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Dubrovin Frobenius manifolds, Hurwitz spaces, and Extended Jacobi groups

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CHAPTER 1

Introduction

Dubrovin Frobenius manifold is a geometric interpretation of a remarkable system of differential equations, called WDVV equations [12]. Since the beginning of the nineties, there has been a continuous exchange of ideas from fields that are not trivially related to each other, such as: Topological quantum field theory, non-linear waves, singularity theory, random matrices theory, integrable systems, and Painleve equations. Dubrovin Frobenius manifolds theory is a bridge between them.

1.1. Topological quantum field theory

The connections made by Dubrovin Frobenius manifolds theory work because all the mentioned theories are related with some WDVV equation. In [12], Dubrovin showed that many constructions of Topological field theories (TFT) can be deduced from the geometry of Dubrovin-Frobenius manifolds. For instance, one of the main objects to be computed in TFT are correlation functions, which are mean values of physical quantities. Since in TFT it is possible to have infinite many correlation functions, an efficient way to compute all of them is encoding all the correlators in a single function, called partition function. In [21], Konsevitch proved that a partition function of a specific Quantum gravity theory can be obtained from the solution of KdV hierarchy, which is an example of integrable hierarchy, i.e. an infinitely list of integrable partial differential equations. This discovery opened a new field of research in mathematical physics, because for this case, it was found an effective way to compute exactly all the correlation functions due to its integrable system nature. In [16], Dubrovin and Zhang constructed a method to derive an integrable hierarchy from the data of Dubrovin Frobenius manifold, furthermore, in many important examples, they were able to relate these integrable hierarchies with partition functions of some TFT.

1.2. Orbit space of reflection groups and its extensions

In [12], Dubrovin points out that, WDVV solutions with certain good analytic properties are related with partition functions of TFT. Afterwards, Dubrovin conjectured that WDVV solutions with certain good analytic properties are in one to one correspondence with discrete groups. This conjecture is supported in ideas which come from singularity theory, because in this setting there exist an integrable systems/ discrete group correspondence. Furthermore, in minimal models such as Gepner chiral rings there exist a correspondence between physical models and discrete groups. In [20], Hertling proved that a particular class of Dubrovin-Frobenius manifold, called polynomial Dubrovin-Frobenius manifold is isomorphic to orbit space of a finite Coxeter group, which is a space such that its geometric structure is invariant under the finite Coxeter group. In [8], [9],[11], [12], [14], [15], [35], there are many examples of WDVV solutions that are associated with orbit spaces of natural extensions of finite Coxeter groups such as extended affine Weyl groups, and Jacobi groups. Therefore, the construction of Dubrovin Frobenius manifolds on orbit spaces of reflection groups and their extensions is a prospective project of the classification of WDVV solutions. In addition, WDDV solutions arising from orbit spaces may also have some applications in TFT or some combinatorial problem, because previously this relation was demonstrated in some examples such as the orbit space of the finite Coxeter group A_1, and the extended affine Weyl group A_1 [13], [16].

1.3. Hurwtiz space/ Orbit space correspondence

There are several other non-trivial connections that Dubrovin Frobenius manifolds theory can make. For example, Hurwitz spaces is the one of the main source of examples of Dubrovin Frobenius manifolds. Hurwitz spaces $H_{g,n_0,n_1,..,n_m}$ are moduli space of coverings over $\mathbb{CP}^1$ with a fixed ramification profile. More specifically, $H_{g,n_0,n_1,..,n_m}$ is moduli space of pairs

$$\{C_g, \lambda : C_g \mapsto \mathbb{CP}^1\}$$

where C_g is a compact Riemann surface of genus g, λ is meromorphic function with poles in

$$\lambda^{-1}(\infty) = \{\infty_0, \infty_1, .., \infty_m\}.$$

Moreover, λ has degree n_i+1 near ∞_i. Hurwitz space with a choice of specific Abelian differential, called quasi-momentum or primary differential, gives rise to a Dubrovin Frobenius manifold, see chapter 3 for details. In some examples, Dubrovin Frobenius structures of Hurwitz spaces are isomorphic to Dubrovin Frobenius manifolds associated with orbit spaces of suitable groups. For instance, the orbit space of the finite Coxeter group A_n is isomorphic to the Hurwitz space $H_{0,n}$, furthermore, orbit space of the extended affine Weyl group $\tilde{A}_n$ and of the Jacobi group $\mathscr{J}(A_n)$ are isomorphic to the Hurwitz spaces $H_{0,n-1,0}$ and $H_{1,n}$ respectively. Motivated by these examples, we construct the following diagram

$$\begin{tikzcd}
H_{0,n} \cong \text{Orbit space of } A_n \arrow[r, "1"] \arrow[d, "2"'] & H_{0,n-1,0} \cong \text{Orbit space of } \tilde{A}_n \arrow[d, "4"] \\
H_{1,n} \cong \text{Orbit space of } \mathscr{J}(A_n) \arrow[r, "3"] & H_{1,n-1,0} \cong ?
\end{tikzcd}$$

From the Hurwitz space side, the vertical lines 2 and 4 mean that we increase the genus by 1, and the horizontal line means that we split one pole of order $n+1$ in a simple pole and a pole of order n. From the orbit space side, the vertical line 2 means that we are doing an extension from the finite Coxeter group A_n to the Jacobi group $\mathscr{J}(A_n)$, the horizontal line 1 means that we are extending the Orbit space of A_n to the extended affine Weyl group $\tilde{A}_n$. Therefore, one might ask if the line 3 and 4 would imply an orbit space interpretation of the Hurwitz space $H_{1,n-1,0}$. The main goal of this thesis is to define a new class of groups such that its orbit space carries Dubrovin-Frobenius structure of $H_{1,n-1,0}$. The new group is called extended affine Jacobi group A_n, and denoted by $\mathscr{J}(\tilde{A}_n)$. This group is an extension of the Jacobi group $\mathscr{J}(A_n)$, and of the extended affine Weyl group $\tilde{A}_n$.

1.4. Thesis results

The main goal of this thesis is to derive the Dubrovin Frobenius structure of the Hurwitz space $H_{1,n-1,0}$ from the data of the group $\mathscr{J}(\tilde{A}_n)$. First of all, we define the group $\mathscr{J}(\tilde{A}_n)$. Recall that the group A_n acts on $\mathbb{C}^n$ by permutations, then the group $\mathscr{J}(\tilde{A}_n)$ is an extension of the group A_n in the following sense:

Proposition 1.4.1. The group $\mathscr{J}(\tilde{A}_n) \ni (w,t,\gamma)$ acts on $\Omega := \mathbb{C} \oplus \mathbb{C}^{n+2} \oplus \mathbb{H} \ni (u,v,\tau)$ as follows

$$\begin{aligned} &w(u,v,\tau) = (u, w \bullet v, \tau) \\ &t(u,v,\tau) = (u- <\lambda, v>_{\tilde{A}_n} - \frac{1}{2} <\lambda,\lambda>_{\tilde{A}_n} \tau, v + \lambda\tau + \mu, \tau) \\ &\gamma(u,v,\tau) = (u + \frac{c<v,v>_{\tilde{A}_n}}{2(c\tau+d)}, \frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}) \end{aligned} \tag{1.1}$$

where $w \in A_n$ acts by permutations in the first $n+1$ variables of $\mathbb{C}^{n+2} \ni v = (v_0, v_1, .., v_n, v_{n+1})$,

$$t = (\lambda, \mu) \in \mathbb{Z}^{2n+4},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

$$<v,v>_{\tilde{A}_n} = \sum_{i=0}^{n} v_i^2 \Big|_{\sum_{i=0}^{n} v_i = 0} - n(n+1)v_{n+1}^2.$$

See section 8.1 for details.

In order to define any geometric structure in the orbit space of $\mathscr{J}(\tilde{A}_n)$, first it is necessary to define a notion of invariant $\mathscr{J}(\tilde{A}_n)$ sections of the orbit space of $\mathscr{J}(\tilde{A}_n)$. For this purpose, we generalise the ring of invariant functions used in [8], [9] for the group $\mathscr{J}(A_n)$, which are called Jacobi forms. This notion was first defined in [18] by Eichler and Zagier for the group $\mathscr{J}(A_1)$, and further it was generalised for the group $\mathscr{J}(A_n)$ in [34] by Wirthmuller. Furthermore, an explicit base of generators were derived in [8], [9] by Bertola. The Jacobi forms used in this thesis are defined by

DEFINITION 1.4.1. The weak $\mathscr{J}(\tilde{A}_n)$ -invariant Jacobi forms of weight $k \in \mathbb{Z}$, order $l \in \mathbb{N}$, and index $m \in \mathbb{N}$ are functions φ on $\Omega = \mathbb{C} \oplus \mathbb{C}^{n+2} \oplus \mathbb{H} \ni (u, v_0, v_1, ..., v_{n+1}, \tau) = (u, v, \tau)$ which satisfy

$$
\begin{aligned}
&\varphi(w(u,v,\tau)) = \varphi(u,v,\tau), \quad A_n \text{ invariant condition}\\
&\varphi(t(u,v,\tau)) = \varphi(u,v,\tau)\\
&\varphi(\gamma(u,v,\tau)) = (c\tau+d)^{-k}\varphi(u,v,\tau)\\
&E\varphi(u,v,\tau) := -\frac{1}{2\pi i}\frac{\partial}{\partial u}\varphi(u,v,\tau) = m\varphi(u,v,\tau), \quad \text{Euler vector field}
\end{aligned}
\tag{1.2}
$$

Moreover, the weak $\tilde{A}_n$ -invariant Jacobi forms are meromorphic in the variable v_{n+1} with poles on a fixed divisor, in contrast with the Jacobi forms of the group $\mathscr{J}(A_n)$,which are holomorphic in each variable, see details in the definition 8.2.1. The ring of weak $\tilde{A}_n$ -invariant Jacobi forms gives the notion of Euler vector field. Indeed, the vector field E defined in the last equation of (1.2) measures the degree of a Jacobi form, which coincides with the index. The differential geometry of the orbit space of the group $\mathscr{J}(\tilde{A}_n)$ should be understood as the space such that its sections are written in terms of Jacobi forms. Then, in order for this statement to make sense, we prove a Chevalley type theorem, which is

THEOREM 1.4.2. The trigraded algebra of weak $\mathscr{J}(\tilde{A}_n)$ -invariant Jacobi forms $J_{\bullet,\bullet,\bullet}^{\mathscr{J}(\tilde{A}_n)} = \bigoplus_{k,l,m} J_{k,l,m}^{\tilde{A}_n}$ is freely generated by $n+1$ fundamental Jacobi forms $(\varphi_0, \varphi_1, , \varphi_2, .., , \varphi_n)$ over the graded ring $E_{\bullet,\bullet}$

$$
J_{\bullet,\bullet,\bullet}^{\mathscr{J}(\tilde{A}_n)} = E_{\bullet,\bullet}\left[\varphi_0, \varphi_1, , \varphi_2, .., , \varphi_n\right], \tag{1.3}
$$

where

$$
E_{\bullet,\bullet} = J_{\bullet,\bullet,0} \quad \text{is the ring of coefficients.}
$$

More specifically, the ring of function $E_{\bullet,\bullet}$ is the space of coefficients $f(v_{n+1},\tau)$ such that for fixed τ, the functions $v_{n+1} \mapsto f(v_{n+1},\tau)$ is an elliptic function.

Moreover, $(\varphi_0, \varphi_1, , \varphi_2, .., \varphi_n)$ are given by

COROLLARY 1.4.2.1. The functions $(\varphi_0, \varphi_1, .., \varphi_n)$ obtained by the formula

$$
\begin{aligned}
\lambda^{\tilde{A}_n} &= e^{2\pi i u}\frac{\prod_{i=0}^{n}\theta_1(z - v_i + v_{n+1},\tau)}{\theta_1^n(z,\tau)\theta_1(z+(n+1)v_{n+1})}\\
&= \varphi_n \wp^{n-2}(z,\tau) + \varphi_{n-1}\wp^{n-3}(z,\tau) + ... + \varphi_2\wp(z,\tau)\\
&+ \varphi_1\left[\zeta(z,\tau) - \zeta(z+(n+1)v_{n+1},\tau) + \zeta((n+1)v_{n+1})\right] + \varphi_0
\end{aligned}
\tag{1.4}
$$

are Jacobi forms of weight $0, -1, -2, .., -n$, respectively, of index 1, and of order 0. Here θ_1 is the Jacobi theta 1 function, ζ is the Weiestrass zeta function, and $\wp$ is the Weiestrass p function.

The corollary 1.4.2.1 realises the functions $(\varphi_0, \varphi_1, , \varphi_2, .., , \varphi_n, v_{n+1}, \tau)$ as coordinates of the orbit space of $\mathscr{J}(\tilde{A}_n)$. The unit vector field, with respect the Frobenius product defined in theorem 1.4.7 see below, is chosen to be

$$e = \frac{\partial}{\partial \varphi_0}, \tag{1.5}$$

because φ_0 is the basic generator with maximum weight, see the sections 8.2, 8.3 for details. The last ingredient we need to construct is the flat pencil metric associated with the orbit space of $\mathscr{J}(\tilde{A}_n)$, which is two compatible flat metrics g^* and η^* such that

$$g^* + \lambda \eta^*$$

is also flat, and the linear combination of its Christoffel symbols

$$\Gamma^{ij}_{k\ g^*} + \lambda \Gamma^{ij}_{k\ \eta^*}$$

is the Christoffel symbol of the flat pencil of metrics (see section 4.4 for the details). The natural candidate to be one of the metrics of the pencil is the invariant metric of the group $\mathscr{J}(\tilde{A}_n)$. This metric is given by

$$g^* = \sum_{i,j} A^{-1}_{ij} \frac{\partial}{\partial v_i} \otimes \frac{\partial}{\partial v_j} - n(n+1) \frac{\partial}{\partial v_{n+1}} \otimes \frac{\partial}{\partial v_{n+1}} + \frac{\partial}{\partial \tau} \otimes \frac{\partial}{\partial u} + \frac{\partial}{\partial u} \otimes \frac{\partial}{\partial \tau}, \tag{1.6}$$

where

$$A_{ij} = \begin{pmatrix} 2 & 1 & 1 & ... & 1 \\ 1 & 2 & 1 & ... & 1 \\ 1 & 1 & 2 & ... & 1 \\ 1 & 1 & 1 & ... & 1 \\ 1 & 1 & 1 & ... & 2 \end{pmatrix}$$

This metric is called intersection form. The second metric is given by

$$\eta^* := Lie_e g^*,$$

and it is denoted by Saito metric due to K.Saito, who was the first to define this metric for the case of finite Coxeter group [28]. One of the main technical problems of the thesis is to prove that the Saito metric η^* is flat. For this purpose, we construct a generating function for the coefficients of the metric η^* in the coordinates $(\varphi_0, \varphi_1, , \varphi_2, .., , \varphi_n, v_{n+1}, \tau)$. We prove the following.

Corollary 1.4.2.2. Let $\eta^*(d\varphi_i, d\varphi_j)$ be given by

$$\eta^*(d\varphi_i, d\varphi_j) := \frac{\partial g^*(d\varphi_i, d\varphi_j)}{\partial \varphi_0}. \tag{1.7}$$

The coefficient $\eta^*(d\varphi_i, d\varphi_j)$ is recovered by the generating formula

$$
\begin{aligned}
&\sum_{k,j=0}^{n+1} \frac{(-1)^{k+j}}{(k-1)!(j-1)!}\tilde{\eta}^*(d\varphi_i, d\varphi_j)\wp(v)^{(k-2)}\wp(v')^{(j-2)} = \\
&= 2\pi i(D_\tau\lambda(v) + D_\tau\lambda(v')) + \frac{1}{2}\frac{\wp'(v)+\wp'(v')}{\wp(v)-\wp(v')}[\frac{d\lambda(v')}{dv'} - \frac{d\lambda(v)}{dv}] \\
&- \frac{1}{n(n+1)}\frac{\partial}{\partial\varphi_0}\left(\frac{\partial\lambda(p)}{\partial v_{n+1}}\right)\frac{\partial\lambda(p')}{\partial v_{n+1}} - \frac{1}{n(n+1)}\frac{\partial\lambda(p)}{\partial v_{n+1}}\frac{\partial}{\partial\varphi_0}\left(\frac{\partial\lambda(p')}{\partial v_{n+1}}\right) \\
&+ \frac{1}{n(n+1)}\sum_{k,j=0}^{n}\frac{(-1)^{k+j}}{(k-1)!(j-1)!}\frac{\partial}{\partial\varphi_0}\left(\frac{\partial\varphi_j}{\partial v_{n+1}}\right)\frac{\partial\varphi_k}{\partial v_{n+1}} \\
&+ \frac{1}{n(n+1)}\sum_{k,j=0}^{n}\frac{(-1)^{k+j}}{(k-1)!(j-1)!}\frac{\partial\varphi_j}{\partial v_{n+1}}\frac{\partial}{\partial\varphi_0}\left(\frac{\partial\varphi_k}{\partial v_{n+1}}\right),
\end{aligned}
\tag{1.8}
$$

where

$$
\begin{aligned}
\tilde{\eta}^*(d\varphi_i, d\varphi_j) &= \eta^*(d\varphi_i, d\varphi_j), \quad i,j \neq 0, \\
\tilde{\eta}^*(d\varphi_0, d\varphi_j) &= \eta^*(d\varphi_i, d\varphi_j) + 4\pi i k_j\varphi_j.
\end{aligned}
\tag{1.9}
$$

See section 8.4 for the definition of D_τ and for further details. Thereafter, we extract the coefficients $\eta^*(d\varphi_i, d\varphi_j)$ from the generating function (1.8)

THEOREM 1.4.3. The coefficients $\eta^*(d\varphi_i, d\varphi_j)$ can be obtained by the formula,

$$
\begin{aligned}
&\tilde{\eta}^*(d\varphi_i, d\varphi_j) = (i+j-2)\varphi_{i+j-2}, \quad i,j \neq 0 \\
&\tilde{\eta}^*(d\varphi_i, d\varphi_0) = 0, \quad i \neq 0, \quad i \neq 1, \\
&\tilde{\eta}^*(d\varphi_1, d\varphi_j) = 0. \quad j \neq 0, \\
&\tilde{\eta}^*(d\varphi_1, d\varphi_0) = \wp((n+1)v_{n+1})\varphi_1.
\end{aligned}
\tag{1.10}
$$

Inspired by the construction of the flat coordinates of the Saito metric done in [**28**], we construct explicitly the coordinates $t^0, t^1, t^2, .., t^n, v_{n+1}, \tau$ by the following formulae

$$
\begin{aligned}
t^\alpha &= \frac{n}{n+1-\alpha}(\varphi_n)^{\frac{n+1-\alpha}{n}}(1+\Phi_{n-\alpha})^{\frac{n+1-\alpha}{n}}, \\
t^0 &= \varphi_0 - \frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\varphi_1 + 4\pi i g_1(\tau)\varphi_2,
\end{aligned}
\tag{1.11}
$$

where

$$
\begin{aligned}
(1+\Phi_i)^{\frac{n+1-\alpha}{n}} &= \sum_{d=0}^{\infty}\binom{\frac{n+1-\alpha}{n}}{d}\Phi_i^d, \\
\Phi_i^d &= \sum_{i_1+i_2+..+i_d=i}\frac{\varphi_{(n-i_1)}}{\varphi_n}....\frac{\varphi_{(n-i_d)}}{\varphi_n}.
\end{aligned}
\tag{1.12}
$$

The Theorem 1.4.3 together with the formulae (1.11), and some extra auxiliary technical lemmas, implies the flatness of the Saito metric η.

THEOREM 1.4.4. Let $(t^0, t^1, t^2, .., t^n)$ be defined in (1.11), and η^* the Saito metric. Then,

$$
\begin{aligned}
&\eta^*(dt^\alpha, dt^{n+3-\beta}) = -(n+1)\delta_{\alpha\beta}, \quad 2 \leq \alpha, \beta \leq n \\
&\eta^*(dt^1, dt^\alpha) = 0, \\
&\eta^*(dt^0, dt^\alpha) = 0, \\
&\eta^*(dt^i, d\tau) = -2\pi i \delta_{i0}, \\
&\eta^*(dt^i, dv_{n+1}) = -\frac{\delta_{i1}}{n+1}.
\end{aligned} \tag{1.13}
$$

Moreover, the coordinates $t^0, t^1, t^2, .., t^n, v_{n+1}, \tau$ are the flat coordinates of η^*. See details in sections 8.5 and 8.6.

A remarkable fact to point out is that, even though the intersection form g^* and their Levi Civita connection are $\mathscr{J}(\tilde{A}_n)$ invariant sections, its coefficients g^{ij}, Γ^{ij}_k are not Jacobi forms, but they live in an extension of the Jacobi forms ring. Hence, we have the following lemma

LEMMA 1.4.5. The coefficients of the intersection form $g^{\alpha\beta}$ and its Christoffel symbol $\Gamma^{\alpha\beta}_\gamma$ on the coordinates $t^0, t^1, .., t^n, v_{n+1}, \tau$ belong to the ring $\widetilde{E}_{\bullet,\bullet}[t^0, t^1, .., t^n, \frac{1}{t^n}]$, where $\widetilde{E}_{\bullet,\bullet}$ is a suitable extension of the ring $E_{\bullet,\bullet}$.

