g,$$

while, using Lemma 13.7,

$$\Delta_{\mathcal{C}}(\Gamma) = G_{(0)} \otimes G_{(1)}{}^{<1>} \otimes G_{(1)}{}^{<2>} \otimes G^{-1} = G \otimes G^{-1} \otimes G \otimes G^{-1} = \Gamma \otimes \Gamma.$$

Thus $(\Phi \otimes \Phi)(\Delta_{\mathcal{C}}(\Gamma)) = g \otimes g = \Delta(\Phi(\Gamma))$. Similarly,

$$\Delta(\Phi(\Xi)) = \Delta(x) = 1 \otimes x + x \otimes g,$$

while, using Lemma 13.7 in the third step,

$$\begin{aligned}
\Delta_{\mathcal{C}}(\Xi) &= \Delta_{\mathcal{C}}(X \otimes G^{-1}) - \Delta_{\mathcal{C}}(1 \otimes XG^{-1}) \\
&= X_{(0)} \otimes X_{(1)}{}^{<1>} \otimes X_{(1)}{}^{<2>} \otimes G^{-1} - 1 \otimes 1 \otimes 1 \otimes XG^{-1} \\
&= 1 \otimes x^{<1>} \otimes x^{<2>} \otimes G^{-1} + X \otimes g^{<1>} \otimes g^{<2>} \otimes G^{-1} - 1 \otimes 1 \otimes 1 \otimes XG^{-1} \\
&= 1 \otimes \Big(1 \otimes X - XG^{-1} \otimes G\Big) \otimes G^{-1} + X \otimes G^{-1} \otimes G \otimes G^{-1} - 1 \otimes 1 \otimes 1 \otimes XG^{-1} \\
&= 1 \otimes 1 \otimes \Big(X \otimes G^{-1} - 1 \otimes XG^{-1}\Big) + \Big(X \otimes G^{-1} - 1 \otimes XG^{-1}\Big) \otimes G \otimes G^{-1} \\
&= 1 \otimes \Xi + \Xi \otimes \Gamma.
\end{aligned}$$

Thus $(\Phi \otimes \Phi)(\Delta_{\mathcal{C}}(\Xi)) = 1 \otimes x + x \otimes g = \Delta(\Phi(\Xi))$. Finally: $\epsilon_{\mathcal{C}}(\Gamma) = 1 = \epsilon(g)$ and $\epsilon_{\mathcal{C}}(\Xi) = 0 = \epsilon(x)$. This concludes the proof. □

The group of characters of the Taft algebra T_N is the cyclic group $\mathbb{Z}_N$: indeed any character ϕ must be such that $\phi(x) = 0$, while $\phi(g)^N = \phi(g^N) = \phi(1) = 1$. Then for the group of gauge transformations of the Galois object A_s, the same as the group of bisections of the bialgebroid $\mathcal{C}(A_s, T_N)$, due to Proposition 13.8 we have,

$$\mathrm{Aut}_{T_N}(A_s) \simeq \mathcal{B}(\mathcal{C}(A_s, T_N)) = \mathrm{Char}(T_N) = \mathbb{Z}_N. \tag{13.18}$$

On the other hand, elements F of $\mathrm{Aut}^{ext}_{T_N}(A_s) \simeq \mathcal{B}^{ext}(\mathcal{C}(A_s, T_N))$, due to equivariance $F(a)_{(0)} \otimes F(a)_{(1)} = F(a_{(0)}) \otimes a_{(1)}$ for any $a \in A_s$, can be given as a block diagonal matrix

$$F = \mathrm{diag}(M_1, M_2, \ldots, M_{N-1}, M_N)$$

with each M_j a $N \times N$ invertible lower triangular matrix

$$M_j = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ b_{21} & a_{N-1} & 0 & \ldots & 0 & 0 \\ b_{31} & b_{32} & a_{N-2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ b_{N-1,1} & b_{N-1,2} & \ddots & \ddots & a_2 & 0 \\ b_{N1} & b_{N2} & \ldots & b_{N,N-2} & b_{N,N-1} & a_1 \end{bmatrix}$$

All matrices M_j have in common the diagonal elements a_j (ciclic permuted) which are all different from zero for the invertibility of M_j. For the subgroup $\mathrm{Aut}_{T_N}(A_s)$ the M_j are diagonal as well with $a_k = (a_1)^k$ and $(a_1)^N = 1$ so that $M_j \in \mathbb{Z}_N$. The reason all M_j share the same diagonal elements (up to permutation) is the following: firstly, the 'diagonal' form of the coaction of G in (13.15) imply that the image $F(G^k)$ is proportional to G^k, say $F(G^k) = \alpha_k G^k$ for some constant α_k. Then, do to the first term in the coaction of X in (13.15), the 'diagonal' component along the basis element $X^l G^k$ of the image $F(X^l G^k)$ is given again by α_k for any possible value of the index l.

Let us illustrate the construction for the cases of $N = 2, 3$. Firstly, $F(1) = 1$ since F is unital. When $N = 2$, on the basis $\{1, X, G, XG\}$, the equivariance $F(a)_{(0)} \otimes F(a)_{(1)} =$

$F(a_{(0)}) \otimes a_{(1)}$ for the coaction (13.15) becomes

$$\begin{aligned} F(X)_{(0)} \otimes F(X)_{(1)} &= 1 \otimes x + F(X) \otimes g, \\ F(G)_{(0)} \otimes F(G)_{(1)} &= F(G) \otimes g, \\ F(XG)_{(0)} \otimes F(XG)_{(1)} &= F(G) \otimes xg + F(XG) \otimes 1. \end{aligned}$$

Next, write $F(a) = f_1(a) + f_2(a)\, X + f_3(a)\, G + f_4(a)\, XG$, for complex numbers $f_k(a)$. And compute $F(a)_{(0)} \otimes F(a)_{(1)} = f_1(a)\, 1 \otimes 1 + f_2(a)\, (1 \otimes x + X \otimes g) + f_3(a)\, G \otimes g + f_4(a)\, (G \otimes xg + XG \otimes 1)$. Then comparing generators, the equivariance gives

$$\begin{aligned} & f_1(X) = f_4(X) = 0 \\ & f_1(G) = f_2(G) = f_4(G) = 0 \\ & f_2(XG) = f_3(XG) = 0, \end{aligned}$$

while the remaining coefficients are related by the system of equations

$$\begin{aligned} f_2(X)\, (1 \otimes x + X \otimes g) + f_3(X)\, G \otimes g &= 1 \otimes x + F(X) \otimes g, \\ f_3(G)\, G \otimes g &= F(G) \otimes g, \\ f_1(XG)\, 1 \otimes 1 + f_4(XG)\, (G \otimes xg + XG \otimes 1) &= F(G) \otimes xg + F(XG) \otimes 1. \end{aligned}$$

One readily finds solutions

$$\begin{aligned} & f_2(X) = 1, \qquad f_3(X) = \gamma, \qquad f_1(XG) = \beta \\ & f_3(G) = f_4(XG) = \alpha \end{aligned}$$

with α, β, γ arbitrary complex numbers. Thus a generic element F of $\mathrm{Aut}^{ext}_{T_2}(A_s)$ can be represented by the matrix:

$$F : \begin{pmatrix} 1 \\ XG \\ G \\ X \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & \gamma & 1 \end{pmatrix} \begin{pmatrix} 1 \\ XG \\ G \\ X \end{pmatrix}. \tag{13.19}$$

Asking F to be invertible requires $\alpha \neq 0$. On the other hand, any $F \in \mathrm{Aut}_{T_2}(A_s)$ is an algebra map and so is determined by its values on the generators G, X. From $F(G) = \alpha G$ and $F(X) = \gamma G + X$: requiring $s = F(X^2) = (\gamma G + X)^2 = \gamma + (GX + XG) + s$ yields $\gamma = 0$; then $\beta + \alpha XG = F(XG) = \alpha XG$ yields $\beta = 0$; and $1 = F(G^2) = (\alpha G)^2$ leads to $\alpha^2 = 1$. Thus $F(X) = X$ and $F(G) = \alpha G$, with $\alpha^2 = 1$ and we conclude that $\mathrm{Aut}_{T_2}(A_s) \simeq \mathbb{Z}_2$.

When $N = 3$, a similar, if longer computation, gives for $\mathrm{Aut}^{exp}_{T_3}(A_s)$ an eight parameter group with its elements F of the form

$$F : \begin{pmatrix} 1 \\ XG^2 \\ X^2G \\ G^2 \\ XG \\ X^2 \\ G \\ X \\ X^2G^{-1} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta & -q\delta & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & -q\gamma & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \theta & -q\beta & \alpha_2 \end{pmatrix} \begin{pmatrix} 1 \\ XG^2 \\ X^2G \\ G^2 \\ XG \\ X^2 \\ G \\ X \\ X^2G^{-1} \end{pmatrix}. \tag{13.20}$$

One needs $\alpha_j \neq 0$, $j = 1, 2$ for invertibility. By going as before, for any $F \in \mathrm{Aut}_{T_N}(A_s)$ one starts from it values on the generators, $F(G) = \alpha_1 G$ and $F(X) = \gamma G + X$, to conclude that F is a diagonal matrix (in particular $F(X)=X$) with $\alpha_2=(\alpha_1)^2$ and $1=(\alpha_1)^3$; thus $\mathrm{Aut}_{T_3}(A_s)\simeq\mathbb{Z}_3$.

14. Crossed module structures on bialgebroids

Isomorphisms of (a usual) groupoid with natural transformations between them form a strict 2-groupoid. In particular, automorphisms of the groupoid with its natural transformations, form a strict 2-group or, equivalently, a crossed module (cf. [28], Definition 3.21). The crossed module combines automorphisms of the groupoid and bisections since the latter are the natural transformations from the identity functor to automorphisms. The crossed module involves the product on bisections and the composition on automorphisms, and the group homomorphism from bisections to automorphisms together with the action of automorphisms on bisections by conjugation. Any bisection σ is the 2-arrow from the identity morphism to an automorphism Ad_σ, and the composition of bisections can be viewed as the horizontal composition of 2-arrows.

In this section we quantise this construction for the Ehresmann–Schauenburg bialgebroid of a Hopf–Galois extension. We construct a crossed module for the bisections and the automorphisms of the bialgebroid. Notice that we do not need the antipode of bialgebroid, that is we do not need to defined the crossed module on Hopf algebroid and the crossed module on bialgebroid is a generalization of the crossed module on groupoid. In the next section, The construction can also be repeated for extended bisections.

14.1. **Automorphisms and crossed modules.** Recall that a crossed module is the data (M,N,μ,α) of two groups M, N together with a group morphism $\mu:M\to N$ and a group morphism $\alpha:N\to\mathrm{Aut}(M)$ such that, denoting $\alpha_n:M\to M$ for every $n\in N$, the following conditions are satisfied:

(1) $\mu(\alpha_n(m))=n\mu(m)n^{-1}$, for any $n\in N$ and $m\in M$;

(2) $\alpha_{\mu(m)}(m')=mm'm^{-1}$, for any $m,m'\in M$.

Then, with the definition of the automorphism group of a bialgebroid as given in Definition 6.8, we aim at proving the following.

Theorem 14.1. *Given a Hopf–Galois extension $B=A^{coH}\subseteq A$, let $\mathcal{C}(A,H)$ be the corresponding left Ehresmann–Schauenburg bialgebroid, and assume B is in the centre of A. Then there is a group morphism $Ad:\mathcal{B}(\mathcal{C}(A,H))\to\mathrm{Aut}(\mathcal{C}(A,H))$ and an action $\triangleright$ of $\mathrm{Aut}(\mathcal{C}(A,H))$ on $\mathcal{B}(\mathcal{C}(A,H))$ that give a crossed module structure to $\big(\mathcal{B}(\mathcal{C}(A,H)),\mathrm{Aut}(\mathcal{C}(A,H))\big)$.*

We give the proof in a few lemmas.

Lemma 14.2. *Given a Hopf–Galois extension $B=A^{coH}\subseteq A$, let $\mathcal{C}(A,H)$ be the corresponding left Ehresmann–Schauenburg bialgebroid. Assume B belongs to the centre of A. For any bisection $\sigma\in\mathcal{B}(\mathcal{C}(A,H))$, denote $ad_\sigma=\sigma\circ s\in\mathrm{Aut}(B)$ and let F_σ be the associated gauge element in $\mathrm{Aut}_H(A)$ (see (12.17)). Define $Ad_\sigma:\mathcal{C}(A,H)\to\mathcal{C}(A,H)$ by*

$$\begin{aligned}Ad_\sigma(a\otimes\tilde a)&:=F_\sigma(a)\otimes F_\sigma(\tilde a)\\&=\sigma(a_{(0)}\otimes a_{(1)}{}^{<1>})a_{(1)}{}^{<2>}\otimes\sigma(\tilde a_{(0)}\otimes\tilde a_{(1)}{}^{<1>})\tilde a_{(1)}{}^{<2>}.\end{aligned}\tag{14.1}$$

Then the pair (Ad_σ,ad_σ) is an automorphism of $\mathcal{C}(A,H)$.

Proof. Since F_σ is an algebra automorphism, so is Ad_σ. Then, for any $b\in B$,

$$Ad_\sigma(t(b))=Ad_\sigma(1\otimes b)=1\otimes\sigma(b\otimes 1)=t(ad_\sigma(b))$$

and

$$Ad_\sigma(s(b)) = Ad_\sigma(b \otimes 1) = \sigma(b \otimes 1) \otimes 1 = s(ad_\sigma(b)).$$

So conditions (i) and (ii) of Definition 6.8 are satisfied. For condition (iii), using H-equivariance of F_σ, with $a \otimes \tilde{a} \in \mathcal{C}(A,H)$ we get

$$(\Delta_{\mathcal{C}(A,H)} \circ Ad_\sigma)(a \otimes \tilde{a}) = F_\sigma(a_{(0)}) \otimes a_{(1)}{}^{<1>} \otimes_B a_{(1)}{}^{<2>} \otimes F_\sigma(\tilde{a}). \tag{14.2}$$

On the other hand,

$$\big((Ad_\sigma \otimes_B Ad_\sigma) \circ \Delta_{\mathcal{C}(A,H)}\big)(a \otimes \tilde{a}) = F_\sigma(a_{(0)}) \otimes F_\sigma(a_{(1)}{}^{<1>}) \otimes_B F_\sigma(a_{(1)}{}^{<2>}) \otimes F_\sigma(\tilde{a}). \tag{14.3}$$

Now, for any $F \in \mathrm{Aut}_H(A)$, given $h \in H$, one has

$$F(h^{<1>}) \otimes_B F(h^{<2>}) = h^{<1>} \otimes_B h^{<2>}, \qquad \text{for any} \quad h \in H. \tag{14.4}$$

By applying the canonical map χ and using equivariance of F we compute,

$$\begin{aligned}\chi(F(h^{<1>}) \otimes_B F(h^{<2>})) &= F(h^{<1>})F(h^{<2>})_{(0)} \otimes F(h^{<2>})_{(1)} \\ &= F(h^{<1>})F(h^{<2>}{}_{(0)}) \otimes h^{<2>}{}_{(1)} \\ &= F(1_A) \otimes h = 1_A \otimes h\end{aligned}$$

using (7.7). Being χ an isomorphism we get the relation (14.4). Using this for the right hand sides of (14.2) and (14.3) shows that they coincide and condition (iii) is satisfied. Finally,

$$(\epsilon \circ Ad_\sigma)(a \otimes \tilde{a}) = \epsilon(F_\sigma(a) \otimes F_\sigma(\tilde{a})) = F_\sigma(a\tilde{a}) = (\sigma \circ s)(a\tilde{a}) = ad_\sigma \circ \epsilon(a \otimes \tilde{a}).$$

This finishes the proof. □

Remark 14.3. The map Ad_σ in (14.1) can also be written in the following useful ways:

$$\begin{aligned}Ad_\sigma(a \otimes \tilde{a}) &= \sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})a_{(1)}{}^{<2>} \otimes a_{(2)}{}^{<1>}\sigma^{-1}(a_{(2)}{}^{<2>} \otimes \tilde{a}) \\ &= \sigma((a \otimes \tilde{a})_{(1)})\,(a \otimes \tilde{a})_{(2)}\,(\sigma \circ s) \circ \sigma^{-1}((a \otimes \tilde{a})_{(3)}))\end{aligned} \tag{14.5}$$

Indeed, for $a \otimes a' \in \mathcal{C}(A,H)$, by inserting (7.8) and using the definition of the inverse ϕ^{-1}, we compute,

$$\begin{aligned}Ad_\sigma(a \otimes \tilde{a}) &= \sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})a_{(1)}{}^{<2>} \otimes \sigma(\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>})\tilde{a}_{(1)}{}^{<2>} \\ &= \sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})a_{(1)}{}^{<2>} \otimes a_{(2)}{}^{<1>}a_{(2)}{}^{<2>}\sigma(\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>})\tilde{a}_{(1)}{}^{<2>} \\ &= \sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})a_{(1)}{}^{<2>} \otimes a_{(2)}{}^{<1>}\sigma(\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}{}^{<1>})a_{(2)}{}^{<2>}\tilde{a}_{(1)}{}^{<2>} \\ &= \sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})a_{(1)}{}^{<2>} \otimes a_{(2)}{}^{<1>}(\sigma \circ s) \circ \sigma^{-1}(a_{(2)}{}^{<2>} \otimes \tilde{a})\big) \\ &= \sigma((a \otimes \tilde{a})_{(1)})\,(a \otimes \tilde{a})_{(2)}\,(\sigma \circ s) \circ \sigma^{-1}((a \otimes \tilde{a})_{(3)})\end{aligned}$$

It is easy to see that $Ad_\sigma \circ Ad_\tau = Ad_{\tau * \sigma}$ for any $\sigma_1, \sigma_2 \in \mathcal{B}(\mathcal{C}(A,H))$, while $(Ad_\sigma)^{-1} = Ad_{\sigma^{-1}}$ and $Ad_\epsilon = id_{\mathcal{C}(A,H)}$. And, of course $ad_\sigma \circ ad_\tau = ad_{\tau*\sigma}$, with $(ad_\sigma)^{-1} = ad_{\sigma^{-1}}$ and $ad_\epsilon = id_B$. Thus Ad is a group morphism $Ad : \mathcal{B}(\mathcal{C}(A,H)) \to \mathrm{Aut}(\mathcal{C}(A,H))$.

