



Towards the determination of the dimension of the critical surface in asymptotically safe gravity



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ABSTRACT

We compute the beta functions of Higher Derivative Gravity within the Functional Renormalization Group approach, going beyond previously studied approximations. We find that the presence of a nontrivial Newtonian coupling induces, in addition to the free fixed point of the one-loop approximation, also two nontrivial fixed points, of which one has the right signs to be free from tachyons. Our results are consistent with earlier suggestions that the dimension of the critical surface for pure gravity is three.

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1. Introduction

Higher Derivative Gravity (HDG) is the theory of gravity based on the metric as the carrier of degrees of freedom, with an action containing terms of order zero, one and two in the curvature. It contains both dimensionful couplings (the cosmological and Newton constant) and dimensionless ones (the coefficients of the HD terms). When treated perturbatively in the latter, it is renormalizable [1], but not unitary. Following some earlier attempts [2,3], its one-loop beta functions were correctly derived for the first time in [4]; for more details and generalizations, see [5,6]. Depending on the signs of the couplings, the theory can be asymptotically free, but it has ghosts and/or tachyons. There has been recently a revival of interest in this theory, and proposals to get around its problems in various ways [7–17].

In the asymptotic safety approach to quantum gravity, one tries to construct a continuum limit around an interacting fixed point (FP) [18]. The main tool to investigate the gravitational renormalization group has been the Functional Renormalization Group Equation (FRGE), as applied for the first time to gravity by Martin Reuter [19]. It defines a flow on the theory space consisting of all diffeomorphism invariant functionals of the metric. One expects that at an interacting gravitational FP, infinitely many gravitational couplings will be nonzero. In spite of this complication, much evidence for the existence of such a FP has been collected so far [20,21].

In the context of asymptotic safety, when one uses the FRGE, there is never the need to postulate the form of the bare action to be used in the path integral. Instead, one directly calculates the flow of the effective action as a function of an external “coarse-graining” scale, or IR cutoff, k . In this context, the action of HDG can be used as an ansatz for the running effective action. We will call this the “HDG truncation”. It tracks the flow of the theory in a five-dimensional “theory space” parametrized by the couplings: \mathcal{V} , Z_N , λ , ξ and ρ , defined below. The beta functions of HDG have been studied from this point of view in several papers. They were obtained in a one-loop approximation to the FRGE in [22–25]. In these calculations, the beta functions of the HD couplings are asymptotically free, in agreement with the old perturbative results, but the flow of the dimensionful couplings looks very similar to the one of the Einstein-Hilbert truncation, and exhibits a nontrivial FP for the cosmological and Newton constant. To go beyond one loop, one has to keep terms involving the beta functions in the r.h.s. of the flow equation, and then solve these algebraic equations for the beta functions. We highlight this process in Section 3.1. This produces non-linearities that amount to resummations of infinitely many loop diagrams. This has been calculated in [26,27] on a generic Einstein manifold, and a fully interacting FP was found, but these calculations were limited to one or two, out of the three HD couplings. This may seem to be sufficient, since one of the three couplings is the coefficient of the Euler term, that does not contribute to the local dynamics. Unfortunately, as we shall see in Sect.2.1, on an Einstein manifold one computes the beta function of certain linear combinations of the three couplings, and it is actually impossible to identify the beta function of the two dynamically interesting ones: there is an unknown mixing with the

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beta function of the Euler term. To compute the beta functions of all the independent couplings is the main task of this paper.

The main motivation for this is the determination of the dimension of the UV critical surface. There is evidence from the $f(R)$ truncations that the scaling exponents at the nontrivial fixed point are not too different from the classical ones, so that couplings with positive mass dimension remain relevant and couplings with negative mass dimension remain irrelevant FP [28–32]. The marginal coupling of the R^2 term becomes relevant, so altogether, in this truncation, the dimension of the critical surface seems to be three. An attempt to include different tensor structures has been made in [31], where actions of the form $f_1(R_{\mu\nu}R^{\mu\nu}) + Rf_2(R_{\mu\nu}R^{\mu\nu})$ are studied, leading to the same conclusion. A limitation of these calculations is that, on a spherical background, it is not possible to properly disentangle independent couplings with the same number of curvatures. The case of Ricci tensor squared and scalar curvature squared actions on an Einstein manifold, has already been cited above [27]. While more general than spheres, Einstein manifolds are still not general enough to distinguish all invariants. With this limitation, it was found again that the dimension of the critical surface is three. This suggests that some linear combination of the HD couplings may be an irrelevant operator. It seems possible, and even likely, that the dimension of the critical surface in pure gravity is determined entirely by the fate of the HD couplings, since they are not expected to remain marginal at an interacting FP.¹ We find that of the three dimensionless couplings, one becomes relevant, one irrelevant and one – the coefficient of the Euler term – remains marginal. The beta function of the Euler term is related to the a -function. The a -theorem states that when two fixed points are joined by an RG trajectory, the value of a at the IR fixed point is lower than the one at the UV fixed point. We find some evidence that this may hold also in gravity.

