Scuola Internazionale Superiore di Studi Avanzati - Trieste

DOCTORAL THESIS

Index Theorems and Soft Theorems

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ACADEMIC YEAR 2019 – 2020

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Dedicated to

my mother for making me who I am
and my father for supporting me all the way...

Abstract

This thesis deals with Index theorems and Soft theorems for gravitini.

In the first part, we derive the Atiyah-Patodi-Singer (APS) index theorem using supersymmetric quantum mechanics. We relate the APS η-invariant to the temperature dependence of the noncompact Witten index. It turns out that the temperature derivative of the Witten index depends solely on the asymptotic boundary of the noncompact target space. We also compute the elliptic genus of some noncompact superconformal field theories, namely $N = (2, 2)$ cigar and $N = (4, 4)$ TaubNUT. This elliptic genera is the completion of a mock Jacobi form. The holomorphic anomaly of this mock Jacobi form again depends on the boundary theory as in the case of the Witten index. We show that the APS index theorem can then be related to the completion of a mock Jacobi form via noncompact Witten index.

In the second part, we derive the leading order soft theorem for multiple soft gravitini. We compute it in an arbitrary theory of supergravity with an arbitrary number of finite energy particles. Our results are valid at all orders in perturbation theory in more than three dimensions. We also comment on the infrared (IR) divergences in supergravity. It turns out that the leading order soft theorem is unaffected by the IR divergences.

Acknowledgements

First and foremost, I would like to thank my supervisor, Prof. Atish Dabholkar, for his teachings and advice that helped me to grow as a physicist. He always encouraged me to talk to seminar speakers and other faculty members and to apply to various schools and conferences, which gave me the confidence to ask 'stupid' questions. Besides physics, discussions with him on politics and life were always very fruitful.

Next, I would like to thank my friends and collaborators, Francesca Ferrari and Arnab Rudra (Arnab da), for countless discussions. Arnab's office was always open for me whenever I ran into any doubt. He taught me how to ask basic questions. Discussions with Francesca gave me new mathematical insights. Her helpful and always inviting nature gave me an example to follow.

In the beginning of this year, I went to the Perimeter institute (PI) as a visiting graduate fellow. I had the opportunity to interact with Prof. Davide Gaiotto. I would like to thank him for his time and patience, even during the COVID lockdown. The work at PI was supported by the visiting graduate fellowship provided by the institute.

I would also like to thank my friends and colleagues, Marco Celoria, Fabrizio Del Monte, Harini Desiraju, Francesco Di Filippo, Marco Gorghetto, Zainab Nazari, Francesco Sagarlata (especially for the amazing food he cooked), Giovanni Tambalo, Vicharit Yingcharoenrat and Ida Zadeh for useful discussions.

Next is Joy, for always being there during the highs and the lows and for listening to my "Pta hai..Pta hai.." stories sometimes patiently and other times not.

During these wonderful four years, I made many friends who I will cherish the whole life. I would like to thank all the Nadaan Parindey's for making me feel like at home away from home, for the fantastic Indian food, and for the long nights at Waikiki. They were always there to celebrate the good moments and to cheer me up during the bad ones.

Last but not the least, I would like to thank my parents, my bhai and bhabhi Diwakar and Rupanshi for being my backbone and always supporting me and to Diwanshi for always bringing a smile to my face.

Preface

This thesis is divided into two separate parts. Each part is based on the following publications:

1. A. Dabholkar, D. Jain and A. Rudra, APS η-invariant, path integrals, and mock modularity, JHEP 11 (2019) 080 [arXiv:1905.05207].

2. D. Jain and A. Rudra, Leading soft theorem for multiple gravitini, JHEP 06 (2019) 004, [arXiv:1811.01804].

Contents

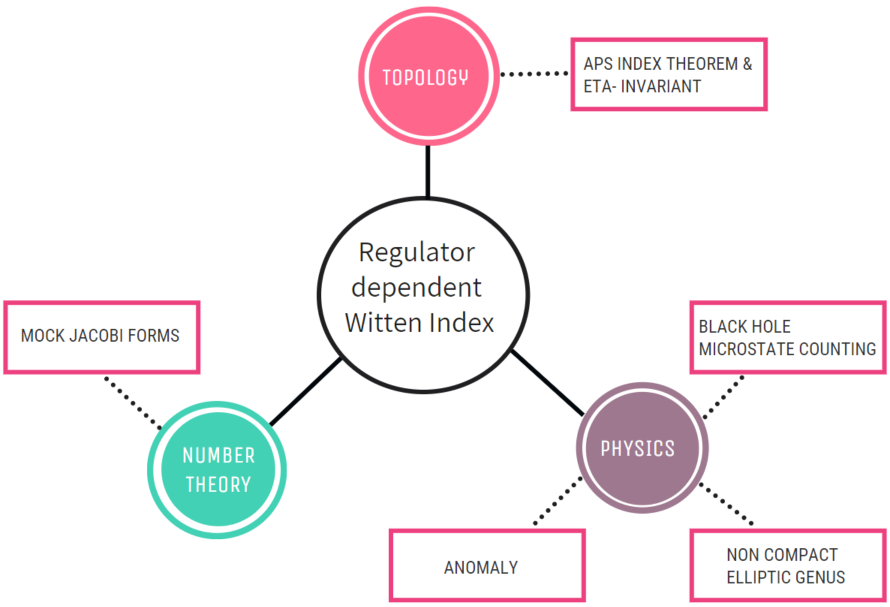
PART I

Index theorems

Chapter 1

Introduction

In this part of the thesis, we establish a connection between three distinct fields: topology, number theory, and physics. In particular, we relate the Atiyah-Patodi-Singer η-invariant [1] with mock Jacobi forms [2] and supersymmetric path integrals. The link between these three is provided by the *temperature-dependent* Witten index of a noncompact theory.



For a supersymmetric quantum field theory in a D-dimensional spacetime, the Witten index [3] is defined by

$$W(\beta) := \operatorname*{Tr}_{\mathcal{H}} \left[(-1)^F e^{-\beta H}\right] \tag{1.1}$$

where $\beta = 1/T$ is the inverse temperature, H is the Hamiltonian, F is the fermion number (which is zero for bosons and one for fermions) and $\mathcal{H}$ is the Hilbert space of the theory. This trace can be related to a supersymmetric Euclidean path integral with periodic boundary conditions for all fields in the Euclidean time direction as in §2.3.

If the quantum field theory is compact in the sense that the spectrum of the Hamiltonian is discrete, then the Witten index is independent of the inverse temperature β. We elaborate on this in §2.2. The states with nonzero energy come in Bose-Fermi pairs and do not contribute to the Witten index [3]. Only the zero energy states graded by $(-1)^F$ contribute, and consequently, the Witten index is independent of β. More precisely, it is a topological invariant of the target space. This is the case, for example, for a supersymmetric sigma model with a compact target space. If the field space is noncompact and the spectrum is continuous, then the above argument can fail because now instead of a discrete indexed sum, one has an integral over a continuum of scattering states. In general, the bosonic density of states in this continuum may not precisely cancel the fermionic density of states, and the noncompact Witten index can be temperature-dependent. This temperature dependence helps us to relate it to the η-invariant and to mock Jacobi forms.

It was shown in [4, 5] that for an appropriate sigma model on a compact target space, the Witten index in the zero temperature ($\beta \to \infty$) limit can be related to some of the well-known topological invariants such as the Euler character or the Dirac index of the target manifold. We define these topological invariants in §2.1. We review their argument and re-derive the Dirac index of a compact manifold without boundary in §2.3. Since for a compact target space, the Witten index is temperature independent; one can evaluate it in a much simpler high temperature ($\beta \to 0$) limit. In this limit, one can use the heat kernel expansion to prove the Atiyah-Singer index theorem (index theorem for differential operators on a compact manifold without boundary) [6]. Evaluating the path integral corresponding to the Witten index in this high-temperature semiclassical limit gives another derivation of the index theorem [4, 5].

For a compact target space with *boundary*, the boundary conditions play an important role and there is a correction to the index theorem. This correction is known as the APS η-invariant. We relate the index on a compact manifold with a *boundary* to the Witten index on a noncompact manifold. We use the temperature dependence of this noncompact Witten index to compute the APS η-invariant. The temperature-dependent piece is no longer topological but is nevertheless 'semi-topological' in the sense that it is independent of any deformations that do not change the asymptotics of the target space. This yields a new proof of the APS index theorem. The η-invariant is nonzero precisely because the noncompact Witten index is temperature-dependent.

The relation to mock modularity arises similarly as a consequence of the noncompactness of the target manifold for a superconformal field theory on a base space Σ, which is a 2-torus with a complex structure parameter τ. The elliptic genus of an SCFT is a generalization of the Witten index that counts the right-moving ground states with arbitrary left-moving excitations. It is *a priori* a function of τ and $\bar{\tau}$. For a compact SCFT, by an argument similar to the above, it is independent of the 'right-moving temperature' and hence of $\bar{\tau}$, and is a (weakly) holomorphic Jacobi form. Once again, for a noncompact SCFT with $\widehat{\mathcal{M}}$ as the target space, this argument fails. There is a 'holomorphic anomaly' because the right-moving bosonic density of states does not precisely cancel the right-moving fermionic density of states. In this case, the elliptic genus is no longer a Jacobi form but is rather a completion of a mock Jacobi form—a new mathematical object introduced in [2]. The holomorphic anomaly is once again governed by the temperature dependence of the 'noncompact right-moving Witten index', which depends solely on the asymptotics of the target space. Hence it can also be related to the torus one-point function of the super-current in the asymptotic boundary theory [7]. We use this relation to compute the holomorphic anomaly for certain examples.

An advantage of mapping the APS index to the Witten index on the noncompact manifold $\widehat{\mathcal{M}}$ is that it becomes easier to obtain its path integral representation. Defining a path integral measure on a target space with a boundary is, in general, rather complicated. Even for a very simple system like a particle in a box, the path integral formulation was achieved relatively recently [8–13]. For a path integral on a manifold $\widehat{\mathcal{M}}$ without a boundary, even if it is noncompact, one can use the canonical measure. The path integral facilitates computations using supersymmetric localization. We derive the APS result for certain type of manifolds by relating it to a Callias-Bott-Seeley [14, 15] index theorem as we explain in §3.4.2. A path integral representation also makes the modular invariance manifest making it easier to see the connection with mock modularity.

Apart from their intrinsic importance in differential topology, the index theorems play an important role in physics. The Dirac index on a manifold $\mathcal{M}$ is related to the Chiral anomaly for fermions living on that manifold [16]. Hence by computing the Dirac index, one can tell if the chiral symmetry can be gauged or not. Index theorems are also used in finding the dimension of instanton moduli space [17]. The η-invariant also has a number of interesting physics applications, for example, in the analysis of global gravitational anomalies [18], in fermion fractionization [19, 20] , in relation to spectral flow in quantum chromodynamics [21, 22], and more recently in the description of symmetry-protected phases of topological insulators (see [23] for a recent review). Similarly, apart from their intrinsic interest in number theory [24, 25], mock modular forms and their cousins have come to play an important role in the physics of quantum black holes, quantum holography and wall-crossing [2, 26–34], in umbral moonshine [35, 36], in the context of WRT invariants [37–40] , and more generally in the context of elliptic genera of noncompact SCFTs [41–46]. We expect our results will have useful implications in these diverse contexts.

This part of the thesis is organized as follows. In Chapter 2, we review the mathematical definition of the Dirac index, and the supersymmetric quantum mechanics (SQM) on a compact target space. We then re-derive the Dirac index using path integral in SQM. In Chapter 3, we describe the Atiyah-Patodi-Singer construction for a compact manifold with a *boundary*. We explain the APS boundary conditions and relate the APS index to the noncompact Witten index. Then we use this formalism to present a proof of the APS theorem in §3.3.3. As an example, we compute the η-invariant for a finite cigar. We then move on to superconformal field theories with a two-dimensional base space in Chapter 4. We review the definitions of mock Jacobi forms in §4.1 and discuss their connection with noncompact non-linear sigma models (NLSM). We then review the renormalization group flow of gauged linear sigma models (GLSM) to non-linear sigma models in §4.3. We focus on the noncompact NLSM's which arise in the RG flow of GLSM's with Stückelberg fields. In the end, we compute the holomorphic anomaly for certain examples, namely $N = (2, 2)$ cigar SCFT and $N = (4, 4)$ TaubNUT SCFT, using different methods. First, we compute it by taking the $\bar{\tau}$ derivative of the elliptic genus computed from GLSM, and then we perform a direct computation in the boundary theory and use the Gaiotto Johnson-Freyd (GJF) anomaly equation.

The examples considered in this work are simple but sufficiently nontrivial and illustrative. Our results indicate that these interesting connections are a rather general consequence of noncompactness. Supersymmetric methods have been used successfully to obtain a path integral derivation of the Atiyah-Singer index theorem for a compact target manifold without boundary, but to our knowledge, no such derivation exists for a manifold with a boundary[1]. Using our formulation in terms of a noncompact Witten index, it should be possible to obtain a more complete path integral derivation of the APS index theorem, for example, even for the manifolds that do not have product form near the boundary [48] and the manifolds with torsion [49]. In such cases, the continuum touches zero modes, and hence the separation of contributions from discrete states and continuum becomes difficult. We would like to address this issue in future works.

We also make some comments on the relation between mock modularity and the presence of Stückelberg field in the GLSM. It would be nice to make this connection more precise.

[1]Indeed, this was posed by Atiyah a decade ago as a problem for the future [47].

Chapter 2

Index theorems and Supersymmetry

In this chapter, we first define various index theorems and then set up the notation for supersymmetric quantum mechanics (SUSY QM). We later give a derivation of the Atiyah-Singer (AS) index theorem using supersymmetric quantum mechanics. This will set up the stage for the next chapters, where we derive Atiyah-Patodi-Singer (APS) index theorem using SUSY QM. Since we are interested in deriving the results from a physics perspective, we will not dive deeply into mathematical technicalities.

2.1 Index Theorems

A generic differential operator $\mathcal{D}$ on a manifold $\mathcal{M}$, is defined as a map between sections of vector bundles. The differential operators $\mathcal{D}$ and its dual $\mathcal{D}^\dagger$ are defined as:

$$\mathcal{D} \quad : \quad \Gamma(\mathcal{M}, E) \to \Gamma(\mathcal{M}, F) \tag{2.1}$$

$$\mathcal{D}^\dagger \quad : \quad \Gamma(\mathcal{M}, F) \to \Gamma(\mathcal{M}, E) \tag{2.2}$$

where E, F are vector bundles over $\mathcal{M}$ and $\Gamma(\mathcal{M}, E)$, $\Gamma(\mathcal{M}, F)$ are sections of vector bundles E and F respectively. For the index theorems, we will be interested in the zero eigenvectors (kernels) of these operators:

$$\text{Ker}\mathcal{D} \quad \equiv \quad \{s \in \Gamma(\mathcal{M}, E); \mathcal{D}s = 0\} \tag{2.3}$$

$$\text{Ker}\mathcal{D}^\dagger \quad \equiv \quad \{s \in \Gamma(\mathcal{M}, F); \mathcal{D}^\dagger s = 0\} \tag{2.4}$$

The index is defined as:

$$\text{Ind}\mathcal{D} = \dim \text{Ker}\mathcal{D} - \dim \text{Ker}\mathcal{D}^\dagger \tag{2.5}$$

This analytical index is well-defined for the elliptic operators [1] which satisfy Fredholm condition[2]. An elliptic operator whose kernel and co-kernel are finite-dimensional is known as Fredholm operator. It is well-known that the elliptic operators on a *compact* manifold $\mathcal{M}$ are Fredholm operators. Some of the examples of such operators are: Laplacian, Dirac operator, de-Rham operator, etc.

It has been proved that this analytical index on compact manifolds without boundary, is a topological quantity; more precisely, it can be expressed in terms of integral of certain characteristic class over $\mathcal{M}$. This interplay between analysis and topology is the main ingredient of the index theorems.

The index theorems can be broadly classified into three categories depending on the manifold $\mathcal{M}$:

- **Atiyah-Singer index theorem (AS):** These are the index theorems defined for differential operators that live on a compact manifold $\mathcal{M}$ without boundary [6]. In this chapter, we will review this case.

- **Atiyah- Patodi-Singer index theorem (APS):** These are the index theorems defined for differential operators that live on a compact manifold $\mathcal{M}$ with boundary [50, 51]. In such cases, boundary conditions play a significant role. We will discuss these in chapter 3.

- **Callias index theorem:** These are the index theorems defined for differential operators that live on a non-compact manifold $\mathcal{M}$. In these cases, imposing Fredholm condition turns out to be a bit tricky. The index of the Dirac operator has been defined and calculated for only a few types of non-compact manifolds [14].

[1] Elliptic operators are defined by the condition that the coefficients of the highest-order derivatives appearing in that operator, are positive. Hence the principal symbol is invertible.

[2] For non-Fredholm operators, the index becomes a difference between two-infinite quantities and hence is not well-defined.

2.1.1 Dirac Index

In this thesis, we will only consider the index of Dirac operator ($\not{D} = \gamma^\mu D_\mu$). To define the Dirac index, we consider a spin bundle over a compact, even-dimensional ($2n$), orientable manifold $\mathcal{M}$. The spin-group is generated by $2n$ dimensional γ-matrices satisfying:

$$(\gamma^i)^\dagger = \gamma^i \tag{2.6}$$

$$\{\gamma^i, \gamma^j\} = -2g^{ij} \tag{2.7}$$

$$\bar{\gamma} = i^n \gamma^1 \gamma^2 .... \gamma^{2n} = \begin{pmatrix} -\mathbb{I}_{n\times n} & 0 \\ 0 & \mathbb{I}_{n\times n} \end{pmatrix} \tag{2.8}$$

$$(\bar{\gamma})^2 = \mathbb{I}_{2n\times 2n} \tag{2.9}$$

Since $(\bar{\gamma})^2 = \mathbb{I}$, the eigenvalues of $\bar{\gamma}$ are ± 1. It is called chirality matrix. Consider a Dirac spinor $\Psi(x)$ living on sections $\Delta(\mathcal{M})$ of the spin bundle. In the basis in which $\bar{\gamma}$ is diagonal, it can be written as:

$$\Psi = \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix} , \tag{2.10}$$

where the $\pm$ denote the chiralities i.e. $\bar{\gamma}\Psi^\pm = \pm\Psi^\pm$. Using this, the sections $\Delta(\mathcal{M})$ can also be separated into two eigenspaces:

$$\Delta(\mathcal{M}) = \Delta^+(\mathcal{M}) \oplus \Delta^-(\mathcal{M}) \tag{2.11}$$

such that $\psi^\pm \in \Delta^\pm(\mathcal{M})$. The Dirac operator in curved space is given by:

$$\gamma^\mu D_\mu \Psi = \gamma^\mu \left(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b \right) \Psi \tag{2.12}$$

where $\omega_{\mu ab}$ is the spin connection and $\gamma^a = \gamma^\mu e^a_\mu$. Here we have introduced orthonormal basis e^a_μ. In even dimensions, we can choose γ-matrices such that the Dirac operator can be written as:

$$\not{D} = \begin{pmatrix} 0 & L \\ L^\dagger & 0 \end{pmatrix} \tag{2.13}$$

where

$$L \ : \ \Delta^+(\mathcal{M}) \to \Delta^-(\mathcal{M}) \tag{2.14}$$
$$L^\dagger \ : \ \Delta^-(\mathcal{M}) \to \Delta^+(\mathcal{M}) \tag{2.15}$$

The index $\mathcal{I}$ of the Dirac operator on the manifold $\mathcal{M}$ is then defined as:

$$\mathcal{I} = \text{dim Ker}L - \text{dim Ker}L^\dagger = n_+ - n_- \tag{2.16}$$

where n_+ is the number of zero modes of L with positive chirality and n_- is the number of zero modes of $L^\dagger$ with negative chirality. The AS index theroem [6] states that the index of Dirac operator is given by:

$$\mathcal{I} = \int_{\mathcal{M}} \alpha(x)|_{\text{Vol.}} = \int_{\mathcal{M}} \prod_{\alpha=1}^{n} \frac{\frac{1}{2}x_\alpha}{\sinh(\frac{1}{2}x_\alpha)} \tag{2.17}$$

where x_α are the skew eigenvalues of the matrix R_{ij} which is a two-form constructed out of Riemann tensor i.e. $R_{ij} = \frac{1}{4\pi}R_{ijkl}dx^k \wedge dx^l$. In particular, in 4D we obtain:

$$\mathcal{I} = -\frac{1}{24}\int_{\mathcal{M}} \frac{tr\, R \wedge R}{16\pi^2} \tag{2.18}$$

where tr is the trace over the indices i, j of the two-form R_{ij}.

Regulating the Index

We note that the equation (2.16) can be re-written as follows:

$$\mathcal{I} = \text{dim Ker}L - \text{dim Ker}L^\dagger \tag{2.19}$$
$$= \text{dim Ker}L^\dagger L - \text{dim Ker}LL^\dagger \tag{2.20}$$
$$= \lim_{\beta\to\infty} \text{Tr}\left(e^{-\beta L^\dagger L} - e^{-\beta LL^\dagger}\right) \tag{2.21}$$
$$= \lim_{\beta\to\infty} \text{Tr}\left(\bar{\gamma}e^{-\beta \not{D}^2}\right) \tag{2.22}$$

where we have introduced a regulator β. In the second last equation, we use the fact that in the $\beta \to \infty$ limit, only the zero modes of $LL^\dagger$ and $L^\dagger L$ will contribute. The last equation is obtained by using the fact that $\bar{\gamma}$ commutes with $\not{D}^2$ and it has eigenvalues ± 1. Notice that (2.22) is the quantity one computes to calculate the Chiral Anomaly in a quantum field theory.

We note that on a *compact* manifold $\mathcal{M}$ *without boundary*, the index is a topological quantity and is independent of the regulator β. Using this, one can evaluate it in the much simpler $\beta \to 0$ limit using the heat kernel expansion. But this regulator independence fails on a *non- compact* manifold. This β dependence plays a key role in deriving the APS index theorem, which is the subject matter of Chapter 3.

2.2 Supersymmetric Quantum Mechanics

In 1983, the Atiyah-Singer index theorem was proved using Supersymmetric Quantum Mechanics (SQM) [3–5]. We will go through that proof in detail in the next section. In this section, we introduce SQM and the Witten index, which will be crucial in deriving the AS index theorem.

Consider a particle moving on a compact manifold $\mathcal{M}$ with the lagrangian given by:

$$\mathcal{L} = \frac{1}{2} \int dt \left[g_{ij}(x) \frac{dx^i}{dt} \frac{dx^j}{dt} \right] \tag{2.23}$$

where g_{ij} is the metric on $\mathcal{M}$, which we refer to as target space. Here x^i's are the maps from the worldline (t) to the target space ($\mathcal{M}$). We can now supersymmetrize the system i.e. we add a fermionic super-partner ψ^i corresponding to every bosonic coordinate x^i. The Lorentzian action of this system is given by:

$$I = \frac{1}{2} \int dt \left[g_{ij}(x) \frac{dx^i}{dt} \frac{dx^j}{dt} + i\, \psi^a \left(\delta_{ab} \frac{d\psi^b}{dt} + \omega_{akb} \frac{dx^k}{dt} \psi^b \right) \right] , \tag{2.24}$$

This can be obtained as a specialization of the worldsheet action (2.65) which we derive using $(1,1)$ superspace in Appendix 2.3.2, with $Q_+ = Q_-$ and by setting

$$F^i = 0 \,, \qquad h = 0 \,, \qquad \psi^a_- = 0 \,, \qquad \frac{\partial}{\partial \sigma} = 0 \,, \qquad \psi^a_+ = \psi^a \,. \tag{2.25}$$

We have defined $\psi^a = e^a_i \psi^i$ using the vielbein and ω_{akb} is the spin connection on $\mathcal{M}$. The conjugate variables are

$$\pi_a := \frac{\partial L}{\partial \dot{\psi}^a} = \frac{i}{2} \psi_a \qquad p_i := \frac{\partial L}{\partial \dot{x}^i} = \dot{x}_i + \frac{i}{2} \psi^a \omega_{iab} \psi^b \tag{2.26}$$

where the dot refers to t-derivative. The nonvanishing canonical commutation relations are

$$\{\psi^a, \psi^b\} = \delta^{ab}\,, \qquad [x^i, p_j] = i\delta^i_j\,. \tag{2.27}$$

The Hilbert space $\mathcal{H}$ furnishes a Dirac representation of $2n$-dimensional γ-matrices with $\sqrt{2}\psi^j = -i\gamma^j$. The chirality matrix $\bar{\gamma}$ defined in (2.9) can be identified with $(-1)^F$ where F is the fermion number. It acts on the fields as follows:

$$(-1)^F x^i = x^i \qquad\qquad (-1)^F\psi^i = -\psi^i \tag{2.28}$$

For a review see [52] which uses slightly different conventions.

Above lagrangian is invariant under the following supersymmetry transformations:

$$\delta x^i = i\epsilon\psi^i\,, \qquad\qquad \delta\psi^i = -\epsilon\dot{x}^i\,. \tag{2.29}$$

where ϵ is the Grassmann variable parametrizing SUSY transformations. This system has one real supercharge, in other words, it is $N = 1/2$ SUSY QM. The corresponding Noether supercharge is given by:

$$\epsilon Q \;\; = \;\; -\epsilon\sqrt{2}\psi^i\dot{x}_i\,, \tag{2.30}$$

Upon quantization, we get

$$Q = \gamma^i D_i = \not{D} \qquad \text{with} \qquad D_i = \partial_i + \frac{1}{4}\omega_{iab}\gamma^a\gamma^b\,. \tag{2.31}$$

which is the Dirac operator on manifold $\mathcal{M}$. The canonical commutations imply the commutation relations

$$\{Q, Q\} = 2H\,, \qquad [H, Q] = 0\,, \qquad \{Q, (-1)^F\} = 0 \tag{2.32}$$

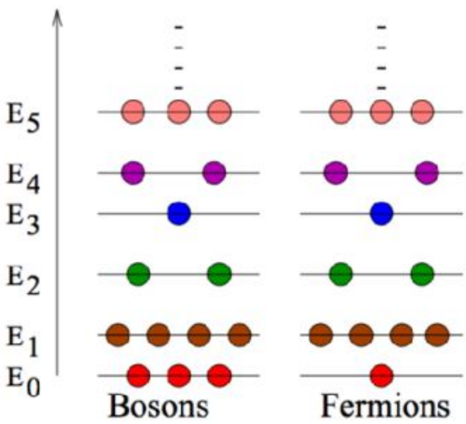
where H is the worldline Hamiltonian. Since the worldline supercharge takes the same form as the Dirac operator on the target space $\mathcal{M}$, there is a one to one correspondence between the worldline quantities and the target space ones.

$$\begin{aligned} \not{D} &\leftrightarrow Q \\ \not{D}^2 &\leftrightarrow H \\ \bar{\gamma} &\leftrightarrow (-1)^F \end{aligned}$$

Hence positive and negative chirality spinors on the target space correspond to positive and negative eigenstates of the operator $(-1)^F$ in the Hilbert space $\mathcal{H}$ of SQM. The Dirac index $\mathcal{I}$ is then naturally identified with the Witten index of the supersymmetric quantum mechanics i.e.

$$\mathcal{I} = \lim_{\beta\to\infty} \text{Tr}\left(\bar{\gamma} e^{-\beta \not{D}^2}\right) \leftrightarrow \lim_{\beta\to\infty} \operatorname*{Tr}_{\mathcal{H}} (-1)^F \exp^{-\beta H} = \lim_{\beta\to\infty} W(\beta) \tag{2.33}$$

For a compact manifold the eigenvalues of H are discrete. It then follows from (2.32) that if $|E,+\rangle$ is a bosonic eigenstate with energy eigenvalue $E > 0$, then $\frac{1}{\sqrt{E}} Q|E,+\rangle := |E,-\rangle$ is a fermionic eigenstate with the same energy eigenvalue.



Hence, eigenstates with non-zero eigenvalues of H come in Bose-Fermi pairs and cancel out in the trace. The Witten index, in this case, receives contribution only from the ground states and is a topological invariant [3, 6]. Hence $W(\beta)$ is independent of β when the target space is *compact*.

2.3 Derivation of Atiyah Singer index theorem

We have already introduced all the basic ingredients required to give a physicist's derivation of the Atiyah-Singer index theorem. We will now compute the Witten index in SUSY QM introduced above, with an even-dimensional[3] $(2n)$ target space $\mathcal{M}$. The Witten index of the worldline theory has a path integral representation

$$\begin{aligned} W(\beta) &= \int dx\, \langle x|(-1)^F e^{-\beta H}|x\rangle \\ &= \int_{PBC} [dX]\, \exp\left(-S[X,\beta]\right)\,, \end{aligned} \tag{2.34}$$

where the path integral is over superfield configurations that are periodic in Euclidean time with period β, hence the Euclidean base space Σ is a circle of radius β. The measure $[dX]$ is induced from the supermeasure[4] on the supermanifold $s\mathcal{M}$ introduced after (2.64). The Euclidean time τ is related to the Lorentzian time t as usual by Wick rotation $t = -i\tau$ and the Euclidean action is:

$$S[X,\beta] = \frac{1}{2}\int_0^\beta d\tau \left[g_{ij}(x)\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} + \psi^a\left(\delta_{ab}\frac{d\psi^b}{d\tau} + \omega_{kab}\frac{dx^k}{d\tau}\psi^b\right)\right] \tag{2.35}$$

$$= \frac{1}{2}\int_0^\beta d\tau \left[g_{ij}(x)\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} + g_{ij}(x)\psi^i\left(\frac{d}{d\tau}\psi^j + \Gamma^j{}_{kl}\frac{dx^k}{d\tau}\psi^l\right)\right] \tag{2.36}$$

where Γ^j_{kl} are the Christoffel symbols given by:

$$\Gamma^j_{kl} = \frac{1}{2}g^{jm}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right)$$

As we already argued, the Witten index on a *compact* target space is independent of β, we can compute the above path integral in $\beta \to 0$ limit. The path integral can be evaluated by following the steps given below:

1. In $\beta \to 0$ limit, the path integral is dominated by the saddle points determined by the following equations:

$$\frac{dx^i}{d\tau} = 0 = \frac{d\psi^i}{d\tau}$$

[3]In odd dimensions, the Dirac index vanishes. This can be seen from equation (2.54).

[4]It is well-known that the supermeasure is flat even if the manifold $\mathcal{M}$ is curved because the factor of $\sqrt{g}$ in the bosonic measure $dx := d^{2n}x\,\sqrt{g}$ cancels against a similar factor in the fermionic measure $d\psi := d^{2n}\psi\,\frac{1}{\sqrt{g}}$.

2. We then compute the quadratic action for small fluctuations around the saddle point.
3. Lastly, we integrate over small fluctuations to evaluate the full path integral.

Since we restrict ourself to the constant modes of both fermions(ψ_0) and bosons (x_0). The calculation is much easier in Riemann normal coordinates (RNC), defined by,

$$g_{ij}(x_0) = \delta_{ij}\,, \qquad \partial_k g_{ij}(x_0) = 0\,. \tag{2.37}$$

Considering small fluctuations around the constant path and expanding the fields in RNC,

$$x^i(\tau) = x_0^i + \xi^i(\tau)\,, \qquad \psi^i(\tau) = \psi_0^i + \eta^i(\tau)\,. \tag{2.38}$$

We can now compute the Lagrangian up-to terms quadratic in fluctuations:

$$\begin{aligned}
g_{ij}(x)\dot{x}^i\dot{x}^j &= g_{ij}(x_0)\dot{\xi}^i\dot{\xi}^j \\
g_{ij}(x)\psi^i\dot{\psi}^j &= g_{ij}(x_0)\eta^i\dot{\eta}^j \\
g_{ij}\psi^i\Gamma^j{}_{kl}\dot{\xi}^k\psi^l &= \delta_{ij}\partial_m\Gamma^j{}_{kl}\xi^m\dot{\xi}^k\psi_0^i\psi_0^l \\
&= \frac{1}{2}\delta_{ij}(\partial_m\Gamma^j{}_{kl} - \partial_k\Gamma^j{}_{ml})\xi^m\dot{\xi}^k\psi_0^i\psi_0^l \\
&= \frac{1}{2}R_{ilmk}\xi^m\dot{\xi}^k\psi_0^i\psi_0^l
\end{aligned}$$

Hence the Lagrangian (upto quadratic terms in fluctuations) is given by:

$$\mathfrak{L} = \frac{1}{2}\dot{\xi}^i\dot{\xi}^i + \frac{1}{2}\eta^i\dot{\eta}^i + \frac{1}{4}R_{ijkl}\psi_0^i\psi_0^j\xi^k\dot{\xi}^l \tag{2.39}$$

The Witten index then becomes,

$$W(\beta) = \mathcal{N}_f \int \prod_{i=1}^{2n} dx_0^i d\psi_0^i \;\mathrm{FD}(x_0^i, \psi_0^i) \tag{2.40}$$

where $\mathcal{N}_f$ is the fermionic normalization, $2n$ is the dimension of the manifold $\mathcal{M}$ and $\mathrm{FD}(x_0^i, \psi_0^i)$ denotes the fluctuation determinant computed in §2.3.2

2.3.1 Fermionic normalization

The chirality matrix $\bar{\gamma}$ can be expressed in terms of the fermionic zero modes using $\sqrt{2}\psi_0^i = -i\gamma^i$. Hence we obtain:

$$\bar{\gamma} = i^n\gamma^1 \dots \gamma^{2n} = (-2i)^n \psi_0^1 \dots \psi_0^{2n} \tag{2.41}$$

Also, $\text{Tr}\bar{\gamma}^2 = \text{Tr}\mathbf{1} = 2^n$. Hence,

$$\begin{aligned}
\text{Tr}\bar{\gamma}^2 &= \text{Tr}(-1)^F(-2i)^n\psi_0^1 \dots \psi_0^{2n} \\
2^n &= \mathcal{N}_f \int \prod_{i=1}^{2n} d\psi_0^i (-2i)^n \psi_0^1 \dots \psi_0^{2n} \\
\implies \mathcal{N}_f &= (-i)^n = (-i)^{D/2}
\end{aligned} \tag{2.42}$$

where $D = 2n$ is the dimension of the target space and we have used the convention given in (2.58) for Grassmann integrals. Notice that even though we derive this normalization for only even D but it can be used to define normalization for arbitrary D. For $D = 1$, we do not get any -ve sign from the Grassmann integral and hence we get $\mathcal{N}_f = (i)^{1/2}$.

2.3.2 Fluctuation determinant

We have periodic boundary conditions for all the fields along the Euclidean time direction i.e

$$\xi^i(\tau + \beta) = \xi^i(\tau) \qquad \eta^i(\tau + \beta) = \eta^i(\tau) \tag{2.43}$$

Hence, we can mode expand the fluctuations as follows:

$$\xi^i(\tau) = \sum_{m\neq 0} \xi_m^i \exp\left[\frac{2\pi i}{\beta} m\tau\right], \qquad \eta^i(\tau) = \sum_{m\neq 0} \eta_m^i \exp\left[\frac{2\pi i}{\beta} m\tau\right] \tag{2.44}$$

The fluctuation determinant is given by:

$$\text{FD}(x_0^i, \psi_0^i) = \prod_{i=1}^{2n} \prod_{m\neq 0} \int d\xi_m^i \, d\eta_m^i \exp\left[-S_E(\xi_m^i, \eta_m^i; x_0^i, \psi_0^i)\right] \tag{2.45}$$

Using these expansion, the bosonic part of action (2.39) can be expressed as follows:

$$S_b = -\sum_{\alpha=1}^{n}\sum_{m>0}\begin{bmatrix}\xi_{-m}^{2\alpha+1} & \xi_{-m}^{2\alpha+2}\end{bmatrix}\begin{bmatrix}\frac{(2\pi m)^2}{\beta} & y_\alpha(2\pi m)\\ -y_\alpha(2\pi m) & \frac{(2\pi m)^2}{\beta}\end{bmatrix}\begin{bmatrix}\xi_m^{2\alpha+1}\\ \xi_m^{2\alpha+2}\end{bmatrix} \tag{2.46}$$

where $y_\alpha(\alpha = 1, \cdots, n)$ are the skew-eigenvalues of $R_{ij} := \frac{1}{2}R_{ijkl}\psi_0^k\psi_0^l$. The bosonic fluctuation determinant then becomes:

$$\begin{aligned} Z_b &= \int \prod_{i=1}^{2n}\prod_{m\neq 0} d\xi_m^i exp^{-\sum_{\alpha=1}^{n}\frac{1}{2}\xi^T A_\alpha \xi} && (2.47)\\ &= \prod_{i=1}^{2n}\prod_{m>0}\prod_{\alpha=1}^{n}(\det(A_\alpha/2\pi))^{-1} && (2.48)\end{aligned}$$

where A_α is the matrix given in (2.46). Finally we obtain

$$\begin{aligned} Z_b &= \prod_{i=1}^{2n}\prod_{\alpha=1}^{n}\prod_{m=1}^{\infty}\left(\frac{2\pi m^2}{\beta}\right)^{-2}\left(1-\frac{(\frac{1}{2}y_\alpha\beta)^2}{\pi^2m^2}\right)^{-1} && (2.49)\\ &= \prod_{i=1}^{2n}\frac{1}{\sqrt{2\pi\beta}}\prod_{\alpha=1}^{n}\frac{\frac{i}{2}y_\alpha\beta}{\sinh(\frac{i}{2}y_\alpha\beta)} && (2.50)\end{aligned}$$

where we have used zeta function regularization to compute the first product. The fermionic part of fluctuations give:

$$Z_f = \int\prod_{i=1}^{2n}\prod_{m=1}^{\infty} d\eta_m^i d\eta_{-m}^i\, e^{-\frac{1}{2}(2\pi i m)\eta_m\eta_{-m}} = 1 \tag{2.51}$$

Here again, we have used zeta function regularization to compute the determinant. Plugging in all the factors, we obtain the following expression for the Witten index:

$$W(\beta) = (-i)^n\int\prod_{\mu=1}^{2n}\frac{dx_0^i}{\sqrt{2\pi\beta}}d\psi_0^i\prod_{\alpha=1}^{n}\frac{\frac{i}{2}y_\alpha\beta}{\sinh(\frac{i}{2}y_\alpha\beta)} \tag{2.52}$$

Since y_α's are bilinear in ψ_0, and the integrand is even under $y_\alpha \to -y_\alpha$, the Taylor series of the integrand only contain terms with even powers of y_α. Hence, the number of ψ_0's is a multiple of 4, and the integral vanishes unless the dimension of the manifold is a multiple of 4. Due to the fermionic zero-mode integral, only one term from the expansion of $\sinh$ contributes.

Let us consider manifolds $\mathcal{M}$ with dimension $2n = 4k$. The term in Taylor series that contributes to the integral contains i^n. This combined with i from the fermionic normalization gives, i^{2n}, and since n is even, this gives 1. We may as well replace,

$$\left(-\frac{i}{2\pi\beta}\right)^n \frac{\frac{i}{2}y_\alpha\beta}{\sinh(\frac{i}{2}y_\alpha\beta)} \longrightarrow \frac{y_\alpha/4\pi}{\sinh(y_\alpha/4\pi)} \tag{2.53}$$

Here we explicitly demonstrate that the answer is independent of the regulator β which matches with the general argument given above. The Witten index/ Dirac index is given by:

$$W(\beta) = \int_{\mathcal{M}} \prod_{\alpha=1}^{n} \frac{y_\alpha/4\pi}{\sinh(y_\alpha/4\pi)} \tag{2.54}$$

which matches (2.17). Notice that the definition of x_α and y_α differ by $1/2\pi$.