Next, given an automorphism (Φ, φ) of $\mathcal{C}(A,H)$ with inverse $(\Phi^{-1}, \varphi^{-1})$, we define an action of (Φ, φ) on the group of bisections $\mathcal{B}(\mathcal{C}(A,H))$ as follow:

$$\Phi \triangleright \sigma := \varphi^{-1} \circ \sigma \circ \Phi, \tag{14.6}$$

for any $\sigma \in \mathcal{B}(\mathcal{C}(A,H))$. The result is an algebra map since it is a composition of algebra map. Moreover, for any $b \in B$, $(\Phi \triangleright \sigma)(t(b)) = \varphi^{-1}(\sigma(t(\varphi(b)))) = \varphi^{-1}(\varphi(b)) = b$, so that $(\Phi \triangleright \sigma) \circ t = \mathrm{id}_B$; while $(\Phi \triangleright \sigma)(s(b)) = \varphi^{-1}\big((\sigma \circ s)(\varphi(b))\big)$, so that $(\Phi \triangleright \sigma) \circ s \in \mathrm{Aut}(B)$. And one checks that

$$(\Phi \triangleright \sigma)^{-1} = \Phi \triangleright \sigma^{-1} = \varphi^{-1} \circ \sigma^{-1} \circ \Phi. \tag{14.7}$$

Lemma 14.4. *Given any automorphism (Φ, φ), the action defined in (14.6) is a group automorphism of $\mathcal{B}(\mathcal{C}(A, H))$.*

Proof. Let $\sigma, \tau \in \mathcal{B}(\mathcal{C}(A, H))$, and $c \in \mathcal{C}(A, H)$, we compute:

$$\begin{aligned}(\Phi \triangleright \tau) * (\Phi \triangleright \sigma)(c) &= (\Phi \triangleright \sigma)(s(\Phi \triangleright \tau(c_{(1)})))(\Phi \triangleright \sigma)(c_{(2)}) \\ &= \big(\varphi^{-1} \circ \sigma \circ \Phi \circ s \circ \varphi^{-1} \circ \tau \circ \Phi(c_{(1)})\big)\big(\varphi^{-1} \circ \sigma \circ \Phi(c_{(2)})\big) \\ &= \big(\varphi^{-1} \circ \sigma \circ s \circ \tau \circ \Phi(c_{(1)})\big)\big(\varphi^{-1} \circ \sigma \circ \Phi(c_{(2)})\big) \\ &= \varphi^{-1}\big(\sigma \circ s \circ \tau(\Phi(c_{(1)}))\, \sigma(\Phi(c_{(2)}))\big) \\ &= \varphi^{-1} \circ (\tau * \sigma) \circ \Phi(c) \\ &= \Phi \triangleright (\tau * \sigma)(c),\end{aligned}$$

where the last but one step uses condition (iii) of Definition 6.8. Also,

$$\Phi \triangleright \epsilon = \varphi^{-1} \circ \epsilon \circ \Phi = \varphi^{-1} \circ \varphi \circ \epsilon = \epsilon.$$

Finally, for any two automorphism (Φ, φ) and (Ψ, ψ) of $\mathcal{C}(A, H)$, we have

$$\Phi \triangleright (\Psi \triangleright (\sigma)) = \varphi^{-1} \circ \psi^{-1} \circ \sigma \circ \Psi \circ \Phi = (\psi\varphi)^{-1} \circ \sigma \circ \Psi \circ \Phi = (\Psi \circ \Phi) \triangleright \sigma.$$

In particular $\Phi^{-1} \triangleright (\Phi \triangleright (\sigma)) = \sigma$ and so the action is an automorphism of $\mathcal{B}(\mathcal{C}(A, H))$. $\square$

Lemma 14.5. *For any automorphism (Φ, φ), and any $\sigma \in \mathcal{B}(\mathcal{C}(A, H))$ we have*

$$Ad_{\Phi \triangleright \sigma} = \Phi^{-1} \circ Ad_\sigma \circ \Phi.$$

Proof. With $a \otimes \tilde{a} \in \mathcal{C}(A, H)$, from (14.5) we get

$$(Ad_\sigma \circ \Phi)(a \otimes \tilde{a}) = \sigma((\Phi(a \otimes \tilde{a}))_{(1)})\, (\Phi(a \otimes \tilde{a}))_{(2)}\, (\sigma \circ s) \circ \sigma^{-1}((\Phi(a \otimes \tilde{a}))_{(3)}), \qquad (14.8)$$

while, using (14.6) and (14.7), we have

$$\begin{aligned}Ad_{\Phi \triangleright \sigma}(a \otimes \tilde{a}) &= (\Phi \triangleright \sigma)((a \otimes \tilde{a})_{(1)})\, (a \otimes \tilde{a})_{(2)}\, \big((\Phi \triangleright \sigma) \circ s\big) \circ (\Phi \triangleright \sigma)^{-1}((a \otimes \tilde{a})_{(3)}) \\ &= \varphi^{-1}\big(\sigma(\Phi((a \otimes \tilde{a})_{(1)}))\big)\, (a \otimes \tilde{a})_{(2)}\, \big((\Phi \triangleright \sigma) \circ s\big) \circ \big(\varphi^{-1} \circ \sigma^{-1}\big)(\Phi((a \otimes \tilde{a})_{(3)})).\end{aligned}$$

Since Φ is a bimodule map: $\Phi(b(a \otimes \tilde{a})\tilde{b}) = \varphi(b)\Phi(a \otimes \tilde{a})\varphi(\tilde{b})$, for all $b, \tilde{b} \in B$, we get,

$$(\Phi \circ Ad_{\Phi \triangleright \sigma})(a \otimes \tilde{a}) = \sigma(\Phi((a \otimes \tilde{a})_{(1)}))\, \Phi((a \otimes \tilde{a})_{(2)})\, (\sigma \circ s) \circ \sigma^{-1}(\Phi((a \otimes \tilde{a})_{(3)})). \qquad (14.9)$$

That the right hand sides of (14.8) and (14.9) are equal follows from the equavariance condition (iii) of Definition 6.8. $\square$

Lemma 14.6. *Let σ, $\tau \in \mathcal{B}(\mathcal{C}(A, H))$, then $Ad_\tau \triangleright \sigma = \tau * \sigma * \tau^{-1}$.*

Proof. With $a \otimes \tilde{a} \in \mathcal{C}(A, H)$, using the definition (14.5) we compute

$$\begin{aligned}Ad_\tau \triangleright \sigma(a \otimes \tilde{a}) &= (ad_\tau^{-1} \circ \sigma)(Ad_\tau(a \otimes \tilde{a})) \\ &= ((\tau \circ s)^{-1} \circ \sigma)\big(\tau((a \otimes \tilde{a})_{(1)})\, (a \otimes \tilde{a})_{(2)}\, (\tau \circ s) \circ \tau^{-1}((a \otimes \tilde{a})_{(3)})\big) \\ &= (\tau^{-1} \circ s)\Big((\sigma \circ s)\big(\tau((a \otimes \tilde{a})_{(1)})\big)\, \sigma((a \otimes \tilde{a})_{(2)})\, (\sigma \circ t) \circ (\tau \circ s \circ \tau^{-1})((a \otimes \tilde{a})_{(3)})\Big) \\ &= (\tau^{-1} \circ s)\Big((\sigma \circ s)\big(\tau((a \otimes \tilde{a})_{(1)})\big)\, \sigma((a \otimes \tilde{a})_{(2)})\, (\tau \circ s) \circ \tau^{-1}((a \otimes \tilde{a})_{(3)})\Big) \\ &= (\tau^{-1} \circ s)\Big((\tau * \sigma)((a \otimes \tilde{a})_{(1)})\Big)\, \tau^{-1}((a \otimes \tilde{a})_{(2)}) \\ &= \tau * \sigma * \tau^{-1}(a \otimes \tilde{a}),\end{aligned}$$

where we used $(\tau \circ s)^{-1} = \tau^{-1} \circ s$, $\sigma \circ t = \mathrm{id}_B$ and definition (12.10) for the product. $\square$

Taken together the previous lemmas establish that a crossed module structure to $\big(\mathcal{B}(\mathcal{C}(A, H)), \mathrm{Aut}(\mathcal{C}(A, H)), Ad, \triangleright\big)$, which is the content of Theorem 14.1.

14.2. **CoInner authomorphisms of bialgebroids.** Given a Hopf algebra H and a character $\phi : H \to \mathbb{C}$, one defines a Hopf algebra automorphisms (see [31, page 3807]) by

$$\mathrm{coinn}(\phi) : H \to H, \qquad \mathrm{coinn}(\phi)(h) := \phi(h_{(1)})h_{(2)}\phi(S(h_{(3)})), \tag{14.10}$$

for any $h \in H$. Recall that for a character $\phi^{-1} = \phi \circ S$. The set $\mathrm{CoInn}(H)$ of co-inner authomorphisms of H is a normal subgroup of the group $\mathrm{Aut}_{\mathrm{Hopf}}(H)$ of Hopf algebra automorphisms (this is just $\mathrm{Aut}(H)$ if one sees view H as a bialgebroid over $\mathbb{C}$).

We know from the previous sections that for a Galois object A of a Hopf algebra H, the corresponding bialgebroid $\mathcal{C}(A, H)$ is a Hopf algebra. Also, the group of gauge transformations of the Galois object which is the same as the group of bisections can be identified with the group of characters of $\mathcal{C}(A, H)$ (see (13.5)). It turns out that these groups are also isomorphic to $\mathrm{CoInn}(\mathcal{C}(A, H))$. We have the following lemma:

Lemma 14.7. *For a Galois object A of a Hopf algebra H, let $\mathcal{C}(A, H)$ be the corresponding bialgebroid of A. If $\phi \in \mathcal{B}(\mathcal{C}(A, H)) = \mathrm{Char}(\mathcal{C}(A, H))$, then $Ad_\phi = \mathrm{coinn}(\phi)$.*

Proof. Let $\phi \in \mathrm{Char}(\mathcal{C}(A, H))$; then $\phi^{-1} = \phi \circ S_{\mathcal{C}}$. Substituting the latter in (14.5), for $a \otimes a' \in \mathcal{C}(A, H)$, we get

$$\begin{aligned} Ad_\phi(a \otimes \tilde{a}) &= \phi((a \otimes \tilde{a})_{(1)})\, (a \otimes \tilde{a})_{(2)}\, (\phi \circ S_{\mathcal{C}})((a \otimes \tilde{a})_{(3)}) \\ &= \mathrm{coinn}(\phi)(a \otimes \tilde{a}), \end{aligned}$$

as claimed. □

Example 14.8. Let us consider again the Taft algebra T_N of Section 13.5. We know from Proposition 13.8 that for any T_N-Galois object A_s the bialgebroid $\mathcal{C}(T_N, A_s)$ is isomorphic to T_N and bisections of $\mathcal{C}(T_N, A_s)$ are the same as characters of T_N the group of which is isomorphic to $\mathbb{Z}_N$. A generic character is a map $\phi_r : T_N \to \mathbb{C}$, given on generators x and g by $\phi_r(x) = 0$ and $\phi_r(g) = r$ for r a N-root of unity $r^N = 1$. The corresponding automorphism $Ad_{\phi_r} = \mathrm{coinn}(\phi_r)$ is easily found to be on generators given by

$$\mathrm{coinn}(\phi_r)(g) = g, \qquad \mathrm{coinn}(\phi_r)(x) = r^{-1}x\,.$$

It is known (cf. [32], Lemma 2.1) that $\mathrm{Aut}(T_N) \simeq \mathrm{Aut}_{\mathrm{Hopf}}(T_N) \simeq \mathbb{C}^\times$: Indeed, given $r \in \mathbb{C}^\times$, one defines an authomorphism $F_r : T_N \to T_N$ by $F_r(x) := rx$ and $F_r(g) := g$. Thus $Ad : \mathrm{Char}(T_N) \to \mathrm{Aut}(T_N)$ is the injection sending ϕ_r to $F_{r^{-1}}$.

Moreover, for $F \in \mathrm{Aut}(T_N)$ and $\phi \in \mathrm{Char}(T_N)$, one checks that $Ad_{F \triangleright \phi}(x) = Ad_\phi(x)$ and $Ad_{F \triangleright \phi}(g) = Ad_\phi(g)$. Thus, as a crossed module, the action of $\mathrm{Aut}(T_N)$ on $\mathrm{Char}(T_N)$ is trival and the crossed module $(\mathrm{Char}(\mathcal{C}(T_N, A_s)), \mathrm{Aut}(\mathcal{C}(T_N, A_s)), Ad, \mathrm{id})$ is isomorphic to $(\mathbb{Z}_N, \mathbb{C}^\times, i, \mathrm{id})$, with inclusion $i : \mathbb{Z}_N \to \mathbb{C}^\times$ given by $i(r) := e^{-i2r\pi/N}$ and $\mathbb{C}^\times$ acting trivially on $\mathbb{Z}_N$.

14.3. **Crossed module structures on extended bisections.** In parallel with the crossed module structure on bialgebroid automorphisms and bisections, there is a similar structure on the set of 'extended' bialgebroid automorphisms and extended bisections.

Given a left bialgebroid $(\mathcal{L}, \Delta, \epsilon, s, t)$ be a left bialgebroid over the algebra B. An extended automorphism of $\mathcal{L}$ is a pair (Φ, φ) with $\varphi : B \to B$ an algebra map and a unital invertible linear map $\Phi : \mathcal{L} \to \mathcal{L}$, obeying the properties $(i) - (iv)$ of Definition 6.8.

So, an extended automorphism is not required in general to be an algebra map while still satisfying all other properties of an automorphism. In particular we still have the bimodule property: $\Phi(ba\tilde{b}) = \varphi(b)\Phi(a)\varphi(\tilde{b})$. We denote by $\mathrm{Aut}^{ext}(\mathcal{L})$ the group (by composition) of extended automorphisms of $\mathcal{L}$. There is an analogous of Theorem 14.1:

Theorem 14.9. *Given a Hopf–Galois extension $B = A^{coH} \subseteq A$, let $\mathcal{C}(A,H)$ be the corresponding left Ehresmann–Schauenburg bialgebroid, and assume B be in the centre of A. Then there is a group morphism $Ad : \mathcal{B}^{ext}(\mathcal{C}(A,H)) \to \mathrm{Aut}^{ext}(\mathcal{C}(A,H))$ and an action $\triangleright$ of $\mathrm{Aut}^{ext}(\mathcal{C}(A,H))$ on $\mathcal{B}^{ext}(\mathcal{C}(A,H))$ such that the group of extended automorphisms $\mathrm{Aut}^{ext}(\mathcal{C}(A,H))$ and of extended bisections $\mathcal{B}^{ext}(\mathcal{C}(A,H))$, form a crossed module.*

This result is established in parallel and similarly to the proof of Theorem 14.1. Here we shall only point to the differences in the definitions and the proofs.

Thus, under the hypothesis of Theorem 14.9, for any bisection $\sigma \in \mathcal{B}^{ext}(\mathcal{C}(A,H))$, define $Ad_\sigma : \mathcal{C}(A,H) \to \mathcal{C}(A,H)$, for any $a \otimes \tilde{a} \in \mathcal{C}(A,H)$, by

$$\begin{aligned} Ad_\sigma(a \otimes \tilde{a}) &:= (\sigma(a_{(0)} \otimes a_{(1)}{}^{<1>})a_{(1)}{}^{<2>}) \otimes \big(a_{(2)}{}^{<1>}(\sigma \circ s) \circ \sigma^{-1}(a_{(2)}{}^{<2>} \otimes \tilde{a})\big), \\ &= \sigma((a \otimes \tilde{a})_{(1)})\, (a \otimes \tilde{a})_{(2)}\, (\sigma \circ s) \circ \sigma^{-1}((a \otimes \tilde{a})_{(3)}) \end{aligned} \tag{14.11}$$

in parallel with (14.5). Then the pair of map $(Ad_\sigma, ad_\sigma = \sigma \circ s)$ is an extended automorphism of $\mathcal{C}(A,H)$. In particular we have, for any $c = a \otimes \tilde{a} \in \mathcal{C}(A,H)$,

$$\begin{aligned} \Delta_{\mathcal{C}}(Ad_\sigma(c)) &= \sigma(c_{(1)})\, c_{(2)} \otimes_B c_{(3)}\, (\sigma \circ s) \circ \sigma^{-1}(c_{(4)}) \\ &= \sigma(c_{(1)})\, c_{(2)}\, (\sigma \circ s) \circ \sigma^{-1}(c_{(3)}) \otimes_B \sigma(c_{(4)})\, c_{(5)}\, (\sigma \circ s) \circ \sigma^{-1}(c_{(6)}) \end{aligned}$$

where the 2nd step use $(\sigma \circ s) \circ \sigma^{-1}(c_{(1)})\, \sigma(c_{(2)}) = \epsilon(c)$. With the latter, we have also,

$$\begin{aligned} (\epsilon \circ Ad_\sigma)(c) &= \sigma(c_{(1)})\, \epsilon(c_{(2)})\, (\sigma \circ s) \circ \sigma^{-1}(c_{(3)}) \\ &= \sigma(c_{(1)})\, (\sigma \circ s) \circ \sigma^{-1}(c_{(2)}) \\ &= (\sigma \circ s)\, \epsilon(c) \\ &= ad_\sigma(\epsilon(c)). \end{aligned}$$

Moreover, for two extended bisections σ and τ we have, for $c \in \mathcal{C}(A,H)$,

$$\begin{aligned} Ad_\sigma \circ Ad_\tau(c) &= Ad_\sigma\big(\tau(c_{(1)})\, c_{(2)}\, (\tau \circ s) \circ \tau^{-1}(c_{(3)})\big) \\ &= s\big(ad_\sigma(\tau(c_{(1)}))\big)\, Ad_\sigma(c_{(2)})\, t\,\big(ad_\sigma \circ (\tau \circ s) \circ \tau^{-1}(c_{(3)})\big) \\ &= s\Big(\sigma\big(s(\tau(c_{(1)}))\big)\Big)\Big(\sigma(c_{(2)})\, c_{(3)}\, \big(\sigma(s(\sigma^{-1}(c_{(4)})))\big)\Big)\, t \circ \big(\sigma \circ s) \circ (\tau \circ s) \circ \tau^{-1}(c_{(5)})\big) \\ &= \Big(\sigma\big(s(\tau(c_{(1)}))\big)\sigma\big(c_{(2)}\big)\Big)\, c_{(3)}\, \Big((\sigma \circ s \circ \tau \circ s)\big((\tau^{-1} \circ s) \circ \sigma^{-1}(c_{(4)})\, \tau^{-1}(c_{(5)})\big)\Big) \\ &= (\tau * \sigma)(c_{(1)})\, c_{(2)}\, \big((\tau * \sigma) \circ s)(\sigma^{-1} * \tau^{-1}(c_{(3)})\big) \\ &= Ad_{\tau * \sigma}(c), \end{aligned}$$

with the 2nd step using Ad_σ is a B-bimodule map. One also shows $ad_\sigma \circ ad_\tau = ad_{\tau*\sigma}$ and $(Ad_\epsilon, ad_\epsilon) = (\mathrm{id}_{\mathcal{C}(A,H)}, \mathrm{id}_B)$. Therefore (Ad_σ, ad_σ) is invertible with inverse $(Ad_{\sigma^{-1}}, ad_{\sigma^{-1}})$.

When σ is an algebra map, (14.11) reduces to (14.1) (or equivalently to(14.5)).

If (Φ, φ) is an extended automorphism of $\mathcal{C}(A,H)$ with inverse $(\Phi^{-1}, \varphi^{-1})$ the formula (14.6) is an action of (Φ, φ) on $\mathcal{B}^{ext}(\mathcal{C}(A,H))$, a group automorphism of $\mathcal{B}^{ext}(\mathcal{C}(A,H))$.