In the present paper we try to shed some light on these issues by computing the beta functions of all the HD couplings beyond the one-loop approximation, taking the anomalous dimensions into account. We shall do this by using the “Universal RG Machine” to compute the r.h.s. of the FRGE on an arbitrary background. This is a technique based on non-diagonal heat kernel coefficients that can be used to evaluate functional traces involving covariant derivatives acting on a function of a Laplacian. The Universal RG Machine has been introduced, and applied to the Einstein-Hilbert truncation, in [34]. Later it was used to calculate the one-loop beta functions in HDG [35]. Technical details are given in [36]. Here we bring that program one step forward by evaluating the full beta functions of HDG, including the anomalous dimensions. The main steps of the calculation are outlined in Sect.2, and in Sect.3 we describe the results. We find three fixed points, of which one has vanishing higher derivative couplings, while the others are fully interacting. In principle, any of these could be a viable UV fixed point. To have a viable theory, one would also have to prove unitarity. For the first of these fixed points, one could apply the arguments developed in perturbation theory [7–17]. For the remaining ones, the issue is more involved and will require a detailed study of the two point function.

2. Beta functions

2.1. Why Einstein backgrounds are not enough

Let us momentarily concentrate on the HD terms, that we can write as $\mathcal{L}_{HD} = \alpha R^2 + \beta R^2_{\mu\nu} + \gamma R^2_{\mu\nu\rho\lambda}$. Due to the fact that the

Gauss-Bonnet combination $E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$ is topological, one of these couplings is uninteresting as far as local dynamics is concerned. It is therefore more meaningful to write the Lagrangian as

$$\mathcal{L}_{HD} = \frac{1}{2\lambda}C^2 + \frac{1}{\xi}R^2 - \frac{1}{\rho}E \quad (2.1)$$

where

$$\frac{1}{\xi} = \frac{3\alpha + \beta + \gamma}{3}, \quad \frac{1}{2\lambda} = \frac{\beta + 4\gamma}{2}, \quad -\frac{1}{\rho} = -\frac{\beta + 2\gamma}{2}, \quad (2.2)$$

and $C^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$ is the square of the Weyl tensor. We are mainly interested in the beta functions of λ and ξ . Calculations are simpler on an Einstein background. In this case $E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and $C^2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R^2/6$, so

$$\mathcal{L}_{HD} = \left(\frac{1}{\xi} - \frac{1}{12\lambda}\right)R^2 + \left(\frac{1}{2\lambda} - \frac{1}{\rho}\right)E. \quad (2.3)$$

This implies that if we expand the r.h.s. of the functional RG equation on an Einstein background, and we interpret the coefficients of R^2 and $E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ as beta functions, we can read off the beta functions of two combinations of λ , ξ , ρ but we are unable to unambiguously identify β_λ and β_ξ . To do this, we need an additional independent equation, that in turn requires a more general background. This is what we do in this paper.

All calculations will be based on the Euclidean action

$$S = \int d^4x \sqrt{-g} [\mathcal{V} - Z_N R + \mathcal{L}_{HD}], \quad (2.4)$$

where $Z_N = \frac{1}{16\pi G}$, G being Newton’s constant, $\mathcal{V} = 2\Lambda Z_N$ and Λ is the cosmological constant. Sometimes we shall use the combinations

$$\omega \equiv -\frac{3\lambda}{\xi}, \quad \theta \equiv \frac{\lambda}{\rho}. \quad (2.5)$$

2.2. Remark on the topological term

Before embarking in calculations, we can make a general remark on the Gauss-Bonnet term, that actually holds *independently of the truncation*. Due to the topological character of the term E , its coefficient $1/\rho$ does not appear in the Hessian and therefore does not appear in the r.h.s. of the flow equation. Thus the beta function of ρ must have the form

$$\beta_\rho = -\frac{1}{16\pi^2}a\rho^2, \quad (2.6)$$

where a is a function of all the other couplings, but not of ρ itself. In the search of a fixed point one can solve first the equations of all the other couplings, which are also independent of ρ . When these fixed point values are inserted in (2.6), a becomes just a number. The UV behavior of ρ is determined by the value of this number. If $a = 0$, ρ could reach any value in the UV. If $a > 0$ ($a < 0$), when all other couplings are very close to a fixed point, it will run logarithmically to zero from above (below).