In this way the Dirac index can be derived by computing the Witten index in a SQM. One can compute various other topological indices of the target manifold by computing different quantities in SQM. Below we list such topological indices/ signatures and the corresponding quantities to be computed in the SQM :

Index computation from SQM		
Index	Number of Supercharges (complex)	Expression (to be computed in SUSY)
Euler number	N = 1	$Tr(-1)^F e^{-\beta H}$
Hirzebruch signature	N = 1	$TrQ_5 e^{-\beta H}$ [5]
Dirac operator	N = 1/2	$Tr(-1)^F e^{-\beta H}$
Dolbeault index	N = 2	$Tr(-1)^F e^{-\beta H}$
G-index ($d_\lambda = d + i\lambda(K)$)	N =1 with central charge	$Tr(-1)^F e^{-\beta H}$

[5] The operator Q_5 implements $\psi^i_\alpha \to (\bar{\gamma}\psi^i_\alpha)$ symmetry.

Appendix

2.A $(1,1)$ SUSY from superspace

We were interested in the supersymmetric path integrals for D-dimensional quantum field theories with $D = 2, 1, 0$. In this appendix, we write down the most general action for a system with $(1,1)$ SUSY in $1+1$ D. The worldline and world point actions can be obtained by taking various limits of the action derived in this appendix.

We use indices $\{\alpha, \beta, \ldots\}$ with values in $\{0,1\}$ to label the components of a worldsheet vector and $\{A, B, \ldots\}$ with values in $\{+,-\}$ to label the components of a worldsheet spinor. On the field space, we use $1 \leq i, j, \ldots \leq 2n$ as the coordinate indices $1 \leq a, b, \ldots \leq 2n$ as the tangent space indices. We use the worldsheet metric to be $\eta_{\alpha\beta} = \text{diag}(-,+)$.

A convenient basis for the two dimensional Dirac matrices is

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{2.55}$$

which satisfy $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$. The worldsheet chirality $\bar{\rho}$ and charge conjugation matrix C are

$$\bar{\rho} = -\rho_0\rho_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathcal{C}_{AB} = \rho^0_{AB} \tag{2.56}$$

A two-dimensional Majorana spinor is a two-component real spinor. A Majorana spinor is obeys $\bar{\psi} = \psi^\dagger \mathcal{C}$ where $\mathcal{C} = \rho^0$ in our case. Hence we obtain:

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \qquad \psi = \psi^* \tag{2.57}$$

We use the superspace $s\Sigma$ with real superspace coordinates $\{\sigma^\alpha, \theta_A\}$ to write down the supersymmetric lagrangian. We use the following convention for superspace derivatives and integrals

$$\frac{\partial}{\partial\theta^A}\theta^B = \delta^B_A, \qquad \frac{\partial}{\partial\bar{\theta}^A}\bar{\theta}^B = \delta^B_A \qquad \text{and} \qquad \int d\theta d\bar{\theta}\ \bar{\theta}\theta = 1. \tag{2.58}$$

The supercharge here is a two-dimensional Majorana spinor.

$$Q = \begin{pmatrix} Q_- \\ Q_+ \end{pmatrix} \qquad Q = Q^* \tag{2.59}$$

In superspace, the components of the supercharge are given by

$$Q_A = \frac{\partial}{\partial\bar{\theta}^A} + i(\rho^\alpha\theta)_A\partial_\alpha \tag{2.60}$$

which satisfy the $\mathcal{N} = (1,1)$ supersymmetry algebra

$$\{Q_A, Q_B\} = 2i(\rho^\alpha\partial_\alpha)_{AB} \tag{2.61}$$

To write actions invariant under the supersymmetry, one needs a supercovariant derivative. Supercovariant derivative is invariant under supersymmetry and it is defined by:

$$\mathcal{D}_A = \frac{\partial}{\partial\bar{\theta}^A} - i(\rho^\alpha\theta)_A\partial_\alpha \tag{2.62}$$

It satisfies the following anticommutations

$$\{\mathcal{D}_A, \mathcal{D}_B\} = -2i(\rho^\alpha\partial_\alpha)_{AB} \tag{2.63}$$

2.A.1 SUSY lagrangian

We consider non-linear sigma models with Euclidean base space Σ, which can be a 2-torus T^2, a circle S^1, and a point. We refer to these as the worldsheet, worldline, and world point respectively. All our examples can be obtained by reductions of Euclidean Wick-rotated version of a $1+1$ dimensional worldsheet with $(1,1)$ supersymmetry, which we describe below.

Consider real superfields $\{X^i(\sigma,\theta)\}$ with the following superspace expansion

$$X^i(\sigma) = x^i(\sigma) + \bar{\theta}\psi^i(\sigma) + \frac{1}{2}\bar{\theta}\theta F^i(\sigma) \tag{2.64}$$

where $\{x^i\}$ are the coordinates of $2n$ dimensional real target manifold $\mathcal{M}$, ψ^i_A are real Grassmann fields and F^i are auxiliary fields. The components of the superfield can be thought of as the coordinates of a supermanifold $s\mathcal{M}$. The Lorentzian action is given by:

$$I(X) \;=\; -\frac{1}{2\pi\alpha'}\int_{s\Sigma} d^2\sigma\, d^2\theta\,\left[\frac{1}{2}g_{ij}(X)\bar{\mathcal{D}}X^i\mathcal{D}X^j + 2\,h(X)\right] \tag{2.65}$$

where $g_{ij}(x)$ is the metric on $\mathcal{M}$, $h(X)$ is the superpotential, and $\mathcal{D}$ is the superspace covariant derivative on the (base) superspace. We have introduced α' for easy comparison with other normalizations in the literature. The action is invariant under diffeomorphisms in the target space $\mathcal{M}$. It is also invariant under translations of t generated by the Hamiltonian H and translations of σ generated by P as well as under the $\mathbb{Z}_2$ action of $(-1)^F$:

$$\psi^i \to -\,\psi^i\,, \qquad x^i \to x^i\,. \tag{2.66}$$

Moreover, it is invariant under the $(1,1)$ supersymmetry generated by a real constant spinor ϵ_A under which the superfield transforms as $\delta X = (\bar{\epsilon}Q)X$ and its components transform as

$$\begin{aligned} \delta x^i &= \bar{\epsilon}\psi^i \\ \delta\psi^i &= (-i\gamma^\alpha\partial_\alpha x^i + F^i)\epsilon \\ \delta F^i &= -i\bar{\epsilon}\gamma^\alpha\partial_\alpha\psi^i\,. \end{aligned} \tag{2.67}$$

With $\alpha' = 1$, the action (2.65) in superfield components is given by

$$\begin{aligned} I \;=\; & -\frac{1}{2\pi}\int_\Sigma d^2\sigma\Big[\frac{1}{2}g_{ij}\left(\partial_\alpha x^i\partial^\alpha x^j - i\bar{\psi}^i\nabla\!\!\!\!/\,\psi^j - F^iF^j\right) \\ & + \frac{1}{4}\partial_k\partial_l g_{ij}(x)(\bar{\psi}^k\psi^l)(\bar{\psi}^i\psi^j) \;-\; \frac{1}{4}\partial_k g_{ij}\bar{\psi}^i\psi^j F^k + \frac{1}{4}\partial_k g_{ij}(F^i\bar{\psi}^j + F^j\bar{\psi}^i)\psi^k \\ & + \frac{\partial h}{\partial x^i}F^i - \frac{1}{2}\frac{\partial^2 h}{\partial x^i\partial x^j}(\bar{\psi}^i\psi^j)\Big] \end{aligned} \tag{2.68}$$

where the covariant derivative

$$\nabla_\alpha\psi^i = \partial_\alpha\psi^i + \Gamma^i{}_{jk}\,\partial_\alpha x^j\psi^k \tag{2.69}$$

is defined using the Christoffel symbols $\Gamma^i_{jk}(x)$ in the target space. When the superpotential is zero, eliminating the auxiliary fields yield the familiar quadratic fermionic term involving the Riemann curvature tensor [53, 54]. It is convenient to introduce an orthonormal basis of forms, $e^a = e^a_i dx^i$, using the vielbein e^a_i and the inverse vielbein $e^i{}_a$. The metric can then be expressed as $g_{ij} = e^a_i e^b_j \delta_{ab}$, and one can define the spin connection $\omega^a{}_{kb}$ associated with the Christoffel symbols.

2.A.2 Central Extension

In the presence of a central charge, the SUSY algebra modifies as follows:

$$\{\widehat{Q}_A, \widehat{Q}_B\} = 2i(\rho^\alpha \partial_\alpha)_{AB} + 2i(\bar{\rho})_{AB} \tag{2.70}$$

where $\bar{\rho}_{AB}$ is symmetric. The super-charge and the covariant derivative then become:

$$\widehat{Q}_A = Q_A + i(\bar{\rho}\,\theta)_A Z \qquad , \qquad \widehat{\mathcal{D}}_A = \mathcal{D}_A - i(\bar{\rho}\,\theta)_A Z \tag{2.71}$$

The presence of the Killing vector in the target space allows a central term in the supersymmetry algebra. The central charge is related to the Killing vector as $K_i(x(\sigma)) = Zx_i(\sigma)$, where x_i is the scalar field sitting in the chiral superfield . The action (2.68) modifies to

$$I \longrightarrow I + \frac{1}{2\pi} \int d^2\sigma \left(g_{ij} K^i K^j + i\bar{\psi}^i D_j K_i \bar{\rho} \psi_j \right) \tag{2.72}$$

where D_j is a covariant derivative on $\mathcal{M}$ [53]. For an off-shell formulation see [54].

Chapter 3

Atiyah-Patodi-Singer Index Theorem

In this chapter, we derive one of the main results of this thesis. We derive the Atiyah-Patodi-Singer index theorem using Supersymmetric Quantum Mechanics. In the next section, we elaborate on the APS index theorem and η-invariant, and then in the subsequent sections, we will give a new proof of APS index theorem using SQM.

3.1 APS Boundary Conditions

Consider a compact manifold $\mathcal{M}$ with a single boundary[1] $\partial\mathcal{M} = \mathcal{N}$ where $\mathcal{N}$ is a compact, connected, oriented manifold with no boundary as shown below:

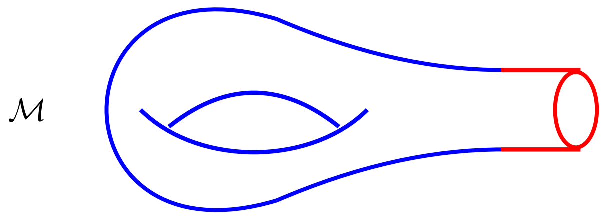


FIGURE 3.1.1: Manifold $\mathcal{M}$ with a collar $\mathcal{N} \times \mathbb{I}$ shown in red.

We wish to compute the index of Dirac operator on such a manifold. To have a well- defined index, we need to impose boundary conditions that preserve both self-adjointness of the Dirac operator and the chirality of the eigenvectors. Usual local boundary conditions like Dirichlet or Neumann do define a self-adjoint Dirac

[1] It is easy to generalize the analysis to a manifold with multiple boundaries.

operator. However, because of the reflection at the boundary, such local boundary conditions mix the positive and negative chirality modes and do not allow one to define the index. To preserve chirality, it is necessary to impose the nonlocal Atiyah-Patodi-Singer boundary conditions [1] explained below.

We assume that $\mathcal{M}$ has a product metric in the 'collar' region $\mathcal{N} \times \mathbb{I}$ near the boundary (Figure 3.1.1). We define local coordinates $\{y^m\,; m = 1, 2, \ldots, 2n-1\}$ on $\mathcal{N}$ and $u \leq 0$ on the interval $\mathbb{I}$ and the boundary is located at $u = 0$. The metric, near the boundary takes the form:

$$ds^2 \;=\; du^2 + g_{mn}|_{\mathcal{N}}\, dy^m dy^n\,, \tag{3.1}$$

The Dirac operator near the boundary becomes

$$\not{D} = \gamma^u \partial_u + \gamma^m D_m\,. \tag{3.2}$$

where $m = 1, ...2n-1$. It can be expressed as

$$\not{D} = \gamma^u(\partial_u \;+ \bar{\gamma}\mathcal{B}) \tag{3.3}$$

where $\bar{\gamma}$ is the chirality matrix on $\mathcal{M}$ and $\mathcal{B} = \widehat{\gamma}^m D_m$ is the boundary Dirac operator with $\widehat{\gamma}^m$ defined by

$$\gamma^m = (\gamma^u \bar{\gamma})\widehat{\gamma}^m \tag{3.4}$$

which satisfy the same Clifford algebra as the original γ matrices:

$$\{\gamma^m, \gamma^n\} = -2\, g^{mn}\,, \qquad\qquad \{\widehat{\gamma}^m, \widehat{\gamma}^n\} = -2\, g^{mn}\,. \tag{3.5}$$

The eigenvalue equation for the Dirac operator near the boundary takes the form

$$\begin{pmatrix} 0 & L \\ L^\dagger & 0 \end{pmatrix} \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix} = \sqrt{E} \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix} \tag{3.6}$$

where $L = \partial_u + \mathcal{B}$. The eigenfunctions can be written as

$$\Psi_-(u, y) = \sum_\lambda \Psi_-^\lambda(u)\, e_\lambda(y) \tag{3.7}$$

$$\Psi_+(u, y) = \sum_\lambda \Psi_+^\lambda(u)\, e_\lambda(y) \tag{3.8}$$

where $\{e_\lambda(y)\}$ are the complete set of eigenmodes of $\mathcal{B}$ with eigenvalue $\{\lambda\}$. For each mode we obtain:

$$\left(\frac{d}{du} + \lambda\right) \Psi_+^\lambda(u) = \sqrt{E}\Psi_-^\lambda(u)$$
$$\left(-\frac{d}{du} + \lambda\right) \Psi_-^\lambda(u) = \sqrt{E}\Psi_+^\lambda(u)\,. \tag{3.9}$$

To motivate the APS boundary conditions consider a noncompact 'trivial' extension $\widehat{\mathcal{M}}$ obtained by gluing a semi-infinite cylinder $\mathcal{N} \times \mathbb{R}^+$ (where $\mathbb{R}^+$ is the half line $u \geq 0$) to the original manifold (Figure 3.1.2). Near the boundary, the zero

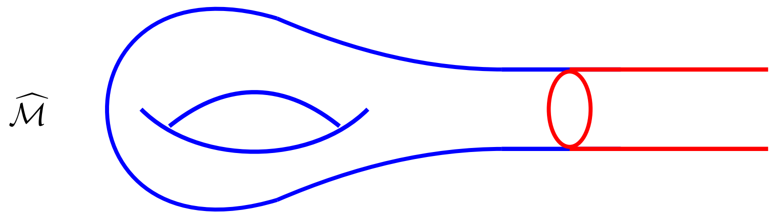


FIGURE 3.1.2: The noncompact $\widehat{\mathcal{M}}$ is a trivial extension of $\mathcal{M}$ obtained by gluing $\mathcal{N} \times \mathbb{R}^+$.

energy solutions of (3.9) on $\mathcal{M}$ have the form:

$$\Psi_\pm^\lambda(u) = \exp\left(\mp\lambda\, u\right)\Psi_\pm^\lambda(0) \tag{3.10}$$

One can ask which of these solutions can be extended to square-integrable or L_2-normalizable solutions on the noncompact manifold $\widehat{\mathcal{M}}$. Since u is positive on the semi-infinite cylinder, the solutions are normalizable if the argument of the exponent is negative. This is consistent with the APS boundary condition [50]. One sets the exponentially growing mode to zero which amounts to Dirichlet boundary condition for half the modes:

$$\Psi_+^\lambda(0) = 0 \qquad \forall \quad \lambda < 0$$
$$\Psi_-^\lambda(0) = 0 \qquad \forall \quad \lambda > 0\,. \tag{3.11}$$

For the remaining half, one uses Robin boundary conditions obtained by using equations (3.9) and (3.11):

$$\begin{aligned} \frac{d\Psi_+^\lambda}{du}(0) + \lambda\Psi_+^\lambda(0) &= 0 \qquad \forall \quad \lambda > 0 \\ -\frac{d\Psi_-^\lambda}{du}(0) + \lambda\Psi_-^\lambda(0) &= 0 \qquad \forall \quad \lambda < 0\,. \end{aligned} \tag{3.12}$$

These boundary conditions are consistent with supersymmetry as one can see from (3.9). By construction, imposing the APS boundary condition on $\mathcal{M}$ is equivalent to requiring L_2-normalizability for the solutions of Dirac equation on the noncompact extension $\widehat{\mathcal{M}}$.

3.2 APS Index Theorem and η-invariant

The APS index theorem states that the index of Dirac operator with APS boundary conditions on the compact Riemannian manifold $\mathcal{M}$ with boundary $\mathcal{N}$ is given by

$$\mathcal{I} = \int_{\mathcal{M}} \alpha(x) - \frac{1}{2}(\eta \pm h) \tag{3.13}$$

where $\int_{\mathcal{M}} \alpha(x)$ is the Atiyah-Singer term which is present also in the compact case, η is the Atiyah-Patodi-Singer η-invariant and h is the number of zero modes of the boundary operator $\mathcal{B}$.

The η-invariant is a measure of the spectral asymmetry which is equal to the regularized difference in the number of modes with positive and negative eigenvalues of the boundary operator $\mathcal{B}$ on $\mathcal{N}$. Let $\{\lambda\}$ be the set of eigenvalues of $\mathcal{B}$, then

$$\eta = \sum_{\lambda \neq 0} \mathrm{sgn}(\lambda) \qquad . \tag{3.14}$$

Here sgn is the sign function which is defined as

$$\begin{aligned} \mathrm{sgn}(\lambda) &= 1 && \text{for} \quad \lambda > 0 \\ \mathrm{sgn}(\lambda) &= -1 && \text{for} \quad \lambda < 0 \\ \mathrm{sgn}(\lambda) &= \text{regulator dependent} && \text{for} \quad \lambda = 0 \end{aligned} \tag{3.15}$$

The zero eigenvalue of $\mathcal{B}$ can be treated by slightly deforming the boundary operator but the answer depends on the direction in which one approaches zero. This sign ambiguity (the sign in front of h) is also present in the original APS formula in equation (3.13). The η-invariant for a large class of boundary manifolds have been computed by Hitchin in [55].

This infinite sum can be regularized in many ways. A natural regularization that arises from the path integral derivation is (3.35)

$$\widehat{\eta}(\beta) := \sum_{\lambda \neq 0} \text{sgn}(\lambda)\, \text{erfc}\left(|\lambda|\sqrt{\beta}\right). \tag{3.16}$$

Another regularization used in the original APS paper [1] is the ζ-function regularization

$$\eta_{\text{APS}}(s) = \sum_{\lambda \neq 0} \frac{\lambda}{|\lambda|^{s+1}} = \sum_{\lambda \neq 0} \frac{\text{sgn}(\lambda)}{|\lambda|^{s}}. \tag{3.17}$$

The two regularization schemes are related by a Mellin transform

$$\eta_{\text{APS}}(s) = \frac{s\sqrt{\pi}}{\Gamma(\frac{s+1}{2})} \int_0^\infty d\beta \beta^{\frac{s}{2}-1} \widehat{\eta}(\beta). \tag{3.18}$$

The APS η-invariant ($\eta_{\text{APS}}(s)$) for an operator $\mathcal{B}$ can be expressed in terms of the Riemann zeta function as follows [56]:

$$\eta_{\text{APS}} = \frac{\zeta_1(s) - \zeta_2(s)}{2^{-s} - 1} \tag{3.19}$$

where $\zeta_1(s)$ and $\zeta_2(s)$ are the Riemann zeta functions corresponding to the operators $\frac{3}{2}|\mathcal{B}| + \frac{1}{2}\mathcal{B}$ and $\frac{3}{2}|\mathcal{B}| - \frac{1}{2}\mathcal{B}$ respectively. So naviely it looks that η_{APS} has a pole at $s = 0$ but it can be checked that for the boundary Dirac operator, the residue at this pole vanishes [50]. Hence $\eta_{\text{APS}}(s)$ is analytic near $s = 0$ and the η-invariant can be defined as:

$$\eta = \lim_{s \to 0} \eta_{\text{APS}}(s). \tag{3.20}$$

It is expected that the answer is independent of the regularization up to local counter-terms that are implicit in the definition of a path integral.

The factor of half in front of η in (3.13) has the following consequence. As one varies the metric on $\mathcal{N}$, the eigenvalues of $\mathcal{B}$ can pass through a zero, and η would change by ± 2. The index then changes by ∓ 1 as expected for an integer. It also shows that neither the index nor the η-invariant are strictly topological and can change under smooth deformations of the boundary. They are nevertheless semi-topological, in the sense, they change only if the asymptotic data near the boundary is changed to alter the spectrum of $\mathcal{B}$.

3.3 APS index theorem & SQM

As shown in the last chapter, the index $\mathcal{I}$ of the Dirac operator equals the Witten index of SQM in the zero temperature limit i.e. in $\beta \to \infty$ limit:

$$\mathcal{I} := W(\infty) . \tag{3.21}$$

For a compact manifold $\mathcal{M}$ without boundary or with a boundary and APS boundary conditions, the Dirac operator is self-adjoint. It has a discrete spectrum, and its eigenvectors span the Hilbert space $\mathcal{H}$. As a result, the Witten index is independent of β and, in particular,

$$W(\infty) = W(0) . \tag{3.22}$$

This equality is an essential step in the proofs of both the Atiyah-Singer and the Atiyah-Patodi-Singer index theorems because one can then evaluate the Witten index in the much simpler $\beta \to 0$ limit using the high-temperature expansion of heat kernels.

For a compact target space without a boundary, the Witten index has a path integral representation, which is the starting point to obtain a derivation of the AS index theorem. Similarly, for a manifold with boundary, one would like to find a path integral representation, so as to apply localization. But there is an obvious difficulty in this case. In general, path integral formulation is much more subtle for a target space with a boundary. For this reason, it is convenient to map the problem to the computation of the Witten index $\widehat{W}$ of a noncompact manifold $\widehat{\mathcal{M}}$ without boundary. This will also lead to a 'spectral theoretic' reformulation of the APS theorem.

3.3.1 Noncompact Witten index

The APS boundary conditions imply that for every solution of the Dirac operator on $\mathcal{M}$, there is a L_2-normalizable solution of the extended Dirac operator on $\widehat{\mathcal{M}}$.

One can, therefore, aim to express the Dirac index in terms of the noncompact Witten index $\widehat{W}(\infty)$ at zero temperature, which admits a more straightforward path integral representation. One immediate problem with this idea is that the spectrum of Dirac Hamiltonian $\widehat{H}$ on $\mathcal{M}$ is expected to contain delta-function normalizable scattering states with a continuous spectrum in addition to the L_2-normalizable states. It is not clear then that the operator $(-1)^F e^{-\beta\widehat{H}}$ is 'trace class' in the conventional sense because it may not have a convergent trace after including the scattering states. Thus, even before developing the path integral for $\widehat{W}(\beta)$, it is necessary to give a proper definition for it in the canonical formulation that correctly generalizes (1.1).

A natural formalism for this purpose is provided by 'rigged Hilbert space' or 'Gel'fand triplet' which generalizes the Von Neumann formulation of quantum mechanics based on a Hilbert space [57, 58]. An advantage of this formalism is that one can discuss the spectral theory of operators with a continuous spectrum with 'generalized' eigenvectors, which may not be square-integrable. We review some of the relevant concepts as they apply in the present context.

The first Von Neumann axiom states that every physical system is represented (up to a phase) by a vector in a Hilbert space $\mathcal{H}$ with the unit norm. This is essential for the Born interpretation because the total probability of outcomes of measurements for any physical system must be unity. The second axiom requires that every physical observable corresponds to a self-adjoint operator on $\mathcal{H}$. This, however, is not always possible. A simple counterexample is a free particle on a line $\mathbb{R}$ with the Hamiltonian $H = p^2$. The self-adjoint operator corresponding to H on $\mathcal{H}$ has no normalizable eigenvectors, so the set of eigenvalues of this operator is empty. On the other hand, on physical grounds, one expects the free particle to have continuous energy with a sensible classical limit. To deal with such more general physical situations, it is necessary to relax the second axiom and represent physical observables by operators defined on a domain in a rigged Hilbert space using a Gel'fand triplet rather than on a domain in a Hilbert space.

For a quantum particle on a real line, the Gel'fand triplet consists of a Hilbert space $\mathcal{H}$, the Schwartz space $\mathcal{S}$, and the conjugate Schwartz space $\mathcal{S}^\times$. The Hilbert space $\mathcal{H}$ is isomorphic to the space $L_2(dx, \mathbb{R})$ of square-integrable wave functions on $\mathbb{R}$:

$$\mathcal{H} = \{|\psi\rangle\} \qquad \text{with} \qquad \langle\psi|\psi\rangle := \int dx\, \psi^*(x)\psi(x) < \infty\,; \tag{3.23}$$

The Schwartz space is the space of infinitely differentiable 'test functions' with exponential fall off. The conjugate Schwartz space $\mathcal{S}^\times$ is the set $\{|\phi\rangle\}$ such that

$$|\phi\rangle \in \mathcal{S}^\times \qquad \text{if} \qquad \langle\psi|\phi\rangle < \infty \qquad \forall \quad |\psi\rangle \in \mathcal{S}\,. \tag{3.24}$$

The Gel'fand triplet provides a rigorous way to define the bra and ket formulation of Dirac and offers a way to discuss the spectral theory of operators with continuous eigenvalues [57, 58]. The notion of the Schwartz space is motivated by the fact that it is left invariant by unbounded operators like the position operator x. The conjugate Schwartz space $\mathcal{S}^\times$ is where objects like the Dirac delta distribution $\delta(x)$ and plane waves e^{ipx} reside. The elements of $\mathcal{S}^\times$ need not have finite inner product with themselves and hence may not be square-integrable, but they have finite overlap with 'test functions' belonging to $\mathcal{S}$.

Consider now a self-adjoint Hamiltonian H defined on a domain $\mathcal{S} \subset \mathcal{H}$. One can define the conjugate Hamiltonian $H^\times$ acting on $|\phi\rangle \in \mathcal{S}^\times$ by the equation

$$\langle\psi|H^\times|\phi\rangle = \langle H\psi|\phi\rangle \qquad \forall \quad |\psi\rangle \in \mathcal{S} \tag{3.25}$$

With this definition, the eigenvalue equation for $H^\times$

$$H^\times|E\rangle = E|E\rangle\,, \qquad |E\rangle \in \mathcal{S}^\times \tag{3.26}$$

should be interpreted in terms of the overlap with test functions:

$$\langle\psi|H^\times|E\rangle = E\langle\psi|E\rangle \qquad \forall \quad |\psi\rangle \in \mathcal{S}\,. \tag{3.27}$$

A 'generalized eigenvector' $|E\rangle$ may lie outside the Hilbert space $\mathcal{H}$ and may not be normalizable. This means that it cannot be prepared in any experimental setup. Nevertheless, the set $\{|E\rangle\}$ provides a complete basis in the sense that any state in $\mathcal{H}$ can be expanded in terms of $\{|E\rangle\}$. This is the content of the Gel'fand-Maurin spectral theorem [58, 59].

For the example of a free particle discussed earlier, the operator $H^\times$ has the same formal expression as H as a differential operator:

$$H^\times = -\frac{d^2}{dx^2}\,. \tag{3.28}$$

However, the domain $D(H^\times)$ is much larger than the domain $D(H)$. This extension of the Hamiltonian is diagonalizable in the larger space $\mathcal{S}^\times$ with generalized eigenfunctions $\{e^{ipx}\}$ and eigenvalues $\{p^2\}$. In any laboratory with a finite extent, one can never experimentally realize an exact plane wave but only a wave packet that is sufficiently close to the energy eigenfunction. Nevertheless, the plane waves form a complete basis in the sense that a square-integrable function in $\mathcal{H}$ can be Fourier-expanded in terms of plane waves. We denote the total space of generalized eigenvectors of $H^\times$ by Sp, which may contain both the square-integrable bound states with discrete energies as well as nonnormalizable scattering states with continuous energies.

Usually, one can gloss over these niceties essentially because of the locality. A particle on an infinite line is an extreme idealization in a universe which may be finite. One expects that measurements of local quantities such as scattering cross-sections in a particle physics experiment in a laboratory should not be affected by boundary conditions imposed at the end of the universe. One should arrive at the same physical conclusions whether one uses periodic or Dirichlet boundary conditions in a large box, as one indeed finds in textbook computations.

In the present situation, we are interested in global properties that depend sensitively on the boundary conditions. For example, one cannot impose Dirichlet boundary conditions while preserving supersymmetry. The Gel'fand triplet provides an appropriate formulation to discuss the scattering states without the need to put any boundary conditions to 'compactify' space. With these preliminaries, one can define the noncompact Witten index by

$$\widehat{W}(\beta) := \operatorname*{Tr}_{Sp(\hat{H})} \left[(-1)^F e^{-\beta\widehat{H}}\right] \tag{3.29}$$

On the face of it, this definition is still not completely satisfactory. Even though the spectrum Sp over which one traces now has a precise meaning, it is not clear that the trace thus defined actually converges. For example, for a free particle, the heat kernel is well-defined[2]

$$K(x, y; \beta) = \langle x|e^{-\beta H}|y\rangle = \frac{1}{\sqrt{4\pi\beta}} \exp\left[-\frac{(x-y)^2}{4\beta}\right]. \tag{3.30}$$

[2]In what follows, we will use H instead of $H^\times$ when there is no ambiguity to unclutter the notation.

However, if we try to define a trace then there is the usual 'volume' divergence:

$$\int dx \langle x|e^{-\beta H}|x\rangle = \frac{1}{\sqrt{4\pi\beta}} \int dx \to \infty\,. \tag{3.31}$$

One might worry that the noncompact Witten index is also similarly divergent. Fortunately, the Witten index is a supertrace or, equivalently, a trace over the *difference* between two heat kernels corresponding to the bosonic and fermionic Hamiltonians H_+ and H_- respectively. If there is a gap between the ground states and the scattering states, then the difference between two Hamiltonians vanish as $r \to \infty$. As a result, the volume divergent contribution cancels in the supertrace. In the path integral representation, this corresponds to the fact that the supertrace involves integrals over the 'fermionic zero modes' in addition to the 'bosonic zero-mode' x. Under suitable conditions, the fermionic Berezin integration localizes the bosonic integral to a compact region on the real line to yield a finite answer. In particular, the path integral receives vanishing contribution from the asymptotic infinity in field space. We elaborate on this point in §3.4.1.

3.3.2 APS index & Non-compact Witten index

Using the above framework, one can now express the index of the Dirac operator on the original manifold $\mathcal{M}$ with a boundary in terms of a noncompact Witten index $\widehat{W}(\beta)$ on $\widehat{\mathcal{M}}$. Assuming that the continuum states in *Sp* are separated from the ground states by a gap, at zero temperature only the L_2-normalizable ground states contribute to $\widehat{W}(\infty)$. Since these states are in one-one correspondence with the states in the original Hilbert space $\mathcal{H}$ on $\mathcal{M}$ with APS boundary conditions, we conclude

$$\mathcal{I} = \widehat{W}(\infty)\,. \tag{3.32}$$

In the limit of $\beta \to 0$, one can evaluate the Witten index by using the short proper time expansion of the heat kernels to obtain a local expression. It must correspond to the Atiyah-Singer term but now evaluated over $\widehat{\mathcal{M}}$:

$$\widehat{W}(0) = \int_{\widehat{\mathcal{M}}} \alpha = \int_{\mathcal{M}} \alpha \tag{3.33}$$

where the second equality follows from the fact that the topological index density vanishes on the half-cylinder $\mathcal{N} \times \mathbb{R}^+$. We can therefore write

$$\mathcal{I} = \widehat{W}(0) + (\widehat{W}(\infty) - \widehat{W}(0)) \, . \tag{3.34}$$

The term in the parenthesis is no longer zero and is in fact related to the η-invariant as will show in §3.3.3. It is convenient to consider a regularized quantity

$$\widehat{\eta}(\beta) = 2(\widehat{W}(\beta) - \widehat{W}(\infty)) \tag{3.35}$$

which in the limit $\beta \to 0$ reduces to the bracket in (3.34). This provides a natural regularization described earlier in (3.16). With these identifications, equation (3.34) can be viewed as the statement of the APS theorem; the discussion above together with §3.3.3 can be viewed as a derivation of the APS result. The noncompact Witten index is, in general, β-dependent because at finite temperature, the scattering states also contribute. The bosonic and fermionic density of states in this continuum may not be exactly equal and need not cancel precisely. The η-invariant of the boundary manifold $\mathcal{N}$ thus measures the failure of the Witten index of the noncompact manifold $\widehat{W}$ to be temperature independent[3].

3.3.3 Scattering theory and the APS theorem

There is a simpler way to compute $\widehat{W}(0)$ that makes this connection with the bulk Atiyah-Singer term (3.33) more manifest and easier to relate it to a path integral. One can simply double the manifold to $\overline{\mathcal{M}}$ by gluing its copy as in Figure **3.3.1** as was suggested in [1]. Since $\overline{\mathcal{M}}$ is a compact manifold without a boundary, its index does not have any contribution from the η-invariant. Moreover, by the reasoning before (3.33) the $\beta \to 0$ expansion is local and gives the Atiyah-Singer index density. In summary,

$$\widehat{W}(0) = \frac{1}{2}\overline{W}(0) = \int_{\mathcal{M}} \alpha \tag{3.36}$$

To prove the APS theorem, we would like to show that the term in the parentheses in (3.34) equals the η-invariant. We note that the spectrum $Sp(H)$ of the Hamiltonian on $\widehat{\mathcal{M}}$ is a direct sum of the discrete spectrum of bound states $Sp_b(H)$

[3] This was noticed earlier in [60] in a particular example.

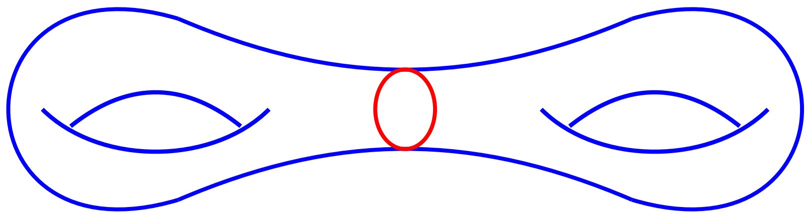


FIGURE 3.3.1: The doubled compact manifold $\overline{\mathcal{M}}$ without boundary.

and the continuum spectrum of scattering states $Sp_s(H)$. Therefore the Witten index admits a spectral decomposition

$$\widehat{W}(\beta) = \operatorname*{Tr}_{SP_b}\left[(-1)^F e^{-\beta\widehat{H}}\right] + \operatorname*{Tr}_{SP_s}\left[(-1)^F e^{-\beta\widehat{H}}\right] \tag{3.37}$$

Since the continuum of states is separated from the zero energy states, it is clear that the first term can be identified with $\widehat{W}(\infty)$. To prove (3.35), we thus need to show that the contribution from the continuum equals the η-invariant:

$$2\operatorname*{Tr}_{SP_s}\left[(-1)^F e^{-\beta\widehat{H}}\right] = \widehat{\eta}(\beta)\,. \tag{3.38}$$

We show this by relating the supertrace to the difference in the density of bosonic and fermionic scattering states[4] on $\widehat{\mathcal{M}}$ which in turn can be related to the difference in phase shifts.

Asymptotically, the metric on $\widehat{\mathcal{M}}$ has the form (3.1) with $0 < u < \infty$. We can use separation of variables to first diagonalize the operator $\mathcal{B}$ on $\mathcal{N}$ with eigenvalues $\{\lambda\}$. The Dirac operator on manifold $\widehat{\mathcal{M}}$ can be expressed in terms of eigenvalues λ of the boundary operator $\mathcal{B}$ as in (3.9). Here, we assume that the boundary operator does not have any zero eigenvalue. The asymptotic form of the scattering wave

[4]Index theory on non-compact manifold and its relation to scattering theory has been considered earlier to compute threshold bound states [61–64] without making the connection to APS index theorem and η-invariant. The relation between η-invariant and scattering theory was observed earlier in special cases in [65–67].

functions is then

$$\begin{aligned} \psi_+^{\lambda k}(u) &\sim c_+^\lambda \left[e^{iku} + e^{i\delta_+^\lambda(k) - iku}\right] \\ \psi_-^{\lambda k}(u) &\sim c_-^\lambda \left[e^{iku} + e^{i\delta_-^\lambda(k) - iku}\right] \end{aligned} \tag{3.39}$$

where $\delta_\pm^\lambda(k)$ are the phase shifts. The trace (3.38) over scattering states can be expressed as

$$2\sum_\lambda \int dk \left[\rho_+^\lambda(k) - \rho_-^\lambda(k)\right] e^{-\beta E(k)} \tag{3.40}$$

where $\rho_+^\lambda(k)$ and $\rho_-^\lambda(k)$ are the densities of bosonic and fermionic states of the theory in λ subsector. Using a standard result (3.128) from scattering theory, which we review in §3.B, we can relate the difference in the density of states to the difference in phase shifts

$$\rho_+^\lambda(k) - \rho_-^\lambda(k) = \frac{1}{\pi}\frac{d}{dk}\left[\delta_+^\lambda(k) - \delta_-^\lambda(k)\right] . \tag{3.41}$$

In general, the individual phase shifts and density of states are nontrivial functions of k that depend on the details of the manifold $\widehat{\mathcal{M}}$. After all, they contain all the information about the S-matrix. The exact form of the scattering states similarly has a complicated functional dependence on u. Generically, it would be impossible to compute any of them exactly. Remarkably, the *difference* between the phase shifts is determined entirely by the asymptotic data as a consequence of supersymmetry relation (3.9) in the asymptotic region. By substituting the asymptotic wave-functions (3.39) into (3.9) we obtain

$$c_+^\lambda \sqrt{E}\left[e^{iku} + e^{i2\delta_+^\lambda} e^{-iku}\right] = c_-^\lambda\left[(-ik+\lambda)e^{iku} + e^{i2\delta_-^\lambda}(ik+\lambda)e^{-iku}\right] \tag{3.42}$$

with $E = k^2 + \lambda^2$. This implies

$$\frac{c_+^\lambda}{c_-^\lambda} = \frac{(-ik+\lambda)}{\sqrt{E}}, \qquad \frac{e^{i2\delta_+^\lambda}}{e^{i2\delta_-^\lambda}} = -\frac{(ik+\lambda)}{(ik-\lambda)} \tag{3.43}$$

and therefore,

$$2\delta_+^\lambda(k) - 2\delta_-^\lambda(k) = -i\ln\left(\frac{ik+\lambda}{ik-\lambda}\right) + \pi \tag{3.44}$$

in each eigensubspace with eigenvalue λ. Now we use the equation (3.44) in (3.128) to obtain:

$$\rho_+^\lambda(k) - \rho_-^\lambda(k) = -\frac{\lambda}{\pi(k^2+\lambda^2)} \tag{3.45}$$

After summing over all λ we obtain

$$\begin{aligned} 2\operatorname*{Tr}_{SP_s}\left[(-1)^F e^{-\beta\widehat{H}}\right] &= \sum_\lambda \int_0^\infty dk\ \left[\rho_+^\lambda(k) - \rho_-^\lambda(k)\right]\ e^{-\beta(k^2+\lambda^2)} \\ &= \sum_\lambda \operatorname{sgn}(\lambda)\operatorname{erfc}\left(|\lambda|\sqrt{\frac{\beta}{2}}\right) \end{aligned} \tag{3.46}$$

This is precisely the regulated expression (3.35) for the η-invariant of the boundary operator. We have thus proven

$$\mathcal{I} = \widehat{W}(0) + (\widehat{W}(\infty) - \widehat{W}(0)) = \int_{\mathcal{M}} \alpha - \frac{1}{2}\eta \tag{3.47}$$

which is the Atiyah-Patodi-Singer index theorem.

3.4 The η-invariant and path integrals

Given the definition of the noncompact Witten index in §3.3.1, one can use its path-integral representation and use localization methods to compute it. In §3.4.3, we show how this works for a two-dimensional finite cigar with a boundary by relating its index to the Witten index of the infinite cigar. In this simple example, one can explicitly evaluate the Witten index using localization and compare it with the η-invariant obtained from operator methods.