We only check $F \triangleright \sigma$ is well defined as an extended bisection since the rest goes as in the previous section. For $a \otimes \tilde{a} \in \mathcal{C}(A,H)$ and $b \in B$, a direct computation yields:

$$\begin{aligned} (\Phi \triangleright \sigma)((a \otimes \tilde{a}) \triangleleft b) &= (\Phi \triangleright \sigma)(a \otimes \tilde{a})\, b \\ (\Phi \triangleright \sigma)(b \triangleright (a \otimes \tilde{a})) &= (\Phi \triangleright \sigma)(s(b))\, (\Phi \triangleright \sigma)(a \otimes \tilde{a}). \end{aligned}$$

Finally, with similar computation as those of Lemma 14.5 and Lemma 14.6 one shows that for any extended automorphism (Φ, φ), and any $\sigma \in \mathcal{B}^{ext}(\mathcal{C}(A,H))$ one has

$$Ad_{\Phi \triangleright \sigma} = \Phi^{-1} \circ Ad_\sigma \circ \Phi.$$

And that, with σ, $\tau \in \mathcal{B}^{ext}(\mathcal{C}(A,H))$, one has

$$Ad_\tau \triangleright \sigma = \tau * \sigma * \tau^{-1}.$$

Example 14.10. Consider a H-Galois object A and let $\mathcal{C}(A,H)$ be the corresponding bialgebroid, a Hopf algebra itself. Given an extended bisection $\sigma \in \mathcal{B}^{ext}(\mathcal{C}(A,H)) \simeq \mathrm{Char}^{ext}(H)$, the expression (14.11) reduces to

$$Ad_\sigma(a \otimes \tilde{a}) = \sigma((a \otimes \tilde{a})_{(1)})\,(a \otimes \tilde{a})_{(2)}\,\sigma^{-1}((a \otimes \tilde{a})_{(3)})$$

In analogy with (14.10) to which it reduces when ϕ is a character (and Lemma 14.7) we may thing of this unital invertible coalgebra map as defining an extended coinner authomorphism of $\mathcal{C}(A,H)$, $\mathrm{coinn}(\sigma)(c) := Ad_\sigma(c) = \sigma(c_{(1)})c_{(2)}\sigma^{-1}(c_{(3)})$.

In Example 14.8 we constructed an Abelian crossed module for the Taft algebras. The following example present a non-Abelian crossed module for Taft algebras with respect to the extended characters and extended automorphisms.

Example 14.11. We know from Section 13.5 that the Schauenburg bialgebroid $\mathcal{C}(A_s,T_N)$ of any Galois object A_s for the Taft algebra T_N, is isomorphic to T_N itself. Thus $\mathrm{Aut}^{ext}(\mathcal{C}(A_s,T_N)) \simeq \mathrm{Aut}^{ext}(T_N)$ is the group of unital invertible coalgebra maps: maps $\Phi : T_N \to T_N$ such that $\Phi(h_{(1)}) \otimes \Phi(h_{(2)}) = \Phi(h)_{(1)} \otimes \Phi(h)_{(2)}$ for any $h \in T_N$ with $\Phi(1) = 1$.

Let us illustrate this for the case $N = 2$. The coproduct of T_2 on the generators x, g will then require the following condition for an automorphism Φ:

$$\begin{aligned}
\Phi(g)_{(1)} \otimes \Phi(g)_{(2)} &= \Phi(g) \otimes \Phi(g) \\
\Phi(x)_{(1)} \otimes \Phi(x)_{(2)} &= 1 \otimes \Phi(x) + \Phi(x) \otimes g \\
\Phi(xg)_{(1)} \otimes \Phi(xg)_{(2)} &= g \otimes \Phi(xg) + \Phi(xg) \otimes 1\,.
\end{aligned} \tag{14.12}$$

A little algebra then shows that

$$\begin{aligned}
\Phi(g) &= g \\
\Phi(x) &= c\,(g-1) + a_2\,x \\
\Phi(xg) &= b\,(1-g) + a_1\,xg
\end{aligned} \tag{14.13}$$

for arbitrary parameters $b, c \in \mathbb{C}$ and $a_1, a_2 \in \mathbb{C}^\times$ (for Φ to be invertible). As in (13.19) we can represent Φ as a matrix:

$$\Phi : \begin{pmatrix} 1 \\ xg \\ g \\ x \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ b & a_1 & -b & 0 \\ 0 & 0 & 1 & 0 \\ -c & 0 & c & a_2 \end{pmatrix} \begin{pmatrix} 1 \\ xg \\ g \\ x \end{pmatrix}. \tag{14.14}$$

One checks that matrices M_Φ of the form above form a group: $\mathrm{Aut}^{ext}(T_N) \simeq \mathrm{Aut}_{\mathrm{Hopf}}(T_N)$.

Given $\sigma \in \mathrm{Char}^{ext}(T_2)$ we shall denote $\sigma_a = \sigma(a) \in \mathbb{C}$ for $a \in \{1, x, g, xg\}$. For the convolution inverse σ^{-1}, from the condition $\sigma * \sigma^{-1} = \epsilon$ we get on the basis that

$$\begin{cases}
\sigma_1 = (\sigma^{-1})_1 = 1, \\
\sigma_g(\sigma^{-1})_g = 1, \\
\sigma_g(\sigma^{-1})_x + \sigma_x = 0, \\
\sigma_g(\sigma^{-1})_{xg} + \sigma_{xg} = 0
\end{cases} \tag{14.15}$$

from which we solve

$$\begin{cases} (\sigma^{-1})_g = (\sigma_g)^{-1}, \\ (\sigma^{-1})_x = -\sigma_x(\sigma_g)^{-1}, \\ (\sigma^{-1})_{xg} = -\sigma_{xg}(\sigma_g)^{-1} \end{cases} \tag{14.16}$$

Then computing $Ad_\sigma(h) = \sigma(h_{(1)})h_{(2)}\sigma^{-1}(h_{(3)})$ leads to

$$Ad_\sigma \begin{pmatrix} 1 \\ xg \\ g \\ x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \sigma_{xg} & \sigma_g & -\sigma_{xg} & 0 \\ 0 & 0 & 1 & 0 \\ -\sigma_x(\sigma_g)^{-1} & 0 & \sigma_x(\sigma_g)^{-1} & (\sigma_g)^{-1} \end{pmatrix}. \tag{14.17}$$

We see that the matrix (14.17) is of the form (14.14) with the restriction that $a_2 = a_1^{-1}$ so that Ad_ϕ has determinant 1. Clearly, the image of $\mathrm{Char}^{ext}(T_2)$ form a subgroup of $\mathrm{Aut}^{ext}(\mathcal{C}(A_s, T_2)) \simeq \mathrm{Aut}^{ext}(T_N)$. Moreover, $Ad : \mathrm{Char}^{ext}(T_2) \to \mathrm{Aut}^{ext}(T_2)$ is an injective map. Finally, the action $Ad_{\Phi \triangleright \sigma}$ will have as matrix just the product:

$$M_{Ad_{\Phi\triangleright\sigma}} = M_{\Phi^{-1}}\, M_{Ad_\sigma}\, M_\Phi \tag{14.18}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1^{-1}[\sigma_{xg} + b(\sigma_g - 1)] & \sigma_g & -a_1^{-1}[\sigma_{xg} + b(\sigma_g - 1)] & 0 \\ 0 & 0 & 1 & 0 \\ -a_2^{-1}[\sigma_x\,(\sigma_g)^{-1} + c((\sigma_g)^{-1} - 1)] & 0 & a_2^{-1}[\sigma_x\,(\sigma_g)^{-1} + c((\sigma_g)^{-1} - 1)] & (\sigma_g)^{-1} \end{pmatrix}$$

We conclude that as a crossed module the action on $\mathrm{Char}^{ext}(T_2)$ is not trivial.

Part 3. On coherent Hopf 2-algebras

15. Coherent Hopf-2-algebras

In [25], the researcher constructs a strict quantum 2-group, which can be viewed as a linear extension of strict 2-group. In [15], the quantum 2-groups are given by a crossed module and a crossed comodule of Hopf algebra. Both of these papers constructed strict quantum 2-group, while here we generalize them. When we consider the strict case, the coherent Hopf 2-algebra in the following will become the dual case of the strict quantum 2-group in [15].

In Section 2 we explain a special case of coherent 2-groups, whose morphisms and objects form a quasigroup, since usually we are interested in a more strict case where all the objects have strict inverse and the unit object of the monoidal category is also strict. Under this condition, we have a more interesting property of the associator. Therefore, base on the idea of 2-arrows quantisation we can construct a coherent quantum 2-group, which is also called coherent Hopf 2-algebra.

Definition 15.1. A *coherent Hopf 2-algebra* consists of a commutative Hopf coquasigroup $(B, m_B, 1_B, \Delta_B, \epsilon_B, S_B)$ and a Hopf coquasigroup $(H, m, 1_H, \blacktriangle, \epsilon_H, S_H)$, together with a central Hopf algebroid $(H, m, 1_H, \Delta, \epsilon, S)$ over B. Moreover, there is an algebra map (called coassociator) $\alpha : H \to B \otimes B \otimes B$. Denote the image of α by $\alpha(h) =: h^{\tilde{1}} \otimes h^{\tilde{2}} \otimes h^{\tilde{3}}$ for any $h \in H$, and Sweedler notation for both the coproduct of Hopf coquasigroup and Hopf algebroid, $\blacktriangle(h) =: h_{(1)} \otimes h_{(2)}$, $\Delta(h) =: h^{(1)} \otimes h^{(2)}$, such that all the structures above satisfy the following axioms:

(i) The underlying algebra of the Hopf coquasigroup $(H, m, 1_H, \blacktriangle, \epsilon_H, S_H)$ and the Hopf algebroid $(H, m, 1_H, \Delta, \epsilon, S)$ coincide with each other.
(ii) $\epsilon : H \to B$ is a morphism of Hopf coquasigroups.
(iii) $s, t : B \to H$ are morphisms of Hopf coquasigroups.
(iv) The two coproducts Δ and $\blacktriangle$ have the following cocommutation relation:

$$(\Delta \otimes \Delta) \circ \blacktriangle = (id_H \otimes \tau \otimes id_H) \circ (\blacktriangle \otimes_B \blacktriangle) \circ \Delta, \tag{15.1}$$

where $\tau : H \otimes H \to H \otimes H$ is given by $\tau(h \otimes g) := g \otimes h$.

(v)

$$\alpha \circ t = (\Delta_B \otimes id_B) \circ \Delta_B, \qquad \alpha \circ s = (id_B \otimes \Delta_B) \circ \Delta_B \tag{15.2}$$

(vi)

$$\begin{cases} \epsilon_B(h^{\tilde{1}})1_B \otimes h^{\tilde{2}} \otimes h^{\tilde{3}} & = 1_B \otimes \epsilon(h_{(1)}) \otimes \epsilon(h_{(2)}) \\ h^{\tilde{1}} \otimes \epsilon_B(h^{\tilde{2}})1_B \otimes h^{\tilde{3}} & = \epsilon(h_{(1)}) \otimes 1_B \otimes \epsilon(h_{(2)}) \\ h^{\tilde{1}} \otimes h^{\tilde{2}} \otimes \epsilon_B(h^{\tilde{3}})1_B & = \epsilon(h_{(1)}) \otimes \epsilon(h_{(2)}) \otimes 1_B. \end{cases} \tag{15.3}$$

(vii)

$$\begin{cases} h^{\tilde{1}} S_B(h^{\tilde{2}}) \otimes h^{\tilde{3}} & = S_B(h^{\tilde{1}})h^{\tilde{2}} \otimes h^{\tilde{3}} = 1_B \otimes \epsilon(h) \\ h^{\tilde{1}} \otimes S_B(h^{\tilde{2}})h^{\tilde{3}} & = h^{\tilde{1}} \otimes h^{\tilde{2}} S_B(h^{\tilde{3}}) = \epsilon(h) \otimes 1_B. \end{cases} \tag{15.4}$$

(viii) Let $\star$ be the convolution product corresponding to the Hopf algebroid coproduct, we have

$$((s \otimes s \otimes s) \circ \alpha) \star ((\blacktriangle \otimes id_H) \circ \blacktriangle) = ((id_H \otimes \blacktriangle) \circ \blacktriangle) \star ((t \otimes t \otimes t) \circ \alpha) \tag{15.5}$$

More precisely,

$$
\begin{aligned}
&s(h^{(1)\tilde{1}})h^{(2)}{}_{(1)(1)} \otimes s(h^{(1)\tilde{2}})h^{(2)}{}_{(1)(2)} \otimes s(h^{(1)\tilde{3}})h^{(2)}{}_{(2)}\\
=&h^{(1)}{}_{(1)}t(h^{(2)\tilde{1}}) \otimes h^{(1)}{}_{(2)(1)}t(h^{(2)\tilde{2}}) \otimes h^{(1)}{}_{(2)(2)}t(h^{(2)\tilde{3}}),
\end{aligned}
$$

(ix) The 3-cocycle condition:

$$((\epsilon\otimes\alpha)\circ\blacktriangle)\star((id_B\otimes\Delta_B\otimes id_B)\circ\alpha)\star((\alpha\otimes\epsilon)\circ\blacktriangle) = ((id_B\otimes id_B\otimes\Delta_B)\circ\alpha)\star((\Delta_B\otimes id_B\otimes id_B)\circ\alpha). \tag{15.6}$$

More precisely,

$$
\begin{aligned}
&\epsilon(h^{(1)}{}_{(1)})h^{(2)\tilde{1}}h^{(3)}{}_{(1)}{}^{\tilde{1}} \otimes h^{(1)}{}_{(2)}{}^{\tilde{1}}h^{(2)\tilde{2}}{}_{(1)}h^{(3)}{}_{(1)}{}^{\tilde{2}} \otimes h^{(1)}{}_{(2)}{}^{\tilde{2}}h^{(2)\tilde{2}}{}_{(2)}h^{(3)}{}_{(1)}{}^{\tilde{3}} \otimes h^{(1)}{}_{(2)}{}^{\tilde{3}}h^{(2)\tilde{3}}\epsilon(h^{(3)}{}_{(2)})\\
=&h^{(1)\tilde{1}}h^{(2)\tilde{1}}{}_{(1)} \otimes h^{(1)\tilde{2}}h^{(2)\tilde{1}}{}_{(2)} \otimes h^{(1)\tilde{3}}{}_{(1)}h^{(2)\tilde{2}} \otimes h^{(1)\tilde{3}}{}_{(2)}h^{(2)\tilde{3}}.
\end{aligned}
$$

A coherent Hopf 2-algebra is called a *strict Hopf 2-algebra*, if H and B are coassociative (H and B are Hopf algebras), and $\alpha = (\epsilon \otimes \epsilon \otimes \epsilon) \circ (\blacktriangle \otimes id_H) \circ \blacktriangle$.

Now let's explain why Definition 15.1 is a quantisation of a coherent 2-group (see Definition 4.4 and Proposition 4.7), whose objects form a quasigroup. First, the morphisms and their composition form a groupoid, which corresponds to a Hopf algebroid, and the tensor product of objects and morphisms forms a quasigroup, which corresponds to a Hopf coquasigroup.

By the definition of monoidal category, we can see that axiom (ii), (iii) and (iv) are natural, since the source and target maps from objects to morphisms preserve the tensor product, and the identity map from objects to morphisms also preserves the tensor product. The interchange law corresponds to condition (iv). The source and target of the morphism $\alpha_{g,h,k}$ is $(gh)k$ and $g(hk)$, which corresponds to condition (v). Since $\alpha_{1,g,h} = \alpha_{g,1,h} = \alpha_{g,h,1} = id_{gh}$, we have condition (vi). Because $\alpha_{g,g^{-1},h} = \alpha_{g^{-1},g,h} = id_h$ and $\alpha_{h,g,g^{-1}} = \alpha_{h,g^{-1},g} = id_h$, we have the corresponding (vii). The naturality of α corresponds to condition (viii), the pentagon corresponds to condition (ix). Here we will still call Definition 15.1 'coherent Hopf 2-algebra' even though it only corresponds to a special case of "quantum" coherent 2-group.

Remark 15.2. For a strict Hopf 2-algebra, we can see the morphisms s, t and ϵ are morphisms of Hopf algebras, and (v), (vi), (vii) are automatically satisfied. For (viii), we have

$$
\begin{aligned}
&((s \otimes s \otimes s) \circ \alpha) \star ((\blacktriangle \otimes id_H) \circ \blacktriangle)(h)\\
=&s(\epsilon(h^{(1)}{}_{(1)}))h^{(2)}{}_{(1)} \otimes s(\epsilon(h^{(1)}{}_{(2)}))h^{(2)}{}_{(2)} \otimes s(\epsilon(h^{(1)}{}_{(3)}))h^{(2)}{}_{(3)}\\
=&(\blacktriangle \otimes id_H) \circ \blacktriangle(h)\\
=&h^{(1)}{}_{(1)}t(\epsilon(h^{(2)}{}_{(1)})) \otimes h^{(1)}{}_{(2)}t(\epsilon(h^{(2)}{}_{(2)})) \otimes h^{(1)}{}_{(3)}t(\epsilon(h^{(2)}{}_{(3)}))\\
=&((id_H \otimes \blacktriangle) \circ \blacktriangle) \star ((t \otimes t \otimes t) \circ \alpha)(h).
\end{aligned}
$$

For (ix) we can see the left hand side is

$$
\begin{aligned}
&\epsilon(h^{(1)}{}_{(1)})h^{(2)\tilde{1}}h^{(3)}{}_{(1)}{}^{\tilde{1}} \otimes h^{(1)}{}_{(2)}{}^{\tilde{1}}h^{(2)\tilde{2}}{}_{(1)}h^{(3)}{}_{(1)}{}^{\tilde{2}} \otimes h^{(1)}{}_{(2)}{}^{\tilde{2}}h^{(2)\tilde{2}}{}_{(2)}h^{(3)}{}_{(1)}{}^{\tilde{3}} \otimes h^{(1)}{}_{(2)}{}^{\tilde{3}}h^{(2)\tilde{3}}\epsilon(h^{(3)}{}_{(2)})\\
=&\epsilon(h^{(1)}{}_{(1)})\epsilon(h^{(2)}{}_{(1)})\epsilon(h^{(3)}{}_{(1)}) \otimes \epsilon(h^{(1)}{}_{(2)})\epsilon(h^{(2)}{}_{(2)})\epsilon(h^{(3)}{}_{(2)})\\
&\otimes\epsilon(h^{(1)}{}_{(3)})\epsilon(h^{(2)}{}_{(3)})\epsilon(h^{(3)}{}_{(3)}) \otimes \epsilon(h^{(1)}{}_{(4)})\epsilon(h^{(2)}{}_{(4)})\epsilon(h^{(3)}{}_{(4)}),
\end{aligned}
$$

while the right hand side is

$$
\begin{aligned}
&h^{(1)\tilde{1}}h^{(2)\tilde{1}}{}_{(1)} \otimes h^{(1)\tilde{2}}h^{(2)\tilde{1}}{}_{(2)} \otimes h^{(1)\tilde{3}}{}_{(1)}h^{(2)\tilde{2}} \otimes h^{(1)\tilde{3}}{}_{(2)}h^{(2)\tilde{3}}\\
=&\epsilon(h^{(1)}{}_{(1)})\epsilon(h^{(2)}{}_{(1)}) \otimes \epsilon(h^{(1)}{}_{(2)})\epsilon(h^{(2)}{}_{(2)}) \otimes \epsilon(h^{(1)}{}_{(3)})\epsilon(h^{(2)}{}_{(3)}) \otimes \epsilon(h^{(1)}{}_{(4)})\epsilon(h^{(2)}{}_{(4)}),
\end{aligned}
$$

using the fact that $\epsilon(h^{(1)})\epsilon(h^{(2)}) = \epsilon(s(\epsilon(h^{(1)})))\epsilon(h^{(2)}) = \epsilon(s(\epsilon(h^{(1)}))h^{(2)}) = \epsilon(h)$, we get that the left and right hand side of (ix) are equal.

Remark 15.3. In general, for any Hopf algebroid over an algebra B, the base algebra B is not necessary commutative. However, in order to give a nice definition of coherent Hopf 2-algebras, we need the maps ϵ, s, t to be Hopf algebra maps, since only in that case condition (iv), (viii) and (ix) make sense. As a result, we assume that the Hopf algebroid H is a central Hopf algebroid, so the base algebra B is a commutative algebra.

