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Landau-Fermi liquids without quasiparticles

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Landau-Fermi liquid theory is conventionally believed to hold whenever the interacting single-particle density of states develops a δ -like component at the Fermi surface, which is associated with quasiparticles. Here we show that a microscopic justification can be actually achieved under more general circumstances, even in case coherent quasiparticles are totally missing and the interacting single-particle density of states vanishes at the chemical potential as consequence of a pole singularity in the self-energy.

I. INTRODUCTION

Landau's Fermi liquid theory [1, 2] is a cornerstone of modern quantum many body physics, and represents by now a chief paradigm for interacting Fermi liquids at low temperatures, even strongly interacting ones as ³He [3] and heavy fermions [4].

Its microscopic justification relies on the hypothesis [5, 6] that the interacting single particle density of states (DOS), $\mathcal{A}(\epsilon, \mathbf{k})$, for energy $\epsilon \rightarrow 0$ and momentum approaching the Fermi surface, $\mathbf{k} \rightarrow \mathbf{k}_F$, defined through the Luttinger theorem [7, 8], becomes a δ -function

$$\mathcal{A}(\epsilon \rightarrow 0, \mathbf{k} \rightarrow \mathbf{k}_F) \simeq z_{\mathbf{k}} \delta(\epsilon - \epsilon_*(\mathbf{k})), \quad (1)$$

where the weight $z_{\mathbf{k}} < 1$ is the quasiparticle residue, which measures how much of a quasiparticle is contained in the physical single-particle excitation, and $\epsilon_*(\mathbf{k})$ the quasiparticle dispersion that vanishes at $\mathbf{k} = \mathbf{k}_F$. The validity of Eq. (1) can be verified order by order in perturbation theory, as we shall later discuss.

The Landau-Fermi liquid theory for a bulk of interacting fermions was phenomenologically extended by Nozières [9] to describe the Kondo regime of a quantum impurity model, what is commonly referred to as a *local* Landau-Fermi liquid, and later justified microscopically, see, e.g., Ref. [10]. However, such generalisation to quantum impurities poses a puzzle that is the actual motivation of the present work, and which we now discuss through a specific example.

Let us consider the model of two Anderson impurities, each hybridised with its own bath, and coupled to each other by an antiferromagnetic exchange J [11–13]. This model has a quantum critical point at $J = J_*$ [14–16] that separates the phase at $J < J_*$, where each impurity is Kondo screened by its bath, from the phase at $J > J_*$, where the two impurities lock by means of J into a spin-singlet state, no more available to Kondo screening. Both phases at $J < J_*$ and $J > J_*$ are local Fermi-liquids in Nozières' sense. However, the Fermi liquid behaviour at $J > J_*$ emerges from a state characterised by the impurity DOS that vanishes quadratically approaching the chemical potential, $\mathcal{A}(\epsilon) \sim \epsilon^2$, i.e., without displaying the peculiar Abrikosov-Suhl resonance, which reflects a diverging impurity self-energy, $\Sigma(\epsilon) \sim 1/\epsilon$ [12, 13]. Despite such singular behaviour, apparently at odds with a

Fermi liquid, one can still justify the latter microscopically [12], which raises the question whether it is possible to follow backward the path from bulk to local Fermi liquids with singular self-energies. Should that be indeed the case, it would imply that a microscopic justification of the Landau-Fermi liquid theory can be achieved with a less stringent requirement than Eq. (1).

This is actually the main outcome of the present work, which is organised as follows. In Sect. II we briefly recall the microscopic justification of Eq. (1), hence of the conventional derivation of Landau-Fermi liquid theory, which we rederive in Sect. III under a more general hypothesis, which includes Eq. (1) as a particular case. The results are exploited to obtain the Landau-Fermi liquid expressions of the dynamical susceptibilities in Sect. IV, which allows deriving a kinetic equation for the Wigner quasi-probability distribution of 'quasiparticles' in Sect. V. Section VI is devoted to concluding remarks.

II. CONVENTIONAL FERMIL LIQUID HYPOTHESIS

Let us recall the general expression of the Green's function in Matsubara frequencies $i\epsilon = i\pi(2n+1)T$, with $n \in \mathbb{Z}$ and T the temperature,

$$G(i\epsilon, \mathbf{k}) = \frac{1}{i\epsilon - \epsilon_{\mathbf{k}} - \Sigma(i\epsilon, \mathbf{k})}, \quad (2)$$

where $\epsilon_{\mathbf{k}}$ is the non-interacting dispersion relation measured relative to the chemical potential μ , and $\Sigma(i\epsilon, \mathbf{k})$ the self-energy. Its continuation in the complex plane, $G(\zeta, \mathbf{k})$ with $\zeta \in \mathbb{C}$, is analytic everywhere but on the real axis, where it generally develops a branch cut

$$\begin{aligned} G(\epsilon + i\eta, \mathbf{k}) - G(\epsilon - i\eta, \mathbf{k}) &\equiv G_+(\epsilon, \mathbf{k}) - G_-(\epsilon, \mathbf{k}) \\ &= -2i \operatorname{Im} G_+(\epsilon, \mathbf{k}) \equiv -2\pi i \mathcal{A}(\epsilon, \mathbf{k}), \end{aligned} \quad (3)$$

with $\epsilon \in \mathbb{R}$, η an infinitesimal positive real number, and $\mathcal{A}(\epsilon, \mathbf{k}) \geq 0$ the single particle DOS at momentum \mathbf{k} , satisfying

$$\int d\epsilon \mathcal{A}(\epsilon, \mathbf{k}) = 1. \quad (4)$$

$G_+(\epsilon, \mathbf{k})$ and $G_-(\epsilon, \mathbf{k})$ in (3) are, respectively, the retarded and advanced Green's functions. It is thus possible to write

$$G(\zeta, \mathbf{k}) = \int d\omega \frac{\mathcal{A}(\omega, \mathbf{k})}{\zeta - \omega}. \quad (5)$$

Similarly, the self-energy in the complex frequency plane, $\Sigma(\zeta, \mathbf{k})$, is also analytic but on the real axis. As before, for $\epsilon \in \mathbb{R}$, and $\eta > 0$ infinitesimal,

$$\Sigma(\epsilon \pm i\eta, \mathbf{k}) \equiv \Sigma_{\pm}(\epsilon, \mathbf{k}) = \text{Re}\Sigma_+(\epsilon, \mathbf{k}) \pm i \text{Im}\Sigma_+(\epsilon, \mathbf{k}), \quad (6)$$

define retarded, $\Sigma_+(\epsilon, \mathbf{k})$, and advanced, $\Sigma_-(\epsilon, \mathbf{k})$, components of the self-energy. It follows that

$$\mathcal{A}(\epsilon, \mathbf{k}) = \frac{1}{\pi} \frac{-\text{Im}\Sigma_+(\epsilon, \mathbf{k})}{(\epsilon - \epsilon_{\mathbf{k}} - \text{Re}\Sigma_+(\epsilon, \mathbf{k}))^2 + \text{Im}\Sigma_+(\epsilon, \mathbf{k})^2}, \quad (7)$$

thus $\text{Im}\Sigma_+(\epsilon, \mathbf{k}) \leq 0$.

In a conventional Fermi liquid, the Fermi surface (FS), $\mathbf{k} = \mathbf{k}_F$, is defined through [8]

$$\epsilon_{\mathbf{k}_F} + \text{Re}\Sigma_+(0, \mathbf{k}_F) = 0, \quad (8)$$

while the quasiparticle dispersion by

$$\epsilon_*(\mathbf{k}) - \epsilon_{\mathbf{k}} - \text{Re}\Sigma_+(\epsilon_*(\mathbf{k}), \mathbf{k}) = 0, \quad (9)$$

so that, by definition, $\epsilon_*(\mathbf{k}_F) = 0$. The important observation is that, order by order in perturbation theory, the following result holds for \mathbf{k} close to the FS [17]

$$-\text{Im}\Sigma_+(\epsilon \rightarrow 0, \mathbf{k}) = \gamma(\mathbf{k}) \epsilon^2 + \text{O}(\epsilon^4). \quad (10)$$