This lemma 1.4.5 is important because it gives a tri-graded ring structure for the coefficients $g^{\alpha\beta}$ and $\Gamma^{\alpha\beta}_\gamma$. In particular, the lemma 1.4.5 implies that $g^{\alpha\beta}$ and $\Gamma^{\alpha\beta}_\gamma$ are eigenfunctions of the Euler vector field given by the last equation (1.2). Using this fact, one can prove that $g^{\alpha\beta}$ and $\Gamma^{\alpha\beta}_\gamma$ are at most linear in the variable t^0, and this fact together with Theorem 1.4.4 proves that g^*, η^* form a flat pencil of metrics, and consequently we can prove

LEMMA 1.4.6. Let the intersection form be (1.6), the unit vector field be (1.5), and the Euler vector field be given by the last equation of (1.2). Then, there exists a function

$$
F(t^0, t^1, t^2, .., t^n.v_{n+1}, \tau) = -\frac{(t^0)^2\tau}{4\pi i} + \frac{t^0}{2}\sum_{\alpha,\beta\neq 0,\tau} \eta_{\alpha\beta} t^\alpha t^\beta + G(t^1, t^2, .., t^n, v_{n+1}, \tau), \tag{1.14}
$$

such that

$$
\begin{aligned}
&Lie_E F = 2F + \text{quadratic terms}, \\
&Lie_E\left(F^{\alpha\beta}\right) = g^{\alpha\beta}, \\
&\frac{\partial^2 G(t^1, t^2, .., t^n, v_{n+1}, \tau)}{\partial t^\alpha \partial t^\beta} \in \widetilde{E}_{\bullet,\bullet}[t^1, t^2, .., t^n, \frac{1}{t^n}],
\end{aligned} \tag{1.15}
$$

where

$$
F^{\alpha\beta} = \eta^{\alpha\alpha'}\eta^{\beta\beta'}\frac{\partial F^2}{\partial t^{\alpha'}\partial t^{\beta'}}. \tag{1.16}
$$

Using the lemma 1.4.6 with some more technical results, we obtain our final result

THEOREM 1.4.7. A suitable covering of the orbit space $\left(\mathbb{C}\oplus\mathbb{C}^{n+1}\oplus\mathbb{H}\right)/\mathscr{J}(\tilde{A}_n)$ with the intersection form (1.6), unit vector field (1.5), and Euler vector field given by the last equation of (1.2) has a Dubrovin Frobenius manifold structure. Moreover, a suitable covering of $\mathbb{C}\oplus\mathbb{C}^{n+1}\oplus\mathbb{H}/\mathscr{J}(\tilde{A}_n)$ is isomorphic as Dubrovin Frobenius manifold to a suitable covering of the Hurwitz space $H_{1,n-1,0}$.

See section 8.11 for details. In particular, in chapter 6, we derive explicitly the WDVV solution associated with the orbit space of $\mathscr{J}(\tilde{A}_1)$,which is given by

$$F\left(t^1,t^2,t^3,t^4\right)=\frac{i}{4\pi}\left(t^1\right)^2t^4-2t^1t^2t^3-\left(t^2\right)^2\log\left(t^2\frac{\theta_1'\left(0,t^4\right)}{\theta_1\left(2t^3,t^4\right)}\right). \tag{1.17}$$

The results of this thesis are important because of the following

(1) The Hurwitz spaces $H_{1,n-1,0}$ are classified by the group $\mathscr{J}(\tilde{A}_n)$, hence we increase the knowledge of the WDVV/ discrete group correspondence. In particular the WDVV solutions associated with this orbit spaces contains a kind of elliptic function in an exceptional variable, which is exotic in theory of WDVV solutions, since most of the known examples are polynomial or polynomial with exponential function. Recently, the case $\mathscr{J}(\tilde{A}_1)$ attracted the attention of experts due to its application in integrable systems [**17**], [**19**], [**26**].
(2) It is well known that Hurwitz spaces are related to matrix models, then, if one derives the associated matrix model of the Hurwitz spaces $H_{1,n-1,0}$, we would immediately classify it by the group $\mathscr{J}(\tilde{A}_n)$.
(3) Even though the orbit space of $\mathscr{J}(\tilde{A}_n)$ is locally isomorphic as Dubrovin Frobenius manifold to the Hurwtiz space $H_{1,n-1,0}$, these two spaces are not necessarily the same. Indeed, the Dubrovin Frobenius manifold associated to Hurwtiz spaces is a local construction, because it is defined in a domain of a solution of a Darboux-Egoroff system. On another hand, orbit spaces are somehow global objects, because their ring of invariant function are polynomial over a suitable ring. In addition, the notion of invariant functions gives information about the non-cubic part of the WDDV solution associated with the orbit space, see the last equation of (1.15) for instance.
(4) The orbit space construction of the group $\mathscr{J}(\tilde{A}_n)$ can be generalised to the other classical finite Coxeter groups such as B_n, D_n. Hence, these orbit spaces could give rise to a new class of Dubrovin Frobenius manifolds. Furthermore, the associated integrable hierarchies of this new class of Dubrovin Frobenius manifolds could have applications in Gromov Witten theory and combinatorics.

The thesis is organised in the following way. In chapter 2, we recall the basics definitions of Dubrovin-Frobenius manifolds. In chapter 3, we recall the Dubrovin Frobenius manifold construction on Hurwitz spaces. In chapter 4, the construction of Dubrovin Frobenius manifolds on the orbit space of the finite Coxeter group A_n is considered in order to give a gentle introduction to this framework. In chapter 5, based in the work done in [**8**] and [**9**], we consider a Dubrovin Frobenius manifolds arising from more involved group called Jacobi groups, from the section 5.4 to section 5.9, we use an alternative approach, which is more closely related with the methods done in [**28**] and [**11**], to construct the Dubrovin Frobenius structure in the orbit space of Jacobi groups. In chapter 6, we defined extended affine Jacobi group $\mathscr{J}(\tilde{A}_1)$ and , we construct Dubrovin-Frobenius structure on the orbit spaces of $\mathscr{J}(\tilde{A}_1)$ and compute its Free-energy. Furthermore, we show that the orbit space of the group $\mathscr{J}(\tilde{A}_1)$ is isomorphic ,as Dubrovin-Frobenius manifold, to the Hurwitz-Frobenius manifold $\tilde{H}_{1,0,0}$ [**12**], [**17**] ,[**29**]. See theorem 6.3.4 for details. In chapter 7, we describe Dubrovin Frobenius manifold of the orbit space $\mathscr{J}(\tilde{A}_1)$. In chapter 8, we generalise the results of chapter 6 for the group $\mathscr{J}(\tilde{A}_n)$.

CHAPTER 2

Review of Dubrovin-Frobenius manifolds

2.1. Basic definitions

We recall the basic definitions of Dubrovin Frobenius manifold, for more details [12].

Definition 2.1.1. A Frobenius Algebra $\mathscr{A}$ is an unital, commutative, associative algebra endowed with an invariant non degenerate bilinear pairing

$$\eta(,) : \mathscr{A} \otimes \mathscr{A} \mapsto \mathbb{C},$$

invariant in the following sense:

$$\eta(A \bullet B, C) = \eta(A, B \bullet C),$$

$\forall A, B, C \in \mathscr{A}$.

Definition 2.1.2. M is smooth or complex Dubrovin-Frobenius manifold of dimension n if a structure of Frobenius algebra is specified on any tangent plane T_tM at any point $t \in M$, smoothly depending on the point such that:

(1) The invariant inner product $\eta(,)$ is a flat metric on M. The flat coordinates of $\eta(,)$ will be denoted by $(t^1, t^2, .., t^n)$.

(2) The unity vector field e is covariantly constant w.r.t. the Levi-Civita connection ∇ for the metric $\eta(,)$

$$\nabla e = 0 \tag{2.1}$$

(3) Let

$$c(u, v, w) := \eta(u \bullet v, w) \tag{2.2}$$

(a symmetric 3-tensor). We require the 4-tensor

$$(\nabla_z c)(u, v, w) \tag{2.3}$$

to be symmetric in the four vector fields u, v, w, z.

(4) A vector field E must be determined on M such that:

$$\nabla\nabla E = 0 \tag{2.4}$$

and that the corresponding one-parameter group of diffeomorphisms acts by conformal transformations of the metric η, and by rescalings on the Frobenius algebras T_tM . Equivalently:

(2.5) $$[E,e] = -e,$$

(2.6) $$\begin{aligned}\mathscr{L}_E\eta(X,Y) &:= E\eta(X,Y) - \eta([E,X],Y) - \eta(X,[E,Y]) \\ &= (2-d)\eta(X,Y),\end{aligned}$$

(2.7) $$\begin{aligned}\mathscr{L}_E c(X,Y,Z) &:= Ec(X,Y,Z) - c([E,X],Y,Z) - c(X,[E,Y],Z) \\ &\quad - c(X,Y,[E,Z]) = (3-d)c(X,Y,Z).\end{aligned}$$

The Euler vector E can be represented as follows:

LEMMA 2.1.1. If the grading operator $Q := \nabla E$ is diagonalizable, then E can be represented as:

(2.8) $$E = \sum_{i=1}^{n}((1-q_i)t_i + r_i)\partial_i$$

We now define scaling exponent as follows:

DEFINITION 2.1.3. A function $\varphi : M \mapsto \mathbb{C}$ is said to be quasi-homogeneous of scaling exponent d_φ, if it is a eigenfunction of Euler vector field:

(2.9) $$E(\varphi) = d_\varphi \varphi$$

DEFINITION 2.1.4. The function $F(t), t = (t^1, t^2, .., t^n)$ is a solution of WDVV equation if its third derivatives

(2.10) $$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$$

satisfy the following conditions:

(1) $$\eta_{\alpha\beta} = c_{1\alpha\beta}$$

is constant nondegenerate matrix.

(2) The function
$$c^\gamma_{\alpha\beta} = \eta^{\gamma\delta} c_{\alpha\beta\delta}$$
is structure constant of associative algebra.

(3) F(t) must be quasi-homogeneous function
$$F(c^{d_1}t^1, .., c^{d_n}t^n) = c^{d_F} F(t^1, .., t^n)$$
for any nonzero c and for some numbers $d_1, ..., d_n, d_F$.

LEMMA 2.1.2. Any solution of WDVV equations with $d_1 \neq 0$ defined in a domain $t \in M$ determines in this domain the structure of a Dubrovin-Frobenius manifold. Conversely, locally any Dubrovin-Frobenius manifold is related to some solution of WDVV equations.

2.1.1. Intersection form.

Definition 2.1.5. Let $x = \eta(X,), y = \eta(Y,) \in \Gamma(T^*M)$ where $X, Y \in \Gamma(TM)$. An induced Frobenius algebra is defined on $\Gamma(T^*M)$ by:

$$x \bullet y = \eta(X \bullet Y,).$$

Definition 2.1.6. The intersection form is a bilinear pairing in T^*M defined by:

$$(\omega_1, \omega_2)^* := \iota_E(\omega_1 \bullet \omega_2)$$

where $\omega_1, \omega_2 \in T^*M$ and $\bullet$ is the induced Frobenius algebra product in the cotangent space. the intersection form will be denoted by g^* .

Remark 1: Let $x = \eta(X,), y = \eta(Y,) \in \Gamma(T^*M)$. Then:

$$g^*(x, y) = \eta(X \bullet Y, E) = c(X, Y, E).$$

Remark 2: It is possible to prove that the tensor g^* defines a bilinear form on the tangent bundle that is almost everywhere non degenerate [**12**].

Proposition 2.1.3. The metric g^* is flat, and $\forall \lambda \in \mathbb{C}$, the contravariant metric $\eta^*(,) + \lambda g^*(,)$ is flat, and the contravariant connection is $\nabla^\eta + \lambda \nabla^g$, where ∇^η, ∇^g are the contravariant connections of η^* and g^* respectively. The family of metrics $\eta^*(,) + \lambda g^*(,)$ is called Flat pencil of metrics.

Lemma 2.1.4. The induced metric η^* on the cotangent bundle T^*M can be written as Lie derivative with respect the unit vector field e of the intersection form g^*. i.e

$$\eta^* = \mathscr{L}_e g^*. \tag{2.11}$$

Lemma 2.1.5. The correspondent WDVV solution $F(t^1, .., t^n)$ of the Dubrovin-Frobenius manifold works as potential function for g^*. More precisely:

$$g^*(dt^i, dt^j) = (1 + d - q_i - q_j)\nabla_{(dt^i)^\sharp}\nabla_{(dt^j)^\sharp}F. \tag{2.12}$$

where the form $(dt^j)^\sharp$ is the image of dt^j by the isomorphism induced by the metric η.

2.1.2. Reconstruction. Suppose that given a Dubrovin-Frobenius manifold M, only the following data are known: intersection form g^*, unit vector field e, Euler vector field E. From the previous lemmas we can reconstruct the Dubrovin-Frobenius manifold by setting:

$$\eta^* = \mathscr{L}_e g^*. \tag{2.13}$$

Then, we find the flat coordinates of η as homogeneous functions, and the structure constants by imposing:

$$g^*(dt^i, dt^j) = (1 + d - q_i - q_j)\nabla_{(dt^i)^\sharp}\nabla_{(dt^j)^\sharp}F. \tag{2.14}$$

Therefore, it is possible to compute the Free-energy by integration. Of course, we may have obstructions when, $1 + d = q_i + q_j$.

2.1.3. Monodromy of Dubrovin-Frobenius manifold. The intersection form g of a Dubrovin-Frobenius manifold is a flat almost everywhere nondegenerate metric. Let

$$\Sigma = \{t \in M : det(g) = 0\}$$

Hence, the linear system of differential equations determining g^*-flat coordinates

$$g^{\alpha\epsilon}(t)\frac{\partial^2 x}{\partial t^\beta \partial t^\epsilon} + \Gamma^{\alpha\epsilon}_\beta(t)\frac{\partial x}{\partial t^\epsilon} = 0$$

has poles, and consequently its solutions $x_a(t^1, .., t^n)$ are multivalued, where $(t^1, .., t^n)$ are flat coordinates of η. The analytical continuation of the solutions $x_a(t^1, .., t^n)$ has monodromy corresponding to loops around Σ. This gives rise to a monodromy representation of $\pi_1(M \setminus \Sigma)$, which is called Monodromy of the Dubrovin-Frobenius manifold.

2.1.4. Dubrovin Connection. In the theory of Dubrovin Frobenius manifold, there is another way to associate a monodromy group on it. Consider the following deformation of the Levi-Civita connection defined on a Dubrovin Frobenius manifold M

$$\tilde{\nabla}_u v := \nabla_u v + zu \bullet v, \quad u, v \in \Gamma(TM), \tag{2.15}$$

where ∇ is the Levi-Civita connection of the metric η, $\bullet$ is the Frobenius product, and $z \in \mathbb{CP}^1$. Then, the following connection defined in $M \times \mathbb{CP}^1$

$$\begin{aligned} \tilde{\nabla}_u v &:= \nabla_u v + zu \bullet v, \\ \tilde{\nabla}_{\frac{d}{dz}}\frac{d}{dz} = 0, &\quad \tilde{\nabla}_v \frac{d}{dz} = 0, \\ \tilde{\nabla}_{\frac{d}{dz}} v &= \partial_z v + E \bullet v - \frac{1}{z}\mu(v). \end{aligned} \tag{2.16}$$

where μ is the diagonal matrix be given by

$$\mu_{\alpha\beta} = (q_\alpha - \frac{d}{2})\delta_{\alpha\beta}. \tag{2.17}$$

The monodromy representation arise by considering the solutions of the flat coordinate systems

$$\tilde{\nabla} d\tilde{t} = 0. \tag{2.18}$$

After doing some Gauge transformations in the system (2.18), and writing it in matricidal form. The system 2.18 takes the form

$$\begin{aligned} \frac{dY}{dt^\alpha} &= zC_\alpha Y, \\ \frac{dY}{dz} &= \Big(U + \frac{\mu}{z}\Big) Y, \end{aligned} \tag{2.19}$$

where

$$C_{\alpha}{}^{\gamma}_{\beta} = c^{\gamma}_{\alpha\beta}, \quad U^{\gamma}_{\beta} = E^{\epsilon} c^{\gamma}_{\beta\epsilon}. \tag{2.20}$$

2.2. Semisimple Dubrovin Frobenius manifolds

DEFINITION 2.2.1. A Frobenius algebra is called semisimple if it does not have nilpotent, i.e. if $a \neq 0$ implies

$$a^m \neq 0, \quad \text{for any} \quad m \in \mathbb{Z}. \tag{2.21}$$

LEMMA 2.2.1. [12] Let A be a semisimple Frobenius algebra, then there exist a base $e_1, e_2, .., e_n$ of A, such that the Frobenius product $\bullet$ in this base is described by

$$e_i \bullet e_j = \delta_{ij} e_i. \tag{2.22}$$

DEFINITION 2.2.2. A point in a Dubrovin Frobenius manifold is called semisimple, if the Frobenius algebra in its tangent space is semisimple.

REMARK 2.2.1. Note that semisimplicity is an open condition.

LEMMA 2.2.2. [12] In a neighbourhood of a semi semisimple point, there exist local coordinates $(u_1, u_2, .., u_n)$ such that

$$\frac{\partial}{\partial u_i} \bullet \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i}. \tag{2.23}$$

The coordinates $(u_1, u_2, .., u_n)$ are called canonical coordinates.

LEMMA 2.2.3. [12] Let M a semisimple Dubrovin Frobenius manifold, on the canonical coordinates $(u_1, u_2, .., u_n)$ the intersection form, Euler vector field, and unit vector field can be written as

$$\begin{aligned} g^{ii} &= u_i \eta^{ii} \delta_{ij}, \\ e &= \sum_{i=1}^{n} \frac{\partial}{\partial u_i}, \\ E &= \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i}. \end{aligned} \tag{2.24}$$

PROPOSITION 2.2.4. [12] In a neighborhood of a semisimple point all the roots $(u_1, u_2, .., u_n)$ of the characteristic equation

$$det(g^{\alpha\beta} - u\eta^{\alpha\beta}) = 0 \tag{2.25}$$

are simple. They are canonical coordinates in this neighbourhood . Conversely, if the roots of the characteristic equation are simple in a point $p \in M$, then $p \in M$ is a semisimple point on the Frobenius manifold and $(u_1, u_2, .., u_n)$ are canonical coordinates in the neighbourhood of this point.

Definition 2.2.3. [**12**] A diagonal metric on a n-dimensional manifold

$$\eta = \sum_{i=1}^{n} \eta_{ii} du_i^2 \tag{2.26}$$

is called potential, if there exist a function $U(u_1, u_2, .., u_n)$ such that

$$\eta_{ii} = \frac{\partial U}{\partial u_i}. \tag{2.27}$$

Definition 2.2.4. [**12**] A potential diagonal flat metric η on a n-dimensional manifold is called Darbou-Egoroff metric.

Lemma 2.2.5. [**12**] Let be η a diagonal potential metric on a n-dimensional manifold

$$\eta = \sum_{i=1}^{n} \eta_{ii} du_i^2. \tag{2.28}$$

Then, the metric (2.28) is Darboux-Egoroff iff its rotational coefficients β_{ij}

$$\beta_{ij} = \frac{\partial_j \sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \tag{2.29}$$

satisfy the system of equations

$$\begin{aligned} \partial_k \beta_{ij} &= \beta_{ik} \beta_{kj}, \\ \sum_{k=1}^{n} \partial_k \beta_{ij} &= 0. \end{aligned} \tag{2.30}$$

CHAPTER 3

Review of Dubrovin-Frobenius manifold on Hurwitz spaces

3.1. Hurwitz spaces

The main reference of this section are [**12**] and [**29**].

Definition 3.1.1. The Hurwitz space $H_{g,n_0,\dots,n_m}$ is the moduli space of curves C_g of genus g endowed with a N branched covering, $\lambda : C_g \mapsto \mathbb{C}P^1$ of $\mathbb{C}P^1$ with $m+1$ branching points over $\infty \in \mathbb{C}P^1$ of branching degree $n_i + 1, i = 0, ..., m$.

Definition 3.1.2. Two pairs (C_g, λ) and $(\tilde{C}_g, \tilde{\lambda})$ are said Hurwitz-equivalent if there exist an analytic isomorphic $F : C_g \mapsto \tilde{C}_g$ such that

$$\lambda \circ F = \tilde{\lambda}. \tag{3.1}$$

Roughly speaking, Hurwitz spaces $H_{g,n_0,\dots,n_m}$ are moduli spaces of meromorphic functions which realise a Riemann surface of genus g C_g as covering over $\mathbb{C}P^1$ with a fixed ramification profile.

