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Long time dynamics of Hamiltonian PDEs: linear Klein-Gordon and water waves equations

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To my family
and my friends

I like to keep my issues strong,
It's always darkest before the dawn...
Florence + the Machine - "Shake It Out"

Contents

Chapter 1

Introduction

The question of the long time behaviour for solutions of evolutionary linear and nonlinear partial differential equations (PDEs) is a major problem in the analysis of dispersive equations arising in physical models, as in quantum mechanics and fluid dynamics, for instance. When the motions take place on compact domains, like the periodic torus $\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d$, and if the equations have an Hamiltonian structrure, deeper insights have been obtained in the last decades by regarding such equations as infinite dimensional dynamical systems and combining PDEs tools with the classical dynamical system theory, as KAM (Kolmogorov-Arnold-Moser) theory and Birkhoff normal form, and with analytical techniques, as pseudo-/paradifferential calculus and the Newton-Nash-Moser implicit function theory.

This thesis addresses some questions concerning the stability of the dynamics for three dispersive partial differential equations evolving in one dimensional space periodic variable. Assume to deal with the Cauchy problem associated to a general dispersive PDE,

$$\begin{cases} u_t = \mathcal{N}u + \mathcal{P}(u)\,, \quad u = u(t,x)\,,\ (t,x) \in [0,T] \times \mathbb{T} \\ u(0,x) = u_0(x) \in H^s(\mathbb{T}) \end{cases},$$

where $\mathcal{N}$ is an unbounded linear operator with purely imaginary discrete spectrum, $\mathcal{P}(u)$ is a nonlinear function in u, eventually depending also on its derivatives, and $H^s(\mathbb{T})$ is the Sobolev space of regularity $s \geqslant 0$ on the torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. We consider the following questions:

- *Growth in time for Sobolev norms of the solutions*: for a global solution $u \in \mathcal{C}([0,\infty), H^s(\mathbb{T}))$, provide time dependent, or eventually uniform in time, upper or lower bounds for the evolution of the Sobolev norm $\|u(t,\,\cdot\,)\|_{H^s(\mathbb{T})}$;

- *Existence of time quasi-periodic solutions*: determine a rationally independent frequency vector $\omega \in \mathbb{R}^\nu\backslash\{0\}$, $\nu \geqslant 1$, namely $\omega \cdot \ell \neq 0$ for any $\ell \in \mathbb{Z}^\nu\backslash\{0\}$, and a time quasi-periodic solution $u(t,x) = U(\varphi,x)|_{\varphi=\omega t}$, $\varphi \in \mathbb{T}^\nu$, with a proper selection of the initial data;

- *Long time existence of local well-posed solutions*: for any initial datum $u_0(x)$ satisfying $\|u_0\|_{H^s(\mathbb{T})} \leqslant \epsilon$, determine lower bounds for time of existence $T_\epsilon > 0$ such that the solution $u(t,x)$ stays small with the same size of u_0, that is $\sup_{t\in[0,T_\epsilon]} \|u(t,\,\cdot\,)\|_{H^s(\mathbb{T})} \leqslant C\epsilon$.

In particular, we provide positive answers for the following problems:

1. **Reducibility for the fast driven linear Klein Gordon equation ([93], Chapter 3):** existence of a bounded invertible map that reduces the quasi-periodically forced linear Klein-Gordon equation on the interval $x \in [0,\pi]$

$$u_{tt} - u_{xx} + \mathtt{m}^2 u + V(\omega t, x) u = 0\,, \quad u(t,0) = u(t,\pi) = 0\,, \tag{1.0.1}$$

to a constant-coefficient, diagonal in Fourier system in the regime of *fast oscillations* $|\omega| \gg 1$ and almost conservation of the Sobolev norms (Theorem 1.1 and Corollary 1.2);

2. **Traveling quasi-periodic water waves with constant vorticity ([41], Chapter 4):** existence of Cantor families of small amplitude, *traveling quasi-periodic* solutions for the 2-dimensional space periodic gravity capillary water waves system with constant scalar vorticity $\gamma \in \mathbb{R}$

$$\begin{cases} \eta_t = G(\eta)\psi + \gamma\eta\eta_x \\ \psi_t = -g\eta - \dfrac{\psi_x^2}{2} + \dfrac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} + \kappa\Big(\dfrac{\eta_x}{\sqrt{1+\eta_x^2}}\Big)_x + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi \end{cases} \tag{1.0.2}$$

and their linear stability (Theorem 1.8, see Appendix B for the derivation of (1.0.2));

3. **Long time existence of periodic gravity-capillary water waves ([38], Chapter 5):** time of existence of magnitude ϵ^{-2} for solutions with initial data of size ϵ of the 2-dimensional space periodic irrotational gravity capillary water waves equations

$$\begin{cases} \eta_t = G(\eta)\psi \\ \psi_t = -g\eta - \dfrac{\psi_x^2}{2} + \dfrac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} + \kappa\Big(\dfrac{\eta_x}{\sqrt{1+\eta_x^2}}\Big)_x \end{cases} \tag{1.0.3}$$

for any value of the parameters $(\kappa, g, \mathtt{h})$, even in presence of finitely many 3-waves resonances (Theorem (1.9) and Theorem (1.10)).

1.1 Main results

In this section we describe the detailed statements of the results briefly listed before, each one followed by some comments about the novelties of the theorems.

1.1.1 Reducibility for the fast driven linear Klein Gordon equation

We consider a linear Klein-Gordon equation with quasi-periodic driving

$$u_{tt} - u_{xx} + \mathtt{m}^2 u + V(\omega t, x) u = 0 \ , \qquad x \in [0, \pi] \ , \ \ t \in \mathbb{R} \ , \tag{1.1.1}$$

with spatial Dirichlet boundary conditions $u(t,0) = u(t,\pi) = 0$.
The potential $V : \mathbb{T}^\nu \times [0,\pi] \to \mathbb{R}$, is quasi-periodic in time with a frequency vector $\omega \in \mathbb{R}^\nu \backslash \{0\}$. The main feature of this driving is that it is not perturbative in size, but we require it to be fast oscillating, namely $|\omega| \gg 1$.

The goal is to provide, for any frequency ω belonging to a Cantor set of large measure, a reducibility result for the system (1.1.1). That is, we construct a change of coordinates which conjugates equation (1.1.1) into a diagonal, time independent one. Up to our knowledge, this is the first result of reducibility in an infinite dimensional setting in which the perturbation is not assumed to be small in size, but only fast oscillating.

The potential driving $V(\omega t, x)$ is treated as a smooth function $V : \mathbb{T}^\nu \times [0,\pi] \ni (\varphi, x) \mapsto V(\varphi, x) \in \mathbb{R}$, $\nu \geqslant 1$, which satisfies two assumptions:

(V1) The even extension in x of $V(\varphi, x)$ on the torus $\mathbb{T} \simeq [-\pi, \pi]$, which we still denote by V, is smooth in both variables and it extends analytically in φ in a proper complex neighbourhood of $\mathbb{T}^\nu$ of width $\rho > 0$. In particular, for any $\beta \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, there is a constant $C_{\beta,\rho} > 0$ such that
$$|\partial_x^\beta V(\varphi, x)| \leqslant C_{\beta,\rho} \quad \forall \, x \in \mathbb{T} \ , \ |\mathrm{Im}\, \varphi| \leqslant \rho \ ;$$

(V2) $\int_{\mathbb{T}^\nu} V(\varphi, x) \, \mathrm{d}\varphi = 0$ for any $x \in [0, \pi]$.

To state precisely our main result, equation (1.1.1) has to be rewritten as a Hamiltonian system. We introduce the new variables

$$\psi := B^{1/2} u + \mathrm{i} B^{-1/2} \partial_t u \ , \qquad \overline{\psi} := B^{1/2} u - \mathrm{i} B^{-1/2} \partial_t u \ ,$$

where

$$B := \sqrt{-\Delta + \mathtt{m}^2} \ ; \tag{1.1.2}$$

note that the operator B is invertible also when $\mathtt{m} = 0$, since we consider Dirichlet boundary conditions. In the new variables equation (1.1.1) is equivalent to

$$\mathrm{i} \partial_t \psi(t) = B \psi(t) + \frac{1}{2} B^{-1/2} V(\omega t) B^{-1/2} (\psi(t) + \overline{\psi}(t)) \ . \tag{1.1.3}$$

Taking (1.1.3) coupled with its complex conjugate, we obtain the following system

$$\mathrm{i}\partial_t \psi(t) = \mathbf{H}(t)\psi(t) \ , \quad \mathbf{H}(t) := \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} + \frac{1}{2} B^{-1/2} V(\omega t, x) B^{-1/2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \ , \tag{1.1.4}$$

where, abusing notation, we denote $\psi(t) \equiv \left(\begin{smallmatrix} \psi(t) \\ \overline{\psi(t)} \end{smallmatrix}\right)$ the vector with the components $\psi, \overline{\psi}$. The phase space for (1.1.4) is $\mathcal{H}^r \times \mathcal{H}^r$, where, for $r \geqslant 0$,

$$\mathcal{H}^r := \Big\{ \psi(x) = \sum_{m \in \mathbb{N}} \psi_m \sin(mx), \ x \in [0,\pi] \ : \ \|\psi\|^2_{\mathcal{H}^r} := \sum_{m \in \mathbb{N}} \langle m \rangle^{2r} |\psi_m|^2 < \infty \Big\} \ . \tag{1.1.5}$$

Here we have used the notation $\langle m \rangle := (1 + |m|^2)^{\frac{1}{2}}$, which will be kept throughout Chapter 3. We define the ν-dimensional annulus of size $\mathtt{M} > 0$ by

$$R_{\mathtt{M}} := \overline{B_{2\mathtt{M}}(0)} \backslash B_{\mathtt{M}}(0) \subset \mathbb{R}^\nu \ ,$$

where $B_M(0)$ denotes the ball of center zero and radius M in the Euclidean topology of $\mathbb{R}^\nu$.

Theorem 1.1. (Reducibility for the fast driven Klein-Gordon equation) *Consider the system* (1.1.4) *and assume (***V1***) and (***V2***). Fix arbitrary $r, \mathtt{m} \geqslant 0$ and $\alpha \in (0,1)$. Fix also an arbitrary $\gamma_* > 0$ sufficiently small.*
Then there exist $\mathtt{M}_* > 1$, $C > 0$ and, for any $\mathtt{M} \geqslant \mathtt{M}_*$, a subset $\Omega^\alpha_\infty = \Omega^\alpha_\infty(\mathtt{M}, \gamma_*)$ in $R_{\mathtt{M}}$, fulfilling

$$\frac{\mathrm{meas}(R_{\mathtt{M}} \backslash \Omega^\alpha_\infty)}{\mathrm{meas}(R_{\mathtt{M}})} \leqslant C\gamma_* ,$$

such that the following holds true. For any frequency vector $\omega \in \Omega^\alpha_\infty$, there exists an operator $\mathcal{T}(\omega t; \omega)$, bounded in $\mathcal{L}(\mathcal{H}^r \times \mathcal{H}^r)$, quasi-periodic in time and analytic in a shrunk neighbourhood of $\mathbb{T}^\nu$ of width $\rho/8$, such that the change of coordinates $\psi = \mathcal{T}(\omega t; \omega) w$ conjugates (1.1.4) *to the diagonal time-independent system*

$$\mathrm{i}\dot{w}(t) = \mathbf{H}^{\infty,\alpha} w(t) \ , \quad \mathbf{H}^{\infty,\alpha} := \begin{pmatrix} \mathcal{D}^{\infty,\alpha} & 0 \\ 0 & -\mathcal{D}^{\infty,\alpha} \end{pmatrix} \ , \ \mathcal{D}^{\infty,\alpha} = \mathrm{diag}\left\{ \lambda^\infty_j(\omega) \ : \ j \in \mathbb{N} \right\}. \tag{1.1.6}$$

The transformation $\mathcal{T}(\omega t; \omega)$ is close to the identity, in the sense that there exists $C_r > 0$ independent of $\mathtt{M}$ such that

$$\|\mathcal{T}(\omega t; \omega) - \mathbb{1}\|_{\mathcal{L}(\mathcal{H}^r \times \mathcal{H}^r)} \leqslant \frac{C_r}{\mathtt{M}^{\frac{1-\alpha}{2}}} \ . \tag{1.1.7}$$

The new eigenvalues $(\lambda^\infty_j(\omega))_{j \in \mathbb{N}}$ are real, Lipschitz in ω, and admit the asymptotics, for $j \in \mathbb{N}$,

$$\lambda^\infty_j(\omega) := \lambda^\infty_j(\omega, \alpha) := \lambda_j + \varepsilon^\infty_j(\omega, \alpha) \ , \quad \varepsilon^\infty_j(\omega, \alpha) \sim O\left(\frac{1}{\mathtt{M} j^\alpha} \right) \ , \tag{1.1.8}$$

where $\lambda_j = \sqrt{j^2 + \mathtt{m}^2}$ *are the eigenvalues of the operator* B.

The proof of Theorem 1.1 is the content of Chapter 3. Let us make some comments:

1) Back to the original coordinates, the equation (1.1.1) is reduced to

$$\partial_{tt} u + \left(\mathcal{D}^{\infty,\alpha}\right)^2 u = 0\,;$$

2) The parameter α, which one chooses and fixes in the real interval $(0,1)$, influences the asymptotic expansion of the final eigenvalues, as one can read from (1.1.8). Also the construction of the set of the admissible frequency vectors heavily depends on this parameter;

3) In Theorem 1.1 we can take also $\mathtt{m} = 0$; indeed, with Dirichlet boundary conditions, the unperturbed eigenvalues λ_j are simple, integers and their corrections are small (see (1.1.8)). This means that it is enough to move the frequency vector ω for avoiding resonances;

4) The assumptions of Theorem 1.1 can be weakened, for example asking only Sobolev regularity for $V(\varphi, x)$, dropping (**V2**) or using periodic boundary conditions instead of the Dirichlet ones. The result still holds and it is addresses in a forthcoming paper [92].

Let us denote by $\mathcal{U}_\omega(t,\tau)$ the propagator generated by (1.1.4) such that $\mathcal{U}_\omega(\tau,\tau) = \mathbb{1}$ for any $\tau \in \mathbb{R}$. An immediate consequence of Theorem 1.1 is that we have a Floquet decomposition:

$$\mathcal{U}_\omega(t,\tau) = \mathcal{T}(\omega t;\omega)^* \circ \mathrm{e}^{-\mathrm{i}(t-\tau)\mathbf{H}^{\infty,\alpha}} \circ \mathcal{T}(\omega\tau;\omega)\ . \tag{1.1.9}$$

Another consequence of (1.1.9) is that, for any $r \geqslant 0$, the norm $\|\mathcal{U}_\omega(t,0)\varphi_0\|_{\mathcal{H}^r\times\mathcal{H}^r}$ is bounded uniformly in time:

Corollary 1.2. (Almost conservation of the Sobolev norms) *Let* $\mathtt{M} \geqslant \mathtt{M}_*$ *and* $\omega \in \Omega^\alpha_\infty$. *For any* $r \geqslant 0$ *one has*

$$c_r\,\|\psi_0\|_{\mathcal{H}^r\times\mathcal{H}^r} \leqslant \|\mathcal{U}_\omega(t,0)\psi_0\|_{\mathcal{H}^r\times\mathcal{H}^r} \leqslant C_r\,\|\psi_0\|_{\mathcal{H}^r\times\mathcal{H}^r}\,, \quad \forall\, t \in \mathbb{R}\ , \forall\, \psi_0 \in \mathcal{H}^r \times \mathcal{H}^r, \tag{1.1.10}$$

for some $c_r > 0, C_r > 0$.
More precisely, there exists a constant $c_r' > 0$ *such that, if the initial data* $\psi_0 \in \mathcal{H}^r \times \mathcal{H}^r$, *then*

$$\left(1 - \frac{c_r'}{\mathtt{M}^{\frac{1-\alpha}{2}}}\right)\|\psi_0\|_{\mathcal{H}^r\times\mathcal{H}^r} \leqslant \|\mathcal{U}_\omega(t,0)\psi_0\|_{\mathcal{H}^r\times\mathcal{H}^r} \leqslant \left(1 + \frac{c_r'}{\mathtt{M}^{\frac{1-\alpha}{2}}}\right)\|\psi_0\|_{\mathcal{H}^r\times\mathcal{H}^r}\,, \quad \forall\, t \in \mathbb{R}\ .$$

Remark 1.3. Corollary 1.2 shows that, if the frequency ω is chosen in the Cantor set Ω^α_∞, no phenomenon of growth of Sobolev norms can happen. On the contrary, if ω is chosen resonant, one can construct drivings which provoke norm explosion with exponential rate. For an overview of the literature, we remind to the discussion in Section 1.2.2.

The main ideas for the proof of Theorem 1.1 will be presented in Section 2.1.

1.1.2 Traveling quasi-periodic water waves with constant vorticity

We consider the Euler equations of hydrodynamics for a 2-dimensional perfect, incompressible, inviscid fluid with *constant vorticity* γ, under the action of gravity and capillary forces at the free surface. The fluid fills an ocean with depth $\mathtt{h} > 0$ (eventually infinite) and with space periodic boundary conditions, namely it occupies the region

$$\mathcal{D}_{\eta,\mathtt{h}} := \big\{(x,y) \in \mathbb{T} \times \mathbb{R} \ : \ -\mathtt{h} \leqslant y < \eta(t,x)\big\} , \quad \mathbb{T} := \mathbb{T}_x := \mathbb{R}/(2\pi\mathbb{Z}) . \tag{1.1.11}$$

In case of a fluid with constant vorticity $v_x - u_y =: \gamma \in \mathbb{R}$, the velocity field is the sum of the Couette flow $\binom{-\gamma y}{0}$, which carries all the vorticity γ of the fluid, and an irrotational field, expressed as the gradient of a harmonic function Φ, called the *generalized velocity potential.*

Denoting by $\psi(t,x)$ the evaluation of the generalized velocity potential at the free interface $\psi(t,x) := \Phi(t,x,\eta(t,x))$, one recovers Φ by solving the elliptic problem

$$\Delta\Phi = 0 \ \text{ in } \mathcal{D}_{\eta,\mathtt{h}} , \quad \Phi = \psi \ \text{ at } y = \eta(t,x) , \quad \Phi_y \to 0 \ \text{ as } y \to -\mathtt{h} . \tag{1.1.12}$$

The third condition in (1.1.12) means the impermeability property of the bottom

$$\Phi_y(t,x,-\mathtt{h}) = 0 , \ \text{if } \mathtt{h} < \infty , \qquad \lim_{y\to-\infty} \Phi_y(t,x,y) = 0 , \ \text{if } \mathtt{h} = +\infty .$$

Imposing that the fluid particles at the free surface remain on it along the evolution (kinematic boundary condition), and that the pressure of the fluid plus the capillary forces at the free surface is equal to the constant atmospheric pressure (dynamic boundary condition), the time evolution of the fluid is determined by the following system of equations

$$\begin{cases} \eta_t = G(\eta)\psi + \gamma\eta\eta_x \\ \psi_t = -g\eta - \dfrac{\psi_x^2}{2} + \dfrac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} + \kappa\Big(\dfrac{\eta_x}{\sqrt{1+\eta_x^2}}\Big)_x + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi . \end{cases} \tag{1.1.13}$$

Here $g > 0$ is the gravity, $\kappa > 0$ is the surface tension coefficient and $G(\eta)$ is the Dirichlet-Neumann operator

$$G(\eta)\psi := G(\eta,\mathtt{h})\psi := \sqrt{1+\eta_x^2}\,(\partial_{\vec{n}}\Phi)|_{y=\eta(x)} = (-\Phi_x\eta_x + \Phi_y)|_{y=\eta(x)} . \tag{1.1.14}$$

It is well known since Calderon that the Dirichlet-Neumann operator is a pseudo-differential operator with principal operator given by the Fourier multiplier

$$G(0) := G(0,\mathtt{h}) = \begin{cases} D\tanh(\mathtt{h}D) & \text{if } \mathtt{h} < \infty \\ |D| & \text{if } \mathtt{h} = +\infty , \end{cases} \qquad \text{where} \qquad D := \frac{1}{\mathrm{i}}\partial_x , \tag{1.1.15}$$

with symbol

$$G_j(0) := G_j(0,\mathtt{h}) = \begin{cases} j\tanh(\mathtt{h}j) & \text{if } \mathtt{h} < \infty \\ |j| & \text{if } \mathtt{h} = +\infty\,. \end{cases} \tag{1.1.16}$$

Actually, we have $G(\eta) - G(0) \in \mathrm{OP}S^{-\infty}$, see for instance [44, 13].

The irrotational model $\gamma = 0$ was formulated by Zakharov[174] and Craig, Sulem [68] , while Constantin, Ivanov, Prodanov [58] and Wahlén [163] provided the system (1.1.13) for any $\gamma \in \mathbb{R}$. The derivation of the equations (1.1.13) is available in Appendix B.

The water waves equations (1.1.13) are a Hamiltonian system. Indeed, as observed in the irrotational case $\gamma = 0$ by Zakharov [174] and later in presence of any $\gamma \in \mathbb{R}$ by Wahlén [163], they are equivalent to write

$$\eta_t = \nabla_\psi H(\eta,\psi)\,, \quad \psi_t = (-\nabla_\eta + \gamma\partial_x^{-1}\nabla_\psi)H(\eta,\psi)\,, \tag{1.1.17}$$

where ∇ denotes the L^2-gradient, with Hamiltonian

$$H(\eta,\psi) = \frac{1}{2}\int_{\mathbb{T}} \Big(\psi\, G(\eta)\psi + g\eta^2\Big)\,\mathrm{d}x + \kappa\int_{\mathbb{T}} \sqrt{1+\eta_x^2}\,\mathrm{d}x + \frac{\gamma}{2}\int_{\mathbb{T}} \Big(-\psi_x\eta^2 + \frac{\gamma}{3}\eta^3\Big)\,\mathrm{d}x\,. \tag{1.1.18}$$

For any nontrivial value of the vorticity $\gamma \neq 0$, the system (1.1.17) is endowed with a non canonical Poisson structure: it will be discussed with more details in Section 4.1.1.

The equations (1.1.13) enjoy two important symmetries. First, they are time reversible: we say that a solution of (1.1.13) is *reversible* if

$$\eta(-t,-x) = \eta(t,x)\,, \quad \psi(-t,-x) = -\psi(t,x)\,. \tag{1.1.19}$$

Second, since the bottom of the fluid domain is flat, the equations (1.1.13) are *invariant by space translations* and, by Noether Theorem, it implies that the momentum $\int_{\mathbb{T}} \eta_x(x)\psi(x)\,\mathrm{d}x$ is a prime integral of (1.1.13).

The variables (η,ψ) of system (1.1.13) belong to some Sobolev space $H_0^s(\mathbb{T}) \times \dot{H}^s(\mathbb{T})$ for some s large. Here $H_0^s(\mathbb{T})$, $s \in \mathbb{R}$, denotes the Sobolev space of functions with zero average

$$H_0^s(\mathbb{T}) := \Big\{ u \in H^s(\mathbb{T}) \; : \; \int_{\mathbb{T}} u(x)\mathrm{d}x = 0 \Big\}$$

and $\dot{H}^s(\mathbb{T})$, $s \in \mathbb{R}$, the corresponding homogeneous Sobolev space, namely the quotient space obtained by identifying all the $H^s(\mathbb{T})$ functions which differ only by a constant. For simplicity of notation, we shall denote the equivalent class $[\psi] = \{\psi + c \,:\, c \in \mathbb{R}\}$ just by ψ. This choice of the phase space is allowed because $\int_{\mathbb{T}} \eta(t,x)\,\mathrm{d}x$ is a prime integral and the right hand side of (1.1.13) depends only on η and $\psi - \frac{1}{2\pi}\int_{\mathbb{T}} \psi\,\mathrm{d}x$.

A fundamental role is played by the system obtained linearizing (1.1.13) at the equilibrium $(\eta, \psi) = (0, 0)$, namely

$$\begin{cases} \partial_t \eta &= G(0)\psi \\ \partial_t \psi &= -(g - \kappa \partial_x^2)\eta + \gamma \partial_x^{-1} G(0)\psi \, . \end{cases} \tag{1.1.20}$$

The Dirichlet-Neumann operator at the flat surface $\eta = 0$ is the Fourier multiplier defined in (1.1.15), (1.1.16). The linear frequencies are given by

$$\Omega_j := \Omega_j(\kappa) = \Omega_j(\kappa, \mathtt{h}, g, \gamma) := \sqrt{\Big(\kappa j^2 + g + \frac{\gamma^2}{4}\frac{G_j(0)}{j^2}\Big) G_j(0)} + \frac{\gamma}{2}\frac{G_j(0)}{j}, \quad j \in \mathbb{Z}\backslash\{0\} \, . \tag{1.1.21}$$

Note that the map $j \mapsto \Omega_j(\kappa)$ is not even due to the vorticity term $\frac{\gamma}{2} G_j(0)/j$, which is odd in j.

In the Euclidean case $x \in \mathbb{R}$, all the solutions of (1.1.20) disperse to 0 as $t \to \infty$. On the contrary, on the compact periodic domain $x \in \mathbb{T}$, all the solutions of the linear system (1.1.20) are either periodic, quasi-periodic or almost periodic in time, with linear frequencies $\Omega_j(\kappa)$.

As we will show in Section 4.1.2, all reversible solutions (see (1.1.19)) of (1.1.20) are

$$\begin{aligned} \begin{pmatrix} \eta(t,x) \\ \psi(t,x) \end{pmatrix} &= \sum_{n \in \mathbb{N}} \begin{pmatrix} M_n \rho_n \cos(nx - \Omega_n(\kappa)t) \\ P_n \rho_n \sin(nx - \Omega_n(\kappa)t) \end{pmatrix} \\ &\quad + \sum_{n \in \mathbb{N}} \begin{pmatrix} M_n \rho_{-n} \cos(nx + \Omega_{-n}(\kappa)t) \\ P_{-n} \rho_{-n} \sin(nx + \Omega_{-n}(\kappa)t) \end{pmatrix} , \end{aligned} \tag{1.1.22}$$

where $\rho_n \geqslant 0$ are arbitrary amplitudes and M_n, $P_{\pm n}$ are the real coefficients

$$M_j := \left(\frac{G_j(0)}{\kappa j^2 + g + \frac{\gamma^2}{4}\frac{G_j(0)}{j^2}} \right)^{\frac{1}{4}} , \ j \in \mathbb{Z}\backslash\{0\} \, , \quad P_{\pm n} := \frac{\gamma}{2}\frac{M_n}{n} \pm M_n^{-1}, \ n \in \mathbb{N} \, .$$

Note that the map $j \mapsto M_j$ is even. Furthermore, note that the functions in (1.1.22) are linear superposition of plane waves traveling either to the right or to the left.

Remark 1.4. Actually, (1.1.22) contains also standing waves, for example when the vorticity $\gamma = 0$ (which implies $\Omega_{-n}(\kappa) = \Omega_n(\kappa)$, $P_{-n} = -P_n$) and $\rho_{-n} = \rho_n$, giving solutions even in x. This is the well known superposition effect of waves with the same amplitude, frequency and wavelength traveling in opposite directions.

We first provide the notion of quasi-periodic traveling wave.

Definition 1.5. (Quasi-periodic traveling wave) We say that $(\eta(t,x), \psi(t,x))$ is a time quasi-periodic *traveling* wave with irrational frequency vector $\omega = (\omega_1, \ldots, \omega_\nu) \in \mathbb{R}^\nu$, $\nu \in \mathbb{N}$, i.e. $\omega \cdot \ell \neq 0$ for any $\ell \in \mathbb{Z}^\nu\backslash\{0\}$, and "wave vectors" $(j_1, \ldots, j_\nu) \in \mathbb{Z}^\nu$, if there exist functions

$(\breve\eta, \breve\psi) : \mathbb{T}^\nu \to \mathbb{R}^2$ such that

$$\begin{pmatrix} \eta(t,x) \\ \psi(t,x) \end{pmatrix} = \begin{pmatrix} \breve\eta(\omega_1 t - j_1 x, \dots, \omega_\nu t - j_\nu x) \\ \breve\psi(\omega_1 t - j_1 x, \dots, \omega_\nu t - j_\nu x) \end{pmatrix} . \tag{1.1.23}$$

Remark 1.6. If $\nu = 1$, such functions are time periodic and indeed stationary in a moving frame with speed ω_1/j_1. On the other hand, if the number of frequencies ν is $\geqslant 2$, the waves (1.1.23) cannot be reduced to steady waves by any appropriate choice of the moving frame.

We shall construct traveling quasi-periodic solutions of (1.1.13) with a diophantine frequency vector ω belonging to an open bounded subset Ω in $\mathbb{R}^\nu$, namely, for some $\upsilon \in (0,1)$, $\tau > \nu - 1$,

$$\mathtt{DC}(\upsilon,\tau) := \Big\{ \omega \in \Omega \subset \mathbb{R}^\nu \ : \ |\omega \cdot \ell| \geqslant \upsilon \langle \ell \rangle^{-\tau} \ , \ \forall\, \ell \in \mathbb{Z}^\nu \backslash \{0\} \Big\} , \quad \langle \ell \rangle := \max\{1, |\ell|\} . \tag{1.1.24}$$

Regarding regularity, we will prove the existence of quasi-periodic traveling waves $(\breve\eta, \breve\psi)$ belonging to some Sobolev space

$$H^s(\mathbb{T}^\nu, \mathbb{R}^2) = \Big\{ \breve f(\varphi) = \sum_{\ell \in \mathbb{Z}^\nu} f_\ell \, e^{\mathrm{i}\ell\cdot\varphi} \ , \quad f_\ell \in \mathbb{R}^2 \ \ : \| \breve f \|_s^2 := \sum_{\ell \in \mathbb{Z}^\nu} |f_\ell|^2 \langle \ell \rangle^{2s} < \infty \Big\} . \tag{1.1.25}$$

Fixed finitely many arbitrary *distinct* natural numbers

$$\mathbb{S}^+ := \{ \overline{n}_1, \dots, \overline{n}_\nu \} \subset \mathbb{N} \ , \quad 1 \leqslant \overline{n}_1 < \ldots < \overline{n}_\nu \, , \tag{1.1.26}$$

and signs

$$\Sigma := \{ \sigma_1, \dots, \sigma_\nu \}, \quad \sigma_a \in \{-1, 1\} \, , \quad a = 1, \dots, \nu \, , \tag{1.1.27}$$

consider the reversible quasi-periodic traveling wave solutions of the linear system (1.1.20) given by

$$\begin{aligned} \begin{pmatrix} \eta(t,x) \\ \psi(t,x) \end{pmatrix} = & \sum_{a \in \{1,\dots,\nu \,:\, \sigma_a = +1\}} \begin{pmatrix} M_{\overline{n}_a} \sqrt{\xi_{\overline{n}_a}} \cos(\overline{n}_a x - \Omega_{\overline{n}_a}(\kappa) t) \\ P_{\overline{n}_a} \sqrt{\xi_{\overline{n}_a}} \sin(\overline{n}_a x - \Omega_{\overline{n}_a}(\kappa) t) \end{pmatrix} \\ & + \sum_{a \in \{1,\dots,\nu \,:\, \sigma_a = -1\}} \begin{pmatrix} M_{\overline{n}_a} \sqrt{\xi_{-\overline{n}_a}} \cos(\overline{n}_a x + \Omega_{-\overline{n}_a}(\kappa) t) \\ P_{-\overline{n}_a} \sqrt{\xi_{-\overline{n}_a}} \sin(\overline{n}_a x + \Omega_{-\overline{n}_a}(\kappa) t) \end{pmatrix} \end{aligned} \tag{1.1.28}$$

where $\xi_{\pm\overline{n}_a} > 0$, $a = 1, \dots, \nu$. The frequency vector of (4.0.6) is $\vec\Omega(\kappa) := (\Omega_{\sigma_a \overline{n}_a}(\kappa))_{a=1,\dots,\nu} \in \mathbb{R}^\nu$.

Remark 1.7. If $\sigma_a = +1$, we select in (4.0.6) a right traveling wave, whereas, if $\sigma_a = -1$, a left traveling one. By (4.0.4), the linear solutions (4.0.6) are genuinely traveling waves: superposition of identical waves traveling in opposite direction, generating standing waves, does not happen.

The result in Theorem 1.8 shows that the linear solutions (4.0.6) can be continued to quasi-periodic traveling wave solutions of the nonlinear water waves equations (1.1.13), for most values of the surface tension $\kappa \in [\kappa_1, \kappa_2]$, with a frequency vector $\widetilde\Omega := (\widetilde\Omega_{\sigma_a \overline{n}_a})_{a=1,\dots,\nu}$, close to $\vec\Omega(\kappa) :=$

$(\Omega_{\sigma_a \overline{n}_a}(\kappa))_{a=1,\ldots,\nu}$. Here is the precise statement.

Theorem 1.8. (KAM for traveling gravity-capillary water waves with constant vorticity) *Consider finitely many tangential sites* $\mathbb{S}^+ \subset \mathbb{N}$ *as in* (4.0.4) *and signs* Σ *as in* (4.0.5). *Then there exist* $\overline{s} > 0$, $\varepsilon_0 \in (0,1)$ *such that, for every* $|\xi| \leqslant \varepsilon_0^2$, $\xi := (\xi_{\sigma_a \overline{n}_a})_{a=1,\ldots,\nu} \in \mathbb{R}^\nu_+$, *the following hold:*

1. *there exists a Cantor-like set* $\mathcal{G}_\xi \subset [\kappa_1, \kappa_2]$ *with asymptotically full measure as* $\xi \to 0$, *i.e.* $\lim_{\xi \to 0} |\mathcal{G}_\xi| = \kappa_2 - \kappa_1$;

2. *for any* $\kappa \in \mathcal{G}_\xi$, *the gravity-capillary water waves equations* (1.1.13) *have a reversible quasi-periodic traveling wave solution (according to Definition 1.5) of the form*

$$\begin{aligned}\begin{pmatrix}\eta(t,x)\\ \psi(t,x)\end{pmatrix} &= \sum_{a\in\{1,\ldots,\nu\}\colon \sigma_a=+1} \begin{pmatrix} M_{\overline{n}_a}\sqrt{\xi_{\overline{n}_a}}\cos(\overline{n}_a x - \widetilde{\Omega}_{\overline{n}_a}(\kappa)t)\\ P_{\overline{n}_a}\sqrt{\xi_{\overline{n}_a}}\sin(\overline{n}_a x - \widetilde{\Omega}_{\overline{n}_a}(\kappa)t)\end{pmatrix} \\ &\quad + \sum_{a\in\{1,\ldots,\nu\}\colon \sigma_a=-1} \begin{pmatrix} M_{\overline{n}_a}\sqrt{\xi_{-\overline{n}_a}}\cos(\overline{n}_a x + \widetilde{\Omega}_{-\overline{n}_a}(\kappa)t)\\ P_{-\overline{n}_a}\sqrt{\xi_{-\overline{n}_a}}\sin(\overline{n}_a x + \widetilde{\Omega}_{-\overline{n}_a}(\kappa)t)\end{pmatrix} + r(t,x)\end{aligned} \tag{1.1.29}$$

where

$$r(t,x) = \breve{r}(\widetilde{\Omega}_{\sigma_1\overline{n}_1}(\kappa)t - \sigma_1\overline{n}_1 x, \ldots, \widetilde{\Omega}_{\sigma_\nu\overline{n}_\nu}(\kappa)t - \sigma_\nu\overline{n}_\nu x)\,, \quad \breve{r} \in H^{\overline{s}}(\mathbb{T}^\nu, \mathbb{R}^2)\,, \quad \lim_{\xi\to 0}\frac{\|\breve{r}\|_{\overline{s}}}{\sqrt{|\xi|}} = 0\,,$$

with a Diophantine frequency vector $\widetilde{\Omega} := (\widetilde{\Omega}_{\sigma_a\overline{n}_a})_{a=1,\ldots,\nu} \in \mathbb{R}^\nu$, *depending on* κ, ξ, *and satisfying* $\lim_{\xi\to 0}\widetilde{\Omega} = \vec{\Omega}(\kappa)$. *In addition these quasi-periodic solutions are linearly stable.*

The proof of Theorem 1.8 is the content of Chapter 4. Let us make some comments.

1) Theorem 1.8 holds for any value of the vorticity γ, so in particular it guarantees existence of quasi-periodic traveling waves also for irrotational fluids, i.e. $\gamma = 0$. In this case the solutions (4.0.9) do not reduce to those in [44], which are standing, i.e. even in x. If the vorticity $\gamma \neq 0$, one does not expect the existence of standing wave solutions since the water waves vector field (1.1.13) does not leave invariant the subspace of functions even in x.

2) Theorem 1.8 produces time quasi-periodic solutions of the Euler equation with a velocity field which is a small perturbation of the Couette flow $\binom{-\gamma y}{0}$. Indeed, from the solution $(\eta(t,x), \psi(t,x))$ in (4.0.9), one recovers the generalized velocity potential $\Phi(t,x,y)$ by solving the elliptic problem (1.1.12) and finally constructs the velocity field $\binom{u(t,x,y)}{v(t,x,y)} = \binom{-\gamma y}{0} + \nabla\Phi(t,x,y)$. The time quasi-periodic potential $\Phi(t,x,y)$ has size $O(\sqrt{|\xi|})$, as $\eta(t,x)$ and $\psi(t,x)$. Our perturbation of the Couette flow, however, is not a shear flow anymore. For the nonlinear 2D Euler equations, it was proved by Bedrossian-Masmoudi [28] the asymptotic stability of shear flow solutions near the Couette flow on $\mathbb{T} \times \mathbb{R}$ in the classes of Gevrey regularity $s \in (1/2, 1]$, while

Deng-Masmoudi [79] showed the instability of the same solutions when the Gevrey regularity is strictly less than 1/2.

3) In the case $\nu = 1$ the solutions constructed in Theorem 1.8 reduce to steady periodic traveling waves, which can be obtained by an application of the Crandall-Rabinowitz theorem, see e.g. [139, 162, 164].

4) Theorem 1.8 selects initial data giving raise to global in time solutions (4.0.9) of the water waves equations (1.1.13). So far, no results about global existence for (1.1.13) with periodic boundary conditions are known. The available results concern local well posedness with a general vorticity, see e.g. the work of Coutand and Shkoller [63], and a ε^{-2} existence for initial data of size ε in the case of constant vorticity by Ifrim and Tataru [113].

5) With the choice (4.0.4)-(4.0.5) the unperturbed frequency vector $\vec{\Omega}(\kappa) = (\Omega_{\sigma_a \overline{n}_a}(\kappa))_{a=1,\dots,\nu}$ is diophantine for most values of the surface tension κ and for all values of vorticity, gravity and depth. It follows by the more general results of Sections 4.3 and 4.4.2. This may not be true for an arbitrary choice of the linear frequencies $\Omega_j(\kappa)$, $j \in \mathbb{Z}\backslash\{0\}$. For example, in the case $\mathtt{h} = +\infty$, the vector

$$\vec{\Omega}(\kappa) = \big(\Omega_{-n_3}(\kappa), \Omega_{-n_2}(\kappa), \Omega_{-n_1}(\kappa), \Omega_{n_1}(\kappa), \Omega_{n_2}(\kappa), \Omega_{n_3}(\kappa)\big)$$

is resonant, for all the values of κ, also taking into account the restrictions on the indexes for the search of traveling waves, see Section 4.2.5. Indeed, recalling (4.0.3) and that, for $\mathtt{h} = +\infty$, $G_j(0, \mathtt{h}) = |j|$, we have, for $\ell = \big(- \ell_{n_3}, -\ell_{n_2}, -\ell_{n_1}, \ell_{n_1}, \ell_{n_2}, \ell_{n_3}\big)$ that the system

$$\vec{\Omega}(\kappa) \cdot \vec{\ell} = \gamma(\ell_{n_1} + \ell_{n_2} + \ell_{n_3}) = 0\,, \quad n_1\ell_{n_1} + n_2\ell_{n_2} + n_3\ell_{n_3} = 0\,,$$

has integer solutions. In this case the possible existence of quasi-periodic solutions of the water waves system (1.1.13) depends on the frequency modulation induced by the nonlinear terms.

6) Comparison with [44]. There are significant differences with respect to [44], which proves the existence of quasi-periodic *standing* waves for *irrotational* fluids, not only in the result –the solutions of Theorem 1.8 are *traveling* waves of fluids with *constant vorticity*– but also in the techniques.

(i) The first difference –which is a novelty of this result– is a new formulation of degenerate KAM theory exploiting the "momentum conservation", namely the invariance under space translations of the Hamilton equations. The degenerate KAM theory approach for PDEs has been developed by Bambusi, Berti, and Magistrelli [21], and then in [44], [13], in order to prove the non-trivial dependence of the linear frequencies with respect to a parameter –in our case the surface tension κ–, see the "Transversality" Proposition 4.53. A key assumption used in [21], [44], [13] is that the linear frequencies are simple (because of Dirichlet boundary conditions in [21] and Neumann boundary conditions in [44], [13]). This is not true for traveling waves. In order to overcome this difficulty we strongly exploit the invariance of the equations (1.1.13) under space translations, which ultimately implies the restrictions to the indexes (4.3.8)-(4.3.10). In this way, assuming that the moduli of the tangential sites are all different as in (4.0.4), cfr. with item 5), we can remove some otherwise possibly degenerate case. This requires to keep trace along all the proof of the "momentum conservation property" that we characterize in different ways in Section 4.2.5. The momentum conservation law has been used in several KAM results for semilinear PDEs since the works Geng and You [95, 96], who provided Birkhoff normal forms and quasi-periodic solutions for the nonlinear Schrödinger equation on tori of dimension one and higher, see also [124, 153, 142, 107, 89] and references therein. The present result gives a new application in the context of degenerate KAM theory (with additional difficulties arising by the quasi-linear nature of the water waves equations).

(*ii*) Other significant differences with respect to [44] arise in the reduction in pseudodifferential orders (Section 4.6) of the quasi-periodic linear operators obtained along the Nash-Moser iteration. In particular we mention that we have to preserve the Hamiltonian nature of these operators (at least until Section 4.6.4). Otherwise it would appear a time dependent operator at the order $|D|^{1/2}$, of the form $\mathrm{i}a(\varphi)\mathcal{H}|D|^{\frac{1}{2}}$, with $a(\varphi) \in \mathbb{R}$ independent of x, compatible with the reversible structure, which can not be eliminated. Note that the operator $\mathrm{i}a(\varphi)\mathcal{H}|D|^{\frac{1}{2}}$ is not Hamiltonian (unless $a(\varphi) = 0$). Note also that the above difficulty was not present in [44] dealing with standing waves, because an operator of the form $\mathrm{i}a(\varphi)\mathcal{H}|D|^{\frac{1}{2}}$ does not map even functions into even functions. In order to overcome this difficulty we have to perform symplectic changes of variables (at least until Section 4.6.4), and not just reversible as in [44, 13]. We finally mention that we perform as a first step in Section 4.6.1 a quasi-periodic time reparametrization to avoid otherwise a technical difficulty in the conjugation of the remainders obtained by the Egorov theorem in Section 4.6.3. This difficulty was not present in [44], since it arises conjugating the additional pseudodifferential term due to vorticity, see Remark 4.70.

7) Another novelty of our result is to exploit the momentum conservation also to prove that the obtained quasi-periodic solutions are indeed quasi-periodic traveling waves, according to Definition 1.5. This requires to check that the approximate solutions constructed along the Nash-Moser iteration of Section 4.8 (and Section 4.5) are indeed traveling waves. Actually this approach shows that the preservation of the momentum condition along the Nash-Moser-KAM iteration is equivalent to the construction of embedded invariant tori which support quasi-periodic traveling waves, namely of the form $u(\varphi, x) = U(\varphi - \vec{\jmath}x)$ (see Definition 4.12), or equivalently, in action-angle-normal variables, which satisfy (4.2.60). We expect this method can be used to obtain quasi-periodic traveling waves for other PDE's which are translation invariant.

In Section 2.2 we further describe the main details for the proof of Theorem 1.8.

1.1.3 Long time existence of periodic gravity-capillary water waves

We consider the Euler equations of hydrodynamics for a 2-dimensional perfect, incompressible, inviscid and irrotational fluid under the action of gravity and capillary forces at the free surface. The fluid fills an ocean with depth $\mathtt{h} > 0$ (eventually infinite) and with space periodic boundary conditions, namely it occupies the region $\mathcal{D}_{\eta,\mathtt{h}}$ defined in (1.1.11). Since the fluid is irrotational and incompressible, the velocity field is the gradient of an harmonic function Φ, called velocity potential, which solves the same problem (1.1.12) as in Section 1.1.2.

Imposing that the fluid particles at the free surface remain on it along the evolution (kinematic boundary condition), and that the pressure of the fluid plus the capillary forces at the free surface is equal to the constant atmospheric pressure (dynamic boundary condition), in the variable $\eta(t,x)$ and $\psi(t,x) := \Phi(t,x,\eta(t,x))$, the time evolution of the fluid is determined, according to Zahkarov [174] and Craig, Sulem [68], by the following system of equations

$$\begin{cases} \eta_t = G(\eta)\psi \\ \psi_t = -g\eta - \dfrac{\psi_x^2}{2} + \dfrac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} + \kappa\Big(\dfrac{\eta_x}{\sqrt{1+\eta_x^2}}\Big)_x . \end{cases} \tag{1.1.30}$$

Here $g > 0$ is the gravity, $\kappa > 0$ is the surface tension coefficient and $G(\eta)$ is the Dirichlet-Neumann operator $G(\eta)\psi = (-\Phi_x\eta_x + \Phi_y)|_{y=\eta(x)}$.

As for the equations (1.1.13), the system (1.1.30) is Hamiltonian. Indeed, setting $\gamma = 0$ in (1.1.17)-(1.1.18), we have that (η,ψ) are canonical variables and

$$\eta_t = \nabla_\psi H(\eta,\psi), \quad \psi_t = -\nabla_\eta H(\eta,\psi), \tag{1.1.31}$$

where ∇ denotes the L^2-gradient, with Hamiltonian

$$H(\eta,\psi) = \frac{1}{2}\int_{\mathbb{T}} \Big(\psi\, G(\eta)\psi + g\eta^2\Big)\,\mathrm{d}x + \kappa\int_{\mathbb{T}} \sqrt{1+\eta_x^2}\,\mathrm{d}x . \tag{1.1.32}$$

The system obtained linearizing (1.1.30) at the equilibrium $(\eta,\psi) = (0,0)$, namely

$$\begin{cases} \partial_t\eta &= G(0)\psi \\ \partial_t\psi &= -(g - \kappa\partial_x^2)\eta . \end{cases} \tag{1.1.33}$$

The Dirichlet-Neumann operator at the flat surface $\eta = 0$ is the Fourier multiplier defined in (1.1.15), (1.1.16). The linear frequencies are given by

$$\Omega_j := \Omega_j(\kappa) = \Omega_j(\kappa,\mathtt{h},g) := \sqrt{(\kappa j^2 + g)G_j(0)}, \quad j \in \mathbb{Z}\backslash\{0\} . \tag{1.1.34}$$

The main goal is to prove that, for *any* value of $(\kappa, g, \mathtt{h})$, $\kappa > 0$, the gravity-capillary water waves system (1.1.30) is conjugated to its Birkhoff normal form, up to cubic remainders that satisfy energy estimates (Theorem 1.9), and that all the solutions of (1.1.30), with initial data of size ϵ in a sufficiently smooth Sobolev space, exist and remain in an ϵ-ball of the same Sobolev space up times of order ϵ^{-2}, see Theorem 1.10. Let us state precisely these results.

Assume that, for s large enough and some $T > 0$, we have a classical solution

$$(\eta, \psi) \in C^0([-T, T]; H_0^{s+\frac{1}{4}} \times \dot{H}^{s-\frac{1}{4}}) \tag{1.1.35}$$

of the Cauchy problem for (1.1.30). The existence of such a solution, at least for small enough T, is guaranteed by local well-posedness theory, see the literature at the end of this chapter.

Theorem 1.9. (Cubic Birkhoff normal form) *Let $\kappa > 0$, $g \geqslant 0$ and $\mathtt{h} \in (0, +\infty]$. There exist $s \gg 1$ and $0 < \overline{\epsilon} \ll 1$, such that, if (η, ψ) is a solution of (1.1.30) satisfying (5.0.4) with*

$$\sup_{t \in [-T,T]} \left(\|\eta\|_{H_0^{s+\frac{1}{4}}} + \|\psi\|_{\dot{H}^{s-\frac{1}{4}}} \right) \leqslant \overline{\epsilon}\,, \tag{1.1.36}$$

then there exists a bounded and invertible linear operator $\mathfrak{B}(\eta, \psi) : H_0^{s+\frac{1}{4}} \times \dot{H}^{s-\frac{1}{4}} \to \dot{H}^s$, which depends (nonlinearly) on (η, ψ), such that

$$\begin{aligned} \|\mathfrak{B}(\eta,\psi)\|_{\mathcal{L}(H_0^{s+\frac{1}{4}} \times \dot{H}^{s-\frac{1}{4}}, \dot{H}^s)} + \|(\mathfrak{B}(\eta,\psi))^{-1}\|_{\mathcal{L}(\dot{H}^s, H_0^{s+\frac{1}{4}} \times \dot{H}^{s-\frac{1}{4}})} \leqslant \\ 1 + C(s)(\|\eta\|_{H_0^{s+\frac{1}{4}}} + \|\psi\|_{\dot{H}^{s-\frac{1}{4}}})\,, \end{aligned} \tag{1.1.37}$$

and the variable $z := \mathfrak{B}(\eta, \psi)[\eta, \psi]$ satisfies the equation

$$\partial_t z = \mathrm{i}\Omega(D) z + \mathrm{i}\partial_{\overline{z}} H^{(3)}_{\mathrm{BNF}}(z, \overline{z}) + \mathcal{X}^+_{\geqslant 3} \tag{1.1.38}$$

where:

1. *$\Omega(D)$ is the Fourier multiplier $u(x) = \sum_{j \neq 0} u_j e^{\mathrm{i} jx} \mapsto \Omega(D) u(x) := \sum_{j \neq 0} \Omega_j u_j e^{\mathrm{i} jx}$, where the symbol Ω_j is defined in (1.1.34), and $\partial_{\overline{z}}$ is defined in (5.4.3);*

2. *the Hamiltonian $H^{(3)}_{\mathrm{BNF}}(z, \overline{z})$ has the form*

$$H^{(3)}_{\mathrm{BNF}}(z, \overline{z}) = \sum_{\substack{\sigma_1 j_1 + \sigma_2 j_2 + \sigma_2 j_3 = 0,\, \sigma_i = \pm\,, \\ \sigma_1 \Omega_{j_1} + \sigma_2 \Omega_{j_2} + \sigma_3 \Omega_{j_3} = 0, j_i \in \mathbb{Z} \setminus \{0\}}} H^{\sigma_1, \sigma_2, \sigma_3}_{j_1, j_2, j_3} z^{\sigma_1}_{j_1} z^{\sigma_2}_{j_2} z^{\sigma_3}_{j_3} \tag{1.1.39}$$

where $z_j^+ := z_j$, $z_j^- := \overline{z_j}$ and z_j denotes the j-th Fourier coefficient of the function z (see

(5.1.2)*), and the coefficients*

$$H^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3} := \frac{\mathrm{i}\sigma_2}{8\sqrt{\pi}}\big(\sigma_1\sigma_3 j_1 j_3 + G_{j_1}(0)G_{j_3}(0)\big)\frac{\Lambda(j_2)}{\Lambda(j_1)\Lambda(j_3)} \tag{1.1.40}$$

with $\Lambda(j)$ *defined in* (5.2.2) *and* $G_j(0) := j\tanh(\mathtt{h}j)$*;*

3. $\mathcal{X}^+_{\geqslant 3} := \mathcal{X}^+_{\geqslant 3}(\eta,\psi,z,\overline{z})$ *satisfies* $\|\mathcal{X}^+_{\geqslant 3}\|_{\dot H^{s-\frac32}} \leqslant C(s)\|z\|^3_{\dot H^s}$ *and the "energy estimate"*

$$\mathrm{Re}\int_{\mathbb{T}} |D|^s\mathcal{X}^+_{\geqslant 3}\cdot\overline{|D|^s z}\,\mathrm{d}x \leqslant C(s)\|z\|^4_{\dot H^s}\,. \tag{1.1.41}$$

The main point of Theorem 1.9 is the construction of the bounded and invertible transformation $\mathfrak{B}(\eta,\psi)$ in (1.1.37) which recasts the irrotational water waves system (1.1.30) in the Birkhoff normal form (1.1.38), where the cubic vector field satisfies the energy estimate (1.1.41). We remark that Craig and Sulem [69] constructed a bounded and symplectic transformation that conjugates (1.1.30) to its cubic Birkhoff normal form, but the cubic terms of the transformed vector field do not satisfy energy estimates.

We underline that, for general values of gravity, surface tension and depth $(g,\kappa,\mathtt{h})$, the "resonant" Birkhoff normal form Hamiltonian $H^{(3)}_{\mathrm{BNF}}$ in (1.1.39) is non zero, because the system

$$\sigma_1\Omega_{j_1} + \sigma_2\Omega_{j_2} + \sigma_3\Omega_{j_3} = 0\,, \qquad \sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 = 0\,, \tag{1.1.42}$$

for $\sigma_j = \pm$, may possess integer solutions $j_1, j_2, j_3 \neq 0$, known as 3-waves resonances In absence of these resonances, existence results have been obtained for times of size ϵ^{-2} by Totz and Wu [161] for 1D pure gravity waves, by Ifrim and Tataru [112] for pure capillarity waves, while an $\epsilon^{-\frac53+}$ result on $\mathbb{T}^2$ has been provided by Ionescu and Pusateri [116]. The resonant Hamiltonian $H^{(3)}_{\mathrm{BNF}}$ gives rise to a complicated dynamics, which, in fluid mechanics, is responsible for the phenomenon of the Wilton ripples. Nevertheless we are able to prove the following long time stability result.

Theorem 1.10. (Quadratic life span) *For any value of* $(\kappa,g,\mathtt{h})$*,* $\kappa>0$*,* $g\geqslant 0$*,* $\mathtt{h}\in(0,+\infty]$*, there exists* $s_0>0$ *and, for all* $s\geqslant s_0$*, there are* $\epsilon_0>0$*,* $c>0$*,* $C>0$*, such that, for any* $0<\epsilon\leqslant\epsilon_0$*, any initial data*

$$(\eta_0,\psi_0)\in H_0^{s+\frac14}(\mathbb{T},\mathbb{R})\times\dot H^{s-\frac14}(\mathbb{T},\mathbb{R}) \quad \textit{with} \quad \|\eta_0\|_{H_0^{s+\frac14}} + \|\psi_0\|_{\dot H^{s-\frac14}} \leqslant \epsilon\,, \tag{1.1.43}$$

there exists a unique classical solution (η,ψ) *of* (1.1.30) *belonging to*

$$C^0\Big([-T_\epsilon,T_\epsilon],H_0^{s+\frac14}(\mathbb{T},\mathbb{R})\times\dot H^{s-\frac14}(\mathbb{T},\mathbb{R})\Big) \quad \textit{with} \quad T_\epsilon\geqslant c\epsilon^{-2}\,,$$

satisfying $(\eta,\psi)_{|t=0} = (\eta_0,\psi_0)$. *Moreover*

$$\sup_{t\in[-T_\epsilon,T_\epsilon]} \left(\|\eta\|_{H_0^{s+\frac{1}{4}}} + \|\psi\|_{\dot{H}^{s-\frac{1}{4}}} \right) \leqslant C\epsilon\,. \tag{1.1.44}$$

The proofs of Theorem 1.9 and Theorem 1.10 are provided in Chapter 5. We describe some key points concerning the proof of these results:

1) The long time existence Theorem 1.10 is deduced by the complete conjugation of the water waves vector field (1.1.30) to its Birkhoff normal form up to degree 3, Theorem 1.9, and not just on the construction of modified energies.

2) Since the gravity-capillary dispersion relation $\sim |\xi|^{\frac{3}{2}}$ is superlinear, the equations (1.1.30) can be reduced, as in the work of Berti and Delort [37], to a paradifferential system with constant coefficient symbols, up to smoothing remainders (see Proposition 5.11). At the beginning of Section 5.3 we remark that, thanks to the x-translation invariance of the equations, the symbols in (5.2.10) of the quadratic paradifferential vector fields are actually zero. For this reason, in Section 5.3, it just remains to perform a Poincaré- Birkhoff normal form on the quadratic smoothing vector fields, see Proposition 5.14.

3) Despite the fact that our transformations are non-symplectic (as in [37] and in the result of Berti, Feola, Pusateri [39]), we prove, in Section 5.4.1, using a normal form identification argument (simpler than in [39]), that the quadratic Poincaré-Birkhoff normal form term in (5.3.9) coincides with the Hamiltonian vector field $\mathrm{i}\partial_{\overline{z}}H^{(3)}_{BNF}$ with Hamiltonian (5.0.8).

4) The Hamiltonian $H^{(2)}_{\mathbb{C}}(z) := \int_{\mathbb{T}} \Omega(D)z\cdot\overline{z}\,\mathrm{d}x$ is a prime integral of the resonant Birkhoff normal form $\partial_t z = \mathrm{i}\Omega(D)z + \mathrm{i}\partial_{\overline{z}}H^{(3)}_{BNF}(z,\overline{z})$. Moreover, since (5.0.11) admits at most finitely many integer solutions (Lemma 5.15) the Hamiltonian $H^{(3)}_{BNF}(z,\overline{z}) = H^{(3)}_{BNF}(z_L,\overline{z}_L)$ where $z_L := \sum_{0<|j|\leqslant \mathtt{C}} z_j e^{\mathrm{i}jx}$, for some finite $\mathtt{C}>0$. Therefore, any solution $z(t)$ of the Birkhoff normal form satisfies, for any $s\geqslant 0$,

$$\|z_L(t)\|^2_{\dot{H}^s} \lesssim_s \|z_L(t)\|^2_{L^2} \lesssim H^{(2)}_{\mathbb{C}}(z_L(t)) = H^{(2)}_{\mathbb{C}}(z_L(0))\,, \quad \forall t\in\mathbb{R}\,,$$

and $\|z(t)\|^2_{\dot{H}^s}$ remains bounded for all times. Finally we deduce the energy estimate (5.4.27) for the solution of the whole system (5.0.7), where we take into account the effect of $\mathcal{X}^+_{\geqslant 3}$, which implies stability for all $|t|\leqslant c\epsilon^{-2}$.

Further details for the proofs of Theorems 1.9, 1.10 will be illustrated in Section 2.3.

1.2 Historical background

In the rest of this introductory chapter we outline the main mathematical ideas behind these problems and their historical developments. In particular we present the main works and recent contributions about the KAM for PDEs theory, the growth of Sobolev norms and some literature about the water waves equations.

1.2.1 KAM for PDEs

The classical KAM theory, named after the works of Kolmogorov [128], Arnold [10, 11] and Moser [146, 147], concerns the persistence of invariant tori, Lagrangian or lower dimensional, that support time quasi-periodic solutions for finite dimensional nearly-integrable Hamiltonian and reversible systems. The generalization of such results to a PDE goes under the name of "KAM for PDEs" theory.

When looking for quasi-periodic motions, already for finite dimensional systems, the main difficulty arises from the presence of *small divisors* in an iterative scheme. These small divisors enter as denominators in the coefficients of the Fourier expansion for the solution of the homological equation at each step of the KAM iteration and affect the convergence of the iterative scheme. For instance, denoting by $\omega \in \mathbb{R}^{\nu}\backslash\{0\}$ the frequency of oscillation on the invariant torus, the set

$$\{\omega \cdot \ell \,:\, \ell \in \mathbb{Z}^{\nu}\backslash\{0\}\}$$

accumulates to 0. This issue can be solved by imposing non-resonance *Diophantine* conditions of the form

$$|\omega \cdot \ell| \geqslant \upsilon |\ell|^{-\tau}\,, \quad \forall\, \ell \in \mathbb{Z}^{\nu}\backslash\{0\}\,,$$

for some $\upsilon \in (0,1)$ and $\tau > \nu - 1$. Such conditions control the way the small divisors accumulate to zero and are sufficient for the convergence of the scheme.

The investigation of periodic and quasi-periodic solution for PDEs, seen as lower dimensional invariant tori for infinite dimensional dynamical systems, started in the 90's. The two main approaches for overcoming the small divisors difficulties are:

- **normal form KAM methods**;
- **Newton Nash-Moser implicit function iterative scheme**.

The first strategy was proposed initially by Kuksin [129] and Wayne [166] for bounded perturbations of parameter dependent, one dimensional Schrödinger and wave equations with Dirichlet boundary conditions, extended by Kuksin-Pöschel [132] and Pöschel [151] to parameter independent nonlinear Schödinger and nonlinear wave equations. In this method the Hamiltonian is moved into a normal form with an invariant torus at the origin by using canonical transformations that reduce step by step the size of the perturbation, extracting the effective contribution to the perturbed frequencies of the motion. Here the small divisors arise in the solutions of the so-called *homological equations* of each step of the iteration. Such equations are constant coefficients linear PDEs and to solve them one needs to impose second order non-resonance Melnikov conditions, for instance of the form

$$|\omega \cdot \ell + \Omega(j) \pm \Omega(j')| \geqslant \upsilon \langle \ell \rangle^{-\tau} \ ,$$

where $\Omega(j)$ denotes the normal frequency of the motion with respect to the tangential frequencies on the aimed invariant torus. The final KAM invariant torus will be *reducible*, in the sense that the linearized equation at it will be diagonal and constant coefficients. Geng, Xu and You [94] proved a KAM theorem for quasi-periodic solutions of the cubic Schrödinger equation on the two dimensional periodic domain $\mathbb{T}^2$. In higher dimensions, Eliasson and Kuksin [84] introduced a modified KAM scheme for the existence of quasi-periodic solutions of the nonlinear Schrödinger equation on $\mathbb{T}^d$ with an external convolution potential, while Procesi and Procesi [153] extended the result for the completely resonant cubic Schrödinger equation.

The second method was introduced first by Craig and Wayne [70] for the search of periodic solutions of a nonlinear wave equation with periodic boundary conditions, extended then by Bourgain for the existence of quasi-periodic solutions of the nonlinear Schrödinger and wave equations in one dimension [47]. In these cases, the presence of clusters of normal frequencies seems incompatible with the KAM methods as in [129], since the second order Melnikov conditions are violated. After a Lyapunov-Schmidt decomposition, the search of invariant tori is reduced to solve some nonlinear functional equations for the embedded torus. By means of a quadratic Newton-type scheme, the solutions are obtained as the limit of a sequence of approximate solutions. This scheme requires to invert the linearized operator at any approximate solution and in order to achieve this, a priori, only first order non-resonance Melnikov conditions are needed, which are roughly of the form

$$|\omega \cdot \ell + \Omega(j)| \geqslant \upsilon \langle \ell \rangle^{-\tau} \ .$$

As a drawback of having imposed only these conditions, the PDEs to solve at any step have variable coefficients and, therefore, this method alone does not provide information for the linear stability for the solutions. In one dimension, the Nash-Moser approach was extended, still for periodic solutions, by Berti and Bolle in [30, 31] for completely resonant nonlinear wave equations with Dirichlet boundary conditions, both with analytic and differentiable nonlinearities, see also Gentile, Mastropietro and Procesi [97]. The higher dimensional case was first treated by Bourgain in [50] in the search of time quasi-periodic solutions for the nonlinear Schrödinger equation on $\mathbb{T}^2$, followed by the results on the nonlinear wave equations on $\mathbb{T}^d$, $d \geqslant 2$, for time periodic [48] and quasi-periodic solutions [52]. The solutions provided by Bourgain are all extremely regular, at least analytic. The extension of the Nash-Moser scheme to finite Sobolev regularity in higher dimensions was considered by Berti and Bolle for quasi-periodic solutions on $\mathbb{T}^d$ of the wave equation [32] and of the nonlinear Schrödinger equation [33] with an external potential. We also mention the work of Berti and Procesi [45] and Berti, Corsi, Procesi [36], where an abstract Nash-Moser theorem for nonlinear Schrödinger and nonlinear wave equations on compact Lie groups was provided, and the recent result by Berti and Bolle [35], with an updated literature for the "KAM for PDEs" theory in the references therein.

In another work by Berti and Bolle [34], the two approaches were unified in the framework of autonomous Hamiltonian PDEs. The main idea is to search the invariant tori as zeroes of the nonlinear functional with a Nash-Moser iterative scheme and to provide a symplectic change of coordinates such that, at each approximate solution of the iteration, the tangential and the normal directions are approximately decoupled. This reduces the problem to the study of the quasi-periodically forced linearized equation in the normal directions. This approximate splitting is actually sharper than the classical Lyapunov-Schmidt reduction in range and bifurcation equations, since the dynamics on the tangential and normal modes is preserved by the Hamiltonian structure. Therefore, the search of an invariant torus is equivalent to prove the existence of a KAM normal form around the torus itself.

All the result mentioned so far concern PDEs with bounded nonlinearities, namely that do not contain any derivative of the unknown. When the nonlinearity is unbounded, a priori, the symplectic transformation at each step of the KAM iteration may loses derivatives and the convergence is in general out of reach. The first KAM results for PDEs with unbounded perturbations were provided by Kuksin [131] and Kappeler, Pöschel [125] for Hamiltonian, analytic perturbations of the KdV equation on the torus. The goal was to prove the existence of solutions bifurcating from Cantor families of finite gap solutions of KdV. The main issue is that the Hamiltonian vector field generated by the perturbation is unbounded of order 1. At the same time, the frequencies of KdV grow as $\sim j^3$, hence the difference $|j^3 - j'^3| \geqslant \frac{1}{2}(j^2 + j'^2)$ for any $j \neq j'$, so that KdV gains two derivatives. This smoothing effect on the small divisors is sufficient to produce a bounded transformation at each step of the KAM iteration. The diagonal terms related to $j = j'$ are not removed by the transformation and therefore are inserted into the normal form. As a consequence, the scalar homological equations have variable coefficients and they can be solved via the *Kuksin Lemma*, introduced in [130].

An improved version of the Kuksin Lemma was introduced by Liu and Yuan in [137] (see also [136]) for proving the existence of quasi-periodic KAM tori for the derivative nonlinear Schrödinger equation (see also the work of Zhang, Gao and Yuan [178]) and the perturbed Benjamin-Ono equation with periodic boundary conditions.

The problem of finding periodic and quasi-periodic solutions further increases in difficulty when the PDEs is not just *semilinear*, that is, when a nonlinearity contains strictly less derivatives than the linear part, but it is *quasi-linear* or even *fully nonlinear*. The first results in this direction are the works of Plotnikov and Toland [150] and Iooss, Plotnikov and Toland [121], for the existence of 2D periodic standing waves with finite and infinite depth, respectively, and Iooss-

Plotnikov [119, 120] for 3D periodic traveling waves for the pure gravity water waves equations. The main difficulty of these results comes from the fully nonlinear nature of the equations, since the linear dispersion relations grow as $\sim |j|^{\frac{1}{2}}$ and the nonlinearity arises from the convective transport term of the Euler equations. The periodic solutions are constructed with a Nash-Moser theorem and the descent method for regularizing the linearized vector field. For a forced quasi-linear Kirchhoff equation, whose nonlinearity is space-independent, time periodic solutions were obtained by Baldi [12] on a bounded domain in $\mathbb{R}^d$ with Dirichlet boundary condition and on the periodic domain $\mathbb{T}^d$. The methods involved in his analysis are tailored to the peculiarity of the nonlinearity and are hardly generalizable to other systems.

The first breakthrough result for time quasi-periodic solutions for quasi-linear and fully nonlinear PDEs are due to Baldi, Berti and Montalto for some quasi-linear and fully nonlinear perturbations of the forced Airy equation [14], of the autonomous KdV [15] and of the autonomous modified KdV [16]. These results are obtained with a Nash-Moser iteration as stated in [34], where the analysis of the linearized operator is inspired by the descent regularization procedure introduced by Plotnikov and Toland [150] via pseudodifferential calculus and combined with the KAM reducibility scheme. About the water waves problem, Berti and Montalto [44], for the gravity-capillary case, and Baldi, Berti, Haus, Montalto [13], for the pure gravity case, proved the existence of one dimensional, quasi-periodic standing waves. Within this context, in our result in Theorem 1.8 a further obstacle arises in this analysis: indeed, the search of quasi-periodic traveling waves forces to work with periodic boundary conditions with no parity restrictions, which induce possibly double eigenvalues at the unperturbed stage. This is solved by a proper choice of the tangential modes and by exploiting the conservation of the x-translation invariance.

The regularization method was applied also by Feola and Procesi [91], who considered a class of fully nonlinear forced and reversible Schrödinger equations on the torus $\mathbb{T}$ and proved existence and stability of quasi-periodic solutions. We refer also the work of Giuliani [101] for quasi-linear perturbations of generalized KdV equations, the result by Feola, Giuliani and Procesi [89] for Hamiltonian perturbations of the Degasperis-Procesi equation and a recent work of Berti, Kappeler and Montalto [42, 43], who provided the existence of finite dimensional invariant tori of any size for perturbations of the defocusing NLS and of KdV, respectively. We mention also the work of Corsi and Montalto [62] for the forced Kirchhoff equation on $\mathbb{T}^d$, which, however, does not make any use of the regularization in decreasing orders and, instead, applies a multiscale approach as in [50, 32, 33, 36] for bounded semilinear PDEs.

Other results of the KAM theory applied to PDEs are presented in Section 1.2.2, where the problem of the reducibility for linear PDEs and the related literature is discussed, and in Section 1.2.3 with the discussion about the traveling and standing quasi-periodic water waves.

1.2.2 Floquet theory and growth of Sobolev norms

The classical Floquet theory concerns the problem of conjugating a time periodic, linear differential equation to a, possibly diagonal, linear system with constant coefficient:

$$\begin{cases} \dot{x} = X(t)x := (A + F(t))x \\ F(t) = F(t+T) \in \mathbb{R}^{n\times n} \end{cases} \quad \overset{x=\Phi_1(t)y}{\longmapsto} \quad \begin{cases} \dot{y} = A^+ y \\ A^+ = \Phi_1(t)^{-1}X(t)\Phi_1(t) - \int_0^1 \Phi_\tau(t)^{-1}\dot{X}(t)\Phi_\tau(t)\,\mathrm{d}\tau\,. \end{cases}$$

The eigenvalues of the constant coefficients vector field A^+ are called *Floquet exponents* and they give information about the stability of the dynamics of the original system. From a mathematical point of view, Theorem 1.1 is part of the attempts to extend the previous classical Floquet theory, together with the generalizations to time quasi-periodic systems, to evolutionary PDEs. In this latter case the only available results nowadays deal with systems which are small perturbations of a diagonal operator, i.e. of the form $D + \epsilon V(\omega t)$, where D is diagonal, ϵ small and ω in some Cantor set. Here the literature splits essentially in two parts:

• **Perturbations with bounded operators.** A preliminary result in this direction is the work of Combescure [56], where she showed that the spectrum of the one dimensional harmonic oscillator perturbed with a time periodic bounded operator is still pure point. The matrix elements of the perturbation have high power law decay with respect to the Hermite basis and a KAM diagonalization procedure is implemented for non-resonant values of the periodic frequency. Then, Duclos and Stovicek [81] proved that the Floquet operator for a pure point Hamiltonian with the gap between the eigenvalues growing as n^α, with $\alpha > 0$, perturbed with a non-resonant small potential has still pure point spectrum. This result is obtained by improving the off-diagonal decay of the perturbation with finitely many adiabatic transformations and then concluding with a KAM reduction as in [56].

The KAM theorems of Kuksin and Pöschel [132, 151, 152] for the Schrödinger and wave equations in one dimension with Dirichlet boundary conditions discussed in Section 1.2.1 are actually the first KAM reducibility results for nonlinear PDEs, as the KAM invariant tori are reducible. We refer also to the work Chierchia and You [54] for the nonlinear wave equation in one dimension with periodic boundary conditions. The reducibility in higher dimensions has been first established in the works of Eliasson and Kuksin on the d-dimensional torus $\mathbb{T}^d$ for a linear Schrödinger equation perturbed with a time quasi-periodic potential [83] and for quasi-periodic solutions of the nonlinear Schrödinger equation [84]. Similar results were provided for other models, from the harmonic oscillator by Grébert and Thomann [106], Grébert and Paturel [105] and Liang, Wang [165], to the Klein-Gordon equation on the torus by Fang, Han and Wang [85] and on higher dimensional spheres by Grébert and Paturel [104]. A KAM reducibility with bounded perturbation was provided also by Corsi, Haus and Procesi [61] for quasi-periodic solutions of Hamiltonian PDEs on compact Lie groups.

• **Perturbations with unbounded operators.** When the perturbation is not a bounded operator between Sobolev spaces, the question of the reducibility becomes more delicate. The first who tackled this problem were Bambusi and Graffi [23]: they provided a KAM reducibility for a one-dimensional anharmonic oscillator perturbed by a time quasi-periodic potential with unbounded growth in the space variable. The variable coefficients homological equations in the scheme can be solved with the help of the Kuksin Lemma. When the Kuksin Lemma is not available, the problem gets much harder. Nevertheless, Berti, Biasco and Procesi [29] were able to prove the existence of KAM tori and the related reducibility for the reversible derivative Klein-Gordon equation on the torus $\mathbb{T}$, which has asymptotically linear frequencies. In the work of Montalto [145] a KAM reducibility is provided for the linearized Kirchhoff equation on $\mathbb{T}^d$: here the perturbation has the same linear order of the leading operator, but it is space-independent. For other results with unbounded perturbations, we refer to Bambusi [20] for the one dimensional harmonic and anharmonic oscillator and to Bambusi, Grébert, Maspero, Robert [26] for the d-dimensional harmonic oscillator.

When it is not possible to obtain reducibility for systems with Hamiltonian of the form $H_0 + V(t)$, in some cases one could deduce dynamical properties via an "almost reducibility"; that is, the original Hamiltonian is conjugated to one of the form $H_0 + Z(t) + R(t)$, where $Z(t)$ commutes with H_0, whereas $R(t)$ is an arbitrary smoothing operator, see Bambusi, Grébert, Maspero, Robert [25]. This normal form ensures upper bounds on the speed of transfer of energy from low to high frequencies; e.g. it implies that the Sobolev norms of each solution grows at most as t^ϵ when $t \to \infty$, for any arbitrary small $\epsilon > 0$. This procedure (or a close variant of it), was applied for Schrödinger-type systems also by Delort [72, 73], Maspero and Robert [141] and Montalto [144].

There are also examples in which the authors engineered periodic drivings aimed to transfer energy from low to high frequencies and leading to unbounded growth of Sobolev norms. For instance, Bourgain constructed bounded, smooth, time periodic potentials on the torus $\mathbb{T}$ forcing the linear wave equation [49] and the linear Schrödinger equation [51] in such a way that, for a choice of the initial datum, the trajectory is not relatively compact in any Sobolev space $H^s(\mathbb{T})$, with $s > 0$, so that in particular the solution is neither almost periodic in time. Recently, Haus and Maspero [109] considered the semiclassical Schrödinger equation on $\mathbb{R}^d$ with an anharmonic trapping potential and a time dependent pertubartion. They showed the existence of a solution with Sobolev norm growing in time, up to the validity of the semiclassical time scales, starting from an unbounded trajectory of the associated classical system and using the semiclassical approximation on coherent states. Other examples of the mechanism for the growth of the Sobolev norms of linear PDEs were provided by Delort [74] for the one-dimensional harmonic oscillator and, in an abstract setting, by Maspero [140]. Very recently, other solutions for the two dimensional harmonic oscillator exhibiting the actual logarithmic growth in the Sobolev norms of [141] were found by Thomann [159], based on the analysis of the linear Lowest Landau Level equation with a time dependent potential, and by Faou, Raphaël [87], modulating the resonant bubbles solutions.

When the magnitude of the perturbation becomes relevant, ideally unstable trajectories are predominant and the analysis of the dynamics gets much harder. Nevertheless, a time periodic (or quasi-periodic) perturbation oscillating with a sufficiently large frequency may avoid resonance effects and create stable motions. Such periodically driven systems have a great interest in physics, both theoretically and experimentally. Indeed, these systems often exhibit a rich and surprising behaviour, like the Kapitza pendulum [123], where the fast periodic driving stabilizes the otherwise unstable equilibrium point in which the pendulum is upside-down. More recently, a lot of attention was dedicated to fast periodically driven many-body systems [82, 102, 127, 122]; here the interest is the possibility of engineering periodic drivings for realizing novel quantum states of matter; this procedure, commonly called "Floquet engineering" [53], has been implemented in several physical systems, including cold atoms, graphenes and crystals.

In order to mathematically deal with perturbations that are periodic in time and fast oscillating, in a series of works [1, 2, 3] Abanin, De Roeck, Ho and Huveneers developed an adapted normal form that generalizes the classical Magnus expansion [138]. Such a normal form, which from now on we call *Magnus normal form*, allows to extract a time independent Hamiltonian (usually called the *effective* Hamiltonian), which approximates well the dynamics up to some finite but very long times. In [3], the authors apply the Magnus normal form to the study of some quantum many-body systems (for instance, spin chains on finite subsets of the lattice $\mathbb{Z}^d$) with a fast periodic driving and extract an effective Hamiltonian which approximate well the dynamics for exponentially long (in $|\omega|$) times. The principal operator in [3] may be of dimension very large, depending on the number of the interacting particles and on the subset of the lattice), but still finite, so that all the involved operators are bounded. We point out that, on the contrary, Theorem 1.1 is an infinite dimensional analysis and already the principal operator is unbounded, therefore we have to take care of controlling an eventual loss of derivatives which are not present in [3]. We quote here also the work of Corsi and Genovese [60] about the long time dynamics of quantum spin chains in the thermodynamics limit, perturbed by a small, time periodic potential with a large frequency of oscillation.

1.2.3 The water waves problem

The analysis of the water waves problem is dated back to the works of Laplace (1776) and Lagrange (1781, 1786), just some years after the derivation of the equations for hydrodynamics by Euler in 1757. These early works concerned mostly the linearized dynamics in some different regimes and the deduction of the respective dispersion relations for various kind of waves. Starting from the 19th century, new improvements were provided by Grestner (1802), with the very first nonlinear exact solution and the contributions by Cauchy (1827) and Poisson (1818) on the initial value problem. The most groundbreaking and influential papers for the early theory of water waves are the works of Green, Kelland, Airy and Earnshaw (1838-1844). For a detailed summary of the early history and related bibliography, we refer to the article by Craik [64]. We now present an overview of some results both contemporary and from the last century.

Local well-posedness. Local existence results for the initial value problem of the pure gravity water waves equations within a Sobolev class go back to the pioneering works of Nalimov [148], Yosihara [173], Craig [65] in one space dimension and with smallness assumptions on the initial data. Beale, Hou and Lowengrub [27] proved that the linearization of the 2D water-wave problem is well-posed if a Taylor sign condition is added to the problem formulation, thus preventing Rayleigh-Taylor instabilities. The full nonlinear well-posedness, that is, without any smallness assumption, is due to Wu in dimension one [169] and in dimension two [170] in the case of infinite depth.

In presence of surface tension, Ambrose [8] and Ambrose, Masmoudi [9] proved local well-posedness of the 2D water waves problem replacing the Taylor sign condition. We quote also the previous work of Beyer-Günther [46] for the motion of a liquid drop in presence of capillary forces on the boundary. For some recent results about gravity-capillary and pure gravity waves we refer to the monograph of Lannes [134] and the works of Coutand and Shkoller [63] for rotational fluids, Shatah and Zeng [156] on non-simply connected domains, Christianson, Hur and Staffilani [55] for the Strichartz estimates and Alazard, Burq and Zuily [5] with the use of paradifferential calculus. Clearly, specializing these results for initial data of size ϵ, the solutions exist and stay regular for times of order ϵ^{-1}.

Global well-posedness on Euclidean domains. In the case $x \in \mathbb{R}^d$ and the initial data sufficiently fast decaying at infinity, global in time solutions have been constructed exploiting the dispersive effects of the system. The first global in time solutions were proved in $d = 2$ by Germain, Masmoudi and Shatah [98] and Wu [172] for gravity water waves, by Germain, Masmoudi and Shatah [99] for the pure capillary problem and by Deng, Ionescu, Pausader, Pusateri [78] for gravity-capillary water waves. In $d = 1$ an almost global existence result for gravity waves was proved by Wu [171], improved to global regularity by Ionescu and Pusateri [114], Alazard and Delort [6], Hunter and Ifrim,Tataru [110, 111]. For capillary waves, global regularity was proved by Ionescu and Pusateri [115] and Ifrim, Tataru [112].

Normal forms. For space periodic water waves, there are no dispersive effects that can lead directly to a control of the solutions for all times using the decay in time. Indeed, considering also the quasi-linear nature of the equations that prevent the use of semilinear techniques, no global regularity results for water waves in periodic settings are known. A major obstacle to this end is the presence of *resonances.* We consider, for instance, a monomial nonlinearity of degree $N \geqslant 2$ of the form $u^{\sigma_1}...u^{\sigma_N} = \sum_{\sigma_1 j_1+...+\sigma_N j_N=j} u_{j_1}^{\sigma_1}...u_{j_N}^{\sigma_N} e^{\mathrm{i}jx}$, where $\sigma_1, ..., \sigma_N = \pm 1$,

$u^+(x) := u(x)$, $u^-(x) := \overline{u}(x)$. Then, the monomial terms that are resonant with the linear dynamics correspond to those $j, j_1, ..., j_N \in \mathbb{Z}$ such that

$$\Omega(j) = \sigma_1 \Omega(j_1) + ... + \sigma_N \Omega(j_N) ,$$

where $\Omega(j)$ denotes the linear frequency. When there is no equality, but the difference of the two sides is close to zero, we talk about *quasi-resonances*, which give rise to a *small divisors* issue.

A partial substitute for global regularity is to prove extended life span results and the existence of solutions with initial data of magnitude ϵ for longer times can be proved via normal form theories. In absence of resonances, existence results were obtained for times of size ϵ^{-2} by Totz and Wu [161] for $1d$ pure gravity waves, by Ifrim and Tataru [112] for pure capillarity waves and by Harrop-Griffiths, Ifrim, Tataru [108] for 1D gravity waves over a flat bottom. If $x \in \mathbb{T}^2$, we refer to the work of Ionescu and Pusateri [116] for an $\epsilon^{-\frac{5}{3}+}$ result. The extended life span of these results is proved with energy estimates and integration by parts.

For nonlinear dispersive PDEs on a periodic domain, the long time existence problem can be tackled also with a Birkhoff normal form procedure. The basic idea of this approach is to reduce the size of the nonlinearity near the origin. One looks for a change of coordinates that removes all the monomials up to a certain degree of the nonlinearity that are non-resonant with respect to the linear dynamics. The transformation is well defined if one imposes non-resonance conditions on the small divisors to ensure the boundedness of the map. Then the extension of the life span is achievable once the resonant contribution and the remaining terms in the nonlinearity are analyzed. In the semilinear setting, for Hamiltonian PDEs the first results of this kind were provided by Bambusi [19], Bambusi and Grebért [24], Delort and Szeftel [76, 77] and Bambusi, Delort, Grebért, Szeftel [22], while for reversible PDEs we refer to the work of Faou and Grebért [86]. The extension to quasi-linear PDEs was first provided by Delort for quasi-linear perturbations of the Klein-Gordon equation on the one dimensional torus [73] and on the d-dimensional sphere [75].

Back to the water waves problem, the first application of the Birkhoff normal form is due to Berti and Delort. In [37], the authors proved an almost global existence result for periodic gravity-capillary water waves, even in $x \in \mathbb{T}$, for times $O(\epsilon^{-N})$ for almost all values of (g, κ). The restriction on the parameters (g, κ) arises to verify the absence of N-waves interactions at any N. The restriction to even in x solutions arises because the transformations in [37] are reversibility preserving but not symplectic.

The only ϵ^{-3} existence result for parameter independent water waves on the torus was proved by Berti, Feola and Pusateri in [39] and it is based on the complete integrability of the fourth order Birkhoff normal form for $1d$ pure gravity water waves in infinite depth, proving a conjecture of Zakharov and Dyachenko [175]. For another long time existence result via Birkhoff normal form, see [17, 90].

Time and space periodic traveling waves which are steady in a moving frame. The literature concerning steady traveling wave solutions is huge, and we refer to [57] for an extended presentation. Here we only mention that, after the pioneering work of Stokes [157], the first rigorous construction of small amplitude space periodic steady traveling waves goes back to the 1920's with the papers of Nekrasov [149], Levi-Civita [135] and Struik [158], in case of irrotational bi-dimensional flows under the action of pure gravity. Later Zeidler [176] considered the effect of capillarity. In the presence of vorticity, the first result is due to Gerstner [100] in 1802, who gave an explicit example of periodic traveling wave, in infinite depth, and with a particular non-zero vorticity. One has to wait the work of Dubreil-Jacotin [80] in 1934 for the first existence results of small amplitude, periodic traveling waves with general (Hölder continuous, small) vorticity, and, later, the works of Goyon [103] and Zeidler [177] in the case of large vorticity. More recently we point out the works of Wahlén [162] for capillary-gravity waves and non-constant vorticity, and of Martin [139] and Wahlén [163] for constant vorticity.

All these results deal with two dimensional water waves, and can ultimately be deduced by the Crandall-Rabinowitz bifurcation theorem from a simple eigenvalue. We also mention that these local bifurcation results can be extended to global branches of steady traveling waves by applying the methods of global bifurcation theory. We refer to Keady and Norbury [126], Toland [160], McLeod [143] for irrotational flows and Constantin, Strauss [59] for fluids with non-constant vorticity.

In the case of three dimensional irrotational fluids, bifurcations of small amplitude traveling waves periodic in space were proved by Reeder and Shinbrot [154], Craig and Nicholls [66, 67] , for both gravity-capillary waves (with a variational bifurcation arguments ła Weinstein-Moser) and by Iooss and Plotnikov [119, 120] for gravity waves (this is a small divisor problem). In a moving frame, these solutions look as steady bi-periodic waves.

Time periodic standing waves. Bifurcations of time periodic standing water waves were obtained in a series of pioneering paper by Plotnikov and Toland [150] and by the works of Iooss, Plotnikov and Toland [121, 117, 118, 119] for pure gravity waves, and by Alazard, Baldi [4] for gravity-capillary fluids (see the previous discussion in Section 1.2.1). Standing waves are even in the space variable and so they do not travel in space. There is a huge difference with the results of the previous group: the construction of time periodic standing waves involves small divisors. Thus, the proofs are based on Nash-Moser implicit function techniques, with the descent method on the linearized operators, and not only on the classical implicit function theorem.

Time quasi-periodic standing waves. The first results in this direction were obtained recently by Berti and Montalto [44] for the gravity-capillary system and by Baldi, Berti, Haus, Montalto [13] for the gravity water waves. Both papers deal with irrotational fluids and the proofs require the Nash-Moser iteration as in the time periodic case coupled with KAM techniques in the reduction of the linearized water waves vector field at any approximate solution. Moreover,

[13] has to face another problem coming from the sublinearity of the linear dispersion relation, which leads to second order non-resonance Melnikov conditions that lose space derivatives: this difficulty is overcome by reducing in pseudodifferential order the linearized vector field up to a sufficiently smoothing order.

Quasi-periodic solutions in fluid dynamics. The first work on time quasi-periodic solutions for Euler equation is due to Crouseilles and Faou [71], who constructed, for a fluid on $\mathbb{T}^2$ with a piece-wise linear shear flow, scalar vorticity solutions that consist of localized waves traveling in the orthogonal direction with respect to the propagation of the flow. The solutions are very explicit and, in particular, the result is not of KAM-type, since no small divisors arise.

Very recently, Baldi and Montalto [18] proved the existence of quasi-periodic solutions for the Euler equation with a small reversible quasi-periodic in time forcing term on the three-dimensional torus $\mathbb{T}^3$. The solutions that they provide are small perturbations of the constant vector fields satisfying Diophantine conditions.

Feola and Giuliani [88] showed in a recent result the existence of small amplitude quasi-periodic traveling waves for the 2-dimensional, irrotational pure gravity water waves with infinite depth. In the construction of these traveling solution, they use the same choice of the tangential sites as in our Theorem (1.8) and they have to preserve the x-translation as well. The lack of parameter to move in order to avoid the resonances is overcome by a weak Birkhoff normal form.

We conclude by quoting also two numerical works by Wilkening and Zhao [167, 168] about *spatially quasi-periodic* gravity-capillary water waves in infinite depth. In particular, they studied numerical traveling-type solutions on a one dimensional domain with multiple space frequencies and investigate the presence of resonances when parameters vary.

Chapter 2

Ideas of the proofs

In this chapter we outline the strategies for the proofs of the results presented in Sections 1.1.1, 1.1.2, 1.1.3, discussing the mathematical key ideas step by step for each result.

2.1 Ideas of the proof of Theorem 1.1

The result in Theorem 1.1 is proved in two steps. First, we need to transform the system (1.1.4) into a perturbative framework. Then, we perform a KAM reducibility scheme once proper non-resonance conditions are imposed.

The Magnus normal form. The very first transformation that we perform, adapted to fast oscillating systems, moves the non-perturbative equation (1.1.4) into a perturbative one where the size of the transformed quasi-periodic potential is as small as the module of the frequency vector is large. Sketchily, we perform a change of coordinates which conjugates

$$\begin{cases} \mathbf{H}(t) = \mathbf{H}_0 + \mathbf{W}(\omega t) \\ \text{"size}(\mathbf{W}) \sim 1\text{"} \end{cases} \rightsquigarrow \begin{cases} \widetilde{\mathbf{H}}(t) = \mathbf{H}_0 + \mathbf{V}(\omega; \omega t) \\ \text{"size}(\mathbf{V}) \sim |\omega|^{-1}\text{"} \end{cases} . \tag{2.1.1}$$

This change of coordinates, called below Magnus normal form, is an extension to quasi-periodic systems of the one performed by Abanin, De Roeck, Ho and Huveneers in [3]. Note that $\mathbf{H}_0$ is the same on both sides of (2.1.1) provided $\int_{\mathbb{T}^\nu} \mathbf{W}(\theta)\, \mathrm{d}\theta = 0$, which is fulfilled in our case thanks to Assumption **(V2)**. The price to pay is that, in principle, it is not clear that the new perturbation is sufficiently regularizing to fit in a standard KAM scheme, as the new perturbation $\mathbf{V}(\omega; \omega t)$ could increase in order. Here it is essential to employ pseudodifferential calculus, thanks to which we control the order (as a pseudodifferential operator) of the new perturbation, and prove that it is actually enough regular for the KAM iteration. For presenting better this point, assume that we have a one-dimensional vector field of the form $H(t) = H_0 + W(\omega t)$, with $H_0 \in \mathrm{OP}S^\mu$, $\mu \geqslant 1$, and $W(\omega t) \in \mathrm{OP}S^0$ for any $t \in \mathbb{R}$ (for the definition of the classes of pseudodifferential operators $\mathrm{OP}S^\mu$, see Definition 3.4). The Magnus transform is generated by an operator $X(\omega;\omega t)$ solving the homological equation

$$-\omega \cdot \partial_\varphi X(\omega;\omega t) + W(\omega t) = 0 \quad \Rightarrow \quad X(\omega;\omega t) = (\omega \cdot \partial_\varphi)^{-1} W(\omega t)\,,$$

which is well defined, assuming $\widehat{W}(0) := (2\pi)^{-\nu} \int_{\mathbb{T}^\nu} W(\varphi) e^{\mathrm{i}\,\ell\cdot\varphi}\,\mathrm{d}\varphi = 0$. This implies that the new perturbation is given by

$$V(\omega;\omega t) = \mathrm{i}[X(\omega;\omega t), H_0] + \ldots \in \mathrm{OP}S^{\mu-1}\,,$$

where the dots stand for lower order terms. When $\mu = 1$, as for our Klein-Gordon model, we therefore obtain a bounded operator in $\mathrm{OP}S^0$ that will be as small in size as the generator X is. For $\mu > 1$, instead, the perturbation V increases in order and the problem of the reducibility is still open. All the details for our computation are provided in Section 3.2.

A remark on the classes of pseudodifferential operators. We add here a brief observation about the pseudodifferential calculus presented in Chapter 3 and Chapter 4. Indeed, the classes of φ-independent symbols S^m of order $m \in \mathbb{R}$ are defined in the same way, see Definition 3.2 and Definition 4.14. In particular, we say that a function $a(x,j)$ is a symbol of order m if it is the restriction to $\mathbb{R} \times \mathbb{Z}$ of a function $a(x,\xi)$ which is $\mathcal{C}^\infty$-smooth on $\mathbb{R} \times \mathbb{R}$, 2π-periodic in x, and satisfies

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-\beta}\,, \quad \forall \alpha,\beta \in \mathbb{N}_0\,.$$

Also the corresponding pseudodifferential operators are defined by the same quantization:

$$u(x) = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx} \;\mapsto\; \left(a(x,D_x)u\right)(x) := \sum_{j\in\mathbb{Z}} a(x,j) u_j e^{\mathrm{i}jx}\,.$$

When the dependence on the angle $\varphi \in \mathbb{T}^\nu$ is considered, for the KAM reducibility of Section 3.4 we want the regularity with respect to φ to be analytic. Therefore, in Chapter 3 we shall control a symbol $a(\varphi,x,\xi)$ with respect to the following seminorm (see Def. 3.3):

$$\wp_\varrho^{m,\rho}(a) := \sup_{|\mathrm{Im}\,\varphi| \leqslant \rho} \sum_{\alpha+\beta \leqslant \varrho} \sup_{(x,\xi)\in\mathbb{R}\times\mathbb{R}} \langle \xi \rangle^{-m+\beta} |\partial_x^\alpha \partial_\xi^\beta a(\varphi,x,\xi)|\,, \quad \varrho \in \mathbb{N}_0\,,$$

assuming that the function $a : \mathbb{T}^\nu \times \mathbb{T} \times \mathbb{R}$ admits an analytic extension with respect to φ on the torus $\mathbb{T}^\nu_\rho := \{\varphi = \vartheta + \mathrm{i}\,\phi \in \mathbb{C}^\nu \,:\, \vartheta \in \mathbb{T}^\nu\,,\ |\psi| \leqslant \rho\}$ for some $\rho > 0$.

On the other hand, it is more desired in all the analysis of Chapter 4 to control together the regularity of the variables $(\varphi,x) \in \mathbb{T}^{\nu+1}$ in Sobolev classes. In particular, we shall check the boundedness of the operator with respect to the following norm on the symbols (see Def. 4.15):

$$\|a\|_{m,s,\alpha} := \max_{0\leqslant\beta\leqslant\alpha} \sup_{\xi\in\mathbb{R}} \|\partial_\xi^\beta a(\cdot,\cdot,\xi)\|_s \,\langle\xi\rangle^{-m+\beta}\,.$$

In Chapter 4 we keep the same notation of the classes S^m and $\mathrm{OP}S^m$ also for φ-dependent symbols and operators, see Definition 4.15 (where also the differentiable dependence on parameters is included). In Chapter 3, instead, since the radius of analyticity varies in the reduction, we denote the analytic classes of symbols and operators by S^m_ρ and $\mathrm{OP}S^m_\rho$, respectively, see Definitions 3.3., 3.4.

Balanced Melnikov conditions and KAM reducibility. After the Magnus normal form, we perform a KAM reducibility scheme in order to remove the time dependence from the coefficients of the equation. For briefly presenting the KAM reduction scheme, consider the system

$$\mathrm{i}\,\dot\psi(t) = \mathbf{H}(\omega t)\psi(t)\,, \quad \mathbf{H}(\omega t) := \mathbf{A}(\omega) + P(\omega;\omega t)\,,$$

where the frequency vector ω varies in some set $\Omega\subset\mathbb{R}^\nu\backslash\{0\}$, with $\mathtt{M}\leqslant|\omega|\leqslant 2\mathtt{M}$, the time independent operator $\mathbf{A}(\omega)$ is diagonal with respect to the Fourier basis and $\mathbf{P}(\omega;\omega t)$ is the time quasi-periodic perturbation. The goal is to square the size of the latter by conjugating the Hamiltonian $\mathbf{H}(\omega t)$ via a transformation generated by $\mathrm{i}\,\mathbf{X}(\omega t)$. The transformed Hamiltonian is given by

$$\mathbf{H}^+(\omega t) := \mathbf{A} + \mathbf{P} + \mathrm{i}\,[\mathbf{X},\mathbf{A}] - \omega\cdot\partial_\varphi\mathbf{X} + \mathbf{R}\,,$$

where $\mathbf{R}$ is the remainder resulting from the commutator expansion. The generator is required to solve the homological equation

$$-\omega\cdot\partial_\varphi\mathbf{X} + \mathrm{i}[\mathbf{X},\mathbf{A}] + \mathbf{P} = \mathbf{Z}\,,$$

where $\mathbf{Z}$ is the new time independent contribution to the normal form. By solving this equation with respect to the Fourier basis representation of the linear operators, we face the presence of small divisors of the form

$$\omega\cdot\ell + \lambda_j \pm \lambda_{j'}\,,$$

where λ_j are the eigenvalues of $\mathbf{A}$. One needs to impose second order non-resonance Melnikov conditions and, for instance, might ask for lower bounds on the denominators $\omega\cdot\ell+\lambda_j-\lambda_{j'}$ of the form, for some $\gamma,\tau>0$,

$$|\omega\cdot\ell + \lambda_j - \lambda_{j'}| \geqslant \frac{\gamma}{\langle\ell\rangle^\tau}\frac{\langle j-j'\rangle}{|\omega|}\,, \quad \forall\,(\ell,j,j')\in\mathbb{Z}^\nu\times\mathbb{N}\times\mathbb{N},\ \ (\ell,j,j')\neq(0,j,j)\,. \tag{2.1.2}$$

Such conditions are violated for a set of frequencies of relative measure bounded by $C\gamma$, where C is a constant independent of $|\omega|$ (we remark that the conditions in (2.1.2) are violated on a set of relative measure $\sim \gamma|\omega|$, which is as large as the size of the frequency vector).
This classical version of the Melnikov conditions is useless in our context: indeed, after the Magnus normal form, the new perturbation has size $\sim |\omega|^{-1}$, whereas the small denominators in (2.1.2) have size $\sim |\omega|$; so the two of them compensate each others, and the KAM step cannot reduce in size. To overcome the problem, rather than (2.1.2), we impose new *balanced* Melnikov conditions, in which we balance a partial loss in size (in the denominator) with a gain in regularity (in the numerator) in (2.1.2). More precisely, we show that for any $\alpha \in [0,1]$ one can impose

$$|\omega \cdot \ell + \lambda_j - \lambda_{j'}| \geqslant \frac{\gamma}{\langle \ell \rangle^\tau} \frac{\langle j - j' \rangle^\alpha}{|\omega|^\alpha}, \quad \forall\, (\ell, j, j') \in \mathbb{Z}^\nu \times \mathbb{N} \times \mathbb{N}, \quad (\ell, j, j') \neq (0, j, j) \tag{2.1.3}$$

for a set of ω's in $R_{\mathtt{M}}$ of large relative measure. By choosing $0 < \alpha < 1$, the left hand side of (2.1.3) is larger than the corresponding one in (2.1.2), and the KAM transformation reduces in size. However, note that the choice of α influences the regularizing effect given by $\langle j \pm j' \rangle^\alpha$ in the right hand side of (2.1.3); ultimately, this modifies the asymptotic expansion of the final eigenvalues, as one can see in (1.1.8).

2.2 Ideas of the proof for Theorem 1.8

The proof of Theorem 1.8 for the existence of the small amplitude quasi-periodic traveling wave solutions of (1.1.13) is inspired by the approach used in [44, 13]. However, there are some major novelties and difficulties in our analysis that differ from the previous works:

- As we look for traveling wave solutions, we need to take care that at each step of the procedure we end up with maps that send traveling waves into themselves and that the invariance by space translations is preserved;
- Unlike [44, 13], which prove the existence of standing waves, the solutions that we construct are not even in x. This implies the presence of nearly double eigenvalues (when $\gamma = 0$, purely double) already at the linear level. Hence, the non-resonance conditions hold on a set of parameter of large measure only when coupled with the restrictions on the Fourier sites due to the conservation of the momentum (related, by Noether Theorem, to the x-translation invariance aforementioned) and with a proper choice of the tangential sites;
- The Hamiltonian structure of the system (1.1.13) is needed in the regularization of the linearized vector field in order to avoid the presence of terms which are non-Hamiltonian and that cannot be eliminated otherwise. Without exploiting this property, some transformations may be not close to the identity and the reduction would not be possible.

We list and discuss in the following the main points of the scheme. Throughout this presentation, instead with the coordinates (η, ψ) which we have used for stating Theorem 1.8, we work with the *Wahlén variable* (η, ζ), where

$$\zeta := \psi - \frac{\gamma}{2}\partial_x^{-1}\eta \,.$$

This variables are the Darboux coordinates in which the symplectic form becomes the canonical one, see Section 4.1.1, and they are the one that we use in the rest of the analysis.

Nash-Moser Theorem of hypothetical conjugation Rescaling the variables $(\eta, \zeta) \mapsto (\varepsilon\eta, \varepsilon\zeta)$ and introducing action-angle variables on the tangential sites (see Section 4.1.3), the system (1.1.13) becomes the Hamiltonian system generated by

$$H_\varepsilon = \vec{\Omega}(\kappa) \cdot I + \tfrac{1}{2}(\mathbf{\Omega}_W w, w)_{L^2} + \varepsilon P \,, \tag{2.2.1}$$

where $\vec{\Omega}(\kappa) \in \mathbb{R}^\nu$ is defined in (4.0.8), $\mathbf{\Omega}_W$ is the quadratic form of the linearized vector field around the equilibrium $(\eta, \zeta) = (0, 0)$, $w \in \mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma}$ is the coordinate in the normal subspace and P is the nonlinear perturbation (see Section 4.4).

The expected quasi-periodic solutions of the autonomous Hamiltonian system generated by H_ε will have shifted frequencies $\widetilde{\Omega}_j(\kappa)$ -to be determined- close to the linear frequencies $\Omega_j(\kappa)$ in (4.0.3). It is convenient to introduce the family of Hamiltonians

$$H_\alpha = \alpha \cdot I + \tfrac{1}{2}(\mathbf{\Omega}_W w, w)_{L^2} + \varepsilon P$$

parametrized by the "counter terms" $\alpha \in \mathbb{R}^\nu$: this allows to use the frequencies $\omega \in \mathbb{R}^\nu$ as parameters to move for proving the non-resonance conditions. In this spirit, a quasi-periodic traveling solution is searched as a ν-dimensional embedded torus of the form

$$i : \mathbb{T}^\nu \to \mathbb{T}^\nu \times \mathbb{R}^\nu \times \mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma} \,, \quad \varphi \mapsto (\theta(\varphi), I(\varphi), w(\varphi)) \,,$$

close to the trivial embedding $(\varphi, 0, 0)$, which is a zero for the nonlinear operator

$$\begin{aligned} \mathcal{F}(i, \alpha, \omega, \kappa, \varepsilon) &:= \omega \cdot \partial_\varphi i(\varphi) - X_{H_\alpha}(i(\varphi)) \\ &= \begin{pmatrix} \omega \cdot \partial_\varphi \theta(\varphi) & -\alpha - \varepsilon \partial_I P(i(\varphi)) \\ \omega \cdot \partial_\varphi I(\varphi) & +\varepsilon \partial_\theta P(i(\varphi)) \\ \omega \cdot \partial_\varphi w(\varphi) & -\Pi^\angle_{\mathbb{S}^+,\Sigma} J(\mathbf{\Omega}_W w(\varphi) + \varepsilon \nabla_w P(i(\varphi))) \end{pmatrix} \end{aligned}$$

satisfying the traveling condition

$$\theta(\varphi - \vec{\jmath}\varsigma) = \theta(\varphi) - \vec{\jmath}\varphi \,, \quad I(\varphi - \vec{\jmath}\varphi) = I(\varphi) \,, \quad w(\varphi - \vec{\jmath}\varsigma) = \tau_\varsigma w(\varphi) \,, \quad \forall \varsigma \in \mathbb{R} \,,$$

where $\vec{\jmath} = (\sigma_a \overline{n}_a)_{a=1,\dots,\nu} \in \mathbb{Z}^\nu \backslash \{0\}$ and $\tau_\varsigma w(\varphi, x) := w(\varphi, x+\varsigma)$. The embedding are search for all the parameters $(\omega, \kappa) \in \mathbb{R}^\nu \times [\kappa_1, \kappa_2]$ in the Cantor set $\mathcal{C}_\infty^\upsilon$ (defined explicitly in (4.4.15)-(4.4.18)) which requires the parameters (ω, κ), in addition to the Diophantine condition

$$|\omega \cdot \ell| \geqslant 8\upsilon \langle \ell \rangle^{-\tau} \quad \forall\, \ell \in \mathbb{Z}^\nu \backslash \{0\}\,, \quad \langle \ell \rangle := \max\{1, |\ell|\}\,,$$

the first and second non-resonance Melnikov conditions, each one coupled with the corresponding momentum condition on the Fourier sites. For instance,

$$\begin{cases} |\omega \cdot \ell + \mu_j^\infty(\omega, \kappa) - \mu_{j'}^\infty(\omega, \kappa)| \geqslant 4\upsilon \langle |j|^{\frac{3}{2}} - |j'|^{\frac{3}{2}} \rangle \langle \ell \rangle^{-\tau}\,, \\ \quad \forall\, \ell \in \mathbb{Z}^\nu,\ \ j, j' \in \mathbb{S}_0^c,\ (\ell, j, j') \neq (0, j, j) \quad \textbf{with}\ \ \vec{\jmath} \cdot \ell + j - j' = 0\,. \end{cases}$$

where $\mu_j^\infty(\omega, \kappa)$ are the "final eigenvalues" in (4.4.13), which are defined for all $(\omega, \kappa) \in \mathbb{R}^\nu \times [\kappa_1, \kappa_2]$ by means of the abstract Whitney extension theorem (see Appendix B in [13] for details), and $\mathbb{S}_0^c := \mathbb{Z} \backslash (\mathbb{S} \cup \{0\})$, where

$$\mathbb{S} := \{\overline{\jmath}_a := \sigma_a \overline{n}_a \,:\, a = 1, \dots, \nu\} \subset \mathbb{Z} \backslash \{0\}\,, \quad \sigma_a \in \Sigma\,,\ \overline{n}_a \in \mathbb{S}^+$$

and $\mathbb{S}^+$, Σ are defined in (1.1.26),(1.1.27), respectively. In particular, the set of the tangential sites $\mathbb{S}$ is characterized by

$$\jmath, \jmath' \in \mathbb{S} \quad \Rightarrow \quad |\jmath| \neq |\jmath'| \tag{2.2.2}$$

This choice of the tangential sites, together with the momentum condition, is not just technical, but prevents the rising of resonances already at the unperturbed level of the linear frequencies. For instance, for $\nu = 6$, the vector

$$\vec{\Omega}(\kappa) = \big(\Omega_{-5}(\kappa), \Omega_{-4}(\kappa), \Omega_{-3}(\kappa), \Omega_3(\kappa), \Omega_4(\kappa), \Omega_5(\kappa)\big) \in \mathbb{R}^6$$

is resonant or quasi-resonant for any value of κ. Indeed, when $\mathtt{h} = +\infty$, it is fully resonant, since, for $\ell = (-\ell_5, -\ell_4, -\ell_3, \ell_3, \ell_4, \ell_5)$, the system

$$\vec{\Omega}(\kappa) \cdot \ell = \gamma(\ell_3 + \ell_4 + \ell_5) = 0\,, \quad 3\ell_3 + 4\ell_4 + 5\ell_5 = 0\,,$$

has the integer solution $\ell_3 = \ell_5 = 1$, $\ell_4 = -2$, whereas, for $\mathtt{h} < \infty$, the system

$$\vec{\Omega}(\kappa) \cdot \ell = \gamma(\ell_3 \tanh(3\mathtt{h}) + \ell_4 \tanh(4\mathtt{h}) + \ell_5 \tanh(5\mathtt{h})) = 0\,, \quad 3\ell_3 + 4\ell_4 + 5\ell_5 = 0\,,$$

may have integer solutions for some values of $\mathtt{h}$ and, for any other fixed value of $\mathtt{h}$, there exist integer combinations such that $\vec{\Omega}(\kappa) \cdot \ell$ is arbitrarily close to zero.

The final traveling embedding $i_\infty(\varphi) = i_\infty(\varphi, \omega, \kappa, \varepsilon)$ and "final counter term" α_∞ of Theorem

4.55 will be produced by the Nash-Moser iteration of Theorem 4.95, which relies on the analysis of the linearized operator $\mathrm{d}_{i,\alpha}\mathcal{F}$ at any approximate solution performed in Sections 4.5-4.7 (see also discussion below). The traveling torus i_∞, as well as the counter term α_∞ and the final eigenvalues μ_j^∞ are $\mathcal{C}^{k_0}$ differentiable with respect to the parameters (ω, κ), where the value of k_0 is determined by the transversality of the unperturbed linear frequencies (see Proposition 4.53).

Concerning the proofs of Theorems 4.55, 4.95, as well as the construction of the approximate inverse in Section 4.5 (modulo the reduction on the normal directions of the linearized operator), the arguments that we use are modelled on the ones in [44, 13]. A very important difference, which is one of the novelties of our result, is that we need to check at each step of the procedure that we are dealing with quasi-periodic traveling embeddings and that the operators in the construction of the approximate inverse are momentum preserving. This is essential not only for ending up with a quasi-periodic traveling wave solution, but also for applying the degenerate KAM theory, which needs the conservation of the momentum, as we discuss right below.

Transversality and degenerate KAM theory In order to prove the existence of quasi-periodic solutions of the system with Hamiltonian H_ε in (2.2.1) and not only of the system generated by the modified Hamiltonian H_α, with $\alpha = \alpha_\infty(\omega, \kappa, \varepsilon)$, we have to show that the curve of the unperturbed linear tangential frequencies

$$[\kappa_1, \kappa_2] \ni \kappa \mapsto \vec{\Omega}(\kappa) \in \mathbb{R}^\nu$$

intersects the image $\alpha_\infty(\mathcal{C}_\infty^\upsilon)$ for "most" values of $\kappa \in [\kappa_1, \kappa_2]$. Setting

$$\vec{\Omega}_\varepsilon(\kappa) := \alpha_\infty^{-1}(\vec{\Omega}(\kappa), \kappa)\,,$$

where $\alpha_\infty^{-1}(\,\cdot\,, \kappa)$ is the inverse of $\alpha_\infty(\,\cdot\,, \kappa)$ at a fixed $\kappa \in [\kappa_1, \kappa_2]$, if the vector $(\vec{\Omega}_\varepsilon(\kappa), \kappa)$ belongs to $\mathcal{C}_\infty^\upsilon$, then Theorem 4.55 implies the existence of a quasi-periodic solution of the system with Hamiltonian H_ε with Diophantine frequency $\vec{\Omega}_\varepsilon(\kappa)$.

In Theorem 4.56 we state that the set of values of $\kappa \in [\kappa_1, \kappa_2]$ for which the vector $(\vec{\Omega}_\varepsilon(\kappa), \kappa)$ belongs to $\mathcal{C}_\infty^\upsilon$ is of large measure. Using that the linear frequencies as maps $\kappa \mapsto \Omega_j(\kappa)$ are analytic in $[\kappa_1, \kappa_2]$, we are able to implement the degenerate KAM theory. Formulated by Bambusi, Berti and Magistrelli [21] and used in [44, 13] in the case of simple eigenvalues, our analysis differs from these previous works since we deal with periodic boundary conditions and we add the conservation of the momentum. In particular, we show that the linear frequencies are non-degenerate, in the sense that no curves of linear combinations of $\Omega_j(\kappa)$ with distinct modules

$$[\kappa_1, \kappa_2] \ni \kappa \mapsto c_1 \Omega_{j_1}(\kappa) + ... + c_N \Omega_{j_N}(\kappa)\,, \quad (c_1, .., c_N) \in \mathbb{R} \backslash \{0\}\,, \quad |j_a| \neq |j_b| \ \text{ for } a \neq b\,,$$

are contained in any hyperplane of $\mathbb{R}^N$. This is proved in Lemma 4.50 using a Vandermorde determinant argument. In Proposition 4.53, this qualitative property is translated into quantitative estimates for the non-degeneracy. For instance, for a second order Melnikov non-resonance condition we prove that

$$\begin{cases} \max_{0 \leqslant n \leqslant m_0} |\partial_\kappa^n (\vec{\Omega}(\kappa) \cdot \ell + \Omega_j(\kappa) - \Omega_{j'}(\kappa))| \geqslant \rho_0 \langle \ell \rangle \\ \vec{\jmath} \cdot \ell + j - j' = 0 \,, \ \ell \in \mathbb{Z}^\nu \,, \ j, j' \in \mathbb{S}_0^c \,, \ (\ell, j, j') \neq (0, j, j) \,, \end{cases} \tag{2.2.3}$$

where ρ_0 is the "amount of non-degeneracy" and m_0 is the "index of non-degeneracy". The main difference with respect to [44] is that we need to impose the momentum condition $\vec{\jmath} \cdot \ell + j + j' = 0$ for proving the transversality. In particular, it turns out to be essential during the proof when we have to show that the system

$$\vec{\Omega}(\kappa) \cdot \ell + \Omega_j(\kappa) - \Omega_{j'}(\kappa) = 0 \,, \quad \forall \, \kappa \in [\kappa_1, \kappa_2] \,, \quad \vec{\jmath} \cdot \ell + j - j' = 0 \,. \tag{2.2.4}$$

has no nontrivial solutions. Without the momentum restriction $\vec{\jmath} \cdot \ell + j - j' = 0$, this is false. Indeed, by picking $j = -\bar{\jmath}_1$ and $j' = -\bar{\jmath}_2$, assuming $\jmath_1 \neq \jmath_2$ (the case $j = j'$ is treated in the Diophantine condition), the non-degeneracy of the frequencies $\Omega_j(\kappa)$ on the interval $[\kappa_1, \kappa_2]$ implies that, for $\ell = (\ell_1, .., \ell_\nu) \in \mathbb{Z}^\nu$,

$$\vec{\Omega}(\kappa) \cdot \ell + \Omega_j(\kappa) - \Omega_{j'}(\kappa) = 0 \,, \ \forall \, \kappa \in [\kappa_1, \kappa_2] \quad \Rightarrow \quad \ell_1 + 1 = 0 \,, \ \ell_2 - 1 = 0 \,, \ \ell_3 = ... \ell_\nu = 0 \,.$$

The vector $\ell = (-1, 1, 0, .., 0) \in \mathbb{Z}^\nu$ would be an acceptable solution for system (2.2.4) if the momentum condition is not taken into account. Instead, we have

$$\vec{\jmath} \cdot \ell + j - j' = (\ell_1 - 1)\bar{\jmath}_1 + (\ell_2 + 1)\bar{\jmath}_2 = -2(\bar{\jmath}_1 - \bar{\jmath}_2) = 0 \,,$$

which leads to $\bar{\jmath}_1 = \bar{\jmath}_2$, contradicting the assumption $\bar{\jmath}_1 \neq \bar{\jmath}_2$.

The transversality conditions in Proposition 4.53 are stable under perturbations that are small in $\mathcal{C}^{k_0}$-norm, where $k_0 = m_0 + 2$, still coupled with the momentum conditions. In particular, this holds when the perturbation is given by the correction to the linear frequencies obtained at the end on the Nash-Moser iteration on the Floquet exponents $\mu_j^\infty(\kappa)$ in (4.4.13), see Lemma 4.58. Then, provided proper estimates on the resonant sets in Lemmata 4.57, 4.59, it is possible to prove the measure estimates of Theorem 4.56. The momentum condition is fundamental also in the proof of Lemmata 4.57, 4.59. This is essentially due to the expansion of the final eigenvalues. Indeed, recalling (4.4.13)

$$\mu_j^\infty(\omega, \kappa) = \mathtt{m}_{\frac{3}{2}}^\infty(\omega, \kappa) \Omega_j(\kappa) + \mathtt{m}_1^\infty(\omega, \kappa) j + \mathtt{m}_{\frac{1}{2}}^\infty(\omega, \kappa) \, |j|^{\frac{1}{2}} + \mathfrak{r}_j^\infty(\omega, \kappa) \,, \tag{2.2.5}$$

we note that the contribution at the first order is not trivial. In [44] they obtain the same expansion with the exception of $\mathtt{m}_1^\infty = 0$, which is due to the parity conditions that they impose for obtaining standing waves. The nontrivial term at order one that is present in our analysis could lead to troubles in the measure estimates if the momentum conditions are not imposed.

Reduction of the linearized operators. The construction of the solutions via the Nash-Moser Theorem mostly relies on showing that the linearized operator $\mathrm{d}_{i,\alpha}\mathcal{F}$ obtained at each step of the iterative scheme admits an *almost approximate inverse* satisfying tame estimates in Sobolev spaces with loss of derivatives, see Theorem 4.65. By *approximate inverse* we mean an operator which is an exact inverse at any traveling wave solution of $\mathcal{F}(i,\alpha) = 0$. The adjective *almost* refers to the fact that at the $\mathtt{n}$-th step of the Nash-Moser we shall require only finitely many non-resonance conditions, therefore the remaining operator supported on high Fourier frequencies of magnitude $O(N_\mathtt{n})$ and thus can be estimated as $O(N_\mathtt{n}^{-\mathrm{a}})$ for some $\mathrm{a} > 0$ (in suitable norms).

In Section 4.5 the almost approximate inverse is constructed under the ansatz that the linearized operator restricted on the normal directions $\mathcal{L}_\omega$, defined in (4.5.33), is almost invertible on traveling wave functions. By Lemma 4.66, the operator $\mathcal{L}_\omega$ is a finite rank perturbation of the restriction to the normal subspace $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ of

$$\begin{aligned}\mathcal{L} = \omega\cdot\partial_\varphi &+ \begin{pmatrix} \partial_x \widetilde{V} + G(\eta)B & -G(\eta) \\ g - \kappa\partial_x c\partial_x + B\widetilde{V}_x + BG(\eta)B & \widetilde{V}\partial_x - BG(\eta)\end{pmatrix} \\ &+ \frac{\gamma}{2}\begin{pmatrix} -G(\eta)\partial_x^{-1} & 0 \\ \partial_x^{-1}G(\eta)B - BG(\eta)\partial_x^{-1} - \frac{\gamma}{2}\partial_x^{-1}G(\eta)\partial_x^{-1} & -\partial_x^{-1}G(\eta)\end{pmatrix},\end{aligned} \tag{2.2.6}$$

where the functions $B, \widetilde{V}, c$ are given in (4.6.11), (4.6.13), which is obtained linearizing the water waves equations (1.1.13) in the Wahlén coordinates at a quasi-periodic traveling wave approximate solution (η,ζ) and changing ∂_t with the directional derivative $\omega\cdot\partial_\varphi$. The goal of Sections 4.6, 4.7 is to reduce the operator $\mathcal{L}_\omega$ to a constant coefficient, Fourier diagonal one so that it can be inverted on traveling wave functions once first order Melnikov conditions are satisfied. The reduction consists of two main blocks:

1. Symmetrization and diagonalization of the operator $\mathcal{L}$ up to smoothing operators;
2. Restriction of the normal subspace and KAM reducibility.

All the transformations performed in Sections 4.6.1-4.6.6, 4.7 are time quasi-periodic change of variables acting in phase spaces of functions in x that are momentum preserving. Therefore, they preserve dynamical system structure of the conjugated linear operators, which in particular will maps quasi-periodic traveling wave functions into themselves.

All these changes of variables are bounded and satisfy tame estimates between Sobolev spaces. As a consequence, the estimates that we shall obtain on the inverse of the final constant coefficient diagonal operator directly provide good tame estimates for the inverse of the operator $\mathcal{L}_\omega$ in (4.5.33).

Another difference with respect to the reduction in [44] is that we need to preserve the Hamiltonian nature of $\mathcal{L}$ at least until the symmetrization of the highest order (Section 4.6.3) in order to avoid operators of the form $\mathrm{i}a(\varphi)\mathcal{H}|D_x|^{\frac{1}{2}}$, where $a(\varphi) \in \mathbb{R}$ and $\mathcal{H}$ denotes the Hilbert transform, see (4.2.19). The latter operator is not present in [44] because it does not map even functions into themselves and therefore it is incompatible with this symmetry. For overcoming this issue, we require all the transformations in 4.6.1-4.6.3 to be symplectic, so that the conjugated operators are Hamiltonian. From Section 4.6.4 on, this property will be not preserved anymore.

We also note that the original system $\mathcal{L}$ is reversible and that all the transformations that we perform are reversibility preserving. The preservation of this property ensures that in the final system the Floquet exponents are real valued. Under this respect, the linear stability of the quasi-periodic traveling wave solutions in Theorem 1.8 is obtained as a consequence of the reversible nature of the water waves equations.

In the following we summarize step by step each part of the reduction. The main tool in Sections 4.6.3-4.6.6 is the pseudodifferential calculus: in order to employ it, it is convenient to ignore the projection on the normal subspace $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ and to perform a regularization procedure on the whole space before the KAM reducibility of Section 4.7, see Remark 4.67. Then, in Section 4.6.7, we project back on the subspace $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$.

1. Quasi-periodic reparametrization of time. The very first transformation that we perform in Section 4.6.1 is a quasi-periodic reparametrization of the time variable of the form

$$\vartheta := \varphi + \omega p(\varphi) \quad \Leftrightarrow \quad \varphi = \vartheta + \omega \breve{p}(\vartheta)\,, \tag{2.2.7}$$

where $p(\varphi)$ is the real $\mathbb{T}^\nu$-periodic traveling function, in the sense that $p(\varphi - \vec{\jmath}\varsigma) = p(\varphi)$ for any $\varsigma \in \mathbb{R}$. In this way the operator $\mathcal{L}$ is transformed into the Hamiltonian, momentum preserving operator

$$\begin{aligned}\mathcal{L}_0 = \omega\cdot\partial_\vartheta &+ \frac{1}{\rho}\begin{pmatrix} \partial_x \widetilde{V} + G(\eta)B & -G(\eta) \\ g - \kappa\partial_x c\partial_x + B\widetilde{V}_x + BG(\eta)B & \widetilde{V}\partial_x - BG(\eta)\end{pmatrix} \\ &+ \frac{1}{\rho}\frac{\gamma}{2}\begin{pmatrix} -G(\eta)\partial_x^{-1} & 0 \\ \partial_x^{-1}G(\eta)B - BG(\eta)\partial_x^{-1} - \frac{\gamma}{2}\partial_x^{-1}G(\eta)\partial_x^{-1} & -\partial_x^{-1}G(\eta)\end{pmatrix}\,,\end{aligned} \tag{2.2.8}$$

where $\rho(\vartheta)$ depends on $p(\varphi)$ and all the functions $\widetilde{V}, B, c$ are meant to be reparametrized according to (2.2.7). The function p will be chosen at the end of Section 4.6.3 to set the coefficient at the highest order to be constant also in the angle variable. In this way we avoid technical difficulties arising in the application of the Egorov Proposition 4.20 in Section 4.6.3.