2.3. Expansion and gauge fixing

We split the metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is an arbitrary background. For details of the expansion of the action, we refer to [24]. The gauge-fixing and ghost action can be written

$$\begin{aligned} \mathcal{L}_{GF+FP}/\sqrt{\bar{g}} = & -\frac{1}{2a}\chi_\mu Y^{\mu\nu}\chi_\nu + iZ_{gh}\bar{c}^\mu \Delta_{\mu\nu}^{(gh)}c^\nu \\ & + \frac{1}{2}Z_Y b_\mu Y^{\mu\nu}b_\nu + Z_Y \bar{\zeta}_\mu Y^{\mu\nu}\zeta_\mu, \end{aligned} \quad (2.7)$$

¹ So far the only indication that things could be more complicated comes from work in progress by Kluth and Litim on actions of the form $f_1(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) + Rf_2(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$, where a term cubic in curvature seems to become relevant [33].

where \bar{c}_μ, c_μ are complex ghosts and b_μ is a real commuting field, $\bar{\zeta}_\mu, \zeta_\mu$ are complex anticommuting fields, and

$$\begin{aligned}\chi_\mu &\equiv \bar{\nabla}^\lambda h_{\lambda\mu} + b\bar{\nabla}_\mu h, \\ \Delta_{\mu\nu}^{(gh)} &\equiv g_{\mu\nu}\bar{\nabla}^2 + (2b+1)\bar{\nabla}_\mu\bar{\nabla}_\nu + \bar{R}_{\mu\nu}, \\ Y_{\mu\nu} &\equiv \bar{g}_{\mu\nu}\bar{\nabla}^2 + c\bar{\nabla}_\mu\bar{\nabla}_\nu - f\bar{\nabla}_\nu\bar{\nabla}_\mu,\end{aligned}\quad (2.8)$$

where a, b, c and f are gauge parameters. There is some freedom in how we choose the wave function renormalisations Z_{gh} and Z_Y since they can be rescaled while keeping $Z_{gh}^2 Z_Y = 1/a$ fixed without affecting the path integral. In our calculations we fix

$$Z_{gh} = 1, \quad Z_Y = 1/a \quad (2.9)$$

We make the usual gauge choice

$$a = \lambda, \quad b = -\frac{1+4\omega}{4+4\omega}, \quad c = \frac{2}{3}(1+\omega), \quad f = 1, \quad (2.10)$$

leading to a minimal fourth order operator for the fluctuations. The operators in (2.7) are then

$$\begin{aligned}\Delta_{\mu\nu}^{(gh)} &\equiv g_{\mu\nu}\bar{\nabla}^2 - \sigma_{gh}\bar{\nabla}_\mu\bar{\nabla}_\nu + \bar{R}_{\mu\nu}, \\ Y_{\mu\nu} &\equiv \bar{g}_{\mu\nu}\bar{\nabla}^2 - \sigma_Y\bar{\nabla}_\mu\bar{\nabla}_\nu - R_{\mu\nu},\end{aligned}\quad (2.11)$$

with

$$\begin{aligned}\sigma_{gh} &= -1 - 2b = -\frac{1-2\omega}{2(1+\omega)}; \\ \sigma_Y &= 1 - 2\frac{\gamma - \alpha}{\beta + 4\gamma} = \frac{1-2\omega}{3}.\end{aligned}\quad (2.12)$$

We note that the cancellation between unphysical degrees of freedom becomes exact in the ‘‘Landau gauge’’ limit $a \rightarrow 0$, which happens to be satisfied in the asymptotically free regime.

Then, the quadratic terms in the action can be written in the form [24]

$$\mathcal{L}^{(2)} = h_{\mu\nu} K^{\mu\nu\rho\sigma} \mathcal{O}_{\rho\sigma}{}^{\alpha\beta} h_{\alpha\beta}, \quad (2.13)$$

where the operator \mathcal{O} is

$$\mathcal{O} = \Delta^2 + V_{\rho\lambda}\bar{\nabla}^\rho\bar{\nabla}^\lambda + U, \quad (2.14)$$

with $\Delta = -\bar{\nabla}^2$, $U = K^{-1}W$ and we write

$$\begin{aligned}K &= \frac{\beta + 4\gamma}{4} \left(\mathbb{I} + \frac{4\alpha + \beta}{\gamma - \alpha} \mathbb{P} \right), \\ K^{-1} &= \frac{4}{\beta + 4\gamma} \left(\mathbb{I} - \frac{4\alpha + \beta}{3\alpha + \beta + \gamma} \mathbb{P} \right),\end{aligned}\quad (2.15)$$

where \mathbb{I} is the identity in the space of symmetric tensor and \mathbb{P} is a projector

$$\begin{aligned}\mathbb{I}_{\mu\nu,\alpha\beta} &\equiv \delta_{\mu\nu,\alpha\beta} = \frac{1}{2}(\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta} + \bar{g}_{\mu\beta}\bar{g}_{\nu\alpha}), \\ \mathbb{P}^{\mu\nu}{}_{\rho\sigma} &\equiv P^{\mu\nu}{}_{\rho\sigma} = \frac{1}{4}\bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}.\end{aligned}\quad (2.16)$$

The coefficients $V_{\rho\lambda}$ and U are functions of the curvatures, \mathcal{V} and Z_N , for whose form we refer again to [24].