In this section, we will formulate a path integral that directly computes the η-invariant without the bulk Atiyah-Singer piece. This can be achieved as follows. As we have observed in §3.3.3, given a manifold with boundary $\mathcal{M}$ such that its metric is of the product form near the boundary, we can *trivially* extend the manifold to a noncompact manifold $\widehat{\mathcal{M}}$. In $\widehat{\mathcal{M}}$, the η-invariant gets contribution only from the scattering states of $\widehat{\mathcal{M}}$. Since the scattering states just depend on asymptotics, the scattering states of $\widehat{\mathcal{M}}$ are the same as those of $\mathbb{R}^+ \times \mathcal{N}$ with APS boundary condition at the origin. We use this physical picture to find a path integral representation to compute η-invariant. First, we will explain the space-time picture, and then we will map it to a worldline computation.

The Dirac operator on the half line is given by $\widetilde{\not{D}} = \gamma^u(\partial_u + \bar{\gamma}\mathcal{B})$. We can diagonalize the boundary operator as in (3.7), (3.8). Effectively, for each eigenvalue λ of the boundary operator $\mathcal{B}$, we have a supersymmetric quantum mechanics on a half line. The APS boundary condition is essentially Dirichlet boundary condition for one chirality and Robin boundary condition for the other chirality. To obtain a path integral representation with these boundary conditions, it is more convenient to 'double' the manifold $\mathbb{R}^+ \times \mathcal{N}$ to obtain a noncompact cylinder $\widetilde{\mathcal{M}} := \mathbb{R} \times \mathcal{N}$ (see Figure **3.4.1**) without any boundary. We extend it in a manner that is consistent with the APS boundary conditions. The manifold $\widetilde{\mathcal{M}} := \mathbb{R} \times \mathcal{N}$ possess parity symmetry

$$P: \qquad u \to -u \qquad , \qquad \psi_\pm \to -\psi_\pm \tag{3.48}$$

that is consistent with supersymmetry and leaves the supercharge invariant. The path integral on the original manifold $\mathbb{R}^+ \times \mathcal{N}$ with APS boundary conditions can thus be obtained by considering the path integral on the manifold $\widetilde{\mathcal{M}} := \mathbb{R} \times \mathcal{N}$ projected onto P invariant states. This is obtained by the insertion of the following operator

$$\frac{1}{2}\Big[1 + P\Big] \tag{3.49}$$

in the path integral. Here P is the parity operator. Invariance under the reflection of u keeps only parity-even wave functions in the trace for one chirality, effectively imposing Dirichlet boundary condition on the half-line. Supersymmetry ensures that the other chirality satisfies the Robin boundary condition as required by the APS boundary conditions. See, for example, [10, 68] for a more detailed discussion.

Since we are interested in the operator $\widetilde{\not{D}} = \gamma^u(\partial_u + \bar{\gamma}\mathcal{B})$ on the half-line, the extension of this operator should transform as an eigen-operator under parity. Given u transform as in (3.48) we are left with the choice

$$\mathcal{B} \to -\mathcal{B}. \tag{3.50}$$

This ensures that Dirac operator as a whole is invariant. The extended Dirac operator on the doubled cylinder thus takes the form

$$\widetilde{\not{D}} = \gamma^u(\partial_u + \varepsilon(u)\bar{\gamma}\mathcal{B}) \tag{3.51}$$

FIGURE 3.4.1: The doubled noncompact cylinder $\widetilde{\mathcal{M}} = \mathcal{N} \times \mathbb{R}$.

instead of (3.2) where $\varepsilon(u)$ is a step function with a discontinuity at $u = 0$. One can also take $\varepsilon(u)$ to be a smooth smearing function which interpolates between -1 to 1 as u varies from $-\infty$ to $+\infty$ to obtain a smooth Dirac operator. One example of such function is $\tanh(au)$. This does not change our conclusions because the η-invariant does not change under deformations that do not change the asymptotics.

Now we return to the worldline picture. For each eigenvalue of $\mathcal{B}$, we can map this problem to a worldline path integral problem. The effect of the eigenvalue of the boundary operator can be incorporated by adding a superpotential $h(u)$ in the Hamiltonian, whose derivative w.r.t. u is given by:

$$h'(u) = \varepsilon(u)\lambda$$

Notice that we need at least two real supercharges in the theory to add a superpotential term. Hence, in this case, we will be working with $N = 1$ SUSY (one complex supercharge). It reduces the problem to computing Witten index $\widetilde{W}(\beta)$ for an SQM with a superpotential $h'(u) = \varepsilon(u)\lambda$ on a target space $\mathbb{R}$. The Witten index can now be computed using path integral. With this construction, we conclude

$$\widehat{\eta}(\beta) = \widetilde{\eta}(\beta) = 2(\widetilde{W}(\beta) - \widetilde{W}(\infty)) \tag{3.52}$$

It is straightforward to write a path integral representation for $\widetilde{\eta}(\beta)$ on the noncompact cylinder $\widetilde{\mathcal{M}}$ which is much simpler than the path integral on $\widehat{\mathcal{M}}$. In §3.4.2, we compute it using localization and relate it to Callias index [14, 15].

3.4.1 Supersymmetric worldpoint integral

Some of the essential points about a noncompact path integral can be illustrated by a 'world point' path-integral where the base space Σ is a point and the target space $\mathcal{M}$ is the real line $-\infty < u < \infty$. We discuss this example first before proceeding to localization. The supersymmetric worldpoint action is given by

$$S(u, F, \psi_-, \psi_+) = \frac{1}{2}F^2 + iF\, h'(u) + ih''(u)\, \psi_-\psi_+ \tag{3.53}$$

where

$$h'(u) := \frac{dh}{du}, \qquad h''(u) = \frac{d^2h}{du^2}. \tag{3.54}$$

The action can be obtained from the Euclidean continuation[5] of (2.68) by setting

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial\sigma} = 0, \qquad g_{11}(u) = 1. \tag{3.55}$$

The SUSY transformations can also be obtained from (2.67) by setting above terms to zero. Notice that in this case we have two real supercharges Q_+ and Q_-.

The path integral is now just an ordinary superintegral with flat measure

$$W(\beta) = -i\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dF \int d\psi_-\, d\psi_+\, \exp\left[-\beta S(U)\right]. \tag{3.56}$$

where $-i$ comes from fermionic normalization as expalined in (2.42). A particularly interesting special case is

$$h'(u) = \lambda \tanh(au), \tag{3.57}$$

for real λ. Integrating out the fermions and the auxiliary field F gives

$$W(\beta) = -\sqrt{\frac{\beta}{2\pi}}\int_{-\infty}^{\infty} dx\, h''(u) \exp\left[-\frac{\beta}{2}(h'(u))^2\right] \tag{3.58}$$

One can change variables

$$y = \sqrt{\frac{\beta}{2}}h'(u), \quad dy = \sqrt{\frac{\beta}{2}}h''(u)dx \tag{3.59}$$

As u goes from $-\infty$ to ∞, $y(u)$ is monotonically increasing or decreasing depending on if λ is positive or negative; the inverse function $u(y)$ is single-valued, and the

[5]Note that in the Euclidean continuation $F \to iF$. So, the limit $\frac{\partial}{\partial\tau} = 0$ of the Euclidean and the Lorentzian actions gives different actions for the supersymmetric integral.

integral reduces to

$$\begin{aligned} W(\beta) &= -\frac{1}{\sqrt{\pi}} \int_{-\sqrt{\frac{\beta}{2}}\lambda}^{\sqrt{\frac{\beta}{2}}\lambda} dy\, e^{-y^2} \\ &= -\mathrm{sgn}(\lambda)\, \mathrm{erf}\left(\sqrt{\frac{\beta}{2}}|\lambda|\right) \end{aligned} \tag{3.60}$$

where erf(z) is the Error-function given by (3.130), not to be confused with the complimentary Error function erfc(z).

This world point integral illustrates several important points.

1. The integral has a volume divergence without the fermionic integrations because $h'(u)$ is bounded above for large $|u|$. The inclusion of fermions effectively limits the integrand to the region close to the origin where $h'(u)$ varies and makes the integral finite.
2. In the limit $\lambda \to 0$, the action reduces to that of a free superparticle. In this case, the integral is of the form $\infty \times 0$ and is ill-defined. Regularizing with λ yields different answers depending on whether we approach 0 from the positive or negative side. This is related to the jump in η-invariant when an eigenvalue of the boundary operator $\mathcal{B}$ crosses a zero in a spectral flow, as explained before figure **3.4.2**.
3. The answer depends only on the asymptotic behavior of $h'(u)$ at $\pm\infty$ and is independent of any deformations that do not change the asymptotics. In particular, one would obtain the same result in the limit $a \to \infty$ in (3.57), when $h'(u)$ can be expressed in terms of the Heaviside step function:

$$h'(u) = \lambda\Big[\theta(u) - \theta(-u)\Big]\,. \tag{3.61}$$

4. The error function (3.60), which appears naturally in this integral, makes its appearance in the proof of the APS theorem [1, 50] and also in the definition of the completion (4.12) of a mock modular form and, in particular, in (4.59). This is not a coincidence. The two turn out to be related through a path integral which localizes precisely to the ordinary super integral considered above. For this reason, this example is particularly important for our discussions of the η-invariant and its connection to mock modularity.

The world point integral does not have an operator interpretation in terms of a trace over a Hilbert space. To see the connection with the canonical formalism, we consider in §3.4.2 the worldline version corresponding to the path integral for a supersymmetric quantum mechanics, and relate it to Callias-Seeley-Bott index theorem [14, 15] . After localization, the worldline path integral will reduce to the world point integral considered above.

3.4.2 Callias index theorem and the η-invariant

In this section, we compute the η-invariant by computing path integral for SQM with target space $\widetilde{\mathcal{M}}$. As discussed in §3.3.3 we can use separation of variables to first diagonalize the operator $\mathcal{B}$ on $\mathcal{N}$ with eigenvalues $\{\lambda\}$ but now for the entire manifold $\widetilde{\mathcal{M}}$. For each eigenvalue λ, the problem reduces to a supersymmetric quantum mechanics with a one-dimensional target space and a superpotential $h(u)$. The path integral for this problem can be readily written down and has been considered earlier in [69]. The action for SQM can be obtained as a specialization of (2.68) with target space $\mathbb{R}$ and by setting

$$\frac{\partial}{\partial\sigma} = 0\,. \tag{3.62}$$

The Euclidean action for the components of the superfield U is

$$S = \int_0^\beta d\tau \left[\frac{1}{2}\dot{u}^2 + \frac{1}{2}\psi_-\dot{\psi}_- + \frac{1}{2}\psi_+\dot{\psi}_+ + \frac{1}{2}F^2 + ih'(u)F + ih''(u)\psi_-\psi_+\right] \tag{3.63}$$

with $h'(u) = \lambda \tanh u$ (where $\lambda \neq 0$). The SUSY transformations for this case can again be obtained from (2.67) by taking appropriate limits. To compute the η-invariant we need to evaluate the projected Witten index (3.49)

$$\begin{aligned}
\widetilde{W}(\beta) &= \frac{1}{2}\text{tr}\left[(-1)^F(1+P(-1)^F)\right] = \frac{1}{2}\text{tr}\left[(-1)^F\right] + \frac{1}{2}\text{tr}\left[P\right] \\
&= \frac{1}{2}\widetilde{W}_1(\beta) + \frac{1}{2}\widetilde{W}_2(\beta)
\end{aligned} \tag{3.64}$$

The path integral for the first term is the same as before with periodic boundary conditions for bosons and fermions. In the path integral for the second term both the bosons and the fermions are anti-periodic because of the insertion of $P(-1)^F$.

Given a path integral representation for the Witten index (2.34), one can use a technique called supersymmetric *localization* to compute it. We elaborate on this technique in the appendix 3.A. To use localization, we deform the lagrangian by adding Q_+V (where Q_+ is a real supercharge) to it and V is given by:

$$\begin{aligned} V &= \psi_+(Q_+\psi_+) = i\psi_+\dot{u} \\ Q_+V &= \dot{u}^2 + \psi_+\dot{\psi}_+ \\ Q_+^2V &= 0 \end{aligned}$$

After using localization, the path intergal localizes to the solutions of $Q_+V = 0$ i.e. the constant modes of u and ψ_+. Fluctuations around the constant modes are given by

$$u = u_0 + \frac{1}{\sqrt{\xi}}\tilde{u} \qquad \psi_+ = \psi_{+0} + \frac{1}{\sqrt{\xi}}\eta \tag{3.65}$$

with $\tilde{u}$ and η satisfying periodic boundary conditions. So we have,

$$\widetilde{W}_1(\beta) = -i\int du_0[dF][d\tilde{u}][d\psi_-]d\psi_{+0}[d\eta]\exp\left(-S[X_0,\beta] - \xi(Q_+V)^{[2]}\right)$$

where $(Q_+V)^{[2]}$ are the quadratic fluctuations around the localization locus. Expanding the ψ_- in modes and after evaluating the non-zero mode integrals we obtain

$$\widetilde{W}_1(\beta) = -i\int \frac{du_0}{\sqrt{2\pi\beta}}\, d\psi_{-0}\, d\psi_{+0}\exp\left(-\frac{\beta}{2}(h'(u_0))^2 - i\beta(h''(u_0)\psi_{-0}\psi_{+0}\right) \tag{3.66}$$

The factor of $\frac{1}{\sqrt{2\pi\beta}}$ comes from the determinants. The integral (3.66) is identical to the world point superintegral (3.58). Hence we obtain

$$\widetilde{W}_1(\beta) = -\mathrm{sgn}(\lambda)\,\mathrm{erf}\left(|\lambda|\sqrt{\frac{\beta}{2}}\right) \tag{3.67}$$

We use localization to compute the second piece as well. In this case, the path-integral localizes to $u = 0 = \psi_+$. Small fluctuations around the saddle point are given by

$$u = 0 + \frac{1}{\sqrt{\xi}}\bar{u} \qquad \psi_+ = 0 + \frac{1}{\sqrt{\xi}}\bar{\eta} \tag{3.68}$$

Here $\bar{u}$ and $\bar{\eta}$ satisfy *anti*-periodic boundary condition.

$$\widetilde{W}_2[\beta] = -i\int [dF][d\bar{u}][d\psi_-][d\bar{\eta}]\exp\left(-S[X_0,\beta] - \xi(Q_+V)^{[2]}\right) = 1 \tag{3.69}$$

So the full answer is given by (3.64)

$$\widetilde{W}(\beta) = -\frac{1}{2}\mathrm{sgn}(\lambda)\,\mathrm{erf}\left(|\lambda|\sqrt{\frac{\beta}{2}}\right) + \frac{1}{2} \tag{3.70}$$

We have performed the computation for a single eigenvalue λ. We get (3.70) for each eigenvalue. From (3.52) we obtain the η-invariant to be

$$\widetilde{\eta}(\beta) = \sum_\lambda \mathrm{sgn}(\lambda)\left[1 - \mathrm{erf}\left(|\lambda|\sqrt{\frac{\beta}{2}}\right)\right] = \sum_\lambda \mathrm{sgn}(\lambda)\mathrm{erfc}\left(|\lambda|\sqrt{\frac{\beta}{2}}\right) \tag{3.71}$$

which reproduces the expression (3.46) for $\widehat{\eta}(\beta)$ obtained from scattering theory. Hence

$$\widetilde{\eta}(\beta) = \widehat{\eta}(\beta) \tag{3.72}$$

In conclusion, the Witten index for the worldline quantum mechanics is temperature-dependent as a consequence of the noncompactness of the target manifold and this temperature dependence helps to compute the η-invariant.

3.4.3 The η-invariant of a finite cigar

It is instructive to apply the general formulation developed in earlier sections to an explicit computation for a simple and illustrative example where $\mathcal{M}$ is a two dimensional *finite* cigar with metric

$$ds^2 = k\,(dr^2 + \tanh^2 r\, d\theta^2) \tag{3.73}$$

where θ is a periodic with period 2π and $0 \le r \le r_c$. The manifold has a boundary at $r = r_c$ with a product form $\mathcal{N} \times \mathbb{I}$ where $\mathcal{N}$ is the circle parametrized by θ with radius $\sqrt{k}$. The non-zero Christoffel symbols are

$$\Gamma_{r\theta\theta} = -\frac{1}{2}k\,\partial_r(\tanh^2 r) \qquad \Gamma_{\theta\theta r} = \Gamma_{\theta r\theta} = \frac{1}{2}k\,\partial_r(\tanh^2 r) \tag{3.74}$$

The orthonormal forms and the nonzero vielbeins are

$$\begin{aligned} e^1 &= \sqrt{k}dr \qquad & e^2 &= \sqrt{k}\tanh(r)d\theta \\ e^1_r &= \sqrt{k} \qquad & e^2_\theta &= \sqrt{k}\tanh(r) \end{aligned} \tag{3.75}$$

The cigar has a Killing isometry under translations of θ with the Killing vector

$$K^i = (0,1) \qquad K_i = g_{ij}K^j = (0, k\tanh^2 r)\,. \tag{3.76}$$

The $N = (0,1)$ supersymmetric action can be obtained from (2.68) by setting

$$F^i = 0 \qquad \psi^i_- = 0 \qquad \psi_+ = \psi^i\,. \tag{3.77}$$

The Lorentzian action is given by:

$$I = \frac{1}{4\pi}\int d^2\sigma\, g_{ij}\Big(\partial_\tau X^i\partial_\tau X^j - \partial_\sigma X^i\partial_\sigma X^j + i\psi^i D_{\tau-\sigma}\psi^j\Big) \tag{3.78}$$

We dimensionally reduce along the worldsheet σ direction to convert the action on the 2-torus to a collection of actions on a circle. Scherk-Schwarz reduction [70] along the sigma direction using the Killing vector gives

$$X^i(\sigma + 2\pi) \;=\; X^i(\sigma) + 2\pi w K^i \tag{3.79}$$

where w is the winding number. We have

$$\partial_\sigma X^i = wK^i \quad \text{and} \quad \partial_\sigma\psi^i = -w\partial_j K^i\psi^j \tag{3.80}$$

where the derivative of ψ^i is deduced from the transformation of the superfield $X^i = x^i + \bar\theta\psi^i + \bar\theta\theta F^i$ under the Killing symmetry. Using (3.80) in action (4.92) and integrating over the σ direction we obtain the following Euclidean action after a

Wick rotation:

$$S = \frac{1}{2}\int d\tau\Big(G_{ij}\partial_\tau X^i\partial_\tau X^j + G_{ij}w^2K^iK^j + G_{ij}\psi^iD_\tau\psi^j - iw\psi^iK_{ij}\psi^j\Big) \tag{3.81}$$

After plugging (3.73) in the action (3.81) we get:

$$\begin{aligned} S[\beta;k,w] &= \int_0^\beta d\tau\frac{1}{2}\Big(k\dot{r}^2 + k\tanh^2 r\dot{\theta}^2 + w^2k\tanh^2 r + k\psi^r\dot{\psi^r} - k\psi^r\partial_r(\tanh^2 r)\dot{\theta}\psi^\theta \\ &\quad + k\tanh^2 r\psi^\theta\dot{\psi^\theta} - iw\psi^\theta\partial_r(k\tanh^2 r)\psi^r\Big) \end{aligned} \tag{3.82}$$

Our goal is to evaluate the path integral on the infinite cigar and then connect it to the η-invariant for a finite cigar. We use localization to compute the Witten index of infinite cigar. To perform localization, we deform the action by adding QV to it, where V is given by:

$$V = G_{rr}\psi^r\delta\psi^r = k\psi^r\dot{r} \tag{3.83}$$

This localizes the integral to constant modes of r and ψ^r. We have:

$$\widehat{W}(\beta) = -i\int dr_0 d\psi_0^r[d\theta][d\psi^\theta]\,\exp\left[-\int_0^\beta d\tau L(r_0,\psi_0^r,\theta,\psi^\theta) - \xi\int_0^\beta QV^{[2]}\right] \tag{3.84}$$

where $QV^{[2]}$ are the quadratic fluctuations around the localization locus. We expand

$$r = r_0 + \frac{1}{\sqrt{\xi}}\chi \qquad \psi^r = \psi_0^r + \frac{1}{\sqrt{\xi}}\eta^r \tag{3.85}$$

so that the quadratic fluctuations are given by

$$\xi\int_0^\beta d\tau\, QV^{[2]} = \int_0^\beta d\tau(k\dot{\chi}^2 + k\eta^r\dot{\eta^r})\,. \tag{3.86}$$

The transformation (3.85) has a unit Jacobian. We can now mode expand θ and ψ^θ and we have

$$\theta(\tau) = \frac{2\pi p\tau}{\beta} + \sum_m \theta_m e^{2\pi i m\tau/\beta} \qquad \psi^\theta(\tau) = \psi_0^\theta + \sum_m \psi_m^\theta e^{2\pi i m\tau/\beta} \tag{3.87}$$

After integrating out the fluctuations and non-zero modes of θ and ψ^θ, we obtain:

$$\begin{aligned}\widehat{W}(\beta;k,w) = & -\frac{2\pi i}{2\pi\beta}\int dr_0 d\theta_0 d\psi_0^r d\psi_0^\theta \sum_p \exp\left[-\int_0^\beta d\tau\, S(r_0,\psi_0^r,\psi_0^\theta)\right] \\ = & -\frac{i}{\beta}\sum_p\int_0^\infty dr_0\frac{1}{2}k\partial_r(\tanh^2 r)\Big|_{r_0}\left(-iw+\frac{2\pi p}{\beta}\right)e^{-\frac{1}{2}\beta k\tanh^2(r_0)\left((\frac{2\pi p}{\beta})^2+w^2\right)}\end{aligned} \tag{3.88}$$

The factor of $\frac{1}{2\pi\beta}$ comes from the determinants as before. Substituting $y=\frac{1}{2}\beta k\tanh^2 r_0$, we obtain

$$\begin{aligned}\widehat{W}(\beta;k,w) &= -\frac{i}{\beta}\sum_{p\neq 0}\int_0^{\frac{1}{2}\beta k}[dy]\left(-iw+\frac{2\pi p}{\beta}\right)\exp\left[-y\left(\left(\frac{2\pi p}{\beta}\right)^2+w^2\right)\right] \\ &= -\frac{i}{\beta}\sum_{p\neq 0}\frac{1}{(iw+\frac{2\pi p}{\beta})}\left[e^{-\frac{1}{2}\beta k\left((\frac{2\pi p}{\beta})^2+w^2\right)}-1\right]\end{aligned} \tag{3.89}$$

Notice that the second term is log-divergent. This happens because at the tip of the cigar an infinite number of winding modes become massless leading to a divergence for this term. This is a consequence of the fact that winding number is strictly not a conserved quantum number at $r=0$ as we have assumed. We can deal with it by regularizing the Witten index $\widehat{W}(2\pi\tau_2)$ near $r=0$ by putting an ϵ cutoff in the r integral in (3.88) and then taking $\epsilon\to 0$ in the end. With this regularization, the contribution from the last term in (3.89) vanishes. After Poisson resumming the first term, with respect to p (see equation (3.136)) we obtain,

$$\widehat{W}(\beta)=\sum_n e^{-\beta n w}\left[\quad -\frac{1}{2}\text{sgn}\left(\frac{n}{k}-w\right)\text{erfc}\left(\sqrt{\frac{k\beta}{2}}\left|\frac{n}{k}-w\right|\right)+\text{sgn}(\beta n)\,\Theta\left[w\left(\frac{n}{k}-w\right)\right]\right] \tag{3.90}$$

where Θ is the Heaviside step function [6]. It is easy to check that $\widehat{W}(\infty)$ vanishes. Using (3.35), we formally obtain

$$\widehat{\eta}(0)=2(\widehat{W}(0)-\widehat{W}(\infty))=\sum_n\text{sgn}\left(w-\frac{n}{k}\right) \tag{3.91}$$

[6]The $e^{-\beta n w}$ is due to the presence of the non-zero central charge of cigar supersymmetric quantum mechanics.

which can be regularized as in (3.16).

It is instructive to compare this result with a target space computation of the spectral asymmetry of the boundary operator $\mathcal{B}$ living on the boundary located at $r = r_c$. Using the inverse vielbeins from (3.75), the Dirac operator near the boundary takes the form

$$\begin{aligned} i\not{D} &= \gamma^r(i\partial_r - w\,K_r) + \gamma^\theta(i\partial_\theta - w\,K_\theta) \\ &= i\gamma^r\left[\partial_r - \frac{1}{\tanh r}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}(i\partial_\theta - w\,k\tanh^2 r)\right] \end{aligned} \tag{3.92}$$

For large r_c, the boundary manifold is a circle S^1. Identifying r with u and comparing with (3.2) we find the boundary operator

$$\mathcal{B} = -(i\partial_\theta - w\,k)\,. \tag{3.93}$$

Here we assume that wk is not an integer in order to avoid zero eigenvalue of the boundary operator. The η-invariant of this operator can be computed readily. Since θ direction is periodic, the eigenfunctions are given by the set

$$\{e^{-in\theta}|n \in \mathbb{Z}\} \tag{3.94}$$

with eigenvalues

$$\{wk - n|n \in \mathbb{Z}\} \tag{3.95}$$

The radius of the cigar is $\sqrt{k}$. As long as k is not an integer, the boundary operator $\mathcal{B}$ has no zero modes. The η-invariant is then given by:

$$\eta = \sum_{n\in\mathbb{Z}} \mathrm{sgn}(w\,k - n) = \sum_{n\in\mathbb{Z}} \mathrm{sgn}\left(w - \frac{n}{k}\right) \tag{3.96}$$

where in the last step we have used the fact that k is positive. This matches with the η invariant computed from the path integral (3.91).

In the infinite sum (3.96), one can absorb the integer part $\lfloor wk \rfloor$ of wk into n, and hence the η-invariant is expected to depend only on the fractional part $\langle wk \rangle$ of wk defined by

$$\langle wk \rangle = wk - \lfloor wk \rfloor$$

where $\lfloor wk \rfloor$ is the greatest integer less than wk. The regularized version of the η-invariant is

$$\eta(s) = -\sum_{n=1}^{\infty} \frac{1}{(n - \langle wk \rangle)^s} + \sum_{n=0}^{\infty} \frac{1}{(n + \langle wk \rangle)^s} . \tag{3.97}$$

The η-invariant can now be expressed in terms of the modified ζ function

$$\zeta(s,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s} \quad , \quad \zeta(0,q) = -q + \frac{1}{2} \tag{3.98}$$

to obtain

$$\eta(0) = -\zeta(0, 1 - \langle wk \rangle) + \zeta(0, \langle wk \rangle) = 1 - 2\langle wk \rangle \tag{3.99}$$

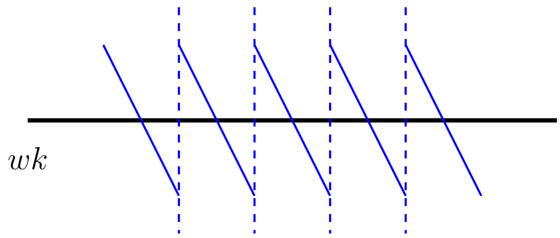


FIGURE 3.4.2: Spectral asymmetry

Note that for k and w both integers, the η-invariant vanishes. As one varies k, the η-invariant changes, and every time k crosses an integer, it jumps by -2. This is as expected from level-crossing because when k is an integer, the boundary operator has a zero eigenvalue. There is an ambiguity in the APS theorem about the sign of the contributions from zero (see discussion below (3.15)). This behavior is plotted in Figure **3.4.2** with η on the y-axis and wk on the x-axis.

Hence we conclude that the APS index theorem consists of two pieces, the AS piece and the η-invariant piece. Both these pieces can be computed using different SUSY QM.

- The AS piece can be computed using a $N = 1/2$ SQM with target space given by $\overline{\mathcal{M}}$ as shown in figure 3.3.1.
- The η-invariant piece can be computed using a $N = 1$ SQM with target space $\mathbb{R}$ and a superpotential $h(u)$.

Appendix

3.A Localization

Here we review the technique of localization to compute supersymmetric path integrals. Our discussion is restricted to supersymmetric quantum mechanics though it can be extended to supersymmetric quantum field theories.

Consider a SUSY partition function given by:

$$Z[\beta] = \int_{\text{PBC}} [\mathcal{D}\Phi] \, e^{-S_E} \tag{3.100}$$

$$S_E[\Phi] = \int_0^\beta d\tau \int [d\theta] \mathfrak{L}_E \tag{3.101}$$

where Φ is a superfield and PBC implies we impose the periodic boundary condition for all fields along the τ direction. S_E the Euclidean action and the measure $[\mathcal{D}\Phi]$ are both invariant under SUSY. Now we deform the above path integral by a Q- exact term i.e.

$$Z[\beta, t] = \int [\mathcal{D}\Phi] \, e^{-S[\Phi] - tQ\mathcal{V}[\Phi]} \tag{3.102}$$

We assume :

- $Q\mathcal{V} \geq 0$, so that adding $Q\mathcal{V}$ to the action does not blow up the path integral.
- $Q^2\mathcal{V} = 0$

If the Q exact term satisfies above conditions, one can show that if the field space is compact without boundary, (3.102) is independent of t. When the field space is non-compact or have a boundary, there is a subtlety (we will comment about this later). More precisely, the derivative w.r.t t is given by:

$$\begin{aligned}\frac{d}{dt}Z[t] &= \int[\mathcal{D}\Phi]\,Q\mathcal{V}[\Phi]\,e^{-S[\Phi]-tQV[\Phi]} \\ &= \int[\mathcal{D}\Phi]\,Q\left[\mathcal{V}[\Phi]\,e^{-S[\Phi]-tQ\mathcal{V}[\Phi]}\right]\end{aligned}$$

where in the second step we have used the fact that $QS[\Phi]=0$ and $Q^2\mathcal{V}=0$.
Now we have

$$\frac{d}{dt}Z[t] = \int[\mathcal{D}\Phi]\,Q\,[\mathcal{A}(\Phi)] \qquad , \qquad \mathcal{A}(\Phi)\equiv\mathcal{V}[\Phi]\,e^{-S[\Phi]-tQ\mathcal{V}[\Phi]} \tag{3.103}$$

Now

$$Q\,[\mathcal{A}(\Phi)] = \left[\frac{d}{d\Phi}\mathcal{A}(\Phi)\right]Q\Phi = \frac{d}{d\Phi}\Big[\mathcal{A}(\Phi)Q\Phi\Big] - \mathcal{A}(\Phi)\Big[\frac{d}{d\Phi}(Q\Phi)\Big] \tag{3.104}$$

Now under infinitesimal transformation

$$\Phi\to\Phi' = \Phi+\epsilon Q\Phi \tag{3.105}$$

Hence the jacobian of this transformation

$$\frac{d\Phi'}{d\Phi} = 1+\epsilon\frac{d}{d\Phi}(Q\Phi) \tag{3.106}$$

If the symmetry generated by Q is non-anomalous then the jacobian should be 1 and hence

$$\frac{d}{d\Phi}(Q\Phi) = 0 \tag{3.107}$$

and hence

$$Q\,[\mathcal{A}(\Phi)] = \frac{d}{d\Phi}\Big[\mathcal{A}(\Phi)Q\Phi\Big] \tag{3.108}$$

i.e.

$$\frac{d}{dt}Z[t] = \int[\mathcal{D}\Phi]\,\frac{d}{d\Phi}\Big[\mathcal{V}[\Phi]\,e^{-S[\Phi]-tQV[\Phi]}Q\Phi\Big] \tag{3.109}$$

If the field space is compact without boundary, then the above integral vanishes.

$$\frac{d}{dt}Z[t] = 0 \tag{3.110}$$

$Z[t]$ is independent of t i.e.

$$Z[t=0] = Z[t=\infty]$$

In the limit $t \to \infty$, only the configurations for which $Q\mathcal{V} = 0$ contribute. So an infinite dimensional path integral gets localized to finite dimensional integral. We obtain

$$Z = \int \mathcal{D}\varphi_0 \, e^{-S[\varphi_0]} \frac{1}{\text{SDet}} \tag{3.111}$$

where SDet comes from the quadratic fluctuations around $Q\mathcal{V} = 0$ configurations. Basically, the saddle point approximation becomes exact.

One canonical choice for V is

$$V = \sum (Q\psi^i)^\dagger \psi_i \tag{3.112}$$

Then the fixed points are essentially

$$Q\psi = 0 \qquad , \qquad (Q\psi)^\dagger = 0 \tag{3.113}$$

These are essentially BPS configurations.

When the field space has a boundary:

In the case when field space is compact *with boundary*, we can use localization if we choose boundary condition in such a way that the t-derivative vanishes i.e.

$$\left[(\psi^\dagger Q^\dagger \psi) \, e^{-S[\Phi] - tQV[\Phi]} Q\Phi \right] \Bigg|_{x=a}^{x=b} = 0 \tag{3.114}$$

When the field space is noncompact:

In the case when field space is non-compact, we can use localization when t-derivative vanishes, this can be obtained by choosing $\mathcal{V}$ such that:

$$\mathcal{V}|_{\Phi\to\infty} = 0 \tag{3.115}$$

3.B Scattering theory

Here we review the relation between density of states and the phase shifts in a scattering theory. Consider the scattering problem for the following Hamiltonian (See for example [71])

$$H = H_0 + V e^{-\epsilon|t|} \tag{3.116}$$

where we have added an adiabatic switching factor for the interaction V so that in the far past and and in the far future one obtains the free Hamiltonian H_0. The time evolution operator in the Dirac picture is given by

$$U_D(t,t') = e^{iH_0 t}\, U(t,t')\, e^{-iH_0 t'} \tag{3.117}$$

where $U(t,t')$ is the time evolution operator of the Heisenberg picture. The Dirac evolution operator satisfies the Schrödinger equation

$$i\frac{d}{dt}U_D(t,t') = V_D(t)\, U_D(t,t')\,, \qquad \text{with} \qquad V_D(t) = e^{iH_0 t}\, V e^{-\epsilon|t|}\, e^{-iH_0 t} \tag{3.118}$$

with the initial condition $U_D(t,t) = \mathbf{1}$. The solution is given by

$$U_D(t,t') = \mathbf{1} - i\int_{t'}^{t} dt'' V(t'')U_D(t'',t') \tag{3.119}$$

We can now define the 'Möller operators'

$$U_\pm = U_D(0,\pm\infty) \tag{3.120}$$

Consider an energy eigenstate $|\phi_E\rangle$ of the free Hamiltonian H_0. Using the Möller operators one can obtain the eigenstate of the full Hamiltonian:

$$|\psi_E^\pm\rangle = U_\pm|\phi_E\rangle \tag{3.121}$$

where $|\psi_E^-\rangle$ are the *in*-states that resemble the free eigenstates in the far past and $|\psi_E^-\rangle$ are the *out*-states that resemble the free eigenstates in the far future. Solving

(3.119) recursively gives the Dyson series expansion

$$U_{\pm}|\phi_E\rangle = |\phi_E\rangle + \frac{V}{E - H_0 \mp i\epsilon}|\phi_E\rangle + \left(\frac{V}{E - H_0 \mp i\epsilon}\right)^2 |\phi_E\rangle + \dots \tag{3.122}$$

This geometric series can be easily summed to obtain

$$U_{\pm}|\phi_E\rangle = \frac{E - H_0 \mp i\epsilon}{E - H \mp i\epsilon}|\phi_E\rangle \tag{3.123}$$

It follows that $|\psi_E^{\pm}\rangle$ satisfy the Lippman-Schwinger equations

$$|\psi_E^{\pm}\rangle = |\phi_E\rangle + \frac{V}{E - H_0 \mp i\epsilon}|\phi_E\rangle \,. \tag{3.124}$$

The S-matrix in the interaction picture is just the time evolution operator $U_D(\infty, -\infty)$ which can be expressed in terms of the Möller operators as

$$S = U_+^\dagger U_- \,. \tag{3.125}$$

The derivative of the S-matrix is given by

$$\frac{d \ln S}{dE} = S^{-1}\frac{dS}{dE} = S^\dagger \frac{dS}{dE} \tag{3.126}$$

and using the above formula it is possible show that

$$\frac{d \ln S}{dE} = 2\pi i \rho(E) = 2\pi i \left[\delta(E - H) - \delta(E - H_0)\right] \tag{3.127}$$

The density of states is then given by the so called 'Krein-Friedel-Lloyd' formula:

$$\rho(E) = \frac{1}{2\pi i}\mathrm{Tr}\left(S^\dagger \frac{dS}{dE}\right) \,. \tag{3.128}$$

If the S-matrix is diagonal, then in each one-dimensional subspace we obtain

$$S(E) = e^{i2\delta(E)} \,, \qquad \rho(E) = \frac{1}{\pi}\frac{d\delta(E)}{dE} \,. \tag{3.129}$$

3.C Error function and incomplete Gamma function

The error function and the complementary error functions are defined by

$$\text{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z dy\, e^{-y^2}\,, \qquad \text{erfc}(z) := \frac{2}{\sqrt{\pi}} \int_z^\infty dy\, e^{-y^2}\,. \tag{3.130}$$

They satisfy the following relation

$$\text{erfc}(z) = 1 - \text{erf}(z) \tag{3.131}$$

Note that $\text{erf}(z)$ is an odd function in z i.e.

$$\text{erf}(-z) = -\text{erf}(z)\,. \tag{3.132}$$

For the purpose of this paper it is convenient to use the expression

$$\begin{aligned} \text{erf}(z) &= \text{sgn}(z)\,\text{erf}(|z|)\,, \\ \text{erfc}(z) &= 1 - \text{sgn}(z)\,\text{erf}(|z|) \end{aligned} \tag{3.133}$$

for $z \in \mathbb{R}$ to make contact with the η-invariant.

The upper incomplete Gamma function is defined by

$$\Gamma(s,x) = \int_x^\infty t^{s-1}\, e^{-t}\, dt\,, \qquad x \geq 0\,. \tag{3.134}$$

A special case that we encounter is

$$\Gamma\left(\frac{1}{2}, x\right) = \sqrt{\pi}\,\text{erfc}(\sqrt{x})\,. \tag{3.135}$$

One of the integrals (involving error function) which is useful in our computation is the following

$$\begin{aligned} f(m) &= -\frac{i}{\beta} \int dp \frac{1}{(iw + \frac{2\pi p}{\beta})} \left[\exp^{-\frac{1}{2}\beta k \left(\frac{(2\pi p)^2}{\beta^2}\right) - 2\pi i n\cdot p} \right] \\ &= -\frac{1}{2}\text{sgn}\left(\frac{n}{k} - w\right)\, \text{erfc}\left(\sqrt{\frac{\beta k}{2}} \left|w - \frac{n}{k}\right|\right) e^{-\beta n w + \frac{\beta k}{2} w^2} + \text{sgn}(\beta n)\, \Theta\left[w\left(\frac{n}{k} - w\right)\right]\, e^{-\beta n w} \end{aligned} \tag{3.136}$$

Chapter 4

Mock Modularity & Elliptic Genera

In this chapter, we discuss the relationship between elliptic genus of SCFT's in 2D with noncompact target space and well-known mathematical objects called *mock Jacobi forms*. We also notice that the modular completion of these objects have the same structure as the APS index theorem derived in the last chapter. Hence we observe a relation between number theory (mock modular forms) and topology (index theorems). We also review the derivation of *holomorphic anomaly* equation for non-linear sigma models [7] on noncompact target space. We then use these results to compute the holomorphic anomaly for various examples.

In the section below, we introduce the basic mathematical objects required for the discussions in subsequent sections. We follow the notations of [2].

4.1 Mock Jacobi forms

Before introducing mock Jacobi forms, it is useful to recall the definitions of modular forms and Jacobi forms. In the following subsections, we give definitions and transformation properties of these objects.

4.1.1 Modular forms

A *modular form* $f(\tau)$ of weight k is a holomorphic function on $\mathbb{H}$ (the upper half plane) that transforms as:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{4.1}$$

where $\tau \in \mathbb{H}$ and $SL(2,\mathbb{Z})$ is a group of matrices with integer entries satisfying $ad-bc=1$. From the above definition one can see that $f(\tau)$ is periodic in $\tau \to \tau+1$, hence it can be expanded as:

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n q^n \qquad \text{with} \qquad q := e^{2\pi i\tau} \tag{4.2}$$

It is bounded as $\text{Im}(\tau) \to \infty$. If $a_0 = 0$, then the modular form vanishes at infinity and is known as *cusp form*. If the Fourier coefficients of $f(\tau)$ have the behavior $a_n = 0$ for $n < -N$ with $N \geq 0$ such that at infinity, $f(\tau)$ grows as $O(q^{-N})$, such a function is known as a *weakly holomorphic modular form*.