It follows that, expanding (7) for $\epsilon \simeq \epsilon_*(\mathbf{k})$, one finds

$$\begin{aligned} \mathcal{A}(\epsilon, \mathbf{k}) &\simeq \frac{Z_*(\epsilon_*(\mathbf{k}), \mathbf{k})}{\pi} \\ &\frac{\gamma_*(\mathbf{k}) \epsilon_*(\mathbf{k})^2}{(\epsilon - \epsilon_*(\mathbf{k}))^2 + \gamma_*(\mathbf{k})^2 \epsilon_*(\mathbf{k})^4} \\ &\xrightarrow{\mathbf{k} \rightarrow \mathbf{k}_F} Z_*(\epsilon_*(\mathbf{k}), \mathbf{k}) \delta(\epsilon - \epsilon_*(\mathbf{k})), \end{aligned} \quad (11)$$

thus Eq. (1) with $z_{\mathbf{k}} = Z_*(\epsilon_*(\mathbf{k}), \mathbf{k})$, where the formal definition of the quasiparticle residue reads

$$Z_*(\epsilon, \mathbf{k}) \equiv \left(1 - \frac{\partial \text{Re}\Sigma_+(\epsilon, \mathbf{k})}{\partial \epsilon} \right)^{-1}, \quad (12)$$

and $\gamma_*(\mathbf{k}) = Z_*(\epsilon_*(\mathbf{k}), \mathbf{k}) \gamma(\mathbf{k})$. In conclusion, one can safely write for $\mathbf{k} \sim \mathbf{k}_F$

$$\mathcal{A}(\epsilon, \mathbf{k}) \simeq z_{\mathbf{k}} \delta(\epsilon - \epsilon_*(\mathbf{k})) + \mathcal{A}_{\text{inc}}(\epsilon, \mathbf{k}), \quad (13)$$

with $\mathcal{A}_{\text{inc}}(\epsilon, \mathbf{k})$ a smooth function that carries the rest $1 - z_{\mathbf{k}}$ of the spectral weight, see Eq. (4), and describes

'incoherent' excitations as opposed to the 'coherent' δ -function component. It follows, through (5), that

$$\begin{aligned} G(i\epsilon, \mathbf{k}) &\simeq \frac{Z_*(\epsilon_*(\mathbf{k}), \mathbf{k})}{i\epsilon - \epsilon_*(\mathbf{k})} + \int d\omega \frac{\mathcal{A}_{\text{inc}}(\omega, \mathbf{k})}{i\epsilon - \omega} \\ &\equiv G_{\text{coh}}(i\epsilon, \mathbf{k}) + G_{\text{inc}}(i\epsilon, \mathbf{k}). \end{aligned} \quad (14)$$

We observe that $G_{\text{coh}} = Z_* G_0$, where $G_0(i\epsilon, \mathbf{k})$ is the Green's function of non-interacting electrons, the quasiparticles, with dispersion $\epsilon_*(\mathbf{k})$. Equation (13) coincides with the equation (2.15) of Ref. [5]. Starting from that, we could retrace all steps of that work, as well as of the second of the series, Ref. [6], and thus recover microscopically the Landau-Fermi liquid theory.

However, the fact that each term in perturbation theory satisfies Eq. (10) does not guarantee that the sum of the perturbation series shares the same property.

III. FERMI LIQUID THEORY REVISED

Hereafter, we will reconsider the microscopic justification of Landau-Fermi liquid theory relaxing the hypothesis (13), or, equivalently, (10). For that, we shall have in mind a system of electrons, coupled to each other by a short range interaction [18], with annihilation operators $c_{a\mathbf{k}}$, where a includes all quantum numbers but momentum. The Green's function and the self-energy will be in general matrices in the a -space, or, if such basis is properly chosen, diagonal in a . In what follows, whenever not necessary, we discard the label a . Moreover, for further simplification, we shall not take into account the possible emergence of non trivial topological properties [19, 20], which has constituted one of the most notable extensions of Fermi liquids in recent years.

We shall here assume that the electron system has gapless single particle excitations, namely it is a normal metal, and, in place of (10), that the following condition is satisfied:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} Z_*(\epsilon, \mathbf{k}) \left(-\text{Im}\Sigma_+(\epsilon, \mathbf{k}) \right) &\equiv \lim_{\epsilon \rightarrow 0} \gamma_*(\epsilon, \mathbf{k}) \\ &= \lim_{\epsilon \rightarrow 0} \gamma_*(\mathbf{k}) \epsilon^2 \rightarrow 0, \end{aligned} \quad (15)$$

with $Z_*(\epsilon, \mathbf{k})$ defined by Eq. (12). Equation (15) is far less stringent than (10). It is evidently satisfied if the conventional Fermi liquid hypothesis (10) holds, but also in the extreme case of $\Sigma_+(\epsilon, \mathbf{k})$ singular at $\epsilon = 0$, and yet leading to a continuous DOS without a true gap. We can consider, for instance, the bulk counterpart of the impurity self-energy in the two-impurity model at $J > J_*$ [12, 13] mentioned in the Introduction. Specifically, we can assume

$$\Sigma_+(\epsilon, \mathbf{k}) \simeq \frac{\Delta(\mathbf{k})^2}{\epsilon + i\eta} - i\Gamma(\mathbf{k}), \quad (16)$$

with $\Delta(\mathbf{k}) \in \mathbb{R}$ and $\Gamma(\mathbf{k}) \geq 0$. In such case,

$$\begin{aligned} \text{Re } \Sigma_+(\epsilon, \mathbf{k}) &\simeq \frac{\Delta(\mathbf{k})^2}{\epsilon}, \\ \text{Im } \Sigma_+(\epsilon, \mathbf{k}) &\simeq -\pi \Delta(\mathbf{k})^2 \delta(\epsilon) - \Gamma(\mathbf{k}), \end{aligned} \quad (17)$$

thus the quasiparticle residue

$$Z_*(\epsilon, \mathbf{k}) \simeq \frac{\epsilon^2}{\Delta(\mathbf{k})^2 + \epsilon^2} \xrightarrow{\epsilon \rightarrow 0} 0, \quad (18)$$

vanishes at the chemical potential, so does the particle DOS

$$\mathcal{A}(\epsilon, \mathbf{k}) \simeq \frac{1}{\pi} \epsilon^2 \frac{\Gamma(\mathbf{k})}{\Delta(\mathbf{k})^4} \xrightarrow{\epsilon \rightarrow 0} 0. \quad (19)$$

Nonetheless, Eq. (15) is satisfied with $\gamma_*(\mathbf{k}) = \Gamma(\mathbf{k})/\Delta(\mathbf{k})^2$.

We intentionally did not specify any precise \mathbf{k} -dependence of $\Sigma_+(\epsilon, \mathbf{k})$ in (16), so to maintain the discussion as general as possible. However, we just mention that the pseudo-gapped behaviour of the particle DOS (19) could, e.g., represent an ‘unsuccessful superconductor’, or, more generally, an ‘unsuccessful insulator’, where the abundance of low energy excitations responsible of the finite $\Gamma(\mathbf{k})$ prevents the opening of a true gap $\sim 2\Delta(\mathbf{k})$.

For later use, we define the ‘quasiparticle’ DOS through

$$\mathcal{A}_{\text{qp}}(\epsilon, \mathbf{k}) \equiv \frac{\mathcal{A}(\epsilon, \mathbf{k})}{Z_*(\epsilon, \mathbf{k})}, \quad (20)$$

and the ‘quasiparticle’ group velocity as

$$\mathbf{v}_*(\epsilon, \mathbf{k}) \equiv Z_*(\epsilon, \mathbf{k}) \left(\frac{\partial \epsilon_{\mathbf{k}}}{\partial \mathbf{k}} + \frac{\partial \text{Re } \Sigma_+(\epsilon, \mathbf{k})}{\partial \mathbf{k}} \right), \quad (21)$$

where, differently from the standard definition, the ϵ -dependence of the quasiparticle residue $Z_*(\epsilon, \mathbf{k})$ is retained. We note that, when the conventional Fermi liquid hypothesis (13) holds, then, for small ϵ , $\mathcal{A}_{\text{qp}}(\epsilon, \mathbf{k}) \simeq \delta(\epsilon - \epsilon_*(\mathbf{k}))$, thus describing a genuine coherent quasiparticle, and the on-shell group velocity $\mathbf{v}_*(\epsilon_*(\mathbf{k}), \mathbf{k}) = \partial \epsilon_*(\mathbf{k})/\partial \mathbf{k}$. However, even in the singular case of Eq. (16), the ‘quasiparticle’ DOS of Eq. (20) is finite at the chemical potential $\epsilon = 0$, though not δ -like, despite the particle DOS (19) vanishes.