The ‘‘beta functional’’ of the theory is the sum of three contributions coming from gravitons, ghosts and the new ghost b_μ :

$$\dot{\Gamma}_k = T_g + T_{gh} + T_Y. \quad (2.17)$$

In order to write these terms more explicitly, we have to choose a cutoff for each of them. For a one-loop calculation, where the couplings in the r.h.s. of the equation are treated as fixed, it was most convenient to think of the cutoff as a function of the whole

operator \mathcal{O} , Δ_{gh} or Y respectively (so-called type III cutoff). In this paper we will not ignore the running of the couplings that may be present in the cutoff, so it is best to minimize their presence. This is achieved by choosing the cutoff to be a function of Δ only (so-called type I cutoff). The one-loop calculation with this cutoff has been done before in [35].

2.4. Graviton contribution

We choose the graviton cutoff to have the form $\mathcal{R} = KR_k(\Delta^2)$, where $R_k(\Delta^2) = (k^4 - \Delta^2)\theta(k^4 - \Delta^2)$ and we define as usual $P_k(\Delta^2) = \Delta^2 + R_k(\Delta^2) = k^4\theta(k^4 - \Delta^2)$. Note that it is convenient to view R_k as a function of Δ^2 , although of course one could also view it as a function of Δ . Then, writing the kinetic operator as $\Delta^2 + V + U$, the graviton contribution to the FRGE is

$$T_g = \frac{1}{2}\text{Tr} \frac{\partial_t[KR_k(\Delta^2)]}{K[\mathcal{O} + R_k(\Delta^2)]} = \frac{1}{2}\text{Tr} \frac{\partial_t R_k(\Delta^2) + \eta_K R_k(\Delta^2)}{P_k(\Delta^2) + V + U}, \quad (2.18)$$

where we defined

$$\eta_K = K^{-1} \frac{dK}{dt}. \quad (2.19)$$

Note that η_K is a tensor. From (2.15) we find

$$\eta_K = \eta_1 \mathbb{I} + \eta_P \mathbb{P}, \quad (2.20)$$

where

$$\eta_1 = -\frac{\dot{\lambda}}{\lambda}, \quad \eta_P = -\frac{\xi\dot{\lambda} - \lambda\dot{\xi}}{\lambda(3\lambda - \xi)}. \quad (2.21)$$

We divide V and U into various terms: $V = V_0 + V_1$ and $U = U_0 + U_1 + U_2$, where the subscript counts the power of curvature, and the remaining dimension is carried either by \mathcal{V} or Z_N :

$$\begin{aligned}V_0 &\sim Z_N \nabla \nabla; & V_1 &\sim R \nabla \nabla; & U_0 &\sim \mathcal{V}; \\ U_1 &\sim Z_N R; & U_2 &\sim R^2.\end{aligned}$$

We now have to decide how to expand the fraction in (2.18). Since we want to compute the beta functions of all the couplings in (2.4), we need to expand to second order in curvatures. It would be natural to assume that $\sqrt{\mathcal{V}} \sim Z_N \sim R$ (which implies also $\Lambda \sim R$), but such an expansion would miss important features, as we shall discuss below. It is possible without too much effort to keep the full dependence on \mathcal{V} , and we shall do so. We will therefore not expand in U_0 . It is much harder to keep all dependence on Z_N , therefore we will expand in V_0, V_1, U_1 and U_2 , to first order in Z_N/k^2 , independently of curvatures.² This corresponds to considering a trans-Planckian regime. If one considers the Einstein-Hilbert part of the action, it correspond to a *strong gravity* expansion. See [37] for a recent discussion. Keeping only terms up to linear order in Z_N we thus have to evaluate:

$$\begin{aligned}T^{\text{grav}} &= \frac{1}{2}\text{Tr} \left[\frac{\partial_t R_k(\Delta) + \eta_K R_k(\Delta)}{P_k(\Delta) + U_0} \right. \\ &\quad \times \left(1 - \frac{1}{P_k(\Delta) + U_0} (V_0 + V_1 + U_1 + U_2) \right. \\ &\quad \left. + \frac{1}{P_k(\Delta) + U_0} V_0 \frac{1}{P_k(\Delta) + U_0} V_1 \right. \\ &\quad \left. + \frac{1}{P_k(\Delta) + U_0} V_1 \frac{1}{P_k(\Delta) + U_0} V_0 \right.\end{aligned}$$

² Note that we wrote $\mathcal{V} = 2Z_N\Lambda$ and treated Λ as an independent coupling, the expansion in Z_N would also entail an expansion in Λ . This is not what we do here.

$$\begin{aligned} & + \frac{2V_0U_2}{(P_k(\Delta) + U_0)^2} + \frac{V_1^2}{(P_k(\Delta) + U_0)^2} + \frac{2V_1U_1}{(P_k(\Delta) + U_0)^2} \\ & + \frac{3V_0V_1^2}{(P_k(\Delta) + U_0)^3} \Big]. \end{aligned} \quad (2.22)$$

In the last two lines we have written the terms only in a schematic way, without paying attention to their order: to be precise one has to write out several terms where the projectors \mathbb{P} appear in different positions. (For details, we refer the reader to the ancillary file on the arXiv page.)