We will denote the vector space over $\mathbb{C}$ of holomorphic modular forms of weight k by M_k.

Some important examples of modular forms of half-integral weights, that will be useful later are:

$$\vartheta(\tau) = \sum_{n\in\mathbb{Z}} \epsilon(n) q^{\lambda n^2} \qquad \epsilon \text{ is some even periodic function} \tag{4.3}$$

$$\vartheta^{(1)}(\tau) = \sum_{n\in\mathbb{Z}} n\epsilon(n) q^{\lambda n^2} \qquad \epsilon \text{ is some odd periodic function} \tag{4.4}$$

where $\lambda \in \mathbb{Q}_+$. The first series is called *theta series* and it is a modular form of weight 1/2 under some congruence subgroup Γ of $SL(2,\mathbb{Z})$. The second one is called *unary theta series* and it is a modular form of weight 3/2 under Γ.

4.1.2 Jacobi forms

A *Jacobi form* of weight k and index m is a holomorphic function $\varphi(\tau, z)$ from $\mathbb{H} \times \mathbb{C}$ to $\mathbb{C}$ which is modular in τ and elliptic in z. It transforms under the modular group as:

$$\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} \varphi(\tau, z) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{4.5}$$

and under the translations in z by $\mathbb{Z}\tau + \mathbb{Z}$ as;

$$\varphi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+2\lambda z)} \varphi(\tau, z) \qquad \forall \qquad \lambda, \mu \in \mathbb{Z} \tag{4.6}$$

From above equations we notice that $\varphi(\tau, z)$ is periodic in both $\tau \to \tau + 1$ and $z \to z + 1$. Hence it can be Fourier expanded as follows:

$$\varphi(\tau, z) = \sum_{n,r} c(n, r) q^n y^r, \qquad q := e^{2\pi i\tau}, y := e^{2\pi iz}. \tag{4.7}$$

Equation (4.6) is then gives:

$$c(n, r) = C(4nm - r^2, r) \qquad \text{where } C(\Delta, r) \text{ depends only on } r \text{ (mod } 2m). \tag{4.8}$$

where $\Delta = 4nm - r^2$. A Jacobi form is called *weakly holomorphic Jacobi form* if it satisfies the condition that $c(n, r) = 0$ unless $n \geq n_0$ for some negative integer n_0.

Jacobi forms admit an important expansion in terms of theta functions which will be useful later. Due to transformation property (4.6), a Jacobi form $\varphi(\tau, z)$ has the following Fourier expansion

$$\varphi(\tau, z) = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau) \vartheta_{m,\ell}(\tau, z) \tag{4.9}$$

where

$$\vartheta_{m,\ell}(\tau, z) := \sum_{n \in \mathbb{Z}} q^{(\ell+2mn)^2/4m} y^{\ell+2mn} \tag{4.10}$$

is weight 1/2, index m Jacobi theta-function and h_ℓ is a modular form of weight $k - 1/2$.

4.1.3 Mock Modular forms

A *mock modular form* of weight $v \in \frac{1}{2}\mathbb{Z}$ is the first member of a pair of functions (h, g) where

- h is a holomorphic functions in τ with exponential growth at all cusps but it is not modular.
- The function $g(\tau)$ is a holomorphic modular form of weight $2 - v$. It is called the *shadow* of h.
- The sum $\hat{h}(\tau, \bar{\tau}) = h(\tau) + g^*(\tau, \bar{\tau})$ transforms like a modular function of weight v. It is called the *completion* of h.

Here the function g^* is the solution of following differential equation:

$$(4\pi\tau_2)^v \frac{\partial g^*}{\partial \bar{\tau}} = -2\pi i \, \overline{g(\tau)} \tag{4.11}$$

Given that $g(\tau)$ has the Fourier expansion $g(\tau) = \sum_{n\geq 0} \; b_n \, q^n$, we fix the choice of g^* by setting

$$g^*(\tau,\bar{\tau}) = \bar{b}_0 \, \frac{(4\pi\tau_2)^{-v+1}}{v-1} + \sum_{n>0} \; n^{v-1} \, \bar{b}_n \; \Gamma(1-v, 4\pi n\tau_2) \; q^{-n} \,, \tag{4.12}$$

where $\tau_2 = \mathrm{Im}(\tau)$ and $\Gamma(1-v,x)$ denotes the incomplete gamma function defined in (3.134), and where the first term must be replaced by $-\bar{b}_0 \; \log(4\pi\tau_2)$ if $v=1$.

Note that the series in (4.12) converges despite the exponentially large factor q^{-n} because $\Gamma(1-v,x) = O(x^{-v}e^{-x})$. If we assume either that $v>1$ or that $b_0 = 0$, then we can define g^* alternatively by the integral

$$g^*(\tau,\bar{\tau}) = \left(\frac{i}{2\pi}\right)^{v-1} \int_{-\bar{\tau}}^{i\infty} (z+\tau)^{-v} \; \overline{g(-\bar{z})} \; dz \; . \tag{4.13}$$

It is called the *non- holomorphic Eichler integral*. Notice that this integrand is holomorphic in z, hence the integral is independent of the path chosen.

Since h is holomorphic in τ, equation (4.11) implies that the completion satisfies:

$$(4\pi\tau_2)^v \frac{\partial \widehat{h}}{\partial \bar{\tau}} = -2\pi i \, \overline{g(\tau)} \tag{4.14}$$

This is also known as the *holomorphic anomaly equation*.

Notice that the equation for the completion of a mock modular form $\hat{h}(\tau,\bar{\tau}) = h(\tau) + g^*(\tau,\bar{\tau})$ looks very similar to the APS index theorem $\mathcal{I} = \int \alpha + \eta$. In both cases, the first piece is present for the compact target space as well and is independent of β $(\bar{\tau})$ and the second piece contains the β $(\bar{\tau})$ dependence. As we will show later, the second piece in both cases, depends only on the asymptotic boundary.

4.1.4 Mock Jacobi forms

A *pure mock Jacobi from* of weight v and index m is a holomorphic function φ on $\mathbb{H}\times\mathbb{C}$ that satisfies the elliptic transformation property (4.6) and hence has a Fourier expansion in terms of ϑ-functions as in (4.9). But the modular property is now weakened. The coefficients h_ℓ appearing in the ϑ- expansion (4.9) now are mock modular forms rather than modular forms of weight $v - \frac{1}{2}$. We can again complete this mock Jacobi form such that it transforms nicely under modular transformations. The *completed* function is given by:

$$\widehat{\varphi}(\tau, z) = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} \widehat{h}_\ell(\tau)\vartheta_{m,\ell}(\tau, z) \tag{4.15}$$

If g_ℓ denotes the shadow of h_ℓ, then we have:

$$\widehat{\varphi}(\tau, z) = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau)\vartheta_{m,\ell}(\tau, z) + \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} g^*_\ell(\tau)\vartheta_{m,\ell}(\tau, z) \tag{4.16}$$

where the first term is the *mock Jacobi form* φ. The holomorphic anomaly equation in this case is given by:

$$(\tau_2)^{v-1/2}\frac{\partial}{\partial\bar{\tau}}\widehat{\varphi}(\tau, \bar{\tau}, z) = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} \overline{g_\ell(\tau)}\,\vartheta_{m,\ell}(\tau, z) \tag{4.17}$$

The objects that will appear in the computations in this chapter are a generalization of the mock Jacobi forms known as *mixed mock Jacobi forms* [1]. A completion of a *mixed mock Jacobi* form admits the following theta expansion:

$$\widehat{\phi}(\tau, \bar{\tau}|z) = f(\tau, z) \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} \widehat{h}_\ell(\tau, \bar{\tau})\,\vartheta_{m,\ell}(\tau|z) \tag{4.18}$$

where $f(\tau, z)$ is a Jacobi form of weight u and index α. The theta coefficients are the *completion of a vector valued mock modular form* of weight $(v - u - \frac{1}{2})$. Using the completion of $\hat{h}_\ell$, (4.18) can be written as

$$\widehat{\phi}(\tau, \bar{\tau}|z) = \phi(\tau, z) + f(\tau, z) \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} g^*_\ell(\tau, \bar{\tau})\,\vartheta_{m,\ell}(\tau|z) \tag{4.19}$$

where

$$\phi(\tau, z) = f(\tau, z) \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau)\,\vartheta_{m,\ell}(\tau|z) \tag{4.20}$$

[1]Our definitions are a slight variant of the definitions in [2].

4.2 Elliptic genera for Gauge theories

Elliptic genus (EG) is a generalization of the Witten index to theories with $2D$ base space. When the two-dimensional theory has a geometric description, the elliptic genus also gives topological information of the target space. For a theory with $N = (2,2)$ supersymmetry on a worldsheet, it is defined as:

$$\chi(\tau, z, u) = \mathrm{Tr}_{RR}(-1)^F q^{H_L} \bar{q}^{H_R} y^{J_L} \prod_a x_a^{K_a} \tag{4.21}$$

where the trace is taken over Ramond-Ramond sector i.e. we have periodic boundary conditions for the fermions, F is the fermion number and $q = e^{2\pi i\tau}$ where τ is the modular parameter of the worldsheet torus. Here H_R and H_L are the right and left moving hamiltonians respectively, they are given by: $2H_L = H + iP$ and $2H_R = H - iP$ where H and P generate time and space translations respectively. J_L is the generator of left-moving R symmetry and K_a are the generators of flavour symmetry. Since H_L and H_R are related to the hamiltonian and the momentum generators, the EG has a path integral representation. The presence of additional insertions has the effect of twisting the boundary conditions along the time direction for the fields charged under J_L and K_a. We also have:

$$y = e^{2\pi i z}, \qquad x_a = e^{2\pi i u_a}. \tag{4.22}$$

where z and u_a are related to the background gauge fields associated with R-symmetry (A^R) and the flavor symmetry (A^a) respectively.

$$z = \oint_t A^R - \tau \oint_\sigma A^R, \qquad u_a = \oint_t A^a - \tau \oint_\sigma A^a. \tag{4.23}$$

where t, σ are the temporal and spatial cycles of the worldsheet torus.

In a superconformal field theory, $H_L = L_0 - c/24$ and $H_R = \bar{L}_0 - \bar{c}/24$ and hence the elliptic genus is given by:

$$\chi(\tau, z) = \mathop{\mathrm{Tr}}_{\mathcal{H}} (-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} y^{J_L} \prod_a x_a^{K_a} \tag{4.24}$$

where L_0 and $\bar{L}_0$ are the left and right-moving Virasoro generators respectively, c and $\bar{c}$ are the central charges. The fermion number can be written as $F_R + F_L$ where

F_L and F_R are the right and left moving fermion numbers respectively. Since J_L and K_a commute with the right-moving fermion number F_R, the elliptic genus can be thought of as a right-moving Witten index multiplied with contributions from the left moving excited states.

Since the elliptic genus has a path integral formulation, for $N = (2,2)$ theory, it can be computed using *localization*. In this thesis, we only consider $2D$ gauge linear sigma models (GLSM) with $U(1)$ vector, chiral and stuckelberg multiplets. Our notations and conventions for $2D$ supersymmetric gauge theories are explained in Appendix 4.A. For $N = (2,2)$ theories with vector and chiral multiplets, the EG was computed in [72, 73]. We will not dwell on the details of the computation; we will summarize their results below.

For these theories, the action of vector and chiral multiplets is Q-exact, and hence using the argument used in §3.A for localization, one can show that the path integral is independent of the gauge coupling. Hence we can compute it in the $e \to 0$ limit, and in this limit, we can minimize the actions of the vector and the chiral multiplet and compute the fluctuations around it. For the theories with $U(1)$ gauge fields, after localization, we obtain:

$$\chi = -\sum_{u_i \in \mathfrak{M}_+} \oint_{u=u_i} du\, Z_{\text{1-loop}}(u) = \sum_{u_i \in \mathfrak{M}_-} \oint_{u=u_i} du\, Z_{\text{1-loop}}(u) \tag{4.25}$$

where u is the holonomy of the gauge field (A) given by:

$$u = \oint_t A_t dt - \tau \oint_\sigma A_\sigma d\sigma \tag{4.26}$$

We get a loop integral over u instead of the integral over the whole plane $\int d^2u$ because the integral over the gaugino zero modes give a total derivative in $\bar{u}$. The integral over the auxiliary field D determines the contour of the loop integral. More details about this computation can be found in [72, 73]. Z_{1-loop} is the contribution from one-loop determinants of fluctuations of various multiplets around the localizing saddle. It receives contributions from both vector and chiral multiplets. As we will see below, the determinants coming from the chiral multiplets have poles in the u variable. To compute the EG, we sum over a certain set of poles u_i. For the $U(1)$ case, this set is determined by the sign of the charge of the chiral superfield under the gauge group. The set of poles $\mathfrak{M}_{\text{sin}}$ split into two groups: $\mathfrak{M}_+$ for $Q_i > 0$

and $\mathfrak{M}_-$ for $Q_i < 0$ i.e.

$$\mathfrak{M}_{\text{sin}} = \mathfrak{M}_+ \cup \mathfrak{M}_- \tag{4.27}$$

For higher rank groups, one computes the *Jeffrey-Kirwan* residues. For more details, look at [73]. The one-loop determinants for various multiplets are summarized below.

4.2.1 Elliptic Genus of $N = (2,2)$ theories

We consider $N = (2,2)$ theories with a $U(1)$ vector multiplet and chiral multiplet Φ_i with R-charge R and flavor charge Q. After localization, we need to integrate over zero modes and compute the contributions from one-loop determinants. Various contributions are listed below:

Vector multiplets

The $(2,2)$ vector multiplet consists of a gauge field (v), a complex scalar (σ), a Dirac fermion (λ_+, λ_-) and an auxillary field (D) i.e. $V = (v_\mu, \sigma, \bar{\sigma}, \lambda_+, \lambda_-, D)$. The zero modes of the gauge field will give integration over holonomies u. The gauge field in $2D$ have no dynamical degree of freedom (gauge field in d dim has $d-2$ dynamical d.o.f). So we do not have any contributions from v_μ in the partition function. Other way to see this is that the contribution due to ghosts cancels the one from v_μ. Hence the one-loop contributions from the scalar σ which is not charged under the R-symmetry and the fermion (the left moving fermion λ has an R-charge -1) is given by:

$$\left(\frac{i\eta^3(\tau)}{\theta_1(\tau, y^{-1})}\right)$$

where $\eta(\tau)$ and ϑ_1 is defined in Appendix 4.B.

Chiral Multiplet:

The $(2,2)$ chiral multiplet consists of a complex scalar (ϕ), a Dirac fermion (ψ_+, ψ_-) and auxillary field (F) i.e. $\Phi = (\phi, \bar{\phi}, \psi_+, \psi_-, F, \bar{F})$. The contribution from a chiral multiplet Φ with R-charge R and flavor charge Q is:

$$\prod_{m,n} \frac{(m + n\tau + (1 - R/2)z + Qu)(m + n\tau + (R/2)\bar{z} + Q\bar{u})}{|m + n\tau + (R/2)z + Qu|^2 + iQD}$$

For $D = 0$ it becomes:

$$\frac{\theta_1(\tau, y^{R/2-1}x^Q)}{\theta_1(\tau, y^{R/2}x^Q)}$$

where $Qu = \sum_a Q^a u_a$. We have $R/2$ instead of R because of the normalization of the R-current. We follow the notations of [72]. Also if the scalar has an left-moving R- charge R/2, the left moving fermion will have R-charge $R/2 - 1$ and the right moving fermion will have R-charge $R/2$.

4.3 NLSM from Gauge theories

In this section, we will review the RG flow of $2D$ GLSM's to nonlinear sigma models on a certain class of manifolds. This was first studied in [74]. The NLSM in the IR can be found by solving for the supersymmetric vacua and then by looking at fluctuations around it. In the infrared, all the massive modes decouple, and classical theory reduces to that of the massless modes only. It can be shown that this theory is a nonlinear sigma model with the target space given by the vacuum manifold. The target space in these cases can either be *compact* or *noncompact*. In this chapter, we will be mostly interested in the latter case, but for the sake of completeness, we will discuss the *compact* case below.

4.3.1 NLSM's with compact target space

We will only consider GLSM's without any superpotential for matter fields turned on. These models give us NLSM's on *toric manifolds*. We will not dwell on the general details but will show this using a simple example. More details can be found in [75]. Consider $N = (2,2)$ GLSM consisting of N chiral superfields Φ_i, k abelian gauge superfields V_a. As shown in (4.103) of Appendix 4.A, the lagrangian is given by:

$$L = \int d^4\theta \left(\sum_{i=1}^{N} \bar{\Phi}_i e^{\sum_{a=1}^{k} Q_i^a V_a} \Phi_i - \frac{1}{2e^2} \sum_{a=1}^{k} \bar{\Sigma}_a \Sigma_a \right) + \frac{1}{2} \left(\int d^2\tilde{\theta} \sum_{a=1}^{k} -t_a \Sigma_a + c.c \right) \tag{4.28}$$

where Σ_a is the twisted chiral superfield associated to the field strength of V_a and $t_a = r_a - i\vartheta_a$ where r_a is the FI parameter and ϑ_a is the theta angle for gauge field V_a. Here Q_i^a is the charge of i^{th} chiral superfield under the gauge field V_a.

The supersymmetric vacua can be found by minimizing the potential energy, which is given by:

$$U = \sum_{a=1}^{k} \left(\sum_{i=1}^{N} |\sigma_a|^2 |\phi_i|^2 + \frac{e^2}{2} \left(\sum_{i=1}^{N} Q_i^a |\phi_i|^2 - r_a \right)^2 \right) \tag{4.29}$$

For $r_a > 0$, U is minimized by :

$$\sigma_a = 0\,, \qquad \sum_{i=1}^{N} Q_i^a |\phi_i|^2 = r_a \tag{4.30}$$

where ϕ_i is the scalar fields sitting in chiral multiplet. This is equivalent to solving the constraints imposed by setting D-terms to zero. The set of all supersymmetric vacua modulo the gauge group action forms the vacuum manifold. In this case, the vacuum manifold is a toric manifold given by:

$$X_{vac} = \left\{ (\phi_1, ..., \phi_n) \,|\, \sum_{i=1}^{N} Q_i^a |\phi^i|^2 = r_a (\forall a) \right\} \Big/ U(1)^k \tag{4.31}$$

For $k = 1$, $Q_i^a > 0$: The vacuum manifold is $\mathbb{CP}^{N-1}$.

When the charges Q_i^a are either all positive or negative, from (4.31), one can conclude that the vacuum manifold will be *compact* but when the some of the charges Q_i^a are positive, and some are negative, the vacuum manifold will be *noncompact*.

The modes ϕ_i along the vacuum manifold are massless. The gauge field and the modes transverse to the vacuum manifold acquire mass equal to $e\sqrt{2r_a}$. In the limit $e \to \infty$, all the massive modes decouple, and the classical theory reduces to that of massless modes only. It can be identified as a nonlinear sigma model with the vacuum manifold as the target space.

4.3.2 NLSM's with noncompact target space

As we saw in the previous subsection, we can obtain an NLSM with a *noncompact* target space when some of the chiral superfields are positively charged, and some are negatively charged under the gauge field. The elliptic genus in such cases is well defined only when some fugacities are turned on. It turns out to be a *vector-valued Jacobi form*, for example, for $\mathbb{C}/\mathbb{Z}_2$ as the target space [33]. After turning on the fugacities, the spectrum of right moving Hamiltonian becomes discrete and hence we obtain a holomorphic result.

There is another way to obtain noncompact vacuum manifolds, i.e., by adding Stückelberg fields to the GLSM [33]. The $(2,2)$ actions with Stückelberg fields is derived in the Appendix 4.A. As we will see in the examples below, the elliptic genus of these theories falls in the category of *mock Jacobi forms*.

Consider $N=(2,2)$ GLSM consisting of N chiral superfields Φ_i, k abelian gauge superfields V_a and M Stückelberg superfields (where $M \leq k$) P_l. The lagrangian is given by:

$$L = \int d^4\theta \quad \left(\sum_{i=1}^{N} \bar{\Phi}_i e^{\sum_{a=1}^{k} Q_i^a V_a} \Phi_i - \frac{1}{2e^2} \sum_{a=1}^{k} \bar{\Sigma}_a \Sigma_a + \sum_{l=1}^{M} \frac{k_l}{4} (P_l + \bar{P}_l + V_l)^2 \right) \\ + \frac{1}{2} \left(\int d^2\tilde{\theta} \sum_{a=M+1}^{k} t_a \Sigma_a + c.c \right) \tag{4.32}$$

Notice that the FI parameter for the gauge fields that couple to Stückelberg field can be absorbed in the definition of P_l and ϑ_a can be removed by a $U(1)$ R-rotation. Therefore, a takes values from $M+1$ to k in the last term of the action. The vacuum manifold of the above GLSM can be found by solving D-term equations. The D-term equations give:

$$\sum_{i=1}^{N} Q_i^a |\phi^i|^2 = -k_a \text{Re}(p_a) \qquad a = 1, 2....M \tag{4.33}$$

$$\sum_{i=1}^{N} Q_i^a |\phi^i|^2 = r_a \qquad a = M+1, ..., k \tag{4.34}$$

where ϕ_i and p_a are the scalar fields sitting in chiral and Stückelberg superfield respectively.

As an example, let us focus on the case with $N=(2,2)$ SUSY and the following field content : one $U(1)$ vector superfield, one chiral superfield with charge $+1$ under the gauge field and one Stückelberg superfield which transforms additively under the gauge field. Hence we have $N=k=M=1$ in (4.32). In this case, equations (4.33) give:

$$|\phi|^2 = -k\text{Re}(p) \tag{4.35}$$

We will only derive the scalar kinetic term so that we can read-off the target space metric. The fermionic counterpart can be derived using supersymmetry. Since scalar of the vector multiplet is set to zero in the vacuum manifold, we are left with the following scalar terms in the GLSM action:

$$-D^\mu\bar{\phi}D_\mu\phi + D|\phi|^2 - \frac{k}{2}D^\mu\bar{P}D_\mu P + \frac{k}{2}D(P+\bar{P}) + \frac{1}{2e^2}D^2 \tag{4.36}$$

where $D_\mu\phi = \partial_\mu\phi + iv_\mu\phi$ but $D_\mu P = \partial_\mu P + iv_\mu$. Notice that here P is the lowest component (scalar) of the Stückelberg superfield. Using $\phi = \rho e^{i\theta}$ and $P = p + i\eta$ the kinetic term becomes:

$$\begin{aligned}
D^\mu\bar{\phi}D_\mu\phi &= (\partial^\mu - iv^\mu)\rho e^{-i\theta}(\partial_\mu + iv_\mu)\rho e^{i\theta} \\
&= \partial^\mu\rho\partial_\mu\rho + \rho^2(\partial_\mu\theta + v_\mu)^2 \\
\frac{k}{2}D^\mu\bar{P}D_\mu P &= \frac{k}{2}\left((\partial^\mu(p-i\eta) - iv^\mu)(\partial_\mu(p+i\eta) + iv_\mu)\right) \\
&= \frac{k}{2}(\partial^\mu p\partial_\mu p + v^\mu v_\mu)
\end{aligned}$$

where in the last step we used gauge transformation to set $\eta = 0$. Notice that under gauge transformations, various fields transform as follows:

$$v_\mu \to v_\mu - \partial_\mu\alpha \qquad \theta \to \theta + \alpha \qquad P \to P + i\alpha \tag{4.37}$$

The $D = 0$ equation and the E.O.M of v_μ gives:

$$\rho^2 = -kp \qquad v_\mu = -\frac{2\rho^2\partial_\mu\theta}{(2\rho^2 + k)}$$

Substituting these in the kinetic terms we get:

$$\begin{aligned}
\partial^\mu\rho\partial_\mu\rho + \frac{k}{2}\partial^\mu p\partial_\mu p &= (\partial_\mu\rho)^2 + \frac{k}{2}\left(\frac{4\rho^2}{k^2}(\partial_\mu\rho)^2\right) \\
&= (\partial_\mu\rho)^2\left(1 + \frac{2\rho^2}{k}\right)
\end{aligned}$$

$$\begin{aligned}\rho^2(\partial_\mu\theta+v_\mu)^2+\frac{k}{2}v^\mu v_\mu &= \rho^2(\partial_\mu\theta)^2\left(1-\frac{1}{1+k/2\rho^2}\right)^2+\frac{k}{2}(\partial_\mu\theta)^2\left(\frac{1}{1+k/2\rho^2}\right)^2 \\ &= \left(\frac{\partial_\mu\theta}{1+k/2\rho^2}\right)^2\left(\frac{k^2}{4\rho^2}+\frac{k}{2}\right) \\ &= \frac{\rho^2}{1+\frac{2\rho^2}{k}}(\partial_\mu\theta)^2\end{aligned}$$

Finally the bosonic action of massless modes is given by:

$$S=-\frac{1}{2\pi}\int d^2x\left(1+\frac{2\rho^2}{k}\right)(\partial_\mu\rho)^2+\frac{\rho^2}{1+\frac{2\rho^2}{k}}(\partial_\mu\theta)^2 \tag{4.38}$$

which is a NLSM with the target space metric given by:

$$ds^2=\left(1+\frac{2\rho^2}{k}\right)\frac{d\rho^2}{2}+\frac{\rho^2 d\theta^2}{2\left(1+\frac{2\rho^2}{k}\right)} \tag{4.39}$$

If one sets $\rho=\sqrt{\frac{k}{2}}\sinh r$, we obtain:

$$ds^2=k\left(\cosh^2 r dr^2+\tanh^2 r\, d\theta^2\right) \tag{4.40}$$

As $r\to\infty$, the above metric approaches a cylinder. It topologically looks like semi-infinite cigar. It is still not conformal and it undergoes further RG flow. As explained in [76], at the end of RG flow we obtain a $SL(2,\mathbb{R})_k/U(1)$ coset SCFT with central charge $c=3+\frac{6}{k}$. The target space metric is given by:

$$ds^2=k\left(dr^2+\tanh^2 r\, d\theta^2\right) \tag{4.41}$$

This is same as the metric of cigar given in (3.73) for which we computed η-invariant in the last chapter. We also have a non-trivial background 'spacetime' dilaton

$$\Phi_d(r)=\Phi_{d0}-\log\cosh r \tag{4.42}$$

which ensures that the theory is conformal even though the target space is not Ricci flat.

Similarly, one can obtain a whole class of noncompact NLSM's by adding multiple Stückelberg fields. In such cases, the elliptic genus is expected to be a *higher* *depth mock modular form* [33], which is outside the scope of this thesis.

4.4 Holomorphic Anomaly

In this chapter, we will only concentrate on the gauge theories which flow to superconformal field theories (SCFT) in the infrared as elaborated in the previous section.

For a compact SCFT with central charge c, the elliptic genus is 'modular in τ' and 'elliptic in z'. The modular invariance of the elliptic genus follows from its path integral representation. The path integral is diffeomorphism invariant when regulated covariantly using a covariant regulator, such as a short proper time cutoff. It is also Weyl invariant for a conformal field theory on the flat worldsheet. Consequently, it is invariant under the mapping class group $SL(2,\mathbb{Z})$, which is the group of global diffeomorphisms of the torus worldsheet modulo Weyl transformations. Similarly, the elliptic transformation properties of the elliptic genus follow from the spectral flow [77] of the left-moving superconformal field theory. Hence the elliptic genus is a *weak Jacobi form* of weight $w = 0$ and index $m = c/6$. The theta expansion (4.9) can be understood [78] physically by bosonizing the $U(1)$ R-symmetry current J_L.

For an SCFT with compact target space, the spectrum of $\bar{L}_0$ is discrete and is paired by supersymmetry. Hence elliptic genus can be thought of as the right moving Witten index multiplied by left moving oscillators. Because of supersymmetry, only the right-moving ground states contribute to the elliptic genus, and hence it is independent of $\bar{\tau}$. This is essentially the same argument we used to show that the Witten index is independent of β. The holomorphic elliptic genus thus counts right-moving ground states with arbitrary left-moving oscillators.

For a noncompact target space, this argument fails. Therefore, the noncompact elliptic genus need not be holomorphic. However, it is clear from its path integral representation that it must nevertheless have modular and elliptic transformation properties of a Jacobi form. As we will see in subsequent sections, the elliptic genus is given instead by the *completion of mock Jacobi form*.

The *holomorphic anomaly equation* or the $\bar{\tau}$ derivative of the elliptic genus for noncompact NLSM was derived in [7]. We will review the derivation below. Consider an NLSM with a noncompact target space $\mathcal{M}$ whose asymptotic boundary is a compact manifold $\mathcal{N}$. The $\bar{\tau}$ derivative of the elliptic genus is given by:

$$\frac{\partial}{\partial \bar{\tau}} \chi_{\mathcal{M}} = -\langle 2\pi i \bar{T}(z) \rangle_{\mathcal{M}} \tag{4.43}$$

where $\bar{T}$ is the energy momentum tensor of the base space (worldsheet). Since the worldsheet theory is a supersymmetric theory, the energy momentum tensor can be related to the supercurrent $\bar{G}$ i.e. $2i\bar{T} = \{Q, \bar{G}\}$. Hence we obtain:

$$\frac{\partial}{\partial \bar{\tau}} \chi_{\mathcal{M}} = -\langle \pi \{Q, \bar{G}(z)\} \rangle_{\mathcal{M}} = -\frac{e^{i\pi/4}}{\sqrt{4\tau_2 \alpha'} \eta(\tau)} \langle \bar{G}(z) \rangle_{\mathcal{N}} \tag{4.44}$$

where in the last equality we have used the fact that the supercharge acts as an exterior derivative and later we used Stokes theorem. We will call this equation as the *'GJF Anomaly'* equation. The normalization factors were explained in [79], they can be understood as follows:

The nonlinear sigma model on the full manifold $\mathcal{M}$ has an extra boson $\phi_\perp$ which describes the direction normal to the boundary $\mathcal{N}$ and the corresponding fermion $\psi_\perp$. When we use the Stokes theorem, we need to take these fields into account separately. The left moving part of $\phi_\perp$ gives a factor of $1/\eta(\tau)$, and its right moving part cancels the contribution due to $\psi_\perp$. The integration of zero-mode of $\psi_\perp$ gives a factor of $\sqrt{i} = e^{i\pi/4}$ as explained in (2.42). The zero-mode of $\phi_\perp$ (momentum integral) gives $\frac{1}{\sqrt{4\tau_2\alpha'}}$. For a *compact* target space without boundary, the holomorphic anomaly (4.44) vanishes due to the Stokes theorem. Hence the elliptic genus is holomorphic in τ.

In the next two sections, we will compute the holomorphic anomaly for some noncompact target spaces. We will use different methods discussed above to do these computations and will compare their results.

4.5 Holomorphic anomaly of $N = (2,2)$ Cigar

In this section we will look at our first example: $N = (2,2)$ SCFT with cigar target space whose metric is given in (4.41). As we saw in §4.3.2, this theory appears in the IR limit of a $N = (2,2)$ GLSM with $U(1)$ gauge multiplet, a chiral multiplet and a Stückelberg field. This SCFT also has a representation as a coset conformal field theory[2] of $SL(2,\mathbb{R})/U(1)$ WZW model at level k. The elliptic genus of this theory was first computed in [42] and for the $\mathbb{Z}_N$-orbifold of this theory was computed in [41, 80] using the path integral for the coset theory. It was re-derived in [46, 68] using canonical methods and in [44, 45] using localization in the GLSM.

We will compute the holomorphic anomaly of this theory using three different methods. Firstly, we will use the Witten index of SQM with cigar target space computed in §3.4.3 to compute the elliptic genus and the holomorphic anomaly of the SCFT. Secondly, we will review the computation of the elliptic genus and the anomaly from the GLSM [44, 45]. Finally, we will compute the holomorphic anomaly directly by computing $\langle \bar{G} \rangle$ in the boundary theory [7] and using (4.44).

4.5.1 SQM computation

The computation of η-invariant for a finite cigar can be used to compute the full elliptic genus for a $N = (2,2)$ SCFT on an infinite cigar [3]. Notice that for the cigar case, the R-symmetry generator J_L appearing in the definition of elliptic genus (4.21) commutes with the right moving supercharge. We do not turn on any additional flavor symmetries, so $x_a = 1$. Hence for the right movers, the computation reduces to computing the noncompact Witten index computed in §3.4.3. We show that our results match with the ones obtained in [42]. The full elliptic genus is given by

$$\begin{aligned} \widehat{\chi}(\tau,\bar{\tau}|z) &= \operatorname*{Tr}_{\mathcal{H}} (-1)^F e^{-2\pi\tau_2(L_0+\bar{L}_0)} e^{2\pi i \tau_1 (L_0-\bar{L}_0)} e^{2\pi i z J_L} \\ &= \widehat{W}(2\pi\tau_2) \cdot \mathcal{Z}_{\text{oscill}}\, e^{2\pi i \tau_1 m w}\, e^{2\pi i z J_L} \end{aligned} \tag{4.45}$$

where $\widehat{W}(2\pi\tau_2)$ is the Witten index with $\beta = 2\pi\tau_2$, $\mathcal{Z}_{\text{oscill}}$ is the contribution coming from left-moving oscillators and n, w are KK momenta and winding respectively along the cigar θ direction. The contribution coming from the oscillators is given by

$$\mathcal{Z}_{\text{oscill}} = \prod_{n=1}^{\infty} \left[\frac{(1-q^n y)(1-q^n y^{-1})}{(1-q^n)^2} \right] \tag{4.46}$$

[2] For the noncompact $SL(2,\mathbb{R})$ WZW model, the parameter k need not in general be an integer.

[3] Since the dilaton couples to the worldsheet curvature; it plays no role if the worldsheet is a torus as in our case.

We get a contribution of $(2i \sin \pi z)$ from the zero modes of left moving fermions since they are charged under $U(1)_R$. Finally we obtain:

$$i\frac{\theta_1(\tau, z)}{\eta(\tau)^3} \tag{4.47}$$

Using equation (3.90) and the contribution from left movers, we conclude that the elliptic genus for the cigar is given by

$$\begin{aligned} \widehat{\chi}(\tau, \bar{\tau}|z) \quad = \quad & -i\frac{\vartheta_1(\tau, z)}{\eta^3(\tau)} \sum_w \sum_n \left[\frac{1}{2}\text{sgn}\left(\frac{n}{k} - w\right) \text{erfc}\left(\sqrt{k\pi\tau_2}\left|w - \frac{n}{k}\right|\right) \right. \\ & \left. -\text{sgn}(\beta n)\, \Theta\left[w\left(\frac{n}{k} - w\right)\right]\right] q^{-(n-wk)^2/4k} q^{(n+wk)^2/4k} y^{J_L} \end{aligned} \tag{4.48}$$

To obtain the above expression we have dropped the last term in (3.90) using the following reasoning. At the tip of the cigar an infinite number of winding modes become massless leading to a divergence for this term. This is a consequence of the fact that winding number is strictly not a conserved quantum number at $r = 0$ as we have assumed. We can deal with it by regularizing the Witten index $\widehat{W}(2\pi\tau_2)$ near $r = 0$ by putting an ϵ cutoff in the r integral in (3.88) and then taking $\epsilon \to 0$ in the end. With this regularization, the contribution from the last term in (3.90) vanishes and we get (somewhat surprisingly) the correct answer by this slightly heuristic procedure. In any case, this affects only the holomorphic piece and not the holomorphic anomaly which is our main interest. Since the holomorphic anomaly is determined by the scattering states, winding-number in the asymptotic region is a good quantum number for our purposes. As a result the holomorphic anomaly is not affected by this regularization.

Note that on the cigar, the R-current is given by

$$J_L = i\sqrt{\frac{1}{k}}\partial\theta - i\psi_r\psi_\theta \tag{4.49}$$

and as a consequence not only the fermions but bosons are also charged under R-symmetry. With this normalization[4] [42], the left-moving fermions have charge -1 and the bosons have charge $1/k$. In terms of the left moving momenta, the R-current is given by $J_L = \sqrt{1/k}\, p_L$. The left and right moving momentas are given

[4]We use $\alpha' = 1$ so that asymptotic radius R of the cigar is $\sqrt{k}$ while [42] uses $\alpha' = 2$.

by:

$$p_L = \left(\frac{n}{R} + wR\right) \qquad , \qquad p_R = \left(\frac{n}{R} - wR\right) \tag{4.50}$$

The expression for the elliptic genus is non-holomorphic, but it is modular if k is an integer. More precisely, it transforms as a *completion* of a mock Jacobi form.

The second piece of (4.48) is the holomorphic piece. It can be re-written as:

$$\widehat{\chi}_h(\tau,\bar{\tau}|z) = i\frac{\vartheta_1(\tau,z)}{\eta^3(\tau)}\left[\sum_{w\geq 0}\sum_{n-wk\geq 0} - \sum_{w<0}\sum_{n-wk<0}\right] q^{nw} y^{\frac{n+wk}{k}} \tag{4.51}$$

Notice that

$$\ell = (n+wk) = (n-wk) \bmod 2k\,. \tag{4.52}$$

Or equivalently

$$n - wk = \ell + 2ks \qquad , \qquad n + wk = \ell + 2ks'\,. \tag{4.53}$$

Hence the holomorphic piece can be written as

$$\widehat{\chi}_h(\tau,z) = i\frac{\vartheta_1(\tau,z)}{\eta^3(\tau)}\left[\sum_{w\geq 0}\sum_{\ell\geq 0} - \sum_{w<0}\sum_{\ell<0}\right] q^{w^2k+w\ell} y^{\frac{\ell+2kw}{k}} = -i\frac{\vartheta_1(\tau,z)}{\eta^3(\tau)}\mathcal{A}_{1,k}\left(\tau,\frac{z}{k}\right) \tag{4.54}$$

where $\mathcal{A}_{1,k}$ is the Appell-Lerch sum given by:

$$\mathcal{A}_{1,k}(\tau,z) = \sum_{t\in\mathbb{Z}} \frac{q^{kt^2} y^{2kt}}{1 - y\,q^t}\,. \tag{4.55}$$

Above result matches with the one obtained in [42].

To find the shadow, let us focus only on the non-holomorphic piece $\widehat{\chi}_{nh}(\tau,\bar{\tau}|z)$ which can be re-written by replacing the sum over by n and w by the sum over s and s' using (4.5.1)

$$\begin{aligned} & -i\frac{\vartheta_1(\tau,z)}{\eta^3}\sum_{\ell\in\mathbb{Z}/2k\mathbb{Z}}\sum_{s,s'} \quad \frac{1}{2}\mathrm{sgn}\,(\ell+2ks)\,\mathrm{erfc}\left(|\ell+2ks|\sqrt{\frac{\pi\tau_2}{k}}\right) \\ & \qquad\qquad q^{-(\ell+2ks)^2/4k} q^{(\ell+2ks')^2/4k} y^{\frac{\ell+2ks'}{k}} \\ = & -i\frac{\vartheta_1(\tau,z)}{\eta(\tau)^3}\sum_{\ell\in\mathbb{Z}/2k\mathbb{Z}}\sum_{r=\ell+2k\mathbb{Z}} \quad \frac{1}{2}\mathrm{sgn}\,(r)\,\mathrm{erfc}\left(|r|\sqrt{\frac{\pi\tau_2}{k}}\right) q^{-r^2/4k}\vartheta_{k,\ell}\left(\tau,\frac{z}{k}\right) \end{aligned} \tag{4.56}$$

Combining the holomorphic (4.54) and non-holomorphic (4.56) contribution, the EG of the cigar is given by

$$\widehat{\chi}(\tau,\bar{\tau}|z) = -i\frac{\vartheta_1(\tau,z)}{\eta^3(\tau)}\widehat{\mathcal{A}}_{1,k}\left(\tau,\frac{z}{k}\right) \tag{4.57}$$

where $\widehat{\mathcal{A}}_{1,k}(\tau,z)$ is the *completion* of the Appell-Lerch sum and is given by

$$\widehat{\mathcal{A}}_{1,k}(\tau,\bar{\tau};z) = \mathcal{A}_{1,k}(\tau,z) + \sum_{\mathbb{Z}/2m\mathbb{Z}} g^*_\ell(\tau,\bar{\tau})\,\vartheta_{k,l}(\tau,z) \tag{4.58}$$

with

$$g^*_\ell(\tau,\bar{\tau}) \;=\; -\frac{1}{2}\sum_{r=\ell+2k\mathbb{Z}} \mathrm{sgn}\,(r)\,\mathrm{erfc}\left(|r|\sqrt{\frac{\pi\tau_2}{k}}\right)\,q^{-r^2/4k} \tag{4.59}$$

Comparing (4.56) and (4.19) we conclude that our elliptic genus is a mixed mock Jacobi form with

$$f(\tau,z) = -i\frac{\vartheta_1(\tau,z)}{\eta^3(\tau)} \tag{4.60}$$

and with $w = 0$ $u = -1$ and hence $v = 1/2$. The total index is $m = \frac{1}{2} + \frac{1}{k}$ which matches[5] with the expected index $m = c/6$ where c is the central charge of the coset.