A. Preliminaries

Given a generic density operator, with Fourier transform

$$\rho_Q(\mathbf{q}) = \sum_{\mathbf{k}} c_{\mathbf{k}}^\dagger \Lambda_Q^{(0)}(\mathbf{k}, \mathbf{k} + \mathbf{q}) c_{\mathbf{k} + \mathbf{q}} = \rho_Q(-\mathbf{q})^\dagger, \quad (22)$$

where $\Lambda_Q^{(0)}$ is the bare vertex associated with that density, the corresponding density-density response function in the Matsubara formalism reads, see Appendix B,

$$\chi_Q(i\omega, \mathbf{q}) = \frac{T}{V} \sum_{\mathbf{k}\epsilon} G(i\epsilon + i\omega, \mathbf{k} + \mathbf{q}) \Lambda_Q^{(0)}(\mathbf{k}, \mathbf{k} + \mathbf{q})^\dagger G(i\epsilon, \mathbf{k}) \Lambda_Q(i\epsilon + i\omega, \mathbf{k} + \mathbf{q}, i\epsilon, \mathbf{k}; i\omega, \mathbf{q}), \quad (23)$$

where V is the number of sites, T the temperature, and Λ_Q the fully interacting density vertex. Henceforth, we shall be interested just in the long wavelength, low frequency and low temperature value of Eq. (23). For that, we start analysing the behaviour of the kernel

$$R(i\epsilon + i\omega, \mathbf{k} + \mathbf{q}; i\epsilon, \mathbf{k}) = G(i\epsilon + i\omega, \mathbf{k} + \mathbf{q}) G(i\epsilon, \mathbf{k}), \quad (24)$$

at small $\omega > 0$ and $q = |\mathbf{q}|$, in the sense of a distribution in ϵ , and thus consider the convolution [5]

$$C_{\mathbf{k}}(i\omega, \mathbf{q}) = T \sum_{\epsilon} R(i\epsilon + i\omega, \mathbf{k} + \mathbf{q}; i\epsilon, \mathbf{k}) F(i\epsilon), \quad (25)$$

with a generic function $F(i\epsilon)$ that decays sufficiently fast for $|\epsilon| \rightarrow \infty$.

In the conventional case, each Green’s function can be

written as in Eq. (14), thus the kernel $R = GG$ is simply

$$\begin{aligned} R &= G_{\text{coh}} G_{\text{coh}} + \dots \equiv G_{\text{coh}} G_{\text{coh}} + R_{\text{inc}} \\ &\equiv \Delta + R_{\text{inc}} = Z_*^2 G_0 G_0 + R_{\text{inc}}, \end{aligned} \quad (26)$$

where, we recall, G_0 is the Green’s function of non-interacting particles with dispersion $\epsilon_*(\mathbf{k})$. It follows that the expression of Δ in the sense of a distribution can be readily obtained through the well known expression of the Lindhard function,

$$G_0 G_0 = \frac{\delta_{\epsilon,0}}{T} \frac{f(\epsilon_*(\mathbf{k})) - f(\epsilon_*(\mathbf{k} + \mathbf{q}))}{i\omega - \epsilon_*(\mathbf{k} + \mathbf{q}) + \epsilon_*(\mathbf{k})}, \quad (27)$$

where $f(\epsilon)$ is the Fermi distribution function (compare, e.g., with Eq. (2.23) in Ref. [5]). The main property of $Z_*^2 G_0 G_0$, which is actually at the hearth of Landau-Fermi liquid theory, is the non-analytic behaviour in the

origin $\omega = q = 0$, unlike R_{inc} that is assumed to be analytic.

If we replace condition (10) with (15), we cannot anymore use Eq. (14), and thus Eq. (27), to determine the analytic properties of the kernel R . However, for that purpose, we can instead follow the derivation of the local Landau-Fermi liquid theory in quantum impurity models [10, 12]. We thus consider a contour in the complex frequency plane, $i\epsilon \rightarrow \zeta \in \mathbb{C}$, which runs clockwise at infinity. Assuming that the integrand vanishes faster than

$1/\zeta$ at infinity,

$$\begin{aligned} 0 &= \oint \frac{d\zeta}{2\pi i} f(\zeta) R(\zeta + i\omega, \mathbf{k} + \mathbf{q}; \zeta, \mathbf{k}) F(\zeta) \\ &= C_{\mathbf{k}}(i\omega, \mathbf{q}) + \dots, \end{aligned}$$

where the dots take into account the singularities of R and F . In particular, R has generically two horizontal branch cuts, the real axis $\zeta = \epsilon$ and the axis $\zeta = -i\omega + \epsilon$, with $\epsilon \in \mathbb{R}$, which merge into a single one, just the real axis, when $\omega = 0$. The contribution of the horizontal strip $-\omega < \text{Im}\zeta < 0$ may not be analytic at $\omega = q = 0$: it trivially vanishes if $\omega \rightarrow 0$ first than $q \rightarrow 0$, so called q -limit, but it may not in the opposite ω -limit. On the contrary, the contributions from $\text{Im}\zeta \geq 0$ and $\text{Im}\zeta \leq -\omega$ do not have any apparent reason of non analyticity. The strip contribution reads

$$\begin{aligned} C_{\mathbf{k}}^{\text{sing}}(i\omega, \mathbf{q}) &= T \sum_{\epsilon=\epsilon_0-i\omega}^{-\epsilon_0} G(i\epsilon + i\omega, \mathbf{k} + \mathbf{q}) G(i\epsilon, \mathbf{k}) F(i\epsilon) \\ &= - \int \frac{d\epsilon}{2\pi i} f(\epsilon) \left[G_+(\epsilon, \mathbf{k}) G_-(\epsilon - i\omega, \mathbf{k}) F(\epsilon - i\omega) - G_+(\epsilon + i\omega, \mathbf{k}) G_-(\epsilon, \mathbf{k}) F(\epsilon) \right], \end{aligned} \quad (28)$$

where $\epsilon_0 = \pi T$ is the lowest fermionic Matsubara fre-

quency, which, after the analytic continuation $i\omega \rightarrow \omega + i\eta$, with $\eta > 0$ infinitesimal, becomes

$$\begin{aligned} C_{\mathbf{k}}^{\text{sing}}(\omega, \mathbf{q}) &= - \int \frac{d\epsilon}{2\pi i} f(\epsilon) \left[G_+(\epsilon, \mathbf{k}) G_-(\epsilon - \omega, \mathbf{k}) F(\epsilon - \omega) - G_+(\epsilon + \omega, \mathbf{k}) G_-(\epsilon, \mathbf{k}) F(\epsilon) \right] \\ &= \int \frac{d\epsilon}{2\pi i} \left(f(\epsilon) F(\epsilon) - f(\epsilon + \omega) F(\epsilon + \omega) \right) G_+(\epsilon + \omega, \mathbf{k}) G_-(\epsilon, \mathbf{k}) \\ &\simeq \int \frac{d\epsilon}{2\pi i} \left(- \frac{\partial f(\epsilon)}{\partial \epsilon} \right) \omega F(\epsilon) G_+(\epsilon + \omega, \mathbf{k}) G_-(\epsilon, \mathbf{k}) \\ &= \int \frac{d\epsilon}{2\pi i} \left(- \frac{\partial f(\epsilon)}{\partial \epsilon} \right) F(\epsilon) \left(G_-(\epsilon, \mathbf{k}) - G_+(\epsilon + \omega, \mathbf{k} + \mathbf{q}) \right) \\ &\quad \frac{\omega}{\omega + i\eta - (\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}) - \left(\Sigma_+(\epsilon + \omega, \mathbf{k} + \mathbf{q}) - \Sigma_-(\epsilon, \mathbf{k}) \right)}. \end{aligned} \quad (29)$$

For small ω and q , recalling that $\text{Re}\Sigma_- = \text{Re}\Sigma_+$ while

$\text{Im}\Sigma_- = -\text{Im}\Sigma_+$, and through equations (12), (15) and (21), we can write

$$\frac{\omega}{\omega + i\eta - (\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}) - \left(\Sigma_+(\epsilon + \omega, \mathbf{k} + \mathbf{q}) - \Sigma_-(\epsilon, \mathbf{k}) \right)} \simeq Z_*(\epsilon, \mathbf{k}) \frac{\omega}{\omega + i\eta - \mathbf{v}_*(\epsilon, \mathbf{k}) \cdot \mathbf{q} + 2i\gamma_*(\epsilon, \mathbf{k})}. \quad (30)$$

Since the derivative of the Fermi distribution function in (29) implies that $\epsilon \sim T \sim 0$, if we assume Eq. (15)

valid we can safely neglect $\gamma_*(\epsilon, \mathbf{k})$ in (30) if either ω or

$\mathbf{v}_*(T, \mathbf{k}) \cdot \mathbf{q}$ are much greater than $\gamma_*(\mathbf{k}) T^2$. Coming back to (29), and noting that

$$G_-(\epsilon, \mathbf{k}) - G_+(\epsilon + \omega, \mathbf{k} + \mathbf{q}) \simeq 2\pi i \mathcal{A}(\epsilon, \mathbf{k}) + O(q, \omega),$$

we can finally write

$$C_{\mathbf{k}}^{sing}(\omega, \mathbf{q}) = \frac{1}{V} \sum_{\mathbf{k}} \int d\epsilon \tilde{\Delta}(\epsilon + \omega, \mathbf{k} + \mathbf{q}; \epsilon, \mathbf{k}) F(\epsilon), \quad (31)$$

having defined the distribution kernel

$$\tilde{\Delta}(\epsilon + \omega, \mathbf{k} + \mathbf{q}; \epsilon, \mathbf{k}) = -\frac{\partial f(\epsilon)}{\partial \epsilon} \mathcal{A}_{qp}(\epsilon, \mathbf{k}) Z_*(\epsilon, \mathbf{k})^2 \frac{\omega}{\omega - \mathbf{v}_*(\epsilon, \mathbf{k}) \cdot \mathbf{q}}, \quad (32)$$

which is indeed non analytic at $\omega = q = 0$, where $\mathcal{A}_{qp}(\epsilon, \mathbf{k})$ is defined by (20). In other words, the non analytic behaviour persists even if R in Eq. (25) cannot be written as in (26) in terms of non-interacting Green's functions, provided Eq. (15) holds.