2.5. Ghost contribution

To some extent, it is possible to treat Δ_{gh} and Y together. Both operators are non-minimal, and of the form $\Delta\delta_\mu^\nu + \sigma\bar{\nabla}_\mu\bar{\nabla}^\nu + B_\mu^\nu$ (note the overall sign is reversed), where σ is a constant defined in (2.12) and $B_\mu^\nu = s\bar{R}_\mu^\nu$, where $s = -1$ for Δ_{gh} and $s = 1$ for Y . In the standard one-loop calculations, one can use the known heat kernel coefficients for this type of operators. In contrast to [22–24] and coherently with the treatment of gravitons, we use a type I cutoff also for the ghosts. This type of cutoff for ghosts had been used before in [35]. The novelty of our calculation is that we also take into account the contributions due to the anomalous dimensions

$$\eta_{gh} = 0, \quad \eta_Y = -\beta_\lambda/\lambda. \quad (2.23)$$

The type I cutoff has the form³

$$\mathcal{R}_{k\nu}^\mu = Z\delta_\mu^\nu R_k(\Delta), \quad (2.24)$$

where Z is given by (2.9), (2.10). Adding the cutoff, the kinetic operator (aside from the factor Z) becomes $P_k(\Delta)\delta_\mu^\nu + \sigma\bar{\nabla}_\mu\bar{\nabla}^\nu + B_\mu^\nu$. In the flow equation one needs the inverse of this operator. We refer to [35] for some technical details. The evaluation of the traces to second order in curvatures is rather laborious. In the end we arrive at the following

$$\begin{aligned} T_{gh} = & -\frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \left\{ \left[3 - \frac{2}{\sigma_{gh}} - \frac{2}{\sigma_{gh}^2} \log(1 - \sigma_{gh}) \right] k^4 \right. \\ & - \frac{1}{12\sigma_{gh}^2} \left[\frac{3\sigma_{gh}(2 + \sigma_{gh}(7 - 5\sigma_{gh}))}{\sigma_{gh} - 1} \right. \\ & - 2(3 - 2\sigma_{gh}) \log(1 - \sigma_{gh}) \Big] k^2 \bar{R} \\ & - \frac{11}{90} \bar{R}^2_{\mu\nu\rho\lambda} + \frac{43 - 2\sigma_{gh}(13 + \sigma_{gh})}{45(1 - \sigma_{gh})^2} \bar{R}^2_{\mu\nu} \\ & \left. + \left[\frac{5}{18} + \frac{1}{6(1 - \sigma_{gh})^2} \right] \bar{R}^2 \right\}. \end{aligned} \quad (2.25)$$

Note the appearance of $\log(1 - \sigma_{gh}) = -\log(2(1 + \omega)/3)$, which forces us to consider only the domain $\omega > -1$. For Y :

$$\begin{aligned} T_Y = & -\frac{1}{2} \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \left\{ \left[3 - \frac{2}{\sigma_Y} - \frac{2}{\sigma_Y^2} \log(1 - \sigma_Y) \right. \right. \\ & + \eta_Y \left(\frac{2 - \sigma_Y + \sigma_Y^2}{2\sigma_Y^2} + \frac{(1 - \sigma_Y)}{\sigma_Y^3} \log(1 - \sigma_Y) \right) \Big] k^4 \\ & \left. + \left[-\frac{2 + \sigma_Y}{4\sigma_Y} - \frac{3 + 2\sigma_Y}{6\sigma_Y^2} \log(1 - \sigma_Y) \right. \right. \end{aligned}$$

$$\begin{aligned} & + \eta_Y \left(\frac{6 - \sigma_Y}{12\sigma_Y^2} + \frac{3 - 2\sigma_Y - \sigma_Y^2}{6\sigma_Y^3} \log(1 - \sigma_Y) \right) \Big] k^2 \bar{R} \\ & - \frac{11}{90} \left(1 + \frac{\eta_Y}{2} \right) \bar{R}^2_{\mu\nu\rho\lambda} + \left[\frac{43}{45} \right. \\ & + \eta_Y \left(\frac{20 - 20\sigma_Y - 39\sigma_Y^2 + 29\sigma_Y^3}{120\sigma_Y^2(\sigma_Y - 1)} \right. \\ & - \frac{1 - \sigma_Y - 2\sigma_Y^2}{12\sigma_Y^3} \log(1 - \sigma_Y) \Big] \bar{R}^2_{\mu\nu} \\ & - \left[\frac{2}{9} + \eta_Y \left(\frac{4 + \sigma_Y^2 + \sigma_Y^3 - 3\sigma_Y^4}{48(-1 + \sigma_Y)\sigma_Y^2} \right. \right. \\ & \left. \left. - \frac{2 - \sigma_Y - 2\sigma_Y^2}{24\sigma_Y^3} \log(1 - \sigma_Y) \right) \right] \bar{R}^2 \Big\}. \end{aligned} \quad (2.26)$$

Both agree with [35] if we put $\eta_Y = 0$.