We can now compute the holomorphic anomaly by taking the $\bar{\tau}$ derivative of (4.57). We obtain:

$$(4\pi\tau_2)^{1/2}\,\frac{\partial\widehat{\chi}(\tau,\bar{\tau}|z)}{\partial\bar{\tau}} = -\sqrt{\frac{\pi}{2k}}\frac{\vartheta_1(\tau,z)}{\eta^3(\tau)}\sum_{\ell\in\mathbb{Z}/2k\mathbb{Z}}\overline{\vartheta^{(1)}_{k,\ell}(\tau)}\vartheta_{k,\ell}\left(\tau,\frac{z}{k}\right) \tag{4.61}$$

where $\vartheta^{(1)}_{k,\ell}$ is the unary theta function which is defined as

$$\vartheta^{(1)}_{k,\ell}(\tau) = \frac{1}{2\pi i}\frac{d}{dz}\vartheta_{k,\ell}(\tau,z)\Big|_{z=0} = \sum_{r\equiv\ell\,(\mathrm{mod}\;2k)} r\,q^{r^2/4k} \tag{4.62}$$

Hence we find that shadow vector is given by:

$$g_\ell(\tau) = -\frac{1}{\sqrt{8\pi k}}\vartheta^{(1)}_{k,\ell}(\tau) \tag{4.63}$$

The shadow vector $\{g_\ell(\tau)\}$ is an unary theta series as in (4.62). In this case with

[5] Note that $\widehat{\mathcal{A}}_{1,k}(\tau,z)$ is a Jacobi form with index k but $\widehat{\mathcal{A}}_{1,k}\left(\tau,\frac{z}{k}\right)$ has index $\frac{1}{k}$ because of the rescaling of z. The theta function $\vartheta_1(\tau,z)$ transforms with index $\frac{1}{2}$.

$v = \frac{1}{2}$, the incomplete gamma function in (4.12) can be expressed in terms of the complementary error function $\mathrm{erfc}(x)$ using (3.135).

4.5.2 GLSM computation

In this subsection, we will compute the holomorphic anomaly for cigar SCFT by computing the elliptic genus of the UV GLSM. This computation was first done in [45]. We will re-derive those results below. Since the elliptic genus is a topological quantity, both the IR and the UV theories are expected to give the same results.

As we saw in §4.3.2, $N = (2,2)$ GLSM with a $U(1)$ gauge multiplet (V), a chiral multiplet (Φ) with charge $+1$ under the gauge symmetry and a Stückelberg multiplet (P) flow to cigar SCFT in the IR. We can now compute the elliptic genus of the GLSM using localization as explained in §4.2.1.

Stückelberg Multiplet:

The presence of the Stückelberg multiplet changes some of the details of the localization computation. This happens because the action of this multiplet is not Q-exact and hence we cannot naively minimize it. But we can still take the $e \to 0$ limit because the action of the vector multiplet is Q-exact. After integrating over the vector multiplet, we are essentially left with a P- multiplet coupled to the zero modes (u, λ_-^0) of the vector multiplet. Its action is given by[6]:

$$\begin{aligned} S = \quad & \frac{1}{4\pi}\int d^2\sigma \frac{k}{2}\big((-\partial^\mu \bar{p} - iu^\mu)(\partial_\mu p + iu_\mu) + i\bar{\chi}_-(\partial_0+\partial_1)\chi_- + i\bar{\chi}_+(\partial_0-\partial_1)\chi_+ \\ & + D(p+\bar{p}) + |F_P|^2 + i\chi_+^0\lambda_-^0 + i\bar{\chi}_+^0\bar{\lambda}_-^0\big) \end{aligned} \tag{4.64}$$

Notice that there are no left moving zero modes because they are charged under the R-symmetry and hence the boundary conditions for the left moving fermions are twisted and they do not allow any zero modes.

We can now perform the integral over fermion zero-modes, but the answer now differs from the one discussed in §4.2.1. Due to the coupling of zero-modes χ_+^0 of the P-multiplet with the gaugino zero modes, we get a factor of $k^2/4$ from the fermion zero mode integral and hence we are still left with the u-plane integral $\int d^2u$. Let us now look at contributions from various multiplets.

[6] We are using $\alpha' = 1$ units as compared to $\alpha' = 2$ units usually used in the literature [45]. Also, $\int d^2\sigma = \int d\sigma^1 d\sigma^2$, where σ^1 and σ^2 parametrize the worldsheet time and space directions.

Consider the zero modes of the boson p. From (4.64), we notice that the zero mode of Re(p) couples to D and integral over it gives $\delta(D)/k\tau_2$. The zero-mode of $\varphi_P = \sqrt{k}\text{Im}(p)$ lives on a circle of radius $\sqrt{k}$ and is additively charged under the gauge field. It satisfies periodic boundary conditions i.e.

$$\varphi_P(\sigma^1 + 2\pi, \sigma^2) = \varphi_P(\sigma^1, \sigma^2) + 2\pi w\sqrt{k} + \frac{2\pi z}{k} \tag{4.65}$$

$$\varphi_P(\sigma^1 + 2\pi\tau_1, \sigma^2 + 2\pi\tau_2) = \varphi_P(\sigma^1, \sigma^2) + 2\pi m\sqrt{k} + \frac{2\pi z}{k} \tag{4.66}$$

where $w, m \in \mathbb{Z}$ and the $\frac{2\pi z}{k}$ comes from the twist in the boundary conditions due to the R symmetry. Notice that Im(p) carries $\frac{1}{k}$ charge under the R-symmetry to cancel the anomaly in the R-symmetry[76].

Hence we can mode expand it as follows:

$$\varphi_P(\sigma^1, \sigma^2) = \sigma^1 w\sqrt{k} + \sigma^2 \left(m + \frac{z}{k} - w\tau_1\right)\frac{\sqrt{k}}{\tau_2} \tag{4.67}$$

The bosonic and fermionic oscillators of the P- multiplet give:

$$i\frac{\vartheta_1(\tau, z)}{\eta(\tau)^3}$$

After taking into account the normalizations due to fermionic and bosonic zero modes, the full contribution from the P-multiplet is given by:

$$i\frac{k}{\tau_2}\frac{\vartheta_1(\tau, z)}{\eta(\tau)^3}\delta(D)\sum_{m,w\in\mathbb{Z}} e^{-\frac{\pi k}{2\tau_2}|m+w\tau+u+\frac{z}{k}|^2} \tag{4.68}$$

Vector multiplet: From (4.2.1), we see that the $U(1)$ vector multiplet gives:

$$\frac{i\eta^3(\tau)}{\theta_1(\tau, y^{-1})} = -\frac{i\eta^3(\tau)}{\theta_1(\tau, z)}$$

Chiral multiplet: In this theory we have one chiral multiplet with charge $+1$ under the gauge symmetry and the boson is uncharged under the R-symmetry. This implies that the left moving fermion has R-charge -1. Using this in (4.2.1) we obtain:

$$\prod_{m,w} \frac{(m+w\tau+z+u)(m+w\tau+\bar{u})}{|m+w\tau+u|^2+iD}$$

Using (4.5.2), (4.5.2) and (4.5.2) and integrating over the auxillary field D, we obtain:

$$\chi(\tau,z) = -ik\int \frac{d^2u}{\tau_2}\frac{\theta_1(\tau,-z+u+m+w\tau)}{\theta_1(\tau,u+m+w\tau)}\left(\sum_{m,w} e^{-\frac{k\pi}{2\tau_2}|m+w\tau+u+\frac{z}{k}|^2}\right) \tag{4.69}$$

The factor of $-i$ comes from the normalization of fermionic zero modes. The above integrand is invariant under $u \to u+m+w\tau$ where $m,w \in \mathbb{Z}$. We can use this to reduce the integration over the whole plane to coset $E = \mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})$.

$$\begin{aligned}\chi(\tau,z) &= -ik\int_E \frac{d^2u}{\tau_2}\frac{\theta_1(\tau,-z+u)}{\theta_1(\tau,u)}\left(\sum_{m,w} e^{2\pi izw}e^{-\frac{k\pi}{2\tau_2}|m+w\tau+u+\frac{z}{k}|^2}\right)\\ &= \int_E \frac{dud\bar{u}}{i\tau_2}\varphi(\tau,z,u)H_k(\tau,u),\end{aligned} \tag{4.70}$$

where

$$\varphi(\tau,z,u) = \frac{\theta_1(\tau,-z+u)}{\theta_1(\tau,u)} \qquad H_k(\tau,u,z) = k\sum_{m,w} e^{2\pi izw}e^{-\frac{k\pi}{2\tau_2}|m+w\tau+u+\frac{z}{k}|^2} \tag{4.71}$$

where the $e^{2\pi izw}$ factor comes by using transformation properties of the ϑ-function given in (4.114).

Holomorphic Anomaly

Now we have the full elliptic genus, we can take the $\bar{\tau}$ derivative to obtain the *holomorphic anomaly* equation. Notice that in equation (4.70), the $\bar{\tau}$ dependence just sits in the factor $H_k(\tau,u)$. Naively, it looks that the measure also contains $\bar{\tau}$ dependence, but that's not the case. We can write $u = u_1+\tau u_2$ where $u_1,u_2 \in [0,1]$, then the measure becomes

$$\frac{dud\bar{u}}{i\tau_2} = \frac{(du_1+\tau_1 du_2+i\tau_2 du_2)\wedge(du_1+\tau_1 du_2-i\tau_2 du_2)}{i\tau_2} = 2du_1du_2$$

Hence there is no $\bar{\tau}$ dependence in the measure. Using the fact that H_k obeys the heat equation $\partial_{\bar{\tau}}H_k(\tau,a\tau+b) = \frac{i}{2\pi k}\partial_{\bar{u}}^2 H_k(\tau,u)|_{u=a\tau+b}$. Hence we get,

$$\partial_{\bar\tau}\chi = \frac{i}{2\pi k}\int_E du d\bar u\, \partial_{\bar u}\left(\frac{1}{i\tau_2}\varphi(\tau,z,u)\partial_{\bar u}H_k(\tau,u,z)\right) \tag{4.72}$$

$$= \frac{1}{2\pi k}\oint_{\partial E} du\, \frac{1}{\tau_2}\varphi(\tau,z,u)\partial_{\bar u}H_k(\tau,u,z) \tag{4.73}$$

The above integrand has a pole at $u = 0$. After computing the residue at the pole, we obtain:

$$\partial_{\bar\tau}\chi = \frac{ik}{4\tau_2^2}\frac{\vartheta_1(\tau,z)}{\eta(\tau)^3}\sum_{m,w}\left(m+\tau w+\frac{z}{k}\right)e^{2\pi i z w}e^{-\frac{k\pi}{2\tau_2}\left|m+\tau w+\frac{z}{k}\right|^2} \tag{4.74}$$

After poisson resumming w.r.t. m, we obtain:

$$\partial_{\bar\tau}\chi(\tau,\bar\tau|z) = -\frac{1}{2\sqrt{2k\tau_2}}\frac{\vartheta_1(\tau,z)}{\eta(\tau)^3}\sum_{n,w}(n-wk)q^{\frac{(n+wk)^2}{4k}}\bar q^{\frac{(n-wk)^2}{4k}}y^{\frac{n}{k}+w}$$

$$(4\pi\tau_2)^{1/2}\frac{\partial\chi(\tau,\bar\tau|z)}{\partial\bar\tau} = -\sqrt{\frac{\pi}{2k}}\frac{\vartheta_1(\tau,z)}{\eta(\tau)^3}\sum_{\ell\in\mathbb{Z}/2k\mathbb{Z}}\overline{\vartheta^{(1)}_{k,\ell}(\tau)}\vartheta_{k,\ell}\left(\tau,\frac{z}{k}\right) \tag{4.75}$$

As expected, this answer matches equation (4.61), the anomaly obtained from the SQM related to the IR theory.

4.5.3 GJF Anomaly

In this subsection, we will compute the holomorphic anomaly of the cigar SCFT directly without computing the full elliptic genus. We will use equation (4.44) which relates the holomorphic anomaly to the expectation value of supercurrent in the boundary theory [7]. From (3.73), we observe that the target space of the cigar SCFT at asymptotic infinity ($r\to\infty$) looks like $\mathbb{R}\times S^1$, where the radius of S^1 is $\sqrt{k}$. We can now compute the expectation value of the supercurrent $\langle\bar G\rangle$ in the boundary theory i.e. the NLSM with S^1 target space. The supercurrent is given by:

$$\bar G(z) = \sqrt{2}i\psi^\theta\bar\partial\theta \tag{4.76}$$

There are following contributions in $\langle\bar G\rangle$:

- Zero mode integral $d\psi_0^\theta$ is soaked by ψ^θ appearing in $\bar G$ and hence we just get a factor of $\sqrt{i}$ due to fermionic normalization.

- Non-zero modes of ψ^θ and the S^1 boson give:

$$i\frac{\vartheta_1(\tau,z)}{\eta(\tau)^2}$$

- Zero mode of the S^1 boson, contributes to $\bar{\partial}\theta$ and gives:

$$\langle\bar{\partial}\theta\rangle = \langle -i\frac{p_R}{2}\rangle = -\frac{i}{2\sqrt{k}}\sum_{n,w}(n-wk)q^{\frac{(n+wk)^2}{4k}}\bar{q}^{\frac{(n-wk)^2}{4k}}y^{\frac{n}{k}+w}$$

where we get $y^{\frac{n}{k}+w}$ because the $U(1)_R$ symmetry acts as translation symmetry along the θ direction. Finally we obtain

$$\langle\bar{G}\rangle = -\sqrt{i}\frac{i}{\sqrt{2k}}\frac{\vartheta_1(\tau,z)}{\eta(\tau)^2}\sum_{n,w}(n-wk)q^{\frac{(n+wk)^2}{4k}}\bar{q}^{\frac{(n-wk)^2}{4k}}y^{\frac{n}{k}+w} \tag{4.77}$$

Using (4.44) we obtain:

$$\begin{aligned}\partial_{\bar{\tau}}\chi &= -\frac{1}{\sqrt{4\tau_2}\sqrt{2k}}\frac{\vartheta_1(\tau,z)}{\eta(\tau)^3}\sum_{n,w}(n-wk)q^{\frac{(n+wk)^2}{4k}}\bar{q}^{\frac{(n-wk)^2}{4k}}y^{\frac{n}{k}+w}\\ (4\pi\tau_2)^{1/2}\frac{\partial\chi(\tau,\bar{\tau}|z)}{\partial\bar{\tau}} &= -\sqrt{\frac{\pi}{2k}}\frac{\vartheta_1(\tau,z)}{\eta(\tau)^3}\sum_{\ell\in\mathbb{Z}/2k\mathbb{Z}}\overline{\vartheta^{(1)}_{k,\ell}(\tau)}\vartheta_{k,\ell}\left(\tau,\frac{z}{k}\right)\end{aligned} \tag{4.78}$$

which matches (4.61) and (4.5.2).

4.6 Holomorphic anomaly of $N=(4,4)$ Taub-NUT

In this section we will look at another example where mock modularity plays an important role. We consider $N=(4,4)$ GLSM which in the IR flows to SCFT with TaubNUT target space. The Dirac index for TaubNUT was computed in [81]. We will compute the holomorphic anomaly of this model first by computing the elliptic genus of the GLSM [29] and then by using GJF anomaly equation.

4.6.1 GLSM computation

In this section, we describe the $N=(4,4)$, $U(1)$ gauged linear sigma model that flows to a non-linear sigma model with TaubNUT target space in the IR. This model is discussed in [29, 82]. The field content of this theory, written in $N = 2$ language, is as follows:

- **Vector Multiplet**: We have a $N = (4,4)$ $U(1)$ vector multiplet which decomposes into a $N = (2,2)$ vector multiplet $V = (v_\mu, \sigma, \lambda_\pm)$ and a neutral chiral multiplet $\Phi = (\phi, \tilde{\lambda}_\pm)$.

- **Hypermultiplet**: This multiplet is charged under the gauge group $U(1)$ and it decomposes into two $N = (2,2)$ chiral multiplets $Q = (q, \psi)$ and $\tilde{Q} = (\tilde{q}, \tilde{\psi})$ with electric charges $+1$ and -1.

- **Twisted hypermultiplet**: It is not charged under the gauge group and it decomposes into a $N = 2$ chiral multiplet $\Psi = (r_1, r_2, \chi_\pm)$ and a Stückelberg chiral multiplet $\Gamma = (r_3, \gamma, \tilde{\chi}_\pm)$.

This theory has 8 real supercharges (4 left moving and 4 right moving). It has the R-symmetry group $SU(2)_1 \times SU(2)_2 \times SU(2)_3$. The supercharges $(Q_-^{a\alpha}, Q_+^{a\dot{\alpha}})$ transform in $(2,1,2)_- \oplus (2,2,1)_+$ representation. Let us write $\{Q_1, Q_2, Q_R\}$ for the Cartan generators of the R-symmetry group and Q_f for the Cartan of $U(1)_f$ flavour symmetry. It turns out that there are two right moving supercharges which are neutral under $Q_1 - Q_2$, Q_R and Q_f. Hence we can define an EG, which preserves right-moving super-symmetry as follows:

$$\chi(\tau, z, \xi_1, \xi_2) = \mathrm{Tr}_{RR}(-1)^F q^{L_0} \bar{q}^{\bar{L}_0} e^{-2\pi i z Q_R} e^{-2\pi i \xi_1 Q_f} e^{-2\pi i \xi_2 (Q_1 - Q_2)} \tag{4.79}$$

The elliptic genus of this GLSM was computed in [29] using localization [7]. It is given by:

$$\begin{aligned}
\chi(\tau, \xi_1, \xi_2, z) &= g^2 \int_{E(\tau)} \frac{du d\bar{u}}{\tau_2} \frac{\vartheta_1(\tau, u+\xi_1+z)\vartheta_1(\tau, u+\xi_1-z)}{\vartheta_1(\tau, u+\xi_1+\xi_2)\vartheta_1(\tau, u+\xi_1-\xi_2)} \sum_{n,w} e^{-\frac{g^2\pi}{2\tau_2}|u+n+\tau w|^2} \\
&= \int_{E(\tau)} \frac{du d\bar{u}}{\tau_2} \varphi(\tau, z, u+\xi_1, \xi_2) H_g(\tau, u),
\end{aligned} \tag{4.80}$$

where

$$\varphi(\tau, z, u, \xi_2) = \frac{\vartheta_1(u+z)\vartheta_1(u-z)}{\vartheta_1(u+\xi_2)\vartheta_1(u-\xi_2)} \qquad H_g(\tau, u) = g^2 \sum_{n,w} e^{-\frac{g^2\pi}{2\tau_2}|u+n+\tau w|^2} \tag{4.81}$$

where g^2 is the coupling in front of twisted hypermultiplet. The ratio of ϑ- functions comes from the $N = (2,2)$ chiral multiplets sitting in the vector, hyper and

[7] Our notations are a bit different from the one in [29], they use $\alpha' = 2$.

twisted hypermultiplets. The exponential part comes from the zero-mode contribution of the boson in the Stückelberg multiplet. The oscillator contribution from $(2,2)$ vector and the Stückelberg multiplet cancel each other as in the cigar case. We will now compute the holomorphic anomaly. It was also computed in [29] but there is a small error in their computation which we correct below.

Holomorphic Anomaly

To compute the holomorphic anomaly, we compute the $\bar{\tau}$ derivative of (4.80). Notice that in equation (4.80), the $\bar{\tau}$ dependence just sits in the factor $H_g(\tau, u)$. The measure is independent of $\bar{\tau}$ as shown in (4.5.2). Similarly to the cigar case, H_g obeys the heat equation $\partial_{\bar{\tau}} H_g(\tau, a\tau + b) = \frac{i}{2\pi g^2}\partial_{\bar{u}}^2 H_g(\tau, u)|_{u=a\tau+b}$. Hence we get,

$$\partial_{\bar{\tau}}\chi = \frac{i}{2\pi g^2}\int_{E^\epsilon(\tau)} du d\bar{u}\, \partial_{\bar{u}}\left(\frac{1}{\tau_2}\varphi(\tau, z, u+\xi_1, \xi_2)\partial_{\bar{u}} H_g(\tau, u)\right) \tag{4.82}$$

$$\partial_{\bar{\tau}}\chi = \frac{i}{2\pi g^2}\oint_{\partial E^\epsilon(\tau)} du\, \frac{1}{\tau_2}\varphi(\tau, z, u+\xi_1, \xi_2)\partial_{\bar{u}} H_g(\tau, u) \tag{4.83}$$

The above integrand has two poles at $u = -\xi_1 - \xi_2$ and $u = -\xi_1 + \xi_2$. The contribution from both the residues is computed below:
Residue at $u = -\xi_1 - \xi_2$:

$$\begin{aligned} &\frac{1}{2\pi}\frac{\vartheta_1(-\xi_2+z)\vartheta_1(-\xi_2-z)}{\eta(\tau)^3\vartheta_1(-2\xi_2)}\partial_{\bar{u}} H_g(\tau, u)|_{u=-\xi_1-\xi_2} \\ =\ &\frac{1}{2\pi}\frac{\vartheta_1(z-\xi_2)\vartheta_1(z+\xi_2)}{\eta(\tau)^3\vartheta_1(2\xi_2)}\left(-\frac{g^4\pi}{\tau_2}\sum_{n,w}(-\xi_1-\xi_2+n+\tau w)e^{-\frac{g^2\pi}{2\tau_2}|-\xi_1-\xi_2+n+\tau w|^2}\right) \end{aligned}$$

Residue at $u = -\xi_1 + \xi_2$:

$$-\frac{1}{2\pi}\frac{\vartheta_1(z-\xi_2)\vartheta_1(z+\xi_2)}{\eta(\tau)^3\vartheta_1(2\xi_2)}\left(-\frac{g^4\pi}{2\tau_2}\sum_{n,w}(-\xi_1+\xi_2+n+\tau w)e^{-\frac{g^2\pi}{2\tau_2}|-\xi_1+\xi_2+n+\tau w|^2}\right)$$

Finally we obtain

$$\begin{aligned} \partial_{\bar{\tau}}\chi = \frac{g^2}{4\tau_2^2}\frac{\vartheta_1(z-\xi_2)\vartheta_1(z+\xi_2)}{\eta(\tau)^3\vartheta_1(2\xi_2)} &\left(\sum_{n,w}(-\xi_1-\xi_2+n+\tau w)e^{-\frac{g^2\pi}{2\tau_2}|-\xi_1-\xi_2+n+\tau w|^2} - \right. \\ &\left.(-\xi_1+\xi_2+n+\tau w)e^{-\frac{g^2\pi}{2\tau_2}|-\xi_1+\xi_2+n+\tau w|^2}\right) \end{aligned} \tag{4.84}$$

Poisson resumming w.r.t variable n gives:

$$\partial_{\bar\tau}\chi = \frac{i}{2g\sqrt{2\tau_2}}\frac{\vartheta_1(z-\xi_2)\vartheta_1(z+\xi_2)}{\eta(\tau)^3\vartheta_1(2\xi_2)} \sum_{m,w}(-m+wg^2)q^{\frac{(m+wg^2)^2}{4g^2}}\bar{q}^{\frac{(m-wg^2)^2}{4g^2}}\left(e^{2\pi i(-\xi_1-\xi_2)m}+e^{2\pi i(\xi_1-\xi_2)m}\right) \quad (4.85)$$

Notice that the above equation has a pole in $\xi_2 \to 0$ limit. For $g^2 = 1$ i.e. at the self-dual radius, we can replace the sum over by m and w by the sum over s and s' by noting that

$$\ell = (w+m) = (w-m) \bmod 2\,. \quad (4.86)$$

Or equivalently

$$w-m = \ell+2s \quad , \quad w+m = \ell+2s'\,. \quad (4.87)$$

Hence we have

$$\partial_{\bar\tau}\chi = \frac{i}{2\sqrt{2\tau_2}}\frac{\vartheta_1(z-\xi_2)\vartheta_1(z+\xi_2)}{\eta(\tau)^3\vartheta_1(2\xi_2)}\sum_{\ell\in\mathbb{Z}/2\mathbb{Z}}\sum_{s,s'}(l+2s)q^{\frac{(\ell+2s')^2}{4}}\bar{q}^{\frac{(\ell+2s)^2}{4}} \left(e^{-2\pi i(\xi_1+\xi_2)(s'-s)}+e^{2\pi i(\xi_1-\xi_2)(s'-s)}\right) \quad (4.88)$$

The first term in the bracket can be expressed as follows:

$$e^{-2\pi i(\xi_1+\xi_2)(s'-s)} = e^{-2\pi i\left(\frac{\xi_1+\xi_2}{2}\right)((l+2s')-(l+2s))} = y_1^{l+2s'}\bar{y}_1^{l+2s} \quad (4.89)$$

where $y_1 = e^{\pi i(-\xi_1-\xi_2)}$ and the fugacities ξ_1 and ξ_2 are real. Hence the holomorphic anomaly equation can be expressed as follows:

$$\partial_{\bar\tau}\chi = \frac{i}{2\sqrt{2\tau_2}}\frac{\vartheta_1(z-\xi_2)\vartheta_1(z+\xi_2)}{\eta(\tau)^3\vartheta_1(2\xi_2)}\sum_{\ell\in\mathbb{Z}/2\mathbb{Z}} \vartheta_{1,\ell}\left(\tau,\frac{-\xi_1-\xi_2}{2}\right)\overline{\vartheta^{(1)}_{1,\ell}\left(\tau,\frac{-\xi_1-\xi_2}{2}\right)} +\vartheta_{1,\ell}\left(\tau,\frac{\xi_1-\xi_2}{2}\right)\overline{\vartheta^{(1)}_{1,\ell}\left(\tau,\frac{\xi_1-\xi_2}{2}\right)} \quad (4.90)$$

where $\vartheta^{(1)}(\tau,\xi) = \partial_z\vartheta(\tau,z)|_{z=\xi}$. For a generic $g^2 \in \mathbb{Z}$, the answer does not seem to decompose into the products of $\vartheta(\tau,z)$ and $\overline{\theta^{(1)}(\tau,z)}$.

4.6.2 GJF anomaly

The $N = (4, 4)$ GLSM described above flows to an NLSM with the Taub-NUT target space. This can be seen by minimizing the scalar potential to find the vacuum manifold as in §4.3.2. By minimizing the scalar potential, we get the following constraints [82]:

$$F_{01} = \sigma = \phi = 0\,, \qquad |q|^2 - |\tilde{q}|^2 = r_3\,, \qquad r_1 + ir_2 = 2q\tilde{q}\,. \tag{4.91}$$

The low energy physics is described by a NLSM on the vacuum moduli space. Notice that we have four complex scalars $(q, \tilde{q}, r_1 + ir_2, r_3 + i\gamma)$ in the matter multiplets of the GLSM. We get three real equations (4.91) by minimizing the scalar potential and one degree of freedom is eliminated using $U(1)$ gauge symmetry. Hence in the IR, we obtain a $N = (4, 4)$ supersymmetric theory with four dimensional target space (TaubNUT) parametrized by these four bosons. The target space metric in radial coordinates, is given by:

$$ds^2 = \left(\frac{r-m}{r+m}\right) dr^2 + (r^2 - m^2)(d\theta^2 + sin^2\theta d\phi^2) + 4m^2 \left(\frac{r-m}{r+m}\right)(d\psi^2 + cos\theta d\phi)^2 \tag{4.92}$$

where $|q|^2 + |\tilde{q}|^2 = r = \sqrt{r_1^2 + r_2^2 + r_3^2}$, $\psi \in [0, 4\pi)$, $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$ and m is related to the coupling of Stückelberg field in the GLSM i.e. $g^2 = 16m^2$. As we can see from (4.92), the radius of S^1 (of periodicity 2π) at infinity is $4m$, hence g^2 is the radius-squared at infinity. The charges $\{Q_1 - Q_2, Q_R, Q_f\}$ of various fields that describe the IR physics, are given in 4.6.1. Notice that unlike the cigar case, the Stückelberg field in this case is not charged under the R symmetry. This is because sum of charges of chirals vanish and hence there is no anomaly in the R symmetry. The flavor symmetries act on ψ direction of Taub-NUT as shift symmetries.

We can use the *GJF anomaly* equation (4.44) to compute the anomaly directly without computing the full elliptic genus. For this, we need to compute the torus one-point function of the supercurrent $\langle \bar{G} \rangle$ in the boundary theory. The supercurrent is given by:

$$\bar{G}(z) = i\sqrt{2} g_{ij} \bar{\chi}^i \bar{\partial} X^j \tag{4.93}$$

The target space (4.92) in $r \to \infty$ limit, looks like $\mathbb{R} \times S^2 \times S^1$ where the radius of S^2 grows with r and it goes to infinity as $r \to \infty$, giving us $\mathbb{R}^3 \times S^1$. In such cases, the elliptic genus is usually not well-defined if we do not turn on any fugacities.

fields	$Q_1 - Q_2$	Q_R	Q_f
q	-1	0	+1
$\tilde{q}$	-1	0	-1
ψ_+	-1	0	+1
ψ_-	0	-1	+1
$\tilde{\psi}_+$	-1	0	-1
$\tilde{\psi}_-$	0	-1	-1
r_1	-2	0	0
r_2	-2	0	0
r_3	0	0	0
γ	0	0	0
χ_+	-2	0	0
$\tilde{\chi}_+$	0	0	0
χ_-	-1	-1	0
$\tilde{\chi}_-$	-1	1	0

TABLE 4.6.1: Charges of various fields of TaubNUT GLSM.

We turn on global charges, and the elliptic genus is now defined as (4.79). Since the chiral superfields are charged under $Q_1 - Q_2$ and Q_R, the boundary of the target space is now two copies of $\mathbb{R}^2 \times S^1$. These global symmetries lift the bosonic zero modes along $\mathbb{R}^2$. Hence we are effectively left with S^1 boundary and oscillator contributions from other modes.

The fugacities ξ_1 and ξ_2 twist the boundary conditions for two of the boundary fermions and hence they do not have any zero mode. We have only one fermionic zero mode in the boundary theory (the super-partner of the boundary S^1). After integrating the fermionic zero mode, we get

$$\langle \bar{G} \rangle|_{boundary} \quad = \quad i\sqrt{2} Z_{oscill.} \langle g_{\psi\psi} \bar{\partial}\psi \rangle \tag{4.94}$$

where $Z_{oscill.}$ is the contribution coming from bosonic and fermionic non-zero modes. Various contributions are given below:

- Oscillator contribution from fermions χ_-

$$\prod_n (1 - q^n e^{-2\pi i \xi_2} e^{-2\pi i z})(1 - q^n e^{2\pi i \xi_2} e^{2\pi i z})$$

- Oscillator contribution from fermions $\tilde{\chi}_-$

$$\prod_n (1 - q^n e^{-2\pi i \xi_2} e^{2\pi i z})(1 - q^n e^{2\pi i \xi_2} e^{-2\pi i z})$$

- Oscillator contribution from fermions χ_+

$$\prod_n (1 - \bar{q}^n e^{-2(2\pi i \xi_2)})(1 - \bar{q}^n e^{2(2\pi i \xi_2)})$$

- Oscillator contribution from fermions $\tilde{\chi}_+$

$$\prod_n (1 - \bar{q}^n)$$

- Oscillator contribution from boson along $\mathbb{R}^2$

$$\prod_n \frac{1}{1 - q^n e^{-2(2\pi i \xi_2)}} \frac{1}{1 - q^n e^{2(2\pi i \xi_2)}} \frac{1}{1 - \bar{q}^n e^{-2(2\pi i \xi_2)}} \frac{1}{1 - \bar{q}^n e^{2(2\pi i \xi_2)}}$$

- Oscillator contribution from boson along S^1

$$\prod_n \frac{1}{1 - q^n} \frac{1}{1 - \bar{q}^n}$$

After adding the zero mode contributions of these fermions and bosons, total contribution from s^2 modes is given by :

$$(i)^{3/2} i \frac{\vartheta_1(\tau, z + \xi_2)\vartheta_1(\tau, z - \xi_2)}{\vartheta_1(\tau, 2\xi_2)\eta(\tau)^2} \tag{4.95}$$

where $i^{3/2}$ comes from the normalization of three boundary fermions. We can now compute the contribution due to boundary S^1's. We have two copies of S^1 because both chirals have different charges under the flavor symmetries. These give the following contribution to the supercurrent.

- Zero modes of one S^1:

$$\langle \bar{\partial}\psi \rangle = \langle -\frac{i p_R}{2} \rangle = \frac{1}{2g} \sum_{m,w} (m - w g^2) q^{\frac{(m+wg^2)^2}{4g^2}} \bar{q}^{\frac{(m-wg^2)^2}{4g^2}} e^{2\pi i(-\xi_1 - \xi_2)n}$$

- Zero modes of second S^1:

$$\langle \bar{\partial}\psi \rangle = \frac{1}{2g} \sum_{m,w} (m - wg^2) q^{\frac{(m+wg^2)^2}{4g^2}} \bar{q}^{\frac{(m-wg^2)^2}{4g^2}} e^{2\pi i(\xi_1 - \xi_2)n}$$

  The $e^{2\pi i(\xi_1-\xi_2)n}$ factor appears because we are computing flavored elliptic genus and the fugacities ξ_1 and ξ_2 act as shift symmetries of the boundary S^1 and hence they couple to the momentum $p_L + p_R$.

Putting all the contributions together and using (4.44), we obtain:

$$\begin{aligned} \frac{\partial \chi}{\partial \bar{\tau}} &= -\frac{e^{i\pi/4}}{\sqrt{4\tau_2}\eta(\tau)} i\sqrt{2} \frac{1}{\sqrt{2g}} \frac{(i)^{3/2} i \vartheta_1(\tau, z+\xi_2)\vartheta_1(\tau, z-\xi_2)}{\vartheta_1(\tau, 2\xi_2)\eta(\tau)^2} \\ &\quad \sum_{m,w} (m - wg^2) q^{\frac{(m+wg^2)^2}{4g^2}} \bar{q}^{\frac{(m-wg^2)^2}{4g^2}} \left(e^{2\pi i(-\xi_1-\xi_2)n} + e^{2\pi i(\xi_1-\xi_2)n}\right) \qquad (4.96) \\ \frac{\partial \chi}{\partial \bar{\tau}} &= -\frac{i}{2g\sqrt{2\tau_2}} \frac{\vartheta_1(z-\xi_2)\vartheta_1(z+\xi_2)}{\eta(\tau)^3 \vartheta_1(2\xi_2)} \\ &\quad \sum_{m,w} (m - wg^2) q^{\frac{(m+wg^2)^2}{4g^2}} \bar{q}^{\frac{(m-wg^2)^2}{4g^2}} \left(e^{2\pi i(-\xi_1-\xi_2)m} + e^{2\pi i(\xi_1-\xi_2)m}\right) \qquad (4.97) \end{aligned}$$

As expected, this matches the GLSM result (4.6.1).

Appendix

4.A $(2,2)$ Supersymmetry

Here we review our conventions for superfields in $N = (2,2)$ superspace [8] We follow the same conventions as [75]. We denote the bosonic coordinates by x^0, x^1 and the fermionic coordinates by θ^+, θ^-, $\bar{\theta}^+$ and $\bar{\theta}^-$. The spinors θ^+ are right-moving and θ^- are left moving. The bosonic coordinates span flat Minkowski space with metric diag$(-1,1)$. The fermionic coordinates are related to each other by complex conjugation i.e. $(\theta^\pm)^\dagger = \bar{\theta}^\pm$. The supersymmetry generators are given by:

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm\partial_\pm\,, \qquad \bar{Q}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} - i\theta^\pm\partial_\pm \tag{4.98}$$

where $\partial_\pm = (\partial_0 \pm \partial_1)/2$. The SUSY generators obey:

$$\{Q_\pm, \bar{Q}_\pm\} = -2i\partial_\pm \tag{4.99}$$

We also define the following superspace derivatives

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\bar{\theta}^\pm\partial_\pm\,, \qquad \bar{D}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_\pm \tag{4.100}$$

which anticommute with $Q_\pm$, $\bar{Q}_\pm$. We can now define various superfield and their supersymmetric lagrangians:

Chiral Superfield: It is defined as:

$$\bar{D}_\pm\Phi = 0$$

Hence, it can be expanded as:

$$\Phi = \phi(y^\pm) + \theta^\alpha\psi_\alpha(y^\pm) + \theta^+\bar{\theta}^+F(y^\pm)$$

[8] These conventions differ from the one's used in Appendix 2.3.2.

where $y^{\pm} = x^{\pm} - i\theta^{\pm}\bar{\theta}^{\pm}$. Here ϕ is a complex boson and $\psi_{\pm}$ is a right (left)-moving weyl fermion and F is an auxiliary field. The action is given by:

$$S_{kin}^{\text{chi}} = \frac{1}{4\pi} \int d^2x d^4\theta \, \bar{\Phi}\Phi \tag{4.101}$$

where $\bar{\Phi}$ is the complex conjugate of chiral superfield and it obeys $D_{\pm}\bar{\Phi} = 0$. It is known as *anti-chiral superfield*. We can also add a superpotential term to the action which is obtained by integrating a holomorphic function ($W(\Phi)$) of the chiral superfield on half superspace i.e.

$$S_{W}^{\text{chi}} = \int d^2x d^2\theta \, W(\Phi) + \text{c.c.} \tag{4.102}$$

A *twisted chiral superfield* U is a superfield that satisfies:

$$\bar{D}_{+}U = D_{-}U = 0$$

Vector Superfield: It is a real scalar superfield V which transforms as

$$V \to V + i(\bar{A} - A)$$

under gauge transformation. Here A is a chiral superfield. We can use the gauge transformation to eliminate some degrees of freedom and expand V in Wess-Zumino gauge as follows:

$$\begin{aligned} V = \quad & \theta^{-}\bar{\theta}^{-}(v_0 - v_1) + \theta^{+}\bar{\theta}^{+}(v_0 + v_1) - \theta^{-}\bar{\theta}^{+}\sigma - \theta^{+}\bar{\theta}^{-}\bar{\sigma} \\ & i\theta^{-}\theta^{+}(\bar{\theta}^{-}\bar{\lambda}_{-} + \bar{\theta}^{+}\bar{\lambda}_{+}) + i\bar{\theta}^{+}\bar{\theta}^{-}(\theta^{-}\lambda_{-} + \theta^{+}\lambda_{+}) + \theta^{-}\theta^{+}\bar{\theta}^{+}\bar{\theta}^{-}D \end{aligned}$$

where v_{μ} is a 2d gauge field, $\lambda_{\pm}$ are right (left) moving fermions, σ is a complex scalar and D is an auxiliary field. In the Wess-Zumino gauge, we still have a residual gauge symmetry:

$$v_{\mu}(x) \to v_{\mu}(x) - \partial_{\mu}\alpha(x)$$

where $\alpha_{\mu}(x)$ is the lowest component of the chiral superfield A. Under the gauge transformations, the chiral superfield of charge q transforms as:

$$\Phi \to e^{iqA}\Phi$$

The field strength Σ is a *twisted chiral* superfield given by:

$$\begin{aligned}\Sigma &= \bar{D}_+ D_- V \\ &= \sigma(\tilde{y}) + i\theta^+\bar{\lambda}_+(\tilde{y}) - i\bar{\theta}^-\lambda_-(\tilde{y}) + \theta^+\bar{\theta}^-(D(\tilde{y}) - iv_{01}(\tilde{y}))\end{aligned}$$

where $\tilde{y}^\pm = x^\pm \mp i\theta^\pm\bar{\theta}^\pm$ and v_{01} is the gauge field strength given by:

$$v_{01} = \partial_0 v_1 - \partial_1 v_0$$

The supersymmetric lagrangian of a chiral superfield with charge q under the vector superfields is given by:

$$L = \int d^4\theta \left(\bar{\Phi}e^{qV}\Phi - \frac{1}{2e^2}\bar{\Sigma}\Sigma\right) + \frac{1}{2}\left(-t\int d^2\tilde{\theta}\Sigma + c.c\right) \tag{4.103}$$

where the last term is the twisted superpotential term with $t = r - i\vartheta$. Here r is the Fayet-Iliopoulous (FI) parameter and ϑ is the theta angle for gauge field V.