Going back to Eq. (25), we end up with the following expression

$$R \equiv \tilde{\Delta} + \tilde{R}_{inc}, \quad (33)$$

where \tilde{R}_{inc} is analytic at the origin, and all non-analyticities are hidden in $\tilde{\Delta}$. Specifically,

$$\begin{aligned} \lim_{\omega \rightarrow 0} \lim_{q \rightarrow 0} \tilde{\Delta}(i\epsilon + i\omega, \mathbf{k} + \mathbf{q}; i\epsilon, \mathbf{k}) &\equiv \tilde{\Delta}^\omega(i\epsilon, \mathbf{k}) \neq 0, \\ \lim_{q \rightarrow 0} \lim_{\omega \rightarrow 0} \tilde{\Delta}(i\epsilon + i\omega, \mathbf{k} + \mathbf{q}; i\epsilon, \mathbf{k}) &\equiv \tilde{\Delta}^q(i\epsilon, \mathbf{k}) = 0. \end{aligned} \quad (34)$$

We further define

$$\Delta(i\epsilon + i\omega, \mathbf{k} + \mathbf{q}; i\epsilon, \mathbf{k}) \equiv \tilde{\Delta}(i\epsilon + i\omega, \mathbf{k} + \mathbf{q}; i\epsilon, \mathbf{k}) - \tilde{\Delta}^\omega(i\epsilon, \mathbf{k}), \quad (35)$$

whose expression on the real axis is

$$\Delta(\epsilon + \omega, \mathbf{k} + \mathbf{q}; \epsilon, \mathbf{k}) = -\frac{\partial f(\epsilon)}{\partial \epsilon} \mathcal{A}_{qp}(\epsilon, \mathbf{k}) Z_*(\epsilon, \mathbf{k})^2$$

$$\begin{aligned} \chi_Q(\omega, \mathbf{q}) &= -\frac{1}{V} \sum_{\mathbf{k}} \int d\epsilon \left(-\frac{\partial f(\epsilon)}{\partial \epsilon} \right) \mathcal{A}_{qp}(\epsilon, \mathbf{k}) \frac{\mathbf{v}_*(\epsilon, \mathbf{k}) \cdot \mathbf{q}}{\omega - \mathbf{v}_*(\epsilon, \mathbf{k}) \cdot \mathbf{q} + i\eta} \\ &\quad - \frac{1}{V^2} \sum_{\mathbf{k}\mathbf{k}'} \int d\epsilon d\epsilon' \left(-\frac{\partial f(\epsilon)}{\partial \epsilon} \right) \left(-\frac{\partial f(\epsilon')}{\partial \epsilon'} \right) \mathcal{A}_{qp}(\epsilon, \mathbf{k}) \frac{\mathbf{v}_*(\epsilon, \mathbf{k}) \cdot \mathbf{q}}{\omega - \mathbf{v}_*(\epsilon, \mathbf{k}) \cdot \mathbf{q} + i\eta} \\ &\quad \sum_{aa'} q_a(\mathbf{k}) q_{a'}(\mathbf{k}') A_{a,a',a',a}(\epsilon \mathbf{k}, \epsilon' \mathbf{k}'; \omega \mathbf{q}) \mathcal{A}_{qp}(\epsilon', \mathbf{k}') \frac{\mathbf{v}_*(\epsilon', \mathbf{k}') \cdot \mathbf{q}}{\omega - \mathbf{v}_*(\epsilon', \mathbf{k}') \cdot \mathbf{q} + i\eta}, \end{aligned} \quad (41)$$

assuming $\sum_a q_a(\mathbf{k})^2 = 1$ the normalisation of the conserved quantity, and having defined the ‘quasiparticle’

where now

$$\begin{aligned} \lim_{\omega \rightarrow 0} \lim_{q \rightarrow 0} \Delta(i\epsilon + i\omega, \mathbf{k} + \mathbf{q}; i\epsilon, \mathbf{k}) &\equiv \Delta^\omega(i\epsilon, \mathbf{k}) = 0, \\ \lim_{q \rightarrow 0} \lim_{\omega \rightarrow 0} \Delta(i\epsilon + i\omega, \mathbf{k} + \mathbf{q}; i\epsilon, \mathbf{k}) &\equiv \Delta^q(i\epsilon, \mathbf{k}) \\ &= -\tilde{\Delta}^\omega(i\epsilon, \mathbf{k}) \neq 0, \end{aligned} \quad (37)$$

and, consequently,

$$R \equiv \Delta + R_{inc}, \quad R_{inc} = \tilde{R}_{inc} + \tilde{\Delta}^\omega. \quad (38)$$

The quantities Δ and $\tilde{\Delta}$ coincide, respectively, with those in equations (2.23) and (2.33) of Ref. [5]. Therefore, from this point on, we can simply follow all steps of Ref. [5], which we shall not repeat here, but just sketch in Appendix B, and jump directly to the final results in the following sections. [21]

IV. DYNAMIC SUSCEPTIBILITIES

We now assume that the interacting Hamiltonian admits a conserved quantity Q . It follows that, in the basis in which the corresponding single-particle operator is diagonal, i.e.,

$$Q = \int d\mathbf{r} \rho_Q(\mathbf{r}) \equiv \sum_a \sum_{\mathbf{k}} q_a(\mathbf{k}) c_{\mathbf{k}a}^\dagger c_{\mathbf{k}a}, \quad (39)$$

with $\rho_Q(\mathbf{r})$ the density operator corresponding to Q , the Green's function is diagonal, too. For simplicity, we shall assume that the Green's function is actually independent of a . A smoothly varying external field $h_Q(t, \mathbf{r})$, with Fourier component $h_Q(\omega, \mathbf{q})$, is coupled to $\rho_Q(\mathbf{r})$, adding to the Hamiltonian the time-dependent perturbation $\delta H(t) = \int d\mathbf{r} h_Q(t, \mathbf{r}) \rho_Q(\mathbf{r})$. At linear order in h_Q , the variation of the expectation value of $\rho_Q(\mathbf{q})$ is

$$\delta \langle \rho_Q(\mathbf{q}) \rangle = \chi_Q(\omega, \mathbf{q}) h_Q(\omega, \mathbf{q}), \quad (40)$$

where the linear response function Eq. (23) is readily obtained [5], see Appendix B for the explicit derivation, and reads, after analytic continuation $i\omega \rightarrow \omega + i\eta$,

scattering amplitude

$$\begin{aligned} A_{a,b;b,a}(\epsilon \mathbf{k}, \epsilon' \mathbf{p}; \omega \mathbf{q}) &= Z_*(\epsilon, \mathbf{k}) Z_*(\epsilon', \mathbf{p}) \\ \Gamma_{a,b;b,a}(\epsilon + i\omega \mathbf{k} + \mathbf{q}, \epsilon' \mathbf{p}; \epsilon' + i\omega \mathbf{p} + \mathbf{q}, \epsilon \mathbf{k}), \end{aligned} \quad (42)$$

as opposed to the particle one Γ , with q and ω limits, respectively, $A_{a,b;b,a}^q(\epsilon \mathbf{k}, \epsilon' \mathbf{p})$ and $A_{a,b;b,a}^\omega(\epsilon \mathbf{k}, \epsilon' \mathbf{p})$. The thermodynamic susceptibility κ_Q is related to the q -limit of the dynamical one χ_Q , specifically,

$$\begin{aligned} \kappa_Q = \chi_Q^q &= \frac{1}{V} \sum_{\mathbf{k}} \int d\epsilon \left(-\frac{\partial f(\epsilon)}{\partial \epsilon} \right) \mathcal{A}_{\text{qp}}(\epsilon, \mathbf{k}) \\ &\quad - \frac{1}{V^2} \sum_{\mathbf{kp}} \int d\epsilon d\epsilon' \left(-\frac{\partial f(\epsilon)}{\partial \epsilon} \right) \left(-\frac{\partial f(\epsilon')}{\partial \epsilon'} \right) \\ &\quad \quad \mathcal{A}_{\text{qp}}(\epsilon, \mathbf{k}) \mathcal{A}_{\text{qp}}(\epsilon', \mathbf{p}) \\ &\quad \quad \sum_{ab} q_a(\mathbf{k}) q_b(\mathbf{p}) A_{a,b;b,a}^q(\epsilon \mathbf{k}, \epsilon' \mathbf{p}). \end{aligned}$$