(1) In condition (iv), $\blacktriangle \otimes_B \blacktriangle : H \otimes_B H \to (H \otimes H) \otimes_{B\otimes B} (H \otimes H)$ is well defined since $H \otimes H$ has $B \otimes B$-bimodule structure: $(b \otimes b') \triangleright (h \otimes h') = s(b)h \otimes s(b')h'$ and $(h \otimes h') \triangleleft (b \otimes b') = t(b)h \otimes t(b')h'$, for any $b \otimes b' \in B \otimes B$ and $h \otimes h' \in H \otimes H$. Indeed, for any $b \in B$ and h, $h' \in H$ we have

$$\begin{aligned}(\blacktriangle \otimes_B \blacktriangle)(h \otimes_B b \triangleright h') =&(\blacktriangle \otimes_B \blacktriangle)(h \otimes_B s(b)h')\\ =&(h_{(1)} \otimes h_{(2)}) \otimes_{B\otimes B} (s(b)_{(1)}g_{(1)} \otimes s(b)_{(2)}g_{(2)})\\ =&(h_{(1)} \otimes h_{(2)}) \otimes_{B\otimes B} (s(b_{(1)})g_{(1)} \otimes s(b_{(2)})g_{(2)})\\ =&(t(b_{(1)})h_{(1)} \otimes t(b_{(2)})h_{(2)}) \otimes_{B\otimes B} (g_{(1)} \otimes g_{(2)})\\ =&(t(b)_{(1)}h_{(1)} \otimes t(b)_{(2)}h_{(2)}) \otimes_{B\otimes B} (g_{(1)} \otimes g_{(2)})\\ =&(\blacktriangle \otimes_B \blacktriangle)(h \triangleleft b \otimes_B h'),\end{aligned}$$

where in the 2nd step we use that $\blacktriangle$ is an algebra map, and in the 3rd step we use the fact that s is a coalgebra map. Clearly, the map $id_H \otimes \tau \otimes id_H : h \otimes h' \otimes_{B\otimes B} g \otimes g' \mapsto (h \otimes_B g) \otimes (h' \otimes_B g')$ is also well defined for any $h \otimes h', g \otimes g' \in H \otimes H$. Concretely, (iv) can be written as

$$h_{(1)}{}^{(1)} \otimes_B h_{(1)}{}^{(2)} \otimes h_{(2)}{}^{(1)} \otimes_B h_{(2)}{}^{(2)} = h^{(1)}{}_{(1)} \otimes_B h^{(2)}{}_{(1)} \otimes h^{(1)}{}_{(2)} \otimes_B h^{(2)}{}_{(2)}, \tag{15.7}$$

for any $h \in H$.

(3) By using condition (v) and the fact that s, t are bialgebra maps, we get that (viii) is well defined over the balanced tensor product $\otimes_B$, since $(s \otimes s \otimes s) \circ \alpha \circ t = (\blacktriangle \otimes id_H) \circ \blacktriangle \circ s$ and $(t \otimes t \otimes t) \circ \alpha \circ s = (id_H \otimes \blacktriangle) \circ \blacktriangle \circ t$.

(4) The left hand side of (15.6) is well defined since:

$$\begin{aligned}((\epsilon \otimes \alpha) \circ \blacktriangle)(t(b)) =&\epsilon(t(b_{(1)})) \otimes \alpha(t(b_{(2)})) = b_{(1)} \otimes b_{(2)(1)(1)} \otimes b_{(2)(1)(2)} \otimes b_{(2)(2)}\\ =&(id_B \otimes \Delta_B \otimes id_B) \circ \alpha(s(b)).\end{aligned}$$

$$\begin{aligned}(id_B \otimes \Delta_B \otimes id_B) \circ \alpha(t(b)) =&b_{(1)(1)} \otimes b_{(1)(2)(1)} \otimes b_{(1)(2)(2)} \otimes b_{(2)} = \alpha(s(b_{(1)})) \otimes \epsilon(s(b_{(2)}))\\ =&((\alpha \otimes \epsilon) \circ \blacktriangle)(s(b)).\end{aligned}$$

The right hand side of (15.6) is also well defined, indeed,

$$((id_B \otimes id_B \otimes \Delta_B) \circ \alpha)(t(b)) = b_{(1)(1)} \otimes b_{(1)(2)} \otimes b_{(2)(1)} \otimes b_{(2)(2)} = ((\Delta_B \otimes id_B \otimes id_B) \circ \alpha)(s(b)).$$

Proposition 15.4. *Given a coherent Hopf 2-algebra as in Definition 15.1, the antipodes have the following property:*

(i) $\Delta \circ S_H = (S_H \otimes_B S_H) \circ \Delta$.

(ii) *S is a coalgebra map on $(H, \blacktriangle, \epsilon_H)$. In other words, $\blacktriangle \circ S = (S \otimes S) \circ \blacktriangle$ and $\epsilon_H \circ S = \epsilon_H$.*

(iii) *If H is commutative, $S \circ S_H = S_H \circ S$.*

Proof. For (i), let $h \in H$,

$$\begin{aligned}
& S_H(h^{(1)}) \otimes_B S_H(h^{(2)}) \\
=& (S_H({h_{(1)(1)}}^{(1)}) \otimes_B S_H({h_{(1)(1)}}^{(2)}))(\Delta(h_{(1)(2)}S_H(h_{(2)}))) \\
=& (S_H({h_{(1)(1)}}^{(1)}) \otimes_B S_H({h_{(1)(1)}}^{(2)}))({h_{(1)(2)}}^{(1)} \otimes_B {h_{(1)(2)}}^{(2)})((S_H(h_{(2)}))^{(1)} \otimes_B (S_H(h_{(2)}))^{(2)}) \\
=& (S_H({h_{(1)}}^{(1)}{}_{(1)}) \otimes_B S_H({h_{(1)}}^{(2)}{}_{(1)}))({h_{(1)}}^{(1)}{}_{(2)} \otimes_B {h_{(1)}}^{(2)}{}_{(2)})((S_H(h_{(2)}))^{(1)} \otimes_B (S_H(h_{(2)}))^{(2)}) \\
=& (\epsilon_H({h_{(1)}}^{(1)}) \otimes_B \epsilon_H({h_{(1)}}^{(2)}))((S_H(h_{(2)}))^{(1)} \otimes_B (S_H(h_{(2)}))^{(2)}) \\
=& \epsilon_B(\epsilon(s(\epsilon({h_{(1)}}^{(1)}))h_{(1)}{}^{(2)}))((S_H(h_{(2)}))^{(1)} \otimes_B (S_H(h_{(2)}))^{(2)}) \\
=& \epsilon_H(h_{(1)})((S_H(h_{(2)}))^{(1)} \otimes_B (S_H(h_{(2)}))^{(2)}) \\
=& (S_H(h))^{(1)} \otimes_B (S_H(h))^{(2)}
\end{aligned}$$

For (ii), on the one hand we have

$$\begin{aligned}
& (S(h^{(1)}{}_{(1)}) \otimes S(h^{(1)}{}_{(2)}))(h^{(2)}{}_{(1)} \otimes h^{(2)}{}_{(2)})(S(h^{(3)})_{(1)} \otimes S(h^{(3)})_{(2)}) \\
=& (S(h^{(1)}{}_{(1)}) \otimes S(h^{(1)}{}_{(2)}))(h^{(2)(1)}{}_{(1)} \otimes h^{(2)(1)}{}_{(2)})(S(h^{(2)(2)})_{(1)} \otimes S(h^{(2)(2)})_{(2)}) \\
=& (S(h^{(1)}{}_{(1)}) \otimes S(h^{(1)}{}_{(2)}))(\blacktriangle(h^{(2)(1)}S(h^{(2)(2)}))) \\
=& (S(h^{(1)}{}_{(1)}) \otimes S(h^{(1)}{}_{(2)}))(s(\epsilon(h^{(2)}))_{(1)} \otimes s(\epsilon(h^{(2)}))_{(2)}) \\
=& (S(h^{(1)}{}_{(1)}) \otimes S(h^{(1)}{}_{(2)}))(s(\epsilon(h^{(2)}{}_{(1)})) \otimes s(\epsilon(h^{(2)}{}_{(2)}))) \\
=& S(h^{(1)}{}_{(1)}t(\epsilon(h^{(2)}{}_{(1)}))) \otimes S(h^{(1)}{}_{(2)}t(\epsilon(h^{(2)}{}_{(2)}))) \\
=& S({h_{(1)}}^{(1)}t(\epsilon({h_{(1)}}^{(2)}))) \otimes S({h_{(2)}}^{(1)}t(\epsilon({h_{(2)}}^{(2)}))) \\
=& S(h_{(1)}) \otimes S(h_{(2)})
\end{aligned}$$

on the other hand we have

$$\begin{aligned}
& (S(h^{(1)}{}_{(1)}) \otimes S(h^{(1)}{}_{(2)}))(h^{(2)}{}_{(1)} \otimes h^{(2)}{}_{(2)})(S(h^{(3)})_{(1)} \otimes S(h^{(3)})_{(2)}) \\
=& (S(h^{(1)(1)}{}_{(1)}) \otimes S(h^{(1)(1)}{}_{(2)}))(h^{(1)(2)}{}_{(1)} \otimes h^{(1)(2)}{}_{(2)})(S(h^{(2)})_{(1)} \otimes S(h^{(2)})_{(2)}) \\
=& (S({h^{(1)}{}_{(1)}}^{(1)}) \otimes S({h^{(1)}{}_{(2)}}^{(1)}))({h^{(1)}{}_{(1)}}^{(2)} \otimes {h^{(1)}{}_{(2)}}^{(2)})(S(h^{(2)})_{(1)} \otimes S(h^{(2)})_{(2)}) \\
=& t(\epsilon(h^{(1)}{}_{(1)})) \otimes t(\epsilon(h^{(1)}{}_{(2)}))((S(h^{(2)}))_{(1)} \otimes S(h^{(2)})_{(2)}) \\
=& (t(\epsilon(h^{(1)}))S(h^{(2)}))_{(1)} \otimes (t(\epsilon(h^{(1)}))S(h^{(2)}))_{(2)} \\
=& \blacktriangle(S(s(\epsilon(h^{(1)}))h^{(2)})) \\
=& \blacktriangle(S(h)),
\end{aligned}$$

and

$$\epsilon_H(S(h)) = \epsilon_H(S(h^{(1)}))\epsilon_H(h^{(2)}) = \epsilon_H(S(h^{(1)})h^{(2)}) = \epsilon_H(t(\epsilon(h))) = \epsilon_H(h).$$

For (iii), on the one hand we have

$$\begin{aligned}
& S(S_H(h^{(1)}))S_H(h^{(2)})S_H(S(h^{(3)})) = S(S_H(h^{(1)}))S_H(h^{(2)}S(h^{(3)})) \\
=& S(S_H(h^{(1)}))S_H(s(\epsilon(h^{(2)}))) = S(S_H(h^{(1)})))s(S_B(\epsilon(h^{(2)}))) \\
=& S(S_H(h^{(1)}))t(S_B(\epsilon(h^{(2)})))) = S(S_H(h^{(1)})S_H(t(\epsilon(h^{(2)})))) \\
=& S(S_H(h^{(1)}t(\epsilon(h^{(2)})))) = S(S_H(h))
\end{aligned}$$

where in the first step we use the fact that H is commutative. On the other hand we have

$$
\begin{aligned}
&S(S_H(h^{(1)}))S_H(h^{(2)})S_H(S(h^{(3)})) = S((S_H(h^{(1)}))^{(1)})(S_H(h^{(1)}))^{(2)}S_H(S(h^{(2)}))\\
=&t(\epsilon(S_H(h^{(1)})))S_H(S(h^{(2)})) = S_H(t(\epsilon(h^{(1)})))S_H(S(h^{(2)}))\\
=&S_H(S(s(\epsilon(h^{(1)}))h^{(2)})) = S_H(S(h)),
\end{aligned}
$$

where the first step uses (i) of this Proposition.

□

Remark 15.5. $S_H \otimes_B S_H$ is well defined, since for any $b \in B$ and $h, h' \in H$ we have

$$
\begin{aligned}
(S_H \otimes_B S_H)(h \otimes_B b \triangleright h') =&(S_H \otimes_B S_H)(h \otimes_B s(b)h')\\
=&S_H(h) \otimes_B S_H(s(b)h') = S_H(h) \otimes_B S_H(h')S_H(s(b))\\
=&S_H(h) \otimes_B S_H(h')s(S_B(b)) = S_H(h) \otimes_B s(S_B(b))S_H(h')\\
=&t(S_B(b))S_H(h) \otimes_B S_H(h') = S_H(h)t(S_B(b)) \otimes_B S_H(h')\\
=&S_H(h)S_H(t(b)) \otimes_B S_H(h') = S_H(h \triangleleft b) \otimes_B S_H(h')\\
=&(S_H \otimes_B S_H)(h \triangleleft b \otimes_B h'),
\end{aligned}
$$

where in the 4th and 8th steps we use the fact that s, t are Hopf algebra maps; the 5th and 7th steps use the fact that the image of s, t belongs to the centre of H.

16. Crossed comodule of Hopf coquasigroups

We know a strict 2-group is equivalent to a crossed module, so it is natural to construct a quantum 2-group in terms of a crossed comodule of Hopf algebra [15]. In this section we show that if the base algebra is commutative, a crossed comodule of Hopf algebra is a strict Hopf 2-algebra. Moreover, we will make a generalisation of it in terms of Hopf coquasigroups, which corresponds to a coherent Hopf 2-algebra.

Definition 16.1. A *crossed comodule of Hopf coquasigroup* consists of a coassociative pair (A, B, ϕ) , such that the following conditions are satisfied:

(1) A is a left B comodule coalgebra and left B comodule algebra, that is:
   (i) A is a left comodule of B with coaction $\delta : A \to B \otimes A$, here we use the Sweedler index notation: $\delta(a) = a^{(-1)} \otimes a^{(0)}$;
   (ii) For any $a \in A$,
$$a^{(-1)} \otimes {a^{(0)}}_{(1)} \otimes {a^{(0)}}_{(2)} = {a_{(1)}}^{(-1)}{a_{(2)}}^{(-1)} \otimes {a_{(1)}}^{(0)} \otimes {a_{(2)}}^{(0)};\tag{16.1}$$
   (iii) For any $a \in A$,
$$\epsilon_A(a) = a^{(-1)}\epsilon_A(a^{(0)});\tag{16.2}$$
   (iv) δ is an algebra map.

(2) For any $b \in B$,
$$\phi(b)^{(-1)} \otimes \phi(b)^{(0)} = b_{(1)(1)}S_B(b_{(2)}) \otimes \phi(b_{(1)(2)}) = b_{(1)}S_B(b_{(2)(2)}) \otimes \phi(b_{(2)(1)});\tag{16.3}$$

(3) For any $a \in A$,
$$\phi(a^{(-1)}) \otimes a^{(0)} = a_{(1)}S_A(a_{(3)}) \otimes a_{(2)}.\tag{16.4}$$

If B is coassociative we call the crossed comodule of Hopf coquasigroup a *crossed comodule of Hopf algebra.*

Lemma 16.2. *Let (A, B, ϕ, δ) be a crossed comodule of Hopf coquasigroup, if B is commutative, then the tensor product $H := A \otimes B$ is a Hopf coquasigroup, with factorwise tensor product multiplication, and unit $1_A \otimes 1_B$. The coproduct is defined by $\blacktriangle(a \otimes b) := a_{(1)} \otimes a_{(2)}{}^{(-1)} b_{(1)} \otimes a_{(2)}{}^{(0)} \otimes b_{(2)}$, for any $a \otimes b \in A \otimes B$, the counit is defined by $\epsilon_H(a \otimes b) := \epsilon_A(a)\epsilon_B(b)$. The antipode is given by $S_H(a \otimes b) := S_A(a^{(0)}) \otimes S_B(a^{(-1)}b)$. Moreover, if B is coassociative, then H is a Hopf algebra.*

Proof. $A \otimes B$ is clearly an unital algebra. Now we show it is also a Hopf coquasigroup:

$$\begin{aligned}((id_H \otimes \epsilon_H) \circ \blacktriangle)(a \otimes b) =& a_{(1)} \otimes a_{(2)}{}^{(-1)} b_{(1)} \epsilon_A(a_{(2)}{}^{(0)}) \epsilon_B(b_{(2)}) \\ =& a_{(1)} \otimes a_{(2)}{}^{(-1)} b \epsilon_A(a_{(2)}{}^{(0)}) \\ =& a_{(1)} \otimes \epsilon_A(a_{(2)}) b \\ =& a \otimes b,\end{aligned}$$

where the 3rd step we use the fact that A is comodule coalgebra,

$$\begin{aligned}((\epsilon_H \otimes id_H) \circ \blacktriangle)(a \otimes b) =& \epsilon_A(a_{(1)}) \epsilon_B(a_{(2)}{}^{(-1)} b_{(1)}) a_{(2)}{}^{(0)} \otimes b_{(2)} \\ =& a \otimes b.\end{aligned}$$

Now we show H is also a bialgebra:

$$\begin{aligned}\blacktriangle(aa' \otimes bb') =& a_{(1)} a'_{(1)} \otimes a_{(2)}{}^{(-1)} a'_{(2)}{}^{(-1)} b_{(1)} b'_{(1)} \otimes a_{(2)}{}^{(0)} a'_{(2)}{}^{(0)} \otimes b_{(2)} b'_{(2)} \\ =& a_{(1)} a'_{(1)} \otimes a_{(2)}{}^{(-1)} b_{(1)} a'_{(2)}{}^{(-1)} b'_{(1)} \otimes a_{(2)}{}^{(0)} a'_{(2)}{}^{(0)} \otimes b_{(2)} b'_{(2)} \\ =& \blacktriangle(a \otimes b) \blacktriangle(a' \otimes b'),\end{aligned}$$

here we use the fact that B is a commutative algebra in the 2nd step. So $\blacktriangle$ and ϵ_H are clearly algebra maps, thus H is a bialgebra. Now check the antipode S_H for $h = a \otimes b$,

$$\begin{aligned}& h_{(1)(1)} \otimes S_H(h_{(1)(2)}) h_{(2)} \\ =& a_{(1)(1)} \otimes a_{(1)(2)}{}^{(-1)} a_{(2)}{}^{(-1)}{}_{(1)} b_{(1)(1)} \otimes S_A(a_{(1)(2)}{}^{(0)(0)}) a_{(2)}{}^{(0)} \otimes S_B(a_{(1)(2)}{}^{(0)(-1)} a_{(2)}{}^{(-1)}{}_{(2)} b_{(1)(2)}) b_{(2)} \\ =& a_{(1)(1)} \otimes a_{(1)(2)}{}^{(-1)} a_{(2)}{}^{(-1)}{}_{(1)} b \otimes S_A(a_{(1)(2)}{}^{(0)(0)}) a_{(2)}{}^{(0)} \otimes S_B(a_{(1)(2)}{}^{(0)(-1)} a_{(2)}{}^{(-1)}{}_{(2)}) \\ =& a_{(1)(1)} \otimes a_{(1)(2)}{}^{(-1)}{}_{(1)} a_{(2)}{}^{(-1)}{}_{(1)} b \otimes S_A(a_{(1)(2)}{}^{(0)}) a_{(2)}{}^{(0)} \otimes S_B(a_{(1)(2)}{}^{(-1)}{}_{(2)} a_{(2)}{}^{(-1)}{}_{(2)}) \\ =& a_{(1)(1)} \otimes S_A(a_{(1)(2)})^{(-1)}{}_{(1)} a_{(2)}{}^{(-1)}{}_{(1)} b \otimes S_A(a_{(1)(2)})^{(0)} a_{(2)}{}^{(0)} \otimes S_B(S_A(a_{(1)(2)})^{(-1)}{}_{(2)} a_{(2)}{}^{(-1)}{}_{(2)}) \\ =& a \otimes b \otimes 1_A \otimes 1_B\end{aligned}$$

where in the second step we use the fact that B is commutative, the fourth step use the fact that $a^{(-1)} \otimes S_A(a^{(0)}) = S_A(a)^{(-1)} \otimes S_A(a)^{(0)}$, indeed,

$$\begin{aligned}& a^{(-1)} \otimes S_A(a^{(0)}) \\ =& a_{(1)(1)}{}^{(-1)} a_{(1)(2)}{}^{(-1)} S_A(a_{(2)})^{(-1)} \otimes S_A(a_{(1)(1)}{}^{(0)}) a_{(1)(2)}{}^{(0)} S_A(a_{(2)})^{(0)} \\ =& a_{(1)}{}^{(-1)} S_A(a_{(2)})^{(-1)} \otimes S_A(a_{(1)}{}^{(0)}{}_{(1)}) a_{(1)}{}^{(0)}{}_{(2)} S_A(a_{(2)})^{(0)} \\ =& a_{(1)}{}^{(-1)} S_A(a_{(2)})^{(-1)} \otimes \epsilon_A(a_{(1)}{}^{(0)}) S_A(a_{(2)})^{(0)} \\ =& S_A(a)^{(-1)} \otimes S_A(a)^{(0)},\end{aligned}$$

where in the second and third steps we use the comodule coalgebra property. The other axioms of Hopf coquasigroups are similar. Thus H is a Hopf coquasigroup.