Example 1:

A generic point of the Hurwitz space $H_{0,n}$ is

$$H_{0,n} = \{\lambda(p, x_0, x_1, x_2, .., x_n) = \prod_{i=0}^{n}(p - x_i) : \sum_{i=0}^{n} x_i = 0\} \tag{3.2}$$

Example 2:

A generic point of the Hurwitz space $H_{0,n-1,0}$ is

$$H_{0,n} = \{\lambda(p, a_2, a_3, .., a_{n+1}, a_{n+2}) = p^n + a_2 p^{n-2} + ... + a_n p + a_{n+1} + \frac{a_{n+2}}{p}.\} \tag{3.3}$$

Example 3:

A generic point of the Hurwitz space $H_{1,n}$ is

$$H_{1,n} = \{\lambda(p, u, v_0, v_1, .., v_n, \tau) = e^{-2\pi i u}\frac{\prod_{i=0}^{n}\theta_1(p - v_i, \tau)}{\theta_1^{n+1}(v, \tau)} : \sum_{i=0}^{n} v_i = 0\} \tag{3.4}$$

here θ_1 is the Jacobi theta function, see (3.25).

Example 4:
A generic point of the Hurwitz space is $H_{1,n-1,0}$ is
(3.5)

$$H_{1,n-1,0} = \{\lambda(p, u, v_0, v_1, .., v_n, v_{n+1}, \tau) = e^{-2\pi iu} \frac{\prod_{i=0}^{n} \theta_1(p - v_i, \tau)}{\theta_1^n(v, \tau)\theta_1(v + (n+1)v_{n+1}, \tau)} : \sum_{i=0}^{n} v_i = -(n+1)v_{n+1}\}$$

The covering $\widetilde{H} = \tilde{H}_{g,n_0,...,n_m}$ consist of the set of points

$$(C_g; \lambda; k_0, ..., k_m; a_1, ..., a_g, b_1, ..., b_g) \in \tilde{H}_{g,n_0,...,n_m}$$

where C_g, λ are the same as above, $a_1, ..., a_g, b_1, ..., b_g \in H_1(C_g, \mathbb{Z})$ are the canonical symplectic basis, and $k_0, ..., k_m$ are roots of λ near $\infty_0, \infty_1, ..., \infty_m$ of the orders $n_0 + 1, n_1 + 1, ..., n_m + 1$. resp.,

$$k_i^{n_i+1}(P) = \lambda(P), \quad \text{P near } \infty_i.$$

3.2. Bidifferential W

Definition 3.2.1. [29] Let $P, Q \in C_g$. The meromorphic Bidifferential W is given by

$$W(P, Q) = d_P d_Q \log E(P, Q), \tag{3.6}$$

where $E(P, Q)$ is the prime form on the Riemann surface C_g. Alternatively, it can be characterised by the following properties

(1) symmetric meromorphic differential in $C_g \times C_g$, with second order pole on $P = Q$ with biresidue 1
(2)

$$\begin{aligned} &\int_{a_k} W(P, Q) = 0; \\ &\int_{b_k} W(P, Q) = 2\pi i \omega_k(P) \end{aligned} \tag{3.7}$$

where $\{\omega_k(P)\}$ are the normalized base of holomorphic differentials, i.e. $\int_{a_j} \omega_k(P) = \delta_{ij}$

The dependence of the bidifferential W on branch points of the Riemann surface is given by the Rauch variational formulas

$$\frac{\partial W(P, Q)}{\partial u_i} = \frac{1}{2} W(P, P_i)W(P_i, Q), \tag{3.8}$$

where $W(P, P_i)$ is the evaluation of $W(P, Q)$ at $Q = Pj$ with respect to the standard local parameter $x_j(Q) = \sqrt{\lambda - \lambda(Q)}$

$$W(P, P_i) = \left. \frac{W(P, Q)}{dx_j(Q)} \right|_{Q=P_j} \tag{3.9}$$

A remarkable consequence of the Rauch variational formula is that it induce a flat metric in the Hurwitz space. Indeed, Proposition 3.2.1. [29] Let be the metric

$$ds_W^2 = \sum_{i=1}^{n} \left(\oint_l h(Q) W(Q, P_i) \right)^2 (du_i)^2, \tag{3.10}$$

where l is a smooth contour in the Riemann surface such that $P_i \notin l$, and $h(Q)$ is a smooth function independent of $\{u_i\}$. Then, the rotational coefficients of (3.10) satisfies a Darboux-Egoroff system in lemma 2.2.5.

For particular choices of the function $h(Q)$ the metrics (3.10) coincides with the metrics induced by the primary differentials see section 3.3 and [12] for details. This fact is remarkable, because, it shows that from only the data of the Hurwtiz space, one can construct a flat metric for the desired Dubrovin Frobenius manifold.

3.3. Reconstruction of Dubrovin Frobenius manifold

Over the space $\widetilde{H}_{g,n_0,\dots,n_m}$, it is possible to introduce a Dubrovin-Frobenius structure by taking as canonical coordinates the ramification points $(u_1, u_2, ..u_n)$ of $H_{g,n_0,\dots,n_m}$. The Dubrovin-Frobenius structure is specified by the following objects:

$$\text{multiplication} \quad \partial_i \bullet \partial_j = \delta_{ij}\partial_i, \text{where} \quad \partial_i = \frac{\partial}{\partial u_i}, \tag{3.11}$$

$$\text{Euler vector field} \quad E = \sum_i u_i \partial_i, \tag{3.12}$$

$$\text{unit vector field} \quad e = \sum_i \partial_i, \tag{3.13}$$

and the metric η defined by the formula

$$ds_\phi^2 = \sum \text{res}_{P_i} \frac{\phi^2}{d\lambda} (du_i)^2, \tag{3.14}$$

where ϕ is some primary differential of the underlying Riemann surface C_g. Note that the Dubrovin-Frobenius manifold structure depends on the meromorphic function λ, and on the primary differential ϕ. The list of possible primary differential ϕ is in [12].

Consider a multivalued function p on C by taking the integral of ϕ

$$p(P) = v.p \int_{\infty_0}^{P} \phi$$

The principal value is defined by omitting the divergent part, when necessary, because ϕ may be divergent at ∞_0, as function of the local parameter k_0. Indeed the primary differentials defined on [12] may diverge as functions of k_i.

$$\phi = dp.$$

Let $\tilde{H}_\phi$ be the open domain in $\tilde{H}$ specifying by the condition

$$\phi(P_i) \neq 0.$$

THEOREM 3.3.1. [**12**] For any primary differential ϕ of the list in [**12**] the multiplication (3.11), the unity (3.13), the Euler vector field (3.12), and the metric (3.14) determine a structure of Dubrovin Frobenius manifold on $\tilde{H}_\phi$. The corresponding flat coordinates $t_A, A = 1, ..., N$ consist of the five parts

$$t_A = (t^{i,\alpha}, i = 0, ..m, \alpha = 1, .., n_i; p^i, q^i, i = 1, .., m; r^i, s^i, i = 1, ..g) \tag{3.15}$$

where

$$t^{i,\alpha} = \mathrm{res}_{\infty_i} k_i^{-\alpha} p d\lambda \quad i = 0, ..m, \alpha = 1, .., n_i \tag{3.16}$$

$$p^i = v.p \int_{\infty_0}^{\infty_i} dp \quad i = 1, ..m. \tag{3.17}$$

$$q^i = -\mathrm{res}_{\infty_i} \lambda dp \quad i = 1, ..m. \tag{3.18}$$

$$r^i = \int_{a_i} dp \quad i = 1, ..g. \tag{3.19}$$

$$s^i = -\frac{1}{2\pi i} \int_{b_i} \lambda dp \quad i = 1, ..g. \tag{3.20}$$

Moreover, function $\lambda = \lambda(p)$ is the superpotential of this Dubrovin Frobenius manifold, i.e. we have the following formulas to compute the metric $\eta = \langle , \rangle$, the intersection form $g^* = (,)$ and the structure constants c.

$$\langle \partial', \partial'' \rangle = -\sum \mathrm{res}_{d\lambda = 0} \frac{\partial'(\lambda) \partial''(\lambda)}{d\lambda} dp \tag{3.21}$$

$$(\partial', \partial'') = -\sum \mathrm{res}_{dLog\lambda = 0} \frac{\partial'(Log\lambda) \partial''(Log\lambda)}{dLog\lambda} dp \tag{3.22}$$

$$c(\partial', \partial'', \partial''') = -\sum \mathrm{res}_{d\lambda = 0} \frac{\partial'(\lambda) \partial''(\lambda) \partial'''(\lambda)}{d\lambda} dp \tag{3.23}$$

REMARK 3.3.1. The Dubrovin Frobenius structure on Hurwitz spaces depend on a choice of suitable primary differentials. Dropping this suitable choice implies that we typically lose the quasi homogeneous condition of the WDVV equation or the fact the unit is covariant constant.

3.3.1. Example $\tilde{H}_{1,0,0}$. $H_{1,0,0}$ is the space of elliptic functions with 2 simple poles, i.e:

$$\lambda(p,a,b,c,\tau) = a + b\left[\frac{\theta_1'(p-c|\tau)}{\theta_1(p-c|\tau)} - \frac{\theta_1'(p+c|\tau)}{\theta_1(p+c|\tau)}\right] \tag{3.24}$$

where

$$\theta_1(v,\tau) = 2\sum_{n=0}^{\infty}(-1)^n e^{\pi i\tau(n+\frac{1}{2})^2}\sin((2n+1)v) \tag{3.25}$$

We take the holomorphic primary differential dp. Applying the Theorem (3.3.1) in this case we get that the flat coordinates for the metric η are exactly (a,b,c,τ). Furthermore, using formula (3.23) we get the following formula (page 28 of [**13**]).

$$F(a,b,c,\tau) = \frac{i}{4\pi}a^2\tau - 2abc - b^2\log\left(b\frac{\theta_1'(0,\tau)}{\theta_1(2c,\tau)}\right) \tag{3.26}$$

Remark: There is a typo in the last term of the expression in the paper [**13**]. The expression (3.26) in a correct form can be found in [**17**], and [**19**]. Further, we derive the expression (3.26) by using orbit space techniques, see section 6.3.

CHAPTER 4

Review of Dubrovin Frobenius manifolds on the orbit space of A_n

The main goal of this section is to introduce the construction of Dubrovin Frobenius manifolds on orbit spaces. For this purpose, we start with the orbit space of the finite Coxeter group A_n. This example is particularly important, because:

(1) It gives a intrinsic description of the differential geometry of the universal unfolding of the simple singularity $\lambda = p^{n+1}$ [**11**]

$$M = \{\lambda(p, a_2, a_3, .., a_{n+1}) = p^{n+1} + a_2 p^{n-1} + a_3 p^{n-2} + .. + a_n p + a_{n+1}\}$$

or equivalently to of Hurwitz space $H_{0,n}$.

(2) It describes the topological minimal model associated with the A_n group [**11**].

This section is a resume of [**11**] and [**28**] which will works as gentle introduction to the orbit space construction. Indeed, the techniques used in this section will be further adapted to be applied in Chapter 6 and 8.

4.1. Finite Coxeter group A_n

Step 1:

The first step of the orbit space construction is the definition of the desired group action. In this particular case, the definition of the A_n action on $\mathbb{C}^n$.

Let A_n be a finite Coxeter group that acts on a lattice $(L^{A_n}, <,>_{A_n})$ with a bilinear form $<,>_{A_n}$, where L^{A_n} is defined below

$$L^{A_n} = \{z = (z_0, z_1, .., z_n) \in \mathbb{Z}^{n+1} : \sum_{i=0}^{n} z_i = 0\},$$

The bilinear pairing $<,>_{A_n}$ is the Euclidean metric restricted to the condition $\sum_{i=0}^{n} z_i = 0$. More explicitly,

$$< z, z >_{A_n} = z^T \begin{pmatrix} 2 & 1 & 1 & ... & 1 \\ 1 & 2 & 1 & ... & 1 \\ 1 & 1 & 2 & ... & 1 \\ 1 & 1 & 1 & ... & 1 \\ 1 & 1 & 1 & ... & 2 \end{pmatrix} z = 2\sum_{i=0}^{n-1} z_i^2 + 2\sum_{i>j} z_i z_j$$

Recall that A_n acts on L^{A_n} by permutations:

$$w_i(z_0, z_1, z_2, .., z_n) = (z_{i_0}, z_{i_1}, ..., z_{i_n}).$$

Moreover, the A_n group acts on the complexification of $L_{A_n}, <,>_{A_n}$

$$L^{A_n} \otimes \mathbb{C} = \{v = (x_0, x_1, .., x_n) \in \mathbb{Z}^{n+1} : \sum_{i=0}^{n} x_i = 0\},$$

by permutation.

Note that, we can identify $L^{A_n} \otimes \mathbb{C}$ with $\mathbb{C}^n$ by the maps

$$(v_0, .., v_{n-1}) \mapsto (v_0, .., v_{n-1}, -\sum_{i=0}^{n} v_i),$$
$$(v_0, .., v_{n-1}, v_n) \mapsto (v_0, .., v_{n-1}).$$

4.2. Invariant ring of A_n

Step 2: The second step consist in the description of a suitable ring of invariant functions. In this particular case, the ring of invariant functions is the ring of polynomials which are invariant under the A_n action. The main result of this section is the Chevalley theorem. This theorem realise the ring of the A_n invariant polynomials as the finite generated ring of the symmetric polynomials $a_2, a_3, ..., a_{n+1}$ in the variables $v_0, v_1, v_2, .., v_n$. In practice, this fact allow us to use a base of symmetric polynomials as coordinates for the orbit space, furthermore, it gives a global description of the orbit space of the A_n, by considering as

$$\mathrm{Spec}\mathbb{C}[a_2, a_3, .., a_n.a_{n+1}] = \mathbb{C}^n.$$

Definition 4.2.1. The Invariant ring of A_n are homogeneous polynomials g on $\Omega = \mathbb{C}^n \ni (v_0, v_1, .., v_n)$ satisfying

$$g(w(v_0, v_1, .., v_n)) = g(v_0, v_1, .., v_n), \quad w \in A_n \quad (A_n \text{ invariant condition}) \tag{4.1}$$

Examples:

The elementary symmetric polynomials

$$a_1(v_0, v_1, .., v_n) = \sum_{i=0}^{n} v_i = 0,$$
$$a_2(v_0, v_1, .., v_n) = \sum_{i=0}^{n} v_i v_j \Big|_{\sum_{i=0}^{n} v_i = 0},$$
$$.$$
$$.$$
$$a_{n+1}(v_0, v_1, .., v_n) = \prod_{i=0}^{n} v_i \Big|_{\sum_{i=0}^{n} v_i = 0}.$$

THEOREM 4.2.1. **Chevalley theorem**[10] The Invariant ring of A_n is free module of rank $n+1$ over $\mathbb{C}$, with generators $a_2, a_3, .., a_{n+1}$. Namely:

$$\textbf{Invariant } A_n \textbf{ polynomials} = \mathbb{C}[a_2, a_3, .., a_n.a_{n+1}]$$

REMARK 4.2.1. The basis for generators is not unique, indeed fixing a base, one could derive another base by doing a weighted polynomial transformation. However, the degree d_i of the homogeneous polynomials are invariant, in particular they are called Coxeter numbers.

THEOREM 4.2.2. There exist a formula for a specific basis of generators given by

$$\lambda^{A_n}(p) = \prod_{i=0}^{n}(p - v_i) = p^{n+1} + a_2 p^{n-1} + ... + a_n p + a_{n+1}, \tag{4.2}$$

where $\sum_{i=0}^{n} v_i = 0$.

REMARK 4.2.2. It is well-know that one can associated any generating function with a recursive operator. In the case of (4.2), the recursive operator can be obtained by the following obeservation

$$\lambda^{A_n}(p) = \left[e^{p\frac{\partial}{\partial x}}\left(\prod_{i=0}^{n}(x - v_i)\right)\right]\Bigg|_{x=0}$$

Then, we can obtain $a_2, .., a_{n+1}$ by doing the following transformation in a_{n+1}

$$a_{n+1} = \prod_{i=0}^{n} v_i \Bigg|_{\sum_{i=0}^{n} v_i = 0} \mapsto \hat{a}_{n+1} = \prod_{i=0}^{n}(p - v_i)\Bigg|_{\sum_{i=0}^{n} v_i = 0},$$

and by applying the recursive operator $\frac{\partial}{\partial x}$ in $\hat{a}_{n+1}$ in the following sense

$$\begin{aligned}
\hat{a}_{n+1}|_{x=0} &= a_{n+1}, \\
\frac{\partial}{\partial x}(\hat{a}_{n+1})\Big|_{x=0} &= a_n, \\
\frac{\partial^2}{\partial x^2}(\hat{a}_{n+1})\Big|_{x=0} &= a_{n-1}, \\
&. \\
&. \\
\frac{\partial^{n-1}}{\partial x^{n-1}}(\hat{a}_{n+1})\Big|_{x=0} &= a_2, \\
\frac{\partial^{n}}{\partial x^{n}}(\hat{a}_{n+1})\Big|_{x=0} &= a_1 = 0, \\
\frac{\partial^{n+1}}{\partial x^{n+1}}(\hat{a}_{n+1})\Big|_{x=0} &= 1,
\end{aligned}$$

Note that the operator $\frac{\partial}{\partial x}$ can be interpreted as a vector field in a enlargement of the orbit space of A_n on the coordinates $(v_0, v_1, .., v_n, x)$. Sometimes is also useful write this vector field $\frac{\partial}{\partial x}$ the following coordinates

$$z_i = v_i + x,$$

as

$$\frac{\partial}{\partial x} = \sum_{i=0}^{n} \frac{\partial}{\partial z_i}. \tag{4.3}$$

The relation (4.3) will be useful in the next chapter.

4.3. Geometric structure of the orbit space of A_n

Step 3: In this section, we introduce the minimal geometric data to reconstruct a Dubrovin Frobenius manifold structure as it was already announced in the subsection 2.1.2. The geometric structure on a orbit space of A_n must be invariant A_n sections, therefore, we need to construct an intersection form, Euler vector field, and unit vector field which are A_n invariant.

Definition 4.3.1. The metric g is the following tensor:

$$g = \sum_{i=0}^{n} dv_i^2|_{\sum_{i=0}^{n} v_i=0} dv_i dv_j = g_{ij} dv_i dv_j, \tag{4.4}$$

where g^{ij} is

$$(g_{ij}) = \begin{pmatrix} 2 & 1 & 1 & ... & 1 \\ 1 & 2 & 1 & ... & 1 \\ 1 & 1 & 2 & ... & 1 \\ 1 & 1 & 1 & ... & 1 \\ 1 & 1 & 1 & ... & 2 \end{pmatrix} \tag{4.5}$$

Definition 4.3.2. The intersection form g^* is given by

$$g^* = g^{ij} \frac{\partial}{\partial v_i} \otimes \frac{\partial}{\partial v_j} \tag{4.6}$$

where $g^{ij} = {g_{ij}}^{-1}$.

Proposition 4.3.1. The intersection form (4.6) is A_n invariant.

Proof. The intersection form (4.6) is A_n invariant iff (4.4) also is. But is a particular restriction of the Euclidean metric, which is clearly A_n invariant. □

Definition 4.3.3. Let be d_i the degree of the polynomial a_i, then the Euler vector field E is given by

$$E = \sum d_i a_i \frac{\partial}{\partial a_i} \tag{4.7}$$

DEFINITION 4.3.4. The Unit vector field e is given by

$$e = \frac{\partial}{\partial a_{n+1}} \tag{4.8}$$

LEMMA 4.3.2. The unit vector field (4.8) and Euler vector field (4.7) are A_n invariant sections.

PROOF. Both (4.8) and (4.7) are written in terms of invariant A_n polynomials.

□

4.4. Differential geometry preliminaries

In order to derive the Dubrovin Frobenius manifolds, we recall some results related with Riemannian geometry of the contravariant "metric" g^{ij}. By metric, I mean symmetric, bilinear, non-degenerate. In coordinates, let the metric

$$g_{ij}dx^i dx^j$$

and its induced contravariant metric

$$g^{ij}\frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}$$

The Levi Civita connection is uniquely specified by

$$\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma^s_{ki} g_{sj} - \Gamma^s_{kj} g_{is} = 0, \tag{4.9}$$

or

$$\nabla_k g^{ij} = \partial_k g^{ij} - \Gamma^i_{ks} g^{sj} - \Gamma^j_{ks} g^{is} = 0, \tag{4.10}$$

and

$$\Gamma^k_{ij} = \Gamma^k_{ji}. \tag{4.11}$$

The Christoffel symbol can be written as

$$\Gamma^k_{ij} = g^{ks}\left(\partial_i g_{sj} + \partial_j g_{is} - \partial_s g_{ij}\right).$$

But, for our purpose it will be more convenient to use

$$\Gamma^{ij}_k := -g^{is}\Gamma^j_{sk}. \tag{4.12}$$

Then, the equations (4.9), (4.10), and (4.11) are equivalent to

$$\begin{aligned} \partial_k g^{ij} &= \Gamma^{ij}_k + \Gamma^{ji}_k, \\ g^{is}\Gamma^{jk}_s &= g^{js}\Gamma^{ik}_s. \end{aligned} \tag{4.13}$$

Introducing the operators

$$\begin{aligned} \nabla^i &= g^{is}\nabla_s, \\ \nabla^i \xi_k &= g^{is}\partial_s \xi_k + \Gamma^{is}_k \xi_s. \end{aligned} \tag{4.14}$$

The curvature tensor R^l_{ijk} of the metric measures noncommutativity of the operators ∇_ior, equivalently ∇^i

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\,\xi_k = R^l_{ijk}\xi_l \tag{4.15}$$

where

$$R^l_{ijk} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^l_{is}\Gamma^s_{jk} - \Gamma^l_{js}\Gamma^s_{ik} \tag{4.16}$$

We say that the metric is flat if the curvature of it vanishes. For a flat metric local flat coordinates $p_1, ..., p_n$ exist such that in these coordinates the metric is constant and the components of the Levi-Civita connection vanish. Conversely, if a system of flat coordinates for a metric exists then the metric is flat. The flat coordinates are determined uniquely up to an affine transformation with constant coefficients. They can be found from the following system

$$\nabla^i\partial_k p = g^{is}\partial_s\partial_k p + \Gamma^{is}_k\partial_s p = 0. \tag{4.17}$$

The correspondent Riemman tensor for the contravariant metric g^{ij} can be written as

$$R^{ijk}_l := g^{is}g^{jt}R^k_{stl} = g^{is}\left(\partial_s\Gamma^{jk}_l - \partial_l\Gamma^{jk}_s\right) + \Gamma^{ij}_s\Gamma^{sk}_l - \Gamma^{ik}_s\Gamma^{sj}_l. \tag{4.18}$$

The aim of this section is to construct a Dubrovin Frobenius structure on the orbit space of A_n. The strategy to achieve this goal is based on the derivation of a WDVV solution from the geometric data of the orbit space A_n. More specifically, the WDVV solution will be derived from the flat pencil structure which the orbit space of A_n naturally has.