Since the local quasiparticle DOS is, by definition,

$$\mathcal{A}_{\text{qp}}(\epsilon) \equiv \frac{1}{V} \sum_{\mathbf{k}} \mathcal{A}_{\text{qp}}(\epsilon, \mathbf{k}), \quad (43)$$

upon defining the Landau A_Q parameter in channel Q as

$$\begin{aligned} A_Q \mathcal{A}_{\text{qp}}(0) &\equiv \frac{1}{V^2} \sum_{\mathbf{kp}} \sum_{ab} q_a(\mathbf{k}) q_b(\mathbf{p}) \\ &\quad \int d\epsilon d\epsilon' \left(-\frac{\partial f(\epsilon)}{\partial \epsilon} \right) \left(-\frac{\partial f(\epsilon')}{\partial \epsilon'} \right) \\ &\quad \quad \mathcal{A}_{\text{qp}}(\epsilon, \mathbf{k}) \mathcal{A}_{\text{qp}}(\epsilon', \mathbf{p}) A_{a,b;b,a}^q(\epsilon \mathbf{k}, \epsilon' \mathbf{p}). \end{aligned} \quad (44)$$

we finally obtain, at zero temperature,

$$\kappa_Q = \mathcal{A}_{\text{qp}}(0) (1 - A_Q), \quad (45)$$

which is the standard Landau-Fermi liquid expression, but derived under the more general assumption (15). In fact, the expression (45) holds also in the case Eq. (16) of a singular self-energy, yielding vanishing quasiparticle residue and particle DOS at the chemical potential. Nonetheless, the ‘quasiparticle’ DOS is finite so as the zero temperature thermodynamic susceptibility. We also remark that a finite Landau A_Q parameter in spite of a vanishing quasiparticle residue implies, through Eq. (42), that Γ is singular at the chemical potential.

We emphasise that the rather simple expression (41) of the linear response functions, which looks like that of weakly interacting (quasi)particles, holds only for density operators that refer to conserved quantities, for which one can use the Ward-Takahashi identity. Otherwise, the response functions contain additional observable-dependent parameters, see Eq. (3.9) in Ref. [5] and Eq. (B13) in Appendix B; specifically, an additional constant that corresponds to the ω -limit of the response function, vanishing for conserved quantities,

and the ω -limit of vertex corrections. The meaning of such difference is that only for conserved quantities we are guaranteed that the matrix element coupling the external field to the density of physical particles is the same as that one coupling to the density of quasiparticles, while for generic observables this ought not to be the case.

V. LANDAU-BOLTZMANN EQUATION

The expression (41) of the linear response functions allows easily deriving a corresponding Boltzmann kinetic equation, which we believe worth showing explicitly.

We first associate to the expression (36) of Δ for real frequencies the components of a matrix $\hat{K}(\omega, \mathbf{q})$ in frequency, momentum, and quantum number a space, through

$$\begin{aligned} K_{\epsilon \mathbf{k} a, \epsilon' \mathbf{k}' a'}(\omega, \mathbf{q}) &= \delta(\epsilon - \epsilon') \delta_{\mathbf{k}, \mathbf{k}'} \delta_{a, a'} \\ &\quad \Delta(\epsilon + \omega, \mathbf{k} + \mathbf{q}; \epsilon, \mathbf{k}). \end{aligned} \quad (46)$$

Next we formally write

$$\begin{aligned} \delta \langle \rho_Q(\mathbf{q}) \rangle &= \chi_Q(\omega, \mathbf{q}) h_Q(\omega, \mathbf{q}) \\ &\equiv \frac{1}{V} \sum_{\mathbf{ak}} \int d\epsilon q_a(\mathbf{k}) \delta n_{\epsilon \mathbf{k} a}(\omega, \mathbf{q}), \end{aligned} \quad (47)$$

where $\delta n_{\epsilon \mathbf{k} a}(\omega, \mathbf{q})$ are the components of the vector $\delta \mathbf{n}(\omega, \mathbf{q})$, which, through Eq. (41), satisfies

$$\delta \mathbf{n} = -\hat{K} \left[1 + \hat{A} \hat{K} \right] \mathbf{V}, \quad (48)$$

or, equivalently,

$$\left[1 + \hat{A} \hat{K} \right]^{-1} \hat{K}^{-1} \delta \mathbf{n} = -\mathbf{V}, \quad (49)$$

having defined \hat{A} the matrix with elements $A_{\epsilon \mathbf{k} a, \epsilon' \mathbf{k}' a'}(\omega, \mathbf{q}) = A_{a, a'; a', a}(\epsilon \mathbf{k}, \epsilon' \mathbf{k}'; \omega, \mathbf{q})$, and \mathbf{V} the vector with components $V_{\epsilon \mathbf{k} a}(\omega, \mathbf{q}) = q_a(\mathbf{k}) h_Q(\omega, \mathbf{q})$. We next introduce the standard Landau’s f parameters through

$$\left[1 + \hat{A} \hat{K} \right]^{-1} \equiv \left[1 - \hat{f} \hat{K} \right], \quad (50)$$

so that Eq. (49) becomes

$$\left[1 - \hat{f} \hat{K} \right] \hat{K}^{-1} \delta \mathbf{n} = \hat{K}^{-1} \delta \mathbf{n} - \hat{f} \delta \mathbf{n} = -\mathbf{V}. \quad (51)$$

Multiplying both sides of Eq. (51) by $(\omega - \mathbf{v}_*(\epsilon, \mathbf{k}) \cdot \mathbf{q}) \hat{K}$ we finally obtain the equation

$$\begin{aligned}
0 &= \left(\omega - \mathbf{v}_*(\epsilon, \mathbf{k}) \cdot \mathbf{q} \right) \delta n_{\epsilon \mathbf{k} a}(\omega, \mathbf{q}) \\
&\quad + \frac{\partial f(\epsilon)}{\partial \epsilon} \mathcal{A}_{\text{qp}}(\epsilon, \mathbf{k}) \mathbf{v}_*(\epsilon, \mathbf{k}) \cdot \mathbf{q} \left\{ \frac{1}{V} \sum_{\mathbf{k}' a'} \int d\epsilon' f_{\epsilon \mathbf{k} a, \epsilon' \mathbf{k}' a'} \delta n_{\epsilon' \mathbf{k}' a'}(\omega, \mathbf{q}) - q_a(\mathbf{k}) h_Q(\omega, \mathbf{q}) \right\} \\
&\equiv \omega \delta n_{\epsilon \mathbf{k} a}(\omega, \mathbf{q}) - \mathbf{v}_*(\epsilon, \mathbf{k}) \cdot \mathbf{q} \delta \bar{n}_{\epsilon \mathbf{k} a}(\omega, \mathbf{q}) - q_a(\mathbf{k}) \frac{\partial f(\epsilon)}{\partial \epsilon} \mathcal{A}_{\text{qp}}(\epsilon, \mathbf{k}) \mathbf{v}_*(\epsilon, \mathbf{k}) \cdot \mathbf{q} h_Q(\omega, \mathbf{q}),
\end{aligned} \tag{52}$$

where we assumed that the dependence of f upon ω and \mathbf{q} is negligible when they are both small, and, by definition,

$$\delta \bar{n} = \left[1 - \hat{K}^q \hat{f} \right] \delta n. \tag{53}$$

It follows that the inverse Fourier transform $\delta n_{\epsilon \mathbf{k} a}(t, \mathbf{r})$ satisfies

$$\begin{aligned}
0 &= \delta \dot{n}_{\epsilon \mathbf{k} a}(t, \mathbf{r}) + \mathbf{v}_*(\epsilon, \mathbf{k}) \cdot \nabla \delta \bar{n}_{\epsilon \mathbf{k} a}(t, \mathbf{r}) \\
&\quad + q_a(\mathbf{k}) \frac{\partial f(\epsilon)}{\partial \epsilon} \mathcal{A}_{\text{qp}}(\epsilon, \mathbf{k}) \mathbf{v}_*(\epsilon, \mathbf{k}) \cdot \nabla h_Q(t, \mathbf{r}),
\end{aligned} \tag{54}$$

which can be interpreted as the standard Landau-Boltzmann kinetic equation once we identify $\delta n_{\epsilon \mathbf{k} a}(t, \mathbf{r})$ and $\delta \bar{n}_{\epsilon \mathbf{k} a}(t, \mathbf{r})$, respectively, with the deviations from global and local equilibrium, see chapter 1 in Ref. [22], of the Wigner quasi probability distribution of quasiparticles.