2. Linearized good unknown of Alinhac. In Section 4.6.2 we introduce the linearized good unknown of Alinhac, as in [4, 44, 13]. This is indeed the same change of variables introduced by Lannes [133, 134] for proving energy estimates for the local existence theory. In these new variables, relabeling $\vartheta \rightsquigarrow \varphi$, the linearized operator (2.2.8) becomes the more symmetric operator

$$\mathcal{L}_1 = \omega \cdot \partial_\varphi + \frac{1}{\rho}\begin{pmatrix} \partial_x \widetilde{V} & -G(\eta) \\ g + a - \kappa \partial_x c \partial_x & \widetilde{V}\partial_x \end{pmatrix} - \frac{1}{\rho}\frac{\gamma}{2}\begin{pmatrix} G(\eta)\partial_x^{-1} & 0 \\ \frac{\gamma}{2}\partial_x^{-1}G(\eta)\partial_x^{-1} & \partial_x^{-1}G(\eta) \end{pmatrix}, \tag{2.2.9}$$

where the Dirichlet-Neumann operator admits the expansion

$$G(\eta) = G(0) + \mathcal{R}_G \,,$$

with $G(0)$ defined in (1.1.15) and $\mathcal{R}_G$ an OP$S^{-\infty}$ smoothing operator, see for instance [44, 13]. The operator in (2.2.9) is Hamiltonian and momentum preserving.

3. Symmetrization and reduction of the highest order. The goal of Section 4.6.3 is to symmetrize and reduce to constant coefficient the leading order of (2.2.9), so that it is conjugate to the momentum preserving linear operator of the form

$$\mathcal{L}_4 = \omega \cdot \partial_\varphi + \frac{1}{\rho}\begin{pmatrix} -\frac{\gamma}{2}G(0)\partial_x^{-1} & -m_{\frac{3}{2}}(\varphi)\omega(\kappa, D) \\ m_{\frac{3}{2}}(\varphi)\omega(\kappa, D) & -\frac{\gamma}{2}G(0)\partial_x^{-1} \end{pmatrix} + \ldots, \tag{2.2.10}$$

where $\omega(\kappa, D) := \sqrt{\kappa\, D^2 G(0) + g\, G(0) - (\frac{\gamma}{2}\partial_x^{-1}G(0))^2}$, $m_{\frac{3}{2}}(\varphi)$ is a function close to 1 depending only on $\varphi \in \mathbb{T}^\nu$ and the dots stand for lower order operators, smaller in size (see (4.6.82) for the complete expression). In particular, in the complex unknowns $(h, \overline{h})$ via the map in (4.1.24), the first component of the operator in (2.2.10) reads

$$(h, \overline{h}) \mapsto \omega \cdot \partial_\varphi h + \mathrm{i}\mathtt{m}_{\frac{3}{2}}\Omega(\kappa, D)h + a_1^{(d)}\partial_x h + R_5^{(d)}h + R_5^{(o)}\overline{h} \tag{2.2.11}$$

(which corresponds to (4.6.90) neglecting the projector $\mathrm{i}\boldsymbol{\Pi}_0$), where $\mathtt{m}_{\frac{3}{2}} := m_{\frac{3}{2}}(\varphi)/\rho(\varphi) \in \mathbb{R}$ is now purely constant, choosing properly the function $p(\varphi)$ of the reparametrization (2.2.7), and $R_5^{(d)}, R_5^{(o)}$ are φ-dependent families of pseudodifferential operators of order 0. We shall the former operator "diagonal" and the latter "off-diagonal", we respect to the variables $(h, \overline{h})$.

In order to transform (2.2.8) into the linear operator (2.2.10), we first introduce a change of variable induced by a diffeomorphism of $\mathbb{T}_x$ of the form $y = x + \beta(\varphi, x)$. Conjugating $\mathcal{L}_1$ by the symplectic change of variables

$$u(\varphi, x) \mapsto (\mathcal{E}u)(\varphi, x) := \sqrt{1 + \beta_x(\varphi, x)}(\mathcal{B}u)(\varphi, x)\,, \quad (\mathcal{B}u)(\varphi, x) := u(\varphi, x + \beta(\varphi, x))\,, \tag{2.2.12}$$

we obtain the operator of the form

$$\mathcal{L}_2 = \omega \cdot \partial_\varphi + \frac{1}{\rho} \begin{pmatrix} -\frac{\gamma}{2} G(0) \partial_y^{-1} & -a_2 G(0) a_2 \\ -\kappa a_2 \partial_y a_3 \partial_y a_2 + g - \left(\frac{\gamma}{2}\right)^2 \partial_y^{-1} G(0) \partial_y^{-1} & -\frac{\gamma}{2} \partial_y^{-1} G(0) \end{pmatrix} + \dots \tag{2.2.13}$$

where a_2, a_3 are quasi-periodic traveling wave functions defined by

$$a_2 = \mathcal{B}^{-1}(\sqrt{1+\beta_x}), \quad a_3 = \mathcal{B}^{-1}\big(c(1+\beta_x)\big)$$

and the dots stand for lower order terms, smaller in size. In particular, $\mathcal{L}_2$ is momentum preserving, because $\beta(\varphi, x)$ is a quasi-periodic traveling function, and Hamiltonian, since $\mathcal{E}$ is symplectic.

Then, conjugating (2.2.13) with the symplectic maps

$$\mathcal{Q} = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} M(D) & 0 \\ 0 & M^{-1}(D) \end{pmatrix}, \tag{2.2.14}$$

where $q(\varphi, x)$ is a quasi-periodic traveling function close to 1 and the symbol of the Fourier multiplier $M(D)$ is defined in (4.0.7) (we are still neglecting in this exposition the action on the zeroth mode, see the correct definition of the map $\widetilde{\mathcal{M}}$ in (4.6.79)), we obtain the operator in (2.2.10). Proper choices of the functions $\beta(\varphi, x)$ and $q(\varphi, x)$ allow to end up with the function $m_{\frac{3}{2}}(\varphi)$ independent of $x \in \mathbb{T}$.

As already mentioned, we require the transformations in (2.2.12), (2.2.14) to be symplectic in order to avoid the rising of an operator of the form $(h, \overline{h}) \mapsto \mathrm{i}\, b(\varphi) \mathcal{H} |D|^{\frac{1}{2}} h$ in (2.2.11), which cannot be deleted by any transformation.

Furthermore, comparing with the reduction in [44], the conjugation with the map in (2.2.12) leads to purely pseudodifferential remainders. Indeed, when we deal with the operator $\partial_x^1 G(0) \partial_x^{-1}$, which is due to the vorticity γ and so is not present in [44], we require to expand the conjugated operator $\mathcal{E}^{-1} \circ \partial_x^{-1} \circ \mathcal{E}$ in homogeneity up to a pseudodifferential remainder of arbitrary lower order $-N$ provided by Proposition 4.20, which is a slight modification of Proposition 2.28 in [43]. The choice of the order $-N$ is fixed at the end of Section 4.6.4.

4. Symmetrization up to smoothing remainders. In Section 4.6.4 we reduce the off-diagonal term $R_5^{(o)}$ to a pseudodifferential operator with very negative order, i.e. we conjugate the above operator to another one of the form (see Lemma 4.75)

$$(h, \overline{h}) \mapsto \omega \cdot \partial_\varphi h + \mathrm{i}\, \mathtt{m}_{\frac{3}{2}} \Omega(\kappa, D) h + a_1^{(d)} \partial_x h + R_6^{(d)} h + R_6^{(o)} \overline{h}, \tag{2.2.15}$$

where $R_6^{(d)} \in \mathrm{OP}S^0$ and $R_6^{(o)} \in \mathrm{OP}S^{-M}$ for a constant M large enough fixed in Section 4.7, in view of the KAM reducibility scheme. The operator in (2.2.15) is still momentum preserving, whereas the Hamiltonianity is not preserved anymore.

5. Reduction of the order 1. In Section 4.6.5 we reduce to constant coefficients the operator $a_1^{(d)}(\varphi,x)\partial_x$. First, we conjugate the operator (2.2.15) by the time-1 flow of the pseudodifferential PDE

$$\partial_\tau u = \mathrm{i}\, B(\varphi,x)|D|^{\frac{1}{2}}\,,$$

where $b(\varphi,x)$ is a small function to be determine. This kind of transformation – which are "semi-Fourier integral operator", namely pseudodifferential operators of type $(\frac{1}{2},\frac{1}{2})$ in Hörmander's notation – was introduced in [4] and studied as flows in [44]. Choosing appropriately the function $b(\varphi,x) = b_1(\varphi,x) + b_2(\varphi)$ and translating the x-variable with respect to a φ-dependent function $\varrho(\varphi)$, see (4.6.136), (4.6.142), (4.6.146), the final outcome is a momentum preserving linear operator of the form (see (4.6.147))

$$(h,\overline{h}) \mapsto \omega\cdot\partial_\varphi h + \mathrm{i}\,\mathtt{m}_{\frac{3}{2}}\Omega(\kappa,D)h + \mathtt{m}_1\partial_x h + \mathrm{i}\, a_3^{(d)}|D|^{\frac{1}{2}}h + R_8^{(d)}h + \mathcal{T}_8(h,\overline{h})\,, \tag{2.2.16}$$

where $\mathtt{m}_1 \in \mathbb{R}$ is a small constant, $a_3^{(d)}(\varphi,x)$ is a small traveling wave function, $R_8^{(d)} \in \mathrm{OP}S^0$ and the linear operator $\mathcal{T}_8$ is small, smoothing and satisfies tame estimates in Sobolev spaces, see (4.6.151). Moreover, the φ-dependent function $b_2(\varphi)$ is determined so that the x-average of the function $a_3^{(d)}$ is independent of $\varphi \in \mathbb{T}^\nu$.

6. Reduction of the order 1/2. In Section 4.6.6 we reduce to constant coefficient the operator $\mathrm{i}\, a_3(\varphi,x)|D|^{\frac{1}{2}}$. We conjugate the operator (2.2.16) by the time-1 bounded flow of the PDE

$$\partial_\tau u = \mathrm{i}\, b_3(\varphi,x)\mathcal{H}\,,$$

where $b_3(\varphi,x)$ is a small function defined in (4.6.166) and $\mathcal{H}$ is the Hilbert transform. The final outcome is the momentum preserving operator, see (4.6.169)

$$(h,\overline{h}) \mapsto \omega\cdot\partial_\varphi h + \mathrm{i}\,\mathtt{m}_{\frac{3}{2}}\Omega(\kappa,D)h + \mathtt{m}_1\partial_x h + \mathrm{i}\,\mathtt{m}_{\frac{1}{2}}|D|^{\frac{1}{2}}h + R_9^{(d)}h + \mathcal{T}_9(h,\overline{h})\,, \tag{2.2.17}$$

where $\mathtt{m}_{\frac{1}{2}} \in \mathbb{R}$ is a small constant, $R_9^{(d)} \in \mathrm{OP}S^0$ and the linear operator $\mathcal{T}_9$ is small, smoothing and satisfies tame estimates in Sobolev spaces, see (4.6.171).

7. KAM reducibility. In Section 4.6.7 it is showed the conjugation of $\mathcal{L}_\omega$ to a quasi-periodic momentum preserving operator of the form

$$(h,\overline{h}) \mapsto \omega\cdot\partial_\varphi h + \mathrm{i}\,\mathtt{m}_{\frac{3}{2}}\Omega(\kappa,D)h + \mathtt{m}_1\partial_x h + \mathrm{i}\,\mathtt{m}_{\frac{1}{2}}|D|^{\frac{1}{2}}h + R_\perp^{(d)}h + R_\perp^{(o)}h\,. \tag{2.2.18}$$

where the linear operators

$$R_\perp^{(d)}\,,\ [R_\perp^{(d)},\partial_x]\,,\ \partial_{\varphi_m}^{s_0}R_\perp^{(d)}\,,\ \partial_{\varphi_m}^{s_0}[R_\perp^{(d)},\partial_x]\,,$$
$$\partial_{\varphi_m}^{s_0+\mathtt{b}}R_\perp^{(d)}\,,\ \partial_{\varphi_m}^{s_0+\mathtt{b}}[R_\perp^{(d)},\partial_x]\,,\ m=1,\dots,\nu\,,$$

and similarly $R_{\perp}^{(o)}$, satisfy tame estimates in Sobolev spaces for some $\mathtt{b} = \mathtt{b}(\tau, k_0) \in \mathbb{N}$ large enough, fixed in (4.7.6), see Lemma 4.84. Such conditions hold under the assumption that M (the order of regularization in Section 4.6.4) is chosen large enough as in (4.7.5) (essentially $M = O(\mathtt{b})$). This is the property that compensates, along the KAM iteration, the loss of derivatives in φ and x produced by the small divisors in the second order non-resonance Melnikov conditions.

We follow the same KAM reducibility scheme in [44], which reduces essentially to prove that the solution of the homological equation produced at each step of the KAM iteration is closed in the class of $\mathcal{D}^{k_0}$-modulo-tame operators (see Definition 4.2.39), which is the content of Lemma 4.88. In addition to the scheme in [44], we show that the generator of each iteration is a momentum preserving operator so that the conjugated operator is momentum preserving, as well.

2.3 Ideas of the proof of Theorem 1.10

Paradifferential reduction up to smoothing remainders. The first step in order to prove Theorems 1.9 and 1.10 is to write (1.1.30) (with $\gamma = 0$) in paradifferential form, to symmetrize it, and reduce to paradifferential symbols which are constant in x, see Proposition 5.11. These results are proved in Berti-Delort [37] (up to minor details).

We define the Fourier multiplier Λ of order $-1/4$ as $\Lambda := \Lambda(D) := \big(G(0)\big)^{\frac14}\big(g + \kappa D^2\big)^{-\frac14}$, which is equivalent to the Fourier multiplier $M(D)$ with symbol given by (4.0.7) in the case $\gamma = 0$, and we consider the complex function

$$u := \frac{1}{\sqrt{2}}\Lambda\omega + \frac{\mathrm{i}}{\sqrt{2}}\Lambda^{-1}\eta\,, \quad \eta = \frac{1}{\mathrm{i}\sqrt{2}}\Lambda(u - \overline{u})\,, \quad \omega = \frac{1}{\sqrt{2}}\Lambda^{-1}(u + \overline{u}) \tag{2.3.1}$$

where Λ^{-1} acts on functions modulo constants in itself. Then Proposition 5.10 shows that, in the variable $U := \binom{u}{\overline{u}}$ defined in (2.3.1), the equations (1.1.30) assumes the form (5.2.6), which in particular can be read as

$$\partial_t U = \mathrm{i}\,\Omega(D)EU + \mathrm{i}\,M(U;t)U$$

where $\Omega(D) = \mathrm{Op}^{\mathrm{BW}}(\Omega(\xi))$, $\Omega(\xi) \in \widetilde{\Gamma}_0^{\frac32}$ is the dispersion relation symbol defined in (4.0.3) (see also Definition 5.3, where the classes $\widetilde{\Gamma}_0^m$ are defined), and $M(U;t)$ is a real-to-real map in $\Sigma\mathcal{M}_{K,1,1}^{m_1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ for some $m_1 \geqslant 3/2$ (see Definition 5.8), using that paradifferential operators and smoothing remainders are maps, see (4.2.6) in [37]

Since the linear dispersion relation in (4.0.3) is superlinear, the complex system of Proposition 5.10 can be transformed into a paradifferential diagonal system with a symbol constant in x, up to smoothing terms, which is the content of Proposition 5.11. In particular, we end up with the system in the variable $Z = \binom{z}{\bar z}$ of the form

$$\partial_t Z = \mathrm{i}\, \mathrm{Op}^{\mathrm{BW}}\big((1+\underline{\zeta}(U;t))\Omega(\xi)E + H(U;t,\xi)\big)Z + \mathrm{i}\, R(U;t)[Z] \tag{2.3.2}$$

where the real function $\underline{\zeta}(U;t)$ and the diagonal matrix of symbols $H(U;t,\xi)$ of order 1 are independent of $x \in \mathbb{T}$, whereas $R(U;t)$ is a matrix of smoothing operators.

Quadratic Poincaré-Birkhoff normal form and 3-waves interaction. In Section 5.3 we transform the paralinearized reduced system (2.3.2) into its quadratic Poincaré-Birkhoff normal form. Moreover, noting that the contribution at the quadratic order of the operator $z \mapsto \mathrm{Op}^{\mathrm{BW}}(\underline{\zeta}(U)\Omega(\xi))Z$ is zero due to the conservation of momentum (see Lemma 5.12), we have only to transform the quadratic term of the smoothing operator $R(U) = -\mathrm{i}\mathtt{R}_1(U) - \mathrm{i}\mathtt{R}_{\geqslant 2}(U)$, where $\mathtt{R}_1(U)$ is homogeneous of degree 1 in U. The goal is therefore to provide a bounded invertible map that transforms (2.3.2) into the system in the new variable $Y = \binom{y}{\bar y}$ of the form

$$\partial_t Y = \mathrm{i}\Omega(D)EY + \mathtt{R}_1^{\mathrm{res}}(Y)[Y] + \mathcal{X}_{\geqslant 3}(U,Y) \tag{2.3.3}$$

where $\mathtt{R}_1^{\mathrm{res}}(Y)$ is Poincaré-Birkhoff resonant, according to Definition (5.13), and $\mathcal{X}_{\geqslant 3}(U,Y)$ is a cubic remainder that contributes to the energy estimates (5.0.10) of Theorem 1.9.

The non-resonant term are removed in the process by solving the homological equation

$$\mathtt{G}_1(\mathrm{i}\Omega(D)EU) + \big[\mathtt{G}_1(U), \mathrm{i}\Omega(D)E\big] + \mathtt{R}_1(U) = \mathtt{R}_1^{\mathrm{res}}(U)\,, \tag{2.3.4}$$

where $\mathtt{G}_1(U)$ is a smoothing generator of the transformation. The equation (2.3.4) is solved in Lemma 5.18, which requires a lower bound on the non-resonant three waves interaction of the linear frequencies, that is, on those Fourier sites $n_1, n_2, n_3 \in \mathbb{Z}\backslash\{0\}$ such that

$$n_1 + \sigma n_2 + \sigma' n_3 = 0\,, \quad \Omega(n_1) + \sigma\Omega(n_2) + \sigma'\Omega(n_3) \neq 0\,.$$

The restriction $n_1 + \sigma n_2 + \sigma' n_3 = 0$ on the Fourier sites is due to the invariance of the equations (1.1.30) with respect to space translations, which we also call, as in Section 2.2, the "momentum condition". In Lemma 5.15 we show that, on such non-resonant sites, an uniform lower bound holds and that there are only *finitely many* triplets of Fourier sites that are resonant:

$$\begin{cases} \sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 = 0 \\ \sigma_1 \Omega_{j_1} + \sigma_2 \Omega_{j_2} + \sigma_3 \Omega_{j_3} = 0 \end{cases} \quad \Longrightarrow \quad \max(|j_1|,|j_2|,|j_3|) < \mathtt{C}\,. \tag{2.3.5}$$

This key fact that the resonant contribution to the dynamics is confined only on finitely many interactions is fundamental for proving the energy estimates that lead to Theorem 1.10. A more complicated and chaotic dynamics takes place when the system is restricted to these resonant sites, which is responsible in fluid dynamics for the phenomenon of the Wilton ripples. For instance, we refer to the work of Craig and Sulem [69], where they study some cases of resonant triads that give rise to quasi-periodic solutions, either stable or unstable.

Normal form identification and energy estimates. In Section 5.4.1 we perform a normal form uniqueness argument that allows to identify the quadratic resonant vector field $\mathtt{R}_1^{\rm res}(Y)[Y]$ in (2.3.3) as the cubic resonant Hamiltonian vector field obtained by the formal Birkhoff normal form construction in [69]. Our identification argument is the easier version of the one used by Berti, Feola and Pusateri in [39, 40] for the pure gravity case. A related strategy was first implemented by Feola, Giuliani and Procesi in [89] for proving the existence of small amplitude, quasi-periodic solution for small quasi-linear Hamiltonian perturbations of the Degasperis-Procesi equation on the torus $\mathbb{T}$.

The first step is to expand in homogeneity the invertible maps provided in Propositions 5.10, 5.11, 5.14 see Lemma 5.20. With such expansion, we compute how the Hamiltonian vector field (1.1.30) (with $\gamma = 0$), up to cubic and higher degrees of homogeneity, are transformed by the previous maps truncated are the quadratic degree with a Lie commutator expansion, obtaining the vector field

$$X_{H_{\mathbb{C}}^{(2)}} + X_{H_{\mathbb{C}}^{(3)}} + [\![\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2, X_{H_{\mathbb{C}}^{(2)}}]\!] + \cdots \,, \tag{2.3.6}$$

where $X_{H_{\mathbb{C}}^{(2)}}$, $X_{H_{\mathbb{C}}^{(3)}}$ are the Hamiltonian vector field generated by the the quadratic and the cubic contribution of the Hamiltonian (1.1.18) (with $\gamma = 0$) in complex coordinates, see (5.4.6), whereas $\mathtt{S}_2^{\mathbb{C}}$ and $\mathtt{T}_2$ are the quadratic vector field of the transformations of Proposition (5.10), 5.14, respectively, and $[\![\cdot\,, \cdot]\!]$ is the nonlinear commutator defined in (5.4.14). By construction, if we project on the cubic resonant Fourier modes the vector field in (2.3.6), we conclude that

$$\mathtt{R}_1^{\rm res}(Y)[Y] = \Pi_{\ker}(X_{H_{\mathbb{C}}^{(3)}}) \overset{(5.4.6)}{=} X_{H_{\rm BNF}^{(3)}} \,,$$

where $H_{\rm BFN}^{(3)}$ is the cubic resonant Hamiltonian in (1.1.39), 1.1.40, as claimed in Theorem 1.9.

The quadratic life span of Theorem 1.10 is proved by the energy estimate argument in Section 5.4.2. By (2.3.5), we consider for any function $z \in \dot{H}^s(\mathbb{T})$ the splitting between low and high modes

$$z = z_L + z_H \,, \quad z_L := \Pi_L z = \sum_{0<|j|\leqslant \mathtt{C}} z_j e^{\mathrm{i} jx} \,, \quad z_H := \Pi_H z := \sum_{|j|\geqslant \mathtt{C}} z_j e^{\mathrm{i} jx} \,.$$

The system (1.1.38) reads in $z = z_L + z_H$ as

$$\begin{cases} \dot z_L = \mathrm{i}\Omega(D) z_L + \mathrm{i}\partial_{\overline{z}} H_{\rm BNF}^{(3)}(z_L, \overline{z}_L) + \Pi_L\big(\mathcal{X}_{\geqslant 3}^+(U,Z)\big) \\ \dot z_H = \mathrm{i}\Omega(D) z_H \qquad\qquad\qquad\qquad\quad\;\; + \Pi_H\big(\mathcal{X}_{\geqslant 3}^+(U,Z)\big) \,, \end{cases}$$

having that $H^{(3)}_{\mathrm{BNF}}(z,\overline{z}) = H^{(3)}_{\mathrm{BNF}}(z_L,\overline{z}_L)$. If we ignore the cubic remainders, we have that the dynamics of z_H is linear and preserves the Sobolev norm H^s. For the low modes, using that, by construction of the Birkhoff normal form, the Hamiltonian $H^{(3)}_{\mathrm{BNF}}$ Poisson commutes with the quadratic Hamiltonian $H^{(2)}_{\mathbb{C}}$, that is

$$\{H^{(3)}_{\mathrm{BNF}}, H^{(2)}_{\mathbb{C}}\} = 0\,,$$

we have that the evolution of z_L is constant for $H^{(2)}_{\mathbb{C}}$, which controls the L^2-norm and any Sobolev norm H^s on the finitely many modes. Therefore, it is possible to show the energy estimates in the H^s-norm of Lemma 5.21 with respect to the equivalent norm

$$\|z\|_s^2 := H^{(2)}_{\mathbb{C}}(z_L) + \|z_H\|^2_{H^s(\mathbb{T})}\,, \qquad \text{where} \qquad H^{(2)}_{\mathbb{C}}(z_L) := \sum_{0<|j|\leqslant \mathtt{C}} \Omega_j z_j \overline{z}_j\,.$$

Theorem (1.10), finally, follows by a standard bootstrap argument.

Chapter 3

Reducibility for a linear Klein-Gordon equation with a fast driven potential

We consider a linear Klein-Gordon equation with quasi-periodic driving

$$u_{tt} - u_{xx} + \mathtt{m}^2 u + V(\omega t, x)u = 0 \ , \qquad x \in [0, \pi] \ , \quad t \in \mathbb{R} \ , \tag{3.0.1}$$

with spatial Dirichlet boundary conditions $u(t, 0) = u(t, \pi) = 0$.
The potential $V : \mathbb{T}^\nu \times [0, \pi] \to \mathbb{R}$, is quasi-periodic in time with a fast oscillating frequency vector $\omega \in \mathbb{R}^\nu \backslash \{0\}$, namely $|\omega| \gg 1$.
The goal is to provide, for any frequency ω belonging to a Cantor set of large measure, a reducibility result for the system (3.0.1). That is, we construct a change of coordinates which conjugates equation (3.0.1) into a diagonal, time independent one.

We recall the assumptions on the potential driving $V(\omega t, x)$:

(**V1**) The even extension in x of $V(\varphi, x) : \mathbb{T}^\nu \times [0, \pi] \to \mathbb{R}$ on the torus $\mathbb{T} \simeq [-\pi, \pi]$, which we still denote by V, is smooth in both variables and it extends analytically in φ in a proper complex neighbourhood of $\mathbb{T}^\nu$ of width $\rho > 0$. In particular, for any $\beta \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, there is a constant $C_{\beta,\rho} > 0$ such that

$$|\partial_x^\beta V(\varphi, x)| \leqslant C_{\beta,\rho} \quad \forall \, x \in \mathbb{T} \ , \ |\mathrm{Im}\, \varphi| \leqslant \rho \ ;$$

(**V2**) $\int_{\mathbb{T}^\nu} V(\varphi, x) \, \mathrm{d}\varphi = 0$ for any $x \in [0, \pi]$.

We introduce the new variables $\psi := B^{1/2}u + \mathrm{i}B^{-1/2}\partial_t u$ and $\overline{\psi} := B^{1/2}u - \mathrm{i}B^{-1/2}\partial_t u$, where $B := \sqrt{-\Delta + \mathtt{m}^2}$ as in (1.1.2). In the new variables equation (3.0.1) is equivalent to the following system

$$\mathrm{i}\partial_t\psi(t) = \mathbf{H}(t)\psi(t) \ , \quad \mathbf{H}(t) := \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} + \frac{1}{2} B^{-1/2} V(\omega t, x) B^{-1/2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \ , \tag{3.0.2}$$

where, abusing notation, we denoted $\psi(t) \equiv \left(\begin{smallmatrix} \psi(t) \\ \overline{\psi}(t) \end{smallmatrix}\right)$ the vector with the components $\psi, \overline{\psi}$. The phase space for (1.1.4) is $\mathcal{H}^r \times \mathcal{H}^r$, where $\mathcal{H}^r$, $r \geqslant 0$, is defined in (1.1.5). Here we have used the notation $\langle m \rangle := (1 + |m|^2)^{\frac{1}{2}}$, which will be kept throughout this chapter. We define the ν-dimensional annulus of size $\mathtt{M} > 0$ by

$$R_{\mathtt{M}} := \overline{B_{2\mathtt{M}}(0)} \backslash B_{\mathtt{M}}(0) \subset \mathbb{R}^\nu \ ;$$

here we denoted by $B_M(0)$ the ball of center zero and radius M in the Euclidean topology of $\mathbb{R}^\nu$.

Theorem 3.1. *Consider the system* (1.1.4) *and assume (***V1***) and (***V2***). Fix arbitrary* $r, \mathtt{m} \geqslant 0$ *and* $\alpha \in (0,1)$*. Fix also an arbitrary* $\gamma_* > 0$ *sufficiently small.*
Then there exist $\mathtt{M}_* > 1$*,* $C > 0$ *and, for any* $\mathtt{M} \geqslant \mathtt{M}_*$*, a subset* $\Omega_\infty^\alpha = \Omega_\infty^\alpha(\mathtt{M}, \gamma_*)$ *in* $R_{\mathtt{M}}$*, fulfilling*

$$\frac{\mathrm{meas}(R_{\mathtt{M}} \backslash \Omega_\infty^\alpha)}{\mathrm{meas}(R_{\mathtt{M}})} \leqslant C\gamma_*,$$

such that the following holds true. For any frequency vector $\omega \in \Omega_\infty^\alpha$*, there exists an operator* $\mathcal{T}(\omega t; \omega)$*, bounded in* $\mathcal{L}(\mathcal{H}^r \times \mathcal{H}^r)$*, quasi-periodic in time and analytic in a shrunk neighbourhood of* $\mathbb{T}^\nu$ *of width* $\rho/8$*, such that the change of coordinates* $\psi = \mathcal{T}(\omega t; \omega) w$ *conjugates* (1.1.4) *to the diagonal time-independent system*

$$\mathrm{i}\dot{w}(t) = \mathbf{H}^{\infty,\alpha} w(t) \ , \quad \mathbf{H}^{\infty,\alpha} := \begin{pmatrix} \mathcal{D}^{\infty,\alpha} & 0 \\ 0 & -\mathcal{D}^{\infty,\alpha} \end{pmatrix} \ , \ \mathcal{D}^{\infty,\alpha} = \mathrm{diag}\left\{ \lambda_j^\infty(\omega) \ \middle| \ j \in \mathbb{N} \right\}. \tag{3.0.3}$$

The transformation $\mathcal{T}(\omega t; \omega)$ *is close to the identity, in the sense that there exists* $C_r > 0$ *independent of* $\mathtt{M}$ *such that*

$$\| \mathcal{T}(\omega t; \omega) - \mathbb{1} \|_{\mathcal{L}(\mathcal{H}^r \times \mathcal{H}^r)} \leqslant \frac{C_r}{\mathtt{M}^{\frac{1-\alpha}{2}}} \ . \tag{3.0.4}$$

The new eigenvalues $(\lambda_j^\infty(\omega))_{j \in \mathbb{N}}$ *are real, Lipschitz in* ω*, and admit the asymptotics, for* $j \in \mathbb{N}$*,*

$$\lambda_j^\infty(\omega) := \lambda_j^\infty(\omega, \alpha) := \lambda_j + \varepsilon_j^\infty(\omega, \alpha) \ , \quad \varepsilon_j^\infty(\omega, \alpha) \sim O\left(\frac{1}{\mathtt{M} j^\alpha} \right) \ , \tag{3.0.5}$$

where $\lambda_j = \sqrt{j^2 + \mathtt{m}^2}$ *are the eigenvalues of the operator* B*.*

The rest of the chapter concerns the proof of Theorem 3.1. The relative functional setting is presented in Section 3.1. In particular, we define the family of pseudodifferential operators analytically depending on the variable $\varphi \in \mathbb{T}^\nu$ and the classes of linear operator with finite s-decay. In Section 3.2 we show how the system is transformed into a perturbative setting via the Magnus normal form. In Section 3.3 we prove the balance second order non-resonance Melnikov conditions that are needed for the KAM reducibility scheme of Section 3.4.

3.1 Functional setting

Given a set $\Omega \subset \mathbb{R}^\nu$ and a Fréchet space $\mathcal{F}$, the latter endowed with a system of seminorms $\{\| \cdot \|_n \; : \; n \in \mathbb{N}\}$, we define for a function $f : \Omega \ni \omega \mapsto f(\omega) \in \mathcal{F}$ the quantities

$$|f|^{\infty}_{n,\Omega} := \sup_{\omega\in\Omega} \|f(\omega)\|_n \, , \qquad |f|^{\rm Lip}_{n,\Omega} := \sup_{\substack{\omega_1,\omega_2\in\Omega \\ \omega_1\neq\omega_2}} \frac{\|f(\omega_1)-f(\omega_2)\|_n}{|\omega_1-\omega_2|}. \tag{3.1.1}$$

Given $\mathtt{w} \in \mathbb{R}_+$, we denote by $\mathrm{Lip}_{\mathtt{w}}(\Omega, \mathcal{F})$ the space of functions from Ω into $\mathcal{F}$ such that

$$\|f\|^{\mathrm{Lip}(\mathtt{w})}_{n,\Omega} := |f|^{\infty}_{n,\Omega} + \mathtt{w}|f|^{\rm Lip}_{n,\Omega} < \infty \; . \tag{3.1.2}$$

3.1.1 Pseudodifferential operators

The main tool for the construction of the Magnus transform in Section 3.2 is the calculus with pseudodifferential operators acting on the scale of the standard Sobolev spaces on the torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, which is defined for any $r \in \mathbb{R}$ as

$$H^r(\mathbb{T}) := \Big\{ \psi(x) = \sum_{j\in\mathbb{Z}} \psi_j e^{\mathrm{i}jx}, \; x \in \mathbb{T} \; : \; \|\psi\|^2_{H^r(\mathbb{T})} := \sum_{j\in\mathbb{Z}} \langle j \rangle^{2r} |\psi_j|^2 < \infty \Big\}. \tag{3.1.3}$$

Definition 3.2. We say that a function $f : \mathbb{R}\times\mathbb{Z} \to \mathbb{R}$, $(x, j) \mapsto f(x, j)$ is a *pseudodifferential symbol* of order $m \in \mathbb{R}$ if it is the restriction of a function $f(x, \xi)$, which is $\mathcal{C}^\infty$ on $\mathbb{R}\times\mathbb{R}$, 2π-periodic in x, so that, for any $\alpha, \beta \in \mathbb{N}_0$, there exists $C_{\alpha,\beta} \geqslant 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta f(x,\xi)| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-\beta} \; , \quad \forall\, x \in \mathbb{R} \, .$$

In this case, we write $f \in S^m$.

We endow S^m with the family of seminorms

$$\wp^m_\varrho(f) := \sum_{\alpha+\beta\leqslant\varrho} \sup_{(x,\xi)\in\mathbb{R}\times\mathbb{R}} \langle \xi \rangle^{-m+\beta} |\partial_x^\alpha \, \partial_\xi^\beta f(x,\xi)| \; , \quad \varrho \in \mathbb{N}_0 \; .$$

Analytic families of pseudodifferential operators. We will consider in our discussion also symbols depending real analytically on the variable $\theta \in \mathbb{T}^\nu$. To define them, we need to introduce the complex neighbourhood of the torus

$$\mathbb{T}^\nu_\rho := \{a + \mathrm{i}b \in \mathbb{C}^\nu \, : \, a \in \mathbb{T}^\nu \, , \; |b| \leqslant \rho\} \, .$$

Definition 3.3. Given $m \in \mathbb{R}$ and $\rho > 0$, a function $f : \mathbb{T}^\nu \times \mathbb{T} \times \mathbb{Z} \to \mathbb{R}$, $(\varphi, x, j) \mapsto f(\varphi, x, j)$, is called a symbol of class S^m_ρ if it is the restriction of a function $f(\varphi, x, \xi)$, which is $\mathcal{C}^\infty$ on $\mathbb{T}^\nu \times \mathbb{R} \times \mathbb{R}$, that extends analytically in φ on $\mathbb{T}^\nu_\rho$ and such that

$$|\partial_x^\alpha \partial_\xi^\beta f(\varphi, x, \xi)| \leqslant C_{\alpha,\beta} \, \langle \xi \rangle^{m-\beta} \; , \quad \forall \, x \in \mathbb{R} \; , \; \forall \, \varphi \in \mathbb{C}^\nu, \; |\mathrm{Im} \, \varphi| \leqslant \rho \, , \quad \forall \, \alpha, \beta \in \mathbb{N}_0 \, .$$

We endow the class S^m_ρ with the family of seminorms

$$\wp^{m,\rho}_\varrho(f) := \sup_{|\mathrm{Im}\,\varphi| \leqslant \rho} \; \sum_{\alpha+\beta \leqslant \varrho} \; \sup_{(x,\xi) \in \mathbb{R}\times\mathbb{R}} \langle \xi \rangle^{-m+\beta} \, |\partial_x^\alpha \, \partial_\xi^\beta f(\varphi, x, \xi)| \; , \quad \varrho \in \mathbb{N}_0 \; .$$

We associate to a symbol $f \in S^m_\rho$ the operator $f(\varphi, x, D_x)$ acting on 2π-periodic functions by standard quantization

$$\psi(x) = \sum_{j \in \mathbb{Z}} \psi_j e^{\mathrm{i}jx} \; \mapsto \; \left(f(\varphi, x, D_x)\psi \right)(x) := \sum_{j \in \mathbb{Z}} f(\varphi, x, j) \psi_j e^{\mathrm{i}jx} \; ; \tag{3.1.4}$$

here $D_x = D := \mathrm{i}^{-1}\partial_x$ is the Hörmander derivative.

Definition 3.4. We say that $F \in \mathrm{OP}S^m_\rho$ if it is a pseudodifferential operator with symbol of class S^m_ρ, i.e. if there exists a symbol $f \in S^m_\rho$ such that $F = f(\varphi, x, D_x)$.
If F does not depend on φ, we simply write $F \in \mathrm{OP}S^m$.

Remark 3.5. For any $\sigma \in \mathbb{R}$, the operator $\langle D \rangle^\sigma \equiv (1 - \partial_{xx})^{\frac{\sigma}{2}}$ is in $\mathrm{OP}S^\sigma$.

As usual we give to $\mathrm{OP}S^m_\rho$ a Fréchet structure by endowing it with the seminorms of the symbols. Finally we define the class of pseudodifferential operators depending on a Lipschitz way on an external parameter.

Definition 3.6. We denote by $\mathrm{Lip}_{\mathtt{w}}(\Omega, \mathrm{OP}S^m_\rho)$ the space of pseudodifferential operators whose symbols belong to $\mathrm{Lip}_{\mathtt{w}}(\Omega, S^m_\rho)$ and by $\left(\wp^{n,\rho}_j(\cdot)^{\mathrm{Lip}(\mathtt{w})}_\Omega \right)_{j \in \mathbb{N}}$ the corresponding seminorms.

Remark 3.7. Let $F \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathrm{OP}S^m_\rho)$ and $G \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathrm{OP}S^n_\rho)$. Then the symbolic calculus implies that $FG \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathrm{OP}S^{m+n}_\rho)$ and $[F, G] \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathrm{OP}S^{m+n-1}_\rho)$, with the quantitative bounds

$$\begin{aligned} &\forall j \;\; \exists N \text{ s.t.} \;\; \wp^{m+n,\rho}_j(FG)^{\mathrm{Lip}(\mathtt{w})}_\Omega \leqslant C_1 \wp^{m,\rho}_N(F)^{\mathrm{Lip}(\mathtt{w})}_\Omega \, \wp^{n,\rho}_N(G)^{\mathrm{Lip}(\mathtt{w})}_\Omega \; , \\ &\forall j \;\; \exists N \text{ s.t.} \;\; \wp^{m+n-1,\rho}_j([F, G])^{\mathrm{Lip}(\mathtt{w})}_\Omega \leqslant C_2 \wp^{m,\rho}_N(F)^{\mathrm{Lip}(\mathtt{w})}_\Omega \, \wp^{n,\rho}_N(G)^{\mathrm{Lip}(\mathtt{w})}_\Omega \; . \end{aligned}$$

Parity preserving operators. The space $\mathcal{H}^0$ of (1.1.5) is naturally identified with the subspace of $H^0(\mathbb{T}) := L^2(\mathbb{T})$ of odd functions. Therefore it makes sense to work with pseudodifferential operators preserving the parity. Before describing them, we recall the orthogonal decomposition of the periodic L^2-functions on $\mathbb{T}$:

$$L^2(\mathbb{T}) = L^2_{even}(\mathbb{T}) \oplus L^2_{odd}(\mathbb{T})$$

where, for $u(x) = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx} \in L^2(\mathbb{T})$, we have for any $j \in \mathbb{Z}$,

$$u \in L^2_{even}(\mathbb{T}) \;\Leftrightarrow\; u_{-j} = u_j \quad \text{and} \quad u \in L^2_{odd}(\mathbb{T}) \;\Leftrightarrow\; u_{-j} = -u_j. \tag{3.1.5}$$

Definition 3.8. We denote by $\mathcal{P}S^m_\rho$ the class of symbols $f \in S^m_\rho$ satisfying the property

$$f(\varphi, x, j) = f(\varphi, -x, -j) \quad \forall\, \varphi \in \mathbb{T}^\nu\,,\; x \in \mathbb{T}\,,\; j \in \mathbb{Z}\,. \tag{3.1.6}$$

We denote by $\mathcal{P}\mathrm{OP}S^m_\rho$ the subset of $\mathrm{OP}S^m_\rho$ of parity preserving operators, that is, those operators $A \in \mathrm{OP}S^m_\rho$ such that $A(L^2_{even}) \subseteq L^2_{even}$ and $A(L^2_{odd}) \subseteq L^2_{odd}$.

Lemma 3.9. *Let $F \in \mathrm{OP}S^m_\rho$ with symbol $f \in S^m_\rho$. Then $F \in \mathcal{P}\mathrm{OP}S^m_\rho$ if and only if $f \in \mathcal{P}S^m_\rho$.*

Proof. It is easy to check that $F(L^2_{odd}(\mathbb{T})) \subseteq L^2_{odd}(\mathbb{T})$ if and only if the symbol $f(x, j)$ of F fulfills $\mathrm{Im}[(f(x, j) - f(-x, -j))e^{\mathrm{i}jx}] \equiv 0$. Similarly $F(L^2_{even}(\mathbb{T})) \subseteq L^2_{even}(\mathbb{T})$ if and only if $\mathrm{Re}[(f(x, j) - f(-x, -j))e^{\mathrm{i}jx}] \equiv 0$. □

Remark 3.10. For all $\sigma \in \mathbb{R}$, the operator $\langle D\rangle^\sigma \in \mathcal{P}\mathrm{OP}S^\sigma$, whereas, by the assumption (**V1**), we have $V \in \mathcal{P}\mathrm{OP}S^0_\rho$.

Remark 3.11. Parity preserving operators are closed under composition and commutators.

Remark 3.12. For $\mathtt{m} = 0$ and $\sigma > 0$, we define $B^{-\sigma}\psi := \sum_{j\neq 0} \frac{1}{|j|^\sigma}\psi_j e^{\mathrm{i}jx}$ for any $\psi \in L^2(\mathbb{T})$; clearly $B^{-\sigma} \in \mathcal{P}\mathrm{OP}S^{-\sigma}$. Note that $BB^{-1}\psi = B^{-1}B\psi = \psi - \psi_0$. However, the restriction $B|_{\mathcal{H}^0}$ of B to the phase space (1.1.5) is invertible (since the phase space contains only functions with zero average) and B^{-1} is its inverse.

3.1.2 Matrix representation and operator matrices

For the KAM reducibility, a second and wider class of operators without a pseudodifferential structure is needed on the scale of Hilbert spaces $(\mathcal{H}^r)_{r\in\mathbb{R}}$, as defined as in (1.1.5). Moreover, let $\mathcal{H}^\infty := \cap_{r\in\mathbb{R}}\mathcal{H}^r$ and $\mathcal{H}^{-\infty} := \cup_{r\in\mathbb{R}}\mathcal{H}^r$. If A is a linear operator, we denote by A^* the adjoint of A with respect to the scalar product of $\mathcal{H}^0$, while we denote by $\overline{A}$ the conjugate operator: $\overline{A}\psi := \overline{A\overline{\psi}}\ \forall\psi \in D(A)$.

Matrix representation of operators. To any linear operator $A\colon \mathcal{H}^\infty \to \mathcal{H}^{-\infty}$ we associate its matrix of coefficients $(A_m^{m'})_{m,m'\in\mathbb{N}}$ on the basis $(\hat{\mathbf{e}}_m := \sin(mx))_{m\in\mathbb{N}}$, defined for $m, m' \in \mathbb{N}$ as

$$A_m^{m'} := \langle A\hat{\mathbf{e}}_{m'}, \hat{\mathbf{e}}_m\rangle_{\mathcal{H}^0} \ .$$

Remark 3.13. If A is a bounded operator, the following implications hold:

$$A = A^* \iff A_m^{m'} = \overline{A_{m'}^m} \quad \forall\, m, m' \in \mathbb{N} \ ;$$
$$\overline{A} = A^* \iff A_m^{m'} = A_{m'}^m \quad \forall\, m, m' \in \mathbb{N} \ .$$

A useful norm we can put on the space of such operators is in the following:

Definition 3.14. Given a linear operator $A\colon \mathcal{H}^\infty \to \mathcal{H}^{-\infty}$ and $s \in \mathbb{R}$, we say that A has finite s-decay norm provided

$$|A|_s := \Big(\sum_{h\in\mathbb{N}_0} \langle h\rangle^{2s} \sup_{|m-m'|=h} |A_m^{m'}|^2 \Big)^{1/2} < \infty \ . \tag{3.1.7}$$

One has the following:

Lemma 3.15 (Algebra of the s-decay)**.** *For any $s > \frac{1}{2}$ there is a constant $C_s > 0$ such that*

$$|AB|_s \leqslant C_s\, |A|_s\, |B|_s \,. \tag{3.1.8}$$

The proof of the Lemma is an easy variant of the one in [33] we sketch it in Appendix A.3.

Remark 3.16. If $A : \mathcal{H}^\infty \to \mathcal{H}^{-\infty}$ has finite s-decay norm with $s > \frac{1}{2}$, then for any $r \in [0, s]$, A extends to a bounded operator $\mathcal{H}^r \to \mathcal{H}^r$. Moreover, by tame estimates, one has the quantitative bound $\|A\|_{\mathcal{L}(\mathcal{H}^r)} \leqslant C_{r,s}|A|_s$.

Next, we consider operators depending analytically on angles $\varphi \in \mathbb{T}^\nu$.

Definition 3.17. Let A be a φ-depending operator, $A\colon \mathbb{T}^\nu \to \mathcal{L}(\mathcal{H}^\infty, \mathcal{H}^{-\infty})$. Given $s \geqslant 0$ and $\rho > 0$, we say that $A \in \mathcal{M}_{\rho,s}$ if one has

$$|A|_{\rho,s} := \sum_{\ell\in\mathbb{Z}^\nu} e^{\rho|\ell|} |\widehat{A}(\ell)|_s < \infty \ , \quad \text{where} \quad \widehat{A}(\ell) := \frac{1}{(2\pi)^\nu} \int_{\mathbb{T}^\nu} A(\varphi)\, e^{-\mathrm{i}\ell\cdot\varphi}\, \mathrm{d}\varphi \ . \tag{3.1.9}$$

Remark 3.18. If A is a φ-depending bounded operator, the following implications hold:

$$A = A^* \iff [\widehat{A}(\ell)]^* = \widehat{A}(-\ell)\ \forall\, \ell \in \mathbb{Z}^\nu \iff \widehat{A}_m^{m'}(\ell) = \overline{\widehat{A}_{m'}^m(-\ell)}\ \forall\, \ell \in \mathbb{Z}^\nu,\ \forall\, m, m' \in \mathbb{N}$$
$$\overline{A} = A^* \iff [\widehat{A}(\ell)]^* = \overline{\widehat{A}(\ell)}\ \forall\, \ell \in \mathbb{Z}^\nu \iff \widehat{A}_m^{m'}(\ell) = \widehat{A}_{m'}^m(\ell)\ \forall\, \ell \in \mathbb{Z}^\nu,\ \forall\, m, m' \in \mathbb{N}$$

If $\Omega \ni \omega \mapsto A(\omega) \in \mathcal{M}_{\rho,s}$ is a Lipschitz map, we write $A \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathcal{M}_{\rho,s})$, provided

$$|A|^{\mathrm{Lip}(\mathtt{w})}_{\rho,s,\Omega} := \sup_{\omega\in\Omega} |A(\omega)|_{\rho,s} + \mathtt{w} \sup_{\omega_1\neq\omega_2\in\Omega} \frac{|A(\omega_1) - A(\omega_2)|_{\rho,s}}{|\omega_1 - \omega_2|} < \infty \ . \tag{3.1.10}$$

Remark 3.19. For any $s > \frac{1}{2}$ and $\rho > 0$, the spaces $\mathcal{M}_{\rho,s}$ and $\mathrm{Lip}_{\mathtt{w}}(\Omega, \mathcal{M}_{\rho,s})$ are closed with respect to composition, with

$$|AB|_{\rho,s} \leqslant C_s \, |A|_{\rho,s} \, |B|_{\rho,s} \,, \qquad |AB|^{\mathrm{Lip}(\mathtt{w})}_{\rho,s,\Omega} \leqslant C_s \, |A|^{\mathrm{Lip}(\mathtt{w})}_{\rho,s,\Omega} \, |B|^{\mathrm{Lip}(\mathtt{w})}_{\rho,s,\Omega} \,.$$

This follows from Lemma 3.15 and the algebra properties for analytic functions.

Operator matrices. We are going to meet matrices of operators of the form

$$\mathbf{A} = \begin{pmatrix} A^d & A^o \\ -\overline{A^o} & -\overline{A^d} \end{pmatrix} \,, \tag{3.1.11}$$

where A^d and A^o are linear operators belonging to the class $\mathcal{M}_{\rho,s}$. Actually, the operator A^d on the diagonal will have different decay properties than the element on the anti-diagonal A^o. Therefore, we introduce classes of operator matrices in which we keep track of these differences.

Definition 3.20. Given an operator matrix $\mathbf{A}$ of the form (3.1.11), $\alpha, \beta \in \mathbb{R}$, $\rho > 0$,$s \geqslant 0$, we say that A belongs to $\mathcal{M}_{\rho,s}(\alpha, \beta)$ if

$$[A^d]^* = A^d \,, \qquad [A^o]^* = \overline{A^o} \tag{3.1.12}$$

and one also has

$$\langle D\rangle^{\alpha} \, A^d \,,\; A^d \, \langle D\rangle^{\alpha} \in \mathcal{M}_{\rho,s} \,, \tag{3.1.13}$$

$$\langle D\rangle^{\beta} \, A^o \,,\; A^o \, \langle D\rangle^{\beta} \in \mathcal{M}_{\rho,s} \,, \tag{3.1.14}$$

$$\langle D\rangle^{\sigma} \, A^{\delta} \, \langle D\rangle^{-\sigma} \in \mathcal{M}_{\rho,s} \,, \quad \forall \sigma \in \{\pm\alpha, \pm\beta, 0\} \,, \quad \forall \delta \in \{d, o\} \,. \tag{3.1.15}$$

We endow $\mathcal{M}_{\rho,s}(\alpha, \beta)$ with the norm

$$\begin{aligned} |\mathbf{A}|^{\alpha,\beta}_{\rho,s} :=& |\langle D\rangle^{\alpha} A^d|_{\rho,s} + |A^d \langle D\rangle^{\alpha}|_{\rho,s} + |\langle D\rangle^{\beta} A^o|_{\rho,s} + |A^o \langle D\rangle^{\beta}|_{\rho,s} \\ &+ \sum_{\substack{\sigma\in\{\pm\alpha,\pm\beta,0\} \\ \delta\in\{d,o\}}} |\langle D\rangle^{\sigma} A^{\delta} \langle D\rangle^{-\sigma}|_{\rho,s} \,, \end{aligned} \tag{3.1.16}$$

with the convention that, in case of repetition (when $\alpha = \beta$, $\alpha = 0$ or $\beta = 0$), the same terms are not summed twice. When $\mathbf{A}$ is independent of $\varphi \in \mathbb{T}^{\nu}$, we use the norm $|\mathbf{A}|^{\alpha,\beta}_{s}$, defined as (3.1.16), but replacing $|\cdot|_{\rho,s}$ with the s-decay norm $|\cdot|_s$ defined in (3.1.7).

Let us motivate the properties describing the class $\mathcal{M}_{\rho,s}(\alpha,\beta)$:

- Condition (3.1.12) is equivalent to ask that $\mathbf{A}$ is the Hamiltonian vector field of a real valued quadratic Hamiltonian, see e.g. [145] for a discussion;
- Conditions (3.1.13) and (3.1.14) control the decay properties for the coefficient of the coefficients of the matrices associated to A^d and A^o: indeed the matrix coefficients of $\langle D\rangle^\alpha A \langle D\rangle^\beta$ are given by

$$[\widehat{\langle D\rangle^\alpha A \langle D\rangle^\beta}]_m^{m'}(k) = \langle m\rangle^\alpha \, \widehat{A}_m^{m'}(k) \, \langle m'\rangle^\beta \ ,$$

  therefore decay (or growth) properties for the matrix coefficients of the operator A are implied by the boundedness of the norms $|\cdot|_{\rho,s}$;
- Condition (3.1.15) is just for simplifying some computations below.

Remark 3.21. Let $0 < \rho' \leqslant \rho$, $0 \leqslant s' \leqslant s$ $\alpha \geqslant \alpha'$, $\beta \geqslant \beta'$. Then $\mathcal{M}_{\rho,s}(\alpha,\beta) \subseteq \mathcal{M}_{\rho',s'}(\alpha',\beta')$ with the quantitative bound $|\mathbf{A}|_{\rho',s'}^{\alpha',\beta'} \leqslant |\mathbf{A}|_{\rho,s}^{\alpha,\beta}$.

Finally, if $A^d(\omega)$ and $A^o(\omega)$ depend in a Lipschitz way on a parameter ω, we introduce the Lipschitz norm

$$|\mathbf{A}|_{\rho,s,\alpha,\beta,\Omega}^{\mathrm{Lip}(\mathtt{w})} := \sup_{\omega\in\Omega} |\mathbf{A}(\omega)|_{\rho,s}^{\alpha,\beta} + \mathtt{w} \sup_{\omega_1\neq\omega_2\in\Omega} \frac{|\mathbf{A}(\omega_1) - \mathbf{A}(\omega_2)|_{\rho,s}^{\alpha,\beta}}{|\omega_1 - \omega_2|} \ . \tag{3.1.17}$$

If such a norm is finite, we write $\mathbf{A} \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathcal{M}_{\rho,s}(\alpha,\beta))$.

Embedding of parity preserving pseudodifferential operators. The introduction of the classes $\mathcal{M}_{\rho,s}(\alpha,\beta)$ is due to the fact that they are closed with respect the KAM reducibility scheme, for a proper choice of α and β. In the next lemma we show how parity preserving pseudodifferential operators embed in such classes.

Lemma 3.22 (Embedding)**.** *Given $\alpha,\beta,\rho > 0$, consider $F \in \mathcal{P}\mathrm{OP}S_\rho^{-\alpha}$ and $G \in \mathcal{P}\mathrm{OP}S_\rho^{-\beta}$. Assume that*

$$F^* = F \ , \qquad G^* = \overline{G} \ ,$$

(where the adjoint is with respect to the scalar product of $\mathcal{H}^0$). Define the operator matrix

$$\mathbf{A} := \begin{pmatrix} F & G \\ -\overline{G} & -\overline{F} \end{pmatrix} \ . \tag{3.1.18}$$

Then, for any $s \geqslant 0$ and $0 < \rho' < \rho$, one has $\mathbf{A} \in \mathcal{M}_{\rho',s}(\alpha,\beta)$. Moreover, there exist $C, c > 0$ such that

$$|\mathbf{A}|_{\rho',s}^{\alpha,\beta} \leqslant \frac{C}{(\rho-\rho')^\nu} \left(\wp_{s+c}^{-\alpha,\rho}(F) + \wp_{s+c}^{-\beta,\rho}(G) \right) . \tag{3.1.19}$$

Finally, if $F \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathcal{P}\mathrm{OP}S^{-\alpha}_{\rho})$*,* $G \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathcal{P}\mathrm{OP}S^{-\beta}_{\rho})$*, one has* $\mathbf{A} \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathcal{M}_{\rho',s}(\alpha, \beta))$ *and* (3.1.19) *holds with the corresponding weighted Lipschitz norms.*

The proof is available in Appendix A.

Commutators and flows. These classes of matrices enjoy also closure properties under commutators and flow generation. We define the adjoint operator

$$\mathrm{ad}_{\mathbf{X}}(\mathbf{V}) := \mathrm{i}[\mathbf{X}, \mathbf{V}] \ ; \tag{3.1.20}$$

note the multiplication by the imaginary unit in the definition of the adjoint map.

Lemma 3.23 (Commutator)**.** *Let* $\alpha, \rho > 0$ *and* $s > \frac{1}{2}$*. Assume* $\mathbf{V} \in \mathcal{M}_{\rho,s}(\alpha, 0)$ *and* $\mathbf{X} \in \mathcal{M}_{\rho,s}(\alpha, \alpha)$*. Then* $\mathrm{ad}_{\mathbf{X}}(\mathbf{V})$ *belongs to* $\mathcal{M}_{\rho,s}(\alpha, \alpha)$ *with the quantitative bound*

$$\left| \mathrm{ad}_{\mathbf{X}}(\mathbf{V}) \right|^{\alpha,\alpha}_{\rho,s} \leqslant 2\, C_s \, |\mathbf{X}|^{\alpha,\alpha}_{\rho,s} \; |\mathbf{V}|^{\alpha,0}_{\rho,s} \ ; \tag{3.1.21}$$

here C_s *is the algebra constant of* (3.1.7)*. Moreover, if* $\mathbf{V} \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathcal{M}_{\rho,s}(\alpha, 0))$ *and* $\mathbf{X} \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathcal{M}_{\rho,s}(\alpha, \alpha))$*, then* $\mathrm{ad}_{\mathbf{X}}(\mathbf{V}) \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathcal{M}_{\rho,s}(\alpha, \alpha))$*, with*

$$|\mathrm{ad}_{\mathbf{X}}(\mathbf{V})|^{\mathrm{Lip}(\mathtt{w})}_{\rho,s,\alpha,\alpha,\Omega} \leqslant 2\, C_s \, |\mathbf{X}|^{\mathrm{Lip}(\mathtt{w})}_{\rho,s,\alpha,\alpha,\Omega} \, |\mathbf{V}|^{\mathrm{Lip}(\mathtt{w})}_{\rho,s,\alpha,0,\Omega} \ . \tag{3.1.22}$$

Also the proof of this lemma is postponed to Appendix A.

Lemma 3.24 (Flow)**.** *Let* $\alpha, \rho > 0$*,* $s > \frac{1}{2}$*. Assume* $\mathbf{V} \in \mathcal{M}_{\rho,s}(\alpha, 0)$*,* $\mathbf{X} \in \mathcal{M}_{\rho,s}(\alpha, \alpha)$*. Then the followings hold true:*

(i) For any $r \in [0, s]$ *and any* $\varphi \in \mathbb{T}^{\nu}$*, the operator* $e^{\mathrm{i}\mathbf{X}(\varphi)} \in \mathcal{L}(\mathbf{H}^r)$*, with the standard operator norm uniformly bounded in* φ*;*

(ii) The operator $e^{\mathrm{i}\mathbf{X}}\, \mathbf{V}\, e^{-\mathrm{i}\mathbf{X}}$ *belongs to* $\mathcal{M}_{\rho,s}(\alpha, 0)$*, while* $e^{\mathrm{i}\mathbf{X}}\, \mathbf{V}\, e^{-\mathrm{i}\mathbf{X}} - \mathbf{V}$ *belongs to* $\mathcal{M}_{\rho,s}(\alpha, \alpha)$ *with the quantitative bounds:*

$$\begin{aligned} \left| e^{\mathrm{i}\mathbf{X}}\, \mathbf{V}\, e^{-\mathrm{i}\mathbf{X}} \right|^{\alpha,0}_{\rho,s} &\leqslant e^{2\, C_s |\mathbf{X}|^{\alpha,\alpha}_{\rho,s}} \, |\mathbf{V}|^{\alpha,0}_{\rho,s} \ ; \\ \left| e^{\mathrm{i}\mathbf{X}}\, \mathbf{V}\, e^{-\mathrm{i}\mathbf{X}} - \mathbf{V} \right|^{\alpha,\alpha}_{\rho,s} &\leqslant 2\, C_s e^{2\, C_s |\mathbf{X}|^{\alpha,\alpha}_{\rho,s}} \, |\mathbf{X}|^{\alpha,\alpha}_{\rho,s} \, |\mathbf{V}|^{\alpha,0}_{\rho,s} \ . \end{aligned} \tag{3.1.23}$$

Analogous assertions hold for $\mathbf{V} \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathcal{M}_{\rho,s}(\alpha, 0))$ *and* $\mathbf{X} \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathcal{M}_{\rho,s}(\alpha, \alpha))$*.*

The proof of this lemma is a standard application of (3.1.21) and the remark that the operator norm is controlled by the $|\cdot|^{\alpha,\alpha}_{\rho,s}$-norm (see also Remark 3.16).

3.2 The Magnus normal form

To begin with, we recall the Pauli matrices notation. Let us introduce

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.2.1}$$

and, moreover, define

$$\boldsymbol{\sigma}_4 := \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \mathbf{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{0} := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Using Pauli matrix notation, equation (1.1.4) reads as

$$\begin{aligned} \mathrm{i}\dot{\psi}(t) =& \mathbf{H}(t)\psi(t) := (\mathbf{H}_0 + \mathbf{W}(\omega t))\psi(t) \ , \\ &\mathbf{H}_0 := B\boldsymbol{\sigma}_3, \quad \mathbf{W}(\omega t) := \frac{1}{2} B^{-1/2} V(\omega t) B^{-1/2} \boldsymbol{\sigma}_4 \ . \end{aligned} \tag{3.2.2}$$

Note that, by assumption (**V1**), one has $V \in \mathcal{P}\mathrm{OP}S^0_\rho$ (see Remark 3.10); therefore the properties of the pseudodifferential calculus and of the associated symbols (see Remarks 3.7 and 3.11) imply that

$$B \in \mathcal{P}\mathrm{OP}S^1 \quad \text{and} \quad B^{-1/2} V B^{-1/2} \in \mathcal{P}\mathrm{OP}S^{-1}_\rho \tag{3.2.3}$$

(in case $\mathtt{m} = 0$, we use Remark 3.12 to define $B^{-1/2}$). The difficulty in treating equation (3.2.2) is that it is not perturbative in the size of the potential, so standard KAM techniques do not apply directly.

To deal with this problem, we perform a change of coordinates, adapted to fast oscillating systems, which puts (3.2.2) in a perturbative setting. We refer to this procedure as Magnus normal form. The Magnus normal form is achieved in the following way: the change of coordinates $\psi(t) = e^{-\mathrm{i}\mathbf{X}(\omega;\omega t)} w(t)$ conjugates (3.2.2) to $\mathrm{i}\partial_t w(t) = \widetilde{\mathbf{H}}(t) w(t)$, where the Hamiltonian $\widetilde{\mathbf{H}}(t)$ is given by (see [20, Lemma 3.2])

$$\widetilde{\mathbf{H}}(t) = e^{-\mathbf{X}(\omega;\omega t)} \mathbf{H}(t) e^{\mathbf{X}(\omega;\omega t)} - \int_0^1 e^{-s\mathbf{X}(\omega;\omega t)} \dot{\mathbf{X}}(\omega;\omega t) e^{s\mathbf{X}(\omega;\omega t)} \, \mathrm{d}s \tag{3.2.4}$$

$$= \mathbf{H}_0 + \mathrm{i}[\mathbf{X}, \mathbf{H}_0] + \mathbf{W} - \dot{\mathbf{X}} + \mathrm{i}[\mathbf{X}, \ldots] \ . \tag{3.2.5}$$

In (3.2.5) we wrote, informally, $[\mathbf{X}, \ldots]$ to remark that all the non written terms are commutators with $\mathbf{X}$. Then one chooses $\mathbf{X}$ to solve $\mathbf{W} - \dot{\mathbf{X}} = 0$; if the frequency ω is large and nonresonant, then $\mathbf{X}$ has size $|\omega|^{-1}$, and the new equation (3.2.5) is now perturbative in size. The price to pay is the appearance of $\mathrm{i}[\mathbf{X}, \mathbf{H}_0]$, which is small in size but possibly unbounded as operator. We control this term by employing pseudodifferential calculus and the properties of the commutators.

With this informal introduction, the main result of the section is the following:

Theorem 3.25 (Magnus normal form). *For any $0 < \gamma_0 < 1$, there exist a set $\Omega_0 \subset R_{\mathtt{M}} \subset \mathbb{R}^\nu$ and a constant $c_0 > 0$ (independent of $\mathtt{M}$), with*

$$\frac{\text{meas}(R_{\mathtt{M}} \backslash \Omega_0)}{\text{meas}(R_{\mathtt{M}})} \leqslant c_0 \gamma_0, \tag{3.2.6}$$

such that the following holds true. For any $\omega \in \Omega_0$ and any weight $\mathtt{w} > 0$, there exists a time dependent change of coordinates $\psi(t) = e^{-\mathrm{i}\mathbf{X}(\omega;\omega t)} w(t)$, where

$$\mathbf{X}(\omega;\omega t) = X(\omega;\omega t)\boldsymbol{\sigma}_4 \ , \qquad X \in \text{Lip}_{\mathtt{w}}(\Omega_0, \mathcal{P}\text{OP}S^{-1}_{\rho/2}) \ ,$$

that conjugates equation (3.2.2) to

$$\mathrm{i}\dot{w}(t) = \widetilde{\mathbf{H}}(t) w(t), \quad \widetilde{\mathbf{H}}(t) := \mathbf{H}_0 + \mathbf{V}(\omega;\omega t) \ , \tag{3.2.7}$$

where

$$\mathbf{V}(\omega;\varphi) = \begin{pmatrix} V^d(\omega;\varphi) & V^o(\omega;\varphi) \\ -\overline{V}^o(\omega;\varphi) & -\overline{V}^d(\omega;\varphi) \end{pmatrix}, \quad \text{with} \quad [V^d]^* = V^d \ , \quad [V^o]^* = \overline{V^o} \tag{3.2.8}$$

and

$$V^d \in \text{Lip}_{\mathtt{w}}(\Omega_0, \mathcal{P}\text{OP}S^{-1}_{\rho/2}) \ , \quad V^o \in \text{Lip}_{\mathtt{w}}(\Omega_0, \mathcal{P}\text{OP}S^{0}_{\rho/2}) \ . \tag{3.2.9}$$

Furthermore, for any $\varrho \in \mathbb{N}_0$, there exists $C_\varrho > 0$ such that

$$\wp_\varrho^{-1,\rho/2}(V^d)^{\text{Lip}(\mathtt{w})}_{\Omega_0} + \wp_\varrho^{0,\rho/2}(V^o)^{\text{Lip}(\mathtt{w})}_{\Omega_0} \leqslant \frac{C_\varrho}{\mathtt{M}}. \tag{3.2.10}$$

Proof. The proof is splitted into two parts, one for the formal algebraic construction, the other for checking that the operators that we have found possess the right pseudodifferential properties we are looking for.

Step I). Expanding (3.2.4) in commutators we have

$$\widetilde{\mathbf{H}}(t) = \mathbf{H}_0 + \mathrm{i}[\mathbf{X}, \mathbf{H}_0] - \tfrac{1}{2}[\mathbf{X}, [\mathbf{X}, \mathbf{H}_0]] + \mathbf{W} - \dot{\mathbf{X}} + \mathbf{R} \ , \tag{3.2.11}$$

where the remainder $\mathbf{R}$ of the expansion is given in integral form by

$$\begin{aligned} \mathbf{R} := & \int_0^1 \frac{(1-s)^2}{2} e^{-s\mathbf{X}} \text{ad}^3_{\mathbf{X}}(\mathbf{H}_0) e^{s\mathbf{X}} \, \mathrm{d}s \\ & + \mathrm{i} \int_0^1 e^{-s\mathbf{X}} [\mathbf{X}, \mathbf{W}] e^{s\mathbf{X}} \, \mathrm{d}s - \mathrm{i} \int_0^1 (1-s) e^{-s\mathbf{X}} [\mathbf{X}, \dot{\mathbf{X}}] e^{s\mathbf{X}} \, \mathrm{d}s. \end{aligned} \tag{3.2.12}$$

From the properties of the Pauli matrices, we note that $\boldsymbol{\sigma}_4^2 = \mathbf{0}$. This means that the terms in (3.2.12) involving $\mathbf{W}$ and $\dot{\mathbf{X}}$ are null, and the remainder is given only by

$$\mathbf{R} = \int_0^1 \frac{(1-s)^2}{2} e^{-s\mathbf{X}} \mathrm{ad}^3_{\mathbf{X}}(\mathbf{H}_0) e^{s\mathbf{X}} \, \mathrm{d}s. \tag{3.2.13}$$

We ask $\mathbf{X}$ to solve the homological equation

$$\mathbf{0} = \mathbf{W} - \dot{\mathbf{X}} = \left(\frac{1}{2} B^{-1/2} V(\omega t) B^{-1/2} - \dot{X}(\omega; \omega t) \right) \boldsymbol{\sigma}_4. \tag{3.2.14}$$

Expanding in Fourier coefficients with respect to the angles, its solution is actually given by

$$\begin{aligned} \hat{X}(\omega; \ell) &= \frac{1}{2\mathrm{i}\, \omega \cdot \ell} B^{-1/2} \hat{V}(\ell) B^{-1/2}, \qquad \text{for } \ell \in \mathbb{Z}^\nu \backslash \{0\}, \\ \hat{X}(\omega; 0) &\equiv \mathbf{0} \end{aligned} \tag{3.2.15}$$

where the second of (3.2.15) is a consequence of (**V2**). It remains to compute the terms in (3.2.4) and (3.2.13) involving $\mathbf{H}_0$. Using again the structure of the Pauli matrices, we get:

$$\mathrm{ad}_{\mathbf{X}}(\mathbf{H}_0) := \mathrm{i}[X\boldsymbol{\sigma}_4, B\boldsymbol{\sigma}_3] = \mathrm{i}XB(\mathbf{1} - \boldsymbol{\sigma}_1) - \mathrm{i}BX(\mathbf{1} + \boldsymbol{\sigma}_1) = \mathrm{i}[X, B]\mathbf{1} - \mathrm{i}[X, B]_{\mathrm{a}}\boldsymbol{\sigma}_1 \ , \tag{3.2.16}$$

where we have denoted by $[X, B]_{\mathrm{a}} := XB + BX$ the anticommutator. Similarly one has

$$\begin{aligned} \mathrm{ad}^2_{\mathbf{X}}(\mathbf{H}_0) &:= -[X\boldsymbol{\sigma}_4, [X\boldsymbol{\sigma}_4, B\boldsymbol{\sigma}_3]] \\ &\overset{(3.2.16)}{=} -([X\boldsymbol{\sigma}_4, [X, B]\mathbf{1}] - [X\boldsymbol{\sigma}_4, [X, B]_{\mathrm{a}}\boldsymbol{\sigma}_1]) \\ &= -([X, [X, B]] - [X, [X, B]_{\mathrm{a}}]_{\mathrm{a}})\boldsymbol{\sigma}_4 \\ &= 4XBX\boldsymbol{\sigma}_4 \ ; \end{aligned} \tag{3.2.17}$$

thus

$$\mathrm{ad}^3_{\mathbf{X}}(\mathbf{H}_0) \overset{(3.2.17)}{=} 4\mathrm{i}[X\boldsymbol{\sigma}_4, XBX\boldsymbol{\sigma}_4] = \mathbf{0} \ . \tag{3.2.18}$$

This shows that $\mathbf{R} \equiv \mathbf{0}$ and, imposing (3.2.15) in (3.2.4), we obtain

$$\tilde{\mathbf{H}}(t) = \mathbf{H}_0 + \mathbf{V}(\omega t; \omega) \ , \tag{3.2.19}$$

with

$$\begin{aligned} V^d(\theta; \omega) &:= \mathrm{i}[X(\theta; \omega), B] + 2X(\theta; \omega)BX(\theta; \omega) \ , \\ V^o(\theta; \omega) &:= -\mathrm{i}[X(\theta; \omega), B]_{\mathrm{a}} + 2X(\theta; \omega)BX(\theta; \omega) \ . \end{aligned} \tag{3.2.20}$$

Step II). We show now that X, V^d and V^o, defined in (3.2.15) and (3.2.20) respectively, are pseudodifferential operators in the proper classes, provided ω is sufficiently nonresonant. First consider **X**. For $\gamma_0 > 0$ and $\tau_0 > \nu - 1$, define the set of Diophantine frequency vectors

$$\Omega_0 := \Omega_0(\gamma_0, \tau_0) := \left\{ \omega \in R_{\mathtt{M}} \, : \, |\omega \cdot \ell| \geqslant \frac{\gamma_0}{\langle \ell \rangle^{\tau_0}} \mathtt{M} \quad \forall \, \ell \in \mathbb{Z}^\nu \backslash \{0\} \right\} . \tag{3.2.21}$$

We will prove in Proposition 3.28 below that

$$\frac{\text{meas}(R_{\mathtt{M}} \backslash \Omega_0)}{\text{meas}(R_{\mathtt{M}})} \leqslant c_0 \gamma_0 \tag{3.2.22}$$

for some constant $c_0 > 0$ independent of $\mathtt{M}$ and γ_0. This fixes the set Ω_0 and proves (3.2.6).
We show now that $X \in \text{Lip}_{\mathtt{w}}(\Omega_0, \mathcal{P}\text{OP}S^{-1}_{\rho/2})$. First note that, by Lemma A.1(i) (in Appendix A) and Remark 3.11, one has $B^{-1/2} \widehat{V}(\ell) B^{-1/2} \in \mathcal{P}\text{OP}S^{-1}$ (both B and V are independent from ω) with

$$\wp^{-1}_{\varrho}(B^{-1/2} \widehat{V}(\ell) B^{-1/2}) \leqslant 4 e^{-\rho |\ell|} \, \wp^{-1,\rho}_{\varrho}(B^{-1/2} V B^{-1/2}) \leqslant 4 e^{-\rho|\ell|} \, C_\varrho .$$

Provided $\omega \in \Omega_0$, it follows that

$$\wp^{-1}_{\varrho}(\widehat{X}(\cdot; \ell))^{\infty}_{\Omega_0} \leqslant \frac{1}{2} \Big[\sup_{\omega \in \Omega_0} \frac{1}{|\omega \cdot \ell|} \Big] \, \wp^{-1}_{\varrho}(B^{-1/2} \widehat{V}(\ell) B^{-1/2}) \leqslant \frac{4 \langle \ell \rangle^{\tau_0}}{\gamma_0 \, \mathtt{M}} e^{-\rho|\ell|} C_\varrho .$$

To compute the Lipschitz norm, it is convenient to use the notation

$$\Delta_\omega f(\omega) = f(\omega + \Delta \omega) - f(\omega) \ , \tag{3.2.23}$$

with ω, $\omega + \Delta\omega \in \Omega_0$, $\Delta \omega \neq 0$. In this way one gets

$$|\Delta_\omega \widehat{X}(\omega; \ell)| \leqslant \frac{|\Delta\omega|}{2 \, |\omega \cdot \ell| \, |(\omega + \Delta\omega) \cdot \ell|} |B^{-1/2} \widehat{V}(\ell) B^{-1/2}| \ \Rightarrow \ \wp^{-1}_{\varrho}(\widehat{X}(\cdot; \ell))^{\text{Lip}}_{\Omega_0} \leqslant \frac{4 \langle \ell \rangle^{2\tau_0}}{(\gamma_0 \mathtt{M})^2} e^{-\rho|\ell|} C_\varrho \ .$$

As a consequence, $X(\omega; \varphi) = \sum_\ell \widehat{X}(\omega; \ell) e^{\mathrm{i} \ell \cdot \varphi}$ is a pseudodifferential operator in the class $\text{Lip}_{\mathtt{w}}(\Omega_0, \mathcal{P}\text{OP}S^{-1}_{\rho/2})$ (see Lemma A.1(ii) in Appendix A for details) fulfilling

$$\wp^{-1,\rho/2}_{\varrho}(X)^{\text{Lip}(\mathtt{w})}_{\Omega_0} \leqslant \Big(\frac{1}{\gamma_0 \mathtt{M}} + \frac{\mathtt{w}}{\gamma_0^2 \mathtt{M}^2} \Big) \frac{C_\varrho}{\rho^{2\tau_0 + \nu}} \leqslant \frac{\max(1, \mathtt{w})}{\mathtt{M}} \, \frac{\widetilde{C}_\varrho}{\rho^{2\tau_0 + \nu}} \ . \tag{3.2.24}$$

It follows by Remark 3.11 that $V^d \in \text{Lip}_{\mathtt{w}}(\Omega_0, \mathcal{P}\text{OP}S^{-1}_{\rho/2})$ and $V^o \in \text{Lip}_{\mathtt{w}}(\Omega_0, \mathcal{P}\text{OP}S^{0}_{\rho/2})$ with the claimed estimates (3.2.10).
Finally, V is a real selfadjoint operator, simply because it is a real bounded potential, and therefore $V^* = V = \overline{V}$. It follows by Remark 3.18 and the explicit expression (3.2.15) that $X^* = X = \overline{X}$. Using these properties one verifies by a direct computation that $[V^d]^* = V^d$ and

$[V^o]^* = V^o$. Estimate (3.2.24) and the symbolic calculus of Remark 3.11 give (3.2.10). □

Remark 3.26. Everything works with the more general assumptions $V \in \mathcal{P}\mathrm{OP}S^0_\rho$.

Remark 3.27. Pseudodifferential calculus is used to guarantee that V^d has order -1 while V^o has order 0 (see (3.2.9)). Without this information it would be problematic to apply the standard KAM iteration of Kuksin [129], which requires the eigenvalues to have an asymptotic of the form $j+\mathcal{O}(j^\delta)$ with $\delta < 0$. In principle one might circumvent this problem by using the ideas of [14, 91] to regularize the order of the perturbation. However, in our context this smoothing procedure is tricky, since it produces terms of size $|\omega|$, which are very large and therefore unacceptable for our purposes.

Proposition 3.28. *For $\gamma_0 > 0$ and $\tau_0 > \nu - 1$, the set Ω_0 defined in* (3.2.21) *fulfills* (3.2.22).

Proof. For any $k \in \mathbb{Z}^\nu \backslash \{0\}$, define the sets $\mathcal{G}^\ell := \{\omega \in R_{\mathtt{M}} \,:\, |\omega \cdot \ell| < \frac{\gamma_0}{\langle \ell \rangle^{\tau_0}} \mathtt{M}\}$. By Lemma 3.30 $|\mathcal{G}^\ell| \lesssim \frac{\gamma_0}{|\ell|^{\tau_0+1}} \mathtt{M}^\nu$. Therefore the set $\mathcal{G} := \bigcup_{\ell \neq 0} \mathcal{G}^\ell$ has measure bounded by $|\mathcal{G}| \leqslant C \gamma_0 \mathtt{M}^\nu$, which proves the claim. □

3.3 Balanced unperturbed Melnikov conditions

As we shall see, in order to perform a converging KAM scheme, we must be able to impose second order Melnikov conditions, namely bounds from below of quantities like $\omega \cdot k + \lambda_i \pm \lambda_{j'}$, where the λ_j's are the eigenvalues of the operator B defined in (1.1.2). Explicitly,

$$\lambda_j := \sqrt{j^2 + \mathtt{m}^2} = j + \frac{c_j(\mathtt{m})}{j} \, , \qquad c_j(\mathtt{m}) := j(\sqrt{j^2 + \mathtt{m}^2} - j). \tag{3.3.1}$$

One can check that $0 \leqslant c_j(\mathtt{m}) \leqslant \mathtt{m}^2$ for any $j \in \mathbb{N}$. We introduce the notation of the indexes sets:

$$\mathcal{I}^+ := \mathbb{Z}^\nu \times \mathbb{N} \times \mathbb{N} \, , \qquad \mathcal{I}^- := \{(\ell, j, j') \in \mathcal{I}^+ \,:\, (\ell, j, j') \neq (0, j, j)\} \, . \tag{3.3.2}$$

Furthermore, we define the relative measure of a measurable set Ω as

$$\mathrm{m}_r(\Omega) := \frac{|\Omega|}{|R_{\mathtt{M}}|} \equiv \frac{|\Omega|}{\mathtt{M}^\nu \, (2^\nu - 1) c_\nu} \tag{3.3.3}$$

where $|\mathcal{C}|$ is the Lebesgue measure of the set $\mathcal{C}$ and c_ν is the volume of the unitary ball in $\mathbb{R}^\nu$. The main result of this section is the following theorem.

Theorem 3.29 (Balanced Melnikov conditions)**.** *Fix $0 \leqslant \alpha \leqslant 1$ and assume that $\mathtt{M} \geqslant \mathtt{M}_0 := \min\{\mathtt{m}^2, \langle \mathtt{m} \rangle^{1/\alpha}\}$ if $\alpha \in [0,1]$. Then, for $0 < \widetilde{\gamma} \leqslant \min\{\gamma_0^{3/2}, 1/8\}$ and $\widetilde{\tau} \geqslant 2\nu + 3$, the set*

$$\mathcal{U}_\alpha := \Big\{ \omega \in \Omega_0 \,:\, |\omega \cdot \ell + \lambda_j \pm \lambda_{j'}| \geqslant \frac{\widetilde{\gamma}}{\langle \ell \rangle^{\widetilde{\tau}}} \frac{\langle j \pm j' \rangle^\alpha}{\mathtt{M}^\alpha} \quad \forall \, (\ell, j, j') \in \mathcal{I}^\pm \Big\} \tag{3.3.4}$$

is of large relative measure, that is

$$\mathrm{m}_r(\Omega_0 \backslash \mathcal{U}_\alpha) \leqslant C\,\widetilde{\gamma}^{1/3}, \tag{3.3.5}$$

where $C > 0$ is independent of $\mathtt{M}$ *and* $\widetilde{\gamma}$.

We will use several times the following standard estimate.

Lemma 3.30. *Fix $\ell \in \mathbb{Z}^\nu \backslash \{0\}$ and let $R_{\mathtt{M}} \ni \omega \mapsto \varsigma(\omega) \in \mathbb{R}$ be a Lipschitz function fulfilling $|\varsigma|^{\mathrm{Lip}}_{R_{\mathtt{M}}} \leqslant \mathtt{c}_0 < |\ell|$. Define $f(\omega) = \omega \cdot \ell + \varsigma(\omega)$. Then, for any $\delta \geqslant 0$, the measure of the set $A := \{\omega \in R_{\mathtt{M}} : |f(\omega)| \leqslant \delta\}$ satisfies the upper bound*

$$|A| \leqslant \frac{2\delta}{|k| - \mathtt{c}_0}(4\mathtt{M})^{\nu-1}\ . \tag{3.3.6}$$

Proof. Take $\omega_1 = \omega + \epsilon\ell$, with ϵ sufficiently small so that $\omega_1 \in R_{\mathtt{M}}$.
Then $\dfrac{|f(\omega_1) - f(\omega)|}{|\omega_1 - \omega|} \geqslant |\ell| - |\varsigma|^{\mathrm{Lip}}_{R_{\mathtt{M}}} \geqslant |\ell| - \mathtt{c}_0$ and the estimate follows by Fubini theorem. □

In the rest of the section we write $a \lesssim b$, meaning that $a \leqslant Cb$ for some numerical constant $C > 0$ independent of the relevant parameters.

The result of Theorem 3.29 is carried out in two steps. The first one is the following lemma.

Lemma 3.31. *Fix $0 \leqslant \alpha \leqslant 1$. There exist $\widetilde{\gamma}_1 > 0$ and $\tau_1 > \nu + \alpha$ such that the set*

$$\mathcal{T}_1 := \Big\{\omega \in \Omega_0 \,:\, |\omega \cdot \ell + j| \geqslant \frac{\widetilde{\gamma}_1}{\langle \ell \rangle^{\tau_1}} \frac{\langle j \rangle^\alpha}{\mathtt{M}^\alpha} \quad \forall\, (\ell, j) \in \mathbb{Z}^{\nu+1} \backslash \{0\}\Big\} \tag{3.3.7}$$

has relative measure $\mathrm{m}_r(\Omega_0 \backslash \mathcal{T}_1) \leqslant C_1 \widetilde{\gamma}_1$, where $C_1 > 0$ is independent of $\mathtt{M}$ *and* $\widetilde{\gamma}_1$.

Proof. If $\ell = 0$ and $j \neq 0$, the estimate in (3.3.7) holds. The same is true if $\ell \neq 0$ and $j = 0$. Therefore, let both ℓ and j be different from zero. For $|j| > 4\mathtt{M}\,|\ell|$, the inequality in (3.3.7) holds true taking $\widetilde{\gamma}_1 \leqslant \frac{1}{2}$. Indeed:

$$|\omega \cdot \ell + j| \geqslant |j| - |\omega|\,|\ell| \geqslant |j| - 2\mathtt{M}\,|\ell| \geqslant \frac{|j|}{2} \geqslant \frac{1}{2}\,|j|^\alpha \geqslant \frac{\widetilde{\gamma}_1}{\langle \ell \rangle^{\tau_1}\,\mathtt{M}^\alpha}\,|j|^\alpha\ .$$

Then, consider the case $1 \leqslant |j| \leqslant 4\mathtt{M}\,|\ell|$ (so, only a finite number of $\ell \in \mathbb{Z}\backslash\{0\}$). For fixed ℓ and j, define the set

$$\mathcal{G}^\ell_j := \Big\{\omega \in R_{\mathtt{M}} \,:\, |\omega \cdot \ell + j| \leqslant \frac{\widetilde{\gamma}_1}{\langle \ell \rangle^{\tau_1}} \frac{|j|^\alpha}{\mathtt{M}^\alpha}\Big\}\,. \tag{3.3.8}$$

By Lemma 3.30, the measure of each set can be estimated by

$$|\mathcal{G}^\ell_j| \lesssim \mathtt{M}^{\nu-1} \frac{\widetilde{\gamma}_1}{\langle \ell \rangle^{\tau_1}} \frac{|j|^\alpha}{\mathtt{M}^\alpha} \frac{1}{|\ell|} \lesssim \widetilde{\gamma}_1 \mathtt{M}^{\nu-1-\alpha} \frac{|j|^\alpha}{\langle \ell \rangle^{\tau_1+1}}\,. \tag{3.3.9}$$

Let $\mathcal{G}_1 := \Omega_0 \cap \bigcup\{\mathcal{G}_j^\ell \,:\, (\ell, j) \in \mathbb{Z}^{\nu+1}\backslash\{0\},\ |j| \leqslant 4\mathtt{M}\,|\ell|\}$. Then

$$\begin{aligned} |\mathcal{G}_1| &\leqslant \sum_{\ell\in\mathbb{Z}^\nu\backslash\{0\}} \sum_{\substack{j\in\mathbb{Z}\backslash\{0\}\\ |j|\leqslant 4\mathtt{M}|\ell|}} \left|\mathcal{G}_j^\ell\right| \overset{(3.3.9)}{\lesssim} \widetilde{\gamma}_1 \mathtt{M}^{\nu-1-\alpha} \sum_{\ell\neq 0} \sum_{|j|\leqslant 4\mathtt{M}|\ell|} \frac{|j|^\alpha}{\langle \ell\rangle^{\tau_1+1}} \\ &\lesssim \widetilde{\gamma}_1 \mathtt{M}^{\nu-1-\alpha} \sum_{\ell\neq 0} \frac{1}{\langle\ell\rangle^{\tau_1+1}} (4\mathtt{M}\,|\ell|)^{\alpha+1} \lesssim \widetilde{\gamma}_1 \mathtt{M}^\nu \sum_{\ell\neq 0} \frac{1}{\langle \ell\rangle^{\tau_1-\alpha}} \lesssim \widetilde{\gamma}_1\mathtt{M}^\nu \end{aligned} \tag{3.3.10}$$

provided $\tau_1 > \nu + \alpha$. It follows that the relative measure of $\mathcal{G}_1$ is given by

$$\mathrm{m}_r(\mathcal{G}_1) \leqslant C_1 \widetilde{\gamma}_1 \ , \tag{3.3.11}$$

where $C_1 > 0$ is independent of $\mathtt{M}$ and $\widetilde{\gamma}_1$. The thesis follows, since $\mathcal{T}_1 = \Omega_0 \backslash \mathcal{G}_1$. □

Remark 3.32. In case $\mathtt{m} = 0$, Lemma 3.31 implies Theorem 3.29.

From now on assume that $\mathtt{m} > 0$. The second step is the next lemma.

Lemma 3.33. *There exist* $0 < \widetilde{\gamma}_2 \leqslant \min\{\gamma_0, \widetilde{\gamma}_1/2\}$ *and* $\tau_2 \geqslant \tau_1 + \nu + 1$ *such that the set*

$$\mathcal{T}_2 := \left\{\omega \in \mathcal{T}_1 \,:\, |\omega\cdot\ell + \lambda_j \pm \lambda_{j'}| \geqslant \frac{\widetilde{\gamma}_2}{\langle\ell\rangle^{\tau_2}} \frac{\langle j \pm j'\rangle^\alpha}{\mathtt{M}^\alpha} \quad \forall\,(\ell, j, j') \in \mathcal{I}^\pm \right\} \tag{3.3.12}$$

fulfills $\mathrm{m}_r(\mathcal{T}_1\backslash\mathcal{T}_2) \leqslant C_2 \dfrac{\widetilde{\gamma}_2}{\widetilde{\gamma}_1}$, *where* $C_2 > 0$ *is independent of* $\mathtt{M}$, $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$.

Proof. Let $(\ell, j, j') \in \mathcal{I}^\pm$. We can rule out some cases for which the inequality in (3.3.12) is already satisfied when $\omega \in \mathcal{T}_1 \subset \Omega_0$:

- For $\pm = +$ and $\ell = 0$, we have

$$\lambda_j + \lambda_{j'} = j + j' + \frac{c_j(\mathtt{m})}{j} + \frac{c_{j'}(\mathtt{m})}{j'} \geqslant j + j' \geqslant \frac{\widetilde{\gamma}_2}{\mathtt{M}^\alpha} \langle j + j'\rangle^\alpha \ ;$$

- For $\pm = -$ and $\ell \neq 0$, $j = j'$, we have $|\omega\cdot\ell| \geqslant \dfrac{\gamma_0}{\langle\ell\rangle^{\tau_0}}\mathtt{M}$;

- For $\pm = -$ and $\ell = 0$, $j \neq j'$, $\alpha \in (0, 1]$, it holds that

$$|\lambda_j - \lambda_{j'}| = \left| \int_{j'}^{j} \frac{x}{\sqrt{x^2 + \mathtt{m}^2}}\,\mathrm{d}x \right| \geqslant \frac{1}{\langle\mathtt{m}\rangle} |j' - j| \geqslant \frac{\widetilde{\gamma}_2}{\mathtt{M}^\alpha} \langle j - j'\rangle^\alpha \ .$$

  When $\alpha = 0$, the estimate is trivially verified.

Therefore, for the rest of this argument, let $\ell \neq 0$ and $j \neq j'$. Assume first that $|j \pm j'| \geqslant 8\mathtt{M}\,|\ell|$.

In this case, one has:

$$\left|\omega \cdot \ell + \lambda_j \pm \lambda_{j'}\right| \geqslant \left|j \pm j'\right| - \left|\frac{c_j(\mathtt{m})}{j} \pm \frac{c_{j'}(\mathtt{m})}{j'}\right| - |\omega \cdot \ell| \geqslant \left|j \pm j'\right| - 4\mathtt{M}\,|\ell| \geqslant \frac{1}{2}\left|j \pm j'\right| \ .$$

Let now $|j \pm j'| < 8\mathtt{M}\,|\ell|$. In the region $j < j'$ assume

$$j\,\langle j \pm j'\rangle^{\alpha} \geqslant \mathtt{R}(\ell) := \frac{4\mathtt{m}^2\mathtt{M}^{\alpha}\,\langle \ell\rangle^{\tau_1}}{\widetilde{\gamma}_1}\ , \tag{3.3.13}$$

where $\widetilde{\gamma}_1$ and τ_1 are the ones of Lemma 3.31. So, for $\omega \in \mathcal{T}_1$, we get

$$\begin{aligned} \left|\omega \cdot \ell + \lambda_j \pm \lambda_{j'}\right| &\geqslant \left|\omega \cdot \ell + j \pm j'\right| - \left|\frac{c_j(\mathtt{m})}{j} \pm \frac{c_{j'}(\mathtt{m})}{j'}\right| \\ &\geqslant \frac{\widetilde{\gamma}_1}{\langle \ell\rangle^{\tau_1}}\frac{\langle j \pm j'\rangle^{\alpha}}{\mathtt{M}^{\alpha}} - \frac{2\mathtt{m}^2}{j} \overset{(3.3.13)}{\geqslant} \frac{\widetilde{\gamma}_1}{2\,\langle \ell\rangle^{\tau_1}}\frac{\langle j \pm j'\rangle^{\alpha}}{\mathtt{M}^{\alpha}}. \end{aligned} \tag{3.3.14}$$

Thus, we consider just those j and j' with $j\,\langle j \pm j'\rangle^{\alpha} < \mathtt{R}(\ell)$. The symmetric argument shows that we can take those $j' < j$ for which $j'\,\langle j \pm j'\rangle^{\alpha} < \mathtt{R}(\ell)$.
Like in the proof of Lemma 3.31, consider the set

$$\mathcal{G}_{j,j'}^{\ell,\pm} := \left\{\omega \in R_{\mathtt{M}}\ :\ |\omega \cdot \ell + \lambda_j \pm \lambda_{j'}| < \frac{\widetilde{\gamma}_2}{\langle \ell\rangle^{\tau_2}}\frac{\langle j \pm j'\rangle^{\alpha}}{\mathtt{M}^{\alpha}}\right\} \tag{3.3.15}$$

defined for those $\ell \neq 0$ and $j \neq j'$ in the regions

$$\mathcal{P}^{\pm} := \{\left|j \pm j'\right| < 8\mathtt{M}\,|\ell|\} \cap \Big(\{j\,\langle j \pm j'\rangle^{\alpha} < \mathtt{R}(\ell),\ j < j'\} \cup \{j'\,\langle j \pm j'\rangle^{\alpha} < \mathtt{R}(\ell),\ j' < j\}\Big)\ . \tag{3.3.16}$$

Using Lemma 3.30, the estimate for its Lebesgue measure is

$$|\mathcal{G}_{j,j'}^{\ell,\pm}| \lesssim \widetilde{\gamma}_2\mathtt{M}^{\nu-1-\alpha}\frac{\langle j \pm j'\rangle^{\alpha}}{|\ell|^{\tau_2+1}}\ . \tag{3.3.17}$$

Define $\mathcal{G}_2^{\pm} := \mathcal{T}_1 \cap \bigcup\{\mathcal{G}_{j,j'}^{\ell,\pm} \, : \, (\ell, j, j') \in \mathcal{P}^{\pm}\}$. By symmetry of the summand, we estimate

$$
\begin{aligned}
|\mathcal{G}_2^-| &\leqslant \sum_{(\ell,j,j')\in\mathcal{P}^-} |\mathcal{G}_{j,j'}^{\ell,-}| \overset{(3.3.17)}{\lesssim} \widetilde{\gamma}_2 \mathtt{M}^{\nu-1-\alpha} \sum_{(\ell,j,j')\in\mathcal{P}^-} \frac{\langle j-j'\rangle^{\alpha}}{|\ell|^{\tau_2+1}} \\
&\lesssim \widetilde{\gamma}_2 \mathtt{M}^{\nu-1-\alpha} \sum_{\ell\neq 0} \sum_{\substack{j<j' \\ j\langle j-j'\rangle^{\alpha} < \mathtt{R}(\ell)}} \sum_{|j-j'|<8\mathtt{M}|\ell|} \frac{\langle j-j'\rangle^{\alpha}}{|\ell|^{\tau_2+1}} \\
&\lesssim \widetilde{\gamma}_2 \mathtt{M}^{\nu-1-\alpha} \sum_{\ell\neq 0} \sum_{\substack{j'-j=:h>0 \\ h<8\mathtt{M}|\ell|}} \sum_{j<\mathtt{R}(\ell)\langle h\rangle^{-\alpha}} \frac{\langle h\rangle^{\alpha}}{|\ell|^{\tau_2+1}} \\
&\overset{(3.3.13)}{\lesssim} \frac{\widetilde{\gamma}_2}{\widetilde{\gamma}_1} \mathtt{M}^{\nu-1} \sum_{\ell\neq 0} \sum_{h<8\mathtt{M}|\ell|} \frac{1}{|\ell|^{\tau_2+1-\tau_1}} \lesssim \frac{\widetilde{\gamma}_2}{\widetilde{\gamma}_1} \mathtt{M}^{\nu} \sum_{\ell\neq 0} \frac{1}{|\ell|^{\tau_2-\tau_1}} \leqslant \frac{\widetilde{\gamma}_2}{\widetilde{\gamma}_1} \mathtt{M}^{\nu}
\end{aligned}
\tag{3.3.18}
$$

provided $\tau_2 > \tau_1 + \nu$. The same computation holds for $\mathcal{G}_2^+$. We conclude that

$$
\mathrm{m}_r(\mathcal{T}_1 \backslash \mathcal{T}_2) \leqslant \mathrm{m}_r(\mathcal{G}_2^- \cap \mathcal{G}_2^+) \leqslant C_2 \frac{\widetilde{\gamma}_2}{\widetilde{\gamma}_1} \, , \tag{3.3.19}
$$

where $C_2 > 0$ is independent of $\mathtt{M}$, $\widetilde{\gamma}_1, \widetilde{\gamma}_2$. □

Proof of Theorem 3.29. Take $\widetilde{\gamma}_1 = \widetilde{\gamma}^{1/3}$, $\widetilde{\gamma}_2 = \widetilde{\gamma}^{2/3}$ with some $\widetilde{\gamma} > 0$ sufficiently small so that $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ fulfill the assumptions of the previous lemmas. Similarly, choose $\tau_1 = \nu + 2$ and $\tau_2 = 2\nu + 3$. By definition, $\mathcal{U}_\alpha = \mathcal{T}_2 \subset \Omega_0$. Since $\Omega_0 \backslash \mathcal{U}_\alpha = (\Omega_0 \backslash \mathcal{T}_1) \cup (\mathcal{T}_1 \backslash \mathcal{T}_2)$, we get by Lemma 3.31 and Lemma 3.33 that

$$
\mathrm{m}_r(\Omega_0 \backslash \mathcal{U}_\alpha) \leqslant C_1 \widetilde{\gamma}_1 + C_2 \frac{\widetilde{\gamma}_2}{\widetilde{\gamma}_1} \leqslant C \widetilde{\gamma}^{1/3} \, , \qquad C = 2\,(C_1 + C_2) \, .
$$

□

3.4 The KAM reducibility transformation

The new potential $\mathbf{V}(\omega; \omega t)$ that we have found in Theorem 3.25 is perturbative, in the sense that the smallness of its norm is controlled by the size $\mathtt{M}$ of the frequency vector ω. Thus, we are now ready to attack with a KAM reduction scheme in analytical regularity, presenting first the algebraic construction of the single iteration, then quantifying it via the norms and seminorms that we have introduced in Section 3.1. The complete result for this reduction transformation, together with its iterative lemma, is proved at the end of this section.

3.4.1 Preparation for the KAM iteration

For the KAM scheme it is more convenient to work with operators of type $\mathcal{M}_{\rho,s}$. Of course, as we have seen in Section 3.1, pseudodifferential operators analytic in φ belong to such a class.

Lemma 3.34. *Fix an arbitrary $s_0 > 1/2$ and put $\rho_0 := \rho/4$. Then the operator $\mathbf{V}(\omega)$ defined in* (3.2.8) *belongs to* $\mathrm{Lip}_{\mathtt{w}}(\Omega_0, \mathcal{M}_{\rho_0,s_0}(1,0))$ *with the quantitative bound*

$$|\mathbf{V}|^{\mathrm{Lip}(\mathtt{w})}_{\rho_0,s_0,1,0,\Omega_0} \leqslant \frac{C}{\mathtt{M}}; \tag{3.4.1}$$

here $C > 0$ is independent of $\mathtt{M}$.

Proof. It is sufficient to apply the embedding Lemma 3.22 and (3.2.10). □

3.4.2 General step of the reduction

Consider the system

$$\mathrm{i}\dot{w}(t) = \mathbf{H}(t)w(t), \quad \mathbf{H}(t) := \mathbf{A}(\omega) + \mathbf{P}(\omega;\omega t), \tag{3.4.2}$$

where the frequency vector ω varies in some set $\Omega \subset \mathbb{R}^\nu$, $\mathtt{M} \leqslant |\omega| \leqslant 2\mathtt{M}$; the time-independent operator $\mathbf{A}(\omega)$ is diagonal, with

$$\mathbf{A}(\omega) = \begin{pmatrix} A(\omega) & 0 \\ 0 & -A(\omega) \end{pmatrix}, \quad A(\omega) := \mathrm{diag}\{\lambda_j^-(\omega) \,:\, j \in \mathbb{N}\} \subset (0,\infty)^{\mathbb{N}} ; \tag{3.4.3}$$

and the quasi-periodic perturbation $\mathbf{P}(\omega;\omega t)$ has the form

$$\mathbf{P}(\omega;\omega t) = \begin{pmatrix} P^d(\omega;\omega t) & P^o(\omega;\omega t) \\ -\overline{P^o}(\omega;\omega t) & -\overline{P^d}(\omega;\omega t) \end{pmatrix}, \quad P^d = [P^d]^* , \quad \overline{P^o} = [P^o]^* . \tag{3.4.4}$$

The goal is to square the size of the perturbation (see Lemma 3.37) and we do it by conjugating the Hamiltonian $\mathbf{H}(t)$ through a transformation $w := e^{-\mathrm{i}\mathbf{X}^+(\omega;\omega t)}z$ of the form

$$\mathbf{X}^+(\omega;\omega t) = \begin{pmatrix} X^d(\omega;\omega t) & X^o(\omega;\omega t) \\ -\overline{X^o}(\omega;\omega t) & -\overline{X^d}(\omega;\omega t) \end{pmatrix} , \quad X^d = [X^d]^*, \ \ \overline{X^o} = [X^o]^* , \tag{3.4.5}$$

so that the transformed Hamiltonian, as in (3.2.4), is

$$\mathbf{H}^+(t) := e^{-\mathbf{X}^+(\omega;\omega t)}\mathbf{H}(t)e^{\mathbf{X}^+(\omega;\omega t)} - \int_0^1 e^{-s\mathbf{X}^+(\omega;\omega t)}\dot{\mathbf{X}}^+(\omega;\omega t)e^{s\mathbf{X}^+(\omega;\omega t)}\,\mathrm{d}s . \tag{3.4.6}$$

Its expansion in commutators is given by

$$\begin{aligned}\mathbf{H}^+(t) &= \mathbf{A} + \mathbf{P} + \mathrm{i}[\mathbf{X}^+, \mathbf{A}] - \dot{\mathbf{X}}^+ + \mathbf{R},\\ \mathbf{R} &:= e^{-\mathbf{X}^+}\mathbf{A}e^{\mathbf{X}^+} - (\mathbf{A} + \mathrm{i}[\mathbf{X}^+, \mathbf{A}]) + e^{-\mathbf{X}^+}\mathbf{P}e^{\mathbf{X}^+} - \mathbf{P} - \left(\int_0^1 e^{-s\mathbf{X}^+}\dot{\mathbf{X}}^+ e^{s\mathbf{X}^+}\,\mathrm{d}s - \dot{\mathbf{X}}^+\right) .\end{aligned} \tag{3.4.7}$$

We ask now $\mathbf{X}^+$ to solve the "quantum" homological equation:

$$\mathrm{i}[\mathbf{X}^+(\varphi), \mathbf{A}] - \omega \cdot \partial_v f \mathbf{X}^+(\varphi) + \Pi_N \mathbf{P}(\varphi) = \mathbf{Z} \tag{3.4.8}$$

where $\Pi_N \mathbf{P}(\omega;\varphi) := \sum_{|\ell|\leqslant N} \widehat{P}(\omega;\ell)e^{\mathrm{i}\ell\cdot\varphi}$ is the projector on the frequencies smaller than N, while $\mathbf{Z}$ is the diagonal, time independent part of P^d:

$$\mathbf{Z} = \mathbf{Z}(\omega) := \begin{pmatrix} Z(\omega) & 0 \\ 0 & -Z(\omega)\end{pmatrix} , \quad Z = \mathrm{diag}\{\widehat{(P^d)}_j^j(\omega;0) \;:\; j \in \mathbb{N}\} . \tag{3.4.9}$$

With this choice, the new Hamiltonian becomes $\mathbf{H}(t)^+ = \mathbf{A}^+ + \mathbf{P}(\omega t)^+$ with

$$\mathbf{A}^+ = \mathbf{A} + \mathbf{Z}, \qquad \mathbf{P}^+ := \Pi_N^\perp \mathbf{P} + \mathbf{R} , \qquad \Pi_N^\perp \mathbf{P} := (\mathbb{1} - \Pi_N)\mathbf{P}. \tag{3.4.10}$$

In order to solve equation (3.4.8), note that it reads block-wise as

$$\begin{cases} \mathrm{i}[X^d, A] - \omega \cdot \partial_\theta X^d + P^d = Z \\ -\mathrm{i}[X^o, A]_{\mathrm{a}} - \omega \cdot \partial_\theta X^o + P^o = 0 \end{cases} . \tag{3.4.11}$$

Expanding both with respect to the exponential basis of B (for the space) and in Fourier in angles (for the time), we get the solutions

$$\widehat{(X^d)}_j^{j'}(\omega;\ell) := \begin{cases} \dfrac{1}{\mathrm{i}(\omega\cdot\ell + \lambda_j^-(\omega) - \lambda_{j'}^-(\omega))}\widehat{(P^d)}_j^{j'}(\omega;\ell) & (\ell, j, j') \in \mathcal{I}_N^- \\ 0 & \text{otherwise} \end{cases} , \tag{3.4.12}$$

$$\widehat{(X^o)}_j^{j'}(\omega;\ell) := \begin{cases} \dfrac{1}{\mathrm{i}(\omega\cdot\ell + \lambda_j^-(\omega) + \lambda_{j'}^-(\omega))}\widehat{(P^o)}_j^{j'}(\omega;\ell) & (\ell, j, j') \in \mathcal{I}_N^+ \\ 0 & \text{otherwise} \end{cases} , \tag{3.4.13}$$

where, following the notation in (3.3.2), we have defined

$$\mathcal{I}_N^\pm := \{(\ell, j, j') \in \mathcal{I}^\pm \;:\; |\ell| \leqslant N\} . \tag{3.4.14}$$

Remark that $A^+(\omega) = \mathrm{diag}\{\lambda_j^+(\omega) \;:\; j \in \mathbb{N}\}$ with $\lambda_j^+(\omega) := \lambda_j^-(\omega) + \widehat{(P^d)}_j^j(\omega;0)$.

3.4.3 Estimates for the general step

Both for well-posing the solutions (3.4.12) and (3.4.13) and ensuring convergence of the norms, second order Melnikov conditions are required to be imposed. In particular, we choose the frequency vector from the following set

$$\Omega^{+} := \left\{ \omega \in \Omega \, : \, |\omega \cdot \ell + \lambda_j^{-}(\omega) \pm \lambda_{j'}^{-}(\omega)| \geqslant \frac{\gamma}{2\langle N\rangle^{\tau}} \frac{\langle j \pm j'\rangle^{\alpha}}{\mathtt{M}^{\alpha}} \, , \, \forall\, (\ell, j, j') \in \mathcal{I}_N^{\pm} \right\} \tag{3.4.15}$$

with $\gamma, \tau > 0$ to be fixed later on. Here $\mathcal{I}_N^{\pm}$ has been defined in (3.4.14).
The fact that Ω^{+} is actually a set of large measure, that is $\mathrm{m}_r(\Omega \backslash \Omega^{+}) = O(\gamma)$, will be clear as a direct consequence of Lemma 3.42 of Section 3.4.4.
From now on, we choose as Lipschitz weight $\mathtt{w} := \gamma/\mathtt{M}^{\alpha}$ and, abusing notation, we denote

$$\mathrm{Lip}_{\gamma}(\Omega, \mathcal{F}) := \mathrm{Lip}_{\gamma/\mathtt{M}^{\alpha}}(\Omega, \mathcal{F}) \, .$$

Furthermore, **we fix once for all** $s_0 > 1/2$ **and** $\alpha \in (0,1)$.
For $\mathbf{V} \in \mathrm{Lip}_{\gamma}(\Omega, \mathcal{M}_{\rho,s_0}(\alpha, 0))$, we write

$$|\mathbf{V}| := |\mathbf{V}|_{s_0}^{\alpha,0} \; , \qquad |\mathbf{V}|_{\rho} := |\mathbf{V}|_{\rho,s_0}^{\alpha,0} \, , \qquad |\mathbf{V}|_{\rho,\Omega}^{\mathrm{Lip}(\gamma)} := |\mathbf{V}|_{\rho,s_0,\alpha,0,\Omega}^{\mathrm{Lip}(\gamma/\mathtt{M}^{\alpha})} \equiv |\mathbf{V}|_{\rho,\Omega}^{\infty} + \frac{\gamma}{\mathtt{M}^{\alpha}} |\mathbf{V}|_{\rho,\Omega}^{\mathrm{Lip}} \; ,$$

whereas, for $\mathbf{V} \in \mathrm{Lip}_{\gamma}(\Omega, \mathcal{M}_{\rho,s_0}(\alpha, \alpha))$, we denote

$$|||\mathbf{V}|||_{\rho} := |\mathbf{V}|_{\rho,s_0}^{\alpha,\alpha} \; , \qquad |||\mathbf{V}|||_{\rho,\Omega}^{\mathrm{Lip}(\gamma)} := |\mathbf{V}|_{\rho,s_0,\alpha,\alpha,\Omega}^{\mathrm{Lip}(\gamma/\mathtt{M}^{\alpha})} := |||\mathbf{V}|||_{\rho,\Omega}^{\infty} + \frac{\gamma}{\mathtt{M}^{\alpha}} |||\mathbf{V}|||_{\rho,\Omega}^{\mathrm{Lip}} \, .$$

Remark 3.35. Note that $|\mathbf{V}|_{\rho_0,\Omega_0}^{\mathrm{Lip}(\gamma)} \leqslant |||\mathbf{V}|||_{\rho_0,\Omega_0}^{\mathrm{Lip}(\gamma)}$.

Now, we provide the estimate on the generator $\mathbf{X}^{+}$ of the previous transformation. For sake of simplicity during the forthcoming proof, as short notation we define

$$\mathrm{g}_{j,j'}^{\ell,\pm}(\omega) := \omega \cdot \ell + \lambda_j^{-}(\omega) \pm \lambda_{j'}^{-}(\omega) \, , \quad (\ell, j, j') \in \mathcal{I}_N^{\pm} \, . \tag{3.4.16}$$

Lemma 3.36. *Assume that:*

(a) $\mathbf{P} \in \mathrm{Lip}_{\gamma}(\Omega, \mathcal{M}_{\rho,s_0}(\alpha, 0))$, *with an arbitrary* $\rho > 0$*;*

(b) There exists $0 < \mathtt{C} \leqslant 1$ *such that for any* $j \in \mathbb{N}$, $\omega, \Delta\omega \in \Omega^{+}$ *one has*

$$|\Delta_{\omega} \lambda_j^{-}(\omega)| \leqslant \mathtt{C} \, |\Delta\omega| \, . \tag{3.4.17}$$

Let $\mathbf{X}^{+} = \mathbf{X}^{+}(\omega; \omega t)$ *be defined by* (3.4.12) *and* (3.4.13)*. Then* $\mathbf{X}^{+} \in \mathrm{Lip}_{\gamma}(\Omega^{+}, \mathcal{M}_{\rho,s_0}(\alpha, \alpha))$ *with* *the quantitative bound*

$$\|\|\mathbf{X}^+\|\|_{\rho,\Omega^+}^{\mathrm{Lip}(\gamma)} \leqslant 16\,\langle N\rangle^{2\tau+1}\,\frac{\mathtt{M}^\alpha}{\gamma}\,|\mathbf{P}|_{\rho,\Omega}^{\mathrm{Lip}(\gamma)}\,. \tag{3.4.18}$$

Proof. We start with the seminorm $\|\|\mathbf{X}^+\|\|_{\rho,\Omega^+}^{\infty}$. Fix $\omega \in \Omega^+$ and $|\ell| \leqslant N$. Then, when $j \neq j'$, we have

$$|\widehat{(X^d)_j^{j'}}(\omega;\ell)| \leqslant \frac{1}{|\mathtt{g}_{j,j'}^{\ell,-}(\omega)|}|\widehat{(P^d)_j^{j'}}(\omega;\ell)| \leqslant \frac{2\,\langle N\rangle^\tau\,\mathtt{M}^\alpha}{\gamma}\frac{|\widehat{(P^d)_j^{j'}}(\omega;\ell)|}{\langle j-j'\rangle^\alpha} \tag{3.4.19}$$

and similarly, for any $j,l \in \mathbb{N}$

$$|\widehat{(X^o)_j^{j'}}(\omega;\ell)| \leqslant \frac{2\,\langle N\rangle^\tau\,\mathtt{M}^\alpha}{\gamma}\frac{|\widehat{(P^o)_j^{j'}}(\omega;\ell)|}{\langle j+j'\rangle^\alpha}\,. \tag{3.4.20}$$

From assumption (a), all the terms $|\,\langle D\rangle^\alpha\,\widehat{P^d}(\omega;\ell)|_{s_0}$, $|\widehat{P^d}(\omega;\ell)\,\langle D\rangle^\alpha\,|_{s_0}$, $|\,\langle D\rangle^\sigma\,\widehat{P^\delta}(\omega;\ell)\,\langle D\rangle^{-\sigma}\,|_{s_0}$ (with $\sigma = \pm\alpha, 0$, $\delta = d, o$) are bounded. In order to bound $\|\|\widehat{\mathbf{X}^+}(\omega;\ell)\|\|$, what we have to prove is that we can control also the terms

$$|\,\langle D\rangle^\alpha\,\widehat{X^\delta}(\omega;\ell)|_{s_0},\quad |\widehat{X^\delta}(\omega;\ell)\,\langle D\rangle^\alpha\,|_{s_0},\quad |\,\langle D\rangle^\sigma\,\widehat{X^\delta}(\omega;\ell)\,\langle D\rangle^{-\sigma}\,|_{s_0}\,.$$

The seminorms involving the diagonal term X^d can be easily handled, since, by (3.4.19), they are essentially bounded by the same seminorms for P^d. The similar bound in (3.4.20) is enough also when we consider the terms $|\,\langle D\rangle^\sigma\,\widehat{X^o}(\omega;\ell)\,\langle D\rangle^{-\sigma}\,|_{s_0}$. Consider now the term $\langle D\rangle^\alpha\,\widehat{X^o}(\omega;\ell)$. Applying again (3.4.20), we get

$$\begin{aligned} |(\langle D\rangle^\alpha\,\widehat{X^o}(\omega;\ell))_j^{j'}| = |\,\langle j'\rangle^\alpha\,\widehat{(X^o)_j^{j'}}(\omega;\ell)| &\leqslant \frac{2\,\langle N\rangle^\tau\,\mathtt{M}^\alpha}{\gamma}\frac{\langle j'\rangle^\alpha}{\langle j+j'\rangle^\alpha}|\widehat{(P^o)_j^{j'}}(\omega;\ell)| \\ &\leqslant \frac{2\,\langle N\rangle^\tau\,\mathtt{M}^\alpha}{\gamma}|\widehat{(P^o)_j^{j'}}(\omega;\ell)|\,. \end{aligned} \tag{3.4.21}$$

The same bound holds for $|(\widehat{X^o}(\omega;\ell)\,\langle D\rangle^\alpha)_j^{j'}|$. We obtain that

$$\|\|\mathbf{X}^+\|\|_{\rho,\Omega^+}^{\infty} \leqslant \frac{2\,\langle N\rangle^\tau\,\mathtt{M}^\alpha}{\gamma}\,|\mathbf{P}|_{\rho,\Omega}^{\infty}\,.$$

We deal now with the estimates on the Lipschitz seminorm $\|\|\mathbf{X}^+\|\|_{\rho,\Omega^+}^{\mathrm{Lip}}$. Using the notation (3.2.23) we have, for $\delta = d, o$:

$$\Delta_\omega\widehat{(X^\delta)_j^{j'}}(\omega;\ell) = -\frac{\mathrm{i}\,\Delta_\omega(\mathtt{g}_{j,j'}^{\ell,\pm}(\omega))}{\mathtt{g}_{j,j'}^{\ell,\pm}(\omega+\Delta\omega)\mathtt{g}_{j,j'}^{\ell,\pm}(\omega)}\widehat{(P^\delta)_j^{j'}}(\omega;\ell) + \frac{\mathrm{i}}{\mathtt{g}_{j,j'}^{\ell,\pm}(\omega+\Delta\omega)}\Delta_\omega\widehat{(P^\delta)_j^{j'}}(\omega;\ell)\,. \tag{3.4.22}$$

By the assumption in (3.4.17), we have that

$$|\Delta_\omega(\mathtt{g}_{j,j'}^{\ell,\pm}(\omega))| = |\Delta\omega\cdot\ell + \Delta_\omega(\lambda_j^- \pm \lambda_{j'}^-)| \overset{(3.4.17)}{\leqslant} |\ell|\,|\Delta\omega| + 2\mathtt{C}\,|\Delta\omega| \leqslant \langle N\rangle\,|\Delta\omega| \qquad (3.4.23)$$

uniformly for every $j, j' \in \mathbb{N}$ and $\ell \in \mathbb{Z}^\nu$, $|\ell| \leqslant N$. Therefore, we can estimate (3.4.22) by

$$|\Delta_\omega\widehat{(X^\delta)}_j^{j'}(\omega;\ell)| \leqslant \frac{8\langle N\rangle^{2\tau+1}\mathtt{M}^{2\alpha}\,|\Delta\omega|}{\gamma^2}\frac{|\widehat{(P^\delta)}_j^{j'}(\omega;\ell)|}{\langle j\pm j'\rangle^{2\alpha}} + \frac{2\langle N\rangle^{\tau}\,\mathtt{M}^{\alpha}}{\gamma}\frac{|\Delta_\omega\widehat{(P^\delta)}_j^{j'}(\omega;\ell)|}{\langle j\pm j'\rangle^{\alpha}}\,, \qquad (3.4.24)$$

from which one deduces the claimed estimate (3.4.18). □

Lemma 3.37. *Let* $\mathbf{P} \in \mathrm{Lip}_\gamma(\Omega, \mathcal{M}_{\rho,s_0}(\alpha,0))$. *Assume* (3.4.17) *and, for some fixed* $C_{s_0} > 0$,

$$C_{s_0}\,16\,\langle N\rangle^{2\tau+1}\,\frac{\mathtt{M}^\alpha}{\gamma}\,|\mathbf{P}|_{\rho,\Omega}^{\mathrm{Lip}(\gamma)} < 1\ . \qquad (3.4.25)$$

Then $\mathbf{P}^+ = \Pi_N^\perp\mathbf{P} + \mathbf{R}$, *defined as in* (3.4.10), *belongs to* $\mathrm{Lip}_\gamma(\Omega^+, \mathcal{M}_{\rho^+,s_0}(\alpha,0))$ *for any* $\rho^+ \in (0,\rho)$, *with bounds*

$$\big|\Pi_N^\perp\mathbf{P}\big|_{\rho^+,\Omega}^{\mathrm{Lip}(\gamma)} \leqslant e^{-(\rho-\rho^+)N}\,|\mathbf{P}|_{\rho,\Omega}^{\mathrm{Lip}(\gamma)}\ , \quad |||\mathbf{R}|||_{\rho,\Omega^+}^{\mathrm{Lip}(\gamma)} \leqslant C_{s_0}\,2^9\,\frac{\mathtt{M}^\alpha}{\gamma}\,\langle N\rangle^{2\tau+1}\,(|\mathbf{P}|_{\rho,\Omega}^{\mathrm{Lip}(\gamma)})^2\ . \qquad (3.4.26)$$

Proof. The estimate on $\Pi_N^\perp\mathbf{P}$ follows by using that it contains only high frequencies. To estimate the remainder $\mathbf{R}$, use (3.4.7),(3.4.8) to write it as

$$\mathbf{R} = \int_0^1 (1-s)e^{-s\mathbf{X}^+}\mathrm{ad}_{\mathbf{X}^+}(\mathbf{Z}-\mathbf{P})e^{s\mathbf{X}^+}\,\mathrm{d}s + \int_0^1 e^{-s\mathbf{X}^+}\mathrm{ad}_{\mathbf{X}^+}(\mathbf{P})e^{s\mathbf{X}^+}\,\mathrm{d}s\ . \qquad (3.4.27)$$

Then, apply Lemma 3.24 and Lemma 3.36. □

Remark 3.38. Defining the quantities

$$\eta := \frac{\mathtt{M}^\alpha}{\gamma}\,|\mathbf{P}|_{\rho,\Omega}^{\mathrm{Lip}(\gamma)}, \qquad \eta^+ := \frac{\mathtt{M}^\alpha}{\gamma}\,\big|\mathbf{P}^+\big|_{\rho^+,\Omega^+}^{\mathrm{Lip}(\gamma)}$$

and choosing $N = -(\rho-\rho^+)^{-1}\ln\eta$, Lemma 3.37 implies that

$$\eta^+ \leqslant (e^{-(\rho-\rho^+)N} + \langle N\rangle^{2\tau+1}\,\eta)\eta \leqslant \Big(1 + \frac{1}{(\rho-\rho^+)^{2\tau+1}}\big(\ln\frac{1}{\eta}\big)^{2\tau+1}\Big)\eta^2\ . \qquad (3.4.28)$$

3.4.4 Iterative Lemma and KAM reduction

Once that the general step has been illustrated, we are ready for setting our iterative scheme. The Hamiltonian the iteration starts with is the one that we have found after the Magnus normal form in Section 3.2:

$$\mathbf{H}^{(0)}(t) = \mathbf{H}_0^{(0)} + \mathbf{V}^{(0)}(\omega;\omega t) \ , \qquad |\mathbf{V}^{(0)}|_{\rho_0,\Omega_0}^{\mathrm{Lip}(\gamma)} \leqslant \frac{C}{\mathtt{M}} \ , \tag{3.4.29}$$

where $\mathbf{H}_0^{(0)} := \mathbf{H}_0$ and $\mathbf{V}^{(0)} := \mathbf{V}$ as in Theorem 3.25. All the iterated objects are constructed from the transformation in Sections 3.4.2, 3.4.3 by setting for $n \geqslant 0$

$$\mathbf{H}^{(n)}(t) := \mathbf{A}(\omega) + \mathbf{P}(\omega;\omega t) \ , \quad \mathbf{A} := \mathbf{H}_0^{(n)} \ , \quad \mathbf{P} := \mathbf{V}^{(n)}$$
$$\mathbf{Z}^{(n)} := \mathbf{Z} \ , \quad \mathbf{X}^{(n)} := \mathbf{X} \ , \quad \mathbf{R}^{(n)} := \mathbf{R}.$$

Given reals $\gamma, \rho_0, \eta_0 > 0$ and a sequence of nested sets $\{\Omega_n\}_{n\geqslant 1}$, we fix the parameters

$$\delta_n := \frac{3}{\pi^2(1+n^2)}\rho_0, \qquad \rho_{n+1} := \rho_n - \delta_n, \qquad \eta_n := \frac{\mathtt{M}^\alpha}{\gamma}|\mathbf{V}^{(n)}|_{\rho_n,\Omega_n}^{\mathrm{Lip}(\gamma)}, \qquad N_n := -\frac{1}{\delta_n}\ln \eta_n$$

Proposition 3.39 (Iterative Lemma). *Fix $\tau > 0$. There exists $\mathtt{k}_0 \equiv \mathtt{k}_0(\tau, \delta_0) > 0$ such that for any $0 < \gamma < \widetilde{\gamma}$, any $\mathtt{M} > 0$ for which*

$$\eta_0 := \frac{\mathtt{M}^\alpha}{\gamma}|\mathbf{V}^{(0)}|_{\rho_0,\Omega_0}^{\mathrm{Lip}(\gamma)} \leqslant \mathtt{k}_0 e^{-1} \ , \tag{3.4.30}$$

the following items hold true for any $n \in \mathbb{N}$:

(i) Setting Ω_0 as in (3.2.21)*, we have recursively for $n \geqslant 0$*

$$\Omega_{n+1} := \left\{ \omega \in \Omega_n \ : \ |\omega \cdot \ell + \lambda_j^{(n)}(\omega) \pm \lambda_{j'}^{(n)}(\omega)| \geqslant \frac{\gamma}{2\,N_n^\tau}\frac{\langle j \pm j' \rangle^\alpha}{\mathtt{M}^\alpha} \ , \quad \forall\, (\ell, j, j') \in \mathcal{I}_{N_n}^{\pm} \right\} \ ;$$

(ii) For every $\omega \in \Omega_n$, the operator $\mathbf{X}^{(n)}(\omega;\,\cdot\,) \in \mathrm{Lip}_\gamma(\Omega_n, \mathcal{M}_{\rho_{n-1},s_0}(\alpha,\alpha))$ and

$$|||\mathbf{X}^{(n)}|||_{\rho_{n-1},\Omega_n}^{\mathrm{Lip}(\gamma)} \leqslant \sqrt{\eta_0}\, e^{\frac{1}{2}(1-(\frac{3}{2})^{n-1})} \ . \tag{3.4.31}$$

The change of coordinates $e^{\mathrm{i}\mathbf{X}^{(n)}}$ conjugates $\mathbf{H}^{(n-1)}$ to $\mathbf{H}^{(n)} = \mathbf{H}_0^{(n)} + \mathbf{V}^{(n)}$ such that:

(iii) The Hamiltonian $\mathbf{H}_0^{(n)}(\omega)$ is diagonal and time independent, $\mathbf{H}_0^{(n)}(\omega) = \mathrm{diag}\{\lambda_j^{(n)}(\omega)\}_{j\in\mathbb{N}}\boldsymbol{\sigma}_3$, and the functions $\lambda_j^{(n)}(\omega) = \lambda_j^{(n)}(\omega, \mathtt{M}, \alpha)$ are defined over all Ω_0, fulfilling

$$|\lambda_j^{(n)} - \lambda_j^{(n-1)}|_{\Omega_0}^{\mathrm{Lip}} \leqslant \eta_0\, e^{1-(\frac{3}{2})^{n-1}} \ ; \tag{3.4.32}$$

(iv) The new perturbation $\mathbf{V}^{(n)} \in \mathrm{Lip}_\gamma(\Omega_n, \mathcal{M}_{\rho_n,s_0}(\alpha,0))$ and

$$\eta_n := \frac{\mathtt{M}^\alpha}{\gamma}|\mathbf{V}^{(n)}|_{\rho_n,\Omega_n}^{\mathrm{Lip}(\gamma)} \leqslant \eta_0\, e^{1-(\frac{3}{2})^n} \ . \tag{3.4.33}$$

Proof. We argue by induction. For $n = 0$, one requires (3.4.30). Now, assume that the statements hold true up to a fixed $n \in \mathbb{N}$. Define Ω_{n+1} as in item (i). In order to apply Lemma 3.36 and Lemma 3.37, we need to check that the assumptions in (3.4.17) and (3.4.25) are verified, respectively. First, note that, by item (iii),

$$|\lambda_j^{(n)}|_{\Omega_0}^{\mathrm{Lip}} \leqslant \sum_{m=1}^{n} |\lambda_j^{(m)} - \lambda_j^{(m-1)}|_{\Omega_0}^{\mathrm{Lip}} + |\lambda_j|_{\Omega_0}^{\mathrm{Lip}} \leqslant \eta_0\, e \sum_{m=1}^{\infty} e^{-\left(\frac{3}{2}\right)^{m-1}} \leqslant \eta_0\, e, \tag{3.4.34}$$

so that (3.4.17) is satisfied, provided simply $\eta_0\, e \leqslant 1$.