Definition 4.4.1. [**11**] Two metrics (g^*, η^*) form a flat pencil if:

(1) The metric

$$g^{ij}_\lambda := g^{ij} + \lambda\eta^{ij}, \tag{4.19}$$

is flat for arbitrary λ.

(2) The Levi-Civita connection of the metric (4.19) has the form

$$\Gamma^{ij}_{k,\lambda} := \Gamma^{ij}_{k,g} + \lambda\Gamma^{ij}_{k,\eta}$$

where $\Gamma^{ij}_{k,g}$, $\Gamma^{ij}_{k,\eta}$ are the Levi-Civita connection of g^*, and η^* respectively.

The main source of flat pencil metric is the following lemma

Lemma 4.4.1. [**11**] If for a flat metric g^* in some coordinate $a_2, a_3, .., a_{n+1}$ both the coefficients of the metric g^{ij} and Levi Civita connection Γ^{ij}_k are linear in the coordinate a_{n+1}, and if $det(g^*) \neq 0$, then, the metric

$$g^{ij} + \lambda\frac{\partial g^{ij}}{\partial a_{n+1}} \tag{4.20}$$

form a flat pencil. The corresponding Levi-Civita connection have the form

$$\Gamma^{ij}_{k,g} := \Gamma^{ij}_{k}, \quad \Gamma^{ij}_{k,\eta} := \frac{\partial \Gamma^{ij}_{k}}{\partial a_{n+1}} \tag{4.21}$$

Hence, our goal is first to construct a flat globally well defined metric in the orbit space of A_n such that there exist coordinates $a_2, a_3, .., a_{n+1}$ in which both the metric and its Christoffel symbols are at most linear on a_{n+1}.

4.5. The Saito metric η

Step 4:

This section will be devoted to construct the flat pencil metric on the orbit space of A_n. The first flat metric was already constructed in (4.6), therefore, this section will concentrate in the construction of the second flat metric. The second metric as the lemma 4.4.1 suggests is given by

$$Lie_{\frac{\partial}{\partial a_{n+1}}} g^{ij} := \eta^{ij}.$$

Hence, we will derive the coefficients of the metric (4.6) in the coordinates $a_2, a_3, .., a_{n+1}$ and from it we derive the coefficients of the second metric of the flat pencil.

Proposition 4.5.1. [28] The coefficient of $g^*(da_p, da_q)$ is recovered by the generating formula

$$\begin{aligned} &\sum_{i,j=0}^{n+1} g^*(da_i, da_j) u^{n+1-i} w^{n+1-j} = \\ &= -\frac{1}{n+1} \frac{d\lambda(w)}{dw} \frac{d\lambda(u)}{du} + \frac{1}{u-w} \left(\lambda(w) \frac{d\lambda(u)}{du} - \frac{d\lambda(w)}{dw} \lambda(u) \right). \end{aligned} \tag{4.22}$$

Before proving it, we state the following corollary.

Corollary 4.5.1.1. [28] Let be η^* defined by

$$\eta^* := Lie_{\frac{\partial}{\partial a_{n+1}}} g^* : \tag{4.23}$$

Then, the coefficient of η^* in the coordinates $a_2, a_3, .., a_{n+1}$ is recovered by the formula

$$\eta^*(da_i, da_j) := \frac{\partial g^*(da_i, da_j)}{\partial a_{n+1}} = -(2n + 4 - i - j)\, a_{i+j-n-3}. \tag{4.24}$$

The metric η^* is called Saito metric due to Saito, who is the first one that defined such metric [28].

Proof.

$$\sum_{i,j=0}^{n+1} \frac{\partial g^*(da_i, da_j)}{\partial a_{n+1}} u^{n+1-i} w^{n+1-j} = \frac{1}{u-w} \left(\frac{d\lambda(u)}{du} - \frac{d\lambda(w)}{dw} \right) \tag{4.25}$$

Note that

$$\lambda(p) = \sum_{i=0}^{n+1} a_i p^{n+1-i}$$
$$\frac{\partial\lambda(p)}{dp} = \sum_{i=0}^{n+1}(n+1-i)a_i p^{n-i} \tag{4.26}$$

Then substituting (4.26) in (4.25)

$$\begin{aligned}\sum_{i,j=0}^{n+1}\frac{\partial g^*(da_i,da_j)}{\partial a_{n+1}}u^{n+1-i}w^{n+1-j} &= \frac{1}{u-w}[\sum_{i=0}^{n+1}(n+1-i)a_i(u^{n-i}-w^{n-i})] \\ &= \sum_{i=0}^{n+1}\sum_{j=0}^{n-1-i}(n+1-i)a_i u^{n-1-i-j}w^j \\ &= \sum_{i=0}^{n+1}\sum_{j=0}^{n-1}(n+1-i+j)a_{i-j}u^{n-1-i}w^j \\ &= \sum_{i=2}^{n+3}\sum_{j=2}^{n+1}(2n+4-i-j)a_{i+j-n-3}u^{n+1-i}w^{n+1-j}\end{aligned} \tag{4.27}$$

□

In order to prove the proposition 4.5.1 it will be necessary to prove some auxiliary lemmas. At first steep, it will be required to extended the metric $g^*(da_i, da_j)$ on the space $\mathbb{C}^n \oplus \mathbb{C} \ni (v_0, v_1, .., v_n, p)$, the extension goes as follows

$$\tilde{g}^* = \sum_{i=0}^{n+1}\frac{\partial}{\partial x_i}\otimes\frac{\partial}{\partial x_i} \tag{4.28}$$

and also we extended $\lambda(p)$ as

$$\tilde{\lambda}(p) = \prod_{i=0}^{n}(p-p_i) = p^{n+1} + p_1p^n + p_2p^{n-1} + ... + p_np + p_{n+1}, \tag{4.29}$$

by forgetting the condition $\sum_{i=0}^n v_i = 0$. Then, we have the following relation between $(p_1, p_2, p_3, .., p_n, p_{n+1})$ and $(a_2, a_3, .., a_n, a_{n+1})$

$$p_i(v_0, ..., v_n)|_{\sum_{i=0}^n v_i=0} = a_i(v_0, ..., v_n). \tag{4.30}$$

Then, we can state

Lemma 4.5.2. [**28**] Let the extended intersection form (4.28), and the extended generating function (4.29). Then, the following identity holds

$$\sum_{i,j=0}^{n+1}\tilde{g}^*(dp_i,dp_j)u^{n+1-i}w^{n+1-j} = \frac{1}{u-w}[\tilde{\lambda}(v)\frac{d\tilde{\lambda}(u)}{du} - \frac{d\tilde{\lambda}(w)}{dw}\tilde{\lambda}(u)] \tag{4.31}$$

PROOF.

$$
\begin{aligned}
\sum_{i,j=0}^{n+1} \tilde{g}^*(dp_i, dp_j) u^{n+1-i} w^{n+1-j} &= \tilde{g}^*(d\tilde{\lambda}(u), d\tilde{\lambda}(w)) \\
&= \sum_{i,j=0}^{n+1} \tilde{g}^*(dp_i, dp_j) \frac{\partial \tilde{\lambda}(u)}{\partial p_i} \frac{\partial \tilde{\lambda}(w)}{\partial p_j} \\
&= \sum_{i,j=0}^{n+1} \frac{1}{p - v_i} \frac{1}{p - v_j} \tilde{\lambda}(u) \tilde{\lambda}(w) \\
&= \frac{1}{u - w} \sum_{i,j=0}^{n+1} [\frac{1}{p - v_i} - \frac{1}{p - v_j}] \tilde{\lambda}(u) \tilde{\lambda}(w) \\
&= \frac{1}{u - w} [\tilde{\lambda}(w) \frac{d\tilde{\lambda}(u)}{du} - \frac{d\tilde{\lambda}(w)}{dw} \tilde{\lambda}(u)]
\end{aligned} \tag{4.32}
$$

□

LEMMA 4.5.3. [28] Let the extended intersection form (4.28), and the extended generating function be given by (4.29). Then, the following identity holds

$$
\tilde{g}^*(dp_1, dp_i) = -(n + 2 - i) p_{i-1} \tag{4.33}
$$

PROOF.

$$
\begin{aligned}
\tilde{g}^*(dp_1, d\tilde{\lambda}(v)) = \sum_{l,m=0}^{n+1} \frac{\partial p_1}{\partial v_l} \frac{\partial \tilde{\lambda}(w)}{\partial v_m} &= \sum_{m=0}^{n+1} \frac{-1}{p - v_m} \tilde{\lambda}(w) = \tilde{\lambda}'(w) \\
= \sum_{m=0}^{n+1} -(n + 1 - m) p_i w^{n-m} &= \sum_{m=0}^{n+1} \tilde{g}^*(dp_1, dp_i) w^{n+1-m} \\
= \sum_{m=1}^{n+2} -(n + 2 - m) p_i w^{n+1-m}&
\end{aligned} \tag{4.34}
$$

□

LEMMA 4.5.4. [28] Let be P, Q two polynomials in $\Omega \ni (v_0, v_1, .., v_n)$, and $\tilde{P}, \tilde{Q}$ two polynomials with variable in $\hat{\Omega} = \Omega \oplus \mathbb{C} \ni (v_0, v_1, .., v_n, p)$ such that

$$
\begin{aligned}
\tilde{P} &= P + p_1 f_1, \\
\tilde{Q} &= Q + p_1 f_2,
\end{aligned}
$$

where f_1, f_2 are polynomial functions in $p_1, p_2, .., p_{n+1}$. Then,

$$
\tilde{g}^*(d\tilde{P}, d\tilde{Q}) = g^*(dP, dQ) + p_1 h_1 h_2.
$$

where h_1, h_2 are polynomial functions in $p_1, p_2, .., p_{n+1}$.

PROOF. We have the orthogonal decomposition of $d\tilde{P}$

$$d\tilde{P} = \sum_{k=0}^{n+1} \frac{\partial \tilde{P}}{\partial v_k} dv_k$$
$$= \eta_{\tilde{P}} + \frac{1}{n+1}\tilde{g}^*(dp_1, d\tilde{P})dp_1$$

where

$$\eta_{\tilde{P}} = \sum_{k=0}^{n+1} \frac{\partial \tilde{P}}{\partial v_k} d\left(v_k - \frac{1}{n+1}p_1\right)$$

Similarly for $d\tilde{Q}$

$$d\tilde{Q} = \sum_{k=0}^{n+1} \frac{\partial \tilde{Q}}{\partial v_k} dv_k$$
$$= \eta_{\tilde{Q}} + \frac{1}{n+1}\tilde{g}^*(dp_1, d\tilde{Q})dp_1$$

where

$$\eta_{\tilde{Q}} = \sum_{k=0}^{n+1} \frac{\partial \tilde{Q}}{\partial v_k} d\left(v_k - \frac{1}{n+1}p_1\right)$$

Since, $\tilde{g}^*(dp_1, dp_1) = n+1$,

$$\begin{aligned}\tilde{g}^*(d\tilde{P}, d\tilde{Q}) &= \tilde{g}^*(\eta_{\tilde{P}}, \eta_{\tilde{Q}}) + \frac{1}{n+1}\tilde{g}^*(dp_1, d\tilde{Q})\tilde{g}^*(\eta_{\tilde{P}}, dp_1) \\ &+ \frac{1}{n+1}\tilde{g}^*(dp_1, d\tilde{P})\tilde{g}^*(\eta_{\tilde{Q}}, dp_1) + \frac{n+1}{(n+1)^2}\tilde{g}^*(dp_1, d\tilde{Q})\tilde{g}^*(d\tilde{P}, dp_1) \\ &= \tilde{g}^*(\eta_{\tilde{P}}, \eta_{\tilde{Q}}) + \frac{1}{n+1}\tilde{g}^*(dp_1, d\tilde{Q})\tilde{g}^*(d\tilde{P}, dp_1)\end{aligned}$$

□

PROOF. **of proposition 4.5.1** Note that

$$a_i = p_i - \left(\frac{n+2-i}{n+1}\right)p_1 p_{i-1}, \quad i = 2, .., n+1.$$

Then, applying the lemmas 4.5.4, 4.5.3, we have

$$\begin{aligned}
\tilde{g}^*(da_i, da_j) &= \tilde{g}^*(dp_i, dp_j) - \left(\frac{n+2-j}{n+1}\right) p_1 \tilde{g}^*(dp_i, dp_{j-1}) - \left(\frac{n+2-j}{n+1}\right) p_{j-1} \tilde{g}^*(dp_i, dp_1) \\
&- \left(\frac{n+2-i}{n+1}\right) p_1 \tilde{g}^*(dp_j, dp_{i-1}) - \left(\frac{n+2-i}{n+1}\right) p_{i-1} \tilde{g}^*(dp_j, dp_1) \\
&+ \left(\frac{n+2-i}{n+1}\right)\left(\frac{n+2-j}{n+1}\right) p_1^2 \tilde{g}^*(dp_{i-1}, dp_{j-1}) \\
&+ \left(\frac{n+2-i}{n+1}\right)\left(\frac{n+2-j}{n+1}\right) p_1 p_{j-1} \tilde{g}^*(dp_1, dp_{i-1}) \\
&+ \left(\frac{n+2-i}{n+1}\right)\left(\frac{n+2-j}{n+1}\right) p_1 p_{i-1} \tilde{g}^*(dp_1, dp_{j-1}) \\
&+ \left(\frac{n+2-i}{n+1}\right)\left(\frac{n+2-j}{n+1}\right) p_{i-1} p_{j-1} \tilde{g}^*(dp_1, dp_1) \\
&= \tilde{g}^*(dp_i, dp_j) - \left(\frac{n+2-i}{n+1}\right)\left(\frac{n+2-j}{n+1}\right) p_{i-1} p_{j-1} + p_1 h,
\end{aligned}$$

where h is some polynomial in the variables $p_1, p_2, .., p_{n+1}$. Then,

(4.35)

$$\sum_{i,j=0}^{n+1} \tilde{g}^*(da_i, da_j) u^{n+1-i} w^{n+1-j} = -\frac{1}{n+1} \frac{d\lambda(w)}{dw} \frac{d\lambda(u)}{du} + \frac{1}{u-v}\left(\lambda(w)\frac{d\lambda(u)}{du} - \frac{d\lambda(w)}{dw}\lambda(u)\right) + p_1 h.$$

□

4.6. Flat coordinates of the Saito metric η

Step 5:

This section will be dedicated to prove that the Saito metric η is flat and non degenerate. This fact, implies the existence of the hypothesis of the lemma 4.4.1. In practice, we will construct the flat coordinates of the Saito metric η as follows.

Let be $t^1, t^2, .., t^n$ given by the following generating function

$$p(k) = k - \frac{1}{n+1}\left(\frac{t^n}{k} + \frac{t^{n-1}}{k^2} + .. + \frac{t^2}{k^{n-2}} + \frac{t^1}{k^n}\right) + O\left(\frac{1}{k^{n+1}}\right) \tag{4.36}$$

defined by the following condition

$$p(k)^{n+1} + a_2 p(k)^{n-1} + a_3 p(k)^{n-2} + ... + a_n p(k) + a_{n+1} = k^{n+1}$$

Lemma 4.6.1. The functions $t^1, t^2, .., t^n$ be defined in (4.36) can be obtained by the formula

$$t^\alpha = \frac{n+1}{n+1-\alpha} \operatorname*{res}_{p=\infty}\left(\lambda^{\frac{n+1-\alpha}{n+1}}(p) dp\right). \tag{4.37}$$

Proof. Consider the integration by parts

$$\frac{n+1}{n+1-\alpha} \int \left(\lambda^{\frac{n+1-\alpha}{n+1}}(p) dp\right) = p\lambda^{\frac{n+1-\alpha}{n+1}} - \int p \lambda^{\frac{-\alpha}{n+1}} d\lambda. \tag{4.38}$$

Lemma proved.

□

LEMMA 4.6.2. The functions $t^1, t^2, .., t^n$ be defined in (4.36) can be obtained by the formula

$$t^\alpha = -\operatorname*{res}_{\lambda=\infty}\left(p(\lambda)\lambda^{\frac{-\alpha}{n+1}}d\lambda\right). \tag{4.39}$$

PROOF. Let $k = \lambda^{\frac{1}{n+1}}$, then

$$\begin{aligned} p(\lambda)\lambda^{\frac{-\alpha}{n+1}}d\lambda &= \left(k - \frac{1}{n+1}\left(\frac{t^n}{k} + \frac{t^{n-1}}{k^2} + .. + \frac{t^2}{k^{n-2}} + \frac{t^1}{k^n}\right) + O\left(\frac{1}{k^{n+1}}\right)\right) k^{-\alpha}\,(n+1)\,k^n dk \\ &= \left((n+1)k^{n+1-\alpha} - \sum_{\beta=1}^{n} \frac{t^\beta}{k^{1-\alpha-\beta}} + O\left(\frac{1}{k^{\alpha+1}}\right)\right) dk. \end{aligned}$$

Hence, the residue is different from 0, when $\alpha = \beta$, resulting in this way the desired result.

□

LEMMA 4.6.3. [28] Let the functions $t^1, t^2, .., t^n$ be defined in (4.36), then

$$t^\alpha = \frac{n+1}{n+1-\alpha}\left(1 + A_{n+2-\alpha}\right)^{\frac{n+1-\alpha}{n+1}}, \tag{4.40}$$

where

$$\begin{aligned} (1 + A_i)^{\frac{n+1-\alpha}{n+1}} &= \sum_{d=0}^{\infty} \binom{\frac{n+1-\alpha}{n+1}}{k} A_i^d, \\ A_i^d &= \sum_{i_1+i_2+..+i_d=i} a_{i_1}...a_{i_d}. \end{aligned} \tag{4.41}$$

PROOF.

$$
\begin{aligned}
t^\alpha &= \frac{n+1}{n+1-\alpha}\operatorname*{res}_{p=\infty}\left(\lambda^{\frac{n+1-\alpha}{n+1}}(p)dp\right)\\
&= \frac{n+1}{n+1-\alpha}\operatorname*{res}_{p=\infty}\left(p^{n+1}+a_2p^{n-1}+a_3p^{n-2}+..+a_np+a_{n+1}\right)^{\frac{n+1-\alpha}{n+1}}dp\\
&= \frac{n+1}{n+1-\alpha}\operatorname*{res}_{p=\infty}p^{n+1-\alpha}\left(1+\frac{a_2}{p^2}+\frac{a_3}{p^3}+..+\frac{a_n}{p^n}+\frac{a_{n+1}}{p^{n+1}}\right)^{\frac{n+1-\alpha}{n+1}}dp\\
&= \frac{n+1}{n+1-\alpha}\operatorname*{res}_{p=\infty}p^{n+1-\alpha}\sum_{d=0}^{\infty}\binom{\frac{n+1-\alpha}{n+1}}{d}\left(\frac{a_2}{p^2}+\frac{a_3}{p^3}+..+\frac{a_n}{p^n}+\frac{a_{n+1}}{p^{n+1}}\right)^{d}dp\\
&= \frac{n+1}{n+1-\alpha}\operatorname*{res}_{p=\infty}p^{n+1-\alpha}\sum_{d=0}^{\infty}\binom{\frac{n+1-\alpha}{n+1}}{d}\sum_{j_1+..+j_n=d}\frac{d!}{j_1!j_2!..j_n!}\left(\frac{a_2}{p^2}\right)^{j_1}\left(\frac{a_3}{p^3}\right)^{j_2}..\left(\frac{a_n}{p^n}\right)^{j_{n-1}}\left(\frac{a_{n+1}}{p^{n+1}}\right)^{j_n}dp\\
&= \frac{n+1}{n+1-\alpha}\operatorname*{res}_{p=\infty}p^{n+1-\alpha-2j_1-3j_2-..-(n+1)j_n}\sum_{d=0}^{\infty}\binom{\frac{n+1-\alpha}{n+1}}{d}\sum_{j_1+..+j_n=d}\frac{d!}{j_1!j_2!..j_n!}a_2^{j_1}..a_{n+1}^{j_n}dp\\
&= \frac{n+1}{n+1-\alpha}\sum_{d=0}^{\infty}\binom{\frac{n+1-\alpha}{n+1}}{d}\sum_{\substack{j_1+..+j_n=d\\ 2j_1+3j_2+..+(n+1)j_n=n+2-\alpha}}\frac{d!}{j_1!j_2!..j_n!}a_2^{j_1}..a_{n+1}^{j_n}\\
&= \frac{n+1}{n+1-\alpha}\sum_{d=0}^{\infty}\binom{\frac{n+1-\alpha}{n+1}}{d}\sum_{i_1+..+i_d=n+2-\alpha}a_{i_1}..a_{i_d}\\
&= \frac{n+1}{n+1-\alpha}\sum_{d=0}^{\infty}\binom{\frac{n+1-\alpha}{n+1}}{d}A^d_{n+2-\alpha}\\
&= \frac{n+1}{n+1-\alpha}\left(1+A_{n+2-\alpha}\right)^{\frac{n+1-\alpha}{n+1}}.
\end{aligned}
$$

$\square$

LEMMA 4.6.4. [28] Let the functions $a_2, a_3, .., a_n, a_{n+1}$ be defined in (4.2), then

$$
\begin{aligned}
a_i &= \frac{n+1}{n+1-i}\left(\frac{1}{1-\frac{T_i}{n+1}}\right)^{n+1-i}, \quad i\neq n+1,\\
a_{n+1} &= (n+1)\log\left(1-\frac{T_{n+1}}{n+1}\right),
\end{aligned}
\tag{4.42}
$$

where

$$
T_i^d = \sum_{i_1+..i_d=i} t_{i_1}...t_{i_d}. \tag{4.43}
$$

LEMMA 4.6.5. Let T_i^d be defined in (4.43), then

$$
\begin{aligned}
\frac{\partial T_i^d}{\partial t_j} &= dT_{i-j}^{d-1},\\
T_i^d &= \sum_{i_1+..i_m=i} T_{i_1}^{d_1}...T_{i_m}^{d_m}, \quad d=d_1+..+d_m.
\end{aligned}
\tag{4.44}
$$

Here $T_i^0=\delta_{i0}$.