In addition, the interpretation in terms of a semiclassical kinetic equation requires that the group velocity defined in Eq. (21) is equivalent to

$$\mathbf{v}_*(\epsilon, \mathbf{k}) \equiv \frac{\partial \epsilon_*(\epsilon, \mathbf{k})}{\partial \mathbf{k}}, \tag{55}$$

where $\epsilon_*(\epsilon, \mathbf{k})$ must be identified with the quasiparticle energy, and that the derivative with respect to \mathbf{k} of the Wigner distribution at equilibrium must correspond to

$$\frac{\partial n_{\epsilon \mathbf{k} a}^0}{\partial \mathbf{k}} = \frac{\partial f(\epsilon)}{\partial \epsilon} \mathcal{A}_{\text{qp}}(\epsilon, \mathbf{k}) \mathbf{v}_*(\epsilon, \mathbf{k}). \tag{56}$$

We shall explicitly prove the last equality in the Appendix. Through Eqs. (55) and (56) we can therefore rewrite Eq. (54) as

$$\begin{aligned}
0 &= \delta \dot{n}_{\epsilon \mathbf{k} a}(t, \mathbf{r}) + \frac{\partial \epsilon_*(\epsilon, \mathbf{k})}{\partial \mathbf{k}} \cdot \nabla \delta \bar{n}_{\epsilon \mathbf{k} a}(t, \mathbf{r}) \\
&\quad + q_a(\mathbf{k}) \frac{\partial n_{\epsilon \mathbf{k} a}^0}{\partial \mathbf{k}} \cdot \nabla h_Q(t, \mathbf{r}),
\end{aligned} \tag{57}$$

which has now truly the form of the conventional Landau-Boltzmann kinetic equation, and entails a Landau's energy functional in absence of the external field

$$\begin{aligned}
F[\delta n] &= \sum_{\mathbf{k} a} \int d\mathbf{r} d\epsilon \left\{ \epsilon_*(\epsilon, \mathbf{k}) \delta n_{\epsilon \mathbf{k} a}(t, \mathbf{r}) \right. \\
&\quad \left. + \frac{1}{2V} \sum_{\mathbf{k}' a'} \int d\epsilon' f_{a a'}(\epsilon \mathbf{k}, \epsilon' \mathbf{k}') \right\}
\end{aligned} \tag{58}$$

$$\left. \delta n_{\epsilon \mathbf{k} a}(t, \mathbf{r}) \delta n_{\epsilon' \mathbf{k}' a'}(t, \mathbf{r}) \right\}.$$

We end remarking that for a conventional Fermi liquid, where $\mathcal{A}_{\text{qp}}(\epsilon, \mathbf{k}) = \delta(\epsilon - \epsilon_*(\mathbf{k}))$, one can readily integrate over ϵ both sides of Eq. (57) and recover the standard Landau-Boltzmann kinetic equation for the integrated $\delta n_{\mathbf{k} a}(t, \mathbf{r}) = \int d\epsilon \delta n_{\epsilon \mathbf{k} a}(t, \mathbf{r})$. However, Eq. (57) remain valid also when $\mathcal{A}_{\text{qp}}(\epsilon, \mathbf{k}) \neq \delta(\epsilon - \epsilon_*(\mathbf{k}))$, in which case the dependence of the quasiparticle DOS and group velocity $\mathbf{v}_*(\epsilon, \mathbf{k})$ on the frequency ϵ , which may also be rather non trivial, must be explicitly taken into account.

VI. CONCLUSIONS

We have shown that the Landau-Fermi liquid low-temperature expressions of the dynamical susceptibilities in the long wavelength limit and small frequency, as well as the corresponding Boltzmann kinetic equation, can be microscopically justified even if the interacting single-particle Green's function does not have a quasiparticle pole near the chemical potential.

This result may not come as a surprise. For instance, also one dimensional Luttinger liquids [23–26], despite not fulfilling Eq. (1), have dynamical susceptibilities similar to Fermi liquids in the long wavelength and low frequency limit. Specifically, in Luttinger liquids such behaviour arises as a consequence of an emerging symmetry that ensures, asymptotically, the independent conservation of electron densities at the two different Fermi points, which could be a mere one dimensional feature, or hide a more fundamental link between Luttinger and Fermi liquids [27].

What is remarkable of our results is that a Fermi liquid behaviour emerges even in the worst case of a self-energy with a pole singularity at the chemical potential, which might look the furthest possible from a conventional Landau-Fermi liquid. We did not consider explicitly any model self-energy, but the extreme case of Eq. (16), where the main assumption (15) is verified, and thus a Landau-Fermi liquid description holds. However, given the generality of that assumption, it is well possible that purported non-Fermi liquid properties sometimes observed in correlated materials might be actually reconciled with the broader Fermi liquid scenario we have here uncovered.

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Appendix A: Luttinger theorem and quasiparticle equilibrium distribution

While previously we defined the energies $\epsilon_{\mathbf{k}}$ relative to the chemical potential μ , in this appendix we move back to absolute units, so that the Green's function

$$G(i\epsilon, \mathbf{k}) \rightarrow \frac{1}{i\epsilon - \epsilon_{\mathbf{k}} + \mu - \Sigma(i\epsilon, \mathbf{k})}, \quad (\text{A1})$$

depends on μ , as well as the self-energy does, though we shall not indicate such explicit dependence.

According to the Luttinger theorem [8] the (conserved) number of a -particles per site can be written as

$$\rho_a = \frac{1}{V} \sum_{\mathbf{k}} \int \frac{d\epsilon}{\pi} f(\epsilon) \frac{\partial \delta_a(\epsilon, \mathbf{k})}{\partial \epsilon}, \quad (\text{A2})$$

where, dropping the label a whenever not needed,

$$\begin{aligned} \delta(\epsilon, \mathbf{k}) &= \pi + \text{Im} \ln G_+(\epsilon, \mathbf{k}) \\ &= \tan^{-1} \frac{-\text{Im} G_+(\epsilon, \mathbf{k})}{-\text{Re} G_+(\epsilon, \mathbf{k})}, \end{aligned} \quad (\text{A3})$$

is the many-body phase shift. The topological content of Eq. (A2) has attracted considerable interest over the years [28–32]. By definition, $\delta(\epsilon, \mathbf{k}) \in [0, \pi]$, and vanishes at $\epsilon \rightarrow -\infty$, while reaches π at $\epsilon \rightarrow \infty$, consistently with each momentum state accommodating at most a single electron species. We remark that $\delta(\epsilon, \mathbf{k})$ is in general not monotonous, and may jump back and forth between 0 and π . The derivative of Eq. (A2) with respect to μ_a corresponds to the thermodynamic compressibility of the species a , and reads

$$\kappa_a \equiv \frac{\partial \rho_a}{\partial \mu_a} = \frac{1}{V} \sum_{\mathbf{k}} \int \frac{d\epsilon}{\pi} f(\epsilon) \frac{\partial^2 \delta_a(\epsilon, \mathbf{k})}{\partial \epsilon \partial \mu_a}. \quad (\text{A4})$$

Thermodynamic stability requires $\kappa_a \geq 0$. We may state that the a electron species is metallic if $\kappa_a > 0$, while is insulating if $\kappa_a = 0$. Note that at low temperatures the integral involves energies within a small window of order T around $\epsilon = 0$. Let us discuss the behaviour of $\delta(\epsilon, \mathbf{k})$ at vanishing temperatures in different cases.