3. Results

3.1. Beta functions

For the study of the flow, the dimensionful couplings \mathcal{V} and Z_N have to be replaced by their dimensionless counterparts $\tilde{\mathcal{V}} = \mathcal{V}/k^4$ and $\tilde{Z}_N = Z_N/k^2$, or the related quantities $\tilde{G} = Gk^2$, $\tilde{\Lambda} = \Lambda/k^2$. The beta functions are too complicated to be written here (they are given in a Mathematica notebook [38]), but they simplify in two cases. Expanding for small λ we obtain the universal one-loop beta functions

$$\beta_\lambda = -\frac{133\lambda^2}{160\pi^2} + O(\lambda^3) \quad (3.1)$$

$$\beta_\omega = -\frac{\lambda(200\omega^2 + 1098\omega + 25)}{960\pi^2} + O(\lambda^2) \quad (3.2)$$

$$\beta_\theta = \frac{7(56 - 171\theta)}{1440\pi^2} \lambda + O(\lambda^2) \quad (3.3)$$

while the non-universal beta functions for \tilde{G} and $\tilde{\Lambda}$ agree with those found in the one-loop calculation [35] at $\lambda = 0$. Explicitly they are given by

$$\beta_{\tilde{G}} = 2\tilde{G} + \tilde{G}^2 \left[-\frac{c_1}{72\pi(1 - 2\omega)} + \frac{c_2 \log(\frac{2(1+\omega)}{3})}{12\pi(1 - 2\omega)^2} \right] + O(\lambda) \quad (3.4)$$

$$\beta_{\tilde{\Lambda}} = -2\tilde{\Lambda} + \frac{\tilde{G}}{72\pi} \left[\frac{c_3 + \tilde{\Lambda}c_4}{1 - 2\omega} + \frac{6(c_5 + \tilde{\Lambda}c_6) \log(\frac{2(1+\omega)}{3})}{(1 - 2\omega)^2} \right] + O(\lambda) \quad (3.5)$$

with the coefficients $c_1 = 35 - 2\omega(109 + 176\omega)$, $c_2 = 65 + 4\omega(7 + 2\omega)$, $c_3 = 162 - 540\omega$, $c_4 = -35 + 218\omega + 352\omega^2$, $c_5 = 6 - 96\omega - 48\omega^2$, $c_6 = 65 + 28\omega + 8\omega^2$.

Our calculation differs from one-loop calculations in that we take into account the anomalous dimensions. For example, we see η_Y appearing explicitly in (2.26), which gives contributions to the beta functions of all the couplings. Equation (2.23) tells us that η_Y is proportional to β_λ . Thus, comparing the terms proportional to C^2 on both sides of the FRGE, we obtain a relation of the form $\beta_i = B_i + C_{ij}\beta_j$. At one loop one just keeps $\beta_i = B_i$. Solving the algebraic equations gives beta functions that contain contributions with arbitrarily high loop order.

However, from the definitions, the anomalous dimensions at a fixed point are known a priori to be

$$\eta_1 = 0; \quad \eta_p = 0; \quad \eta_Y = 0. \quad (3.6)$$

³ We observe that the calculation of the ghost contributions is considerably simpler with a so-called type-II cutoff $\mathcal{R}_{k\nu}^\mu = Z\delta_\mu^\nu R_k(\Delta + B)$. The use of this alternative scheme for the ghosts would lead to only small quantitative differences in the final results for the fixed points and we shall not discuss this in detail.

Table 1
Fixed points in the approximation $\tilde{\mathcal{V}} = 0$.