When B is coassociative, for any $a\otimes b\in A\otimes B$, we also have

$$
\begin{aligned}
((id_H\otimes\blacktriangle)\circ\blacktriangle)(a\otimes b) =&(id_H\otimes\blacktriangle)(a_{(1)}\otimes {a_{(2)}}^{(-1)}b_{(1)}\otimes {a_{(2)}}^{(0)}\otimes b_{(2)})\\
=&a_{(1)}\otimes {a_{(2)}}^{(-1)}b_{(1)}\otimes {{a_{(2)}}^{(0)}}_{(1)}\otimes {{{a_{(2)}}^{(0)}}_{(2)}}^{(-1)}b_{(2)}\otimes {{{a_{(2)}}^{(0)}}_{(2)}}^{(0)}\otimes b_{(3)}\\
=&a_{(1)}\otimes {a_{(2)}}^{(-1)}{a_{(3)}}^{(-1)}b_{(1)}\otimes {a_{(2)}}^{(0)}\otimes {a_{(3)}}^{(0)(-1)}b_{(2)}\otimes {a_{(3)}}^{(0)(0)}\otimes b_{(3)}\\
=&a_{(1)}\otimes {a_{(2)}}^{(-1)}{{a_{(3)}}^{(-1)}}_{(1)}b_{(1)}\otimes {a_{(2)}}^{(0)}\otimes {{a_{(3)}}^{(-1)}}_{(2)}b_{(2)}\otimes {a_{(3)}}^{(0)}\otimes b_{(3)}\\
=&(\blacktriangle\otimes id_H)(a_{(1)}\otimes {a_{(2)}}^{(-1)}b_{(1)}\otimes {a_{(2)}}^{(0)}\otimes b_{(2)})\\
=&((\blacktriangle\otimes id_H)\circ\blacktriangle)(a\otimes b),
\end{aligned}
$$

where in the 3rd step we use the fact that A is a comodule coalgebra and in the 4th step we use the fact that A is a left B comodule. So $(H,\blacktriangle,\epsilon_H)$ is coassociative. □

From the proof above we can also see that even if A is a Hopf coquasigroup, we can also get a Hopf coquasigroup $A\otimes B$, with the same coproduct, counit and antipode.

Lemma 16.3. *Let (A,B,ϕ,δ) be a crossed comodule of Hopf coquasigroup. If B is commutative and the image of ϕ belongs to the center of A, then $H=A\otimes B$ is a central Hopf algebroid over B, such that the source, target and counit (of the bialgebroid structure) are bialgebra map.*

Proof. We can see that H is a tensor product algebra. The source and target maps $s,t:B\to H$ are given by $s(b):=\phi(b_{(1)})\otimes b_{(2)}$, and $t(b):=1_A\otimes b$, for any $b\in B$. The counit map $\epsilon:H\to B$ is defined to be $\epsilon(a\otimes b):=\epsilon_A(a)b$, and the left bialgebroid coproduct is defined to be $\Delta(a\otimes b):=(a_{(1)}\otimes 1_B)\otimes_B(a_{(2)}\otimes b)$. The antipode is given by $S(a\otimes b):=S_A(a)\phi(b_{(1)})\otimes b_{(2)}$. Now we show all the structure above forms a left bialgebroid structure on H. First we can see that s,t are algebra maps, so H is a $B\otimes B$-ring. Now we show H is a B-coring. Here the B-bimodule structure on H is given by $b'\triangleright(a\otimes b)\triangleleft b''=s(b')t(b'')(a\otimes b)$ for $a\otimes b\in H$, $b',b''\in B$. So we have

$$
\begin{aligned}
\epsilon(b'\triangleright(a\otimes b)\triangleleft b'') =&\epsilon(s(b')t(b'')(a\otimes b))=\epsilon_A(\phi({b'}_{(1)})a){b'}_{(2)}b''b\\
=&\epsilon_B({b'}_{(1)})\epsilon_A(a){b'}_{(2)}b''b=b'\epsilon(a\otimes b)b'',
\end{aligned}
$$

where we use the fact that ϕ is a bialgebra map in the 3rd step. Clearly, ϵ is an algebra map from $A\otimes B$ to B. We also have

$$
\begin{aligned}
(\epsilon\otimes\epsilon)(\blacktriangle(a\otimes b)) =&(\epsilon\otimes\epsilon)(a_{(1)}\otimes {a_{(2)}}^{(-1)}b_{(1)}\otimes {a_{(2)}}^{(0)}\otimes b_{(2)})\\
=&\epsilon_A(a_{(1)}){a_{(2)}}^{(-1)}b_{(1)}\otimes\epsilon_A({a_{(2)}}^{(0)})b_{(2)}\\
=&a^{(-1)}b_{(1)}\otimes\epsilon_A(a^{(0)})b_{(2)}\\
=&\epsilon_A(a)b_{(1)}\otimes b_{(2)}\\
=&\Delta_B(\epsilon(a\otimes b)),
\end{aligned}
$$

where for the 3rd step we use the fact that A is a comodule algebra. So we can see that ϵ is a bialgebra map from $A\otimes B$ to B. We can also show s and t are also bialgebra maps, since we have

$$\begin{aligned}
\blacktriangle(s(b)) =&\blacktriangle(\phi(b_{(1)})\otimes b_{(2)})\\
=&\phi(b_{(1)})_{(1)}\otimes \phi(b_{(1)})_{(2)}{}^{(-1)}b_{(2)(1)}\otimes \phi(b_{(1)})_{(2)}{}^{(0)}\otimes b_{(2)(2)}\\
=&\phi(b_{(1)(1)})\otimes \phi(b_{(1)(2)})^{(-1)}b_{(2)(1)}\otimes \phi(b_{(1)(2)})^{(0)}\otimes b_{(2)(2)}\\
=&\phi(b_{(1)(1)})\otimes \phi(b_{(1)(2)(1)})^{(-1)}b_{(1)(2)(2)}\otimes \phi(b_{(1)(2)(1)})^{(0)}\otimes b_{(2)(2)}\\
=&\phi(b_{(1)(1)})\otimes b_{(1)(2)(1)(1)(1)}S_B(b_{(1)(2)(1)(2)})b_{(1)(2)(2)}\otimes \phi(b_{(1)(2)(1)(1)(2)})\otimes b_{(2)(2)}\\
=&\phi(b_{(1)(1)})\otimes b_{(1)(2)(1)}\otimes \phi(b_{(1)(2)(2)})\otimes b_{(2)(2)}\\
=&\phi(b_{(1)(1)})\otimes b_{(1)(2)}\otimes \phi(b_{(2)(1)})\otimes b_{(2)(2)}\\
=&(s\otimes s)(\Delta_B(b))
\end{aligned}$$

where in the 4th and 7th steps we use the axiom of coassociative pairing, and in the 5th step we use (16.3). We also have

$$\blacktriangle(t(b)) =\blacktriangle(1\otimes b) = 1\otimes b_{(1)}\otimes 1\otimes b_{(2)} = (t\otimes t)(\Delta_B(b)),$$

for any $b\in B$. So s and t are algebra maps. We can also show Δ is a B-bimodule map:

$$\begin{aligned}
\Delta(b'\triangleright(a\otimes b)) =&\Delta(\phi(b'_{(1)})a\otimes b'_{(2)}b)\\
=&(\phi(b'_{(1)})_{(1)}a_{(1)}\otimes 1)\otimes_B(\phi(b'_{(1)})_{(2)}a_{(2)}\otimes b'_{(2)}b)\\
=&(\phi(b'_{(1)(1)})a_{(1)}\otimes 1)\otimes_B(\phi(b'_{(1)(2)})a_{(2)}\otimes b'_{(2)}b)\\
=&(\phi(b'_{(1)})a_{(1)}\otimes 1)\otimes_B(\phi(b'_{(2)(1)})a_{(2)}\otimes b'_{(2)(2)}b)\\
=&(\phi(b'_{(1)})a_{(1)}\otimes 1)\otimes_B s(b'_{(2)})(a_{(2)}\otimes b)\\
=&(\phi(b'_{(1)})a_{(1)}\otimes 1)t(b'_{(2)})\otimes_B(a_{(2)}\otimes b)\\
=&(\phi(b'_{(1)})a_{(1)}\otimes b'_{(2)})\otimes_B(a_{(2)}\otimes b)\\
=&b'\triangleright\Delta(a\otimes b),
\end{aligned}$$

where in the fourth step we use the axiom of coassociative pairing. We also have

$$\Delta((a\otimes b)\triangleleft b') =\Delta(a\otimes bb') = (a_{(1)}\otimes 1)\otimes_B(a_{(2)}\otimes bb') = \Delta(a\otimes b)\triangleleft b'$$

for any $a\otimes b\in A\otimes B$ and $b'\in B$. Δ is clearly coassociative, and we also have

$$(id_H\otimes_B\epsilon)\circ\Delta(a\otimes b) = a\otimes b\otimes_B 1_H,$$

and

$$(\epsilon\otimes_B id_H)\circ\Delta(a\otimes b) = \epsilon_A(a_{(1)})\otimes 1\otimes_B(a_{(2)}\otimes b) = 1_H\otimes_B a\otimes b$$

by straightforward computation. Up to now we have already shown that H is a B-coring. Clearly, Δ is also an algebra map from H to $H\times_B H$. Given $a\otimes b, a'\otimes b'\in H$, we have $\epsilon((a\otimes b)(a'\otimes b')) = \epsilon_A(aa')bb'$, and $\epsilon((a\otimes b)t(\epsilon(a'\otimes b'))) = \epsilon(a\otimes b\epsilon_A(a')b') = \epsilon_A(aa')bb'$. We also have

$$\begin{aligned}
\epsilon((a\otimes b)s(\epsilon(a'\otimes b'))) =&\epsilon((a\otimes b)(\phi(\epsilon_A(a')b'_{(1)})\otimes b'_{(2)}))\\
=&\epsilon_A(aa')bb',
\end{aligned}$$

thus ϵ is a left character and H is therefore a left bialgebroid. Since the image of ϕ belongs to the center of A, we can check that $S(t(b')(a\otimes b)s(b'')) = t(b'')S(a\otimes b)s(b')$ for any

$a \otimes b \in H$ and $b', b'' \in B$:

$$\begin{aligned} S(t(b')(a\otimes b)s(b'')) =& S(a\phi(b''_{(1)}) \otimes b'bb''_{(2)}) \\ =& S_A(a\phi(b''_{(1)}))\phi(b'_{(1)}b_{(1)}b''_{(2)(1)}) \otimes b'_{(2)}b_{(2)}b''_{(2)(2)} \\ =& S_A(a)\phi(b_{(1)})\phi(b'_{(1)}) \otimes b''b_{(2)}b'_{(2)} \\ =& t(b'')S(a\otimes b)s(b'), \end{aligned}$$

where the 3rd step uses the fact that B is commutative and its image of ϕ belongs to the center of A. Now we can see that

$$S(a_{(1)} \otimes 1)(a_{(2)} \otimes b) = S_A(a_{(1)})a_{(2)} \otimes b = (t \circ \epsilon)(a \otimes b),$$

and

$$(a_{(1)} \otimes 1)S(a_{(2)} \otimes b) = a_{(1)}S_A(a_{(2)})\phi(b_{(1)}) \otimes b_{(2)} = (s \circ \epsilon)(a \otimes b)$$

So H is a Hopf algebroid. □

Lemma 16.4. *For A, B and $H = A \otimes B$ as above, we have*

$$(\Delta \otimes \Delta) \circ \blacktriangle = (id_H \otimes \tau \otimes id_H) \circ (\blacktriangle \otimes_B \blacktriangle) \circ \Delta.$$

Proof. Let $h = a \otimes b \in H$, on the left hand side we have

$$(\Delta \otimes \Delta) \circ \blacktriangle(h) = a_{(1)} \otimes 1 \otimes_B a_{(2)} \otimes a_{(3)}{}^{(-1)}b_{(1)} \otimes a_{(3)}{}^{(0)}{}_{(1)} \otimes 1 \otimes_B a_{(3)}{}^{(0)}{}_{(2)} \otimes b_{(2)},$$

on the right hand side we have

$$\begin{aligned} &(H \otimes \tau \otimes H) \circ (\blacktriangle \otimes_B \blacktriangle) \circ (\Delta(h)) \\ =& a_{(1)} \otimes a_{(2)}{}^{(-1)} \otimes_B a_{(3)} \otimes a_{(4)}{}^{(-1)}b_{(1)} \otimes a_{(2)}{}^{(0)} \otimes 1 \otimes_B a_{(4)}{}^{(0)} \otimes b_{(2)} \\ =& a_{(1)} \otimes 1 \otimes_B \phi(a_{(2)}{}^{(-1)}{}_{(1)})a_{(3)} \otimes a_{(2)}{}^{(-1)}{}_{(2)}a_{(4)}{}^{(-1)}b_{(1)} \otimes a_{(2)}{}^{(0)} \otimes 1 \otimes_B a_{(4)}{}^{(0)} \otimes b_{(2)} \\ =& a_{(1)} \otimes 1 \otimes_B \phi(a_{(2)}{}^{(-1)})a_{(3)} \otimes a_{(2)}{}^{(0)(-1)}a_{(4)}{}^{(-1)}b_{(1)} \otimes a_{(2)}{}^{(0)(0)} \otimes 1 \otimes_B a_{(4)}{}^{(0)} \otimes b_{(2)} \\ =& a_{(1)} \otimes 1 \otimes_B a_{(2)(1)}S_A(a_{(2)(3)})a_{(3)} \otimes a_{(2)(2)}{}^{(-1)}a_{(4)}{}^{(-1)}b_{(1)} \otimes a_{(2)(2)}{}^{(0)} \otimes 1 \otimes_B a_{(4)}{}^{(0)} \otimes b_{(2)} \\ =& a_{(1)} \otimes 1 \otimes_B a_{(2)} \otimes a_{(3)}{}^{(-1)}a_{(4)}{}^{(-1)}b_{(1)} \otimes a_{(3)}{}^{(0)} \otimes 1 \otimes_B a_{(4)}{}^{(0)} \otimes b_{(2)} \\ =& a_{(1)} \otimes 1 \otimes_B a_{(2)} \otimes a_{(3)}{}^{(-1)}b_{(1)} \otimes a_{(3)}{}^{(0)}{}_{(1)} \otimes 1 \otimes_B a_{(3)}{}^{(0)}{}_{(2)} \otimes b_{(2)}, \end{aligned}$$

where in the second step we use the balanced tensor product over B, the fourth step uses (16.4), and in the last step we use the fact that A is a comodule coalgebra of B. □

Since for strict Hopf 2-algebra all the axioms of coassociator are trivial, we can conclude:

Theorem 16.5. *Let (A, B, ϕ, δ) be a crossed comodule of Hopf algebra, if B is commutative and the image of ϕ belongs to the center of B, then $H = A \otimes B$ is a strict Hopf 2-algebra with the structure maps given by:*

$$\begin{aligned} \blacktriangle(a \otimes b) =& a_{(1)} \otimes a_{(2)}{}^{(-1)}b_{(1)} \otimes a_{(2)}{}^{(0)} \otimes b_{(2)}, \\ \epsilon_H(a \otimes b) =& \epsilon_A(a)\epsilon_B(b), \\ S_H(a \otimes b) =& S_A(a^{(0)}) \otimes S_B(a^{(-1)}b), \\ s(b) =& \phi(b_{(1)}) \otimes b_{(2)}, \\ t(b) =& 1 \otimes b, \\ \Delta(a \otimes b) =& a_{(1)} \otimes 1 \otimes_B a_{(2)} \otimes b, \\ \epsilon(a \otimes b) =& \epsilon_A(a)b, \\ S(a \otimes b) =& S_A(a)\phi(b_{(1)}) \otimes b_{(2)}. \end{aligned}$$

Here are some examples of crossed comodule of Hopf algebras:

Example 16.6. Let $\phi : B \to A$ be a surjective morphism of Hopf algebras, where A is commutative, such that for any $i \in I := ker(\phi)$, $i_{(1)}S_B(i_{(3)}) \otimes i_{(2)} \in B \otimes I$. Thus we can define $\delta : A \to B \otimes A$ by $\delta([a]) := a_{(1)}S_B(a_{(3)}) \otimes [a_{(2)}]$, where $[a]$ denote the image of ϕ. We can see that A is a comodule coalgebra and comodule algebra of B, since A is commutative. Moreover, (16.3) and (16.4) are also satisfied. Therefore (A, B, ϕ, δ) is a crossed comodule of Hopf algebra.

Example 16.7. Let $G \hookrightarrow H \twoheadrightarrow E$ be a short exact sequence of Hopf algebras with injection $i : G \to H$, surjection $\pi : H \to E$ and G is commutative, such that $h_{(1)} \otimes \pi(h_{(2)}) = h_{(2)} \otimes \pi(h_{(1)})$ for any $h \in H$. For any $k \in H$, we can see $k_{(1)}S_H(k_{(3)}) \otimes k_{(2)} \in i(G) \otimes H$, since $k_{(1)}S_H(k_{(3)}) \otimes k_{(2)} \in ker(\pi) \otimes H$. Therefore, we can define a coaction $\delta : H \to G \otimes H$ by $\delta(h) := h_{(1)}S_H(h_{(3)}) \otimes h_{(2)}$ (here we identify G and its image under i). We can see that the H is a G-comodule algebra and comodule coalgebra. (16.3) and (16.4) are also satisfied. Thus (H, G, i, δ) is a crossed comodule of Hopf algebra. Moreover, if the image of i belongs to the centre of H, then (H, G, i, δ) forms a strict Hopf 2-algebra.