1. Systems with a single-particle gap

Suppose that the single-particle DOS, $\mathcal{A}(\epsilon, \mathbf{k}) = -\pi \text{Im} G_+(\epsilon, \mathbf{k})$, vanishes for ϵ in a whole interval $X_{\text{ins}}(\mathbf{k})$,

$\forall \mathbf{k}$, which includes $\epsilon = 0$ and is definitely much wider than the temperature. By the Kramers-Krönig relations it follows that $-\text{Re} G_+(\epsilon, \mathbf{k})$ must cross zero with positive slope for $\epsilon = \epsilon_{\text{root}}(\mathbf{k}) - \mu \in X_{\text{ins}}(\mathbf{k})$. Correspondingly, the phase shift for $\epsilon \in X_{\text{ins}}(\mathbf{k})$ reads

$$\delta(\epsilon, \mathbf{k}) = \pi \theta(\epsilon_{\text{root}}(\mathbf{k}) - \mu - \epsilon). \quad (\text{A5})$$

We can envisage two different cases. If $\epsilon_{\text{root}}(\mathbf{k})$ is not pinned at the chemical potential, Eq. (A2) at zero temperature simplifies into

$$\rho = \frac{1}{V} \sum_{\mathbf{k}} \theta(\epsilon_{\text{root}}(\mathbf{k}) - \mu), \quad (\text{A6})$$

which implies that the total density corresponds to the volume that contains all \mathbf{k} such that $\epsilon_{\text{root}}(\mathbf{k}) > \mu$, and thus enclosed by the Luttinger surface (LS) defined through $\epsilon_{\text{root}}(\mathbf{k}) = \mu$ [28, 30], or, equivalently,

$$\text{Re} G_+(0, \mathbf{k}) = 0 \quad \forall \mathbf{k} \in \text{LS}. \quad (\text{A7})$$

This is, e.g., the case of a BCS superconductor, where $\epsilon_{\text{root}}(\mathbf{k}) = -\epsilon_{-\mathbf{k}} + 2\mu = -\epsilon_{\mathbf{k}} + 2\mu$, so that the LS is just the non-interacting Fermi surface, and the compressibility (A4) is equal to the non-interacting one.

It may instead happen that $\epsilon_{\text{root}}(\mathbf{k})$ is pinned at the chemical potential, i.e., $\epsilon_{\text{root}}(\mathbf{k}) = \mu$, $\forall \mathbf{k}$, so that $\delta(\epsilon, \mathbf{k})$ jumps from π to 0 right at $\epsilon = 0$. In this case, Eq. (A2) becomes, at zero temperature,

$$\begin{aligned} \rho_a &= \frac{1}{V} \sum_{\mathbf{k}} \int \frac{d\epsilon}{\pi} f(\epsilon - \mu) \frac{\partial \delta_a(\epsilon, \mathbf{k})}{\partial \epsilon} \\ &= -f(0) + \frac{1}{V} \sum_{\mathbf{k}} \int \frac{d\epsilon}{\pi} f(\epsilon - \mu) \frac{\partial \delta_a(\epsilon, \mathbf{k})}{\partial \epsilon} \\ &= -f(0) + \frac{1}{V} \sum_{\mathbf{k}} \frac{\delta_a(\mu^-, \mathbf{k})}{\pi} = \frac{1}{2}, \end{aligned} \quad (\text{A8})$$

where $f \dots$ is the Cauchy principal value of the integral, which implies that the state a is half-filled. Moreover, the compressibility (A4) vanishes, as expected for an insulator. Such circumstance in which $\epsilon_{\text{root}}(\mathbf{k}) = \mu$ defines, e.g., a Mott insulator, and entails a self energy with a pole at $\epsilon = 0$.

2. Systems with gapless single-particle excitations satisfying Eq. (15)

Gapless single-particle excitations correspond to a DOS $\mathcal{A}(\epsilon, \mathbf{k})$ smooth and finite in a finite interval around $\epsilon = 0$ and for momenta \mathbf{k} within regions of the Brillouin zone with non-zero measure. We further assume the validity of Eq. (15), which allowed us recovering the Landau-Fermi liquid theory. In this case it is straightforward to show [5] that the compressibility as defined in (A4) has the same expression as that of Eq. (45), and

thus is finite as expected for a metallic state. It is therefore tempting to make the association

$$n_{\epsilon\mathbf{k}a}^0 \stackrel{?}{=} f(\epsilon) \frac{1}{\pi} \frac{\partial\delta_a(\epsilon, \mathbf{k})}{\partial\epsilon}, \quad (\text{A9})$$

between the equilibrium distribution of ‘quasiparticles’ and the derivative of the many-body phase shift. However, such equivalence, though building a suggestive link to quantum impurity models [33], is dubious, since the right hand side of Eq. (A9) may be negative or even singular, as it is the case for the self-energy in Eq. (16). However, the Landau hypothesis of adiabatic evolution [1] only refers to low energy excitation, with no reference to the ground state. In other words, what really matters is the variation of the ‘quasiparticles’ distribution with respect to the equilibrium one, the latter playing no role in the theory. Therefore, the most correct association is not Eq. (A9) but rather

$$\begin{aligned} \delta(n_{\epsilon\mathbf{k}a}^0) &\equiv \delta\left(f(\epsilon) \frac{1}{\pi} \frac{\partial\delta_a(\epsilon, \mathbf{k})}{\partial\epsilon}\right) \\ &= \delta\left(-\frac{\partial f(\epsilon)}{\partial\epsilon} \frac{\delta_a(\epsilon, \mathbf{k})}{\pi}\right), \end{aligned} \quad (\text{A10})$$

where $\delta(\dots)$ denotes the variation with respect to internal or thermodynamic variables, and the last expression on the right hand side is obtained after integration by part of Eq. (A2).

Indeed, by the definition of $\delta(\epsilon, \mathbf{k})$ in Eq. (A3) it is straightforward to show that

$$\frac{\partial n_{\epsilon\mathbf{k}a}^0}{\partial\mathbf{k}} = \frac{1}{\pi} \left(-\frac{\partial f(\epsilon)}{\partial\epsilon}\right) \frac{\partial\delta_a(\epsilon, \mathbf{k})}{\partial\mathbf{k}}, \quad (\text{A11})$$

$$\frac{\partial n_{\epsilon\mathbf{k}a}^0}{\partial T} = \left(-\frac{\partial f(\epsilon)}{\partial\epsilon}\right) \mathcal{A}_{\text{qp}}(\epsilon, \mathbf{k}) \left[\frac{\epsilon}{T} - \frac{1}{V} \sum_{\mathbf{k}'a'} \int d\epsilon' \left(-\frac{\partial f(\epsilon')}{\partial\epsilon'}\right) \frac{\epsilon'}{T} A_{a,a'}^q(\epsilon\mathbf{k}, \epsilon'\mathbf{k}') \mathcal{A}_{\text{qp}}(\epsilon', \mathbf{k}')\right]. \quad (\text{A17})$$

If we instead consider the derivative with respect to T of the local equilibrium Wigner distribution (53), its expression greatly simplifies by making use of Eq. (50) that relates A^q to the Landau f -parameters, leading to

$$\frac{\partial \bar{n}_{\epsilon\mathbf{k}a}^0}{\partial T} = -\frac{\partial f(\epsilon)}{\partial\epsilon} \frac{\epsilon}{T} \mathcal{A}_{\text{qp}}(\epsilon, \mathbf{k}). \quad (\text{A18})$$

which is actually the Eq. (56) that we assumed to interpret Eq. (54) as a genuine Boltzmann kinetic equation. Another important derivative of the equilibrium distribution that is required to study the response to a temperature gradient is

$$\frac{\partial n_{\epsilon\mathbf{k}a}^0}{\partial T} = \left(-\frac{\partial f(\epsilon)}{\partial\epsilon}\right) \left[\frac{\epsilon}{T} \frac{\partial\delta_a(\epsilon, \mathbf{k})}{\partial\epsilon} + \frac{\partial\delta_a(\epsilon, \mathbf{k})}{\partial T}\right], \quad (\text{A12})$$

where

$$\frac{\partial\delta_a(\epsilon, \mathbf{k})}{\partial T} = \text{Im} \left[G_+(\epsilon, \mathbf{k}) \frac{\partial\Sigma_+(\epsilon, \mathbf{k})}{\partial T} \right]. \quad (\text{A13})$$

Inspection of the perturbative expansion of the self-energy in terms of skeleton diagrams leads to the following result

$$\begin{aligned} \frac{\partial\Sigma_a(\epsilon, \mathbf{k})}{\partial T} &= \frac{1}{V} \sum_{\mathbf{k}a'} \int d\epsilon' \frac{\partial f(\epsilon')}{\partial T} \\ &\Gamma_{a,a';a',a}^q(\epsilon\mathbf{k}, \epsilon'\mathbf{k}'; \epsilon'\mathbf{k}', \epsilon\mathbf{k}) \mathcal{A}(\epsilon', \mathbf{k}'), \end{aligned} \quad (\text{A14})$$

where Γ^q is the q -limit of the reducible vertex, which implies that

$$\begin{aligned} \frac{\partial\Sigma_{+a}(\epsilon, \mathbf{k})}{\partial T} &= Z_*(\epsilon, \mathbf{k})^{-1} \frac{1}{V} \sum_{\mathbf{k}'a'} \int d\epsilon' \frac{\partial f(\epsilon')}{\partial T} \\ &A_{a,b}^q(\epsilon\mathbf{k}, \epsilon'\mathbf{k}') \mathcal{A}_{\text{qp}}(\epsilon', \mathbf{k}'). \end{aligned} \quad (\text{A15})$$

In conclusion, through (A13), we find

$$\begin{aligned} \frac{\partial\delta_a(\epsilon, \mathbf{k})}{\partial T} &= -\mathcal{A}_{\text{qp}a}(\epsilon, \mathbf{k}) \frac{\pi}{V} \sum_{\mathbf{k}'a'} \int d\epsilon' \frac{\partial f(\epsilon')}{\partial T} \\ &A_{a,a'}^q(\epsilon\mathbf{k}, \epsilon'\mathbf{k}') \mathcal{A}_{\text{qp}}(\epsilon', \mathbf{k}'), \end{aligned} \quad (\text{A16})$$

and thus Eq. (A12) becomes

Appendix B: Formal derivation of Eq. (41)

The Landau-Fermi liquid expression of the linear response function Eq. (41) has been thoroughly derived for instance in Ref. [5]. However, for completeness, here we shall briefly sketch such derivation.