LEMMA 4.6.6. [28] Let

$$g_k(T_i) = \sum_{d=0}^{\infty} g_{ik} T_i^d \tag{4.45}$$

a formal power series in T_i be defined in (4.43), then

$$\begin{aligned} \frac{\partial g_k(T_i)}{\partial t_j} &= g_k'(T_{i-j}), \\ g_1(T_i)...g_m(T_i) &= \sum_{i_1+..+i_m=i} g_1(T_{i_1})..g_m(T_{i_m}). \end{aligned} \tag{4.46}$$

Here the symbol $'$ in the right hand side of the first equation of (4.46) means derivative with respect the formal variable T_i.

THEOREM 4.6.7. [28] Let $(t^1, t^2, .., t^n)$ be defined in (4.36), and η^* be defined in (4.23). Then,

$$\eta^*(dt^\alpha, dt^{n+3-\beta}) = -(n+1)\delta_{\alpha\beta}. \tag{4.47}$$

PROOF. Consider the metric η^* in the coordinates $(t^1, t^2, .., t^n)$

$$\eta^*(da_i, da_j) = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \frac{\partial a_i}{\partial t_\alpha}\frac{\partial a_j}{\partial t_\beta}\eta^*(dt^\alpha, dt^\beta) \tag{4.48}$$

Moreover, consider $\frac{\partial a_i}{\partial t_\alpha}$, and use the first line of the equation (4.46)

$$\frac{\partial a_i}{\partial t_\alpha} = \left(\frac{1}{1-\frac{T_{i-\alpha}}{n+1}}\right)^{n+2-i}.$$

Then,

$$\sum_{\alpha=1}^{n} \frac{\partial a_i}{\partial t_\alpha}\frac{\partial a_{n+3-j}}{\partial t_{n+3-\alpha}} = \frac{1}{\left(1-\frac{T_{i-\alpha}}{n+1}\right)^{n+2-i}}\frac{1}{\left(1-\frac{T_{\alpha-j}}{n+1}\right)^{j-1}}. \tag{4.49}$$

Using the second of the equation (4.46) in (4.49)

$$\begin{aligned} \sum_{\alpha=1}^{n} \frac{\partial a_i}{\partial t_\alpha}\frac{\partial a_{n+3-j}}{\partial t_{n+3-\alpha}} &= \frac{1}{\left(1-\frac{T_{i-j}}{n+1}\right)^{n+1-i+j}} \\ &= \left(\frac{n+1-i+j}{n+1}\right) a_{i-j} \end{aligned} \tag{4.50}$$

Substituting (4.50) in (4.48)

$$\sum_{\alpha=1}^{n} \frac{\partial a_i}{\partial t_\alpha}\frac{\partial a_{n+3-j}}{\partial t_{n+3-\alpha}} = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \frac{\partial a_i}{\partial t_\alpha}\frac{\partial a_{n+3-j}}{\partial t_{n+3-\beta}}\delta_{\alpha\beta}. \tag{4.51}$$

On another hand, using equation (4.24), we have

$$\begin{aligned}\eta^*(da_i, da_{n+3-j}) &= -(n+1-i+j)a_{i-j} \\ &= \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \frac{\partial a_i}{\partial t_\alpha}\frac{\partial a_{n+3-j}}{\partial t_{n+3-\beta}}\eta^*(dt^\alpha, dt^{n+3-\beta}) \\ &= (n+1)\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \frac{\partial a_i}{\partial t_\alpha}\frac{\partial a_{n+3-j}}{\partial t_{n+3-\beta}}\delta_{\alpha\beta}\end{aligned}$$

Then, we obtain

$$\eta^*(dt^\alpha, dt^{n+3-\beta}) = -\,(n+1)\,\delta_{\alpha\beta}.$$

□

COROLLARY 4.6.7.1. The Saito metric η is non degenerate.

4.7. The action of Euler vector in the geometric data

Step 6:

We realise the geometric data of the orbit space of A_n as eigenfunctions of the Euler vector field. The eigenvalues of the Euler vector field introduce a notion of degree in the geometric data of the orbit space of A_n.

DEFINITION 4.7.1. A function f is quasi homogeneous of degree d if it is an eigenfunction of the Euler vector field (4.7) with eigenvalue d, i.e.

$$E(f) = df.$$

LEMMA 4.7.1. Let $a_2, .., a_{n+1}$ be defined in (4.2), and E theEuler vector field be defined in (4.7), then polynomials a_i has degree d_i, i.e

$$E(a_i) = d_i a_i \tag{4.52}$$

PROOF.

$$E(a_i) = \sum_{j=2}^{n+1} d_j a_j \frac{\partial a_i}{\partial a_i} = d_i a_i. \tag{4.53}$$

□

LEMMA 4.7.2. [11] Let $a_2, .., a_{n+1}$ be defined in (4.2), and E the Euler vector field be defined in (4.7), then coefficients of the metric g^{ij}, and its Christoffel symbols in the coordinates $a_2, .., a_{n+1}$ are A_n invariant polynomials, furthermore,

$$\begin{aligned}E(g^{ij}) &= (d_i + d_j - 2)g^{ij} \\ E(\Gamma_k^{ij}) &= (d_i + d_j - d_k - 2)\Gamma_k^{ij}.\end{aligned} \tag{4.54}$$

Corollary 4.7.2.1. [11] The functions $g^{ij}(a)$ and $\Gamma_k^{ij}(a)$ are at most linear on a_{n+1}.

Corollary 4.7.2.2. [11] There exist homogeneous polynomials $t^1, t^2, .., t^n$, with degrees $d_1, ..., d_n$ respectively such that the matrix

$$\eta^{\alpha\beta} = \delta_{\alpha+\beta,n+1}. \tag{4.55}$$

Moreover, the Euler vector field in this coordinates becomes

$$E = \sum_{\alpha}^{n} d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha}. \tag{4.56}$$

Corollary 4.7.2.3. [11] The orbit space of the group A_n carries a flat pencil metric $g^{ij}(a)$, $\eta^{ij}(a)$, with $\eta^{ij}(a)$ is polynomialy invertible globally on the orbit space of A_n.

4.8. Construction of WDVV solution

Step 7:

The main aim of this section is to extract a WDVV equation from the data of the group A_n.

The following lemma shows that flat pencil structure is almost the same as Dubrovin Frobenius structure.

Lemma 4.8.1. [11] For a flat pencil metric $g^{\alpha\beta}, \eta^{\alpha\beta}$ there exist a vector field $f = f^\gamma \partial_\gamma$ such that the tensor

$$\Delta^{\alpha\beta\gamma} = \eta^{\alpha\delta}\Gamma_{\delta,g}^{\beta\gamma} - g^{\alpha\delta}\Gamma_{\delta,\eta}^{\beta\gamma} \tag{4.57}$$

and the metric $g^{\alpha\beta}$ have the following form

$$\begin{aligned} \Delta^{\alpha\beta\gamma} &= \eta^{\alpha\mu}\eta^{\beta\nu}\partial_\mu\partial_\nu f^\gamma \\ g^{\alpha\beta} &= \eta^{\alpha\mu}\partial_\mu f^\beta + \eta^{\beta\nu}\partial_\nu f^\alpha + c\eta^{\alpha\beta} \end{aligned} \tag{4.58}$$

for some constant c. The vector field f should satisfy

$$\Delta_\epsilon^{\alpha\beta}\Delta_\delta^{\epsilon\gamma} = \Delta_\epsilon^{\alpha\gamma}\Delta_\delta^{\epsilon\beta} \tag{4.59}$$

where

$$\begin{aligned} \Delta_\gamma^{\alpha\beta} &= \eta_{\gamma\epsilon}\Delta^{\alpha\beta\epsilon} = \eta^{\alpha\mu}\partial_\mu\partial_\gamma f^\beta \\ (\eta^{\alpha\epsilon}g^{\beta\delta} &- g^{\alpha\epsilon}\eta^{\beta\delta})\partial_\epsilon\partial_\delta f^\gamma = 0 \end{aligned} \tag{4.60}$$

Conversely, for any metric $\eta^{\alpha\beta}$ and for f solution of the system (4.59) and (4.60) the metrics $\eta^{\alpha\beta}$ and $g^{\alpha\beta}$ form a flat pencil metric.

Lemma 4.8.2. [11] Let $t^1, t^2, .., t^n$ be defined in (4.36), then

$$\begin{aligned} g^{n\alpha} &= d_\alpha t^\alpha, \\ \Gamma_\beta^{n\alpha} &= (d_\alpha - 1)\delta_\beta^\alpha. \end{aligned} \tag{4.61}$$

Lemma 4.8.3. [11] Let $t^1, ..., t^n$ be the Saito flat coordinates on the space of orbits of a finite Coxeter group A_n and

(4.62)
$$\eta^{\alpha\beta} = \frac{\partial g^{\alpha\beta}}{\partial t^1}$$

be the correspondent constant Saito metric. Then there exists a quasihomogeneous polynomial F (t) of the degree $2(n+1)+2$ such that

(4.63)
$$g^{\alpha\beta} = \frac{(d_\alpha + d_\beta - 2)}{2}\eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu F.$$

The polynomial F(t) determines on the space of orbits a Dubrovin Frobenius structure

(4.64)
$$c^\gamma_{\alpha\beta} = \eta^{\gamma\lambda}\partial_\lambda\partial_\beta\partial_\alpha F.$$

with the structure constants the unity

(4.65)
$$e = \frac{\partial}{\partial t^1}$$

and the invariant inner product η.

4.9. Mirror symmetry between the orbit space of A_n and the Hurwitz space $H_{0,n}$

Theorem 4.9.1. The Dubrovin Frobenius structure of the orbit space A_n is isomorphic as Dubrovin Frobenius manifold to the Hurwitz space $H_{0,n}$.

Proof. Both the orbit space A_n and the Hurwitz space $H_{0,n}$ have the same intersection form, Euler vector, unit vector field. From this data, one can reconstruct the WDVV solution by using the relation

(4.66)
$$F^{\alpha\beta} = \eta^{\alpha\alpha'}\eta^{\beta\beta'}\frac{\partial^2 F}{\partial t^{\alpha'}\partial t^{\beta'}} = \frac{g^{\alpha\beta}}{\deg g^{\alpha\beta}}.$$

Theorem proved. ☐

Remark 4.9.1. The Dubrovin Frobenius structure in the orbit space of A_n and in the Hurwitz space $H_{0,n}$ is globally well-defined. i.e. the structure constant (4.64), the Saito metric η (4.23) , the unit vector field (4.8) , and the Euler vector field (4.7) are globally well-defined. However, the flat coordinates of the intersection form (4.6) is multivalued due to the fact that the intersection form (4.6) is not everywhere non degenerate, see subsection 2.1.3 for details. Note that the monodromy of the flat coordinates of the intersection form (4.6) is exactly the group A_n. Moreover, we generate a n-sheeted covering over the orbit space of A_n by fixing a chart in the orbit space of A_n, i.e. choosing a representative to each orbit, and after that we act the group A_n in this space.Therefore, the flat coordinates of the intersection form are globally well defined in each sheet of this covering, and fixing a sheet solves the problem of the multivalueness of the flat coordinates of the intersection form.

Remark 4.9.2. The result of the Theorem 4.9.1 implies that the orbit space of A_n is isomoprphic to the Hurwitz space $H_{0,n}$. In the remark 4.9.1, we defined a Dubrovin Frobenius manifold in each sheet of the associate covering over the orbit space of A_n, then, one could ask what happens in the Hurwitz space side. Indeed, the Hurwitz space $H_{0,n}$ is also associated with a n-sheeted covering, and we can fix sheet by choosing a root of λ near ∞, which is equivalent to fix a root of unity of $z^n = 1$.

CHAPTER 5

Review of Dubrovin Frobenius manifolds on the orbit space of $\mathscr{J}(A_n)$

This chapter is a summary of the work done in [**8**] and [**9**], which generalises the constructions done in chapter 4 for a suitable extension of the A_n group.

5.1. Ordinary Jacobi group $\mathscr{J}(A_n)$

Step 1:
The first step of the orbit space construction, as it was previously done for the A_n, is the definition of the desired group action. In this particular case, the definition of the $\mathscr{J}(A_n)$ action on $\mathbb{C} \oplus \mathbb{C}^n \oplus \mathbb{H}$, see [**8**] for details.

Consider the action of the group A_n on $(L^{A_n}, \langle , \rangle_{A_n})$ done in section 4.1. Then, consider the following group $L^{A_n} \times L^{A_n} \times \mathbb{Z}$ with the following group operation

$$\forall(\lambda, \mu, k), (\tilde{\lambda}, \tilde{\mu}, \tilde{k}) \in L^{A_n} \times L^{A_n} \times \mathbb{Z}$$
$$(\lambda, \mu, k) \bullet (\tilde{\lambda}, \tilde{\mu}, \tilde{k}) = (\lambda + \tilde{\lambda}, \mu + \tilde{\mu}, k + \tilde{k} + \langle \lambda, \tilde{\lambda} \rangle_{A_n})$$

Note that $<,>_{A_n}$ is invariant under A_n group, then A_n acts on $L^{A_n} \times L^{A_n} \times \mathbb{Z}$. Hence, we can take the semidirect product $A_n \ltimes (L^{A_n} \times L^{A_n} \times \mathbb{Z})$ given by the following product.

$$\forall(w, \lambda, \mu, k), (\tilde{w}, \tilde{\lambda}, \tilde{\mu}, \tilde{k}) \in A_n \times L^{A_n} \times L^{A_n} \times \mathbb{Z}$$
$$(w, \lambda, \mu, k) \bullet (\tilde{w}, \tilde{\lambda}, \tilde{\mu}, \tilde{k}) = (w\tilde{w}, w\lambda + \tilde{\lambda}, w\mu + \tilde{\mu}, k + \tilde{k} + \langle \lambda, \tilde{\lambda} \rangle_{A_n})$$

Denoting $W(A_n) := A_n \ltimes (L^{A_n} \times L^{A_n} \times \mathbb{Z})$, we can define

Definition 5.1.1. The Jacobi group $\mathscr{J}(A_n)$ is defined as a semidirect product $W(A_n) \rtimes SL_2(\mathbb{Z})$. The group action of $SL_2(\mathbb{Z})$ on $W(A_n)$ is defined as

$$Ad_\gamma(w) = w$$
$$Ad_\gamma(\lambda, \mu, k) = (a\mu - b\lambda, -c\mu + d\lambda, k + \frac{ac}{2}\langle \mu, \mu \rangle_{A_n} - bc\langle \mu, \lambda \rangle_{A_n} + \frac{bd}{2}\langle \lambda, \lambda \rangle_{A_n})$$

for $(w, t = (\lambda, \mu, k)) \in W(\tilde{A}_n), \gamma \in SL_2(\mathbb{Z})$. Then the multiplication rule is given as follows

$$(w, t, \gamma) \bullet (\tilde{w}, \tilde{t}, \tilde{\gamma}) = (w\tilde{w}, t \bullet Ad_\gamma(w\tilde{t}), \gamma\tilde{\gamma})$$

Recall the following identification $\mathbb{Z}^n \cong L^{A_n}, \mathbb{C}^n \cong L^{A_n} \otimes \mathbb{C}$ done in 4.1.

Then the action of Jacobi group $\mathscr{J}(A_n)$ on $\Omega := \mathbb{C} \oplus \mathbb{C}^n \oplus \mathbb{H}$ is given as follows

PROPOSITION 5.1.1. [8] The group $\mathscr{J}(A_n) \ni (w,t,\gamma)$ acts on $\Omega := \mathbb{C} \oplus \mathbb{C}^n \oplus \mathbb{H} \ni (u,v,\tau)$ as follows:

$$\begin{aligned}
&w(u,v,\tau) = (u, wv, \tau),\\
&t(u,v,\tau) = (u - \langle \lambda, v\rangle_{A_n} - \frac{1}{2}\langle \lambda, \lambda\rangle_{A_n}\tau, v + \lambda\tau + \mu, \tau),\\
&\gamma(u,v,\tau) = (u + \frac{c\langle v,v\rangle_{A_n}}{2(c\tau+d)}, \frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}).
\end{aligned} \tag{5.1}$$

5.2. Jacobi forms of $\mathscr{J}(A_n)$

Step 2:

The main goal of this section is construct notion of invariant ring which generalise the symmetric polynomials in chapter 4. The ring of invariant suitable for Jacobi groups is called ring of Jacobi forms. This notion was first defined in [18] for the Jacobi group A_1, and further generalise in [34]. In [8] and [9], Jacobi forms were used in the context of Dubrovin Frobenius manifolds.

DEFINITION 5.2.1. [8] The weak Jacobi forms of $\mathscr{J}(A_n)$ of weight k, and index m are functions on $\Omega = \mathbb{C} \oplus \mathbb{C}^n \oplus \mathbb{H} \ni (u,v,\tau)$ which are holomorphic on (u,v,τ) and satisfy

$$\begin{aligned}
&\varphi\left(w(u,v,\tau)\right) = \varphi(u,v,\tau), \quad A_n \text{ invariant condition}\\
&\varphi(t(u,v,\tau)) = \varphi(u,v,\tau)\\
&\varphi(\gamma(u,v,\tau)) = (c\tau+d)^{-k}\varphi(u,v,\tau)\\
&E\varphi(u,v,\tau) := \frac{1}{2\pi i}\frac{\partial}{\partial u}\varphi(u,v,\tau) = m\varphi(u,v,\tau)
\end{aligned} \tag{5.2}$$

Moreover,

(1) φ is locally bounded functions on v as $\Im(\tau) \mapsto +\infty$ (weak condition).

The space of Invariant functions of $\mathscr{J}(A_n)$ of weight k, and index m is denoted by $J^{A_n}_{k,m}$.

DEFINITION 5.2.2. $J^{\mathscr{J}(A_n)}_{\bullet,\bullet} = \bigoplus_{k,m} J^{A_n}_{k,m}$.

REMARK 5.2.1. The condition $E\varphi(u,v,\tau) = m\varphi(u,v,\tau)$ implies that $\varphi(u,v,\tau)$ has the following form

$$\varphi(u,v,,\tau) = f(v,\tau)e^{2\pi i m u}$$

and the function $f(v,\tau)$ has the following transformation law

$$\begin{aligned}
&f(w(v,\tau)) = f(v,\tau),\\
&f(t(v,\tau)) = e^{-2\pi i m(\langle\lambda,v\rangle + \frac{\langle\lambda,\lambda\rangle}{2}\tau)}f(v,\tau),\\
&f(\gamma(v,\tau)) = (c\tau+d)^{-k}e^{2\pi i m(\frac{c\langle v,v\rangle}{(c\tau+d)})}f(v,\tau).
\end{aligned} \tag{5.3}$$

The functions $f(v,\tau)$ are more closely related with the definition of Jacobi form of Eichler-Zagier type [18]. The coordinate u works as kind of automorphic correction in this functions $f(v,\tau)$.