After integrating out the fermionic superspace coordinates and the auxiliary fields we finally obtain:

$$\begin{aligned}L = \quad & -D^\mu\bar{\phi}D_\mu\phi + i\bar{\psi}_-(D_0 + D_1)\psi_- + i\bar{\psi}_+(D_0 - D_1)\psi_+ \\ & -\frac{e^2}{2}(|\phi|^2 - r)^2 - |\sigma^2||\phi|^2 - \bar{\psi}_-\sigma\psi_+ - \bar{\psi}_+\bar{\sigma}\psi_- \\ & -i\bar{\phi}\lambda_-\psi_+ + i\bar{\phi}\lambda_+\psi_- + i\bar{\psi}_+\bar{\lambda}_-\phi - i\bar{\psi}_-\bar{\lambda}_+\phi + \theta v_{01} \\ & \frac{1}{2e^2}(-\partial^\mu\bar{\sigma}\partial_\mu\sigma + i\bar{\lambda}_-(\partial_0 + \partial_1)\lambda_- + i\bar{\lambda}_+(\partial_0 - \partial_1)\lambda_+ + v_{01}^2)\end{aligned} \tag{4.104}$$

where $D_\mu = \partial_\mu + iqv_\mu$.

Stückelberg superfield: It is a chiral superfield P which transforms additively under the gauge transformation i.e.

$$P \to P + iA\,, \qquad \bar{P} \to \bar{P} - i\bar{A} \tag{4.105}$$

and has the following action:

$$S_{\text{stu}} = \frac{1}{4\pi}\int d^2x d^2\theta\,\frac{k}{4}(P + \bar{P} + V)^2 \tag{4.106}$$

The FI term and the ϑ term can now be absorbed in P. We take the gauge group to be $U(1)$, and from (4.105), we observe that ImP is periodically identified. The full supersymmetric Lagrangian with a chiral superfield Φ of charge q, a Stückelberg superfield P, and a $U(1)$ gauge superfield V is given by:

$$L = \int d^4\theta \left(\bar{\Phi} e^{qV} \Phi - \frac{1}{2e^2} \bar{\Sigma}\Sigma + \frac{k}{4}(P + \bar{P} + V)^2 \right) \tag{4.107}$$

The chiral superfield P can be gauged away entirely, and we are left with Φ and a massive vector superfield V. Alternatively, we can choose the Wess-Zumino gauge for V and retain P. In this case, after integrating out the fermionic coordinates, we obtain:

$$\begin{aligned} L = \quad & -D^\mu \bar{\phi} D_\mu \phi + i\bar{\psi}_-(D_0 + D_1)\psi_- + i\bar{\psi}_+(D_0 - D_1)\psi_+ D|\phi|^2 + |F|^2 \\ & -|\sigma^2||\phi|^2 - \bar{\psi}_- \sigma \psi_+ - \bar{\psi}_+ \bar{\sigma} \psi_- - i\bar{\phi}\lambda_- \psi_+ + i\bar{\phi}\lambda_+ \psi_- + i\bar{\psi}_+ \bar{\lambda}_- \phi - i\bar{\psi}_- \bar{\lambda}_+ \phi \\ & \frac{1}{2e^2}[-\partial^\mu \bar{\sigma} \partial_\mu \sigma + i\bar{\lambda}_-(\partial_0 + \partial_1)\lambda_- + i\bar{\lambda}_+(\partial_0 - \partial_1)\lambda_+ + v_{01}^2 + D^2] \\ & \frac{k}{2}\left[(-\partial^\mu \bar{p} + iv^\mu)(\partial_\mu p + iv_\mu) + i\bar{\chi}_-(\partial_0 + \partial_1)\chi_- + i\bar{\chi}_+(\partial_0 - \partial_1)\chi_+ + D(p + \bar{p})\right. \\ & \left. + |F_P|^2 - |\sigma|^2 + i\chi_+ \lambda_- + i\bar{\chi}_+ \bar{\lambda}_- - i\chi_- \lambda_+ - i\bar{\chi}_- \bar{\lambda}_+ \right] \end{aligned} \tag{4.108}$$

4.B Eta and theta functions

The Dedekind eta function is a modular form of weight $1/2$ which is defined as:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

where $q = e^{2\pi i \tau}$ and $\tau \in \mathbb{H}$. Due to its modular properties, it satisfies:

$$\eta(\tau + 1) = e^{i\pi/12}\eta(\tau)\,, \qquad \eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau)\,. \tag{4.109}$$

In the computation of elliptic genus, the Jacobi theta function plays an important role. It has the following product representation:

$$\begin{aligned}\vartheta_1(\tau|z) &= -iq^{\frac{1}{8}}y^{\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^{n-1}) \qquad (4.110)\\ &= -iq^{\frac{1}{8}}y^{\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n)(1-y^{-1})\\ &= -iq^{\frac{1}{8}}(y^{\frac{1}{2}}-y^{-\frac{1}{2}})\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n) \qquad (4.111)\end{aligned}$$

where $y = e^{2\pi iz}$. There are various variations in the definition of ϑ-functions. In some cases, there is no $-i$ factor in front. We use the above definition. It has another representation:

$$\vartheta_1(\tau|z) = -i\sum_{n\in Z}(-1)^n y^{n+\frac{1}{2}}q^{\frac{1}{2}(n+\frac{1}{2})^2} \qquad (4.112)$$

We list some of the useful properties of ϑ_1 below:

$$\vartheta_1(\tau|z+n\tau+m) = (-1)^{n+m}q^{-\frac{n^2}{2}}y^{-n}\vartheta_1(\tau|z) \qquad (4.113)$$

where $n, m \in \mathbb{Z}$. Using this we obtain:

$$\frac{\vartheta_1(\tau|z+m+w\tau)}{\vartheta_1(\tau|m+w\tau)} = e^{-2\pi iwz}\frac{\vartheta_1(\tau|z)}{\vartheta_1(\tau)} \qquad (4.114)$$

The modular properties of $\vartheta_1(\tau|z)$ give:

$$\vartheta_1(\tau+1|z) = e^{\pi i/4}\vartheta_1(\tau|z)\,, \qquad \vartheta_1(-\frac{1}{\tau}|\frac{z}{\tau}) = -i\sqrt{-i\tau}e^{i\pi\frac{z^2}{\tau}}\vartheta_1(\tau|z)\,. \qquad (4.115)$$

Also,

$$\theta_1(-z;\tau) = -\theta_1(z;\tau)$$

The first derivative at $z = 0$ is useful:

$$\frac{1}{2\pi i}\frac{\partial}{\partial z}\vartheta_1(\tau|z)|_{z=0} = -iq^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)^3 = -i\eta(\tau)^3 \qquad (4.116)$$

Conclusions and Outlook

In this part of the thesis, we re-derived the Dirac index theorem for manifolds with boundary and explored its relation with mock modularity. The main results are summarized as follows:

- The Atiyah-Patodi-Singer (APS) index theorem for manifolds with a product metric near the boundary can be derived by computing the Witten index for a non-compact target space. This non-compact manifold is obtained by trivially extending the original manifold with boundary as explained in Chapter3.
- The APS index theorem consists of two terms: the Atiyah- Singer (AS) piece and the η-invariant piece. Both of these pieces can be computed using different supersymmetric quantum mechanics (SQM).
  - The AS piece is derived using an $N = 1/2$ SQM with target space $\overline{\mathcal{M}}$ as shown in figure 3.3.1.
  - The η-invariant is determined by using an $N = 1$ SQM with target space $\mathbb{R}$ and a superpotential $h(u)$ which depends on the eigenvalues of the boundary operator. Hence the η-invariant piece depends only on the boundary.
- The η-invariant is also related to the temperature dependence of the non-compact Witten index, and hence it is related to the difference in density of states of bosons and fermions. This difference in the density of states, in turn, is connected to the scattering theory.
- The elliptic genus of a superconformal field theory (SCFT) with a non-compact target space is non-holomorphic in τ. The $\bar{\tau}$ dependence is related to the temperature dependence of the Witten index of the right-movers. Similarly to the Witten index, it also depends only on the asymptotic boundary of the target space.
- We computed the holomorphic anomaly for the $N = (2, 2)$ Cigar SCFT and the $N = (4, 4)$ TaubNUT SCFT using different methods: First from the GLSM

and then from the boundary theory of the NLSM using the *GJF anomaly* equation. For the cigar case, we also compared the results with the SQM computation.

There are certain unanswered questions yet to be explored. Some of them are listed below:

- We re-derived the APS index theorems for manifolds with a product metric near the boundary. For the manifolds with a non-product metric near the boundary, there is a correction in the APS index theorem [48]. We do not have a derivation of this correction term from SQM. In this case, the main issue is that the continuum touches the zero modes, and hence the contributions from the discrete states and the scattering states cannot be separated as in (3.37). It will be interesting to understand this issue.

- We noticed that the elliptic genus for SCFT's whose target space has a non-product metric at asymptotic infinity vanishes unless some fugacities are turned on. After turning on the fugacities, not all non-compact manifolds give mock objects. In the examples we explored, only the GLSM's with Stückelberg fields seem to give non-zero holomorphic anomaly. It would be interesting to explore this co-relation and make a more precise statement.

- It would also be interesting to recover different examples of mock modular forms such as higher depth mock modular forms [31, 34, 83–85] just by looking at GLSM's with multiple Stückelberg fields.

PART II

Soft theorems for Gravitini

Chapter 5

Introduction

The study of soft theorems in quantum field theory has a long history dating back to work done by Bloch, Nordsieck, and Low [86–88]. In 1965, Weinberg extended this work to show that infrared divergences in a quantum theory of gravity can be removed in the same way as in quantum electrodynamics [89]. A particle whose momenta in the center of mass frame is much lower than other particles is called a soft particle in a scattering event. Soft theorems demonstrate the relation between the S-matrix of hard particles with and without soft particles. Soft theorems capture certain universal features of the theory.

In the last few years, the interest in soft theorems has been renewed because of its connection to asymptotic symmetries. Asymptotic symmetries are the symmetries whose action on fields does not vanish at infinity. This is why they are also known as "large gauge symmetries". In the early 2010s, Strominger and his collaborators [90–94] found that soft theorems are essentially the Ward identities associated with asymptotic symmetries. Further studies revealed an interesting relation between the Bondi–Metzner–Sachs (BMS) group (the asymptotic symmetry group corresponding to large diffeomorphisms) and the gravitational memory effect [95]. These studies established the relation between three seemingly different phenomena - Asymptotic symmetry, soft theorems, and memory effect. In subsequent papers, the study of asymptotic symmetry was extended to higher than four dimensions [94, 96, 97] but the understanding of the same in arbitrary dimensions is far from being complete.

There is an approach to derive soft theorems independently of the spacetime dimension. It relies on Feynman diagrammatic techniques. In this approach, one starts from a specific Lagrangian and then computes only a subclass of Feynman diagrams, contributing to the (sub-)leading soft theorem(s). The new impetus to this direction is Sen's work [98, 99]. His method relies on covariantization of one particle irreducible (1PI) effective action with respect to the soft field and computes the diagrams which contribute to the soft theorem. This powerful method was used to compute the sub-sub-leading soft graviton theorem [100] and also to compute multiple (sub-)leading soft graviton theorem [101]. It has been noted that the soft-photon theorem is universal at leading order [89, 102], and the soft-graviton theorem is universal, not only in the leading order but also in the sub-leading order [103]. Much work has been done on soft theorems with multiple soft particles, soft theorems in string theory, and memory effect in the last decade [98, 104–111].

In this part of the thesis, we derive soft theorems for gravitini. One of the primary motivations for this work is that in four and higher dimensions, the theories of massless particles are severely constrained by Coleman-Mandula theorem [112]. Massless particles with spin > 2 cannot couple minimally; they only couple through the field strength. So the only particles which possess gauge invariance and can have minimal coupling have spin $1, 3/2$ and 2. We already have a complete understanding of soft photon and soft graviton theorem. However, we still do not have many results about soft gluon and soft gravitino theorem. Hence we tried to attack the latter problem.

These computations involve a subtlety because the leading soft factors do not commute, and their commutator is also leading order in soft momenta. At the level of complexity, the soft gravitino theorem is more subtle than soft photon or graviton but significantly less subtle than that of the gluon. This is because even though the commutator of two soft factors is non-vanishing, the commutator of three soft factors vanishes in the case of a gravitino but not in the case of a gluon. However, for specific types of theories, the soft gluon theorem can be conveniently computed using Cachazo, He, and Yuan (CHY) formalism [113–116]. This advantage is not currently available for soft gravitino/photino. In this work, we derived the leading order soft theorem for gravitino in a general quantum field theory with local supersymmetry and in an arbitrary number of dimensions. The soft gravitino operator is a fermionic soft operator. Though a lot is known about bosonic soft theorems, the available literature for the fermionic soft theorem is significantly little. Single Soft photino theorem was computed in [117].

The soft theorems for the case of a single and double soft gravitino for four-dimensional supergravity theories were computed for a particular model in [118–120]. The result for single soft gravitino in $D = 4$ can also be obtained from asymptotic symmetry [121, 122]. We generalize the result to the case with an arbitrary number of soft gravitini. In our work, we follow Sen's covariantization approach [98–100]. This method's advantage is that it is valid for arbitrary theories, to all orders in perturbation theory and in arbitrary dimensions, as long as there is no infrared divergence.

This part of the thesis is organized as follows: In Chapter 6, we set up the notation and derive the Feynman rules required to compute the leading soft gravitini theorem. In Chapter 7, we explicitly derive the soft theorem with one, two, and three external soft gravitini. We then use these results to get a soft theorem for an arbitrary number of soft gravitini. In Chapter8, we look at the possible infrared divergences that can affect our results. It turns out that our results are valid for $D \geq 4$.

Chapter 6

Feynman Rules

In this chapter, we derive the relevant Feynman rules for computing the leading order soft gravitini theorem.

6.1 Set-up

Our starting point is a globally supersymmetric Lagrangian invariant under some number of Majorana supersymmetry [1]. So the usual (dimension-dependent) restriction for the existence of a globally supersymmetric Lagrangian applies. The matter content of the theory is some reducible super-multiplet. We do not assume anything about the multiplet in which matter fields are sitting.

Let Φ_m be any quantum field that transforms under some reducible representation of the Poincare group, supersymmetry, and the internal symmetry group(s). The transform of the fields under the global supersymmetry is given by

$$\Phi_m \longrightarrow (Q_\alpha)_m{}^n \Phi_n \tag{6.1}$$

Q_α are supersymmetry generators. They satisfy the following algebra

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\gamma^\mu_{\alpha\beta} P_\mu \tag{6.2}$$

Here P_μ is the momentum generator. The indices α, β are the collection of all possible spinor indices, not the indices for the minimal spinor (of that dimension). So, in a theory of more than one supersymmetry, Q_α are the collection of all the supercharges. Gamma matrices are in Majorana representation and are symmetric in the spinor indices.

[1] From Coleman-Mandula theorem, the maximum number of super-charges is 32.

We start with the 1PI effective action. The kinetic term is given by:

$$S = \frac{1}{2}\int \frac{d^d p_1}{(2\pi)^D}\frac{d^d p_2}{(2\pi)^D}\Phi_m(p_1)\mathcal{K}^{mn}(p_2)\Phi_n(p_2)(2\pi)^D\delta^{(D)}(p_1+p_2) \tag{6.3}$$

where $\mathcal{K}^{mn}(p_2)$ is a combination of derivative operators. The kinetic term is invariant under global supersymmetry transformation. This implies

$$\mathcal{K}^{m_1 m_3}(Q_\alpha)_{m_3}{}^{m_2} + \mathcal{K}^{m_3 m_2}(Q_\alpha)_{m_3}{}^{m_1} \;=\; 0 \tag{6.4}$$

6.1.1 Propagator

Let us assume the propagator has the following form:

$$\Xi(q)(q^2+M^2)^{-1} \tag{6.5}$$

where $\Xi(q)$ is defined as

$$\Xi(q) = i(q^2+M^2)\mathcal{K}^{-1}(q) \tag{6.6}$$

and M is some arbitrary mass parameter [2] . From (6.4) we get,

$$\Xi^{m_1 m_3}(Q_\alpha)_{m_3}{}^{m_2} + \Xi^{m_3 m_2}(Q_\alpha)_{m_3}{}^{m_1} \;=\; 0 \tag{6.7}$$

We write down two more relations which would be useful later

$$\begin{aligned}
\mathcal{K}^{m_1 m_2}(-p)\,\Xi_{m_2 m_3}(-p) &= i(p^2+M^2)\,\delta^{m_1}{}_{m_3} \\
\frac{\partial \mathcal{K}^{m_1 m_2}(-p)}{\partial p_\mu}\Xi_{m_2 m_3}(-p) &= -\mathcal{K}^{m_1 m_2}(-p)\frac{\partial \Xi_{m_2 m_3}(-p)}{\partial p_\mu} + 2ip^\mu\,\delta^{m_1}{}_{m_3}
\end{aligned} \tag{6.8}$$

6.1.2 Covariant derivative

In super-gravity theories, the super-covariant derivative [123] is given by

$$\mathcal{D}_a = E_a{}^\mu\left(\partial_\mu - i\,\kappa\Psi_\mu{}^\alpha Q_\alpha - i\,\kappa\frac{1}{2}\omega_\mu{}^{cd}\mathcal{J}_{cd}\right) \tag{6.9}$$

[2]We have already used M for number of soft-particles. Since the mass-parameter does not appear extensively, we also M for mass-parameter.

Here κ is the gravitational coupling constant, ω_μ^{cd} is the spin connection and $\mathcal{J}_{cd}$ are the angular momentum generators. The local-supersymmetry transformation of the vielbein e^a_μ (the inverse of E^μ_a appearing above) and the gravitino $\Psi_{\mu\alpha}$ are given by

$$\delta e^a_\mu = \frac{1}{2}\theta\gamma^a\Psi_\mu \tag{6.10a}$$

$$\delta\Psi_{\mu\alpha} = \mathcal{D}_\mu\,\theta_\alpha = \partial_\mu\,\theta_\alpha + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\theta_\alpha \tag{6.10b}$$

Here θ_α is the local supersymmetry parameter. Now consider a small fluctuation

$$E_a{}^\mu = \delta_a{}^\mu - \zeta_a{}^\mu e^{ik\cdot x} \tag{6.11a}$$

$$\Psi_\mu{}^\alpha = \epsilon_\mu{}^\alpha e^{ik\cdot x} \tag{6.11b}$$

So at the linear order of fluctuations, we get the following expression for the super-covariant derivative

$$D_a = \partial_a - \kappa\,\zeta_a{}^\mu\partial_\mu - i\kappa\,\epsilon^\alpha_a Q_\alpha - i\frac{\kappa}{2}\omega_a{}^{cd}\mathcal{J}_{cd} \tag{6.12}$$

6.2 Soft gravitino - Matter Vertex

The coupling of one soft gravitino to the matter fields at linear order can be found by covariantizing the derivative in (6.3). Due the interaction with gravitino, the momenta of hard particle changes by $\delta q = -\kappa\,\epsilon^\alpha_\mu Q_\alpha$.

The coupling of gravitino with the matter field can then be found just from the quadratic part of the 1PI effective action by making the following changes in (6.3) [100]:

- $\delta^{(D)}(p_1+p_2)$ gets replaced by $\delta^{(D)}(p_1+p_2+k)$ where k is the momenta of soft gravitino.
- The change in kinetic operator $\mathcal{K}^{mn}$ due to shift in momenta has to be substituted.

So we get

$$S^{(L)} = \frac{1}{2}\int \frac{d^d p_1}{(2\pi)^D}\frac{d^d p_2}{(2\pi)^D}\Phi_m(p_1)\left[-\frac{\partial\mathcal{K}(p_2)}{\partial p_{2\mu}}\kappa\epsilon^{\alpha}_{\mu}Q_{\alpha}\right]^{mn}\Phi_n(p_2)(2\pi)^D\delta^{(D)}(p_1+p_2+k) \tag{6.13}$$

So the $\Phi_m - \epsilon^{\alpha}_{\mu} - \Phi_n$ vertex is given by:

$$-\left[i\kappa\,\frac{\partial\mathcal{K}(p_i)}{\partial\,p_{i\mu}}\epsilon^{\alpha}_{\mu}Q_{\alpha}\right]^{mn} \tag{6.14}$$

Since we compute only the S-matrix elements, all the particles satisfy on-shell and transversality condition . The external particle of polarization $\varepsilon_{i,m}$ and momenta p_i satisfies the following conditions:

$$\varepsilon_{i,m}\mathcal{K}^{mn}(q) = 0 \tag{6.15a}$$

$$p_i^2 + M_i^2 = 0 \tag{6.15b}$$

6.3 Gravitino - Graviton - Gravitino Vertex

When we have more than one soft gravitino, we need to consider the minimal coupling of gravitino with graviton. At the leading order, the graviton coupling to any matter field can be again be found by covariantizing the derivatives appearing in the kinetic operator. From equation (6.12), we see that there are two changes in the kinetic operator: one due to the shift in momenta due to $\kappa\zeta^{\mu}_{a}\partial_{\mu}$ term and the other due to the spin connection term. Hence we have:

$$\begin{aligned} S^{(L)} = &\frac{1}{2}\int \frac{d^D k_1}{(2\pi)^D}\frac{d^D k_2}{(2\pi)^D}(2\pi)^D\delta^{(D)}(k_1+k_2+p) \\ &\Phi_m(k_1)\left[-\zeta_{\mu\nu}k_2^{\nu}\frac{\partial}{\partial k_{2\mu}}\mathcal{K}^{mn}(k_2)+\frac{1}{2}(p_b\zeta_{a\mu}-p_a\zeta_{b\mu})\frac{\partial}{\partial k_{2\mu}}\mathcal{K}^{mp}(k_2)(\mathcal{J}^{ab})_p{}^n\right]\Phi_n(k_2) \end{aligned} \tag{6.16}$$

where $\zeta_{\mu\nu}$ is the graviton polarization.

The kinetic term for the gravitino, in the harmonic gauge, is given by

$$\mathcal{K}^{\mu\alpha;\nu\beta}(p) = (p_\rho\gamma^\rho)^{\alpha\beta}\eta^{\mu\nu} \tag{6.17}$$

The angular momentum generator is

$$(\mathcal{J}^{ab})_{\mu,\alpha}{}^{\nu,\beta} = (\mathcal{J}_V^{ab})_\mu{}^\nu\,\delta_\alpha{}^\beta + (\mathcal{J}_S^{ab})_\alpha{}^\beta\delta_\mu{}^\nu \tag{6.18}$$

where $\mathcal{J}_V^{ab}$ and $\mathcal{J}_S^{ab}$ are angular momentum generator in vector and spinor representations respectively.

$$(\mathcal{J}_V^{ab})_\mu{}^\nu = \delta^a{}_\mu\eta^{b\nu} - \delta^b{}_\mu\eta^{a\nu} \tag{6.19a}$$

$$(\mathcal{J}_S^{ab})_\alpha{}^\beta = -\frac{1}{2}(\gamma^{ab})_\alpha{}^\beta \qquad \gamma^{ab} \equiv \frac{1}{2}(\gamma^a\gamma^b - \gamma^b\gamma^a) \tag{6.19b}$$

Our gamma matrix convention is given in (7.67). Our convention is that all the particles are incoming; the gravitino has momentum k_1 and k_2 and the graviton has momenta p. The momentum conservation implies

$$p + k_1 + k_2 = 0 \tag{6.20}$$

Substituting equations (6.17) and (6.18) in the equation (6.16), we find that the gravitino-graviton-gravitino vertex $(\mathcal{V}^{\mu\nu;\mu_1\mu_2})^{\alpha\beta}$ is given by

$$-i\kappa\Big[k_2^{\mu_2}(\gamma^{\mu_1})^{\alpha\beta}\eta^{\mu\nu} + \frac{1}{4}(p_d\,\delta_c^{\mu_2} - p_c\,\delta_d^{\mu_2})(\gamma^{\mu_1}\gamma^{cd})^{\alpha\beta}\eta^{\mu\nu} + (p^\mu\eta^{\nu\mu_2} - p^\nu\eta^{\mu\mu_2})(\gamma^{\mu_1})^{\alpha\beta}\Big] \tag{6.21}$$

6.4 Gravitino - Graviphoton - Gravitino Vertex

In case of extended supersymmetries, one can have central charges in the supersymmetry algebra. The supersymmetry algebra in (6.2) modifies to

$$\Big\{Q_\alpha, Q_\beta\Big\} = -\frac{1}{2}\gamma^\mu_{\alpha\beta}P_\mu - \frac{1}{2}\mathcal{Z}_{\alpha\beta}\mathcal{U} \tag{6.22}$$

$\mathcal{U}$ is (are) the generator(s) of $U(1)$ symmetry(-ies) generated by the central charge(s). As explained below equation (6.2), α, β are some (ir-)reducible spinor indices. In this language the existence of central charge is equivalent to the condition that there exists an element(s) $Z_\alpha{}^\beta$ in the Clifford algebra such that, $\mathcal{Z}_{\alpha\beta}$ satisfies

$$\mathcal{Z}_{\alpha\beta} = \mathcal{Z}_{\beta\alpha} \tag{6.23}$$

In general, there can be higher form central charges. For example, in $D = 11$, the supersymmetry algebra is of the form

$$\Big\{Q_\alpha, Q_\beta\Big\} = -\frac{1}{2}\gamma^\mu_{\alpha\beta}P_\mu + \gamma^{\mu_1\mu_2\mu_3}_{\alpha\beta}A_{\mu_1\mu_2\mu_3} \tag{6.24}$$

But for our purpose, we ignore any higher form central charges. This is because the higher form central charges can only minimally couple to extended objects (of appropriate dimensions), whereas here we are considering the scattering of point-like states only.

In this case the commutator of two soft operators in (7.19) is modified as follows

$$[\mathcal{S}_u, \mathcal{S}_v] = -\frac{\kappa^2}{2}\sum_{i=1}^{N}\left[\epsilon^{(u);\alpha}_\mu(\not{p}_{i;\alpha\beta} + e_i\,\mathcal{Z}_{\alpha\beta})\epsilon^{(v);\beta}_\nu \frac{p_i^\mu}{p_i\cdot k_u}\frac{p_i^\nu}{p_i\cdot k_v}\right] \tag{6.25}$$

When we gauge the global supersymmetry with a central charge to get supergravity, we get a $U(1)^N$ gauge symmetry generated by spin 1 bosons (graviphoton) present in the graviton multiplet. These graviphotons couple to the gravitino and to any matter which carries the central charge. The coupling of the graviphoton to gravitino is entirely fixed by supersymmetry and is related to that of the graviton.

The gravitino-gravitino-graviphoton three point function $(\widetilde{\mathcal{V}}^{\mu\nu;\mu_1})_{\alpha\beta}$ is given by

$$-i\kappa\Big[k_2^{\mu_1}(\mathcal{Z})^{\alpha\beta}\eta^{\mu\nu}\Big] - \frac{i\kappa}{2}\Big[[(k_1+k_2)_c\,\delta^{\mu_1}_d](\mathcal{Z}\gamma^{cd})^{\alpha\beta}\eta^{\mu\nu}\Big] + i\kappa\Big[(k_2^\mu\,\eta^{\mu_1\nu} - k_1^\nu\,\eta^{\mu_1\mu})(\mathcal{Z})^{\alpha\beta}\Big] \tag{6.26}$$

We have now derived all the relevant vertices required to compute the leading order soft theorem for gravitini. In the next chapter, we will explicitly compute the soft factors.

Chapter 7

Soft Gravitino Theorem

We have now reached a point where we can use the Feynman rules derived in the previous chapter to compute the leading soft theorems for gravitini. In this chapter, we explicitly compute the leading soft theorem for one, two, and three external gravitini. We then use these results to write down the expression for soft theorem with multiple external soft gravitini. Before dwelling into the computation, there is a brief note on Feynman diagram conventions.

7.1 Note on Feynman diagrams

We use a red double-arrowed line for soft-gravitino, a blue wavy line to denote soft gravitons, a violet wavy line for graviphoton, Cyan double arrowed[1] line for hard fermionic particles (including hard gravitini) and black line to denote hard bosonic particles.

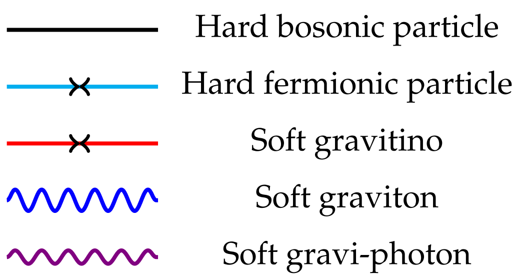


FIGURE 7.1.1: Conventions for Feynman diagrams

[1]We use a double arrowed line for Majorana particles because they are their own anti-particle; they only have $\mathbb{Z}_2$ charge.

7.2 Single soft gravitino

In this section, we compute the leading order contribution to the soft gravitino theorem due to one soft gravitino. This result for $D = 4$ was first derived in [119] and was reproduced from the analysis of asymptotic symmetries in [121, 122]. The only diagram that contributes to this process is depicted in figure 7.2.1.

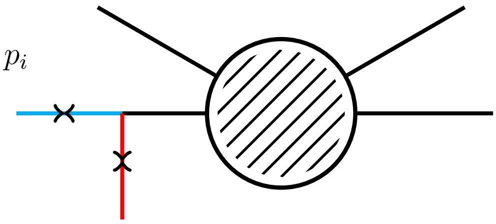


FIGURE 7.2.1: Feynman diagram for single soft gravitino

The expression for the propagator is given in equation (6.5). In this diagram, the propagator carries momenta $p_i + k$ and M_i is the mass of the i-th particle. Let us denote the corresponding propagator by $\Xi_{m_i n_i}(p_i+k)$. The contribution to figure 7.2.1 is given by:

$$\begin{aligned}\Gamma^{m_1..m_N}_{N+1}(\{p_i\},k) &= \left[i\kappa \sum_{i=1}^{N} \left(\frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\mu}} \epsilon^\alpha_\mu Q_\alpha \right)^{m_i n_i} \frac{\Xi_{n_i \tilde{n}_i}}{(p_i+k)^2 + M_i^2} \right] \Gamma^{m_1..m_{i-1}\tilde{n}_i m_{i+1}..m_N}_{N}(\{p_i\}) \\ &= \left[i\kappa \sum_{i=1}^{N} \left(\frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\mu}} \epsilon^\alpha_\mu Q_\alpha \right)^{m_i n_i} \frac{\Xi_{n_i \tilde{n}_i}}{(2\, p_i \cdot k)} \right] \Gamma^{m_1..m_{i-1}\tilde{n}_i m_{i+1}..m_N}_{N}(\{p_i\}) \quad (7.1)\end{aligned}$$

where in the second step, we have used the on-shell condition (6.15b) for external hard particle and the fact that gravitino is soft. Now we will use (6.7) and (6.8) to simplify the expression

$$\begin{aligned}\left(\frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\mu}} \epsilon^\alpha_\mu \, Q_\alpha \right)^{m_i n_i} \Xi_{n_i \tilde{n}_i} &= \epsilon^\alpha_\mu \left(\frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\mu}} Q_\alpha \, \Xi \right)^{m_i}{}_{\tilde{n}_i} = -\epsilon^\alpha_\mu \left(\frac{\partial \mathcal{K}(-p_i)}{\partial p_{i\mu}} \Xi \, Q_\alpha \right)^{m_i}{}_{\tilde{n}_i} \\ &= -\epsilon^\alpha_\mu \left(-\mathcal{K}(-p_i) \frac{\partial\, \Xi}{\partial p_{i\mu}} \, Q_\alpha + 2\, i\, p_i^\mu \, Q_\alpha \right)^{m_i}{}_{\tilde{n}_i} \quad (7.2)\end{aligned}$$

From first step to second step we have used (6.7) and from second step to third step we have used (6.8). Now the first term drops out because of the on-shell condition

(6.15a). Hence we obtain [119]

$$\Gamma^{m_1\ldots m_N}_{N+1}(\{p_i\},k) \;=\; \left[\kappa\sum_{i=1}^{N}\left(\frac{p_i^\mu\,\epsilon_\mu^\alpha}{p_i\cdot k}Q_\alpha\right)^{m_i}{}_{\tilde n_i}\right]\Gamma_N^{m_1\ldots m_{i-1}\tilde n_i m_{i+1}\ldots m_N}(\{p_i\}) \tag{7.3}$$

Soft operator We define the soft operator $\mathcal{S}_u$ [119] as

$$\mathcal{S}_u = \kappa\sum_{i=1}^{N}\left(\frac{p_i^\mu\epsilon_\mu^{(u)\,\alpha}}{p_i\cdot k_u}Q_\alpha\right) \tag{7.4}$$

where u labels the soft gravitino. So the above result can be re-written as:

$$\Gamma^{m_1\ldots m_N}_{N+1}(\{p_i\},k) = \left[\;\mathcal{S}^{m_i}{}_{\tilde n_i}\right]\Gamma_N^{m_1\ldots m_{i-1}\tilde n_i m_{i+1}\ldots m_N}(\{p_i\}) \tag{7.5}$$

7.2.1 Gauge invariance

As a consistency check, we check the gauge invariance of equation (7.3). We put pure gauge polarization for the gravitino

$$\epsilon_{\alpha\mu} = k_\mu\,\theta_\alpha \tag{7.6}$$

Here θ_α is a Majorana spinor. For pure gauge gravitino the amplitude should vanish. From (7.3), we obtain

$$\theta^\alpha\sum_{i=1}^{N}(Q_\alpha)^{m_i}{}_{\tilde n_i}\Gamma_N^{m_1\ldots m_{i-1}\tilde n_i m_{i+1}\ldots m_N}(p_i) = 0 \tag{7.7}$$

This is the Ward-identity for the global super-symmetry.

7.3 Two soft gravitini

Now we will consider the amplitude with N hard particles and two soft gravitini. In this case, the order of soft limits can affect the result. In this work, we took the simultaneous soft limit. More about this is explained in subsection 7.3.3.

There are essentially four different types of Feynman diagrams which can contribute in this case:

1. The class of diagrams where the two soft gravitini are attached to different external legs (figure 7.3.1). These diagrams are easy to evaluate. The computation for these type of diagrams is essentially the same as single soft gravitino.

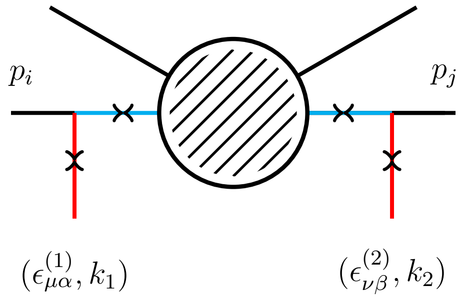


FIGURE 7.3.1: Feynman diagram for double soft gravitini - I

The contribution from figure 7.3.1 is given by

$$\kappa^2 \sum_{i=1}^{N} \frac{\epsilon^{(1);\alpha}_{\mu} p_i^{\mu}}{p_i \cdot k_1} Q_\alpha \sum_{j=1;j\neq i}^{N} \frac{\epsilon^{(2);\beta}_{\nu} p_i^{\nu}}{p_j \cdot k_2} Q_\beta \Gamma(\{p_i\}) \tag{7.8}$$

2. The class of diagrams where both of the soft gravitini are attached to the same external leg. There are three types of such diagrams - (figure 7.3.2, figure 7.3.3, figure 7.3.4). Figure 7.3.2, Figure 7.3.3 denote the diagrams where the soft gravitino directly attaches the same hard-particles. These two diagrams differ only in the order of attaching to the hard particle. Figure 7.3.4 captures the process when the soft gravitini combine to give a soft graviton, and then the soft graviton attaches to the hard particles.