We prove now that (3.4.25) is fulfilled. We have

$$\langle N_n \rangle^{2\tau+1}\, \eta_n \leqslant \Big(\frac{1+n^2}{\delta_0}\Big)^{2\tau+1} \eta_n^{\frac{1}{2}} \overset{(3.4.33)}{\leqslant} (\eta_0\, e)^{\frac{1}{2}}\, e^{-\frac{1}{2}\left(\frac{3}{2}\right)^n} \Big(\frac{1+n^2}{\delta_0}\Big)^{2\tau+1} \leqslant \frac{1}{2 \cdot 16 \cdot C_{s_0}}$$

as long as $\eta_0\, e$ is sufficiently small (depending only on δ_0, τ). Therefore we can apply Lemma 3.36 and Lemma 3.37 with $\mathbf{P} = \mathbf{V}^{(n)}$ and define $\mathbf{X}^{(n+1)} \in \mathrm{Lip}_\gamma(\Omega_{n+1}, \mathcal{M}_{\rho_n, s_0}(\alpha, \alpha))$, the new eigenvalues

$$\lambda_j^{(n+1)}(\omega) := \lambda_j^{(n)}(\omega) + (\widehat{V^{d,(n)}})_j^j(\omega; 0) \qquad \forall\, j \in \mathbb{N} \tag{3.4.35}$$

and the new perturbation $\mathbf{V}^{(n+1)}$. We are left only with the quantitative estimates.
We start with item (iv). By Remark 3.38, one has

$$\eta_{n+1} \leqslant \Big(1 + \frac{1}{\delta_n^{2\tau+1}} \big(\ln \frac{1}{\eta_n}\big)^{2\tau+1}\Big) \eta_n^2 \leqslant 2 \Big(\frac{1+n^2}{\delta_0}\Big)^{2\tau+1} (\eta_0\, e)^{\frac{7}{4}} e^{-\frac{7}{4}\left(\frac{3}{2}\right)^n} . \tag{3.4.36}$$

Thus, (3.4.33) is satisfied at the iteration $n+1$ provided again that $\eta_0\, e$ is sufficiently small (depending only on δ_0, τ). For item (iii), it is sufficient to note that

$$|\lambda_j^{(n+1)} - \lambda_j^{(n)}|_{\Omega_n}^{\mathrm{Lip}} = |(\widehat{V^{d,(n)}})_j^j(\cdot; 0)|_{\Omega_n}^{\mathrm{Lip}} \leqslant |\mathbf{V}^{(n)}|_{\rho_n, \Omega_n}^{\mathrm{Lip}} \leqslant \frac{\mathtt{M}^\alpha}{\gamma} |\mathbf{V}^{(n)}|_{\rho_n, \Omega_n}^{\mathrm{Lip}(\gamma)} \overset{(3.4.33)}{\leqslant} \eta_0\, e^{1-\left(\frac{3}{2}\right)^n}. \tag{3.4.37}$$

Now, by Kirszbraun theorem, we can extend the functions $\lambda_j^{(n)}(\omega, \mathtt{M})$ to all Ω_0 preserving their Lipschitz constant; this proves (iii). Item (ii) is proved in the same lines, using (3.4.18) and the inductive assumption; we skip the details. □

A consequence of the iterative lemma is the following result.

Corollary 3.40 (Final eigenvalues)**.** *Fix $\tau > \tilde{\tau}$ (of Theorem 3.29). Assume* (3.4.30)*. Then for every $\omega \in \Omega_0$ and for every $j \in \mathbb{N}$, the sequence $\{\lambda_j^{(n)}(\,\cdot\,, \mathtt{M}, \alpha)\}_{n \geqslant 1}$ is a Cauchy sequence. We denote by $\lambda_j^\infty(\omega, \mathtt{M}, \alpha)$ its limit, which is given by $\lambda_j^\infty(\omega) = \lambda_j + \varepsilon_j^\infty(\omega)$ and one has the estimate*

$$\sup_{j \in \mathbb{N}} |j^\alpha \varepsilon_j^\infty|_{\Omega_0}^{\mathrm{Lip}(\gamma)} \leqslant \frac{\gamma}{\mathtt{M}^\alpha} \eta_0\, e\ . \tag{3.4.38}$$

Proof. By (3.4.35) we have $\varepsilon_j^\infty(\omega) := \sum_{n=0}^\infty \widehat{(V^{(n),d})}_j^j(\omega;0)$. The thesis follows using

$$|j^\alpha \widehat{(V^{(n),d})}_j^j(\omega;0)| \leqslant |\langle D\rangle^\alpha \widehat{V^{(n),d}}(\omega;0)|_{s_0} \leqslant |\mathbf{V}^{(n)}|_{\rho_n,\Omega_n}^{\mathrm{Lip}(\gamma)} \overset{(3.4.33)}{\leqslant} \frac{\gamma}{\mathtt{M}^\alpha}\eta_0\, e^{1-(\frac{3}{2})^n}. \tag{3.4.39}$$

□

Corollary 3.41 (Iterated flow). *Fix an arbitrary $r \in [0, s_0]$; under the same assumptions of Corollary 3.40, for any $\omega \in \cap_n \Omega_n$ and $\varphi \in \mathbb{T}^n$, the sequence of transformations*

$$\mathcal{W}^n(\omega;\varphi) := e^{-\mathrm{i}\mathbf{X}^{(1)}(\omega;\varphi)} \circ \cdots \circ e^{-\mathrm{i}\mathbf{X}^{(n)}(\omega;\varphi)} \tag{3.4.40}$$

is a Cauchy sequence in $\mathcal{L}(\mathcal{H}^r \times \mathcal{H}^r)$ fulfilling

$$\|\mathcal{W}^n(\omega;\varphi) - \mathbb{1}\|_{\mathcal{L}(\mathcal{H}^r\times\mathcal{H}^r)} \leqslant \sqrt{\eta_0\, e}\,\Sigma\; e^{\sqrt{\eta_0\, e}\,\Sigma} \tag{3.4.41}$$

where $\Sigma := \sum_{q=0}^\infty e^{-\frac{1}{2}(\frac{3}{2})^q}$. We denote by $\mathcal{W}^\infty(\omega;\varphi)$ its limit in $\mathcal{L}(\mathcal{H}^r \times \mathcal{H}^r)$.

Proof. The convergence of the transformations is a standard argument, whereas the control of the operator norm $\mathcal{L}(\mathcal{H}^r \times \mathcal{H}^r)$ follows from Remark 3.16; we skip the details. □

Since for any $j \in \mathbb{N}$ the sequence $\{\lambda_j^{(n)}\}_{n\geqslant 1}$ converges to a well defined Lipschitz function λ_j^∞ defined on Ω_0, we can now impose second order Melnikov conditions only on the final frequencies.

Lemma 3.42 (Measure estimates). *Consider the set*

$$\Omega_{\infty,\alpha} := \left\{\omega \in \mathcal{U}_\alpha \;:\; |\omega\cdot k + \lambda_j^\infty(\omega) \pm \lambda_{j'}^\infty(\omega)| \geqslant \frac{\gamma}{\langle \ell\rangle^\tau}\frac{\langle j \pm j'\rangle^\alpha}{\mathtt{M}^\alpha}, \quad \forall\,(\ell,j,j') \in \mathcal{I}^\pm\right\}. \tag{3.4.42}$$

Then $\Omega_{\infty,\alpha} \subseteq \cap_n\Omega_n$. Furthermore, taking $\tau > \nu + \alpha + \frac{\tilde\tau}{\alpha}$, $\gamma \in [0, \tilde\gamma/2]$ and $\mathtt{M} \geqslant \mathtt{M}_0$ (defined in Theorem 3.29), there exists a constant $C_\infty > 0$, independent of $\mathtt{M}$ and γ, such that

$$\mathrm{m}_r(\mathcal{U}_\alpha \backslash \Omega_{\infty,\alpha}) \leqslant C_\infty\gamma\,. \tag{3.4.43}$$

Proof. The proof that $\Omega_{\infty,\alpha} \subseteq \cap_n\Omega_n$ is standard, see e.g. Lemma 7.6 of [142].
To prove the measure estimate, let $\omega \in \mathcal{U}_\alpha$ and $(\ell,j,j') \in \mathcal{I}^\pm$. We can rule out the cases as at the beginning of Lemma 3.33 with essentially the same arguments. Thus, we restrict to consider all $(\ell,j,j') \in \mathcal{I}^\pm$ for which $\ell \neq 0$ and $j \neq j'$. Furthermore, if $|j \pm j'| \geqslant 16\mathtt{M}\,|\ell|$, we get again that $|\omega\cdot\ell + \lambda_j^\infty(\omega) \pm \lambda_{j'}^\infty(\omega)| \geqslant \frac{1}{2}|j\pm j'|$ (recall $\mathtt{M} > \mathtt{m}^2$). So, we can work in the regions $|j \pm j'| < 16\mathtt{M}\,|\ell|$.
Now, for $j < j'$ satisfying

$$j\,\langle j \pm j'\rangle \geqslant \left(\frac{2\eta_0\, e\,\langle \ell\rangle^{\tilde\tau}}{c(\gamma,\tilde\gamma)}\right)^{\frac{1}{\alpha}} =: \tilde{\mathtt{R}}(\ell), \tag{3.4.44}$$

where $c(\gamma,\widetilde{\gamma}) := \frac{\widetilde{\gamma}}{\gamma} - 1 > 1$ (recall that $\widetilde{\gamma}/2 > \gamma$), we have (using also (3.4.38))

$$|\omega\cdot\ell + \lambda_j^\infty(\omega) \pm \lambda_{j'}^\infty(\omega)| \geqslant |\omega\cdot\ell + \lambda_j \pm \lambda_{j'}| - |\varepsilon_j^\infty(\omega)| - |\varepsilon_{j'}^\infty(\omega)|$$
$$\geqslant \frac{\widetilde{\gamma}}{\langle\ell\rangle^{\widetilde{\tau}}}\frac{\langle j\pm j'\rangle^\alpha}{\mathtt{M}^\alpha} - 2\frac{\gamma}{\mathtt{M}^\alpha}\frac{\eta_0 e}{j^\alpha} \geqslant \frac{\gamma}{\mathtt{M}^\alpha}\frac{\langle j\pm j'\rangle^\alpha}{\langle\ell\rangle^{\widetilde{\tau}}}\ .$$

Therefore, we can further restrict to consider just those $j < j'$ satisfying $j\,\langle j\pm j'\rangle < \widetilde{\mathtt{R}}(\ell)$. The symmetric argument leads to work in the sector $j' < l$ under the condition $j'\,\langle j'\pm j\rangle < \widetilde{\mathtt{R}}(\ell)$. Now, define the set

$$\mathcal{G}_{j,j'}^{\ell,\pm} := \Big\{\omega\in R_{\mathtt{M}}\ :\ |\omega\cdot\ell + \lambda_j^\infty(\omega) \pm \lambda_{j'}^\infty(\omega)| < \frac{\gamma}{\langle k\rangle^\tau}\frac{\langle j\pm l\rangle^\alpha}{\mathtt{M}^\alpha}\Big\} \tag{3.4.45}$$

for those $\ell\neq 0$ and $j\neq j'$ in the region

$$\mathcal{R}^\pm := \{|j\pm j'| < 16\mathtt{M}\,|\ell|\}\cap(\{j\,\langle j\pm j'\rangle < \widetilde{\mathtt{R}}(\ell),\ j<j'\}\cup\{j'\,\langle j\pm j'\rangle < \widetilde{\mathtt{R}}(\ell),\ j'<j\})\,; \tag{3.4.46}$$

Recall that $f_{\ell,j,j'}^\pm(\omega) := \omega\cdot\ell + \lambda_j^\infty(\omega) \pm \lambda_{j'}^\infty(\omega)$ are Lipschitz functions on $R_{\mathtt{M}}$. For $\ell\neq 0$, since $|\lambda_{j'}^\infty|_{R_{\mathtt{M}}}^{\mathrm{Lip}} < |\ell|/4$, by Lemma 3.30 we get

$$|\mathcal{G}_{j,j'}^{\ell,\pm}| \lesssim \mathtt{M}^{\nu-\alpha}\gamma\frac{\langle j\pm j'\rangle^\alpha}{|\ell|^{\tau+1}}\ .$$

Define $\mathcal{G}_\infty^\pm := \bigcup\{\mathcal{G}_{j,j'}^{\ell,\pm}\ :\ (\ell,j,j')\in\mathcal{R}^\pm\}\cap\mathcal{U}_\alpha$. We have

$$|\mathcal{G}_\infty^-| \lesssim 2\gamma\,\mathtt{M}^{\nu-\alpha}\sum_{\ell\neq0}\sum_{\substack{j<j'\\ j\langle j-j'\rangle<\widetilde{\mathtt{R}}(\ell)}}\sum_{|j-j'|<16\mathtt{M}|\ell|}\frac{\langle j-j'\rangle^\alpha}{|\ell|^{\tau+1}} \lesssim \gamma\,\mathtt{M}^{\nu-\alpha}\sum_{\ell\neq0}\sum_{\substack{j'-j=:h>0\\ |h|<16\mathtt{M}|\ell|}}\sum_{j<\widetilde{\mathtt{R}}(\ell)\langle h\rangle^{-1}}\frac{\langle h\rangle^\alpha}{|\ell|^{\tau+1}}$$
$$\lesssim \frac{\gamma}{c(\gamma,\widetilde{\gamma})^{\frac{1}{\alpha}}}\mathtt{M}^\nu\sum_{\ell\neq0}\frac{1}{|\ell|^{\tau+1-\alpha-\frac{\widetilde{\tau}}{\alpha}}} \lesssim \frac{\gamma}{c(\gamma,\widetilde{\gamma})^{\frac{1}{\alpha}}}\mathtt{M}^\nu \lesssim \gamma\,\mathtt{M}^\nu\ ,$$

taking $\tau+1-\alpha-\frac{\widetilde{\tau}}{\alpha} > \nu$. The same computation holds for $\mathcal{G}_\infty^+$, and proves (3.4.43). □

Theorem 3.43 (KAM reducibility). *Fix $\alpha\in(0,1)$, $s_0 > 1/2$, and $\tau > \nu+1+\alpha+\frac{\widetilde{\tau}}{\alpha}$. For any $0<\gamma<\widetilde{\gamma}$, there exists $\mathtt{M}_* = \mathtt{M}_*(\mathtt{m},\alpha,\gamma,\rho_0) > 0$ such that for any $\mathtt{M}\geqslant\mathtt{M}_*$ the following holds true. There exist functions $\{\lambda_j^\infty(\omega,\mathtt{M},\alpha)\}_{j\in\mathbb{N}}$, defined and Lipschitz in ω in the set $R_{\mathtt{M}}$ such that:*

(i) The set $\Omega_{\infty,\alpha} = \Omega_{\infty,\alpha}(\gamma,\tau,\mathtt{M})\subset R_{\mathtt{M}}$ defined in (3.4.42) fulfills $\mathrm{m}_r(R_{\mathtt{M}}\backslash\Omega_\infty)\leqslant C(\gamma+\widetilde{\gamma}^{1/3}+\gamma_0)$, where γ_0 is defined in Theorem 3.25 and $\widetilde{\gamma}$ in Theorem 3.29.

(ii) For each $\omega\in\Omega_{\infty,\alpha}$ there exists a change of coordinates $w = \mathcal{W}^\infty(\omega;\omega t)\phi$ which conjugates

equation (3.2.7) *to a constant-coefficient diagonal one:*

$$\mathrm{i}\dot{\phi} = \mathbf{H}^\infty \phi \ , \quad \mathbf{H}^\infty = \mathbf{H}^\infty(\omega, \alpha) = \operatorname{diag}\{\lambda_j^\infty(\omega, \alpha) \ : \ j \in \mathbb{N}\}\boldsymbol{\sigma}_3 \ . \tag{3.4.47}$$

Furthermore for any $r \in [0, s_0]$ *one has*

$$\|\mathcal{W}^\infty - \mathbb{1}\|_{\mathcal{L}(\mathcal{H}^r \times \mathcal{H}^r)} \leqslant \sqrt{\eta_0\, e}\, \Sigma\ e^{\sqrt{\eta_0\, e}\,\Sigma} \ . \tag{3.4.48}$$

Proof. Having fixed α, s_0 and τ, we can produce the constant $\mathrm{k}_0(\delta_0, \tau)$ of the iterative Lemma 3.39. Having fixed also $0 < \gamma < \widetilde{\gamma}$, we produce $\mathtt{M}_* > 0$ in such a way that for every $\mathtt{M} \geqslant \mathtt{M}_*$, the estimate (3.4.30) is fulfilled. We can now apply the iterative Lemma 3.39, Corollary 3.40 and Lemma 3.42 to get the result. □

3.4.5 A final remark

The KAM reducibility scheme that we have presented has transformed Equation (3.2.7) into (3.4.47), where the asymptotic for the final eigenvalues are given, using Equation (3.4.38), by

$$\lambda_j^\infty(\omega, \alpha) - \lambda_j \sim O\left(\frac{\eta_0}{\mathtt{M}^\alpha j^\alpha}\right) \overset{(3.4.29)}{\sim} O\left(\frac{1}{\mathtt{M} j^\alpha}\right) \ . \tag{3.4.49}$$

One can argue that the asymptotic $\lambda_j^\infty(\alpha) - \lambda_j \sim O(\mathtt{M}^{-1} j^{-\alpha})$ is not that satisfying, since the pertubation $\mathbf{V}^{(0)}$ at the beginning of the KAM scheme belongs to the class $\mathcal{M}_{\rho_0, s_0}(1, 0)$ and so its diagonal elements have a smoothing effect of order 1 which could be expected to be preserved in the effective Hamiltonian.

Actually, it is possible to modify our reducibility scheme for achieving this result: we explain now briefly how to do it. After the Magnus normal form, we conjugate system (3.2.7) through $e^{-\mathrm{i}\mathbf{Y}(\omega t)}$, where

$$\mathbf{Y}(\omega t) := \begin{pmatrix} 0 & Y^o(\omega t) \\ -\overline{Y^o}(\omega t) & 0 \end{pmatrix} \tag{3.4.50}$$

so that Y^o solves the homological equation

$$-\mathrm{i}[Y^o(\theta), B]_{\mathrm{a}} + V^o(\theta) - \omega \cdot \partial_\theta Y^o(\theta) = 0 \ \Rightarrow\ \widehat{(Y^o)_j^{j'}}(\ell) := \frac{\widehat{(V^o)_j^{j'}}(\ell)}{\mathrm{i}(\omega \cdot \ell + \lambda_j + \lambda_{j'})} \quad \forall\, \ell, j, j' \ . \tag{3.4.51}$$

We ask now the frequency vector ω to belong to $\mathcal{U}_1 \cap \mathcal{U}_0$ (see (3.3.4)). In this way one gets (in the same lines of the proof of Lemma 3.36) that $\mathbf{Y} \in \mathrm{Lip}_{\gamma/\mathtt{M}}(\mathcal{U}_1, \mathcal{M}_{\widetilde{\rho}_0, s_0}(1, 1))$, since we have chosen $\omega \in \mathcal{U}_1$, with the bound

$$|\mathbf{Y}|^{\mathrm{Lip}(\gamma/\mathtt{M})}_{\widetilde{\rho}_0, s_0, 1, 1} \leqslant C\frac{\mathtt{M}}{\gamma} |\mathbf{V}^{(0)}|^{\mathrm{Lip}(\gamma/\mathtt{M})}_{\rho_0, s_0, 1, 1} \leqslant C\frac{\mathtt{M}}{\gamma} |\mathbf{V}^{(0)}|^{\mathrm{Lip}(\gamma/\mathtt{M})}_{\rho_0, s_0, 1, 0} \overset{(3.4.1)}{\leqslant} \widetilde{C} \ . \tag{3.4.52}$$

The new perturbation

$$\widetilde{\mathbf{V}^{(0)}}(\omega t) := \begin{pmatrix} V^d(\omega t) & 0 \\ 0 & -\overline{V^d}(\omega t) \end{pmatrix} + \int_0^1 (1-s) e^{-s\mathbf{Y}(\omega t)} \mathrm{ad}_{\mathbf{Y}(\omega t)}[\mathbf{V}^{(0)}(\omega t)] e^{s\mathbf{Y}(\omega t)}\, \mathrm{d}s \qquad (3.4.53)$$

belongs to the class $\mathrm{Lip}_{\gamma/\mathtt{M}}(\mathcal{U}_1, \mathcal{M}_{\widetilde{\rho}_0, s_0}(1,1))$ fulfilling estimate (3.4.1).
Thus, one can perform a KAM reducibility scheme as in Section 3.4.3–3.4.4, with $\alpha = 0$ in (3.4.15), the perturbations appearing in the iterations in the class $\mathrm{Lip}_{\gamma/\mathtt{M}^0}(\widetilde{\Omega_n}, \mathcal{M}_{\widetilde{\rho}_n, s_0}(1,1))$ and the new final eigenvalues $\widetilde{\lambda_j^\infty}$ satisfying the non-resonance conditions

$$|\omega \cdot \ell + \widetilde{\lambda_j^\infty} \pm \widetilde{\lambda_{j}^\infty}| \geqslant \frac{\gamma}{\langle \ell \rangle^\tau}, \qquad \forall\, (\ell, j, j') \in \mathcal{I}^\pm \ . \qquad (3.4.54)$$

In particular, we obtain better asymptotics on the final eigenvalues, that is $\widetilde{\lambda_j^\infty} - \lambda_j \sim O(\mathtt{M}^{-1} j^{-1})$. The price that we pay for this result is that the preliminary change of coordinate $e^{-\mathrm{i}\mathbf{Y}(\omega t)}$ is not a transformation close to identity, as the generator $\mathbf{Y}(\omega t)$ is just a bounded operator and not small in size, see (3.4.52). The main consequence is that the effective dynamics of the original system, as Corollary 1.2 is no more valid. In this case, it is possible to conclude just that the Sobolev norms stay uniformly bounded in time and do not grow, but in general their (almost-)conservation is lost.

Chapter 4

Traveling quasi-periodic gravity capillary water waves with constant vorticity

We consider the space periodic gravity-capillary water waves equations with constant vorticity

$$\begin{cases} \eta_t = G(\eta)\psi + \gamma\eta\eta_x \\ \psi_t = -g\eta - \dfrac{\psi_x^2}{2} + \dfrac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} + \kappa\Big(\dfrac{\eta_x}{\sqrt{1+\eta_x^2}}\Big)_x + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi\,. \end{cases} \tag{4.0.1}$$

The variable $\eta(t,x)$ denotes the free boundary of the two dimensional fluid domain $\mathcal{D}_{\eta,\mathtt{h}}$ defined in (1.1.11), whereas $\psi(t,x)$ is the trace at the free boundary $y=\eta(t,x)$ of the generalized velocity potential $\Phi(t,x,y)$ solving (1.1.12). Here $g>0$ is the gravity, $\kappa>0$ is the surface tension coefficient and $G(\eta)$ is the Dirichlet-Neumann operator defined in (1.1.14), with linear principal part $G(0)$ defined in (1.1.15), (1.1.16). The derivation of the equations (1.1.13) is available in Appendix B.

The water waves equations (1.1.13) are a Hamiltonian system on the phase space $H_0^s(\mathbb{T})\times\dot{H}^s(\mathbb{T})$, endowed with a non canonical Poisson structure: it will be discussed with more details in Section 4.1.1.

The system obtained linearizing (1.1.13) at the equilibrium $(\eta,\psi)=(0,0)$ is given by

$$\begin{cases} \partial_t\eta &= G(0)\psi \\ \partial_t\psi &= -(g-\kappa\partial_x^2)\eta + \gamma\partial_x^{-1}G(0)\psi\,. \end{cases} \tag{4.0.2}$$

The linear frequencies are given by

$$\Omega_j := \Omega_j(\kappa) = \Omega_j(\kappa, \mathtt{h}, g, \gamma) := \sqrt{\Big(\kappa j^2 + g + \frac{\gamma^2}{4}\frac{G_j(0)}{j^2}\Big)G_j(0)} + \frac{\gamma}{2}\frac{G_j(0)}{j}, \quad j \in \mathbb{Z}\backslash\{0\}\,. \tag{4.0.3}$$

Note that the map $j \mapsto \Omega_j(\kappa)$ is not even due to the vorticity term $\frac{\gamma}{2}G_j(0)/j$, which is odd in j.

Fixed finitely many arbitrary *distinct* natural numbers

$$\mathbb{S}^+ := \{\overline{n}_1, \ldots, \overline{n}_\nu\} \subset \mathbb{N}\,, \quad 1 \leqslant \overline{n}_1 < \ldots < \overline{n}_\nu\,, \tag{4.0.4}$$

and signs

$$\Sigma := \{\sigma_1, \ldots, \sigma_\nu\}, \quad \sigma_a \in \{-1, 1\}\,, \quad a = 1, \ldots, \nu\,, \tag{4.0.5}$$

we consider the reversible quasi-periodic traveling wave solutions of the linear system (1.1.20) given by

$$\begin{aligned}\begin{pmatrix}\eta(t,x)\\ \psi(t,x)\end{pmatrix} = &\sum_{a\in\{1,\ldots,\nu\,:\,\sigma_a=+1\}} \begin{pmatrix} M_{\overline{n}_a}\sqrt{\xi_{\overline{n}_a}}\cos(\overline{n}_a x - \Omega_{\overline{n}_a}(\kappa)t)\\ P_{\overline{n}_a}\sqrt{\xi_{\overline{n}_a}}\sin(\overline{n}_a x - \Omega_{\overline{n}_a}(\kappa)t)\end{pmatrix}\\ + &\sum_{a\in\{1,\ldots,\nu\,:\,\sigma_a=-1\}} \begin{pmatrix} M_{\overline{n}_a}\sqrt{\xi_{-\overline{n}_a}}\cos(\overline{n}_a x + \Omega_{-\overline{n}_a}(\kappa)t)\\ P_{-\overline{n}_a}\sqrt{\xi_{-\overline{n}_a}}\sin(\overline{n}_a x + \Omega_{-\overline{n}_a}(\kappa)t)\end{pmatrix}\end{aligned} \tag{4.0.6}$$

where $\xi_{\pm\overline{n}_a} > 0$, $a = 1, \ldots, \nu$, and M_n and $P_{\pm n}$ are the real coefficients

$$M_j := \left(\frac{G_j(0)}{\kappa j^2 + g + \frac{\gamma^2}{4}\frac{G_j(0)}{j^2}}\right)^{\frac14}\,,\ j \in \mathbb{Z}\backslash\{0\}\,, \quad P_{\pm n} := \frac{\gamma}{2}\frac{M_n}{n} \pm M_n^{-1},\ n \in \mathbb{N}\,. \tag{4.0.7}$$

. The frequency vector of (4.0.6) is

$$\vec{\Omega}(\kappa) := (\Omega_{\sigma_a\overline{n}_a}(\kappa))_{a=1,\ldots,\nu} \in \mathbb{R}^\nu\,. \tag{4.0.8}$$

A more general definition of quasi-periodic traveling wave is given in Definition 1.5.

We shall construct traveling quasi-periodic solutions of (1.1.13) with a Diophantine frequency vector $\omega \in \mathtt{DC}(\upsilon, \tau)$ belonging to an open bounded subset Ω in $\mathbb{R}^\nu$ for some $\upsilon \in (0,1)$, $\tau > \nu - 1$, as in (1.1.24)

Regarding regularity, we will prove the existence of quasi-periodic traveling waves $(\breve{\eta}, \breve{\psi})$ belonging to some Sobolev space $H^s(\mathbb{T}^\nu, \mathbb{R}^2)$ defined in (1.1.25).

The result in Theorem (4.1) shows that the linear solutions (4.0.6) can be continued to quasi-periodic traveling wave solutions of the nonlinear water waves equations (4.0.1), for most values of the surface tension $\kappa \in [\kappa_1, \kappa_2]$, with a frequency vector $\widetilde{\Omega} := (\widetilde{\Omega}_{\sigma_a\overline{n}_a})_{a=1,\ldots,\nu}$, close to $\vec{\Omega}(\kappa) := (\Omega_{\sigma_a\overline{n}_a}(\kappa))_{a=1,\ldots,\nu}$. Here is the precise statement.

Theorem 4.1. (KAM for traveling gravity-capillary water waves with constant vorticity) *Consider finitely many tangential sites* $\mathbb{S}^+ \subset \mathbb{N}$ *as in* (4.0.4) *and signs* Σ *as in* (4.0.5). *Then there exist* $\overline{s} > 0$, $\varepsilon_0 \in (0,1)$ *such that, for every* $|\xi| \leqslant \varepsilon_0^2$, $\xi := (\xi_{\sigma_a \overline{n}_a})_{a=1,\ldots,\nu} \in \mathbb{R}^\nu_+$, *the following hold:*

1. *there exists a Cantor-like set* $\mathcal{G}_\xi \subset [\kappa_1, \kappa_2]$ *with asymptotically full measure as* $\xi \to 0$, *i.e.* $\lim_{\xi\to 0} |\mathcal{G}_\xi| = \kappa_2 - \kappa_1$;

2. *for any* $\kappa \in \mathcal{G}_\xi$, *the gravity-capillary water waves equations* (1.1.13) *have a reversible quasi-periodic traveling wave solution (according to Definition 1.5) of the form*

$$\begin{aligned}\begin{pmatrix}\eta(t,x)\\ \psi(t,x)\end{pmatrix} &= \sum_{a\in\{1,\ldots,\nu\}:\,\sigma_a=+1} \begin{pmatrix} M_{\overline{n}_a}\sqrt{\xi_{\overline{n}_a}}\cos(\overline{n}_a x - \widetilde{\Omega}_{\overline{n}_a}(\kappa)t)\\ P_{\overline{n}_a}\sqrt{\xi_{\overline{n}_a}}\sin(\overline{n}_a x - \widetilde{\Omega}_{\overline{n}_a}(\kappa)t)\end{pmatrix}\\ &\quad+ \sum_{a\in\{1,\ldots,\nu\}:\,\sigma_a=-1} \begin{pmatrix} M_{\overline{n}_a}\sqrt{\xi_{-\overline{n}_a}}\cos(\overline{n}_a x + \widetilde{\Omega}_{-\overline{n}_a}(\kappa)t)\\ P_{-\overline{n}_a}\sqrt{\xi_{-\overline{n}_a}}\sin(\overline{n}_a x + \widetilde{\Omega}_{-\overline{n}_a}(\kappa)t)\end{pmatrix} + r(t,x)\end{aligned} \tag{4.0.9}$$

where

$$r(t,x) = \breve{r}(\widetilde{\Omega}_{\sigma_1\overline{n}_1}(\kappa)t - \sigma_1\overline{n}_1 x, \ldots, \widetilde{\Omega}_{\sigma_\nu\overline{n}_\nu}(\kappa)t - \sigma_\nu\overline{n}_\nu x)\,, \quad \breve{r} \in H^{\overline{s}}(\mathbb{T}^\nu, \mathbb{R}^2)\,, \quad \lim_{\xi\to 0}\frac{\|\breve{r}\|_{\overline{s}}}{\sqrt{|\xi|}} = 0\,,$$

with a Diophantine frequency vector $\widetilde{\Omega} := (\widetilde{\Omega}_{\sigma_a\overline{n}_a})_{a=1,\ldots,\nu} \in \mathbb{R}^\nu$, *depending on* κ, ξ, *and satisfying* $\lim_{\xi\to 0}\widetilde{\Omega} = \vec{\Omega}(\kappa)$. *In addition these quasi-periodic solutions are linearly stable.*

The rest of this chapter concerns the proof of Theorem 4.1.

In Section 4.1 we start by describing the Hamiltonian structure of equations (4.0.1) together with the choice of the Wahlén coordinates and the solution of the linearized system around the trivial equilibrium. Then we provide a splitting of the phase space that allows to introduce the normal subspace and the action-angle coordinates on the tangential one. Section 4.2 is devoted to the functional setting required for the proof of Theorem 3.1. In particular, we define the quasi-periodic traveling wave functions the φ-dependent families of momentum preserving linear operators, together with their properties. The rest of the functional setting, in particular the pseudodifferential norms and the class of $\mathcal{D}^{k_0}$-tame operators, are quoted almost verbatim from [44, 13]. In Section 4.3 we prove the non-degeneracy of the unperturbed linear frequencies and the transversality of the non-resonance conditions coupled with corresponding momentum conditions. In Section 4.4 we state the Nash-Moser theorem and we prove that the non-resonance conditions on the final eigenvalues hold on a set of parameter of large measure. In Section 4.5 we construct the approximate inverse at each approximate quasi-periodic traveling wave embedding, under the ansatz of the almost invertibility of the linearized vector field restricted on the normal directions. Sections 4.6 and 4.7 are devoted to the reduction to constant coefficients up to bounded remainders and to the KAM reducibility scheme of the linearized vector field projected on the normal direction, in order to provide estimates for its almost inverted operator. In Section 4.8 the Nash-Moser Theorem 4.55 and the convergence of the Nash-Moser iteration are proved. In particular, we check that each approximate torus is reversible and traveling.

4.1 Hamiltonian structure and linearization at the origin

In this section we describe the Hamiltonian structure of the water waves equations (4.0.1), their symmetries and the solutions of the linearized system (1.1.20) at the equilibrium.

4.1.1 Hamiltonian structure

The Hamiltonian formulation of the water waves equations (4.0.1) with non-zero constant vorticity was obtained by Constantin-Ivanov-Prodanov [58] and Wahlén [163] in the case of finite depth. For irrotational flows it reduces to the classical Craig-Sulem-Zakharov formulation in [174], [68].

On the phase space $H_0^1(\mathbb{T}) \times \dot{H}^1(\mathbb{T})$, endowed with the non canonical Poisson tensor

$$J_M(\gamma) := \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & \gamma\partial_x^{-1} \end{pmatrix}, \tag{4.1.1}$$

we consider the Hamiltonian

$$H(\eta,\psi) = \frac{1}{2}\int_{\mathbb{T}} \big(\psi G(\eta)\psi + g\eta^2\big)\,\mathrm{d}x + \kappa\int_{\mathbb{T}} \sqrt{1+\eta_x^2}\,\mathrm{d}x + \frac{\gamma}{2}\int_{\mathbb{T}} \Big(-\psi_x\eta^2 + \frac{\gamma}{3}\eta^3\Big)\,\mathrm{d}x\,. \tag{4.1.2}$$

Such Hamiltonian is well defined on $H_0^1(\mathbb{T}) \times \dot{H}^1(\mathbb{T})$ since $G(\eta)[1] = 0$ and $\int_{\mathbb{T}} G(\eta)\psi\,\mathrm{d}x = 0$.

It turns out [58, 163] that equations (4.0.1) are the Hamiltonian system generated by $H(\eta,\psi)$ with respect to the Poisson tensor $J_M(\gamma)$, namely

$$\partial_t \begin{pmatrix} \eta \\ \psi \end{pmatrix} = J_M(\gamma) \begin{pmatrix} \nabla_\eta H \\ \nabla_\psi H \end{pmatrix} \tag{4.1.3}$$

where $(\nabla_\eta H, \nabla_\psi H) \in \dot{L}^2(\mathbb{T}) \times L_0^2(\mathbb{T})$ denote the L^2-gradients.

Remark 4.2. The non canonical Poisson tensor $J_M(\gamma)$ in (4.1.1) has to be regarded as an operator from (subspaces of) $(L_0^2 \times \dot{L}^2)^* = \dot{L}^2 \times L_0^2$ to $L_0^2 \times \dot{L}^2$, that is

$$J_M(\gamma) = \begin{pmatrix} 0 & \mathrm{Id}_{L_0^2 \to L_0^2} \\ -\mathrm{Id}_{\dot{L}^2 \to \dot{L}^2} & \gamma\partial_x^{-1} \end{pmatrix}.$$

The operator ∂_x^{-1} maps a dense subspace of L_0^2 in $\dot{L}^2$. For sake of simplicity, throughout the chapter we may omit this detail. Above the dual space $(L_0^2 \times \dot{L}^2)^*$ with respect to the scalar product in L^2 is identified with $\dot{L}^2 \times L_0^2$.

The Hamiltonian (4.1.2) enjoys several symmetries that we now describe.

Reversible structure. Defining on the phase space $H_0^1(\mathbb{T}) \times \dot{H}^1(\mathbb{T})$ the involution

$$\mathcal{S}\begin{pmatrix}\eta\\ \psi\end{pmatrix} := \begin{pmatrix}\eta^\vee\\ -\psi^\vee\end{pmatrix}, \quad \eta^\vee(x) := \eta(-x), \tag{4.1.4}$$

the Hamiltonian (4.1.2) is invariant under $\mathcal{S}$, that is $H \circ \mathcal{S} = H$, or, equivalently, the water waves vector field X defined in the right hand side on (4.0.1) satisfies

$$X \circ \mathcal{S} = -\mathcal{S} \circ X. \tag{4.1.5}$$

This property follows noting that the Dirichlet-Neumann operator satisfies

$$G(\eta^\vee)[\psi^\vee] = (G(\eta)[\psi])^\vee . \tag{4.1.6}$$

Translation invariance. Since the bottom of the fluid domain (1.1.11) is flat (or in case of infinite depth there is no bottom), the water waves equations (4.0.1) are invariant under space translations. Specifically, defining the translation operator

$$\tau_\varsigma \colon u(x) \mapsto u(x+\varsigma), \qquad \varsigma \in \mathbb{R}, \tag{4.1.7}$$

the Hamiltonian (4.1.2) satisfies $H \circ \tau_\varsigma = H$ for any $\varsigma \in \mathbb{R}$, or, equivalently, the water waves vector field X defined in the right hand side on (4.0.1) satisfies

$$X \circ \tau_\varsigma = \tau_\varsigma \circ X, \quad \forall \varsigma \in \mathbb{R}. \tag{4.1.8}$$

In order to verify this property, note that the Dirichlet-Neumann operator satisfies

$$\tau_\varsigma \circ G(\eta) = G(\tau_\varsigma \eta) \circ \tau_\varsigma, \quad \forall \varsigma \in \mathbb{R}. \tag{4.1.9}$$

Wahlén coordinates. The variables (η, ψ) are not Darboux coordinates, in the sense that the Poisson tensor (4.1.1) is not the canonical one for values of the vorticity $\gamma \neq 0$. Wahlén [163] noted that in the variables (η, ζ), where ζ is defined by

$$\zeta := \psi - \frac{\gamma}{2}\partial_x^{-1}\eta, \tag{4.1.10}$$

the symplectic form induced by $J_M(\gamma)$ becomes the canonical one. Indeed, under the linear transformation of the phase space $H^1_0 \times \dot{H}^1$ into itself defined by

$$\begin{pmatrix}\eta\\ \psi\end{pmatrix} = W\begin{pmatrix}\eta\\ \zeta\end{pmatrix}, \quad W := \begin{pmatrix}\mathrm{Id} & 0\\ \frac{\gamma}{2}\partial_x^{-1} & \mathrm{Id}\end{pmatrix}, \quad W^{-1} := \begin{pmatrix}\mathrm{Id} & 0\\ -\frac{\gamma}{2}\partial_x^{-1} & \mathrm{Id}\end{pmatrix}, \tag{4.1.11}$$

the Poisson tensor $J_M(\gamma)$ is transformed into the canonical one,

$$W^{-1}J_M(\gamma)(W^{-1})^* = J, \quad J := \begin{pmatrix}0 & \mathrm{Id}\\ -\mathrm{Id} & 0\end{pmatrix}. \tag{4.1.12}$$

Here W^* and $(W^{-1})^*$ are the adjoints maps from (a dense subspace of) $\dot{L}^2 \times L^2_0$ into itself, and the Poisson tensor J acts from (subspaces of) $\dot{L}^2 \times L^2_0$ to $L^2_0 \times \dot{L}^2$. Then the Hamiltonian (4.1.2) becomes

$$\mathcal{H} := H \circ W, \quad \text{i.e.} \quad \mathcal{H}(\eta,\zeta) := H\Big(\eta, \zeta + \frac{\gamma}{2}\partial_x^{-1}\eta\Big), \tag{4.1.13}$$

and the Hamiltonian equations (4.1.3) (i.e. (4.0.1)) are transformed into

$$\partial_t\begin{pmatrix}\eta\\ \zeta\end{pmatrix} = X_{\mathcal{H}}(\eta,\zeta), \quad X_{\mathcal{H}}(\eta,\zeta) := J\begin{pmatrix}\nabla_\eta\mathcal{H}\\ \nabla_\zeta\mathcal{H}\end{pmatrix}(\eta,\zeta). \tag{4.1.14}$$

By (4.1.12), the symplectic form of (4.1.14) is the standard one,

$$\mathcal{W}\left(\begin{pmatrix}\eta_1\\ \zeta_1\end{pmatrix}, \begin{pmatrix}\eta_2\\ \zeta_2\end{pmatrix}\right) = \left(J^{-1}\begin{pmatrix}\eta_1\\ \zeta_1\end{pmatrix}, \begin{pmatrix}\eta_2\\ \zeta_2\end{pmatrix}\right)_{L^2} = (-\zeta_1,\eta_2)_{L^2} + (\eta_1,\zeta_2)_{L^2}, \tag{4.1.15}$$

where J^{-1} is the symplectic operator

$$J^{-1} = \begin{pmatrix}0 & -\mathrm{Id}\\ \mathrm{Id} & 0\end{pmatrix} \tag{4.1.16}$$

regarded as a map from $L^2_0 \times \dot{L}^2$ into $\dot{L}^2 \times L^2_0$. Note that $JJ^{-1} = \mathrm{Id}_{L^2_0\times\dot{L}^2}$ and $J^{-1}J = \mathrm{Id}_{\dot{L}^2\times L^2_0}$. The Hamiltonian vector field $X_{\mathcal{H}}(\eta,\zeta)$ in (4.1.14) is characterized by the identity

$$d\mathcal{H}(\eta,\zeta)[\widehat{u}] = \mathcal{W}\big(X_{\mathcal{H}}(\eta,\zeta), \widehat{u}\big), \quad \forall \widehat{u} := \begin{pmatrix}\widehat{\eta}\\ \widehat{\zeta}\end{pmatrix}.$$

The transformation W defined in (4.1.11) is reversibility preserving, namely it commutes with the involution $\mathcal{S}$ in (4.1.4) (see Definition 4.31 below), and thus also the Hamiltonian $\mathcal{H}$ in (4.1.13) is invariant under the involution $\mathcal{S}$, as well as H in (4.1.2). For this reason we look for solutions

$(\eta(t,x),\zeta(t,x))$ of (4.1.14) which are reversible, i.e. see (1.1.19),

$$\begin{pmatrix}\eta\\ \zeta\end{pmatrix}(-t) = \mathcal{S}\begin{pmatrix}\eta\\ \zeta\end{pmatrix}(t)\,. \tag{4.1.17}$$

The corresponding solutions $(\eta(t,x),\psi(t,x))$ of (4.0.1) induced by (4.1.11) are reversible as well.

We finally note that the transformation W defined in (4.1.11) commutes with the translation operator τ_ς, therefore the Hamiltonian $\mathcal{H}$ in (4.1.13) is invariant under τ_ς, as well as H in (4.1.2). By Noether theorem, the horizontal momentum $\int_{\mathbb{T}}\zeta\eta_x\,\mathrm{d}x$ is a prime integral of (4.1.14).

4.1.2 Linearization at the equilibrium

In this section we study the linear system (1.1.20) and prove that its reversible solutions have the form (1.1.22).

In view of the Hamiltonian (4.1.2) of the water waves equations (4.0.1), also the linear system (1.1.20) is Hamiltonian and it is generated by the quadratic Hamiltonian

$$H_L(\eta,\psi) := \frac{1}{2}\int_{\mathbb{T}}\left(\psi G(0)\psi + g\eta^2 + \kappa\eta_x^2\right)\mathrm{d}x = \frac{1}{2}\left(\mathbf{\Omega}_L\begin{pmatrix}\eta\\ \psi\end{pmatrix},\begin{pmatrix}\eta\\ \psi\end{pmatrix}\right)_{L^2}.$$

Thus, recalling (4.1.3), the linear system (1.1.20) is

$$\partial_t\begin{pmatrix}\eta\\ \psi\end{pmatrix} = J_M(\gamma)\mathbf{\Omega}_L\begin{pmatrix}\eta\\ \psi\end{pmatrix}, \qquad \mathbf{\Omega}_L := \begin{pmatrix}-\kappa\partial_x^2+g & 0\\ 0 & G(0)\end{pmatrix}. \tag{4.1.18}$$

The linear operator $\mathbf{\Omega}_L$ acts from (a dense subspace) of $L_0^2\times\dot{L}^2$ to $\dot{L}^2\times L_0^2$. In the Wahlén coordinates (4.1.11), the linear Hamiltonian system (1.1.20), i.e. (4.1.18), transforms into the linear Hamiltonian system

$$\begin{gathered}\partial_t\begin{pmatrix}\eta\\ \zeta\end{pmatrix} = J\mathbf{\Omega}_W\begin{pmatrix}\eta\\ \zeta\end{pmatrix},\\ \mathbf{\Omega}_W := W^*\mathbf{\Omega}_L W = \begin{pmatrix}-\kappa\partial_x^2+g-\left(\frac{\gamma}{2}\right)^2\partial_x^{-1}G(0)\partial_x^{-1} & -\frac{\gamma}{2}\partial_x^{-1}G(0)\\ \frac{\gamma}{2}G(0)\partial_x^{-1} & G(0)\end{pmatrix}\end{gathered} \tag{4.1.19}$$

generated by the quadratic Hamiltonian

$$\mathcal{H}_L(\eta,\zeta) := (H_L\circ W)(\eta,\zeta) = \frac{1}{2}\left(\mathbf{\Omega}_W\begin{pmatrix}\eta\\ \zeta\end{pmatrix},\begin{pmatrix}\eta\\ \zeta\end{pmatrix}\right)_{L^2}. \tag{4.1.20}$$

The linear operator $\mathbf{\Omega}_W$ acts from (a dense subspace) of $L_0^2\times\dot{L}^2$ to $\dot{L}^2\times L_0^2$. The linear system (4.1.19) is the Hamiltonian system obtained by linearizing (4.1.14) at the equilibrium $(\eta, \zeta) = (0, 0)$. We want to transform (4.1.19) in diagonal form by using a symmetrizer and then introducing complex coordinates. We first conjugate (4.1.19) under the symplectic transformation (with respect to the standard symplectic form $\mathcal{W}$ in (4.1.15)) of the phase space

$$\begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \mathcal{M} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $\mathcal{M}$ is the diagonal matrix of self-adjoint Fourier multipliers

$$\mathcal{M} := \begin{pmatrix} M(D) & 0 \\ 0 & M(D)^{-1} \end{pmatrix}, \quad M(D) := \left(\frac{G(0)}{\kappa D^2 + g - \frac{\gamma^2}{4} \partial_x^{-1} G(0) \partial_x^{-1}} \right)^{1/4}, \tag{4.1.21}$$

with the real valued symbol M_j defined in (4.0.7). The map $\mathcal{M}$ is reversibility preserving.

Remark 4.3. In (4.1.21) the Fourier multiplier $M(D)$ acts in H_0^1. On the other hand, with a slight abuse of notation, $M(D)^{-1}$ denotes the Fourier multiplier operator in $\dot{H}^1$ defined as

$$M(D)^{-1}[\zeta] := \big[\sum_{j \neq 0} M_j^{-1} \zeta_j e^{\mathrm{i} jx} \big], \qquad \zeta(x) = \sum_{j \in \mathbb{Z}} \zeta_j e^{\mathrm{i} jx}.$$

where $[\zeta]$ is the element in $\dot{H}^1$ with representant $\zeta(x)$.

By a direct computation, the Hamiltonian system (4.1.19) assumes the symmetric form

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = J \boldsymbol{\Omega}_S \begin{pmatrix} u \\ v \end{pmatrix}, \quad \boldsymbol{\Omega}_S := \mathcal{M}^* \boldsymbol{\Omega}_W \mathcal{M} = \begin{pmatrix} \omega(\kappa, D) & -\frac{\gamma}{2} \partial_x^{-1} G(0) \\ \frac{\gamma}{2} G(0) \partial_x^{-1} & \omega(\kappa, D) \end{pmatrix}, \tag{4.1.22}$$

where

$$\omega(\kappa, D) := \sqrt{\kappa D^2 \, G(0) + g \, G(0) - \Big(\frac{\gamma}{2} \partial_x^{-1} G(0) \Big)^2}. \tag{4.1.23}$$

Remark 4.4. To be precise, the Fourier multiplier operator $\omega(\kappa, D)$ in the top left position in (4.1.22) maps H_0^1 into $\dot{H}^1$ and the one in the bottom right position maps $\dot{H}^1$ into H_0^1. The operator $\partial_x^{-1} G(0)$ acts on $\dot{H}^1$ and $G(0) \partial_x^{-1}$ on H_0^1.

Now we introduce complex coordinates by the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{C} \begin{pmatrix} z \\ \overline{z} \end{pmatrix}, \qquad \mathcal{C} := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{Id} & \mathrm{Id} \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}, \quad \mathcal{C}^{-1} := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{Id} & \mathrm{i} \\ \mathrm{Id} & -\mathrm{i} \end{pmatrix}. \tag{4.1.24}$$

In these variables, the Hamiltonian system (4.1.22) becomes the diagonal system

$$\partial_t \begin{pmatrix} z \\ \overline{z} \end{pmatrix} = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \boldsymbol{\Omega}_D \begin{pmatrix} z \\ \overline{z} \end{pmatrix}, \quad \boldsymbol{\Omega}_D := \mathcal{C}^* \boldsymbol{\Omega}_S \mathcal{C} = \begin{pmatrix} \Omega(\kappa, D) & 0 \\ 0 & \overline{\Omega}(\kappa, D) \end{pmatrix}, \tag{4.1.25}$$

where

$$\Omega(\kappa, D) := \omega(\kappa, D) + \mathrm{i}\,\frac{\gamma}{2}\partial_x^{-1}G(0) \tag{4.1.26}$$

is the Fourier multiplier with symbol $\Omega_j(\kappa)$ defined in (4.0.3) and $\overline{\Omega}(\kappa, D)$ is defined by

$$\overline{\Omega}(\kappa, D)z := \overline{\Omega(\kappa, D)\overline{z}}\,, \quad \overline{\Omega}(\kappa, D) = \omega(\kappa, D) - \mathrm{i}\,\frac{\gamma}{2}\partial_x^{-1}G(0)\,.$$

Note that $\overline{\Omega}(\kappa, D)$ is the Fourier multiplier with symbol $\{\Omega_{-j}(\kappa)\}_{j\in\mathbb{Z}\setminus\{0\}}$.

Remark 4.5. We regard the system (4.1.25) in $\dot{H}^1 \times \dot{H}^1$.

The diagonal system (4.1.25) amounts to the scalar equation

$$\partial_t z = -\mathrm{i}\Omega(\kappa, D)z\,, \quad z(x) = \sum_{j\in\mathbb{Z}\setminus\{0\}} z_j e^{\mathrm{i}jx}\,, \tag{4.1.27}$$

and, writing (4.1.27) in the exponential Fourier basis, to the infinitely many decoupled harmonic oscillators

$$\dot{z}_j = -\mathrm{i}\Omega_j(\kappa)z_j\,, \quad j \in \mathbb{Z}\setminus\{0\}\,. \tag{4.1.28}$$

Note that, in these complex coordinates, the involution $\mathcal{S}$ defined in (4.1.4) reads as the map

$$\begin{pmatrix} z(x) \\ \overline{z(x)} \end{pmatrix} \mapsto \begin{pmatrix} \overline{z(-x)} \\ z(-x) \end{pmatrix} \tag{4.1.29}$$

that we may read just as the scalar map $z(x) \mapsto \overline{z(-x)}$. Moreover, in the Fourier coordinates introduced in (4.1.27), it amounts to

$$z_j \mapsto \overline{z_j}\,, \quad \forall j \in \mathbb{Z}\setminus\{0\}\,. \tag{4.1.30}$$

In view of (4.1.28) and (4.1.30) every *reversible* solution (which is characterized as in (4.1.17)) of (4.1.27) has the form

$$z(t, x) := \frac{1}{\sqrt{2}} \sum_{j\in\mathbb{Z}\setminus\{0\}} \rho_j\, e^{-\mathrm{i}\,(\Omega_j(\kappa)t - j\,x)} \quad \text{with} \quad \rho_j \in \mathbb{R}\,. \tag{4.1.31}$$

Let us see the form of these solutions back in the original variables (η, ψ). First, by (4.1.21), (4.1.24),

$$\begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \mathcal{M}\,\mathcal{C}\begin{pmatrix} z \\ \overline{z} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} M(D) & M(D) \\ -\mathrm{i}M(D)^{-1} & \mathrm{i}M(D)^{-1} \end{pmatrix}\begin{pmatrix} z \\ \overline{z} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} M(D)(z + \overline{z}) \\ -\mathrm{i}M(D)^{-1}(z - \overline{z}) \end{pmatrix}, \tag{4.1.32}$$

and the solutions (4.1.31) assume the form

$$\begin{pmatrix}\eta(t,x)\\ \zeta(t,x)\end{pmatrix} = \sum_{n\in\mathbb{N}} \begin{pmatrix} M_n\rho_n\cos(nx-\Omega_n(\kappa)t)\\ M_n^{-1}\rho_n\sin(nx-\Omega_n(\kappa)t)\end{pmatrix} + \sum_{n\in\mathbb{N}} \begin{pmatrix} M_n\rho_{-n}\cos(nx+\Omega_{-n}(\kappa)t)\\ -M_n^{-1}\rho_{-n}\sin(nx+\Omega_{-n}(\kappa)t)\end{pmatrix}.$$

Back to the variables (η,ψ) with the change of coordinates (4.1.11) one obtains formula (1.1.22).

Decomposition of the phase space in Lagrangian subspaces invariant under (4.1.19)**.** We express the Fourier coefficients $z_j \in \mathbb{C}$ in (4.1.27) as

$$z_j = \frac{\alpha_j + \mathrm{i}\beta_j}{\sqrt{2}}, \quad (\alpha_j,\beta_j)\in\mathbb{R}^2, \quad j\in\mathbb{Z}\backslash\{0\}.$$

In the new coordinates $(\alpha_j,\beta_j)_{j\in\mathbb{Z}\backslash\{0\}}$, we write (4.1.32) as (recall that $M_j = M_{-j}$)

$$\begin{pmatrix}\eta(x)\\ \zeta(x)\end{pmatrix} = \sum_{j\in\mathbb{Z}\backslash\{0\}} \begin{pmatrix} M_j(\alpha_j\cos(jx)-\beta_j\sin(jx))\\ M_j^{-1}(\beta_j\cos(jx)+\alpha_j\sin(jx))\end{pmatrix} \tag{4.1.33}$$

with

$$\begin{aligned}\alpha_j &= \frac{1}{2\pi}\Big(M_j^{-1}(\eta,\cos(jx))_{L^2} + M_j(\zeta,\sin(jx))_{L^2}\Big),\\ \beta_j &= \frac{1}{2\pi}\Big(M_j(\zeta,\cos(jx))_{L^2} - M_j^{-1}(\eta,\sin(jx))_{L^2}\Big).\end{aligned} \tag{4.1.34}$$

The symplectic form (4.1.15) then becomes

$$2\pi\sum_{j\in\mathbb{Z}\backslash\{0\}} \mathrm{d}\alpha_j\wedge\mathrm{d}\beta_j.$$

Each 2-dimensional subspace in the sum (4.1.33), spanned by $(\alpha_j,\beta_j)\in\mathbb{R}^2$ is therefore a symplectic subspace. The quadratic Hamiltonian $\mathcal{H}_L$ in (4.1.20) reads

$$2\pi\sum_{j\in\mathbb{Z}\backslash\{0\}} \frac{\Omega_j(\kappa)}{2}(\alpha_j^2+\beta_j^2). \tag{4.1.35}$$

In view of (4.1.33), the involution $\mathcal{S}$ defined in (4.1.4) reads

$$(\alpha_j,\beta_j)\mapsto(\alpha_j,-\beta_j), \quad \forall j\in\mathbb{Z}\backslash\{0\}, \tag{4.1.36}$$

and the translation operator τ_ς defined in (4.1.7) as

$$\begin{pmatrix}\alpha_j\\ \beta_j\end{pmatrix}\mapsto\begin{pmatrix}\cos(j\varsigma) & -\sin(j\varsigma)\\ \sin(j\varsigma) & \cos(j\varsigma)\end{pmatrix}\begin{pmatrix}\alpha_j\\ \beta_j\end{pmatrix}, \quad \forall j\in\mathbb{Z}\backslash\{0\}. \tag{4.1.37}$$

We may also enumerate the independent variables $(\alpha_j,\beta_j)_{j\in\mathbb{Z}\setminus\{0\}}$ as $\big(\alpha_{-n},\beta_{-n},\alpha_n,\beta_n\big)$, $n\in\mathbb{N}$. Thus the phase space $\mathfrak{H} := L_0^2\times\dot{L}^2$ of (4.1.14) decomposes as the direct sum

$$\mathfrak{H} = \sum_{n\in\mathbb{N}} V_{n,+}\oplus V_{n,-}$$

of 2-dimensional Lagrangian symplectic subspaces

$$V_{n,+} := \left\{ \begin{pmatrix}\eta\\ \zeta\end{pmatrix} = \begin{pmatrix} M_n(\alpha_n\cos(nx)-\beta_n\sin(nx))\\ M_n^{-1}(\beta_n\cos(nx)+\alpha_n\sin(nx))\end{pmatrix}, (\alpha_n,\beta_n)\in\mathbb{R}^2\right\}, \tag{4.1.38}$$

$$V_{n,-} := \left\{ \begin{pmatrix}\eta\\ \zeta\end{pmatrix} = \begin{pmatrix} M_n(\alpha_{-n}\cos(nx)+\beta_{-n}\sin(nx))\\ M_n^{-1}(\beta_{-n}\cos(nx)-\alpha_{-n}\sin(nx))\end{pmatrix}, (\alpha_{-n},\beta_{-n})\in\mathbb{R}^2\right\}, \tag{4.1.39}$$

which are invariant for the linear Hamiltonian system (4.1.19), namely $J\boldsymbol{\Omega}_W : V_{n,\sigma}\mapsto V_{n,\sigma}$ (for a proof see e.g. remark 4.11). The symplectic projectors $\Pi_{V_{n,\sigma}}$, $\sigma\in\{\pm\}$, on the symplectic subspaces $V_{n,\sigma}$ are explicitly provided by (4.1.33) and (4.1.34) with $j = n\sigma$.

Note that the involution $\mathcal{S}$ defined in (4.1.4) and the translation operator τ_ς in (4.1.7) leave the subspaces $V_{n,\sigma}$, $\sigma\in\{\pm\}$, invariant.

4.1.3 Tangential and normal subspaces of the phase space

We decompose the phase space $\mathfrak{H}$ of (4.1.14) into a direct sum of *tangential* and *normal* Lagrangian subspaces $\mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}$ and $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$. Note that the main part of the solutions (4.0.9) that we shall obtain in Theorem 1.8 is the component in the tangential subspace $\mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}$, whereas the component in the normal subspace $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ is much smaller.

Recalling the definition of the sets $\mathbb{S}^+$ and Σ defined in (4.0.4) respectively (4.0.5), we split

$$\mathfrak{H} = \mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}\oplus\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma} \tag{4.1.40}$$

where $\mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}$ is the finite dimensional *tangential subspace*

$$\mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma} := \sum_{a=1}^{\nu} V_{\overline{n}_a,\sigma_a} \tag{4.1.41}$$

and $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ is the *normal subspace* defined as its symplectic orthogonal

$$\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma} := \sum_{a=1}^{\nu} V_{\overline{n}_a,-\sigma_a}\oplus\sum_{n\in\mathbb{N}\setminus\mathbb{S}^+}\big(V_{n,+}\oplus V_{n,-}\big). \tag{4.1.42}$$

Both the subspaces $\mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}$ and $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ are Lagrangian. We denote by $\Pi^{\intercal}_{\mathbb{S}^+,\Sigma}$ and $\Pi^{\angle}_{\mathbb{S}^+,\Sigma}$ the symplectic projections on the subspaces $\mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}$ and $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$, respectively. Since $\mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}$ and $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ are symplectic orthogonal, the symplectic form $\mathcal{W}$ in (4.1.15) decomposes as

$$\mathcal{W}(v_1 + w_1, v_2 + w_2) = \mathcal{W}(v_1, v_2) + \mathcal{W}(w_1, w_2)\,, \quad \forall v_1, v_2 \in \mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}\,, \ w_1, w_2 \in \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}\,.$$

The symplectic projections $\Pi^{\intercal}_{\mathbb{S}^+,\Sigma}$ and $\Pi^{\angle}_{\mathbb{S}^+,\Sigma}$ satisfy the following properties:

Lemma 4.6. *We have that*

$$\Pi^{\intercal}_{\mathbb{S}^+,\Sigma}\, J = J\big(\Pi^{\intercal}_{\mathbb{S}^+,\Sigma}\big)^*\,, \qquad \big(\Pi^{\intercal}_{\mathbb{S}^+,\Sigma}\big)^*\, J^{-1} = J^{-1}\, \Pi^{\intercal}_{\mathbb{S}^+,\Sigma}\,, \tag{4.1.43}$$

$$\Pi^{\angle}_{\mathbb{S}^+,\Sigma}\, J = J\,\big(\Pi^{\angle}_{\mathbb{S}^+,\Sigma}\big)^*\,, \qquad \big(\Pi^{\angle}_{\mathbb{S}^+,\Sigma}\big)^*\, J^{-1} = J^{-1}\Pi^{\angle}_{\mathbb{S}^+,\Sigma}\,. \tag{4.1.44}$$

Proof. Since the subspaces $\mathfrak{H}^{\intercal} := \mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}$ and $\mathfrak{H}^{\angle} := \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ are symplectic orthogonal, we have, recalling (4.1.15), that

$$(J^{-1}v, w)_{L^2} = (J^{-1}w, v)_{L^2} = 0, \qquad \forall v \in \mathfrak{H}^{\intercal}\,,\ \forall w \in \mathfrak{H}^{\angle}\,.$$

Thus, using the projectors $\Pi^{\intercal} := \Pi^{\intercal}_{\mathbb{S}^+,\Sigma}$, $\Pi^{\angle} := \Pi^{\angle}_{\mathbb{S}^+,\Sigma}$, we have that

$$(J^{-1}\Pi^{\intercal}v, \Pi^{\angle}w)_{L^2} = (J^{-1}\Pi^{\angle}w, \Pi^{\intercal}v)_{L^2} = 0\,, \quad \forall v, w \in \mathfrak{H}\,,$$

and, taking adjoints, $((\Pi^{\angle})^* J^{-1}\Pi^{\intercal}v, w)_{L^2} = ((\Pi^{\intercal})^* J^{-1}\Pi^{\angle}w, v)_{L^2} = 0$ for any $v, w \in \mathfrak{H}$, so that

$$(\Pi^{\angle})^* J^{-1}\Pi^{\intercal} = 0 = (\Pi^{\intercal})^* J^{-1}\Pi^{\angle}\,. \tag{4.1.45}$$

Now inserting the identity $\Pi^{\angle} = \mathrm{Id} - \Pi^{\intercal}$ in (4.1.45), we get

$$J^{-1}\Pi^{\intercal} = (\Pi^{\intercal})^* J^{-1}\Pi^{\intercal} = (\Pi^{\intercal})^* J^{-1}$$

proving the second identity of (4.1.43). The first identity of (4.1.43) follows applying J to the left and to the right of the second identity. The identity (4.1.44) follows in the same way. □

Note that the restricted symplectic form $\mathcal{W}|_{\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}}$ is represented by the symplectic structure

$$J_{\angle}^{-1} : \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma} \to \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}\,, \qquad J_{\angle}^{-1} := \Pi^{L^2}_{\angle}\, J^{-1}_{|\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}}\,, \tag{4.1.46}$$

where $\Pi^{L^2}_{\angle}$ is the L^2-projector on the subspace $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$. Indeed

$$\mathcal{W}|_{\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}}(w, \hat{w}) = (J_{\angle}^{-1}w, \widehat{w})_{L^2} = (J^{-1}w, \widehat{w})_{L^2}, \quad \forall\, w, \hat{w} \in \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}\,.$$

We also denote the associated (restricted) Poisson tensor

$$J_\angle : \mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma} \to \mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma}\,, \quad J_\angle := \Pi^\angle_{\mathbb{S}^+,\Sigma}\, J_{|\mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma}}\,. \tag{4.1.47}$$

In the next lemma we prove that $J_\angle^{-1}$ and $J_\angle$ are each other inverses.

Lemma 4.7. $J_\angle^{-1}\, J_\angle = J_\angle\, J_\angle^{-1} = \mathrm{Id}_{\mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma}}$.

Proof. Let $v \in \mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma}$. By (4.1.46) and (4.1.47), for any $h \in \mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma}$ one has

$$\begin{aligned}(J_\angle^{-1}\, J_\angle\, v, h)_{L^2} &= (J^{-1}\Pi^\angle_{\mathbb{S}^+,\Sigma}\, Jv, \Pi^{L^2}_\angle h)_{L^2} = -(\Pi^\angle_{\mathbb{S}^+,\Sigma}\, Jv,\, J^{-1}h)_{L^2} \\ &= -(Jv, (\Pi^\angle_{\mathbb{S}^+,\Sigma})^* J^{-1}h)_{L^2} \overset{(4.1.44)}{=} -(Jv, J^{-1}\,\Pi^\angle_{\mathbb{S}^+,\Sigma} h)_{L^2} = (v,h)_{L^2}\,.\end{aligned}$$

The proof that $J_\angle J_\angle^{-1} = \mathrm{Id}_{\mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma}}$ is similar. □

Lemma 4.8. $\Pi^\angle_{\mathbb{S}^+,\Sigma} J \Pi^{L^2}_\angle = \Pi^\angle_{\mathbb{S}^+,\Sigma} J$.

Proof. For any $u, h \in \mathfrak{H}$ we have, using Lemma 4.6,

$$\begin{aligned}(\Pi^\angle_{\mathbb{S}^+,\Sigma} J \Pi^{L^2}_\angle u, h)_{L^2} &= -(\Pi^{L^2}_\angle u, J(\Pi^\angle_{\mathbb{S}^+,\Sigma})^* h)_{L^2} = -(\Pi^{L^2}_\angle u, \Pi^\angle_{\mathbb{S}^+,\Sigma} Jh)_{L^2} \\ &= -(u, \Pi^\angle_{\mathbb{S}^+,\Sigma} Jh)_{L^2} = (J(\Pi^\angle_{\mathbb{S}^+,\Sigma})^* u, h)_{L^2} = (\Pi^\angle_{\mathbb{S}^+,\Sigma} Ju, h)_{L^2}\end{aligned}$$

implying the lemma. □

Action-angle coordinates. We introduce action-angle coordinates on the tangential subspace $\mathfrak{H}^\intercal_{\mathbb{S}^+,\Sigma}$ defined in (4.1.41). Given the sets $\mathbb{S}^+$ and Σ defined in (4.0.4) and (4.0.5), we define the set

$$\mathbb{S} := \{\bar{\jmath}_1, \ldots, \bar{\jmath}_\nu\} \subset \mathbb{Z}\backslash\{0\}\,, \quad \bar{\jmath}_a := \sigma_a \overline{n}_a\,, \quad a = 1, \ldots, \nu\,, \tag{4.1.48}$$

and the action-angle coordinates $(\theta_j, I_j)_{j \in \mathbb{S}}$, by the relations

$$\alpha_j = \sqrt{\frac{1}{\pi}(I_j + \xi_j)}\cos(\theta_j)\,,\ \beta_j = -\sqrt{\frac{1}{\pi}(I_j + \xi_j)}\sin(\theta_j)\,, \quad \xi_j > 0\,,\ |I_j| < \xi_j\,,\ \forall\, j \in \mathbb{S}\,. \tag{4.1.49}$$

In view of (4.1.40)-(4.1.42), we represent any function of the phase space $\mathfrak{H}$ as

$$\begin{aligned}A(\theta, I, w) &:= v^\intercal(\theta, I) + w\,, \\ &:= \frac{1}{\sqrt{\pi}}\sum_{j \in \mathbb{S}}\left[\begin{pmatrix} M_j\sqrt{I_j+\xi_j}\cos(\theta_j) \\ -M_j^{-1}\sqrt{I_j+\xi_j}\sin(\theta_j)\end{pmatrix}\cos(jx) + \begin{pmatrix} M_j\sqrt{I_j+\xi_j}\sin(\theta_j) \\ M_j^{-1}\sqrt{I_j+\xi_j}\cos(\theta_j)\end{pmatrix}\sin(jx)\right] + w \\ &= \frac{1}{\sqrt{\pi}}\sum_{j \in \mathbb{S}}\left[\begin{pmatrix} M_j\sqrt{I_j+\xi_j}\cos(\theta_j - jx) \\ -M_j^{-1}\sqrt{I_j+\xi_j}\sin(\theta_j - jx)\end{pmatrix}\right] + w\end{aligned} \tag{4.1.50}$$

where $\theta := (\theta_j)_{j\in\mathbb{S}} \in \mathbb{T}^\nu$, $I := (I_j)_{j\in\mathbb{S}} \in \mathbb{R}^\nu$ and $w \in \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$.

Remark 4.9. In these coordinates the solutions (4.0.6) of the linear system (1.1.20) simply read as $Wv^\intercal(\vec{\Omega}(\kappa)t, 0)$, where $\vec{\Omega}(\kappa) := (\Omega_j(\kappa))_{j\in\mathbb{S}}$ is given in (4.0.8).

In view of (4.1.50), the involution $\mathcal{S}$ in (4.1.4) reads

$$\vec{\mathcal{S}} : (\theta, I, w) \mapsto (-\theta, I, \mathcal{S}w) \ , \tag{4.1.51}$$

the translation operator τ_ς in (4.1.7) reads

$$\vec{\tau}_\varsigma : (\theta,\, I,\, w) \mapsto (\theta - \vec{\jmath}\varsigma,\, I,\, \tau_\varsigma w), \quad \forall \varsigma \in \mathbb{R}\,, \tag{4.1.52}$$

where

$$\vec{\jmath} := (j)_{j\in\mathbb{S}} = (\bar{\jmath}_1, \ldots, \bar{\jmath}_\nu) \in \mathbb{Z}^\nu \backslash \{0\}\,, \tag{4.1.53}$$

and the symplectic 2-form (4.1.15) becomes

$$\mathcal{W} = \sum_{j\in\mathbb{S}} (\mathrm{d}\theta_j \wedge \mathrm{d}I_j) \oplus \mathcal{W}|_{\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}}\,. \tag{4.1.54}$$

We also note that $\mathcal{W}$ is exact, namely

$$\mathcal{W} = d\Lambda\,, \qquad \text{where} \qquad \Lambda_{(\theta,I,w)}[\widehat{\theta}, \widehat{I}, \widehat{w}] := -\sum_{j\in\mathbb{S}} I_j \widehat{\theta}_j + \tfrac{1}{2}\left(J_\angle^{-1} w, \widehat{w}\right)_{L^2} \tag{4.1.55}$$

is the associated Liouville 1-form (the operator $J_\angle^{-1}$ is defined in (4.1.46)).

Given a Hamiltonian $K\colon \mathbb{T}^\nu \times \mathbb{R}^\nu \times \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma} \to \mathbb{R}$, the associated Hamiltonian vector field (with respect to the symplectic form (4.1.54)) is

$$X_K := \big(\partial_I K, -\partial_\theta K, J_\angle \nabla_w K\big) = \big(\partial_I K, -\partial_\theta K, \Pi^{\angle}_{\mathbb{S}^+,\Sigma} J \nabla_w K\big)\,, \tag{4.1.56}$$

where $\nabla_w K$ denotes the L^2 gradient of K with respect to $w \in \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$. Indeed, the only nontrivial component of the vector field X_K is the last one, which we denote by $[X_K]_w \in \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$. It fulfills

$$(J_\angle^{-1}[X_K]_w, \widehat{w})_{L^2} = \mathrm{d}_w K[\widehat{w}] = (\nabla_w K, \widehat{w})_{L^2}\,, \quad \forall\, \widehat{w} \in \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}\,, \tag{4.1.57}$$

and (4.1.56) follows by Lemma 4.7. We remark that along the paper we only consider Hamiltonians such that the L^2-gradient $\nabla_w K$ defined by (4.1.57), as well as the Hamiltonian vector field $\Pi^{\angle}_{\mathbb{S}^+,\Sigma} J \nabla_w K$, maps spaces of Sobolev functions into Sobolev functions (not just distributions), with possible loss of derivatives.

Tangential and normal subspaces in complex variables. Each 2-dimensional symplectic subspace $V_{n,\sigma}$, $n \in \mathbb{N}$, $\sigma = \pm 1$, defined in (4.1.38)-(4.1.39) is isomorphic, through the linear map $\mathcal{MC}$ defined in (4.1.32), to the complex subspace

$$\mathbf{H}_j := \left\{ \begin{pmatrix} z_j e^{\mathrm{i}jx} \\ \overline{z_j} e^{-\mathrm{i}jx} \end{pmatrix}, \ z_j \in \mathbb{C} \right\}, \qquad \text{with} \qquad j = n\sigma \in \mathbb{Z}\,.$$

Denoting by Π_j the L^2-projection on $\mathbf{H}_j$, we have that $\Pi_{V_{n,\sigma}} = \mathcal{MC}\,\Pi_j\,(\mathcal{MC})^{-1}$. Thus $\mathcal{MC}$ is an isomorphism between the tangential subspace $\mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}$ defined in (4.1.41) and

$$\mathbf{H}_{\mathbb{S}} := \left\{ \begin{pmatrix} z \\ \overline{z} \end{pmatrix} \ : \ z(x) = \sum_{j \in \mathbb{S}} z_j e^{\mathrm{i}jx} \right\}$$

and between the normal subspace $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ defined in (4.1.42) and

$$\mathbf{H}^{\perp}_{\mathbb{S}_0} := \left\{ \begin{pmatrix} z \\ \overline{z} \end{pmatrix} \ : \ z(x) = \sum_{j \in \mathbb{S}_0^c} z_j e^{\mathrm{i}jx} \in L^2 \right\}, \quad \mathbb{S}_0^c := \mathbb{Z} \backslash (\mathbb{S} \cup \{0\})\,. \tag{4.1.58}$$

Denoting by $\Pi^{\intercal}_{\mathbb{S}}$, $\Pi^{\perp}_{\mathbb{S}_0}$, the L^2-orthogonal projections on the subspaces $\mathbf{H}_{\mathbb{S}}$ and $\mathbf{H}^{\perp}_{\mathbb{S}_0}$, we have that

$$\Pi^{\intercal}_{\mathbb{S}^+,\Sigma} = \mathcal{MC}\,\Pi^{\intercal}_{\mathbb{S}}\,(\mathcal{MC})^{-1}\,, \quad \Pi^{\angle}_{\mathbb{S}^+,\Sigma} = \mathcal{MC}\,\Pi^{\perp}_{\mathbb{S}_0}\,(\mathcal{MC})^{-1}\,. \tag{4.1.59}$$

The following lemma, used in Section 4.4, is an easy corollary of the previous analysis.

Lemma 4.10. *We have that* $(v^{\intercal}, \mathbf{\Omega}_W w)_{L^2} = 0$*, for any* $v^{\intercal} \in \mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}$ *and* $w \in \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$.

Proof. Write $v^{\intercal} = \mathcal{MC} z^{\intercal}$ and $\mathcal{MC} z^{\perp}$ with $z^{\intercal} \in \mathbf{H}_{\mathbb{S}}$ and $z^{\perp} \in \mathbf{H}^{\perp}_{\mathbb{S}_0}$. Then, by (4.1.22) and (4.1.25),

$$(v^{\intercal}, \mathbf{\Omega}_W w)_{L^2} = \big(\mathcal{MC} z^{\intercal}, \mathbf{\Omega}_W \mathcal{MC} z^{\perp}\big)_{L^2} = \big(z^{\intercal}, \mathbf{\Omega}_D z^{\perp}\big)_{L^2} = 0\,,$$

since $\mathbf{\Omega}_D$ preserves the subspace $\mathbf{H}^{\perp}_{\mathbb{S}_0}$. □

Remark 4.11. The same proof of Lemma 4.10 actually shows that $(v_{n,-\sigma}, \mathbf{\Omega}_W v_{n,\sigma})_{L^2} = 0$ for any $v_{n,\pm\sigma} \in V_{n,\pm\sigma}$, for any $n \in \mathbb{N}$, $\sigma = \pm 1$. Thus $\mathcal{W}(v_{n,-\sigma}, J\mathbf{\Omega}_W v_{n,\sigma}) = (v_{n,-\sigma}, J^{-1} J\mathbf{\Omega}_W v_{n,\sigma})_{L^2} = 0$ which shows that $J\mathbf{\Omega}_W$ maps $V_{n,\sigma}$ in itself.

Notation. For $a \lesssim_s b$ means that $a \leqslant C(s) b$ for some positive constant $C(s)$. We denote $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

4.2 Functional setting

Along this chapter we consider functions $u(\varphi, x) \in L^2\left(\mathbb{T}^{\nu+1}, \mathbb{C}\right)$ depending on the space variable $x \in \mathbb{T} = \mathbb{T}_x$ and the angles $\varphi \in \mathbb{T}^\nu = \mathbb{T}^\nu_\varphi$ (so that $\mathbb{T}^{\nu+1} = \mathbb{T}^\nu_\varphi \times \mathbb{T}_x$) which we expand in Fourier series as

$$u(\varphi, x) = \sum_{j \in \mathbb{Z}} u_j(\varphi) e^{\mathrm{i}\, jx} = \sum_{\ell \in \mathbb{Z}^\nu, j \in \mathbb{Z}} u_{\ell,j} e^{\mathrm{i}(\ell \cdot \varphi + jx)} . \tag{4.2.1}$$

We also consider real valued functions $u(\varphi, x) \in \mathbb{R}$, as well as vector valued functions $u(\varphi, x) \in \mathbb{C}^2$ (or $u(\varphi, x) \in \mathbb{R}^2$). When no confusion appears, we denote simply by L^2, $L^2(\mathbb{T}^{\nu+1})$, $L^2_x := L^2(\mathbb{T}_x)$, $L^2_\varphi := L^2(\mathbb{T}^\nu)$ either the spaces of real/complex valued, scalar/vector valued, L^2-functions.

A crucial role is played by the following subspace of functions of (φ, x).

Definition 4.12. (Quasi-periodic traveling waves) Let $\vec{\jmath} := (\bar{\jmath}_1, \dots, \bar{\jmath}_\nu) \in \mathbb{Z}^\nu$ be the vector defined in (4.1.53). A function $u(\varphi, x)$ is called a *quasi-periodic traveling* wave if it has the form $u(\varphi, x) = U(\varphi - \vec{\jmath} x)$ where $U : \mathbb{T}^\nu \to \mathbb{C}^K$, $K \in \mathbb{N}$, is a $(2\pi)^\nu$-periodic function.

Comparing with Definition 1.5, we find convenient to call *quasi-periodic traveling* wave both the function $u(\varphi, x) = U(\varphi - \vec{\jmath} x)$ and the function of time $u(\omega t, x) = U(\omega t - \vec{\jmath} x)$.

Quasi-periodic traveling waves are characterized by the relation

$$u(\varphi - \vec{\jmath} \varsigma, \cdot) = \tau_\varsigma u \quad \forall \varsigma \in \mathbb{R} , \tag{4.2.2}$$

where τ_ς is the translation operator in (4.1.7). Product and composition of quasi-periodic traveling waves is a quasi-periodic traveling wave. Expanded in Fourier series as in (4.2.1), a quasi-periodic traveling wave has the form

$$u(\varphi, x) = \sum_{\ell \in \mathbb{Z}^\nu, j \in \mathbb{Z}, j + \vec{\jmath} \cdot \ell = 0} u_{\ell,j} e^{\mathrm{i}(\ell \cdot \varphi + jx)} , \tag{4.2.3}$$

namely, comparing with Definition 4.12,

$$u(\varphi, x) = U(\varphi - \vec{\jmath} x) , \quad U(\psi) = \sum_{\ell \in \mathbb{Z}^\nu} U_\ell e^{\mathrm{i}\ell \cdot \psi} , \quad U_\ell = u_{\ell, -\vec{\jmath} \cdot \ell} . \tag{4.2.4}$$

The traveling waves $u(\varphi, x) = U(\varphi - \vec{\jmath} x)$ where $U(\cdot)$ belongs to the Sobolev space $H^s(\mathbb{T}^\nu, \mathbb{C}^K)$ in (1.1.25) (with values in $\mathbb{C}^K$, $K \in \mathbb{N}$), form a subspace of the Sobolev space

$$H^s(\mathbb{T}^{\nu+1}) = \Big\{ u = \sum_{(\ell,j) \in \mathbb{Z}^{\nu+1}} u_{\ell,j}\, e^{\mathrm{i}(\ell \cdot \varphi + jx)} \, : \, \|u\|_s^2 := \sum_{(\ell,j) \in \mathbb{Z}^{\nu+1}} |u_{\ell,j}|^2 \langle \ell, j \rangle^{2s} < \infty \Big\} \tag{4.2.5}$$

where $\langle \ell, j\rangle := \max\{1, |\ell|, |j|\}$. Note the equivalence of the norms (use (4.2.4))

$$\|u\|_{H^s(\mathbb{T}^\nu_\varphi \times \mathbb{T}_x)} \simeq_s \|U\|_{H^s(\mathbb{T}^\nu)} .$$

For $s \geqslant s_0 := \left[\frac{\nu+1}{2}\right] + 1 \in \mathbb{N}$ one has $H^s(\mathbb{T}^{\nu+1}) \subset C(\mathbb{T}^{\nu+1})$, and $H^s(\mathbb{T}^{\nu+1})$ is an algebra. Along the chapter we denote by $\| \ \|_s$ both the Sobolev norms in (1.1.25) and (4.2.5).

For $K \geqslant 1$ we define the smoothing operator Π_K on the traveling waves

$$\Pi_K : u = \sum_{\ell\in\mathbb{Z}^\nu,\, j\in\mathbb{S}_0^c,\, j+\vec{\jmath}\cdot\ell=0} u_{\ell,j} e^{\mathrm{i}(\ell\cdot\varphi+jx)} \mapsto \Pi_K u = \sum_{\langle\ell\rangle\leqslant K,\, j\in\mathbb{S}_0^c,\, j+\vec{\jmath}\cdot\ell=0} u_{\ell,j} e^{\mathrm{i}(\ell\cdot\varphi+jx)} , \tag{4.2.6}$$

and $\Pi_K^\perp := \mathrm{Id} - \Pi_K$. Note that, writing a traveling wave as in (4.2.4), the projector Π_K in (4.2.6) is equal to

$$(\Pi_K u)(\varphi, x) = U_K(\varphi - \vec{\jmath}x) , \quad U_K(\psi) := \sum_{\ell\in\mathbb{Z}^\nu,\, \langle\ell\rangle\leqslant K} U_\ell e^{\mathrm{i}\ell\cdot\psi} .$$

Whitney-Sobolev functions. We consider families of Sobolev functions $\lambda \mapsto u(\lambda) \in H^s(\mathbb{T}^{\nu+1})$ and $\lambda \mapsto U(\lambda) \in H^s(\mathbb{T}^\nu)$ which are k_0-times differentiable in the sense of Whitney with respect to the parameter $\lambda := (\omega, \kappa) \in F \subset \mathbb{R}^\nu \times [\kappa_1, \kappa_2]$ where $F \subset \mathbb{R}^{\nu+1}$ is a closed set. The case that we encounter is when ω belongs to the closed set of Diophantine vectors $\mathtt{DC}(\upsilon, \tau)$ defined in (1.1.24). We refer to Definition 2.1 in [13], for the definition of a Whitney-Sobolev function $u : F \to H^s$ where H^s may be either the Hilbert space $H^s(\mathbb{T}^\nu \times \mathbb{T})$ or $H^s(\mathbb{T}^\nu)$. Here we mention that, given $\upsilon \in (0,1)$, we can identify a Whitney-Sobolev function $u : F \to H^s$ with k_0 derivatives with the equivalence class of functions $f \in W^{k_0,\infty,\upsilon}(\mathbb{R}^{\nu+1}, H^s)/\sim$ with respect to the equivalence relation $f \sim g$ when $\partial_\lambda^j f(\lambda) = \partial_\lambda^j g(\lambda)$ for all $\lambda \in F$, $|j| \leqslant k_0 - 1$, with equivalence of the norms

$$\|u\|_{s,F}^{k_0,\upsilon} \sim_{\nu,k_0} \|u\|_{W^{k_0,\infty,\upsilon}(\mathbb{R}^{\nu+1},H^s)} := \sum_{|\alpha|\leqslant k_0} \upsilon^{|\alpha|} \|\partial_\lambda^\alpha u\|_{L^\infty(\mathbb{R}^{\nu+1},H^s)} .$$

The key result is the Whitney extension theorem, which associates to a Whitney-Sobolev function $u : F \to H^s$ with k_0-derivatives a function $\tilde{u} : \mathbb{R}^{\nu+1} \to H^s$, $\tilde{u}$ in $W^{k_0,\infty}(\mathbb{R}^{\nu+1}, H^s)$ (independently of the target Sobolev space H^s) with an equivalent norm. For sake of simplicity in the notation we often denote $\| \ \|_{s,F}^{k_0,\upsilon} = \| \ \|_s^{k_0,\upsilon}$.

Thanks to this equivalence, all the tame estimates which hold for Sobolev spaces carry over for Whitney-Sobolev functions. For example the following classical tame estimate for the product holds: (see e.g. Lemma 2.4 in [13]): for all $s \geqslant s_0 > (\nu+1)/2$,

$$\|uv\|_s^{k_0,\upsilon} \leqslant C(s, k_0)\|u\|_s^{k_0,\upsilon}\|v\|_{s_0}^{k_0,\upsilon} + C(s_0, k_0)\|u\|_{s_0}^{k_0,\upsilon}\|v\|_s^{k_0,\upsilon} . \tag{4.2.7}$$

Moreover the following estimates hold for the smoothing operators defined in (4.2.6): for any traveling wave u

$$\|\Pi_K u\|_s^{k_0,\upsilon} \leqslant K^\alpha \|u\|_{s-\alpha}^{k_0,\upsilon}, \ 0 \leqslant \alpha \leqslant s, \quad \|\Pi_K^\perp u\|_s^{k_0,\upsilon} \leqslant K^{-\alpha} \|u\|_{s+\alpha}^{k_0,\upsilon}, \ \alpha \geqslant 0. \tag{4.2.8}$$

We also state a standard Moser tame estimate for the nonlinear composition operator, see e.g. Lemma 2.6 in [13],

$$u(\varphi, x) \mapsto \mathtt{f}(u)(\varphi, x) := f(\varphi, x, u(\varphi, x)).$$

Since the variables $(\varphi, x) =: y$ have the same role, we state it for a generic Sobolev space $H^s(\mathbb{T}^d)$.

Lemma 4.13. (Composition operator) *Let $f \in \mathcal{C}^\infty(\mathbb{T}^d \times \mathbb{R}, \mathbb{R})$. If $u(\lambda) \in H^s(\mathbb{T}^d)$ is a family of Sobolev functions satisfying $\|u\|_{s_0}^{k_0,\upsilon} \leqslant 1$, then, for all $s \geqslant s_0 := (d+1)/2$,*

$$\|\mathtt{f}(u)\|_s^{k_0,\upsilon} \leqslant C(s, k_0, f)\big(1 + \|u\|_s^{k_0,\upsilon}\big).$$

If $f(\varphi, x, 0) = 0$ then $\|\mathtt{f}(u)\|_s^{k_0,\upsilon} \leqslant C(s, k_0, f)\|u\|_s^{k_0,\upsilon}$.

Diophantine equation. If ω is a Diophantine vector in $\mathtt{DC}(\upsilon, \tau)$, see (1.1.24), then the equation $\omega \cdot \partial_\varphi v = u$, where $u(\varphi, x)$ has zero average with respect to φ, has the periodic solution

$$(\omega \cdot \partial_\varphi)^{-1} u := \sum_{\ell \in \mathbb{Z}^\nu \setminus \{0\}, j \in \mathbb{Z}} \frac{u_{\ell,j}}{\mathrm{i}\,\omega \cdot \ell} e^{\mathrm{i}(\ell \cdot \varphi + jx)}.$$

For all $\omega \in \mathbb{R}^\nu$, we define its extension

$$(\omega \cdot \partial_\varphi)^{-1}_{\rm ext} u(\varphi, x) := \sum_{(\ell, j) \in \mathbb{Z}^{\nu+1}} \frac{\chi(\omega \cdot \ell \upsilon^{-1} \langle \ell \rangle^\tau)}{\mathrm{i}\omega \cdot \ell} u_{\ell,j} e^{\mathrm{i}(\ell \cdot \varphi + jx)}, \tag{4.2.9}$$

where $\chi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is an even positive $\mathcal{C}^\infty$ cut-off function such that

$$\chi(\xi) = \begin{cases} 0 & \text{if } \ |\xi| \leqslant \frac{1}{3} \\ 1 & \text{if } \ |\xi| \geqslant \frac{2}{3} \end{cases}, \qquad \partial_\xi \chi(\xi) > 0 \quad \forall\, \xi \in (\tfrac{1}{3}, \tfrac{2}{3}). \tag{4.2.10}$$

Note that $(\omega \cdot \partial_\varphi)^{-1}_{\rm ext} u = (\omega \cdot \partial_\varphi)^{-1} u$ for all $\omega \in \mathtt{DC}(\upsilon, \tau)$. Moreover, if $u(\varphi, x)$ is a quasi-periodic traveling wave with zero average with respect to φ, then, by (4.2.3), we see that $(\omega \cdot \partial_\varphi)^{-1}_{\rm ext} u(\varphi, x)$ is a quasi-periodic traveling wave. The following estimate holds

$$\|(\omega \cdot \partial_\varphi)^{-1}_{\rm ext} u\|_{s, \mathbb{R}^{\nu+1}}^{k_0,\upsilon} \leqslant C(k_0) \upsilon^{-1} \|u\|_{s+\mu, \mathbb{R}^{\nu+1}}^{k_0,\upsilon}, \quad \mu := k_0 + \tau(k_0 + 1). \tag{4.2.11}$$

and, for $F \subseteq \mathtt{DC}(\upsilon, \tau) \times \mathbb{R}_+$, one has $\|(\omega \cdot \partial_\varphi)^{-1} u\|_{s,F}^{k_0,\upsilon} \leqslant C(k_0) \upsilon^{-1} \|u\|_{s+\mu,F}^{k_0,\upsilon}$.

Linear operators. We consider φ-dependent families of linear operators $A : \mathbb{T}^\nu \mapsto \mathcal{L}(L^2(\mathbb{T}_x))$, $\varphi \mapsto A(\varphi)$, acting on subspaces of $L^2(\mathbb{T}_x)$, either real or complex valued. We also regard A as an operator (which for simplicity we denote by A as well) that acts on functions $u(\varphi, x)$ of space and time, that is

$$(Au)(\varphi, x) := (A(\varphi)u(\varphi, \cdot))(x). \tag{4.2.12}$$

The action of an operator A as in (4.2.12) on a scalar function $u(\varphi, x) \in L^2$ expanded as in (4.2.1) is

$$Au(\varphi, x) = \sum_{j,j' \in \mathbb{Z}} A_j^{j'}(\varphi) u_{j'}(\varphi) e^{\mathrm{i}\, jx} = \sum_{j,j' \in \mathbb{Z}} \sum_{\ell, \ell' \in \mathbb{Z}^\nu} A_j^{j'}(\ell - \ell') u_{\ell', j'} e^{\mathrm{i}(\ell \cdot \varphi + jx)}. \tag{4.2.13}$$

We identify an operator A with its matrix $\big(A_j^{j'}(\ell - \ell')\big)_{j,j' \in \mathbb{Z}, \ell, \ell' \in \mathbb{Z}^\nu}$, which is Töplitz with respect to the index ℓ. We always consider Töplitz operators as in (4.2.12), (4.2.13).

Real operators. A linear operator A is *real* if $A = \overline{A}$, where $\overline{A}$ is defined by $\overline{A}(u) := \overline{A(\overline{u})}$. Equivalently A is *real* if it maps real valued functions into real valued functions. We represent a real operator acting on (η, ζ) belonging to (a subspace of) $L^2(\mathbb{T}_x, \mathbb{R}^2)$ by a matrix

$$\mathcal{R} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4.2.14}$$

where A, B, C, D are real operators acting on the scalar valued components $\eta, \zeta \in L^2(\mathbb{T}_x, \mathbb{R})$.

The change of coordinates (4.1.24) transforms the real operator $\mathcal{R}$ into a complex one acting on the variables $(z, \overline{z})$, given by the matrix

$$\begin{gathered} \mathbf{R} := \mathcal{C}^{-1} \mathcal{R} \mathcal{C} = \begin{pmatrix} \mathcal{R}_1 & \mathcal{R}_2 \\ \overline{\mathcal{R}}_2 & \overline{\mathcal{R}}_1 \end{pmatrix}, \\ \mathcal{R}_1 := \frac{1}{2} \{(A + D) - \mathrm{i}(B - C)\}, \quad \mathcal{R}_2 := \frac{1}{2} \{(A - D) + \mathrm{i}(B + C)\}. \end{gathered} \tag{4.2.15}$$

A matrix operator acting on the complex variables $(z, \overline{z})$ of the form (4.2.15), we call it *real*. We shall also consider real operators $\mathbf{R}$ of the form (4.2.15) acting on subspaces of L^2.

Lie expansion. Let $X(\varphi)$ be a linear operator with associated flow $\Phi^\tau(\varphi)$ defined by

$$\begin{cases} \partial_\tau \Phi^\tau(\varphi) = X(\varphi) \Phi^\tau(\varphi) \\ \Phi^0(\varphi) = \mathrm{Id}, \end{cases} \qquad \tau \in [0, 1].$$

Let $\Phi(\varphi) := \Phi^\tau(\varphi)_{|\tau=1}$ denote the time-1 flow. Given a linear operator $A(\varphi)$, the conjugated operator

$$A^+(\varphi) := \Phi(\varphi)^{-1} A(\varphi)\Phi(\varphi)$$

admits the Lie expansion, for any $M \in \mathbb{N}_0$,

$$\begin{aligned} A^+(\varphi) &= \sum_{m=0}^{M} \frac{(-1)^m}{m!} \mathrm{ad}^m_{X(\varphi)}(A(\varphi)) + R_M(\varphi)\,, \\ R_M(\varphi) &= \frac{(-1)^{M+1}}{M!} \int_0^1 (1-\tau)^M\, (\Phi^\tau(\varphi))^{-1} \mathrm{ad}^{M+1}_{X(\varphi)}(A(\varphi))\Phi^\tau(\varphi)\, \mathrm{d}\tau\,, \end{aligned} \tag{4.2.16}$$

where $\mathrm{ad}_{X(\varphi)}(A(\varphi)) := [X(\varphi), A(\varphi)] = X(\varphi)A(\varphi) - A(\varphi)X(\varphi)$ and $\mathrm{ad}^0_{X(\varphi)} := \mathrm{Id}$.

In particular, for $A = \omega \cdot \partial_\varphi$, since $[X(\varphi), \omega\cdot\partial_\varphi] = -(\omega\cdot\partial_\varphi X)(\varphi)$, we obtain

$$\begin{aligned} \Phi(\varphi)^{-1} \circ \omega\cdot\partial_\varphi \circ \Phi(\varphi) = \omega\cdot\partial_\varphi &+ \sum_{m=1}^{M} \frac{(-1)^{m+1}}{m!} \mathrm{ad}^{m-1}_{X(\varphi)}(\omega\cdot\partial_\varphi X(\varphi)) \\ &+ \frac{(-1)^M}{M!} \int_0^1 (1-\tau)^M (\Phi^\tau(\varphi))^{-1} \mathrm{ad}^M_{X(\varphi)}(\omega\cdot\partial_\varphi X(\varphi))\Phi^\tau(\varphi)\, \mathrm{d}\tau\,. \end{aligned} \tag{4.2.17}$$

For matrices of operators $\mathbf{X}(\varphi)$ and $\mathbf{A}(\varphi)$ as in (4.2.15), the same formula (4.2.16) holds.

4.2.1 Pseudodifferential calculus

In this section we report fundamental notions of pseudodifferential calculus, following [44].

Definition 4.14. (ΨDO) A *pseudodifferential* symbol $a(x,j)$ of order m is the restriction to $\mathbb{R}\times\mathbb{Z}$ of a function $a(x,\xi)$ which is $\mathcal{C}^\infty$-smooth on $\mathbb{R}\times\mathbb{R}$, 2π-periodic in x, and satisfies

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leqslant C_{\alpha,\beta}\langle\xi\rangle^{m-\beta}\,, \quad \forall \alpha,\beta\in\mathbb{N}_0\,.$$

We denote by S^m the class of symbols of order m and $S^{-\infty} := \cap_{m\geqslant 0} S^m$. To a symbol $a(x,\xi)$ in S^m we associate its quantization acting on a 2π-periodic function $u(x) = \sum_{j\in\mathbb{Z}} u_j\, e^{\mathrm{i}jx}$ as

$$[\mathrm{Op}(a)u](x) := \sum_{j\in\mathbb{Z}} a(x,j) u_j\, e^{\mathrm{i}jx}\,.$$

We denote by $\mathrm{OP}S^m$ the set of pseudodifferential operators of order m and $\mathrm{OP}S^{-\infty} := \bigcap_{m\in\mathbb{R}} \mathrm{OP}S^m$. For a matrix of pseudodifferential operators

$$\mathbf{A} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad A_i \in \mathrm{OP}S^m, \quad i = 1,\dots,4 \tag{4.2.18}$$

we say that $\mathbf{A} \in \mathrm{OP}S^m$.

When the symbol $a(x)$ is independent of ξ, the operator $\mathrm{Op}(a)$ is the multiplication operator by the function $a(x)$, i.e. $\mathrm{Op}(a) : u(x) \mapsto a(x)u(x)$. In such a case we also denote $\mathrm{Op}(a) = a(x)$.

We shall use the following notation, used also in [4, 44, 13]. For any $m \in \mathbb{R}\backslash\{0\}$, we set

$$|D|^m := \mathrm{Op}\big(\chi(\xi)|\xi|^m\big),$$

where χ is an even, positive $\mathcal{C}^\infty$ cut-off satisfying (4.2.10). We also identify the Hilbert transform $\mathcal{H}$, acting on the 2π-periodic functions, defined by

$$\mathcal{H}(e^{\mathrm{i}jx}) := -\mathrm{i}\,\mathrm{sign}\,(j)e^{\mathrm{i}jx} \quad \forall j \neq 0, \quad \mathcal{H}(1) := 0, \tag{4.2.19}$$

with the Fourier multiplier $\mathrm{Op}(-\mathrm{i}\,\mathrm{sign}\,(\xi)\chi(\xi))$. Similarly we regard the operator

$$\partial_x^{-1}\left[e^{\mathrm{i}jx}\right] := -\,\mathrm{i}\,j^{-1}\,e^{\mathrm{i}jx} \quad \forall\, j \neq 0, \quad \partial_x^{-1}[1] := 0, \tag{4.2.20}$$

as the Fourier multiplier $\partial_x^{-1} = \mathrm{Op}\left(-\mathrm{i}\,\chi(\xi)\xi^{-1}\right)$ and the projector π_0, defined on the 2π-periodic functions as

$$\pi_0 u := \frac{1}{2\pi}\int_{\mathbb{T}} u(x)\,dx\,, \tag{4.2.21}$$

with the Fourier multiplier $\mathrm{Op}\big(1-\chi(\xi)\big)$. Finally we define, for any $m \in \mathbb{R}\backslash\{0\}$,

$$\langle D\rangle^m := \pi_0 + |D|^m := \mathrm{Op}\big((1-\chi(\xi)) + \chi(\xi)|\xi|^m\big).$$

We shall consider families of pseudodifferential operators with a symbol $a(\lambda;\varphi,x,\xi)$ which is k_0-times differentiable with respect to a parameter $\lambda := (\omega,\kappa)$ in an open subset $\Lambda_0 \subset \mathbb{R}^\nu \times [\kappa_1,\kappa_2]$. Note that $\partial_\lambda^k A = \mathrm{Op}\left(\partial_\lambda^k a\right)$ for any $k \in \mathbb{N}_0^{\nu+1}$.

We recall the pseudodifferential norm introduced in Definition 2.11 in [44].

Definition 4.15. (Weighted ΨDO norm) Let $A(\lambda) := a(\lambda;\varphi,x,D) \in \mathrm{OP}S^m$ be a family of pseudodifferential operators with symbol $a(\lambda;\varphi,x,\xi) \in S^m$, $m \in \mathbb{R}$, which are k_0-times differentiable with respect to $\lambda \in \Lambda_0 \subset \mathbb{R}^{\nu+1}$. For $\upsilon \in (0,1)$, $\alpha \in \mathbb{N}_0$, $s \geqslant 0$, we define

$$\|A\|_{m,s,\alpha}^{k_0,\upsilon} := \sum_{|k|\leqslant k_0} \upsilon^{|k|} \sup_{\lambda\in\Lambda_0} \|\partial_\lambda^k A(\lambda)\|_{m,s,\alpha}$$

where $\|A(\lambda)\|_{m,s,\alpha} := \max_{0\leqslant\beta\leqslant\alpha} \sup_{\xi\in\mathbb{R}} \|\partial_\xi^\beta a(\lambda;\cdot,\cdot,\xi)\|_s \,\langle\xi\rangle^{-m+\beta}$. For a matrix of pseudodifferential operators $\mathbf{A} \in \mathrm{OP}S^m$ as in (4.2.18), we define $\|\mathbf{A}\|_{m,s,\alpha}^{k_0,\upsilon} := \max_{i=1,\dots,4} \|A_i\|_{m,s,\alpha}^{k_0,\upsilon}$.

Given a function $a(\lambda;\varphi,x) \in \mathcal{C}^\infty$ which is k_0-times differentiable with respect to λ, the weighted norm of the corresponding multiplication operator is

$$\|\mathrm{Op}(a)\|_{0,s,\alpha}^{k_0,\upsilon} = \|a\|_s^{k_0,\upsilon}\,, \quad \forall \alpha \in \mathbb{N}_0\,. \tag{4.2.22}$$

Composition of pseudodifferential operators. If $\mathrm{Op}(a)$, $\mathrm{Op}(b)$ are pseudodifferential operators with symbols $a \in S^m$, $b \in S^{m'}$, $m, m' \in \mathbb{R}$, then the composition operator $\mathrm{Op}(a)\mathrm{Op}(b)$ is a pseudodifferential operator $\mathrm{Op}(a\#b)$ with symbol $a\#b \in S^{m+m'}$. It admits the asymptotic expansion: for any $N \geqslant 1$

$$(a\#b)(\lambda;\varphi,x,\xi) = \sum_{\beta=0}^{N-1} \frac{1}{\mathrm{i}^\beta \beta!} \partial_\xi^\beta a(\lambda;\varphi,x,\xi)\partial_x^\beta b(\lambda;\varphi,x,\xi) + (r_N(a,b))(\lambda;\varphi,x,\xi) \tag{4.2.23}$$

where $r_N(a,b) \in S^{m+m'-N}$. The following result is proved in Lemma 2.13 in [44].

Lemma 4.16. (Composition) *Let $A = a(\lambda;\varphi,x,D)$, $B = b(\lambda;\varphi,x,D)$ be pseudodifferential operators with symbols $a(\lambda;\varphi,x,\xi) \in S^m$, $b(\lambda;\varphi,x,\xi) \in S^{m'}$, $m, m' \in \mathbb{R}$. Then $A \circ B \in \mathrm{OP}S^{m+m'}$ satisfies, for any $\alpha \in \mathbb{N}_0$, $s \geqslant s_0$,*

$$\begin{aligned} \|AB\|_{m+m',s,\alpha}^{k_0,\upsilon} \lesssim_{m,\alpha,k_0} & \; C(s)\,\|A\|_{m,s,\alpha}^{k_0,\upsilon}\,\|B\|_{m',s_0+|m|+\alpha,\alpha}^{k_0,\upsilon} \\ & + C(s_0)\,\|A\|_{m,s_0,\alpha}^{k_0,\upsilon}\,\|B\|_{m',s+|m|+\alpha,\alpha}^{k_0,\upsilon}\,. \end{aligned} \tag{4.2.24}$$

Moreover, for any integer $N \geqslant 1$, the remainder $R_N := \mathrm{Op}(r_N)$ in (4.2.23) *satisfies*

$$\begin{aligned} \|\mathrm{Op}(r_N(a,b))\|_{m+m'-N,s,\alpha}^{k_0,\upsilon} \lesssim_{m,N,\alpha,k_0} & \; C(s)\,\|A\|_{m,s,N+\alpha}^{k_0,\upsilon}\,\|B\|_{m',s_0+|m|+2N+\alpha,N+\alpha}^{k_0,\upsilon} \\ & + C(s_0)\,\|A\|_{m,s_0,N+\alpha}^{k_0,\upsilon}\,\|B\|_{m',s+|m|+2N+\alpha,N+\alpha}^{k_0,\upsilon}\,. \end{aligned} \tag{4.2.25}$$

Both (4.2.24)-(4.2.25) *hold with the constant $C(s_0)$ interchanged with $C(s)$.*
Analogous estimates hold if $\mathbf{A}$ and $\mathbf{B}$ are matrix operators of the form (4.2.18).

The commutator between two pseudodifferential operators $\mathrm{Op}(a) \in \mathrm{OP}S^m$ and $\mathrm{Op}(b) \in \mathrm{OP}S^{m'}$ is a pseudodifferential operator in $\mathrm{OP}S^{m+m'-1}$ with symbol $a \star b \in S^{m+m'-1}$, namely $[\mathrm{Op}(a), \mathrm{Op}(b)] = \mathrm{Op}\,(a \star b)$, that admits, by (4.2.23), the expansion

$$\begin{aligned} & a \star b = -\mathrm{i}\,\{a,b\} + \widetilde{r}_2(a,b)\,, \quad \widetilde{r}_2(a,b) := r_2(a,b) - r_2(b,a) \in S^{m+m'-2}\,, \\ & \text{where} \quad \{a,b\} := \partial_\xi a \partial_x b - \partial_x a \partial_\xi b\,, \end{aligned} \tag{4.2.26}$$

is the Poisson bracket between $a(x,\xi)$ and $b(x,\xi)$. As a corollary of Lemma 4.16 we have:

Lemma 4.17. (Commutator) *Let $A = \mathrm{Op}(a)$ and $B = \mathrm{Op}(b)$ be pseudodifferential operators with symbols $a(\lambda;\varphi,x,\xi) \in S^m$, $b(\lambda;\varphi,x,\xi) \in S^{m'}$, $m, m' \in \mathbb{R}$. Then the commutator $[A,B] :=$*

$AB - BA \in \mathrm{OP}S^{m+m'-1}$ *satisfies*

$$\begin{aligned}\|[A,B]\|^{k_0,\upsilon}_{m+m'-1,s,\alpha} \lesssim_{m,m',\alpha,k_0} & C(s)\,\|A\|^{k_0,\upsilon}_{m,s+|m'|+\alpha+2,\alpha+1}\,\|B\|^{k_0,\upsilon}_{m',s_0+|m|+\alpha+2,\alpha+1} \\ & + C(s_0)\,\|A\|^{k_0,\upsilon}_{m,s_0+|m'|+\alpha+2,\alpha+1}\,\|B\|^{k_0,\upsilon}_{m',s+|m|+\alpha+2,\alpha+1}\,. \end{aligned} \tag{4.2.27}$$

Finally we consider the exponential of a pseudodifferential operator of order 0. The following lemma follows as in Lemma 2.12 of [43] (or Lemma 2.17 in [44]).

Lemma 4.18. (Exponential map) *If $A := \mathrm{Op}(a(\lambda;\varphi,x,\xi))$ is in OPS^0, then e^A is in OPS^0 and for any $s \geqslant s_0$, $\alpha \in \mathbb{N}_0$, there is a constant $C(s,\alpha) > 0$ so that*

$$\|e^A - \mathrm{Id}\|^{k_0,\upsilon}_{0,s,\alpha} \leqslant \|A\|^{k_0,\upsilon}_{0,s+\alpha,\alpha}\exp\big(C(s,\alpha)\|A\|^{k_0,\upsilon}_{0,s_0+\alpha,\alpha}\big)\,.$$

The same holds for a matrix $\mathbf{A}$ of the form (4.2.18) in $\mathrm{OP}S^0$.

Egorov Theorem. Consider the family of φ-dependent diffeomorphisms of $\mathbb{T}_x$ defined by

$$y = x + \beta(\varphi,x) \qquad \Longleftrightarrow \qquad x = y + \breve{\beta}(\varphi,y)\,, \tag{4.2.28}$$

where $\beta(\varphi,x)$ is a small smooth function, and the induced operators

$$(\mathcal{B}u)(\varphi,x) := u(\varphi, x+\beta(\varphi,x))\,, \quad (\mathcal{B}^{-1}u)(\varphi,y) := u(\varphi, y+\breve{\beta}(\varphi,y))\,. \tag{4.2.29}$$

Lemma 4.19. (Composition) *Let $\|\beta\|^{k_0,\upsilon}_{2s_0+k_0+2} \leqslant \delta(s_0,k_0)$ small enough. Then the composition operator $\mathcal{B}$ satisfies the tame estimates, for any $s \geqslant s_0$,*

$$\|\mathcal{B}u\|^{k_0,\upsilon}_s \lesssim_{s,k_0} \|u\|^{k_0,\upsilon}_{s+k_0} + \|\beta\|^{k_0,\upsilon}_s\|u\|^{k_0,\upsilon}_{s_0+k_0+1}\,,$$

and the function $\breve{\beta}$ defined in (4.2.28) by the inverse diffeomorphism satisfies $\|\breve{\beta}\|^{k_0,\upsilon}_s \lesssim_{s,k_0} \|\beta\|^{k_0,\upsilon}_{s+k_0}$.

The following result is a small variation of Proposition 2.28 of [43].

Proposition 4.20. (Egorov) *Let $N \in \mathbb{N}$, $\mathtt{q}_0 \in \mathbb{N}_0$, $S > s_0$ and assume that $\partial^k_\lambda\beta(\lambda;\cdot,\cdot)$ are $\mathcal{C}^\infty$ for all $|k| \leqslant k_0$. There exist constants $\sigma_N, \sigma_N(\mathtt{q}_0) > 0$, $\delta = \delta(S,N,\mathtt{q}_0,k_0) \in (0,1)$ such that, if $\|\beta\|^{k_0,\upsilon}_{s_0+\sigma_N(\mathtt{q}_0)} \leqslant \delta$, then the conjugated operator $\mathcal{B}^{-1}\circ\partial^m_x\circ\mathcal{B}$, $m \in \mathbb{Z}$, is a pseudodifferential operator of order m with an expansion of the form*

$$\mathcal{B}^{-1}\circ\partial^m_x\circ\mathcal{B} = \sum_{i=0}^N p_{m-i}(\lambda;\varphi,y)\partial_y^{m-i} + \mathcal{R}_N(\varphi)$$

with the following properties:

1. The principal symbol of p_m *is*

$$p_m(\lambda;\varphi,y) = \Big([1+\beta_x(\lambda;\varphi,x)]^m\Big)|_{x=y+\breve\beta(\lambda;\varphi,y)}$$

where $\breve\beta(\lambda;\varphi,y)$ *has been introduced in* (4.2.28)*. For any* $s \geqslant s_0$ *and* $i=1,\dots,N$*,*

$$\|p_m-1\|_s^{k_0,\upsilon}\,,\ \|p_{m-i}\|_s^{k_0,\upsilon} \lesssim_{s,N} \|\beta\|_{s+\sigma_N}^{k_0,\upsilon}\,. \tag{4.2.30}$$

2. For any $\mathtt{q}\in\mathbb{N}_0^\nu$ *with* $|\mathtt{q}|\leqslant \mathtt{q}_0$*,* $n_1,n_2\in\mathbb{N}_0$ *with* $n_1+n_2+\mathtt{q}_0 \leqslant N+1-k_0-m$*, the operator* $\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathcal{R}_N(\varphi)\langle D\rangle^{n_2}$ *is* $\mathcal{D}^{k_0}$*-tame with a tame constant satisfying, for any* $s_0\leqslant s\leqslant S$*,*

$$\mathfrak{M}_{\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathcal{R}_N(\varphi)\langle D\rangle^{n_2}}(s) \lesssim_{S,N,\mathtt{q}_0} \|\beta\|_{s+\sigma_N(\mathtt{q}_0)}^{k_0,\upsilon}\,. \tag{4.2.31}$$

3. Let $s_0<s_1$ *and assume that* $\|\beta_j\|_{s_1+\sigma_N(\mathtt{q}_0)}\leqslant\delta$*,* $j=1,2$*. Then* $\|\Delta_{12}p_{m-i}\|_{s_1}\lesssim_{s_1,N}\|\Delta_{12}\beta\|_{s_1+\sigma_N}$*,* $i=0,\dots,N$*, and, for any* $|\mathtt{q}|\leqslant\mathtt{q}_0$*,* $n_1,n_2\in\mathbb{N}_0$ *with* $n_1+n_2+\mathtt{q}_0\leqslant N-m$*,*

$$\|\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\Delta_{12}\mathcal{R}_N(\varphi)\langle D\rangle^{n_2}\|_{\mathcal{B}(H^{s_1})} \lesssim_{s_1,N,n_1,n_2} \|\Delta_{12}\beta\|_{s_1+\sigma_N(\mathtt{q}_0)}\,.$$

Finally, if $\beta(\varphi,x)$ *is a quasi-periodic traveling wave, then* $\mathcal{B}$ *is momentum preserving (we refer to Definition 4.38 and Lemma 4.44), as well as the conjugated operator* $\mathcal{B}^{-1}\circ\partial_x^m\circ\mathcal{B}$*, and each function* p_{m-i}*,* $i=0,\dots,N$*, is a quasi-periodic traveling wave.*

Dirichlet-Neumann operator. We remind the following decomposition of the Dirichlet-Neumann operator proved in Proposition 2.37 of [44], in the case of infinite depth, and in Appendix A of [13], for finite depth.

Proposition 4.21. (Dirichlet-Neumann operator) *Assume that* $\partial_\lambda^k\eta(\lambda,\cdot,\cdot)$ *is* $\mathcal{C}^\infty(\mathbb{T}^\nu\times\mathbb{T}_x)$ *for all* $|k|\leqslant k_0$*. There exists* $\delta(s_0,k_0)>0$ *such that, if* $\|\eta\|_{2s_0+2k_0+1}^{k_0,\upsilon}\leqslant\delta(s_0,k_0)$*, then the Dirichlet-Neumann operator* $G(\eta)=G(\eta,\mathtt{h})$ *may be written as*

$$G(\eta,\mathtt{h}) = G(0,\mathtt{h}) + \mathcal{R}_G(\eta) \tag{4.2.32}$$

where $\mathcal{R}_G(\eta):=\mathcal{R}_G(\eta,\mathtt{h})\in \mathrm{OP}S^{-\infty}$ *is an integral operator with* $\mathcal{C}^\infty$ *Kernel* K_G *which satisfies, for all* $m,s,\alpha\in\mathbb{N}$*, the estimate*

$$\|\mathcal{R}_G(\eta)\|_{-m,s,\alpha}^{k_0,\upsilon}\leqslant C(s,m,\alpha,k_0)\|K_G\|_{s+m+\alpha}^{k_0,\upsilon}\leqslant C(s,m,\alpha,k_0)\|\eta\|_{s+s_0+2k_0+m+\alpha+3}^{k_0,\upsilon}\,. \tag{4.2.33}$$

Let $s_1>2s_0+1$*. There exists* $\delta(s_1)>0$ *such that the map* $\{\|\eta\|_{s_1+6}<\delta(s_1)\}\to H^{s_1}(\mathbb{T}^\nu\times\mathbb{T}\times\mathbb{T})$*,* $\eta\mapsto K_G(\eta)$ *is* C^1 *with bounded derivative.*

We conclude by recalling the estimate for the Dirichlet-Neumann operator and for its first and second variation in η, see Lemma 2.41 in [44].