Example 16.8. Let A, B be two Hopf algebras and let A be cocommutative, such that A is a comodule algebra and comodule coalgebra of B. Define $\phi : B \to A$ by $\phi(b) := \epsilon_B(b)1_A$. Clearly, ϕ is a Hopf algebra map, (16.3) and (16.4) are also satisfied, since A is cocommutative. Therefore (A, B, ϕ, δ) is a crossed comodule of Hopf algebra.

Example 16.9. Let A, B be two cocommutative Hopf algebras, and $\phi : B \to A$ be a Hopf algebra map. Define $\delta : A \to B \otimes A$ by $\delta(a) := 1_B \otimes a$. Clearly, A is a comodule algebra and comodule coalgebra of B, and (16.3) and (16.4) are also satisfied, since A and B are cocommutative. Therefore (A, B, ϕ, δ) is a crossed comodule of Hopf algebra.

17. Quasi coassociative Hopf coquasigroups

In this section we will construct a crossed comodule of Hopf coquasigroup as a generalisation of Example 16.6, and then construct a coherent Hopf 2-algebra. First we define a quasi coassociative Hopf coquasigroup, which can be viewed as a quantum quasiassociative quasigroup.

Definition 17.1. Let (C, B, ϕ) be a coassociative pair. We call the Hopf coquasigroup B *quasi coassociative* corresponding to (C, B, ϕ), if:

- $\phi : B \to C$ is a surjective morphism of Hopf coquasigroups.
- For any $i \in I_B := ker(\phi)$,

$$\begin{cases} i_{(1)(1)}S_B(i_{(2)}) \otimes i_{(1)(2)} & \in B \otimes I_B, \\ i_{(1)}S_B(i_{(2)(2)}) \otimes i_{(2)(1)} & \in B \otimes I_B. \end{cases} \tag{17.1}$$

- $I \subseteq ker(\beta)$, where $\beta : B \to B \otimes B \otimes B$ is the coassociator (5.7).

Since ϕ is surjective, then for any element x in C, there is an element $c \in B$, such that $x = [c] := \phi(c)$. If B is a quasi coassociative, by (17.1) there is a linear map $Ad : C \to B \otimes C$, $Ad([c]) := c_{(1)}S_B(c_{(3)}) \otimes [c_{(2)}] := c_{(1)(1)}S_B(c_{(2)}) \otimes [c_{(2)(1)}] = c_{(1)}S_B(c_{(2)(2)}) \otimes [c_{(1)(2)}]$ for any $[c] \in C$ (the last equality hold because of Proposition 5.6). We can see in the following that Ad is a comodule map and C is a comodule coalgebra of B. For any $b \in B$, there is an important result of Proposition 5.6:

$$b_{(1)(1)}S_B(b_{(1)(3)})b_{(2)} \otimes [b_{(1)(2)}] = b_{(1)(1)(1)}S_B(b_{(1)(2)})b_{(2)} \otimes [b_{(1)(1)(2)}] = b_{(1)} \otimes [b_{(2)}]. \tag{17.2}$$

Similarly,

$$S_B(b_{(1)})b_{(2)(1)}S_B(b_{(2)(3)}) \otimes [b_{(2)(2)}] = S_B(b_{(2)}) \otimes [b_{(1)}]. \tag{17.3}$$

Since $I \subseteq ker(\beta)$, there is a linear map $\tilde{\beta} : C \to B \otimes B \otimes B$ given by $\tilde{\beta}([b]) := \beta(b)$, which is denoted by $\beta(b) = b^{\hat{1}} \otimes b^{\hat{2}} \otimes b^{\hat{3}}$.

Lemma 17.2. *Let B be a quasi coassociative Hopf coquasigroup corresponding to (C, B, ϕ). If B is commutative, then the Hopf coquasigroup B and Hopf algebra C together with the maps $Ad : C \to B \otimes C$ and the quotient map $\phi : B \to C$ form a crossed comodule of Hopf coquasigroup.*

Proof. We first prove Ad is a comodule map:

$$c_{(1)(1)}S_B(c_{(3)(2)}) \otimes c_{(1)(2)}S_B(c_{(3)(1)}) \otimes [c_{(2)}] = c_{(1)}S_B(c_{(3)}) \otimes c_{(2)(1)}S_B(c_{(2)(3)}) \otimes [c_{(2)(2)}], \tag{17.4}$$

for which it is sufficient to show

$$\begin{aligned}
&c_{(1)(1)(1)}S_B(c_{(1)(3)(2)})c_{(2)} \otimes c_{(1)(1)(2)}S_B(c_{(1)(3)(1)}) \otimes [c_{(1)(2)}]\\
=&c_{(1)(1)}S_B(c_{(1)(3)})c_{(2)} \otimes c_{(1)(2)(1)}S_B(c_{(1)(2)(3)}) \otimes [c_{(1)(2)(2)}].
\end{aligned}$$

On the one hand we have

$$\begin{aligned}
&c_{(1)(1)(1)}S_B(c_{(1)(3)(2)})c_{(2)} \otimes c_{(1)(1)(2)}S_B(c_{(1)(3)(1)}) \otimes [c_{(1)(2)}]\\
=&c_{(1)(1)(1)}S_B(c_{(1)(3)})_{(1)}c_{(2)(1)(1)} \otimes c_{(1)(1)(2)}S_B(c_{(1)(3)})_{(2)}c_{(2)(1)(2)}S_B(c_{(2)(2)}) \otimes [c_{(1)(2)}]\\
=&c_{(1)(1)(1)(1)}S_B(c_{(1)(1)(3)})_{(1)}c_{(1)(2)(1)} \otimes c_{(1)(1)(1)(2)}S_B(c_{(1)(1)(3)})_{(2)}c_{(1)(2)(2)}S_B(c_{(2)}) \otimes [c_{(1)(1)(2)}]\\
=&c_{(1)(1)(1)} \otimes c_{(1)(1)(2)}S_B(c_{(2)}) \otimes [c_{(1)(2)}]\\
=&c_{(1)(1)} \otimes c_{(1)(2)}S_B(c_{(2)(2)}) \otimes [c_{(2)(1)}],
\end{aligned}$$

where in the first step we use the definition of a Hopf coquasigroup, in the second and last step we use Proposition 5.6, and in the third step we use (17.2). On the other hand we have

$$\begin{aligned}
&c_{(1)(1)}S_B(c_{(1)(3)})c_{(2)} \otimes c_{(1)(2)(1)}S_B(c_{(1)(2)(3)}) \otimes [c_{(1)(2)(2)}]\\
=&c_{(1)} \otimes c_{(2)(1)}S_B(c_{(2)(3)}) \otimes [c_{(2)(2)}]\\
=&c_{(1)(1)(1)} \otimes c_{(1)(1)(2)}S_B(c_{(1)(2)})c_{(2)(1)}S_B(c_{(2)(3)}) \otimes [c_{(2)(2)}]\\
=&c_{(1)(1)} \otimes c_{(1)(2)}S_B(c_{(2)(1)})c_{(2)(2)(1)}S_B(c_{(2)(2)(3)}) \otimes [c_{(2)(2)(2)}]\\
=&c_{(1)(1)} \otimes c_{(1)(2)}S_B(c_{(2)(2)}) \otimes [c_{(2)(1)}],
\end{aligned}$$

where in the first and last step we use (17.2) and (17.3), the second step uses the definition of a Hopf coquasigroup, and the third step uses Proposition (5.6). So we have

$$\begin{aligned}
&c_{(1)(1)}S_B(c_{(3)(2)}) \otimes c_{(1)(2)}S_B(c_{(3)(1)}) \otimes [c_{(2)}]\\
=&c_{(1)(1)(1)(1)}S_B(c_{(1)(1)(3)(2)})c_{(1)(2)}S_B(c_{(2)}) \otimes c_{(1)(1)(1)(2)}S_B(c_{(1)(1)(3)(1)}) \otimes [c_{(1)(1)(2)}]\\
=&c_{(1)(1)(1)}S_B(c_{(1)(1)(3)})c_{(1)(2)}S_B(c_{(2)}) \otimes c_{(1)(1)(2)(1)}S_B(c_{(1)(1)(2)(3)}) \otimes [c_{(1)(1)(2)(2)}]\\
=&c_{(1)}S_B(c_{(3)}) \otimes c_{(2)(1)}S_B(c_{(2)(3)}) \otimes [c_{(2)(2)}],
\end{aligned}$$

where in the first and last step we use the definition of Hopf coquasigroup, and the second step uses the relation we just proved above. We can see that Ad is an algebra map, since B is commutative. Now let's show that Ad is a comodule coalgebra map: On the one hand

$$[c]^{(-1)} \otimes [c]^{(0)}{}_{(1)} \otimes [c]^{(0)}{}_{(2)} = c_{(1)}S_B(c_{(3)}) \otimes [c_{(2)(1)}] \otimes [c_{(2)(2)}].$$

On the other hand

$$\begin{aligned}&[c]_{(1)}{}^{(-1)}[c]_{(2)}{}^{(-1)} \otimes [c]_{(1)}{}^{(0)} \otimes [c]_{(2)}{}^{(0)}\\ =&c_{(1)(1)}S_B(c_{(1)(3)})c_{(2)(1)}S_B(c_{(2)(3)}) \otimes [c_{(1)(2)}] \otimes [c_{(2)(2)}]\\ =&c_{(1)}S_B(c_{(3)}) \otimes [c_{(2)(1)}] \otimes [c_{(2)(2)}],\end{aligned}$$

where the last step uses Proposition 5.6. And

$$\epsilon_C([c]) = \epsilon_B(c) = c_{(1)}S_B(c_{(3)})\epsilon_B(c_{(2)}) = [c]^{(-1)}\epsilon_C([c]^{(0)}).$$

(16.3) and (16.4) are given by the construction of Ad.

□

Now we want to construct a coherent 2-group in terms of the crossed comodule (C, B, ϕ, Ad) we just considered above. In the following we always assume B to be commutative. Compare to Definition 15.1, the first Hopf coquasigroup is B. The second Hopf coquasigroup is $H := C \otimes B$, with the canonical unit and factorwise multiplication. The coproduct, counit and antipode are defined by the following:

$$\blacktriangle([c] \otimes b) := [c]_{(1)} \otimes [c]_{(2)}{}^{(-1)}b_{(1)} \otimes [c]_{(2)}{}^{(0)} \otimes b_{(2)}, \tag{17.5}$$

$$\epsilon_H([c] \otimes b) := \epsilon_B(c)\epsilon_B(b), \tag{17.6}$$

$$S_H([c] \otimes b) := [S_B(c_{(1)(2)})] \otimes S_B(c_{(1)})c_{(2)(2)}S_B(b) = S_C([c]^{(0)}) \otimes S_B([c]^{(-1)}b). \tag{17.7}$$

By Lemma 16.2 and Lemma 17.2, we have

$H = C \otimes B$ is a Hopf coquasigroup.

Then we construct a Hopf algebroid structure on H by Lemma 16.3 with the source and target maps $s, t : B \to H$ given by

$$s(b) := [b_{(1)}] \otimes b_{(2)}, \quad \text{and} \quad t(b) := 1_C \otimes b, \tag{17.8}$$

for any $b \in B$. The Hopf algebroid coproduct is given by

$$\Delta([c] \otimes b) := ([c_{(1)}] \otimes 1_B) \otimes_B ([c_{(2)}] \otimes b), \tag{17.9}$$

and the counit is given by

$$\epsilon([c] \otimes b) := \epsilon_B(c)b. \tag{17.10}$$

The antipode is

$$S([c] \otimes b) := [S_B(c)b_{(1)}] \otimes b_{(2)}. \tag{17.11}$$

Using Lemma 16.4, we can also get the cocommutation relation of coproducts:

$$(\Delta \otimes \Delta) \circ \blacktriangle = (id_H \otimes \tau \otimes id_H) \circ (\blacktriangle \otimes_B \blacktriangle) \circ \Delta.$$

The coassociator $\alpha : H \to B \otimes B \otimes B$ is given by

$$\alpha([c] \otimes b) := \beta(c)(b_{(1)(1)} \otimes b_{(1)(2)} \otimes b_{(2)}) = c^{\hat{1}}b_{(1)(1)} \otimes c^{\hat{2}}b_{(1)(2)} \otimes c^{\hat{3}}b_{(2)}. \tag{17.12}$$

This is well defined, since B is quasi coassociative with $I \subseteq ker(\beta)$. By using (5.8) we can check condition (v) of Definition 15.1:

$$\alpha(t(b)) = b_{(1)(1)} \otimes b_{(1)(2)} \otimes b_{(2)}$$

and

$$\begin{aligned}\alpha(s(b)) =&(b_{(1)})^{\hat{1}}b_{(2)(1)(1)} \otimes (b_{(1)})^{\hat{2}}b_{(2)(1)(2)} \otimes (b_{(1)})^{\hat{3}}b_{(2)(2)}\\ =&b_{(1)} \otimes b_{(2)(1)} \otimes b_{(2)(2)},\end{aligned}$$

by using (5.7). For condition (vi) of Definition 15.1, we can see

$$\epsilon_B(h^{\tilde{1}})1_B\otimes h^{\tilde{2}}\otimes h^{\tilde{3}}=1_B\otimes c_{(1)(1)}S_B(c_{(2)})_{(1)}b_{(1)}\otimes c_{(1)(2)}S_B(c_{(2)})_{(2)}b_{(2)}=1_B\otimes\epsilon(h_{(1)})\otimes\epsilon(h_{(2)}),$$

where $h=[c]\otimes b$, and the rest of condition (vi) of Definition 15.1 can be checked similarly. For condition (vii) we can see

$$\begin{aligned}h^{\tilde{1}}S_B(h^{\tilde{2}})\otimes h^{\tilde{3}}=&c_{(1)(1)}S_B(c_{(2)})_{(1)(1)}S_B(c_{(1)(2)(1)}S_B(c_{(2)})_{(1)(2)})\otimes c_{(1)(2)(2)}S_B(c_{(2)})_{(2)}b\\=&c_{(1)(1)}S_B(c_{(1)(2)(1)})S_B(c_{(2)})_{(1)(1)}S_B(S_B(c_{(2)})_{(1)(2)})\otimes c_{(1)(2)(2)}S_B(c_{(2)})_{(2)}b\\=&1_B\otimes\epsilon(h),\end{aligned}$$

where we use the fact that B is commutative and the rest of (vii) is similar. Now let's check (viii) and (ix).

Lemma 17.3. *For any $h\in H$, we have*

$$\begin{aligned}&s(h^{(1)\tilde{1}})h^{(2)}{}_{(1)(1)}\otimes s(h^{(1)\tilde{2}})h^{(2)}{}_{(1)(2)}\otimes s(h^{(1)\tilde{3}})h^{(2)}{}_{(2)}\\=&h^{(1)}{}_{(1)}t(h^{(2)\tilde{1}})\otimes h^{(1)}{}_{(2)(1)}t(h^{(2)\tilde{2}})\otimes h^{(1)}{}_{(2)(2)}t(h^{(2)\tilde{3}}).\end{aligned}$$

Proof. Let $h=[c]\otimes b$. The left hand side of the equation above is:

$$\begin{aligned}&(s((c_{(1)})^{\hat{1}})[c_{(2)(1)(1)}]\otimes[c_{(2)(1)(2)}]^{(-1)}[c_{(2)(2)}]^{(-1)}{}_{(1)}b_{(1)(1)})\otimes(s((c_{(1)})^{\hat{2}})[c_{(2)(1)(2)}]^{(0)}\otimes[c_{(2)(2)}]^{(-1)}{}_{(2)}b_{(1)(2)})\\&\otimes(s((c_{(1)})^{\hat{3}})[c_{(2)(2)}]^{(0)}\otimes b_{(2)}),\end{aligned}$$

while the right hand side of the equation is

$$\begin{aligned}&[c_{(1)(1)}]\otimes[c_{(1)(2)}]^{(-1)}(c_{(2)})^{\hat{1}}b_{(1)(1)}\otimes[c_{(1)(2)}]^{(0)}{}_{(1)}\otimes[c_{(1)(2)}]^{(0)}{}_{(2)}{}^{(-1)}(c_{(2)})^{\hat{2}}b_{(1)(2)}\\&\otimes[c_{(1)(2)}]^{(0)}{}_{(2)}{}^{(0)}\otimes(c_{(2)})^{\hat{3}}b_{(2)}.\end{aligned}$$

So it is sufficient to show

$$\begin{aligned}&(s((c_{(1)})^{\hat{1}})[c_{(2)(1)(1)}]\otimes[c_{(2)(1)(2)}]^{(-1)}[c_{(2)(2)}]^{(-1)}{}_{(1)})\otimes(s((c_{(1)})^{\hat{2}})[c_{(2)(1)(2)}]^{(0)}\otimes[c_{(2)(2)}]^{(-1)}{}_{(2)})\\&\otimes(s((c_{(1)})^{\hat{3}})[c_{(2)(2)}]^{(0)}\otimes 1)\\=&[c_{(1)(1)}]\otimes[c_{(1)(2)}]^{(-1)}(c_{(2)})^{\hat{1}}\otimes[c_{(1)(2)}]^{(0)}{}_{(1)}\otimes[c_{(1)(2)}]^{(0)}{}_{(2)}{}^{(-1)}(c_{(2)})^{\hat{2}}\otimes[c_{(1)(2)}]^{(0)}{}_{(2)}{}^{(0)}\otimes(c_{(2)})^{\hat{3}}.\end{aligned}$$

By the definition of Hopf coquasigroup, this is equivalent to

$$\begin{aligned}&(s((c_{(1)(1)(1)})^{\hat{1}})[c_{(1)(1)(2)(1)(1)}]\otimes[c_{(1)(1)(2)(1)(2)}]^{(-1)}[c_{(1)(1)(2)(2)}]^{(-1)}{}_{(1)}c_{(1)(2)(1)(1)}S_B(c_{(2)})_{(1)(1)})\\&\otimes(s((c_{(1)(1)(1)})^{\hat{2}})[c_{(1)(1)(2)(1)(2)}]^{(0)}\otimes[c_{(1)(1)(2)(2)}]^{(-1)}{}_{(2)}c_{(1)(2)(1)(2)}S_B(c_{(2)})_{(1)(2)})\\&\otimes(s((c_{(1)(1)(1)})^{\hat{3}})[c_{(1)(1)(2)(2)}]^{(0)}\otimes c_{(1)(2)(2)}S_B(c_{(2)})_{(2)})\\=&[c_{(1)(1)(1)(1)}]\otimes[c_{(1)(1)(1)(2)}]^{(-1)}(c_{(1)(1)(2)})^{\hat{1}}c_{(1)(2)(1)(1)}S_B(c_{(2)})_{(1)(1)}\\&\otimes[c_{(1)(1)(1)(2)}]^{(0)}{}_{(1)}\otimes[c_{(1)(1)(1)(2)}]^{(0)}{}_{(2)}{}^{(-1)}(c_{(1)(1)(2)})^{\hat{2}}c_{(1)(2)(1)(2)}S_B(c_{(2)})_{(1)(2)}\\&\otimes[c_{(1)(1)(1)(2)}]^{(0)}{}_{(2)}{}^{(0)}\otimes(c_{(1)(1)(2)})^{\hat{3}}c_{(1)(2)(2)}S_B(c_{(2)})_{(2)}.\end{aligned}$$