REMARK 5.2.2. Note that the ring of Jacobi forms of $\mathscr{J}(A_n)$ is not exactly invariant under the action of $\mathscr{J}(A_n)$. Indeed, the first two equations of (8.2) show that Jacobi forms are invariant under the first to action of (5.1), but the third equation of (8.2) gives a modular behaviour to the ring of Jacobi forms.

The main result of section is the following.

The ring of A_n invariant Jacobi forms is polynomial over a suitable ring $M_\bullet := J_{\bullet,0}^{\mathscr{J}(A_n)}$ on suitable generators $\varphi_0, \varphi_2, ..\varphi_{n+1}$.
Before state precisely the theorem, I will define the objects $M_\bullet, \varphi_0, \varphi_2, ..\varphi_{n+1}$.

The ring $M_\bullet := J_{\bullet,0}^{\mathscr{J}(A_n)}$ is the space of Jacobi forms of index 0, by definition.

LEMMA 5.2.1. The sub-ring $J_{\bullet,0,}^{\mathscr{J}(A_n)}$ is equal to $M_\bullet := \bigoplus M_k$, where M_k is the space of modular forms of weight k for the full group $SL_2(\mathbb{Z})$.

PROOF. Using the Remark 5.2.1 , we have that functions $\varphi(u,v,\tau) \in J_{\bullet,0}^{\mathscr{J}(A_n)}$ can not depend on u, then $\varphi(u,v,\tau) = \varphi(v,\tau)$. Moreover, for fixed τ the functions $v_i \mapsto \varphi(v,\tau))$ are holomorphic elliptic function for any i. Therefore, by Liouville theorem, these functions are constant in v. Then, $\varphi = \varphi(\tau)$ are standard holomorphic modular forms. □

At this stage, we can state the main theorem of this section

THEOREM 5.2.2. [34] The ring of A_n invariant Jacobi forms is free module of rank $n+1$ over the ring of modular forms, i.e. there exist Jacobi forms $\varphi_0, \varphi_2, .., \varphi_{n+1}$ such that

$$J_{\bullet,\bullet}^{\mathscr{J}(A_n)} = M_\bullet[\varphi_0, \varphi_2, .., \varphi_{n+1}].$$

An explicit base of generators were derived in [8]. The strategy done by Bertola in [8] was starting with a basic Jacobi form of A_n, which was constructed in [34] as

$$\varphi_{n+1} = e^{2\pi i u} \prod_{i=0}^{n} \frac{\theta_1(v_i,\tau)}{\theta_1'(0,\tau)}, \tag{5.4}$$

thereafter, Bertola defined a recursive operator to generate the other basic Jacobi forms. For this purpose, it is necessary to enlarge the domain of the Jacobi forms from $\mathbb{C} \oplus \mathbb{C}^n \oplus \mathbb{H} \ni (u, v_0, v_1, .., v_n, \tau)$ to $\mathbb{C} \oplus \mathbb{C}^{n+1} \oplus \mathbb{H} \ni (u, v_0, v_1, .., v_n, p, \tau)$. In addition, we lift the Jacobi forms defined in $\mathbb{C} \oplus \mathbb{C}^n \oplus \mathbb{H}$ to $\mathbb{C} \oplus \mathbb{C}^{n+1} \oplus \mathbb{H}$ as

$$\varphi(u, v_0, v_1, v_2, .., v_n, \tau) \mapsto \hat{\varphi}(p) := \varphi(u, v_0 + p, v_1 + p, .., v_n + p, \tau).$$

A convenient way to do computation in these extended Jacobi forms is by using the following coordinates

$$
\begin{aligned}
s &= u - \frac{g_1(\tau)p^2}{n+1}, \\
z_i &= v_i + p, \quad i = 1, .., n+1. \\
\tau &= \tau.
\end{aligned} \tag{5.5}
$$

The bilinear form $\langle v, v\rangle_{\tilde{A}_1}$ is extended to

$$
\langle (z_1, z_2, , ..z_n, z_{n+1}), (z_1, z_2, .., z_n, z_{n+1})\rangle_E = \sum_{i=1}^{n+1} z_i^2, \tag{5.6}
$$

or equivalently,

$$
\langle (v_0, v_1, .., v_n, p), (v_0, v_1, .., v_n, p)\rangle_E = \sum A_{ij} v_i v_j + (n+1)p^2. \tag{5.7}
$$

The action of the Jacobi group A_n in this extended space is

$$
\begin{aligned}
\hat{w}_E(u, v, p, \tau) &= (u, w(v), p, \tau), \\
t_E(u, v, p, \tau) &= \left(u - \langle \lambda, v\rangle_E - \frac{1}{2}\langle \lambda, \lambda\rangle_E \tau + k, v + p + \lambda\tau + \mu, \tau \right), \\
\gamma_E(u, v, p, \tau) &= \left(u + \frac{c\langle v, v\rangle_E}{2(c\tau + d)}, \frac{v}{c\tau + d}, \frac{p}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right).
\end{aligned} \tag{5.8}
$$

Proposition 5.2.3. [8] Let $\varphi \in J_{k,m}^{\mathscr{J}(A_n)}$, and $\hat{\varphi}$ the correspondent extended Jacobi form. Then,

$$
\frac{\partial}{\partial p}(\hat{\varphi})\bigg|_{p=0} \in J_{k-1,m}^{\mathscr{J}(A_n)}. \tag{5.9}
$$

Proof. (1) A_n**-invariant**

The vector field $\frac{\partial}{\partial p}$ in coordinates $s, z_1, z_2, .., z_n, z_{n+1}, \tau$ reads

$$
\frac{\partial}{\partial p} = \sum_{i=1}^{n+1} \frac{\partial}{\partial z_i} - \frac{2\pi i g_1(\tau)p}{n+1}\frac{\partial}{\partial u} \tag{5.10}
$$

Moreover, in the coordinates $s, z_1, z_2, .., z_n, z_{n+1}, \tau$ the A_n group acts by permutation on the variables $\{z_i\}$. Then

$$
\begin{aligned}
\frac{\partial}{\partial p}(\varphi(s, z_{i_1}, z_{i_2}, , ..z_{i_n}, z_{n+1}, \tau))\bigg|_{p=0} &= \left(\sum_{i=1}^{n+1} \frac{\partial}{\partial z_i}\right) \left(\varphi(s, z_{i_1}, z_{i_2}, , ..z_{i_n}, z_{i_{n+1}}, \tau)\right)\Big|_{p=0} \\
&= \left(\sum_{i=1}^{n+1} \frac{\partial \varphi}{\partial z_i}\right) \left(s, z_{i_0}, z_{i_1}, , ..z_{i_n}, z_{i_{n+1}}, \tau\right)\Big|_{p=0} \\
&= \left(\sum_{i=1}^{n+1} \frac{\partial \varphi}{\partial z_i}\right) (s, z_0, z_1, , ..z_n, z_{n+1}, \tau)\big|_{p=0}.
\end{aligned}
$$

(2) **Translation invariant**

$$\frac{\partial}{\partial p}\left(\varphi(u-\langle\lambda,v\rangle_E-\langle\lambda,\lambda\rangle_E, v+p+\lambda\tau+\mu,\tau)\right)\Big|_{p=0}$$
$$=\frac{\partial}{\partial p}\langle\lambda,v\rangle_E\Big|_{p=0}\varphi(u,v,\tau)+\frac{\partial\varphi}{\partial p}\left(u-\langle\lambda,v\rangle_{\tilde{A}_1}-\frac{1}{2}\langle\lambda,\lambda\rangle_{A_n}\tau+k, v+\lambda\tau+\mu,\tau\right)$$
$$=\frac{\partial\varphi}{\partial p}\left(u-\langle\lambda,v\rangle_{A_n}-\frac{1}{2}\langle\lambda,\lambda\rangle_{A_n}\tau+k, v+\lambda\tau+\mu,\tau\right)$$
$$=\frac{\partial\varphi}{\partial p}(u,v,\tau)\Big|_{p=0}$$

(3) $SL_2(\mathbb{Z})$ **equivariant**

$$\frac{\partial}{\partial p}\left(\varphi(u+\frac{c\langle v,v\rangle_E}{2(c\tau+d)},\frac{v}{c\tau+d},\frac{p}{c\tau+d},\frac{a\tau+b}{c\tau+d})\right)\Big|_{p=0}$$
$$=\frac{c}{2(c\tau+d)}\frac{\partial}{\partial p}\langle v,v\rangle_E\Big|_{p=0}\varphi(u,v,\tau)+\frac{1}{c\tau+d}\frac{\partial\varphi}{\partial p}\left(u+\frac{c\langle v,v\rangle_{A_n}}{2(c\tau+d)},\frac{v}{c\tau+d},\frac{p}{c\tau+d},\frac{a\tau+b}{c\tau+d})\right)$$
$$=\frac{1}{c\tau+d}\frac{\partial\varphi}{\partial p}\left(u+\frac{c\langle v,v\rangle_{A_n}}{2(c\tau+d)},\frac{v}{c\tau+d},\frac{p}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)$$
$$=\frac{1}{(c\tau+d)^k}\ \frac{\partial\varphi}{\partial p}(u,v,\tau)\Big|_{p=0}.$$

Then,

$$\frac{\partial\varphi}{\partial p}\left(u+\frac{c\langle v,v\rangle_E}{2(c\tau+d)},\frac{v}{c\tau+d},\frac{p}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)\Big|_{p-0}=\frac{1}{(c\tau+d)^{k-1}}\ \frac{\partial\varphi}{\partial p}(u,v,\tau)\Big|_{p=0}.$$

(4) **Index 1**

$$\frac{1}{2\pi i}\frac{\partial}{\partial u}\frac{\partial}{\partial p}\hat{\varphi}=\frac{1}{2\pi i}\frac{\partial}{\partial p}\frac{\partial}{\partial u}\hat{\varphi}=\frac{\partial}{\partial p}\hat{\varphi}.$$

□

COROLLARY 5.2.3.1. [8] The generators of the algebra $J_{\bullet,\bullet}^{\mathscr{J}(A_n)}$ are given by the following generating function

$$\left[e^{z\frac{\partial}{\partial p}}\left(e^{2\pi iu}\prod_{i=1}^{n+1}\frac{\theta_1(z_i)}{\theta_1'(0)}\right)\right]\Big|_{p=0}=\varphi_{n+1}+\varphi_n z+\varphi_{n-1}z^2+...+\varphi_2 z^{n-1}+\varphi_0 z^{n+1}+O(z^{n+2}) \tag{5.11}$$

where

$$\varphi_j:=\frac{\partial^{n+1-j}}{\partial p^{n+1-j}}\left(\hat{\varphi}_{n+1}\right)\Big|_{p=0} \tag{5.12}$$

PROOF. Acting $\frac{\partial}{\partial p}$ k times in φ_{n+1}, we have

$$\left[\frac{\partial^k}{\partial^k p}\left(e^{2\pi iu}\prod_{i=1}^{n+1}\frac{\theta_1(z_i)}{\theta_1'(0)}\right)\right]\Big|_{p=0}\in J_{-n-1+k,1}^{\mathscr{J}(A_n)}.$$

□

Corollary 5.2.3.2. [8] The generating function can be written as

$$\left[e^{z\frac{\partial}{\partial p}}\left(e^{2\pi iu}\prod_{i=1}^{n+1}\frac{\theta_1(z_i)}{\theta_1'(0)}\right)\right]\Bigg|_{p=0} = e^{-2\pi i(-u+(n+1)g_1(\tau)z^2)}\prod_{i=0}^{n}\frac{\theta_1(z-v_i)}{\theta_1'(0)}. \tag{5.13}$$

The next lemma is one of the main points of [8] and [9], because this lemma identify the orbit space of the group $\mathscr{J}(A_n)$ with the Hurwitz space $H_{1,n}$. This relationship is possible due to the construction of the generating function of the Jacobi forms of type A_n, which can be completed to be the Landau-Ginzburg superpotential of $H_{1,n}$ as follows

$$e^{-2\pi i(-u+(n+1)g_1(\tau)z^2)}\prod_{i=0}^{n}\frac{\theta_1(z-v_i)}{\theta_1'(0)} \mapsto e^{-2\pi iu}\frac{\prod_{i=0}^{n}\theta_1(z-v_i,\tau)}{\theta_1^{n+1}(v,\tau)}. \tag{5.14}$$

Lemma 5.2.4. There is a local biholomorphism between $\Omega/\mathscr{J}(A_n)$ and $H_{1,n,}$, i.e the space of elliptic functions with 1 pole of order n, and one simple pole.

Proof. The correspondence is realized by the map:

$$[(u,v_0,v_1,..,v_{n-1},\tau)] \longleftrightarrow \lambda(z) = e^{-2\pi iu}\frac{\prod_{i=0}^{n}\theta_1(z-v_i,\tau)}{\theta_1^{n+1}(v,\tau)} \tag{5.15}$$

Note that this map is well defined and one to one. Indeed:

(1) **Well defined**
Note that proof that the map does not depend on the choice of the representant of $[(u,v_0,v_1,..,v_{n-1},v_{n+1},\tau)]$ is equivalent to prove that the function (5.15) is invariant under the action of $\mathscr{J}(A_n)$. Indeed

(2) A_n **invariant**
The A_n group acts on (5.15) by permuting its roots, thus (5.15) remais invariant under this operation.

(3) **Translation invariant**
Recall that under the translation $v \mapsto v+m+n\tau$, the Jacobi theta function transform as [8], [33]:

$$\theta_1(v_i+\mu_i+\lambda_i\tau,\tau) = (-1)^{\lambda_i+\mu_i}e^{-2\pi i(\lambda_i v_i+\frac{\lambda_i^2}{2}\tau)}\theta_1(v_i,\tau) \tag{5.16}$$

Then substituting the transformation (5.16) into (5.15), we conclude that (5.15) remains invariant.

(4) $SL_2(\mathbb{Z})$ **invariant**
Under $SL_2(\mathbb{Z})$ action the following function transform as

$$\frac{\theta_1\left(\frac{v_i}{c\tau+d},\frac{a\tau+d}{c\tau+d}\right)}{\theta_1'\left(0,\frac{a\tau+d}{c\tau+d}\right)} = \exp\left(\frac{\pi icv_i^2}{c\tau+d}\right)\frac{\theta_1(v_i,\tau)}{\theta_1'(0,\tau)} \tag{5.17}$$

Then substituting the transformation (5.17) into (5.15), we conclude that (5.15) remains invariant.

(5) **Injectivity**

Two elliptic functions are equal if they have the same zeros and poles with multiplicity.

(6) **Surjectivity**

Any elliptic function can be written as rational functions of Weierstrass sigma function up to a multiplication factor [**33**]. By using the formula to relate Weierstrass sigma function and Jacobi theta function

$$\sigma(v_i,\tau)=\frac{\theta_1(v_i,\tau)}{\theta_1'(0,\tau)}\exp(-2\pi i g_1(\tau)v_i^2) \tag{5.18}$$

where $g_1(\tau)$ is a specific normalization of Eisenstein 2 [**8**]. Hence, we get the desire result.

□

At this stage, it is possible to show the relation between the Jacobi forms $\varphi_0,..,\varphi_{n+1}$ and the elementary symmetric polynomials $a_2,a_3,..,a_{n+1}$.

Proposition 5.2.5. [**8**] Let the Jacobi forms $\varphi_0,..,\varphi_{n+1}$ be defined by (5.11), then the lowest term of the Taylor expansion in the variables $\{v_i\}$ are given by

$$\begin{aligned}
\varphi_{n+1}&=a_{n+1}+O(||v||^n),\\
\varphi_n&=a_n+O(||v||^{n-1})\\
\varphi_{n-1}&=a_{n-1}+O(||v||^{n-2})\\
&.\\
&.\\
\varphi_2&=a_2+O(||v||^4)\\
\varphi_1&=a_1=0\\
\varphi_0&=a_0+O(||v||^2)
\end{aligned} \tag{5.19}$$

where $a_2,a_3,.,a_{n+1}$ are defined in (4.2)

Proof. Expanding the $\theta_1(v_i,\tau)$ in φ_{n+1}, we obtain

$$\varphi_{n+1}=a_{n+1}+O(||v||^n). \tag{5.20}$$

Applying the operator (5.10) in φ_{n+1} and using (4.3), we get the desired result.

□

Corollary 5.2.5.1. [**8**] The Jacobi forms $\varphi_0,..,\varphi_{n+1}$ be defined by (5.11) are algebraically independent.

PROOF. The Jacobi forms $\varphi_0, .., \varphi_{n+1}$ are algebraically independent iff $a_2, a_3, .., a_{n+1}$ are algebraically independent due to 5.19. The functions $a_2, a_3, .., a_{n+1}$ are algebraically independent due to Chevalley theorem 4.2.1.

□

COROLLARY 5.2.5.2. The functions $(\varphi_0, \varphi_2, .., \varphi_{n+1})$ obtained by the formula

$$\begin{aligned} \lambda &= e^{-2\pi i u} \frac{\prod_{i=0}^{n} \theta_1(z - v_i, \tau)}{\theta_1^{n+1}(z,\tau))} \\ &= \varphi_{n+1}\wp^{n-1}(z,\tau) + \varphi_n \wp^{n-2}(z,\tau) + ... + \varphi_2 \wp(z,\tau) + \varphi_0, \end{aligned} \tag{5.21}$$

are Jacobi forms of weight $0, -2, .., -n-1$ respectively, index 1.

PROOF. Let us prove each item separated.

(1) **A_n invariant, translation invariant**
The l.h.s of (5.21) are A_n invariant, and translation invariant by the lemma (5.2.4). Then, by the uniqueness of Laurent expansion of λ, we have that φ_i are A_n invariant, and translation invariant.

(2) **$SL_2(\mathbb{Z})$ equivariant**
The l.h.s of (5.21) are $SL_2(\mathbb{Z})$ invariant, but the Weierstrass functions of the r.h.s have the following transformation law

$$\wp^{(k-2)}(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k \wp^{(k-2)}(z,\tau). \tag{5.22}$$

Then, φ_k must have the following transformation law

$$\varphi_k(u + \frac{c\langle v, v\rangle_{A_n}}{2(c\tau+d)}, \frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}) = (c\tau+d)^{-k}\varphi_k(u,v,\tau). \tag{5.23}$$

(3) **Index** 1

$$\frac{1}{2\pi i}\frac{\partial}{\partial u}\lambda = \lambda. \tag{5.24}$$

Then

$$\frac{1}{2\pi i}\frac{\partial}{\partial u}\varphi_i = \varphi_i. \tag{5.25}$$

□

5.3. Intersection form, Unit vector field, Euler vector field and Bertola's reconstruction process

Step 3:
This section focus on the Bertola approach to construct a Dubrovin Frobenius structure on the orbit space of $\mathscr{J}(A_n)$. The first step of this process is to build the invariant $\mathscr{J}(A_n)$ invariant

sections : intersection form, Unit vector field and Euler vector field, and from this data, Bertola reconstruct the Dubrovin Frobenius structure of the Hurwitz space $H_{1,n}$.

Remark 5.3.1. $\mathscr{J}(A_n)$ invariant sections means invariant under the first to action of (5.1) and modular with respect the third action of (5.1).

Definition 5.3.1. [8] Let bilinear pairing be given in coordinates $(u, v_0, v_1, .., v_n, \tau)$ by

$$
\begin{aligned}
g &= \sum_{i=0}^{n} dv_i^2|_{\sum_{i=0}^n v_i=0} + 2dud\tau \\
&= \sum_{i,j} A_{ij} dv_i dv_j + 2dud\tau,
\end{aligned} \tag{5.26}
$$

where matrix A_{ij} is equal to the matrix g_{ij} in (4.5). The intersection form is given by

$$
g^* = \sum_{i,j} A_{ij}^{-1} \frac{\partial}{\partial v_i} \otimes \frac{\partial}{\partial v_j} + \frac{\partial}{\partial u} \otimes \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \tau} \otimes \frac{\partial}{\partial u}. \tag{5.27}
$$

Proposition 5.3.1. [8] The intersection form (5.27) is invariant under the first two actions of (5.1), and behaves a modular form of weight 2 under the last action of (5.1).

Note that (5.27) is modular with respect the $SL_2(\mathbb{Z})$, but this does not means that its coefficients on some coordinates have the same modular behaviour. Indeed, taking the coordinates $\varphi_0, \varphi_2, .., \varphi_{n+1}, \tau$, we have that if under the $SL_2(\mathbb{Z})$

$$
\varphi_i \mapsto \frac{\varphi_i}{(c\tau + d)^i},
$$

then,

$$
d\varphi_i \mapsto \frac{d\varphi_i}{(c\tau + d)^i} - \frac{ic\varphi_i d\tau}{(c\tau + d)^{i+1}}.
$$

Since (5.26) and (5.27) behaves as modular, the coefficient g_{ij} and g^{ij} have to transform in a non modular way to cancel the cancel the non modular contribution of $d\varphi_i$ and $\frac{\partial}{\partial \varphi_i}$. For this purpose, Bertola considered the following metric.