Lemma 4.22 (Tame estimates for $G(\eta)$). *There it $\delta(s_0,k_0)>0$ such that, if $\|\eta\|_{3s_0+2k_0+5}^{k_0,\upsilon} \leqslant \delta(s_0,k_0)$, then, for any $s \geqslant s_0$,*

$$\begin{aligned} \|(G(\eta)-G(0))\zeta\|_s^{k_0,\upsilon} &\lesssim_{s,k_0} \|\eta\|_{s+2s_0+2k_0+3}^{k_0,\upsilon}\|\zeta\|_{s_0}^{k_0,\upsilon} + \|\eta\|_{3s_0+2k_0+3}^{k_0,\upsilon}\|\zeta\|_s^{k_0,\upsilon}, \\ \|G'(\eta)[\widehat{\eta}]\zeta\|_s^{k_0,\upsilon} &\lesssim_{s,k_0} \|\psi\|_{s+2}^{k_0,\upsilon}\|\widehat{\eta}\|_{s_0+1}^{k_0,\upsilon} + \|\psi\|_{s_0+2}^{k_0,\upsilon}\|\widehat{\eta}\|_{s+1}^{k_0,\upsilon} \\ &\quad + \|\eta\|_{s+2s_0+2k_0+4}^{k_0,\upsilon}\|\psi\|_{s_0+2}^{k_0,\upsilon}\|\widehat{\eta}\|_{s_0+1}^{k_0,\upsilon}, \\ \|G''(\eta)[\widehat{\eta},\widehat{\eta}]\zeta\|_s^{k_0,\upsilon} &\lesssim_{s,k_0} \|\psi\|_{s+3}^{k_0,\upsilon}(\|\widehat{\eta}\|_{s_0+2}^{k_0,\upsilon})^2 + \|\psi\|_{s_0+3}^{k_0,\upsilon}\|\widehat{\eta}\|_{s_0+2}^{k_0,\upsilon}\|\widehat{\eta}\|_{s+2}^{k_0,\upsilon} \\ &\quad + \|\eta\|_{s+2s_0+2k_0+4}^{k_0,\upsilon}\|\psi\|_{s_0+3}^{k_0,\upsilon}(\|\widehat{\eta}\|_{s_0+2}^{k_0,\upsilon})^2 . \end{aligned} \tag{4.2.34}$$

4.2.2 $\mathcal{D}^{k_0}$-tame and modulo-tame operators

We present the notion of tame and modulo tame operators introduced in [44]. Let $A := A(\lambda)$ be a linear operator as in (4.2.12), k_0-times differentiable with respect to the parameter λ in the open set $\Lambda_0 \subset \mathbb{R}^{\nu+1}$.

Definition 4.23. ($\mathcal{D}^{k_0}$-σ-tame) Let $\sigma \geqslant 0$. A linear operator $A := A(\lambda)$ is $\mathcal{D}^{k_0}$-σ-tame if there exists a non-decreasing function $[s_0,S] \to [0,+\infty)$, $s \mapsto \mathfrak{M}_A(s)$, with possibly $S=+\infty$, such that, for all $s_0 \leqslant s \leqslant S$ and $u \in H^{s+\sigma}$,

$$\sup_{|k|\leqslant k_0}\sup_{\lambda\in\Lambda_0} \upsilon^{|k|}\|(\partial_\lambda^k A(\lambda))u\|_s \leqslant \mathfrak{M}_A(s_0)\,\|u\|_{s+\sigma} + \mathfrak{M}_A(s)\,\|u\|_{s_0+\sigma} . \tag{4.2.35}$$

We say that $\mathfrak{M}_A(s)$ is a *tame constant* of the operator A. The constant $\mathfrak{M}_A(s) = \mathfrak{M}_A(k_0,\sigma,s)$ may also depend on k_0,σ but we shall often omit to write them. When the "loss of derivatives" σ is zero, we simply write $\mathcal{D}^{k_0}$-tame instead of $\mathcal{D}^{k_0}$-0-tame. For a matrix operator as in (4.2.15), we denote the tame constant $\mathfrak{M}_{\mathbf{R}}(s) := \max\{\mathfrak{M}_{\mathcal{R}_1}(s), \mathfrak{M}_{\mathcal{R}_2}(s)\}$.

Note that the tame constants $\mathfrak{M}_A(s)$ are not uniquely determined. An immediate consequence of (4.2.35) is that $\|A\|_{\mathcal{L}(H^{s_0+\sigma},H^{s_0})} \leqslant 2\mathfrak{M}_A(s_0)$. Also note that, representing the operator A by its matrix elements $(A_j^{j'}(\ell-\ell'))_{\ell,\ell'\in\mathbb{Z}^\nu, j,j'\in\mathbb{Z}}$ as in (4.2.13), we have for all $|k|\leqslant k_0$, $j'\in\mathbb{Z}$, $\ell'\in\mathbb{Z}^\nu$,

$$\upsilon^{2|k|}\sum_{\ell,j}\langle \ell,j\rangle^{2s}\left|\partial_\lambda^k A_j^{j'}(\ell-\ell')\right|^2 \leqslant 2\big(\mathfrak{M}_A(s_0)\big)^2\langle \ell',j'\rangle^{2(s+\sigma)} + 2(\mathfrak{M}_A(s))^2\langle \ell',j'\rangle^{2(s_0+\sigma)} . \tag{4.2.36}$$

The class of $\mathcal{D}^{k_0}$-σ-tame operators is closed under composition.

Lemma 4.24. (Composition, Lemma 2.20 in [44]) *Let A,B be respectively $\mathcal{D}^{k_0}$-σ_A-tame and $\mathcal{D}^{k_0}$-σ_B-tame operators with tame constants respectively $\mathfrak{M}_A(s)$ and $\mathfrak{M}_B(s)$. Then the composed* *operator* $A \circ B$ *is* $\mathcal{D}^{k_0}$-$(\sigma_A + \sigma_B)$*-tame with tame constant*

$$\mathfrak{M}_{AB}(s) \leqslant C(k_0) \left(\mathfrak{M}_A(s)\mathfrak{M}_B(s_0 + \sigma_A) + \mathfrak{M}_A(s_0)\mathfrak{M}_B(s + \sigma_A) \right) .$$

It is proved in Lemma 2.22 in [44] that the action of a $\mathcal{D}^{k_0}$-σ-tame operator $A(\lambda)$ on a Sobolev function $u = u(\lambda) \in H^{s+\sigma}$ is bounded by

$$\|Au\|_s^{k_0,\upsilon} \lesssim_{k_0} \mathfrak{M}_A(s_0)\|u\|_{s+\sigma}^{k_0,\upsilon} + \mathfrak{M}_A(s)\|u\|_{s_0+\sigma}^{k_0,\upsilon} . \tag{4.2.37}$$

Pseudodifferential operators are tame operators. We use in particular the following lemma:

Lemma 4.25. (Lemma 2.21 in [44]) *Let* $A = a(\lambda; \varphi, x, D) \in \mathrm{OP}S^0$ *be a family of pseudodifferential operators satisfying* $\|A\|_{0,s,0}^{k_0,\upsilon} < \infty$ *for* $s \geqslant s_0$*. Then* A *is* $\mathcal{D}^{k_0}$*-tame with a tame constant* $\mathfrak{M}_A(s)$ *satisfying, for any* $s \geqslant s_0$*,*

$$\mathfrak{M}_A(s) \leqslant C(s)\|A\|_{0,s,0}^{k_0,\upsilon} . \tag{4.2.38}$$

The same statement holds for a matrix operator $\mathbf{R}$ *as in* (4.2.15)*.*

In view of the KAM reducibility scheme of Section 4.7 we also consider the stronger notion of $\mathcal{D}^{k_0}$-modulo-tame operator, that we need only for operators with loss of derivative $\sigma = 0$. We first recall the notion of *majorant operator*: given a linear operator A acting as in (4.2.13), we define the majorant operator $|A|$ by its matrix elements $(|A_j^{j'}(\ell - \ell')|)_{\ell,\ell' \in \mathbb{Z}^\nu, j,j' \in \mathbb{Z}}$.

Definition 4.26. ($\mathcal{D}^{k_0}$-modulo-tame) A linear operator $A = A(\lambda)$ is $\mathcal{D}^{k_0}$*-modulo-tame* if there exists a non-decreasing function $[s_0, S] \to [0, +\infty]$, $s \mapsto \mathfrak{M}_A^\sharp(s)$, such that for all $k \in \mathbb{N}_0^{\nu+1}$, $|k| \leqslant k_0$, the majorant operator $|\partial_\lambda^k A|$ satisfies, for all $s_0 \leqslant s \leqslant S$ and $u \in H^s$,

$$\sup_{|k| \leqslant k_0} \sup_{\lambda \in \Lambda_0} \upsilon^{|k|} \| |\partial_\lambda^k A| u\|_s \leqslant \mathfrak{M}_A^\sharp(s_0) \left\| u \right\|_s + \mathfrak{M}_A^\sharp(s) \left\| u \right\|_{s_0} . \tag{4.2.39}$$

The constant $\mathfrak{M}_A^\sharp(s)$ is called a *modulo-tame constant* for the operator A. For a matrix of operators as in (4.2.15), we denote the modulo-tame constant $\mathfrak{M}_{\mathbf{R}}^\sharp(s) := \max\{\mathfrak{M}_{\mathcal{R}_1}^\sharp(s), \mathfrak{M}_{\mathcal{R}_2}^\sharp(s)\}$.

If A, B are $\mathcal{D}^{k_0}$-modulo-tame operators with $|A_j^{j'}(\ell)| \leqslant |B_j^{j'}(\ell)|$, then $\mathfrak{M}_A^\sharp(s) \leqslant \mathfrak{M}_B^\sharp(s)$. A $\mathcal{D}^{k_0}$-modulo-tame operator is also $\mathcal{D}^{k_0}$-tame and $\mathfrak{M}_A(s) \leqslant \mathfrak{M}_A^\sharp(s)$.

In view of the next lemma, given a linear operator A acting as in (4.2.13), we define the operator $\langle \partial_\varphi \rangle^{\mathtt{b}} A$, $\mathtt{b} \in \mathbb{R}$, whose matrix elements are $\langle \ell - \ell' \rangle^{\mathtt{b}} A_j^{j'}(\ell - \ell')$.

Lemma 4.27. (Sum and composition, Lemma 2.25 in [44]) *Let* A*,* B*,* $\langle \partial_\varphi \rangle^{\mathtt{b}} A$*,* $\langle \partial_\varphi \rangle^{\mathtt{b}} B$

be $\mathcal{D}^{k_0}$-modulo-tame operators. Then $A+B$, $A\circ B$ and $\langle\partial_\varphi\rangle^{\mathtt{b}}(AB)$ are $\mathcal{D}^{k_0}$-modulo-tame with

$$
\begin{aligned}
&\mathfrak{M}^\sharp_{A+B}(s) \leqslant \mathfrak{M}^\sharp_A(s) + \mathfrak{M}^\sharp_B(s)\\
&\mathfrak{M}^\sharp_{AB}(s) \leqslant C(k_0)\big(\mathfrak{M}^\sharp_A(s)\mathfrak{M}^\sharp_B(s_0) + \mathfrak{M}^\sharp_A(s_0)\mathfrak{M}^\sharp_B(s)\big)\\
&\mathfrak{M}^\sharp_{\langle\partial_\varphi\rangle^{\mathtt{b}}(AB)}(s) \leqslant C(\mathtt{b})C(k_0)\big(\mathfrak{M}^\sharp_{\langle\partial_\varphi\rangle^{\mathtt{b}}A}(s)\mathfrak{M}^\sharp_B(s_0) + \mathfrak{M}^\sharp_{\langle\partial_\varphi\rangle^{\mathtt{b}}A}(s_0)\mathfrak{M}^\sharp_B(s)\\
&\qquad\qquad + \mathfrak{M}^\sharp_A(s)\mathfrak{M}^\sharp_{\langle\partial_\varphi\rangle^{\mathtt{b}}B}(s_0) + \mathfrak{M}^\sharp_A(s_0)\mathfrak{M}^\sharp_{\langle\partial_\varphi\rangle^{\mathtt{b}}B}(s)\big)\,.
\end{aligned}
$$

The same statement holds for matrix operators $\mathbf{A}$, $\mathbf{B}$ as in (4.2.15).

By Lemma 4.27 we deduce the following result, cfr. Lemma 2.20 in [43].

Lemma 4.28. (Exponential) *Let A and $\langle\partial_\varphi\rangle^{\mathtt{b}}A$ be $\mathcal{D}^{k_0}$-modulo-tame and assume that $\mathfrak{M}^\sharp_A(s_0)\leqslant 1$. Then the operators $e^{\pm A}-\mathrm{Id}$ and $\langle\partial_\varphi\rangle^{\mathtt{b}}e^{\pm A}-\mathrm{Id}$ are $\mathcal{D}^{k_0}$-modulo-tame with modulo-tame constants satisfying*

$$
\mathfrak{M}^\sharp_{e^{\pm A}-\mathrm{Id}}(s) \lesssim_{k_0} \mathfrak{M}^\sharp_A(s)\,,\quad \mathfrak{M}^\sharp_{\langle\partial_\varphi\rangle^{\mathtt{b}}e^{\pm A}-\mathrm{Id}}(s) \lesssim_{k_0,\mathtt{b}} \mathfrak{M}^\sharp_{\langle\partial_\varphi\rangle^{\mathtt{b}}A}(s) + \mathfrak{M}^\sharp_A(s)\mathfrak{M}^\sharp_{\langle\partial_\varphi\rangle^{\mathtt{b}}A}(s_0)\,.
$$

Given a linear operator A acting as in (4.2.13), we define the *smoothed operator* $\Pi_N A$, $N\in\mathbb{N}$ whose matrix elements are

$$
(\Pi_N A)_j^{j'}(\ell-\ell') := \begin{cases} A_j^{j'}(\ell-\ell') & \text{if } \langle\ell-\ell'\rangle\leqslant N\\ 0 & \text{otherwise}\,.\end{cases} \tag{4.2.40}
$$

We also denote $\Pi_N^\perp := \mathrm{Id}-\Pi_N$. It is proved in Lemma 2.27 in [44] that

$$
\mathfrak{M}^\sharp_{\Pi_N^\perp A}(s) \leqslant N^{-\mathtt{b}}\mathfrak{M}^\sharp_{\langle\partial_\varphi\rangle^{\mathtt{b}}A}(s)\,,\quad \mathfrak{M}^\sharp_{\Pi_N^\perp A}(s)\leqslant \mathfrak{M}^\sharp_A(s)\,. \tag{4.2.41}
$$

The same estimate holds with a matrix operator $\mathbf{R}$ as in (4.2.15).

4.2.3 Tame estimates for the flow of pseudo-PDEs

We report in this section some results concerning tame estimates for the flow $\Phi(\tau)$ of the pseudo-PDE Cauchy problem

$$
\begin{cases}\partial_\tau u = \mathrm{i}\,\mathrm{Op}\big(\mathtt{a}(\varphi,x)\,|\xi|^{\frac12}\big)u\\ u(0,\varphi,x)=u_0(\varphi,x)\end{cases}\,,\quad \varphi\in\mathbb{T}^\nu\ \ x\in\mathbb{T}\,, \tag{4.2.42}
$$

where $\mathtt{a}(\varphi,x)=\mathtt{a}(\lambda;\varphi,x)$ is a real valued function that is $\mathcal{C}^\infty$ with respect to the variables (φ,x) and k_0-times differentiable with respect to the parameters $\lambda=(\omega,\kappa)$. The function $\mathtt{a}=\mathtt{a}(i)$ may also depend on the "approximate" torus $i(\varphi)$. The flow operator $\Phi(\tau)=\Phi(\lambda;\varphi,\tau)$ satisfies the equation

$$\begin{cases} \partial_\tau \Phi(\tau) = \mathrm{i}\, \mathrm{Op}\big(\mathsf{a}(\varphi,x)\,|\xi|^{\frac12}\big)\Phi(\tau) \\ \Phi(0) = \mathrm{Id}\,. \end{cases} \tag{4.2.43}$$

Since the function $\mathsf{a}(\varphi,x)$ is real valued, usual energy estimates imply that the flow $\Phi(\tau)$ is a bounded operator mapping H^s_x to H^s_x. Moreover, since (4.2.43) is an autonomous equation, the flow $\Phi(\varphi,\tau)$ satisfies the group property

$$\Phi(\varphi,\tau_1+\tau_2) = \Phi(\varphi,\tau_1)\circ\Phi(\varphi,\tau_2)\,, \quad \Phi(\varphi,\tau)^{-1} = \Phi(\varphi,-\tau)\,. \tag{4.2.44}$$

Since $\mathsf{a}(\lambda\ \cdot)$ is k_0-times differentiable with respect to the parameter λ, then $\Phi(\lambda;\varphi,\tau)$ is k_0-times differentiable with respect to λ as well. Note also that $\Phi^{-1}(\tau) = \Phi(-\tau) = \overline{\Phi}(\tau)$ because there operators solve the same Cauchy problem. Moreover, if $\mathsf{a}(\varphi,x)$ is odd(φ,x), then the real operator

$$\boldsymbol{\Phi}(\varphi,x) := \begin{pmatrix} \Phi(\varphi,x) & 0 \\ 0 & \overline{\Phi}(\varphi,x) \end{pmatrix}$$

is reversibility preserving by Lemma 4.32.
The operator $\partial_\lambda^k\partial_\varphi^\beta\Phi$ loses $|D_x|^{\frac{|\beta|+|k|}{2}}$ derivatives, which in (4.2.46) are compensated by $\langle D\rangle^{-m_1}$ on the left hand side and $\langle D\rangle^{-m_2}$ on the right hand side, with $m_1,m_2\in\mathbb{R}$ satisfying $m_1+m_2 = \frac12(|\beta|+|k|)$. The following proposition provides tame estimates in the Sobolev spaces $H^s_{\varphi,x}$.

Proposition 4.29. *Let $\beta_0,k_0\in\mathbb{N}_0$. For any $\beta,k\in\mathbb{N}_0^\nu$ with $|\beta|\leqslant\beta_0$, $|k|\leqslant k_0$, for any $m_1,m_2\in\mathbb{R}$ with $m_1+m_2=\frac12(|\beta|+|k|)$ and for any $s\geqslant s_0:=(\nu+1)/2$, there exist constants $\sigma(|\beta|,|k|,m_1,m_2)>0$, $\delta(s,m_1)>0$ such that, if*

$$\|\mathsf{a}\|_{2s_0+|m_1|+2}\leqslant\delta(s,m_1)\,, \quad \|\mathsf{a}\|^{k_0,\upsilon}_{s_0+\sigma(\beta_0,k_0,m_1,m_2)}\leqslant 1\,, \tag{4.2.45}$$

then the flow $\Phi(\tau)=\Phi(\lambda;\varphi,\tau)$ of (4.2.42) satisfies

$$\begin{aligned} &\sup_{\tau\in[0,1]}\|\langle D\rangle^{-m_1}\partial_\lambda^k\partial_\varphi^\beta\Phi(\tau)\langle D\rangle^{-m_2}h\|_s \\ &\qquad\lesssim_{s,\beta_0,k_0,m_1,m_2}\upsilon^{-|k|}\big(\|h\|_s+\|\mathsf{a}\|^{k_0,\upsilon}_{s+\sigma(|\beta|,|k|,m_1,m_2)}\|h\|_{s_0}\big)\,, \end{aligned}\tag{4.2.46}$$

$$\begin{aligned} &\sup_{\tau\in[0,1]}\|\partial_\lambda^k(\Phi(\tau)-\mathrm{Id})h\|_s \\ &\qquad\lesssim_s\upsilon^{-|k|}\big(\|\mathsf{a}\|^{k_0,\upsilon}_{s_0}\|h\|_{s+\frac{|k|+1}{2}}+\|\mathsf{a}\|^{k_0,\upsilon}_{s+s_0+k_0+\frac32}\|h\|_{s_0+\frac{|k|+1}{2}}\big)\,. \end{aligned}\tag{4.2.47}$$

Proof. See Proposition 2.37 in [13] and Appendix A in [44]. □

We consider also the dependence of the flow Φ with respect to the torus $i=i(\varphi)$ and the estimates for the adjoint operator Φ^*.

Proposition 4.30. *Let $s_1 > s_0$ and $\beta_0 \in \mathbb{N}_0$. For any $\beta \in \mathbb{N}_0^\nu$, $|\beta| \leqslant \beta_0$, and for any $m_1, m_2 \in \mathbb{R}$ satisfying $m_1 + m_2 = \frac{1}{2}(|\beta| + 1)$, there exists a constant $\sigma(|\beta|) = \sigma(|\beta|, m_1, m_2) > 0$ such that, if $\|\mathtt{a}\|_{s_1+\sigma(\beta_0)} \leqslant \delta(s)$ with $\delta(s) > 0$ small enough, then the following estimate holds:*

$$\sup_{\tau\in[0,1]} \| \langle D\rangle^{-m_1} \partial_\varphi^\beta \Delta_{12}\Phi(\tau) \langle D\rangle^{-m_2} \|_{s_1} \lesssim_{s_1} \|\Delta_{12}\mathtt{a}\|_{s_1+\sigma(|\beta|)} \|h\|_{s_1} , \tag{4.2.48}$$

where $\Delta_{12}\Phi := \Phi(i_2) - \Phi(i_1)$ and $\Delta_{12}\mathtt{a} := \mathtt{a}(i_2) - \mathtt{a}(i_1)$. Moreover, for any $k \in \mathbb{N}_0^{\nu+1}$, $|k| \leqslant k_0$ and for all $s \geqslant s_0$,

$$\|(\partial_\lambda^k \Phi^*)h\|_s \lesssim_s \upsilon^{-|k|} \big(\|h\|_{s+\frac{|k|}{2}} + \|\mathtt{a}\|^{k_0,\upsilon}_{s+s_0+|k|+\frac{3}{2}} \|h\|_{s_0+\frac{|k|}{2}} \big) ,$$
$$\|\partial_\lambda^k (\Phi^* - \mathrm{Id}) h\|_s \lesssim_s \upsilon^{-|k|} \big(\|\mathtt{a}\|^{k_0,\upsilon}_{s_0} \|h\|_{s+\frac{|k|+1}{2}} + \|\mathtt{a}\|^{k_0,\upsilon}_{s+s_0+|k|+2} \|h\|_{s_0+\frac{|k|+1}{2}} \big) .$$

Finally, for all $s \in [s_0, s_1]$, one has $\|\Delta_{12}\Phi^* h\|_s \lesssim_s \|\Delta_{12}\mathtt{a}\|_{s+s_0+\frac{1}{2}} \|h\|_{s+\frac{1}{2}}$.

Proof. See Lemma 2.38 in [13] and Appendix A in [44]. □

4.2.4 Hamiltonian and Reversible operators

Along the reduction of the linearized operators we shall exploit both the Hamiltonian and reversible structure, that we now present.

Hamiltonian operators. A matrix operator $\mathcal{R}$ as in (4.2.14) is Hamiltonian if the matrix

$$J^{-1}\mathcal{R} = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -C & -D \\ A & B \end{pmatrix}$$

is self-adjoint, namely $B^* = B$, $C^* = C$, $A^* = -D$ and A, B, C, D are real.

Correspondingly, a matrix operator as in (4.2.15) is Hamiltonian if

$$\mathcal{R}_1^* = -\mathcal{R}_1 , \quad \mathcal{R}_2^* = \overline{\mathcal{R}_2} . \tag{4.2.49}$$

Symplectic operators. A φ-dependent family of linear operators $\mathcal{R}(\varphi)$, $\varphi \in \mathbb{T}^\nu$, as in (4.2.14) is *symplectic* if

$$\mathcal{W}(\mathcal{R}(\varphi)u, \mathcal{R}(\varphi)v) = \mathcal{W}(u, v) \quad \forall u, v \in L^2(\mathbb{T}_x, \mathbb{R}^2) , \tag{4.2.50}$$

where the symplectic 2-form $\mathcal{W}$ is defined in (4.1.15).

Reversible and reversibility preserving operators. Let $\mathcal{S}$ be an involution as in (4.1.4) acting on the real variables $(\eta, \zeta) \in \mathbb{R}^2$, or as in (4.1.51) acting on the action-angle-normal variables (θ, I, w), or as in (4.1.29) acting in the $(z, \overline{z})$ complex variables introduced in (4.1.24).

Definition 4.31. (Reversibility) A φ-dependent family of operators $\mathcal{R}(\varphi)$, $\varphi \in \mathbb{T}^\nu$, is

- *reversible* if $\mathcal{R}(-\varphi) \circ \mathcal{S} = -\mathcal{S} \circ \mathcal{R}(\varphi)$ for all $\varphi \in \mathbb{T}^\nu$;
- *reversibility preserving* if $\mathcal{R}(-\varphi) \circ \mathcal{S} = \mathcal{S} \circ \mathcal{R}(\varphi)$ for all $\varphi \in \mathbb{T}^\nu$.

Since in the complex coordinates $(z, \overline{z})$ the involution $\mathcal{S}$ defined in (4.1.4) reads as in (4.1.29), an operator $\mathbf{R}(\varphi)$ as in (4.2.15) is reversible, respectively anti-reversible, if, for any $i = 1, 2$,

$$\mathcal{R}_i(-\varphi) \circ \mathcal{S} = -\mathcal{S} \circ \mathcal{R}_i(\varphi) \,, \quad \text{resp.} \ \ \mathcal{R}_i(-\varphi) \circ \mathcal{S} = \mathcal{S} \circ \mathcal{R}_i(\varphi) \,, \tag{4.2.51}$$

where, with a small abuse of notation, we still denote $(\mathcal{S}u)(x) = \overline{u(-x)}$. Moreover, recalling that in the Fourier coordinates such involution reads as in (4.1.30), we obtain the following lemma.

Lemma 4.32. *A φ-dependent family of operators $\mathbf{R}(\varphi)$, $\varphi \in \mathbb{T}^\nu$, as in* (4.2.15) *is*

- *reversible if, for any $i = 1, 2$,*

$$(\mathcal{R}_i)_j^{j'}(-\varphi) = -\overline{(\mathcal{R}_i)_j^{j'}(\varphi)} \quad \forall\, \varphi \in \mathbb{T}^\nu \,, \quad i.e. \ (\mathcal{R}_i)_j^{j'}(\ell) = -\overline{(\mathcal{R}_i)_j^{j'}(\ell)} \quad \forall\, \ell \in \mathbb{Z}^\nu \,; \tag{4.2.52}$$

- *reversibility preserving if, for any $i = 1, 2$,*

$$(\mathcal{R}_i)_j^{j'}(-\varphi) = \overline{(\mathcal{R}_i)_j^{j'}(\varphi)} \quad \forall\, \varphi \in \mathbb{T}^\nu \,, \quad i.e. \ (\mathcal{R}_i)_j^{j'}(\ell) = \overline{(\mathcal{R}_i)_j^{j'}(\ell)} \quad \forall\, \ell \in \mathbb{Z}^\nu \,. \tag{4.2.53}$$

Note that the composition of a reversible operator with a reversibility preserving operator is reversible. The flow generated by a reversibility preserving operator is reversibility preserving. If $\mathcal{R}(\varphi)$ is reversibility preserving, then $(\omega \cdot \partial_\varphi \mathcal{R})(\varphi)$ is reversible.

We shall say that a linear operator of the form $\omega \cdot \partial_\varphi + A(\varphi)$ is reversible if $A(\varphi)$ is reversible. Conjugating the linear operator $\omega \cdot \partial_\varphi + A(\varphi)$ by a family of invertible linear maps $\Phi(\varphi)$, we get the transformed operator

$$\begin{aligned} &\Phi^{-1}(\varphi) \circ \big(\omega \cdot \partial_\varphi + A(\varphi)\big) \circ \Phi(\varphi) = \omega \cdot \partial_\varphi + A_+(\varphi) \,, \\ &A_+(\varphi) := \Phi^{-1}(\varphi)\,(\omega \cdot \partial_\varphi \Phi(\varphi)) + \Phi^{-1}(\varphi) A(\varphi) \Phi(\varphi) \,. \end{aligned} \tag{4.2.54}$$

The conjugation of a reversible operator with a reversibility preserving operator is reversible.

Lemma 4.33. *A pseudodifferential operator* $\mathrm{Op}(a(\varphi, x, \xi))$ *is reversible, respectively reversibility preserving, if and only if its symbol satisfies*

$$a(-\varphi, -x, \xi) = -\overline{a(\varphi, x, \xi)} \,, \quad \textit{resp.} \quad a(-\varphi, -x, \xi) = \overline{a(\varphi, x, \xi)} \,. \tag{4.2.55}$$

Proof. If the symbols a satisfies (4.2.55), then, recalling the complex form of the involution $\mathcal{S}$ in (4.1.29)-(4.1.30), we deduce that $\mathrm{Op}(a(\varphi,x,\xi))$ is reversible, respectively anti-reversible. The vice versa follows using that $a(\varphi,x,j)=e^{-\mathrm{i}jx}\mathrm{Op}(a(\varphi,x,\xi))[e^{\mathrm{i}jx}]$. $\square$

Remark 4.34. Let $A(\varphi)=R(\varphi)+T(\varphi)$ be a reversible operator. Then $A(\varphi)=R_+(\varphi)+T_+(\varphi)$ where both operators

$$R_+(\varphi):=\tfrac{1}{2}(R(\varphi)-\mathcal{S}R(-\varphi)\mathcal{S})\,,\quad T_+(\varphi):=\tfrac{1}{2}(T(\varphi)-\mathcal{S}T(-\varphi)\mathcal{S})\,,$$

are reversible. If $R(\varphi)=\mathrm{Op}(r(\varphi,x,\xi))$ is pseudodifferential, then

$$R_+(\varphi)=\mathrm{Op}(r_+(\varphi,x,\xi))\,,\quad r_+(\varphi,x,\xi):=\tfrac{1}{2}(r(\varphi,x,\xi)-\overline{r(-\varphi,-x,\xi)})$$

and the pseudodifferential norms of $\mathrm{Op}(r)$ and $\mathrm{Op}(r_+)$ are equivalent. If $T(\varphi)$ is a tame operator with a tame constant $\mathfrak{M}_T(s)$, then $T_+(\varphi)$ is a tame operator as well with an equivalent tame constant.

Definition 4.35. (Reversible and anti-reversible function) A function $u(\varphi,\cdot)$ is called

$$\textit{Reversible}\ \text{ if }\ \mathcal{S}u(\varphi,\cdot)=u(-\varphi,\cdot)\ \text{(cfr.(4.1.17))};\quad \textit{Anti}-\textit{reversible}\ \text{ if }\ -\mathcal{S}u(\varphi,\cdot)=u(-\varphi,\cdot)\,.$$

The same definition holds in the action-angle-normal variables (θ,I,w) with the involution $\vec{\mathcal{S}}$ defined in (4.1.51) and in the $(z,\overline{z})$ complex variables with the involution in (4.1.29).

A reversibility preserving operator maps reversible, respectively anti-reversible, functions into reversible, respectively anti-reversible, functions.

Lemma 4.36. *Let X be a reversible vector field, according to* (4.1.5)*, and $u(\varphi,x)$ be a reversible quasi-periodic function. Then the linearized operator $\mathrm{d}_uX(u(\varphi,\cdot))$ is reversible, according to Definition 4.31.*

Proof. Differentiating (4.1.5) we get $(\mathrm{d}_uX)(\mathcal{S}u)\circ\mathcal{S}=-\mathcal{S}(\mathrm{d}_uX)(u)$ and use $\mathcal{S}u(\varphi,\cdot)=u(-\varphi,\cdot)$. $\square$

We also note the following lemma.

Lemma 4.37. *The projections $\Pi^{\intercal}_{\mathbb{S}^+,\Sigma}$, $\Pi^{\angle}_{\mathbb{S}^+,\Sigma}$ defined in Section 4.1.3 commute with the involution $\mathcal{S}$ defined in* (4.1.4)*, i.e. are reversibility preserving. The orthogonal projectors $\Pi_{\mathbb{S}}$ and $\Pi^{\perp}_{\mathbb{S}_0}$ commute with the involution in* (4.1.29)*, i.e. are reversibility preserving.*

Proof. The involution $\mathcal{S}$ defined in (4.1.4) maps $V_{n,\pm}$ into itself, acting as in (4.1.36). Then, by the decomposition (4.1.33), each projector $\Pi_{V_{n,\sigma}}$ commutes with $\mathcal{S}$. $\square$

4.2.5 Momentum preserving operators

The following definition is crucial in the construction of the traveling waves.

Definition 4.38. (Momentum preserving) A φ-dependent family of linear operators $A(\varphi)$, $\varphi \in \mathbb{T}^\nu$, is *momentum preserving* if

$$A(\varphi - \vec{\jmath}\varsigma) \circ \tau_\varsigma = \tau_\varsigma \circ A(\varphi)\,, \quad \forall\, \varphi \in \mathbb{T}^\nu\,,\ \varsigma \in \mathbb{R}\,, \tag{4.2.56}$$

where the translation operator τ_ς is defined in (4.1.7). A linear matrix operator $\mathbf{A}(\varphi)$ of the form (4.2.14) or (4.2.15) is *momentum preserving* if each of its components is momentum preserving.

Momentum preserving operators are closed under several operations.

Lemma 4.39. *Let $A(\varphi), B(\varphi)$ be momentum preserving operators. Then:*

(i) (Composition)*: $A(\varphi) \circ B(\varphi)$ is a momentum preserving operator.*

(ii) (Adjoint)*: the adjoint $(A(\varphi))^*$ is momentum preserving.*

(iii) (Inversion)*: If $A(\varphi)$ is invertible then $A(\varphi)^{-1}$ is momentum preserving.*

(iv) (Flow)*: Assume that*

$$\partial_t \Phi^t(\varphi) = A(\varphi)\Phi^t(\varphi)\,, \quad \Phi^0(\varphi) = \mathrm{Id}\,, \tag{4.2.57}$$

has a unique propagator $\Phi^t(\varphi)$ for any $t \in [0,1]$. Then $\Phi^t(\varphi)$ is momentum preserving.

Proof. Item (i) follows directly by (4.2.56). Item (ii), respectively (iii), follows by taking the adjoint, respectively the inverse, of (4.2.56) and using that $\tau_\varsigma^* = \tau_{-\varsigma} = \tau_\varsigma^{-1}$. Finally, item (iv) holds because $\tau_\varsigma^{-1}\Phi^t(\varphi - \vec{\jmath}\varsigma)\tau_\varsigma$ solves the same Cauchy in (4.2.57). □

We shall say that a linear operator of the form $\omega \cdot \partial_\varphi + A(\varphi)$ is momentum preserving if $A(\varphi)$ is momentum preserving. In particular, conjugating a momentum preserving operator $\omega \cdot \partial_\varphi + A(\varphi)$ by a family of invertible linear momentum preserving maps $\Phi(\varphi)$, we obtain the transformed operator $\omega \cdot \partial_\varphi + A_+(\varphi)$ in (4.2.54) which is momentum preserving.

Lemma 4.40. *Let $A(\varphi)$ be a momentum preserving linear operator and u a quasi-periodic traveling wave, according to Definition 4.12. Then $A(\varphi)u$ is a quasi-periodic traveling wave.*

Proof. It follows by Definition 4.38 and by the characterization of traveling waves in (4.2.2). □

Lemma 4.41. *Let X be a vector field translation invariant, according to* (4.1.8)*. Let u be a quasi-periodic traveling wave. Then the linearized operator $\mathrm{d}_u X(u(\varphi,\cdot))$ is momentum preserving.*

Proof. Differentiating (4.1.8) we get $(\mathrm{d}_u X)(\tau_\varsigma u) \circ \tau_\varsigma = \tau_\varsigma (\mathrm{d}_u X)(u)$, $\varsigma \in \mathbb{R}$. Then, apply (4.2.2). □

We now provide a characterization of the momentum preserving property in Fourier space.

Lemma 4.42. *Let φ-dependent family of operators $A(\varphi)$, $\varphi \in \mathbb{T}^\nu$, is momentum preserving if and only if the matrix elements of $A(\varphi)$, defined by* (4.2.13)*, fulfill*

$$A_j^{j'}(\ell) \neq 0 \quad \Rightarrow \quad \vec{\jmath} \cdot \ell + j - j' = 0\,, \quad \forall\, \ell \in \mathbb{Z}^\nu, \quad j, j' \in \mathbb{Z}\,. \tag{4.2.58}$$

Proof. By (4.2.13) we have, for any function $u(x)$,

$$\tau_\varsigma(A(\varphi)u) = \sum_{j,j' \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}^\nu} A_j^{j'}(\ell) e^{\mathrm{i}j\varsigma} u_{j'} e^{\mathrm{i}(\ell \cdot \varphi + jx)}$$

and

$$A(\varphi - \vec{\jmath}\varsigma)[\tau_\varsigma u] = \sum_{j,j' \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}^\nu} A_j^{j'}(\ell) e^{-\mathrm{i}\ell \cdot \vec{\jmath}\varsigma} e^{\mathrm{i}j'\varsigma} u_{j'} e^{\mathrm{i}(\ell \cdot \varphi + jx)}\,.$$

Therefore (4.2.56) is equivalent to (4.2.58). □

We characterize the symbol of a pseudodifferential operator which is momentum preserving:

Lemma 4.43. *A pseudodifferential operator $A(\varphi, x, D) = \mathrm{Op}(a(\varphi, x, \xi))$ is momentum preserving if and only if its symbol satisfies*

$$a(\varphi - \vec{\jmath}\varsigma, x, \xi) = a(\varphi, x + \varsigma, \xi)\,, \quad \forall\, \varsigma \in \mathbb{R}\,. \tag{4.2.59}$$

Proof. If the symbol a satisfies (4.2.59), then, for all $\varsigma \in \mathbb{R}$,

$$\tau_\varsigma \circ \mathrm{Op}(a(\varphi, x, \xi)) = \mathrm{Op}(a(\varphi, x + \varsigma, \xi)) \circ \tau_\varsigma = \mathrm{Op}(a(\varphi - \vec{\jmath}\varsigma, x, \xi)) \circ \tau_\varsigma\,,$$

proving that $\tau_\varsigma \circ A(\varphi, x, D) = A(\varphi - \vec{\jmath}\varsigma, x, D) \circ \tau_\varsigma$. The vice versa follows using that $a(\varphi, x, \xi) = e^{-\mathrm{i}\xi x} A(\varphi, x, D)[e^{\mathrm{i}\xi x}]$. □

If a symbol $a(\varphi, x, \xi)$ satisfies (4.2.59), then $(\omega \cdot \partial_\varphi a)(\varphi, x, \xi)$ satisfies (4.2.59) as well.

Lemma 4.44. *If $\beta(\varphi, x)$ is a quasi-periodic traveling wave, then the operator $\mathcal{B}(\varphi)$ defined in* (4.2.29) *is momentum preserving.*

Proof. We have $\mathcal{B}(\varphi - \vec{\jmath}\varsigma)[\tau_\varsigma u] = u(x + \beta(\varphi - \vec{\jmath}\varsigma, x) + \varsigma) = u(x + \varsigma + \beta(\varphi, x + \varsigma)) = \tau_\varsigma\big(\mathcal{B}(\varphi)u\big)$. □

We also note the following lemma.

Lemma 4.45. *The symplectic projections $\Pi^{\intercal}_{\mathbb{S}^+,\Sigma}$, $\Pi^{\angle}_{\mathbb{S}^+,\Sigma}$, the L^2-projections $\Pi^{L^2}_{\angle}$ and $\Pi_{\mathbb{S}}$, $\Pi^{\perp}_{\mathbb{S}_0}$ defined in Section 4.1.3 commute with the translation operators τ_ς defined in* (4.1.7)*, i.e. are momentum preserving.*

Proof. Recall that the translation τ_ς maps $V_{n,\pm}$ into itself, acting as in (4.1.37). Consider the L^2-orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_\angle \oplus \mathfrak{H}_\angle^\perp$, setting $\mathfrak{H}_\angle := \mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma}$ for brevity:

$$u = \Pi^{L^2}_{\mathfrak{H}_\angle} u + \Pi^{L^2}_{\mathfrak{H}_\angle^\perp} u\,, \quad \Pi^{L^2}_{\mathfrak{H}_\angle} u \in \mathfrak{H}_\angle\,, \quad \Pi^{L^2}_{\mathfrak{H}_\angle^\perp} u \in \mathfrak{H}_\angle^\perp\,.$$

Applying τ_ς we get $\tau_\varsigma u = \tau_\varsigma \Pi^{L^2}_{\mathfrak{H}_\angle} u + \tau_\varsigma \Pi^{L^2}_{\mathfrak{H}_\angle^\perp} u$. As shown above, τ_ς maps $\mathfrak{H}_\angle$ into itself for all ς. Thus also the L^2-orthogonal subspace $\mathfrak{H}_\angle^\perp$ is invariant under the action of τ_ς and we conclude, by the uniqueness of the orthogonal decomposition, that $\tau_\varsigma \Pi^{L^2}_{\mathfrak{H}_\angle} u = \Pi^{L^2}_{\mathfrak{H}_\angle} \tau_\varsigma u$, $\tau_\varsigma \Pi^{L^2}_{\mathfrak{H}_\angle^\perp} u = \Pi^{L^2}_{\mathfrak{H}_\angle^\perp} \tau_\varsigma u$. □

The next lemma concerns the Dirichlet-Neumann operator.

Lemma 4.46. *The Dirichlet-Neumann operator $G(\overline{\eta}, \mathtt{h})$, evaluated at a quasi-periodic traveling wave $\overline{\eta}(\varphi, x)$, is momentum preserving.*

Proof. It follows by (4.1.9) and the characterization in (4.2.2) of the quasi-periodic traveling wave $\overline{\eta}(\varphi, x)$. □

Quasi-periodic traveling waves in action-angle-normal coordinates. We now discuss how the momentum preserving condition reads in the coordinates (θ, I, w) introduced in (4.1.50). Recalling (4.1.52), if $u(\varphi, x)$ is a quasi-periodic traveling wave with action-angle-normal components $(\theta(\varphi), I(\varphi), w(\varphi, x))$, the condition $\tau_\varsigma u = u(\varphi - \vec{\jmath}\varsigma, \cdot)$ becomes

$$\begin{pmatrix} \theta(\varphi) - \vec{\jmath}\varsigma \\ I(\varphi) \\ \tau_\varsigma w(\varphi, \cdot) \end{pmatrix} = \begin{pmatrix} \theta(\varphi - \vec{\jmath}\varsigma) \\ I(\varphi - \vec{\jmath}\varsigma) \\ w(\varphi - \vec{\jmath}\varsigma, \cdot) \end{pmatrix}, \quad \forall \varsigma \in \mathbb{R}\,. \tag{4.2.60}$$

As we look for $\theta(\varphi)$ of the form $\theta(\varphi) = \varphi + \Theta(\varphi)$, with a $(2\pi)^\nu$-periodic function $\Theta : \mathbb{R}^\nu \mapsto \mathbb{R}^\nu$, $\varphi \mapsto \Theta(\varphi)$, the traveling wave condition becomes

$$\begin{pmatrix} \Theta(\varphi) \\ I(\varphi) \\ \tau_\varsigma w(\varphi, \cdot) \end{pmatrix} = \begin{pmatrix} \Theta(\varphi - \vec{\jmath}\varsigma) \\ I(\varphi - \vec{\jmath}\varsigma) \\ w(\varphi - \vec{\jmath}\varsigma, \cdot) \end{pmatrix}, \quad \forall \varsigma \in \mathbb{R}\,. \tag{4.2.61}$$

Definition 4.47. (Traveling wave variation) We call a traveling wave variation $g(\varphi) = (g_1(\varphi), g_2(\varphi), g_3(\varphi, \cdot)) \in \mathbb{R}^\nu \times \mathbb{R}^\nu \times \mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma}$ a function satisfying (4.2.61), i.e.

$$g_1(\varphi) = g_1(\varphi - \vec{\jmath}\varsigma), \quad g_2(\varphi) = g_2(\varphi - \vec{\jmath}\varsigma), \quad \tau_\varsigma g_3(\varphi) = g_3(\varphi - \vec{\jmath}\varsigma), \ \forall \varsigma \in \mathbb{R}\,,$$

or equivalently $D\vec{\tau}_\varsigma g(\varphi) = g(\varphi - \vec{\jmath}\varsigma)$ for any $\varsigma \in \mathbb{R}$, where $D\vec{\tau}_\varsigma$ is the differential of $\vec{\tau}_\varsigma$, namely

$$D\vec{\tau}_\varsigma \begin{pmatrix} \Theta \\ I \\ w \end{pmatrix} = \begin{pmatrix} \Theta \\ I \\ \tau_\varsigma w \end{pmatrix}, \quad \forall \varsigma \in \mathbb{R}.$$

According to Definition 4.38, a linear operator acting in $\mathbb{R}^\nu \times \mathbb{R}^\nu \times \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ momentum preserving if

$$A(\varphi - \vec{\jmath}\varsigma) \circ D\vec{\tau}_\varsigma = D\vec{\tau}_\varsigma \circ A(\varphi), \quad \forall \varsigma \in \mathbb{R}. \tag{4.2.62}$$

Similarly to Lemma 4.40, one proves the following result:

Lemma 4.48. *Let $A(\varphi)$ be a momentum preserving linear operator acting on $\mathbb{R}^\nu \times \mathbb{R}^\nu \times \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ and $g \in \mathbb{R}^\nu \times \mathbb{R}^\nu \times \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ be a traveling wave variation. Then $A(\varphi)g(\varphi)$ is a traveling wave variation.*

4.3 Transversality of linear frequencies

In this section we extend the KAM theory approach of [44], [21] in order to deal with the linear frequencies $\Omega_j(\kappa)$ defined in (4.0.3). The main novelty is the use of the momentum condition in the proof of Proposition 4.53. We shall also exploit that the tangential sites $\mathbb{S} := \{ \bar{\jmath}_1, \ldots, \bar{\jmath}_\nu \} \subset \mathbb{Z}\backslash\{0\}$ defined in (4.1.48), have all distinct modulus $|\bar{\jmath}_a| = \bar{n}_a$, see assumption (4.0.4).

We first introduce the following definition.

Definition 4.49. A function $f = (f_1, \ldots, f_N) : [\kappa_1, \kappa_2] \to \mathbb{R}^N$ is *non-degenerate* if, for any $c \in \mathbb{R}^N\backslash\{0\}$, the scalar function $f \cdot c$ is not identically zero on the whole interval $[\kappa_1, \kappa_2]$.

From a geometric point of view, if f is non-degenerate it means that the image of the curve $f([\kappa_1, \kappa_2]) \subset \mathbb{R}^N$ is not contained in any hyperplane of $\mathbb{R}^N$.

We shall use in the sequel that the maps $\kappa \mapsto \Omega_j(\kappa)$ are analytic in $[\kappa_1, \kappa_2]$. We decompose

$$\Omega_j(\kappa) = \omega_j(\kappa) + \frac{\gamma}{2}\frac{G_j(0)}{j}, \quad \omega_j(\kappa) := \sqrt{\kappa\, G_j(0) j^2 + g\, G_j(0) + \left(\frac{\gamma}{2}\frac{G_j(0)}{j}\right)^2}. \tag{4.3.1}$$

Note that the dependence on κ of $\Omega_j(\kappa)$ enters only through $\omega_j(\kappa)$, because $\frac{G_j(0)}{j}$ is independent of κ. Note also that $j \mapsto \omega_j(\kappa)$ is even in j, whereas the component due to the vorticity $j \mapsto \frac{\gamma}{2}\frac{G_j(0)}{j}$ is odd. Moreover this term is, in view of (1.1.16), uniformly bounded in j.

Lemma 4.50. (Non-degeneracy-I) *The following frequency vectors are non-degenerate:*

1. $\vec{\Omega}(\kappa) := (\Omega_j(\kappa))_{j \in \mathbb{S}} \in \mathbb{R}^\nu$;

2. $\big(\vec{\Omega}(\kappa), \sqrt{\kappa}\big) \in \mathbb{R}^{\nu+1}$*;*

3. $\big(\vec{\Omega}(\kappa), \Omega_j(\kappa)\big) \in \mathbb{R}^{\nu+1}$*, for any* $j \in \mathbb{Z}\backslash(\{0\} \cup \mathbb{S} \cup (-\mathbb{S}))$*;*

4. $\big(\vec{\Omega}(\kappa), \Omega_j(\kappa), \Omega_{j'}(\kappa)\big) \in \mathbb{R}^{\nu+2}$*, for any* $j, j' \in \mathbb{Z}\backslash(\{0\} \cup \mathbb{S} \cup (-\mathbb{S}))$ *and* $|j| \neq |j'|$.

Proof. Let

$$\widetilde{\Omega}_j(\kappa) := \begin{cases} \Omega_j(\kappa) & \text{for } j \neq 0 \\ \sqrt{\kappa} & \text{for } j = 0\,, \end{cases} \qquad \widetilde{\omega}_j(\kappa) := \begin{cases} \omega_j(\kappa) & \text{for } j \neq 0 \\ \sqrt{\kappa} & \text{for } j = 0\,. \end{cases} \tag{4.3.2}$$

Recalling (4.3.1), we have that, for any $j \in \mathbb{Z}$,

$$\partial_\kappa \widetilde{\omega}_j(\kappa) = \lambda_j(\kappa)\widetilde{\omega}_j(\kappa)\,, \quad \lambda_j(\kappa) := \begin{cases} \dfrac{G_j(0)j^2}{2\Big(\kappa\, G_j(0)j^2 + g\, G_j(0) + \Big(\frac{\gamma}{2}\frac{G_j(0)}{j}\Big)^2\Big)} & \text{for } j \neq 0 \\ \frac{1}{2\kappa} & \text{for } j = 0\,. \end{cases} \tag{4.3.3}$$

Moreover $\partial_\kappa \lambda_j(\kappa) = -2\lambda_j(\kappa)^2$, for any $j \in \mathbb{Z}$, and therefore, for any $n \in \mathbb{N}$,

$$\partial_\kappa^n \widetilde{\omega}_j(\kappa) = \widetilde{c}_n \lambda_j(\kappa)^n \widetilde{\omega}_j(\kappa)\,, \quad \widetilde{c}_n := c_1 \cdot \ldots \cdot c_n\,, \quad c_n := 3 - 2n\,. \tag{4.3.4}$$

We now prove items 2 and 3, i.e. the non-degeneracy of the vector $\big(\vec{\Omega}(\kappa), \widetilde{\Omega}_j(\kappa)\big) \in \mathbb{R}^{\nu+1}$ for any $j \in \mathbb{Z}\backslash(\mathbb{S} \cup (-\mathbb{S}))$, where $\widetilde{\Omega}_j(\kappa)$ is defined in (4.3.2). Items 1 and 4 follow similarly. For this purpose, by analyticity, it is sufficient to find one value of $\kappa \in [\kappa_1, \kappa_2]$ so that the determinant of the $(\nu+1) \times (\nu+1)$ matrix

$$\mathcal{A}(\kappa) := \begin{pmatrix} \partial_\kappa \Omega_{\bar{\jmath}_1}(\kappa) & \cdots & \partial_\kappa \Omega_{\bar{\jmath}_\nu}(\kappa) & \partial_\kappa \widetilde{\Omega}_j(\kappa) \\ \vdots & \ddots & \vdots & \vdots \\ \partial_\kappa^{\nu+1} \Omega_{\bar{\jmath}_1}(\kappa) & \cdots & \partial_\kappa^{\nu+1} \Omega_{\bar{\jmath}_\nu}(\kappa) & \partial_\kappa^{\nu+1} \widetilde{\Omega}_j(\kappa) \end{pmatrix}$$

is not zero. We actually show that $\det \mathcal{A}(\kappa) \neq 0$ for any $\kappa \in [\kappa_1, \kappa_2]$. By (4.3.2)-(4.3.4) and the multilinearity of the determinant function, we get

$$\det \mathcal{A}(\kappa) = C(\kappa) \det \begin{pmatrix} 1 & \cdot & 1 & 1 \\ \lambda_{\bar{\jmath}_1}(\kappa) & \cdot & \lambda_{\bar{\jmath}_\nu}(\kappa) & \lambda_j(\kappa) \\ \vdots & \ddots & \vdots & \vdots \\ \lambda_{\bar{\jmath}_1}(\kappa)^\nu & \cdot & \lambda_{\bar{\jmath}_\nu}(\kappa)^\nu & \lambda_j(\kappa)^\nu \end{pmatrix} =: C(\kappa) \det \mathcal{B}(\kappa)$$

where

$$C(\kappa) := \prod_{q=1}^{\nu+1} \widetilde{c}_q \cdot \prod_{p \in \{\bar{\jmath}_1, \ldots, \bar{\jmath}_\nu, j\}} \lambda_p(\kappa)\widetilde{\omega}_p(\kappa) \neq 0\,, \quad \forall\, \kappa \in [\kappa_1, \kappa_2]\,.$$

Since $\mathcal{B}(\kappa)$ is a Vandermorde matrix, we conclude that

$$\det \mathcal{A}(\kappa) = C(\kappa) \prod_{p,p' \in \{\bar{\jmath}_1,\ldots,\bar{\jmath}_\nu,j\}, p<p'} \big(\lambda_p(\kappa) - \lambda_{p'}(\kappa)\big) .$$

Now, the fact that $\det \mathcal{A}(\kappa) \neq 0$ for any $\kappa \in [\kappa_1, \kappa_2]$ is a consequence from the following

Claim: *For any $p, p' \in \{\bar{\jmath}_1, \ldots, \bar{\jmath}_\nu, j\}$, $p \neq p'$, one has $\lambda_p(\kappa) \neq \lambda_{p'}(\kappa)$ for any $\kappa \in [\kappa_1, \kappa_2]$.*

Proof of the Claim: If $p' = 0$ and $p \neq 0$, the claim follows because, by (4.3.3),

$$\lambda_p(\kappa) = \frac{1}{2\Big(\kappa + \frac{g}{p^2} + \frac{\gamma^2}{4}\frac{G_p(0)}{p^4}\Big)} < \frac{1}{2\kappa} = \lambda_0(\kappa) .$$

Consider now the case $p, p' \neq 0$. We now prove that the map $p \mapsto \lambda_p(\kappa)$ is strictly monotone on $(0, +\infty)$. In case of finite depth, $G_p(0) = p \tanh(\mathtt{h}p)$, and

$$\partial_p \lambda_p(\kappa) = \frac{1}{2\left(\kappa + \frac{g}{p^2} + \frac{\gamma^2}{4}\frac{\tanh(\mathtt{h}p)}{p^3}\right)^2} \left\{ \frac{2g}{p^3} + \frac{\gamma^2}{4} \frac{3\tanh(\mathtt{h}p) - (1 - \tanh^2(\mathtt{h}p))\mathtt{h}p}{p^4} \right\} .$$

The function $f(y) := 3\tanh(y) - (1-\tanh^2(y))y$ is positive for any $y > 0$. Indeed $f(y) \to 0$ as $y \to 0$, and it is strictly monotone increasing for $y > 0$, since $f'(y) = 2(1-\tanh^2(y))(1+y\tanh(y)) > 0$. We deduce that $\partial_p \lambda_p(\kappa) > 0$, also if the depth $\mathtt{h} = +\infty$. Since the function $p \mapsto \lambda_p(\kappa)$ is even we have proved that that it is strictly monotone decreasing on $(-\infty, 0)$ and increasing in $(0, +\infty)$. Thus, if $\lambda_p(\kappa) = \lambda_{p'}(\kappa)$ then $p = -p'$. But this case is excluded by the assumption (4.0.4) and the condition $j \notin \mathbb{S} \cup (-\mathbb{S})$, which together imply $|p| \neq |p'|$. □

Note that in items 3 and 4 of Lemma 4.50 we require that j and j' do not belong to $\{0\} \cup \mathbb{S} \cup (-\mathbb{S})$. In order to deal in Proposition 4.53 when j and j' are in $\mathbb{S} \cup (-\mathbb{S})$, we need also the following lemma. It is actually a direct consequence of the proof of Lemma 4.50, noting that $\Omega_j(\kappa) - \omega_j(\kappa)$ is independent of κ.

Lemma 4.51. (Non-degeneracy-II) *Let $\vec{\omega}(\kappa) := \big(\omega_{\bar{\jmath}_1}(\kappa), \ldots, \omega_{\bar{\jmath}_\nu}(\kappa)\big)$. The following vectors are non-degenerate:*

1. *$(\vec{\omega}(\kappa), 1) \in \mathbb{R}^{\nu+1}$;*
2. *$(\vec{\omega}(\kappa), \omega_j(\kappa), 1) \in \mathbb{R}^{\nu+2}$, for any $j \in \mathbb{Z} \setminus (\{0\} \cup \mathbb{S} \cup (-\mathbb{S}))$.*

For later use, we provide the following asymptotic estimate of the linear frequencies.

Lemma 4.52. (Asymptotics) *For any $j \in \mathbb{Z} \setminus \{0\}$, we have*

$$\omega_j(\kappa) = \sqrt{\kappa}\,|j|^{\frac{3}{2}} + \frac{c_j(\kappa)}{\sqrt{\kappa}\,|j|^{\frac{1}{2}}} , \tag{4.3.5}$$

where, for any $n \in \mathbb{N}_0$, there exists a constant $C_{n,\mathtt{h}} > 0$ such that

$$\sup_{\substack{j \in \mathbb{Z}\setminus\{0\} \\ \kappa \in [\kappa_1,\kappa_2]}} \left| \partial_\kappa^n \frac{c_j(\kappa)}{\sqrt{\kappa}} \right| \leqslant C_{n,\mathtt{h}} \,. \tag{4.3.6}$$

Proof. By (4.3.1) we deduce (4.3.5) with

$$c_j(\kappa) := \frac{\kappa |j| (G_j(0) - |j|) + \frac{g\, G_j(0)}{|j|} \Big(1 + \big(\frac{\gamma}{2}\big)^2 \frac{G_j(0)}{g|j|^2} \Big)}{1 + \sqrt{1 + \frac{G_j(0)-|j|}{|j|} + \frac{g\, G_j(0)}{\kappa |j|^3} \Big(1 + \big(\frac{\gamma}{2}\big)^2 \frac{G_j(0)}{g|j|^2} \Big)}} \,.$$

Then (4.3.6) follows exploiting that (both for finite and infinite depth) the quantities $|j|(G_j(0) - |j|)$ and $G_j(0)/|j|$ are uniformly bounded in j, see (1.1.16). □

The next proposition is the key of the argument. We remind that $\vec{\jmath} = (\bar{\jmath}_1, \ldots, \bar{\jmath}_\nu)$ denotes the vector in $\mathbb{Z}^\nu$ of tangential sites introduced in (4.1.53).

Proposition 4.53. (Transversality) *There exist $m_0 \in \mathbb{N}$ and $\rho_0 > 0$ such that, for any $\kappa \in [\kappa_1, \kappa_2]$, the following hold:*

$$\max_{0 \leqslant n \leqslant m_0} |\partial_\kappa^n \vec{\Omega}(\kappa) \cdot \ell| \geqslant \rho_0 \langle \ell \rangle \,, \quad \forall\, \ell \in \mathbb{Z}^\nu \setminus \{0\} \,; \tag{4.3.7}$$

$$\begin{cases} \max\limits_{0 \leqslant n \leqslant m_0} |\partial_\kappa^n (\vec{\Omega}(\kappa) \cdot \ell + \Omega_j(\kappa))| \geqslant \rho_0 \langle \ell \rangle \\ \vec{\jmath} \cdot \ell + j = 0 \,, \quad \ell \in \mathbb{Z}^\nu \,,\ j \in \mathbb{S}_0^c \,; \end{cases} \tag{4.3.8}$$

$$\begin{cases} \max\limits_{0 \leqslant n \leqslant m_0} |\partial_\kappa^n (\vec{\Omega}(\kappa) \cdot \ell + \Omega_j(\kappa) - \Omega_{j'}(\kappa))| \geqslant \rho_0 \langle \ell \rangle \\ \vec{\jmath} \cdot \ell + j - j' = 0 \,, \quad \ell \in \mathbb{Z}^\nu \,,\ j, j' \in \mathbb{S}_0^c \,,\ (\ell, j, j') \neq (0, j, j) \,; \end{cases} \tag{4.3.9}$$

$$\begin{cases} \max\limits_{0 \leqslant n \leqslant m_0} |\partial_\kappa^n (\vec{\Omega}(\kappa) \cdot \ell + \Omega_j(\kappa) + \Omega_{j'}(\kappa))| \geqslant \rho_0 \langle \ell \rangle \\ \vec{\jmath} \cdot \ell + j + j' = 0 \,,\ \ell \in \mathbb{Z}^\nu \,,\ j, j' \in \mathbb{S}_0^c \,. \end{cases} \tag{4.3.10}$$

We call ρ_0 the amount of non-degeneracy and m_0 the index of non-degeneracy.

Proof. We prove separately (4.3.7)-(4.3.10). In this proof we set for brevity $\mathfrak{K} := [\kappa_1, \kappa_2]$.
Proof of (4.3.7). By contradiction, assume that for any $m \in \mathbb{N}$ there exist $\kappa_m \in \mathfrak{K}$ and $\ell_m \in \mathbb{Z}^\nu \setminus \{0\}$ such that

$$\left| \partial_\kappa^n \vec{\Omega}(\kappa_m) \cdot \frac{\ell_m}{\langle \ell_m \rangle} \right| < \frac{1}{\langle m \rangle} \,, \quad \forall\, 0 \leqslant n \leqslant m \,. \tag{4.3.11}$$

The sequences $(\kappa_m)_{m \in \mathbb{N}} \subset \mathfrak{K}$ and $(\ell_m / \langle \ell_m \rangle)_{m \in \mathbb{N}} \subset \mathbb{R}^\nu \setminus \{0\}$ are both bounded. By compactness, up to subsequences $\kappa_m \to \bar{\kappa} \in \mathfrak{K}$ and $\ell_m / \langle \ell_m \rangle \to \bar{c} \neq 0$. Therefore, in the limit for $m \to +\infty$, by (4.3.11) we get $\partial_\kappa^n \vec{\Omega}(\bar{\kappa}) \cdot \bar{c} = 0$ for any $n \in \mathbb{N}_0$. By the analyticity of $\vec{\Omega}(\kappa)$, we deduce that the function $\kappa \mapsto \vec{\Omega}(\kappa) \cdot \overline{c}$ is identically zero on $\mathfrak{K}$, which contradicts Lemma 4.50-1.

Proof of (4.3.8). We divide the proof in 4 steps.

STEP 1. Recalling (4.3.1) and Lemma 4.52, we have that, for any $\kappa \in \mathfrak{K}$,

$$|\vec{\Omega}(\kappa) \cdot \ell + \Omega_j(\kappa)| \geqslant |\Omega_j(\kappa)| - |\vec{\Omega}(\kappa) \cdot \ell| \geqslant \sqrt{\kappa_1}\, |j|^{\frac{3}{2}} - C\langle \ell \rangle \geqslant \langle \ell \rangle$$

whenever $|j|^{\frac{3}{2}} \geqslant C_0 \langle \ell \rangle$, for some $C_0 > 0$. In this cases (4.3.8) is already fulfilled with $n = 0$. Hence we restrict in the sequel to indexes $\ell \in \mathbb{Z}^\nu$ and $j \in \mathbb{S}_0^c$ satisfying

$$|j|^{\frac{3}{2}} < C_0 \langle \ell \rangle . \tag{4.3.12}$$

STEP 2. By contradiction, we assume that, for any $m \in \mathbb{N}$, there exist $\kappa_m \in \mathfrak{K}$, $\ell_m \in \mathbb{Z}^\nu$ and $j_m \in \mathbb{S}_0^c$, with $|j_m|^{\frac{3}{2}} < C_0 \langle \ell_m \rangle$, such that, for any $n \in \mathbb{N}_0$ with $n \leqslant m$,

$$\begin{cases} \left| \partial_\kappa^n \big(\vec{\Omega}(\kappa) \cdot \frac{\ell_m}{\langle \ell_m \rangle} + \frac{1}{\langle \ell_m \rangle} \Omega_{j_m}(\kappa) \big)_{|\kappa = \kappa_m} \right| < \frac{1}{\langle m \rangle} \\ \vec{\jmath} \cdot \ell_m + j_m = 0 . \end{cases} \tag{4.3.13}$$

Up to subsequences $\kappa_m \to \overline{\kappa} \in \mathfrak{K}$ and $\ell_m / \langle \ell_m \rangle \to \overline{c} \in \mathbb{R}^\nu$.

STEP 3. We consider first the case when the sequence $(\ell_m)_{m \in \mathbb{N}} \subset \mathbb{Z}^\nu$ is bounded. Up to subsequences, we have definitively that $\ell_m = \overline{\ell} \in \mathbb{Z}^\nu$. Moreover, since j_m and ℓ_m satisfy (4.3.12), also the sequence $(j_m)_{m \in \mathbb{N}}$ is bounded and, up to subsequences, definitively $j_m = \overline{\jmath} \in \mathbb{S}_0^c$. Therefore, in the limit $m \to \infty$, from (4.3.13) we obtain

$$\partial_\kappa^n \big(\vec{\Omega}(\kappa) \cdot \overline{\ell} + \Omega_{\overline{\jmath}}(\kappa) \big)_{|\kappa = \overline{\kappa}} = 0 \ , \ \forall\, n \in \mathbb{N}_0 \, , \quad \vec{\jmath} \cdot \overline{\ell} + \overline{\jmath} = 0 \, .$$

By analyticity this implies

$$\vec{\Omega}(\kappa) \cdot \overline{\ell} + \Omega_{\overline{\jmath}}(\kappa) = 0 \, , \ \forall\, \kappa \in \mathfrak{K} \, , \quad \vec{\jmath} \cdot \overline{\ell} + \overline{\jmath} = 0 \, . \tag{4.3.14}$$

We distinguish two cases:

- Let $\overline{\jmath} \notin -\mathbb{S}$. By (4.3.14) the vector $\big(\vec{\Omega}(\kappa), \Omega_{\overline{\jmath}}(\kappa)\big)$ is degenerate according to Definition 4.49 with $c := (\overline{\ell}, 1) \neq 0$. This contradicts Lemma 4.50-3.

- Let $\overline{\jmath} \in -\mathbb{S}$. With no loss of generality suppose $\overline{\jmath} = -\overline{\jmath}_1$. Then, denoting $\overline{\ell} = (\overline{\ell_1}, \ldots, \overline{\ell_\nu})$, system (4.3.14) reads, for any $\kappa \in \mathfrak{K}$,

$$\begin{cases} (\overline{\ell_1} + 1) \omega_{\overline{\jmath}_1}(\kappa) + \sum_{a=2}^\nu \overline{\ell_a} \omega_{\overline{\jmath}_a}(\kappa) + \frac{\gamma}{2} \Big((\overline{\ell_1} - 1) \frac{G_{\overline{\jmath}_1}(0)}{\overline{\jmath}_1} + \sum_{a=2}^\nu \overline{\ell_a} \frac{G_{\overline{\jmath}_a}(0)}{\overline{\jmath}_a} \Big) = 0 \\ (\overline{\ell_1} - 1) \overline{\jmath}_1 + \sum_{a=2}^\nu \overline{\ell_a}\, \overline{\jmath}_a = 0 \, . \end{cases} \tag{4.3.15}$$

By Lemma 4.51 the vector $(\vec{\omega}(\kappa), 1)$ is non-degenerate, which is a contradiction for $\gamma \neq 0$. If $\gamma = 0$ we only deduce $\overline{\ell_1} = -1$ and $\overline{\ell_2} = \ldots = \overline{\ell_\nu} = 0$. Inserting these values in the momentum condition in (4.3.15), we get $2\overline{\jmath}_1 = 0$. This is a contradiction with $\overline{\jmath}_1 \neq 0$.

STEP 4. We consider now the case when the sequence $(\ell_m)_{m\in\mathbb{N}}$ is unbounded. Up to subsequences $|\ell_m| \to \infty$ as $m \to \infty$ and $\lim_{m\to\infty} \ell_m/\langle \ell_m\rangle =: \overline{c} \neq 0$. By (4.3.1) and (4.3.5), for any $n \in \mathbb{N}_0$,

$$\partial_\kappa^n \frac{1}{\langle \ell_m\rangle}\Omega_{j_m}(\kappa_m) = \partial_\kappa^n \Big(\frac{1}{\langle \ell_m\rangle}\sqrt{\kappa}\,|j_m|^{\frac{3}{2}} + \frac{c_{j_m}(\kappa)}{\langle \ell_m\rangle \sqrt{\kappa}\,|j_m|^{\frac{1}{2}}} + \frac{\gamma}{2\langle \ell_m\rangle}\frac{G_{j_m}(0)}{j_m}\Big)_{|\kappa=\kappa_m}$$
$$\overset{(4.3.6)}{\to} \overline{d}(\partial_\kappa^n\sqrt{\kappa})_{|\kappa=\overline{\kappa}}\,, \text{ for } m \to \infty\,,$$

with $\overline{d} := \lim_{m\to\infty} |j_m|^{\frac{3}{2}}/\langle \ell_m\rangle \in \mathbb{R}$. Note that $\overline{d}$ is finite because j_m and ℓ_m satisfy (4.3.12). Therefore (4.3.13) becomes, in the limit $m \to \infty$,

$$\partial_\kappa^n \big(\vec{\Omega}(\kappa)\cdot \overline{c} + \overline{d}\sqrt{\kappa}\big)_{|\kappa=\overline{\kappa}} = 0\,, \quad \forall\, n \in \mathbb{N}_0\,.$$

By analyticity, this implies that $\vec{\Omega}(\kappa)\cdot\overline{c} + \overline{d}\sqrt{\kappa} = 0$ for any $\kappa \in \mathfrak{K}$. This contradicts the non-degeneracy of the vector $(\vec{\Omega}(\kappa), \sqrt{\kappa})$ in Lemma 4.50-2, since $(\overline{c}, \overline{d}) \neq 0$.

Proof of (4.3.9). We split again the proof into 4 steps.

STEP 1. By Lemma 4.52, for any $\kappa \in \mathfrak{K}$,

$$|\vec{\Omega}(\kappa)\cdot\ell + \Omega_j(\kappa) - \Omega_{j'}(\kappa)| \geqslant |\Omega_j(\kappa) - \Omega_{j'}(\kappa)| - |\vec{\Omega}(\kappa)\cdot\ell| \geqslant \sqrt{\kappa_1}\big|\,|j|^{\frac{3}{2}} - |j'|^{\frac{3}{2}}\big| - C\langle\ell\rangle \geqslant \langle\ell\rangle$$

whenever $|\,|j|^{\frac{3}{2}} - |j'|^{\frac{3}{2}}| \geqslant C_1\langle\ell\rangle$ for some $C_1 > 0$. In this case (4.3.9) is already fulfilled with $n = 0$. Thus we restrict to indexes $\ell \in \mathbb{Z}^\nu$ and $j, j' \in \mathbb{S}_0^c$, such that

$$\big|\,|j|^{\frac{3}{2}} - |j'|^{\frac{3}{2}}\big| < C_1\langle\ell\rangle\,. \tag{4.3.16}$$

Furthermore we may assume $j_m \neq j'_m$ because the case $j_m = j'_m$ is included in (4.3.7).

STEP 2. By contradiction, we assume that, for any $m \in \mathbb{N}$, there exist $\kappa_m \in \mathfrak{K}$, $\ell_m \in \mathbb{Z}^\nu$ and $j_m, j'_m \in \mathbb{S}_0^c$, satisfying (4.3.16), such that, for any $0 \leqslant n \leqslant m$,

$$\begin{cases} \big|\partial_\kappa^n\big(\vec{\Omega}(\kappa)\cdot\frac{\ell_m}{\langle\ell_m\rangle} + \frac{1}{\langle\ell_m\rangle}\big(\Omega_{j_m}(\kappa) - \Omega_{j'_m}(\kappa)\big)\big)_{|\kappa=\kappa_m}\big| < \frac{1}{\langle m\rangle} \\ \vec{\jmath}\cdot\ell_m + j_m - j'_m = 0\,. \end{cases} \tag{4.3.17}$$

Up to subsequences $\kappa_m \to \overline{\kappa} \in \mathfrak{K}$ and $\ell_m/\langle\ell_m\rangle \to \overline{c} \in \mathbb{R}^\nu$.

STEP 3. We start with the case when $(\ell_m)_{m\in\mathbb{N}} \subset \mathbb{Z}^\nu$ is bounded. Up to subsequences, we have definitively that $\ell_m = \overline{\ell} \in \mathbb{Z}^\nu$. Moreover, if $|j_m| \neq |j'_m|$, there is $c > 0$ such that

$$c\big(|j_m|^{\frac{1}{2}} + |j'_m|^{\frac{1}{2}}\big) \leqslant \big||j_m|^{\frac{3}{2}} - |j'_m|^{\frac{3}{2}}\big| < C_1\langle \ell_m \rangle \leqslant C\,, \qquad \forall m \in \mathbb{N}\,,$$

If $j_m = -j'_m$ we deduce by the momentum relation that $|j_m| = |j'_m| \leqslant C\langle \ell_m \rangle \leqslant C$, and we conclude that in any case the sequences $(j_m)_{m\in\mathbb{N}}$ and $(j'_m)_{m\in\mathbb{N}}$ are bounded. Up to subsequences, we have definitively that $j_m = \overline{\jmath}$ and $j'_m = \overline{\jmath}'$, with $\overline{\jmath}, \overline{\jmath}' \in \mathbb{S}_0^c$ and such that

$$\jmath \neq \overline{\jmath}'\,. \tag{4.3.18}$$

Therefore (4.3.17) becomes, in the limit $m \to \infty$,

$$\partial_\kappa^n \big(\vec{\Omega}(\kappa) \cdot \overline{\ell} + \Omega_{\overline{\jmath}}(\kappa) - \Omega_{\overline{\jmath}'}(\kappa)\big)_{|\kappa=\overline{\kappa}} = 0\,,\ \forall\, n \in \mathbb{N}_0\,, \quad \vec{\jmath} \cdot \overline{\ell} + \overline{\jmath} - \overline{\jmath}' = 0\,.$$

By analyticity, we obtain that

$$\vec{\Omega}(\kappa) \cdot \overline{\ell} + \Omega_{\overline{\jmath}}(\kappa) - \Omega_{\overline{\jmath}'}(\kappa) = 0\,,\ \forall\, \kappa \in \mathfrak{K}\,, \quad \vec{\jmath} \cdot \overline{\ell} + \overline{\jmath} - \overline{\jmath}' = 0\,. \tag{4.3.19}$$

We distinguish several cases:

- Let $\overline{\jmath}, \overline{\jmath}' \notin -\mathbb{S}$ and $|\overline{\jmath}| \neq |\overline{\jmath}'|$. By (4.3.19) the vector $(\vec{\Omega}(\kappa), \Omega_{\overline{\jmath}}(\kappa), \Omega_{\overline{\jmath}'}(\kappa))$ is degenerate with $c := (\overline{\ell}, 1, -1) \neq 0$, contradicting Lemma 4.50-4.

- Let $\overline{\jmath}, \overline{\jmath}' \notin -\mathbb{S}$ and $\overline{\jmath}' = -\overline{\jmath}$. In view of (4.3.1), system (4.3.19) becomes

$$\begin{cases} \vec{\omega}(\kappa) \cdot \overline{\ell} + \frac{\gamma}{2}\Big(\sum_{a=1}^{\nu} \overline{\ell_a} \frac{G_{\overline{\jmath}_a}(0)}{\overline{\jmath}_a} + 2\frac{G_{\overline{\jmath}}(0)}{\overline{\jmath}}\Big) = 0\,, & \forall \kappa \in \mathfrak{K}\,, \\ \vec{\jmath} \cdot \overline{\ell} + 2\overline{\jmath} = 0\,. \end{cases} \tag{4.3.20}$$

  By Lemma 4.51 the vector $(\vec{\omega}(\kappa), 1)$ is non-degenerate, which is a contradiction for $\gamma \neq 0$. If $\gamma = 0$ the first equation in (4.3.20) implies $\overline{\ell} = 0$. Then the momentum condition implies $2\overline{\jmath} = 0$, which is a contradiction with $\overline{\jmath} \neq 0$.

- Let $\overline{\jmath}' \notin -\mathbb{S}$ and $\overline{\jmath} \in -\mathbb{S}$. With no loss of generality suppose $\overline{\jmath} = -\overline{\jmath}_1$. In view of (4.3.1), the first equation in (4.3.19) implies that, for any $\kappa \in \mathfrak{K}$

$$(\overline{\ell_1} + 1)\omega_{\overline{\jmath}_1}(\kappa) + \sum_{a=2}^{\nu} \overline{\ell_a}\omega_{\overline{\jmath}_a}(\kappa) - \omega_{\overline{\jmath}'}(\kappa) + \frac{\gamma}{2}\Big((\overline{\ell_1} - 1)\frac{G_{\overline{\jmath}_1}(0)}{\overline{\jmath}_1} + \sum_{a=2}^{\nu} \overline{\ell_a}\frac{G_{\overline{\jmath}_a}(0)}{\overline{\jmath}_a} - \frac{G_{\overline{\jmath}'}(0)}{\overline{\jmath}'}\Big) = 0\,.$$

  By Lemma 4.51 the vector $\big(\vec{\omega}(\kappa), \omega_{\overline{\jmath}'}(\kappa), 1\big)$ is non-degenerate, which is a contradiction.

- Last, let $\overline{\jmath}, \overline{\jmath}' \in -\mathbb{S}$ and $\overline{\jmath} \neq \overline{\jmath}'$, by (4.3.18). With no loss of generality suppose $\overline{\jmath} = -\overline{\jmath}_1$ and

$\overline{\jmath}' = -\overline{\jmath}_2$. Then (4.3.19) reads

$$\begin{cases} (\overline{\ell_1}+1)\omega_{\overline{\jmath}_1}(\kappa) + \big(\overline{\ell_2}-1\big)\,\omega_{\overline{\jmath}_2} + \sum_{a=3}^{\nu} \overline{\ell_a}\omega_{\overline{\jmath}_a}(\kappa) \\ +\frac{\gamma}{2}\Big((\overline{\ell_1}-1)\frac{G_{\overline{\jmath}_1}(0)}{\overline{\jmath}_1} + (\overline{\ell_2}+1)\frac{G_{\overline{\jmath}_2}(0)}{\overline{\jmath}_2} + \sum_{a=3}^{\nu} \overline{\ell_a}\frac{G_{\overline{\jmath}_a}(0)}{\overline{\jmath}_a}) \Big) = 0\,, \ \forall \kappa \in \mathfrak{K}\,, \\ (\overline{\ell_1}-1)\overline{\jmath}_1 + (\overline{\ell_2}+1)\overline{\jmath}_2 + \sum_{a=3}^{\nu} \overline{\ell_a}\,\overline{\jmath}_a = 0\,. \end{cases} \tag{4.3.21}$$

By Lemma 4.51 the vector $(\vec{\omega}(\kappa), 1)$ is non-degenerate, which is a contradiction for $\gamma \neq 0$. If $\gamma = 0$ the first equation in (4.3.21) implies that $\overline{\ell_1} = -1$, $\overline{\ell_2} = 1$, $\overline{\ell_3} = \ldots = \overline{\ell_\nu} = 0$. Inserting these values in the momentum condition we obtain $-2\overline{\jmath}_1 + 2\overline{\jmath}_2 = 0$. This contradicts $\overline{\jmath} \neq \overline{\jmath}'$.

Step 4. We finally consider the case when $(\ell_m)_{m\in\mathbb{N}}$ is unbounded. Up to subsequences $|\ell_m| \to \infty$ as $m \to \infty$ and $\lim_{m\to\infty} \ell_m/\langle \ell_m \rangle =: \overline{c} \neq 0$. In addition, by (4.3.16), up to subsequences

$$\lim_{m\to\infty} \frac{|j_m|^{\frac{3}{2}} - |j_m'|^{\frac{3}{2}}}{\langle \ell_m \rangle} = \overline{d}_1 \in \mathbb{R}\,. \tag{4.3.22}$$

By (4.3.1) and (4.3.5) we have, for any n,

$$\begin{aligned} &\partial_\kappa^n \frac{1}{\langle \ell_m \rangle}\Big(\Omega_{j_m}(\kappa) - \Omega_{j_m'}(\kappa) \Big)_{|\kappa=\kappa_m} = \partial_\kappa^n \Big(\frac{\sqrt{\kappa}}{\langle \ell_m \rangle} \big(|j_m|^{\frac{3}{2}} - |j_m'|^{\frac{3}{2}} \big) \\ &+ \frac{1}{\langle \ell_m \rangle \sqrt{\kappa}} \Big(\frac{c_{j_m}(\kappa)}{|j_m|^{\frac{1}{2}}} - \frac{c_{j_m'}(\kappa)}{|j_m'|^{\frac{1}{2}}} \Big) + \frac{\gamma}{2\langle \ell_m \rangle} \Big(\frac{G_{j_m}(0)}{j_m} - \frac{G_{j_m'}(0)}{j_m'} \Big)_{|\kappa=\kappa_m} \Big) \to \overline{d}_1 \partial_\kappa^n (\sqrt{\kappa})_{|\kappa=\overline{\kappa}} \end{aligned}$$

using (4.3.22) and $\langle \ell_m \rangle \to \infty$. Therefore (4.3.17) becomes, in the limit $m \to \infty$,

$$\partial_\kappa^n \big(\vec{\Omega}(\kappa) \cdot \overline{c} + \overline{d}_1 \sqrt{\kappa} \big)_{|\kappa=\overline{\kappa}} = 0\,, \quad \forall n \in \mathbb{N}_0\,.$$

By analyticity this implies $\vec{\Omega}(\kappa) \cdot \overline{c} + \overline{d}_1 \sqrt{\kappa} = 0$, for all $\kappa \in \mathfrak{K}$. Thus $(\vec{\Omega}(\kappa), \sqrt{\kappa})$ is degenerate with $c = (\overline{c}, \overline{d}_1) \neq 0$, contradicting Lemma 4.50-2.

Proof of (4.3.10). The proof is similar to that for (4.3.9) and we omit it. □

4.4 Nash-Moser theorem and measure estimates

Under the rescaling $(\eta, \zeta) \mapsto (\varepsilon\eta, \varepsilon\zeta)$, the Hamiltonian system (4.1.14) transforms into the Hamiltonian system generated by

$$\mathcal{H}_\varepsilon(\eta, \zeta) := \varepsilon^{-2} \mathcal{H}(\varepsilon\eta, \varepsilon\zeta) = \mathcal{H}_L(\eta, \zeta) + \varepsilon P_\varepsilon(\eta, \zeta)\,, \tag{4.4.1}$$

where $\mathcal{H}$ is the water waves Hamiltonian (4.1.13) expressed in the Wahlén coordinates (4.1.11), $\mathcal{H}_L$ is defined in (4.1.20) and

$$\begin{aligned} P_\varepsilon(\eta,\zeta) := & \frac{1}{2\varepsilon}\int_{\mathbb{T}} \big(\zeta + \tfrac{\gamma}{2}\partial_x^{-1}\eta\big)\big(G(\varepsilon\eta)-G(0)\big)\big(\zeta+\tfrac{\gamma}{2}\partial_x^{-1}\eta\big)\,\mathrm{d}x \\ & + \frac{\kappa}{\varepsilon^3}\int_{\mathbb{T}}\big(\sqrt{1+\varepsilon^2\eta_x^2}-1-\tfrac{\varepsilon^2}{2}\eta_x^2\big)\,\mathrm{d}x + \frac{\gamma}{2}\int_{\mathbb{T}}\Big(-\big(\zeta+\tfrac{\gamma}{2}\partial_x^{-1}\eta\big)_x\eta^2+\tfrac{\gamma}{3}\eta^3\Big)\,\mathrm{d}x\,. \end{aligned} \tag{4.4.2}$$

We now study the Hamiltonian system generated by the Hamiltonian $\mathcal{H}_\varepsilon(\eta,\zeta)$, in the action-angle and normal coordinates (θ, I, w) defined in Section 4.1.3. Thus we consider the Hamiltonian $H_\varepsilon(\theta, I, w)$ defined by

$$H_\varepsilon := \mathcal{H}_\varepsilon \circ A = \varepsilon^{-2}\mathcal{H}\circ\varepsilon A \tag{4.4.3}$$

where A is the map defined in (4.1.50). The associated symplectic form is given in (4.1.54).

By Lemma 4.10 (see also (4.1.35), (4.1.49)), in the variables (θ, I, w) the quadratic Hamiltonian $\mathcal{H}_L$ defined in (4.1.20) simply reads, up to a constant,

$$\mathcal{N} := \mathcal{H}_L \circ A = \vec{\Omega}(\kappa)\cdot I + \tfrac{1}{2}\left(\mathbf{\Omega}_W w, w\right)_{L^2}$$

where $\vec{\Omega}(\kappa) \in \mathbb{R}^\nu$ is defined in (4.0.8) and $\mathbf{\Omega}_W$ in (4.1.19). Thus the Hamiltonian H_ε in (4.4.3) is

$$H_\varepsilon = \mathcal{N} + \varepsilon P \qquad \text{with} \qquad P := P_\varepsilon \circ A\,. \tag{4.4.4}$$

We look for an embedded invariant torus

$$i : \mathbb{T}^\nu \to \mathbb{R}^\nu \times \mathbb{R}^\nu \times \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}\,, \quad \varphi \mapsto i(\varphi) := (\theta(\varphi), I(\varphi), w(\varphi))\,,$$

of the Hamiltonian vector field $X_{H_\varepsilon} := (\partial_I H_\varepsilon, -\partial_\theta H_\varepsilon, \Pi^{\angle}_{\mathbb{S}^+,\Sigma} J\nabla_w H_\varepsilon)$ filled by quasi-periodic solutions with Diophantine frequency vector $\omega \in \mathbb{R}^\nu$ (which satisfies also first and second order Melnikov non-resonance conditions, see (4.4.15)-(4.4.17)).

4.4.1 Nash-Moser theorem of hypothetical conjugation

The quasi-periodic solutions for the Hamiltonian system generated by the Hamiltonian in (4.4.3) are expected to have a shifted frequency vector close the unperturbed linear frequency vector $\vec{\Omega}(\kappa)$ defined in (4.0.8). It is therefore convenient to introduce a "counterterm" $\alpha \in \mathbb{R}^\nu$ and consider the family of modified Hamiltonians

$$H_\alpha := \mathcal{N}_\alpha + \varepsilon P\,, \quad \mathcal{N}_\alpha := \alpha\cdot I + \tfrac{1}{2}\left(w, \mathbf{\Omega}_W w\right)_{L^2}\,. \tag{4.4.5}$$

In particular, when $\alpha = \vec{\Omega}(\kappa)$, we have $H_\alpha = H_\varepsilon$.

Then we look for a zero (i, α) of the nonlinear operator

$$\begin{aligned}\mathcal{F}(i,\alpha) &:= \mathcal{F}(\omega,\kappa,\varepsilon;i,\alpha) := \omega\cdot\partial_\varphi i(\varphi) - X_{H_\alpha}(i(\varphi)) \\ &= \begin{pmatrix} \omega\cdot\partial_\varphi\theta(\varphi) & -\alpha-\varepsilon\partial_I P(i(\varphi)) \\ \omega\cdot\partial_\varphi I(\varphi) & +\varepsilon\partial_\theta P(i(\varphi)) \\ \omega\cdot\partial_\varphi w(\varphi) & -\Pi^{\angle}_{\mathbb{S}^+,\Sigma} J(\mathbf{\Omega}_W w(\varphi)+\varepsilon\nabla_w P(i(\varphi)))\end{pmatrix}.\end{aligned} \tag{4.4.6}$$

If $\mathcal{F}(i,\alpha)=0$, then the embedding $\varphi\mapsto i(\varphi)$ is an invariant torus for the Hamiltonian vector field X_{H_α}, filled with quasi-periodic solutions with frequency ω.

Each Hamiltonian H_α in (4.4.5) is invariant under the involution $\vec{\mathcal{S}}$ and the translations $\vec{\tau}_\varsigma$, $\varsigma\in\mathbb{R}$, defined respectively in (4.1.51) and in (4.1.52):

$$H_\alpha\circ\vec{\mathcal{S}} = H_\alpha\,, \qquad H_\alpha\circ\vec{\tau}_\varsigma = H_\alpha\,, \quad \forall\,\varsigma\in\mathbb{R}\,. \tag{4.4.7}$$

We look for a reversible traveling torus embedding $\varphi\mapsto i(\varphi) = (\theta(\varphi), I(\varphi), w(\varphi))$, namely satisfying

$$\vec{\mathcal{S}} i(\varphi) = i(-\varphi)\,, \qquad \vec{\tau}_\varsigma i(\varphi) = i(\varphi - \vec{\jmath}\varsigma)\,, \quad \forall\,\varsigma\in\mathbb{R}\,. \tag{4.4.8}$$

Lemma 4.54. *The operator $\mathcal{F}(\cdot,\alpha)$ maps a reversible, respectively traveling, wave into an anti-reversible, respectively traveling, wave variation, according to Definition 4.47.*

Proof. It follows directly by (4.4.6) and (4.4.7). □

The norm of the periodic components of the embedded torus

$$\mathfrak{I}(\varphi) := i(\varphi) - (\varphi,0,0) := (\Theta(\varphi), I(\varphi), w(\varphi))\,, \quad \Theta(\varphi) := \theta(\varphi) - \varphi\,, \tag{4.4.9}$$

is $\|\mathfrak{I}\|_s^{k_0,\upsilon} := \|\Theta\|_{H^s_\varphi}^{k_0,\upsilon} + \|I\|_{H^s_\varphi}^{k_0,\upsilon} + \|w\|_s^{k_0,\upsilon}$, where

$$k_0 := m_0 + 2 \tag{4.4.10}$$

and $m_0\in\mathbb{N}$ is the index of non-degeneracy provided by Proposition 4.53, which only depends on the linear unperturbed frequencies. Thus, k_0 is considered as an absolute constant and we will often omit to write the dependence of the various constants with respect to k_0. We look for quasi-periodic solutions of frequency ω belonging to a δ-neighbourhood (independent of ε)

$$\Omega := \big\{\omega\in\mathbb{R}^\nu \ : \ \mathrm{dist}\big(\omega, \vec{\Omega}[\kappa_1,\kappa_2]\big) < \delta\big\}\,, \quad \delta>0\,,$$

of the curve $\vec{\Omega}[\kappa_1,\kappa_2]$ defined by (4.0.8).

Theorem 4.55. (Nash-Moser) *There exist positive constants $\mathrm{a}_0, \varepsilon_0, C$ depending on $\mathbb{S}$, k_0 and $\tau\geqslant 1$ such that, for all $\upsilon=\varepsilon^{\mathrm{a}}$, $\mathrm{a}\in(0,\mathrm{a}_0)$ and for all $\varepsilon\in(0,\varepsilon_0)$, there exist*

1. *a k_0-times differentiable function*

$$\alpha_\infty : \Omega \times [\kappa_1, \kappa_2] \mapsto \mathbb{R}^\nu \,,$$
$$\alpha_\infty(\omega,\kappa) := \omega + r_\varepsilon(\omega,\kappa) \quad \text{with} \quad |r_\varepsilon|^{k_0,\upsilon} \leqslant C\varepsilon\upsilon^{-1}\,; \tag{4.4.11}$$

2. *a family of embedded reversible traveling tori $i_\infty(\varphi)$ (cfr. (4.4.8)), defined for all $(\omega,\kappa) \in \Omega \times [\kappa_1,\kappa_2]$, satisfying*
$$\| i_\infty(\varphi) - (\varphi,0,0)\|_{s_0}^{k_0,\upsilon} \leqslant C\varepsilon\upsilon^{-1}\,; \tag{4.4.12}$$

3. *a sequence of k_0-times differentiable functions $\mu_j^\infty : \mathbb{R}^\nu \times [\kappa_1,\kappa_2] \to \mathbb{R}$, $j \in \mathbb{S}_0^c = \mathbb{Z}\setminus(\mathbb{S}\cup\{0\})$, of the form*

$$\mu_j^\infty(\omega,\kappa) = \mathtt{m}_{\frac{3}{2}}^\infty(\omega,\kappa)\Omega_j(\kappa) + \mathtt{m}_1^\infty(\omega,\kappa) j + \mathtt{m}_{\frac{1}{2}}^\infty(\omega,\kappa)\,|j|^{\frac{1}{2}} + \mathfrak{r}_j^\infty(\omega,\kappa)\,, \tag{4.4.13}$$

with $\Omega_j(\kappa)$ defined in (4.0.3), satisfying

$$|\mathtt{m}_{\frac{3}{2}}^\infty - 1|^{k_0,\upsilon},\ |\mathtt{m}_1^\infty|^{k_0,\upsilon},\ |\mathtt{m}_{\frac{1}{2}}^\infty|^{k_0,\upsilon} \leqslant C\varepsilon\,, \quad \sup_{j\in\mathbb{S}_0^c} |\mathfrak{r}_j^\infty|^{k_0,\upsilon} \leqslant C\varepsilon\upsilon^{-1}\,, \tag{4.4.14}$$

such that, for all (ω,κ) in the Cantor-like set

$$\begin{aligned}
\mathcal{C}_\infty^\upsilon := \Big\{ & (\omega,\kappa) \in \Omega \times [\kappa_1,\kappa_2] \ : \ |\omega\cdot\ell| \geqslant 8\upsilon\langle\ell\rangle^{-\tau}\ ,\ \forall\,\ell\in\mathbb{Z}^\nu\setminus\{0\}\,; && (4.4.15)\\
& \big|\omega\cdot\ell + \mu_j^\infty(\omega,\kappa)\big| \geqslant 4\upsilon\,|j|^{\frac{3}{2}}\langle\ell\rangle^{-\tau}\ ,\forall\,\ell\in\mathbb{Z}^\nu,\ j\in\mathbb{S}_0^c \text{ with } \vec{\jmath}\cdot\ell + j = 0\,; && (4.4.16)\\
& \big|\omega\cdot\ell + \mu_j^\infty(\omega,\kappa) - \mu_{j'}^\infty(\omega,\kappa)\big| \geqslant 4\upsilon\,\langle |j|^{\frac{3}{2}} - |j'|^{\frac{3}{2}}\rangle\langle\ell\rangle^{-\tau}\ , && (4.4.17)\\
& \qquad \forall\ell\in\mathbb{Z}^\nu,\ j,j'\in\mathbb{S}_0^c,\ (\ell,j,j')\neq(0,j,j) \text{ with } \vec{\jmath}\cdot\ell + j - j' = 0\,,\\
& \big|\omega\cdot\ell + \mu_j^\infty(\omega,\kappa) + \mu_{j'}^\infty(\omega,\kappa)\big| \geqslant 4\upsilon\,\big(|j|^{\frac{3}{2}} + |j'|^{\frac{3}{2}}\big)\langle\ell\rangle^{-\tau}\ , && (4.4.18)\\
& \qquad \forall\,\ell\in\mathbb{Z}^\nu,\ j,j'\in\mathbb{S}_0^c\,,\ \text{with } \vec{\jmath}\cdot\ell + j + j' = 0 \Big\}\,,
\end{aligned}$$

the function $i_\infty(\varphi) := i_\infty(\omega,\kappa,\varepsilon;\varphi)$ is a solution of $\mathcal{F}(\omega,\kappa,\varepsilon; i_\infty, \alpha_\infty(\omega,\kappa)) = 0$. As a consequence, the embedded torus $\varphi \mapsto i_\infty(\varphi)$ is invariant for the Hamiltonian vector field $X_{H_{\alpha_\infty(\omega,\kappa)}}$ as it is filled by quasi-periodic reversible traveling wave solutions with frequency ω.

We remind that the conditions on the indexes in (4.4.16)-(4.4.17) (where $\vec{\jmath}\in\mathbb{Z}^\nu$ is the vector in (4.1.53)) are due to the fact that we look for traveling wave solutions. These restrictions are essential to prove the measure estimates of the next section.

4.4.2 Measure estimates

By (4.4.11), the function $\alpha_\infty(\,\cdot\,,\kappa)$ from Ω into its image $\alpha_\infty(\Omega,\kappa)$ is invertible and

$$\begin{aligned} &\beta = \alpha_\infty(\omega,\kappa) = \omega + r_\varepsilon(\omega,\kappa) \ \Leftrightarrow \\ &\omega = \alpha_\infty^{-1}(\beta,\kappa) = \beta + \breve{r}_\varepsilon(\beta,\kappa), \quad |\breve{r}_\varepsilon|^{k_0,\upsilon} \leqslant C\varepsilon\upsilon^{-1}. \end{aligned} \tag{4.4.19}$$

Then, for any $\beta \in \alpha_\infty(\mathcal{C}_\infty^\upsilon)$, Theorem 4.55 proves the existence of an embedded invariant torus filled by quasi-periodic solutions with Diophantine frequency $\omega = \alpha_\infty^{-1}(\beta,\kappa)$ for the Hamiltonian

$$H_\beta = \beta \cdot I + \tfrac{1}{2}(w, \mathbf{\Omega}_W w)_{L^2} + \varepsilon P .$$

Consider the curve of the unperturbed tangential frequency vector $\vec{\Omega}(\kappa)$ in (4.0.8). In Theorem 4.56 below we prove that for "most" values of $\kappa \in [\kappa_1,\kappa_2]$ the vector $(\alpha_\infty^{-1}(\vec{\Omega}(\kappa),\kappa),\kappa)$ is in $\mathcal{C}_\infty^\upsilon$, obtaining an embedded torus for the Hamiltonian H_ε in (4.4.3), filled by quasi-periodic solutions with Diophantine frequency vector $\omega = \alpha_\infty^{-1}(\vec{\Omega}(\kappa),\kappa)$, denoted $\widetilde{\Omega}$ in Theorem 1.8. Thus $\varepsilon A(i_\infty(\widetilde{\Omega}t))$, where A is defined in (4.1.50), is a quasi-periodic traveling wave solution of the water waves equations (4.1.14) written in the Wahlén variables. Finally, going back to the original Zakharov variables via (4.1.10) we obtain solutions of (4.0.1). This proves Theorem 1.8 together with the following measure estimate.

Theorem 4.56. (Measure estimates) *Let*

$$\upsilon = \varepsilon^{\mathrm{a}}, \quad 0 < \mathrm{a} < \min\{\mathrm{a}_0, 1/(1+k_0)\}, \quad \tau > m_0(\nu+4), \tag{4.4.20}$$

where m_0 is the index of non-degeneracy given in Proposition 4.53 and $k_0 := m_0 + 2$. Then, for $\varepsilon \in (0,\varepsilon_0)$ small enough, the measure of the set

$$\mathcal{G}_\varepsilon := \big\{\kappa \in [\kappa_1,\kappa_2] \ : \ \big(\alpha_\infty^{-1}(\vec{\Omega}(\kappa),\kappa),\kappa\big) \in \mathcal{C}_\infty^\upsilon\big\} \tag{4.4.21}$$

satisfies $|\mathcal{G}_\varepsilon| \to \kappa_2 - \kappa_1$ as $\varepsilon \to 0$.

The rest of this section is devoted to prove Theorem 4.56. By (4.4.19) we have

$$\vec{\Omega}_\varepsilon(\kappa) := \alpha_\infty^{-1}(\vec{\Omega}(\kappa),\kappa) = \vec{\Omega}(\kappa) + \vec{r}_\varepsilon, \tag{4.4.22}$$

where $\vec{r}_\varepsilon(\kappa) := \breve{r}_\varepsilon(\vec{\Omega}(\kappa),\kappa)$ satisfies

$$|\partial_\kappa^k \vec{r}_\varepsilon(\kappa)| \leqslant C\varepsilon\upsilon^{-(1+k)}, \quad \forall\ |k| \leqslant k_0, \text{ uniformly on } [\kappa_1,\kappa_2]. \tag{4.4.23}$$

We also denote, with a small abuse of notation, for all $j \in \mathbb{S}_0^c$,

$$\mu_j^\infty(\kappa) := \mu_j^\infty\big(\vec{\Omega}_\varepsilon(\kappa),\kappa\big) := \mathtt{m}_{\frac32}^\infty(\kappa)\Omega_j(\kappa) + \mathtt{m}_1^\infty(\kappa) j + \mathtt{m}_{\frac12}^\infty(\kappa)\,|j|^{\frac12} + \mathfrak{r}_j^\infty(\kappa)\,, \tag{4.4.24}$$

where $\mathtt{m}_{\frac32}^\infty(\kappa) := \mathtt{m}_{\frac32}^\infty(\vec{\Omega}_\varepsilon(\kappa),\kappa)$, $\mathtt{m}_1^\infty(\kappa) := \mathtt{m}_1^\infty(\vec{\Omega}_\varepsilon(\kappa),\kappa)$, $\mathtt{m}_{\frac12}^\infty(\kappa) := \mathtt{m}_{\frac12}^\infty(\vec{\Omega}_\varepsilon(\kappa),\kappa)$ and $\mathfrak{r}_j^\infty(\kappa) := \mathfrak{r}_j^\infty(\vec{\Omega}_\varepsilon(\kappa),\kappa)$.

By (4.4.14) and (4.4.23) we have

$$\big|\partial_\kappa^k\big(\mathtt{m}_{\frac32}^\infty(\kappa)-1\big)\big|,\ |\partial_\kappa^k\mathtt{m}_1^\infty(\kappa)|,\ |\partial_\kappa^k\mathtt{m}_{\frac12}^\infty(\kappa)| \leqslant C\varepsilon\upsilon^{-k}, \tag{4.4.25}$$

$$\sup_{j\in\mathbb{S}_0^c}|\partial_\kappa^k\mathfrak{r}_j^\infty(\kappa)| \leqslant C\varepsilon\upsilon^{-1-k}\,, \quad \forall\, 0\leqslant k\leqslant k_0\,. \tag{4.4.26}$$

Recalling (4.4.15)-(4.4.17), the Cantor set in (4.4.21) becomes

$$\begin{aligned}\mathcal{G}_\varepsilon := \Big\{ & \kappa\in[\kappa_1,\kappa_2] \ : \ |\vec{\Omega}_\varepsilon(\kappa)\cdot\ell| \geqslant 8\upsilon\,\langle\ell\rangle^{-\tau}\ ,\ \forall\,\ell\in\mathbb{Z}^\nu\backslash\{0\}\,;\\ & |\vec{\Omega}_\varepsilon(\kappa)\cdot\ell+\mu_j^\infty(\kappa)| \geqslant 4\upsilon|j|^{\frac32}\,\langle\ell\rangle^{-\tau}\ ,\ \forall\,\ell\in\mathbb{Z}^\nu\,,\, j\in\mathbb{S}_0^c\,,\ \text{with } \vec{\jmath}\cdot\ell+j=0\,;\\ & |\vec{\Omega}_\varepsilon(\kappa)\cdot\ell+\mu_j^\infty(\kappa)-\mu_{j'}^\infty(\kappa)| \geqslant 4\upsilon\,\langle|j|^{\frac32}-|j'|^{\frac32}\rangle\langle\ell\rangle^{-\tau}\ ,\\ & \forall\ell\in\mathbb{Z}^\nu,\ j,j'\in\mathbb{S}_0^c,\ (\ell,j,j')\neq(0,j,j) \text{ with } \vec{\jmath}\cdot\ell+j-j'=0\,;\\ & |\vec{\Omega}_\varepsilon(\kappa)\cdot\ell+\mu_j^\infty(\kappa)+\mu_{j'}^\infty(\kappa)| \geqslant 4\upsilon\,\big(|j|^{\frac32}+|j'|^{\frac32}\big)\,\langle\ell\rangle^{-\tau}\ ,\\ & \forall\,\ell\in\mathbb{Z}^\nu,\ j,j'\in\mathbb{S}_0^c \text{ with } \vec{\jmath}\cdot\ell+j+j'=0\Big\}\,.\end{aligned}$$

We estimate the measure of the complementary set

$$\begin{aligned}\mathcal{G}_\varepsilon^c &:= [\kappa_1,\kappa_2]\backslash\mathcal{G}_\varepsilon\\ &= \Big(\bigcup_{\ell\neq0}R_\ell^{(0)}\Big)\cup\Big(\bigcup_{\substack{\ell\in\mathbb{Z}^\nu,\, j\in\mathbb{S}_0^c\\ \vec{\jmath}\cdot\ell+j=0}}R_{\ell,j}^{(I)}\Big)\cup\Big(\bigcup_{\substack{(\ell,j,j')\neq(0,j,j),j\neq j'\\ \vec{\jmath}\cdot\ell+j-j'=0}}R_{\ell,j,j'}^{(II)}\Big)\cup\Big(\bigcup_{\substack{\ell\in\mathbb{Z}^\nu,j,j'\in\mathbb{S}_0^c,\\ \vec{\jmath}\cdot\ell+j+j'=0}}Q_{\ell,j,j'}^{(II)}\Big),\end{aligned} \tag{4.4.27}$$

where the "nearly-resonant sets"

$$R_\ell^{(0)} := \big\{\kappa\in[\kappa_1,\kappa_2] \ : \ |\vec{\Omega}_\varepsilon(\kappa)\cdot\ell| < 8\upsilon\,\langle\ell\rangle^{-\tau}\big\}\,, \tag{4.4.28}$$

$$R_{\ell,j}^{(I)} := \big\{\kappa\in[\kappa_1,\kappa_2] \ : \ |\vec{\Omega}_\varepsilon(\kappa)\cdot\ell+\mu_j^\infty(\kappa)| < 4\upsilon|j|^{\frac32}\,\langle\ell\rangle^{-\tau}\big\}\,, \tag{4.4.29}$$

$$R_{\ell,j,j'}^{(II)} := \big\{\kappa\in[\kappa_1,\kappa_2] \ : \ |\vec{\Omega}_\varepsilon(\kappa)\cdot\ell+\mu_j^\infty(\kappa)-\mu_{j'}^\infty(\kappa)| < 4\upsilon\,\langle|j|^{\frac32}-|j'|^{\frac32}\rangle\langle\ell\rangle^{-\tau}\big\}\,, \tag{4.4.30}$$

$$Q_{\ell,j,j'}^{(II)} := \big\{\kappa\in[\kappa_1,\kappa_2] \ : \ |\vec{\Omega}_\varepsilon(\kappa)\cdot\ell+\mu_j^\infty(\kappa)+\mu_{j'}^\infty(\kappa)| < 4\upsilon\big(|j|^{\frac32}+|j'|^{\frac32}\big)\,\langle\ell\rangle^{-\tau}\big\}\,. \tag{4.4.31}$$

Note that in the third union in (4.4.27) we may require $j\neq j'$ because $R_{\ell,j,j}^{(II)}\subset R_\ell^{(0)}$. In the sequel we shall always suppose the momentum conditions on the indexes ℓ,j,j' written in (4.4.27). Some of the above sets are empty.

Lemma 4.57. *Consider the sets in* (4.4.27)-(4.4.31). *For* $\varepsilon \in (0, \varepsilon_0)$ *small enough, we have that*

1. *If* $R^{(I)}_{\ell,j} \neq \varnothing$ *then* $|j|^{\frac{3}{2}} \leqslant C\langle \ell \rangle$;
2. *If* $R^{(II)}_{\ell,j,j'} \neq \varnothing$ *then* $\big||j|^{\frac{3}{2}} - |j'|^{\frac{3}{2}}\big| \leqslant C\langle \ell \rangle$;
3. *If* $Q^{(II)}_{\ell,j,j'} \neq \varnothing$ *then* $|j|^{\frac{3}{2}} + |j'|^{\frac{3}{2}} \leqslant C\langle \ell \rangle$.

Proof. We provide the proof for $R^{(II)}_{\ell,j,j'}$. If $R^{(II)}_{\ell,j,j'} \neq \varnothing$ then there exists $\kappa \in [\kappa_1, \kappa_2]$ such that

$$\big|\mu_j^\infty(\kappa) - \mu_{j'}^\infty(\kappa)\big| < \frac{4\upsilon \langle |j|^{\frac{3}{2}} - |j'|^{\frac{3}{2}} \rangle}{\langle \ell \rangle^\tau} + |\vec{\Omega}_\varepsilon(\kappa) \cdot \ell| \leqslant 4\upsilon \, \big||j|^{\frac{3}{2}} - |j'|^{\frac{3}{2}}\big| + C\langle \ell \rangle \,. \tag{4.4.32}$$

By (4.4.24) we have

$$\mu_j^\infty(\kappa) - \mu_{j'}^\infty(\kappa) = \mathtt{m}^\infty_{\frac{3}{2}}(\kappa)(\Omega_j(\kappa) - \Omega_{j'}(\kappa)) + \mathtt{m}_1^\infty(\kappa)(j - j') + \mathtt{m}^\infty_{\frac{1}{2}}(\kappa)(|j|^{\frac{1}{2}} - |j'|^{\frac{1}{2}}) + \mathfrak{r}_j^\infty(\kappa) - \mathfrak{r}_{j'}^\infty(\kappa) \,.$$

Then, by (4.4.25)-(4.4.26) with $k = 0$, (4.3.5)-(4.3.6), the momentum condition $j - j' = -\vec{\jmath} \cdot \ell$, and the elementary inequality $||j|^{\frac{3}{2}} - |j'|^{\frac{3}{2}}| \geqslant ||j|^{\frac{1}{2}} - |j'|^{\frac{1}{2}}|$, we deduce the lower bound

$$\begin{aligned} |\mu_j^\infty(\kappa) - \mu_{j'}^\infty(\kappa)| &\geqslant (1 - C\varepsilon)\sqrt{\kappa}\big(\big||j|^{\frac{3}{2}} - |j'|^{\frac{3}{2}}\big| - C\big) - C\varepsilon|\vec{\jmath} \cdot \ell| - C\varepsilon\big||j|^{\frac{1}{2}} - |j'|^{\frac{1}{2}}\big| - C\varepsilon\upsilon^{-1} \\ &\geqslant \tfrac{\sqrt{\kappa}}{2}\, \big||j|^{\frac{3}{2}} - |j'|^{\frac{3}{2}}\big| - C\varepsilon|\ell| - C' - C\varepsilon\upsilon^{-1} \,. \end{aligned} \tag{4.4.33}$$

Combining (4.4.32) and (4.4.33), we deduce $||j|^{\frac{3}{2}} - |j'|^{\frac{3}{2}}| \leqslant C\langle \ell \rangle$, for ε small enough. ☐

In order to estimate the measure of the sets (4.4.28)-(4.4.31) that are nonempty, the key point is to prove that the perturbed frequencies satisfy estimates similar to (4.3.7)-(4.3.10).

Lemma 4.58. (Perturbed transversality) *For* $\varepsilon \in (0, \varepsilon_0)$ *small enough and for all* $\kappa \in [\kappa_1, \kappa_2]$,

$$\max_{0 \leqslant n \leqslant m_0} |\partial_\kappa^n \vec{\Omega}_\varepsilon(\kappa) \cdot \ell| \geqslant \frac{\rho_0}{2}\langle \ell \rangle \,, \quad \forall\, \ell \in \mathbb{Z}^\nu \backslash \{0\} \,; \tag{4.4.34}$$

$$\begin{cases} \max_{0 \leqslant n \leqslant m_0} |\partial_\kappa^n (\vec{\Omega}_\varepsilon(\kappa) \cdot \ell + \mu_j^\infty(\kappa))| \geqslant \frac{\rho_0}{2}\langle \ell \rangle \\ \vec{\jmath} \cdot \ell + j = 0 \,, \quad \ell \in \mathbb{Z}^\nu \,,\ j \in \mathbb{S}_0^c \,; \end{cases} \tag{4.4.35}$$

$$\begin{cases} \max_{0 \leqslant n \leqslant m_0} |\partial_\kappa^n (\vec{\Omega}_\varepsilon(\kappa) \cdot \ell + \mu_j^\infty(\kappa) - \mu_{j'}^\infty(\kappa))| \geqslant \frac{\rho_0}{2}\langle \ell \rangle \\ \vec{\jmath} \cdot \ell + j - j' = 0 \,, \quad \ell \in \mathbb{Z}^\nu \,,\ j, j' \in \mathbb{S}_0^c \,,\ (\ell, j, j') \neq (0, j, j) \,; \end{cases} \tag{4.4.36}$$

$$\begin{cases} \max_{0 \leqslant n \leqslant m_0} |\partial_\kappa^n (\vec{\Omega}_\varepsilon(\kappa) \cdot \ell + \mu_j^\infty(\kappa) + \mu_{j'}^\infty(\kappa))| \geqslant \frac{\rho_0}{2}\langle \ell \rangle \\ \vec{\jmath} \cdot \ell + j + j' = 0 \,, \quad \ell \in \mathbb{Z}^\nu \,,\ j, j' \in \mathbb{S}_0^c \,. \end{cases} \tag{4.4.37}$$

We recall that ρ_0 is the amount of non-degeneracy that has been defined in Proposition 4.53.

Proof. We prove (4.4.36). The proofs of (4.4.34), (4.4.35) and (4.4.37) are similar. By (4.4.24) we have

$$\begin{aligned}&\vec{\Omega}_\varepsilon(\kappa)\cdot\ell+\mu_j^\infty(\kappa)-\mu_{j'}^\infty(\kappa)=\vec{\Omega}(\kappa)\cdot\ell+\vec{r}_\varepsilon(\kappa)\cdot\ell+\Omega_j(\kappa)-\Omega_{j'}(\kappa) \qquad (4.4.38)\\ &+(\mathtt{m}_{\frac{3}{2}}^\infty(\kappa)-1)\big(\Omega_j(\kappa)-\Omega_{j'}(\kappa)\big)+\mathtt{m}_1^\infty(\kappa)(j-j')+\mathtt{m}_{\frac{1}{2}}^\infty(\kappa)\,(|j|^{\frac{1}{2}}-|j'|^{\frac{1}{2}})+\mathfrak{r}_j^\infty(\kappa)-\mathfrak{r}_{j'}^\infty(\kappa)\,.\end{aligned}$$

By Lemma 4.52 we get that, for any $n\in\{0,\dots,m_0\}$,

$$\big|\partial_\kappa^n(\Omega_j(\kappa)-\Omega_{j'}(\kappa))\big|\leqslant C(\kappa)\big||j|^{\frac{3}{2}}-|j'|^{\frac{3}{2}}\big|+C\leqslant C'(\kappa)\langle\ell\rangle\,, \qquad (4.4.39)$$

because, by Lemma 4.57-(2), we can restrict to indexes ℓ,j,j' such that $||j|^{\frac{3}{2}}-|j'|^{\frac{3}{2}}|\leqslant C\langle\ell\rangle$. Furthermore

$$\big||j|^{\frac{1}{2}}-|j'|^{\frac{1}{2}}\big|\leqslant\big||j|^{\frac{3}{2}}-|j'|^{\frac{3}{2}}\big|\leqslant C\langle\ell\rangle\,. \qquad (4.4.40)$$

Therefore, by (5.3.15), (4.4.25), (4.4.26), (4.4.23), (4.4.39), (4.4.40), and the momentum condition $j-j'=-\vec{\jmath}\cdot\ell$, we have that, for any $n\in\{0,\dots,m_0\}$,

$$|\partial_\kappa^n\,(\vec{\Omega}_\varepsilon(\kappa)\cdot\ell+\mu_j^\infty(\kappa)-\mu_{j'}^\infty(\kappa))|\geqslant|\partial_\kappa^n\,(\vec{\Omega}(\kappa)\cdot\ell+\Omega_j(\kappa)-\Omega_{j'}(\kappa))|-C\varepsilon\upsilon^{-(1+m_0)}\langle\ell\rangle\,.$$

Since $\vec{\Omega}(\kappa)\cdot\ell+\Omega_j(\kappa)-\Omega_{j'}(\kappa)$ satisfies (4.3.9), we deduce that

$$\max_{0\leqslant n\leqslant m_0}|\partial_\kappa^n\,(\vec{\Omega}_\varepsilon(\kappa)\cdot\ell+\mu_j^\infty(\kappa)-\mu_{j'}^\infty(\kappa))|\geqslant\rho_0\langle\ell\rangle-C\varepsilon\upsilon^{-(1+m_0)}\langle\ell\rangle\geqslant\tfrac{\rho_0}{2}\langle\ell\rangle$$

for $\varepsilon>0$ small enough. □

As an application of Rüssmann Theorem 17.1 in [155], we deduce the following result:

Lemma 4.59. (Estimates of the resonant sets) *The measure of the sets* (4.4.27)- (4.4.31) *satisfy*

$$|R_\ell^{(0)}|\lesssim(\upsilon\langle\ell\rangle^{-(\tau+1)})^{\frac{1}{m_0}}\,,\quad|R_{\ell,j}^{(I)}|\lesssim\big(\upsilon|j|^{\frac{3}{2}}\langle\ell\rangle^{-(\tau+1)}\big)^{\frac{1}{m_0}}\,,$$
$$|R_{\ell,j,j'}^{(II)}|\lesssim\big(\upsilon\langle|j|^{\frac{3}{2}}-|j'|^{\frac{3}{2}}\rangle\langle\ell\rangle^{-(\tau+1)}\big)^{\frac{1}{m_0}}\,,\quad|Q_{\ell,j,j'}^{(II)}|\lesssim\big(\upsilon\,\big(|j|^{\frac{3}{2}}+|j'|^{\frac{3}{2}}\big)\langle\ell\rangle^{-(\tau+1)}\big)^{\frac{1}{m_0}}\,,$$

and, recalling Lemma 4.57,

$$|R_{\ell,j}^{(I)}|\,,\ |R_{\ell,j,j'}^{(II)}|\,,\ |Q_{\ell,j,j'}^{(II)}|\lesssim(\upsilon\langle\ell\rangle^{-\tau})^{\frac{1}{m_0}}\,.$$

Proof. We estimate $R_{\ell,j,j'}^{(II)}$ defined in (4.4.30). The other cases follow similarly. Defining $f_{\ell,j,j'}(\kappa):=$

$(\vec{\Omega}_\varepsilon(\kappa)\cdot \ell + \mu_j^\infty(\kappa) - \mu_{j'}^\infty(\kappa))\langle \ell\rangle^{-1}$, we write

$$R^{(II)}_{\ell,j,j'} = \big\{ \kappa \in [\kappa_1,\kappa_2] \, : \, \big| f_{\ell,j,j'}(\kappa) \big| < 4\upsilon \langle |j|^{\frac32} - |j'|^{\frac32} \rangle \langle \ell \rangle^{-\tau-1} \big\} \, .$$

By Lemma 4.57 we restrict to indexes satisfying $\big||j|^{\frac32} - |j'|^{\frac32}\big| \leqslant C\langle \ell \rangle$. By (4.4.36),

$$\max_{0\leqslant n\leqslant m_0} \big| \partial_\kappa^n f_{\ell,j,j'}(\kappa) \big| \geqslant \rho_0/2\,, \quad \forall\, \kappa \in [\kappa_1,\kappa_2] \, .$$

In addition, by (4.4.22)-(4.4.26), Lemma 4.52, the momentum condition $|j-j'| = |\vec{\jmath}\cdot\ell|$, and (4.4.40), we deduce that $\max_{0\leqslant n\leqslant k_0} \big|\partial_\kappa^n f_{\ell,j,j'}(\kappa)\big| \leqslant C$ for all $\kappa \in [\kappa_1,\kappa_2]$, provided $\varepsilon\upsilon^{-(1+k_0)}$ is small enough, namely, by (4.4.20) and ε small enough. In particular, $f_{\ell,j,j'}$ is of class $\mathcal{C}^{k_0-1} = \mathcal{C}^{m_0+1}$. Thus Theorem 17.1 in [155] applies. □

Proof of Theorem 4.56 completed. We estimate the measure of all the sets in (4.4.27). By Lemma 4.57 and Lemma 4.59 we have that

$$\Big| \bigcup_{\ell\neq 0} R_\ell^{(0)} \Big| \leqslant \sum_{\ell\neq 0} |R_\ell^{(0)}| \lesssim \sum_{\ell \neq 0} \Big(\frac{\upsilon}{\langle \ell\rangle^{\tau+1}} \Big)^{\frac{1}{m_0}} \, , \tag{4.4.41}$$

$$\Big| \bigcup_{\substack{\ell,\, j\in\mathbb{S}_0^c \\ \vec{\jmath}\cdot\ell+j=0}} R^{(I)}_{\ell,j} \Big| \leqslant \sum_{\substack{|j|\leqslant C\langle\ell\rangle^{\frac23} \\ \vec{\jmath}\cdot\ell+j=0}} |R^{(I)}_{\ell,j}| \lesssim \sum_{|j|\leqslant C\langle \ell\rangle^{\frac23}} \Big(\frac{\upsilon}{\langle\ell\rangle^\tau}\Big)^{\frac{1}{m_0}} \lesssim \sum_{\ell\in\mathbb{Z}^\nu} \frac{\upsilon^{\frac{1}{m_0}}}{\langle \ell\rangle^{\frac{\tau}{m_0}-\frac23}} \, , \tag{4.4.42}$$

$$\Big| \bigcup_{\substack{\ell,\, j, j'\in\mathbb{S}_0^c \\ \vec{\jmath}\cdot\ell+j+j'=0}} Q^{(II)}_{\ell,j,j'} \Big| \leqslant \sum_{|j|,|j'|\leqslant C\langle\ell\rangle^{\frac23}} |Q^{(II)}_{\ell,j,j'}| \lesssim \sum_{|j|,|j'|\leqslant C\langle \ell\rangle^{\frac23}} \Big(\frac{\upsilon}{\langle\ell\rangle^\tau}\Big)^{\frac{1}{m_0}} \lesssim \sum_{\ell\in\mathbb{Z}^\nu} \frac{\upsilon^{\frac{1}{m_0}}}{\langle \ell\rangle^{\frac{\tau}{m_0}-\frac43}} \, . \tag{4.4.43}$$

We are left with estimating the measure of

$$\bigcup_{\substack{(\ell,j,j')\neq(0,j,j), j\neq j' \\ \vec{\jmath}\cdot\ell+j-j'=0}} R^{(II)}_{\ell,j,j'} = \Big(\bigcup_{\substack{\ell, j\in\mathbb{S}_0^c \\ \vec{\jmath}\cdot\ell+2j=0}} R^{(II)}_{\ell,j,-j} \Big) \cup \Big(\bigcup_{\substack{\ell,j,j'\,,\ |j|\neq|j'| \\ \vec{\jmath}\cdot\ell+j-j'=0}} R^{(II)}_{\ell,j,j'} \Big) \, . \tag{4.4.44}$$

By the momentum condition $\vec{\jmath}\cdot\ell+2j=0$ we get $|j|\leqslant C\langle\ell\rangle$, and, by Lemma 4.59,

$$\Big| \bigcup_{\ell,j\in\mathbb{S}_0^c, \vec{\jmath}\cdot\ell+2j=0} R^{(II)}_{\ell,j,-j} \Big| \leqslant \sum_{|j|\leqslant C\langle\ell\rangle} |R^{(II)}_{\ell,j,-j}| \lesssim \sum_{|j|\leqslant C\langle\ell\rangle} \Big(\frac{\upsilon}{\langle\ell\rangle^\tau}\Big)^{\frac{1}{m_0}} \lesssim \sum_{\ell\in\mathbb{Z}^\nu} \frac{\upsilon^{\frac{1}{m_0}}}{\langle \ell\rangle^{\frac{\tau}{m_0}-1}} \, . \tag{4.4.45}$$

Finally we estimate the measure of the second union in (4.4.44). By Lemma 4.57 we can restrict to indexes satisfying $||j|^{3/2} - |j'|^{3/2}| \leqslant C\langle\ell\rangle$. Now, for any $|j|\neq|j'|$, we have

$$\big||j|^{\frac32} - |j'|^{\frac32}\big| = \big||j|^{\frac12} - |j'|^{\frac12}\big| \big(|j| + |j'| + |j|^{\frac12}|j'|^{\frac12}\big) \geqslant \frac{|j|+|j'|+|j|^{\frac12}|j'|^{\frac12}}{|j|^{\frac12}+|j'|^{\frac12}} \geqslant \frac{|j|^{\frac12}+|j'|^{\frac12}}{2} \, ,$$

implying the upper bounds $|j|, |j'| \leqslant C\langle \ell \rangle^2$. Hence

$$\Big| \bigcup_{\substack{\ell,j,j',\,|j|\neq|j'| \\ \vec{\jmath}\cdot\ell+j-j'=0}} R^{(II)}_{\ell,j,j'} \Big| \leqslant \sum_{|j|,|j'|\leqslant C\langle\ell\rangle^2} |R^{(II)}_{\ell,j,j'}| \lesssim \sum_{|j|,|j'|\leqslant C\langle\ell\rangle^2} \Big(\frac{\upsilon}{\langle\ell\rangle^\tau} \Big)^{\frac{1}{m_0}} \lesssim \sum_{\ell\in\mathbb{Z}^\nu} \frac{\upsilon^{\frac{1}{m_0}}}{\langle\ell\rangle^{\frac{\tau}{m_0}-4}} . \tag{4.4.46}$$

As $\frac{\tau}{m_0} - 4 > \nu$ by (4.4.20), all the series in (4.4.41), (4.4.42), (4.4.43), (4.4.45), (4.4.46) are convergent, and we deduce

$$|\mathcal{G}^c_\varepsilon| \leqslant C\upsilon^{\frac{1}{m_0}} .$$

For $\upsilon = \varepsilon^{\mathtt{a}}$ as in (4.4.20), we get $|\mathcal{G}_\varepsilon| \geqslant \kappa_2 - \kappa_1 - C\varepsilon^{\mathtt{a}/m_0}$. The proof of Theorem 4.56 is concluded. □

4.5 Approximate inverse

In order to implement a convergent Nash-Moser scheme that leads to a solution of $\mathcal{F}(i,\alpha) = 0$, where $\mathcal{F}(i,\alpha)$ is the nonlinear operator defined in (4.4.6), we construct an *almost approximate right inverse* of the linearized operator

$$\mathrm{d}_{i,\alpha}\mathcal{F}(i_0,\alpha_0)[\widehat{\imath},\widehat{\alpha}] = \omega\cdot\partial_\varphi\widehat{\imath} - \mathrm{d}_i X_{H_\alpha}(i_0(\varphi))[\widehat{\imath}] - (\widehat{\alpha},0,0) .$$

Note that $\mathrm{d}_{i,\alpha}\mathcal{F}(i_0,\alpha_0) = \mathrm{d}_{i,\alpha}\mathcal{F}(i_0)$ is independent of α_0. We assume that the torus $i_0(\varphi) = (\theta_0(\varphi), I_0(\varphi), w_0(\varphi))$ is reversible and traveling, according to (4.4.8).