The Bethe-Salpeter equation for the particle scattering amplitude Γ , also denoted as two-particle reducible vertex, diagrammatically shown in the top panel of Fig. 1, and for the interacting density-vertex corresponding to

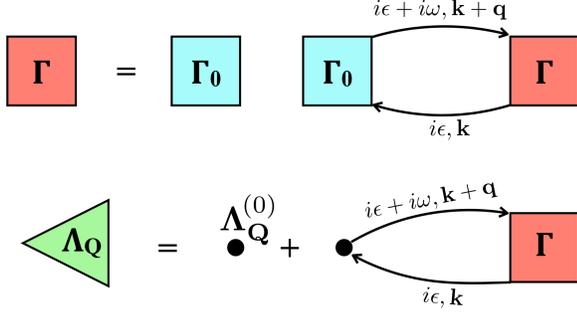


FIG. 1. Bethe-Salpeter equations for the two-particle reducible vertex Γ , top panel, and for the charge density vertex Λ , bottom panel. Γ_0 is the two-particle irreducible vertex, while $\Lambda^{(0)}$ is the non-interacting charge density vertex. The solid arrow lines are fully interacting Green's functions at the indicated frequency and momentum.

a generic observable Q , bottom panel of Fig. 1, can be shortly written as

$$\Gamma = \Gamma_0 + \Gamma_0 \odot R \odot \Gamma, \quad (\text{B1})$$

$$\Lambda_Q = \Lambda_Q^{(0)} + \Lambda_Q^{(0)} \odot R \odot \Gamma, \quad (\text{B2})$$

where Γ_0 is irreducible in the particle-hole channel Q with frequency and momentum transferred ω and \mathbf{q} , respectively, R is the kernel in (24), $\Lambda_Q^{(0)}$ the non-interacting density vertex and \odot stands for a summation over all common internal variables and quantum labels. We take the ω -limit of Eq. (B1),

$$\Gamma^\omega = \Gamma_0 + \Gamma_0 \odot R^\omega \odot \Gamma^\omega, \quad (\text{B3})$$

assuming that Γ_0 is analytic at $\omega = |\mathbf{q}| = 0$ [21], use it to express Γ_0 in terms of Γ^ω and R^ω , i.e.,

$$\begin{aligned} \Gamma_0 &= \Gamma^\omega \odot \left(1 + R^\omega \odot \Gamma^\omega\right)^{-1} \\ &= \left(1 + \Gamma^\omega \odot R\right)^{-1} \odot \Gamma^\omega, \end{aligned} \quad (\text{B4})$$

where, through Eq. (38),

$$R^\omega = \Delta^\omega + R_{\text{inc}} = R_{\text{inc}}, \quad (\text{B5})$$

and replace Γ_0 in Eq. (B1) with Eq. (B4). In this way we find that

$$\Gamma = \Gamma^\omega + \Gamma^\omega \odot \left(R - R^\omega\right) \odot \Gamma = \Gamma^\omega + \Gamma^\omega \odot \Delta \odot \Gamma, \quad (\text{B6})$$

with Δ defined in Eq. (36), which also implies that

$$\Gamma^\omega = \Gamma \odot \left(1 + \Delta \odot \Gamma\right)^{-1}. \quad (\text{B7})$$

Since $R^\omega + \Delta = R$, it also follows that

$$1 + R^\omega \odot \Gamma^\omega = \left(1 + R \odot \Gamma\right) \odot \left(1 + \Delta \odot \Gamma\right)^{-1}. \quad (\text{B8})$$

Similarly, we can take the ω -limit of (B2),

$$\Lambda_Q^\omega = \Lambda_Q^{(0)} + \Lambda_Q^{(0)} \odot R^\omega \odot \Gamma^\omega, \quad (\text{B9})$$

and solve it to find $\Lambda_Q^{(0)}$ as function of all other quantities, which, through Eq. (B8), is

$$\begin{aligned} \Lambda_Q^{(0)} &= \Lambda_Q^\omega \odot \left(1 + R^\omega \odot \Gamma^\omega\right)^{-1} \\ &= \Lambda_Q^\omega \odot \left(1 + \Delta \odot \Gamma\right) \odot \left(1 + R \odot \Gamma\right)^{-1} \\ &= \left(1 + \Gamma \odot R\right)^{-1} \odot \left(1 + \Gamma \odot \Delta\right) \odot \Lambda_Q^\omega. \end{aligned} \quad (\text{B10})$$

We then replace such expression back in Eq. (B2), and readily find that

$$\Lambda_Q = \Lambda_Q^\omega + \Lambda_Q^\omega \odot \Delta \odot \Gamma. \quad (\text{B11})$$

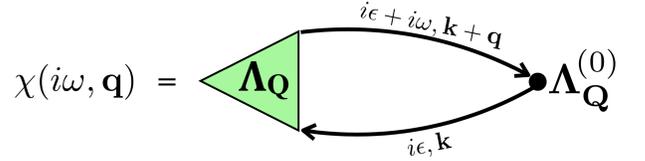


FIG. 2. Diagrammatic representation of the linear response function of the density corresponding to the observable Q . The interacting density vertex, Λ_Q , is related to the non-interacting one, $\Lambda_Q^{(0)}$, through the Bethe-Salpeter equation shown in the bottom panel of Fig. 1.

The linear response function can be shortly written, see Fig. 2, as

$$\chi_Q = \text{Tr} \left(\Lambda_Q \odot R \odot \Lambda_Q^{(0)} \right), \quad (\text{B12})$$

and, using Eq. (B11) and the last equation in (B10), is also equivalent to

$$\begin{aligned} \chi_Q &= \chi_Q^\omega + \text{Tr} \left(\Lambda_Q^\omega \odot \Delta \odot \Lambda_Q^\omega \right) \\ &\quad + \text{Tr} \left(\Lambda_Q^\omega \odot \Delta \odot \Gamma \odot \Delta \odot \Lambda_Q^\omega \right), \end{aligned} \quad (\text{B13})$$

where

$$\chi_Q^\omega = \text{Tr} \left(\Lambda_Q^\omega \odot R^\omega \odot \Lambda_Q^{(0)} \right), \quad (\text{B14})$$

is the ω -limit of the response function.

The Eq. (B13) greatly simplifies if Q is a conserved quantity, whose conservation implies that $\chi_Q^\omega = 0$, and that the Ward-Takahashi identity

$$\Lambda_Q^\omega = \Lambda_Q^{(0)} Z_*^{-1}, \quad (\text{B15})$$

holds. Next, we write

$$\Delta \equiv Z_*^2 \Delta_0, \quad (\text{B16})$$

where, through Eq. (36),

$$\Delta_0(\epsilon + \omega, \mathbf{k} + \mathbf{q}; \epsilon, \mathbf{k}) = -\frac{\partial f(\epsilon)}{\partial \epsilon} \mathcal{A}_{\text{qp}}(\epsilon, \mathbf{k}) \frac{\mathbf{v}_*(\epsilon, \mathbf{k}) \cdot \mathbf{q}}{\omega - \mathbf{v}_*(\epsilon, \mathbf{k}) \cdot \mathbf{q}}, \quad (\text{B17})$$

and define

$$A = Z_* \Gamma Z_*, \quad (\text{B18})$$

so that for a conserved quantity Eq. (B13) reduces to

$$\chi_Q = \text{Tr} \left(\Lambda_Q^{(0)} \odot \Delta_0 \odot \Lambda_Q^{(0)} \right) + \text{Tr} \left(\Lambda_Q^{(0)} \odot \Delta_0 \odot A \odot \Delta_0 \odot \Lambda_Q^{(0)} \right), \quad (\text{B19})$$

which is just Eq. (41) in the main text.

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