Thus it is sufficient to show

$$\begin{aligned}&(s((c_{(1)(1)})^{\hat{1}})[c_{(1)(2)(1)(1)}]\otimes[c_{(1)(2)(1)(2)}]^{(-1)}[c_{(1)(2)(2)}]^{(-1)}{}_{(1)}c_{(2)(1)(1)})\\&\otimes(s((c_{(1)(1)})^{\hat{2}})[c_{(1)(2)(1)(2)}]^{(0)}\otimes[c_{(1)(2)(2)}]^{(-1)}{}_{(2)}c_{(2)(1)(2)})\otimes(s((c_{(1)(1)})^{\hat{3}})[c_{(1)(2)(2)}]^{(0)}\otimes c_{(2)(2)})\\=&[c_{(1)(1)(1)}]\otimes[c_{(1)(1)(2)}]^{(-1)}(c_{(1)(2)})^{\hat{1}}c_{(2)(1)(1)}\\&\otimes[c_{(1)(1)(2)}]^{(0)}{}_{(1)}\otimes[c_{(1)(1)(2)}]^{(0)}{}_{(2)}{}^{(-1)}(c_{(1)(2)})^{\hat{2}}c_{(2)(1)(2)}\otimes[c_{(1)(1)(2)}]^{(0)}{}_{(2)}{}^{(0)}\otimes(c_{(1)(2)})^{\hat{3}}c_{(2)(2)}.\end{aligned}$$

The left hand side is

$$
\begin{aligned}
&(s((c_{(1)(1)})^{\hat{1}})[c_{(1)(2)(1)(1)}] \otimes [c_{(1)(2)(1)(2)}]^{(-1)}[c_{(1)(2)(2)}]^{(-1)}{}_{(1)}c_{(2)(1)(1)})\\
\otimes&(s((c_{(1)(1)})^{\hat{2}})[c_{(1)(2)(1)(2)}]^{(0)} \otimes [c_{(1)(2)(2)}]^{(-1)}{}_{(2)}c_{(2)(1)(2)}) \otimes (s((c_{(1)(1)})^{\hat{3}})[c_{(1)(2)(2)}]^{(0)} \otimes c_{(2)(2)})\\
=&(s((c_{(1)})^{\hat{1}})[c_{(2)(1)(1)}] \otimes [c_{(2)(1)(2)}]^{(-1)}[c_{(2)(2)(1)(1)}]^{(-1)}{}_{(1)}c_{(2)(2)(1)(2)(1)})\\
\otimes&(s((c_{(1)})^{\hat{2}})[c_{(2)(1)(2)}]^{(0)} \otimes [c_{(2)(2)(1)(1)}]^{(-1)}{}_{(2)}c_{(2)(2)(1)(2)(2)}) \otimes (s((c_{(1)})^{\hat{3}})[c_{(2)(2)(1)(1)}]^{(0)} \otimes c_{(2)(2)(2)})\\
=&(s((c_{(1)})^{\hat{1}})[c_{(2)(1)(1)(1)}] \otimes [c_{(2)(1)(1)(2)}]^{(-1)}c_{(2)(1)(2)(1)})\\
\otimes&(s((c_{(1)})^{\hat{2}})[c_{(2)(1)(1)(2)}]^{(0)} \otimes c_{(2)(1)(2)(2)}) \otimes (s((c_{(1)})^{\hat{3}})[c_{(2)(2)(1)}] \otimes c_{(2)(2)(2)})\\
=&(s((c_{(1)})^{\hat{1}})[c_{(2)(1)(1)}] \otimes [c_{(2)(1)(2)(1)(1)}]^{(-1)}c_{(2)(1)(2)(1)(2)})\\
\otimes&(s((c_{(1)})^{\hat{2}})[c_{(2)(1)(2)(1)(1)}]^{(0)} \otimes c_{(2)(1)(2)(2)}) \otimes (s((c_{(1)})^{\hat{3}})[c_{(2)(2)(1)}] \otimes c_{(2)(2)(2)})\\
=&(s((c_{(1)})^{\hat{1}})[c_{(2)(1)(1)}] \otimes c_{(2)(1)(2)(1)(1)}) \otimes (s((c_{(1)})^{\hat{2}})[c_{(2)(1)(2)(1)(2)}] \otimes c_{(2)(1)(2)(2)}) \otimes (s((c_{(1)})^{\hat{3}})[c_{(2)(2)(1)}] \otimes c_{(2)(2)(2)})\\
=&[((c_{(1)})^{\hat{1}})_{(1)}c_{(2)(1)(1)(1)}] \otimes ((c_{(1)})^{\hat{1}})_{(2)}c_{(2)(1)(1)(2)} \otimes [((c_{(1)})^{\hat{2}})_{(1)}c_{(2)(1)(2)(1)}] \otimes ((c_{(1)})^{\hat{2}})_{(2)}c_{(2)(1)(2)(2)}\\
\otimes&[((c_{(1)})^{\hat{3}})_{(1)}c_{(2)(2)(1)}] \otimes ((c_{(1)})^{\hat{3}})_{(2)}c_{(2)(2)(2)}\\
=&[c_{(1)(1)}] \otimes c_{(1)(2)} \otimes [c_{(2)(1)(1)}] \otimes c_{(2)(1)(2)} \otimes [c_{(2)(2)(1)}] \otimes c_{(2)(2)(2)}
\end{aligned}
$$

where for the 1st, 3rd, 5th step we use Proposition 5.6, in the 2nd, 4th step we use (17.2), and the last step uses (5.8). The right hand side is:

$$
\begin{aligned}
&[c_{(1)(1)(1)}] \otimes [c_{(1)(1)(2)}]^{(-1)}(c_{(1)(2)})^{\hat{1}}c_{(2)(1)(1)} \otimes [c_{(1)(1)(2)}]^{(0)}{}_{(1)} \otimes [c_{(1)(1)(2)}]^{(0)}{}_{(2)}{}^{(-1)}(c_{(1)(2)})^{\hat{2}}c_{(2)(1)(2)}\\
\otimes&[c_{(1)(1)(2)}]^{(0)}{}_{(2)}{}^{(0)} \otimes (c_{(1)(2)})^{\hat{3}}c_{(2)(2)}\\
=&[c_{(1)(1)}] \otimes [c_{(1)(2)}]^{(-1)}(c_{(2)(1)})^{\hat{1}}c_{(2)(2)(1)(1)} \otimes [c_{(1)(2)}]^{(0)}{}_{(1)} \otimes [c_{(1)(2)}]^{(0)}{}_{(2)}{}^{(-1)}(c_{(2)(1)})^{\hat{2}}c_{(2)(2)(1)(2)}\\
\otimes&[c_{(1)(2)}]^{(0)}{}_{(2)}{}^{(0)} \otimes (c_{(2)(1)})^{\hat{3}}c_{(2)(2)(2)}\\
=&[c_{(1)(1)}] \otimes [c_{(1)(2)}]^{(-1)}c_{(2)(1)} \otimes [c_{(1)(2)}]^{(0)}{}_{(1)} \otimes [c_{(1)(2)}]^{(0)}{}_{(2)}{}^{(-1)}c_{(2)(2)(1)} \otimes [c_{(1)(2)}]^{(0)}{}_{(2)}{}^{(0)} \otimes c_{(2)(2)(2)}\\
=&[c_{(1)}] \otimes [c_{(2)(1)(1)}]^{(-1)}c_{(2)(1)(2)} \otimes [c_{(2)(1)(1)}]^{(0)}{}_{(1)} \otimes [c_{(2)(1)(1)}]^{(0)}{}_{(2)}{}^{(-1)}c_{(2)(2)(1)} \otimes [c_{(2)(1)(1)}]^{(0)}{}_{(2)}{}^{(0)} \otimes c_{(2)(2)(2)}\\
=&[c_{(1)}] \otimes c_{(2)(1)(1)} \otimes [c_{(2)(1)(2)(1)}] \otimes [c_{(2)(1)(2)(2)}]^{(-1)}c_{(2)(2)(1)} \otimes [c_{(2)(1)(2)(2)}]^{(0)} \otimes c_{(2)(2)(2)}\\
=&[c_{(1)}] \otimes c_{(2)(1)} \otimes [c_{(2)(2)(1)}] \otimes [c_{(2)(2)(2)(1)(1)}]^{(-1)}c_{(2)(2)(2)(1)(2)} \otimes [c_{(2)(2)(2)(1)(1)}]^{(0)} \otimes c_{(2)(2)(2)(2)}\\
=&[c_{(1)}] \otimes c_{(2)(1)} \otimes [c_{(2)(2)(1)}] \otimes c_{(2)(2)(2)(1)(1)} \otimes [c_{(2)(2)(2)(1)(2)}] \otimes c_{(2)(2)(2)(2)}\\
=&[c_{(1)(1)}] \otimes c_{(1)(2)} \otimes [c_{(2)(1)(1)}] \otimes c_{(2)(1)(2)} \otimes [c_{(2)(2)(1)}] \otimes c_{(2)(2)(2)},
\end{aligned}
$$

where in the 1st, 3rd, 5th and last step we use Proposition 5.6, the 2nd step uses (5.8), and the 4th and 6th step use (17.2).

□

Lemma 17.4. *$\alpha : H \to B \otimes B \otimes B$ satisfies the 3-cocycle condition:*

$$
\begin{aligned}
&h^{(1)\tilde{1}}h^{(2)\tilde{1}}{}_{(1)} \otimes h^{(1)\tilde{2}}h^{(2)\tilde{1}}{}_{(2)} \otimes h^{(1)\tilde{3}}{}_{(1)}h^{(2)\tilde{2}} \otimes h^{(1)\tilde{3}}{}_{(2)}h^{(2)\tilde{3}}\\
=&\epsilon(h^{(1)}{}_{(1)})h^{(2)\tilde{1}}h^{(3)}{}_{(1)}{}^{\tilde{1}} \otimes h^{(1)}{}_{(2)}{}^{\tilde{1}}h^{(2)\tilde{2}}{}_{(1)}h^{(3)}{}_{(1)}{}^{\tilde{2}} \otimes h^{(1)}{}_{(2)}{}^{\tilde{2}}h^{(2)\tilde{2}}{}_{(2)}h^{(3)}{}_{(1)}{}^{\tilde{3}} \otimes h^{(1)}{}_{(2)}{}^{\tilde{3}}h^{(2)\tilde{3}}\epsilon(h^{(3)}{}_{(2)}).
\end{aligned}
$$

for any $h \in H$.

Proof. Let $h = [c] \otimes b$, the left hand side is

$$
(c_{(1)})^{\hat{1}}(c_{(2)})^{\hat{1}}{}_{(1)}b_{(1)(1)(1)} \otimes (c_{(1)})^{\hat{2}}(c_{(2)})^{\hat{1}}{}_{(2)}b_{(1)(1)(2)} \otimes (c_{(1)})^{\hat{3}}{}_{(1)}(c_{(2)})^{\hat{2}}b_{(1)(2)} \otimes (c_{(1)})^{\hat{3}}{}_{(2)}(c_{(2)})^{\hat{3}}b_{(2)}
$$

while the right hand side is

$$c_{(1)(1)(1)}S_B(c_{(1)(2)})(c_{(2)})^{\hat{1}}(c_{(3)})^{\hat{1}}b_{(1)(1)(1)} \otimes (c_{(1)(1)(2)})^{\hat{1}}(c_{(2)})^{\hat{2}}{}_{(1)}(c_{(3)})^{\hat{2}}b_{(1)(1)(2)}$$
$$\otimes (c_{(1)(1)(2)})^{\hat{2}}(c_{(2)})^{\hat{2}}{}_{(2)}(c_{(3)})^{\hat{3}}b_{(1)(2)} \otimes (c_{(1)(1)(2)})^{\hat{3}}(c_{(2)})^{\hat{3}}b_{(2)}.$$

Notice that $c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$ can be replaced by $c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}$ or $c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}$, since $[c] \in C$. Now we have

$$(c_{(1)})^{\hat{1}}(c_{(2)})^{\hat{1}}{}_{(1)} \otimes (c_{(1)})^{\hat{2}}(c_{(2)})^{\hat{1}}{}_{(2)} \otimes (c_{(1)})^{\hat{3}}{}_{(1)}(c_{(2)})^{\hat{2}} \otimes (c_{(1)})^{\hat{3}}{}_{(2)}(c_{(2)})^{\hat{3}}$$
$$=(c_{(1)(1)(1)})^{\hat{1}}(c_{(1)(1)(2)})^{\hat{1}}{}_{(1)}c_{(1)(2)(1)(1)(1)}S_B(c_{(2)})_{(1)(1)(1)} \otimes (c_{(1)(1)(1)})^{\hat{2}}(c_{(1)(1)(2)})^{\hat{1}}{}_{(2)}c_{(1)(2)(1)(1)(2)}S_B(c_{(2)})_{(1)(1)(2)}$$
$$\otimes (c_{(1)(1)(1)})^{\hat{3}}{}_{(1)}(c_{(1)(1)(2)})^{\hat{2}}c_{(1)(2)(1)(2)}S_B(c_{(2)})_{(1)(2)} \otimes (c_{(1)(1)(1)})^{\hat{3}}{}_{(2)}(c_{(1)(1)(2)})^{\hat{3}}c_{(1)(2)(2)}S_B(c_{(2)})_{(2)}$$

and

$$c_{(1)(1)(1)}S_B(c_{(1)(2)})(c_{(2)})^{\hat{1}}(c_{(3)})^{\hat{1}} \otimes (c_{(1)(1)(2)})^{\hat{1}}(c_{(2)})^{\hat{2}}{}_{(1)}(c_{(3)})^{\hat{2}}$$
$$\otimes (c_{(1)(1)(2)})^{\hat{2}}(c_{(2)})^{\hat{2}}{}_{(2)}(c_{(3)})^{\hat{3}} \otimes (c_{(1)(1)(2)})^{\hat{3}}(c_{(2)})^{\hat{3}}$$
$$=c_{(1)(1)(1)(1)(1)}S_B(c_{(1)(1)(1)(2)})(c_{(1)(1)(2)})^{\hat{1}}(c_{(1)(1)(3)})^{\hat{1}}c_{(1)(2)(1)(1)(1)}S_B(c_{(2)})_{(1)(1)(1)}$$
$$\otimes (c_{(1)(1)(1)(1)(2)})^{\hat{1}}(c_{(1)(1)(2)})^{\hat{2}}{}_{(1)}(c_{(1)(1)(3)})^{\hat{2}}c_{(1)(2)(1)(1)(2)}S_B(c_{(2)})_{(1)(1)(2)}$$
$$\otimes (c_{(1)(1)(1)(1)(2)})^{\hat{2}}(c_{(1)(1)(2)})^{\hat{2}}{}_{(2)}(c_{(1)(1)(3)})^{\hat{3}}c_{(1)(2)(1)(2)}S_B(c_{(2)})_{(1)(2)}$$
$$\otimes (c_{(1)(1)(1)(1)(2)})^{\hat{3}}(c_{(1)(1)(2)})^{\hat{3}}c_{(1)(2)(2)}S_B(c_{(2)})_{(2)}$$

Thus to show this lemma it is sufficient to show:

$$(c_{(1)(1)})^{\hat{1}}(c_{(1)(2)})^{\hat{1}}{}_{(1)}c_{(2)(1)(1)(1)} \otimes (c_{(1)(1)})^{\hat{2}}(c_{(1)(2)})^{\hat{1}}{}_{(2)}c_{(2)(1)(1)(2)}$$
$$\otimes (c_{(1)(1)})^{\hat{3}}{}_{(1)}(c_{(1)(2)})^{\hat{2}}c_{(2)(1)(2)} \otimes (c_{(1)(1)})^{\hat{3}}{}_{(2)}(c_{(1)(2)})^{\hat{3}}c_{(2)(2)}$$
$$=c_{(1)(1)(1)(1)}S_B(c_{(1)(1)(2)})(c_{(1)(2)})^{\hat{1}}(c_{(1)(3)})^{\hat{1}}c_{(2)(1)(1)(1)} \otimes (c_{(1)(1)(1)(2)})^{\hat{1}}(c_{(1)(2)})^{\hat{2}}{}_{(1)}(c_{(1)(3)})^{\hat{2}}c_{(2)(1)(1)(2)}$$
$$\otimes (c_{(1)(1)(1)(2)})^{\hat{2}}(c_{(1)(2)})^{\hat{2}}{}_{(2)}(c_{(1)(3)})^{\hat{3}}c_{(2)(1)(2)} \otimes (c_{(1)(1)(1)(2)})^{\hat{3}}(c_{(1)(2)})^{\hat{3}}c_{(2)(2)}$$

Using (5.8) the left hand side of the above equation becomes

$$(c_{(1)(1)})^{\hat{1}}(c_{(1)(2)})^{\hat{1}}{}_{(1)}c_{(2)(1)(1)(1)} \otimes (c_{(1)(1)})^{\hat{2}}(c_{(1)(2)})^{\hat{1}}{}_{(2)}c_{(2)(1)(1)(2)}$$
$$\otimes (c_{(1)(1)})^{\hat{3}}{}_{(1)}(c_{(1)(2)})^{\hat{2}}c_{(2)(1)(2)} \otimes (c_{(1)(1)})^{\hat{3}}{}_{(2)}(c_{(1)(2)})^{\hat{3}}c_{(2)(2)}$$
$$=(c_{(1)})^{\hat{1}}(c_{(2)(1)})^{\hat{1}}{}_{(1)}c_{(2)(2)(1)(1)(1)} \otimes (c_{(1)})^{\hat{2}}(c_{(2)(1)})^{\hat{1}}{}_{(2)}c_{(2)(2)(1)(1)(2)}$$
$$\otimes (c_{(1)})^{\hat{3}}{}_{(1)}(c_{(2)(1)})^{\hat{2}}c_{(2)(2)(1)(2)} \otimes (c_{(1)})^{\hat{3}}{}_{(2)}(c_{(2)(1)})^{\hat{3}}c_{(2)(2)(2)}$$
$$=(c_{(1)})^{\hat{1}}c_{(2)(1)(1)} \otimes (c_{(1)})^{\hat{2}}c_{(2)(1)(2)} \otimes (c_{(1)})^{\hat{3}}{}_{(1)}c_{(2)(2)(1)} \otimes (c_{(1)})^{\hat{3}}{}_{(2)}c_{(2)(2)(2)}$$
$$=c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)(1)} \otimes c_{(2)(2)(2)}.$$