In the sequel we shall assume the smallness condition, for some $\mathtt{k} := \mathtt{k}(\tau,\nu) > 0$,

$$\varepsilon\upsilon^{-\mathtt{k}} \ll 1 .$$

We closely follow the strategy presented in [34] and implemented for the water waves equations in [44, 13]. The main novelty is to check that this construction preserves the momentum preserving properties needed for the search of traveling waves. Therefore, along this section we shall focus on this verification. The estimates are very similar to those in [44, 13] and will be proved in Appendix C.2.

First of all, we state tame estimates for the composition operator induced by the Hamiltonian vector field $X_P = (\partial_I P, -\partial_\theta P, \Pi^\angle_{\mathbb{S}^+,\Sigma} J\nabla_w P)$ in (4.4.6).

Lemma 4.60. (Estimates of the perturbation P) *Let $\mathfrak{I}(\varphi)$ in (4.4.9) satisfy $\|\mathfrak{I}\|^{k_0,\upsilon}_{3s_0+2k_0+5} \leqslant 1$. Then, for any $s \geqslant s_0$, $\|X_P(i)\|^{k_0,\upsilon}_s \lesssim_s 1 + \|\mathfrak{I}\|^{k_0,\upsilon}_{s+2s_0+2k_0+3}$, and, for all $\widehat{\imath} := (\widehat{\theta}, \widehat{I}, \widehat{w})$,*

$$\|\mathrm{d}_i X_P(i)[\widehat{\imath}]\|^{k_0,\upsilon}_s \lesssim_s \|\widehat{\imath}\|^{k_0,\upsilon}_{s+1} + \|\mathfrak{I}\|^{k_0,\upsilon}_{s+2s_0+2k_0+4} \|\widehat{\imath}\|^{k_0,\upsilon}_{s_0+1} ,$$
$$\big\|\mathrm{d}^2_i X_P(i)[\widehat{\imath},\widehat{\imath}]\big\|^{k_0,\upsilon}_s \lesssim_s \|\widehat{\imath}\|^{k_0,\upsilon}_{s+1} \|\widehat{\imath}\|^{k_0,\upsilon}_{s_0+1} + \|\mathfrak{I}\|^{k_0,\upsilon}_{s+2s_0+2k_0+5} (\|\widehat{\imath}\|^{k_0,\upsilon}_{s_0+1})^2 .$$

Proof. The proof follows as in Lemma 5.1 of [44], using also the estimates of the Dirichlet-Neumann operator in Lemmata 4.21, 4.22, see Appendix C.2. □

Along this section, we assume the following hypothesis, which is verified by the approximate solutions obtained at each step of the Nash-Moser Theorem 4.95.

- ANSATZ. The map $(\omega,\kappa)\mapsto \mathfrak{I}_0(\omega,\kappa)=i_0(\varphi;\omega,\kappa)-(\varphi,0,0)$ is k_0-times differentiable with respect to the parameters $(\omega,\kappa)\in\mathbb{R}^\nu\times[\kappa_1,\kappa_2]$ and, for some $\mu:=\mu(\tau,\nu)>0$, $\upsilon\in(0,1)$,
$$\|\mathfrak{I}_0\|_{s_0+\mu}^{k_0,\upsilon}+|\alpha_0-\omega|^{k_0,\upsilon}\leqslant C\varepsilon\upsilon^{-1}\,. \tag{4.5.1}$$

As in [34, 44, 13], we first modify the approximate torus $i_0(\varphi)$ to obtain a nearby isotropic torus $i_\delta(\varphi)$, namely such that the pull-back 1-form $i_\delta^*\Lambda$ is closed, where Λ is the Liouville 1-form defined in (4.1.55). We first consider the pull-back 1-form

$$i_0^*\Lambda=\sum_{k=1}^{\nu}a_k(\varphi)\mathrm{d}\varphi_k\,,\quad a_k(\varphi):=-\big([\partial_\varphi\theta_0(\varphi)]^\top I_0(\varphi)\big)_k+\tfrac12\big(J_\angle^{-1}w_0(\varphi),\partial_{\varphi_k}w_0(\varphi)\big)_{L^2}\,, \tag{4.5.2}$$

and its exterior differential

$$i_0^*\mathcal{W}=\mathrm{d}i_0^*\Lambda=\sum_{1\leqslant k<j\leqslant\nu}A_{kj}\mathrm{d}\varphi_k\wedge\mathrm{d}\varphi_j\,,\quad A_{kj}(\varphi):=\partial_{\varphi_k}a_j(\varphi)-\partial_{\varphi_j}a_k(\varphi)\,. \tag{4.5.3}$$

By the formula given in Lemma 5 in [34], we deduce, if ω belongs to $\mathtt{DC}(\upsilon,\tau)$, the estimate

$$\|A_{kj}\|_s^{k_0,\upsilon}\lesssim_s\upsilon^{-1}\big(\|Z\|_{s+\tau(k_0+1)+k_0+1}^{k_0,\upsilon}+\|Z\|_{s_0+1}^{k_0,\upsilon}\|\mathfrak{I}_0\|_{s+\tau(k_0+1)+k_0+1}^{k_0,\upsilon}\big)\,, \tag{4.5.4}$$

where $Z(\varphi)$ is the "error function"

$$Z(\varphi):=\mathcal{F}(i_0,\alpha_0)(\varphi)=\omega\cdot\partial_\varphi i_0(\varphi)-X_{H_{\alpha_0}}(i_0(\varphi))\,. \tag{4.5.5}$$

Note that if $Z(\varphi)=0$, the torus $i_0(\varphi)$ is invariant for $X_{H_{\alpha_0}}$ and the 1-form $i_0^*\Lambda$ is closed, namely the torus $i_0(\varphi)$ is isotropic. We denote below the Laplacian $\Delta_\varphi:=\sum_{k=1}^\nu\partial_{\varphi_k}^2$.

Lemma 4.61. **(Isotropic torus)** *The torus $i_\delta(\varphi):=(\theta_0(\varphi),I_\delta(\varphi),w_0(\varphi))$, defined by*

$$I_\delta(\varphi):=I_0(\varphi)+[\partial_\varphi\theta_0(\varphi)]^{-\top}\rho(\varphi)\,,\quad\rho=(\rho_j)_{j=1,\dots,\nu}\,,\quad\rho_j(\varphi):=\Delta_\varphi^{-1}\sum_{k=1}^{\nu}\partial_{\varphi_k}A_{kj}(\varphi)\,, \tag{4.5.6}$$

is isotropic. Moreover, there is $\sigma := \sigma(\nu,\tau)$ *such that, for all* $s \geqslant s_0$,

$$\|I_\delta - I_0\|_s^{k_0,\upsilon} \lesssim_s \|\mathfrak{I}_0\|_{s+1}^{k_0,\upsilon}\,, \tag{4.5.7}$$

$$\|I_\delta - I_0\|_s^{k_0,\upsilon} \lesssim_s \upsilon^{-1}\big(\|Z\|_{s+\sigma}^{k_0,\upsilon} + \|Z\|_{s_0+\sigma}^{k_0,\upsilon}\|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}\big) \tag{4.5.8}$$

$$\|\mathcal{F}(i_\delta,\alpha_0)\|_s^{k_0,\upsilon} \lesssim_s \|Z\|_{s+\sigma}^{k_0,\upsilon} + \|Z\|_{s_0+\sigma}^{k_0,\upsilon}\|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon} \tag{4.5.9}$$

$$\|\mathrm{d}_i(i_\delta)[\widehat{\imath}]\|_{s_1} \lesssim_{s_1} \|\widehat{\imath}\|_{s_1+1} + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}\|\widehat{\imath}\|_{s_0}^{k_0,\upsilon}\,, \tag{4.5.10}$$

for $s_1 \leqslant s_0 + \mu$ *(cfr.* (4.5.1)*). Furthermore* $i_\delta(\varphi)$ *is a reversible and traveling torus, cfr.* (4.4.8)*.*

Proof. Since $i_0(\varphi)$ is a traveling torus (see (4.2.60)), in order to prove that $i_\delta(\varphi)$ is a traveling torus it is sufficient to prove that $I_\delta(\varphi - \vec{\jmath}\varsigma) = I_\delta(\varphi)$, for any $\varsigma \in \mathbb{R}$. In view of (4.5.6), this follows by checking that $\partial_\varphi\theta_0(\varphi - \vec{\jmath}\varsigma) = \partial_\varphi\theta_0(\varphi)$ and $\rho(\varphi - \vec{\jmath}\varsigma) = \rho(\varphi)$ for any $\varsigma \in \mathbb{R}$. The first identity is a trivial consequence of the fact that $\theta_0(\varphi - \vec{\jmath}\varsigma) = \theta_0(\varphi) - \vec{\jmath}\varsigma$ for any $\varsigma \in \mathbb{R}$, whereas the second one follows once we prove that the functions $a_k(\varphi)$ defined in (4.5.2) satisfy

$$a_k(\varphi - \vec{\jmath}\varsigma) = a_k(\varphi) \quad \forall\,\varsigma \in \mathbb{R}\,, \quad \forall\, k = 1,\ldots,\nu\,. \tag{4.5.11}$$

Using that $i_0(\varphi)$ is a traveling torus, we get, for any $\varsigma \in \mathbb{R}$,

$$\big(\partial_{\varphi_k} w_0(\varphi - \vec{\jmath}\varsigma), J_\angle^{-1} w_0(\varphi - \vec{\jmath}\varsigma)\big)_{L^2} = \big(\partial_{\varphi_k}\tau_\varsigma w_0(\varphi), J_\angle^{-1}\tau_\varsigma w_0(\varphi)\big)_{L^2} = \big(\partial_{\varphi_k} w_0(\varphi), J_\angle^{-1} w_0(\varphi)\big)_{L^2}$$

and, recalling (4.5.2), we deduce (4.5.11). Moreover, since $i_0(\varphi)$ is reversible, in order to prove that $i_\delta(\varphi)$ is reversible as well, it is sufficient to show that $I_\delta(\varphi)$ is even. This follows by (4.5.2), Lemma 4.37 and $\mathcal{S}J^{-1} = -J^{-1}\mathcal{S}$. Finally, the estimates (4.5.7)-(4.5.10) follow e.g. as in Lemma 5.3 in [13] and will be proved in Appendix C.2. □

In the sequel we denote by $\sigma = \sigma(\nu,\tau)$ constants, which may increase from lemma to lemma, which represent "loss of derivatives".

In order to find an approximate inverse of the linearized operator $\mathrm{d}_{i,\alpha}\mathcal{F}(i_\delta)$, we introduce the symplectic diffeomorphism $G_\delta : (\phi, y, \mathtt{w}) \to (\theta, I, w)$ of the phase space $\mathbb{T}^\nu \times \mathbb{R}^\nu \times \mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma}$,

$$\begin{pmatrix}\theta\\ I\\ w\end{pmatrix} := G_\delta\begin{pmatrix}\phi\\ y\\ \mathtt{w}\end{pmatrix} := \begin{pmatrix}\theta_0(\phi)\\ I_\delta(\phi) + [\partial_\phi\theta_0(\phi)]^{-\top} y + [(\partial_\theta\widetilde{w}_0)(\theta_0(\phi))]^\top J_\angle^{-1}\mathtt{w}\\ w_0(\phi) + \mathtt{w}\end{pmatrix}, \tag{4.5.12}$$

where $\widetilde{w}_0(\theta) := w_0(\theta_0^{-1}(\theta))$. It is proved in Lemma 2 of [34] that G_δ is symplectic, because the torus i_δ is isotropic (Lemma 4.61). In the new coordinates, i_δ is the trivial embedded torus $(\phi, y, \mathtt{w}) = (\phi, 0, 0)$.

Lemma 4.62. *The diffeomorphism* G_δ *in* (4.5.12) *is reversibility and momentum preserving, in* *the sense that*

$$\vec{\mathcal{S}} \circ G_\delta = G_\delta \circ \vec{\mathcal{S}} \,, \quad \vec{\tau}_\varsigma \circ G_\delta = G_\delta \circ \vec{\tau}_\varsigma \,, \quad \forall \varsigma \in \mathbb{R} \,, \tag{4.5.13}$$

where $\vec{\mathcal{S}}$ *and* $\vec{\tau}_\varsigma$ *are defined respectively in* (4.1.51), (4.1.52).

Proof. We prove the second identity in (4.5.13), which, in view of (4.5.12), (4.1.52) amounts to

$$\theta_0(\phi) - \vec{\jmath}\varsigma = \theta_0(\phi - \vec{\jmath}\varsigma) \,, \ \forall \varsigma \in \mathbb{R} \,, \tag{4.5.14}$$

$$I_\delta(\phi) + [\partial_\phi \theta_0(\phi)]^{-\top} y + [(\partial_\theta \widetilde{w}_0)(\theta_0(\phi))]^\top J_\angle^{-1} \mathtt{w} \tag{4.5.15}$$
$$= I_\delta(\phi - \vec{\jmath}\varsigma) + [\partial_\phi \theta_0(\phi - \vec{\jmath}\varsigma)]^{-\top} y + [(\partial_\theta \widetilde{w}_0)(\theta_0(\phi - \vec{\jmath}\varsigma))]^\top J_\angle^{-1} \tau_\varsigma \mathtt{w} \,,$$

$$\tau_\varsigma w_0(\phi) + \tau_\varsigma \mathtt{w} = w_0(\phi - \vec{\jmath}\varsigma) + \tau_\varsigma \mathtt{w} \,. \tag{4.5.16}$$

Identities (4.5.14) and (4.5.16) follow because $i_\delta(\varphi)$ is a traveling torus (Lemma 4.61). For the same reason $I_\delta(\phi) = I_\delta(\phi - \vec{\jmath}\varsigma)$ and $\partial_\phi \theta_0(\phi) = \partial_\phi \theta_0(\phi - \vec{\jmath}\varsigma)$ for any $\varsigma \in \mathbb{R}$. Hence, for verifying (4.5.15) it is sufficient to check that $[(\partial_\theta \widetilde{w}_0)(\theta_0(\phi))]^\top = [(\partial_\theta \widetilde{w}_0)(\theta_0(\phi - \vec{\jmath}\varsigma))]^\top \tau_\varsigma$ (we have used that $J_\angle^{-1}$ and τ_ς commute by Lemma 4.45), which in turn follows by

$$\tau_\varsigma \circ (\partial_\theta \widetilde{w}_0)(\theta_0(\phi)) = (\partial_\theta \widetilde{w}_0)(\theta_0(\phi - \vec{\jmath}\varsigma)) \,, \quad \forall \varsigma \in \mathbb{R} \,, \tag{4.5.17}$$

by taking the transpose and using that $\tau_\varsigma^\top = \tau_{-\varsigma} = \tau_\varsigma^{-1}$. We claim that (4.5.17) is implied by $\widetilde{w}_0$ being a traveling wave, i.e.

$$\tau_\varsigma \widetilde{w}_0(\theta, \cdot) = \widetilde{w}_0(\theta - \vec{\jmath}\varsigma) \,, \quad \forall \varsigma \in \mathbb{R} \,. \tag{4.5.18}$$

Indeed, taking the differential of (4.5.18) with respect to θ, evaluating at $\theta = \theta_0(\varphi)$, and using that $\theta_0(\varphi) - \vec{\jmath}\varsigma = \theta_0(\varphi - \vec{\jmath}\varsigma)$ one deduces (4.5.17). It remains to prove (4.5.18). By the definition of $\widetilde{w}_0$, and since w_0 is a traveling wave, we have

$$\widetilde{w}_0(\theta - \vec{\jmath}\varsigma) = w_0(\theta_0^{-1}(\theta - \vec{\jmath}\varsigma)) = w_0(\theta_0^{-1}(\theta) - \vec{\jmath}\varsigma) = \tau_\varsigma w_0(\theta_0^{-1}(\theta)) = \tau_\varsigma \widetilde{w}_0 \,,$$

using also that $\theta_0^{-1}(\theta - \vec{\jmath}\varsigma) = \theta_0^{-1}(\theta) - \vec{\jmath}\varsigma$, which follows by inverting (4.5.14). The proof of the first identity in (4.5.13) follows by (4.5.12), (4.1.51), the fact that i_δ is reversible, Lemma 4.37 and since J^{-1} and $\mathcal{S}$ anti-commute. □

Under the symplectic diffeomorphism G_δ, the Hamiltonian vector field X_{H_α} changes into

$$X_{K_\alpha} = (DG_\delta)^{-1} X_{H_\alpha} \circ G_\delta \qquad \text{where} \qquad K_\alpha := H_\alpha \circ G_\delta \,. \tag{4.5.19}$$

By (4.5.13) and (4.4.7) we deduce that K_α is reversible and momentum preserving, in the sense that

$$K_\alpha \circ \vec{\mathcal{S}} = K_\alpha \,, \quad K_\alpha \circ \vec{\tau}_\varsigma = K_\alpha \,, \quad \forall \varsigma \in \mathbb{R} \,. \tag{4.5.20}$$

The Taylor expansion of K_α at the trivial torus $(\phi, 0, 0)$ is

$$\begin{aligned} K_\alpha(\phi, y, \mathtt{w}) = {} & K_{00}(\phi, \alpha) + K_{10}(\phi, \alpha) \cdot y + (K_{01}(\phi, \alpha), \mathtt{w})_{L^2} + \tfrac{1}{2} K_{20}(\phi) y \cdot y \\ & + (K_{11}(\phi) y, \mathtt{w})_{L^2} + \tfrac{1}{2} (K_{02}(\phi)\mathtt{w}, \mathtt{w})_{L^2} + K_{\geqslant 3}(\phi, y, \mathtt{w}) \,, \end{aligned} \tag{4.5.21}$$

where $K_{\geqslant 3}$ collects all terms at least cubic in the variables $(y, \mathtt{w})$. By (4.4.5) and (4.5.12), the only Taylor coefficients that depend on α are $K_{00} \in \mathbb{R}$, $K_{10} \in \mathbb{R}^\nu$ and $K_{01} \in \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$, whereas the $\nu \times \nu$ symmetric matrix K_{20}, $K_{11} \in \mathcal{L}(\mathbb{R}^\nu, \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma})$ and the linear self-adjoint operator K_{02}, acting on $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$, are independent of it.

Differentiating the identities in (4.5.20) at $(\phi, 0, 0)$, we have (recalling (4.1.51))

$$\begin{aligned} & K_{00}(-\phi) = K_{00}(\phi) \,, \quad K_{10}(-\phi) = K_{10}(\phi) \,, \quad K_{20}(-\phi) = K_{20}(\phi) \,, \\ & \mathcal{S} \circ K_{01}(-\phi) = K_{01}(\phi) \,, \quad \mathcal{S} \circ K_{11}(-\phi) = K_{11}(\phi) \,, \quad K_{02}(-\phi) \circ \mathcal{S} = \mathcal{S} \circ K_{02}(\phi) \,, \end{aligned} \tag{4.5.22}$$

and, recalling (4.1.52) and using that $\tau_\varsigma^\top = \tau_{-\varsigma} = \tau_\varsigma^{-1}$, for any $\varsigma \in \mathbb{R}$,

$$\begin{aligned} & K_{00}(\phi - \vec{\jmath}\varsigma) = K_{00}(\phi) \,, \quad K_{10}(\phi - \vec{\jmath}\varsigma) = K_{10}(\phi) \,, \quad K_{20}(\phi - \vec{\jmath}\varsigma) = K_{20}(\phi) \,, \\ & K_{01}(\phi - \vec{\jmath}\varsigma) = \tau_\varsigma K_{01}(\phi) \,, \quad K_{11}(\phi - \vec{\jmath}\varsigma) = \tau_\varsigma K_{11}(\phi) \,, \quad K_{02}(\phi - \vec{\jmath}\varsigma) \circ \tau_\varsigma = \tau_\varsigma \circ K_{02}(\phi) \,. \end{aligned} \tag{4.5.23}$$

The Hamilton equations associated to (4.5.21) are

$$\begin{cases} \dot\phi = K_{10}(\phi, \alpha) + K_{20}(\phi) y + [K_{11}(\phi)]^\top \mathtt{w} + \partial_y K_{\geqslant 3}(\phi, y, \mathtt{w}) \\ \dot y = -\partial_\phi K_{00}(\phi, \alpha) - [\partial_\phi K_{10}(\phi, \alpha)]^\top y - [\partial_\phi K_{01}(\phi, \alpha)]^\top \mathtt{w} \\ \qquad -\partial_\phi \left(\tfrac{1}{2} K_{20}(\phi) y \cdot y + (K_{11}(\phi) y, \mathtt{w})_{L^2} + \tfrac{1}{2} (K_{02}(\phi)\mathtt{w}, \mathtt{w})_{L^2} + K_{\geqslant 3}(\phi, y, \mathtt{w}) \right) \\ \dot{\mathtt{w}} = J_\angle \left(K_{01}(\phi, \alpha) + K_{11}(\phi) y + K_{02}(\phi) \mathtt{w} + \nabla_{\mathtt{w}} K_{\geqslant 3}(\phi, y, \mathtt{w}) \right) \end{cases} , \tag{4.5.24}$$

where $\partial_\phi K_{10}^\top$ is the $\nu \times \nu$ transposed matrix and $\partial_\phi K_{01}^\top, K_{11}^\top : \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma} \to \mathbb{R}^\nu$ are defined by the duality relation $(\partial_\phi K_{01}[\widehat\phi], \mathtt{w})_{L^2} = \widehat\phi \cdot [\partial_\phi K_{01}]^\top \mathtt{w}$ for any $\widehat\phi \in \mathbb{R}^\nu$, $\mathtt{w} \in \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$. The transpose $K_{11}^\top(\phi)$ is defined similarly.

On an exact solution (that is $Z = 0$), the terms K_{00}, K_{01} in the Taylor expansion (4.5.21) vanish and $K_{10} = \omega$. More precisely, arguing as in Lemma 5.4 in [13], we have

Lemma 4.63. *There is $\sigma := \sigma(\nu,\tau) > 0$, such that, for all $s \geqslant s_0$,*

$$\begin{aligned} & \|\partial_\phi K_{00}(\cdot,\alpha_0)\|_s^{k_0,\upsilon} + \|K_{10}(\cdot,\alpha_0) - \omega\|_s^{k_0,\upsilon} + \|K_{01}(\cdot,\alpha_0)\|_s^{k_0,\upsilon} \\ & \qquad \lesssim_s \|Z\|_{s+\sigma}^{k_0,\upsilon} + \|Z\|_{s_0+\sigma}^{k_0,\upsilon} \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}, \qquad (4.5.25) \\ & \|\partial_\alpha K_{00}\|_s^{k_0,\upsilon} + \|\partial_\alpha K_{10} - \mathrm{Id}\|_s^{k_0,\upsilon} + \|\partial_\alpha K_{01}\|_s^{k_0,\upsilon} \lesssim_s \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}, \qquad (4.5.26) \\ & \|K_{20}\|_s^{k_0,\upsilon} \lesssim_s \varepsilon(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}), \qquad (4.5.27) \\ & \|K_{11}y\|_s^{k_0,\upsilon} \lesssim_s \varepsilon(\|y\|_s^{k_0,\upsilon} + \|y\|_{s_0}^{k_0,\upsilon}\|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}), \ \|K_{11}^\top \mathtt{w}\|_s^{k_0,\upsilon} \lesssim_s \varepsilon(\|\mathtt{w}\|_s^{k_0,\upsilon} + \|\mathtt{w}\|_{s_0}^{k_0,\upsilon}\|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}). \\ & \qquad (4.5.28) \end{aligned}$$

A proof of the lemma is provided in Appendix C.2. Under the linear change of variables

$$DG_\delta(\varphi,0,0)\begin{pmatrix}\widehat{\phi}\\ \widehat{y}\\ \widehat{\mathtt{w}}\end{pmatrix} := \begin{pmatrix}\partial_\phi\theta_0(\varphi) & 0 & 0\\ \partial_\phi I_\delta(\varphi) & [\partial_\phi\theta_0(\varphi)]^{-\top} & [(\partial_\theta\widetilde{w}_0)(\theta_0(\varphi))]^\top J_\angle^{-1}\\ \partial_\phi w_0(\varphi) & 0 & \mathrm{Id}\end{pmatrix}\begin{pmatrix}\widehat{\phi}\\ \widehat{y}\\ \widehat{\mathtt{w}}\end{pmatrix}, \qquad (4.5.29)$$

the linearized operator $\mathrm{d}_{i,\alpha}\mathcal{F}(i_\delta)$ is approximately transformed into the one obtained when one linearizes the Hamiltonian system (4.5.24) at $(\phi,y,\mathtt{w}) = (\varphi,0,0)$, differentiating also in α at α_0 and changing $\partial_t \rightsquigarrow \omega\cdot\partial_\varphi$, namely

$$\begin{pmatrix}\widehat{\phi}\\ \widehat{y}\\ \widehat{\mathtt{w}}\\ \widehat{\alpha}\end{pmatrix} \mapsto \begin{pmatrix}\omega\cdot\partial_\varphi\widehat{\phi} - \partial_\phi K_{10}(\varphi)[\widehat{\phi}] - \partial_\alpha K_{10}(\varphi)[\widehat{\alpha}] - K_{20}(\varphi)\widehat{y} - [K_{11}(\varphi)]^\top\widehat{\mathtt{w}}\\ \omega\cdot\partial_\varphi\widehat{y} + \partial_{\phi\phi}K_{00}(\varphi)[\widehat{\phi}] + \partial_\alpha\partial_\phi K_{00}(\varphi)[\widehat{\alpha}] + [\partial_\phi K_{10}(\varphi)]^\top\widehat{y} + [\partial_\phi K_{01}(\varphi)]^\top\widehat{\mathtt{w}}\\ \omega\cdot\partial_\varphi\widehat{\mathtt{w}} - J_\angle\left(\partial_\phi K_{01}(\varphi)[\widehat{\phi}] + \partial_\alpha K_{01}(\varphi)[\widehat{\alpha}] + K_{11}(\varphi)\widehat{y} + K_{02}(\varphi)\widehat{\mathtt{w}}\right)\end{pmatrix}. \qquad (4.5.30)$$

By (4.5.29), (4.5.1) and Lemma 4.61, the induced composition operator satisfies, for any traveling wave variation $\widehat{\imath} := (\widehat{\phi},\widehat{y},\widehat{\mathtt{w}})$,

$$\|DG_\delta(\varphi,0,0)[\widehat{\imath}]\|_s^{k_0,\upsilon} + \|DG_\delta(\varphi,0,0)^{-1}[\widehat{\imath}]\|_s^{k_0,\upsilon} \lesssim_s \|\widehat{\imath}\|_s^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}\|\widehat{\imath}\|_{s_0}^{k_0,\upsilon}, \qquad (4.5.31)$$

$$\|D^2G_\delta(\varphi,0,0)[\widehat{\imath}_1,\widehat{\imath}_2]\|_s^{k_0,\upsilon} \lesssim_s \|\widehat{\imath}_1\|_s^{k_0,\upsilon}\|\widehat{\imath}_2\|_{s_0}^{k_0,\upsilon} + \|\widehat{\imath}_1\|_{s_0}^{k_0,\upsilon}\|\widehat{\imath}_2\|_s^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}\|\widehat{\imath}_1\|_{s_0}^{k_0,\upsilon}\|\widehat{\imath}_2\|_{s_0}^{k_0,\upsilon}. \qquad (4.5.32)$$

In order to construct an “almost approximate" inverse of (4.5.30), we need that

$$\mathcal{L}_\omega := \Pi_{\mathbb{S}^+,\Sigma}^\angle\left(\omega\cdot\partial_\varphi - JK_{02}(\varphi)\right)|_{\mathfrak{H}_{\mathbb{S}^+,\Sigma}^\angle} \qquad (4.5.33)$$

is "almost invertible" (on traveling waves) up to remainders of size $O(N_{n-1}^{-\mathtt{a}})$, where, for $n \in \mathbb{N}_0$

$$N_n := K_n^p, \quad K_n := K_0^{\chi^n}, \quad \chi = 3/2. \qquad (4.5.34)$$

The $(K_n)_{n\geqslant 0}$ is the scale used in the nonlinear Nash-Moser iteration of Section 4.8 and $(N_n)_{n\geqslant 0}$

is the one in the reducibility scheme of Section 4.7. Let $H^s_\angle(\mathbb{T}^{\nu+1}) := H^s(\mathbb{T}^{\nu+1}) \cap \mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma}$.

(AI) **Almost invertibility of $\mathcal{L}_\omega$:** *There exist positive real numbers σ, $\mu(\mathtt{b})$, $\mathtt{a}$, p, K_0 and a subset $\Lambda_o \subset \mathtt{DC}(\upsilon,\tau) \times [\kappa_1,\kappa_2]$ such that, for all $(\omega,\kappa) \in \Lambda_o$, the operator $\mathcal{L}_\omega$ may be decomposed as*

$$\mathcal{L}_\omega = \mathcal{L}_\omega^< + \mathcal{R}_\omega + \mathcal{R}_\omega^\perp \,, \tag{4.5.35}$$

where, for every traveling wave function $g \in H^{s+\sigma}_\angle(\mathbb{T}^{\nu+1},\mathbb{R}^2)$ and for every $(\omega,\kappa) \in \Lambda_o$, there is a traveling wave solution $h \in H^s_\angle(\mathbb{T}^{\nu+1},\mathbb{R}^2)$ of $\mathcal{L}_\omega^< h = g$ satisfying, for all $s_0 \leqslant s \leqslant S$,

$$\big\|(\mathcal{L}_\omega^<)^{-1} g\big\|_s^{k_0,\upsilon} \lesssim_S \upsilon^{-1}\big(\|g\|_{s+\sigma}^{k_0,\upsilon} + \|g\|_{s_0+\sigma}^{k_0,\upsilon} \|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\sigma}^{k_0,\upsilon} \big) \,. \tag{4.5.36}$$

In addition, if g is anti-reversible, then h is reversible. Moreover, for any $s_0 \leqslant s \leqslant S$, for any traveling wave $h \in \mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma}$, the operators $\mathcal{R}_\omega, \mathcal{R}_\omega^\perp$ satisfy the estimates

$$\begin{aligned}
\|\mathcal{R}_\omega h\|_s^{k_0,\upsilon} &\lesssim_S \varepsilon \upsilon^{-1} N_{n-1}^{-\mathtt{a}} \big(\|h\|_{s+\sigma}^{k_0,\upsilon} + \|h\|_{s_0+\sigma}^{k_0,\upsilon} \|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\sigma}^{k_0,\upsilon} \big) \,, \\
\big\|\mathcal{R}_\omega^\perp h\big\|_{s_0}^{k_0,\upsilon} &\lesssim_S K_n^{-b} \big(\|h\|_{s_0+b+\sigma}^{k_0,\upsilon} + \|h\|_{s_0+\sigma}^{k_0,\upsilon} \|\mathfrak{I}_0\|_{s_0+\mu(\mathtt{b})+\sigma+b} \big) \,, \ \forall\, b > 0 \,, \\
\big\|\mathcal{R}_\omega^\perp h\big\|_s^{k_0,\upsilon} &\lesssim_S \|h\|_{s+\sigma}^{k_0,\upsilon} + \|h\|_{s_0+\sigma}^{k_0,\upsilon} \|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\sigma}^{k_0,\upsilon} \,.
\end{aligned}$$

This assumption shall be verified by Theorem 4.93 at each n-th step of the Nash-Moser nonlinear iteration.

In order to find an almost approximate inverse of the linear operator in (4.5.30) (and so of $\mathrm{d}_{i,\alpha}\mathcal{F}(i_\delta)$), it is sufficient to invert the operator

$$\mathbb{D}\big[\widehat{\phi}, \widehat{y}, \widehat{\mathtt{w}}, \widehat{\alpha}\big] := \begin{pmatrix} \omega \cdot \partial_\varphi \widehat{\phi} - \partial_\alpha K_{10}(\varphi)[\widehat{\alpha}] - K_{20}(\varphi)\widehat{y} - K_{11}^\top(\varphi)\widehat{\mathtt{w}} \\ \omega \cdot \partial_\varphi \widehat{y} + \partial_\alpha \partial_\phi K_{00}(\varphi)[\widehat{\alpha}] \\ \mathcal{L}_\omega^< \widehat{\mathtt{w}} - J_\angle \left(\partial_\alpha K_{01}(\varphi)[\widehat{\alpha}] + K_{11}(\varphi)\widehat{y}\right) \end{pmatrix} \tag{4.5.37}$$

obtained neglecting in (4.5.30) the terms $\partial_\phi K_{10}$, $\partial_{\phi\phi} K_{00}$, $\partial_\phi K_{00}$, $\partial_\phi K_{01}$ (they vanish at an exact solution by Lemma 4.63) and the small remainders $\mathcal{R}_\omega$, $\mathcal{R}_\omega^\perp$ appearing in (4.5.35). We look for an inverse of $\mathbb{D}$ by solving the system

$$\mathbb{D}\big[\widehat{\phi}, \widehat{y}, \widehat{\mathtt{w}}, \widehat{\alpha}\big] = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} , \tag{4.5.38}$$

where (g_1, g_2, g_3) is an anti-reversible traveling wave variation (cfr. Definition 4.47), i.e.

$$g_1(\varphi) = g_1(-\varphi), \qquad g_2(\varphi) = -g_2(-\varphi), \qquad \mathcal{S} g_3(\varphi) = -g_3(-\varphi)\,, \tag{4.5.39}$$

$$g_1(\varphi) = g_1(\varphi - \vec{\jmath}\varsigma), \quad g_2(\varphi) = g_2(\varphi - \vec{\jmath}\varsigma), \quad \tau_\varsigma g_3(\varphi) = g_3(\varphi - \vec{\jmath}\varsigma), \ \forall \varsigma \in \mathbb{R}\,. \tag{4.5.40}$$

We first consider the second equation in (4.5.37)-(4.5.38), that is $\omega \cdot \partial_\varphi \widehat{y} = g_2 - \partial_\alpha \partial_\phi K_{00}(\varphi)[\widehat{\alpha}]$. By (4.5.39) and (4.5.22), the right hand side of this equation is odd in φ. In particular it has zero average and so

$$\widehat{y} := (\omega \cdot \partial_\varphi)^{-1}(g_2 - \partial_\alpha \partial_\phi K_{00}(\varphi)[\widehat{\alpha}])\,. \tag{4.5.41}$$

Since $g_2(\varphi) = g_2(\varphi - \vec{\jmath}\varsigma)$ for any $\varsigma \in \mathbb{R}$ by (4.5.40) and $\partial_\alpha \partial_\phi K_{00}(\varphi)[\widehat{\alpha}]$ satisfies the same property by (4.5.23), we deduce also that

$$\widehat{y}(\varphi - \vec{\jmath}\varsigma) = \widehat{y}(\varphi), \quad \forall \, \varsigma \in \mathbb{R}\,. \tag{4.5.42}$$

Next we consider the third equation $\mathcal{L}_\omega^{<} \widehat{\mathtt{w}} = g_3 + J_\angle(\partial_\alpha K_{01}(\varphi)[\widehat{\alpha}] + K_{11}(\varphi)\widehat{y})$. The right hand side of this equation is a traveling wave by (4.5.40), (4.5.23), (4.5.42) and since $J_\angle = \Pi^\angle_{\mathbb{S}^+,\Sigma} J_{|\mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma}}$ commutes with τ_ς (by Lemma 4.45). Thus, by assumption (AI), there is a traveling wave solution

$$\widehat{\mathtt{w}} := (\mathcal{L}_\omega^{<})^{-1}\big(g_3 + J_\angle(\partial_\alpha K_{01}(\varphi)[\widehat{\alpha}] + K_{11}(\varphi)\widehat{y})\big)\,. \tag{4.5.43}$$

Finally, we solve the first equation in (4.5.38), which, inserting (4.5.41) and (4.5.43), becomes

$$\omega \cdot \partial_\varphi \widehat{\phi} = g_1 + M_1(\varphi)[\widehat{\alpha}] + M_2(\varphi) g_2 + M_3(\varphi) g_3\,, \tag{4.5.44}$$

where

$$\begin{aligned}
M_1(\varphi) &:= \partial_\alpha K_{10}(\varphi) - M_2(\varphi)\partial_\alpha \partial_\phi K_{00}(\varphi) + M_3(\varphi) J_\angle \partial_\alpha K_{01}(\varphi)\,, \\
M_2(\varphi) &:= K_{20}(\varphi)(\omega \cdot \partial_\varphi)^{-1} + K_{11}^\top(\varphi)\, (\mathcal{L}_\omega^{<})^{-1}\, J_\angle K_{11}(\varphi)(\omega \cdot \partial_\varphi)^{-1}\,, \\
M_3(\varphi) &:= K_{11}^\top(\varphi)\, (\mathcal{L}_\omega^{<})^{-1}\,.
\end{aligned}$$

In order to solve (4.5.44), we choose $\widehat{\alpha}$ such that the average in φ of the right hand side is zero. By Lemma 4.63 and (4.5.1), the φ-average of the matrix M_1 satisfies $\langle M_1 \rangle_\varphi = \mathrm{Id} + O(\varepsilon \upsilon^{-1})$. Then, for $\varepsilon \upsilon^{-1}$ small enough, $\langle M_1 \rangle_\varphi$ is invertible and $\langle M_1 \rangle_\varphi^{-1} = \mathrm{Id} + O(\varepsilon \upsilon^{-1})$. Thus we define

$$\widehat{\alpha} := -\langle M_1 \rangle_\varphi^{-1} \left(\langle g_1 \rangle_\varphi + \langle M_2 g_2 \rangle_\varphi + \langle M_3 g_3 \rangle_\varphi \right), \tag{4.5.45}$$

and the solution of equation (4.5.44)

$$\widehat{\phi} := (\omega \cdot \partial_\varphi)^{-1}\big(g_1 + M_1(\varphi)[\widehat{\alpha}] + M_2(\varphi) g_2 + M_3(\varphi) g_3\big)\,. \tag{4.5.46}$$

The property $\widehat{\phi}(\varphi - \vec{\jmath}\varsigma) = \widehat{\phi}(\varphi)$ for any $\varsigma \in \mathbb{R}$ follows by (4.5.23), (4.5.42) and the fact that $\widehat{\mathtt{w}}$ in (4.5.43) is a traveling wave. This proves that $(\widehat{\phi}, \widehat{y}, \widehat{\mathtt{w}})$ is a traveling wave variation, i.e. (4.5.40) holds. Moreover, using (4.5.39), (4.5.22), Lemma 4.37, the fact that J and $\mathcal{S}$ anti-commutes and (AI), one checks that $(\widehat{\phi}, \widehat{y}, \widehat{\mathtt{w}})$ is reversible, i.e.

$$\widehat{\phi}(\varphi) = -\widehat{\phi}(-\varphi), \qquad \widehat{y}(\varphi) = \widehat{y}(-\varphi), \qquad \mathcal{S}\widehat{\mathtt{w}}(\varphi) = \widehat{\mathtt{w}}(-\varphi) . \tag{4.5.47}$$

In conclusion, we have obtained a solution $(\widehat{\phi}, \widehat{y}, \widehat{\mathtt{w}}, \widehat{\alpha})$ of the linear system (4.5.38), and, denoting the norm $\|(\phi, y, \mathtt{w}, \alpha)\|_s^{k_0,\upsilon} := \max\{\|(\phi, y, \mathtt{w})\|_s^{k_0,\upsilon}, |\alpha|^{k_0,\upsilon}\}$, we have:

Proposition 4.64. *Assume* (4.5.1) *(with* $\mu = \mu(\mathtt{b}) + \sigma$*) and* (AI)*. Then, for all* $(\omega, \kappa) \in \Lambda_o$*, for any anti-reversible traveling wave variation* $g = (g_1, g_2, g_3)$ *(i.e. satisfying* (4.5.39)-(4.5.40)*), system* (4.5.38) *has a solution* $\mathbb{D}^{-1}g := (\widehat{\phi}, \widehat{y}, \widehat{\mathtt{w}}, \widehat{\alpha})$*, with* $(\widehat{\phi}, \widehat{y}, \widehat{\mathtt{w}}, \widehat{\alpha})$ *defined in* (4.5.46), (4.5.41), (4.5.43), (4.5.45)*, where* $(\widehat{\phi}, \widehat{y}, \widehat{\mathtt{w}})$ *is a reversible traveling wave variation, satisfying, for any* $s_0 \leqslant s \leqslant S$

$$\|\mathbb{D}^{-1}g\|_s^{k_0,\upsilon} \lesssim_S \upsilon^{-1}\big(\|g\|_{s+\sigma}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\sigma}^{k_0,\upsilon}\|g\|_{s_0+\sigma}^{k_0,\upsilon}\big) . \tag{4.5.48}$$

Proof. The estimate (4.5.48) follows by the explicit expression of the solution in (4.5.41), (4.5.43), (4.5.45), (4.5.46), and Lemma 4.63, (4.5.36), (4.5.1). □

Finally we prove that the operator

$$\mathbf{T}_0 := \mathbf{T}_0(i_0) := (D\widetilde{G}_\delta)(\varphi, 0, 0) \circ \mathbb{D}^{-1} \circ (DG_\delta)(\varphi, 0, 0)^{-1} \tag{4.5.49}$$

is an almost approximate right inverse for $\mathrm{d}_{i,\alpha}\mathcal{F}(i_0)$, where $\widetilde{G}_\delta(\phi, y, \mathtt{w}, \alpha) := (G_\delta(\phi, y, \mathtt{w}), \alpha)$ is the identity on the α-component.

Theorem 4.65. (Almost approximate inverse) *Assume* (AI)*. Then there is* $\overline{\sigma} := \overline{\sigma}(\tau, \nu, k_0) > 0$ *such that, if* (4.5.1) *holds with* $\mu = \mu(\mathtt{b}) + \overline{\sigma}$*, then, for all* $(\omega, \kappa) \in \Lambda_o$ *and for any anti-reversible traveling wave variation* $g := (g_1, g_2, g_3)$ *(i.e. satisfying* (4.5.39)-(4.5.40)*), the operator* $\mathbf{T}_0$ *defined in* (4.5.49) *satisfies, for all* $s_0 \leqslant s \leqslant S$,

$$\|\mathbf{T}_0 g\|_s^{k_0,\upsilon} \lesssim_S \upsilon^{-1}\big(\|g\|_{s+\overline{\sigma}}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\overline{\sigma}}^{k_0,\upsilon}\|g\|_{s_0+\overline{\sigma}}^{k_0,\upsilon}\big) . \tag{4.5.50}$$

Moreover, the first three components of $\mathbf{T}_0 g$ *form a reversible traveling wave variation (i.e. satisfy* (4.5.47) *and* (4.5.40)*). Finally,* $\mathbf{T}_0$ *is an almost approximate right inverse of* $\mathrm{d}_{i,\alpha}\mathcal{F}(i_0)$*, namely*

$$\mathrm{d}_{i,\alpha}\mathcal{F}(i_0) \circ \mathbf{T}_0 - \mathrm{Id} = \mathcal{P}(i_0) + \mathcal{P}_\omega(i_0) + \mathcal{P}_\omega^\perp(i_0) ,$$

where, for any traveling wave variation g, for all $s_0 \leqslant s \leqslant S$,

$$\|\mathcal{P}g\|_s^{k_0,\upsilon} \lesssim_S \upsilon^{-1}\Big(\|\mathcal{F}(i_0,\alpha_0)\|_{s_0+\overline{\sigma}}^{k_0,\upsilon}\|g\|_{s+\overline{\sigma}}^{k_0,\upsilon} + \big(\|\mathcal{F}(i_0,\alpha_0)\|_{s+\overline{\sigma}}^{k_0,\upsilon} + \|\mathcal{F}(i_0,\alpha_0)\|_{s_0+\overline{\sigma}}^{k_0,\upsilon}\|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\overline{\sigma}}^{k_0,\upsilon}\big)\|g\|_{s_0+\overline{\sigma}}^{k_0,\upsilon}\Big)\,, \tag{4.5.51}$$

$$\|\mathcal{P}_\omega g\|_s^{k_0,\upsilon} \lesssim_S \varepsilon\upsilon^{-2}N_{n-1}^{-\mathtt{a}}\big(\|g\|_{s+\overline{\sigma}}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\overline{\sigma}}^{k_0,\upsilon}\|g\|_{s_0+\overline{\sigma}}^{k_0,\upsilon}\big)\,, \tag{4.5.52}$$

$$\|\mathcal{P}_\omega^\perp g\|_{s_0}^{k_0,\upsilon} \lesssim_{S,b} \upsilon^{-1}K_n^{-b}\Big(\|g\|_{s_0+\overline{\sigma}+b}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s_0+\mu(\mathtt{b})+b+\overline{\sigma}}^{k_0,\upsilon}\|g\|_{s_0+\overline{\sigma}}^{k_0,\upsilon}\Big)\,, \quad \forall\, b>0\,, \tag{4.5.53}$$

$$\|\mathcal{P}_\omega^\perp g\|_s^{k_0,\upsilon} \lesssim_S \upsilon^{-1}\big(\|g\|_{s+\overline{\sigma}}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\overline{\sigma}}^{k_0,\upsilon}\|g\|_{s_0+\overline{\sigma}}^{k_0,\upsilon}\big)\,. \tag{4.5.54}$$

The proof of Theorem 4.65 is available in Appendix C.2.

4.6 The linearized operator in the normal subspace

We now write an explicit expression of the linear operator $\mathcal{L}_\omega$ defined in (4.5.33).

Lemma 4.66. *The Hamiltonian operator $\mathcal{L}_\omega$ defined in* (4.5.33)*, acting on the normal subspace $\mathfrak{H}_{\mathbb{S}^+,\Sigma}^\angle$, has the form*

$$\mathcal{L}_\omega = \Pi_{\mathbb{S}^+,\Sigma}^\angle(\mathcal{L}-\varepsilon JR)|_{\mathfrak{H}_{\mathbb{S}^+,\Sigma}^\angle}\,, \tag{4.6.1}$$

where :

1. *$\mathcal{L}$ is the Hamiltonian operator*

$$\mathcal{L} := \omega\cdot\partial_\varphi - J\partial_u\nabla_u\mathcal{H}(T_\delta(\varphi))\,, \tag{4.6.2}$$

where $\mathcal{H}$ is the water waves Hamiltonian in the Wahlén variables defined in (4.1.13)*, evaluated at*

$$T_\delta(\phi) := \varepsilon A(i_\delta(\phi)) = \varepsilon A\left(\theta_0(\phi), I_\delta(\phi), w_0(\phi)\right) = \varepsilon v^\intercal\left(\theta_0(\phi), I_\delta(\phi)\right) + \varepsilon w_0(\phi)\,, \tag{4.6.3}$$

the torus $i_\delta(\varphi) := (\theta_0(\varphi), I_\delta(\varphi), w_0(\varphi))$ is defined in Lemma 4.61 and $A(\theta,I,w)$, $v^\intercal(\theta,I)$ in (4.1.50)*;*

2. *$R(\phi)$ has the finite rank form*

$$R(\phi)[h] = \sum_{j=1}^{\nu}(h,g_j)_{L^2}\,\chi_j\,, \quad \forall\, h\in\mathfrak{H}_{\mathbb{S}^+,\Sigma}^\angle\,, \tag{4.6.4}$$

for functions $g_j,\chi_j\in\mathfrak{H}_{\mathbb{S}^+,\Sigma}^\angle$ which satisfy, for some $\sigma := \sigma(\tau,\nu,k_0)>0$, for all $j=1,\dots,\nu$, *for all* $s \geqslant s_0$,

$$\begin{aligned} \|g_j\|_s^{k_0,\upsilon} + \|\chi_j\|_s^{k_0,\upsilon} &\lesssim_s 1 + \|\mathfrak{I}_\delta\|_{s+\sigma}^{k_0,\upsilon} , \\ \|\mathrm{d}_i g_j[\widehat{\imath}]\|_s + \|\mathrm{d}_i \chi_j[\widehat{\imath}]\|_s &\lesssim_s \|\widehat{\imath}\|_{s+\sigma} + \|\widehat{\imath}\|_{s_0+\sigma} \|\mathfrak{I}_\delta\|_{s+\sigma} . \end{aligned} \tag{4.6.5}$$

The operator $\mathcal{L}_\omega$ *is reversible and momentum preserving.*

Proof. In view of (4.5.21), (4.5.19) and (4.4.5) we have

$$\begin{aligned} K_{02}(\phi) &= \partial_{\mathtt{w}} \nabla_{\mathtt{w}} K_\alpha(\phi, 0, 0) = \partial_{\mathtt{w}} \nabla_{\mathtt{w}} (H_\alpha \circ G_\delta)(\phi, 0, 0) \\ &= \Pi_\angle^{L^2} \mathbf{\Omega}_W|_{\mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma}} + \varepsilon \partial_{\mathtt{w}} \nabla_{\mathtt{w}} (P \circ G_\delta)(\phi, 0, 0) , \end{aligned} \tag{4.6.6}$$

where $\mathbf{\Omega}_W$ is defined in (4.1.19) and G_δ in (4.5.12). Differentiating with respect to $\mathtt{w}$ the Hamiltonian

$$(P \circ G_\delta)(\phi, y, \mathtt{w}) = P\big(\theta_0(\phi), I_\delta(\phi) + L_1(\phi) y + L_2(\phi)\mathtt{w}, w_0(\phi) + \mathtt{w}\big) ,$$

where $L_1(\phi) := [\partial_\phi \theta_0(\phi)]^{-\top}$ and $L_2(\phi) := [\partial_\phi \widetilde{w}_0(\theta_0(\phi))]^\top J_\angle^{-1}$ (see (4.5.12)), we get

$$\partial_{\mathtt{w}} \nabla_{\mathtt{w}} (P \circ G_\delta)(\phi, 0, 0) = \partial_w \nabla_w P(i_\delta(\phi)) + R(\phi) , \tag{4.6.7}$$

where $R(\phi) := R_1(\phi) + R_2(\phi) + R_3(\phi)$ and

$$R_1 := L_2(\phi)^\top \partial_I^2 P(i_\delta(\phi)) L_2(\phi), \quad R_2 := L_2(\phi)^\top \partial_w \partial_I P(i_\delta(\phi)), \quad R_3 := \partial_I \nabla_w P(i_\delta(\phi)) L_2(\phi) .$$

Each operator R_1, R_2, R_3 has the finite rank form (4.6.4) because it is the composition of at least one operator with finite rank $\mathbb{R}^\nu$ in the space variable (for more details see e.g. Lemma 6.1 in [44]) and the estimates (4.6.5) follow by Lemma 4.60. By (4.6.6), (4.6.7), (4.4.4), (4.4.3), (4.4.1), we obtain

$$K_{02}(\phi) = \Pi_\angle^{L^2} \partial_u \nabla_u \mathcal{H}(A(i_\delta(\phi)))_{|\mathfrak{H}^\angle_{\mathbb{S}^+,\Sigma}} + \varepsilon R(\phi) . \tag{4.6.8}$$

In conclusion, by (4.6.8), Lemma 4.8, and since $T_\delta(\phi) = A(i_\delta(\phi))$, we deduce that the operator $\mathcal{L}_\omega$ in (4.5.33) has the form (4.6.1)-(4.6.2). Finally the operator $\Pi^\angle_{\mathbb{S}^+,\Sigma} J K_{02}(\varphi)$ is reversible and momentum preserving, by (4.5.22), (4.5.23), Lemmata 4.37, 4.45, and the fact that J commutes with τ_ς and anti-commutes with $\mathcal{S}$. □

We remark that $\mathcal{L}$ in (4.6.2) is obtained by linearizing the water waves Hamiltonian system (4.1.13), (4.1.14) in the Wahlén variables defined in (4.1.11) at the torus $u = (\eta, \zeta) = T_\delta(\varphi)$ defined in (4.6.3) and changing $\partial_t \rightsquigarrow \omega \cdot \partial_\varphi$. This is equal to

$$\mathcal{L} = \omega \cdot \partial_\varphi - W^{-1} (dX)(W T_\delta(\varphi)) W , \tag{4.6.9}$$

where X is the water waves vector field on the right hand side of (4.0.1). The operator $\mathcal{L}$ acts on (a dense subspace) of the phase space $L^2_0 \times \dot{L}^2$.

In order to compute dX we use the "shape derivative" formula, see e.g. [133],

$$G'(\eta)[\widehat{\eta}]\psi := \lim_{\epsilon \to 0} \tfrac{1}{\epsilon}\big(G(\eta + \epsilon\widehat{\eta})\psi - G(\eta)\psi\big) = -G(\eta)(B\widehat{\eta}) - \partial_x(V\widehat{\eta}) \,, \tag{4.6.10}$$

where

$$B(\eta, \psi) := \frac{G(\eta)\psi + \eta_x \psi_x}{1 + \eta_x^2} \,, \quad V(\eta, \psi) := \psi_x - B(\eta, \psi)\eta_x \,. \tag{4.6.11}$$

It turns out that $(V, B) = (\Phi_x, \Phi_y)$ is the gradient of the generalized velocity potential defined in (1.1.12), evaluated at the free surface $y = \eta(x)$.

Using (4.6.9), (4.0.1), (4.6.10), (4.6.11), the operator $\mathcal{L}$ is

$$\begin{aligned} \mathcal{L} = \omega \cdot \partial_\varphi &+ \begin{pmatrix} \partial_x \widetilde{V} + G(\eta)B & -G(\eta) \\ g - \kappa \partial_x c \partial_x + B\widetilde{V}_x + BG(\eta)B & \widetilde{V}\partial_x - BG(\eta) \end{pmatrix} \\ &+ \frac{\gamma}{2} \begin{pmatrix} -G(\eta)\partial_x^{-1} & 0 \\ \partial_x^{-1}G(\eta)B - BG(\eta)\partial_x^{-1} - \frac{\gamma}{2}\partial_x^{-1}G(\eta)\partial_x^{-1} & -\partial_x^{-1}G(\eta) \end{pmatrix} , \end{aligned} \tag{4.6.12}$$

where

$$\widetilde{V} := V - \gamma\eta \,, \quad c(\eta) := (1 + \eta_x^2)^{-\frac{3}{2}} \,, \tag{4.6.13}$$

and the functions $B := B(\eta, \psi)$, $V := V(\eta, \psi)$, $c := c(\eta)$ in (4.6.12) are evaluated at the reversible traveling wave $(\eta, \psi) := WT_\delta(\varphi)$ where $T_\delta(\varphi)$ is defined in (4.6.3).

Remark 4.67. From now on we consider the operator $\mathcal{L}$ in (4.6.12) acting on (a dense subspace of) the whole $L^2(\mathbb{T}) \times L^2(\mathbb{T})$. In particular we extend the operator ∂_x^{-1} to act on the whole $L^2(\mathbb{T})$ as in (4.2.20). In Sections 4.6.1-4.6.6 we are going to make several transformations, whose aim is to conjugate $\mathcal{L}$ to a constant coefficients Fourier multiplier, up to a pseudodifferential operator of order zero plus a remainder that satisfies tame estimates, both small in size, see $\mathcal{L}_9$ in (4.6.169). Finally, in Section 4.6.7 we shall conjugate the restricted operator $\mathcal{L}_\omega$ in (4.6.1).

Notation. In (4.6.12) and hereafter any function a is identified with the corresponding multiplication operators $h \mapsto ah$, and, where there is no parenthesis, composition of operators is understood. For example, $\partial_x c \partial_x$ means: $h \mapsto \partial_x(c\partial_x h)$.

Lemma 4.68. *The functions* $(\eta, \zeta) = T_\delta(\varphi)$ *and* $B, \widetilde{V}, c$ *defined in* (4.6.11), (4.6.13) *are quasi-periodic traveling waves. The functions* $(\eta, \zeta) = T_\delta(\varphi)$ *are* $(\mathrm{even}(\varphi, x), \mathrm{odd}(\varphi, x))$, B *is* $\mathrm{odd}(\varphi, x)$, $\widetilde{V}$ *is* $\mathrm{even}(\varphi, x)$ *and* c *is* $\mathrm{even}(\varphi, x)$. *The Hamiltonian operator* $\mathcal{L}$ *is reversible and momentum preserving.*

Proof. The function $(\eta, \zeta) = T_\delta(\varphi)$ is a quasi-periodic traveling wave and, using also Lemmata 4.46 and 4.40, we deduce that $B, \widetilde{V}, c$ are quasi-periodic traveling waves. Since $(\eta, \zeta) = T_\delta(\varphi)$ is reversible, we have that (η, ζ) is $(\mathrm{even}(\varphi, x), \mathrm{odd}(\varphi, x))$. Therefore, using also (4.1.6), we deduce that B is odd(φ, x), $\widetilde{V}$ is even(φ, x) and c is even(φ, x). By Lemmata 4.36 and 4.41, the operator $\mathcal{L}$ in (4.6.9) evaluated at the reversible quasi-periodic traveling wave $WT_\delta(\varphi)$ is reversible and momentum preserving. □

For the sequel we will always assume the following ansatz (satisfied by the approximate solutions obtained along the nonlinear Nash-Moser iteration of Section 4.8): for some constants $\mu_0 := \mu_0(\tau, \nu) > 0$, $\upsilon \in (0,1)$, (cfr. Lemma 4.61)

$$\|\mathfrak{I}_0\|_{s_0+\mu_0}^{k_0,\upsilon}\,,\ \|\mathfrak{I}_\delta\|_{s_0+\mu_0}^{k_0,\upsilon} \leqslant 1\,. \tag{4.6.14}$$

In order to estimate the variation of the eigenvalues with respect to the approximate invariant torus, we need also to estimate the variation with respect to the torus $i(\varphi)$ in another low norm $\| \ \|_{s_1}$ for all Sobolev indexes s_1 such that

$$s_1 + \sigma_0 \leqslant s_0 + \mu_0\,, \qquad \text{for some } \ \sigma_0 := \sigma_0(\tau, \nu) > 0\,. \tag{4.6.15}$$

Thus, by (4.6.14), we have

$$\|\mathfrak{I}_0\|_{s_1+\sigma_0}^{k_0,\upsilon}\,,\ \|\mathfrak{I}_\delta\|_{s_1+\sigma_0}^{k_0,\upsilon} \leqslant 1\,.$$

The constants μ_0 and σ_0 represent the *loss of derivatives* accumulated along the reduction procedure of the next sections. What is important is that they are independent of the Sobolev index s. In the following sections we shall denote by $\sigma := \sigma(\tau, \nu, k_0) > 0$, $\sigma_N(\mathtt{q}_0) := \sigma_N(\mathtt{q}_0, \tau, \nu, k_0)$, $\sigma_M := \sigma_M(k_0, \tau, \nu) > 0$, $\aleph_M(\alpha)$ constants (which possibly increase from lemma to lemma) representing losses of derivatives along the finitely many steps of the reduction procedure.

Remark 4.69. In the next sections $\mu_0 := \mu_0(\tau, \nu, M, \alpha) > 0$ will depend also on indexes M, α, whose maximal values will be fixed depending only on τ and ν (and k_0 which is however considered an absolute constant along the paper). In particular M is fixed in (4.7.5), whereas the maximal value of α depends on M, as explained in Remark 4.79.

As a consequence of Moser composition Lemma 4.13 and (4.5.7), the Sobolev norm of the function $u = T_\delta(\varphi)$ defined in (4.6.3) satisfies for all $s \geqslant s_0$

$$\|u\|_s^{k_0,\upsilon} = \|\eta\|_s^{k_0,\upsilon} + \|\zeta\|_s^{k_0,\upsilon} \leqslant \varepsilon C(s)\big(1 + \|\mathfrak{I}_0\|_s^{k_0,\upsilon}\big) \tag{4.6.16}$$

(the map A defined in (4.1.50) is smooth). Similarly, using (4.5.10),

$$\|\Delta_{12} u\|_{s_1} \lesssim_{s_1} \varepsilon \, \|i_2 - i_1\|_{s_1}\,, \qquad \text{where } \ \Delta_{12} u := u(i_2) - u(i_1)\,.$$

We finally recall that $\mathfrak{I}_0 = \mathfrak{I}_0(\omega, \kappa)$ is defined for all $(\omega, \kappa) \in \mathbb{R}^\nu \times [\kappa_1, \kappa_2]$ and that the functions $B, \widetilde{V}$ and c appearing in $\mathcal{L}$ in (4.6.12) are $\mathcal{C}^\infty$ in (φ, x), as $u = (\eta, \zeta) = T_\delta(\varphi)$ is.

4.6.1 Quasi-periodic reparametrization of time

We conjugate the operator $\mathcal{L}$ in (4.6.12) by the change of variables induced by the quasi-periodic reparametrization of time

$$\vartheta := \varphi + \omega p(\varphi) \quad \Leftrightarrow \quad \varphi = \vartheta + \omega \breve{p}(\vartheta)\,, \tag{4.6.17}$$

where $p(\varphi)$ is the real $\mathbb{T}^\nu$-periodic function defined in (4.6.88). Since $\eta(\varphi, x)$ is a quasi-periodic traveling wave, even in (φ, x) (cfr. Lemma 4.68), it results that

$$p(\varphi - \vec{\jmath}\varsigma) = p(\varphi)\,,\ \forall \varsigma \in \mathbb{R}\,, \quad p \text{ is } \mathrm{odd}(\varphi)\,. \tag{4.6.18}$$

Moreover, by (4.6.88), (4.2.11), Lemma 4.13, (4.6.16) and (4.6.14) and Lemma 2.30 in [44], both p and $\breve{p}$ satisfy, for some $\sigma := \sigma(\tau, \nu, k_0) > 0$, the tame estimates, for $s \geqslant s_0$,

$$\|p\|_s^{k_0,\upsilon} + \|\breve{p}\|_s^{k_0,\upsilon} \lesssim_s \varepsilon^2 \upsilon^{-1}\big(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}\big)\,. \tag{4.6.19}$$

Remark 4.70. We perform as a first step the time reparametrization (4.6.17) of $\mathcal{L}$, with a function $p(\varphi)$ which will be fixed only later in Step 4 of Section 4.6.3, to avoid otherwise a technical difficulty in the conjugation of the remainders obtained by the Egorov theorem in Step 1 of Section 4.6.3. We need indeed to apply the Egorov Proposition 4.20 for conjugating the additional pseudodifferential term in (4.6.12) due to vorticity.

Denoting by

$$(\mathcal{P}h)(\varphi, x) := h(\varphi + \omega p(\varphi), x)\,, \quad (\mathcal{P}^{-1}h)(\vartheta, x) := h(\vartheta + \omega \breve{p}(\vartheta), x)\,,$$

the induced diffeomorphism of functions $h(\varphi, x) \in \mathbb{C}^2$, we have

$$\mathcal{P}^{-1} \circ \omega \cdot \partial_\varphi \circ \mathcal{P} = \rho(\vartheta)\omega \cdot \partial_\vartheta\,, \quad \rho(\vartheta) := \mathcal{P}^{-1}(1 + \omega \cdot \partial_\varphi p)\,. \tag{4.6.20}$$

Therefore, for any $\omega \in \mathtt{DC}(\upsilon, \tau)$, we get

$$\begin{aligned} \mathcal{L}_0 := \frac{1}{\rho}\mathcal{P}^{-1}\mathcal{L}\mathcal{P} = \omega \cdot \partial_\vartheta + \frac{1}{\rho} & \begin{pmatrix} \partial_x \widetilde{V} + G(\eta)B & -G(\eta) \\ g - \kappa \partial_x c \partial_x + B\widetilde{V}_x + BG(\eta)B & \widetilde{V}\partial_x - BG(\eta) \end{pmatrix} \\ & + \frac{1}{\rho}\frac{\gamma}{2}\begin{pmatrix} -G(\eta)\partial_x^{-1} & 0 \\ \partial_x^{-1}G(\eta)B - BG(\eta)\partial_x^{-1} - \frac{\gamma}{2}\partial_x^{-1}G(\eta)\partial_x^{-1} & -\partial_x^{-1}G(\eta) \end{pmatrix}\,, \end{aligned} \tag{4.6.21}$$

where $\widetilde{V}, B, c, V$ and $G(\eta)$ are evaluated at $(\eta_p, \psi_p) := \mathcal{P}^{-1}(\eta, \psi)$. For simplicity in the notation we do not report in (4.6.21) the explicit dependence on p, writing for example (cfr. (4.6.13))

$$c = \big(1 + (\mathcal{P}^{-1}\eta)_x^2\big)^{-\frac{3}{2}} = \mathcal{P}^{-1}\big(1+\eta_x^2\big)^{-\frac{3}{2}} . \tag{4.6.22}$$

Lemma 4.71. *The maps $\mathcal{P}$, $\mathcal{P}^{-1}$ are $\mathcal{D}^{k_0}$-(k_0+1)-tame, the maps $\mathcal{P} - \mathrm{Id}$ and $\mathcal{P}^{-1} - \mathrm{Id}$ are $\mathcal{D}^{k_0}$-(k_0+2)-tame, with tame constants satisfying, for some $\sigma := \sigma(\tau,\nu,k_0) > 0$ and for any $s_0 \leqslant s \leqslant S$,*

$$\mathfrak{M}_{\mathcal{P}^{\pm 1}}(s) \lesssim_S 1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon} , \quad \mathfrak{M}_{\mathcal{P}^{\pm 1}-\mathrm{Id}}(s) \lesssim_S \varepsilon^2\upsilon^{-1}\big(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}\big) . \tag{4.6.23}$$

The function ρ defined in (4.6.20) satisfies

$$\rho \text{ is even}(\vartheta) \quad \textit{and} \quad \rho(\vartheta - \vec{\jmath}\varsigma) = \rho(\vartheta) , \ \forall \varsigma \in \mathbb{R} . \tag{4.6.24}$$

The operator $\mathcal{L}_0$ is Hamiltonian, reversible and momentum preserving.

Proof. Estimates (4.6.23) follow by (4.6.19) and Lemma 2.30 in [44], writing $(\mathcal{P} - \mathrm{Id})h = p\int_0^1 \mathcal{P}_\tau(\omega\cdot\partial_\varphi h)\,\mathrm{d}\tau$, where $(\mathcal{P}_\tau h)(\varphi,x) := h(\varphi + \tau\omega p(\varphi), x)$. We deduce (4.6.24) by (4.6.18) and (4.6.20). Denoting $\mathcal{L} = \omega\cdot\partial_\varphi + A(\varphi)$ the operator $\mathcal{L}$ in (4.6.12), then the operator $\mathcal{L}_0$ in (4.6.21) is $\mathcal{L}_0 = \omega\cdot\partial_\vartheta + A_+(\vartheta)$ with $A_+(\vartheta) = \rho^{-1}(\vartheta)A(\vartheta + \breve{p}(\vartheta)\omega)$. It follows that $A_+(\varphi)$ is Hamiltonian, reversible and momentum preserving as $A(\varphi)$ (Lemma 4.68). □

Remark 4.72. The map $\mathcal{P}$ is not reversibility and momentum preserving according to Definitions 4.31, respectively 4.38, but maps (anti)-reversible, respectively traveling, waves, into (anti)-reversible, respectively traveling, waves. Note that the multiplication operator for the function $\rho(\vartheta)$, which satisfies (4.6.24), is reversibility and momentum preserving according to Definitions 4.31 and 4.38.

4.6.2 Linearized good unknown of Alinhac

We conjugate the linear operator $\mathcal{L}_0$ in (4.6.21), where we rename ϑ with φ, by the multiplication matrix operator

$$\mathcal{Z} := \begin{pmatrix} \mathrm{Id} & 0 \\ B & \mathrm{Id} \end{pmatrix} , \qquad \mathcal{Z}^{-1} = \begin{pmatrix} \mathrm{Id} & 0 \\ -B & \mathrm{Id} \end{pmatrix} ,$$

obtaining (in view of (4.2.54))

$$\begin{aligned} \mathcal{L}_1 &:= \mathcal{Z}^{-1}\mathcal{L}_0\mathcal{Z} \\ &= \omega\cdot\partial_\varphi + \frac{1}{\rho}\begin{pmatrix} \partial_x \widetilde{V} & -G(\eta) \\ g + a - \kappa\partial_x c\partial_x & \widetilde{V}\partial_x \end{pmatrix} - \frac{1}{\rho}\frac{\gamma}{2}\begin{pmatrix} G(\eta)\partial_x^{-1} & 0 \\ \frac{\gamma}{2}\partial_x^{-1}G(\eta)\partial_x^{-1} & \partial_x^{-1}G(\eta) \end{pmatrix} , \end{aligned} \tag{4.6.25}$$

where a is the function

$$a := \widetilde{V} B_x + \rho\,(\omega \cdot \partial_\varphi B)\,. \tag{4.6.26}$$

The matrix $\mathcal{Z}$ amounts to introduce, as in [133] and [44, 13], a linearized version of the "good unknown of Alinhac".

Lemma 4.73. *The maps $\mathcal{Z}^{\pm 1} - \mathrm{Id}$ are $\mathcal{D}^{k_0}$-tame with tame constants satisfying, for some $\sigma := \sigma(\tau, \nu, k_0) > 0$, for all $s \geqslant s_0$,*

$$\mathfrak{M}_{\mathcal{Z}^{\pm 1} - \mathrm{Id}}(s)\,,\ \mathfrak{M}_{(\mathcal{Z}^{\pm 1} - \mathrm{Id})^*}(s) \lesssim_s \varepsilon\big(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}\big)\,. \tag{4.6.27}$$

The function a is a quasi-periodic traveling wave $\mathrm{even}(\varphi, x)$*. There is $\sigma := \sigma(\tau, \nu, k_0) > 0$ such that, for all $s \geqslant s_0$,*

$$\|a\|_s^{k_0,\upsilon} + \|\widetilde{V}\|_s^{k_0,\upsilon} + \|B\|_s^{k_0,\upsilon} \lesssim_s \varepsilon\big(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}\big)\,, \quad \|1 - c\|_s^{k_0,\upsilon} \lesssim_s \varepsilon^2\big(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}\big)\,. \tag{4.6.28}$$

Moreover, for any s_1 as in (4.6.15),

$$\|\Delta_{12} a\|_{s_1} + \|\Delta_{12} \widetilde{V}\|_{s_1} + \|\Delta_{12} B\|_{s_1} \lesssim_{s_1} \varepsilon\,\|i_1 - i_2\|_{s_1+\sigma}\,, \tag{4.6.29}$$

$$\|\Delta_{12} c\|_{s_1} \lesssim_{s_1} \varepsilon^2\,\|i_1 - i_2\|_{s_1+\sigma}\,, \tag{4.6.30}$$

$$\|\Delta_{12}(\mathcal{Z}^{\pm 1})h\|_{s_1}, \|\Delta_{12}(\mathcal{Z}^{\pm 1})^* h\|_{s_1} \lesssim_{s_1} \varepsilon\,\|i_1 - i_2\|_{s_1+\sigma}\,\|h\|_{s_1}\,. \tag{4.6.31}$$

The operator $\mathcal{L}_1$ is Hamiltonian, reversible and momentum preserving.

Proof. The estimates (4.6.28) follow by the expressions of $a, \widetilde{V}, B, c$ in (4.6.26), (4.6.11), (4.6.13), (reparametrized by $\mathcal{P}^{-1}$ as in (4.6.22)), Lemmata 4.13, 4.21 and (4.6.23), (4.2.7), (4.2.38) and (4.2.37). The estimate (4.6.27) follows by (4.2.38), (4.2.22), (4.6.28) and since the adjoint $\mathcal{Z}^* = \begin{pmatrix} \mathrm{Id} & B \\ 0 & \mathrm{Id} \end{pmatrix}$. The estimates (4.6.29)-(4.6.31) follow similarly. Since B is a $\mathrm{odd}(\varphi, x)$ quasi-periodic traveling wave, then the operators $\mathcal{Z}^{\pm}$ are reversibility and momentum preserving. □

4.6.3 Symmetrization and reduction of the highest order

The aim of this long section is to conjugate the Hamiltonian operator $\mathcal{L}_1$ in (4.6.25) to the Hamiltonian operator $\mathcal{L}_5$ in (4.6.90) whose coefficient $\mathtt{m}_{\frac{3}{2}}$ of the highest order is constant. This is achieved in several steps. All the transformations of this section are symplectic.

Recalling the expansion (4.2.32) of the Dirichlet-Neumann operator in Lemma 4.21, we first write

$$\mathcal{L}_1 = \omega \cdot \partial_\varphi + \frac{1}{\rho}\begin{pmatrix} -\frac{\gamma}{2} G(0)\partial_x^{-1} & -G(0) \\ -\kappa \partial_x c \partial_x + g - \left(\frac{\gamma}{2}\right)^2 \partial_x^{-1} G(0) \partial_x^{-1} & -\frac{\gamma}{2}\partial_x^{-1} G(0) \end{pmatrix} + \frac{1}{\rho}\begin{pmatrix} \partial_x \widetilde{V} & 0 \\ a & \widetilde{V}\partial_x \end{pmatrix} + \mathbf{R}_1\,, \tag{4.6.32}$$

where

$$\mathbf{R}_1 := -\frac{1}{\rho}\begin{pmatrix} \frac{\gamma}{2}\mathcal{R}_G(\eta)\partial_x^{-1} & \mathcal{R}_G(\eta) \\ \left(\frac{\gamma}{2}\right)^2 \partial_x^{-1}\mathcal{R}_G(\eta)\partial_x^{-1} & \frac{\gamma}{2}\partial_x^{-1}\mathcal{R}_G(\eta) \end{pmatrix} \tag{4.6.33}$$

is a small remainder in $\mathrm{OP}S^{-\infty}$.

Step 1: We first conjugate $\mathcal{L}_1$ with the symplectic change of variable (cfr. (4.2.50))

$$(\mathcal{E}u)(\varphi,x) := \sqrt{1+\beta_x(\varphi,x)}\,(\mathcal{B}u)(\varphi,x)\,, \qquad (\mathcal{B}u)(\varphi,x) := u\,(\varphi, x+\beta(\varphi,x))\ , \tag{4.6.34}$$

induced by a family of φ-dependent diffeomorphisms of the torus $y = x+\beta(\varphi,x)$, where $\beta(\varphi,x)$ is a small function to be determined, see (4.6.68). We denote the inverse diffeomorphism by $x = y + \breve\beta(\varphi,y)$. By direct computation we have that

$$\mathcal{E}^{-1}\widetilde{V}\partial_x\mathcal{E} = \big\{\mathcal{B}^{-1}\big(\widetilde{V}(1+\beta_x)\big)\big\}\partial_y + \tfrac{1}{2}\big\{\mathcal{B}^{-1}\widetilde{V}\beta_{xx}(1+\beta_x)^{-1}\big\}\,, \tag{4.6.35}$$

$$\mathcal{E}^{-1}\partial_x\widetilde{V}\mathcal{E} = \big\{\mathcal{B}^{-1}\big(\widetilde{V}(1+\beta_x)\big)\big\}\partial_y + \{\mathcal{B}^{-1}(\widetilde{V}_x + \tfrac{1}{2}\widetilde{V}\beta_{xx}(1+\beta_x)^{-1})\}\,, \tag{4.6.36}$$

$$\mathcal{E}^{-1}a\mathcal{E} = \{\mathcal{B}^{-1}a\}\,, \tag{4.6.37}$$

$$\begin{aligned}\mathcal{E}^{-1}\partial_x c\partial_x\mathcal{E} &= \mathcal{B}^{-1}(1+\beta_x)^{-\frac{1}{2}}\mathcal{B}\ \mathcal{B}^{-1}\partial_x\mathcal{B}\ \mathcal{B}^{-1}c\mathcal{B}\ \mathcal{B}^{-1}\partial_x\mathcal{B}\ \mathcal{B}^{-1}(1+\beta_x)^{\frac{1}{2}}\mathcal{B} \\ &= \big\{\mathcal{B}^{-1}(1+\beta_x)^{\frac{1}{2}}\big\}\,\partial_y\,\{\mathcal{B}^{-1}(c(1+\beta_x))\}\,\partial_y\,\big\{\mathcal{B}^{-1}(1+\beta_x)^{\frac{1}{2}}\big\}\,,\end{aligned} \tag{4.6.38}$$

$$\mathcal{E}^{-1}\omega\cdot\partial_\varphi\mathcal{E} = \omega\cdot\partial_\varphi + \big\{\mathcal{B}^{-1}\left(\omega\cdot\partial_\varphi\beta\right)\big\}\,\partial_y + \tfrac{1}{2}\{\mathcal{B}^{-1}\big((\omega\cdot\partial_\varphi\beta_x)(1+\beta_x)^{-1}\big)\}\,. \tag{4.6.39}$$

Then we write the Dirichlet-Neumann operator $G(0)$ in (1.1.15) as

$$G(0) = G(0,\mathtt{h}) = \partial_x\mathcal{H}T(\mathtt{h})\,, \tag{4.6.40}$$

where $\mathcal{H}$ is the Hilbert transform in (4.2.19) and

$$T(\mathtt{h}) := \begin{cases} \tanh(\mathtt{h}|D|) = \mathrm{Id} + \mathrm{Op}(r_{\mathtt{h}}) & \text{if } \mathtt{h} < +\infty\,, \qquad r_{\mathtt{h}}(\xi) := -\frac{2}{1+e^{2\mathtt{h}|\xi|\chi(\xi)}} \in S^{-\infty}\,, \\ \mathrm{Id} & \text{if } \mathtt{h} = \infty\,. \end{cases} \tag{4.6.41}$$

We have the conjugation formula (see formula (7.42) in [13])

$$\mathcal{B}^{-1}G(0)\mathcal{B} = \big\{\mathcal{B}^{-1}(1+\beta_x)\big\}\,G(0) + \mathcal{R}_1\,, \tag{4.6.42}$$

where

$$\mathcal{R}_1 := \big\{\mathcal{B}^{-1}(1+\beta_x)\big\}\,\partial_y\left(\mathcal{H}\left(\mathcal{B}^{-1}\mathrm{Op}(r_{\mathtt{h}})\mathcal{B} - \mathrm{Op}(r_{\mathtt{h}})\right) + \left(\mathcal{B}^{-1}\mathcal{H}\mathcal{B} - \mathcal{H}\right)\left(\mathcal{B}^{-1}T(\mathtt{h})\mathcal{B}\right)\right)\ . \tag{4.6.43}$$

The operator $\mathcal{R}_1$ is in $\mathrm{OP}S^{-\infty}$ because both $\mathcal{B}^{-1}\mathrm{Op}(r_{\mathtt{h}})\mathcal{B} - \mathrm{Op}(r_{\mathtt{h}})$ and $\mathcal{B}^{-1}\mathcal{H}\mathcal{B} - \mathcal{H}$ are in $\mathrm{OP}S^{-\infty}$ and there is $\sigma > 0$ such that, for any $m \in \mathbb{N}$, $s \geqslant s_0$, and $\alpha \in \mathbb{N}_0$,

$$\begin{aligned} &\|\mathcal{B}^{-1}\mathcal{H}\mathcal{B} - \mathcal{H}\|_{-m,s,\alpha}^{k_0,\upsilon} \lesssim_{m,s,\alpha,k_0} \|\beta\|_{s+m+\alpha+\sigma}^{k_0,\upsilon} , \\ &\|\mathcal{B}^{-1}\mathrm{Op}(r_{\mathtt{h}})\mathcal{B} - \mathrm{Op}(r_{\mathtt{h}})\|_{-m,s,\alpha}^{k_0,\upsilon} \lesssim_{m,s,\alpha,k_0} \|\beta\|_{s+m+\alpha+\sigma}^{k_0,\upsilon} . \end{aligned} \tag{4.6.44}$$

The first estimate is given by Lemmata C.5, C.5, whereas the second one follows by the fact that $r_{\mathtt{h}} \in S^{-\infty}$ (see (4.6.41)) and Lemmata C.3, C.2, C.1. Therefore by (4.6.42) we obtain

$$\mathcal{E}^{-1}G(0)\mathcal{E} = \{\mathcal{B}^{-1}(1+\beta_x)^{\frac{1}{2}}\}\, G(0)\, \{\mathcal{B}^{-1}(1+\beta_x)^{\frac{1}{2}}\} + \widetilde{\mathcal{R}}_1 , \tag{4.6.45}$$

where

$$\widetilde{\mathcal{R}}_1 := \{\mathcal{B}^{-1}(1+\beta_x)^{-\frac{1}{2}}\}\, \mathcal{R}_1\, \{\mathcal{B}^{-1}(1+\beta_x)^{\frac{1}{2}}\}. \tag{4.6.46}$$

Next we transform $G(0)\partial_x^{-1}$. By (4.6.40) and using the identities $\mathcal{H}\partial_x\partial_x^{-1} = \mathcal{H}$ and $\mathcal{H}T(\mathtt{h}) = G(0)\partial_y^{-1}$ on the periodic functions, we have that

$$\mathcal{E}^{-1}G(0)\partial_x^{-1}\mathcal{E} = \mathcal{E}^{-1}\partial_x\mathcal{H}T(\mathtt{h})\partial_x^{-1}\mathcal{E} = G(0)\partial_y^{-1} + \mathcal{R}_2 , \tag{4.6.47}$$

where

$$\begin{aligned} \mathcal{R}_2 := &\{\mathcal{B}^{-1}(1+\beta_x)^{-\frac{1}{2}}\}\big[\mathcal{H}T(\mathtt{h}), \{\mathcal{B}^{-1}(1+\beta_x)^{\frac{1}{2}}\} - 1\big] + \{\mathcal{B}^{-1}(1+\beta_x)^{-\frac{1}{2}}\} \circ \\ &\circ \big((\mathcal{B}^{-1}\mathcal{H}\mathcal{B} - \mathcal{H})(\mathcal{B}^{-1}T(\mathtt{h})\mathcal{B}) + \mathcal{H}\big(\mathcal{B}^{-1}\mathrm{Op}(r_{\mathtt{h}})\mathcal{B} - \mathrm{Op}(r_{\mathtt{h}})\big)\big)\, \{\mathcal{B}^{-1}(1+\beta_x)^{\frac{1}{2}}\} . \end{aligned} \tag{4.6.48}$$

The operator $\mathcal{R}_2$ is in $\mathrm{OP}S^{-\infty}$ by (4.6.44), (4.6.41) and because the commutator of $\mathcal{H}$ with any smooth function a is in $\mathrm{OP}S^{-\infty}$. In particular, by Lemma C.4, there is $\sigma > 0$ such that, for any $m \in \mathbb{N}$, $s \geqslant s_0$, and $\alpha \in \mathbb{N}_0$,

$$\|[\mathcal{H}T(\mathtt{h}), a]\|_{-m,s,\alpha}^{k_0,\upsilon} \lesssim_{m,s,\alpha,k_0} \|a\|_{s+m+\alpha+\sigma}^{k_0,\upsilon} . \tag{4.6.49}$$

Finally we conjugate $\partial_x^{-1}G(0)\partial_x^{-1}$. By the Egorov Proposition 4.20, we have that, for any $N \in \mathbb{N}$,

$$\mathcal{E}^{-1}\partial_x^{-1}\mathcal{E} = \Big\{\mathcal{B}^{-1}\Big(\frac{1}{1+\beta_x}\Big)\Big\}\partial_y^{-1} + P_{-2,N}^{(1)}(\varphi, x, D) + \mathtt{R}_N , \tag{4.6.50}$$

where $P_{-2,N}^{(1)}(\varphi, x, D) \in \mathrm{OP}S^{-2}$ is

$$P_{-2,N}^{(1)}(\varphi, x, D) := \{\mathcal{B}^{-1}(1+\beta_x)^{-\frac{1}{2}}\}\Big\{\big[p_{-1}\partial_y^{-1}, \mathcal{B}^{-1}(1+\beta_x)^{\frac{1}{2}}\big] + \sum_{j=1}^{N} p_{-1-j}\partial_y^{-1-j}\{\mathcal{B}^{-1}(1+\beta_x)^{\frac{1}{2}}\}\Big\}$$

with functions $p_{-1-j}(\lambda;\varphi,y)$, $j=0,\dots,N$, satisfying (4.2.30) and $\mathtt{R}_N$ is a regularizing operator satisfying the estimate (4.2.31). So, using (4.6.50) and (4.6.47), we obtain

$$\mathcal{E}^{-1}\partial_x^{-1}G(0)\partial_x^{-1}\mathcal{E} = \big(\mathcal{E}^{-1}\partial_x^{-1}\mathcal{E}\big)\big(\mathcal{E}^{-1}G(0)\partial_x^{-1}\mathcal{E}\big) = \partial_y^{-1}G(0)\partial_y^{-1} + P^{(2)}_{-1,N} + \mathtt{R}_{2,N}\,, \tag{4.6.51}$$

where

$$P^{(2)}_{-1,N} := \Big(-\Big\{\mathcal{B}^{-1}\Big(\frac{\beta_x}{1+\beta_x}\Big)\Big\}\partial_y^{-1} + P^{(1)}_{-2,N}(\varphi,x,D)\Big)G(0)\partial_y^{-1} \in \mathrm{OP}S^{-1} \tag{4.6.52}$$

and $\mathtt{R}_{2,N}$ is the regularizing operator

$$\mathtt{R}_{2,N} := (\mathcal{E}^{-1}\partial_x^{-1}\mathcal{E})\mathcal{R}_2 + \mathtt{R}_N G(0)\partial_y^{-1}\,. \tag{4.6.53}$$

The smoothing order $N \in \mathbb{N}$ will be chosen in Section 4.7 during the KAM iteration (see also Remark 4.76).

In conclusion, by (4.6.35)-(4.6.39), (4.6.45), (4.6.47) and (4.6.51) we obtain

$$\begin{aligned}\mathcal{L}_2 := \mathcal{E}^{-1}\mathcal{L}_1\mathcal{E} = \omega\cdot\partial_\varphi + \frac{1}{\rho}&\begin{pmatrix} -\frac{\gamma}{2}G(0)\partial_y^{-1} & -a_2G(0)a_2 \\ -\kappa a_2\partial_y a_3\partial_y a_2 + g - \big(\frac{\gamma}{2}\big)^2\partial_y^{-1}G(0)\partial_y^{-1} & -\frac{\gamma}{2}\partial_y^{-1}G(0)\end{pmatrix} \\ &+ \frac{1}{\rho}\begin{pmatrix} a_1\partial_y + a_4 & 0 \\ a_5 - \big(\frac{\gamma}{2}\big)^2 P^{(2)}_{-1,N} & a_1\partial_y + a_6\end{pmatrix} + \mathbf{R}_2^{\Psi} + \mathbf{T}_{2,N}\,,\end{aligned} \tag{4.6.54}$$

where

$$a_1(\varphi,y) := \mathcal{B}^{-1}\big((1+\beta_x)\widetilde{V} + \big(\omega\cdot\partial_\varphi\beta\big)\big)\,, \tag{4.6.55}$$

$$a_2(\varphi,y) := \mathcal{B}^{-1}\big(\sqrt{1+\beta_x}\big)\,, \quad a_3(\varphi,y) := \mathcal{B}^{-1}\big(c(1+\beta_x)\big)\,, \tag{4.6.56}$$

$$a_4(\varphi,y) := \mathcal{B}^{-1}\Big(\frac{\widetilde{V}\beta_{xx} + (\omega\cdot\partial_\varphi\beta_x)}{2(1+\beta_x)} + \widetilde{V}_x\Big)\,, \quad a_5(\varphi,y) := \mathcal{B}^{-1}a\,, \tag{4.6.57}$$

$$a_6(\varphi,y) := \mathcal{B}^{-1}\Big(\frac{\widetilde{V}\beta_{xx} + (\omega\cdot\partial_\varphi\beta_x)}{2(1+\beta_x)}\Big)\,, \tag{4.6.58}$$

the operator $P^{(2)}_{-1,N} \in \mathrm{OP}S^{-1}$ is defined in (4.6.52) and

$$\mathbf{R}_2^{\Psi} := -\frac{1}{\rho}\begin{pmatrix}\frac{\gamma}{2}\mathcal{R}_2 & \widetilde{\mathcal{R}}_1 \\ 0 & \frac{\gamma}{2}\mathcal{R}_2\end{pmatrix} + \mathcal{E}^{-1}\mathbf{R}_1\mathcal{E}\,, \qquad \mathbf{T}_{2,N} := -\frac{1}{\rho}\Big(\frac{\gamma}{2}\Big)^2\begin{pmatrix}0 & 0 \\ \mathtt{R}_{2,N} & 0\end{pmatrix}\,, \tag{4.6.59}$$

with $\widetilde{\mathcal{R}}_1$, $\mathcal{R}_2$, $\mathtt{R}_{2,N}$ defined in (4.6.46), (4.6.48), (4.6.53) and $\mathbf{R}_1$ in (4.6.33).

Step 2: We now conjugate the operator $\mathcal{L}_2$ in (4.6.54) with the multiplication matrix operator

$$\mathcal{Q} := \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \qquad \mathcal{Q}^{-1} := \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix},$$

where $q(\varphi, y)$ is a real function, close to 1, to be determined. The maps $\mathcal{Q}$ and $\mathcal{Q}^{-1}$ are symplectic (cfr. (4.2.50)). We have that

$$\mathcal{L}_3 := \mathcal{Q}^{-1}\mathcal{L}_2\mathcal{Q} = \omega \cdot \partial_\varphi + \frac{1}{\rho}\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \mathcal{Q}^{-1}(\mathbf{R}_2^\Psi + \mathbf{T}_{2,N})\mathcal{Q}, \tag{4.6.60}$$

where

$$A := q^{-1}\big(-\tfrac{\gamma}{2}G(0)\partial_y^{-1} + a_1\partial_y + a_4\big)q + \rho q^{-1}(\omega \cdot \partial_\varphi q), \tag{4.6.61}$$

$$B := -q^{-1}a_2 G(0) a_2 q^{-1}, \tag{4.6.62}$$

$$C := q\big(-\kappa a_2\partial_y a_3 \partial_y a_2 + g - \left(\tfrac{\gamma}{2}\right)^2 \partial_y^{-1}G(0)\partial_y^{-1} + a_5 - \left(\tfrac{\gamma}{2}\right)^2 P_{-1,N}^{(2)}\big)q, \tag{4.6.63}$$

$$D := q\big(-\tfrac{\gamma}{2}\partial_y^{-1}G(0) + a_1\partial_y + a_6\big)q^{-1} - \rho q^{-1}(\omega \cdot \partial_\varphi q). \tag{4.6.64}$$

We choose the function q so that the coefficients of the highest order terms of the off-diagonal operators B and C satisfy

$$q^{-2}a_2^2 = q^2 a_2^2 a_3 = m_{\frac{3}{2}}(\varphi), \tag{4.6.65}$$

with $m_{\frac{3}{2}}(\varphi)$ independent of x. This is achieved choosing

$$q := \left(\frac{1}{a_3}\right)^{1/4} \tag{4.6.66}$$

and, recalling (4.6.56), the function β so that

$$(1 + \beta_x(\varphi, x))^3 c(\varphi, x) = m(\varphi), \tag{4.6.67}$$

with $m(\varphi)$ independent of x (the function c is defined in (4.6.22)). The solution of (4.6.67) is

$$m(\varphi) := \Big(\frac{1}{2\pi}\int_{\mathbb{T}} c(\varphi, x)^{-1/3}\,\mathrm{d}x\Big)^{-3}, \qquad \beta(\varphi, x) := \partial_x^{-1}\Big(\Big(\frac{m(\varphi)}{c(\varphi, x)}\Big)^{1/3} - 1\Big). \tag{4.6.68}$$

In such a way, by (4.6.56), we obtain (4.6.65) with $m_{\frac{3}{2}}(\varphi) := \sqrt{m(\varphi)}$. By (4.6.68) and (4.6.22) we have

$$m_{\frac{3}{2}}(\varphi) = \mathcal{P}^{-1}\Big(\frac{1}{2\pi}\int_{\mathbb{T}} \sqrt{1 + \eta_x^2(\varphi, x)}\mathrm{d}x\Big)^{-\frac{3}{2}}. \tag{4.6.69}$$

Note that, since by (4.6.65) the function $q^{-1}a_2$ is independent of x, we have

$$B = -q^{-1}a_2 G(0) a_2 q^{-1} = -q^{-2}a_2^2 G(0)\,. \tag{4.6.70}$$

Moreover we have the expansion

$$qa_2\partial_y a_3 \partial_y a_2 q = q^2 a_2^2 a_3 \partial_y^2 + (q^2a_2^2a_3)_y\partial_y + qa_2(a_3(qa_2)_y)_y \overset{(4.6.65)}{=} m_{\frac{3}{2}}(\varphi)\partial_y^2 + a_7\,, \tag{4.6.71}$$

where

$$a_7 := qa_2(a_3(qa_2)_y)_y\,. \tag{4.6.72}$$

In conclusion, the operator $\mathcal{L}_3$ in (4.6.60) is, in view of (4.6.61)-(4.6.64) and (4.6.70), (4.6.71),

$$\begin{aligned}\mathcal{L}_3 = \mathcal{Q}^{-1}\mathcal{L}_2\mathcal{Q} = \omega\cdot\partial_\varphi + \frac{1}{\rho}&\begin{pmatrix} -\frac{\gamma}{2}G(0)\partial_y^{-1} & -m_{\frac{3}{2}}(\varphi)G(0) \\ m_{\frac{3}{2}}(\varphi)\left(-\kappa\partial_y^2 + g - \left(\frac{\gamma}{2}\right)^2\partial_y^{-1}G(0)\partial_y^{-1}\right) & -\frac{\gamma}{2}\partial_y^{-1}G(0)\end{pmatrix} \\ &+ \frac{1}{\rho}\begin{pmatrix} a_1\partial_y + a_8 & 0 \\ a_9 + P^{(3)}_{-1,N} & a_1\partial_y + a_{10}\end{pmatrix} + \mathbf{R}_3^{\Psi} + \mathbf{T}_{3,N}\,,\end{aligned} \tag{4.6.73}$$

where

$$a_8 := a_1 q^{-1} q_y + \rho\, q^{-1}(\omega\cdot\partial_\varphi q) + a_4\,, \quad a_9 := a_5 q^2 + g(q^2 - m_{\frac{3}{2}}) - \kappa a_7\,, \tag{4.6.74}$$

$$a_{10} := -a_1 q^{-1} q_y - \rho\, q^{-1}(\omega\cdot\partial_\varphi q) + a_6\,, \tag{4.6.75}$$

$$P^{(3)}_{-1,N} := -\left(\tfrac{\gamma}{2}\right)^2\left(qP^{(2)}_{-1,N}q + (q^2 - m_{\frac{3}{2}})G(0)\partial_y^{-2} + q[G(0)\partial_y^{-2}, q-1]\right) \in \mathrm{OP}S^{-1}\,, \tag{4.6.76}$$

and $\mathbf{R}_3^{\Psi}, \mathbf{T}_{3,N}$ are the smoothing remainders

$$\mathbf{R}_3^{\Psi} := \frac{1}{\rho}\begin{pmatrix} -\frac{\gamma}{2}q^{-1}[\mathcal{H}T(\mathtt{h}), q-1] & 0 \\ 0 & -\frac{\gamma}{2}q[\mathcal{H}T(\mathtt{h}), q^{-1}-1]\end{pmatrix} + \mathcal{Q}^{-1}\mathbf{R}_2^{\Psi}\mathcal{Q} \in \mathrm{OP}S^{-\infty}\,, \tag{4.6.77}$$

$$\mathbf{T}_{3,N} := \mathcal{Q}^{-1}\mathbf{T}_{2,N}\mathcal{Q}\,. \tag{4.6.78}$$

Step 3: We now conjugate $\mathcal{L}_3$ in (4.6.73), where we rename the space variable y by x, by the symplectic transformation (cfr. (4.2.50))

$$\widetilde{\mathcal{M}} := \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1}\end{pmatrix}\,, \quad \widetilde{\mathcal{M}}^{-1} := \begin{pmatrix} \Lambda^{-1} & 0 \\ 0 & \Lambda\end{pmatrix}\,, \tag{4.6.79}$$

where $\Lambda \in \mathrm{OP}S^{-\frac{1}{4}}$ is the Fourier multiplier

$$\Lambda := \tfrac{1}{\sqrt{g}}\pi_0 + M(D)\,, \quad \text{with inverse} \quad \Lambda^{-1} := \sqrt{g}\pi_0 + M(D)^{-1} \in \mathrm{OP}S^{\frac{1}{4}}\,, \tag{4.6.80}$$

with π_0 defined in (4.2.21) and $M(D)$ in (4.1.21). We have the identity

$$\Lambda\big(-\kappa\partial_x^2 + g - \big(\tfrac{\gamma}{2}\big)^2\partial_x^{-1}G(0)\partial_x^{-1}\big)\Lambda = \Lambda^{-1}G(0)\Lambda^{-1} + \pi_0 = \omega(\kappa, D) + \pi_0\,, \tag{4.6.81}$$

where $\omega(\kappa, D)$ is defined in (4.1.23). In (4.6.80) and (4.6.81) we mean that the symbols of $M(D), M(D)^{-1}$ and $\omega(\kappa, D)$ are extended to 0 at $j = 0$, multiplying them by the cut-off function χ defined in (4.2.10). The factors in front of the projection π_0 in (4.6.80) on the zeroth mode allow the transformation in (4.6.79) to be symplectic. Thus we obtain

$$\begin{aligned}\mathcal{L}_4 := \widetilde{\mathcal{M}}^{-1}\mathcal{L}_3\widetilde{\mathcal{M}} = \omega\cdot\partial_\varphi + \frac{1}{\rho}\begin{pmatrix} -\frac{\gamma}{2}G(0)\partial_x^{-1} & -m_{\frac{3}{2}}(\varphi)\omega(\kappa, D) \\ m_{\frac{3}{2}}(\varphi)\omega(\kappa, D) & -\frac{\gamma}{2}G(0)\partial_x^{-1}\end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \pi_0 & 0\end{pmatrix} \\ + \frac{1}{\rho}\begin{pmatrix} a_1\partial_x + P_0^{(41)} & 0 \\ P_{-\frac{1}{2}}^{(43)} & a_1\partial_x + P_0^{(44)}\end{pmatrix} + \mathbf{R}_4^\Psi + \mathbf{T}_{4,N}\,,\end{aligned} \tag{4.6.82}$$

where

$$P_0^{(41)} := \Lambda^{-1}[a_1\partial_x, \Lambda] + \Lambda^{-1}a_8\Lambda \in \mathrm{OP}S^0, \tag{4.6.83}$$

$$P_{-\frac{1}{2},N}^{(43)} := \Lambda a_9\Lambda + \Lambda P_{-1,N}^{(3)}\Lambda \in \mathrm{OP}S^{-\frac{1}{2}}\,, \tag{4.6.84}$$

$$P_0^{(44)} := \Lambda[a_1\partial_x, \Lambda^{-1}] + \Lambda a_{10}\Lambda^{-1} \in \mathrm{OP}S^0\,, \tag{4.6.85}$$

and $\mathbf{R}_4^\Psi, \mathbf{T}_{4,N}$ are the smoothing remainders

$$\begin{aligned}\mathbf{R}_4^\Psi &:= \begin{pmatrix} 0 & 0 \\ (\rho^{-1}m_{\frac{3}{2}} - 1)\pi_0 & 0\end{pmatrix} + \widetilde{\mathcal{M}}^{-1}\mathbf{R}_3^\Psi\widetilde{\mathcal{M}} \in \mathrm{OP}S^{-\infty}\,, \\ \mathbf{T}_{4,N} &:= \widetilde{\mathcal{M}}^{-1}\mathbf{T}_{3,N}\widetilde{\mathcal{M}} = -\frac{\gamma^2}{4\rho}\begin{pmatrix} 0 & 0 \\ \Lambda q \mathtt{R}_{2,N} q\Lambda & 0\end{pmatrix}.\end{aligned} \tag{4.6.86}$$

Step 4: We finally move in complex coordinates, conjugating the operator $\mathcal{L}_4$ in (4.6.82) via the transformation $\mathcal{C}$ defined in (4.1.24). We use the transformation formula (4.2.15). We choose the function $p(\varphi)$ in (4.6.17) in order to obtain a constant coefficient at the highest order. More precisely we choose the periodic function $p(\varphi)$ such that

$$\frac{m_{\frac{3}{2}}}{\rho} \overset{(4.6.69),(4.6.20)}{=} \mathcal{P}^{-1}\Big(\frac{\big(\frac{1}{2\pi}\int_{\mathbb{T}}\sqrt{1+\eta_x^2(\varphi, x)}\mathrm{d}x\big)^{-\frac{3}{2}}}{1 + \omega\cdot\partial_\varphi p}\Big) = \mathtt{m}_{\frac{3}{2}} \tag{4.6.87}$$

is a real constant independent of φ. Thus, recalling (4.2.9), we define the periodic function

$$p(\varphi) := (\omega \cdot \partial_\varphi)^{-1}_{\rm ext}\Big(\frac{1}{\mathtt{m}_{\frac32}}\Big(\frac{1}{2\pi}\int_{\mathbb{T}}\sqrt{1+\eta_x^2(\varphi,x)}\,{\rm d}x\Big)^{-\frac32} - 1\Big) \tag{4.6.88}$$

and the real constant

$$\mathtt{m}_{\frac32} := \frac{1}{(2\pi)^\nu}\int_{\mathbb{T}^\nu}\Big(\frac{1}{2\pi}\int_{\mathbb{T}}\sqrt{1+\eta_x^2(\varphi,x)}\,{\rm d}x\Big)^{-\frac32}{\rm d}\varphi\,. \tag{4.6.89}$$

Note that (4.6.87) holds for $\omega \in \mathtt{DC}(\upsilon,\tau)$. Moreover, by Lemmata 4.13, 4.68 and (4.6.16), p satisfies (4.6.19) and it is odd in φ. Let

$$\mathbf{\Pi}_0 := -{\rm i}\,\mathcal{C}^{-1}\begin{pmatrix} 0 & 0 \\ \pi_0 & 0\end{pmatrix}\mathcal{C} = \frac12\begin{pmatrix}\pi_0 & \pi_0 \\ -\pi_0 & -\pi_0\end{pmatrix}.$$

Lemma 4.74. *Let $N \in \mathbb{N}$, $\mathtt{q}_0 \in \mathbb{N}_0$. For all $\omega \in \mathtt{DC}(\upsilon,\tau)$, we have that*

$$\begin{aligned}\mathcal{L}_5 &:= \big(\mathcal{E}\mathcal{Q}\widetilde{\mathcal{M}}\mathcal{C}\big)^{-1}\mathcal{L}_1\big(\mathcal{E}\mathcal{Q}\widetilde{\mathcal{M}}\mathcal{C}\big)\\ &= \omega\cdot\partial_\varphi + {\rm i}\mathtt{m}_{\frac32}\mathbf{\Omega}(\kappa,D) + \mathbf{A}_1\partial_x + {\rm i}\mathbf{\Pi}_0 + \mathbf{R}_5^{(0,d)} + \mathbf{R}_5^{(0,o)} + \mathbf{T}_{5,N}\,,\end{aligned} \tag{4.6.90}$$

where:

1. *The operators $\mathcal{E}^{\pm1}$ are $\mathcal{D}^{k_0}$-(k_0+1)-tame, the operators $\mathcal{E}^{\pm1}-{\rm Id}$, $(\mathcal{E}^{\pm1}-{\rm Id})^*$ are $\mathcal{D}^{k_0}$-(k_0+2)-tame and the operators $\mathcal{Q}^{\pm1}$, $\mathcal{Q}^{\pm1}-{\rm Id}$, $(\mathcal{Q}^{\pm1}-{\rm Id})^*$ are $\mathcal{D}^{k_0}$-tame with tame constants satisfying, for some $\sigma := \sigma(\tau,\nu,k_0) > 0$ and for all $s_0 \leqslant s \leqslant S$,*

$$\mathfrak{M}_{\mathcal{E}^{\pm1}}(s) \lesssim_S 1 + \|\mathfrak{I}_0\|^{k_0,\upsilon}_{s+\sigma}\,, \quad \mathfrak{M}_{\mathcal{Q}^{\pm1}}(s) \lesssim_S 1 + \|\mathfrak{I}_0\|^{k_0,\upsilon}_{s+\sigma}\,, \tag{4.6.91}$$

$$\mathfrak{M}_{\mathcal{E}^{\pm1}-{\rm Id}}(s) + \mathfrak{M}_{(\mathcal{E}^{\pm1}-{\rm Id})^*}(s) \lesssim_S \varepsilon^2(1 + \|\mathfrak{I}_0\|^{k_0,\upsilon}_{s+\sigma})\,, \tag{4.6.92}$$

$$\mathfrak{M}_{\mathcal{Q}^{\pm1}-{\rm Id}}(s) + \mathfrak{M}_{(\mathcal{Q}^{\pm1}-{\rm Id})^*}(s) \lesssim_S \varepsilon^2(1 + \|\mathfrak{I}_0\|^{k_0,\upsilon}_{s+\sigma})\,; \tag{4.6.93}$$

2. *the constant $\mathtt{m}_{\frac32} \in \mathbb{R}$ defined in (4.6.89) satisfies $|\mathtt{m}_{\frac32} - 1|^{k_0,\upsilon} \lesssim \varepsilon^2$;*

3. *$\mathbf{\Omega}(\kappa,D)$ is the Fourier multiplier (see (4.1.25), (4.1.26))*

$$\mathbf{\Omega}(\kappa,D) = \begin{pmatrix}\Omega(\kappa,D) & 0 \\ 0 & -\overline{\Omega(\kappa,D)}\end{pmatrix}, \quad \Omega(\kappa,D) = \omega(\kappa,D) + {\rm i}\,\frac{\gamma}{2}\partial_x^{-1}G(0)\,; \tag{4.6.94}$$

4. *the matrix of functions $\mathbf{A}_1$ is*

$$\mathbf{A}_1 := \begin{pmatrix} a_1^{(d)} & 0 \\ 0 & a_1^{(d)}\end{pmatrix}, \tag{4.6.95}$$

for a real function $a_1^{(d)}(\varphi, x)$ *which is a quasi-periodic traveling wave,* even(φ, x)*, satisfying, for some* $\sigma := \sigma(k_0, \tau, \nu) > 0$ *and for all* $s \geqslant s_0$*,*

$$\|a_1^{(d)}\|_s^{k_0,\upsilon} \lesssim_s \varepsilon(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}) ; \tag{4.6.96}$$

5. $\mathbf{R}_5^{(0,d)}$ *and* $\mathbf{R}_5^{(0,o)}$ *are pseudodifferential operators in* $\mathrm{OP}S^0$ *of the form*

$$\mathbf{R}_5^{(0,d)} := \begin{pmatrix} r_5^{(d)}(\varphi, x, D) & 0 \\ 0 & \overline{r_5^{(d)}(\varphi, x, D)} \end{pmatrix}, \quad \mathbf{R}_5^{(0,o)} := \begin{pmatrix} 0 & r_5^{(o)}(\varphi, x, D) \\ \overline{r_5^{(d)}(\varphi, x, D)} & 0 \end{pmatrix}, \tag{4.6.97}$$

reversibility and momentum preserving, satisfying, for some $\sigma_N := \sigma(\tau, \nu, N) > 0$*, for all* $s \geqslant s_0$*,* $\alpha \in \mathbb{N}_0$*,*

$$\|\mathbf{R}_5^{(0,d)}\|_{0,s,\alpha}^{k_0,\upsilon} + \|\mathbf{R}_5^{(0,o)}\|_{0,s,\alpha}^{k_0,\upsilon} \lesssim_{s,N,\alpha} \varepsilon(1 + \|\mathfrak{I}_0\|_{s+\sigma_N+2\alpha}^{k_0,\upsilon}) ; \tag{4.6.98}$$

6. *For any* $\mathtt{q} \in \mathbb{N}_0^\nu$ *with* $|\mathtt{q}| \leqslant \mathtt{q}_0$*,* $n_1, n_2 \in \mathbb{N}_0$ *with* $n_1 + n_2 \leqslant N - (k_0 + \mathtt{q}_0) + \frac{5}{2}$*, the operator* $\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathbf{T}_{5,N}(\varphi)\langle D\rangle^{n_2}$ *is* $\mathcal{D}^{k_0}$*-tame with a tame constant satisfying, for some* $\sigma_N(\mathtt{q}_0) := \sigma_N(\mathtt{q}_0, k_0, \tau, \nu) > 0$ *and for any* $s_0 \leqslant s \leqslant S$*,*

$$\mathfrak{M}_{\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathbf{T}_{5,N}(\varphi)\langle D\rangle^{n_2}}(s) \lesssim_{S,N,\mathtt{q}_0} \varepsilon\big(1 + \|\mathfrak{I}_0\|_{s+\sigma_N(\mathtt{q}_0)}^{k_0,\upsilon}\big) ; \tag{4.6.99}$$

7. *Moreover, for any* s_1 *as in* (4.6.15)*,* $\alpha \in \mathbb{N}_0$*,* $\mathtt{q} \in \mathbb{N}_0^\nu$*, with* $|\mathtt{q}| \leqslant \mathtt{q}_0$*, and* $n_1, n_2 \in \mathbb{N}_0$*, with* $n_1 + n_2 \leqslant N - \mathtt{q}_0 + \frac{3}{2}$*,*

$$\|\Delta_{12}(\mathcal{A})h\|_{s_1} \lesssim_{s_1} \varepsilon \|i_1 - i_2\|_{s_1+\sigma} \|h\|_{s_1+\sigma} , \quad \mathcal{A} \in \{\mathcal{E}^{\pm 1}, (\mathcal{E}^{\pm 1})^*, \mathcal{Q}^{\pm 1} = (\mathcal{Q}^{\pm 1})^*\}, \tag{4.6.100}$$

$$\|\Delta_{12}a_1^{(d)}\|_{s_1} \lesssim_{s_1} \varepsilon \|i_1 - i_2\|_{s_1+\sigma} , \quad |\Delta_{12}\mathtt{m}_{\frac{3}{2}}| \lesssim \varepsilon^2 \|i_1 - i_2\|_{s_1+\sigma} , \tag{4.6.101}$$

$$\|\Delta_{12}\mathbf{R}_5^{(d)}\|_{0,s_1,\alpha} + \|\Delta_{12}\mathbf{R}_5^{(o)}\|_{0,s_1,\alpha} \lesssim_{s_1,N,\alpha} \varepsilon \|i_1 - i_2\|_{s_1+\sigma_N+2\alpha} , \tag{4.6.102}$$

$$\big\|\langle D\rangle^{n_1} \partial_\varphi^{\mathtt{q}}\mathbf{T}_{5,N}(\varphi) \langle D\rangle^{n_2}\big\|_{\mathcal{L}(H^{s_1})} \lesssim_{s_1,N,\mathtt{q}_0} \varepsilon \|i_1 - i_2\|_{s_1+\sigma_N(\mathtt{q}_0)} . \tag{4.6.103}$$

The real operator $\mathcal{L}_5$ *is Hamiltonian, reversible and momentum preserving.*

Proof. By the expression of $\mathcal{L}_4$ in (4.6.82), using (4.2.15), and (4.6.87), we obtain that $\mathcal{L}_5$ has the form (4.6.90). The functions β and q, defined respectively in (4.6.68) and (4.6.66) with a_3 defined in (4.6.56), satisfy, by Lemmata 4.19, 4.13 and (4.6.28), for some $\sigma := \sigma(k_0, \tau, \nu) > 0$ and for all $s \geqslant s_0$,

$$\|\beta\|_s^{k_0,\upsilon} \lesssim_s \varepsilon^2(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}) , \quad \|q^{\pm 1} - 1\|_s^{k_0,\upsilon} \lesssim_s \varepsilon^2(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}) . \tag{4.6.104}$$

The estimates (4.6.91)-(4.6.93) follow by Lemmata 4.24, 4.25, 4.19, (4.6.104) and writing

$$(\mathcal{B}-\mathrm{Id})h = \beta\mathcal{B}_\tau[h_x]\,, \quad \mathcal{B}_\tau[h](\varphi,x) := \int_0^1 h_x(\varphi, x+\tau\beta(\varphi,x))\,\mathrm{d}\tau\,, \tag{4.6.105}$$

$\mathcal{B}^* h(\varphi,y) = (1+\breve{\beta}(\varphi,y))h(\varphi,y+\breve{\beta}(\varphi,y))$, and similar expressions for $\mathcal{B}^{-1}-\mathrm{Id}$, $(\mathcal{B}^{-1})^*$. The estimate for $\mathtt{m}_{\frac{3}{2}}$ follows by (4.6.89), Lemma 4.13 and (4.6.16). The real function $a_1^{(d)}$ in (4.6.95) is

$$a_1^{(d)}(\varphi,x) := \rho(\varphi)^{-1}a_1(\varphi,x)\,,$$

where ρ and a_1 are defined respectively in (4.6.20) and (4.6.55). Recalling Lemmata 4.68 and 4.71, the function $a_1^{(d)}$ is a quasi-periodic traveling wave, even in (φ,x). Moreover, (4.6.96) follows by Lemma 4.13 and (4.6.16), (4.6.19), (4.6.28), (4.6.104). By direct computations, we have

$$\begin{aligned} r_5^{(d)}(\varphi,x,D) &:= \frac{1}{2\rho}\left(P_0^{(41)} + P_0^{(44)} + \mathrm{i}P_{-\frac{1}{2},N}^{(43)} + \gamma(\rho\,\mathtt{m}_{\frac{3}{2}}-1)G(0)\partial_x^{-1}\right)\,, \\ r_5^{(o)}(\varphi,x,D) &:= \frac{1}{2\rho}\left(P_0^{(41)} - P_0^{(44)} + \mathrm{i}P_{-\frac{1}{2},N}^{(43)}\right)\,, \end{aligned} \tag{4.6.106}$$

where $P_0^{(41)}$, $P_{-\frac{1}{2},N}^{(43)}$, $P_0^{(44)}$ are defined in (4.6.83), (4.6.84), (4.6.85) and $\rho\,\mathtt{m}_{\frac{3}{2}} = m_{\frac{3}{2}}(\varphi)$ with $m_{\frac{3}{2}}(\varphi)$ defined in (4.6.69) (cfr. (4.6.87)). Therefore, the estimate (4.6.98) follows by (4.6.74), (4.6.72), (4.6.55), (4.6.56), (4.6.57), (4.6.58), (4.6.76), (4.6.52), (4.6.80), (4.1.21), applying Lemmata 4.16, 4.17, 4.19, 4.13, Proposition 4.20 and estimates (4.6.16), (4.6.19), (4.6.28), (4.6.104). The estimate (4.6.99), where

$$\mathbf{T}_{5,N} := \mathcal{C}^{-1}(\mathbf{R}_4^{\Psi} + \mathbf{T}_{4,N})\mathcal{C}\,,$$

follows by (4.6.86), (4.6.78), (4.6.77), (4.6.59), (4.6.53), (4.6.50), (4.6.48), (4.6.46), (4.6.43), Lemmata 4.24, 4.25, estimates (4.6.44), (4.6.49), Proposition 4.20 and (4.6.91), (4.6.104), Lemmata C.2, C.1, 4.21. The estimates (4.6.100), (4.6.101), (4.6.102), (4.6.103) are proved in the same fashion. Since the transformations $\mathcal{E}$, $\mathcal{Q}$, $\widetilde{\mathcal{M}}$ are symplectic, the operator $\mathcal{L}_4$ is Hamiltonian. Hence the operator $\mathcal{L}_5$ obtained conjugating with $\mathcal{C}$ is Hamiltonian according to (4.2.49). By Lemma 4.68, the functions $\beta(\varphi,x)$ and $q(\varphi,x)$, defined in (4.6.68), (4.6.66) (with a_3 defined in (4.6.56)), are both quasi-periodic traveling waves, respectively odd(φ,x) and even(φ,x). Therefore, the transformations $\mathcal{E}$ and $\mathcal{Q}$ are momentum and reversibility preserving. Moreover, also $\widetilde{\mathcal{M}}$ and $\mathcal{C}$ are momentum and reversibility preserving (writing the involution in complex variables as in (4.1.29)). Hence, since $\mathcal{L}_1$ is momentum preserving and reversible (Lemma 4.73), the operator $\mathcal{L}_5$ is momentum preserving and reversible as well, in particular the operators $\mathbf{R}_5^{(0,d)}$ and $\mathbf{R}_5^{(0,o)}$ in (4.6.97) (e.g. check the definition in (4.6.106), see also Remark 4.34). □

4.6.4 Symmetrization up to smoothing remainders

The goal of this section is to transform the operator $\mathcal{L}_5$ in (4.6.90) into the operator $\mathcal{L}_6$ in (4.6.109) which is block diagonal up to a regularizing remainder. From this step we do not preserve any further the Hamiltonian structure, but only the reversible and momentum preserving one (it is now sufficient for proving Theorem 4.55).