Lemma 5.3.2. [8] Let $\varphi_i \in J_{-k_i,m_i}^{A_n}$ and $\eta(\tau)$ the Dedekind eta function, then the metric given by

$$
\frac{1}{\eta^{2i+2j}} g^*(d\eta^{2i}\varphi_i, d\eta^{2j}\varphi_j) \frac{\partial}{\partial \varphi_i} (\eta^{2i}.) \otimes \frac{\partial}{\partial \varphi_j} (\eta^{2j}.) \tag{5.28}
$$

is invariant under the first two actions of (5.1), and behaves a modular form of weight 2 under the last action of (5.1).

Moreover, the coeffiecients of the metric (5.28) are given by

$$
\begin{aligned}
M(d\varphi_i, d\varphi_j) &:= \frac{1}{\eta^{2i+2j}} g^*(d\eta^{2i}\varphi_i, d\eta^{2j}\varphi_j) \\
&= g^*(d\varphi_i, d\varphi_j) - 4\pi i g_1(\tau)(k_i m_j + k_j m_i)\varphi_i \varphi_j,
\end{aligned} \tag{5.29}
$$

furthermore, $M(d\varphi_i, d\varphi_j) \in J_{-k_i-k_j+2,m_i+m_j}^{A_n}$.

PROOF. The metric (5.28) is is invariant under the first two actions of (5.1), because proposition 5.3.1, and because η do not change under this action. The equivariance with respect the $SL_2(\mathbb{Z})$ follows again from proposition 5.3.1, and from the fact that the transformation laws of η get canceled.

The equation (5.29) follows from the chain rule, from the identity

$$\frac{\eta'}{\eta}(\tau) = g_1(\tau). \tag{5.30}$$

$\square$

The ring of Jacobi forms of $\mathscr{J}(A_n)$ give us the data to build the remaining part of the Dubrovin Frobenius structure. Indeed:

DEFINITION 5.3.2. The Euler vector field with respect the orbit space $\mathscr{J}(\tilde{A}_n)$ is defined by the last equation of (5.2), i.e

$$E := -\frac{1}{2\pi i}\frac{\partial}{\partial u} \tag{5.31}$$

DEFINITION 5.3.3. The Unit vector field with respect the orbit space $\mathscr{J}(A_n)$ is the vector associated to the invariant coordinate φ_0 defined in (5.21) , i.e

$$e := \frac{\partial}{\partial \varphi_0}. \tag{5.32}$$

At this point, we can state the main result of [9].

THEOREM 5.3.3. [9] The Dubrovin Frobenius structure of the orbit space of $\mathscr{J}(A_n)$ is locally isomorphic as Dubrovin Frobenius manifold to the Hurwitz space $H_{1,n}$.

PROOF. Both the orbit space $\mathscr{J}(A_n)$ and the Hurwitz space $H_{1,n}$ has the same intersection form, Euler vector, unit vector field. From this data, one can reconstruct the WDVV solution by using the relation

$$F^{\alpha\beta} = \frac{g^{\alpha\beta}}{\deg g^{\alpha\beta}}. \tag{5.33}$$

Theorem proved, see details in [9]. $\square$

REMARK 5.3.2. Note that even though the Hurwitz space H_1, n is locally isomorphic to the orbit space of $\mathscr{J}(A_n)$, this does not mean that the two constructions are completely equivalent. The Dubrovin Frobenius structure on the Hurwitz space depends on the Hurwitz space, and in choice of a primary differential, and the Dubrovin Frobenius structure on orbit space depends on the data of the group. Moreover, Dubrovin Frobenius structure on the Hurwitz space is a local construction, since it is constructed on a solution of a Darboux-Egoroff system. The orbit space construction instead, have the invariant ring of function, which gives a global picture.

Step 4:

From this point, we will prove the existence of a Dubrovin Frobenius structure in the orbit space of $\mathscr{J}(A_n)$ by using a strategy that is more closed related to the Saito and Dubrovin approach of differential geometry of orbit spaces [**28**], [**11**]. The effort to do this alternative proof worth, because, it would stress even more the fact that the construction of Hurwitz spaces and orbit spaces are independent, furthermore, this construction could be more suitable for the others Jacobi groups associated to the finite Coxeter group. This section will be devoted to construct the flat pencil metric on the orbit space of $\mathscr{J}(A_n)$. The first flat metric was already constructed in (5.27), therefore, this section will concentrate in the construction of the second flat metric. The second metric as the equations (5.36) suggests is given by

$$Lie_{\frac{\partial}{\partial\varphi_0}} g^{ij} := \eta^{ij}.$$

Hence, we will derive the coefficients of the metric (5.27) in the coordinates $\varphi_0, \varphi_2, .., \varphi_{n+1}$ and from it we derive the coefficients of the second metric of the flat pencil.

In order to derive the coefficients of the metric η^*, first, we derive a generating function for the coefficients of g^*.

Definition 5.4.1. [**8**] Let E^k the space of elliptic function of weight k. The elliptic connection $D_\tau : E^K \mapsto E^k$ is linear map defined by

$$D_\tau F(v,\tau) = \partial_\tau F(v,\tau) - 2kg_1(\tau)F(v,\tau) - \frac{1}{2\pi i}\frac{\theta_1'(v,\tau)}{\theta_1(v,\tau)}F'(v,\tau), \tag{5.34}$$

where $F(v,\tau) \in E^k$.

Theorem 5.4.1. [**8**] The coefficient of $M^*(d\varphi_i, d\varphi_j)$ be given (5.29) is recovered by the generating formula

$$\begin{aligned}
&\sum_{k,j=0}^{n+1} \frac{(-1)^{k+j}}{(k-1)!(j-1)!} M^*(d\varphi_i, d\varphi_j)\wp(v)^{(k-2)}\wp(v')^{(j-2)} = \\
&= 2\pi i(\lambda(v')D_\tau\lambda(v) + \lambda(v)D_\tau\lambda(v')) - \frac{1}{n+1}\frac{d\lambda(v)}{dv}\frac{d\lambda(v')}{dv'} \\
&+ \frac{1}{2}\frac{\wp'(v)+\wp'(v')}{\wp(v)-\wp(v')}[\lambda(v)\frac{d\lambda(v')}{dv'} - \frac{d\lambda(v)}{dv}\lambda(v')]
\end{aligned} \tag{5.35}$$

Starting from this point, there exist some original work. For these results, I will not put references.

Corollary 5.4.1.1. Let $\tilde{\eta}^*(d\varphi_i, d\varphi_j)$ and $\eta^*(d\varphi_i, d\varphi_j)$ be given by

$$\begin{aligned}
\tilde{\eta}^*(d\varphi_i, d\varphi_j) &:= \frac{\partial M^*(d\varphi_i, d\varphi_j)}{\partial\varphi_0}, \\
\eta^*(d\varphi_i, d\varphi_j) &:= \frac{\partial g^*(d\varphi_i, d\varphi_j)}{\partial\varphi_0}.
\end{aligned} \tag{5.36}$$

The coefficient of $\tilde{\eta}^*(d\varphi_i, d\varphi_j)$ is recovered by the generating formula

$$\begin{aligned}
&\sum_{k,j=0}^{n+1} \frac{(-1)^{k+j}}{(k-1)!(j-1)!}\tilde{\eta}^*(d\varphi_i, d\varphi_j)\wp(v)^{(k-2)}\wp(v')^{(j-2)} = \\
&= 2\pi i(D_\tau\lambda(v) + D_\tau\lambda(v')) + \frac{1}{2}\frac{\wp'(v)+\wp'(v')}{\wp(v)-\wp(v')}[\frac{d\lambda(v')}{dv'} - \frac{d\lambda(v)}{dv}]
\end{aligned} \tag{5.37}$$

Moreover,

$$\begin{aligned}
\tilde{\eta}^*(d\varphi_i, d\varphi_j) &= \eta^*(d\varphi_i, d\varphi_j), \quad i,j \neq 0 \\
\tilde{\eta}^*(d\varphi_0, d\varphi_j) &= \eta^*(d\varphi_i, d\varphi_j) + 4\pi i k_j \varphi_j
\end{aligned} \tag{5.38}$$

Proof. Just differentiate equation (5.35) with respect φ_0, and use the equation (5.21). $\square$

Corollary 5.4.1.2. The metric $\tilde{\eta}^*$ and η^* defined in (5.36) behave as modular form of weight 2 under the last action of (5.1).

Theorem 5.4.2. Let $\eta^*(d\varphi_i, d\varphi_j)$ defined in (5.36), then its coefficients can be obtained by the formula

$$\begin{aligned}
\eta^*(d\varphi_i, d\varphi_j) &= (i+j-2)\varphi_{i+j-2}, \quad i,j \neq 0 \\
\eta^*(d\varphi_i, d\varphi_0) &= 0, \quad i \neq 0.
\end{aligned} \tag{5.39}$$

Proof. Consider the equation (5.21) written in a concise way as follows

$$\lambda(v) = \sum_{k=0}^{n+1} \frac{(-1)^{n-k}}{(n-k)!}\varphi_{n+1-k}\wp^{n-1-k}(v). \tag{5.40}$$

Substituting (5.40) in (5.37)

$$\begin{aligned}
&\sum_{k,j=0}^{n+1} \frac{(-1)^{k+j}}{(k-1)!(j-1)!}\frac{\partial M^*(d\varphi_i, d\varphi_j)}{\partial \varphi_0}\wp(v)^{(k-2)}\wp(v')^{(j-2)} = \\
&= \sum_{k=0}^{n+1} \frac{(-1)^{n-k}}{(n-k)!}\varphi_{n+1-k}\left[2\pi i\left(D_\tau\wp^{n-1-k}(v) + D_\tau\wp^{n-1-k}(v')\right)\right] \\
&+ \sum_{k=0}^{n+1} \frac{(-1)^{n-k}}{(n-k)!}\varphi_{n+1-k}\left[\left(\zeta(v-v') + \zeta(v') - \zeta(v)\right)\left(\wp^{n-k}(v') - \wp^{n-k}(v')\right)\right]
\end{aligned} \tag{5.41}$$

Expanding the left-hand side of (5.41), we get

$$\begin{aligned}
&\sum_{k,j=0}^{n+1} \frac{(-1)^{k+j}}{(k-1)!(j-1)!}\frac{\partial M^*(d\varphi_i, d\varphi_j)}{\partial \varphi_0}\wp(v)^{(k-2)}\wp(v')^{(j-2)} \\
&= \sum_{k,j=0}^{n+1} \frac{\partial M^*(d\varphi_i, d\varphi_j)}{\partial \varphi_0}\frac{1}{v^k(v')^j} + \text{Other terms},
\end{aligned} \tag{5.42}$$

where "Other terms" in the equation (5.42) means positive powers of either v or v'. For convenience, define

$$(1):=2\pi i\left(D_\tau\wp^{n-1-k}(v)+D_\tau\wp^{n-1-j}(v')\right)$$
$$+\left(\zeta(v-v')+\zeta(v')-\zeta(v)\right)\left(\wp^{n-j}(v')-\wp^{n-k}(v)\right) \tag{5.43}$$

In order to better to compute (5.43), consider the analytical behaviour of the term

$$D_\tau\wp^{n-1-k}(v)=\partial_\tau\wp^{n-1-k}(v)-2(n+1-k)g_1(\tau)\wp^{n-1-k}(v)-\frac{1}{2\pi i}\frac{\theta_1'(v,\tau)}{\theta_1(v,\tau)}\wp^{n-k}(v). \tag{5.44}$$

The term

$$\partial_\tau\wp^{n-1-k}(v)$$

in (5.44) is holomorphic, therefore, it does not contribute for the Laurent tail. The term

$$2(n+1-k)g_1(\tau)\wp^{n-1-k}(v) \tag{5.45}$$

also do not contribute, because the full expression (5.44) behaves as modular form under the $SL_2(\mathbb{Z})$, but (5.45) is clear a quasi-modular form, since it contains $g_1(\tau)$. Hence, (5.45) is canceled with the Laurent tail of

$$\frac{1}{2\pi i}\frac{\theta_1'(v,\tau)}{\theta_1(v,\tau)}\wp^{n-k}(v). \tag{5.46}$$

To sum up, the analytical behavior of (5.44) is essentially given by (5.46). Under this consideration, and by using the equation

$$\begin{aligned}\zeta(v,\tau)&=\frac{\theta_1'(v,\tau)}{\theta_1(v,\tau)}-4\pi i g_1(\tau)v\\&=\frac{1}{v}+O(v^3)\end{aligned} \tag{5.47}$$

the equation (5.43) became

$$
\begin{aligned}
(1) &= -\zeta(v)\wp^{n-k}(v) - \zeta(v')\wp^{n-k}(v') \\
&+ \big(\zeta(v-v') + \zeta(v') - \zeta(v)\big)\left(\wp^{n-k}(v') - \wp^{n-k}(v)\right) + \text{Other terms} \\
&= \zeta(v-v')\left(\wp^{n-k}(v') - \wp^{n-k}(v')\right) - \zeta(v)\wp^{n-k}(v') - \zeta(v')\wp^{n-k}(v) + \text{Other terms} \\
&= \frac{1}{v-v'}\left(\frac{(-1)^{n-k}(n-k-1)!}{v'^{n+2-k}} - \frac{(-1)^{n-k}(n-k-1)!}{v^{n+2-k}}\right) \\
&- \frac{1}{v}\frac{(-1)^{n-k}(n-k-1)!}{v'^{n+2-k}} - \frac{1}{v'}\frac{(-1)^{n-k}(n-k-1)!}{v^{n+2-k}} + \text{Other terms} \\
&= (-1)^{n-k}(n-k-1)!\left(\frac{1}{v-v'}\frac{v^{n+2-k} - v'^{n+2-k}}{(v'v)^{n+2-k}}\right) \\
&- \frac{1}{v}\frac{(-1)^{n-k}(n-k-1)!}{v'^{n+2-k}} - \frac{1}{v'}\frac{(-1)^{n-k}(n-k-1)!}{v^{n+2-k}} + \text{Other terms} \\
&= (-1)^{n-k}(n-k-1)!\left(\sum_{j=0}^{n+1-k}\frac{v^{n+1-k-j}v'^{j}}{(v'v)^{n+2-k}}\right) \qquad (5.48) \\
&- \frac{1}{v}\frac{(-1)^{n-k}(n-k-1)!}{v'^{n+2-k}} - \frac{1}{v'}\frac{(-1)^{n-k}(n-k-1)!}{v^{n+2-k}} + \text{Other terms} \\
&= (-1)^{n-k}(n-k-1)!\left(\sum_{j=0}^{n+1-k}\frac{1}{v^{1+j}v'^{n+2-k-j}}\right) \\
&- \frac{1}{v}\frac{(-1)^{n-k}(n-k-1)!}{v'^{n+2-k}} - \frac{1}{v'}\frac{(-1)^{n-k}(n-k-1)!}{v^{n+2-k}} + \text{Other terms} \\
&= (-1)^{n-k}(n-k-1)!\left(\sum_{j=1}^{n-k}\frac{1}{v^{1+j}v'^{n+2-k-j}}\right) + \text{Other terms} \\
&= (-1)^{n-k}(n-k-1)!\left(\sum_{j=2}^{n+1-k}\frac{1}{v^{j}v'^{n+3-k-j}}\right) + \text{Other terms}
\end{aligned}
$$

Substituting (5.48) in right-hand side of (5.41)

$$
\begin{aligned}
&\sum_{k=0}^{n+1}\frac{(-1)^{n-k}}{(n-k)!}\varphi_{n+1-k}\left[2\pi i\left(D_\tau\wp^{n-1-k}(v)+D_\tau\wp^{n-1-k}(v')\right)\right]\\
&+\sum_{k=0}^{n+1}\frac{(-1)^{n-k}}{(n-k)!}\varphi_{n+1-k}\left[(\zeta(v-v')+\zeta(v')-\zeta(v))\left(\wp^{n-k}(v')-\wp^{n-k}(v')\right)\right]\\
&=\sum_{k=0}^{n+1}\sum_{j=2}^{n+1-k}\frac{(n+1-k)\,\varphi_{n+1-k}}{v^j v'^{n+3-k-j}}+\text{Other terms}\\
&=\sum_{k=0}^{n+1}\sum_{j=2}^{n+1}\frac{(n+1-k+j)\,\varphi_{n+1-k+j}}{v^j v'^{n+3-k}}+\text{Other terms}\\
&=\sum_{k=0}^{n+1}\sum_{j=2}^{n+1}\frac{(k+j)\,\varphi_{k+j}}{v^j v'^{k+2}}+\text{Other terms}\\
&=\sum_{k=2}^{n+3}\sum_{j=2}^{n+1}\frac{(k+j)\,\varphi_{k+j-2}}{v^j v'^{k}}+\text{Other terms}
\end{aligned}
\tag{5.49}
$$

Comparing (5.49) with (5.42), we get the desired result.

□

5.5. Flat coordinates of the Saito metric η

Step 5:

This section will be dedicated to prove that the Saito metric η is flat and non degenerate to complete that hypothesis of the lemma 4.4.1. In practice, we will construct the flat coordinates of the Saito metric η as follows.

Let $t^1, t^2, .., t^n$ be given by the following generating function

$$v(z)=\frac{-1}{n+1}\left(t^{n+1}z+t^n z^2+.....+t^2 z^n+O(z^{n+2})\right). \tag{5.50}$$

Defined by the following condition

$$\lambda(v)=\frac{1}{z^{n+1}}.$$

Moreover,

$$t^1=\varphi_0+4\pi i g_1(\tau)\varphi_2. \tag{5.51}$$

Lemma 5.5.1. The functions $t^2, .., t^{n+1}$ be defined in (5.50) can be obtained by the formula

$$t^\alpha=\frac{n+1}{n+2-\alpha}\operatorname*{res}_{p=\infty}\left(\lambda^{\frac{n+2-\alpha}{n+1}}(p)dp\right). \tag{5.52}$$

Proof. Consider the integration by parts

$$\frac{n+1}{n+2-\alpha}\int\left(\lambda^{\frac{n+2-\alpha}{n+1}}(p)dp\right)=p\lambda^{\frac{n+2-\alpha}{n+1}}-\int p\lambda^{\frac{1-\alpha}{n+1}}d\lambda. \tag{5.53}$$

Lemma proved.

□

LEMMA 5.5.2. The functions $t^2, .., t^n$ be defined in (5.50) can be obtained by the formula

$$t^\alpha = -\operatorname*{res}_{\lambda=\infty}\left(v(z)\lambda^{\frac{1-\alpha}{n+1}}d\lambda\right). \tag{5.54}$$

PROOF. Let

$$z = \left(\frac{1}{\lambda}\right)^{\frac{1}{n+1}},$$

then,

$$\begin{aligned}
v(z)\lambda^{\frac{1-\alpha}{n+1}}d\lambda &= \left(\frac{1}{n+1}\right)\left(t^{n+1}z + t^n z^2 + ... + t^2 z^n + O(z^{n+2})\right) z^{\alpha-1}\,(n+1)\,z^{-n-2}dz \\
&= \left(\sum_{\beta=2}^{n+1} t^\beta z^{n+2-\beta} + O(z^{n+2})\right) z^{\alpha-n-3}dz \\
&= \left(\sum_{\beta=2}^{n+1} t^\beta z^{\alpha-\beta-1} + O(z^{\alpha-n-3})\right) dz.
\end{aligned}$$

Hence, the residue is different from 0, when $\alpha=\beta$, resulting in this way the desired result.