Using the Proposition 5.6 the right hand side becomes

$$
\begin{aligned}
&c_{(1)(1)(1)}S_B(c_{(1)(2)})(c_{(2)(1)})^{\hat{1}}(c_{(2)(2)(1)})^{\hat{1}}c_{(2)(2)(2)(1)(1)(1)}\otimes(c_{(1)(1)(2)})^{\hat{1}}(c_{(2)(1)})^{\hat{2}}{}_{(1)}(c_{(2)(2)(1)})^{\hat{2}}c_{(2)(2)(2)(1)(1)(2)}\\
&\otimes(c_{(1)(1)(2)})^{\hat{2}}(c_{(2)(1)})^{\hat{2}}{}_{(2)}(c_{(2)(2)(1)})^{\hat{3}}c_{(2)(2)(2)(1)(2)}\otimes(c_{(1)(1)(2)})^{\hat{3}}(c_{(2)(1)})^{\hat{3}}c_{(2)(2)(2)(2)}\\
&=c_{(1)(1)(1)}S_B(c_{(1)(2)})(c_{(2)(1)})^{\hat{1}}(c_{(2)(2)(1)(1)})^{\hat{1}}c_{(2)(2)(1)(2)(1)(1)}\otimes(c_{(1)(1)(2)})^{\hat{1}}(c_{(2)(1)})^{\hat{2}}{}_{(1)}(c_{(2)(2)(1)(1)})^{\hat{2}}c_{(2)(2)(1)(2)(1)(2)}\\
&\otimes(c_{(1)(1)(2)})^{\hat{2}}(c_{(2)(1)})^{\hat{2}}{}_{(2)}(c_{(2)(2)(1)(1)})^{\hat{3}}c_{(2)(2)(1)(2)(2)}\otimes(c_{(1)(1)(2)})^{\hat{3}}(c_{(2)(1)})^{\hat{3}}c_{(2)(2)(2)}\\
&=c_{(1)(1)(1)}S_B(c_{(1)(2)})(c_{(2)(1)})^{\hat{1}}c_{(2)(2)(1)(1)}\otimes(c_{(1)(1)(2)})^{\hat{1}}(c_{(2)(1)})^{\hat{2}}{}_{(1)}c_{(2)(2)(1)(2)(1)}\\
&\otimes(c_{(1)(1)(2)})^{\hat{2}}(c_{(2)(1)})^{\hat{2}}{}_{(2)}c_{(2)(2)(1)(2)(2)}\otimes(c_{(1)(1)(2)})^{\hat{3}}(c_{(2)(1)})^{\hat{3}}c_{(2)(2)(2)}\\
&=c_{(1)(1)(1)}S_B(c_{(1)(2)})c_{(2)(1)}\otimes(c_{(1)(1)(2)})^{\hat{1}}c_{(2)(2)(1)(1)}\otimes(c_{(1)(1)(2)})^{\hat{2}}c_{(2)(2)(1)(2)}\otimes(c_{(1)(1)(2)})^{\hat{3}}c_{(2)(2)(2)}\\
&=c_{(1)(1)(1)(1)}S_B(c_{(1)(1)(2)})c_{(1)(2)}\otimes(c_{(1)(1)(1)(2)})^{\hat{1}}c_{(2)(1)(1)}\otimes(c_{(1)(1)(1)(2)})^{\hat{2}}c_{(2)(1)(2)}\otimes(c_{(1)(1)(1)(2)})^{\hat{3}}c_{(2)(2)}\\
&=c_{(1)(1)}\otimes(c_{(1)(2)})^{\hat{1}}c_{(2)(1)(1)}\otimes(c_{(1)(2)})^{\hat{2}}c_{(2)(1)(2)}\otimes(c_{(1)(2)})^{\hat{3}}c_{(2)(2)}\\
&=c_{(1)}\otimes(c_{(2)(1)})^{\hat{1}}c_{(2)(2)(1)(1)}\otimes(c_{(2)(1)})^{\hat{2}}c_{(2)(2)(1)(2)}\otimes(c_{(2)(1)})^{\hat{3}}c_{(2)(2)(2)}\\
&=c_{(1)}\otimes c_{(2)(1)}\otimes c_{(2)(2)(1)}\otimes c_{(2)(2)(2)},
\end{aligned}
$$

where in the 1st, 4th and 6th step we use Proposition 5.6, for the 2nd, 3rd and last step we use (5.8), and the 5th step uses (17.2).

□

As a result of Lemma 17.2, 16.2, 16.3, 17.3 and 17.4 we have

Theorem 17.5. *Let B be a quasi coassociative Hopf coquasigroup corresponding to a coassociative pair (C, B, ϕ). If B is commutative, then $H = C \otimes B$ is a coherent Hopf 2-algebra.*

Since quasi coassociative Hopf coquasigroups are the quantisation of quasiassociative quasigroups, we can see Theorem 17.5 is the 'quantum' case of Theorem 4.8.

18. Finite dimensional coherent Hopf 2-algebras and examples

In [21] there is a dual pairing between bialgebras, we can see that there is also a dual pairing between Hopf coquasigroups and Hopf quasigroups. In this section we will make clear why quasi coassociative Hopf coquasigroup is the correct quantization of quasiassociative quasi group.

Definition 18.1. Given a Hopf quasigroup $(A, \Delta_A, \epsilon_A, m_A, 1_A, S_A)$ and a Hopf coquasigroup $(B, \Delta_B, \epsilon_B, m_B, 1_B, S_B)$. A *dual pairing* between A and B is a bilinear map $\langle \bullet, \bullet\rangle : B \times A \to k$ such that:

- $\langle \Delta_B(b), a \otimes a'\rangle = \langle b, aa'\rangle$ and $\langle b \otimes b', \Delta_A(a)\rangle = \langle bb', a\rangle$.
- $\epsilon_B(b) = \langle b, 1_A\rangle$ and $\epsilon_A(a) = \langle 1_B, a\rangle$.

for any $a, a' \in A$ and $b, b' \in B$. A dual pairing between B and A is called *nondegenerate* if $\langle b, a\rangle = 0$ for all $b \in B$ implies $a = 0$ and if $\langle b, a\rangle = 0$ for all $a \in A$ implies $b = 0$.

Remark 18.2. Given two dual pairings $\langle \bullet, \bullet\rangle_1 : B_1 \times A_1 \to k$ and $\langle \bullet, \bullet\rangle_2 : B_2 \times A_2 \to k$ for two Hopf quasigroups A_1, A_2 and two Hopf coquasigroup B_1, B_2, we can construct a new dual pairing $\langle \bullet, \bullet\rangle : B_1 \otimes B_2 \times A_1 \otimes A_2 \to k$, which is given by $\langle b_1 \otimes b_2, a_1 \otimes a_2\rangle := \langle a_1, b_1\rangle_1\langle a_2, b_2\rangle_2$ for any $a_1 \in A_1$, $a_2 \in A_2$ and $b_1 \in B_1$, $b_2 \in B_2$. Notice that $A_1 \otimes A_2$ is also a Hopf quasigroup with the factorwise (co)product, (co)unit and antipode. Similarly, $B_1 \otimes B_2$ is also a Hopf coquasigroup. Moreover, if both the dual pairings are nondegenerate, then the new pairing is also nondegenerate.

If B is a finite dimensional Hopf coquasigroup, there is a nondegenerate dual pairing between B and its dual algebra $A := \mathrm{Hom}(B, k)$. More precisely, the dual pairing is given by $\langle b, a\rangle := a(b)$ for $b \in B$ and $a \in A$. In this case A is a Hopf quasigroup, with structure given by $aa'(b) := a(b_{(1)})a'(b_{(2)})$, $1_A(b) := \epsilon(b)$, $\Delta_A(a)(b \otimes b') := a(bb')$, $\epsilon_A(a) := a(1_B)$, $S_A(a)(b) := a(S_B(b))$, for any $a \in A$ and $b \in B$. We can see that it satisfies the axioms of a Hopf quasigroup, for example, $(S(a_{(1)})(a_{(2)}a'))(b) = a_{(1)}(S_B(b_{(1)}))a_{(2)}(b_{(2)(1)})a'(b_{(2)(2)}) = a(S_B(b_{(1)})b_{(2)(1)})a'(b_{(2)(2)}) = \epsilon_A(a)a'(b)$, since the pairing of A and B are nondegenerate, we get the required axioms.

Given a finite dimensional coquasigroup B with its dual $A := \mathrm{Hom}(B, k)$, recall the associative elements of a quasigroup form a subquasi group. We have similarly a subset of A:

$$N_A := \{a \in A \,|\, a(uv) = (au)v,\ u(av) = (ua)v,\ u(va) = (uv)a,\quad \forall u, v \in A\}, \tag{18.1}$$

Clearly, N_A is an associative algebra. The elements in N_A can pass though brakets for the multiplication like the associative elements of a quasigroup.

If $\Delta_A(N_A) \subseteq N_A \otimes N_A$, then N_A is a Hopf algebra with the structure inherited from A. In this case there is also a dual pairing between N_A and B by the restriction of dual pairing between A and B, which is not necessarily nondegenerate. From now on we will assume N_A to be a Hopf algebra.

Define

$$I_B := \{b \in B \,|\, \langle b, a\rangle = 0\ \ \forall a \in N_A\}, \tag{18.2}$$

we can see that I_B is an ideal of B, since for any $b \in B$, $a \in N_A$ and $i \in I_B$, $a(bi) = a_{(1)}(b)a_{(2)}(i) = 0$. I_B is also a coideal (i.e. $\Delta_B(i) \in I_B \otimes B + B \otimes I_B$ for any $i \in I_B$), since N_A is an algebra. As a result, the quotient algebra $C := B/I_B$ is a Hopf coquasigroup. We can see that there is also a dual pairing between N_A and C given by $\langle [b], a\rangle := \langle b, a\rangle$, where $b \in B$ and $[b]$ is the image of the quotient map in C, and $a \in N_A$. If $\langle [b], a\rangle = 0$ for any $b \in B$, we get $a = 0$. If $\langle [b], a\rangle = 0$ for any $a \in N_A$, we get $b \in I_B$, so $[b] = 0$. Thus the dual pairing between C and N_A is nondegenerate. Since N_A is associative and the dual pairing is nondegenerate, we get that C is coassociative. As a result, C is a Hopf algebra.

Remark 18.3. Let $\langle \bullet, \bullet\rangle : B \times A \to k$ be a nondegenerate dual pairing between a Hopf coquasigroup B and a Hopf quasigroup A. Recall the linear map $\beta : B \to B \otimes B \otimes B$

$$\beta(b) = b_{(1)(1)}S_B(b_{(2)})_{(1)(1)} \otimes b_{(1)(2)(1)}S_B(b_{(2)})_{(1)(2)} \otimes b_{(1)(2)(2)}S_B(b_{(2)})_{(2)}$$

for any $b \in B$. We can see that β is the dualisation of associator $\beta^* : A \otimes A \otimes A \to A$, which is given by

$$\beta^*(u \otimes v \otimes w) := (u_{(1)}(v_{(1)}w_{(1)}))(S(w_{(2)})(S(v_{(2)})S(u_{(2)}))) \tag{18.3}$$

for any $u, v, w \in A$. Indeed, we can see

$$\langle \beta(b), u \otimes v \otimes w\rangle = \langle b, \beta^*(u \otimes v \otimes w)\rangle.$$

for any $b \in B$. We call A *quasiassociative*, if N_A is invariant under the adjoint action (i.e. $a_{(1)}nS(a_{(2)})) \in N_A$ for any $n \in N_A$ and $a \in A$) and the image of β^* belongs to N_A. We can see that if a quasigroup G is a quasiassociative (see Definition 4.1), then its linear extension kG is a quasiassociative Hopf quasigroup.

Now we will give an explicit example of a coherent Hopf 2-algebra based on the Cayley algebra:

Example 18.4. In [20] the unital basis of Cayley algebras $\mathcal{G}_n := \{\pm e_a \mid a \in \mathbb{Z}_2^n\}$ is a quasigroup, with the product controlled by a 2-cochain $F : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to k^*$, more precisely, $e_a e_b := F(a,b)e_{a+b}$. From now on we also denote $e_a^0 := e_a$ and $e_a^1 := -e_a$, i.e. $\mathcal{G}_n = \{e_a^i \mid a \in \mathbb{Z}_2^n,\ i \in \mathbb{Z}_2\}$, so we have $e_a^i e_b^j = F(a,b)e_{a+b}^{i+j}$. We define $k\mathcal{G}_n$ as the linear extension of $\mathcal{G}_n$, which is a Hopf quasigroup with the coalgebra structure given by $\Delta(u) = u \otimes u$, $\epsilon(u) = 1$, and $S(u) := u^{-1}$ on the basis elements.
As we already know from [20] that

$$k\mathcal{G}_n \simeq \begin{cases} \mathbb{C} & \text{if } n = 1 \\ \mathbb{H} & \text{if } n = 2 \\ \mathbb{O} & \text{if } n = 3. \end{cases}$$

Since $k\mathcal{G}_n$ is a subalgebra of $k\mathcal{G}_m$, for $n \leq m$, we get $N_{k\mathcal{G}_m} \subseteq N_{k\mathcal{G}_n}$. We have

$$N_{k\mathcal{G}_n} \simeq \begin{cases} \mathbb{C} & \text{if } n = 1 \\ \mathbb{H} & \text{if } n = 2 \\ \mathbb{R} & \text{if } n \geq 3, \end{cases}$$

since $N_{k\mathcal{G}_3} = \mathbb{R}$. In [[20], Prop 3.6] we find that $\mathcal{G}_n$ is quasiassociative and the dual of $\mathcal{G}_n$ is a Hopf coquasigroup $B := k[\mathcal{G}_n]$ given by functions on $\mathcal{G}_n$. Let $f_a^i \in k[\mathcal{G}_n]$ be the delta function on each element of $\mathcal{G}_n$, i.e. $f_a^i(e_b^j) = \delta_{a,b}\delta_{i,j}$. We can see $k[\mathcal{G}_n]$ is an algebra with generators $\{f_a^i \mid a \in \mathbb{Z}_2^n,\ i \in \mathbb{Z}_2\}$ subject to the relations:

$$f_a^i f_{a'}^{i'} = \begin{cases} f_a^i & \text{if } a = a' \quad \text{and} \quad i = i' \\ 0 & \text{otherwise} \end{cases}$$

The unit of $k[\mathcal{G}_n]$ is $\sum_{a \in \mathbb{Z}_2^n, i \in \mathbb{Z}_2} f_a^i$. The coproduct, counit and antipode are given by

$$\Delta_B(f_a^i) := \sum_{\substack{b+c=a \\ j+k=i}} F(b,c) f_b^j \otimes f_c^k. \tag{18.4}$$

$$\epsilon_B(f_a^i) := \delta_{a,0}\delta_{i,0}. \tag{18.5}$$

$$S_B(f_j^i) := F(a,a) f_a^i. \tag{18.6}$$

The previous structures make $k[\mathcal{G}_n]$ a Hopf coquasigroup.

Now we can show $k[\mathcal{G}_n]$ is quasi coassociative corresponding to the coassociative pair $(k[\mathcal{G}_0], k[\mathcal{G}_n], \pi)$, where $\pi : k[\mathcal{G}_n] \to k[\mathcal{G}_0]$ is the canonical projection map and $k[\mathcal{G}_0]$ is just the functions on $\{-e_0, e_0\}$. First, we can see that $(k[\mathcal{G}_0], k[\mathcal{G}_n], \pi)$ is a coassociative pair by Definition 5.4. Second, we have

$$x_{(1)(1)} S_B(x_{(2)}) \otimes x_{(1)(2)} \in B \otimes I_B$$
$$x_{(1)} S_B(x_{(2)(2)}) \otimes x_{(2)(1)} \in B \otimes I_B,$$

for any $x \in I_B$. Indeed, let $x \in I_B = ker(\pi)$, then x is a linear combination of f_a^i with $a \neq 0$. Without losing generality (since every map below is linear), assuming $x = f_a^i$, we can see

$$x_{(2)(1)} \otimes x_{(1)} S_B(x_{(2)(2)}) = \sum_{\substack{b+c+d=a \\ j+k+l=i}} F(b,c+d)F(c,d)F(d,d) f_c^k \otimes f_b^j f_d^l, \tag{18.7}$$

the right hand side of the equality is not zero only if $b = d$. As a result c is equal to a, and $x_{(1)(1)} S_B(x_{(2)}) \otimes x_{(1)(2)} \in B \otimes I_B$. Similarly, we also have $x_{(1)} S_B(x_{(2)(2)}) \otimes x_{(2)(1)} \in B \otimes I_B$.

Finally, recall the linear map $\beta : B \to B \otimes B \otimes B$

$$\beta(b) = b_{(1)(1)} S_B(b_{(2)})_{(1)(1)} \otimes b_{(1)(2)(1)} S_B(b_{(2)})_{(1)(2)} \otimes b_{(1)(2)(2)} S_B(b_{(2)})_{(2)}$$

for any $b \in B$.

We can see $I_B \subseteq ker(\beta)$. Indeed, without losing generality, let $x = f_a^i$ with $a \neq 0$, then we have:

$$\beta(x) = \beta(f_a^i) = \sum_{\substack{j+k+l+m+n+p=i \\ b+c+d+e+f+g=a}} F(b+c+d, e+f+g)F(b, c+d)F(c,d)F(e, f+g)F(f,g)$$
$$F(e,e)F(f,f)F(g,g) f_b^j f_g^p \otimes f_c^k f_f^n \otimes f_d^l f_e^m.$$

Since $a \neq 0$, we can see the right hand side of the above equation is zero (by using $b+c+d+e+f+g=a$). So I_B belongs to the kernel of β. By Definition 17.1 we can see $k[\mathcal{G}_n]$ is quasi coassociative corresponding to the coassociative pair $(k[\mathcal{G}_0], k[\mathcal{G}_n], \pi)$.

Thus by Theorem 17.5 there is a coherent Hopf 2-algebra structure, with $C = k[\mathcal{G}_0]$, and $H = k[\mathcal{G}_0] \otimes k[\mathcal{G}_n]$. To be more precise, we give the structure maps:

$$\Delta(f_0^i \otimes f_a^l) = \sum_{j+k=i} f_0^j \otimes 1 \otimes_B f_0^k \otimes f_a^l;$$
$$\epsilon(f_0^i \otimes f_a^l) = \epsilon_B(f_0^i) f_a^l;$$
$$S(f_0^i \otimes f_a^l) = \sum_{i+n=l} f_0^i \otimes f_a^n;$$
$$s(f_a^l) = \sum_{m+n=l} f_0^m \otimes f_a^n;$$
$$t(f_a^l) = 1 \otimes f_a^l.$$

All the above is the structure of Hopf algebroid over $B = k[\mathcal{G}_n]$.

For the Hopf coquasigroup structure on H we have:

$$\blacktriangle(f_0^i \otimes f_a^l) = \sum_{\substack{m+n=l \\ b+c=a \\ j+k=i}} F(b,c) f_0^j \otimes f_b^m \otimes f_0^k \otimes f_c^n;$$
$$\epsilon_H(f_0^i \otimes f_a^l) = \epsilon_B(f_0^i f_a^l) = \delta_{i,0}\delta_{l,0}\delta_{a,0};$$
$$S_H(f_0^i \otimes f_a^l) = F(a,a) f_0^i \otimes f_a^l.$$

Recall $\alpha : H \to B \otimes B \otimes B$ in (17.12), we have

$$\alpha(f_0^i \otimes f_a^l) = \sum_{\substack{k+m+n=l \\ b+c+d=a}} \beta(f_0^i)(F(b+c,d)F(b,c) f_b^k \otimes f_c^m \otimes f_d^n).$$

From the formula of β, we can see that α is controlled by a 3-cocycle corresponding to the 2-cochain F. In fact,

$$\beta(f_0^i) = \sum_{\substack{j,k,l \in \mathbb{Z}^2 \\ b,c,d \in \mathbb{Z}_n^2}} F(b+c+d, b+c+d)F(b, c+d)F(c,d)F(d, c+b)F(c,b)$$
$$F(d,d)F(c,c)F(b,b) f_b^j \otimes f_c^k \otimes f_d^l$$
$$= \sum_{\substack{j,k,l \in \mathbb{Z}^2 \\ b,c,d \in \mathbb{Z}_n^2}} F(b+c+d, b+c+d)\psi(b,c,d)F(d,d)F(c,c)F(b,b) f_b^j \otimes f_c^k \otimes f_d^l,$$

where ψ is the 3-cocycle given by the 2-cochain F,

$$\psi(b,c,d) = \frac{F(b,c+d)F(c,d)}{F(d,c+b)F(c,b)} = F(b,c+d)F(c,d)F(d,c+b)F(c,b),$$

since F takes its value in $\{\pm 1\}$ [20].

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