Lemma 4.75. *Fix $M, N \in \mathbb{N}$, $\mathtt{q}_0 \in \mathbb{N}_0$. There exist real, reversibility and momentum preserving operator matrices $\{\mathbf{X}_m\}_{m=1}^M$ of the form*

$$\mathbf{X}_m := \begin{pmatrix} 0 & \chi_m(\varphi, x, D) \\ \overline{\chi_m(\varphi, x, D)} & 0 \end{pmatrix}, \qquad \chi_m(\varphi, x, \xi) \in S^{-\frac{1}{2}-m}, \tag{4.6.107}$$

such that, conjugating the operator $\mathcal{L}_5$ in (4.6.90) via the map

$$\mathbf{\Phi}_M := e^{\mathbf{X}_1} \circ \cdots \circ e^{\mathbf{X}_M}, \tag{4.6.108}$$

we obtain the real, reversible and momentum preserving operator

$$\begin{aligned} \mathcal{L}_6 := \mathcal{L}_6^{(M)} &:= \mathbf{\Phi}_M^{-1} \mathcal{L}_5 \mathbf{\Phi}_M \\ &= \omega \cdot \partial_\varphi + \mathrm{i}\, \mathtt{m}_{\frac{3}{2}} \mathbf{\Omega}(\kappa, D) + \mathbf{A}_1 \partial_x + \mathrm{i}\mathbf{\Pi}_0 + \mathbf{R}_6^{(0,d)} + \mathbf{R}_6^{(-M,o)} + \mathbf{T}_{6,N}, \end{aligned} \tag{4.6.109}$$

with a block-diagonal operator

$$\mathbf{R}_6^{(0,d)} := \mathbf{R}_{6,M}^{(0,d)} := \begin{pmatrix} r_6^{(d)}(\varphi, x, D) & 0 \\ 0 & \overline{r_6^{(d)}(\varphi, x, D)} \end{pmatrix} \in \mathrm{OP}S^0,$$

and a smoothing off diagonal remainder

$$\mathbf{R}_6^{(-M,o)} := \mathbf{R}_{6,M}^{(-M,o)} := \begin{pmatrix} 0 & r_6^{(o)}(\varphi, x, D) \\ \overline{r_6^{(o)}(\varphi, x, D)} & 0 \end{pmatrix} \in \mathrm{OP}S^{-M} \tag{4.6.110}$$

both reversibility and momentum preserving, which satisfy for all $\alpha \in \mathbb{N}_0$, for some $\sigma_N := \sigma_N(k_0, \tau, \nu, N) > 0$, $\aleph_M(\alpha) > 0$, for all $s \geqslant s_0$,

$$\|\mathbf{R}_6^{(0,d)}\|_{0,s,\alpha}^{k_0,\upsilon} + \|\mathbf{R}_6^{(-M,o)}\|_{-M,s,\alpha}^{k_0,\upsilon} \lesssim_{s,M,N,\alpha} \varepsilon\big(1 + \|\mathfrak{I}_0\|_{s+\sigma_N+\aleph_M(\alpha)}^{k_0,\upsilon}\big). \tag{4.6.111}$$

For any $\mathtt{q} \in \mathbb{N}_0^\nu$ with $|\mathtt{q}| \leqslant \mathtt{q}_0$, $n_1, n_2 \in \mathbb{N}_0$ with $n_1 + n_2 \leqslant N - (k_0 + \mathtt{q}_0) + \frac{5}{2}$, the operator $\langle D\rangle^{n_1} \partial_\varphi^{\mathtt{q}} \mathbf{T}_{6,N}(\varphi) \langle D\rangle^{n_2}$ is $\mathcal{D}^{k_0}$-tame with a tame constant satisfying, for some $\sigma_N(\mathtt{q}_0) :=$

$\sigma_N(k_0, \tau, \nu, \mathtt{q}_0)$, *for any* $s_0 \leqslant s \leqslant S$,

$$\mathfrak{M}_{\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathbf{T}_{6,N}(\varphi)\langle D\rangle^{n_2}}(s) \lesssim_{S,M,N,\mathtt{q}_0} \varepsilon(1 + \|\mathfrak{I}_0\|^{k_0,\upsilon}_{s+\sigma_N(\mathtt{q}_0)+\aleph_M(0)}) . \tag{4.6.112}$$

The conjugation map $\mathbf{\Phi}_M$ *in* (4.6.108) *satisfies, for all* $s \geqslant s_0$,

$$\|\mathbf{\Phi}_M^{\pm 1} - \mathrm{Id}\|^{k_0,\upsilon}_{0,s,0} + \|\big(\mathbf{\Phi}_M^{\pm 1} - \mathrm{Id}\big)^*\|^{k_0,\upsilon}_{0,s,0} \lesssim_{s,M,N} \varepsilon(1 + \|\mathfrak{I}_0\|^{k_0,\upsilon}_{s+\sigma_N+\aleph_M(0)}) . \tag{4.6.113}$$

Furthermore, for any s_1 *as in* (4.6.15), $\alpha \in \mathbb{N}_0$, $\mathtt{q} \in \mathbb{N}_0^\nu$, *with* $|\mathtt{q}| \leqslant \mathtt{q}_0$, *and* $n_1, n_2 \in \mathbb{N}_0$, *with* $n_1 + n_2 \leqslant N - \mathtt{q}_0 + \frac{3}{2}$, *we have*

$$\|\Delta_{12}\mathbf{R}_6^{(0,d)}\|_{0,s_1,\alpha} + \|\Delta_{12}\mathbf{R}_6^{(-M,o)}\|_{-s_1,M,\alpha} \lesssim_{s_1,M,N,\alpha} \varepsilon\,\|i_1 - i_2\|_{s_1+\sigma_N+\aleph_M(\alpha)} , \tag{4.6.114}$$

$$\|\langle D\rangle^{n_1}\,\partial_\varphi^{\mathtt{q}}\Delta_{12}\mathbf{T}_{6,N}\,\langle D\rangle^{n_2}\|_{\mathcal{L}(H^{s_1})} \lesssim_{s_1,M,N,\mathtt{q}_0} \varepsilon\,\|i_1 - i_2\|_{s_1+\sigma_N(\mathtt{q}_0)+\aleph_M(0)} , \tag{4.6.115}$$

$$\|\Delta_{12}\mathbf{\Phi}_M^{\pm 1}\|_{0,s_1,0} + \|\Delta_{12}(\mathbf{\Phi}_M^{\pm 1})^*\|_{0,s_1,0} \lesssim_{s_1,M,N} \varepsilon\,\|i_1 - i_2\|_{s_1+\sigma_N+\aleph_M(0)} . \tag{4.6.116}$$

Proof. The proof is inductive on the index M. The operator $\mathcal{L}_6^{(0)} := \mathcal{L}_5$ satisfy (4.6.111)-(4.6.112) with $\aleph_0(\alpha) := 2\alpha$, by Lemma 4.74. Suppose we have done already M steps obtaining an operator $\mathcal{L}_6^{(M)}$ as in (4.6.109) with a remainder $\mathbf{\Phi}_M^{-1}\mathbf{T}_{5,N}\mathbf{\Phi}_M$, instead of $\mathbf{T}_{6,N}$. We now show how to perform the $(M+1)$-th step. Define the symbol

$$\chi_{M+1}(\varphi, x, \xi) := -\big(2\mathrm{i}\,\mathtt{m}_{\frac{3}{2}}\omega(\kappa, \xi)\big)^{-1} r_{6,M}^{(o)}(\varphi, x, \xi)\chi(\xi) \in S^{-\frac{3}{2}-M} , \tag{4.6.117}$$

where χ is the cut-off function defined in (4.2.10) and $\omega(\kappa, \xi)$ is the symbol (cfr. (4.1.23))

$$\omega(\kappa, \xi) := \sqrt{G(0;\xi)\Big(\kappa\xi^2 + g + \frac{\gamma^2}{4}\frac{G(0;\xi)}{\xi^2}\Big)} \in S^{\frac{3}{2}} , \quad G(0;\xi) := \begin{cases} \chi(\xi)|\xi|\tanh(\mathtt{h}|\xi|) , & \mathtt{h} < +\infty \\ \chi(\xi)|\xi| , & \mathtt{h} = +\infty \\ , . \end{cases}$$

Note that χ_{M+1} in (4.6.117) is well defined because $\omega(\kappa, \xi)$ is positive on the support of $\chi(\xi)$. We conjugate the operator $\mathcal{L}_6^{(M)}$ in (4.6.109) by the flow generated by $\mathbf{X}_{M+1}$ of the form (4.6.107) with $\chi_{M+1}(\varphi, x, \xi)$ defined in (4.6.117). By (4.6.111) and Lemma 4.74-2, for any $s \geqslant s_0$ and $\alpha \in \mathbb{N}_0$,

$$\|\mathbf{X}_{M+1}\|^{k_0,\upsilon}_{-\frac{1}{2}-(M+1),s,\alpha} \lesssim_{s,M,\alpha} \varepsilon\big(1 + \|\mathfrak{I}_0\|^{k_0,\upsilon}_{s+\sigma_N+\aleph_M(\alpha)}\big) . \tag{4.6.118}$$

Therefore, by Lemmata 4.18, 4.16 and the induction assumption (4.6.113) for $\mathbf{\Phi}_M$, the conjugation map $\mathbf{\Phi}_{M+1} := \mathbf{\Phi}_M e^{\mathbf{X}_{M+1}}$ is well defined and satisfies estimate (4.6.113) with $M + 1$. By the Lie expansion (4.2.16) we have

$$\mathcal{L}_6^{(M+1)} := e^{-\mathbf{X}_{M+1}} \mathcal{L}_6^{(M)} e^{\mathbf{X}_{M+1}} = \omega \cdot \partial_\varphi + \mathrm{i}\mathtt{m}_{\frac{3}{2}} \mathbf{\Omega}(\kappa, D) + \mathbf{A}_1 \partial_x + \mathrm{i}\mathbf{\Pi}_0 + \mathbf{R}_{6,M}^{(0,d)} \tag{4.6.119}$$

$$- \big[\mathbf{X}_{M+1}, \mathrm{i}\mathtt{m}_{\frac{3}{2}} \mathbf{\Omega}(\kappa, D)\big] + \mathbf{R}_{6,M}^{(-M,o)} + \mathbf{\Phi}_{M+1}^{-1} \mathbf{T}_{5,N} \mathbf{\Phi}_{M+1}$$

$$- \int_0^1 e^{-\tau \mathbf{X}_{M+1}} \big[\mathbf{X}_{M+1}\,,\, \omega \cdot \partial_\varphi + \mathbf{A}_1^{(d)} \partial_x + \mathrm{i}\mathbf{\Pi}_0 + \mathbf{R}_{6,M}^{(0,d)}\big] e^{\tau \mathbf{X}_{M+1}}\, \mathrm{d}\tau \tag{4.6.120}$$

$$- \int_0^1 e^{-\tau \mathbf{X}_{M+1}} \Big[\mathbf{X}_{M+1}, \mathbf{R}_{6,M}^{(-M,o)}\Big] e^{\tau \mathbf{X}_{M+1}}\, \mathrm{d}\tau \tag{4.6.121}$$

$$+ \int_0^1 (1-\tau) e^{-\tau \mathbf{X}_{M+1}} \Big[\mathbf{X}_{M+1}, \Big[\mathbf{X}_{M+1}, \mathrm{i}\mathtt{m}_{\frac{3}{2}} \mathbf{\Omega}(\kappa, D)\Big]\Big] e^{\tau \mathbf{X}_{M+1}}\, \mathrm{d}\tau\,. \tag{4.6.122}$$

In view of (4.6.107), (4.6.94) and (4.6.110), we have that

$$-\big[\mathbf{X}_{M+1}, \mathrm{i}\mathtt{m}_{\frac{3}{2}} \mathbf{\Omega}(\kappa, D)\big] + \mathbf{R}_{6,M}^{(-M,o)} = \begin{pmatrix} 0 & Z_{M+1} \\ \overline{Z_{M+1}} & 0 \end{pmatrix} =: \mathbf{Z}_{M+1}, ,$$

where, denoting for brevity $\chi_{M+1} := \chi_{M+1}(\varphi, x, \xi)$, it results

$$\begin{aligned} Z_{M+1} = \mathrm{i}\mathtt{m}_{\frac{3}{2}} &\left(\mathrm{Op}(\chi_{M+1}) \omega(\kappa, D) + \omega(\kappa, D) \mathrm{Op}(\chi_{M+1}) \right) \\ &+ \mathtt{m}_{\frac{3}{2}} \tfrac{\gamma}{2} \left[\chi_{M+1}, \partial_x^{-1} G(0)\right] + \mathrm{Op}(r_{6,M}^{(o)})\,. \end{aligned} \tag{4.6.123}$$

By (4.2.23), Lemma 4.16 and since $\chi_{M+1}(\varphi, x, \xi) \in S^{-\frac{3}{2}-M}$ by (4.6.117), we have that

$$\mathrm{Op}(\chi_{M+1}) \omega(\kappa, D) + \omega(\kappa, D) \mathrm{Op}(\chi_{M+1}) = \mathrm{Op}\big(2\omega(\kappa, \xi) \chi_{M+1}(\varphi, x, \xi)\big) + \mathtt{r}_{M+1}, ,$$

where $\mathtt{r}_{M+1}$ is in $\mathrm{OP}S^{-M-1}$. By (4.6.117) and (4.6.123)

$$Z_{M+1} = \mathrm{i}\mathtt{m}_{\frac{3}{2}} \mathtt{r}_{M+1} + \mathtt{m}_{\frac{3}{2}} \tfrac{\gamma}{2} \left[\chi_{M+1}, \partial_x^{-1} G(0)\right] + \mathrm{Op}(r_{6,M}^{(o)}(1 - \chi(\xi))) \in \mathrm{OP}S^{-M-1}\,.$$

The remaining pseudodifferential operators in (4.6.120)-(4.6.122) have order $\mathrm{OP}S^{-M-\frac{3}{2}}$. Therefore the operator $\mathcal{L}_6^{(M+1)}$ in (4.6.119) has the form (4.6.109) at $M+1$ with

$$\mathbf{R}_{6,M+1}^{(0,d)} + \mathbf{R}_{6,M+1}^{(-(M+1),o)} := \mathbf{R}_{6,M}^{(0,d)} + \mathbf{Z}_{M+1} + (4.6.120) + (4.6.121) + (4.6.122) \tag{4.6.124}$$

and a remainder $\mathbf{\Phi}_{M+1}^{-1} \mathbf{T}_{5,N} \mathbf{\Phi}_{M+1}$. By Lemmata 4.16, 4.17, the induction assumption (4.6.111), (4.6.118), (4.6.96), we conclude that $\mathbf{R}_{6,M+1}^{(0,d)}$ and $\mathbf{R}_{6,M+1}^{(-(M+1),o)}$ satisfy (4.6.111) at order $M+1$ for suitable constants $\aleph_{M+1}(\alpha) > \aleph_M(\alpha)$. Moreover the operator $\mathbf{\Phi}_{M+1}^{-1} \mathbf{T}_{5,N} \mathbf{\Phi}_{M+1}$ satisfies (4.6.113) (with $M+1$) by Lemmata 4.24, 4.25 and estimates (4.6.99), (4.6.113). Estimates (4.6.114), (4.6.115), (4.6.116) follow similarly. By (4.6.117), (4.2.51), Lemmata 4.33, 4.43, and the induction assumption that $\mathbf{R}_{6,M}^{(-M,o)}$ is reversible and momentum preserving, we conclude that $\mathbf{X}_{M+1}$ is reversibility and momentum preserving, and so are $e^{\pm\mathbf{X}_{M+1}}$. By the induction assumption $\mathcal{L}_6^{(M)}$ is reversible and momentum preserving, and so $\mathcal{L}_6^{(M+1)}$ is reversible and momentum preserving as well, in particular the terms $\mathbf{R}_{6,M+1}^{(0,d)} + \mathbf{R}_{6,M+1}^{(-(M+1),o)}$ in (4.6.124). □

Remark 4.76. The number of regularizing iterations $M \in \mathbb{N}$ will be fixed by the KAM reduction scheme in Section 4.7, see (4.7.5). Note that it is independent of the Sobolev index s.

So far the operator $\mathcal{L}_6$ of Lemma 4.75 depends on two indexes M, N which provide respectively the order of the regularizing off-diagonal remainder $\mathbf{R}_6^{(-M,o)}$ and of the smoothing tame operator $\mathbf{T}_{6,N}$. From now on we fix

$$N = M\,. \tag{4.6.125}$$

4.6.5 Reduction of the order 1

The goal of this section is to transform the operator $\mathcal{L}_6$ in (4.6.109), with $N = M$ (cfr. (4.6.125)), into the operator $\mathcal{L}_8$ in (4.6.147) whose coefficient in front of ∂_x is a constant. We first eliminate the x-dependence and then the φ-dependence.

Space reduction. First we rewrite the operator $\mathcal{L}_6$ in (4.6.109), with $N = M$, as

$$\mathcal{L}_6 = \omega \cdot \partial_\varphi + \begin{pmatrix} P_6 & 0 \\ 0 & \overline{P_6} \end{pmatrix} + \mathrm{i}\boldsymbol{\Pi}_0 + \mathbf{R}_6^{(-M,o)} + \mathbf{T}_{6,M}\,,$$

having denoted

$$P_6 := P_6(\varphi, x, D) := \mathrm{i}\mathtt{m}_{\frac{3}{2}}\Omega(\kappa, D) + a_1^{(d)}(\varphi, x)\partial_x + r_6^{(d)}(\varphi, x, D)\,. \tag{4.6.126}$$

We conjugate $\mathcal{L}_6$ through the real operator

$$\boldsymbol{\Phi}(\varphi) := \begin{pmatrix} \Phi(\varphi) & 0 \\ 0 & \overline{\Phi}(\varphi) \end{pmatrix} \tag{4.6.127}$$

where $\Phi(\varphi) := \Phi^\tau(\varphi)|_{\tau=1}$ is the time 1-flow of the PDE

$$\begin{cases} \partial_\tau \Phi^\tau(\varphi) = \mathrm{i}A(\varphi)\Phi^\tau(\varphi)\,, \\ \Phi^0(\varphi) = \mathrm{Id}\,, \end{cases} \qquad A(\varphi) := b(\varphi, x)|D|^{\frac{1}{2}}\,, \tag{4.6.128}$$

and $b(\varphi, x)$ is a real, smooth, odd (φ, x), periodic function chosen later, see (4.6.134), (4.6.136), (4.6.142). Usual energy estimates imply that the flow $\Phi^\tau(\varphi)$ of (4.6.128) is a bounded operator is H^s_x. The operator $\partial_\lambda^k \partial_\varphi^\beta \Phi$ loses $|D|^{\frac{|\beta|+|k|}{2}}$ derivatives, which are compensated by $\langle D \rangle^{-m_1}$ on the left hand side and $\langle D\rangle^{-m_2}$ on the right hand side, with $m_1, m_2 \in \mathbb{R}$ satisfying $m_1+m_2 = \frac{1}{2}(|\beta|+|k|)$, according to the tame estimates in the Sobolev spaces $H^s_{\varphi,x}$ of Proposition 4.29 in Appendix 4.2.3. Moreover, since $b(\varphi,x)$ is odd(φ,x), then $b(\varphi,x)|D|^{\frac{1}{2}}$ is reversibility preserving as well as $\boldsymbol{\Phi}(\varphi)$. Finally, note that $\Phi\pi_0 = \pi_0 = \Phi^{-1}\pi_0$, which implies

$$\boldsymbol{\Phi}^{-1}\boldsymbol{\Pi}_0\boldsymbol{\Phi} = \boldsymbol{\Pi}_0\boldsymbol{\Phi}\,. \tag{4.6.129}$$

By the Lie expansion (4.2.16) we have

$$\begin{aligned} \Phi^{-1}P_6\Phi &= P_6 - \mathrm{i}[A,P_6] - \frac{1}{2}[A,[A,P_6]] + \sum_{n=3}^{2M+2}\frac{(-\mathrm{i})^n}{n!}\mathrm{ad}^n_{A(\varphi)}(P_6) + T_M\,,\\ T_M &:= \frac{(-\mathrm{i})^{2M+3}}{(2M+2)!}\int_0^1 (1-\tau)^{2M+2}\Phi^{-\tau}(\varphi)\,\mathrm{ad}^{2M+3}_{A(\varphi)}(P_6)\,\Phi^{\tau}(\varphi)\mathrm{d}\tau\,, \end{aligned} \tag{4.6.130}$$

and, by (4.2.17),

$$\begin{aligned} \Phi^{-1}\circ\omega\cdot\partial_\varphi\circ\Phi &= \omega\cdot\partial_\varphi + \mathrm{i}(\omega\cdot\partial_\varphi A)(\varphi) - \sum_{n=2}^{2M+1}\frac{(-\mathrm{i})^n}{n!}\mathrm{ad}^{n-1}_{A(\varphi)}(\omega\cdot\partial_\varphi A(\varphi)) + T'_M\,,\\ T'_M &:= -\frac{(-\mathrm{i})^{2M+2}}{(2M+1)!}\int_0^1 (1-\tau)^{2M+1}\Phi^{-\tau}(\varphi)\,\mathrm{ad}^{2M+1}_{A(\varphi)}(\omega\cdot\partial_\varphi A(\varphi))\,\Phi^{\tau}(\varphi)\mathrm{d}\tau\,. \end{aligned} \tag{4.6.131}$$

Note that $\mathrm{ad}^{2M+3}_{A(\varphi)}(P_6)$ and $\mathrm{ad}^{2M+1}_{A(\varphi)}(\omega\cdot\partial_\varphi A(\varphi))$ are in $\mathrm{OP}S^{-M}$. The number M will be fixed in (4.7.5). Note also that in the expansions (4.6.130), (4.6.131) the operators have decreasing order and size. The terms of order 1 come from (4.6.130), in particular from $P_6 - \mathrm{i}[A,P_6]$. Recalling (4.6.126), that $A(\varphi) := b(\varphi,x)|D|^{\frac{1}{2}}$, (4.2.26) and that (cfr. (4.3.1), (4.3.5))

$$\Omega(\kappa,\xi) = \sqrt{\kappa}|\xi|^{\frac{3}{2}}\chi(\xi) + r_0(\kappa,\xi)\,, \quad r_0(\kappa,\xi)\in S^0\,, \tag{4.6.132}$$

(the cut-off function χ is defined in (4.2.10)) we deduce that

$$[A,P_6] = \mathrm{i}\tfrac{3}{2}\sqrt{\kappa}\,\mathtt{m}_{\frac{3}{2}}\,b_x\partial_x + \big(\tfrac{1}{2}(a_1^{(d)})_x b - a_1^{(d)}b_x\big)|D|^{\frac{1}{2}} + \mathrm{Op}(r_{b,0}),\,, \tag{4.6.133}$$

where $r_{b,0}\in S^0$ is small with b. As a consequence, the first order term of $P_6 - \mathrm{i}[A,P_6]$ is $(a_1^{(d)} + \frac{3}{2}\sqrt{\kappa}\,\mathtt{m}_{\frac{3}{2}}\,b_x)\partial_x$ and we choose $b(\varphi,x)$ so that it is independent of x: we look for a solution

$$b(\varphi,x) = b_1(\varphi,x) + b_2(\varphi) \tag{4.6.134}$$

of the equation

$$a_1^{(d)}(\varphi,x) + \tfrac{3}{2}\mathtt{m}_{\frac{3}{2}}\sqrt{\kappa}\,b_x(\varphi,x) = \langle a_1^{(d)}\rangle_x(\varphi)\,, \quad \langle a_1^{(d)}\rangle_x(\varphi) := \frac{1}{2\pi}\int_{\mathbb{T}} a_1^{(d)}(\varphi,x)\mathrm{d}x\,. \tag{4.6.135}$$

Therefore

$$b_1(\varphi, x) := -\tfrac{2}{3\mathtt{m}_{\frac{3}{2}}\sqrt{\kappa}}\,\partial_x^{-1}\big(a_1^{(d)}(\varphi, x) - \langle a_1^{(d)}\rangle_x(\varphi)\big)\,. \tag{4.6.136}$$

We now determine $b_2(\varphi)$ by imposing a condition at the order 1/2. We deduce by (4.6.130), (4.6.131), (4.6.126), (4.6.133)-(4.6.135), that

$$\begin{aligned} L_7 := \Phi^{-1}(\varphi)\,(\omega\cdot\partial_\varphi + P_6)\,\Phi(\varphi) = {} & \omega\cdot\partial_\varphi + \mathrm{i}\,\mathtt{m}_{\frac{3}{2}}\Omega(\kappa, D) + \langle a_1^{(d)}\rangle_x(\varphi)\,\partial_x \\ & + \mathrm{i}\,a_2^{(d)}|D|^{\frac{1}{2}} + \mathrm{Op}(r_7^{(d)}) + T_M + T_M',, \end{aligned} \tag{4.6.137}$$

where $a_2^{(d)}(\varphi, x)$ is the real function

$$\begin{aligned} a_2^{(d)} := {} & -\tfrac{1}{2}(a_1^{(d)})_x b_1 + a_1^{(d)}(b_1)_x + \frac{3}{4}\sqrt{\kappa}\,\mathtt{m}_{\frac{3}{2}}\big((b_1)_x^2 - \frac{1}{2}(b_1)_{xx}b_1\big) + (\omega\cdot\partial_\varphi b_1) \\ & - \big(\tfrac{1}{2}(a_1^{(d)})_x + \frac{3}{8}\sqrt{\kappa}\,\mathtt{m}_{\frac{3}{2}}(b_1)_{xx}\big)b_2 + (\omega\cdot\partial_\varphi b_2) \end{aligned} \tag{4.6.138}$$

and

$$\begin{aligned} \mathrm{Op}(r_7^{(d)}) := {} & \mathrm{Op}(-\mathrm{i}r_{b,0} + r_{b,-\frac{1}{2}} + r_6^{(d)}) - \frac{1}{2}\big[b|D|^{\frac{1}{2}}, (\tfrac{1}{2}(a_1^{(d)})_x b - a_1^{(d)}b_x)|D|^{\frac{1}{2}} + \mathrm{Op}(r_{b,0})\big] \\ & + \sum_{n=3}^{M-1}\frac{(-\mathrm{i})^n}{n!}\mathrm{ad}_{A(\varphi)}^n(P_6) - \sum_{n=2}^{M}\frac{(-\mathrm{i})^n}{n!}\mathrm{ad}_{A(\varphi)}^{n-1}(\omega\cdot\partial_\varphi A(\varphi)) \in \mathrm{OP}S^0\,, \end{aligned} \tag{4.6.139}$$

where $r_{b,-\frac{1}{2}} \in S^{-\frac{1}{2}}$ is small in b. In view of Section 4.6.6 we now determine the function $b_2(\varphi)$ so that the space average of the function $a_2^{(d)}$ in (4.6.138) is independent of φ, i.e.

$$\langle a_2^{(d)}\rangle_x(\varphi) = \mathtt{m}_{\frac{1}{2}} \in \mathbb{R}\,, \quad \forall\varphi \in \mathbb{T}^\nu\,. \tag{4.6.140}$$

Noting that the space average $\big\langle\big(\frac{1}{2}(a_1^{(d)})_x + \frac{3}{8}\mathtt{m}_{\frac{3}{2}}\sqrt{\kappa}(b_1)_{xx}\big)b_2(\varphi)\big\rangle_x = 0$ and that $\big\langle\omega\cdot\partial_\varphi b_1\big\rangle_{\varphi,x} = 0$, we get

$$\mathtt{m}_{\frac{1}{2}} := \langle -\tfrac{1}{2}(a_1^{(d)})_x b_1 + a_1^{(d)}(b_1)_x + \frac{3}{4}\sqrt{\kappa}\,\mathtt{m}_{\frac{3}{2}}\big((b_1)_x^2 - \frac{1}{2}(b_1)_{xx}b_1\big)\rangle_{\varphi,x}\,, \tag{4.6.141}$$

$$\begin{aligned} b_2(\varphi) := {} & -(\omega\cdot\partial_\varphi)_{\mathrm{ext}}^{-1}\Big(\big\langle -\tfrac{1}{2}(a_1^{(d)})_x b_1 + a_1^{(d)}(b_1)_x + \\ & + \frac{3}{4}\mathtt{m}_{\frac{3}{2}}\sqrt{\kappa}\big((b_1)_x^2 - \frac{1}{2}(b_1)_{xx}b_1\big) + (\omega\cdot\partial_\varphi b_1)\big\rangle_x - \mathtt{m}_{\frac{1}{2}}\Big)\,. \end{aligned} \tag{4.6.142}$$

Note that (4.6.140) holds for any $\omega \in \mathtt{DC}(\upsilon, \tau)$.

Time reduction. In order to remove the φ-dependence of the coefficient $\langle a_1^{(d)} \rangle_x (\varphi)$ of the first order term of the operator L_7 in (4.6.137), we conjugate L_7 with the map

$$(\mathcal{V}u)(\varphi, x) := u(\varphi, x + \varrho(\varphi)), , \tag{4.6.143}$$

where $\varrho(\varphi)$ is a real periodic function to be chosen, see (4.6.146). Note that $\mathcal{V}$ is a particular case of the transformation $\mathcal{E}$ in (4.6.34) for a function $\beta(\varphi, x) = \varrho(\varphi)$, independent of x. We have that

$$\mathcal{V}^{-1}(\omega \cdot \partial_\varphi)\mathcal{V} = \omega \cdot \partial_\varphi + (\omega \cdot \partial_\varphi \varrho)\partial_x ,$$

whereas the Fourier multipliers are left unchanged and a pseudodifferential operator of symbol $a(\varphi, x, \xi)$ transforms as

$$\mathcal{V}^{-1}\mathrm{Op}(a(\varphi, x, \xi))\mathcal{V} = \mathrm{Op}(a(\varphi, x - \varrho(\varphi), \xi)) . \tag{4.6.144}$$

We choose $\varrho(\varphi)$ such that

$$\omega \cdot \partial_\varphi \varrho(\varphi) + \langle a_1^{(d)} \rangle_x (\varphi) = \mathtt{m}_1 , \qquad \mathtt{m}_1 := \langle a_1^{(d)} \rangle_{\varphi, x} \in \mathbb{R} , \tag{4.6.145}$$

(where $a_1^{(d)}$ is fixed in Lemma 4.74), namely we define

$$\varrho(\varphi) := -(\omega \cdot \partial_\varphi)^{-1}_{\mathrm{ext}} \big(\langle a_1^{(d)} \rangle_x - \mathtt{m}_1 \big) . \tag{4.6.146}$$

Note that (4.6.145) holds for any $\omega \in \mathtt{DC}(\upsilon, \tau)$.

We sum up these two transformations into the following lemma.

Lemma 4.77. *Let $M \in \mathbb{N}$, $\mathtt{q}_0 \in \mathbb{N}_0$. Let $b(\varphi, x) = b_1(\varphi, x) + b_2(\varphi)$ and $\varrho(\varphi)$ be the functions defined respectively in* (4.6.136), (4.6.142), (4.6.146). *Then, conjugating $\mathcal{L}_6$ in* (4.6.109) *via the invertible, real, reversibility preserving and momentum preserving maps $\boldsymbol{\Phi}$, $\mathcal{V}$ defined in* (4.6.127)-(4.6.128) *and* (4.6.143), *we obtain, for any $\omega \in \mathtt{DC}(\upsilon, \tau)$, the real, reversible and momentum preserving operator*

$$\begin{aligned} \mathcal{L}_8 &:= \mathcal{V}^{-1}\boldsymbol{\Phi}^{-1}\mathcal{L}_6\boldsymbol{\Phi}\mathcal{V} \\ &= \omega \cdot \partial_\varphi + \mathrm{i}\,\mathtt{m}_{\frac{3}{2}}\boldsymbol{\Omega}(\kappa, D) + \mathtt{m}_1\partial_x + \mathrm{i}\mathbf{A}_3^{(d)}|D|^{\frac{1}{2}} + \mathrm{i}\boldsymbol{\Pi}_0 + \mathbf{R}_8^{(0,d)} + \mathbf{T}_{8,M} , \end{aligned} \tag{4.6.147}$$

where:

1. *the real constant $\mathtt{m}_1$ defined in* (4.6.145) *satisfies $|\mathtt{m}_1|^{k_0,\upsilon} \lesssim \varepsilon$;*

2. $\mathbf{A}_3^{(d)}$ *is a diagonal matrix of multiplication*

$$\mathbf{A}_3^{(d)} := \begin{pmatrix} a_3^{(d)} & 0 \\ 0 & a_3^{(d)} \end{pmatrix},$$

for a real function $a_3^{(d)}$ *which is a quasi-periodic traveling wave,* $\mathrm{even}(\varphi, x)$*, satisfying*

$$\langle a_3^{(d)} \rangle_x (\varphi) = \mathtt{m}_{\frac{1}{2}} \in \mathbb{R}, \quad \forall \varphi \in \mathbb{T}^\nu, \tag{4.6.148}$$

where $\mathtt{m}_{\frac{1}{2}} \in \mathbb{R}$ *is the constant in* (4.6.141)*, and for some* $\sigma = \sigma(\tau, \nu, k_0) > 0$*, for all* $s \geqslant s_0$*,*

$$\|a_3^{(d)}\|_s^{k_0,\upsilon} \lesssim_s \varepsilon \upsilon^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}); \tag{4.6.149}$$

3. $\mathbf{R}_8^{(0,d)}$ *is a block-diagonal operator*

$$\mathbf{R}_8^{(0,d)} = \begin{pmatrix} r_8^{(d)}(\varphi, x, D) & 0 \\ 0 & \overline{r_8^{(d)}(\varphi, x, D)} \end{pmatrix} \in \mathrm{OP}S^0,$$

that satisfies for all $\alpha \in \mathbb{N}_0$*, for some* $\sigma_M(\alpha) := \sigma_M(k_0, \tau, \nu, \alpha) > 0$ *and for all* $s \geqslant s_0$*,*

$$\|\mathbf{R}_8^{(0,d)}\|_{0,s,\alpha}^{k_0,\upsilon} \lesssim_{s,M,\alpha} \varepsilon \upsilon^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma_M(\alpha)}^{k_0,\upsilon}) ; \tag{4.6.150}$$

4. *For any* $\mathtt{q} \in \mathbb{N}_0^\nu$ *with* $|\mathtt{q}| \leqslant \mathtt{q}_0$*,* $n_1, n_2 \in \mathbb{N}_0$ *with* $n_1 + n_2 \leqslant M - 2(k_0 + \mathtt{q}_0) + \frac{5}{2}$*, the operator* $\langle D \rangle^{n_1} \partial_\varphi^{\mathtt{q}} \mathbf{T}_{8,M}(\varphi) \langle D \rangle^{n_2}$ *is* $\mathcal{D}^{k_0}$*-tame with a tame constant satisfying, for some* $\sigma_M(\mathtt{q}_0) := \sigma_M(k_0, \tau, \nu, \mathtt{q}_0)$*, for any* $s_0 \leqslant s \leqslant S$*,*

$$\mathfrak{M}_{\langle D \rangle^{n_1} \partial_\varphi^{\mathtt{q}} \mathbf{T}_{8,M}(\varphi) \langle D \rangle^{n_2}}(s) \lesssim_{S,M,\mathtt{q}_0} \varepsilon \upsilon^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma_M(\mathtt{q}_0)}^{k_0,\upsilon}); \tag{4.6.151}$$

5. *The operators* $\mathbf{\Phi}^{\pm 1} - \mathrm{Id}$*,* $(\mathbf{\Phi}^{\pm 1} - \mathrm{Id})^*$ *are* $\mathcal{D}^{k_0}$*-*$\frac{1}{2}(k_0+1)$*-tame and the operators* $\mathcal{V}^{\pm 1} - \mathrm{Id}$*,* $(\mathcal{V}^{\pm 1} - \mathrm{Id})^*$ *are* $\mathcal{D}^{k_0}$*-*(k_0+2)*-tame, with tame constants satisfying, for some* $\sigma > 0$ *and for all* $s_0 \leqslant s \leqslant S$*,*

$$\mathfrak{M}_{\mathbf{\Phi}^{\pm 1} - \mathrm{Id}}(s) + \mathfrak{M}_{(\mathbf{\Phi}^{\pm 1} - \mathrm{Id})^*}(s) \lesssim_S \varepsilon \upsilon^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}), \tag{4.6.152}$$

$$\mathfrak{M}_{\mathcal{V}^{\pm 1} - \mathrm{Id}}(s) + \mathfrak{M}_{(\mathcal{V}^{\pm 1} - \mathrm{Id})^*}(s) \lesssim_S \varepsilon \upsilon^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}). \tag{4.6.153}$$

Furthermore, for any s_1 *as in* (4.6.15)*,* $\alpha \in \mathbb{N}_0$*,* $\mathtt{q} \in \mathbb{N}_0^\nu$*, with* $|\mathtt{q}| \leqslant \mathtt{q}_0$*, and* $n_1, n_2 \in \mathbb{N}_0$*, with*

$n_1 + n_2 \leqslant M - 2\mathtt{q}_0 + \frac{1}{2}$, *we have*

$$\|\Delta_{12} a_3^{(d)}\|_{s_1} \lesssim_{s_1} \varepsilon \upsilon^{-1} \|i_1 - i_2\|_{s_1+\sigma} \,, \ |\Delta_{12}\mathtt{m}_1| \lesssim \varepsilon \|i_1 - i_2\|_{s_0+\sigma} \,, \tag{4.6.154}$$

$$\|\Delta_{12}\mathbf{R}_8^{(0,d)}\|_{0,s_1,\alpha} \lesssim_{s_1,M,\alpha} \varepsilon \upsilon^{-1} \|i_1 - i_2\|_{s_1+\sigma_M(\alpha)} \,, \tag{4.6.155}$$

$$\| \langle D\rangle^{n_1} \partial_\varphi^{\mathtt{q}} \Delta_{12}\mathbf{T}_{8,M} \langle D\rangle^{n_2} \|_{\mathcal{L}(H^{s_1})} \lesssim_{s_1,M,\mathtt{q}_0} \varepsilon \upsilon^{-1} \|i_1 - i_2\|_{s_1+\sigma_M(\mathtt{q}_0)} \,, \tag{4.6.156}$$

$$\|\Delta_{12}(\mathcal{A})h\|_{s_1} \lesssim_{s_1} \varepsilon \upsilon^{-1} \|i_1 - i_2\|_{s_1+\sigma} \|h\|_{s_1+\sigma} \,, \quad \mathcal{A} \in \{\mathbf{\Phi}^{\pm 1}, (\mathbf{\Phi}^{\pm 1})^*, \mathcal{V}^{\pm 1}, (\mathcal{V}^{\pm 1})^*\} \,. \tag{4.6.157}$$

Proof. The function $b(\varphi, x) = b_1(\varphi, x) + b_2(\varphi)$, with b_1 and b_2, defined in (4.6.136) and (4.6.142) and the function $\varrho(\varphi)$ in (4.6.146), satisfy, by Lemma 4.19 and (4.6.96),

$$\|b_1\|_s^{k_0,\upsilon} \lesssim_s \varepsilon(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}) \,, \quad \|b\|_s^{k_0,\upsilon} \,, \ \|b_2\|_s^{k_0,\upsilon} \,, \ \|\varrho\|_s^{k_0,\upsilon} \lesssim_s \varepsilon \upsilon^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}) \tag{4.6.158}$$

for some $\sigma > 0$ and for all $s \geqslant s_0$. The estimate $|\mathtt{m}_1|^{k_0,\upsilon} \lesssim \varepsilon$ follows by (4.6.145) and (4.6.96). The function

$$a_3^{(d)}(\varphi, x) := \mathcal{V}^{-1}(a_2^{(d)}) = a_2^{(d)}(\varphi, x - \varrho(\varphi)), ,$$

where $a_2^{(d)}$ is defined in (4.6.138), satisfies (4.6.148) by (4.6.140). Moreover, the estimate (4.6.149) follows by Lemma 4.19 and (4.6.96), (4.6.158). The estimate (4.6.150) for (cfr. (4.6.144))

$$r_8^{(d)}(\varphi, x, D) := \mathcal{V}^{-1} r_7^{(d)}(\varphi, x, D)\mathcal{V} = r_7^{(d)}(\varphi, x - \varrho(\varphi), D)$$

with $r_7^{(d)}$ defined in (4.6.139), follows by Lemmata 4.16, 4.17, 4.19 and (4.6.158), (4.6.111). The smoothing term $\mathbf{T}_{8,M}$ in (4.6.147) is, using also (4.6.129),

$$\mathbf{T}_{8,M} := \mathcal{V}^{-1}\big(\mathbf{\Phi}^{-1}\mathbf{T}_{6,M}\mathbf{\Phi} + \mathrm{i}\mathbf{\Pi}_0(\mathbf{\Phi} - \mathrm{Id}) + \mathbf{\Phi}^{-1}\mathbf{R}_6^{(-M,o)}\mathbf{\Phi}\big)\mathcal{V} + \mathcal{V}^{-1}\begin{pmatrix} T_M + T_M' & 0 \\ 0 & \overline{T_M} + \overline{T_M'} \end{pmatrix}\mathcal{V}$$

with T_M and T_M' defined in (4.6.130), (4.6.131). The estimate (4.6.151) follows by (4.6.126), Lemmata 4.24, 4.25, the tame estimates of $\mathbf{\Phi}$ in Proposition 4.29, and estimates (4.6.96), (4.6.158), (4.6.152), (4.6.112), noting that operators of the form $\partial_\lambda^k \partial_\varphi^{\mathtt{q}} \mathcal{V}^{\pm 1}$ lose $|k| + |\mathtt{q}|$ derivatives. The estimate (4.6.152) follows by Proposition 4.30 and (4.6.158), whereas (4.6.153) follows by the equivalent representation for $\mathcal{V}$ as in (4.6.105), Lemma 4.24 and (4.6.158). The estimates (4.6.154), (4.6.155), (4.6.156), (4.6.157) are proved in the same fashion. By Lemma 4.74, the function $a_1^{(d)}$ is an even(φ, x) quasi-periodic traveling wave, hence the function b_1 in (4.6.136) is a odd(φ, x) quasi-periodic traveling wave, the function b_2 in (4.6.142) is odd in φ and satisfies $b_2(\varphi - \vec{\jmath}\varsigma) = b_2(\varphi)$ for all $\varsigma \in \mathbb{R}$, whereas the function ϱ in (4.6.146) is odd in φ and satisfies $\varrho(\varphi - \vec{\jmath}\varsigma) = \varrho(\varphi)$ for all $\varsigma \in \mathbb{R}$. By Lemmata 4.33, 4.43, and 4.39, the transformations $\mathbf{\Phi}$ and $\mathcal{V}$ are reversibility and momentum preserving. Then the operator $\mathcal{L}_8$ is reversible and momentum preserving. The function $a_3^{(d)}$ is an even(φ, x) quasi-periodic traveling wave. □

4.6.6 Reduction of the order 1/2

The goal of this section is to transform the operator $\mathcal{L}_8$ in (4.6.147) into the operator $\mathcal{L}_9$ in (4.6.169) whose coefficient in front of $|D|^{1/2}$ is a constant. We eliminate the x-dependence and, in view of the property (4.6.148), we obtain that this transformation removes also the φ-dependence.

We first write the operator $\mathcal{L}_8$ in (4.6.147) as

$$\mathcal{L}_8 = \omega \cdot \partial_\varphi + \begin{pmatrix} P_8 & 0 \\ 0 & \overline{P_8} \end{pmatrix} + \mathrm{i}\mathbf{\Pi}_0 + \mathbf{T}_{8,M} \,,$$

where

$$P_8 := \mathrm{i}\mathtt{m}_{\frac{3}{2}} \Omega(\kappa, D) + \mathtt{m}_1 \partial_x + \mathrm{i} a_3^{(d)} |D|^{\frac{1}{2}} + \mathrm{Op}(r_8^{(d)}) \,. \tag{4.6.159}$$

We conjugate $\mathcal{L}_8$ through the real operator

$$\mathbf{\Psi}(\varphi) := \begin{pmatrix} \Psi(\varphi) & 0 \\ 0 & \overline{\Psi}(\varphi) \end{pmatrix} , \tag{4.6.160}$$

where $\Psi(\varphi) := \Psi^\tau(\varphi)|_{\tau=1}$ is the time-1 flow of

$$\begin{cases} \partial_\tau \Psi^\tau(\varphi) = B(\varphi)\Psi^\tau(\varphi) \,, \\ \Psi^0(\varphi) = \mathrm{Id} \,, \end{cases} \qquad B(\varphi) := b_3(\varphi, x)\mathcal{H} \,, \tag{4.6.161}$$

the function $b_3(\varphi, x)$ is a smooth, real, periodic function to be chosen later (see (4.6.166)) and $\mathcal{H}$ is the Hilbert transform defined in (4.2.19). Note that $\Psi \pi_0 = \pi_0 = \Psi^{-1}\pi_0$, so that

$$\mathbf{\Psi}^{-1}\mathbf{\Pi}_0 \mathbf{\Psi} = \mathbf{\Pi}_0 \mathbf{\Psi} \,. \tag{4.6.162}$$

By the Lie expansion in (4.2.16) we have

$$\begin{aligned} \Psi^{-1} P_8 \Psi &= P_8 - [B, P_8] + \sum_{n=2}^{M+1} \frac{(-1)^n}{n!} \mathrm{ad}^n_{B(\varphi)}(P_8) + L_M \,, \\ L_M &:= \frac{(-1)^{M+2}}{(M+1)!} \int_0^1 (1-\tau)^{M+1} \Psi^{-\tau}(\varphi) \, \mathrm{ad}^{M+2}_{B(\varphi)}(P_8) \, \Psi^\tau(\varphi) \mathrm{d}\tau \,, \end{aligned} \tag{4.6.163}$$

and, by (4.2.17),

$$\begin{aligned}\Psi^{-1}\circ\omega\cdot\partial_\varphi\circ\Psi&=\omega\cdot\partial_\varphi+(\omega\cdot\partial_\varphi B(\varphi))-\sum_{n=2}^{M}\frac{(-1)^n}{n!}\mathrm{ad}^{n-1}_{B(\varphi)}(\omega\cdot\partial_\varphi B(\varphi))+L'_M\,,\\ L'_M&:=\frac{(-1)^M}{M!}\int_0^1(1-\tau)^M\Psi^{-\tau}(\varphi)\,\mathrm{ad}^M_{B(\varphi)}(\omega\cdot\partial_\varphi B(\varphi))\,\Psi^{\tau}(\varphi)\mathrm{d}\tau\,.\end{aligned}\tag{4.6.164}$$

The number M will be fixed in (4.7.5). The contributions at order 1/2 come from (4.6.163), in particular from $P_8-[B,P_8]$ (recall (4.6.159)). Since $B=b_3\mathcal{H}$ (see (4.6.161)), by (4.2.26) and (4.6.132) we have

$$\begin{aligned}P_8-[B,P_8]=&\,\mathrm{i}\,\mathtt{m}_{\frac32}\Omega(\kappa,D)+\mathtt{m}_1\partial_x+\mathrm{i}\left(a_3^{(d)}-\tfrac32\mathtt{m}_{\frac32}\sqrt{\kappa}(b_3)_x\right)|D|^{\frac12}\\&+\mathrm{Op}(r_8^{(d)}+r_{b_3,-\frac12})-[B,\mathtt{m}_1\partial_x+\mathrm{i}\,a_3^{(d)}|D|^{\frac12}+\mathrm{Op}(r_8^{(d)})],\,,\end{aligned}\tag{4.6.165}$$

where $\mathrm{Op}(r_{b_3,-\frac12})\in\mathrm{OP}S^{-\frac12}$ is small with b_3. Recalling that, by (4.6.148), the space average $\langle a_3^{(d)}\rangle_x(\varphi)=\mathtt{m}_{\frac12}$ for all $\varphi\in\mathbb{T}^\nu$, we choose the function $b_3(\varphi,x)$ such that $a_3^{(d)}-\frac32\mathtt{m}_{\frac32}\sqrt{\kappa}(b_3)_x=\mathtt{m}_{\frac12}$, namely

$$b_3(\varphi,x):=\tfrac{2}{3\mathtt{m}_{\frac32}\sqrt{\kappa}}\partial_x^{-1}(a_3^{(d)}(\varphi,x)-\langle a_3^{(d)}\rangle_x(\varphi))\,,\quad\langle a_3^{(d)}\rangle_x(\varphi)=\mathtt{m}_{\frac12}\,.\tag{4.6.166}$$

We deduce by (4.6.163)-(4.6.164) and (4.6.165), (4.6.166) that

$$\begin{aligned}L_9&:=\Psi^{-1}(\varphi)(\omega\cdot\partial_\varphi+P_8)\Psi(\varphi)\\&=\omega\cdot\partial_\varphi+\mathrm{i}\,\mathtt{m}_{\frac32}\Omega(\kappa,D)+\mathtt{m}_1\partial_x+\mathrm{i}\,\mathtt{m}_{\frac12}|D|^{\frac12}+\mathrm{Op}(r_9^{(d)})+L_M+L'_M,\,,\end{aligned}\tag{4.6.167}$$

where

$$\begin{aligned}\mathrm{Op}(r_9^{(d)}):=&\,\mathrm{Op}(r_8^{(d)}+r_{b_3,-\frac12})-[B(\varphi),\mathtt{m}_1\partial_x+\mathrm{i}\,a_3^{(d)}|D|^{\frac12}+\mathrm{Op}(r_8^{(d)})]+(\omega\cdot\partial_\varphi B(\varphi))\\&+\sum_{n=2}^{M-1}\frac{(-1)^n}{n!}\mathrm{ad}^n_{B(\varphi)}(P_8)-\sum_{n=2}^{M}\frac{(-1)^n}{n!}\mathrm{ad}^{n-1}_{B(\varphi)}(\omega\cdot\partial_\varphi B(\varphi))\in\mathrm{OP}S^0\,.\end{aligned}\tag{4.6.168}$$

Define the matrix $\boldsymbol{\Sigma}:=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Summing up, we have obtained the following lemma.

Lemma 4.78. *Let $M\in\mathbb{N}$, $\mathtt{q}_0\in\mathbb{N}_0$. Let b_3 be the function defined in* (4.6.166)*. Then, conjugating the operator $\mathcal{L}_8$ in* (4.6.147) *via the invertible, real, reversibility and momentum preserving map $\boldsymbol{\Psi}$ defined in* (4.6.160), (4.6.161)*, we obtain, for any $\omega\in\mathtt{DC}(\upsilon,\tau)$, the real, reversible and momentum* *preserving operator*

$$\mathcal{L}_9 := \mathbf{\Psi}^{-1}\mathcal{L}_8\mathbf{\Psi} = \omega\cdot\partial_\varphi + \mathrm{i}\,\mathtt{m}_{\frac{3}{2}}\mathbf{\Omega}(\kappa,D) + \mathtt{m}_1\partial_x + \mathrm{i}\mathtt{m}_{\frac12}\mathbf{\Sigma}|D|^{\frac12} + \mathrm{i}\mathbf{\Pi}_0 + \mathbf{R}_9^{(0,d)} + \mathbf{T}_{9,M}, , \tag{4.6.169}$$

where

1. *the constant* $\mathtt{m}_{\frac12}$ *defined in* (4.6.141) *satisfies* $|\mathtt{m}_{\frac12}|^{k_0,\upsilon} \lesssim \varepsilon^2$*;*
2. $\mathbf{R}_9^{(0,d)}$ *is a block-diagonal operator*

$$\mathbf{R}_9^{(0,d)} = \begin{pmatrix} r_9^{(d)}(\varphi,x,D) & 0 \\ 0 & \overline{r_9^{(d)}(\varphi,x,D)}\end{pmatrix} \in \mathrm{OP}S^0\,,$$

that satisfies, for some $\sigma_M := \sigma_M(k_0,\tau,\nu) > 0$*, and for all* $s \geqslant s_0$*,*

$$\|\mathbf{R}_9^{(0,d)}\|_{0,s,1}^{k_0,\upsilon} \lesssim_{s,M} \varepsilon\upsilon^{-1}(1+\|\mathfrak{I}_0\|_{s+\sigma_M}^{k_0,\upsilon})\,; \tag{4.6.170}$$

3. *For any* $\mathtt{q}\in\mathbb{N}_0^\nu$ *with* $|\mathtt{q}| \leqslant \mathtt{q}_0$*,* $n_1,n_2\in\mathbb{N}_0$ *with* $n_1+n_2 \leqslant M-2(k_0+\mathtt{q}_0)+\frac52$*, the operator* $\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathbf{T}_{9,M}(\varphi)\langle D\rangle^{n_2}$ *is* $\mathcal{D}^{k_0}$*-tame with a tame constant satisfying, for some* $\sigma_M(\mathtt{q}_0) := \sigma_M(k_0,\tau,\nu,\mathtt{q}_0)$*, for any* $s_0 \leqslant s \leqslant S$*,*

$$\mathfrak{M}_{\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathbf{T}_{9,M}(\varphi)\langle D\rangle^{n_2}}(s) \lesssim_{S,M,\mathtt{q}_0} \varepsilon\upsilon^{-1}(1+\|\mathfrak{I}_0\|_{s+\sigma_M(\mathtt{q}_0)}^{k_0,\upsilon})\,; \tag{4.6.171}$$

4. *The operators* $\mathbf{\Psi}^{\pm1}-\mathrm{Id}$*,* $(\mathbf{\Psi}^{\pm1}-\mathrm{Id})^*$ *are* $\mathcal{D}^{k_0}$*-tame, with tame constants satisfying, for some* $\sigma := \sigma(k_0,\tau,\nu) > 0$ *and for all* $s \geqslant s_0$*,*

$$\mathfrak{M}_{\mathbf{\Psi}^{\pm1}-\mathrm{Id}}(s) + \mathfrak{M}_{(\mathbf{\Psi}^{\pm1}-\mathrm{Id})^*}(s) \lesssim_s \varepsilon\upsilon^{-1}(1+\|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon})\,. \tag{4.6.172}$$

Furthermore, for any s_1 *as in* (4.6.15)*,* $\alpha\in\mathbb{N}_0$*,* $\mathtt{q}\in\mathbb{N}_0^\nu$*, with* $|\mathtt{q}| \leqslant \mathtt{q}_0$*, and* $n_1,n_2\in\mathbb{N}_0$*, with* $n_1+n_2 \leqslant M-2\mathtt{q}_0+\frac12$*, we have*

$$\|\Delta_{12}\mathbf{R}_9^{(0,d)}\|_{0,s_1,1} \lesssim_{s_1,M} \varepsilon\upsilon^{-1}\,\|i_1-i_2\|_{s_1+\sigma_M}\ ,\ |\Delta_{12}\mathtt{m}_{\frac12}| \lesssim \varepsilon^2\,\|i_1-i_2\|_{s_0+\sigma}\ , \tag{4.6.173}$$

$$\|\,\langle D\rangle^{n_1}\,\partial_\varphi^{\mathtt{q}}\Delta_{12}\mathbf{T}_{9,M}\,\langle D\rangle^{n_2}\,\|_{\mathcal{L}(H^{s_1})} \lesssim_{s_1,M,\mathtt{q}_0} \varepsilon\upsilon^{-1}\,\|i_1-i_2\|_{s_1+\sigma_M(\mathtt{q}_0)}\ , \tag{4.6.174}$$

$$\|\Delta_{12}(\mathbf{\Psi}^{\pm1})h\|_{s_1} + \|\Delta_{12}(\mathbf{\Psi}^{\pm1})^*h\|_{s_1} \lesssim_{s_1} \varepsilon\upsilon^{-1}\,\|i_1-i_2\|_{s_1+\sigma}\,\|h\|_{s_1+\sigma}\ . \tag{4.6.175}$$

Proof. The function $b_3(\varphi,x)$ defined in (4.6.166), satisfies, by (4.6.149) and the estimate of $\mathtt{m}_{\frac32}$ given in Lemma 4.74-item-2, for some $\sigma>0$ and for all $s \geqslant s_0$,

$$\|b_3\|_s^{k_0,\upsilon} \lesssim_s \varepsilon\upsilon^{-1}(1+\|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon})\,. \tag{4.6.176}$$

The estimate for $\mathtt{m}_{\frac{1}{2}}$ follows by (4.6.141), (4.2.7) and (4.6.96), (4.6.158). The estimate (4.6.170) follows by (4.6.168), (4.6.159), Lemmata 4.16, 4.17, and (4.6.149), (4.6.150), (4.6.176). By (4.6.147), (4.6.159), (4.6.167), and (4.6.162), the smoothing term $\mathbf{T}_{9,M}$ in (4.6.169) is

$$\mathbf{T}_{9,M} := \boldsymbol{\Psi}^{-1}\mathbf{T}_{8,M}\boldsymbol{\Psi} + \mathrm{i}\boldsymbol{\Pi}_0(\boldsymbol{\Psi} - \mathrm{Id}) + \begin{pmatrix} L_M + L'_M & 0 \\ 0 & \overline{L_M} + \overline{L'_M} \end{pmatrix}$$

with L_M and L'_M introduced in (4.6.163), (4.6.164). The estimate (4.6.171) follows by Lemmata 4.24, 4.25, 4.18, (4.6.159), (4.6.149), (4.6.151), (4.6.176), (4.6.172). The estimate (4.6.172) follows by Lemma 4.25 and (4.6.176). The estimates (4.6.173), (4.6.174), (4.6.175) are proved in the same fashion. By Lemma 4.77, the function $a_3^{(d)}$ is a even(φ, x) quasi-periodic traveling wave. Hence the function b_3 in (4.6.166) is a odd(φ, x) quasi-periodic traveling wave. By Lemmata 4.33, 4.43, and 4.39, the transformation $\boldsymbol{\Psi}$ is reversibility and momentum preserving, therefore the operator $\mathcal{L}_9$ is reversible and momentum preserving. □

Remark 4.79. In Proposition 4.83 we shall estimate $\|[\partial_x, \mathbf{R}_9^{(0,d)}]\|_{0,s,0}^{k_0,\upsilon}$ using (4.6.170) and (4.2.27). In order to control $\|\mathbf{R}_9^{(0,d)}\|_{0,s,1}^{k_0,\upsilon}$ we used the estimates (4.6.98) for finitely many $\alpha \in \mathbb{N}_0$, $\alpha \leqslant \alpha(M)$, depending on M. Furthermore in Proposition 4.83 we shall use (4.6.173)-(4.6.174) only for $s_1 = s_0$.

4.6.7 Conclusion: partial reduction of $\mathcal{L}_\omega$

By Sections 4.6.1-4.6.6, the linear operator $\mathcal{L}$ in (4.6.12) is semi-conjugated, for all $\omega \in \mathtt{DC}(\upsilon, \tau)$, to the real, reversible and momentum preserving operator $\mathcal{L}_9$ defined in (4.6.169), namely

$$\mathcal{L}_9 = \mathcal{W}_2^{-1}\mathcal{L}\mathcal{W}_1 \,, \tag{4.6.177}$$

where

$$\mathcal{W}_1 := \mathcal{P}\mathcal{Z}\mathcal{E}\mathcal{Q}\widetilde{\mathcal{M}}\mathcal{C}\boldsymbol{\Phi}_M\boldsymbol{\Phi}\mathcal{V}\boldsymbol{\Psi}\,, \quad \mathcal{W}_2 := \mathcal{P}\rho\mathcal{Z}\mathcal{E}\mathcal{Q}\widetilde{\mathcal{M}}\mathcal{C}\boldsymbol{\Phi}_M\boldsymbol{\Phi}\mathcal{V}\boldsymbol{\Psi}\,. \tag{4.6.178}$$

Moreover $\mathcal{L}_9$ is defined for all $\omega \in \mathbb{R}^\nu$.

Now we deduce a similar conjugation result for the projected operator $\mathcal{L}_\omega$ in (4.5.33), i.e. (4.6.1), which acts in the normal subspace $\mathfrak{H}_{\mathbb{S}^+,\Sigma}^{\angle}$. We first introduce some notation.

We denote by $\Pi_{\mathbb{S}^+,\Sigma}^{\intercal}$ and $\Pi_{\mathbb{S}^+,\Sigma}^{\angle}$ the projections on the subspaces $\mathfrak{H}_{\mathbb{S}^+,\Sigma}^{\intercal}$ and $\mathfrak{H}_{\mathbb{S}^+,\Sigma}^{\angle}$ defined in Section 4.1.3. In view of Remark 4.67, we denote, with a small abuse of notation, $\Pi_{\mathbb{S}_0^+,\Sigma}^{\intercal} := \Pi_{\mathbb{S}^+,\Sigma}^{\intercal} + \pi_0$, so that $\Pi_{\mathbb{S}_0^+,\Sigma}^{\intercal} + \Pi_{\mathbb{S}^+,\Sigma}^{\angle} = \mathrm{Id}$ on the whole $L^2 \times L^2$. We remind that $\mathbb{S}_0 = \mathbb{S} \cup \{0\}$, where $\mathbb{S}$ is the set defined in (4.1.48). We denote by $\Pi_{\mathbb{S}_0} := \Pi_{\mathbb{S}}^{\intercal} + \pi_0$, where $\Pi_{\mathbb{S}}^{\intercal}$ is defined below (4.1.58) together with the definition of $\Pi_{\mathbb{S}_0}^{\perp}$, so that we have $\Pi_{\mathbb{S}_0} + \Pi_{\mathbb{S}_0}^{\perp} = \mathrm{Id}$.

Lemma 4.80. *Let $M > 0$. There is $\sigma_M > 0$ (depending also on k_0, τ, ν) such that, assuming* (4.6.14) *with $\mu_0 \geqslant \sigma_M$, the following holds: the maps $\mathcal{W}_1$, $\mathcal{W}_2$ defined in* (4.6.178) *have the form*

$$\mathcal{W}_i = \widetilde{\mathcal{M}}\mathcal{C} + \mathcal{R}_i(\varepsilon)\,, \tag{4.6.179}$$

where, for any $i = 1, 2$, for all $s_0 \leqslant s \leqslant S$,

$$\|\mathcal{R}_i(\varepsilon)h\|_s^{k_0,\upsilon} \lesssim_{S,M} \varepsilon\upsilon^{-1}\big(\|h\|_{s+\sigma_M}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\sigma_M}^{k_0,\upsilon}\|h\|_{s_0+\sigma_M}^{k_0,\upsilon}\big)\,. \tag{4.6.180}$$

Moreover, for $\varepsilon\upsilon^{-1} \leqslant \delta(S)$ small enough, the operators

$$\mathcal{W}_1^{\perp} := \Pi_{\mathbb{S}^+,\Sigma}^{\angle}\mathcal{W}_1\Pi_{\mathbb{S}_0}^{\perp}\,, \quad \mathcal{W}_2^{\perp} := \Pi_{\mathbb{S}^+,\Sigma}^{\angle}\mathcal{W}_2\Pi_{\mathbb{S}_0}^{\perp}\,, \tag{4.6.181}$$

are invertible and, for all $s_0 \leqslant s \leqslant S$, $i = 1, 2$,

$$\|(\mathcal{W}_i^{\perp})^{\pm 1}h\|_s^{k_0,\upsilon} \lesssim_{S,M} \|h\|_{s+\sigma_M}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\sigma_M}^{k_0,\upsilon}\|h\|_{s_0+\sigma_M}^{k_0,\upsilon}\,, \tag{4.6.182}$$

$$\|\Delta_{12}(\mathcal{W}_i^{\perp})^{\pm 1}h\|_{s_1} \lesssim_{s_1,M} \varepsilon\upsilon^{-1}\,\|i_1 - i_2\|_{s_1+\sigma_M}\,\|h\|_{s_1+\sigma_M}\,. \tag{4.6.183}$$

The operators $\mathcal{W}_1^{\perp}$, $\mathcal{W}_2^{\perp}$ map (anti)-reversible, respectively traveling, waves, into (anti)-reversible, respectively traveling, waves.

Proof. The formulae (4.6.179) and the estimates (4.6.180) follow by (4.6.178), Lemmata 4.24, 4.25, and (4.2.37), (4.6.23), (4.6.27), (4.6.92), (4.6.93), (4.6.113), (4.6.152), (4.6.153), (4.6.172). The invertibility of each $\mathcal{W}_i^{\perp}$ and the estimates (4.6.182) follow with a perturbative argument as in [14, 13], noting that $\Pi_{\mathbb{S}^+,\Sigma}^{\angle}\widetilde{\mathcal{M}}\mathcal{C}\,\Pi_{\mathbb{S}_0}^{\perp} = \Pi_{\mathbb{S}^+,\Sigma}^{\angle}\mathcal{M}\mathcal{C}\,\Pi_{\mathbb{S}_0}^{\perp}$ are invertible on their ranges with inverses $(\Pi_{\mathbb{S}^+,\Sigma}^{\angle}\mathcal{M}\mathcal{C}\,\Pi_{\mathbb{S}_0}^{\perp})^{-1} = \Pi_{\mathbb{S}_0}^{\perp}(\mathcal{M}\mathcal{C})^{-1}\Pi_{\mathbb{S}^+,\Sigma}^{\angle}$. Since $\mathcal{Z}, \mathcal{E}, \mathcal{Q}, \widetilde{\mathcal{M}}, \mathbf{\Phi}_M, \mathbf{\Phi}, \mathcal{V}, \mathbf{\Psi}$ are reversibility and momentum preserving and using Remark 4.72 and Lemmata 4.37 and 4.45, we deduce that $\mathcal{W}_1^{\perp}$, $\mathcal{W}_2^{\perp}$ map (anti)-reversible, respectively traveling, waves, into (anti)-reversible, respectively traveling, waves. □

Remark 4.81. The time reparametrization $\mathcal{P}$ and the multiplication for the function ρ (which is independent of the space variable), commute with the projections $\Pi_{\mathbb{S}^+,\Sigma}^{\angle}$ and $\Pi_{\mathbb{S}_0}^{\perp}$.

The operator $\mathcal{L}_\omega$ in (4.5.33) (i.e. (4.6.1)) is semi-conjugated to

$$\mathcal{L}_{\perp} := (\mathcal{W}_2^{\perp})^{-1}\mathcal{L}_\omega\mathcal{W}_1^{\perp} = \Pi_{\mathbb{S}_0}^{\perp}\,\mathcal{L}_9\,\Pi_{\mathbb{S}_0}^{\perp} + \mathcal{R}^f \tag{4.6.184}$$

where $\mathcal{R}^f$ is, by (4.6.181), (4.6.177), (4.6.179) (recall that $\widetilde{\mathcal{M}}$ is defined in (4.6.79)-(4.6.80)), and

(4.1.59),

$$\begin{aligned}\mathcal{R}^f &:= (\mathcal{W}_2^{\perp})^{-1}\Pi^{\angle}_{\mathbb{S}^+,\Sigma}\mathcal{R}_2(\varepsilon)\Pi_{\mathbb{S}_0}\mathcal{L}_9\Pi^{\perp}_{\mathbb{S}_0} \\ &\quad - (\mathcal{W}_2^{\perp})^{-1}\Pi^{\angle}_{\mathbb{S}^+,\Sigma}\mathcal{L}\Pi^{\intercal}_{\mathbb{S}_0^+,\Sigma}\mathcal{R}_1(\varepsilon)\Pi^{\perp}_{\mathbb{S}_0} - \varepsilon(\mathcal{W}_2^{\perp})^{-1}\Pi^{\angle}_{\mathbb{S}^+,\Sigma}JR\mathcal{W}_1^{\perp}\,.\end{aligned} \tag{4.6.185}$$

Lemma 4.82. *The operator* $\mathcal{R}^f$ *in* (4.6.185) *has the finite rank form* (4.6.4), (4.6.5). *Moreover, let* $\mathtt{q}_0 \in \mathbb{N}_0$ *and* $M \geqslant 2(k_0+\mathtt{q}_0)-\frac{3}{2}$. *There exists* $\aleph(M,\mathtt{q}_0)>0$ *(depending also on* k_0, τ, ν*) such that, for any* $n_1,n_2 \in \mathbb{N}_0$, *with* $n_1+n_2 \leqslant M-2(k_0+\mathtt{q}_0)+\frac{5}{2}$, *and any* $\mathtt{q}\in\mathbb{N}_0^\nu$, *with* $|\mathtt{q}|\leqslant \mathtt{q}_0$, *the operator* $\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathcal{R}^f\langle D\rangle^{n_2}$ *is* $\mathcal{D}^{k_0}$*-tame, with a tame constant satisfying*

$$\mathfrak{M}_{\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathcal{R}^f\langle D\rangle^{n_2}}(s) \lesssim_{S,M,\mathtt{q}_0} \varepsilon\upsilon^{-1}(1+\|\mathfrak{I}_0\|^{k_0,\upsilon}_{s+\aleph(M,\mathtt{q}_0)})\,, \quad \forall s_0\leqslant s\leqslant S\,, \tag{4.6.186}$$

$$\|\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\Delta_{12}\mathcal{R}^f\langle D\rangle^{n_2}\|_{\mathcal{L}(H^{s_1})} \lesssim_{s_1,M,\mathtt{q}_0} \varepsilon\upsilon^{-1}\|i_1-i_2\|_{s_1+\aleph(M,\mathtt{q}_0)}\,, \tag{4.6.187}$$

for any s_1 *as in* (4.6.15).

Proof. The first two terms in (4.6.185) have the finite rank form (4.6.4) because of the presence of the finite dimensional projector $\Pi_{\mathbb{S}_0}$, respectively $\Pi^{\intercal}_{\mathbb{S}_0^+,\Sigma}$. In the last term, the operator R has the finite rank form (4.6.4). The estimate (4.6.186) follows by (4.6.185), (4.6.178), (4.6.181), (4.6.169), (4.6.4), (4.2.7) and (4.6.180), (4.6.182), (4.6.170), (4.6.171), (4.6.5). The estimate (4.6.187) follows similarly. □

Proposition 4.83. (Reduction of $\mathcal{L}_\omega$ up to smoothing operators) *For all* $(\omega,\kappa)\in\mathtt{DC}(\upsilon,\tau)\times[\kappa_1,\kappa_2]$, *the operator* $\mathcal{L}_\omega$ *in* (4.5.33) *(i.e.* (4.6.1)*) is semi-conjugated via* (4.6.184) *to the real, reversible and momentum preserving operator* $\mathcal{L}_\perp$. *For all* $(\omega,\kappa)\in\mathbb{R}^\nu\times[\kappa_1,\kappa_2]$, *the extended operator defined by the right hand side in* (4.6.184) *has the form*

$$\mathcal{L}_\perp = \omega\cdot\partial_\varphi\mathbb{1}_\perp + \mathrm{i}\,\mathbf{D}_\perp + \mathbf{R}_\perp,\,, \tag{4.6.188}$$

where $\mathbb{1}_\perp$ *denotes the identity map of* $\mathbf{H}^\perp_{\mathbb{S}_0}$ *(cfr.* (4.1.58)*) and*

1. $\mathbf{D}_\perp$ *is the diagonal operator*

$$\mathbf{D}_\perp := \begin{pmatrix}\mathcal{D}_\perp & 0\\ 0 & -\overline{\mathcal{D}_\perp}\end{pmatrix}\,, \quad \mathcal{D}_\perp := \mathrm{diag}_{j\in\mathbb{S}_0^c}\,\mu_j\,, \quad \mathbb{S}_0^c := \mathbb{Z}\backslash(\mathbb{S}\cup\{0\})\,,$$

with eigenvalues $\mu_j := \mathtt{m}_{\frac{3}{2}}\Omega_j(\kappa)+\mathtt{m}_1 j+\mathtt{m}_{\frac{1}{2}}|j|^{\frac{1}{2}}\in\mathbb{R}$, *where the real constants* $\mathtt{m}_{\frac{3}{2}},\mathtt{m}_1,\mathtt{m}_{\frac{1}{2}}$, *defined respectively in* (4.6.89), (4.6.145), (4.6.141), *satisfy*

$$|\mathtt{m}_{\frac{3}{2}}-1|^{k_0,\upsilon}+|\mathtt{m}_1|^{k_0,\upsilon}+|\mathtt{m}_{\frac{1}{2}}|^{k_0,\upsilon}\lesssim\varepsilon\,; \tag{4.6.189}$$

In addition, for some $\sigma > 0$,

$$|\Delta_{12}\mathtt{m}_{\frac{3}{2}}| + |\Delta_{12}\mathtt{m}_1| + |\Delta_{12}\mathtt{m}_{\frac{1}{2}}| \lesssim \varepsilon \|i_1 - i_2\|_{s_0+\sigma} \,. \tag{4.6.190}$$

2. *The operator* $\mathbf{R}_\perp$ *is real, reversible and momentum preserving. Moreover, for any* $\mathtt{q}_0 \in \mathbb{N}_0$, $M > 2(k_0 + \mathtt{q}_0) - \frac{3}{2}$, *there is a constant* $\aleph(M, \mathtt{q}_0) > 0$ *(depending also on* k_0, τ, ν*) such that, assuming* (4.6.14) *with* $\mu_0 \geqslant \aleph(M, \mathtt{q}_0)$, *for any* $s_0 \leqslant s \leqslant S$, $\mathtt{q} \in \mathbb{N}_0^\nu$, *with* $|\mathtt{q}| \leqslant \mathtt{q}_0$, *the operators* $\partial_\varphi^{\mathtt{q}}\mathbf{R}_\perp$, $[\partial_\varphi^{\mathtt{q}}\mathbf{R}_\perp, \partial_x]$ *are* $\mathcal{D}^{k_0}$*-tame with tame constants satisfying*

$$\mathfrak{M}_{\partial_\varphi^{\mathtt{q}}\mathbf{R}_\perp}(s),\ \mathfrak{M}_{[\partial_\varphi^{\mathtt{q}}\mathbf{R}_\perp, \partial_x]}(s) \lesssim_{S,M,\mathtt{q}_0} \varepsilon \upsilon^{-1}(1 + \|\mathfrak{I}_0\|^{k_0,\upsilon}_{s+\aleph(M,\mathtt{q}_0)}) \,. \tag{4.6.191}$$

Moreover, for any $\mathtt{q} \in \mathbb{N}_0^\nu$, *with* $|\mathtt{q}| \leqslant \mathtt{q}_0$,

$$\|\partial_\varphi^{\mathtt{q}}\Delta_{12}\mathbf{R}_\perp\|_{\mathcal{L}(H^{s_0})} + \|\partial_\varphi^{\mathtt{q}}\Delta_{12}[\mathbf{R}_\perp, \partial_x]\|_{\mathcal{L}(H^{s_0})} \lesssim_M \varepsilon \upsilon^{-1} \|i_1 - i_2\|_{s_0+\aleph(M,\mathtt{q}_0)} \,. \tag{4.6.192}$$

Proof. By (4.6.184) and (4.6.169) we deduce (4.6.188) with

$$\mathbf{R}_\perp := \Pi_{\mathbb{S}_0}^\perp(\mathbf{R}_9^{(0,d)} + \mathbf{T}_{9,M})\Pi_{\mathbb{S}_0}^\perp + \mathcal{R}^f \,.$$

The estimates (4.6.189)-(4.6.190) follow by Lemmata 4.74, 4.77, 4.78. The estimate (4.6.191) follows by Lemmata 4.17, 4.25, (4.6.170) and (4.6.171), (4.6.186), choosing $(n_1, n_2) = (1, 0), (0, 1)$. The estimate (4.6.192) follows similarly. The operator $\mathcal{L}_\omega$ in (4.5.33) is reversible and momentum preserving (Lemma 4.66). By Sections 4.6.2-4.6.6, the maps $\mathcal{Z}, \mathcal{E}, \mathcal{Q}, \widetilde{\mathcal{M}}, \mathbf{\Phi}_M, \mathbf{\Phi}, \mathcal{V}, \mathbf{\Psi}$ are reversibility and momentum preserving. Therefore, using also (4.6.18), (4.6.24) and Lemmata 4.37 and 4.45, we deduce that the operator $\mathcal{L}_\perp$ in (4.6.184) is reversible and momentum preserving. Since $\mathrm{i}\,\mathbf{D}_\perp$ is reversible and momentum preserving, we deduce that $\mathbf{R}_\perp$ is reversible and momentum preserving. ☐

4.7 Almost-diagonalization and invertibility of $\mathcal{L}_\omega$

In Proposition 4.83 we obtained the operator $\mathcal{L}_\perp$ in (4.6.188) which is diagonal and constant coefficient up to the bounded operator $\mathbf{R}_\perp(\varphi)$. In this section we complete the diagonalization of $\mathcal{L}_\perp$ implementing a KAM iterative scheme. As starting point, we consider the real, reversible and momentum preserving operator, acting in $\mathbf{H}_{\mathbb{S}_0}^\perp$,

$$\mathbf{L}_0 := \mathbf{L}_0(i) := \mathcal{L}_\perp = \omega \cdot \partial_\varphi \mathbb{1}_\perp + \mathrm{i}\,\mathbf{D}_0 + \mathbf{R}_\perp^{(0)} \,, \tag{4.7.1}$$

defined for all $(\omega, \kappa) \in \mathbb{R}^\nu \times [\kappa_1, \kappa_2]$, with diagonal part (with respect to the exponential basis)

$$\mathbf{D}_0 := \begin{pmatrix} \mathcal{D}_0 & 0 \\ 0 & -\overline{\mathcal{D}_0} \end{pmatrix}, \quad \mathcal{D}_0 := \mathrm{diag}_{j \in \mathbb{S}_0^c} \mu_j^{(0)}, \quad \mu_j^{(0)} := \mathtt{m}_{\frac{3}{2}} \Omega_j(\kappa) + \mathtt{m}_1 j + \mathtt{m}_{\frac{1}{2}} |j|^{\frac{1}{2}}, \tag{4.7.2}$$

where $\mathbb{S}_0^c = \mathbb{Z} \backslash \mathbb{S}_0$, $\mathbb{S}_0 = \mathbb{S} \cup \{0\}$, the real constants $\mathtt{m}_{\frac{3}{2}}$, $\mathtt{m}_1$, $\mathtt{m}_{\frac{1}{2}}$ satisfy (4.6.189)-(4.6.190) and

$$\mathbf{R}_\perp^{(0)} := \mathbf{R}_\perp := \begin{pmatrix} R_\perp^{(0,d)} & R_\perp^{(0,o)} \\ \overline{R_\perp^{(0,o)}} & \overline{R_\perp^{(0,d)}} \end{pmatrix}, \quad R_\perp^{(0,d)} : H_{\mathbb{S}_0}^\perp \to H_{\mathbb{S}_0}^\perp, \, R_\perp^{(0,o)} : H_{-\mathbb{S}_0}^\perp \to H_{\mathbb{S}_0}^\perp, \tag{4.7.3}$$

which is a real, reversible, momentum preserving operator satisfying (4.6.191), (4.6.192). We denote $H_{\pm\mathbb{S}_0}^\perp = \{h(x) = \sum_{j \notin \pm\mathbb{S}_0} h_j e^{\pm \mathrm{i} jx} \in L^2\}$. Note that

$$\overline{\mathcal{D}_0} : H_{-\mathbb{S}_0}^\perp \to H_{-\mathbb{S}_0}^\perp, \quad \overline{\mathcal{D}_0} = \mathrm{diag}_{j \in -\mathbb{S}_0^c} (\mu_{-j}^{(0)}). \tag{4.7.4}$$

Proposition 4.83 implies that the operator $\mathbf{R}_\perp^{(0)}$ satisfies the tame estimates of Lemma 4.84 below by fixing the constant M large enough (which means performing sufficiently many regularizing steps in Section 4.6.4), namely

$$M := \left[2(k_0 + s_0 + \mathtt{b}) - \tfrac{3}{2}\right] + 1 \in \mathbb{N}, \tag{4.7.5}$$

where

$$\mathtt{b} := [\mathtt{a}] + 2 \in \mathbb{N}, \quad \mathtt{a} := 3\tau_1 \geqslant 1, \quad \tau_1 := k_0 + (k_0 + 1)\tau. \tag{4.7.6}$$

These conditions imply the convergence of the iterative scheme (4.7.46)-(4.7.47), see Lemma 4.91. We also set

$$\mu(\mathtt{b}) := \aleph(M, s_0 + \mathtt{b}), \tag{4.7.7}$$

where the constant $\aleph(M, \mathtt{q}_0)$ is given in Proposition 4.83.

Lemma 4.84. (Smallness of $\mathbf{R}_\perp^{(0)}$) *Assume* (4.6.14) *with* $\mu_0 \geqslant \mu(\mathtt{b})$. *Then the operators* $\mathbf{R}_\perp^{(0)}$, $[\mathbf{R}_\perp^{(0)}, \partial_x]$, *and* $\partial_{\varphi_m}^{s_0} \mathbf{R}_\perp^{(0)}$, $[\partial_{\varphi_m}^{s_0} \mathbf{R}_\perp^{(0)}, \partial_x]$, $\partial_{\varphi_m}^{s_0+\mathtt{b}} \mathbf{R}_\perp^{(0)}$, $[\partial_{\varphi_m}^{s_0+\mathtt{b}} \mathbf{R}_\perp^{(0)}, \partial_x]$, $m = 1, \ldots, \nu$, *are* $\mathcal{D}^{k_0}$*-tame and, defining*

$$\mathbb{M}_0(s) := \max\left\{\mathfrak{M}_{\mathbf{R}_\perp^{(0)}}(s), \ \mathfrak{M}_{[\mathbf{R}_\perp^{(0)}, \partial_x]}(s), \ \mathfrak{M}_{\partial_{\varphi_m}^{s_0} \mathbf{R}_\perp^{(0)}}(s), \ \mathfrak{M}_{[\partial_{\varphi_m}^{s_0} \mathbf{R}_\perp^{(0)}, \partial_x]}(s), \, m = 1, \ldots, \nu\right\}, \tag{4.7.8}$$

$$\mathbb{M}_0(s, \mathtt{b}) := \max\left\{\mathfrak{M}_{\partial_{\varphi_m}^{s_0+\mathtt{b}} \mathbf{R}_\perp^{(0)}}(s), \ \mathfrak{M}_{[\partial_{\varphi_m}^{s_0+\mathtt{b}} \mathbf{R}_\perp^{(0)}, \partial_x]}(s), \, m = 1, \ldots, \nu\right\}, \tag{4.7.9}$$

we have, for all $s_0 \leqslant s \leqslant S$,

$$\mathfrak{M}_0(s,\mathtt{b}) := \max\{\,\mathbb{M}_0(s), \mathbb{M}_0(s,\mathtt{b})\,\} \leqslant C(S)\frac{\varepsilon}{\upsilon}(1 + \|\mathfrak{I}_0\|^{k_0,\upsilon}_{s+\mu(\mathtt{b})})\,,\ \mathfrak{M}_0(s_0,\mathtt{b}) \leqslant C(S)\frac{\varepsilon}{\upsilon}\,. \quad (4.7.10)$$

Moreover, for all $\mathtt{q} \in \mathbb{N}_0^\nu$, with $|\mathtt{q}| \leqslant s_0 + \mathtt{b}$,

$$\|\partial_\varphi^{\mathtt{q}} \Delta_{12} \mathbf{R}_\perp^{(0)}\|_{\mathcal{L}(H^{s_0})}\,,\ \|\Delta_{12}[\partial_\varphi^{\mathtt{q}} \mathbf{R}_\perp^{(0)}, \partial_x]\|_{\mathcal{L}(H^{s_0})} \leqslant C(S)\varepsilon\upsilon^{-1}\,\|i_1 - i_2\|_{s_0+\mu(\mathtt{b})}\,. \quad (4.7.11)$$

Proof. Recalling (4.7.8), (4.7.9), the bounds (4.7.10)-(4.7.11) follow by (4.6.191), (4.7.5), (4.7.7), (4.6.192). □

We perform the almost-reducibility of $\mathbf{L}_0$ along the scale

$$N_{-1} := 1\,, \quad N_{\mathtt{n}} := N_0^{\chi^{\mathtt{n}}}\,, \quad \forall\, \mathtt{n} \in \mathbb{N}_0\,, \quad \chi := 3/2\,. \quad (4.7.12)$$

Theorem 4.85. (Almost-diagonalization of $\mathbf{L}_0$: KAM iteration) *There exists $\tau_2(\tau,\nu) > \tau_1(\tau,\nu) + \mathtt{a}$ (with $\tau_1, \mathtt{a}$ defined in (4.7.6)) such that, for all $S > s_0$, there is $N_0 := N_0(S,\mathtt{b}) \in \mathbb{N}$ such that, if*

$$N_0^{\tau_2}\mathfrak{M}_0(s_0,\mathtt{b})\upsilon^{-1} \leqslant 1\,, \quad (4.7.13)$$

then, for all $\bar{\mathtt{n}} \in \mathbb{N}_0$, $\mathtt{n} = 0,1,\ldots,\bar{\mathtt{n}}$:

$(\mathbf{S1})_{\mathtt{n}}$ *There exists a real, reversible and momentum preserving operator*

$$\begin{aligned} \mathbf{L}_{\mathtt{n}} &:= \omega\cdot\partial_\varphi \mathbb{1}_\perp + \mathrm{i}\,\mathbf{D}_{\mathtt{n}} + \mathbf{R}_\perp^{(\mathtt{n})}\,, \\ \mathbf{D}_{\mathtt{n}} &:= \begin{pmatrix} \mathcal{D}_{\mathtt{n}} & 0 \\ 0 & -\overline{\mathcal{D}_{\mathtt{n}}} \end{pmatrix}\,, \quad \mathcal{D}_{\mathtt{n}} := \mathrm{diag}_{j\in\mathbb{S}_0^c}\,\mu_j^{(\mathtt{n})}\,, \end{aligned} \quad (4.7.14)$$

defined for all (ω,κ) in $\mathbb{R}^\nu \times [\kappa_1,\kappa_2]$, where $\mu_j^{(\mathtt{n})}$ are k_0-times differentiable real functions

$$\mu_j^{(\mathtt{n})}(\omega,\kappa) := \mu_j^{(0)}(\omega,\kappa) + \mathfrak{r}_j^{(\mathtt{n})}(\omega,\kappa)\,, \quad \mu_j^{(0)} = \mathtt{m}_{\frac{3}{2}}\,\Omega_j(\kappa) + \mathtt{m}_1\,j + \mathtt{m}_{\frac{1}{2}}\,|j|^{\frac{1}{2}}\,, \quad (4.7.15)$$

satisfying $\mathfrak{r}_j^{(0)} = 0$ and, for $\mathtt{n} \geqslant 1$,

$$|\mathfrak{r}_j^{(\mathtt{n})}|^{k_0,\upsilon} \leqslant C(S,\mathtt{b})\varepsilon\upsilon^{-1}\,, \quad |\mu_j^{(\mathtt{n})} - \mu_j^{(\mathtt{n}-1)}|^{k_0,\upsilon} \leqslant C(S,\mathtt{b})\varepsilon\upsilon^{-1}N_{\mathtt{n}-2}^{-\mathtt{a}}\,, \quad \forall j \in \mathbb{S}_0^c\,. \quad (4.7.16)$$

The remainder

$$\mathbf{R}_\perp^{(\mathtt{n})} := \begin{pmatrix} R_\perp^{(\mathtt{n},d)} & R_\perp^{(\mathtt{n},o)} \\ \overline{R_\perp^{(\mathtt{n},o)}} & \overline{R_\perp^{(\mathtt{n},d)}} \end{pmatrix}\,, \quad R_\perp^{(\mathtt{n},d)} : H_{\mathbb{S}_0}^\perp \to H_{\mathbb{S}_0}^\perp\,,\ R_\perp^{(\mathtt{n},o)} : H_{-\mathbb{S}_0}^\perp \to H_{\mathbb{S}_0}^\perp \quad (4.7.17)$$

is $\mathcal{D}^{k_0}$-modulo-tame: more precisely, the operators $R_\perp^{(\mathtt{n},d)}$, $R_\perp^{(\mathtt{n},o)}$, $\langle\partial_\varphi\rangle^{\mathtt{b}} R_\perp^{(\mathtt{n},d)}$, $\langle\partial_\varphi\rangle^{\mathtt{b}} R_\perp^{(\mathtt{n},o)}$, are $\mathcal{D}^{k_0}$-modulo-tame with modulo-tame constants

$$\begin{aligned}
\mathfrak{M}_{\mathtt{n}}^\sharp(s) &:= \mathfrak{M}^\sharp_{\mathbf{R}_\perp^{(\mathtt{n})}}(s) := \max\{\mathfrak{M}^\sharp_{R_\perp^{(\mathtt{n},d)}}(s), \mathfrak{M}^\sharp_{R_\perp^{(\mathtt{n},o)}}(s)\}\,, \\
\mathfrak{M}_{\mathtt{n}}^\sharp(s,\mathtt{b}) &:= \mathfrak{M}^\sharp_{\langle\partial_\varphi\rangle^{\mathtt{b}}\mathbf{R}_\perp^{(\mathtt{n})}}(s) := \max\{\mathfrak{M}^\sharp_{\langle\partial_\varphi\rangle^{\mathtt{b}}R_\perp^{(\mathtt{n},d)}}(s), \mathfrak{M}^\sharp_{\langle\partial_\varphi\rangle^{\mathtt{b}}R_\perp^{(\mathtt{n},o)}}(s)\}\,,
\end{aligned} \tag{4.7.18}$$

which satisfy, for some constant $C_*(s_0,\mathtt{b})>0$, for all $s_0\leqslant s\leqslant S$,

$$\mathfrak{M}_{\mathtt{n}}^\sharp(s) \leqslant C_*(s_0,\mathtt{b})\mathfrak{M}_0(s,\mathtt{b})N_{\mathtt{n}-1}^{-\mathtt{a}}\,, \quad \mathfrak{M}_{\mathtt{n}}^\sharp(s,\mathtt{b}) \leqslant C_*(s_0,\mathtt{b})\mathfrak{M}_0(s,\mathtt{b})N_{\mathtt{n}-1}\,. \tag{4.7.19}$$

Define the sets $\Lambda_{\mathtt{n}}^\upsilon = \Lambda_{\mathtt{n}}^\upsilon(i)$ by $\Lambda_0^\upsilon := \mathtt{DC}(2\upsilon,\tau)\times[\kappa_1,\kappa_2]$ and, for $\mathtt{n}\geqslant 1$,

$$\begin{aligned}
\Lambda_{\mathtt{n}}^\upsilon := \big\{ & \lambda = (\omega,\kappa)\in\Lambda_{\mathtt{n}-1}^\upsilon \ : \\
& \big|\omega\cdot\ell + \mu_j^{(\mathtt{n}-1)} - \mu_{j'}^{(\mathtt{n}-1)}\big| \geqslant \upsilon\,\langle |j|^{\frac32} - |j'|^{\frac32}\rangle\langle\ell\rangle^{-\tau} \\
& \forall\ |\ell|\leqslant N_{\mathtt{n}-1}\,,\ j,j'\notin\mathbb{S}_0\,,\ (\ell,j,j')\neq(0,j,j),\ \textit{with } \vec{\jmath}\cdot\ell + j - j' = 0\,, \\
& \big|\omega\cdot\ell + \mu_j^{(\mathtt{n}-1)} + \mu_{j'}^{(\mathtt{n}-1)}\big| \geqslant \upsilon\,\big(|j|^{\frac32} + |j'|^{\frac32}\big)\langle\ell\rangle^{-\tau} \\
& \forall\ |\ell|\leqslant N_{\mathtt{n}-1}\,,\ j,j'\notin\mathbb{S}_0\ \textit{with } \vec{\jmath}\cdot\ell + j + j' = 0\big\}\,.
\end{aligned} \tag{4.7.20}$$

For $\mathtt{n}\geqslant 1$ there exists a real, reversibility and momentum preserving map, defined for all $(\omega,\kappa)\in\mathbb{R}^\nu\times[\kappa_1,\kappa_2]$, of the form

$$\mathbf{\Phi}_{\mathtt{n}-1} = e^{\mathbf{X}_{\mathtt{n}-1}}\,, \quad \mathbf{X}_{\mathtt{n}-1} := \begin{pmatrix} X_{\mathtt{n}-1}^{(d)} & X_{\mathtt{n}-1}^{(o)} \\ \overline{X_{\mathtt{n}-1}^{(o)}} & \overline{X_{\mathtt{n}-1}^{(d)}} \end{pmatrix}\,, \quad X_{\mathtt{n}-1}^{(d)} : H_{\mathbb{S}_0}^\perp \to H_{\mathbb{S}_0}^\perp\,, X_{\mathtt{n}-1}^{(o)} : H_{-\mathbb{S}_0}^\perp \to H_{\mathbb{S}_0}^\perp\,,$$

such that, for all $\lambda\in\Lambda_{\mathtt{n}}^\upsilon$, the following conjugation formula holds:

$$\mathbf{L}_{\mathtt{n}} = \mathbf{\Phi}_{\mathtt{n}-1}^{-1}\mathbf{L}_{\mathtt{n}-1}\mathbf{\Phi}_{\mathtt{n}-1}\,. \tag{4.7.21}$$

The operators $\mathbf{X}_{\mathtt{n}-1}$, $\langle\partial_\varphi\rangle^{\mathtt{b}}\mathbf{X}_{\mathtt{n}-1}$, are $\mathcal{D}^{k_0}$-modulo-tame with modulo tame constants satisfying, for all $s_0\leqslant s\leqslant S$,

$$\begin{aligned}
\mathfrak{M}^\sharp_{\mathbf{X}_{\mathtt{n}-1}}(s) &\leqslant C(s_0,\mathtt{b})\upsilon^{-1}N_{\mathtt{n}-1}^{\tau_1}N_{\mathtt{n}-2}^{-\mathtt{a}}\mathfrak{M}_0(s,\mathtt{b})\,, \\
\mathfrak{M}^\sharp_{\langle\partial_\varphi\rangle^{\mathtt{b}}\mathbf{X}_{\mathtt{n}-1}}(s) &\leqslant C(s_0,\mathtt{b})\upsilon^{-1}N_{\mathtt{n}-1}^{\tau_1}N_{\mathtt{n}-2}\mathfrak{M}_0(s,\mathtt{b})\,.
\end{aligned} \tag{4.7.22}$$

$(\mathbf{S2})_{\mathtt{n}}$ *Let $i_1(\omega,\kappa)$, $i_2(\omega,\kappa)$ such that $\mathbf{R}_\perp^{(\mathtt{n})}(i_1)$, $\mathbf{R}_\perp^{(\mathtt{n})}(i_2)$ satisfy (4.7.10), (4.7.11). Then, for all $(\omega,\kappa)\in\Lambda_{\mathtt{n}}^{\upsilon_1}(i_1)\cap\Lambda_{\mathtt{n}}^{\upsilon_2}(i_2)$ with $\upsilon_1,\upsilon_2\in[\upsilon/2,2\upsilon]$,*

$$\|\|\Delta_{12}\mathbf{R}_\perp^{(\mathtt{n})}\|\|_{\mathcal{L}(H^{s_0})} \lesssim_{S,\mathtt{b}} \varepsilon\upsilon^{-1}N_{\mathtt{n}-1}^{-\mathtt{a}}\,\|i_1-i_2\|_{s_0+\mu(\mathtt{b})}\,, \tag{4.7.23}$$

$$\|\|\langle\partial_\varphi\rangle^{\mathtt{b}}\Delta_{12}\mathbf{R}_\perp^{(\mathtt{n})}\|\|_{\mathcal{L}(H^{s_0})} \lesssim_{S,\mathtt{b}} \varepsilon\upsilon^{-1}N_{\mathtt{n}-1}\,\|i_1-i_2\|_{s_0+\mu(\mathtt{b})}\,. \tag{4.7.24}$$

Furthermore, for $\mathtt{n} \geqslant 1$, *for all* $j \in \mathbb{S}_0^c$,

$$|\Delta_{12}(\mathfrak{r}_j^{(\mathtt{n})} - \mathfrak{r}_j^{(\mathtt{n}-1)})| \leqslant C \| |\Delta_{12} \mathbf{R}_\perp^{(\mathtt{n})}| \|_{\mathcal{L}(H^{s_0})}, \tag{4.7.25}$$

$$|\Delta_{12} \mathfrak{r}_j^{(\mathtt{n})}| \leqslant C(S, \mathtt{b}) \varepsilon \upsilon^{-1} \|i_1 - i_2\|_{s_0 + \mu(\mathtt{b})}. \tag{4.7.26}$$

$(\mathbf{S3})_\mathtt{n}$ *Let* i_1, i_2 *be like in* $(\mathbf{S2})_\mathtt{n}$ *and* $0 < \rho < \upsilon/2$. *Then*

$$\varepsilon \upsilon^{-1} C(S) N_{\mathtt{n}-1}^{\tau+1} \|i_1 - i_2\|_{s_0 + \mu(\mathtt{b})} \leqslant \rho \quad \Rightarrow \quad \Lambda_\mathtt{n}^\upsilon(i_1) \subseteq \Lambda_\mathtt{n}^{\upsilon - \rho}(i_2). \tag{4.7.27}$$

Theorem 4.85 implies also that the invertible operator

$$\mathbf{U}_{\overline{\mathtt{n}}} := \mathbf{\Phi}_0 \circ \ldots \circ \mathbf{\Phi}_{\overline{\mathtt{n}}-1}, \quad \overline{\mathtt{n}} \geqslant 1, \tag{4.7.28}$$

has almost diagonalized $\mathbf{L}_0$. We have indeed the following corollary.

Theorem 4.86. (Almost-diagonalization of $\mathbf{L}_0$) *Assume* (4.6.14) *with* $\mu_0 \geqslant \mu(\mathtt{b})$. *For all* $S > s_0$, *there exist* $N_0 = N_0(S, \mathtt{b}) > 0$ *and* $\delta_0 = \delta_0(S) > 0$ *such that, if the smallness condition*

$$N_0^{\tau_2} \varepsilon \upsilon^{-2} \leqslant \delta_0 \tag{4.7.29}$$

holds, where $\tau_2 = \tau_2(\tau, \nu)$ *is defined in Theorem 4.85, then, for all* $\overline{\mathtt{n}} \in \mathbb{N}$ *and for all* $(\omega, \kappa) \in \mathbb{R}^\nu \times [\kappa_1, \kappa_2]$ *the operator* $\mathbf{U}_{\overline{\mathtt{n}}}$ *in* (4.7.28) *is well-defined, the operators* $\mathbf{U}_{\overline{\mathtt{n}}}^{\pm 1} - \mathbb{1}_\perp$ *are* $\mathcal{D}^{k_0}$*-modulo-tame with modulo-tame constants satisfying, for all* $s_0 \leqslant s \leqslant S$,

$$\mathfrak{M}^\sharp_{\mathbf{U}_{\overline{\mathtt{n}}}^{\pm 1} - \mathbb{1}_\perp}(s) \lesssim_S \varepsilon \upsilon^{-2} N_0^{\tau_1} (1 + \|\mathfrak{I}_0\|_{s + \mu(\mathtt{b})}^{k_0, \upsilon}), \tag{4.7.30}$$

where τ_1 *is given by* (4.7.6). *Moreover* $\mathbf{U}_{\overline{\mathtt{n}}}$, $\mathbf{U}_{\overline{\mathtt{n}}}^{-1}$ *are real, reversibility and momentum preserving. The operator* $\mathbf{L}_{\overline{\mathtt{n}}} = \omega \cdot \partial_\varphi \mathbb{1}_\perp + \mathrm{i}\, \mathbf{D}_{\overline{\mathtt{n}}} + \mathbf{R}_\perp^{(\overline{\mathtt{n}})}$, *defined in* (4.7.14) *with* $\mathtt{n} = \overline{\mathtt{n}}$ *is real, reversible and momentum preserving. The operator* $\mathbf{R}_\perp^{(\overline{\mathtt{n}})}$ *is* $\mathcal{D}^{k_0}$*-modulo-tame with a modulo-tame constant satisfying, for all* $s_0 \leqslant s \leqslant S$,

$$\mathfrak{M}^\sharp_{\mathbf{R}_\perp^{(\overline{\mathtt{n}})}}(s) \lesssim_S \varepsilon \upsilon^{-1} N_{\overline{\mathtt{n}}-1}^{-\mathtt{a}} (1 + \|\mathfrak{I}_0\|_{s + \mu(\mathtt{b})}^{k_0, \upsilon}).$$

Moreover, for all (ω, κ) *in* $\Lambda_{\overline{\mathtt{n}}}^\upsilon = \Lambda_{\overline{\mathtt{n}}}^\upsilon(i) = \bigcap_{\mathtt{n}=0}^{\overline{\mathtt{n}}} \Lambda_\mathtt{n}^\upsilon$, *where the sets* $\Lambda_\mathtt{n}^\upsilon$ *are defined in* (4.7.20), *the conjugation formula* $\mathbf{L}_{\overline{\mathtt{n}}} := \mathbf{U}_{\overline{\mathtt{n}}}^{-1} \mathbf{L}_0 \mathbf{U}_{\overline{\mathtt{n}}}$ *holds.*

Proof of Theorem 4.85

The proof of Theorem 4.85 is inductive. We first show that $(\mathbf{S1})_\mathtt{n}$-$(\mathbf{S3})_\mathtt{n}$ hold when $\mathtt{n} = 0$.

The step $\mathtt{n} = 0$. PROOF OF $(\mathbf{S1})_0$. Properties (4.7.14)-(4.7.15), (4.7.17) for $\mathtt{n} = 0$ hold by (4.7.1), (4.7.2), (4.7.3) with $\mathfrak{r}_j^{(0)} = 0$. We now prove that also (4.7.19) for $\mathtt{n} = 0$ holds.

Lemma 4.87. *We have* $\mathfrak{M}_0^\sharp(s), \mathfrak{M}_0^\sharp(s,\mathtt{b}) \lesssim_{s_0,\mathtt{b}} \mathfrak{M}_0(s,\mathtt{b})$.

Proof. Let $R \in \{R_\perp^{(0,d)}, R_\perp^{(0,o)}\}$. We prove that $\langle \partial_\varphi \rangle^{\mathtt{b}} R$ is $\mathcal{D}^{k_0}$-modulo-tame. Using the inequality

$$\langle \ell - \ell' \rangle^{2\mathtt{q}_0} \langle j - j' \rangle^2 \lesssim_{\mathtt{q}_0} 1 + |\ell - \ell'|^{2\mathtt{q}_0} + |j - j'|^2 + |\ell - \ell'|^{2\mathtt{q}_0} |j - j'|^2 ,$$

it follows, recalling (4.2.36), (4.7.10), (the matrix elements of the commutator $[\partial_x, A]$ are $\mathrm{i}(j - j')A_j^{j'}(\ell - \ell')$), that, for any $j' \in \mathbb{S}_0^c$, $\ell' \in \mathbb{Z}^\nu$,

$$\begin{aligned} &\upsilon^{2|k|} \sum_{\ell,j} \langle \ell, j \rangle^{2s} \langle \ell - \ell' \rangle^{2(s_0+\mathtt{b})} \langle j - j' \rangle^2 \left|\partial_\lambda^k R_j^{j'}(\ell - \ell')\right|^2 \\ &\quad \lesssim_{\mathtt{b}} \mathfrak{M}_0(s_0,\mathtt{b})^2 \langle \ell', j' \rangle^{2s} + \mathfrak{M}_0(s,\mathtt{b})^2 \langle \ell', j' \rangle^{2s_0} . \end{aligned} \tag{4.7.31}$$

Let $s_0 \leqslant s \leqslant S$. Then, for any $|k| \leqslant k_0$, by Cauchy-Schwartz inequality, we have

$$\begin{aligned} \| |\langle \partial_\varphi \rangle^{\mathtt{b}} \partial_\lambda^k R| h\|_s^2 &\leqslant \sum_{\ell,j} \langle \ell, j \rangle^{2s} \Big(\sum_{\ell',j'} \langle \ell - \ell' \rangle^{\mathtt{b}} \left|(\partial_\lambda^k R)_j^{j'}(\ell - \ell')\right| |h_{\ell',j'}| \Big)^2 \\ &\leqslant \sum_{\ell,j} \langle \ell, j \rangle^{2s} \Big(\sum_{\ell',j'} \langle \ell - \ell' \rangle^{s_0+\mathtt{b}} \langle j - j' \rangle |(\partial_\lambda^k R)_j^{j'}(\ell - \ell')||h_{\ell',j'}| \frac{1}{\langle \ell - \ell' \rangle^{s_0} \langle j - j' \rangle} \Big)^2 \\ &\lesssim_{s_0} \sum_{\ell,j} \langle \ell, j \rangle^{2s} \sum_{\ell',j'} \langle \ell - \ell' \rangle^{2(s_0+\mathtt{b})} \langle j - j' \rangle^2 |(\partial_\lambda R)_j^{j'}(\ell - \ell')|^2 |h_{\ell',j'}|^2 \\ &\overset{(4.7.31)}{\lesssim_{s_0,\mathtt{b}}} \upsilon^{-2|k|} \sum_{\ell',j'} |h_{\ell',j'}|^2 \left(\mathfrak{M}_0(s_0,\mathtt{b})^2 \langle \ell', j' \rangle^{2s} + \mathfrak{M}_0(s,\mathtt{b})^2 \langle \ell', j' \rangle^{2s_0} \right) . \end{aligned}$$

Therefore, we obtain $\mathfrak{M}^\sharp_{\langle \partial_\varphi \rangle^{\mathtt{b}} R}(s) \lesssim_{s_0,\mathtt{b}} \mathfrak{M}_0(s,\mathtt{b})$ and then $\mathfrak{M}_0^\sharp(s,\mathtt{b}) \lesssim_{s_0,\mathtt{b}} \mathfrak{M}_0(s,\mathtt{b})$. The inequality $\mathfrak{M}_0^\sharp(s) \lesssim_{s_0} \mathfrak{M}_0(s,\mathtt{b})$ follows similarly. ☐

PROOF OF $(\mathbf{S2})_0$. The proof of estimates (4.7.23), (4.7.24) at $\mathtt{n} = 0$ follows by (4.7.11), arguing similarly to Lemma 4.87.

PROOF OF $(\mathbf{S3})_0$. It is trivial since, by definition, $\Lambda_0^\upsilon(i_1) = \mathtt{DC}(2\upsilon,\tau) \times [\kappa_1,\kappa_2] \subset \Lambda_0^{\upsilon-\rho}(i_2)$.

The reducibility step. We now describe the generic inductive step, showing how to transform $\mathbf{L}_{\mathtt{n}}$ into $\mathbf{L}_{\mathtt{n}+1}$ by the conjugation with $\mathbf{\Phi}_{\mathtt{n}}$. For sake of simplicity in the notation, we drop the index $\mathtt{n}$ and we write $+$ instead of $\mathtt{n}+1$, so that we write $\mathbf{L} := \mathbf{L}_{\mathtt{n}}$, $\mathbf{L}_+ := \mathbf{L}_{\mathtt{n}+1}$, $\mathbf{R}_\perp := \mathbf{R}_\perp^{(\mathtt{n})}$, $\mathbf{R}_\perp^{(+)} := \mathbf{R}_\perp^{(\mathtt{n}+1)}$, $N := N_{\mathtt{n}}$, etc. We conjugate $\mathbf{L}$ in (4.7.14) by a transformation of the form

$$\mathbf{\Phi} := e^{\mathbf{X}} , \quad \mathbf{X} := \begin{pmatrix} X^{(d)} & X^{(o)} \\ \overline{X^{(o)}} & \overline{X^{(d)}} \end{pmatrix} , \ X^{(d)} : H_{\mathbb{S}_0}^\perp \to H_{\mathbb{S}_0}^\perp , \ X^{(o)} : H_{-\mathbb{S}_0}^\perp \to H_{\mathbb{S}_0}^\perp , \tag{4.7.32}$$

where $\mathbf{X}$ is a bounded linear operator, chosen below in (4.7.37), (4.7.38). By the Lie expansions (4.2.16)-(4.2.17) we have

$$\begin{aligned}\mathbf{L}_+ := \mathbf{\Phi}^{-1}\mathbf{L}\mathbf{\Phi} = \omega\cdot\partial_\varphi \mathbb{1}_\perp + \mathrm{i}\,\mathbf{D} + ((\omega\cdot\partial_\varphi\mathbf{X}) - \mathrm{i}[\mathbf{X},\mathbf{D}] + \Pi_N\mathbf{R}_\perp) + \Pi_N^\perp\mathbf{R}_\perp \\ - \int_0^1 e^{-\tau\mathbf{X}}[\mathbf{X},\mathbf{R}_\perp]e^{\tau\mathbf{X}}\,\mathrm{d}\tau - \int_0^1 (1-\tau)e^{-\tau\mathbf{X}}[\mathbf{X},(\omega\cdot\partial_\varphi\mathbf{X}) - \mathrm{i}[\mathbf{X},\mathbf{D}]]e^{\tau\mathbf{X}}\,\mathrm{d}\tau\end{aligned} \tag{4.7.33}$$

where Π_N is defined in (4.2.40) and $\Pi_N^\perp := \mathrm{Id} - \Pi_N$. We want to solve the homological equation

$$\omega\cdot\partial_\varphi\mathbf{X} - \mathrm{i}[\mathbf{X},\mathbf{D}] + \Pi_N\mathbf{R}_\perp = [\mathbf{R}_\perp] \tag{4.7.34}$$

where

$$[\mathbf{R}_\perp] := \begin{pmatrix}[R_\perp^{(d)}] & 0 \\ 0 & \overline{[R_\perp^{(d)}]}\end{pmatrix}, \quad [R_\perp^{(d)}] := \mathrm{diag}_{j\in\mathbb{S}_0^c}(R_\perp^{(d)})_j^j(0)\,. \tag{4.7.35}$$

By (4.7.14), (4.7.17) and (4.7.32), the homological equation (4.7.34) is equivalent to the two scalar homological equations

$$\begin{aligned}\omega\cdot\partial_\varphi X^{(d)} - \mathrm{i}(X^{(d)}\mathcal{D} - \mathcal{D}X^{(d)}) + \Pi_N R_\perp^{(d)} &= [R_\perp^{(d)}] \\ \omega\cdot\partial_\varphi X^{(o)} + \mathrm{i}(X^{(o)}\overline{\mathcal{D}} + \mathcal{D}X^{(o)}) + \Pi_N R_\perp^{(o)} &= 0\,.\end{aligned} \tag{4.7.36}$$

Recalling (4.7.14) and since $\overline{\mathcal{D}} = \mathrm{diag}_{j\in-\mathbb{S}_0^c}(\mu_{-j})$, acting in $H_{-\mathbb{S}_0}^\perp$ (see (4.7.4)) the solutions of (4.7.36) are, for all $(\omega,\kappa)\in\Lambda_{\mathtt{n}+1}^{\upsilon}$ (see (4.7.20) with $\mathtt{n}\rightsquigarrow\mathtt{n}+1$)

$$(X^{(d)})_j^{j'}(\ell) := \begin{cases} -\dfrac{(R_\perp^{(d)})_j^{j'}(\ell)}{\mathrm{i}(\omega\cdot\ell + \mu_j - \mu_{j'})} & \text{if } \begin{cases}(\ell,j,j')\neq(0,j,j),\ j,j'\in\mathbb{S}_0^c,\ \langle\ell\rangle\leqslant N \\ \ell\cdot\vec{\jmath} + j - j' = 0\end{cases} \\ 0 & \text{otherwise}\,,\end{cases} \tag{4.7.37}$$

$$(X^{(o)})_j^{j'}(\ell) := \begin{cases} -\dfrac{(R_\perp^{(o)})_j^{j'}(\ell)}{\mathrm{i}(\omega\cdot\ell + \mu_j + \mu_{-j'})} & \text{if } \begin{cases}\forall\,\ell\in\mathbb{Z}^\nu\ j,-j'\in\mathbb{S}_0^c,\ \langle\ell\rangle\leqslant N \\ \ell\cdot\vec{\jmath} + j - j' = 0\end{cases} \\ 0 & \text{otherwise}\,.\end{cases} \tag{4.7.38}$$

Note that, since $-j'\in\mathbb{S}_0^c$, we can apply the bounds (4.7.20) for $(\omega,\kappa)\in\Lambda_{\mathtt{n}+1}^{\upsilon}$.

Lemma 4.88. (Homological equations) *The real operator* $\mathbf{X}$ *defined in* (4.7.32), (4.7.37), (4.7.38), *(which for all* $(\omega,\kappa)\in\Lambda_{\mathtt{n}+1}^{\upsilon}$ *solves the homological equation* (4.7.20)*)) admits an extension to the whole parameter space* $\mathbb{R}^\nu\times[\kappa_1,\kappa_2]$. *Such extended operator is* $\mathcal{D}^{k_0}$*-modulo-tame with a modulo-tame constant satisfying, for all* $s_0\leqslant s\leqslant S$,

$$\mathfrak{M}_{\mathbf{X}}^\sharp(s) \lesssim_{k_0} N^{\tau_1}\upsilon^{-1}\mathfrak{M}^\sharp(s)\,, \quad \mathfrak{M}_{\langle\partial_\varphi\rangle^{\mathtt{b}}\mathbf{X}}^\sharp(s) \lesssim_{k_0} N^{\tau_1}\upsilon^{-1}\mathfrak{M}^\sharp(s,\mathtt{b})\,, \tag{4.7.39}$$

where $\tau_1 := \tau(k_0+1)+k_0$. *If* $\upsilon/2 \leqslant \upsilon_1, \upsilon_2 \leqslant 2\upsilon$, *then, for all* $(\omega,\kappa) \in \Lambda^{\upsilon_1}_{\mathtt{n}+1}(i_1) \cap \Lambda^{\upsilon_2}_{\mathtt{n}+1}(i_2)$,

$$\| |\Delta_{12}\mathbf{X}| \|_{\mathcal{L}(H^{s_0})} \lesssim N^{2\tau}\upsilon^{-1}(\| |\mathbf{R}_\perp(i_2)| \|_{\mathcal{L}(H^{s_0})} \|i_1-i_2\|_{s_0+\mu(\mathtt{b})} + \| |\Delta_{12}\mathbf{R}_\perp| \|_{\mathcal{L}(H^{s_0})}) , \quad (4.7.40)$$

$$\| |\langle\partial_\varphi\rangle^{\mathtt{b}}\Delta_{12}\mathbf{X}| \|_{\mathcal{L}(H^{s_0})} \lesssim$$
$$N^{2\tau}\upsilon^{-1}(\| |\langle\partial_\varphi\rangle^{\mathtt{b}}\mathbf{R}_\perp(i_2)| \|_{\mathcal{L}(H^{s_0})} \|i_1-i_2\|_{s_0+\mu(\mathtt{b})} + \| |\langle\partial_\varphi\rangle^{\mathtt{b}}\Delta_{12}\mathbf{R}_\perp| \|_{\mathcal{L}(H^{s_0})}) . \quad (4.7.41)$$

The operator $\mathbf{X}$ *is reversibility and momentum preserving.*

Proof. We prove that (4.7.39) holds for $X^{(d)}$. The proof for $X^{(o)}$ holds analogously. First, we extend the solution in (4.7.37) to all λ in $\mathbb{R}^\nu \times [\kappa_1,\kappa_2]$ by setting (without any further relabeling) $(X^{(d)})_j^{j'}(\ell) = \mathrm{i}\, g_{\ell,j,j'}(\lambda)(R_\perp^{(d)})_j^{j'}(\ell)$, where

$$g_{\ell,j,j'}(\lambda) := \frac{\chi(f(\lambda)\rho^{-1})}{f(\lambda)} , \quad f(\lambda) := \omega\cdot\ell + \mu_j - \mu_{j'} , \quad \rho := \upsilon\langle\ell\rangle^{-\tau}\langle |j|^{\frac32} - |j'|^{\frac32}\rangle ,$$

and χ is the cut-off function (4.2.10). By (4.7.15), (4.7.16), (4.6.189), (4.7.20), Lemma 4.52, (4.4.40), together with (4.2.10), we deduce that, for any $k_1 \in \mathbb{N}_0^\nu$, $|k_1| \leqslant k_0$,

$$\sup_{|k_1|\leqslant k_0} \left|\partial_\lambda^{k_1} g_{\ell,j,j'}\right| \lesssim_{k_0} \langle\ell\rangle^{\tau_1}\upsilon^{-1-|k_1|} , \quad \tau_1 = \tau(k_0+1)+k_0 ,$$

and we deduce, for all $0 \leqslant |k| \leqslant k_0$,

$$\begin{aligned} |\partial_\lambda^k (X^{(d)})_j^{j'}(\ell)| &\lesssim_{k_0} \sum_{k_1+k_2=k} |\partial_\lambda^{k_1} g_{\ell,j,j'}(\lambda)||\partial_\lambda^{k_2}(R_\perp^{(d)})_j^{j'}(\ell)| \\ &\lesssim_{k_0} \langle\ell\rangle^{\tau_1}\upsilon^{-1-|k|} \sum_{|k_2|\leqslant|k|} \upsilon^{|k_2|}|\partial_\lambda^{k_2}(R_\perp^{(d)})_j^{j'}(\ell)| . \end{aligned} \quad (4.7.42)$$

By (4.7.37) we have that $(X^{(d)})_j^{j'}(\ell) = 0$ for all $\langle\ell\rangle > N$. Therefore, for all $|k| \leqslant k_0$, we have

$$\begin{aligned} \| |\langle\partial_\varphi\rangle^{\mathtt{b}}\partial_\lambda^k X^{(d)}|h\|_s^2 &\leqslant \sum_{\ell,j}\langle\ell,j\rangle^{2s}\Big(\sum_{\langle\ell-\ell'\rangle\leqslant N,j'} |\langle\ell-\ell'\rangle^{\mathtt{b}}\partial_\lambda^k(X^{(d)})_j^{j'}(\ell-\ell')||h_{\ell',j'}|\Big)^2 \\ &\overset{(4.7.42)}{\lesssim_{k_0}} N^{2\tau_1}\upsilon^{-2(1+|k|)}\sum_{|k_2|\leqslant|k|}\upsilon^{2|k_2|}\sum_{\ell,j}\langle\ell,j\rangle^{2s}\Big(\sum_{\ell',j'}|\langle\ell-\ell'\rangle^{\mathtt{b}}\partial_\lambda^{k_2}(R_\perp^{(d)})_j^{j'}(\ell-\ell')||h_{\ell',j'}|\Big)^2 \\ &\lesssim_{k_0} N^{2\tau_1}\upsilon^{-2(1+|k|)}\sum_{|k_2|\leqslant|k|}\upsilon^{2|k_2|}\| |\langle\partial_\varphi\rangle^{\mathtt{b}}\partial_\lambda^{k_2}R_\perp^{(d)}||h|\|_s^2 \\ &\overset{(4.2.39),(4.7.18)}{\lesssim_{k_0}} N^{2\tau_1}\upsilon^{-2(1+|k|)}\big(\mathfrak{M}^\sharp(s,\mathtt{b})^2\|h\|_{s_0}^2 + \mathfrak{M}^\sharp(s_0,\mathtt{b})^2\|h\|_s^2\big) , \end{aligned}$$

and, by Definition 4.26, we conclude that $\mathfrak{M}^\sharp_{\langle\partial_\varphi\rangle^{\mathtt{b}}X^{(d)}}(s) \lesssim_{k_0} N^{\tau_1}\upsilon^{-1}\mathfrak{M}^\sharp(s,\mathtt{b})$. The analogous estimates for $\langle\partial_\varphi\rangle^{\mathtt{b}}X^{(o)}$, $X^{(d)}$, $X^{(o)}$ and (4.7.40), (4.7.41) follow similarly. By induction, the operator $\mathbf{R}_\perp$ is reversible and momentum preserving. Therefore, by (4.7.32), (4.7.37), (4.7.38) and Lemmata 4.32, 4.42, it follows that $\mathbf{X}$ is reversibility and momentum preserving. □

By (4.7.33), (4.7.34), for all $\lambda \in \Lambda^{\upsilon}_{\mathtt{n}+1}$, we have

$$\mathbf{L}_+ = \mathbf{\Phi}^{-1}\mathbf{L}\mathbf{\Phi} = \omega \cdot \partial_\varphi \mathbb{1}_\perp + \mathrm{i}\,\mathbf{D}_+ + \mathbf{R}_\perp^{(+)}\,, \tag{4.7.43}$$

where

$$\begin{aligned} &\mathbf{D}_+ := \mathbf{D} - \mathrm{i}[\mathbf{R}_\perp]\,, \\ &\mathbf{R}_\perp^{(+)} := \Pi_N^\perp \mathbf{R}_\perp - \int_0^1 e^{-\tau \mathbf{X}}[\mathbf{X}, \mathbf{R}_\perp] e^{\tau \mathbf{X}}\,\mathrm{d}\tau + \int_0^1 (1-\tau) e^{-\tau\mathbf{X}}[\mathbf{X}, \Pi_N \mathbf{R}_\perp - [\mathbf{R}_\perp]] e^{\tau\mathbf{X}}\,\mathrm{d}\tau\,. \end{aligned} \tag{4.7.44}$$

The right hand side of (4.7.43)-(4.7.44) define an extension of $\mathbf{L}_+$ to the whole parameter space $\mathbb{R}^\nu \times [\kappa_1, \kappa_2]$, since $\mathbf{R}_\perp$ and $\mathbf{X}$ are defined on $\mathbb{R}^\nu \times [\kappa_1, \kappa_2]$.

The new operator $\mathbf{L}_+$ in (4.7.43) has the same form of $\mathbf{L}$ in (4.7.14) with the non-diagonal remainder $\mathbf{R}_\perp^{(+)}$ which is the sum of a term $\Pi_N^\perp \mathbf{R}_\perp$ supported on high frequencies and a quadratic function of $\mathbf{X}$ and $\mathbf{R}_\perp$. The new normal form $\mathbf{D}_+$ is diagonal:

Lemma 4.89. (New diagonal part) *For all $(\omega, \kappa) \in \mathbb{R}^\nu \times [\kappa_1, \kappa_2]$, the new normal form is*

$$\mathrm{i}\,\mathbf{D}_+ = \mathrm{i}\,\mathbf{D} + [\mathbf{R}_\perp] = \mathrm{i}\begin{pmatrix} \mathcal{D}_+ & 0 \\ 0 & -\overline{\mathcal{D}_+} \end{pmatrix}\,,\quad \mathcal{D}_+ := \mathrm{diag}_{j\in\mathbb{S}_0^c}\, \mu_j^{(+)}\,,\quad \mu_j^{(+)} := \mu_j + \mathtt{r}_j \in \mathbb{R}\,,$$

where each $\mathtt{r}_j$ satisfies, on $\mathbb{R}^\nu \times [\kappa_1, \kappa_2]$,

$$|\mathtt{r}_j|^{k_0,\upsilon} = |\mu_j^{(+)} - \mu_j|^{k_0,\upsilon} \lesssim \mathfrak{M}^\sharp(s_0)\,. \tag{4.7.45}$$

Moreover, given tori $i_1(\omega,\kappa), i_2(\omega,\kappa)$, we have $|\mathtt{r}_j(i_1) - \mathtt{r}_j(i_2)| \lesssim \| |\Delta_{12}\mathbf{R}_\perp| \|_{\mathcal{L}(H^{s_0})}$.

Proof. Recalling (4.7.35), we have that $\mathtt{r}_j := -\mathrm{i}(R_\perp^{(d)})_j^j(0)$, for all $j \in \mathbb{S}_0^c$. By the reversibility of $R_\perp^{(d)}$ and (4.2.52) we deduce that $\mathtt{r}_j \in \mathbb{R}$. Recalling the definition of $\mathfrak{M}^\sharp(s_0)$ in (4.7.18) (with $s = s_0$) and Definition 4.26, we have, for all $0 \leqslant |k| \leqslant k_0$, $\| |\partial_\lambda^k R_\perp^{(d)}| h\|_{s_0} \leqslant 2\upsilon^{-|k|}\mathfrak{M}^\sharp(s_0)\,\|h\|_{s_0}$, and therefore $|\partial_\lambda^k (R_\perp^{(d)})_j^j(0)| \lesssim \upsilon^{-|k|}\mathfrak{M}^\sharp(s_0)$. Hence (4.7.45) follows. The last bound for $|\mathtt{r}_j(i_1) - \mathtt{r}_j(i_2)|$ follows analogously. □

The iterative step. Let $\mathtt{n} \in \mathbb{N}_0$ and assume that the statements $(\mathbf{S1})_\mathtt{n}$-$(\mathbf{S3})_\mathtt{n}$ are true. We now prove $(\mathbf{S1})_{\mathtt{n}+1}$-$(\mathbf{S3})_{\mathtt{n}+1}$. For sake of simplicity in the notation (as in other parts of the paper) we omit to write the dependence on k_0, which is considered as a fixed constant.

Proof of $(\mathbf{S1})_{\mathtt{n}+1}$. The real operator $\mathbf{X}_\mathtt{n}$ defined in Lemma 4.88 is defined for all $(\omega,\kappa) \in \mathbb{R}^\nu \times [\kappa_1, \kappa_2]$ and, by (4.7.39), (4.7.19), satisfies the estimates (4.7.22) at the step $\mathtt{n}+1$. The flow maps $\mathbf{\Phi}_{\mathtt{n}}^{\pm 1} = e^{\pm \mathbf{X}_{\mathtt{n}}}$ are well defined by Lemma 4.28. By (4.7.43), for all $\lambda \in \Lambda_{\mathtt{n}+1}^{\upsilon}$, the conjugation formula (4.7.21) holds at the step $\mathtt{n}+1$. The operator $\mathbf{X}_{\mathtt{n}}$ is reversibility and momentum preserving, and so are the operators $\mathbf{\Phi}_{\mathtt{n}}^{\pm 1} = e^{\pm \mathbf{X}_{\mathtt{n}}}$. By Lemma 4.89, the operator $\mathbf{D}_{\mathtt{n}+1}$ is diagonal with eigenvalues $\mu_j^{(\mathtt{n}+1)} : \mathbb{R}^\nu \times [\kappa_1, \kappa_2] \to \mathbb{R}$, $\mu_j^{(\mathtt{n}+1)} = \mu_j^{(0)} + \mathfrak{r}_j^{(\mathtt{n}+1)}$ with $\mathfrak{r}_j^{(\mathtt{n}+1)} := \mathfrak{r}_j^{(\mathtt{n})} + \mathtt{r}_j^{(\mathtt{n})}$ satisfying, using also (4.7.19), (4.7.16) at the step $\mathtt{n}+1$. The next lemma provides the estimates of the remainder $\mathbf{R}_\perp^{(\mathtt{n}+1)} = \mathbf{R}_\perp^{(+)}$ defined in (4.7.44).