	λ_*	ξ_*	ρ_*	ω_*	\tilde{Z}_{N*}	\tilde{G}_*
FP ₁	0	0	0	-0.02286	0.00833	2.388
FP ₂	29.26	-220.2	0	0.4040	0.01318	1.509
FP ₃	52.61	1672	0	-0.0944	0.00761	2.614

So, in the search of fixed points, one can use simplified beta functions where these values are used: the full expressions for the anomalous dimensions are only needed when one calculates the scaling exponents. It is easy to see that if we had assumed that all terms in V and U are of the same order, namely $\sqrt{\tilde{\mathcal{V}}} \sim Z_N \sim R$, then all the terms containing V_0 and U_1 would not contribute to the beta functions of λ , ξ and ρ . Therefore, these beta functions would not contain Z_N and would be exactly the same as in [35]. This is why it is important to keep the expansion in Z_N separate from the expansion in R .⁴

Even the simplified beta functions with (3.6) are too complicated to be reported in detail. However, we shall see a posteriori that $\tilde{\mathcal{V}}$ is very small at fixed points. If we put $\tilde{\mathcal{V}} = 0$, the equations for the remaining variables become simple enough:

$$\beta_\lambda = -\frac{133}{160\pi^2}\lambda^2 + \tilde{Z}_N\lambda^3\frac{251\xi - 58\lambda}{120\pi^2\xi} \quad (3.7)$$

$$\beta_\xi = -\frac{5(72\lambda^2 - 36\lambda\xi + \xi^2)}{576\pi^2} + \tilde{Z}_N\frac{9720\lambda^3 - 1980\lambda^2\xi + 489\lambda\xi^2 - 14\xi^3}{6480\pi^2} \quad (3.8)$$

$$\beta_\rho = -\frac{49}{180\pi^2}\rho^2 + \tilde{Z}_N\lambda\rho^2\frac{233\xi - 58\lambda}{240\pi^2\xi} \quad (3.9)$$

$$\beta_{\tilde{Z}_N} = \left(-2 + \frac{(30\lambda - \xi)(4\lambda + \xi)}{192\pi^2\xi}\right)\tilde{Z}_N + \frac{-3168\lambda^2 + 654\lambda\xi + 35\xi^2}{1152\pi^2\xi(6\lambda + \xi)} - \frac{72\lambda^2 - 84\lambda\xi + 65\xi^2}{192\pi^2(6\lambda + \xi)^2} \log\left(\frac{2}{3} - \frac{2\lambda}{\xi}\right). \quad (3.10)$$

3.2. Fixed points

Now we recall that already in the one-loop calculation, the beta functions of \tilde{Z}_N (and also $\tilde{\mathcal{V}}$) have a nontrivial fixed point. This nonzero value of Z_N enters in the beta functions of (3.7)-(3.9) in such a way that besides the asymptotically free fixed point, there are now two (and only two) new ones. Their coordinates are given in Table 1.

The first fixed point is found also in the one-loop approximation, and it is a non-trivial fact that it persists also when \tilde{Z}_N is present in the beta functions of λ and ξ .⁵ Note that in the one-loop approximation there is also another fixed point with $\lambda = \xi = 0$, $\omega = -5.467$, which however is excluded by our condition $\omega > -1$ (otherwise it gives a complex \tilde{Z}_N). The remaining two fixed points are “fully interacting”. It is worth noting that if we treat \tilde{Z}_N as an external parameter in the beta functions of λ and ξ , we find that λ_* and ξ_* go to infinity for $\tilde{Z}_N \rightarrow 0$.⁶

⁴ It would obviously be even better not to expand in Z_N at all, but this would be technically much more challenging.

⁵ Actually, this fixed point is best studied using the variable ω instead of ξ . It corresponds to letting λ and ξ go to zero with a particular ratio, and is different from setting e.g. first $\lambda = 0$ and then $\xi = 0$.

⁶ and to zero for $\tilde{Z}_N \rightarrow \infty$, but this is outside the domain of our approximation.

We then come to the solution of the full flow equations, where we take into account also the running of $\tilde{\mathcal{V}}$. There are now more fixed points, and we report in Table 2 the properties of the most interesting ones.

We see that in all cases the fixed point value of $\tilde{\mathcal{V}}$ is very small, justifying the earlier approximation $\mathcal{V} = 0$. In fact, by considering only the beta functions of λ , ξ and \tilde{Z}_N , and treating $\tilde{\mathcal{V}}$ as a parameter, and letting this parameter vary between zero and 0.004575, or 0.006928, we can see that FP₂ and FP₃ change continuously from the values of Table 1 to those of Table 2. We may thus identify the first three fixed points of Table 2 with those of Table 1.

There are several other fixed points with $\lambda = 0$, of which FP₄ is a representative example. We list it here for reasons that will become clear later. There may also exist other non-trivial fixed points with $\lambda \neq 0$, but this would require a more extensive numerical search that we have not undertaken. Besides, these fixed points are probably artifacts of the truncation, as are known to occur in other similar cases.

We note that also \tilde{Z}_{N*} is small, and this justifies *a posteriori* the expansion in \tilde{Z}_N that we used throughout our calculations. If we change variable from \tilde{Z}_N to \tilde{G}_N and set $\lambda = 0$, then as seen from (3.4) there is a fixed point at $\tilde{G} = 0$. On the other hand, if we first set $\tilde{G} = 0$, there is no acceptable fixed point for the dimensionless couplings. In any case, since we have expanded in \tilde{Z}_N , any result near $\tilde{G} = 0$ is unreliable. This is unfortunate, because it means that we cannot check whether there exists a RG trajectory joining one of the fixed points listed above to the standard weak gravity regime in the IR.