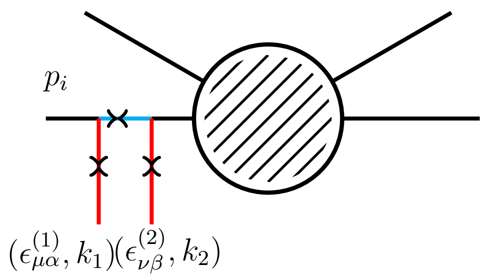


FIGURE 7.3.2: Feynman diagram for double soft gravitini - II

The contribution from the Feynman diagram in figure 7.3.2 is given by

$$\Gamma^{(1)}_{N+2} = \kappa^2 \sum_{i=1}^{N} \frac{\partial \mathcal{K}^{mp}(-p_i)}{\partial p_{i\mu}} \frac{[\epsilon^{(1);\alpha}_{\mu} Q_\alpha \Xi(-p_i - k_1)]_{pq}}{(2p_i \cdot k_1)} \frac{\partial \mathcal{K}^{qr}(-p_i - k_1)}{\partial p_{i\nu}} \frac{[\epsilon^{(2);\beta}_{\nu} Q_\beta \, \Xi(-p_i - k_1 - k_2)]_{rs}}{(2p_i \cdot (k_1 + k_2))} \Gamma_N(\{p_i\}) \tag{7.9}$$

Using (6.7) and (6.8) we can simplify this expression and we get

$$\kappa^2 \sum_{i=1}^{N} \frac{\epsilon^{(1);\alpha}_{\mu} p_i^{\mu}}{p_i \cdot k_1} \frac{\epsilon^{(2);\beta}_{\nu} p_i^{\nu}}{p_i \cdot (k_1 + k_2)} Q_\alpha \, Q_\beta \, \Gamma_N(\{p_i\}) \tag{7.10}$$

The second diagram is given by:

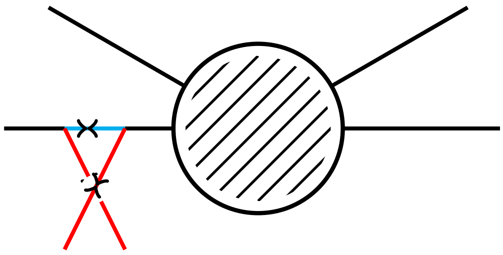


FIGURE 7.3.3: Feynman diagram for double soft gravitini - III

The contribution due to figure 7.3.3 can obtained from equation (7.9) by interchanging $1 \longleftrightarrow 2$

$$\begin{aligned} \Gamma^{(2)}_{N+2}(\{p_i\}, k_1, k_2) &= \kappa^2 \sum_{i=1}^{N} \frac{\epsilon^{(2);\beta}_{\nu} p_i^{\nu}}{p_i \cdot k_2} \frac{\epsilon^{(1);\alpha}_{\mu} p_i^{\mu}}{p_i \cdot (k_1 + k_2)} Q_\beta Q_\alpha \, \Gamma_N(\{p_i\}) \\ &= \kappa^2 \sum_{i=1}^{N} \frac{\epsilon^{(1);\alpha}_{\mu} p_i^{\mu}}{p_i \cdot k_2} \frac{\epsilon^{(2);\beta}_{\nu} p_i^{\nu}}{p_i \cdot (k_1 + k_2)} \left[Q_\alpha Q_\beta + \frac{1}{2} (\not{p}_i)_{\alpha\beta} \right] \Gamma_N(\{p_i\}) \end{aligned} \tag{7.11}$$

The final contribution comes from figure 7.3.4. This diagram denotes the process when two soft gravitini interact first to produce a soft graviton which then attaches to any of the external legs. The contributions from these kinds of processes

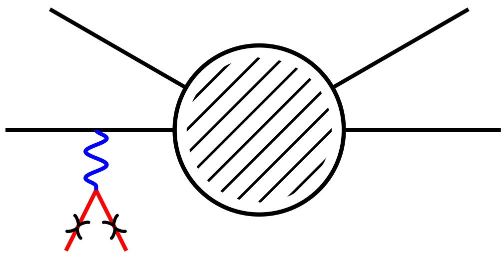


FIGURE 7.3.4: Feynman diagram for double soft gravitini - IV

are given by

$$\Gamma^{(3)}_{N+2}(\{p_i\},k_1,k_2) = -\left[\epsilon^{(1)\,\alpha}_{\mu}(\mathcal{V}^{\mu\nu;\mu_1\mu_2})_{\alpha\beta}\,\epsilon^{(2)\,\beta}_{\nu}\right]\left[\left(\frac{i}{2}\right)\frac{\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}+\eta_{\mu_1\nu_2}\eta_{\mu_2\nu_1}-\frac{2}{D-2}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}}{2(k_1\cdot k_2)}\right]$$
$$\left[-i\kappa\, p_i^{\nu_1}\frac{\partial\mathcal{K}}{\partial p_{i\nu_2}}\frac{\Xi}{2p_i\cdot(k_1+k_2)}\right]\Gamma_N(\{p_i\}) \tag{7.12}$$

where the first square bracket denotes gravitino-gravitino-graviton vertex, the second one is the graviton propagator and the third one is the matter- soft graviton -matter vertex. Using the expression for $(\mathcal{V}^{\mu\nu;\mu_1\mu_2})_{\alpha\beta}$ from (6.21) and simplifying the above expression, we get

$$\Gamma^{(3)}_{N+2}(\{p_i\},k_1,k_2) = \frac{\kappa^2}{2}\sum_{i=1}^{N}\epsilon^{(1)\alpha}_{\mu}(\not{p}_i)_{\alpha\beta}\left[\frac{1}{(p_i\cdot(k_2+k_1))(k_1\cdot k_2)}\right]$$
$$\left[-\eta^{\mu\nu}p_i\cdot k_2-\frac{1}{2}\eta^{\mu\nu}(k_1+k_2)_d\,p_{ie}\gamma^{de}+(k_2^{\mu}p_i^{\nu}-k_1^{\nu}p_i^{\mu})\right]\epsilon^{(2)\beta}_{\nu}\Gamma_N(\{p_i\}) \tag{7.13}$$

After simplifying the second term and using gamma-traceless condition for gravitino, we get

$$\Gamma^{(3)}_{N+2} = \kappa^2\left[\sum_{i=1}^{N}\mathcal{C}_{12}(p_i)\frac{1}{(p_i\cdot(k_2+k_1))}\right]\Gamma_N(\{p_i\}) \tag{7.14}$$

where we have introduced $\mathcal{C}_{12}$ and $\mathcal{C}_{uv}(p_i)$ is defined as follows

$$\mathcal{C}_{uv}(p_i) = \frac{1}{2}\epsilon^{(u)}_{\mu}\not{p}_i\epsilon^{(v)}_{\nu}\left[\frac{1}{2}\frac{\eta^{\mu\nu}p_i\cdot(k_u-k_v)}{k_u\cdot k_v}+\frac{(k_v^{\mu}p_i^{\nu}-k_u^{\nu}p_i^{\mu})}{k_u\cdot k_v}\right] \tag{7.15}$$

From the property of the gamma matrices it follows that $\mathcal{C}_{uv}(p_i)$ is symmetric in its particle indices

$$\epsilon_\mu^{(u)} \not{p}_i \epsilon_\nu^{(v)} = -\epsilon_\mu^{(v)} \not{p}_i \epsilon_\nu^{(u)} \implies \mathcal{C}_{uv}(p_i) = \mathcal{C}_{vu}(p_i) \tag{7.16}$$

Total contribution Now we add the contributions from (7.8), (7.10), (7.11) and (7.14) to get the full answer for two soft gravitini. The total contribution can be written as

$$\Gamma_{N+2}(\{p_i\}, k_1, k_2) = \Big[\mathcal{S}_1\, \mathcal{S}_2 + \mathcal{M}_{12}\Big] \Gamma_N(\{p_i\}) \tag{7.17}$$

we have already defined $\mathcal{S}_u$ in (7.4). $\mathcal{M}_{uv}$ is defined as follows

$$\mathcal{M}_{uv} = \kappa^2 \sum_{i=1}^{N} \frac{1}{2} \frac{\epsilon_\mu^{(u)} \not{p}_i \epsilon_\nu^{(v)}}{p_i \cdot (k_u + k_v)} \left[\frac{p_i^\mu p_i^\nu}{p_i \cdot k_v} + \frac{1}{2} \frac{\eta^{\mu\nu} p_i \cdot (k_u - k_v)}{k_u \cdot k_v} + \frac{(k_v^\mu p_i^\nu - k_u^\nu p_i^\mu)}{k_u \cdot k_v}\right] \tag{7.18}$$

7.3.1 Some properties of $\mathcal{S}_u$ and $\mathcal{M}_{uv}$

- Two soft operators do not commute

$$[\mathcal{S}_u, \mathcal{S}_v] = -\frac{\kappa^2}{2} \sum_{i=1}^{N} \left[\left(\epsilon_\mu^{(u);\alpha} \not{p}_{i;\alpha\beta} \epsilon_\nu^{(v);\beta}\right) \frac{p_i^\mu}{p_i \cdot k_u} \frac{p_i^\nu}{p_i \cdot k_v}\right] \tag{7.19}$$

- While writing the result for two soft gravitini, we could have chosen the other ordering of soft factors but both results should match i.e.

$$\mathcal{S}_u\, \mathcal{S}_v + \mathcal{M}_{uv} = \mathcal{S}_v\, \mathcal{S}_u + \mathcal{M}_{vu} \tag{7.20}$$

  Above equation can be explicitly verified by noting that:

$$\mathcal{M}_{vu} - \mathcal{M}_{uv} \;=\; \kappa^2 \sum_{i=1}^{N} \frac{\epsilon^{(u)} \cdot p_i}{p_i \cdot k_u} \frac{\epsilon^{(v)} \cdot p_i}{p_i \cdot k_v} \left(-\frac{1}{2} \not{p}_i\right) \tag{7.21}$$

  We already computed $\mathcal{S}_u\, \mathcal{S}_v - \mathcal{S}_v\, \mathcal{S}_u$ in (7.19). Hence (7.20) is satisfied.

- Three soft operators satisfy Jacobi identity.

$$[\mathcal{S}_u, [\mathcal{S}_v, \mathcal{S}_w]] + [\mathcal{S}_v, [\mathcal{S}_w, \mathcal{S}_u]] + [\mathcal{S}_w, [\mathcal{S}_u, \mathcal{S}_v]] = 0 \tag{7.22}$$

In this particular case, each term in the above equation is individually zero.

$$[\mathcal{S}_u, [\mathcal{S}_v, \mathcal{S}_w]] = 0 \tag{7.23}$$

This is not true for soft gluon operator(s). Though (7.22) is true for soft gluon operator, (7.23) does not hold for soft gluon operator. This fact makes the computation of the soft factors for multiple soft gluon even more cumbersome.

- Some more properties of $\mathcal{M}_{uv}$ are listed below

$$\begin{aligned} \mathcal{M}_{uv} &\neq \pm\mathcal{M}_{vu} && (7.24a)\\ \mathcal{M}_{u_1v_1}\,\mathcal{M}_{u_2v_2} &= \mathcal{M}_{u_2v_2}\,\mathcal{M}_{u_1v_1} && (7.24b)\\ \mathcal{S}_w\,\mathcal{M}_{uv} &= \mathcal{M}_{uv}\,\mathcal{S}_w && (7.24c) \end{aligned}$$

7.3.2 Gauge invariance

As a consistency check, we check the gauge invariance of the result obtained in (7.17). The right-hand side should vanish when one puts any of the gravitini as a pure gauge. Here we will put $\epsilon^{(2)}$ as a pure gauge and check if RHS vanishes or not.

$$\epsilon^{(2)\alpha}_{\mu} = k_{2\mu}\,\theta^{\alpha}_{2} \tag{7.25}$$

So for pure gauge, the first term in (7.17) vanishes because Q_β directly hits $\Gamma_N(\{p_i\})$ and gives zero due to supersymmetry ward-identity (7.7). The second piece gives:

$$\begin{aligned} \mathcal{M}_{12}(\epsilon^{\mu\alpha}_1, k^{\mu}_2\theta^{\alpha}_2) &= \kappa^2 \sum_{i=1}^{N} \frac{1}{2}\frac{\epsilon^{(1)}_{\mu}\not{p}_i\theta^{(2)}}{p_i\cdot(k_1+k_2)}\left[\frac{p^{\mu}_i p_i\cdot k_2}{p_i\cdot k_2}\right.\\ &\quad\left.+\frac{1}{2}\frac{k^{\mu}_2 p_i\cdot(k_1-k_2)}{k_1\cdot k_2} + \frac{(k^{\mu}_2(k_2\cdot p_i) - k_2\cdot k_1 p^{\mu}_i)}{k_1\cdot k_2}\right]\\ &= \kappa^2 \sum_{i=1}^{N} \frac{1}{2}\epsilon^{(1)}_{\mu}\not{p}_i\theta^{(2)}\left[\frac{1}{2}\frac{k^{\mu}_2}{k_1\cdot k_2}\right] = 0 \end{aligned} \tag{7.26}$$

where in the last step we have used momentum conservation $\sum_{i=1}^{N} p_i = 0$. One should be able to show the gauge invariance when $\epsilon^{(1)}$ is pure gauge. But in this case, first term in (7.17) does not give ward-identity directly and also $\mathcal{M}_{12}$ term does not vanish. But one can check that the sum is gauge invariant. Alternative we can use (7.20) to express the amplitude in the other ordering of soft factors

$$\Gamma_{N+2}(\{p_i\}, k_1, k_2) = \Big[\mathcal{S}_2\,\mathcal{S}_1 + \mathcal{M}_{21}\Big]\Gamma_N(\{p_i\}) \tag{7.27}$$

In this representation, it is obvious that the RHS vanishes for pure-gauge $\epsilon^{(1)}$. In general,

$$\mathcal{M}_{uv}(\epsilon_u^{\mu\,\alpha}, k_v^\mu \theta_v^\alpha) \;=\; 0 \tag{7.28a}$$

$$\mathcal{M}_{uv}(k_u^\mu\,\theta_u^\alpha, \epsilon_v^{\mu\alpha}) \;\neq\; 0 \tag{7.28b}$$

At this point, we would like to emphasize that the combined contribution from figure 7.3.2 and 7.3.3 is not gauge-invariant. Only after adding the contribution from figure 7.3.4 the answer becomes gauge invariant. A different way to state the same result is that massless spin $3/2$ particles that interact with other fields at low momenta requires an interacting massless spin 2 particle at low energy. This point was first elucidated in [119].

Symmetrized form the amplitude The expression for the soft factor in (7.17) is not manifestly symmetric on the gravitini. That form was useful to prove gauge invariance. Now we use (7.22) and (7.23) to write the answer in a form which is manifestly symmetric on the gravitini

$$\begin{aligned} &\Gamma_{N+2}(\{p_i\}, k_1, k_2) \\ =\;& \frac{1}{2}\Big[\mathcal{S}_1\,\mathcal{S}_2 + \mathcal{S}_2\,\mathcal{S}_1 + \mathcal{M}_{12} + \mathcal{M}_{21}\Big]\Gamma_N(\{p_i\}) \\ =\;& \Bigg[\frac{1}{2}(\mathcal{S}_1\,\mathcal{S}_2 + \mathcal{S}_2\,\mathcal{S}_1) \\ &+\kappa^2 \sum_{i=1}^{N} \frac{1}{p_i\cdot(k_1+k_2)}\left[\mathcal{C}_{12}(p_i) + \frac{1}{4}(p_i\cdot\epsilon^{(1)})\not{p}_i(\epsilon^{(2)}\cdot p_i)\frac{p_i\cdot(k_1-k_2)}{(p_i\cdot k_2)(p_i\cdot k_1)}\right]\Bigg]\,\Gamma_N \end{aligned} \tag{7.29}$$

Apart from the last term, other terms are clearly symmetric under the exchange $1 \longleftrightarrow 2$.

7.3.3 Order of the soft limit

When there are more than one soft particles, there are various ways in which one can take the soft limit. Consider the amplitude with N hard particles with momenta $\{p_i\}$ and two soft particles with momenta k_1 and k_2 ($\Gamma_{N+2}(\{p_i\}, k_1, k_2)$).

The consecutive soft limit is then defined as the limit in which the momenta are taken to be soft one after another. So for two soft particles, this can be done in two different ways

$$\lim_{k_1 \to 0} \lim_{k_2 \to 0} \Gamma_{N+2}(\{p_i\}, k_1, k_2) \qquad , \qquad \lim_{k_2 \to 0} \lim_{k_1 \to 0} \Gamma_{N+2}(\{p_i\}, k_1, k_2) \tag{7.30}$$

Alternatively, one can take simultaneous limit where one takes both k_1 and k_2 to zero keeping k_1/k_2 fixed

$$\lim_{k_1, k_2 \to 0} \Gamma_{N+2}(\{p_i\}, k_1, k_2) \tag{7.31}$$

In this work, we have focused on the simultaneous limit in all cases. If the single soft factors mutually commute (i. e. if the generators of the gauge symmetry commute), then the simultaneous limit is the same as the consecutive limit. For example, in the case of a photon, these two limits give the same answer. However, if the symmetry generators do not commute, then these two limits differ. In our case, the supersymmetry generators do not commute. For example, if we take the consecutive limit by taking k_1 to be soft first, then the Feynman diagram in figure 7.3.3 does not contribute because the soft particle (with momentum k_1) in figure 7.3.3 is emitted from an internal line. Hence the total contribution, in this case, is different from the case when we take simultaneous soft limits.

7.4 Three soft gravitini

In this section, we present the explicit computation for three external soft gravitini. This computation is instructive to understand the soft factor for multiple gravitini, described in the next section. In this section, we denote the amplitude with the soft gravitini by Γ_{N+3}, and similarly, we write Γ_N instead of $\Gamma_N(\{p_i\})$ to denote the amplitudes involving only the hard-particles. For three soft gravitini, the different contributions are as follows:

- We first consider the Feynman diagrams where all three gravitini attach to separate external legs (figure 7.4.1). In this case, the contribution will be just the multiplication of individual soft factors.

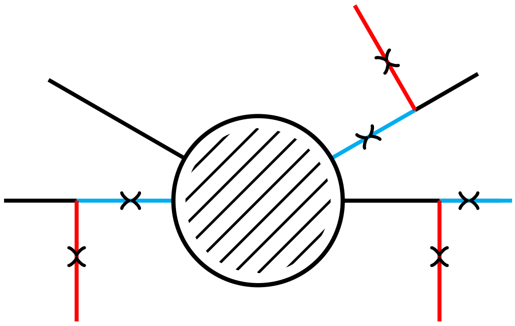


FIGURE 7.4.1: Feynman diagram for three soft gravitini - I

Hence we obtain

$$\Gamma^{(1)}_{N+3} = \kappa^3 \sum_{i=1}^{N} \frac{p_i^\mu \epsilon_\mu^{(1);\alpha_1}}{p_i \cdot k_1} Q_{\alpha_1} \sum_{j=1, j\neq i}^{N} \frac{p_j^\mu \epsilon_\mu^{(2);\alpha_2}}{p_j \cdot k_2} Q_{\alpha_2} \sum_{k=1, k\neq i,j}^{N} \frac{p_k^\mu \epsilon_\mu^{(3)\alpha_3}}{p_k \cdot k_3} Q_{\alpha_3} \Gamma_N(\{p_i\}) \tag{7.32}$$

- Next case is when two gravitini attach to the same leg and the third one on different leg as shown in figure 7.4.2.

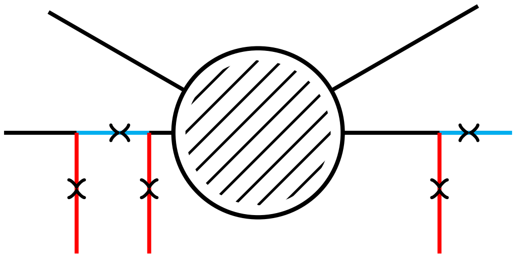


FIGURE 7.4.2: Feynman diagram for three soft gravitini - II

The contribution from such configurations is given by

$$\Gamma^{uv|w;1}_{N+3} = \kappa^3 \sum_{i=1}^{N} \frac{p_i^\mu \epsilon_\mu^{(u)\alpha_u}}{p_i \cdot k_u} Q_{\alpha_u} \frac{p_i^\mu \epsilon_\mu^{(v)\alpha_v}}{p_i \cdot (k_u + k_v)} Q_{\alpha_v} \sum_{j=1, j\neq i}^{N} \frac{p_j^\mu \epsilon_\mu^{(w)\alpha_w}}{p_j \cdot k_w} Q_{\alpha_w} \Gamma_N(\{p_i\}) \tag{7.33}$$

where u, v, w can take values $1, 2, 3$. We can have different contributions depending on the order in which gravitini attach.

- The third possibility consists of the diagrams when all gravitini are being attached to the same external leg (figure 7.4.3).

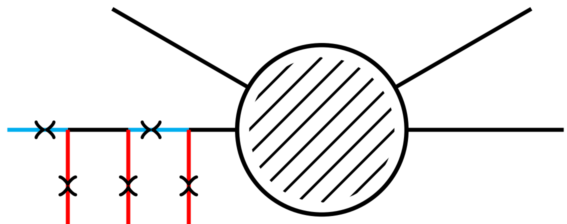


FIGURE 7.4.3: Feynman diagram for three soft gravitini - III

This contribution is given by:

$$\Gamma^{uvw}_{N+3} = \kappa^3 \sum_{i=1}^{N} \frac{\epsilon^{(u)\alpha_u}_{\mu} p_i^{\mu}}{p_i \cdot k_u} \frac{\epsilon^{(v)\alpha_v}_{\nu} p_i^{\nu}}{p_i \cdot (k_u + k_v)} \frac{\epsilon^{(w)\alpha_w}_{\rho} p_i^{\rho}}{p_i \cdot (k_u + k_v + k_w)} Q_{\alpha_u} Q_{\alpha_v} Q_{\alpha_w} \Gamma_N(\{p_i\}) \tag{7.34}$$

We get six such diagrams which can be obtained by interchanging the external soft gravitini.

- Now we consider the diagrams in which any two soft gravitini combine to give a soft graviton, and then this soft graviton attaches to the external leg, the leftover (lonely !) third gravitini directly attaches to the external leg. This can also give rise to two scenarios, i.e., the internal soft graviton and the leftover lonely gravitino can attach to the same hard particles or different hard particles.

  In the case when they attach on separate legs as shown in figure 7.4.4, we just have the multiplication of two factors:

$$\Gamma^{uv|w;2}_{N+3} = \kappa^3 \sum_{i=1}^{N} \left[\frac{\mathcal{C}_{uv}(p_i)}{p_i \cdot (k_u + k_v)} \right] \sum_{j=1, j \neq i}^{N} \frac{\epsilon^{(w)\alpha_w}_{\mu} p_j^{\mu}}{p_j \cdot k_w} Q_{\alpha_w} \Gamma_N(\{p_i\}) \tag{7.35}$$

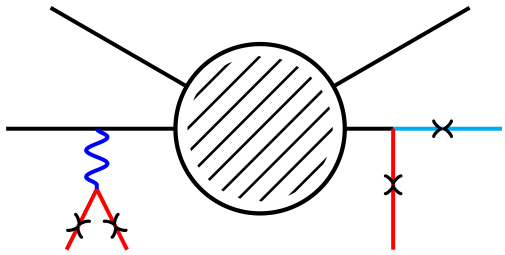


FIGURE 7.4.4: Feynman diagram for three soft gravitini - IV

Since any two gravitini can combine to give the internal soft graviton (and the third one will attach to the separate leg), there are three possibilities.

Now we can have the case when both the internal soft graviton and the left-over soft gravitino attach to same external leg as shown in figure 7.4.5.

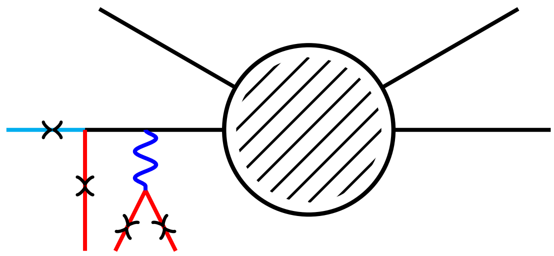


FIGURE 7.4.5: Feynman diagram for three soft gravitini - V

This diagram gives the following contribution:

$$\Gamma^{uv|w;3}_{N+3} = \kappa^3 \sum_{i=1}^{N} \left[\frac{\epsilon^{(w)\alpha_w}_{\mu} p^{\mu}_{j}}{p_j \cdot k_w} Q_{\alpha_w} \frac{\mathcal{C}_{uv}(p_i)}{(p_i \cdot (k_1 + k_2 + k_3))} \right] \Gamma_N(\{p_i\}) \quad (7.36)$$

We will have another diagram in which the graviton attaches to the external leg first and then the gravitino attaches to the external leg i.e.

which gives us:

$$\Gamma^{uv|w;4}_{N+3} = \kappa^3 \sum_{i=1}^{N} \left[\frac{\mathcal{C}_{uv}(p_i)}{p_i \cdot (k_u + k_v)} \frac{\epsilon^{(w)\alpha_w}_{\mu} p^{\mu}_{j}}{p_i \cdot (k_1 + k_2 + k_3)} Q_{\alpha_w} \right] \Gamma_N(\{p_i\}) \quad (7.37)$$

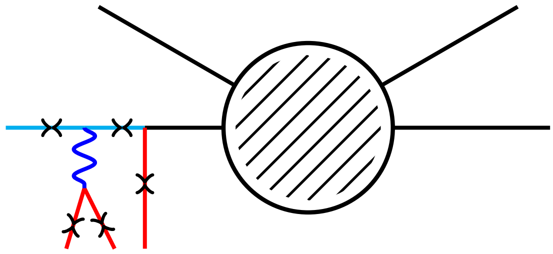


FIGURE 7.4.6: Feynman diagram for three soft gravitini - VI

Adding the contributions from (7.35), (7.36) and (7.37), we get

$$\begin{aligned}\Gamma^{uv|w;4}_{N+3} &= \Gamma^{uv|w;2}_{N+3} + \Gamma^{uv|w;3}_{N+3} + \Gamma^{uv|w;4}_{N+3} \\ &= \kappa^2 \left[\left[\sum_{i=1}^{N} \frac{\mathcal{C}_{uv}(p_i)}{(p_i \cdot (k_u + k_v))}\right] \mathcal{S}_w\right] \Gamma_N(\{p_i\})\end{aligned} \tag{7.38}$$

To write down the contributions from six diagrams shown in figure 7.4.3 we choose a particular ordering i.e. we choose Q_γ (the supercharge appearing with the third gravitini) to be the right-most. The Γ^{123}_{N+3} remains the same

$$\Gamma^{123}_{N+3} = \kappa^3 \sum_{i=1}^{N} \frac{\epsilon^{(1)\alpha}_{\mu} p^{\mu}_i}{p_i \cdot k_1} \frac{\epsilon^{(2)\beta}_{\nu} p^{\nu}_i}{p_i \cdot (k_1 + k_2)} \frac{\epsilon^{(3)\gamma}_{\rho} p^{\rho}_i}{p_i \cdot (k_1 + k_2 + k_3)} Q_\alpha Q_\beta Q_\gamma \, \Gamma_N(\{p_i\}) \tag{7.39}$$

We can bring any other expression into this particular ordering by using (6.2). For example,

$$\begin{aligned}\Gamma^{132}_{N+3} &= \kappa^3 \sum_{i=1}^{N} \frac{\epsilon^{(1)\alpha}_{\mu} p^{\mu}_i}{p_i \cdot k_1} \frac{\epsilon^{(3)\gamma}_{\nu} p^{\nu}_i}{p_i \cdot (k_1 + k_3)} \frac{\epsilon^{(2)\beta}_{\rho} p^{\rho}_i}{p_i \cdot (k_1 + k_2 + k_3)} Q_\alpha Q_\gamma Q_\beta \, \Gamma_N(\{p_i\}) \\ &= \kappa^3 \sum_{i=1}^{N} \frac{\epsilon^{(1)\alpha}_{\mu} p^{\mu}_i}{p_i \cdot k_1} \frac{\epsilon^{(2)\beta}_{\nu} p^{\nu}_i}{p_i \cdot (k_1 + k_3)} \frac{\epsilon^{(3)\gamma}_{\rho} p^{\rho}_i}{p_i \cdot (k_1 + k_2 + k_3)} \left[Q_\alpha Q_\beta Q_\gamma + \frac{1}{2}(\not{p}_i)_{\beta\gamma} Q_\alpha\right] \Gamma_N(\{p_i\})\end{aligned} \tag{7.40}$$

Adding all such contributions, we get:

$$\begin{aligned}\Gamma^{ex}_{N+3} = \kappa^3 \sum_{i=1}^{N} & \frac{\epsilon^{(1)\alpha}_{\mu} p_i^{\mu}}{p_i \cdot k_1} \frac{\epsilon^{(2)\beta}_{\nu} p_i^{\nu}}{p_i \cdot k_2} \frac{\epsilon^{(3)\gamma}_{\rho} p_i^{\rho}}{p_i \cdot k_3} Q_\alpha Q_\beta Q_\gamma \Gamma_N(p_i) \\ + \frac{\kappa^3}{2} \sum_{i=1}^{N} & \left[\frac{\epsilon^{(1)\alpha}_{\mu} p_i^{\mu}}{p_i \cdot k_1} (\not{p}_i)_{\beta\gamma} Q_\alpha \frac{\epsilon^{(2)\beta}_{\nu} p_i^{\nu}}{p_i \cdot (k_2 + k_3)} - \frac{\epsilon^{(1)\alpha}_{\mu} p_i^{\mu}}{p_i \cdot (k_1 + k_3)} \frac{\epsilon^{(2)\beta}_{\nu} p_i^{\nu}}{p_i \cdot k_2} (\not{p}_i)_{\alpha\gamma} Q_\beta \right. \\ & \left. - \frac{\epsilon^{(1)\alpha}_{\mu} p_i^{\mu}}{p_i \cdot (k_3 + k_2)} \frac{\epsilon^{(2)\beta}_{\nu} p_i^{\nu}}{p_i \cdot k_2} (\not{p}_i)_{\alpha\beta} Q_\gamma \right] \frac{\epsilon^{(3)\gamma}_{\rho} p_i^{\rho}}{p_i \cdot k_3} \Gamma_N(\{p_i\}) \qquad (7.41)\end{aligned}$$

Finally, the full result can be written as:

$$\Gamma_{N+3}(\{p_i\}, \{k_u\}) = \Big[\mathcal{S}_1 \, \mathcal{S}_2 \, \mathcal{S}_3 + \mathcal{M}_{12} \, \mathcal{S}_3 + \mathcal{M}_{23} \, \mathcal{S}_1 + \mathcal{M}_{13} \, \mathcal{S}_2 \Big] \Gamma_N(\{p_i\}) \qquad (7.42)$$

where $\mathcal{S}_u$ and $\mathcal{M}_{uv}$ is defined in (7.4) and (7.18) respectively. The above answer matches with the proposed answer (7.45) with $M = 3$.

Rearrangement

We have written the answer for a particular ordering (1-2-3). In this case, we explicitly demonstrate the rearrangement. Let us say that we want to write in the order 1-3-2. We apply the identity (7.20) for $u = 2, v = 3$

$$\begin{aligned}\Gamma_{N+3}(\{p_i\}, \{k_u\}) &= \Big[\mathcal{S}_1 (\mathcal{S}_3 \mathcal{S}_2 - \mathcal{M}_{23} + \mathcal{M}_{32}) + \mathcal{M}_{12} \, \mathcal{S}_3 + \mathcal{M}_{23} \, \mathcal{S}_1 + \mathcal{M}_{13} \, \mathcal{S}_2 \Big] \Gamma_N(\{p_i\}) \\ &= \Big[\mathcal{S}_1 \, \mathcal{S}_3 \, \mathcal{S}_2 + \mathcal{M}_{12} \, \mathcal{S}_3 + \mathcal{M}_{32} \, \mathcal{S}_1 + \mathcal{M}_{13} \, \mathcal{S}_2 \Big] \Gamma_N(\{p_i\}) \qquad (7.43)\end{aligned}$$

7.4.1 Gauge invariance

The gauge invariance of (7.42) is the easiest to show if we put pure gauge polarization for the last one, for example, the third gravitino in (7.42) and the second one in (7.43). Because the answer can always be rearranged to any particular ordering, we can always bring any particular gravitino to be the last entry. Hence it is sufficient to show the gauge invariance for the pure gauge polarization of the last one.

Let us consider (7.42) and pure gauge polarization for the third gravitino. The first and the second term vanish as in equation (7.7) and the third & the fourth term vanish because of (7.28a).

Symmetric form: We can write the answer (7.43) in the form which is manifestly symmetric in all the gravitini

$$\Gamma_{N+3}(\{p_i\},\{k_u\}) = \Bigg[\frac{1}{3!}\mathcal{S}_{(1}\,\mathcal{S}_2\,\mathcal{S}_{3)} + \kappa^2 \sum_{r\neq u\neq s}\sum_{i=1}^{N}\frac{1}{p_i\cdot(k_r+k_u)} \left(\frac{(\epsilon^{(r)}\cdot p_i)\not{p}_i(\epsilon^{(u)}\cdot p_i)p_i\cdot(k_r-k_u)}{4(p_i\cdot k_r)(p_i\cdot k_u)} + \mathcal{C}_{ur}(p_i)\right)\mathcal{S}_s\Bigg]\Gamma_N(\{p_i\}) \tag{7.44}$$

7.5 Arbitrary number of soft gravitini

In this section, we generalize the above results to write down the expression for soft theorem with an arbitrary number of external soft gravitini. In this case, the following type of diagrams can contribute:

- Some of the soft gravitini attach on one external leg and some on another external leg(s), but none of them form pairs to give soft graviton, as shown in figure 7.5.1(a).

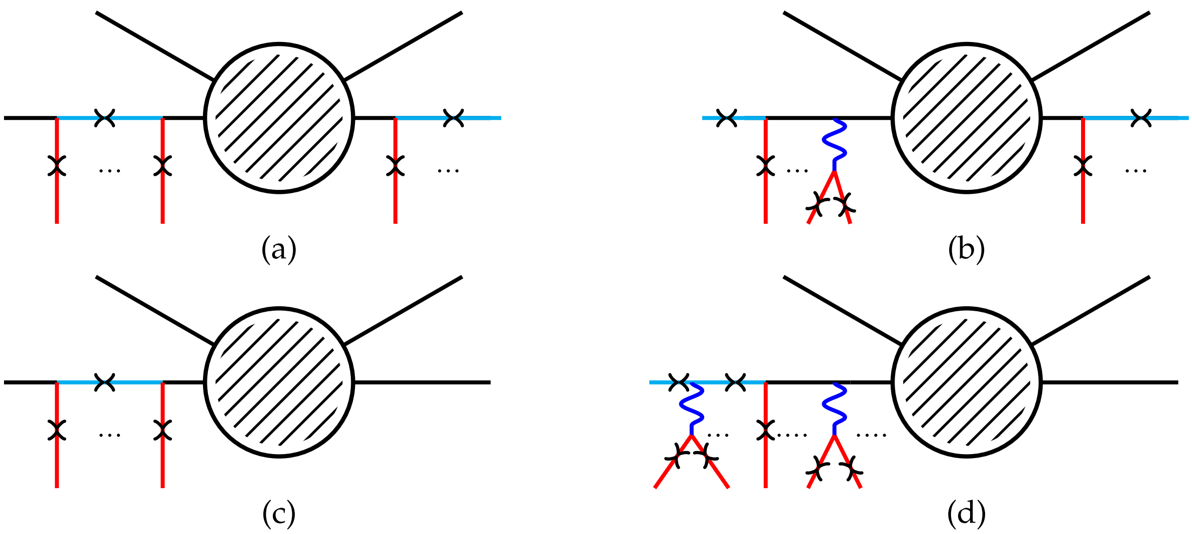


FIGURE 7.5.1: Feynman diagram for multiple soft gravitini

- Next is the case when some of the soft gravitini attach on one external leg and some on another external leg(s) and some form pairs to give soft graviton as shown in figure 7.5.1(b).

- All gravitini attach on the same external leg, but none of them form pairs to give soft graviton as shown in figure 7.5.1(c).

- Some gravitini form pairs and give a soft graviton while some attach directly to external leg as shown in figure 7.5.1(d).

By looking at the pattern followed in two and three soft gravitini case, we propose the following expression for M-soft gravitini.

$$\Gamma_{N+M}(\{p_i\},\{k_{u_i}\}) = \left[\prod_{i=1}^{M}\mathcal{S}_{u_i} + \sum_{A=1}^{\lfloor M/2\rfloor}\prod_{i=1}^{A}\mathcal{M}_{u_i v_i}\prod_{j=1}^{M-2A}\mathcal{S}_{r_j}\right]\Gamma_N(\{p_i\}) \tag{7.45}$$

where $\lfloor M/2\rfloor$ denotes the greatest integer which is less than or equal to $M/2$. Various terms appearing in the above expression are explained below:

1. The first term is very similar to the leading soft factor for multiple soft photons or multiple soft gravitons. The other terms are there because of the fact that soft gravitino factors do not commute. We always write the first factor in a particular order, for example, $\mathcal{S}_{u_1}, ..., \mathcal{S}_{u_M}$ $u_1 < u_2 ... < u_M$ and the particular form of the second term depends on the choice of ordering for the first term. This way of writing in a particular order is also convenient to check gauge invariance.

2. In the second term, A counts the number of pairs of gravitini giving soft gravitons. For each pair, we have a factor of $\mathcal{C}_{uv}$ coming from the gravitino-graviton-gravitino vertex, which combines with a factor coming from the anti-commutation relation (which is used to bring the first term in particular order), to give $\mathcal{M}_{uv}$. The subscripts $\{r_j, u_i, v_i\}$ can take values from $1, ..., M$ and $v_i > u_i$ and r_j's are also ordered with the largest r_j appearing on the right.

The disadvantage of the expression (7.45) is that it depends on the ordering of the external soft gravitini. This expression does not look invariant under change of the ordering, but we show below that it is invariant under rearrangement. We can go to any particular ordering starting from any other ordering. Our strategy is as follows:

1. We first show that any two consecutive entries can be interchanged.
2. By repeating this operation (of interchanging any two consecutive entries) many times, we can obtain any ordering starting from any other ordering[2].

7.5.1 Re-arrangement

In this subesection, we show that any two consecutive terms of equation (7.45) can be interchanged. Consider the i^{th} and $(i+1)^{\text{th}}$ particle. We write the expression (7.45)

$$\begin{aligned}\Gamma_{N+M}(\{p_i\},\{k_u\}) \;=\; & \Big[\mathcal{S}_{u_1}...\mathcal{S}_{u_i}\mathcal{S}_{u_{i+1}}...\mathcal{S}_{u_M} + \mathcal{M}_{u_1u_2}\,\mathcal{S}_{u_3}...\mathcal{S}_{u_i}\mathcal{S}_{u_{i+1}}...\mathcal{S}_{u_M}\\ & \mathcal{M}_{u_2u_3}\,\mathcal{S}_{u_1}...\mathcal{S}_{u_i}\mathcal{S}_{u_{i+1}}...\mathcal{S}_{u_{M-1}}\mathcal{S}_{u_M} + ... + \mathcal{M}_{u_1u_i}\mathcal{S}_{u_2}...\mathcal{S}_{u_{i+1}}...\mathcal{S}_{u_{M-1}}\mathcal{S}_{u_M}\\ & \mathcal{M}_{u_2u_i}\,\mathcal{S}_{u_1}...\mathcal{S}_{u_{i+1}}...\mathcal{S}_{u_{M-1}}\mathcal{S}_{u_M} + ... + \mathcal{M}_{u_iu_{i+1}}\mathcal{S}_{u_1}...\mathcal{S}_{u_{M-1}}\mathcal{S}_{u_M} + \\ & ... + \mathcal{M}_{u_1u_2}...\mathcal{M}_{u_iu_{i+1}}...\mathcal{M}_{u_{M-1}u_M}\Big]\Gamma_N(\{p_i\})\end{aligned} \tag{7.46}$$

Here the i^{th} and $(i+1)^{\text{th}}$ particle can appear only in three different ways

- **Possibility I:** Both the i^{th} and $(i+1)^{\text{th}}$ gravitini appear in the $\mathcal{S}$ factor

$$\Big[A\,\mathcal{S}_{u_i}\mathcal{S}_{u_{i+1}}\,B\Big]\Gamma_N(\{p_i\}) \tag{7.47}$$

  where A and B involves all the other $M-2$ gravitini. The other gravitioni appear as ordered multiplications of $\mathcal{S}_u$ and $\mathcal{M}_{vw}$'s in all possible ways.

- **Possibility II:** Both the i^{th} and $(i+1)^{\text{th}}$ gravitino appear in $\mathcal{M}_{uv}$ together

$$\Big[\widetilde{A}\,\mathcal{M}_{u_iu_{i+1}}\,\widetilde{B}\Big]\Gamma_N(\{p_i\}) \tag{7.48}$$

  Here $\tilde{A}$ and $\tilde{B}$ involves all the other $M-2$ gravitini. Again the other gravitions appear as ordered multiplications of $\mathcal{S}_u$ and $\mathcal{M}_{vw}$'s in all possible ways. This

[2]Theorem 2.1 in this note gives a proof of the above statement.

would imply

$$A = \widetilde{A} \qquad , \qquad B = \widetilde{B} \tag{7.49}$$

So same A and B appear in (7.47) and in (7.48). Adding (7.47) and (7.48) we get

$$\Big[A(\mathcal{S}_{u_i}\mathcal{S}_{u_{i+1}} + \mathcal{M}_{u_i u_{i+1}})B\Big]\Gamma_N(\{p_i\}) \tag{7.50}$$

- **Possibility III:** At least one of them appears as $\mathcal{M}$ and if both of them appear in $\mathcal{M}_{uv}$, they do not appear together. The possibility of both of them to appear together in $\mathcal{M}_{uv}$ has already been taken into account in possibility II.

$$\sum_{j=1, j\neq i, i+1}^{N} \Big[\mathcal{M}_{u_j u_i} C_{i+1}(\epsilon_{u_{i+1}}) + \mathcal{M}_{u_j u_{i+1}} C_i(\epsilon_{u_i})\Big]\Gamma_N(\{p_i\}) \tag{7.51}$$

Here $C_{i+1}(\epsilon_{u_{i+1}})$ is the all possible arrangements of all the gravitini except u_j and u_i and similarly $C_i(\epsilon_{u_i})$ is the all possible arrangements of all the gravitini except u_j and u_{i+1}.

Now if we started with an ordering in which u_{i+1} appeared before u_i then we can repeat the same analysis. Equation (7.51) is same in both cases, but in (7.46) and in (7.47) i and $i+1$ will get interchanged (i.e. $i \longleftrightarrow i+1$). Hence instead of (7.50) we would get

$$\Big[A\left(\mathcal{S}_{u_{i+1}}\mathcal{S}_{u_i} + \mathcal{M}_{u_{i+1} u_i}\right)B\Big]\Gamma_N(\{p_i\}) \tag{7.52}$$

But now we can use (7.20) to see that (7.50) and (7.52) a essentially the same. Hence the final answer is same irrespective of ordering of the soft factors.