Lemma 4.90. *The operators $\mathbf{R}_\perp^{(\mathtt{n}+1)}$ and $\langle \partial_\varphi \rangle^{\mathtt{b}} \mathbf{R}_\perp^{(\mathtt{n}+1)}$ are $\mathcal{D}^{k_0}$-modulo-tame with modulo-tame constants satisfying*

$$\mathfrak{M}_{\mathtt{n}+1}^\sharp(s) \lesssim N_{\mathtt{n}}^{-\mathtt{b}} \mathfrak{M}_{\mathtt{n}}^\sharp(s,\mathtt{b}) + N_{\mathtt{n}}^{\tau_1} \upsilon^{-1} \mathfrak{M}_{\mathtt{n}}^\sharp(s) \mathfrak{M}_{\mathtt{n}}^\sharp(s_0)\,, \tag{4.7.46}$$

$$\mathfrak{M}_{\mathtt{n}+1}^\sharp(s,\mathtt{b}) \lesssim_{\mathtt{b}} \mathfrak{M}_{\mathtt{n}}^\sharp(s,\mathtt{b}) + N_{\mathtt{n}}^{\tau_1} \upsilon^{-1} \big(\mathfrak{M}_{\mathtt{n}}^\sharp(s,\mathtt{b}) \mathfrak{M}_{\mathtt{n}}^\sharp(s_0) + \mathfrak{M}_{\mathtt{n}}^\sharp(s_0,\mathtt{b}) \mathfrak{M}_{\mathtt{n}}^\sharp(s) \big)\,. \tag{4.7.47}$$

Proof. The estimates (4.7.46), (4.7.47) follow by (4.7.44), Lemmata 4.27, , 4.28, (4.2.41) and (4.7.39), (4.7.19), (4.7.6), (4.7.12), (4.7.13). □

Lemma 4.91. *Estimates* (4.7.19) *holds at the step* $\mathtt{n}+1$.

Proof. It follows by (4.7.46), (4.7.47), (4.7.19) at the step $\mathtt{n}$, (4.7.6), the smallness condition (4.7.13) with $N_0 = N_0(s_0, \mathtt{b}) > 0$ large enough and taking $\tau_2 > \tau_1 + \mathtt{a}$. □

Finally $\mathbf{R}_\perp^{(\mathtt{n}+1)}$ is real, reversible and momentum preserving as $\mathbf{R}_\perp^{(\mathtt{n})}$, since $\mathbf{X}_{\mathtt{n}}$ is real, reversibility and momentum preserving. This concludes the proof of $(\mathbf{S1})_{\mathtt{n}+1}$.
PROOF OF $(\mathbf{S2})_{\mathtt{n}+1}$. It follows by similar arguments and we omit it.
PROOF OF $(\mathbf{S3})_{\mathtt{n}+1}$. The proof follows as for $(\mathbf{S4})_{\nu+1}$ of Theorem 7.3 in [44], using $(\mathbf{S2})_{\mathtt{n}}$ and the fact that the momentum condition in (4.7.20) implies $|j - j'| \lesssim N_{\mathtt{n}}$.

Almost invertibility of $\mathcal{L}_\omega$

By (4.6.184) and Theorem 4.86 (where $\mathbf{L}_0 = \mathcal{L}_\perp$) we obtain

$$\mathcal{L}_\omega = \mathbf{W}_{2,\bar{\mathtt{n}}} \mathbf{L}_{\bar{\mathtt{n}}} \mathbf{W}_{1,\bar{\mathtt{n}}}^{-1}\,, \quad \mathbf{W}_{1,\bar{\mathtt{n}}} := \mathcal{W}_1^\perp \mathbf{U}_{\bar{\mathtt{n}}}\,, \quad \mathbf{W}_{2,\bar{\mathtt{n}}} := \mathcal{W}_2^\perp \mathbf{U}_{\bar{\mathtt{n}}}\,, \tag{4.7.48}$$

where the operator $\mathbf{L}_{\bar{\mathtt{n}}}$ is defined in (4.7.14) with $\mathtt{n} = \bar{\mathtt{n}}$. By (4.6.182) and (4.7.30), we have, for some $\sigma := \sigma(\tau, \nu, k_0) > 0$, for any $s_0 \leqslant s \leqslant S$,

$$\| \mathbf{W}_{1,\bar{\mathtt{n}}}^{\pm 1} h \|_s^{k_0,\upsilon}, \| \mathbf{W}_{2,\bar{\mathtt{n}}}^{\pm 1} h \|_s^{k_0,\upsilon} \lesssim_S \| h \|_{s+\sigma}^{k_0,\upsilon} + \| \mathfrak{I}_0 \|_{s+\mu(\mathtt{b})+\sigma}^{k_0,\upsilon} \| h \|_{s_0+\sigma}^{k_0,\upsilon}\,. \tag{4.7.49}$$

In order to verify the almost invertibility assumption (AI) of $\mathcal{L}_\omega$ in Section 4.5, we decompose the operator $\mathbf{L}_{\bar{\mathtt{n}}}$ in (4.7.14) (with $\bar{\mathtt{n}}$ instead of $\mathtt{n}$) as

$$\mathbf{L}_{\bar{\mathtt{n}}} = \mathbf{D}_{\bar{\mathtt{n}}}^{\leqslant} + \mathbf{Q}_\perp^{(\bar{\mathtt{n}})} + \mathbf{R}_\perp^{(\bar{\mathtt{n}})} \tag{4.7.50}$$

where

$$\mathbf{D}_{\bar{\mathtt{n}}}^{\leqslant} := \Pi_{K_{\bar{\mathtt{n}}}}(\omega \cdot \partial_\varphi \mathbb{1}_\perp + \mathrm{i}\,\mathbf{D}_{\bar{\mathtt{n}}})\Pi_{K_{\bar{\mathtt{n}}}} + \Pi_{K_{\bar{\mathtt{n}}}}^\perp, \quad \mathbf{Q}_\perp^{(\bar{\mathtt{n}})} := \Pi_{K_{\bar{\mathtt{n}}}}^\perp(\omega \cdot \partial_\varphi \mathbb{1}_\perp + \mathrm{i}\,\mathbf{D}_{\bar{\mathtt{n}}})\Pi_{K_{\bar{\mathtt{n}}}}^\perp - \Pi_{K_{\bar{\mathtt{n}}}}^\perp, \tag{4.7.51}$$

and the smoothing operator Π_K on the traveling waves is defined in (4.2.6), and $\Pi_K^\perp := \mathrm{Id} - \Pi_K$. The constants $K_{\mathtt{n}}$ in (4.7.51) are $K_{\mathtt{n}} := K_0^{\chi^{\mathtt{n}}}$, $\chi = 3/2$ (cfr. (4.5.34)), and K_0 will be fixed in (4.8.5).

Lemma 4.92. (First order Melnikov non-resonance conditions) *For all $\lambda = (\omega, \kappa)$ in*

$$\Lambda_{\bar{\mathtt{n}}+1}^{\upsilon, I} := \Big\{ \lambda \in \mathbb{R}^\nu \times [\kappa_1, \kappa_2] \,:\, |\omega \cdot \ell + \mu_j^{(\bar{\mathtt{n}})}| \geqslant 2\upsilon \frac{|j|^{\frac{3}{2}}}{\langle \ell \rangle^\tau}, \, \forall\, |\ell| \leqslant K_{\bar{\mathtt{n}}},\, j \in \mathbb{S}_0^c, j + \vec{\jmath} \cdot \ell = 0 \Big\}, \tag{4.7.52}$$

on the subspace of the traveling waves $\tau_\varsigma g(\varphi) = g(\varphi - \vec{\jmath}\varsigma)$, $\varsigma \in \mathbb{R}$, such that $g(\varphi, \cdot) \in \mathbf{H}_{\mathbb{S}_0}^\perp$, the operator $\mathbf{D}_{\bar{\mathtt{n}}}^{\leqslant}$ in (4.7.51) is invertible and there exists an extension of the inverse operator (that we denote in the same way) to the whole $\mathbb{R}^\nu \times [\kappa_1, \kappa_2]$ satisfying the estimate

$$\|(\mathbf{D}_{\bar{\mathtt{n}}}^{\leqslant})^{-1} g\|_s^{k_0,\upsilon} \lesssim_{k_0} \upsilon^{-1} \|g\|_{s+\tau_1}^{k_0,\upsilon}, \quad \tau_1 = k_0 + \tau(k_0 + 1). \tag{4.7.53}$$

Moreover $(\mathbf{D}_{\bar{\mathtt{n}}}^{\leqslant})^{-1} g$ is a traveling wave.

Proof. The estimate (4.7.53) follows arguing as in Lemma 4.88. □

Standard smoothing properties imply that the operator $\mathbf{Q}_\perp^{(\bar{\mathtt{n}})}$ in (4.7.51) satisfies, for any traveling wave $h \in \mathbf{H}_{\mathbb{S}_0}^\perp$, for all $b > 0$,

$$\|\mathbf{Q}_\perp^{(\bar{\mathtt{n}})} h\|_{s_0}^{k_0,\upsilon} \lesssim K_{\bar{\mathtt{n}}}^{-b} \|h\|_{s_0+b+\frac{3}{2}}^{k_0,\upsilon}, \quad \|\mathbf{Q}_\perp^{(\bar{\mathtt{n}})} h\|_s^{k_0,\upsilon} \lesssim \|h\|_{s+\frac{3}{2}}^{k_0,\upsilon}. \tag{4.7.54}$$

By the decompositions (4.7.48), (4.7.50), Theorem 4.86 (note that (4.5.1) and Lemma 4.61 imply (4.6.14)), Proposition 4.83, the fact that $\mathbf{W}_{1,\bar{\mathtt{n}}}$, $\mathbf{W}_{2,\bar{\mathtt{n}}}$ map (anti)-reversible, respectively traveling, waves, into (anti)-reversible, respectively traveling, waves (Lemma 4.80) and estimates (4.7.49), (4.7.53), (4.7.54), (4.2.8) we deduce the following theorem.

Theorem 4.93. (Almost invertibility of $\mathcal{L}_\omega$) *Assume* (4.5.1). *Let* $\mathtt{a}, \mathtt{b}$ *as in* (4.7.6) *and* M *as in* (4.7.5). *Let* $S > s_0$ *and assume the smallness condition* (4.7.29). *Then the almost invertibility* *assumption (AI) in Section 4.5 holds with* Λ_0 *replaced by*

$$\mathbf{\Lambda}^{\upsilon}_{\bar{\mathtt{n}}+1} := \mathbf{\Lambda}^{\upsilon}_{\bar{\mathtt{n}}+1}(i) := \Lambda^{\upsilon}_{\bar{\mathtt{n}}+1} \cap \Lambda^{\upsilon,I}_{\bar{\mathtt{n}}+1}\,, \tag{4.7.55}$$

(see (4.7.20), (4.7.52)*) and, with* $\mu(\mathtt{b})$ *defined in* (4.7.7),

$$\mathcal{L}^{<}_{\omega} := \mathbf{W}_{2,\bar{\mathtt{n}}}\mathbf{D}^{<}_{\bar{\mathtt{n}}}\mathbf{W}^{-1}_{1,\bar{\mathtt{n}}}\,, \quad \mathcal{R}_{\omega} := \mathbf{W}_{2,\bar{\mathtt{n}}}\mathbf{R}^{(\bar{\mathtt{n}})}_{\perp}\mathbf{W}^{-1}_{1,\bar{\mathtt{n}}}\,, \qquad \mathcal{R}^{\perp}_{\omega} := \mathbf{W}_{2,\bar{\mathtt{n}}}\mathbf{Q}^{(\bar{\mathtt{n}})}_{\perp}\mathbf{W}^{-1}_{1,\bar{\mathtt{n}}}\,.$$

4.8 Proof of Theorem 4.55

Theorem 4.55 is a consequence of Theorem 4.95 below. We consider the finite dimensional subspaces of traveling wave variations

$$E_{\mathtt{n}} := \big\{\mathfrak{I}(\varphi) = (\Theta, I, w)(\varphi) \text{ such that (4.2.61) holds } : \ \Theta = \Pi_{\mathtt{n}}\Theta\,,\ I = \Pi_{\mathtt{n}}I\,,\ w = \Pi_{\mathtt{n}}w\big\}$$

where $\Pi_{\mathtt{n}}w := \Pi_{K_{\mathtt{n}}}w$ are defined as in (4.2.6) with K_n in (4.5.34), and we denote with the same symbol $\Pi_{\mathtt{n}}g(\varphi) := \sum_{|\ell|\leqslant K_{\mathtt{n}}} g_\ell e^{\mathrm{i}\ell\cdot\varphi}$. Note that the projector $\Pi_{\mathtt{n}}$ maps (anti)-reversible traveling variations into (anti)-reversible traveling variations.

In view of the Nash-Moser Theorem 4.95 we introduce the constants

$$\mathtt{a}_1 := \max\{6\sigma_1 + 13, \chi(p(\tau+1) + \mu(\mathtt{b}) + 2\sigma_1) + 1\}\,, \quad \mathtt{a}_2 := \chi^{-1}\mathtt{a}_1 - \mu(\mathtt{b}) - 2\sigma_1\,, \tag{4.8.1}$$

$$\mu_1 := 3(\mu(\mathtt{b}) + 2\sigma_1) + 1\,, \quad \mathtt{b}_1 := \mathtt{a}_1 + 2\mu(\mathtt{b}) + 4\sigma_1 + 3 + \chi^{-1}\mu_1\,, \quad \chi = 3/2 \tag{4.8.2}$$

$$\sigma_1 := \max\{\overline{\sigma}, 2s_0 + 2k_0 + 5\}\,, \quad S = s_0 + \mathtt{b}_1\,, \tag{4.8.3}$$

where $\overline{\sigma} = \overline{\sigma}(\tau, \nu, k_0) > 0$ is defined by Theorem 4.65, $2s_0 + 2k_0 + 5$ is the largest loss of regularity in the estimates of the Hamiltonian vector field X_P in Lemma 4.60, $\mu(\mathtt{b})$ is defined in (4.7.7), and $\mathtt{b} = [\mathtt{a}] + 2$ is defined in (4.7.6). The exponent p in (4.5.34) is required to satisfy

$$p\mathtt{a} > \tfrac{1}{2}\mathtt{a}_1 + \tfrac{3}{2}\sigma_1\,. \tag{4.8.4}$$

By (4.7.6), and the definition of $\mathtt{a}_1$ in (4.8.1), there exists $p = p(\tau, \nu, k_0)$ such that (4.8.4) holds, for example we fix

$$p := \frac{3(\mu(\mathtt{b}) + 4\sigma_1 + 1)}{\mathtt{a}}\,.$$

Remark 4.94. The constant $\mathtt{a}_1$ is the exponent in (4.8.9). The constant $\mathtt{a}_2$ is the exponent in the second bound in (4.8.7). The constant μ_1 is the exponent in $(\mathcal{P}3)_{\mathtt{n}}$. The conditions on the constants $\mu_1, \mathtt{b}_1, \mathtt{a}_1$ to allow the convergence of the Nash-Moser scheme in Theorem 4.95 are

$$\mathtt{a}_1 > 6\sigma_1 + 12\,, \quad \mathtt{b}_1 > \mathtt{a}_1 + 2\mu(\mathtt{b}) + 4\sigma_1 + \chi^{-1}\mu_1\,, \quad p\mathtt{a} > \tfrac{1}{2}\mathtt{a}_1 + \tfrac{3}{2}\sigma_1\,,$$

as well as $\mu_1 > 3(\mu(\mathtt{b}) + 2\sigma_1)$. In addition, we require $\mathtt{a}_1 \geqslant \chi(p(\tau+1) + \mu(\mathtt{b}) + 2\sigma_1) + 1$ so that $\mathtt{a}_2 \geqslant p(\tau+1) + \chi^{-1}$, which is used in the proof of Lemma 4.96.

Given a function $W = (\mathfrak{I}, \beta)$ where $\mathfrak{I}$ is the periodic component of a torus as in (4.4.9) and $\beta \in \mathbb{R}^\nu$, we denote $\|W\|_s^{k_0,\upsilon} := \|\mathfrak{I}\|_s^{k_0,\upsilon} + |\beta|^{k_0,\upsilon}$.

Theorem 4.95. (Nash-Moser) *There exist $\delta_0, C_* > 0$ such that, if*

$$K_0^{\tau_3} \varepsilon \upsilon^{-2} < \delta_0\,, \ \tau_3 := \max\{p\tau_2, 2\sigma_1 + \mathtt{a}_1 + 4\}\,, \ K_0 := \upsilon^{-1}\,, \ \upsilon := \varepsilon^{\mathrm{a}}\,, \ 0 < \mathrm{a} < (2+\tau_3)^{-1}\,, \tag{4.8.5}$$

where $\tau_2 = \tau_2(\tau, \nu)$ is given by Theorem 4.85, then, for all $\mathtt{n} \geqslant 0$:

$(\mathcal{P}1)_{\mathtt{n}}$ *There exists a k_0-times differentiable function $\widetilde{W}_{\mathtt{n}} : \mathbb{R}^\nu \times [\kappa_1, \kappa_2] \to E_{\mathtt{n}-1} \times \mathbb{R}^\nu$, $\lambda = (\omega, \kappa) \mapsto \widetilde{W}_{\mathtt{n}}(\lambda) := (\widetilde{\mathfrak{I}}_{\mathtt{n}}, \widetilde{\alpha}_{\mathtt{n}} - \omega)$, for $\mathtt{n} \geqslant 1$, and $\widetilde{W}_0 := 0$, satisfying*

$$\|\widetilde{W}_{\mathtt{n}}\|_{s_0+\mu(\mathtt{b})+\sigma_1}^{k_0,\upsilon} \leqslant C_* \varepsilon \upsilon^{-1}\,. \tag{4.8.6}$$

Let $\widetilde{U}_{\mathtt{n}} := U_0 + \widetilde{W}_{\mathtt{n}}$, where $U_0 := (\varphi, 0, 0, \omega)$. The difference $\widetilde{H}_{\mathtt{n}} := \widetilde{U}_{\mathtt{n}} - \widetilde{U}_{\mathtt{n}-1}$, for $\mathtt{n} \geqslant 1$, satisfies

$$\|\widetilde{H}_1\|_{s_0+\mu(\mathtt{b})+\sigma_1}^{k_0,\upsilon} \leqslant C_* \varepsilon \upsilon^{-1}\,, \quad \|\widetilde{H}_{\mathtt{n}}\|_{s_0+\mu(\mathtt{b})+\sigma_1}^{k_0,\upsilon} \leqslant C_* \varepsilon \upsilon^{-1} K_{\mathtt{n}-1}^{-\mathtt{a}_2}\,, \ \forall\, \mathtt{n} \geqslant 2\,. \tag{4.8.7}$$

The torus embedding $\widetilde{\imath}_{\mathtt{n}} := (\varphi, 0, 0) + \widetilde{\mathfrak{I}}_{\mathtt{n}}$ is reversible and traveling, i.e. (4.4.8) holds.

$(\mathcal{P}2)_{\mathtt{n}}$ *We define*

$$\mathcal{G}_0 := \Omega \times [\kappa_1, \kappa_2]\,, \quad \mathcal{G}_{\mathtt{n}+1} := \mathcal{G}_{\mathtt{n}} \cap \mathbf{\Lambda}_{\mathtt{n}+1}^{\upsilon}(\widetilde{\imath}_{\mathtt{n}})\,, \quad \forall\, \mathtt{n} \geqslant 0\,, \tag{4.8.8}$$

where $\mathbf{\Lambda}_{\mathtt{n}+1}^{\upsilon}(\widetilde{\imath}_{\mathtt{n}})$ is defined in (4.7.55). Then, for all $\lambda \in \mathcal{G}_{\mathtt{n}}$, setting $K_{-1} := 1$, we have

$$\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0}^{k_0,\upsilon} \leqslant C_* \varepsilon K_{\mathtt{n}-1}^{-\mathtt{a}_1}\,. \tag{4.8.9}$$

$(\mathcal{P}3)_{\mathtt{n}}$ (High norms) *For all $\lambda \in \mathcal{G}_{\mathtt{n}}$, we have*

$$\|\widetilde{W}_{\mathtt{n}}\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon} \leqslant C_* \varepsilon \upsilon^{-1} K_{\mathtt{n}-1}^{\mu_1}\,. \tag{4.8.10}$$

Proof. We argue by induction.
STEP 1: Proof of $(\mathcal{P}1,2,3)_0$. They follow by

$$\|\mathcal{F}(U_0)\|_s^{k_0,\upsilon} \overset{(4.4.6),\ \text{Lemma4.60}}{=} O(\varepsilon) \tag{4.8.11}$$

taking C_* large enough and by noting that $i_0 := (\varphi, 0, 0)$ is clearly reversible and traveling, satisfying (4.2.60).

STEP 2: ASSUME THAT $(\mathcal{P}1,2,3)_{\mathtt{n}}$ HOLD FOR SOME $\mathtt{n} \in \mathbb{N}_0$ AND PROVE $(\mathcal{P}1,2,3)_{\mathtt{n}+1}$.
We are going to define the successive approximation $\widetilde{U}_{\mathtt{n}+1}$ by a modified Nash-Moser scheme and prove by induction that the approximate torus $\widetilde{\imath}_{\mathtt{n}+1}$ is a reversible and traveling wave. For that we prove the almost-approximate invertibility of the linearized operator $L_{\mathtt{n}} = L_{\mathtt{n}}(\lambda) := \mathrm{d}_{i,\alpha}\mathcal{F}(\widetilde{\imath}_{\mathtt{n}}(\lambda))$ applying Theorem 4.65 to $L_{\mathtt{n}}(\lambda)$. By (4.8.5), the smallness condition (4.7.29) holds for ε small enough. Therefore Theorem 4.93 holds and we deduce that the inversion assumption (4.5.36) holds for all $\lambda \in \mathbf{\Lambda}^{\upsilon}_{\mathtt{n}+1}$, see (4.7.55). Now we apply Theorem 4.65 to the linearized operator $L_{\mathtt{n}}(\lambda)$ with $\Lambda_o := \mathbf{\Lambda}^{\upsilon}_{\mathtt{n}+1}(\widetilde{\imath}_{\mathtt{n}})$ and $S := s_0 + \mathtt{b}_1$, where $\mathtt{b}_1$ is defined in (4.8.2). It implies the existence of an almost-approximate inverse $\mathbf{T}_{\mathtt{n}} := \mathbf{T}_{\mathtt{n}}(\lambda, \widetilde{\imath}_{\mathtt{n}})$ of the linearized operator $\mathrm{d}_{i,\alpha}\mathcal{F}(\widetilde{\imath}_{\mathtt{n}})$ which satisfies for any anti-reversible traveling wave variation g and for any $s_0 \leqslant s \leqslant s_0 + \mathtt{b}_1$

$$\|\mathbf{T}_{\mathtt{n}} g\|_s^{k_0,\upsilon} \lesssim_{s_0+\mathtt{b}_1} \upsilon^{-1}\big(\|g\|_{s+\sigma_1}^{k_0,\upsilon} + \|\widetilde{\mathfrak{I}}_{\mathtt{n}}\|_{s+\sigma_1+\mu(\mathtt{b})}^{k_0,\upsilon}\|g\|_{s_0+\sigma_1}^{k_0,\upsilon}\big)\,, \tag{4.8.12}$$

$$\|\mathbf{T}_{\mathtt{n}} g\|_{s_0}^{k_0,\upsilon} \lesssim_{s_0+\mathtt{b}_1} \upsilon^{-1}\|g\|_{s_0+\sigma_1}^{k_0,\upsilon}\,. \tag{4.8.13}$$

Moreover, the first three components of $\mathbf{T}_{\mathtt{n}} g$ form a reversible traveling wave variation. For all $\lambda \in \mathbb{R}^\nu \times [\kappa_1, \kappa_2]$ we define the successive approximation

$$U_{\mathtt{n}+1} := \widetilde{U}_{\mathtt{n}} + H_{\mathtt{n}+1}\,, \quad H_{\mathtt{n}+1} := (\widehat{\mathfrak{I}}_{\mathtt{n}+1}, \widehat{\alpha}_{\mathtt{n}+1}) := -\mathbf{\Pi}_{\mathtt{n}}\mathbf{T}_{\mathtt{n}}\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}}) \in E_{\mathtt{n}} \times \mathbb{R}^\nu\,, \tag{4.8.14}$$

where $\mathbf{\Pi}_{\mathtt{n}}$ is defined for any $(\mathfrak{I}, \alpha)$, with $\mathfrak{I}$ a traveling wave variation, by

$$\mathbf{\Pi}_{\mathtt{n}}(\mathfrak{I}, \alpha) := (\Pi_{\mathtt{n}}\mathfrak{I}, \alpha)\,, \quad \mathbf{\Pi}_{\mathtt{n}}^{\perp} := (\Pi_{\mathtt{n}}^{\perp}\mathfrak{I}, 0)\,. \tag{4.8.15}$$

By Lemma 4.54 and since $\widetilde{\imath}_{\mathtt{n}}$ is a reversible traveling wave by induction assumption, we have that $\mathcal{F}(\widetilde{U}_{\mathtt{n}}) = \mathcal{F}(\widetilde{\imath}_{\mathtt{n}}, \widetilde{\alpha}_n)$ is an anti-reversible traveling wave variation, i.e (4.5.39)-(4.5.40) hold. Thus the first three components of $\mathbf{T}_{\mathtt{n}}\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})$ form a reversible traveling wave variation, as well as $\mathbf{\Pi}_{\mathtt{n}}\mathbf{T}_{\mathtt{n}}\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})$. We now show that the iterative scheme in (4.8.14) is rapidly converging. We write

$$\mathcal{F}(U_{\mathtt{n}+1}) = \mathcal{F}(\widetilde{U}_{\mathtt{n}}) + L_{\mathtt{n}} H_{\mathtt{n}+1} + Q_{\mathtt{n}}\,,$$

where $L_{\mathtt{n}} := \mathrm{d}_{i,\alpha}\mathcal{F}(\widetilde{\imath}_{\mathtt{n}})$ and

$$Q_{\mathtt{n}} := Q(\widetilde{U}_{\mathtt{n}}, H_{\mathtt{n}+1})\,, \quad Q(\widetilde{U}_{\mathtt{n}}, H) := \mathcal{F}(\widetilde{U}_{\mathtt{n}} + H) - \mathcal{F}(\widetilde{U}_{\mathtt{n}}) - L_{\mathtt{n}} H\,, \quad H \in E_{\mathtt{n}} \times \mathbb{R}^\nu\,. \tag{4.8.16}$$

Then, by the definition of $H_{\mathtt{n}+1}$ in (4.8.14), we have

$$
\begin{aligned}
\mathcal{F}(U_{\mathtt{n}+1}) &= \mathcal{F}(\widetilde{U}_{\mathtt{n}}) - L_{\mathtt{n}}\mathbf{\Pi}_{\mathtt{n}}\mathbf{T}_{\mathtt{n}}\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}}) + Q_{\mathtt{n}} \\
&= \mathcal{F}(\widetilde{U}_{\mathtt{n}}) - L_{\mathtt{n}}\mathbf{T}_{\mathtt{n}}\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}}) + L_{\mathtt{n}}\Pi_{\mathtt{n}}^{\perp}\mathbf{T}_{\mathtt{n}}\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}}) + Q_{\mathtt{n}} \\
&= (\Pi_{\mathtt{n}} + \Pi_{\mathtt{n}}^{\perp})\mathcal{F}(\widetilde{U}_{\mathtt{n}}) - (\Pi_{\mathtt{n}} + \Pi_{\mathtt{n}}^{\perp})L_{\mathtt{n}}\mathbf{T}_{\mathtt{n}}\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}}) + L_{\mathtt{n}}\Pi_{\mathtt{n}}^{\perp}\mathbf{T}_{\mathtt{n}}\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}}) + Q_{\mathtt{n}} \\
&= \Pi_{\mathtt{n}}^{\perp}\mathcal{F}(\widetilde{U}_{\mathtt{n}}) + R_{\mathtt{n}} + P_{\mathtt{n}} + Q_{\mathtt{n}} \,,
\end{aligned}
\tag{4.8.17}
$$

where

$$
R_{\mathtt{n}} := (L_{\mathtt{n}}\mathbf{\Pi}_{\mathtt{n}}^{\perp} - \Pi_{\mathtt{n}}^{\perp}L_{\mathtt{n}})\mathbf{T}_{\mathtt{n}}\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}}) \,, \quad P_{\mathtt{n}} := \Pi_{\mathtt{n}}(\mathrm{Id} - L_{\mathtt{n}}\mathbf{T}_{\mathtt{n}})\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}}) \,. \tag{4.8.18}
$$

We first note that, for all $\lambda \in \Omega \times [\kappa_1, \kappa_2]$, $s \geqslant s_0$,

$$
\begin{aligned}
\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_s^{k_0,\upsilon} &\lesssim_s \|\mathcal{F}(U_0)\|_s^{k_0,\upsilon} + \|\mathcal{F}(\widetilde{U}_{\mathtt{n}}) - \mathcal{F}(U_0)\|_s^{k_0,\upsilon} \\
&\overset{(4.8.11)}{\lesssim_s} \varepsilon + \|L_{\mathtt{n}}\widetilde{W}_{\mathtt{n}}\|_s^{k_0,\upsilon} \overset{(4.8.3)}{\lesssim_s} \varepsilon + \|\widetilde{W}_{\mathtt{n}}\|_{s+\mu(\mathtt{b})+\sigma_1}^{k_0,\upsilon}
\end{aligned}
\tag{4.8.19}
$$

and, by (4.8.6), (4.8.5), we have

$$
\upsilon^{-1}\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0}^{k_0,\upsilon} \leqslant 1 \,. \tag{4.8.20}
$$

We want to estimate the H^{s_0}-norm of $\mathcal{F}(U_{\mathtt{n}+1})$, decomposed as in (4.8.17), in terms of $\mathcal{F}(\widetilde{U}_{\mathtt{n}})$. First, we need to estimate $\widetilde{H}_{\mathtt{n}+1}$ in $\|\cdot\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon}$. By (4.8.14), we have

$$
\begin{aligned}
\|\widetilde{H}_{\mathtt{n}+1}\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon} &\overset{(4.8.14)}{=} \|\mathbf{\Pi}_{\mathtt{n}}\mathbf{T}_{\mathtt{n}}\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon} \\
&\overset{(4.2.8)}{\lesssim_{s_0+\mathtt{b}_1}} K_{\mathtt{n}}^{\sigma_1}\|\mathbf{T}_{\mathtt{n}}\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\mathtt{b}_1-\sigma_1}^{k_0,\upsilon} \\
&\overset{(4.8.12)}{\lesssim_{s_0+\mathtt{b}_1}} \upsilon^{-1}K_{\mathtt{n}}^{\sigma_1}(\|\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon} + \|\widetilde{\mathfrak{I}}_{\mathtt{n}}\|_{s_0+\mathtt{b}_1+\mu(\mathtt{b})}^{k_0,\upsilon}\|\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\sigma_1}^{k_0,\upsilon}) \\
&\overset{(4.8.19),(4.8.20)}{\lesssim_{s_0+\mathtt{b}_1}} \upsilon^{-1}K_{\mathtt{n}}^{\mu(\mathtt{b})+2\sigma_1}(\varepsilon + \|\widetilde{W}_{\mathtt{n}}\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon}) \,,
\end{aligned}
\tag{4.8.21}
$$

$$
\|\widetilde{H}_{\mathtt{n}+1}\|_{s_0}^{k_0,\upsilon} \overset{(4.8.14),(4.8.13)}{\lesssim_{s_0}} \upsilon^{-1}K_{\mathtt{n}}^{\sigma_1}\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0}^{k_0,\upsilon} \,. \tag{4.8.22}
$$

Now we estimate $Q_{\mathtt{n}}$, $R_{\mathtt{n}}$, $P_{\mathtt{n}}$ in the norm $\|\cdot\|_{s_0}^{k_0,\upsilon}$. By the definition of $Q_{\mathtt{n}}$ in (4.8.16), we have the quadratic estimate

$$
\begin{aligned}
\|Q_{\mathtt{n}}\|_{s_0}^{k_0,\upsilon} &\lesssim_{s_0} \|\mathrm{d}_{i,\alpha}^2\mathcal{F}(\widetilde{U}_{\mathtt{n}})[\widetilde{H}_{\mathtt{n}+1}, \widetilde{H}_{\mathtt{n}+1}]\|_{s_0}^{k_0,\upsilon} \overset{(4.4.6)}{=} \varepsilon\|\mathrm{d}_{i,\alpha}^2 X_P(\widetilde{U}_{\mathtt{n}})[\widetilde{H}_{\mathtt{n}+1}, \widetilde{H}_{\mathtt{n}+1}]\|_{s_0}^{k_0,\upsilon} \\
&\overset{\text{Lemma 4.60}}{\lesssim_{s_0}} \varepsilon(1 + \|\widetilde{\mathfrak{I}}_{\mathtt{n}}\|_{2s_0+2k_0+5}^{k_0,\upsilon})(\|\widetilde{H}_{\mathtt{n}+1}\|_{s_0+2}^{k_0,\upsilon})^2 \\
&\overset{(4.8.3),(4.2.8),(4.8.13)}{\lesssim_{s_0}} \varepsilon\upsilon^{-2}K_{\mathtt{n}}^{2\sigma_1+4}(1 + \|\widetilde{\mathfrak{I}}_{\mathtt{n}}\|_{s_0+\sigma_1}^{k_0,\upsilon})(\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0}^{k_0,\upsilon})^2 \\
&\overset{\varepsilon\upsilon^{-1}\leqslant 1,\,(4.8.6)}{\lesssim_{s_0}} \upsilon^{-1}K_{\mathtt{n}}^{2\sigma_1+4}(\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0}^{k_0,\upsilon})^2 \,.
\end{aligned}
\tag{4.8.23}
$$

According to Theorem 4.65, we write the term $P_{\mathtt{n}}$ in (4.8.18) as

$$\begin{aligned} & P_{\mathtt{n}} = -\Pi_{\mathtt{n}}(L_{\mathtt{n}}\mathbf{T}_{\mathtt{n}} - \mathrm{Id})\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}}) = -P_{\mathtt{n}}^{(1)} - P_{\mathtt{n},\omega} - P_{\mathtt{n},\omega}^{\perp}\,, \\ & P_{\mathtt{n}}^{(1)} := \Pi_{\mathtt{n}}\mathcal{P}(\widetilde{\imath}_{\mathtt{n}})\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\,, \quad P_{\mathtt{n},\omega} := \Pi_{\mathtt{n}}\mathcal{P}_{\omega}(\widetilde{\imath}_{\mathtt{n}})\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\,, \quad P_{\mathtt{n},\omega}^{\perp} := \Pi_{\mathtt{n}}\mathcal{P}_{\omega}^{\perp}(\widetilde{\imath}_{\mathtt{n}})\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\,. \end{aligned}$$

Moreover, by (4.2.8), we have that

$$\begin{aligned} \|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\sigma_1}^{k_0,\upsilon} & \leqslant \|\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\sigma_1}^{k_0,\upsilon} + \|\Pi_{\mathtt{n}}^{\perp}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\sigma_1}^{k_0,\upsilon} \\ & K_{\mathtt{n}}^{\sigma_1}\big(\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0}^{k_0,\upsilon} + K_{\mathtt{n}}^{-\mathtt{b}_1}\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon}\big)\,, \end{aligned} \tag{4.8.24}$$

which implies, together with (4.8.6), (4.8.5), (4.8.19), (4.8.20) and the bounds in Theorem 4.65, the following estimates:

$$\begin{aligned} \|P_{\mathtt{n}}^{(1)}\|_{s_0}^{k_0,\upsilon} & \lesssim_{s_0+\mathtt{b}_1} \upsilon^{-1}(\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\sigma_1}^{k_0,\upsilon}\|\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\sigma_1}^{k_0,\upsilon} \\ & \qquad + (1 + \|\widetilde{\mathfrak{I}}_{\mathtt{n}}\|_{s_0+\mu(\mathtt{b})+\sigma_1}^{k_0,\upsilon})\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\sigma_1}^{k_0,\upsilon}\|\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\sigma_1}^{k_0,\upsilon}) \\ & \lesssim_{s_0+\mathtt{b}_1} \upsilon^{-1}\|(\Pi_{\mathtt{n}} + \Pi_{\mathtt{n}}^{\perp})\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\sigma_1}^{k_0,\upsilon}\|\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\sigma_1}^{k_0,\upsilon} \\ & \lesssim_{s_0+\mathtt{b}_1} \upsilon^{-1}K_{\mathtt{n}}^{2\sigma_1}((\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0}^{k_0,\upsilon})^2 + K_{\mathtt{n}}^{2\sigma_1-\mathtt{b}_1}\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon}\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0}^{k_0,\upsilon}) \\ & \lesssim_{s_0+\mathtt{b}_1} \upsilon^{-1}K_{\mathtt{n}}^{2\sigma_1}(\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0}^{k_0,\upsilon})^2 + \upsilon^{-1}K_{\mathtt{n}}^{3\sigma_1+\mu(\mathtt{b})-\mathtt{b}_1}(\varepsilon + \|\widetilde{W}_{\mathtt{n}}\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon})\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0}^{k_0,\upsilon}\,, \end{aligned} \tag{4.8.25}$$

$$\begin{aligned} \|P_{\mathtt{n},\omega}\|_{s_0}^{k_0,\upsilon} & \lesssim_{s_0+\mathtt{b}_1} \varepsilon\upsilon^{-2}K_{\mathtt{n}-1}^{-p\mathtt{a}}(1 + \|\widetilde{\mathfrak{I}}_{\mathtt{n}}\|_{s_0+\mu(\mathtt{b})+\sigma_1}^{k_0,\upsilon})\|\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\sigma_1}^{k_0,\upsilon} \\ & \lesssim_{s_0+\mathtt{b}_1} \varepsilon\upsilon^{-2}K_{\mathtt{n}-1}^{-p\mathtt{a}}\|\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\sigma_1}^{k_0,\upsilon} \lesssim_{s_0+\mathtt{b}_1} \varepsilon\upsilon^{-2}K_{\mathtt{n}-1}^{-p\mathtt{a}}K_{\mathtt{n}}^{\sigma_1}\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0}^{k_0,\upsilon}\,, \end{aligned} \tag{4.8.26}$$

$$\begin{aligned} \|P_{\mathtt{n},\omega}^{\perp}\|_{s_0}^{k_0,\upsilon} & \lesssim_{s_0+\mathtt{b}_1} \upsilon^{-1}K_{\mathtt{n}}^{\mathtt{b}_1}(\|\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\sigma_1+\mathtt{b}_1}^{k_0,\upsilon} + \|\widetilde{\mathfrak{I}}_{\mathtt{n}}\|_{s_0+\mu(\mathtt{b})+\sigma_1+\mathtt{b}_1}^{k_0,\upsilon}\|\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\sigma_1}^{k_0,\upsilon}) \\ & \lesssim_{\sigma_0+\mathtt{b}_1} \upsilon^{-1}K_{\mathtt{n}}^{\sigma_1-\mathtt{b}_1}\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon} + K_{\mathtt{n}}^{\mu(\mathtt{b})+2\sigma_1-\mathtt{b}_1}\|\widetilde{\mathfrak{I}}_{\mathtt{n}}\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon}\upsilon^{-1}\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0}^{k_0,\upsilon} \\ & \lesssim_{s_0+\mathtt{b}_1} \upsilon^{-1}K_{\mathtt{n}}^{\mu(\mathtt{b})+2\sigma_1-\mathtt{b}_1}(\varepsilon + \|\widetilde{W}_{\mathtt{n}}\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon})\,. \end{aligned} \tag{4.8.27}$$

Now we estimate $R_{\mathtt{n}}$ in (4.8.18). By (4.4.5), (4.4.6) and (4.8.15), we note that, for $H := (\widehat{\mathfrak{I}}, \widehat{\alpha})$,

$$(L_{\mathtt{n}}\mathbf{\Pi}_{\mathtt{n}}^{\perp} - \Pi_{\mathtt{n}}^{\perp}L_{\mathtt{n}})H = \varepsilon\,[\mathrm{d}_i X_P(\widetilde{\imath}_{\mathtt{n}}), \Pi_{\mathtt{n}}^{\perp}]\widehat{\mathfrak{I}}\,.$$

Thus, by Lemma 4.60 and (4.2.8), we have the following estimate

$$\|(L_{\mathtt{n}}\mathbf{\Pi}_{\mathtt{n}}^{\perp} - \Pi_{\mathtt{n}}^{\perp}L_{\mathtt{n}})H\|_{s_0}^{k_0,\upsilon} \lesssim_{s_0+\mathtt{b}_1} \varepsilon K_{\mathtt{n}}^{\sigma_1-\mathtt{b}_1+1}(\|\widehat{\mathfrak{I}}\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon} + \|\widetilde{\mathfrak{I}}_{\mathtt{n}}\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon}\|\widehat{\mathfrak{I}}\|_{s_0+1}^{k_0,\upsilon})\,. \tag{4.8.28}$$

Hence, by (4.8.18), (4.8.28), (4.8.12), (4.8.5), (4.8.6), (4.8.19), (4.8.20) and $\varepsilon\upsilon^{-1} \leqslant 1$, we get

$$\begin{aligned} \|R_{\mathtt{n}}\|_{s_0}^{k_0,\upsilon} & \lesssim_{s_0+\mathtt{b}_1} \varepsilon\,K_{\mathtt{n}}^{\sigma_1-\mathtt{b}_1+1}(\|\mathbf{T}_{\mathtt{n}}\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon} + \|\widetilde{\mathfrak{I}}_{\mathtt{n}}\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon}\|\mathbf{T}_{\mathtt{n}}\Pi_{\mathtt{n}}\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+1}^{k_0,\upsilon}) \\ & \lesssim_{s_0+\mathtt{b}_1} K_{\mathtt{n}}^{3\sigma_1+\mu(\mathtt{b})+2-\mathtt{b}_1}(\varepsilon\upsilon^{-1}\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon} + \varepsilon\|\widetilde{\mathfrak{I}}_{\mathtt{n}}\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon}) \\ & \lesssim_{s_0+\mathtt{b}_1} K_{\mathtt{n}}^{4\sigma_1+2\mu(\mathtt{b})+2-\mathtt{b}_1}(\varepsilon + \|\widetilde{W}_{\mathtt{n}}\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon})\,. \end{aligned} \tag{4.8.29}$$

We obtain, by (4.8.17), (4.8.23), (4.8.25)-(4.8.27), (4.8.29), (4.8.19), (4.8.20),

$$\begin{aligned}\|\mathcal{F}(\widetilde{U}_{\mathtt{n}+1})\|_{s_0}^{k_0,\upsilon} \lesssim_{s_0+\mathtt{b}_1} & \upsilon^{-1} K_{\mathtt{n}}^{\mu_2-\mathtt{b}_1}(\varepsilon + \|\widetilde{W}_{\mathtt{n}}\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon}) \\ & + \upsilon^{-1} K_{\mathtt{n}}^{2\sigma_1+4}(\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0}^{k_0,\upsilon})^2 + \varepsilon \upsilon^{-2} K_{\mathtt{n}-1}^{-p\mathtt{a}} K_{\mathtt{n}}^{\sigma_1} \|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0}^{k_0,\upsilon}\end{aligned} \tag{4.8.30}$$

with $\mu_2 := 4\sigma_1 + 2\mu(\mathtt{b}) + 2$. Moreover, we have the bound

$$\|\widetilde{W}_1\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon} \overset{(4.8.14)}{=} \|H_1\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon} \overset{(4.8.12)}{\lesssim_{s_0+\mathtt{b}_1}} \upsilon^{-1} \|\mathcal{F}(U_0)\|_{s_0+\sigma_1+\mathtt{b}_1}^{k_0,\upsilon} \lesssim_{s_0+\mathtt{b}_1} \varepsilon \upsilon^{-1}, \tag{4.8.31}$$

whereas for $\widetilde{W}_{\mathtt{n}+1} := \widetilde{W}_{\mathtt{n}} + H_{\mathtt{n}+1}$, $\mathtt{n} \geqslant 1$, we have, by (4.8.21),

$$\|\widetilde{W}_{\mathtt{n}+1}\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon} \lesssim_{s_0+\mathtt{b}_1} \upsilon^{-1} K_{\mathtt{n}}^{2\sigma_1+\mu(\mathtt{b})}(\varepsilon + \|\widetilde{W}_{\mathtt{n}}\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon}). \tag{4.8.32}$$

We extend $H_{\mathtt{n}+1}$, defined for $\lambda \in \mathcal{G}_{\mathtt{n}+1}$, to $\widetilde{H}_{\mathtt{n}+1}$ defined for all $\lambda \in \mathbb{R}^\nu \times [\kappa_1, \kappa_2]$, with an equivalent $\| \|_s^{k_0,\upsilon}$-norm. Set $\widetilde{U}_{\mathtt{n}+1} := \widetilde{U}_{\mathtt{n}} + \widetilde{H}_{\mathtt{n}+1}$. Therefore, by (4.8.30), (4.8.31), (4.8.32) and the induction assumption, we conclude that (4.8.6), (4.8.6) (4.8.9), (4.8.10) hold true at the step $\mathtt{n}+1$. Finally, by (4.8.14), (4.4.6), (4.4.7), Theorem 4.65 and the induction assumption on $\widetilde{U}_{\mathtt{n}}$, we have that $\widehat{\mathfrak{I}}_{\mathtt{n}+1}$ satisfies (4.2.61) and so $\widetilde{U}_{\mathtt{n}+1}$ is a quasi-periodic traveling wave. This concludes the proof. □

Proof of Theorem 4.55. Let $\upsilon = \varepsilon^{\mathrm{a}}$, with $0 < \mathrm{a} < \mathrm{a}_0 := 1/(2+\tau_3)$. Then, the smallness condition in (4.8.5) holds for $0 < \varepsilon < \varepsilon_0$ small enough and Theorem 4.95 holds. By (4.8.7), the sequence of functions $\widetilde{W}_{\mathtt{n}} = \widetilde{U}_{\mathtt{n}} - (\varphi, 0, 0, \omega) = (\widetilde{\mathfrak{I}}_{\mathtt{n}}, \widetilde{\alpha}_{\mathtt{n}} - \omega)$ converges to a function $W_\infty :$ $\mathbb{R}^\nu \times [\kappa_1, \kappa_2] \to H_\varphi^{s_0} \times H_\varphi^{s_0} \times H^{s_0} \times \mathbb{R}^\nu$, and we define

$$U_\infty := (i_\infty, \alpha_\infty) := (\varphi, 0, 0, \omega) + W_\infty .$$

The torus i_∞ is reversible and traveling, i.e. (4.4.8) holds. By (4.8.6), (4.8.7), we also deduce

$$\|U_\infty - U_0\|_{s_0+\mu(\mathtt{b})+\sigma_1}^{k_0,\upsilon} \leqslant C_* \varepsilon \upsilon^{-1}, \quad \|U_\infty - \widetilde{U}_{\mathtt{n}}\|_{s_0+\mu(\mathtt{b})+\sigma_1}^{k_0,\upsilon} \leqslant C \varepsilon \upsilon^{-1} K_{\mathtt{n}}^{-\mathtt{a}_2}, \ \forall\, \mathtt{n} \geqslant 1. \tag{4.8.33}$$

In particular (4.4.11)-(4.4.12) hold. By Theorem 4.95-$(\mathcal{P}2)_{\mathtt{n}}$, we deduce that $\mathcal{F}(\lambda; U_\infty(\lambda)) = 0$ for any

$$\lambda \in \bigcap_{\mathtt{n} \in \mathbb{N}_0} \mathcal{G}_{\mathtt{n}} = \mathcal{G}_0 \cap \bigcap_{\mathtt{n} \geqslant 1} \mathbf{\Lambda}_{\mathtt{n}}^{\upsilon}(\widetilde{\imath}_{\mathtt{n}-1}) \overset{(4.7.55)}{=} \mathcal{G}_0 \cap \Big[\bigcap_{\mathtt{n} \geqslant 1} \Lambda_{\mathtt{n}}^{\upsilon}(\widetilde{\imath}_{\mathtt{n}-1})\Big] \cap \Big[\bigcap_{\mathtt{n} \geqslant 1} \Lambda_{\mathtt{n}}^{\upsilon,I}(\widetilde{\imath}_{\mathtt{n}-1})\Big]$$

where $\mathcal{G}_0 := \Omega \times [\kappa_1, \kappa_2]$. To conclude the proof of Theorem 4.55 it remains only to define the μ_j^∞ in (4.4.13) and prove that the set $\mathcal{C}_\infty^\upsilon$ in (4.4.15)-(4.4.17) is contained in $\cap_{\mathtt{n} \geqslant 0} \mathcal{G}_{\mathtt{n}}$. We first define

$$\mathcal{G}_\infty := \mathcal{G}_0 \cap \Big[\bigcap_{\mathtt{n}\geqslant 1} \Lambda_{\mathtt{n}}^{2\upsilon}(i_\infty) \Big] \cap \Big[\bigcap_{\mathtt{n}\geqslant 1} \Lambda_{\mathtt{n}}^{2\upsilon, I}(i_\infty) \Big] . \tag{4.8.34}$$

Lemma 4.96. *$\mathcal{G}_\infty \subseteq \cap_{\mathtt{n}\geqslant 0} \mathcal{G}_{\mathtt{n}}$, where $\mathcal{G}_{\mathtt{n}}$ are defined in* (4.8.8).

Proof. We shall use the inclusion property (4.7.27), with S fixed in (4.8.3). By (4.8.33) we have

$$\varepsilon(2\upsilon)^{-1} C(S) N_0^{\tau+1} \| i_\infty - i_0 \|_{s_0+\mu(\mathtt{b})} \leqslant \varepsilon(2\upsilon)^{-1} C(S) K_0^{p(\tau+1)} C_* \varepsilon \upsilon^{-1} \leqslant \upsilon \,,$$
$$\varepsilon(2\upsilon)^{-1} C(S) N_{\mathtt{n}-1}^{\tau+1} \| i_\infty - \widetilde{\imath}_{\mathtt{n}-1} \|_{s_0+\mu(\mathtt{b})} \leqslant \varepsilon(2\upsilon)^{-1} C(S) K_{\mathtt{n}-1}^{p(\tau+1)} C \varepsilon \upsilon^{-1} K_{\mathtt{n}-1}^{-\mathtt{a}_2} \leqslant \upsilon \,, \quad \forall\, \mathtt{n} \geqslant 2 \,,$$

since $\tau_3 > p(\tau+1)$ (by (4.8.5) and $\tau_2 > \tau_1 = \tau(k_0+1)+k_0$) and $\mathtt{a}_2 > p(\tau+1)$ (see Remark 4.94). Therefore (4.7.27) implies $\Lambda_{\mathtt{n}}^{2\upsilon}(i_\infty) \subset \Lambda_{\mathtt{n}}^{\upsilon}(\widetilde{\imath}_{\mathtt{n}-1})$, $\forall\, \mathtt{n} \geqslant 1$. By similar arguments we deduce that $\Lambda_{\mathtt{n}}^{2\upsilon,I}(i_\infty) \subset \Lambda_{\mathtt{n}}^{\upsilon,I}(\widetilde{\imath}_{\mathtt{n}-1})$. □

Then we define the μ_j^∞ in (4.4.13), where $\mathtt{m}_{\frac{3}{2}}^\infty := \mathtt{m}_{\frac{3}{2}}(i_\infty)$, $\mathtt{m}_1^\infty = \mathtt{m}_1(i_\infty)$, $\mathtt{m}_{\frac{1}{2}}^\infty = \mathtt{m}_{\frac{1}{2}}(i_\infty)$, with $\mathtt{m}_{\frac{3}{2}}, \mathtt{m}_1, \mathtt{m}_{\frac{1}{2}}$ provided in Proposition 4.83. By (4.7.16), the sequence $(\mathfrak{r}_j^{(\mathtt{n})}(i_\infty))_{\mathtt{n}\in\mathbb{N}}$, with $\mathfrak{r}_j^{(\mathtt{n})}$ given by Theorem 4.85-$(\mathbf{S1})_{\mathtt{n}}$ (evaluated at $i = i_\infty$), is a Cauchy sequence in $| \cdot |^{k_0,\upsilon}$. Then we define $\mathfrak{r}_j^\infty := \lim_{\mathtt{n}\to\infty} \mathfrak{r}_j^{(\mathtt{n})}(i_\infty)$, for any $j \in \mathbb{S}_0^c$, which satisfies $|\mathfrak{r}_j^\infty - \mathfrak{r}_j^{(\mathtt{n})}(i_\infty)|^{k_0,\upsilon} \leqslant C\varepsilon\upsilon^{-1} N_{\mathtt{n}-1}^{-\mathtt{a}}$ for any $\mathtt{n} \geqslant 0$. Then, recalling $\mathfrak{r}_j^{(0)}(i_\infty) = 0$ and (4.6.189), the estimates (4.4.14) hold (here $C = C(S)$ with S fixed in (4.8.3)). Finally one checks (see e.g. Lemma 8.7 in [44]) that the Cantor set $\mathcal{C}_\infty^\upsilon$ in (4.4.15)-(4.4.18) satisfies $\mathcal{C}_\infty^\upsilon \subseteq \mathcal{G}_\infty$, with $\mathcal{G}_\infty$ defined in (4.8.34), and Lemma 4.96 implies that $\mathcal{C}_\infty^\upsilon \subseteq \cap_{\mathtt{n}\geqslant 0} \mathcal{G}_{\mathtt{n}}$. This concludes the proof of Theorem 4.55.

Chapter 5

Quadratic life span of periodic gravity-capillary water waves

We consider the space periodic gravity-capillary water waver equations

$$\begin{cases} \eta_t = G(\eta)\psi + \gamma\eta\eta_x \\ \psi_t = -g\eta - \dfrac{\psi_x^2}{2} + \dfrac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} + \kappa\Big(\dfrac{\eta_x}{\sqrt{1+\eta_x^2}}\Big)_x . \end{cases} \tag{5.0.1}$$

The variable $\eta(t,x)$ denotes the free boundary of the two dimensional fluid domain $\mathcal{D}_{\eta,\mathtt{h}}$ defined in (1.1.11), whereas $\psi(t,x)$ is the trace at the free boundary $y = \eta(t,x)$ of the generalized velocity potential $\Phi(t,x,y)$ solving (1.1.12). Here $g > 0$ is the gravity, $\kappa > 0$ is the surface tension coefficient and $G(\eta)$ is the Dirichlet-Neumann operator $G(\eta)\psi = (-\Phi_x\eta_x + \Phi_y)|_{y=\eta(x)}$. As observed by Zakharov [174], the equations (5.0.1) are the Hamiltonian system in (1.1.31), (1.1.32)

The system obtained linearizing (5.0.1) at the equilibrium $(\eta,\psi) = (0,0)$, namely

$$\begin{cases} \partial_t\eta &= G(0)\psi \\ \partial_t\psi &= -(g - \kappa\partial_x^2)\eta . \end{cases} \tag{5.0.2}$$

The associated linear frequencies (see (1.1.34)) are given by

$$\Omega(j) := \Omega_{\kappa,g,\mathtt{h}}(j) := \sqrt{(\kappa j^2 + g)G_j(0)} , \quad j \in \mathbb{Z}\backslash\{0\} . \tag{5.0.3}$$

The main goal is to prove that, for *any* value of $(\kappa, g, \mathtt{h})$, $\kappa > 0$, the gravity-capillary water waves system (5.0.1) is conjugated to its Birkhoff normal form, up to cubic remainders that satisfy energy estimates (Theorem 5.1), and that all the solutions of (5.0.1), with initial data of size ϵ in a sufficiently smooth Sobolev space, exist and remain in an ϵ-ball of the same Sobolev space up times of order ϵ^{-2}, see Theorem 5.2. Let us state precisely these results.

Assume that, for s large enough and some $T > 0$, we have a classical solution

$$(\eta, \psi) \in C^0([-T,T]; H_0^{s+\frac{1}{4}} \times \dot{H}^{s-\frac{1}{4}}) \tag{5.0.4}$$

of the Cauchy problem for (5.0.1). The existence of such a solution, at least for small enough T, is guaranteed by local well-posedness theory, see the literature at the end of this chapter.

Theorem 5.1. (Cubic Birkhoff normal form) *Let $\kappa > 0$, $g \geqslant 0$ and $\mathtt{h} \in (0, +\infty]$. There exist $s \gg 1$ and $0 < \bar{\epsilon} \ll 1$, such that, if (η, ψ) is a solution of (5.0.1) satisfying (5.0.4) with*

$$\sup_{t\in[-T,T]} \left(\|\eta\|_{H_0^{s+\frac{1}{4}}} + \|\psi\|_{\dot{H}^{s-\frac{1}{4}}} \right) \leqslant \bar{\epsilon}\,, \tag{5.0.5}$$

then there exists a bounded and invertible linear operator $\mathfrak{B}(\eta, \psi) : H_0^{s+\frac{1}{4}} \times \dot{H}^{s-\frac{1}{4}} \to \dot{H}^s$, which depends (nonlinearly) on (η, ψ), such that

$$\begin{aligned} \|\mathfrak{B}(\eta,\psi)\|_{\mathcal{L}(H_0^{s+\frac{1}{4}} \times \dot{H}^{s-\frac{1}{4}}, \dot{H}^s)} + \|(\mathfrak{B}(\eta,\psi))^{-1}\|_{\mathcal{L}(\dot{H}^s, H_0^{s+\frac{1}{4}} \times \dot{H}^{s-\frac{1}{4}})} \leqslant \\ 1 + C(s)(\|\eta\|_{H_0^{s+\frac{1}{4}}} + \|\psi\|_{\dot{H}^{s-\frac{1}{4}}})\,, \end{aligned} \tag{5.0.6}$$

and the variable $z := \mathfrak{B}(\eta, \psi)[\eta, \psi]$ satisfies the equation

$$\partial_t z = \mathrm{i}\Omega(D) z + \mathrm{i}\partial_{\bar{z}} H^{(3)}_{\mathrm{BNF}}(z, \bar{z}) + \mathcal{X}^+_{\geqslant 3} \tag{5.0.7}$$

where:

1. *$\Omega(D)$ is the Fourier multiplier with symbol defined in (5.0.3) and $\partial_{\bar{z}}$ is defined in (5.4.3);*

2. *the Hamiltonian $H^{(3)}_{\mathrm{BNF}}(z, \bar{z})$ has the form*

$$H^{(3)}_{\mathrm{BNF}}(z, \bar{z}) = \sum_{\substack{\sigma_1 j_1 + \sigma_2 j_2 + \sigma_2 j_3 = 0,\, \sigma_i = \pm\,, \\ \sigma_1\Omega(j_1) + \sigma_2\Omega(j_2) + \sigma_3\Omega(j_3) = 0,\, j_i \in \mathbb{Z}\setminus\{0\}}} H^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3} z^{\sigma_1}_{j_1} z^{\sigma_2}_{j_2} z^{\sigma_3}_{j_3} \tag{5.0.8}$$

   where $z_j^+ := z_j$, $z_j^- := \overline{z_j}$ and z_j denotes the j-th Fourier coefficient of the function z (see (5.1.2)), and the coefficients

$$H^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3} := \frac{\mathrm{i}\sigma_2}{8\sqrt{\pi}} \left(\sigma_1\sigma_3 j_1 j_3 + G_{j_1}(0) G_{j_3}(0)\right) \frac{\Lambda(j_2)}{\Lambda(j_1)\Lambda(j_3)} \tag{5.0.9}$$

   with $\Lambda(j)$ defined in (5.2.2) and $G_j(0) := j\tanh(\mathtt{h}j)$;

3. $\mathcal{X}^+_{\geqslant 3} := \mathcal{X}^+_{\geqslant 3}(\eta,\psi,z,\overline{z})$ *satisfies* $\|\mathcal{X}^+_{\geqslant 3}\|_{\dot{H}^{s-\frac{3}{2}}} \leqslant C(s)\|z\|^3_{\dot{H}^s}$ *and the "energy estimate"*

$$\mathrm{Re}\int_{\mathbb{T}} |D|^s \mathcal{X}^+_{\geqslant 3} \cdot \overline{|D|^s z}\, \mathrm{d}x \leqslant C(s)\|z\|^4_{\dot{H}^s}\,. \tag{5.0.10}$$

The main point of Theorem 5.1 is the construction of the bounded and invertible transformation $\mathfrak{B}(\eta,\psi)$ in (5.0.6) which recasts the irrotational water waves system (5.0.1) in the Birkhoff normal form (1.1.38), where the cubic vector field satisfies the energy estimate (5.0.10).

For general values of gravity, surface tension and depth $(g,\kappa,\mathtt{h})$, the "resonant" Birkhoff normal form Hamiltonian $H^{(3)}_{\mathrm{BNF}}$ in (5.0.8) is non zero, because the system

$$\sigma_1\Omega(j_1)+\sigma_2\Omega(j_2)+\sigma_3\Omega(j_3)=0\,, \qquad \sigma_1 j_1+\sigma_2 j_2+\sigma_3 j_3=0\,, \tag{5.0.11}$$

for $\sigma_j = \pm$, may possess integer solutions $j_1,j_2,j_3 \neq 0$, known as 3-waves resonances (cases with absence of 3-waves resonances are discussed in remark 5.16). The resonant Hamiltonian $H^{(3)}_{\mathrm{BNF}}$ gives rise to a complicated dynamics, which, in fluid mechanics, is responsible for the phenomenon of the Wilton ripples. Nevertheless we are able to prove the following long time stability result.

Theorem 5.2. (Quadratic life span) *For any value of* $(\kappa,g,\mathtt{h})$, $\kappa>0$, $g\geqslant 0$, $\mathtt{h}\in(0,+\infty]$, *there exists* $s_0>0$ *and, for all* $s\geqslant s_0$, *there are* $\epsilon_0>0$, $c>0$, $C>0$, *such that, for any* $0<\epsilon\leqslant\epsilon_0$, *any initial data*

$$(\eta_0,\psi_0)\in H_0^{s+\frac14}(\mathbb{T},\mathbb{R})\times \dot{H}^{s-\frac14}(\mathbb{T},\mathbb{R}) \quad \textit{with} \quad \|\eta_0\|_{H_0^{s+\frac14}}+\|\psi_0\|_{\dot{H}^{s-\frac14}}\leqslant \epsilon\,, \tag{5.0.12}$$

there exists a unique classical solution (η,ψ) *of* (1.1.30) *belonging to*

$$C^0\Big([-T_\epsilon,T_\epsilon], H_0^{s+\frac14}(\mathbb{T},\mathbb{R})\times \dot{H}^{s-\frac14}(\mathbb{T},\mathbb{R})\Big) \quad \textit{with} \quad T_\epsilon \geqslant c\epsilon^{-2}\,,$$

satisfying $(\eta,\psi)_{|t=0}=(\eta_0,\psi_0)$. *Moreover*

$$\sup_{t\in[-T_\epsilon,T_\epsilon]}\big(\|\eta\|_{H_0^{s+\frac14}}+\|\psi\|_{\dot{H}^{s-\frac14}}\big)\leqslant C\epsilon\,. \tag{5.0.13}$$

The rest of this chapter concerns the proof of Theorem 5.1 and Theorem 5.2. In Section 5.1 the paradifferential calculus of [37] is recalled, in particular the definitions and main properties of paradifferential symbols, smoothing operators and multilinear maps. In Section 5.2 we state the paralinearization in complex form and the paradifferential reduction to constant symbols up to smoothing operators of system (5.0.1) as proved in [37]. In Section 5.3 the paralinearized reduced system is transformed into its quadratic Poincaré-Birkhoff normal form and we show that there are only finitely many 3-waves interactions between the Fourier modes. In Section

5.4 we perform the normal form uniqueness argument and then we prove the energy estimates required for Theorem 5.2.

5.1 Functional Setting and Paradifferential calculus

In this section we recall definitions and results of para-differential calculus following Chapter 3 of [37], where we refer for more information. In the sequel we will deal with parameters

$$s \geqslant s_0 \gg K \gg \rho \gg 1\,.$$

Given an interval $I \subset \mathbb{R}$, symmetric with respect to $t = 0$, and $s \in \mathbb{R}$, we define the space $C_*^K(I, \dot{H}^s(\mathbb{T}, \mathbb{C}^2)) := \bigcap_{k=0}^K C^k\big(I; \dot{H}^{s-\frac{3}{2}k}(\mathbb{T}; \mathbb{C}^2)\big)$ endowed with the norm

$$\sup_{t\in I} \|U(t,\cdot)\|_{K,s} \quad \text{where} \quad \|U(t,\cdot)\|_{K,s} := \sum_{k=0}^K \|\partial_t^k U(t,\cdot)\|_{\dot{H}^{s-\frac{3}{2}k}}\,.$$

With similar meaning we consider $C_*^K(I; \dot{H}^s(\mathbb{T}; \mathbb{C}))$. We denote by $C_{*\mathbb{R}}^K(I, \dot{H}^s(\mathbb{T}, \mathbb{C}^2))$ the subspace of functions U in $C_*^K(I, \dot{H}^s(\mathbb{T}, \mathbb{C}^2))$ such that $U = \binom{u}{\overline{u}}$. Given $r > 0$ we set

$$B_s^K(I; r) := \Big\{ U \in C_*^K(I, \dot{H}^s(\mathbb{T}; \mathbb{C}^2)) : \sup_{t\in I} \|U(t,\cdot)\|_{K,s} < r \Big\}\,. \tag{5.1.1}$$

We expand a 2π-periodic function $u(x)$, with zero average in x, (which is identified with u in the homogeneous space), in Fourier series as

$$u(x) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \hat{u}(n)\frac{e^{\mathrm{i}nx}}{\sqrt{2\pi}}\,, \qquad \hat{u}(n) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{T}} u(x)e^{-\mathrm{i}nx}\,dx\,. \tag{5.1.2}$$

We also use the notation $u_n^+ := u_n := \hat{u}(n)$ and $u_n^- := \overline{u_n} := \overline{\hat{u}(n)}$. We set $u^+(x) := u(x)$ and $u^-(x) := \overline{u(x)}$.

For $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ we denote by Π_n the orthogonal projector from $L^2(\mathbb{T}; \mathbb{C})$ to the subspace spanned by $\{e^{\mathrm{i}nx}, e^{-\mathrm{i}nx}\}$, i.e. $(\Pi_n u)(x) := \hat{u}(n)\frac{e^{\mathrm{i}nx}}{\sqrt{2\pi}} + \hat{u}(-n)\frac{e^{-\mathrm{i}nx}}{\sqrt{2\pi}}$, and we denote by Π_n also the corresponding projector in $L^2(\mathbb{T}, \mathbb{C}^2)$. If $\mathcal{U} = (U_1, \ldots, U_p)$ is a p-tuple of functions, $\vec{n} = (n_1, \ldots, n_p) \in (\mathbb{N}^*)^p$, we set $\Pi_{\vec{n}}\mathcal{U} := (\Pi_{n_1}U_1, \ldots, \Pi_{n_p}U_p)$.

We deal with vector fields X which satisfy the *x-translation invariance* property

$$X \circ \tau_\theta = \tau_\theta \circ X\,, \quad \forall\, \theta \in \mathbb{R}\,, \quad \text{where} \quad \tau_\theta : u(x) \mapsto (\tau_\theta u)(x) := u(x+\theta)\,.$$

Para-differential operators. We first give the definition of the classes of symbols, collecting Definitions 3.1, 3.2 and 3.4 in [37]. Roughly speaking, the class $\widetilde{\Gamma}_p^m$ contains homogeneous symbols of order m and homogeneity p in U, while the class $\Gamma^m_{K,K',p}$ contains non-homogeneous symbols of order m which vanish at degree at least p in U, and that are $(K-K')$-times differentiable in t.

Definition 5.3. (Classes of symbols) Let $m \in \mathbb{R}$, $p, N \in \mathbb{N}$ with $p \leqslant N$, K, K' in $\mathbb{N}$ with $K' \leqslant K$, $r > 0$.
(i) **p-homogeneous symbols.** We denote by $\widetilde{\Gamma}^m_p$ the space of symmetric p-linear maps from $(\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^p$ to the space of C^∞ functions of $(x,\xi) \in \mathbb{T}\times\mathbb{R}$, $\mathcal{U} \to ((x,\xi) \to a(\mathcal{U};x,\xi))$, satisfying the following. There is $\mu > 0$ and, for any $\alpha, \beta \in \mathbb{N}$, there is $C > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(\Pi_{\vec{n}}\mathcal{U};x,\xi)| \leqslant C|\vec{n}|^{\mu+\alpha}\langle\xi\rangle^{m-\beta}\prod_{j=1}^{p}\|\Pi_{n_j}U_j\|_{L^2} \tag{5.1.3}$$

for any $\mathcal{U} = (U_1,\dots,U_p)$ in $(\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^p$, and $\vec{n} = (n_1,\dots,n_p) \in (\mathbb{N}^*)^p$. Moreover we assume that, if for some $(n_0,\dots,n_p) \in \mathbb{N}\times(\mathbb{N}^*)^p$, $\Pi_{n_0}a(\Pi_{n_1}U_1,\dots,\Pi_{n_p}U_p;\cdot) \neq 0$, then there exists a choice of signs $\sigma_0,\dots,\sigma_p \in \{-1,1\}$ such that $\sum_{j=0}^p \sigma_j n_j = 0$. For $p = 0$ we denote by $\widetilde{\Gamma}^m_0$ the space of constant coefficients symbols $\xi \mapsto a(\xi)$ which satisfy (5.1.3) with $\alpha = 0$ and the right hand side replaced by $C\langle\xi\rangle^{m-\beta}$. In addition we require the translation invariance property

$$a(\tau_\theta\mathcal{U};x,\xi) = a(\mathcal{U};x+\theta,\xi)\,, \quad \forall\theta\in\mathbb{R}\,. \tag{5.1.4}$$

(ii) **Non-homogeneous symbols.** Let $p \geqslant 1$. We denote by $\Gamma^m_{K,K',p}[r]$ the space of functions $(U;t,x,\xi)\mapsto a(U;t,x,\xi)$, defined for $U \in B^K_{s_0}(I;r)$, for some large enough s_0, with complex values such that for any $0 \leqslant k \leqslant K-K'$, any $\sigma \geqslant s_0$, there are $C > 0$, $0 < r(\sigma) < r$ and for any $U \in B^K_{s_0}(I;r(\sigma)) \cap C^{k+K'}_*(I,\dot{H}^\sigma(\mathbb{T};\mathbb{C}^2))$ and any $\alpha,\beta\in\mathbb{N}$, with $\alpha \leqslant \sigma - s_0$

$$|\partial_t^k\partial_x^\alpha\partial_\xi^\beta a(U;t,x,\xi)| \leqslant C\langle\xi\rangle^{m-\beta}\|U\|^{p-1}_{k+K',s_0}\|U\|_{k+K',\sigma}\,. \tag{5.1.5}$$

(iii) **Symbols.** We denote by $\Sigma\Gamma^m_{K,K',p}[r,N]$ the space of functions $(U,t,x,\xi) \to a(U;t,x,\xi)$ such that there are homogeneous symbols $a_q \in \widetilde{\Gamma}^m_q$, $q = p,\dots,N-1$, and a non-homogeneous symbol $a_N \in \Gamma^m_{K,K',N}[r]$ such that $a(U;t,x,\xi) = \sum_{q=p}^{N-1} a_q(U,\dots,U;x,\xi) + a_N(U;t,x,\xi)$. We denote by $\Sigma\Gamma^m_{K,K',p}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ the space 2×2 matrices with entries in $\Sigma\Gamma^m_{K,K',p}[r,N]$.

As a consequence of the *momentum* condition (5.1.4) a symbol a_1 in the class $\widetilde{\Gamma}^m_1$, for some $m\in\mathbb{R}$, can be written as

$$a_1(U;x,\xi) = \sum_{j\in\mathbb{Z}\setminus\{0\},\sigma=\pm}(a_1)^\sigma_j(\xi)u^\sigma_j e^{\mathrm{i}\sigma jx} \tag{5.1.6}$$

for some coefficients $(a_1)^\sigma_j(\xi)\in\mathbb{C}$, see [39].

Remark 5.4. A symbol $a_1 \in \widetilde{\Gamma}^m_1$ of the form (5.1.6), independent of x, is actually $a_1 \equiv 0$.

We also define classes of functions in analogy with our classes of symbols.

Definition 5.5. (Functions) Fix $N, p \in \mathbb{N}$ with $p \leqslant N$, $K, K' \in \mathbb{N}$ with $K' \leqslant K$, $r > 0$. We denote by $\widetilde{\mathcal{F}}_p$, resp. $\mathcal{F}_{K,K',p}[r]$, $\Sigma\mathcal{F}_p[r,N]$, the subspace of $\widetilde{\Gamma}^0_p$, resp. $\Gamma^0_p[r]$, resp. $\Sigma\Gamma^0_p[r,N]$, made of those symbols which are independent of ξ. We write $\widetilde{\mathcal{F}}^{\mathbb{R}}_p$, resp. $\mathcal{F}^{\mathbb{R}}_{K,K',p}[r]$, $\Sigma\mathcal{F}^{\mathbb{R}}_p[r,N]$, to denote functions in $\widetilde{\mathcal{F}}_p$, resp. $\mathcal{F}_{K,K',p}[r]$, $\Sigma\mathcal{F}_p[r,N]$, which are real valued.

Paradifferential quantization. Given $p \in \mathbb{N}$ we consider functions $\chi_p \in C^\infty(\mathbb{R}^p \times \mathbb{R};\mathbb{R})$ and $\chi \in C^\infty(\mathbb{R}\times\mathbb{R};\mathbb{R})$, even with respect to each of their arguments, satisfying, for $0 < \delta \ll 1$,

$$\begin{aligned} &\operatorname{supp}\chi_p \subset \{(\xi',\xi) \in \mathbb{R}^p \times \mathbb{R}; |\xi'| \leqslant \delta\langle\xi\rangle\}, \qquad \chi_p(\xi',\xi) \equiv 1 \ \text{ for } \ |\xi'| \leqslant \delta\langle\xi\rangle/2\,,\\ &\operatorname{supp}\chi \subset \{(\xi',\xi) \in \mathbb{R} \times \mathbb{R}; |\xi'| \leqslant \delta\langle\xi\rangle\}, \qquad \chi(\xi',\xi) \equiv 1 \ \text{ for } \ |\xi'| \leqslant \delta\langle\xi\rangle/2\,. \end{aligned}$$

For $p = 0$ we set $\chi_0 \equiv 1$. We assume moreover that $|\partial_\xi^\alpha \partial_{\xi'}^\beta \chi_p(\xi',\xi)| \leqslant C_{\alpha,\beta}\langle\xi\rangle^{-\alpha-|\beta|}$, $\forall\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^p$, and $|\partial_\xi^\alpha \partial_{\xi'}^\beta \chi(\xi',\xi)| \leqslant C_{\alpha,\beta}\langle\xi\rangle^{-\alpha-\beta}$, $\forall\alpha$, $\beta \in \mathbb{N}$.

If $a(x,\xi)$ is a smooth symbol we define its *Weyl* quantization as the operator acting on a 2π-periodic function $u(x)$ (written as in (5.1.2)) as

$$Op^W(a)u = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \Big(\sum_{j\in\mathbb{Z}} \hat{a}\big(k-j, \frac{k+j}{2}\big)\hat{u}(j)\Big) \frac{e^{\mathrm{i}kx}}{\sqrt{2\pi}} \tag{5.1.7}$$

where $\hat{a}(k,\xi)$ is the $k^{th}-$Fourier coefficient of the $2\pi-$periodic function $x \mapsto a(x,\xi)$.

Definition 5.6. (Bony-Weyl quantization) If a is a symbol in $\widetilde{\Gamma}^m_p$, respectively in $\Gamma^m_{K,K',p}[r]$, we set

$$\begin{aligned} a_{\chi_p}(\mathcal{U};x,\xi) &:= \sum_{\vec{n}\in\mathbb{N}^p} \chi_p(\vec{n},\xi)\, a(\Pi_{\vec{n}}\mathcal{U};x,\xi)\,,\\ a_\chi(U;t,x,\xi) &:= \frac{1}{2\pi}\int_{\mathbb{R}} \chi(\xi',\xi)\,\hat{a}(U;t,\xi',\xi)e^{\mathrm{i}\xi' x}d\xi'\,, \end{aligned}$$

where in the last equality $\hat{a}$ stands for the Fourier transform with respect to the x variable, and we define the *Bony-Weyl* quantization of a as

$$\mathrm{Op}^{\mathrm{BW}}(a(\mathcal{U};\cdot)) = Op^W(a_{\chi_p}(\mathcal{U};\cdot)), \qquad \mathrm{Op}^{\mathrm{BW}}(a(U;t,\cdot)) = Op^W(a_\chi(U;t,\cdot))\,.$$

If a is a symbol in $\Sigma\Gamma^m_{K,K',p}[r,N]$, we define its *Bony-Weyl* quantization $\mathrm{Op}^{\mathrm{BW}}(a(U;t,\cdot)) = \sum_{q=p}^{N-1}\mathrm{Op}^{\mathrm{BW}}(a_q(U,\ldots,U;\cdot)) + \mathrm{Op}^{\mathrm{BW}}(a_N(U;t,\cdot))$.

Paradifferential operators act on homogeneous spaces. If a is in $\Sigma\Gamma^m_{K,K',p}[r,N]$, the corresponding para-differential operator is bounded from $\dot{H}^s$ to $\dot{H}^{s-m}$, for all $s \in \mathbb{R}$, see Proposition 3.8 in [37].

Definition 5.6 is independent of the cut-off functions χ_p, χ, up to smoothing operators that we define below (see Definition 3.7 in [37]). Roughly speaking, the class $\widetilde{\mathcal{R}}^{-\rho}_p$ contains smoothing

operators which gain ρ derivatives and are homogeneous of degree p in U, while the class $\mathcal{R}^{-\rho}_{K,K',p}$ contains non-homogeneous ρ-smoothing operators which vanish at degree at least p in U, and are $(K-K')$-times differentiable in t.
Given $(n_1,\dots,n_{p+1})\in\mathbb{N}^{p+1}$ we denote by $\max_2(n_1,\dots,n_{p+1})$ the second largest among the integers $n_1,\dots,n_{p+1}$.

Definition 5.7. (Classes of smoothing operators) Let $N\in\mathbb{N}^*$, $K,K'\in\mathbb{N}$ with $K'\leqslant K\in\mathbb{N}$, $\rho\geqslant 0$ and $r>0$.

(i) **p-homogeneous smoothing operators.** We denote by $\widetilde{\mathcal{R}}^{-\rho}_p$ the space of $(p+1)$-linear maps R from $(\dot H^\infty(\mathbb{T};\mathbb{C}^2))^p\times\dot H^\infty(\mathbb{T};\mathbb{C})$ to $\dot H^\infty(\mathbb{T};\mathbb{C})$, symmetric in $(U_1,\dots,U_p)$, of the form $(U_1,\dots,U_{p+1})\to R(U_1,\dots,U_p)U_{p+1}$ that satisfy the following. There are $\mu\geqslant 0$, $C>0$ such that

$$\|\Pi_{n_0}R(\Pi_{\vec n}\mathcal{U})\Pi_{n_{p+1}}U_{p+1}\|_{L^2}\leqslant C\frac{\max_2(n_1,\dots,n_{p+1})^{\rho+\mu}}{\max(n_1,\dots,n_{p+1})^{\rho}}\prod_{j=1}^{p+1}\|\Pi_{n_j}U_j\|_{L^2}$$

for any $\mathcal{U}=(U_1,\dots,U_p)\in(\dot H^\infty(\mathbb{T};\mathbb{C}^2))^p$, $U_{p+1}\in\dot H^\infty(\mathbb{T};\mathbb{C})$, $\vec n=(n_1,\dots,n_p)\in(\mathbb{N}^*)^p$, any $n_0,n_{p+1}\in\mathbb{N}^*$. Moreover, if

$$\Pi_{n_0}R(\Pi_{n_1}U_1,\dots,\Pi_{n_p}U_p)\Pi_{n_{p+1}}U_{p+1}\neq 0\,, \tag{5.1.8}$$

then there is a choice of signs $\sigma_0,\dots,\sigma_{p+1}\in\{\pm1\}$ such that $\sum_{j=0}^{p+1}\sigma_j n_j=0$. In addition we require the translation invariance property

$$R(\tau_\theta\mathcal{U})[\tau_\theta U_{p+1}]=\tau_\theta\big(R(\mathcal{U})U_{p+1}\big)\,,\quad\forall\theta\in\mathbb{R}\,. \tag{5.1.9}$$

(ii) **Non-homogeneous smoothing operators.** We denote by $\mathcal{R}^{-\rho}_{K,K',N}[r]$ the space of maps $(V,U)\mapsto R(V)U$ defined on $B^K_{s_0}(I;r)\times C^K_*(I,\dot H^{s_0}(\mathbb{T},\mathbb{C}))$ which are linear in the variable U and such that the following holds true. For any $s\geqslant s_0$ there are $C>0$ and $r(s)\in]0,r[$ such that, for any $V\in B^K_{s_0}(I;r)\cap C^K_*(I,\dot H^s(\mathbb{T},\mathbb{C}^2))$, any $U\in C^K_*(I,\dot H^s(\mathbb{T},\mathbb{C}))$, any $0\leqslant k\leqslant K-K'$ and any $t\in I$, we have

$$\begin{aligned}\|\partial_t^k\left(R(V)U\right)(t,\cdot)\|_{\dot H^{s-\frac32 k+\rho}}\leqslant\sum_{k'+k''=k}C\Big(&\|U\|_{k'',s}\|V\|^N_{k'+K',s_0}\\&+\|U\|_{k'',s_0}\|V\|^{N-1}_{k'+K',s_0}\|V\|_{k'+K',s}\Big)\,.\end{aligned} \tag{5.1.10}$$

(iii) **Smoothing operators.** We denote by $\Sigma\mathcal{R}^{-\rho}_{K,K',p}[r,N]$ the space of maps $(V,t,U)\to R(V;t)U$ that may be written as $R(V;t)U=\sum_{q=p}^{N-1}R_q(V,\dots,V)U+R_N(V;t)U$ for some R_q in $\widetilde{\mathcal{R}}^{-\rho}_q$, $q=p,\dots,N-1$ and R_N in $\mathcal{R}^{-\rho}_{K,K',N}[r]$.

We denote by $\Sigma\mathcal{R}^{-\rho}_{K,K',p}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ the space of 2×2 matrices with entries in the class $\Sigma\mathcal{R}^{-\rho}_{K,K',p}[r,N]$.

Below we introduce classes of operators without keeping track of the number of lost derivatives in a precise way (see Definition 3.9 in [37]). The class $\widetilde{\mathcal{M}}_p^m$ denotes multilinear maps that lose m derivatives and are p-homogeneous in U, while the class $\mathcal{M}^m_{K,K',p}$ contains non-homogeneous maps which lose m derivatives, vanish at degree at least p in U, and are $(K-K')$-times differentiable in t.

Definition 5.8. (Classes of maps) Let $p, N \in \mathbb{N}$, with $p \leqslant N$, $N \geqslant 1$, $K, K' \in \mathbb{N}$ with $K' \leqslant K$ and $m \geqslant 0$.

(i) p**-homogeneous maps.** We denote by $\widetilde{\mathcal{M}}_p^m$ the space of $(p+1)$-linear maps M from $(\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^p \times \dot{H}^\infty(\mathbb{T};\mathbb{C})$ to $\dot{H}^\infty(\mathbb{T};\mathbb{C})$ which are symmetric in $(U_1,\dots,U_p)$, of the form $(U_1,\dots,U_{p+1}) \to M(U_1,\dots,U_p)U_{p+1}$ and that satisfy the following. There is $C>0$ such that

$$\|\Pi_{n_0} M(\Pi_{\vec{n}}\mathcal{U})\Pi_{n_{p+1}}U_{p+1}\|_{L^2} \leqslant C(n_0+n_1+\dots+n_{p+1})^m \prod_{j=1}^{p+1} \|\Pi_{n_j}U_j\|_{L^2}$$

for any $\mathcal{U}=(U_1,\dots,U_p) \in (\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^p$, any $U_{p+1} \in \dot{H}^\infty(\mathbb{T};\mathbb{C})$, $\vec{n}=(n_1,\dots,n_p)$ in $(\mathbb{N}^*)^p$, any $n_0, n_{p+1} \in \mathbb{N}^*$. Moreover the properties (5.1.8)-(5.1.9) hold.

(ii) **Non-homogeneous maps.** We denote by $\mathcal{M}^m_{K,K',N}[r]$ the space of maps $(V,u) \mapsto M(V)U$ defined on $B^K_{s_0}(I;r) \times C^K_*(I,\dot{H}^{s_0}(\mathbb{T},\mathbb{C}))$ which are linear in the variable U and such that the following holds true. For any $s \geqslant s_0$ there are $C>0$ and $r(s) \in]0,r[$ such that for any $V \in B^K_{s_0}(I;r) \cap C^K_*(I,\dot{H}^s(\mathbb{T},\mathbb{C}^2))$, any $U \in C^K_*(I,\dot{H}^s(\mathbb{T},\mathbb{C}))$, any $0 \leqslant k \leqslant K-K'$, $t \in I$, we have that $\|\partial_t^k (M(V)U)(t,\cdot)\|_{\dot{H}^{s-\frac{3}{2}k-m}}$ is bounded by the right hand side of (5.1.10).

(iii) **Maps.** We denote by $\Sigma\mathcal{M}^m_{K,K',p}[r,N]$ the space of maps $(V,t,U) \to M(V;t)U$ that may be written as $M(V;t)U = \sum_{q=p}^{N-1} M_q(V,\dots,V)U + M_N(V;t)U$ for some M_q in $\widetilde{\mathcal{M}}_q^m$, $q=p,\dots,N-1$ and M_N in $\mathcal{M}^m_{K,K',N}[r]$. Finally we set $\widetilde{\mathcal{M}}_p := \cup_{m\geqslant 0}\widetilde{\mathcal{M}}_p^m$, $\mathcal{M}_{K,K',p}[r] := \cup_{m\geqslant 0}\mathcal{M}^m_{K,K',p}[r]$, $\Sigma\mathcal{M}_{K,K',p}[r,N] := \cup_{m\geqslant 0}\Sigma\mathcal{M}^m_{K,K',p}[r]$.

We denote by $\Sigma\mathcal{M}^m_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ the space of 2×2 matrices whose entries are maps in $\Sigma\mathcal{M}^m_{K,K',p}[r,N]$. We set $\Sigma\mathcal{M}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C}) := \cup_{m\in\mathbb{R}}\Sigma\mathcal{M}^m_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$.

Given an operator $\mathtt{R}_1$ in $\widetilde{\mathcal{R}}_1^{-\rho}$ (or in $\widetilde{\mathcal{M}}_1^m$), and z^{σ_2}, $\sigma_2=\pm$, the *momentum* condition (5.1.9) implies that

$$\mathtt{R}_1(U)[z^{\sigma_2}] = \sum_{j_1,j_2\in\mathbb{Z}\setminus\{0\},\sigma_1=\pm} (\mathtt{R}_1)^{\sigma_1,\sigma_2}_{j_1,j_2} u^{\sigma_1}_{j_1} z^{\sigma_2}_{j_2} e^{\mathrm{i}(\sigma_1 j_1+\sigma_2 j_2)x} \tag{5.1.11}$$

for some $(\mathtt{R}_1)^{\sigma_1,\sigma_2}_{j_1,j_2} \in \mathbb{C}$, see [39].

Proposition 5.9. (Compositions) *Let $m, m' \in \mathbb{R}$, $N, K, K' \in \mathbb{N}$ with $K' \leqslant K$, $p_1, p_2, p_3 \in \mathbb{N}$, $\rho \geqslant 0$ and $r>0$. Let $a \in \Sigma\Gamma^m_{K,K',p_1}[r,N]$, $R \in \Sigma\mathcal{R}^{-\rho}_{K,K',p_2}[r,N]$ and $M \in \Sigma\mathcal{M}^{m'}_{K,K',p_3}[r,N]$. Then:*

(i) $R(U;t) \circ \mathrm{Op}^{\mathrm{BW}}(a(U;t,x,\xi))$, $\mathrm{Op}^{\mathrm{BW}}(a(U;t,x,\xi)) \circ R(U;t)$ are in $\Sigma\mathcal{R}^{-\rho+m}_{K,K',p_1+p_2}[r,N]$;

(ii) $R(U;t) \circ M(U;t)$ *and* $M(U;t) \circ R(U;t)$ *are smoothing operators in* $\Sigma\mathcal{R}^{-\rho+m'}_{K,K',p_2+p_3}[r,N]$;

(iii) If $R_1 \in \widetilde{\mathcal{R}}^{-\rho}_{p_1}$, $p_1 \geqslant 1$, *then* $R_1(\underbrace{U,\dots,U}_{p_1-1}, M(U;t)U)$ *belongs to* $\Sigma\mathcal{R}^{-\rho+m'}_{K,K',p_1+p_3}[r,N]$.

Proof. See Propositions 3.16, 3.17 in [37]. The translation invariance properties for the composed operators and symbols in items (i)-(ii) follow as in [39]. □

Real-to-real operators. Given a linear operator $R(U)[\cdot]$ acting on $\mathbb{C}$ (it may be a smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,K',1}$ or a map in $\Sigma\mathcal{M}_{K,K',1}$) we associate the linear operator defined by

$$\overline{R}(U)[v] := \overline{R(U)[\overline{v}]}\,, \quad \forall\, v \in \mathbb{C}\,.$$

We say that a matrix of operators acting on $\mathbb{C}^2$ is *real-to-real*, if it has the form

$$R(U) = \begin{pmatrix} R_1(U) & R_2(U) \\ \overline{R_2}(U) & \overline{R_1}(U) \end{pmatrix}. \tag{5.1.12}$$

If $R(U)$ is a real-to-real matrix of operators then, given $V = \binom{v}{\overline{v}}$, the vector $Z := R(U)[V]$ has the form $Z = \binom{z}{\overline{z}}$, i.e. the second component is the complex conjugated of the first one.

Given two linear operators A, B (either two operator-valued matrices acting on $\mathbb{C}^2$ as in (5.1.12)), we denote their commutator by $[A,B] = AB - BA$.

- The notation $A \lesssim_s B$ means that $A \leqslant C(s)B$ for some positive constant $C(s) > 0$.

5.2 Paradifferential reduction to constant symbols up to smoothing operators

The first step in order to prove Theorem 1.9 is to write (5.0.1) in paradifferential form, to symmetrize it, and reduce to paradifferential symbols which are constant in x, see Proposition 5.11. These results are proved in [37] (up to minor details). We denote the horizontal and vertical components of the velocity field at the free interface by

$$\begin{aligned} V &= V(\eta,\psi) := (\partial_x\Phi)(x,\eta(x)) = \psi_x - \eta_x B\,, \\ B &= B(\eta,\psi) := (\partial_y\Phi)(x,\eta(x)) = \frac{G(\eta)\psi + \eta_x\psi_x}{1+\eta_x^2}\,, \end{aligned}$$

and the "good unknown" of Alinhac

$$\omega := \psi - \mathrm{Op}^{\mathrm{BW}}(B(\eta,\psi))\eta\,, \tag{5.2.1}$$

as introduced in Alazard-Metivier [7]. The function $B(\eta,\psi)$ belongs to $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N]$, for any $N>0$ (see Proposition 7.4 in [37]). Then, by the action of a paraproduct, if $\eta\in H_0^{s+\frac{1}{4}}$ and $\psi\in\dot H^{s-\frac{1}{4}}$ then the good unknown ω is in $\dot H^{s-\frac{1}{4}}$.

Define the Fourier multiplier Λ of order $-1/4$ as

$$\Lambda:=\Lambda(D):=\big(D\tanh(hD)\big)^{\frac{1}{4}}\big(g+\kappa D^2\big)^{-\frac{1}{4}} \tag{5.2.2}$$

and consider the complex function

$$u:=\frac{1}{\sqrt{2}}\Lambda\omega+\frac{\mathrm{i}}{\sqrt{2}}\Lambda^{-1}\eta\,,\quad \eta=\frac{1}{\mathrm{i}\sqrt{2}}\Lambda(u-\overline{u})\,,\quad \omega=\frac{1}{\sqrt{2}}\Lambda^{-1}(u+\overline{u}) \tag{5.2.3}$$

where Λ^{-1} acts on functions modulo constants in itself.

Let $K\in\mathbb{N}$. We first remark that, if (η,ψ) solves the gravity-capillary system (5.0.1), then the function u defined in (5.2.3) satisfies, by Proposition 7.9 in [37], for $s\gg K$, as long as u stays in the unit ball of $\dot H^s(\mathbb{T},\mathbb{C})$,

$$\|\partial_t^k u\|_{\dot H^{s-\frac{3}{2}k}}\lesssim_{s,K}\|u\|_{\dot H^s}\,,\quad \forall\,0\leqslant k\leqslant K\,. \tag{5.2.4}$$

As a consequence, if (5.0.5) holds then

$$\sup_{t\in[-T,T]}\|\partial_t^k u\|_{\dot H^{s-\frac{3}{2}k}}\leqslant C_{s,K}\overline{\epsilon}\,,\quad \forall\,0\leqslant k\leqslant K\,. \tag{5.2.5}$$

Proposition 5.10. (Paradifferential complex form of the water waves equations) *Let $N,K\in\mathbb{N}^*$, $\rho>0$. Assume that (η,ψ) solves the gravity-capillary system* (5.0.1) *and satisfy* (5.0.5) *for some $T>0$ and $s\gg K$. Then the function $U:=\left(\begin{smallmatrix}u\\ \overline{u}\end{smallmatrix}\right)$, with u defined in* (5.2.3), *solves*

$$D_tU=\Omega(D)EU+\mathrm{Op}^{\mathrm{BW}}(A(U;t,x,\xi))U+R(U;t)U\,,\quad E:=\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right], \tag{5.2.6}$$

where $D_t:=\frac{1}{\mathrm{i}}\partial_t$ and:

- *$\Omega(D)=\mathrm{Op}^{\mathrm{BW}}(\Omega(\xi))$ where $\Omega(\xi)\in\widetilde\Gamma_0^{\frac{3}{2}}$ is the dispersion relation symbol*

$$\Omega(\xi):=\Omega_{\kappa,g,\mathtt{h}}(\xi):=(\kappa|\xi|^3+g|\xi|)^{\frac{1}{2}}\big(\tanh(\mathtt{h}|\xi|)\big)^{\frac{1}{2}}\,; \tag{5.2.7}$$

- *the matrix of symbols $A(U;t,x,\xi)\in\Sigma\Gamma^1_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ has the form*

$$\begin{aligned}A(U;t,x,\xi)&=\big(\zeta(U;t,x)\Omega(\xi)+\lambda_{\frac{1}{2}}(U;t,x,\xi)\big)\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]\\&\quad+\big(\zeta(U;t,x)\Omega(\xi)+\lambda_{-\frac{1}{2}}(U;t,x,\xi)\big)\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]\\&\quad+\lambda_1(U;t,x,\xi)\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]+\lambda_0(U;t,x,\xi)\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]\end{aligned} \tag{5.2.8}$$

where:

- *the function* $\zeta(U;t,x)$ *is in* $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N]$*;*
- *the symbols* $\lambda_j(U;t,x,\xi)$ *are in* $\Sigma\Gamma^{j}_{K,1,1}[r,N]$, $j=1,0,1/2,-1/2$, *and* $\Im\lambda_j(U;t,x,\xi)$ *are in* $\Sigma\Gamma^{j-1}_{K,1,1}[r,N]$ *for* $j=1,1/2$*;*
- *the matrix of smoothing operators* $R(U;t)$ *is in* $\Sigma\mathcal{R}^{-\rho}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$*;*
- *the operators* $\mathrm{iOp}^{\mathrm{BW}}(A(U;t,x,\xi))$ *and* $\mathrm{i}R(U;t)$ *are real-to-real, according to* (5.1.12).

Proof. It is Corollary 7.7 and Proposition 7.8 in [37]. The only difference is that $U(x)$ is not even in x. The property that the homogeneous components $A_p(U;t,x,\xi)$, $R_p(U;t)$, $p=1,\dots,N$, of the matrices $A(U;t,x,\xi)$, $R(U;t)$ satisfy (5.1.4) and (5.1.9) is checked as in [39]. □

System (5.2.6) has the form

$$D_tU=\Omega(D)EU+M(U;t)U \tag{5.2.9}$$

where $M(U;t)$ is a real-to-real map in $\Sigma\mathcal{M}^{m_1}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ for some $m_1\geqslant 3/2$ (using that paradifferential operators and smoothing remainders are maps, see (4.2.6) in [37]).

As in [37], since the dispersion law (5.2.7) is *super-linear*, system (5.2.6) can be transformed into a paradifferential diagonal system with a symbol constant in x, up to smoothing terms.

Proposition 5.11. **(Reduction to constant coefficients up to smoothing operators)** *Fix* $\rho>0$ *arbitrary. There exist* $s_0>0$, $K':=K'(\rho)$ *such that, for any* $s\geqslant s_0$, *for all* $0<r\leqslant r_0(s)$ *small enough, for all* $K\geqslant K'$ *and any solution* $U\in B^K_s(I;r)$ *of* (5.2.6), *there is a family of real-to-real, bounded, invertible linear maps* $\mathfrak{F}^\theta(U)$, $\theta\in[0,1]$, *such that the function*

$$Z:=\left(\begin{smallmatrix}z\\ \bar z\end{smallmatrix}\right)=(\mathfrak{F}^\theta(U))_{|\theta=1}[U]$$

solves the system

$$D_tZ=\mathrm{Op}^{\mathrm{BW}}\big((1+\underline{\zeta}(U;t))\Omega(\xi)E+H(U;t,\xi)\big)Z+R(U;t)[Z] \tag{5.2.10}$$

where:

- *the function* $\underline{\zeta}(U;t)\in\Sigma\mathcal{F}^{\mathbb{R}}_{K,K',1}[r,N]$ *and the diagonal matrix of symbols* $H(U;t,\xi)\in\Sigma\Gamma^{1}_{K,K',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ *are* independent *of* x*;*
- *the symbol* $\Im H(U;t,\xi)$ *belongs to* $\Sigma\Gamma^{0}_{K,K',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$*;*
- *the operators* $\mathrm{iOp}^{\mathrm{BW}}(H(U;t,\xi))$ *and* $\mathrm{i}R(U;t)$ *are real-to-real, according to* (5.1.12)*;*

- *the map* $\mathfrak{F}^\theta(U)$ *satisfies, for all* $0 \leqslant k \leqslant K - K'$, *for any* $V \in C^{K-K'}_{*\mathbb{R}}(I; \dot{H}^s(\mathbb{T}; \mathbb{C}^2))$,

$$\|\partial_t^k \mathfrak{F}^\theta(U)[V]\|_{\dot{H}^{s-\frac{3}{2}k}} + \|\partial_t^k (\mathfrak{F}^\theta(U))^{-1}[V]\|_{\dot{H}^{s-\frac{3}{2}k}} \leqslant \|V\|_{k,s}\big(1 + C_{s,r,K}\|U\|_{K,s_0}\big) \tag{5.2.11}$$

uniformly in $\theta \in [0,1]$. *Moreover the map* $\mathfrak{F}^\theta(U) = U + \theta M_1(U)[U] + M_{\geqslant 2}(\theta; U)[U]$ *where* $M_1(U)$ *is in* $\widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$ *and* $M_{\geqslant 2}(\theta; U) \in \mathcal{M}_{K,K',2}[r] \otimes \mathcal{M}_2(\mathbb{C})$ *with estimates uniform in* $\theta \in [0,1]$.

Proof. This statement collects the results of Propositions 4.9, 5.1 and 5.5 in [37]. The remainder in (5.2.9) in [37] has the form (5.2.10) expressing $U = (\mathfrak{F}^\theta(U))^{-1}_{|\theta=1} Z$ and using the estimates (5.2.11), which follow by Lemma 3.22 in [37]. Another difference is that $Z(x)$ is not even in x. The x-invariance properties (5.1.4) for the symbols and (5.1.9) for the smoothing operators are checked as in [39]. The last statement follows using Lemma A.2 in [39]. ☐

5.3 Poincaré - Birkhoff normal form at quadratic degree

From this section the analysis strongly differs from [37].

- **Notation**: for simplicity in the sequel we omit to write the dependence on the time t in the symbols, smoothing remainders and maps, writing $a(U; x, \xi)$, $R(U)$, $M(U)$ instead of $a(U; t, x, \xi)$, $R(U; t)$, $M(U; t)$.

The aim of this section is to transform system (5.2.10) into its quadratic Poincaré-Birkhoff normal form, see system (5.3.9). We first observe that the paradifferential vector field in (5.2.10) of quadratic homogeneity is actually zero.

Lemma 5.12. (Quadratic Poincaré-Birkhoff normal form up to smoothing vector fields) *The system* (5.2.10) *with* $N = 2$ *has the form*

$$\partial_t Z = \mathrm{i}\Omega(D) E Z + \mathtt{R}_1(U)[Z] + \widetilde{\mathcal{X}}_{\geqslant 3}(U, Z) \tag{5.3.1}$$

where $\mathtt{R}_1(U) \in \widetilde{\mathcal{R}}^{-\rho}_1 \otimes \mathcal{M}_2(\mathbb{C})$ *and*

$$\widetilde{\mathcal{X}}_{\geqslant 3}(U, Z) = \mathrm{i}\mathrm{Op}^{\mathrm{BW}}\big(\mathcal{H}_{\geqslant 2}(U; \xi)\big) Z + \mathtt{R}_{\geqslant 2}(U)[Z] \tag{5.3.2}$$

where $\mathcal{H}_{\geqslant 2}(U; \xi) \in \Gamma^{3/2}_{K,K',2}[r] \otimes \mathcal{M}_2(\mathbb{C})$ *is a diagonal matrix of symbols independent of* x, *such that*

$$\Im \mathcal{H}_{\geqslant 2}(U; \xi) \in \Gamma^0_{K,K',2}[r] \otimes \mathcal{M}_2(\mathbb{C}), \tag{5.3.3}$$

and $\mathtt{R}_{\geqslant 2}(U) \in \mathcal{R}^{-\rho}_{K,K',2}[r] \otimes \mathcal{M}_2(\mathbb{C})$. *The operators* $\mathtt{R}_1(U)$ *and* $\widetilde{\mathcal{X}}_{\geqslant 3}(U, Z)$ *are real-to-real.*

Proof. We expand in homogeneity the function $\underline{\zeta}(U) = \zeta_1(U) + \zeta_{\geqslant 2}(U)$, $\zeta_1 \in \widetilde{\mathcal{F}}_1^{\mathbb{R}}$, the diagonal matrix of symbols $H(U;\xi) = H_1(U;\xi) + H_{\geqslant 2}(U;\xi)$, $H_1(U;\xi) \in \widetilde{\Gamma}_1^1 \otimes \mathcal{M}_2(\mathbb{C})$, and the smoothing remainder $R(U) = -\mathrm{i}\mathtt{R}_1(U) - \mathrm{i}\mathtt{R}_{\geqslant 2}(U)$, $\mathtt{R}_1(U) \in \widetilde{\mathcal{R}}_1^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$. Since the function $\zeta_1(U)$ and $H_1(U;\xi)$ admit an expansion as (5.1.6) and are independent of x (see Proposition 5.11), Remark 5.4 implies that $\zeta_1(U) = 0$, $H_1(U;\xi) = 0$. This proves (5.3.1)-(5.3.3). □

System (5.3.1) is yet in Poincaré-Birkhoff normal form at degree 2 up to smoothing remainders and the cubic term $\widetilde{\mathcal{X}}_{\geqslant 3}$ in (5.3.2) admits an energy estimate as (5.0.10), since $\mathcal{H}_{\geqslant 2}(U;\xi)$ is independent of x and purely imaginary up to symbols of order 0, see (5.3.3).

The goal is now to transform the quadratic smoothing term $\mathtt{R}_1(U)[Z]$ in (5.3.1) to Poincaré-Birkhoff normal form at degree 2, see Definition 5.13. The remainder $\mathtt{R}_1(U)$ in (5.3.1) is *real-to-real* (i.e. has the form (5.1.12)), satisfies the momentum condition (5.1.9), thus it has the form (5.1.11), and so we write it as

$$\mathtt{R}_1(U) = \begin{pmatrix} (\mathtt{R}_1(U))_+^+ & (\mathtt{R}_1(U))_+^- \\ (\mathtt{R}_1(U))_-^+ & (\mathtt{R}_1(U))_-^- \end{pmatrix}, \ (\mathtt{R}_1(U))_\sigma^{\sigma'} \in \widetilde{\mathcal{R}}_1^{-\rho}, \ (\mathtt{R}_1(U))_\sigma^{\sigma'} = \overline{(\mathtt{R}_1(U))_{-\sigma}^{-\sigma'}}, \tag{5.3.4}$$

for $\sigma, \sigma' = \pm$. For any $\sigma, \sigma' = \pm$ we expand

$$(\mathtt{R}_1(U))_\sigma^{\sigma'} = \sum_{\epsilon=\pm} (\mathtt{R}_{1,\epsilon}(U))_\sigma^{\sigma'}, \tag{5.3.5}$$

where, for $\epsilon = \pm$, and $(\mathtt{R}_{1,\epsilon}(U))_\sigma^{\sigma'} \in \widetilde{\mathcal{R}}_1^{-\rho}$ is the homogeneous smoothing operator

$$(\mathtt{R}_{1,\epsilon}(U))_\sigma^{\sigma'} z^{\sigma'} = \frac{1}{\sqrt{2\pi}} \sum_{j\in\mathbb{Z}\setminus\{0\}} \Big(\sum_{k\in\mathbb{Z}\setminus\{0\}} (\mathtt{R}_{1,\epsilon}(U))_{\sigma,j}^{\sigma',k} z_k^{\sigma'} \Big) e^{\mathrm{i}\sigma j x} \tag{5.3.6}$$

with entries

$$(\mathtt{R}_{1,\epsilon}(U))_{\sigma,j}^{\sigma',k} := \frac{1}{\sqrt{2\pi}} \sum_{\substack{n\in\mathbb{Z}\setminus\{0\} \\ \epsilon n + \sigma' k = \sigma j}} (\mathtt{r}_{1,\epsilon})_{n,k}^{\sigma,\sigma'} u_n^\epsilon, \quad j,k \in \mathbb{Z}\setminus\{0\}, \tag{5.3.7}$$

for suitable scalar coefficients $(\mathtt{r}_{1,\epsilon})_{n,k}^{\sigma,\sigma'} \in \mathbb{C}$. The restriction $\epsilon n + \sigma' k = \sigma j$ is due to the momentum condition.

Definition 5.13. (Poincaré-Birkhoff Resonant smoothing operator) Given a real-to-real, smoothing operator $\mathtt{R}_1(U) \in \widetilde{\mathcal{R}}_1^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$ as in (5.3.4)-(5.3.7), we define the Poincaré-Birkhoff resonant, real-to-real, smoothing operator $\mathtt{R}_1^{\mathrm{res}}(U) \in \widetilde{\mathcal{R}}_1^{-\rho} \otimes \mathcal{M}_2(\mathbb{C})$ with matrix entries

$(\mathtt{R}^{\rm res}_{1,\epsilon}(U))^{\sigma',k}_{\sigma,j}$ defined as in (5.3.7) such that, for any $\epsilon,\sigma,\sigma'=\pm$, $j,k\in\mathbb{Z}\backslash\{0\}$,

$$(\mathtt{R}^{\rm res}_{1,\epsilon}(U))^{\sigma',k}_{\sigma,j}=\frac{1}{\sqrt{2\pi}}\sum_{\substack{n\in\mathbb{Z}\backslash\{0\}\,,\epsilon n+\sigma' k=\sigma j\\ \sigma\Omega(j)-\sigma'\Omega(k)-\epsilon\Omega(n)=0}}(\mathtt{r}_{1,\epsilon})^{\sigma,\sigma'}_{n,k}u^{\epsilon}_{n}\,. \tag{5.3.8}$$

In the next Proposition we conjugate (5.3.1) into its complete quadratic Poincaré-Birkhoff normal form.

Proposition 5.14. (Quadratic Poincaré-Birkhoff normal form) *There exists $\rho_0>0$ such that, for all $\rho\geqslant\rho_0$, $K\geqslant K'$ with $K':=K'(\rho)$ given by Proposition 5.11, there exists $s_0>0$ such that, for any $s\geqslant s_0$, for all $0<r\leqslant r_0(s)$ small enough, and any solution $U\in B^K_s(I;r)$ of the water waves system (5.2.6), there is a family of real-to-real, bounded, invertible linear maps $\mathfrak{C}^\theta(U)$, $\theta\in[0,1]$, such that, if Z solves (5.3.1), then the function*

$$Y:=\left(\begin{smallmatrix}y\\ \overline{y}\end{smallmatrix}\right)=(\mathfrak{C}^\theta(U)[Z])_{|\theta=1}$$

solves

$$\partial_t Y=\mathrm{i}\Omega(D)EY+\mathtt{R}^{\rm res}_1(Y)[Y]+\mathcal{X}_{\geqslant3}(U,Y) \tag{5.3.9}$$

where:

- *E is the matrix in (5.2.6) and $\Omega(D)$ has symbol (5.2.7);*
- *$\mathtt{R}^{\rm res}_1(Y)\in\widetilde{\mathcal{R}}^{-(\rho-\rho_0)}_1\otimes\mathcal{M}_2(\mathbb{C})$ is the real-to-real Poincaré-Birkhoff resonant smoothing operator introduced in Definition 5.13;*
- *$\mathcal{X}_{\geqslant3}(U,Y)$ has the form*

$$\mathcal{X}_{\geqslant3}(U,Y)=\left(\begin{smallmatrix}\mathcal{X}^+_{\geqslant3}(U,Y)\\ \overline{\mathcal{X}^+_{\geqslant3}(U,Y)}\end{smallmatrix}\right):=\mathrm{i}\mathrm{Op}^{\rm BW}(\mathcal{H}_{\geqslant2}(U;\xi))[Y]+\mathfrak{R}_{\geqslant2}(U)[Y] \tag{5.3.10}$$

  where $\mathcal{H}_{\geqslant2}(U;\xi)$ is defined in (5.3.2) and satisfies (5.3.3), while $\mathfrak{R}_{\geqslant2}(U)$ is a matrix of real-to-real smoothing operators in $\mathcal{R}^{-(\rho-\rho_0)}_{K,K',2}[r]\otimes\mathcal{M}_2(\mathbb{C})$;
- *the map $\mathfrak{C}^\theta(U)$ satisfies, for any $0\leqslant k\leqslant K-K'$, $V\in C^{K-K'}_{*\mathbb{R}}(I;\dot{H}^s(\mathbb{T};\mathbb{C}^2))$,*

$$\begin{aligned}&\|\partial_t^k\mathfrak{C}^\theta(U)[V]\|_{\dot{H}^{s-\frac32 k}}+\|\partial_t^k(\mathfrak{C}^\theta(U))^{-1}[V]\|_{\dot{H}^{s-\frac32 k}}\\&\leqslant\|V\|_{k,s}(1+C_{s,r,K}\|U\|_{K,s_0})+C_{s,r,K}\|V\|_{k,s_0}\|U\|_{K,s}\,,\end{aligned} \tag{5.3.11}$$

  uniformly in $\theta\in[0,1]$. Moreover the map $\mathfrak{C}^\theta(U)[V]=V+\theta M_1(U)[V]+M_{\geqslant2}(\theta;U)[V]$ where $M_1(U)$ is in $\widetilde{\mathcal{M}}_1\otimes\mathcal{M}_2(\mathbb{C})$ and $M_{\geqslant2}(\theta;U)\in\mathcal{M}_{K,K',2}[r]\otimes\mathcal{M}_2(\mathbb{C})$ with estimates uniform in $\theta\in[0,1]$.

In order to prove Proposition 5.14 we first provide lower bounds on the "small divisors" which appear in the Poincaré-Birkhoff reduction procedure.

5.3.1 Three waves interactions

We analyze the possible three waves interactions among the linear frequencies (5.2.7). We first notice that they admit an expansion as

$$\Omega(n) = \sqrt{|n| \tanh(\mathtt{h}|n|)(g + \kappa n^2)} = \sqrt{\kappa}|n|^{\frac{3}{2}} + \mathtt{r}(n), \quad |\mathtt{r}(n)| \leqslant C|n|^{-\frac{1}{2}} \tag{5.3.12}$$

for some constant $C := C(\kappa, g, \mathtt{h}) > 0$.

Lemma 5.15. **(3-waves interactions)** *There exist* $\mathtt{c}, \mathtt{C} > 0$ *such that for any* $n_1, n_2, n_3 \in \mathbb{Z}\backslash\{0\}$, $\sigma, \sigma' = \pm$, *such that*

$$n_1 + \sigma n_2 + \sigma' n_3 = 0, \tag{5.3.13}$$

and $\max(|n_1|, |n_2|, |n_3|) \geqslant \mathtt{C}$, *we have*

$$|\Omega(n_1) + \sigma\Omega(n_2) + \sigma'\Omega(n_3)| \geqslant \mathtt{c}. \tag{5.3.14}$$

If $\max(|n_1|, |n_2|, |n_3|) < \mathtt{C}$, *then, either the phase* $\Omega(n_1) + \sigma\Omega(n_2) + \sigma'\Omega(n_3)$ *is zero, or* (5.3.14) *holds.*

Proof. If $\sigma = \sigma' = +$ then the bound (5.3.14) is trivial for all $n_1, n_2, n_3 \in \mathbb{Z}\backslash\{0\}$. Assume $\sigma = -$ and $\sigma' = -$ (the cases $(\sigma, \sigma') = (+, -)$ and $(\sigma, \sigma') = (-, +)$ are the same, up to reordering the indexes). Then, by (5.3.13), we have $n_1 = n_2 + n_3$ and we may suppose that $|n_1| \geqslant |n_2|, |n_3|$, otherwise the bound (5.3.14) is trivial. Without loss of generality we assume $n_1 > 0$, thus, also n_2 and n_3 are positive. In conclusion we assume that $n_1 \geqslant n_2 \geqslant n_3 \geqslant 1$. By (5.3.12),

$$\begin{aligned} |\Omega(n_1) - \Omega(n_2) - \Omega(n_3)| &= |\Omega(n_2 + n_3) - \Omega(n_2) - \Omega(n_3)| \\ &\geqslant \sqrt{\kappa}\big((n_2 + n_3)^{\frac{3}{2}} - n_2^{\frac{3}{2}} - n_3^{\frac{3}{2}}\big) - \frac{3C}{\sqrt{n_3}}. \end{aligned} \tag{5.3.15}$$

Now

$$
\begin{aligned}
(n_2+n_3)^{\frac32}-n_2^{\frac32}-n_3^{\frac32} &= \frac{(n_2+n_3)^3-(n_2^{\frac32}+n_3^{\frac32})^2}{(n_2+n_3)^{\frac32}+n_2^{\frac32}+n_3^{\frac32}} = \frac{3(n_2^2n_3+n_2n_3^2)-2n_2^{\frac32}n_3^{\frac32}}{(n_2+n_3)^{\frac32}+n_2^{\frac32}+n_3^{\frac32}} \\
&= \frac{9(n_2^2n_3+n_2n_3^2)^2-4n_2^3n_3^3}{(n_2+n_3)^{\frac32}+n_2^{\frac32}+n_3^{\frac32}}\frac{1}{3(n_2^2n_3+n_2n_3^2)+2n_2^{\frac32}n_3^{\frac32}} \\
&= \frac{9(n_2^4n_3^2+n_2^2n_3^4)+14n_2^3n_3^3}{\big((n_2+n_3)^{\frac32}+n_2^{\frac32}+n_3^{\frac32}\big)\big(3(n_2^2n_3+n_2n_3^2)+2n_2^{\frac32}n_3^{\frac32}\big)} \\
&\geqslant \frac{9}{16(1+\sqrt{2})}\sqrt{n_2} \geqslant \frac{\sqrt{n_2}}{5}
\end{aligned} \tag{5.3.16}
$$

using that $n_2 \geqslant n_3 \geqslant 1$. By (5.3.15) and (5.3.16) we deduce that the phase

$$
|\Omega(n_1)-\Omega(n_2)-\Omega(n_3)| \geqslant \sqrt{n_2}\Big(\frac{\sqrt{\kappa}}{5}-\frac{3C}{\sqrt{n_2\,n_3}}\Big) \geqslant \sqrt{n_2}\frac{\sqrt{\kappa}}{10} \tag{5.3.17}
$$

if $n_2n_3 \geqslant (30C)^2/\kappa$, in particular, since $n_3 \geqslant 1$, if

$$
n_2 \geqslant C_1 := (30C)^2/\kappa\,.
$$

Recall that $n_1 = n_2+n_3 \leqslant 2n_2$. Therefore $n_2 \geqslant n_1/2$ and we conclude that

$$
n_1 = \max(n_1,n_2,n_3) \geqslant 2C_1 \implies n_2 \geqslant C_1 \implies |\Omega(n_1)-\Omega(n_2)-\Omega(n_3)| \geqslant \sqrt{n_2}\frac{\sqrt{\kappa}}{10}\,.
$$

For the finitely many integers n_1,n_2,n_3 satisfying $\max(|n_1|,|n_2|,|n_3|) \leqslant \mathtt{C} := 2C_1$ such that the phase $\Omega(n_1)-\Omega(n_2)-\Omega(n_3) \neq 0$, the lower bound (5.3.14) is trivial. □

Remark 5.16. The constant $C(\kappa,g,\mathtt{h})$ in (5.3.12) is bounded by $c(\sqrt{\kappa}\,\mathtt{h}^{-2}+g\kappa^{-1/2})$, for some constant $c>0$ independent of $\kappa,g,\mathtt{h}$. Then, there are $\mathtt{h}_0,\kappa_0$ such that, if $\mathtt{h} \geqslant \mathtt{h}_0$, $\kappa > \kappa_0 g$, then (5.3.17) holds, for all $n_1,n_2,n_3 \in \mathbb{Z}\backslash\{0\}$. As a consequence there are no 3-waves interactions, i.e. (5.3.14) holds for all $n_1,n_2,n_3 \in \mathbb{Z}\backslash\{0\}$.

Note that, for some values of the parameters $(\kappa,g,\mathtt{h})$, there could be 3-waves interactions.

5.3.2 Poincaré-Birkhoff normal form of the smoothing quadratic terms

In order to prove Proposition 5.14, we conjugate (5.3.1) with the flow

$$
\partial_\theta \mathfrak{C}^\theta(U) = \mathtt{G}_1(U)\mathfrak{C}^\theta(U)\,, \quad \mathfrak{C}^0(U) = \mathrm{Id}\,, \tag{5.3.18}
$$

with an operator $\mathtt{G}_1(U)$ in $\widetilde{\mathcal{R}}_1^{-\rho}\otimes\mathcal{M}_2(\mathbb{C})$, of the same form of $\mathtt{R}_1(U)$ in (5.3.4)-(5.3.7), to be determined. We introduce the new variable $Y := \big(\begin{smallmatrix} y \\ \bar{y}\end{smallmatrix}\big) = \big(\mathfrak{C}^\theta(U)[Z]\big)_{|\theta=1}$.

Lemma 5.17. *If $\mathtt{G}_1(U)\in\widetilde{\mathcal{R}}_1^{-\rho}\otimes\mathcal{M}_2(\mathbb{C})$ solves the homological equation*

$$\mathtt{G}_1(\mathrm{i}\Omega(D)EU) + \big[\mathtt{G}_1(U), \mathrm{i}\Omega(D)E\big] + \mathtt{R}_1(U) = \mathtt{R}_1^{\rm res}(U)\,, \tag{5.3.19}$$

where $\mathtt{R}_1^{\rm res}(U)$ is the Poincaré-Birkhoff resonant operator in Definition 5.13, then

$$\partial_t Y = \mathrm{i}\Omega(D)EY + \mathtt{R}_1^{res}(U)[Y] + \mathrm{i}\mathrm{Op}^{\rm BW}\big(\mathcal{H}_{\geqslant 2}(U;\xi)\big)Y + \mathtt{R}_{\geqslant 2}(U)[Y] \tag{5.3.20}$$

where $\mathcal{H}_{\geqslant 2}(U;\xi)$ is the same diagonal matrix of symbols in (5.3.2) and $\mathtt{R}_{\geqslant 2}(U)$ is a real-to-real smoothing operator in $\mathcal{R}^{-\rho+m_1}_{K,K',2}[r]\otimes\mathcal{M}_2(\mathbb{C})$ with $m_1\geqslant 3/2$ (fixed below (5.2.9)).

The flow map $\mathfrak{C}^\theta(U)$ in (5.3.18) satisfies (5.3.11) and $\mathfrak{C}^\theta(U) = U+\theta M_1(U)[U]+M_{\geqslant 2}(\theta;U)[U]$ where $M_1(U)$ is in $\widetilde{\mathcal{M}}_1\otimes\mathcal{M}_2(\mathbb{C})$ and $M_{\geqslant 2}(\theta;U)\in\mathcal{M}_{K,K',2}[r]\otimes\mathcal{M}_2(\mathbb{C})$ with estimates uniform in $\theta\in[0,1]$.

Proof. Since $\mathtt{G}_1(U)$ is a smoothing operator then the flow in (5.3.18) is well-posed in Sobolev spaces and satisfies the estimates (5.3.11), as well as the last statement, by e.g. Lemma *A*.3 in [39]. To conjugate (5.3.1) we apply the usual Lie expansion up to the first order (see for instance Lemma *A*.1 in [39]). Denoting $\mathrm{Ad}_{\mathtt{G}_1} := [\mathtt{G}_1, \ ]$, we have

$$\begin{aligned}\mathfrak{C}^1(U)\Omega(D)E(\mathfrak{C}^1(U))^{-1} &= \Omega(D)E + \big[\mathtt{G}_1(U),\Omega(D)E\big] \\ &\quad + \int_0^1 (1-\theta)\mathfrak{C}^\theta(U)\mathrm{Ad}^2_{\mathtt{G}_1(U)}[\Omega(D)E](\mathfrak{C}^\theta(U))^{-1}d\theta\,. \end{aligned}\tag{5.3.21}$$

Using that $\mathtt{G}_1(U)$ belongs to $\widetilde{\mathcal{R}}_1^{-\rho}\otimes\mathcal{M}_2(\mathbb{C})$, Proposition 5.9 and (5.3.11), the integral term in (5.3.21) is a smoothing operator in $\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K',2}[r]\otimes\mathcal{M}_2(\mathbb{C})$. Similarly, we obtain

$$\mathfrak{C}^1(U)\mathrm{Op}^{\rm BW}(\mathcal{H}_{\geqslant 2}(U;\xi))(\mathfrak{C}^1(U))^{-1} = \mathrm{Op}^{\rm BW}(\mathcal{H}_{\geqslant 2}(U;\xi))$$

up to a matrix of smoothing operators in $\mathcal{R}^{-\rho+\frac{3}{2}}_{K,K',2}[r]\otimes\mathcal{M}_2(\mathbb{C})$. Finally

$$\mathfrak{C}^1(U)\big(\mathtt{R}_1(U)+\mathtt{R}_{\geqslant 2}(U)\big)(\mathfrak{C}^1(U))^{-1} = \mathtt{R}_1(U)$$

plus a smoothing operator in $\mathcal{R}^{-\rho}_{K,K',2}[r]\otimes\mathcal{M}_2(\mathbb{C})$.