3.3. Scaling exponents

If we rescale the fluctuation field $h_{\mu\nu}$ by a factor $\sqrt{\lambda}$, so that the prefactor of its kinetic term is canonical, the fixed point FP₁ is seen to be a Gaussian fixed point, and indeed we find that the scaling exponents are given by the canonical dimensions: 4, 2, 0, 0, 0. The scaling exponents of FP₂, listed from more to less relevant, are

$$\theta_{1,2} = 2.35191 \pm 1.67715i, \quad \theta_3 = 1.76672,$$

$$\theta_4 = 0, \quad \theta_5 = -3.20030,$$

while those of FP₃ are

$$\theta_{1,2} = 2.03270 \pm 1.52155i, \quad \theta_3 = 1.23742,$$

$$\theta_4 = 0, \quad \theta_5 = -5.27685.$$

The marginal coupling is ρ , the (inverse of the) coefficient of the topological term. At the non-Gaussian fixed points, we find $\beta_\rho = A\rho^2$ with $A = 0.01736$ at FP₂ and $A = 0.02258$ at FP₃. Thus, at both fixed points, ρ is marginally relevant when it is negative and marginally irrelevant when it is positive. We thus arrive at the conclusion that also in the present approximation, the dimension of the critical surface of pure gravity is three, up to the marginal topological term.

3.4. The a -function

The beta function of ρ is given by (2.6). In an ordinary CFT, the coefficient a appears in the trace anomaly as

$$\langle T^\mu{}_\mu \rangle = \frac{1}{16\pi^2} (cC_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - aE). \quad (3.11)$$

For example, for a free theory with N_S scalars, N_f Dirac fields and N_V gauge fields,

Table 2
Selected fixed points including $\tilde{\nu}$.

	λ_*	ξ_*	ρ_*	ω_*	\tilde{Z}_{N*}	$\tilde{\nu}_*$	\tilde{G}_*	$\tilde{\Lambda}_*$	a
FP ₁	0	0	0	-0.02286	0.00833	0.006487	2.388	0.3894	4.356
FP ₂	24.91	-287.1	0	0.2603	0.01635	0.004575	1.217	0.1399	-2.741
FP ₃	28.24	175.6	0	-0.4825	0.01499	0.006928	1.327	0.2310	-3.566
FP ₄	0	-312.2	0	0	0.009222	0.006092	2.157	0.3303	4.357

$$a = \frac{1}{360}(N_S + 11N_f + 62N_V), \quad c = \frac{1}{120}(N_S + 6N_f + 12N_V). \quad (3.12)$$

According to the a -theorem, if there is a RG trajectory joining two fixed points, a is higher at the UV fixed point [39–41]. This accords to the intuition that a is a measure of the number of degrees of freedom of the theory. There is no known a -theorem for gravity. However, we can view our calculation as a quantum field theory in a curved background, and from this point of view the theorem should be applicable.⁷ At FP₁ we have $a = \frac{196}{45}$. The values of a at the other fixed points can be calculated numerically and are reported in the last column of Table 2.

Since FP₂ and FP₃ have a unique irrelevant direction, there is only one RG trajectory leaving these fixed points, that can be integrated numerically in the direction of increasing $t = \log k$ and ends up (in the UV) at another fixed point. In this way we have found an RG trajectory that goes from FP₁ to FP₃ and one that goes from FP₄ to FP₂. The value of a decreases along these trajectories, in accordance with the theorem. On the other hand, all the fixed points with $\lambda = 0$ have very similar values of a and there is a trajectory that goes from FP₄ to another fixed point with $\lambda = 0$ and a slightly larger value of a , in contradiction to the theorem. Since it is doubtful that these additional fixed points do exist, the meaning of this result is not very clear, and will have to be investigated more carefully in the future.

3.5. Spectrum

The appearance of several non-trivial fixed points is not a novelty in this kind of calculations. Several of these are likely to be spurious, but we do not see any reasons why FP₁ or FP₂ should be rejected *a priori*, or to prefer one over the other. Regarding the spectrum, we recall that in order to avoid tachyons in the expansion around flat space, the action for gravity in Lorentzian signature⁸ must have a negative Weyl squared term and a positive R^2 term. A naive Wick rotation of the linearized action around flat space leads to a Lorentzian action that only differs from the Euclidean one by an overall sign. Therefore, FP₂ has the correct signs to avoid tachyons. Although this is not sufficient to guarantee a healthy theory, it gives us some more room in the search of one.

Note added: After this paper was submitted to the journal the work referred to in footnote 1 has appeared on the arXiv [44].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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⁷ Similar calculations involving gravity have been reported in [42,43].

⁸ we use the Lorentzian signature $-+++$.

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