7.5.2 Gauge invariance

We have already proved that the expression for multiple soft gravitini can be rearranged to any particular ordering. Using this, we can bring any gravitino to be the rightmost. We will show the gauge invariance of the expression only when the rightmost gravitino is pure gauge.

The rightmost gravitino can appear only in two ways:

1. It can appear in $\mathcal{S}_u$. Since it is the rightmost gravitino, it will directly hit the hard-particle's amplitude, and hence it gives zero by (7.7).

2. It can appear in $\mathcal{M}_{uv}$. Again it will always appear as the 2nd index. However, this vanishes because of (7.28a).

7.6 Soft theorems in the presence of Central Charge

As noticed in chapter 6, in the presence of central charge, we get a $U(1)^N$ gauge symmetry generated by graviphotons. These graviphotons also couple to the soft gravitini and whenever we have more than one soft gravitini, the vertex in (6.26) contributes. In particular, consider the case of two soft gravitini. We already evaluated it in section 7.3. In presence of the central charge(s) we have a new contribution from the diagram 7.6.1.

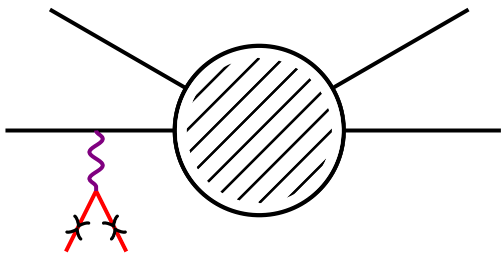


FIGURE 7.6.1: Feynman diagram for double soft gravitini - V

The evaluation of this diagram very similar to the evaluation of the figure 7.3.4 . It is given by:

$$\Gamma^{(4)}_{N+2}(\{p_i\},k_1,k_2) \quad = \quad \left[\epsilon^{(1)\alpha}_{\mu}(\widetilde{\mathcal{V}}^{\mu\nu;\mu_1})_{\alpha\beta}\,\epsilon^{(2)\beta}_{\nu}\right]\left[\frac{i\eta_{\mu_1\mu_2}}{2k_1\cdot k_2}\right]\left[-2\kappa\,e_i\,p_i^{\mu_2}\right]\left[\frac{1}{2p_i\cdot(k_1+k_2)}\right]\Gamma_N(\{p_i\}) \tag{7.53}$$

Here the first square bracket denotes gravitino-gravitino-graviphoton vertex, the second one is the graviphoton propagator and the third one is the matter- soft graviphoton -matter vertex and the last one is the internal propagator. Now we can substitute the explicit expression for $(\widetilde{\mathcal{V}}^{ab;\mu})_{\alpha\beta}$ from equation (6.26) to obtain

$$\Gamma^{(4)}_{N+2} = \frac{(-i\kappa)^2}{2}\sum_{i=1}^{N} e_i\,\epsilon^{(1)}_{\mu\alpha}(\mathcal{Z})^{\alpha\beta}\left[\eta^{\mu\nu}p_i\cdot k_2+\frac{1}{2}\eta^{\mu\nu}(k_1+k_2)_d\,p_{ie}\gamma^{de}-(k_2^\mu p_i^\nu-k_1^\nu p_i^\mu)\right]\epsilon^{(2)}_{\nu\beta}$$
$$\left[\frac{1}{(p_i\cdot(k_2+k_1))(k_1\cdot k_2)}\right]\Gamma_N(\{p_i\}) \tag{7.54}$$

After some simplification we get:

$$\Gamma^{(4)}_{N+2}(\{p_i\},k_1,k_2) = \frac{\kappa^2}{2}\sum_{i=1}^{N} e_i\,\epsilon^{(1)}_{\mu\alpha}(\mathcal{Z})^{\alpha\beta}\left[-\eta^{\mu\nu}p_i\cdot k_2+\frac{1}{2}\eta^{\mu\nu}p_i\cdot(k_1+k_2)+(k_2^\mu p_i^\nu-k_1^\nu p_i^\mu)\right]\epsilon^{(2)}_{\nu\beta}$$
$$\left[\frac{1}{(p_i\cdot(k_2+k_1))(k_1\cdot k_2)}\right]\Gamma_N(\{p_i\}) \tag{7.55}$$

When we add this contribution to the original result, the definition of $\mathcal{C}_{uv}(p_i)$ in (7.15) will be modified as follows:

$$\widetilde{\mathcal{C}}_{uv}(p_i) = \mathcal{C}_{uv}(p_i)+\frac{\kappa^2}{2}e_i\,\epsilon^{(u)}_\mu\,\mathcal{Z}\epsilon^{(v)}_\nu\left[\frac{1}{2}\frac{\eta^{\mu\nu}p_i\cdot(k_u-k_v)}{k_u\cdot k_v}+\frac{(k_v^\mu p_i^\nu-k_u^\nu p_i^\mu)}{k_u\cdot k_v}\right] \tag{7.56}$$

In equation (7.16) we show that $\mathcal{C}_{uv}$ is symmetric in its particle indices. The same property holds for $\widetilde{\mathcal{C}}_{uv}$

$$\widetilde{\mathcal{C}}_{uv}(p_i) = \widetilde{\mathcal{C}}_{vu}(p_i) \tag{7.57}$$

We add the contribution from (7.56) to (7.17) to get the final answer. It is given by

$$\Gamma_{N+2}(\{p_i\},k_1,k_2) = \left[\mathcal{S}_1\,\mathcal{S}_2+\widetilde{\mathcal{M}}_{12}\right]\Gamma_N(\{p_i\}) \tag{7.58}$$

Here we have introduced $\widetilde{\mathcal{M}}_{uv}$. It is defined as

$$\widetilde{\mathcal{M}}_{uv} = \mathcal{M}_{uv}+\frac{\kappa^2}{2}\sum_{i=1}^{N} e_i\,\frac{\epsilon^{(u)}_\mu\,\mathcal{Z}\epsilon^{(v)}_\nu}{p_i\cdot(k_u+k_v)}\left[\frac{p_i^\mu p_i^\nu}{p_i\cdot k_v}+\frac{1}{2}\frac{\eta^{\mu\nu}p_i\cdot(k_u-k_v)}{k_u\cdot k_v}+\frac{(k_v^\mu p_i^\nu-k_u^\nu p_i^\mu)}{k_u\cdot k_v}\right] \tag{7.59}$$

Note that the relations in equations (7.24a), (7.24b), (7.24c) remain the same if we replace $\mathcal{M}_{uv}$ with $\widetilde{\mathcal{M}}_{uv}$. In this particular case we have:

$$\mathcal{S}_u\,\mathcal{S}_v-\mathcal{S}_v\,\mathcal{S}_u = -\widetilde{\mathcal{M}}_{uv}+\widetilde{\mathcal{M}}_{vu} \tag{7.60}$$

7.6.1 Gauge invariance

As explained in the section 7.3.2, it is easier to prove gauge invariance if we put pure gauge polarization for the gravitino adjacent to Γ_N. So we consider pure gauge polarization for the second gravitino

$$\epsilon_2^{\mu\alpha} = k_2^\mu \theta_2^\alpha \tag{7.61}$$

For pure gauge

$$\begin{aligned}
\widetilde{\mathcal{M}}_{uv} &= \frac{\kappa^2}{2}\sum_{i=1}^{N}\frac{1}{p_i\cdot(k_u+k_v)}\left[\epsilon_\mu^{(u)}\not{p}_i\theta^{(v)} + e_i\,\epsilon_\mu^{(u)}\mathcal{Z}\theta^{(v)}\right] \\
&\quad \left[\frac{p_i^\mu p_i\cdot k_v}{p_i\cdot k_v} + \frac{1}{2}\frac{k_v^\mu p_i\cdot(k_u-k_v)}{k_u\cdot k_v} + \frac{(k_v^\mu k_v\cdot p_i - k_v\cdot k_u p_i^\mu)}{k_u\cdot k_v}\right] \\
&= \frac{\kappa^2}{2}\sum_{i=1}^{N}\frac{1}{p_i\cdot(k_u+k_v)}\left[\epsilon_\mu^{(u)}\not{p}_i\theta^{(v)} + e_i\,\epsilon_\mu^{(u)}\mathcal{Z}\theta^{(v)}\right]\left[\frac{1}{2}\frac{k_v^\mu}{k_u\cdot k_v}\right] = 0
\end{aligned} \tag{7.62}$$

where in the last step we have used momentum conservation and (central-)charge conservation

$$\sum_{i=1}^{N} p_i = 0 \quad , \quad \sum_{i=1}^{N} e_i = 0 \tag{7.63}$$

7.7 Presence of soft graviton

Following [98–100] it is easy to include soft graviton into this computation. The vertex for the leading soft graviton ($\zeta_{\mu\nu}P^\mu P^\nu$) commutes with the vertex for soft gravitino and also commutes with the vertex for any other soft graviton. So, in the presence of M_1 soft gravitini and M_2 soft gravitons equation (7.45) is modified as follows

$$\Gamma_{N+M_1+M_2}(\{p_i\},\{k_r\}) = \left[\prod_{j=1}^{M_2}\widetilde{\mathcal{S}}_{u_j}\right]\left[\prod_{i=1}^{M_1}\mathcal{S}_{u_i} + \sum_{A=1}^{\lfloor M_1/2\rfloor}\prod_{i=1}^{A}\mathcal{M}_{u_i v_i}\prod_{j=1}^{M_1-2A}\mathcal{S}_{r_j}\right]\Gamma_N(\{p_i\}) \tag{7.64}$$

$\widetilde{\mathcal{S}}_u$ is the leading soft factor for graviton, given by:

$$\tilde{S}_u = \kappa \sum_{i=1}^{N} \frac{\zeta^{(u)}_{\mu\nu} p_i^\mu p_i^\nu}{p_i \cdot k_u} \tag{7.65}$$

We also notice that the leading order soft gravitini theorem is universal i.e., it is independent of the details of the lagrangian. It has been observed previously that the leading and sub-leading soft factors for multiple gravitons are universal [100]. Notice that these three soft theorems must be inter-related by supersymmetry. One way to argue this is to observe that all these three soft theorems follow from covariantizing the action with respect to the soft field. In supergravity, the structure of the covariant derivative is uniquely fixed by supersymmetry. Hence these theorems must be related to each other by supersymmetry.

Appendix

7.A Notation and convention

Our notation is as follows

Curved space indices	μ, ν, ρ, σ
Tangent space indices	a, b
$SO(d,1)$ spinor indices	α, β
Soft-particle indices	u, v
Hard-particle indices	i, j
Number of Soft-particles	M
Number of Hard-particles	N
Polarization of the graviton	$\zeta_{\mu\nu}$
Polarization of the gravitino	$\epsilon_{\mu\alpha}$

7.A.1 Gamma matrix and spinor convention

We use the following the gamma matrix convention

$$\{\gamma^a, \gamma^b\} = -2\,\eta^{ab} \tag{7.67}$$

and we get

$$[\gamma^a, \gamma^{bc}] = -2\eta^{ab}\gamma^c + 2\eta^{ac}\gamma^b \tag{7.68}$$

The spinors have the following index structure

$$\psi_\alpha \tag{7.69}$$

and the gamma matrix index structure is

$$(\gamma^\mu)_\alpha{}^\beta \tag{7.70}$$

We raise and lower the indices as follows (NW-SE convention)

$$\psi^\alpha = \mathcal{C}^{\alpha\beta}\psi_\beta \qquad , \qquad \psi_\alpha = \psi^\beta \mathcal{C}_{\beta\alpha} \tag{7.71}$$

Here $\mathcal{C}^{\alpha\beta}$ satisfies

$$\mathcal{C}^{\alpha\beta}\mathcal{C}_{\gamma\beta} = \delta^\alpha_\gamma \qquad \mathcal{C}_{\beta\alpha}\mathcal{C}^{\beta\gamma} = \delta^\gamma_\alpha \tag{7.72}$$

$(\gamma^\mu)_{\alpha\beta}$ is given by

$$\gamma^\mu_{\alpha\beta} = (\gamma^\mu)_\alpha{}^\gamma \mathcal{C}_{\gamma\beta} \tag{7.73}$$

7.A.2 Majorana spinor

For two Majorana spinors ψ_1 and ψ_2

$$(\psi_1)^\alpha(\psi_2)_\alpha = (\psi_2)^\alpha(\psi_1)_\alpha \tag{7.74}$$

Chapter 8

Infrared Divergences

In this chapter, we briefly discuss infrared divergences in supergravity theories. In the computation of soft theorems, we used 1PI effective action, but this approach fails when 1PI vertices have IR divergences. The massless particles in loops can potentially give rise to these divergences, hence in the supergravity theories, graviton, gravi-photon, and gravitino[1] can contribute to these divergences. We show below that there are no IR divergences in the 1PI effective action for $D \geq 5$. We will also show that the virtual gravitino does not give rise to IR divergence in any dimension. Hence in $D = 4$, 1PI vertices suffer from IR divergences only due to graviton and graviphoton. However, a more careful analysis shows that the IR divergences do not alter the leading soft gravitino factor.

First, we discuss the case of $D \geq 5$. Then we discuss the case of $D = 4$, which needs more careful analysis. We show that the IR divergences do not alter the leading soft gravitino theorem.

8.1 Infrared divergences in $D \geq 5$

In this section, we wish to check if the approach based on 1PI effective action remains valid in $D \geq 5$ even after taking the soft limit for the external gravitino.

Consider the Feynman diagram in figure 8.1.1(a). If the external momenta are finite, then by naive power-counting, we can see that the amplitude does not have IR divergence for $D \geq 4$. Basically, we have three powers of ℓ in the denominator, one from each of the propagators with momenta $p_i + \ell$, $p_j - \ell$ and ℓ. The last propagator gives one power of ℓ because it is a fermionic particle. In D dimensions, we have D powers of ℓ in the numerator due to the loop integral, and hence the

[1] If there is a massless matter multiplet then in principle it can also contribute to infrared divergence

amplitude goes like ℓ^{D-3} for small loop momentum ℓ. So the diagram is free of IR divergence in $D \geq 4$. Hence any virtual gravitino does not give rise to IR divergences. But when the momenta $k \to 0$, then the propagator carrying momentum $p_i + k + \ell$ gives another power of ℓ and makes the result logarithmically divergent in $D = 4$ but there is no additional divergence in $D \geq 5$. So our results are still valid for $D \geq 5$.

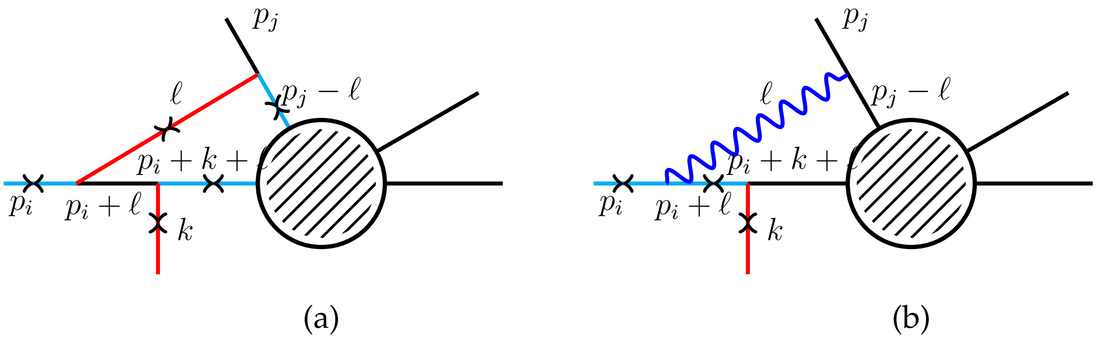


FIGURE 8.1.1: Infrared divergence in supergravity I

Next we consider the Feynman diagram in figure 8.1.1(b). In this case, the internal massless particle is graviton (it can also be photon/graviphoton). We see four powers of ℓ in the denominator from power-counting, one from each of the propagators with momenta $p_i + \ell$, $p_j - \ell$ and two powers of ℓ coming from the graviton propagator. Now in $k \to 0$ limit, the propagator carrying momentum $p_i+k+\ell$ gives another power of ℓ and the diagram is logarithmic divergent in $D = 5$. However, the leading order answer is $O(k^{-1})$, and hence it still holds for $D \geq 5$.

8.2 Infrared divergences in $D = 4$

In $D = 4$, the 1PI effective action suffers from IR divergences due to the presence of graviton and photon in the loop (We already argued that there is no IR divergence in the 1PI vertex due to the presence of gravitino in the loop). So we cannot use it to compute the S-matrix. Nevertheless, one can use the tree level action to derive soft theorems order by order in the perturbation theory. So in four dimensions, we use the tree level action instead of 1PI action in equation (6.3).

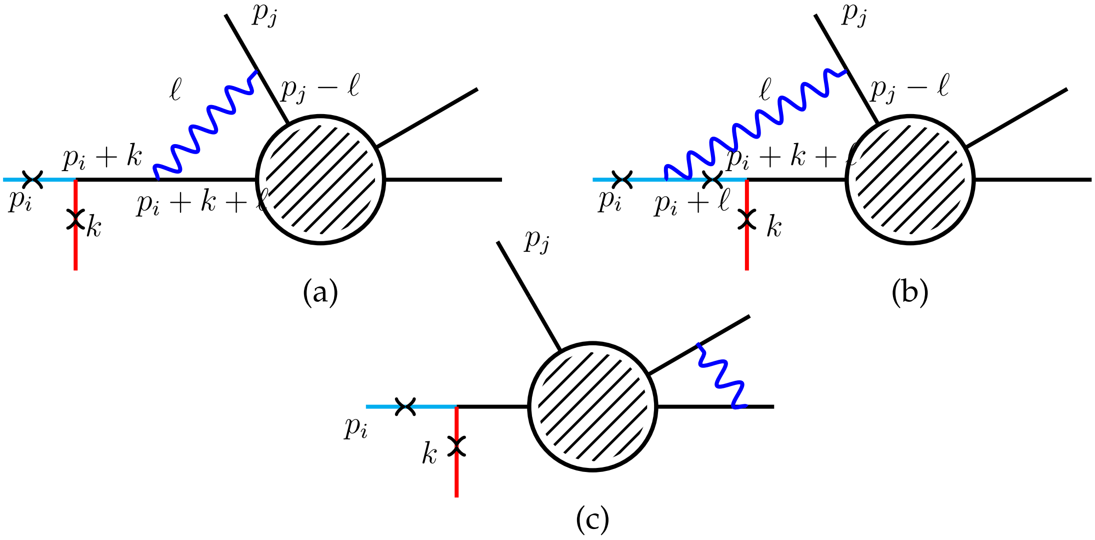


FIGURE 8.1.2: Infrared divergence in supergravity II

Now the question is whether loop corrections can alter the results of leading soft theorems. In the case of the soft graviton (and photon [2]), it has been shown that even though the amplitudes with and without soft particles suffer from IR divergences but at leading order, when one sum over diagrams, the divergence factorizes out and cancels from both sides [89]. In this section, we show that the same result holds for soft gravitino. We will show that the IR divergence due to graviton and graviphoton is the same for amplitudes with and without soft gravitino.

8.2.1 Single real soft gravitino in the presence of virtual graviton

In this subsection, we consider the loop corrections to the soft gravitino factor in $D = 4$ in the presence of a graviton running in the loop. We denote the contributions from these diagrams as $\Gamma^{(i;j,k)}_{N+1}(k, \{p_i\})$; here the superscripts j and k denote the legs to which the virtual graviton attaches and i denote the one to which the soft gravitino attaches. The total contribution is given by

$$\Gamma_{N+1}(k, \{p_i\}) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1;k\neq j}^{N} \Gamma^{(i;j,k)}_{N+1}(k, \{p_i\}) \tag{8.1}$$

[2]with massive matter

First we evaluate $\Gamma^{(i;i,j)}_{N+1}(k, \{p_i\})$. It is given by

$$\Gamma^{(i;i,j)}_{N+1}(k, \{p_i\}) = \left[\widetilde{A}_1(p_i, p_j; k) + \widetilde{A}_2(p_i, p_j; k)\right] \Gamma_N(\{p_i\}) \tag{8.2}$$

$\widetilde{A}_1(p_i, p_j; k)$ and $\widetilde{A}_2(p_i, p_j; k)$ are contributions from diagram (a) and (b) respectively in fig 8.1.2. In small k and small ℓ limit, these contributions are given by

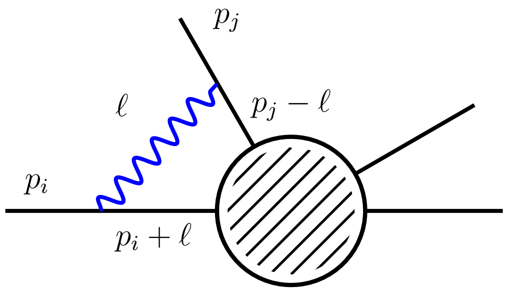


FIGURE 8.2.1: Infrared divergence in Supergravity III

$$\widetilde{A}_1(p_i, p_j; k) = \kappa^3 \beta_{ij}\, (p_i \cdot \epsilon^\alpha Q_\alpha) \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2} \frac{1}{p_i \cdot \ell} \frac{1}{p_j \cdot \ell} \frac{1}{p_i \cdot (k+\ell)} \tag{8.3}$$

$$\widetilde{A}_2(p_i, p_j; k) = \kappa^3 \beta_{ij}\, (p_i \cdot \epsilon^\alpha Q_\alpha) \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2} \frac{1}{p_i \cdot k} \frac{1}{p_j \cdot \ell} \frac{1}{p_i \cdot (k+\ell)} \tag{8.4}$$

where β_{ij} is given by

$$\beta_{ij} = \left(\frac{i}{2}\right) \left[2(p_i \cdot p_j)^2 - p_i^2 p_j^2\right] \tag{8.5}$$

Adding the contributions from (8.3) and (8.4), we get

$$\begin{aligned}\Gamma^{(i;i,j)}_{N+1}(k, \{p_i\}) &= \kappa^2 \beta_{ij} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2} \frac{\kappa\, p_i \cdot \epsilon^\alpha Q_\alpha}{p_i \cdot k} \frac{1}{p_j \cdot \ell} \frac{1}{p_i \cdot \ell} \Gamma_N(\{p_i\} + \mathcal{O}(k^0) \\ &= A(p_i, p_j) \frac{\kappa\, p_i \cdot \epsilon^\alpha Q_\alpha}{p_i \cdot k} \Gamma_N(\{p_i\}) + \mathcal{O}(k^0)\end{aligned} \tag{8.6}$$

where $A(p_i, p_j)$ is the IR divergence that appears in diagram without soft gravitino which is depicted in fig. 8.2.1. It is given by:

$$A(p_i, p_j) = \kappa^2 \beta_{ij} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2} \frac{1}{p_i \cdot \ell} \frac{1}{p_j \cdot \ell} \tag{8.7}$$

The contribution from diagram (c) in fig 8.1.2 is given by [89]

$$\Gamma^{(i;j\neq i,k\neq i)}_{N+1}(k,\{p_i\}) \quad = \quad A(p_j,p_k)\left[\frac{\kappa p_i\cdot\epsilon^\alpha Q_\alpha}{p_i\cdot k}\right]\Gamma_N(\{p_i\}) \tag{8.8}$$

Putting (8.6) and (8.8) in (8.1), we obtain

$$\Gamma_{N+1}(k,\{p_i\}) \quad = \quad \left[\kappa\sum_{i=1}^{N}\frac{p_i\cdot\epsilon^\alpha Q_\alpha}{p_i\cdot k}\right]\left[\sum_{j=1}^{N}\sum_{k=1,\neq j}^{N}A(p_j,p_k)\right]\Gamma_N(\{p_i\}) \tag{8.9}$$

Hence we find that the soft gravitino factor factors out from the IR divergent integral.

Next, we will compute the two loop contributions to IR divergence. The corresponding Feynman diagrams are given in figure 8.2.2.

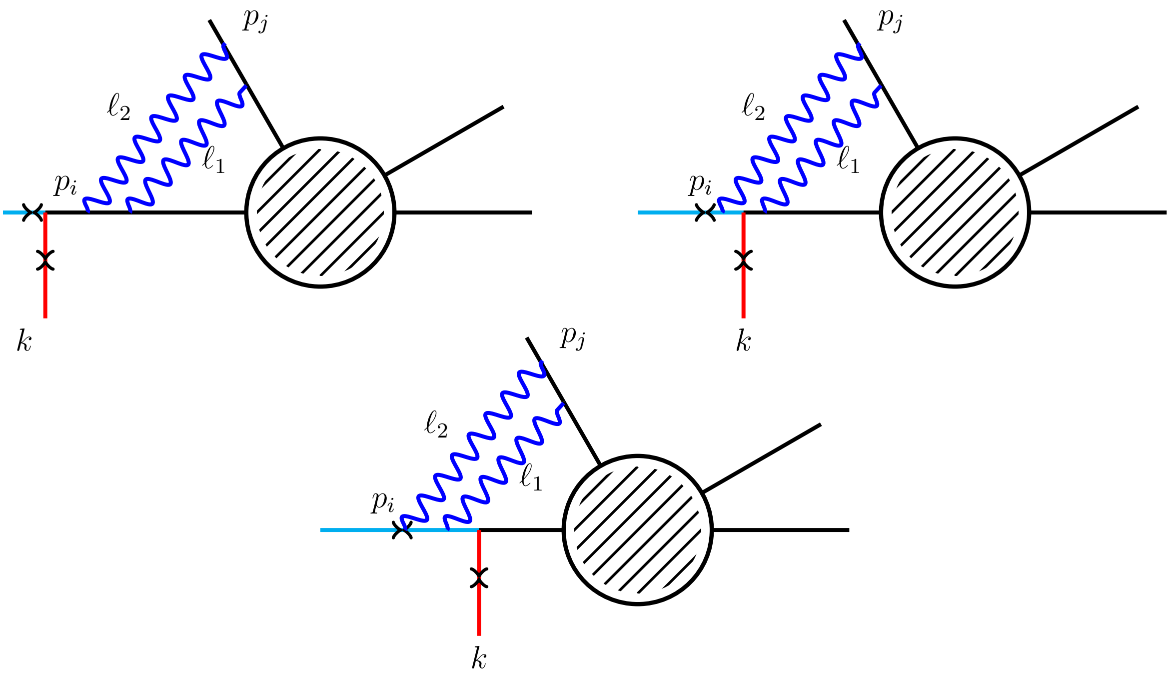


FIGURE 8.2.2: Infrared divergence in Supergravity IV

The contribution from these diagrams are given by

$$\Gamma^{(i;i,j)}_{N+1}(k,\{p_i\}) = \left[\sum_{a=1}^{6}\int d^4\ell_1 d^4\ell_2\, I^{(a)}\right][\kappa\, p_i\cdot\epsilon^\alpha Q_\alpha]\,\Gamma_N(\{p_i\}) \tag{8.10}$$

where $I^{(\alpha)}$ with $\alpha = 1, 2, 3$ are the integrands obtained from the three diagrams shown above. The other three integrands are obtained from the non-planar diagram. The explicit expressions for these integrands are given by

$$\begin{aligned} I^{(1)} &= \kappa^2 \frac{1}{p_i \cdot k} \frac{\beta_{ij}}{\ell_2^2} \frac{1}{p_i \cdot (k + \ell_2)} \frac{\beta_{ij}}{\ell_1^2} \frac{1}{p_i \cdot (k + \ell_1 + \ell_2)} \frac{1}{p_j \cdot \ell_2} \frac{1}{p_j \cdot (\ell_1 + \ell_2)} \\ I^{(2)} &= \kappa^2 \frac{\beta_{ij}}{\ell_2^2} \frac{1}{p_i \cdot l_2} \frac{1}{p_i \cdot (k + \ell_2)} \frac{\beta_{ij}}{p_i \cdot (k + \ell_1 + \ell_2)} \frac{1}{p_j \cdot \ell_2} \frac{1}{\ell_1^2} \frac{1}{p_j \cdot (\ell_1 + \ell_2)} \\ I^{(3)} &= \kappa^2 \frac{\beta_{ij}}{p_i \cdot \ell_2} \frac{\beta_{ij}}{p_i \cdot (\ell_1 + \ell_2)} \frac{1}{p_i \cdot (k + \ell_1 + \ell_2)} \frac{1}{p_j \cdot \ell_2} \frac{1}{\ell_1^2} \frac{1}{\ell_2^2} \frac{1}{p_j \cdot (\ell_1 + \ell_2)} \end{aligned} \tag{8.11}$$

Adding the three contributions above and the contributions from the non-planar diagrams we obtain,

$$I(p_i, p_j; k) = \frac{1}{p_i \cdot k} \int d^4\ell_1 d^4\ell_2 \, I(p_i, p_j; \ell_1, \ell_2) \tag{8.12}$$

where $I(p_i, p_j; \ell_1, \ell_2)$ is given by

$$I(p_i, p_j; \ell_1, \ell_2) = \kappa^2 \frac{1}{p_i \cdot (\ell_1 + \ell_2)} \frac{1}{p_i \cdot \ell_2} \frac{1}{p_j \cdot \ell_2} \frac{\beta_{ij}}{\ell_1^2} \frac{\beta_{ij}}{\ell_2^2} \frac{1}{p_j \cdot \ell_1} \tag{8.13}$$

which is the same two loop integrand we get when there is no soft gravitino. There are other two loop diagrams that we have not depicted here, for example, the diagrams in which two virtual gravitons attach to different legs etc. Adding contribution from those loop diagrams we obtain

$$\Gamma_{N+1}(k, \{p_i\}) = \frac{1}{2} \left[\sum_{j=1}^{N} \sum_{k=1;\neq j}^{N} A(p_j, p_k) \right]^2 \left[\kappa \sum_{i=1}^{N} \frac{p_i \cdot \epsilon^\alpha Q_\alpha}{p_i \cdot k} \right] \Gamma_N(\{p_i\}) \tag{8.14}$$

Note that the soft factor appears just as a multiplicative factor with the infrared divergent piece. One can show that the contribution due to N- virtual soft-gravitons and an external soft gravitino comes out to be

$$\Gamma_{N+1}(k, \{p_i\}) = \left[\kappa \sum_{i=1}^{N} \frac{p_i \cdot \epsilon^\alpha Q_\alpha}{p_i \cdot k} \right] \left[\sum_{N=0}^{\infty} \frac{1}{N!} \left[\sum_{j=1}^{N} \sum_{k=1;k\neq j}^{N} A(p_j, p_k) \right]^N \right] \Gamma_N(\{p_i\}) \tag{8.15}$$

This implies the soft theorem is not affected by the IR divergence.

8.2.2 Single real soft gravitino in presence of virtual graviphoton

In the presence of graviphoton, there are new IR divergent diagrams due to graviphoton running in the loops. These diagrams can be obtained by replacing graviton with graviphoton in the fig 8.1.2 and in the fig 8.2.2. The computation is very similar to the one presented in subsection 8.1. The infrared divergence due to graviphoton is given by

$$\left[\sum_{N=0}^{\infty}\frac{1}{N!}\left[B(p_i,p_j)\right]^N\right] \tag{8.16}$$

where $B(p_i,p_j)$ is given by

$$B(p_i,p_j)=\kappa^2 e_i e_j\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2}\frac{1}{p_i\cdot\ell}\frac{1}{p_j\cdot\ell}\frac{1}{p_i\cdot(k+\ell)} \tag{8.17}$$

In presence of graviphoton, equation (8.18) will be replaced by the following equation

$$\Gamma_{N+1}(k,\{p_i\}) = \left[\kappa\sum_{i=1}^{N}\frac{p_i\cdot\epsilon^\alpha Q_\alpha}{p_i\cdot k}\right]\left[\sum_{N=0}^{\infty}\frac{1}{N!}\left[\sum_{j=1}^{N}\sum_{k=1;k\neq j}^{N}\left(A(p_j,p_k)+B(p_j,p_k)\right)\right]^N\right]\Gamma_N(\{p_i\}) \tag{8.18}$$

Again we can see that the soft factor is not affected by the IR divergence.

8.2.3 Massless matter

Now we concentrate on the particular case when some (or all) of the matter fields are massless [3]. Weinberg in [89] showed that in the presence of massless matter, the IR divergence due to virtual graviton cancels. However, there are irremovable IR divergences in QED with the massless charged matter.

In this case, the IR divergence comes from the presence of virtual graviton and virtual graviphoton. The ones due to virtual graviton cancel due to Weinberg's argument. However, in the presence of graviphoton, there might be some non-removable IR divergences. Graviphoton gauges the symmetries generated by the central charge. The central charge puts a lower bound on the mass of the particle

[3] We are thankful to the unknown referee for pointing out this issue

(the BPS bound). The graviphoton only couples to matter with non-zero central charge and hence with non-zero mass. So there is no irremovable IR divergence in this case.

In the presence of both the vector multiplet(s) and the massless matter multiplet(s) charged under the vector multiplet(s), there are irremovable IR divergences in $D = 4$ due to the photon/gluon (of the vector multiplet) running in the loop. Since there is no vector multiplet in $\mathcal{N} = 8$ supergravity, our analysis implies that there are no irremovable IR divergences in $\mathcal{N} = 8$ supergravity (and in type II string theory).

Conclusions and Outlook

In this part of the thesis, we have computed the leading order soft gravitini theorem for an arbitrary theory of supergravity. We also observe that the leading order soft factor for multiple gravitini is universal, i.e., it is independent of the specific supergravity lagrangian.

Our main result is equation (7.45) which is elaborated below:

$$\Gamma_{M+N}(\{p_i\},\{k_u\}) = \left[\prod_{i=1}^{M}\mathcal{S}_{u_i} + \sum_{A=1}^{\lfloor M/2\rfloor}\prod_{i=1}^{A}\mathcal{M}_{u_iv_i}\prod_{j=1}^{M-2A}\mathcal{S}_{r_j}\right]\Gamma_N(\{p_i\}) + \mathcal{O}(1/k^{M-1}) \tag{8.19}$$

where $\Gamma_{M+N}(\{p_i\},\{k_u\})$ is the amplitude for M soft gravitini and N any other hard particles, p_i are the momenta of hard particles and k_u are momenta of soft particles. Various terms appearing in the RHS of are explained below:

1. $\mathcal{S}_u$ is the soft factor for single soft gravitino. It is given by

$$\mathcal{S}_u \;=\; \kappa\sum_{i=1}^{N}\left(\frac{\epsilon_\mu^{(u)\,\alpha}\,p_i^\mu}{p_i\cdot k_u}Q_\alpha\right) \tag{8.20}$$

Here κ is the gravitational coupling constant. $\epsilon_\mu^{(u)\,\alpha}$ is the polarization of the gravitino. The gravitino polarization (in the harmonic gauge) satisfies the transversality condition and gamma traceless condition:

$$(k_u)^\mu\epsilon_\mu^{(u)\,\alpha} = 0 \qquad , \qquad \gamma^\mu_{\alpha\beta}\epsilon_\mu^{(u)\,\beta} = 0 \tag{8.21}$$

Q_α are the supersymmetry charges/generators. Since $\mathcal{S}_u$ is a product of two grassmann odd quantities, it is grassmann even. Two single soft factors do not commute with each other:

$$\mathcal{S}_u\,\mathcal{S}_v \neq \mathcal{S}_v\,\mathcal{S}_u \tag{8.22}$$

2. Whenever there is more than one gravitino, they can combine pairwise to give a soft graviton which in-turn couples to the hard particles and $\mathcal{M}_{uv}$ encodes these type of contributions. The gravitino-graviton-gravitino vertex is

crucial to make the two soft gravitini amplitude gauge-invariant. A different way to state the same result is that, a massless spin $3/2$ particle that interacts with other fields at low momenta requires an interacting massless spin 2 particle at low energy. The explicit expression for $\mathcal{M}_{uv}$ is given by

$$\mathcal{M}_{uv} = \kappa^2 \sum_{i=1}^{N} \frac{1}{2} \frac{\epsilon_\mu^{(u)} \not{p}_i \epsilon_\nu^{(v)}}{p_i \cdot (k_u + k_v)} \left[\frac{p_i^\mu p_i^\nu}{p_i \cdot k_v} + \frac{1}{2} \frac{\eta^{\mu\nu} p_i \cdot (k_u - k_v)}{k_u \cdot k_v} + \frac{(k_v^\mu p_i^\nu - k_u^\nu p_i^\mu)}{k_u \cdot k_v} \right] \tag{8.23}$$

3. Since the single soft factors for gravitino do not commute, the final expression for arbitrary soft gravitino depends on the choice of ordering. In §7.5.1, we demonstrate that any ordering can be obtained from any other ordering. However, our expression is not manifestly symmetric on the various soft gravitini.

4. The first term is the product of single-soft gravitino factors. The single-soft factors appear in a particular order, and the explicit form of the second piece changes depending on the ordering of soft factors because two soft factors do not commute.

5. In the second term, $\lfloor M/2 \rfloor$ denotes the greatest integer, which is less than or equal to $M/2$ and A counts the number of pairs of gravitini giving a soft graviton. The subscripts $\{r_j, u_i, v_i\}$ take values $1, ..., M$ and $v_i > u_i$ and r_j's are also ordered with the largest r_j appearing on the right.

6. The supersymmetry algebra may have a non-vanishing central charge. In this case, the gravitino super-multiplet contains a graviphoton. In presence of the central charge, there are additional contributions to $\mathcal{M}_{uv}$ due to graviphoton couplings. In this case, the expression of $\mathcal{M}_{uv}$ is modified as follows

$$\widetilde{\mathcal{M}}_{uv} = \mathcal{M}_{uv} + \frac{\kappa^2}{2} \sum_{i=1}^{N} e_i \frac{\epsilon_\mu^{(u)} \mathcal{Z} \epsilon_\nu^{(v)}}{p_i \cdot (k_u + k_v)} \left[\frac{p_i^\mu p_i^\nu}{p_i \cdot k_v} + \frac{1}{2} \frac{\eta^{\mu\nu} p_i \cdot (k_u - k_v)}{k_u \cdot k_v} + \frac{(k_v^\mu p_i^\nu - k_u^\nu p_i^\mu)}{k_u \cdot k_v} \right] \tag{8.24}$$

e_i is the charge of the i^{th} external state under the symmetry generated by the graviphoton. $\mathcal{Z}$ is an element of the Clifford algebra such that $\mathcal{Z}_\alpha{}^\beta$ commutes with all other element of the Clifford algebra.

In presence of a soft graviton, we have to multiply the above expression by soft factors of the graviton. For M_1 soft gravitini and M_2 soft gravitons equation (8.2.3) takes the following form:

$$\Gamma_{N+M_1+M_2}(\{p_i\},\{k_r\}) = \left[\prod_{j=1}^{M_2}\widetilde{\mathcal{S}}_{u_j}\right]\left[\prod_{i=1}^{M_1}\mathcal{S}_{u_i} + \sum_{A=1}^{\lfloor M_1/2\rfloor}\prod_{i=1}^{A}\mathcal{M}_{u_iv_i}\prod_{j=1}^{M_1-2A}\mathcal{S}_{r_j}\right]\Gamma_N(\{p_i\}) \tag{8.25}$$

where $\widetilde{\mathcal{S}}_u$ is the leading soft factor for graviton. It is given by

$$\widetilde{\mathcal{S}}_u = \kappa\sum_{i=1}^{N}\left(\frac{\zeta^{(u)}_{\mu\nu}p_i^\mu p_i^\nu}{p_i\cdot k_u}\right) \tag{8.26}$$

here $\zeta_{\mu\nu}$ is the polarization of soft graviton.

We also observe that the infrared divergences do not affect the leading order results. Hence our results are valid in $D \geq 4$.

There are certain directions in which this work can be extended. Some of them are listed below:

- One is to understand the structure of the sub-leading soft gravitino theorem and its relation to that of sub-leading and sub-subleading soft graviton theorem.
- The approach used in above chapters can also be applied to compute the soft photino theorem in the presence of gravitino, photon, and graviton.
- Another interesting question is to derive the result for multiple soft gravitini from the analysis of asymptotic symmetries and from the CFT living on $\mathcal{I}^\pm$ following [124–127].

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