Next we consider the contribution coming from the conjugation of ∂_t. Applying again a Lie expansion formula (see Lemma $A.1$ in [39]) we get

$$\partial_t \mathfrak{C}^1(U)(\mathfrak{C}^1(U))^{-1} = \partial_t \mathtt{G}_1(U) + \frac{1}{2}\big[\mathtt{G}_1(U), \partial_t \mathtt{G}_1(U)\big] + \frac{1}{2}\int_0^1 (1-\theta)^2 \mathfrak{C}^\theta(U) \mathrm{Ad}^2_{\mathtt{G}_1(U)} \left[\partial_t \mathtt{G}_1(U)\right] (\mathfrak{C}^\theta(U))^{-1} d\theta \,. \tag{5.3.22}$$

Recalling (5.2.9) we have

$$\partial_t \mathtt{G}_1(U) = \mathtt{G}_1(\mathrm{i}\Omega(D)EU + \mathrm{i}M(U)[U]) = \mathtt{G}_1(\mathrm{i}\Omega(D)EU) \tag{5.3.23}$$

up to a term in $\mathcal{R}^{-\rho+m_1}_{K,K',2}[r]\otimes\mathcal{M}_2(\mathbb{C})$, where we used Proposition 5.9. By (5.3.23), the fact that $\mathtt{G}_1(\mathrm{i}\Omega(D)EU)$ is in $\widetilde{\mathcal{R}}^{-\rho+(3/2)}_1 \otimes\mathcal{M}_2(\mathbb{C})$ and (5.3.11), we deduce that the term in (5.3.22) belongs to $\Sigma\mathcal{R}^{-\rho+m_1}_{K,K',2}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Collecting all the previous expansions, and using that $\mathtt{G}_1(U)$ solves (5.3.19), we deduce (5.3.20). □

We now solve the homological equation (5.3.19).

Lemma 5.18. (Homological equation) *Consider $\mathtt{R}_1(U)$ appearing in Lemma 5.12 and recall its expansion* (5.3.4)-(5.3.7). *Let $\mathtt{G}_1(U)$ be an operator of the form* (5.3.4)-(5.3.7) *with coefficients*

$$(\mathtt{g}_{1,\epsilon})^{\sigma,\sigma'}_{n,k} := \frac{(\mathtt{r}_{1,\epsilon})^{\sigma,\sigma'}_{n,k}}{\mathrm{i}\big(\sigma\Omega(j) - \sigma'\Omega(k) - \epsilon\Omega(n)\big)}\,, \tag{5.3.24}$$

for any $\sigma,\sigma',\epsilon = \pm$, $j,n,k\in\mathbb{Z}\backslash\{0\}$, satisfying

$$\sigma j - \sigma' k - \epsilon n = 0\,, \qquad \sigma\Omega(j) - \sigma'\Omega(k) - \epsilon\Omega(n) \neq 0\,, \tag{5.3.25}$$

and $(\mathtt{g}_{1,\epsilon})^{\sigma,\sigma'}_{n,k} := 0$ otherwise. Then $\mathtt{G}_1(U)$ is in $\widetilde{\mathcal{R}}^{-\rho}_1\otimes\mathcal{M}_2(\mathbb{C})$ and solves the homological equation (5.3.19).

Proof. The coefficients in (5.3.24) are well defined by (5.3.25) and, by Lemma 5.15, they satisfy the uniform lower bound $|\sigma\Omega(j) - \sigma'\Omega(k) - \epsilon\Omega(n)| \geqslant \mathtt{c}$. Then the operator $\mathtt{G}_1(U)$ is in $\widetilde{\mathcal{R}}^{-\rho}_1 \otimes \mathcal{M}_2(\mathbb{C})$, see e.g. Lemma 6.5 of [39].

Next, recalling (5.3.4), the homological equation (5.3.19) amounts to the equations

$$(\mathtt{G}_1(\mathrm{i}\Omega(D)EU))^{\sigma'}_{\sigma} + (\mathtt{G}_1(U))^{\sigma'}_{\sigma}\sigma'\mathrm{i}\Omega(D) - \sigma\mathrm{i}\Omega(D)(\mathtt{G}_1(U))^{\sigma'}_{\sigma} + (\mathtt{R}_1(U))^{\sigma'}_{\sigma} = \big(\mathtt{R}^{\mathrm{res}}_1(U)\big)^{\sigma'}_{\sigma}$$

for $\sigma,\sigma' = \pm$, and, setting $\mathtt{F}_1(U) := \mathtt{G}_1(\mathrm{i}\Omega(D)EU)$ to the equations, for any $j,k\in\mathbb{Z}\backslash\{0\}$, $\epsilon = \pm$,

$$\begin{aligned}(\mathtt{F}_{1,\epsilon}(U))^{\sigma',k}_{\sigma,j} + (\mathtt{G}_{1,\epsilon}(U))^{\sigma',k}_{\sigma,j}\big(-\sigma\mathrm{i}\Omega(j) + \sigma'\mathrm{i}\Omega(k)\big) + (\mathtt{R}_{1,\epsilon}(U))^{\sigma',k}_{\sigma,j} \\ = \big(\mathtt{R}^{\mathrm{res}}_{1,\epsilon}(U)\big)^{\sigma',k}_{\sigma,j}\,.\end{aligned} \tag{5.3.26}$$

Expanding $(\mathtt{G}_1(U))_\sigma^{\sigma'}$ as in (5.3.5)-(5.3.7) with entries

$$(\mathtt{G}_{1,\epsilon}(U))_{\sigma,j}^{\sigma',k} = \frac{1}{\sqrt{2\pi}} \sum_{\substack{n\in\mathbb{Z}\setminus\{0\} \\ \epsilon n+\sigma' k=\sigma j}} (\mathtt{g}_{1,\epsilon})_{n,k}^{\sigma,\sigma'} u_n^\epsilon \,, \quad j,k \in \mathbb{Z}\setminus\{0\}\,,$$

we have that $\mathtt{F}_1(U)$ satisfies

$$(\mathtt{F}_{1,\epsilon}(U))_{\sigma,j}^{\sigma',k} = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}\setminus\{0\},\epsilon n+\sigma' k=\sigma j} (\mathtt{g}_{1,\epsilon})_{n,k}^{\sigma,\sigma'} (\mathrm{i}\Omega(n)\epsilon) u_n^\epsilon \,.$$

Hence the left hand side in (5.3.26) has coefficients

$$-(\mathtt{g}_{1,\epsilon})_{n,k}^{\sigma,\sigma'} \mathrm{i}\big(\sigma\Omega(j) - \sigma'\Omega(k) - \epsilon\Omega(n)\big) + (\mathtt{r}_{1,\epsilon})_{n,k}^{\sigma,\sigma'}$$

for $j,k,n \in \mathbb{Z}\setminus\{0\}$ and $\sigma,\sigma',\epsilon = \pm$ with $\epsilon n + \sigma' k = \sigma j$. Recalling Definition 5.13 we deduce that $\mathtt{G}_1(U)$ with coefficients in (5.3.24) solves the homological equation (5.3.19). □

Proof of Proposition 5.14. We apply Lemmata 5.17 and 5.18. The change of variables that transforms (5.3.1) into (5.3.20) is $Y = \mathfrak{C}^\theta(U)Z$ where $\mathfrak{C}^\theta(U)$ is the flow map in (5.3.18) that satisfies (5.3.11) and the last statement in Lemma 5.17. Moreover, using also the last item of Proposition 5.11 we may express

$$\begin{aligned} & Y = (\mathfrak{C}^\theta(U) \circ \mathfrak{F}^\theta(U))_{|\theta=1}[U] = U + \widetilde{\mathtt{M}}(U)[U]\,, \\ & \widetilde{\mathtt{M}}(U) \in \Sigma\mathcal{M}_{K,K',1}^{m_2}[r,2] \otimes \mathcal{M}_2(\mathbb{C})\,,\ m_2 \geqslant 3/2\,. \end{aligned} \tag{5.3.27}$$

Then system (5.3.20) can be written as system (5.3.9) with $\mathcal{X}_{\geqslant 3}(U,Y)$ given in (5.3.10) and

$$\mathfrak{R}_{\geqslant 2}(U) := \mathtt{R}_1^{\mathrm{res}}(U) - \mathtt{R}_1^{\mathrm{res}}(U + \widetilde{\mathtt{M}}(U)[U]) + \mathtt{R}_{\geqslant 2}(U)\,.$$

By (5.3.27) and Proposition 5.9-(*iii*) we have that $\mathfrak{R}_{\geqslant 2}(U) \in \Sigma\mathcal{R}_{K,K',2}^{-(\rho-\rho_0)} \otimes \mathcal{M}_2(\mathbb{C})$ where $\rho_0 := \max\{m_1, m_2\}$. □

5.4 Birkhoff normal form and quadratic life-span of solutions

In this section we prove Theorems 1.9 and 1.10. Let $\mathtt{B},\mathtt{B}^{-1}$ the linear maps defined by the matrices

$$\mathtt{B} := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{i}\Lambda^{-1} & \Lambda \\ -\mathrm{i}\Lambda^{-1} & \Lambda \end{pmatrix}, \quad \mathtt{B}^{-1} := \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i}\Lambda & \mathrm{i}\Lambda \\ \Lambda^{-1} & \Lambda^{-1} \end{pmatrix}, \tag{5.4.1}$$

where Λ is the Fourier multiplier defined in (5.2.2). We now describe the Hamiltonian formalism in the complex symplectic variables $(w,\overline{w}) = \mathtt{B}(\eta,\psi)$ induced by $\mathtt{B}$. A vector field $X(\eta,\psi)$ and a function $H(\eta,\psi)$ assume the form

$$X^{\mathbb{C}} := \mathtt{B}^\star X := \mathtt{B} X \mathtt{B}^{-1}\,, \qquad H_{\mathbb{C}} := H \circ \mathtt{B}^{-1}\,. \tag{5.4.2}$$

We remind that the Poisson bracket between two real functions $H(\eta,\psi)$, $F(\eta,\psi)$ is

$$\{H,F\} = \int_{\mathbb{T}} (\nabla_\eta H \nabla_\psi F - \nabla_\psi H \nabla_\eta F)\,\mathrm{d}x\,.$$

while in the complex variables $(w,\overline{w})$ reads

$$\{F_{\mathbb{C}}, H_{\mathbb{C}}\} := \mathrm{i} \sum_{j\in\mathbb{Z}\setminus\{0\}} \partial_{w_j} H_{\mathbb{C}} \partial_{\overline{w_j}} F_{\mathbb{C}} - \partial_{\overline{w_j}} H_{\mathbb{C}} \partial_{w_j} F_{\mathbb{C}}\,.$$

Given a Hamiltonian $F_{\mathbb{C}}$, expressed in the complex variables $(w,\overline{w})$, the associated Hamiltonian vector field $X_{F_{\mathbb{C}}}$ is

$$X_{F_{\mathbb{C}}} = \begin{pmatrix} \mathrm{i}\partial_{\overline{w}} F_{\mathbb{C}} \\ -\mathrm{i}\partial_w F_{\mathbb{C}} \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}\setminus\{0\}} \begin{pmatrix} \mathrm{i}\partial_{\overline{w_k}} F_{\mathbb{C}}\, e^{\mathrm{i}kx} \\ -\mathrm{i}\partial_{w_k} F_{\mathbb{C}}\, e^{-\mathrm{i}kx} \end{pmatrix}, \tag{5.4.3}$$

that we also identify, using the standard vector field notation, with

$$X_{F_{\mathbb{C}}} = \sum_{k\in\mathbb{Z}\setminus\{0\},\sigma=\pm} \mathrm{i}\sigma \partial_{w_k^{-\sigma}} F_{\mathbb{C}}\, \partial_{w_k^\sigma}\,.$$

If X_F is the Hamiltonian vector field of the Hamiltonian $F := F_{\mathbb{C}} \circ \mathtt{B}$, we have

$$X_F^{\mathbb{C}} := \mathtt{B}^\star X_F = X_{F_{\mathbb{C}}}\,. \tag{5.4.4}$$

The push-forward acts naturally on the commutator of nonlinear vector fields, defined in (5.4.14), namely

$$\mathtt{B}^\star [\![X,Y]\!] = [\![\mathtt{B}^\star X, \mathtt{B}^\star Y]\!] = [\![X^{\mathbb{C}}, Y^{\mathbb{C}}]\!]\,. \tag{5.4.5}$$

Recalling the Taylor expansion of the Hamiltonian (1.1.18) (with $\gamma = 0$),

$$H = H^{(2)} + H^{(3)} + \dots,$$

where (up to a constant)

$$H^{(2)} := \frac{1}{2}\int_{\mathbb{T}} \psi G(0)\psi\,\mathrm{d}x + \frac{g}{2}\int_{\mathbb{T}} \eta^2\,\mathrm{d}x + \frac{\kappa}{2}\int_{\mathbb{T}} \eta_x^2\,\mathrm{d}x\,, \quad H^{(3)} := \frac{1}{2}\int_{\mathbb{T}} \psi\Big(D\eta D - G(0)\eta G(0)\Big)\psi\,\mathrm{d}x\,,$$

and the dots collects all the terms of homogeneity in (η,ψ) greater or equal than 4, in complex coordinates this expansion reads

$$H_{\mathbb{C}} := H \circ \mathtt{B}^{-1} = H_{\mathbb{C}}^{(2)} + H_{\mathbb{C}}^{(3)} + \dots$$

where, recalling (5.4.1), (5.2.7), (5.1.2),

$$H_{\mathbb{C}}^{(2)} = \sum_{j \in \mathbb{Z}\setminus\{0\}} \Omega(j) w_j \overline{w_j}\,, \qquad H_{\mathbb{C}}^{(3)} = \sum_{\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 = 0} H_{j_1,j_2,j_3}^{\sigma_1,\sigma_2,\sigma_3} w_{j_1}^{\sigma_1} w_{j_2}^{\sigma_2} w_{j_3}^{\sigma_3} \tag{5.4.6}$$

and $H_{j_1,j_2,j_3}^{\sigma_1,\sigma_2,\sigma_3}$ are computed in (5.0.9), for $j_1, j_2, j_3 \in \mathbb{Z}\setminus\{0\}$.

5.4.1 Normal form identification and proof of Theorem 1.9

A normal form uniqueness argument allows to identify the quadratic Poincaré-Birkhoff resonant vector field $\mathtt{R}_1^{res}(Y)[Y]$ in (5.3.9) as the cubic resonant Hamiltonian vector field obtained by the formal Birkhoff normal form construction in [69].

Proposition 5.19. (Identification of the quadratic resonant Birkhoff normal form) *The Birkhoff resonant vector field* $\mathtt{R}_1^{\rm res}(Y)[Y]$ *defined in* (5.3.9) *is equal to*

$$\mathtt{R}_1^{\rm res}(Y)[Y] = X_{H_{\rm BNF}^{(3)}} \tag{5.4.7}$$

where $H_{\rm BNF}^{(3)}$ *is the cubic Birkhoff normal form Hamiltonian in* (5.0.8).

The proof follows the ideas developed in Section 7 in [39]. Recalling (5.4), we first expand the water waves Hamiltonian vector field in (5.0.1)-(1.1.32) in degrees of homogeneity

$$X_H = X_1 + X_2 + X_{\geqslant 3} \qquad \text{where} \qquad X_1 := X_{H^{(2)}},\ X_2 := X_{H^{(3)}}\,, \tag{5.4.8}$$

and $X_{\geqslant 3}$ collects the higher order terms. System (5.3.9) has been obtained conjugating (5.0.1) under the map

$$Y = \mathbf{F}^1(U) \circ \mathtt{B} \circ \mathcal{G}\big(\begin{smallmatrix}\eta\\ \psi\end{smallmatrix}\big)\,, \tag{5.4.9}$$

where $\mathcal{G}$ is the good-unknown transformation (see (5.2.1))

$$\big(\begin{smallmatrix}\eta\\ \omega\end{smallmatrix}\big) = \mathcal{G}\big(\begin{smallmatrix}\eta\\ \psi\end{smallmatrix}\big) := \big(\begin{smallmatrix}\eta\\ \psi - {\rm Op}^{\rm BW}(B(\eta,\psi))\eta\end{smallmatrix}\big)\,, \tag{5.4.10}$$

the map $\mathtt{B}$ is defined in (5.4.1) and

$$\mathbf{F}^\theta(U) := \mathfrak{C}^\theta(U) \circ \mathfrak{F}^\theta(U)\,, \quad \theta \in [0,1]\,, \tag{5.4.11}$$

where $\mathfrak{F}^\theta(U)$, $\mathfrak{C}^\theta(U)$ are defined respectively in Propositions 5.11 and 5.14. Note that the variables

$U = \binom{u}{\overline{u}}$ defined by (5.2.3) are equal to $\mathtt{B} \circ \mathcal{G}\binom{\eta}{\psi}$. In order to identify the quadratic vector field in system (5.3.9), we perform a Lie commutator expansion, up to terms of homogeneity at least 3. Notice that the quadratic term in (5.3.9) may arise by only the conjugation of $X_1 + X_2$ under the homogeneous components of the paradifferential transformations $\mathcal{G}$ and $\mathbf{F}^1(U)$, neglecting cubic terms.

We use the following Lemma 5.20 that collects Lemmata A.8, A.9 and A.10 in [39]. The variable U may denote both the couple of complex variables $(u, \overline{u})$ in (5.2.3) or the real variables (η, ψ).

Lemma 5.20 ([39]). **(Lie expansion)** *Consider a map $\theta \mapsto \mathbf{F}^\theta_{\leqslant 2}(U)$, $\theta \in [0,1]$, of the form*

$$\mathbf{F}^\theta_{\leqslant 2}(U) = U + \theta M_1(U)[U] , \qquad M_1(U) \in \widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C}) . \tag{5.4.12}$$

Then:

(i) the family of maps $\mathbf{G}^\theta_{\leqslant 2}(V) := V - \theta M_1(V)[V]$ is such that

$$\mathbf{G}^\theta_{\leqslant 2} \circ \mathbf{F}^\theta_{\leqslant 2}(U) = U + M_{\geqslant 2}(\theta; U)[U] , \quad \mathbf{F}^\theta_{\leqslant 2} \circ \mathbf{G}^\theta_{\leqslant 2}(V) = V + M_{\geqslant 2}(\theta; U)[U] ,$$

where $M_{\geqslant 2}(\theta; U)$ is a polynomial in θ and finitely many monomials $M_p(U)[U]$ for $M_p(U) \in \widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$, $p \geqslant 2$;

(ii) the family of maps $\mathbf{G}^\theta_{\leqslant 2}(V)$ satisfies

$$\partial_\theta \mathbf{G}^\theta_{\leqslant 2}(V) = S(\mathbf{G}^\theta_{\leqslant 2}(V)) + M_{\geqslant 2}(\theta; U)[U] , \quad \mathbf{G}^0_{\leqslant 2}(V) = V ,$$

where $S(U) = S_1(U)[U]$ with $S_1(U) \in \widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$ and $M_{\geqslant 2}(\theta; U)$ is a polynomial in θ and finitely many monomials $M_p(U)[U]$ for maps $M_p(U) \in \widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$, $p \geqslant 2$;

(iii) Let $X(U) = M(U)U$ for some map $M(U) = M_0 + M_1(U)$ where M_0 is in $\widetilde{\mathcal{M}}_0 \otimes \mathcal{M}_2(\mathbb{C})$ and $M_1(U)$ in $\widetilde{\mathcal{M}}_1 \otimes \mathcal{M}_2(\mathbb{C})$. If U solves $\partial_t U = X(U)$, then the function $V := \mathbf{F}^1_{\leqslant 2}(U)$ solves

$$\partial_t V = X(V) + [\![S, X]\!](V) + \cdots , \tag{5.4.13}$$

up to terms of degree of homogeneity greater or equal to 3, where we define the nonlinear commutator

$$[\![S, X]\!](U) := \mathrm{d}_U X(U)[S(U)] - \mathrm{d}_U S(U)[X(U)] . \tag{5.4.14}$$

- **Notation.** Given a homogeneous vector field X, we denote by $\Phi^\star_S X$ the induced (formal) push forward (see (5.4.13))

$$\Phi^\star_S X = X + [\![S, X]\!] + \cdots \tag{5.4.15}$$

where the dots $\cdots$ denote cubic terms.

Proof of Proposition 5.19.

Step 1. The good unknown change of variable $\mathcal{G}$ in (5.4.10). First of all we note that $\mathcal{G}(\eta,\psi)=(\Phi^\theta(\eta,\psi))_{\theta=1}$ where

$$\Phi^\theta\binom{\eta}{\psi}=\binom{\eta}{\psi-\theta \mathrm{Op}^{\mathrm{BW}}(B(\eta,\psi))\eta}\,,\quad \theta\in[0,1]\,.$$

Since $B(\eta,\psi)$ is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,2]$ we have that $\Phi^\theta(\eta,\psi)$ has an expansion as in (5.4.12) up to cubic terms. Hence, by Lemma 5.20-(i)-(ii), we regard the inverse of the map $\mathcal{G}_{\leqslant 2}$, obtained truncating $\mathcal{G}$ up to cubic remainders, as the (formal) time one flow of a quadratic vector field

$$\mathtt{S}_2:=S_1(\eta,\psi)\binom{\eta}{\psi}\,,\quad S_1(\eta,\psi)\in\widetilde{\mathcal{M}}_1\otimes\mathcal{M}_2(\mathbb{C})\,. \tag{5.4.16}$$

By (5.4.8), (5.4.15) and (5.4.16), we get

$$\Phi^\star_{\mathtt{S}_2}(X_1+X_2)=X_1+X_2+[\![\mathtt{S}_2,X_1]\!]+\cdots\,. \tag{5.4.17}$$

Step 2. Complex coordinates. Conjugating (5.4.17) with the linear map $\mathtt{B}$ defined in (5.4.1), we obtain, recalling (5.4.2) and (5.4.5),

$$\mathtt{B}^\star\Phi^\star_{\mathtt{S}_2}(X_1+X_2)=X_1^{\mathbb{C}}+X_2^{\mathbb{C}}+[\![\mathtt{S}_2^{\mathbb{C}},X_1^{\mathbb{C}}]\!]+\cdots \tag{5.4.18}$$

where, by (5.4.4), (5.4.8), (5.4.6),

$$X_1^{\mathbb{C}}=X_{H^{(2)}_{\mathbb{C}}}=\mathrm{i}\sum_{j,\sigma}\sigma\Omega(j)u_j^\sigma\partial_{u_j^\sigma}\,,\quad X_2^{\mathbb{C}}=X_{H^{(3)}_{\mathbb{C}}}\,. \tag{5.4.19}$$

Step 3. The transformation $\mathbf{F}^1$ in (5.4.11)**.** By the last items of Proposition 5.11 and Proposition 5.14, the map $\mathbf{F}^\theta(U)$ has the form (5.4.12) up to cubic terms. Thus, by Lemma 5.20-(i)-(ii), the approximate inverse of the truncated map $\mathbf{F}^1_{\leqslant 2}$ can be regarded as the (formal) time-one flow of a vector field

$$\mathtt{T}_2:=T_1(U)[U]\,,\quad T_1(U)\in\widetilde{\mathcal{M}}_1\otimes\mathcal{M}_2(\mathbb{C})\,. \tag{5.4.20}$$

By (5.4.18), (5.4.19), (5.4.15), we get

$$\Phi^\star_{\mathtt{T}_2}\mathtt{B}^\star\Phi^\star_{\mathtt{S}_2}(X_1+X_2)=X_{H^{(2)}_{\mathbb{C}}}+X_{H^{(3)}_{\mathbb{C}}}+[\![\mathtt{S}_2^{\mathbb{C}}+\mathtt{T}_2,X_{H^{(2)}_{\mathbb{C}}}]\!]+\cdots\,. \tag{5.4.21}$$

Comparing (5.3.9) and (5.4.21) we deduce that

$$\mathtt{R}_1^{\rm res}(Y)[Y] \equiv X_{H_{\mathbb{C}}^{(3)}} + [\![\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2, X_{H_{\mathbb{C}}^{(2)}}]\!] \,. \tag{5.4.22}$$

The vector field $\mathtt{R}_1^{\rm res}(Y)[Y]$ is in Poincaré-Birkhoff normal form, recall Definition 5.13. Therefore, defining the linear operator $\Pi_{\rm ker}$ acting on a quadratic monomial vector field $u_{j_1}^{\sigma_1} u_{j_2}^{\sigma_2} \partial_{u_j^\sigma}$ as

$$\Pi_{\rm ker}\Big(u_{j_1}^{\sigma_1} u_{j_2}^{\sigma_2} \partial_{u_j^\sigma} \Big) := \begin{cases} u_{j_1}^{\sigma_1} u_{j_2}^{\sigma_2} \partial_{u_j^\sigma} & \text{if } \; -\sigma\Omega(j) + \sigma_1\Omega(j_1) + \sigma_2\Omega(j_2) = 0 \\ 0 & \text{otherwise}\,, \end{cases} \tag{5.4.23}$$

we have that

$$\mathtt{R}_1^{\rm res}(Y)[Y] = \Pi_{\rm ker}(\mathtt{R}_1^{\rm res}(Y)[Y]) \,. \tag{5.4.24}$$

In addition, since

$$[\![u_{j_1}^{\sigma_1} u_{j_2}^{\sigma_2} \partial_{u_j^\sigma}, X_{H_{\mathbb{C}}^{(2)}}]\!] = \mathrm{i}\big(\sigma\Omega(j) - \sigma_1\Omega(j_1) - \sigma_2\Omega(j_2)\big) u_{j_1}^{\sigma_1} u_{j_2}^{\sigma_2} \partial_{u_j^\sigma} \,,$$

we deduce

$$\Pi_{\rm ker}[\![\mathtt{S}_2^{\mathbb{C}} + \mathtt{T}_2, X_{H_{\mathbb{C}}^{(2)}}]\!] = 0 \,. \tag{5.4.25}$$

In conclusion, (5.4.24), (5.4.22) and (5.4.25) imply that

$$\mathtt{R}_1^{\rm res}(Y)[Y] = \Pi_{\rm ker}(X_{H_{\mathbb{C}}^{(3)}}) \overset{(5.4.6)}{=} X_{H_{\rm BNF}^{(3)}}$$

where $H_{\rm BNF}^{(3)}$ is the Hamiltonian in (5.0.8). This proves (5.4.7).

Proof of Theorem 1.9. Hypothesis (5.0.5) implies that the variable u defined in (5.2.3) satisfies (5.2.5) and therefore the function $U = \binom{u}{\overline{u}}$ belongs to the ball $B_s^K(I;r)$ (recall (5.1.1)) with $r = C_{s,K}\overline{\epsilon} \ll 1$ and $I = [-T,T]$. By Proposition 5.10 the function U solves system (5.2.6). Then we apply Proposition 5.11 and the Poincaré-Birkhoff Proposition 5.14 with $s \gg K \geqslant K'(\rho)$ and $K'(\rho)$ given by Proposition 5.11, taking $\overline{\epsilon}$ small enough. The map $\mathbf{F}^1(U)$ in (5.4.11) transforms the water waves system (5.2.6) into (5.3.9), which, thanks to Proposition 5.19, is expressed in terms of the Hamiltonian $H_{\rm BNF}^{(3)}$ in (5.0.8) as

$$\partial_t y = \mathrm{i}\Omega(D)y + \mathrm{i}\partial_{\overline{y}} H_{\rm BNF}^{(3)}(y,\overline{y}) + \mathcal{X}_{\geqslant 3}^+$$

where $\mathcal{X}_{\geqslant 3}^+$ is the first component of $\mathcal{X}_{\geqslant 3}(U,Y)$ in (5.3.10). Renaming $y \rightsquigarrow z$, the above equation is (5.0.7). We define $z = \mathfrak{B}(\eta,\psi)[\eta,\psi]$ as the first component of the change of variable (5.4.9), namely of $\mathbf{F}^1(U) \circ \mathtt{B} \circ \mathcal{G}[\eta,\psi]$, with U written in terms of (η,ψ) by (5.2.3), (5.2.1). By (5.3.11) and (5.2.11) with $k=0$, and using that $U \in B_s^K(I;r)$, we get

$$\|z(t)\|_{\dot{H}^s} \sim_s \|u(t)\|_{\dot{H}^s}, \tag{5.4.26}$$

and (5.0.6) follows, using also (5.2.3), (5.2.1). The cubic vector field $\mathcal{X}^+_{\geqslant 3}$ in (5.3.10) satisfies the estimate $\|\mathcal{X}^+_{\geqslant 3}\|_{\dot{H}^{s-\frac{3}{2}}} \lesssim_s \|z\|^3_{\dot{H}^s}$ by Proposition 3.8 in [37] (recall that $\mathcal{H}_{\geqslant 2}(U;\xi) \in \Gamma^{3/2}_{K,K',2}[r] \otimes \mathcal{M}_2(\mathbb{C})$), by (5.1.10) with $k=0$, and (5.2.4), (5.4.26). Moreover, the vector field $\mathcal{X}^+_{\geqslant 3}$ satisfies the energy estimate (5.0.10) since the symbol $\mathcal{H}_{\geqslant 2}(U;\xi)$ is independent of x and purely imaginary up to symbols of order 0, see (5.3.3) (for the detailed argument we refer to Lemma 7.5 in [39]).

5.4.2 Energy estimate and proof of Theorem 1.10

We now deduce Theorem 1.10 by Theorem 1.9 and the following energy estimate for the solution z of the Birkhoff resonant system (5.0.7). By time reversibility, without loss of generality, we may only look at positive times $t>0$.

Lemma 5.21. (Energy estimate) *Fix $s, \bar{\epsilon} > 0$ as in Theorem 1.9 and assume that the solution (η,ψ) of (5.0.1) satifies (5.0.5). Then the solution $z(t)$ of (5.0.7) satisfies*

$$\|z(t)\|^2_{\dot{H}^s} \leqslant C(s)\|z(0)\|^2_{\dot{H}^s} + C(s)\int_0^t \|z(\tau)\|^4_{\dot{H}^s}\, d\tau, \quad \forall t \in [0,T]. \tag{5.4.27}$$

Proof. By Lemma 5.15, the Birkhoff resonant Hamiltonian $H^{(3)}_{\mathrm{BNF}}$ in (5.0.8) depends on finitely many variables $z^{\pm}_{j_1}, z^{\pm}_{j_2}, z^{\pm}_{j_3}$, $j_1, j_2, j_3 \in \mathbb{Z}\backslash\{0\}$, because

$$\begin{cases} \sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 = 0 \\ \sigma_1\Omega(j_1) + \sigma_2\Omega(j_2) + \sigma_3\Omega(j_3) = 0 \end{cases} \quad \Longrightarrow \quad \max(|j_1|,|j_2|,|j_3|) < \mathtt{C}. \tag{5.4.28}$$

For any function $w \in \dot{H}^s(\mathbb{T})$ we define the projector Π_L on low modes, respectively the projector Π_H on high modes, as

$$w_L := \Pi_L w := \frac{1}{\sqrt{2\pi}} \sum_{0<|j|\leqslant \mathtt{C}} w_j e^{\mathrm{i}jx}, \qquad w_H := \Pi_H w := \frac{1}{\sqrt{2\pi}} \sum_{|j|>\mathtt{C}} w_j e^{\mathrm{i}jx}.$$

We write $w = w_L + w_H$ and we define the norm

$$\|w\|^2_s := H^{(2)}_{\mathbb{C}}(w_L) + \|w_H\|^2_{\dot{H}^s}$$

where (see (5.4.6))

$$H^{(2)}_{\mathbb{C}}(w) = \int_{\mathbb{T}} \Omega(D)w \cdot \overline{w}\, dx = \sum_{j\in\mathbb{Z}\backslash\{0\}} \Omega(j) w_j \overline{w_j}. \tag{5.4.29}$$

Since $\Omega(j) > 0$, $\forall j \neq 0$, and w_L is supported on finitely many Fourier modes $0 < |j| \leqslant \mathtt{C}$, we have that, for some constant $C_s > 0$,

$$C_s^{-1}\|w\|_s \leqslant \|w\|_{\dot{H}^s} \leqslant C_s\|w\|_s\,, \tag{5.4.30}$$

i.e. the norms $\|\cdot\|_s$ and $\|\cdot\|_{\dot{H}^s}$ are equivalent. We now prove the estimate (5.4.27) for the equivalent norm $\|\cdot\|_s$.

We first note that, by (5.4.28), $H^{(3)}_{\mathrm{BNF}}(z,\overline{z}) = H^{(3)}_{\mathrm{BNF}}(z_L,\overline{z}_L)$. Therefore $\Pi_H\partial_{\overline{z}}H^{(3)}_{\mathrm{BNF}}(z,\overline{z}) = 0$ and the equation (5.0.7) amounts to the system

$$\begin{cases} \dot{z}_L = \mathrm{i}\Omega(D)z_L + \mathrm{i}\partial_{\overline{z}}H^{(3)}_{\mathrm{BNF}}(z_L,\overline{z}_L) + \Pi_L\big(\mathcal{X}^+_{\geqslant 3}(U,Z)\big) \\ \dot{z}_H = \mathrm{i}\Omega(D)z_H \qquad\qquad\qquad\qquad\quad + \Pi_H\big(\mathcal{X}^+_{\geqslant 3}(U,Z)\big)\,. \end{cases} \tag{5.4.31}$$

Moreover since the Hamiltonian $H^{(3)}_{\mathrm{BNF}}$ in (5.0.8) is in Birkhoff normal form, it Poisson commutes with the quadratic Hamiltonian $H^{(2)}_{\mathbb{C}}$ in (5.4.29), i.e.

$$\{H^{(3)}_{\mathrm{BNF}}, H^{(2)}_{\mathbb{C}}\} = 0\,. \tag{5.4.32}$$

We have

$$\begin{aligned} \partial_t H^{(2)}_{\mathbb{C}}(z_L) &\overset{(5.4.31)}{=} \{H^{(3)}_{\mathrm{BNF}}, H^{(2)}_{\mathbb{C}}\} + 2\mathrm{Re}\int_{\mathbb{T}}\Omega(D)\Pi_L\big(\mathcal{X}^+_{\geqslant 3}(U,Z)\big)\cdot\overline{z}_L dx \\ &\overset{(5.4.32)}{=} 2\mathrm{Re}\int_{\mathbb{T}}\Pi_L\Omega(D)\big(\mathcal{X}^+_{\geqslant 3}(U,Z)\big)\cdot\Pi_L\overline{z}\,dx \lesssim_s \|z\|^4_{\dot{H}^s} \end{aligned} \tag{5.4.33}$$

using that $\|\Pi_L\Omega(D)\mathcal{X}^+_{\geqslant 3}\|_{\dot{H}^0} \lesssim_s \|z\|^3_{\dot{H}^s}$ by item (2) of Theorem 1.9. Moreover, since Π_H and Π_L project on L^2-orthogonal subspaces,

$$\begin{aligned} \partial_t\|z_H\|^2_{\dot{H}^s} &= \partial_t(|D|^s z_H, |D|^s z_H)_{L^2} \overset{(5.4.31)}{=} 2\mathrm{Re}\int_{\mathbb{T}}|D|^s\Pi_H\big(\mathcal{X}^+_{\geqslant 3}(U,Z)\big)\cdot|D|^s\Pi_H\overline{z}\,dx \\ &= 2\mathrm{Re}\int_{\mathbb{T}}|D|^s\mathcal{X}^+_{\geqslant 3}(U,Z)\cdot|D|^s\overline{z}\,dx - 2\mathrm{Re}\int_{\mathbb{T}}|D|^s\Pi_L\big(\mathcal{X}^+_{\geqslant 3}(U,Z)\big)\cdot|D|^s\Pi_L\overline{z}\,dx \\ &\overset{(5.0.10)}{\lesssim_s} \|z\|^4_{\dot{H}^s} + \|\Pi_L\mathcal{X}^+_{\geqslant 3}\|_{\dot{H}^s}\|\Pi_L z\|_{\dot{H}^s} \lesssim_s \|z\|^4_{\dot{H}^s} \end{aligned} \tag{5.4.34}$$

by item (2) of Theorem 1.9. Integrating in t the inequalities (5.4.33), (5.4.34), we deduce

$$\|z(t)\|^2_s \lesssim_s \|z(0)\|^2_s + \int_0^t \|z(\tau)\|^4_{\dot{H}^s}d\tau$$

which, together with the equivalence (5.4.30), implies (5.4.27). □

Conclusion of the Proof of Theorem 1.10. Consider initial data (η_0,ψ_0) satisfying (5.0.12) with $s \gg 1$ given by Theorem 1.9. Classical local existence results imply that

$$(\eta, \psi) \in C^0\big([0, T_{\rm loc}], H_0^{s+\frac{1}{4}}(\mathbb{T}, \mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T}, \mathbb{R})\big)$$

for some $T_{\rm loc} > 0$ and thus (5.0.5) holds with $\bar{\epsilon} = 2\epsilon$ and $T = T_{\rm loc}$. A standard bootstrap argument based on the energy estimate (5.4.27) (see for instance Proposition 7.6 in [39]) implies that the solution $z(t)$ of (5.0.7) can be extended up to a time $T_\epsilon := c_0 \epsilon^{-2}$ for some $c_0 > 0$, and satisfies

$$\sup_{t \in [0, T_\epsilon]} \|z(t)\|_{\dot{H}^s} \lesssim_s \epsilon \, . \tag{5.4.35}$$

We deduce (5.0.13) by (5.4.35), the equivalence (5.4.26), and going back to the original variables (η, ψ) by (5.2.3) and (5.2.1). □

Appendix A

Technical results from Chapter 3

A.1 Properties of pseudodifferential operators

Recall that if F is an operator, we denote by $\widehat{F}(\ell)$ its ℓ-th Fourier coefficient defined as in (3.1.9). If F is a pseudodifferential operator with symbol f, so $\widehat{F}(\ell)$ is, with symbol given by

$$\widehat{f}(\ell, x, j) := \frac{1}{(2\pi)^\nu} \int_{\mathbb{T}^\nu} f(\varphi, x, j)\, e^{-\mathrm{i}\varphi\cdot\ell}\, \mathrm{d}\varphi\,.$$

Lemma A.1. *Let $\rho > 0$ and $\mu \in \mathbb{R}$. The following holds true:*

(i) If $F \in \mathrm{OP}S^\mu_\rho$, then the operator $\widehat{F}(\ell)$ belongs to $\mathrm{OP}S^\mu$ for any $\ell \in \mathbb{Z}^\nu$ and

$$\wp^\mu_\varrho(\widehat{F}(\ell)) \leqslant e^{-\rho|\ell|}\, \wp^{\mu,\rho}_\varrho(F) \qquad \forall\, \varrho \in \mathbb{N}_0\ .$$

(ii) Assume to have for any $\ell \in \mathbb{Z}^\nu$ an operator $\widehat{F}(\ell) \in \mathrm{OP}S^\mu$ fulfilling

$$\wp^\mu_\varrho(\widehat{F}(\ell)) \leqslant \langle\ell\rangle^\tau\ e^{-\rho|\ell|}\, C_\varrho \qquad \forall\, \ell \in \mathbb{Z}^\nu,\ \ \forall\, \varrho \in \mathbb{N}_0\ , \tag{A.1.1}$$

for some $\tau \geqslant 0$, $\rho > 0$ and $C_\varrho > 0$ independent of ℓ. Define the operator $F(\varphi) := \sum_{\ell\in\mathbb{Z}^\nu} \widehat{F}(\ell) e^{\mathrm{i}\varphi\cdot\ell}$. Then, F belongs to $\mathrm{OP}S^\mu_{\rho'}$ for any $0 < \rho' < \rho$ and one has

$$\wp^{\mu,\rho'}_\varrho(F) \leqslant \frac{C_\varrho}{(\rho-\rho')^{\tau+\nu}} \qquad \forall\, \varrho \in \mathbb{N}_0\ .$$

On the classes $\mathrm{Lip}_{\mathtt{w}}(\Omega, \mathcal{P}\mathrm{OP}S^\mu_\rho)$, these assertions extend naturally without any further loss of analyticity.

Proof. (i) By Cauchy estimates, it is well-known the analytic decay for the Fourier coefficients of the symbol $f(\varphi;x,j)$:

$$\left|\widehat{f}(\ell,x,j)\right| \leqslant e^{-\rho|\ell|} \sup_{|\mathrm{Im}\,\varphi|\leqslant\rho} |f(\varphi,x,j)| \ . \tag{A.1.2}$$

Plugging it into Definition 3.2 of $\wp^{\mu}_{\varrho}(\widehat{F}(\ell))$, we get the claim;
(ii) It is possible to control the seminorm $\wp^{\mu,\rho'}_{\varrho}(F)$ in terms of the ones for the Fourier coefficients:

$$\wp^{\mu,\rho'}_{\varrho}(F) \leqslant \sum_{\ell\in\mathbb{Z}^\nu} e^{\rho'|\ell|}\wp^{\mu}_{\varrho}(\widehat{F}(\ell)) \overset{(\mathrm{A.1.1})}{\leqslant} \sum_{\ell\in\mathbb{Z}^\nu} e^{(\rho'-\rho)|\ell|}\langle\ell\rangle^{\tau} C_{\varrho} \leqslant \frac{C_{\varrho}}{(\rho-\rho')^{\tau+\nu}} \ . \tag{A.1.3}$$

□

In the next Proposition we essentially prove that pseudodifferential operators as in Definition 3.6 have matrices which belong to the classes $\mathrm{Lip}_{\mathtt{w}}(\Omega,\mathcal{M}_{\rho,s})$ extended from Definition 3.17.

Proposition A.2. *Let* $F\in\mathrm{Lip}_{\mathtt{w}}(\Omega,\mathcal{P}\mathrm{OP}S^{\mu}_{\rho})$, *with* $\rho>0$. *For any* $0<\rho'<\rho$ *and* $s>\frac{1}{2}$, *the matrix of the operator*

$$\langle D\rangle^{\alpha}\, F\, \langle D\rangle^{\beta}\,, \qquad \alpha+\beta+\mu\leqslant 0\ ,$$

belongs to $\mathrm{Lip}_{\mathtt{w}}(\Omega,\mathcal{M}_{\rho',s})$. *Moreover for any* $s>\frac{1}{2}$, $\forall\alpha+\beta\leqslant-\mu$, *there exists* $\sigma>0$ *such that*

$$|\langle D\rangle^{\alpha}\, F\, \langle D\rangle^{\beta}|^{\mathrm{Lip}(\mathtt{w})}_{\rho',s,\Omega} \leqslant \frac{C}{(\rho-\rho')^{\nu}}\ \wp^{\mu,\rho}_{s+\sigma}(F)^{\mathrm{Lip}(\mathtt{w})}_{\Omega}. \tag{A.1.4}$$

Proof. Since $\langle D\rangle\in\mathcal{P}\mathrm{OP}S^1$ is clearly independent of parameters, without loss of generality let F belong to $\mathcal{P}\mathrm{OP}S^{\mu}_{\rho}$. We start by proving the result in the case $\mu=\alpha=\beta=0$. Let an arbitrary $s>\frac{1}{2}$ be fixed. Then

$$\begin{aligned}\widehat{F}^{n}_{m}(\ell) &:= \frac{1}{(2\pi)^{\nu}}\int_{\mathbb{T}^\nu\times[0,\pi]} f(\varphi,x,D_x)[\sin(mx)]\sin(nx)e^{-\mathrm{i}\ell\cdot\varphi}\,\mathrm{d}\varphi\,\mathrm{d}x\\ &= \frac{1}{2(2\pi)^{\nu}}\int_{\mathbb{T}^\nu\times[-\pi,\pi]} f(\varphi,x,D_x)[\sin(mx)]\sin(nx)e^{-\mathrm{i}\ell\cdot\varphi}\,\mathrm{d}\varphi\,\mathrm{d}x\\ &= \frac{1}{4(2\pi)^{\nu}}\int_{\mathbb{T}^{\nu+1}} f(\varphi,x,m)(e^{\mathrm{i}(m-n)x}-e^{\mathrm{i}(m+n)x})e^{-\mathrm{i}\ell\cdot\varphi}\,\mathrm{d}\varphi\,\mathrm{d}x\ ,\end{aligned} \tag{A.1.5}$$

where $f\in\mathcal{P}S^{\mu}_{\rho}$ is the symbol of F. Consider first the case $m\neq n$. Then, integrating by parts $\widetilde{s}$-times in x, with $\widetilde{s}:=\lfloor s+2\rfloor+1$, and shifting the contour of integration in φ to $\mathbb{T}^\nu-\mathrm{i}\rho\,\mathrm{sgn}(\ell)$ (here $\mathrm{sgn}(\ell):=(\mathrm{sgn}(\ell_1),...,\mathrm{sgn}(\ell_\nu))\in\{-1,1\}^\nu$), one gets that for any $n,m\in\mathbb{N}$, $n\neq m$, $\ell\in\mathbb{Z}^\nu$,

$$|\widehat{F}^{n}_{m}(\ell)| \leqslant e^{-\rho|\ell|}\Big(\frac{1}{|m+n|^{\widetilde{s}}}+\frac{1}{|m-n|^{\widetilde{s}}}\Big) \sup_{\substack{|\mathrm{Im}\,\varphi|<\rho\\ (x,m)\in\mathbb{T}\times\mathbb{N}}} \left|\partial_x^{\widetilde{s}} f(\varphi;x,m)\right| \leqslant \frac{2e^{-\rho|\ell|}}{|m-n|^{\widetilde{s}}}\,\wp^{0,\rho}_{\widetilde{s}}(f)\ .$$

If $m = n$, in a similar way one proves the bound $\sup_{m\in\mathbb{N}} \left|\widehat{F}_m^m(\ell)\right| \leqslant e^{-\rho|\ell|}\, \wp_0^{0,\rho}(f)$. It follows that for any $0 < \rho' < \rho$, one has $|F|_{\rho',s} \leqslant C(\rho-\rho')^{-\nu}\wp_{\tilde{s}}^{0,\rho}(f) < \infty$, which proves (A.1.4) in the case $\alpha = \beta = \mu = 0$. To treat the general case, it is sufficient to note that, by Remarks 3.7, 3.10 and 3.11, the operator $\langle D\rangle^{\alpha} F \langle D\rangle^{\beta} \in \mathcal{P}\mathrm{OP}S^0_\rho$, so we have

$$| \langle D\rangle^{\alpha} F \langle D\rangle^{\beta} |_{\rho',s} \leqslant \frac{C}{(\rho-\rho')^{\nu}}\, \wp_{s+\sigma}^{0,\rho}(\langle D\rangle^{\alpha} F \langle D\rangle^{\beta}) \leqslant \frac{C_{\alpha,\beta}}{(\rho-\rho')^{\nu}}\, \wp_{s+\sigma}^{\mu,\rho}(F). \tag{A.1.6}$$

□

A.2 Proof of Lemma 3.22 (Embedding)

The result of Lemma 3.22 follows by a straightforward application of Proposition A.2 to the operators $F \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathcal{P}\mathrm{OP}S_\rho^{-\alpha})$ and $G \in \mathrm{Lip}_{\mathtt{w}}(\Omega, \mathcal{P}\mathrm{OP}S_\rho^{-\beta})$. Indeed, we obtain

$$|\langle D\rangle^{\sigma} F \langle D\rangle^{-\sigma}|^{\mathrm{Lip}(\mathtt{w})}_{\rho',s,\Omega}, \quad |\langle D\rangle^{\alpha} F|^{\mathrm{Lip}(\mathtt{w})}_{\rho',s,\Omega}, \quad |F\langle D\rangle^{\alpha}|^{\mathrm{Lip}(\mathtt{w})}_{\rho',s,\Omega} \leqslant \frac{C}{(\rho-\rho')^{\nu}} \wp_{s+\sigma}^{-\alpha,\rho}(F)^{\mathrm{Lip}(\mathtt{w})}_{\Omega}\ .$$

The estimates for G are analogous.

A.3 Proof of Lemma 3.15 (Algebra of the s-decay)

Denote by A_e the extension of the operator A on $L^2(\mathbb{T})$ which coincides with A on $L^2_{odd}(\mathbb{T}) \equiv \mathcal{H}^0$ and is identically zero on $L^2_{even}(\mathbb{T})$. Since A_e is parity preserving, one verifies that

$$\langle A_e\, e^{\mathrm{i}mx}, e^{\mathrm{i}m'x}\rangle_{L^2(\mathbb{T})} = \langle A\sin(mx), \sin(m'x)\rangle$$

for any $m, m' \in \mathbb{Z}$. We want to give a proof of the algebra property for the s-decay norm on linear operators, as in Definition 3.14 and Lemma 3.15. For our purposes it is useful to introduce the notation

$$\mathcal{S}_A(h) := \sup_{|m-m'|=h} |A_m^{m'}|$$

for any linear operator $A : \mathcal{H}^\infty \to \mathcal{H}^{-\infty}$, so that (3.1.7) reads as $|A|_s := \left(\sum_{h\in\mathbb{N}_0} |\langle h\rangle^s \mathcal{S}_A(h)|^2\right)^{\frac{1}{2}}$.

Let $A, B : \mathcal{H}^\infty \to \mathcal{H}^{-\infty}$. We compute:

$$\mathcal{S}_{AB}(h) = \sup_{|j-j'|=h} |(AB)_j^{j'}| \leqslant \sup_{|j-j'|=h} \sum_{k\in\mathbb{N}} |A_k^{j'}||B_j^k| \leqslant \sup_{|j-j'|=h} \sum_{k\in\mathbb{N}} \mathcal{S}_A(|k-j'|)\mathcal{S}_B(|j-k|)\ .$$

It follows that

$$\begin{aligned}\langle h\rangle^s \mathcal{S}_{AB}(h) &\leqslant \sup_{|j-j'|=h} \langle j-j'\rangle^s \sum_{k\in\mathbb{N}} \mathcal{S}_A(|k-j'|)\mathcal{S}_B(|j-k|)\\ &= \sup_{|j-j'|=h} \sum_{k\in\mathbb{N}} \langle k-j'\rangle^s \mathcal{S}_A(|k-j'|)\, \langle j-k\rangle^s \mathcal{S}_B(|j-k|) \frac{\langle j-j'\rangle^s}{\langle k-j'\rangle^s \langle j-k\rangle^s}\ .\end{aligned}$$

By Cauchy-Schwartz, we get

$$|\langle h\rangle^s \mathcal{S}_{AB}(h)|^2 \leqslant \sup_{|j-j'|=h} \sum_{k\in\mathbb{N}} \langle k-j'\rangle^{2s} \mathcal{S}_A(|k-j'|)^2 \langle j-k\rangle^{2s} \mathcal{S}_B(|j-k|)^2 \cdot \sum_{k\in\mathbb{N}} \frac{\langle j-j'\rangle^{2s}}{\langle k-j'\rangle^{2s} \langle j-k\rangle^{2s}} \tag{A.3.1}$$

It is easy to see that, for any $a>1$, one has $\sum_{k\in\mathbb{N}} \frac{\langle j-j'\rangle^a}{\langle k-j'\rangle^a \langle j-j'\rangle^a} \leqslant C_a$, thus, we obtain in (A.3.1)

$$\begin{aligned}|\langle h\rangle^s \mathcal{S}_{AB}(h)|^2 &\leqslant C_{2s} \sup_{|j-j'|=h} \sum_{k\in\mathbb{N}} \langle k-j'\rangle^{2s} \mathcal{S}_A(|k-j'|)^2 \langle j-k\rangle^{2s} \mathcal{S}_B(|j-k|)^2\\ &=: C_{2s} \sup_{|j-j'|=h} \mathcal{R}_{AB}(h)\\ &= C_{2s} \max\Big\{ \sup_{\substack{j'\geqslant j\\ j'-j=h}} \mathcal{R}_{AB}(h)\ ,\ \sup_{\substack{j\geqslant j'\\ j-j'=h}} \mathcal{R}_{AB}(h)\Big\}\,.\end{aligned} \tag{A.3.2}$$

Without loss of generality, assume that the maximum is attained in the first region, that is when $j'\geqslant j$. We are now ready to compute:

$$\begin{aligned}|AB|_s^2 &= \sum_{h\geqslant 0} |\langle h\rangle^s \mathcal{S}_{AB}(h)|^2 \lesssim \sum_{h\geqslant 0} \sup_{\substack{j'\geqslant j\\ j'-j=h}} \mathcal{R}_{AB}(h)\\ &= \sum_{h\geqslant 0} \sup_{\substack{j'\geqslant j\\ j'-j=h}} \sum_{1\leqslant k\leqslant j} \langle j'-k\rangle^{2s} \mathcal{S}_A(j'-k)^2 \langle j-k\rangle^{2s} \mathcal{S}_B(j-k)^2\\ &+ \sum_{h\geqslant 0} \sup_{\substack{j'\geqslant j\\ j'-j=h}} \sum_{j+1\leqslant k\leqslant j'} \langle j'-k\rangle^{2s} \mathcal{S}_A(j'-k)^2 \langle k-j\rangle^{2s} \mathcal{S}_B(k-j)^2\\ &+ \sum_{h\geqslant 0} \sup_{\substack{j'\geqslant j\\ j'-j=h}} \sum_{k\geqslant j'+1} \langle k-j'\rangle^{2s} \mathcal{S}_A(k-j')^2 \langle k-j\rangle^{2s} \mathcal{S}_B(k-j)^2\ .\end{aligned} \tag{A.3.3}$$

For these three sums we can perform change of indexes, so that we get

$$\begin{aligned} |AB|_s^2 \lesssim & \sum_{h\geqslant 0}\sum_{h\leqslant a<\infty} \langle a\rangle^{2s}\,\mathcal{S}_A(a)^2\,\langle a-h\rangle^{2s}\,\mathcal{S}_B(a-h)^2 \\ & + \sum_{h\geqslant 0}\sum_{0\leqslant a\leqslant h-1} \langle a\rangle^{2s}\,\mathcal{S}_A(a)^2\,\langle h-a\rangle^{2s}\,\mathcal{S}_B(h-a)^2 \\ & + \sum_{h\geqslant 0}\sum_{a\geqslant 1} \langle a\rangle^{2s}\,\mathcal{S}_A(a)^2\,\langle a+h\rangle^{2s}\,\mathcal{S}_B(a+h)^2 \\ & =: \Sigma_1+\Sigma_2+\Sigma_3 \ . \end{aligned} \tag{A.3.4}$$

We estimate Σ_1 as follows:

$$\begin{aligned} \Sigma_1 &= \sum_{h\geqslant 0}\sum_{h\leqslant a<\infty} \langle a\rangle^{2s}\,\mathcal{S}_A(a)^2\,\langle a-h\rangle^{2s}\,\mathcal{S}_B(a-h)^2 \\ &= \sum_{h\geqslant 0}\sum_{h_1\geqslant 0} \langle h+h_1\rangle^{2s}\,\mathcal{S}_A(h+h_1)^2\,\langle h_1\rangle^{2s}\,\mathcal{S}_B(h_1)^2 \\ &= \sum_{h_1\geqslant 0}\Big(\sum_{h\geqslant 0} \langle h+h_1\rangle^{2s}\,\mathcal{S}_A(h+h_1)^2\Big)\langle h_1\rangle^{2s}\,\mathcal{S}_B(h_1)^2 \\ &\leqslant |A|_s^2\sum_{h_1\geqslant 0}\langle h_1\rangle^{2s}\,\mathcal{S}_B(h_1)^2 = |A|_s^2\,|B|_s^2 \ . \end{aligned} \tag{A.3.5}$$

The same estimate holds for Σ_2 and Σ_3, so that we can conclude

$$|AB|_s^2 \leqslant C_s\,|A|_s^2\,|B|_s^2 \ , \tag{A.3.6}$$

as claimed.

A.4 Proof of Lemma 3.23 (Commutator)

In this section we show the proof of Lemma 3.23. We start with operators independent of $\varphi\in\mathbb{T}^\nu$. Let

$$\mathbf{X} = \begin{pmatrix} X^d & X^o \\ -\overline{X^o} & -\overline{X^d}\end{pmatrix} \ , \qquad \mathbf{V} = \begin{pmatrix} V^d & V^o \\ -\overline{V^o} & -\overline{V^d}\end{pmatrix} \ .$$

One has

$$\mathrm{i}[\mathbf{X},\mathbf{V}] = \mathrm{i}\,(\mathbf{X}\mathbf{V}-\mathbf{V}\mathbf{X}) = \begin{pmatrix} \mathrm{i}Z^d & \mathrm{i}Z^o \\ -\overline{(\mathrm{i}Z^o)} & -\overline{(\mathrm{i}Z^d)}\end{pmatrix} \ ,$$

where

$$Z^d := X^dV^d - X^o\overline{V^o} - V^dX^d + V^o\overline{X^o}, \qquad Z^o := X^dV^o - X^o\overline{V^d} - V^dX^o + V^o\overline{X^d} \ .$$

Omitting for sake of simplicity conjugate operators and labels for diagonal and anti-diagonal elements, by Remark 3.19, the following inequalities hold (here $\sigma = \pm\alpha, 0$):

$$
\begin{aligned}
\left|\langle D\rangle^{\sigma} XV \langle D\rangle^{-\sigma}\right|_s &\leqslant C_s \left|\langle D\rangle^{\sigma} X \langle D\rangle^{-\sigma}\right|_s \left|\langle D\rangle^{\sigma} V \langle D\rangle^{-\sigma}\right|_s \ ; \\
\left|\langle D\rangle^{\alpha} XV\right|_s &\leqslant C_s \left|\langle D\rangle^{\alpha} X\right|_s |V|_s \ ; \\
\left|XV \langle D\rangle^{\alpha}\right|_s &\leqslant C_s \left|X \langle D\rangle^{\alpha}\right|_s \left|\langle D\rangle^{-\alpha} V \langle D\rangle^{\alpha}\right|_s \ ;
\end{aligned}
\tag{A.4.1}
$$

the same for those terms involving VX. All these norms extend easily to the analytic case. Therefore, by the assumption and from the definition in (3.1.16), properties 3.1.13, 3.1.14 and 3.1.15 are satisfied. It remains to show the symmetries conditions in (3.1.12). Note that $(\mathrm{i}Z^d)^* = \mathrm{i}Z^d$ and $(\mathrm{i}Z^o)^* = \overline{\mathrm{i}Z^o}$ if and only if $(Z^d)^* = -Z^d$, $(Z^o)^* = \overline{Z^o}$. We check the condition for Z^d. We have

$$
\begin{aligned}
(Z^d)^* &= (V^d)^*(X^d)^* - (\overline{V^o})^*(X^o)^* - (X^d)^*(V^d)^* + (\overline{X^o})^*(V^o)^* \\
&= V^d X^d - V^o \overline{X^o} - X^d V^d + X^o \overline{V^o}^* = -Z^d \ .
\end{aligned}
\tag{A.4.2}
$$

In the same way one checks that $(Z^o)^* = \overline{Z^o}$. The Lipschitz dependence is easily checked.

Appendix B

Derivation of water waves equations with constant vorticity

The dynamics of the water waves in the two dimensional fluid domain $\mathcal{D}_{\eta,\mathtt{h}}$ defined in (1.1.11) is described by the two components velocity field $u(t,x,y)$ and $v(t,x,y)$, which prescribe the dynamics of the fluid particles inside $\mathcal{D}_{\eta,\mathtt{h}}$, and by the profile of the free surface $\eta(t,x)$. The equation of motions are the mass conservation and Euler equations in two dimensions:

$$\begin{cases} \operatorname{div} \vec{u} = 0 \\ \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} = -\nabla P - g \mathbf{e}_y \end{cases} , \tag{B.0.1}$$

where $P(t,x,y)$ denotes the pressure and g the gravity. Denoting $\vec{u} := \binom{u}{v}$, they read in components as

$$\begin{cases} u_x + v_y = 0 \\ u_t + u u_x + v u_y = -P_x \\ v_t + u v_x + v v_y = -P_y - g \, . \end{cases} \quad \text{in } \mathcal{D}_{\eta,\mathtt{h}} \tag{B.0.2}$$

The boundary conditions that we impose are

$$\begin{cases} v = \eta_t + u \eta_x & \text{at } y = \eta(t,x) \\ v \to 0 & \text{for } y \to -\mathtt{h} \\ P = P_0 - \kappa \Big(\dfrac{\eta_x}{\sqrt{1+\eta_x^2}} \Big)_x & \text{at } y = \eta(t,x) \, . \end{cases} \tag{B.0.3}$$

The first equation in (B.0.3), called kinematic boundary condition, expresses the fact the fluid particles at the free surface remain on it along the evolution. The second condition in (B.0.3) is equivalent to the impermeability condition of the bottom of the ocean:

$$\begin{cases} v(t,x,-\mathtt{h}) = 0\,, & \text{if } \mathtt{h} \text{ is finite}\,, \\ \lim_{y\to-\infty} v(t,x,y) = 0 & \text{if } \mathtt{h} = +\infty\,; \end{cases}$$

The third equation in (B.0.3), called the dynamic boundary condition, describes the presence of capillarity forces at the interface between the water and the air above. The difference between the outer and the inner pressure at the interface is proportional to the mean curvature of the free surface, where $\kappa > 0$ denotes the surface tension coefficient.

By taking the rotor of the Euler equation (B.0.1), we obtain that the scalar vorticity

$$\operatorname{rot}\vec{u} := \omega := v_x - u_y$$

evolves according to the Helmholtz equation

$$\partial_t\omega + (u\partial_x + v\partial_y)\omega = 0\,. \tag{B.0.4}$$

In our model we assume that the scalar vorticity of the vector field $\vec{u}$ is constant:

$$\omega := v_x - u_y = \gamma\,. \tag{B.0.5}$$

Note that by (B.0.4), if the initial vorticity $\omega_{|t=0} = \gamma$ is constant, then $\omega = \gamma$ remains constant at any time t of existence of the solution.

B.1 Helmholtz decomposition of a vector field on $\mathcal{D}_{\eta,\mathtt{h}}$

Inside the fluid, sufficiently away from the small waves of the free surface, for instance when $\|\eta(t,\cdot)\|_{L^\infty(\mathbb{T}_x)} < 1$, the average in the horizontal direction of the vertical component of the velocity field is null.

Lemma B.1. *We have $\int_0^{2\pi} v(t,x,y)\,\mathrm{d}x = 0$ for all times t and $y < -1$.*

Proof. Note that, by the divergence free condition $u_x + v_y = 0$ in (B.0.2) and the 2π-periodicity of u, we have

$$\partial_y \int_0^{2\pi} v(t,x,y)\,\mathrm{d}x = \int_0^{2\pi} -u_x(t,x,y)\,\mathrm{d}x = 0\,.$$

Hence, for any $y \in (-\mathtt{h},-1)$, we obtain $\int_0^{2\pi} v(t,x,y)\,\mathrm{d}x = \lim_{y\to-\mathtt{h}} \int_0^{2\pi} v(t,x,y)\,\mathrm{d}x = 0$ by the impermeability condition in (B.0.3). ☐

We analyze now the structure of vector field with constant scalar vorticity and its decomposition as a sum of an irrotational vector field and divergence-free one. We denote the interior of the fluid domain by $\mathrm{Int}\mathcal{D}_{\eta,\mathtt{h}} := \mathcal{D}_{\eta,\mathtt{h}} \backslash \partial\mathcal{D}_{\eta,\mathtt{h}}$.

Lemma B.2. *Let* $\vec{a}(x,y) := \binom{a(x,y)}{b(x,y)}$ *be a vector field of class* $C^1(\mathrm{Int}\mathcal{D}_{\eta,\mathtt{h}}) \cap C^0(\mathcal{D}_{\eta,\mathtt{h}})$ *with constant scalar vorticity, namely*

$$\mathrm{rot}\,\vec{a} := b_x - a_y = \gamma\,. \tag{B.1.1}$$

Then , for any $y \in (-\mathtt{h}, -1)$*, the function*

$$\mathtt{c} := \frac{1}{2\pi}\int_0^{2\pi} a(x,y)\,\mathrm{d}x + \gamma y \tag{B.1.2}$$

is independent of y *and there exists a* C^2 *function* $\Phi(x,y)$ *defined for any* $(x,y) \in \mathrm{Int}\mathcal{D}_{\eta,\mathtt{h}}$ *such that*

$$\begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix} = \begin{pmatrix} \mathtt{c} \\ 0 \end{pmatrix} + \nabla\Phi(x,y) + \begin{pmatrix} -\gamma y \\ 0 \end{pmatrix}\,. \tag{B.1.3}$$

The function Φ *is uniquely defined up to a constant. Moreover,* Φ *admits an extension to the whole domain* $\mathcal{D}_{\eta,\mathtt{h}}$ *and*

$$\partial_y\Phi(x,-\mathtt{h}) = b(x,-\mathtt{h})\,, \quad \frac{d}{dx}\Phi(x,\eta(x)) = (a(x,\eta(x)) - \mathtt{c} + \gamma\eta(x)) + b(x,\eta(x))\eta_x(x)\,. \tag{B.1.4}$$

Proof. First, we assume the fluid to be irrotational, namely $\gamma = 0$. By the 2π-periodicity of $\vec{a}$, we have

$$\partial_y \int_0^{2\pi} a(x,y)\mathrm{d}x = \int_0^{2\pi} a_y(x,y)\mathrm{d}x = \int_0^{2\pi} b_x(x,y)\mathrm{d}x = b(2\pi,y) - b(0,y) = 0\,,$$

which implies that $\mathtt{c}$ is independent of $y \in (-1,-\mathtt{h})$. We now fix an arbitrary $\bar{\mathtt{h}} \in (1,\mathtt{h})$ and we define the function $\Phi(x,y)$ as the potential

$$\Phi(x,y) := \int_{(0,-\bar{\mathtt{h}})}^{(x,y)} (a(x,y) - \mathtt{c})\,\mathrm{d}x + b(x,y)\mathrm{d}y\,,$$

integrated along any path joining $(0,-\bar{\mathtt{h}})$ and $(x,y) \in \mathbb{T} \times (-\mathtt{h},-1)$, which is the same by the Gauss-Green theorem since in this region it holds that $(a(x,y) - \mathtt{c})_y = b_x$. Choosing a path of integration which is first horizontal from $(0,-\bar{\mathtt{h}})$ to $(x,\bar{\mathtt{h}})$ and then vertical from $(x,-\bar{\mathtt{h}})$ to (x,y) we get

$$\Phi(x,y) = \int_0^x \big(a(s,-\bar{\mathtt{h}}) - \mathtt{c}\big)\,\mathrm{d}s + \int_{-\bar{\mathtt{h}}}^y b(x,s)\,\mathrm{d}s\,, \tag{B.1.5}$$

which is 2π-periodic in x because, exploiting the periodicity of the vector field $(a(x,y), b(x,y))$

and the definition of $\mathtt{c}$ in (B.1.2), one has

$$\Phi(2\pi + x, y) - \Phi(x, y) = \int_0^{2\pi} \big(a(s, -\overline{\mathtt{h}}) - \mathtt{c}\big)\, \mathrm{d}s = 0\,.$$

Moreover, the function $\Phi(x, y)$ in (B.1.5) is allowed to be defined also for every $(x, y) \in \mathrm{Int}\mathcal{D}_{\eta,\mathtt{h}}$, without losing any differentiability and the periodicity in x. By (B.1.5), we have

$$\partial_x \Phi(x, y) = a(x, -\overline{\mathtt{h}}) - \mathtt{c} + \int_{-\overline{\mathtt{h}}}^{y} b_x(x, s)\mathrm{d}s = a(x, -\overline{\mathtt{h}}) - \mathtt{c} + \int_{-\overline{\mathtt{h}}}^{y} a_y(x, s)\mathrm{d}s = a(x, y) - \mathtt{c}$$

for any $(x, y) \in \mathrm{Int}\mathcal{D}_{\eta,\mathtt{h}}$ and similarly $\partial_y \Phi(x, y) = b(x, y)$, proving the representation (B.1.3) for $\gamma = 0$. The formula (B.1.5) defines also an extension to the closed domain $\mathcal{D}_{\eta,\mathtt{h}}$. Therefore, in the case $\gamma = 0$, the relations in (B.1.4) follow.

Let $(\tilde{\mathtt{c}}, \tilde{\Phi})$ be another solution of

$$\begin{pmatrix} a(x, y) \\ b(x, y) \end{pmatrix} = \begin{pmatrix} \tilde{\mathtt{c}} \\ 0 \end{pmatrix} + \nabla \tilde{\Phi}(x, y)\,.$$

Then, by integrating in $x \in \mathbb{T}$, we obtain that $\tilde{\mathtt{c}} = \mathtt{c}$ and $\nabla(\Phi - \tilde{\Phi}) = 0$.

Finally, for any value of the vorticity $\gamma \in \mathbb{R}$, we note that the vector field $\begin{pmatrix} a(x, y) + \gamma y \\ b(x, y) \end{pmatrix}$ is irrotational and we deduce straightforward the representation in (B.1.3) and the relations in (B.1.4). □

Lemma B.3. *Let* $\vec{a}(x, y) = \binom{a(x,y)}{b(x,y)}$ *be a divergence free vector field of class* $C^1(\mathrm{Int}\mathcal{D}_{\eta,\mathtt{h}}) \cap C^0(\mathcal{D}_{\eta,\mathtt{h}})$ *such that*

$$\lim_{y \to -\mathtt{h}} \int_0^{2\pi} b(x, y)\, \mathrm{d}x = 0\,. \tag{B.1.6}$$

Then there exists a potential $\Psi(x, y)$ *defined on* $\mathrm{Int}\mathcal{D}_{\eta,\mathtt{h}}$ *such that*

$$\begin{pmatrix} a(x, y) \\ b(x, y) \end{pmatrix} = \begin{pmatrix} -\partial_y \Psi(x, y) \\ \partial_x \Psi(x, y) \end{pmatrix}\,. \tag{B.1.7}$$

Moreover, $\Psi(x, y)$ *admits an extension to* $\mathcal{D}_{\eta,\mathtt{h}}$*, with*

$$\partial_y \Psi(x, -\mathtt{h}) = -a(x, -\mathtt{h})\,, \quad \frac{d}{dx}\Psi(x, \eta(x)) = b(x, \eta(x)) - a(x, \eta(x))\eta_x(x)\,. \tag{B.1.8}$$

Proof. Since $\mathrm{div}\vec{a} = 0$ the vector field

$$\begin{pmatrix} b(x, y) \\ -a(x, y) \end{pmatrix}$$

is irrotational. It follows from Lemma B.2 with $\gamma = 0$ that there exists $\mathtt{c} \in \mathbb{R}$ and a potential Ψ such that

$$\begin{pmatrix} b(x,y) \\ -a(x,y) \end{pmatrix} = \begin{pmatrix} \mathtt{c} \\ 0 \end{pmatrix} + \nabla\Psi(x,y)\,.$$

By taking the average in x and the limit as $y \to -\mathtt{h}$ of the relation $b(x,y) = \mathtt{c} + \partial_x \Phi(x,y)$, we get

$$c = \lim_{y\to -\mathtt{h}} \frac{1}{2\pi}\int_0^{2\pi} b(x,y)\,\mathrm{d}x \overset{(B.1.6)}{=} 0\,.$$

This proves (B.1.7). □

By Lemma B.2 above we conclude that there exists a function $\mathtt{c}(t)$ and a potential $\Phi(t,x,y)$, 2π-periodic in x, called the generalized velocity potential, such that the velocity field of (B.0.1)-(B.0.3) admits the decomposition

$$\begin{aligned} u(t,x,y) &= \mathtt{c} - \gamma y + \Phi_x(t,x,y) \\ v(t,x,y) &= \Phi_y(t,x,y) \end{aligned} \tag{B.1.9}$$

where

$$\mathtt{c}(t) := \frac{1}{2\pi}\int_0^{2\pi} u(t,x,y)\mathrm{d}x + \gamma y\,, \tag{B.1.10}$$

is independent of y. Actually $\mathtt{c}$ is constant also for any time of existence t. Indeed, by choosing an arbitrary $y \in (-\mathtt{h}, -1)$, we compute that

$$\begin{aligned} \partial_t \mathtt{c}(t) :=& \frac{1}{2\pi}\int_0^{2\pi} \partial_t u(t,x,y)\,\mathrm{d}x \overset{(B.0.2)}{=} -\frac{1}{2\pi}\int_0^{2\pi} \big(\tfrac{1}{2}\partial_x(u^2) + v\partial_y u + \partial_x P\big)\,\mathrm{d}x \\ \overset{(B.0.5)}{=}& -\frac{1}{2\pi}\int_0^{2\pi} v(\partial_x v - \gamma)\,\mathrm{d}x = \frac{\gamma}{2\pi}\int_0^{2\pi} v\,\mathrm{d}x = 0\,, \end{aligned}$$

by Lemma B.1.

B.2 The Zahkarov-Wahlén-Constantin formulation

The equations (B.0.1)-(B.0.3) can be regairded in a space-time frame moving horizontally with an arbitrary constant speed $c \in \mathbb{R}$. A direct computation shows that the new variables

$$\begin{aligned} \widetilde{u}(t,x,y) &:= u(t,x+ct,y) - c\,, \quad & \widetilde{\eta}(t,x) &:= \eta(t,x+ct)\,, \\ \widetilde{v}(t,x,y) &:= v(t,x+ct,y)\,, & \widetilde{P}(t,x,y) &:= P(t,x+ct,y)\,, \end{aligned} \tag{B.2.1}$$

satisfy the same equations (B.0.1)-(B.0.3). This means that we can always add an arbitrary constant c to the horizontal component of the velocity field.

In view of the change of variable (B.2.1) induced by the moving frame, i.e. substituting u, v, η, P with $\widetilde{u}, \widetilde{v}, \widetilde{\eta}, \widetilde{P}$ and setting $c = \mathtt{c}$, where $\mathtt{c}$ is defined in (B.1.10), we can always assume the following decomposition of the velocity field:

$$\begin{cases} u(t,x,y) = -\gamma y + \partial_x \Phi(t,x,y) \\ v(t,x,y) = \partial_y \Phi(t,x,y) \,. \end{cases} \tag{B.2.2}$$

We want to use such decomposition of the velocity field in order to obtain the formulation of the water waves problem with constant vorticity in (4.0.1).

By (B.2.2) and since $\vec{u}$ is divergence free, it follows that

$$\Delta\Phi(t,x,y) = 0 \,. \tag{B.2.3}$$

We also express the boundary conditions (B.0.3) in terms of Φ, obtaining

$$\begin{cases} \eta_t = \Phi_y - \Phi_x \eta_x + \gamma\eta\,\eta_x & \text{at } y = \eta(t,x) \\ \Phi_y \to 0 & \text{for } y \to -\mathtt{h} \,. \end{cases} \tag{B.2.4}$$

We define the trace of the generalized velocity potential at the free boundary

$$\psi(t,x) = \Phi(t;x,y)|_{y=\eta} = \Phi(t;x,\eta(t,x)) \,. \tag{B.2.5}$$

In such a way, given η, ψ, the generalized velocity potential Φ is recovered by solving the elliptic problem

$$\begin{cases} \Delta\Phi = 0 & \text{in } \mathcal{D}_\eta \\ \Phi = \psi & \text{at } y = \eta(t,x) \\ \Phi_y = 0 & \text{at } y = -\mathtt{h} \,. \end{cases} \tag{B.2.6}$$

Defining the Dirichlet-Neumann operator $G(\eta,\mathtt{h})\psi$ as

$$G(\eta,\mathtt{h})\psi := \sqrt{1+\eta_x^2}\,(\partial_{\vec{n}}\Phi)|_{y=\eta(t,x)} = (-\Phi_x\eta_x + \Phi_y)|_{y=\eta(t,x)} \,, \tag{B.2.7}$$

we deduce from (B.2.4) that

$$\eta_t = G(\eta,\mathtt{h})\psi + \gamma\eta\,\eta_x \tag{B.2.8}$$

which is the first equation in (4.0.1).

Remark B.4. We have that

$$G(\eta,\mathtt{h})[1] = 0 \,, \qquad \int_{\mathbb{T}} G(\eta,\mathtt{h})[\psi]\,\mathrm{d}x = 0 \,.$$

Associated to the generalized velocity potential $\Phi(t,x,y)$, we have the so called stream function, which is the potential obtained in Lemma B.3.

Lemma B.5. (Stream function) *There exists a function $\Psi(t,x,y)$ on $\mathcal{D}_{\eta,\mathtt{h}}$ such that*

$$u = \Psi_y\,, \quad v = -\Psi_x\,. \tag{B.2.9}$$

In particular, the function $\widetilde{\Psi} = \Psi + \frac{\gamma y^2}{2}$ solves

$$\Phi_x = \widetilde{\Psi}_y = u + \gamma y\,, \quad \Phi_y = -\widetilde{\Psi}_x = v\,. \tag{B.2.10}$$

Remark B.6. Note that the fluid particles evolve according to the time-dependent Hamiltonian system

$$\begin{cases} \dot{x} = u = \Psi_y = \partial_y(\widetilde{\Psi} - \frac{\gamma}{2}y^2) \\ \dot{y} = v = -\Psi_x = -\partial_x(\widetilde{\Psi} - \frac{\gamma}{2}y^2)\,. \end{cases}$$

To deduce the second equation of water waves, we start again with the Euler equation and use the vectorial identity

$$\vec{u}\cdot\nabla\vec{u} = \nabla\Big(\frac{|u|^2}{2}\Big) - \vec{u}\wedge \mathrm{rot}\vec{u}$$

to write the second equation of (B.0.1) as

$$\partial_t\vec{u} + \nabla\Big(\frac{|u|^2}{2}\Big) - \vec{u}\wedge\mathrm{rot}\vec{u} = -\nabla(P+gy)\,. \tag{B.2.11}$$

In particular, (B.2.11) is equivalent to

$$\partial_t\Big(\nabla\Phi + \begin{pmatrix} -\gamma y \\ 0 \end{pmatrix}\Big) + \nabla\Big(\frac{|\nabla\Psi|^2}{2}\Big) + \gamma\nabla\Psi + \nabla(P+gy) = 0\,.$$

where we have used that $\vec{u} = \big(\begin{smallmatrix} -\gamma y \\ 0 \end{smallmatrix}\big) + \nabla\Phi$, $|\vec{u}|^2 = |\nabla\Psi|^2$ and

$$\vec{u}\wedge\mathrm{rot}\vec{u} = \gamma\begin{pmatrix} v \\ -u \end{pmatrix} = -\gamma\nabla\Psi\,,$$

which follows by (B.0.5) and (B.2.9), Therefore, in the time dependent fluid domain we have that

$$\partial_t\Phi + \frac{|\nabla\Psi|^2}{2} + \gamma\Psi + P + gy = C(t) \tag{B.2.12}$$

for some $C(t)$, which determines the pressure in the fluid. The equation (B.2.12) is a generalization of the Bernoulli theorem for ideal fluid with constant scalar vorticity

Evaluating (B.2.12) at the free surface, and imposing the last dynamic condition in (B.0.3) we obtain that

$$\Phi_t + \frac{|\nabla\Psi|^2}{2} + \gamma\Psi - \kappa\Big(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\Big)_x + g\eta = c(t) \quad \text{at } y = \eta(t,x)\,, \tag{B.2.13}$$

where $c(t) = C(t) - P_0$.

We want to write now the equation in (B.2.13) in terms of η, ψ only. We use the following preliminary lemma. Given a 2π-periodic function $f(x)$ with zero average we define $g := \partial_x^{-1} f$ the unique 2π-periodic function with zero average such that $\partial_x g = f$.

Lemma B.7. *There is $c_0(t)$ such that*

$$\Psi(t,x,\eta(t,x)) = -\tfrac{\gamma}{2}\eta^2 - \partial_x^{-1} G(\eta,\mathtt{h})\psi + c_0(t)\,. \tag{B.2.14}$$

Proof. By Lemma B.5, we have that

$$\begin{aligned}\frac{d}{dx}\big(\Psi(t,x,\eta(t,x)) + \tfrac{\gamma}{2}\eta^2\big) &= \Psi_x(t,x,\eta(t,x)) + \Psi_y(t,x,\eta(t,x))\eta_x + \gamma\eta\eta_x \\ &= -\Phi_y(t,x,\eta(t,x)) + (\Phi_x(t,x,\eta(t,x)) - \gamma\eta(t,x))\,\eta_x + \gamma\eta\eta_x \\ &\overset{(B.2.7)}{=} -G(\eta,\mathtt{h})\psi\,.\end{aligned}$$

By integrating on $[0,x]$, we obtain (B.2.14). □

Remark B.8. The previous computation gives another proof that $\int_{\mathbb{T}} G(\eta,\mathtt{h})\psi\,dx = 0$.

Inverting with respect to Φ_x and Φ_y the following system, see (B.2.5) and (B.2.7),

$$\psi_x = \Phi_x + \Phi_y\eta_x\,, \quad G(\eta,\mathtt{h})\psi = \Phi_y - \Phi_x\eta_x\,, \quad \text{at } y = \eta(t,x)\,, \tag{B.2.15}$$

we get

$$\begin{cases} \Phi_x(x,\eta(x)) = \dfrac{\psi_x - \eta_x G(\eta,\mathtt{h})\psi}{1+\eta_x^2} \\ \Phi_y(x,\eta(x)) = \dfrac{\psi_x\eta_x + G(\eta,\mathtt{h})\psi}{1+\eta_x^2}\,. \end{cases} \tag{B.2.16}$$

By Lemma B.5 we have that, at $y = \eta$,

$$\frac{|\nabla\Psi|^2}{2} = \frac{(\Phi_x - \gamma\eta)^2 + \Phi_y^2}{2} = \gamma^2\frac{\eta^2}{2} + \frac{\Phi_x^2 - 2\gamma\Phi_x\eta + \Phi_y^2}{2}\,. \tag{B.2.17}$$

Moreover, by differentiating (B.2.5) with respect to t, we have, at $y = \eta(t,x)$,

$$\begin{aligned}\psi_t &= \Phi_t + \Phi_y\eta_t \\ &\overset{(B.2.13),(B.2.8)}{=} -\frac{|\nabla\Psi|^2}{2} - \gamma\Psi + \kappa\Big(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\Big)_x - g\eta + c(t) + \Phi_y(G(\eta,\mathtt{h})\psi + \gamma\eta\,\eta_x)\,.\end{aligned} \tag{B.2.18}$$

Now, by inserting (B.2.17), (B.2.18) and (B.2.16) into (B.2.13), we obtain, for some function $\tilde{c}(t)$,

$$\psi_t = -g\eta - \frac{\psi_x^2}{2} + \frac{(\eta_x\psi_x + G(\eta, \mathtt{h})\psi)^2}{2(1+\eta_x^2)} + \kappa\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta, \mathtt{h})\psi + \tilde{c}(t)\,,$$

which is the second equation in (4.0.1).

Appendix C

Technical results from Chapter 4

C.1 Integral operators

The results that we present in this section come from Section 2.3 of [44].

Integral operators with $\mathcal{C}^\infty$ kernels are operators in $\mathrm{OP}S^{-\infty}$, see Definition 4.14. The respective norms are given in Definition 4.15.

Lemma C.1 (Lemma 2.32, [44]). *Let $K = K(\lambda; \cdot) \in \mathcal{C}^\infty(\mathbb{T}^\nu \times \mathbb{T} \times \mathbb{T})$. Then the integral operator*

$$(\mathcal{R}u)(\varphi, x) := \int_{\mathbb{T}} K(\lambda; \varphi, x, y) u(\varphi, y)\, \mathrm{d}y \tag{C.1.1}$$

is in $\mathrm{OP}S^{-\infty}$ and, for all $m, s, \alpha \in \mathbb{N}_0$, it holds that

$$\|\mathcal{R}\|_{-m,s,\alpha}^{k_0,\upsilon} \leqslant C(m, s, \alpha, k_0) \|K\|_{\mathcal{C}^{s+m+\alpha}}^{k_0,\upsilon} . \tag{C.1.2}$$

Proof. The symbol associated to the integral operator (C.1.1) acting on periodic functions is given, for any $j \in \mathbb{Z}$,

$$a(\lambda; \varphi, x, j) := \int_{\mathbb{T}} K(\lambda; \varphi, x, y) e^{\mathrm{i}(y-x)j}\, \mathrm{d}y . \tag{C.1.3}$$

In particular, we consider its extension on $\mathbb{R}$

$$\widetilde{a}(\lambda; \varphi, x, \xi) := \int_{\mathbb{R}} K(\lambda; \varphi, x, y) \theta(y) e^{\mathrm{i}\, y\xi}\, \mathrm{d}y , \tag{C.1.4}$$

where the function $\theta \in \mathcal{D}(\mathbb{R})$ satisfies

$$\mathrm{spt}(\theta) \subset [-\tfrac{4}{3}\pi, \tfrac{4}{3}\pi] , \quad \theta(x) + \theta(x - 2\pi) = 1 \quad \forall\, x \in [0, 2\pi] , \quad \sum_{j \in \mathbb{Z}} \theta(x + 2j\pi) = 1 \quad \forall\, x \in \mathbb{R} .$$

In particular, the Fourier transform $\widehat{\theta}(\xi) \in \mathcal{S}(\mathbb{R})$ satisfies $\widehat{\theta}(0) = 1$ and $\widehat{\theta}(j) = 0$ for any $j \in \mathbb{Z} \backslash \{0\}$.

Indeed,

$$\begin{aligned}\widehat{\theta}(j) :=& \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mathrm{i}\,jx}\theta(x)\,\mathrm{d}x = \frac{1}{2\pi}\Big(\int_{-2\pi}^{0} e^{-\mathrm{i}\,jx}\theta(x)\,\mathrm{d}x + \int_0^{2\pi} e^{-\mathrm{i}\,jx}\theta(x)\,\mathrm{d}x\Big)\\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}\,jx}(\theta(x)+\theta(x-2\pi))\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}\,jx}\,\mathrm{d}x = \delta_{0,j}\,.\end{aligned}$$

The function in (C.1.4) is $\mathcal{C}^\infty$ in (φ, x, ξ) and k_0-times differentiable with respect to λ. Therefore, by the Poisson summation formula, we have that

$$\widetilde{a}(\lambda;\varphi,x,\xi) = \sum_{j\in\mathbb{Z}} a(\lambda;\varphi,x,j)\widehat{\theta}(\xi-j)\,, \tag{C.1.5}$$

so that $\widetilde{a}(\cdot;\cdot,\cdot,j) = a(\cdot;\cdot,\cdot,j)$ for any $j\in\mathbb{Z}$. We show now that $\mathrm{Op}(\widetilde{a}) \in \mathrm{OP}S^{-\infty}$ with the required estimate.

Let $\xi \in (j-\frac{1}{3}, j+\frac{1}{3})$ for some $j\in\mathbb{Z}$. By (C.1.5), (C.1.3) and (4.2.10), for any $m,q,\beta\in\mathbb{N}_0$, $n\in\mathbb{N}_0^\nu$ and $k\in\mathbb{N}_0^{\nu+1}$, with $|k|\leqslant k_0$, we have

$$\begin{aligned}(\mathrm{i}\,\xi)^{m+\beta}\,\partial_\lambda^k\partial_\varphi^n\partial_x^q\partial_\xi^\beta\widetilde{a}(\lambda;\varphi,x,\xi) &= \sum_{j'\in\mathbb{Z}}\partial_\lambda^k\partial_\varphi^n\partial_x^q a(\lambda;\varphi,x,j')(\mathrm{i}\,\xi)^{m+\beta}\partial_\xi\widehat{\theta}(\xi-j')\\ &\leqslant C_\beta\,\partial_\lambda^k\partial_\varphi^n\partial_x^q a(\lambda;\varphi,x,j)(\mathrm{i}j)^{m+\beta}\\ &= C_\beta\sum_{q_1+q_2=q} C_{q_1,q_2}\int_T(\partial_\lambda^k\partial_\varphi^n\partial_x^{q_1}K)(\lambda;\varphi,x,y)\partial_y^{q_2+m+\beta}(e^{\mathrm{i}(y-x)j})\,\mathrm{d}y\\ &= \sum_{q_1+q_2=q} C_{q_1,q_2,m,\beta}\int_T(\partial_\lambda^k\partial_\varphi^n\partial_x^{q_1}\partial_y^{q_2+m+\beta}K)(\lambda;\varphi,x,y)\,e^{\mathrm{i}(y-x)j}\,\mathrm{d}y\,.\end{aligned}$$

It follows that, for any $m,q,\beta\in\mathbb{N}_0$, $n\in\mathbb{N}_0^\nu$ and $k\in\mathbb{N}_0^{\nu+1}$, with $|k|\leqslant k_0$, there exists a constant $C(m,q,\beta)>0$ such that

$$|\partial_\lambda^k\partial_\varphi^n\partial_x^q\partial_\xi^\beta\widetilde{a}(\lambda;\varphi,x,\xi)| \leqslant C(m,q,\beta)\upsilon^{-|k|}\|K\|^{k_0,\upsilon}_{\mathcal{C}^{m+q+\beta+|n|}}\,\langle\xi\rangle^{-m-\beta}\,,$$

from which we obtain, for any $m,s,\beta\in\mathbb{N}_0$ and $k\in\mathbb{N}_0^{\nu+1}$, $|k|\leqslant k_0$, using that $\|u\|_s \simeq \|u\|_{H^s_\varphi L^2_x} + \|u\|_{L^2_\varphi H^s_x}$,

$$\begin{aligned}\|\partial_\xi^\beta\partial_\lambda^k\widetilde{a}(\lambda;\cdot,\cdot,\xi)\|_s\,\langle\xi\rangle^{-m-\beta} &\simeq \Big(\|\partial_\xi^\beta\partial_\lambda^k\widetilde{a}(\lambda:\cdot,\cdot,\xi)\|_{L^2_\varphi L^2_x} + \|\partial_x^s\partial_\xi^\beta\partial_\lambda^k\widetilde{a}(\lambda:\cdot,\cdot,\xi)\|_{L^2_\varphi L^2_x}\\ &\quad+\sup_{|n|=s}\|\partial_\varphi^n\partial_\xi^\beta\partial_\lambda^k\widetilde{a}(\lambda:\cdot,\cdot,\xi)\|_{L^2_\varphi L^2_x}\Big)\langle\xi\rangle^{-m-\beta}\\ &\lesssim_{s,m,\beta}\upsilon^{-|k|}\|K\|^{k_0,\upsilon}_{\mathcal{C}^{s+m+\beta}}\,.\end{aligned}$$

Then, the estimate (C.1.2) follows by Definition 4.15. □

An integral operator transforms into another integral operator under a change of variables

$$Pu(\varphi, x) := u(\varphi, x + p(\varphi, x)) . \tag{C.1.6}$$

Lemma C.2 (Lemma 2.34, [44]). *Let $K = K(\lambda; \cdot) \in \mathcal{C}^\infty(\mathbb{T}^\nu \times \mathbb{T} \times \mathbb{T})$ and $p(\lambda; \cdot) \in \mathcal{C}^\infty(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R})$. There exists $\delta(s_0, k_0) > 0$ such that, if $\|p\|_{2s_0+k_0+1}^{k_0,\upsilon} \leqslant \delta(s_0, k_0)$, then the integral operator $\mathcal{R}$ in* (C.1.1) *transforms into the integral operator $(P^{-1}\mathcal{R}P)u(\varphi, x) = \int_{\mathbb{T}} \breve{K}(\lambda; \varphi, x, y)u(\varphi, y)\,\mathrm{d}y$ with $\mathcal{C}^\infty$ kernel*

$$\breve{K}(\lambda; \varphi, x, z) := \big(1 + \partial_z q(\lambda; \varphi, z)\big) K(\lambda; \varphi, x + q(\lambda; \varphi, x), z + q(\lambda, \varphi, z)) , \tag{C.1.7}$$

where $z \mapsto z + q(\lambda; \varphi, z)$ is the inverse diffeomorphism of $x \mapsto x + p(\lambda; \varphi, x)$. The function $\breve{K}$ satisfies, for all $s \geqslant s_0$,

$$\|\breve{K}\|_s^{k_0,\upsilon} \leqslant C(s, k_0)\big(\|K\|_{s+k_0}^{k_0,\upsilon} + \|p\|_{s+k_0+1}^{k_0,\upsilon}\|K\|_{s_0+k_0+1}^{k_0,\upsilon}\big) . \tag{C.1.8}$$

Proof. We have that

$$(\mathcal{R}P)(\varphi, x) = \int_{\mathbb{T}} K(\lambda; \varphi, x, y)u(\varphi, y + p(\lambda; \varphi, y))\,\mathrm{d}y .$$

By making the change of coordinates $z = y + p(\lambda; \varphi, y)$, we get the operator $(P^{-1}\mathcal{R}P)u(\varphi, x) = \int_{\mathbb{T}} \breve{K}(\lambda; \varphi, x, y)u(\varphi, y)\,\mathrm{d}y$ with $\breve{K}$ as in (C.1.7).

The function $q(\lambda; \varphi, z)$ satisfies $q(\lambda; \varphi, z) + p(\lambda; \varphi, z + q(\lambda; \varphi, z)) = 0$. By a standard implicit function argument, we have that $q(\lambda; \cdot) \in \mathcal{C}^\infty(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R})$, k_0-times differentiable with respect to λ, and satisfies the estimates in Lemma 4.19. Therefore, we get $\breve{K}(\lambda; \cdot) \in \mathcal{C}^\infty(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R})$, k_0-times differentiable with respect to λ, and it satisfies the estimate (C.1.8) by Lemma 4.19 and estimate (4.2.7). □

We provide this estimate for the integral kernel of a family of Fourier multipliers in $\mathrm{OP}S^{-\infty}$.

Lemma C.3 (Lemma 2.28, [44]). *Let $g(\lambda; \varphi, \xi)$ be a family of Fourier multipliers with $\partial_\lambda^k g(\lambda; g, \cdot) \in S^{-\infty}$ for all $k \in \mathbb{N}_0^{\nu+1}$, $|k| \leqslant k_0$. Then the operator $\mathrm{Op}(g)$ is an integral operator with a $\mathcal{C}^\infty$ kernel $K_g = K_g(\lambda; \cdot)$ satisfying $\|K_g\|_{\mathcal{C}^s}^{k_0,\upsilon} \lesssim \|\mathrm{Op}(g)\|_{-1,s+s_0,0}^{k_0,\upsilon} + \|\mathrm{Op}(g)\|_{-s-s_0-1,0,0}^{k_0,\upsilon}$, for all $s \in \mathbb{N}_0$.*

Proof. The operator $\mathrm{Op}(g)$ acting on periodic functions admits the integral representation

$$\begin{aligned} [\mathrm{Op}(g)u](\varphi, x) &= \int_{\mathbb{T}} K_g(\lambda; \varphi, x, y)U(\varphi, y)\,\mathrm{d}y , \\ K_g(\lambda; \varphi, x, y) &:= \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} g(\lambda; \varphi, j)e^{\mathrm{i}(x-y)j} . \end{aligned} \tag{C.1.9}$$

Then the estimate on K_g follows directly from its definition in (C.1.9). □

On 2π-periodic functions the Hilbert transform $\mathcal{H}$, defined as a Fourier multiplier in (4.2.19), acts as

$$\begin{aligned}\mathcal{H}u(x) &= \frac{1}{2\pi}\text{p.v.}\int_{\mathbb{T}} \frac{u(y)}{\tan(\frac{1}{2}(x-y))}\,\mathrm{d}y \\ &:= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\Big\{\int_{x-\pi}^{x-\varepsilon} + \int_{x+\varepsilon}^{x+\pi}\Big\}\int_{\mathbb{T}} \frac{u(y)}{\tan(\frac{1}{2}(x-y))}\,\mathrm{d}y\,.\end{aligned} \tag{C.1.10}$$

The commutator between the Hilbert transform and the multiplication operator by a smooth function a is a regularizing operator in $\mathrm{OP}S^{-\infty}$.

Lemma C.4 (Lemma 2.35, [44]). *Let* $a(\lambda;\,\cdot\,) \in \mathcal{C}^\infty(\mathbb{T}^\nu \times \mathbb{T}_x, \mathbb{R})$. *Then the commutator* $[a,\mathcal{H}]$ *is in* $\mathrm{OP}S^{-\infty}$ *and satisfies, for all* $m, s, \alpha \in \mathbb{N}_0$,

$$\|[a,\mathcal{H}]\|_{-m,s,\alpha}^{k_0,\upsilon} \leqslant C(m,s,\alpha,k_0)\|a\|_{s+s_0+1+m+\alpha}^{k_0,\upsilon}\,. \tag{C.1.11}$$

Proof. By (C.1.10), the commutator

$$(\mathcal{H}a - a\mathcal{H})u(x) = \frac{1}{2\pi}\text{p.v.}\int_T \frac{(a(y)-a(x))u(y)}{\tan(\frac{1}{2}(x-y))}\,\mathrm{d}y = \int_T K(x,y)u(y)\,\mathrm{d}y$$

is an integral operator with $\mathcal{C}^\infty$ kernel given by

$$K(\lambda;\varphi,x,y) := \frac{a(y)-a(x)}{\tan(\frac{1}{2}(x-y))} = \Big(\int_0^1 a_x(\lambda;\varphi,x+t(y-x))\,\mathrm{d}t\Big)\frac{y-x}{\tan(\frac{1}{2}(x-y))}\,.$$

Then we obtain the estimate (C.1.11) by Lemma C.1 and the bound $\|K\|_{\mathcal{C}^s}^{k_0,\upsilon} \lesssim_s \|K\|_{s+s_0}^{k_0,\upsilon} \lesssim_s \|a\|_{s+s_0+1}^{k_0,\upsilon}$ for any $s \geqslant 0$. □

Lemma C.5 (Lemma 2.36, [44]). *Let* $p = p(\lambda;\,\cdot\,) \in \mathcal{C}^\infty(\mathbb{T}^{\nu+1})$ *and* $P = P(\lambda;\,\cdot\,)$ *be the associated change of variable in* (C.1.6). *There exists* $\delta(s_0,k_0) > 0$ *such that, if* $\|p\|_{2s_0+k_0+1}^{k_0,\upsilon} \leqslant \delta(s_0,k_0)$, *then the operator* $P^{-1}\mathcal{H}P - \mathcal{H}$ *is an integral operator with a* $\mathcal{C}^\infty$ *kernel* $K = K(\lambda;\,\cdot\,)$ *satisfying for all* $s \geqslant s_0$,

$$\|K\|_s^{k_0,\upsilon} \leqslant C(s,k_0)\|p\|_{s+k_0+1}^{k_0,\upsilon}\,. \tag{C.1.12}$$

Proof. Changing the variable $z = y + p(\lambda;\varphi,y)$ in (C.1.10), we get

$$(P^{-1}\mathcal{H}P)u(\varphi,x) = \frac{1}{2\pi}\text{p.v.}\int_{\mathbb{T}} \frac{u(\varphi,z)(1+\partial_z q(\lambda;\varphi,z))}{\tan\big(\frac{1}{2}(x-z+q(\lambda;\varphi,x)-q(\lambda;\varphi,z))\big)}\,\mathrm{d}z\,.$$

It follow that the operator $P^{-1}\mathcal{H}P - \mathcal{H}$ is an integral operator with $\mathcal{C}^\infty$ kernel

$$
\begin{aligned}
K(\lambda;\varphi,x,z) &= \frac{1}{2\pi}\Big(\frac{1+\partial_z q(\lambda;\varphi,z)}{\tan\big(\frac12(x-z+q(\lambda;\varphi,x)-q(\lambda;\varphi,z))\big)} - \frac{1}{\tan(\frac12(x-z))}\Big)\\
&= -\frac{1}{\pi}\partial_z \ln\Big(\frac{\sin\big(\frac12(x-z+q(\lambda;\varphi,x)-q(\lambda;\varphi,z))\big)}{\sin(\frac12(x-z))}\Big)\\
&= -\frac{1}{\pi}\partial_z \ln(1+g(\lambda;\varphi,x,z))\,,
\end{aligned}
$$

where the family of $\mathcal{C}^\infty$ functions

$$
\begin{aligned}
g(\lambda;\varphi,x,z) = &\cos\Big(\frac{q(\lambda;\varphi,x)-q(\lambda;\varphi,z)}{2}\Big) - 1\\
&+\cos\Big(\frac{x-z}{2}\Big)\frac{\sin\big(\frac12(q(\lambda;\varphi,x)-q(\lambda;\varphi,z))\big)}{\sin(\frac12(x-z))}
\end{aligned}
$$

satisfies the estimate $\|g\|_s^{k_0,\upsilon} \lesssim_s \|q\|_{s+1}^{k_0,\upsilon} \lesssim_s \|p\|_{s+k_0+1}^{k_0,\upsilon}$ by Lemma 4.19. Then the estimate (C.1.12) follows by Lemma 4.13. □

C.2 Estimates for the approximate inverse

The proofs of this section are a readaptation of the results in Section 5 of [44].

Proof of the estimates in Lemma 4.60 We prove tame estimates for the composition operator induced by the Hamiltonian vector field $X_P = (\partial_I P, -\partial_\theta P, \Pi^{\angle}_{\mathbb{S}^+,\Sigma})J\nabla_w P$ in (4.4.6). By definition (4.4.4), $P = P_\varepsilon \circ A$, where A is defined in (4.1.50) and P_ε in (4.4.2). Hence

$$
X_P = \begin{pmatrix} [\partial_I v^\intercal(\theta,I)]^T \nabla P_\varepsilon(A(\theta,I,w)) \\ -[\partial_\theta v^\intercal(\theta,I)]^T \nabla P_\varepsilon(A(\theta,I,w)) \\ \Pi^{\angle}_{\mathbb{S}^+,\Sigma} J\nabla P_\varepsilon(A(,\theta,I,w)) \end{pmatrix}, \tag{C.2.1}
$$

where $\Pi^{\angle}_{\mathbb{S}^+,\Sigma}$ is the symplectic projection on the normal subspace $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ defined in (4.1.42). Now, $\nabla P_\varepsilon = -JX_{P_\varepsilon}$, where

$$
X_{P_\varepsilon} = \begin{pmatrix} \varepsilon^{-1}(G(\varepsilon\eta)-G(0))(\zeta+\frac{\gamma}{2}\partial_x^{-1}\eta) + \eta\,\eta_x \\ -\frac12((\zeta+\frac{\gamma}{2}\partial_x^{-1}\eta)_x)^2 + \frac{(\varepsilon\eta_x(\zeta+\frac{\gamma}{2}\partial_x^{-1}\eta)_x + G(\varepsilon\eta)(\zeta+\frac{\gamma}{2}\partial_x^{-1}\eta))^2}{2(1+\varepsilon^2\eta_x^2)} + \varepsilon^{-1}\kappa\eta_{xx}((1+\varepsilon^2\eta_x^2)^{-\frac32}-1) \\ +\gamma\eta(\zeta+\frac{\gamma}{2}\partial_x^{-1}\eta)_x + \varepsilon^{-1}\frac{\gamma}{2}\partial_x^{-1}(G(\varepsilon\eta)-G(0))(\zeta+\frac{\gamma}{2}\partial_x^{-1}\eta) + \frac{\gamma}{2}\partial_x^{-1}(\eta\,\eta_x) \end{pmatrix}.
$$

The smallness condition of Lemma 4.22 is fulfilled because

$$
\|\eta\|_{3s_0+2k_0+5}^{k_0,\upsilon} \leqslant \varepsilon\|A(\theta(\cdot),I(\cdot),w(\cdot))\|_{3s_0+2k_0+5}^{k_0,\upsilon} \lesssim_{s_0} \varepsilon(1+\|\mathfrak{I}\|_{3s_0+2k_0+5}^{k_0,\upsilon}) \leqslant \delta(s_0,k_0)
$$

for $\varepsilon > 0$ small enough. Moreover, by Lemma 4.13 and the analyticity of the functions in (4.1.50), we have, for any $\alpha, \beta \in \mathbb{N}_0^\nu$, with $|\alpha| + |\beta| \leqslant 3$, and any $s \geqslant s_0$,

$$\|\partial_\theta^\alpha \partial_I^\beta v^\intercal(\theta(\cdot), I(\cdot))\|_s^{k_0,\upsilon} \lesssim_s 1 + \|\mathfrak{I}\|_s^{k_0,\upsilon} . \tag{C.2.2}$$

Thus, by Lemma 4.22, the interpolation inequality (4.2.7) and (C.2.2), we get

$$\|\nabla P_\varepsilon(A(\theta(\cdot), I(\cdot), w(\cdot)))\|_s^{k_0,\upsilon} \lesssim_s \|A(\theta(\cdot), I(\cdot), w(\cdot))\|_{s+2s_0+2k_0+3}^{k_0,\upsilon} \lesssim_s 1 + \|\mathfrak{I}\|_{s+2s_0+2k_0+3}^{k_0,\upsilon} . \tag{C.2.3}$$

Hence, by (C.2.1), (4.2.7), (C.2.2) and (C.2.3), we conclude that, for any $s \geqslant s_0$,

$$\|X_P(i)\|_s^{k_0,\upsilon} \lesssim_s 1 + \|\mathfrak{I}\|_{s+2s_0+2k_0+3}^{k_0,\upsilon} .$$

The other estimates in Lemma 4.60 for $\mathrm{d}_i X_p$ and $\mathrm{d}_i^2 X_P$ follow by differentiating the expression of X_P in (C.2.1), applying Lemma 4.22 and estimates (4.2.7), (C.2.2), (C.2.3).

Proof of the estimates in Lemma 4.61 We first prove the estimate (4.5.4) for the coefficients $A_{kj}(\varphi)$ defined in (4.5.3). By Lemma 5 in [34], the coefficients satisfy the identity

$$\omega \cdot \partial_\varphi A_{kj} = \mathcal{W}(\partial_\varphi Z(\varphi)\underline{e}_k, \partial_\varphi i_0(\varphi)\underline{e}_j) + \mathcal{W}(\partial_\varphi i_0(\varphi)\underline{e}_k, \partial_\varphi Z(\varphi)\underline{e}_j) ,$$

where $\mathcal{W}$ is the symplectic form in (4.1.15), (4.1.54), $\underline{e}_k$ denotes the k-th versor of $\mathbb{R}^\nu$ and $Z(\varphi)$ is the error function defined in (4.5.5). Then, by (4.2.7) and (4.5.1), we get

$$\|\omega \cdot \partial_\varphi A_{kj}\|_s^{k_0,\upsilon} \lesssim_s \|Z\|_{s+1}^{k_0,\upsilon} + \|Z\|_{s_0+1}^{k_0,\upsilon}\|\mathfrak{I}_0\|_{s+1}^{k_0,\upsilon}$$

and (4.5.4) follows by applying $(\omega \cdot \partial_\varphi)_{\rm ext}^{-1}$ defined in (4.2.9) and the estimate (4.2.11).

Now, the estimate (4.5.7) follows by (4.5.6), (4.5.2), (4.5.3), (4.2.7), and (4.5.1). The estimate (4.5.8) follows by (4.5.6) and (4.5.4). The estimate (4.5.10) follows by (4.5.6), (4.5.2), (4.5.3) and (4.5.1). It remains to show the bound in (4.5.9). We have

$$\begin{aligned} \mathcal{F}(i_\delta, \alpha_0) &= \mathcal{F}(i_0, \alpha_0) + \begin{pmatrix} 0 \\ \omega \cdot \partial_\varphi (I_\delta - I_0) \\ 0 \end{pmatrix} + \varepsilon (X_P(i_\delta) - X_P(i_0)) \\ &= \mathcal{F}(i_0, \alpha_0) + \begin{pmatrix} 0 \\ \omega \cdot \partial_\varphi (I_\delta - I_0) \\ 0 \end{pmatrix} + \varepsilon \int_0^1 \partial_I X_P(\lambda i_\delta + (1-\lambda) i_0) \cdot (I_\delta - I_0) \, \mathrm{d}\lambda . \end{aligned} \tag{C.2.4}$$

By differentiating (4.5.6) and arguing as in [34], we get

$$\begin{aligned}\omega\cdot\partial_\varphi(I_\delta-I_0)&=[\partial_\varphi\theta_0(\varphi)]^{-\top}\omega\cdot\partial_\varphi\rho(\varphi)\\&\qquad-\big([\partial_\varphi\theta_0(\varphi)]^{-\top}(\omega\cdot\partial_\varphi[\partial_\varphi\theta_0(\varphi)]^{\top})[\partial_\varphi\theta_0(\varphi)]^{-\top}\big)\rho(\varphi)\,,\\\omega\cdot\partial_\varphi[\partial_\varphi\theta_0(\varphi)]&=\varepsilon\partial_\varphi(\partial_I P)(i_0(\varphi))+\partial_\varphi Z_1(\varphi)\,.\end{aligned}\tag{C.2.5}$$

Then the estimate (4.5.9) follows by (C.2.4), (C.2.5), (4.4.6), Lemma (4.60) and estimates (4.2.7), (4.5.8), (4.5.1), (4.5.4).

Proof of the estimates in Lemma 4.63 By (4.5.19) and using that $i_\delta(\phi)=G_\delta(\phi,0,0)$, we have that

$$X_{K_{\alpha_0}}(\phi,0,0)=(DG_\delta(\phi,0,0))^{-1}X_{H_{\alpha_0}}(\phi,0,0)=(DG_\delta(\phi,0,0))^{-1}(\omega\cdot\partial_\phi i_\delta(\phi)-Z_\delta(\phi))\,,\tag{C.2.6}$$

where $Z_\delta=(Z_{1,\delta},Z_{2,\delta},Z_{3,\delta}):=\mathcal{F}(i_\delta,\alpha_0)$. Moreover, from (4.5.24) we get

$$X_{K_{\alpha_0}}=\begin{pmatrix}K_{10}(\phi,\alpha_0)\\-\partial_\varphi K_{00}(\phi,\alpha_0)\\J_\angle K_{01}(\phi,\alpha_0)\end{pmatrix}\tag{C.2.7}$$

with $J_\angle$ defined in (4.1.47). By combining (C.2.6), (C.2.7) together with the inverse of the linear operator in (4.5.29) and

$$(DG_\delta(\phi,0,0))^{-1}Di_\delta(\phi)[\omega]=(\omega,0,0)\,,$$

we deduce that

$$\begin{aligned}\partial_\phi K_{00}(\phi,\alpha_0)&=[\partial_\phi\theta_0(\phi)]^{\top}\big(-[\partial_\phi I_\delta(\phi)][\partial_\phi\theta_0(\phi)]^{-1}Z_{1,\delta}+Z_{2,\delta}\\&\qquad+[\partial_\theta\widetilde{w}_0(\theta_0(\phi))]^{\top}J_\angle^{-1}\partial_\phi w_0(\phi)[\partial_\phi\theta_0(\phi)]^{-1}Z_{1,\delta}+[\partial_\theta\widetilde{w}_0(\theta_0(\phi))]^{\top}J_\angle^{-1}Z_{3,\delta}\big)\,,\\K_{10}(\phi,\alpha_0)&=\omega-[\partial_\phi\theta_0(\phi)]^{-1}Z_{1,\delta}\,,\\K_{01}(\phi,\alpha_0)&=J_\angle^{-1}\big(\partial_\phi w_0(\phi)[\partial_\phi\theta_0(\phi)]^{-1}Z_{1,\delta}-Z_{3,\delta}\big)\,.\end{aligned}$$

Then the estimates in (4.5.25) follow by (4.5.7), (4.5.8), (4.2.7), (4.5.1). As in [34, 14], by (4.5.19), (4.5.12), (4.4.5), (4.5.29), it is possible to compute

$$\begin{aligned}&\partial_\alpha K_{00}(\phi)=I_\delta(\phi)\quad\partial_\alpha K_{10}(\phi)=[\partial_\phi\theta_0(\phi)]^{-1}\,,\quad\partial_\alpha K_{01}(\phi)=-J_\angle^{-1}\partial_\theta\widetilde{w}_0(\theta_0(\phi))\,,\\&K_{20}(\phi)=\varepsilon[\partial_\varphi\theta_0(\varphi)]^{-1}\partial_{II}P(i_\delta(\varphi))[\partial_\varphi\theta_0(\varphi)]^{-\top}\,,\\&K_{11}(\varphi)=\varepsilon\big(\partial_I\nabla_w P(i_\delta(\varphi))[\partial_\varphi\theta_0(\varphi)]^{-\top}-J_\angle^{-1}\partial_\theta\widetilde{w}_0(\theta_0(\varphi))\partial_{II}P(i_\delta(\varphi))[\partial_\varphi\theta_0(\varphi)]^{-\top}\big)\,.\end{aligned}$$

Then the estimates (4.5.26)-(4.5.28) follow by Lemmata 4.60, 4.61 and (4.5.1).

Proof of Theorem 4.65 We claim that the first three components of $\mathbf{T}_0 g$, with $\mathbf{T}_0$ defined in (4.5.49), form a reversible traveling wave variation when g is an anti-reversible traveling wave variation. Indeed, differentiating (4.5.13) it follows that $DG_\delta(\varphi, 0, 0)$, thus $(DG_\delta(\varphi, 0, 0))^{-1}$, is reversibility and momentum preserving (cfr. (4.2.62)). In particular these operators map an (anti)-reversible, respectively traveling, waves variation into a (anti)-reversible traveling waves variation (cfr. Lemma 4.48). Moreover, by Proposition 4.64, the operator $\mathbb{D}^{-1}$ maps an anti-reversible traveling wave into a vector whose first three components form a reversible traveling wave. This proves the claim.

We now compute the operators $\mathcal{P}, \mathcal{P}_\omega$ and $\mathcal{P}_\omega^\perp$ and prove that they are defined on traveling waves. By (4.4.6), since $X_\mathcal{N}$ is independent of the action I and i_δ differs from i_0 only in the I-component, see Lemma 4.61, we have

$$\mathrm{d}_{i,\alpha}\mathcal{F}(i_0) - \mathrm{d}_{i,\alpha}\mathcal{F}(i_\delta) = \varepsilon \int_0^1 \partial_I \mathrm{d}_i X_P(\theta_0, I_\delta + \lambda(I_0 - I_\delta), w_0)[I_0 - I_\delta, \Pi[\,\cdot\,]]\,\mathrm{d}\lambda =: \mathcal{E}_0\,, \quad \text{(C.2.8)}$$

where Π throughout this proof denotes the projection $(\widehat{\imath}, \widehat{\alpha}) \mapsto \widehat{\imath}$. Denote by $\mathtt{u} := (\phi, y, \mathtt{w})$ the symplectic coordinates induced by G_δ in (4.5.12). Under the symplectic map G_δ, the nonlinear operator $\mathcal{F}$ in (4.4.6) is transformed into

$$\mathcal{F}(G_\delta(\mathtt{u}(\varphi)), \alpha) = DG_\delta(\mathtt{u}(\varphi))(\omega\cdot\partial_\varphi \mathtt{u}(\varphi) - X_{K_\alpha}(\mathtt{u}(\varphi), \alpha))\,, \quad \text{(C.2.9)}$$

where $K_\alpha = H_\alpha \circ G_\delta$ as in (4.5.19). By differentiating (C.2.9) at the trivial torus $\mathtt{u}_\delta(\varphi) := G_\delta^{-1}(i_\delta)(\varphi) = (\varphi, 0, 0)$ and at $\alpha = \alpha_0$, we get

$$\mathrm{d}_{i,\alpha}\mathcal{F}(i_\delta) = DG_\delta(\mathtt{u}_\delta)(\omega\cdot\partial_\varphi - \mathrm{d}_{\mathtt{u},\alpha}X_{K_\alpha}(u_\delta, \alpha_0))D\widetilde{G}_\delta(\mathtt{u}_\delta)^{-1} + \mathcal{E}_1\,, \quad \text{(C.2.10)}$$

where

$$\mathcal{E}_1 := D^2 G_\delta(\mathtt{u}_\delta)[DG_\delta U(\mathtt{u}_\delta)^{-1}\mathcal{F}(i_\delta, \alpha_0), DG_\delta(\mathtt{u}_\delta)^{-1}\Pi[\,\cdot\,]]\,. \quad \text{(C.2.11)}$$

In expanded form, $\omega\cdot\partial_\varphi - \mathrm{d}_{\mathtt{u},\alpha}X_{K_\alpha}(u_\delta, \alpha_0)$ is provided in (4.5.30). By (4.5.37), (4.5.33) and (4.5.35), we split

$$\omega\cdot\partial_\varphi - \mathrm{d}_{\mathtt{u},\alpha}X_{K_\alpha}(u_\delta, \alpha_0) = \mathbb{D} + R_Z + \mathbb{R}_\omega + \mathbb{R}_\omega^\perp\,, \quad \text{(C.2.12)}$$

where

$$R_Z[\widehat{\phi}, \widehat{y}, \widehat{\mathtt{w}}, \widehat{\alpha}] := \begin{pmatrix} -\partial_\phi K_{10}(\varphi, \alpha_0)[\widehat{\phi}] \\ \partial_{\phi\phi}K_{00}(\varphi, \alpha_0)[\widehat{\phi}] + [\partial_\phi K_{10}(\varphi, \alpha_0)]^\top \widehat{y} + [\partial_\phi K_{01}(\varphi, \alpha_0)]^\top \widehat{\mathtt{w}} \\ -J_\angle \partial_\phi K_{01}(\varphi, \alpha_0)[\widehat{\phi}] \end{pmatrix}$$

and

$$\mathbb{R}_\omega[\widehat{\phi}, \widehat{y}, \widehat{\mathtt{w}}, \widehat{\alpha}] := \begin{pmatrix} 0 \\ 0 \\ \mathcal{R}_\omega[\widehat{\mathtt{w}}] \end{pmatrix}, \qquad \mathbb{R}_\omega^\perp[\widehat{\phi}, \widehat{y}, \widehat{\mathtt{w}}, \widehat{\alpha}] := \begin{pmatrix} 0 \\ 0 \\ \mathcal{R}_\omega^\perp[\widehat{\mathtt{w}}] \end{pmatrix}. \tag{C.2.13}$$

By (C.2.8), (C.2.10), (C.2.11), (C.2.12), we get the decomposition

$$\mathrm{d}_{i,\alpha}\mathcal{F}(i_0) = DG_\delta(\mathtt{u}_\delta) \circ \mathbb{D} \circ D\widetilde{G}_\delta(\mathtt{u}_\delta)^{-1} + \mathcal{E} + \mathcal{E}_\omega + \mathcal{E}_\omega^\perp, \tag{C.2.14}$$

where

$$\begin{aligned} &\mathcal{E} := \mathcal{E}_0 + \mathcal{E}_1 + DG_\delta(\mathtt{u}_\delta) R_Z D\widetilde{G}_\delta(\mathtt{u}_\delta)^{-1}, \\ &\mathcal{E}_\omega := DG_\delta(\mathtt{u}_\delta)\mathbb{R}_\omega D\widetilde{G}_\delta(\mathtt{u}_\delta)^{-1}, \qquad \mathcal{E}_\omega^\perp := DG_\delta(\mathtt{u}_\delta)\mathbb{R}_\omega^\perp D\widetilde{G}_\delta(\mathtt{u}_\delta)^{-1}. \end{aligned} \tag{C.2.15}$$

Applying $\mathbf{T}_0$ defined in (4.5.49) to the right of (C.2.14), since $\mathbb{D} \circ \mathbb{D}^{-1} = \mathrm{Id}$ by Proposition 4.64, we get

$$\mathrm{d}_{i,\alpha}\mathcal{F}(i_0) \circ \mathbf{T}_0 - \mathrm{Id} = \mathcal{P} + \mathcal{P}_\omega + \mathcal{P}_\omega^\perp,$$

where

$$\mathcal{P} = \mathcal{E} \circ \mathbf{T}_0, \quad \mathcal{P}_\omega = \mathcal{E}_\omega \circ \mathbf{T}_0, \quad \mathcal{P}_\omega^\perp = \mathcal{E}_\omega^\perp \circ \mathbf{T}_0.$$

A direct inspection of these formulas shows that $\mathcal{P}, \mathcal{P}_\omega$ and $\mathcal{P}_\omega^\perp$ are defined on traveling wave variations. In particular, note that the operators $\mathbb{R}_\omega$, $\mathbb{R}_\omega^\perp$ in (C.2.13) are defined only if $\widehat{\mathtt{w}}$ is a traveling wave, because the operators $\mathcal{R}_\omega, \mathcal{R}_\omega^\perp$ defined in (AI) act only on a traveling wave. However, note that, if g is a traveling wave variation, the third component of $D\widetilde{G}_\delta(\mathtt{u}_\delta)^{-1}\mathbf{T}_0 g$ is a traveling wave and therefore the operators $\mathcal{E}_\omega, \mathcal{E}_\omega^\perp$ in (C.2.15) are well defined.

By Lemmata 4.60, 4.63, 4.61 and (4.5.1), (4.5.31), (4.5.32), we obtain the estimate

$$\|\mathcal{E}[\widehat{\imath}, \widehat{\alpha}]\|_s^{k_0,\upsilon} \lesssim_s \|Z\|_{s_0+\sigma}^{k_0,\upsilon} \|\widehat{\imath}\|_{s+\sigma}^{k_0,\upsilon} + \|Z\|_{s+\sigma}^{k_0,\upsilon} \|\widehat{\imath}\|_{s_0+\sigma}^{k_0,\upsilon} + \|Z\|_{s_0+\sigma}^{k_0,\upsilon} \|\widehat{\imath}\|_{s_0+\sigma}^{k_0,\upsilon} \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}, \tag{C.2.16}$$

where $Z = \mathcal{F}(i_0, \alpha_0)$, recall (4.5.5). The estimate (4.5.50) follows by (4.5.49), Proposition 4.64 and (4.5.31). The estimate (4.5.51) follows by (4.5.50), (C.2.16), (4.5.1). The estimates (4.5.52), (4.5.54), (4.5.53) follow by the almost invertibility assumption (AI) on $\mathcal{L}_\omega$, see (4.5.35), Lemmata 4.61, 4.63, (4.64) and estimates (4.5.50), (4.5.1), (4.5.31).

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