□

COROLLARY 5.5.2.1. The coordinate t^{n+1} can be written in terms of the coordinates $\varphi_0, \varphi_2, .., \varphi_{n+1}$ as

$$t^{n+1} = (\varphi_{n+1})^{\frac{1}{n+1}}. \tag{5.55}$$

PROOF.

$$\begin{aligned}
t^{n+1} &= \operatorname*{res}_{v=0} \lambda^{\frac{1}{n+1}}(v)dv \\
&= \operatorname*{res}_{v=0}\left(\frac{\varphi_{n+1}}{v^{n+1}} + \frac{\varphi_n}{v^n} + .. + \frac{\varphi_2}{v^2} + O(1)\right)^{\frac{1}{n+1}} dv \\
&= \operatorname*{res}_{v=0} \frac{(\varphi_{n+1})^{\frac{1}{n+1}}}{v}\,(1+O(v))^{\frac{1}{n+1}}\,dv \\
&= (\varphi_{n+1})^{\frac{1}{n+1}}.
\end{aligned}$$

□

LEMMA 5.5.3. Let the functions $t^2, .., t^{n+1}$ be defined in (5.50), then the following identity holds

$$t^\alpha = \frac{n+1}{n+2-\alpha}\,(\varphi_{n+1})^{\frac{n+2-\alpha}{n+1}}\,(1+\Phi_{n+1-\alpha})^{\frac{n+2-\alpha}{n+1}}, \tag{5.56}$$

where

$$(5.57)\qquad \begin{aligned}(1+\Phi_i)^{\frac{n+2-\alpha}{n+1}} &= \sum_{d=0}^{\infty}\binom{\frac{n+2-\alpha}{n+1}}{d}\Phi_i^d,\\ \Phi_i^d &= \sum_{i_1+i_2+..+i_d=i}\frac{\varphi_{(n+1-i_1)}}{\varphi_{n+1}}....\frac{\varphi_{(n+1-i_d)}}{\varphi_{n+1}}.\end{aligned}$$

PROOF.

$$\begin{aligned}
t^\alpha &= \frac{n+1}{n+2-\alpha}\operatorname*{res}_{v=0}\left(\lambda^{\frac{n+2-\alpha}{n+1}}(v)dv\right)\\
&= \frac{n+1}{n+2-\alpha}\operatorname*{res}_{v=0}\left(\frac{\varphi_{n+1}}{v^{n+1}}+\frac{\varphi_n}{v^n}+..+\frac{\varphi_2}{v^2}+O(1)\right)^{\frac{n+2-\alpha}{n+1}}dv\\
&= \frac{n+1}{n+2-\alpha}\operatorname*{res}_{v=0}\left(\frac{\varphi_{n+1}}{v^{n+1}}\right)^{\frac{n+2-\alpha}{n+1}}\left(1+\frac{\varphi_n}{\varphi_{n+1}}v+\frac{\varphi_{n-1}}{\varphi_{n+1}}v^2+..+\frac{\varphi_2}{\varphi_{n+1}}v^{n-1}+O\left(v^{n+1}\right)\right)^{\frac{n+2-\alpha}{n+1}}dv\\
&= \frac{n+1}{n+2-\alpha}\operatorname*{res}_{v=0}\left(\frac{\varphi_{n+1}}{v^{n+1}}\right)^{\frac{n+2-\alpha}{n+1}}\sum_{d=0}^{\infty}\binom{\frac{n+2-\alpha}{n+1}}{d}\left(1+\frac{\varphi_n}{\varphi_{n+1}}v+\frac{\varphi_{n-1}}{\varphi_{n+1}}v^2+..+\frac{\varphi_2}{\varphi_{n+1}}v^{n-1}+O\left(v^{n+1}\right)\right)^{d}dv\\
&= \frac{n+1}{n+2-\alpha}\operatorname*{res}_{v=0}\left(\frac{\varphi_{n+1}}{v^{n+1}}\right)^{\frac{n+2-\alpha}{n+1}}\sum_{d=0}^{\infty}\binom{\frac{n+2-\alpha}{n+1}}{d}\sum_{j_1+..+j_n=d}\frac{d!}{j_1!j_2!..j_n!}\prod_{i=1}^{n-1}\left(\frac{\varphi_{n+1-i}v^i}{\varphi_{n+1}}\right)^{j_i}(O(v^{n+1}))^{j_n}dv\\
&= \frac{n+1}{n+2-\alpha}\operatorname*{res}_{v=0}(\varphi_{n+1})^{\frac{n+2-\alpha}{n+1}}\sum_{d=0}^{\infty}\binom{\frac{n+2-\alpha}{n+1}}{d}\sum_{j_1+..+j_n=d}\frac{d!}{j_1!j_2!..j_n!}\prod_{i=1}^{n-1}\left(\frac{\varphi_{n+1-i}}{\varphi_{n+1}}\right)^{j_i}v^{\left(\sum_{i=1}^{n-1}ij_i-n-2-\alpha\right)}dv\\
&+O(1)\\
&= \frac{n+1}{n+2-\alpha}(\varphi_{n+1})^{\frac{n+2-\alpha}{n+1}}\sum_{d=0}^{\infty}\binom{\frac{n+2-\alpha}{n+1}}{d}\sum_{\substack{j_1+..+j_n=d\\ j_1+2j_2+3j_3+..+(n-1)j_{n-1}=n+1-\alpha}}\frac{d!}{j_1!j_2!..j_n!}\prod_{i=1}^{n-1}\left(\frac{\varphi_{n+1-i}}{\varphi_{n+1}}\right)^{j_i}\\
&= \frac{n+1}{n+2-\alpha}(\varphi_{n+1})^{\frac{n+2-\alpha}{n+1}}\sum_{d=0}^{\infty}\binom{\frac{n+2-\alpha}{n+1}}{d}\sum_{i_1+..+i_d=n+1-\alpha}\frac{\varphi_{(n+1-i_1)}}{\varphi_{n+1}}....\frac{\varphi_{(n+1-i_d)}}{\varphi_{n+1}}\\
&= \frac{n+1}{n+2-\alpha}(\varphi_{n+1})^{\frac{n+2-\alpha}{n+1}}\sum_{d=0}^{\infty}\binom{\frac{n+2-\alpha}{n+1}}{d}\Phi^d_{n+1-\alpha}\\
&= \frac{n+1}{n+2-\alpha}(\varphi_{n+1})^{\frac{n+2-\alpha}{n+1}}(1+\Phi_{n+1-\alpha})^{\frac{n+2-\alpha}{n+1}}.
\end{aligned}$$

□

LEMMA 5.5.4. Let $\varphi_0,\varphi_2,..,\varphi_{n+1}$ and $\lambda(v)$ be defined in (5.21), then

$$(5.58)\qquad k\varphi_k = \operatorname*{res}_{v=0}k\lambda v^{k-1}dv.$$

PROOF.

$$\begin{aligned}\operatorname*{res}_{v=0}k\lambda v^{k-1}dv &= \operatorname*{res}_{v=0}k\left(\frac{\varphi_{n+1}}{v^{n+1}}+\frac{\varphi_n}{v^n}+..+\frac{\varphi_k}{v^k}+..+\frac{\varphi_2}{v^2}+O(1)\right)v^{k-1}dv\\ &= k\varphi_k.\end{aligned}$$

□

LEMMA 5.5.5. Let $\varphi_0, \varphi_2, .., \varphi_{n+1}$ and $\lambda(v)$ be defined in (5.21), then

$$k\varphi_k = -\operatorname*{res}_{\lambda=\infty} v^k d\lambda. \tag{5.59}$$

PROOF. Using formula (5.59) and integration by parts

$$k\varphi_k = \operatorname*{res}_{v=0} k\lambda v^{k-1} dv = -\operatorname*{res}_{\lambda=\infty} v^k d\lambda.$$

□

LEMMA 5.5.6. Let $\varphi_0, \varphi_2, .., \varphi_{n+1}$, $\lambda(v)$ be defined in (5.21) and $(t^2, .., t^{n+1})$ be defined in (5.50) , then

$$k\varphi_k = \frac{(-1)^k}{(n+1)^{k-1}} T^k_{n+1}, \tag{5.60}$$

where

$$T^k_{n+1} = \sum_{i_1+..+i_k=n+1} t^{(n+2-i_1)} ... t^{(n+2-i_k)}. \tag{5.61}$$

PROOF. Let $z := \left(\frac{1}{\lambda}\right)^{\frac{1}{n+1}}$, then by using equation (5.59):

$$\begin{aligned}
-\operatorname*{res}_{\lambda=\infty} v^k d\lambda &= \operatorname*{res}_{z=0} \frac{(n+1)v^k(z)dz}{z^{n+2}} \\
&= \operatorname*{res}_{z=0} \frac{(-1)^k}{(n+1)^{k-1}} \left(t^{n+1}z + t^n z^2 + .. + t^2 z^n + O(z^{n+2})\right) \frac{dz}{z^{n+2}} \\
&= \operatorname*{res}_{z=0} \frac{(-1)^k}{(n+1)^{k-1}} \sum_{j_1+j_2+..+j_n+j_{n+2}=k} \left(t^{n+1}z\right)^{j_1} \left(t^n z^2\right)^{j_2} .. \left(t^2 z^n\right)^{j_n} \left(O(z^{n+2})\right)^{j_{n+2}} \frac{dz}{z^{n+2}} \\
&= \frac{(-1)^k}{(n+1)^{k-1}} \sum_{\substack{j_1+j_2+..+j_n=k \\ j_1+2j_2+3j_3+..+(n)j_n=n+1}} \frac{k!}{j_1!j_2!..j_n!} \left(t^{n+1}\right)^{j_1} \left(t^n\right)^{j_2} .. \left(t^2\right)^{j_n} \\
&= \frac{(-1)^k}{(n+1)^{k-1}} \sum_{i_1+..i_k=n+1} t^{(n+2-i_1)} ... t^{(n+2-i_k)} \\
&= \frac{(-1)^k}{(n+1)^{k-1}} T^k_{n+1}.
\end{aligned}$$

□

LEMMA 5.5.7. Let T^k_{n+1} be defined in (5.61), then

$$\frac{\partial T^k_{n+1}}{\partial t^\alpha} = k T^{k-1}_{\alpha-1}. \tag{5.62}$$

PROOF.

$$\begin{aligned}\frac{\partial T^k_{n+1}}{\partial t^\alpha} &= \frac{\partial}{\partial t^\alpha}\left(\sum_{i_1+..i_k=n+1} t^{(n+2-i_1)}...t^{(n+2-i_k)}\right)\\ &= \sum_{i_1+..i_k=n+1} k\delta_{n+2-i_k,\alpha}t^{(n+2-i_1)}...t^{(n+2-i_{k-1})}\\ &= k\sum_{i_1+..i_{k-1}=\alpha-1} t^{(n+2-i_1)}...t^{(n+2-i_{k-1})}\\ &= kT^{k-1}_{\alpha-1}.\end{aligned}$$

□

THEOREM 5.5.8. Let $(t^2,..,t^{n+1})$ be defined in (5.50), and η^* be defined in (5.36). Then,

(5.63) $$\eta^*(dt^\alpha, dt^{n+3-\beta}) = -(n+1)\delta_{\alpha\beta}.$$

PROOF. If $i,j\neq 0$

(5.64) $$\eta^*(d\varphi_i, d\varphi_j) = \sum_{\alpha=2}^{n+1}\sum_{\beta=2}^{n+1}\frac{\partial\varphi_i}{\partial t^\alpha}\frac{\partial\varphi_j}{\partial t^\beta}\eta^*(dt^\alpha, dt^\beta)$$

Using (5.62) and (5.60), we get

(5.65) $$\frac{\partial\varphi_k}{\partial t^\alpha} = \frac{(-1)^k}{(n+1)^{k-1}}kT^{k-1}_{\alpha-1}.$$

Then,

(5.66) $$\sum_{\alpha=2}^{n+1}\frac{\partial\varphi_i}{\partial t^\alpha}\frac{\partial\varphi_{n+3-j}}{\partial t^{n+3-\alpha}} = \sum_{\alpha=2}^{n+1}\frac{(-1)^{i-j+n+1}}{(n+1)^{i-j+n+1}}T^{i-1}_{\alpha-1}T^{n+2-j}_{n+2-\alpha}.$$

Using the second of the equation (4.46) in (5.66)

(5.67) $$\begin{aligned}\sum_{\alpha=2}^{n+1}\frac{\partial\varphi_i}{\partial t^\alpha}\frac{\partial\varphi_{n+3-j}}{\partial t^{n+3-\alpha}} &= \frac{T^{n+1+i-j}_{n+1}}{(n+1)}\\ &= \frac{(n+1+i-j)}{n+1}\varphi_{(n+1+i-j)}.\end{aligned}$$

Note that

(5.68) $$\sum_{\alpha=2}^{n+1}\frac{\partial\varphi_i}{\partial t^\alpha}\frac{\partial\varphi_{n+3-j}}{\partial t^{n+3-\alpha}} = \sum_{\alpha=2}^{n+1}\sum_{\alpha=2}^{n+1}\frac{\partial\varphi_i}{\partial t^\alpha}\frac{\partial\varphi_{n+3-j}}{\partial t^{n+3-\beta}}\delta_{\alpha\beta}.$$

On another hand, using equation (5.39), we have

$$\begin{aligned}\eta^*(d\varphi_i, d\varphi_{n+3-j}) &= (n+1+i-j)\varphi_{n+1+i-j}\\ &= \sum_{\alpha=2}^{n+1}\sum_{\alpha=2}^{n+1}\frac{\partial\varphi_i}{\partial t^\alpha}\frac{\partial\varphi_{n+3-j}}{\partial t^{n+3-\beta}}\eta^*(dt^\alpha, dt^{n+3-\beta})\\ &= -(n+1)\sum_{\alpha=2}^{n+1}\sum_{\alpha=2}^{n+1}\frac{\partial\varphi_i}{\partial t^\alpha}\frac{\partial\varphi_{n+3-j}}{\partial t^{n+3-\beta}}\delta_{\alpha\beta}.\end{aligned}$$

Then, we obtain

$$\eta^*(dt^\alpha, dt^{n+3-\beta}) = (n+1)\,\delta_{\alpha\beta}.$$

□

LEMMA 5.5.9. The metric

$$\sum_{\alpha=2}^{n+1}\eta^*(dt^\alpha, dt^{n+3-\alpha})dt^\alpha dt^{n+3-\alpha} + 2dt^1 d\tau \tag{5.69}$$

behaves as a modular form of weight 2, under the $SL_2(\mathbb{Z})$ action of (5.1).

PROOF. Under the $SL_2(\mathbb{Z})$ action of (5.1), we have that $t^1, t^2, .., t^{n+1}$ have the following transformation law (see lemma 5.6.1)

$$\begin{aligned}t^1 &\mapsto t^1 + \frac{c}{2(c\tau+d)}\varphi_2 = t^1 + \frac{c}{4(n+1)(c\tau+d)}\sum_{\alpha=2}^{n+1}t^\alpha t^{n+3-\alpha}\\ t^\alpha &\mapsto \frac{t^\alpha}{c\tau+d}, \quad \alpha\neq 1.\end{aligned}$$

Hence, its differentials transform as

$$\begin{aligned}dt^1 &\mapsto dt^1 + \frac{c}{4(n+1)(c\tau+d)}\sum_{\alpha=2}^{n+1}\left(dt^\alpha t^{n+3-\alpha} + t^\alpha dt^{n+3-\alpha}\right)\\ dt^\alpha &\mapsto \frac{dt^\alpha}{c\tau+d} - \frac{ct^\alpha d\tau}{(c\tau+d)^2}, \quad \alpha\neq 1.\end{aligned} \tag{5.70}$$

Substituting (5.70) in (5.69), we get the desired result. □

LEMMA 5.5.10. Let t^1 defined in (5.51), and η^* defined in (5.36). Then,

$$\eta^*(dt^1, dt^\alpha) = 0, \quad \alpha\neq 1. \tag{5.71}$$

PROOF. If $i\neq 0$, using the definition of η^* in equation (5.36) and formula (5.39)

$$\begin{aligned}\eta^*(dt^1, d\varphi_i) &= \eta^*(d\varphi_0, d\varphi_i) + 4\pi i g_1(\tau)\eta^*(d\varphi_2, d\varphi_i) + 4\pi i g_1'(\tau)\eta^*(d\tau, d\varphi_i)\\ &= \eta^*(d\varphi_0, d\varphi_i) + 4\pi i g_1(\tau)\eta^*(d\varphi_2, d\varphi_i)\\ &= \tilde{\eta}^*(d\varphi_0, d\varphi_i) - 4\pi i g_1(\tau)k_i\varphi_i + 4\pi i g_1(\tau)k_i\varphi_i = 0\\ &= -4\pi i g_1(\tau)k_i\varphi_i + 4\pi i g_1(\tau)k_i\varphi_i = 0.\end{aligned}$$

Then,

$$\eta^*(dt^1, dt^\alpha) = \sum_{\alpha=2}^{n+1} \frac{\partial t^\alpha}{\partial \varphi_i} \eta^*(dt^1, d\varphi_i) = 0.$$

□

THEOREM 5.5.11. Let $(t^1, t^2, .., t^{n+1})$ be defined in (5.50), and η^* be defined in (5.36). Then,

$$\eta^*(dt^\alpha, dt^{n+3-\beta}) = -(n+1)\delta_{\alpha\beta}. \tag{5.72}$$

PROOF. The theorem is already proved for $\alpha, \beta \in \{2, .., n+1\}$ in theorem 5.5.8, and for $\alpha = 1$, and $\beta \neq 1$ in the lemma 5.5.10. The only missing part is to prove

$$\eta^*(dt^1, dt^1) = 0. \tag{5.73}$$

Recall that from corollary 5.4.1.2, the metric η^* behaves as modular form of weight 2 under the $SL_2(\mathbb{Z})$ action. Moreover, the same statement is valid for (5.69), because of lemma 5.5.9. However, the tensor $dt^1 \otimes dt^1$ behaves as quasi-modular. Hence, if the coefficient of the component $dt^1 \otimes dt^1$ is different from 0, we have a contradiction with corollary 5.4.1.2.

□

COROLLARY 5.5.11.1. The metric $\eta^*(d\varphi_i, d\varphi_j) := \frac{\partial g^*(d\varphi_i, d\varphi_j)}{\partial \varphi_0}$ is triangular, and non degenerate.

DEFINITION 5.5.1. Let $\eta^* = \eta^{\alpha\beta} \frac{\partial}{\partial t^\alpha} \otimes \frac{\partial}{\partial t^\beta}$ be defined in (5.36). The metric defined by

$$\eta = \eta_{\alpha\beta} dt^\alpha \otimes dt^\beta \tag{5.74}$$

is denoted by η.

5.6. The extended ring of Jacobi forms

The main point of this section is to point out that the flat coordinates of the Saito metric η does not live in the orbit space of $\mathscr{J}(A_n)$, but it lives in suitable covering over it. Therefore, all the geometric data of Dubrovin Frobenius manifold are in suitable extension of the ring of Jacobi forms.

LEMMA 5.6.1. [9] The coordinates $(t^1, t^2, .., t^{n+1}, \tau)$ defined on (5.50) have the following transformation laws under the action of the group $\mathscr{J}(A_n)$: they transform as follows under the third action (5.1)

$$\begin{aligned} t^1 &\mapsto t^1 + \frac{c \sum_{\alpha,\beta \neq 1,\tau} \eta_{\alpha\beta} t^\alpha t^\beta}{2(n+1)(c\tau + d)} \\ t^\alpha &\mapsto \frac{t^\alpha}{c\tau + d}, \quad \alpha \neq 1, \\ \tau &\mapsto \frac{a\tau + b}{c\tau + d} \end{aligned} \tag{5.75}$$

Proof. Note that the term Φ_i^d equation (5.57) has weight $+i$, then using that φ_{n+1} has weight $-n-1$, we have that the weight of t^α for $\alpha \neq 1$ must have weight -1 due to (5.56). The transformation law of t^1 follows from the transformation law of $g_1(\tau)$

$$g_1(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^2 g_1(\tau) + 2c(c\tau+d), \tag{5.76}$$

and by using equation (5.60) for $k=2$.

□

In addition , from the formula (5.56) it is clear that the multivaluedness of $(t^1, .., t^{n+1})$ comes from $(\varphi_{n+1})^{\frac{1}{n+1}}$. Therefore, the coordinates lives in a suitable covering over the orbit space of the group $\mathscr{J}(A_n)$. This covering is obtained by forgetting to act the Coxeter group A_n and the $SL_2(\mathbb{Z})$ action of $\mathscr{J}(A_n)$ on $\mathbb{C} \oplus \mathbb{C}^n \oplus \mathbb{H}$. The only remaining part of the $\mathscr{J}(A_n)$ action are the translations

$$v_i \mapsto v_i + \lambda_i \tau + \mu_i.$$

Hence, the coordinates $(t^1, .., t^{n+1})$ live in n-dimensional tori with fixed symplectic base of the torus homology and with a branching divisor $Y := \{\varphi_{n+1} = 0\}$. Another geometric interpretation of this covering can be done by the use of the following coordinates

Lemma 5.6.2. The equations

$$\begin{aligned} u &= \frac{-1}{2\pi i}\left[\sum_{i=0}^{n} \log \sigma(z_i, \omega, \omega') - \frac{\eta}{\omega}\left(\langle z, z\rangle_{A_n}\right)\right] \\ v_i &= \frac{z_i}{2\omega}, \quad i = 0, 1, .., n, \\ \tau &= \frac{\omega'}{\omega}. \end{aligned} \tag{5.77}$$

determine local coordinates in $H_{1,n}$, where $\eta = \zeta(\omega, \omega, \omega')$ is Weiestrass Zeta function evaluated in ω.

Proof. Expressing the σ function for the lattice generated by ω, ω' in terms of Jacobi theta 1

$$\sigma(z, \omega, \omega') = 2\omega \frac{\theta_1(\frac{z}{2\omega}, \frac{\omega'}{\omega})}{\theta_1'(0, \frac{\omega'}{\omega})} e^{\frac{\eta}{2\omega} z^2}. \tag{5.78}$$

Substituting (5.78), we obtain

$$-2\pi i u = \log\left((2\omega)^{-n-1} \frac{\theta_1'(0,\tau)^{n+1}}{\prod_{i=0}^{n} \theta_1(v_i, \tau)}\right), \tag{5.79}$$

Then, the following equations determines the inverse map of (5.77)

$$(2\omega)^{n+1} = \frac{\theta_1'(0,\tau)^{n+1}}{\prod_{i=0}^{n}\theta_1(v_i,\tau)}e^{-2\pi i u},$$
$$z_i = 2\omega v_i, \quad i = 0,1,..,n, \tag{5.80}$$
$$\omega' = \tau\omega.$$

□

Using the coordinates (5.77), the covering of the orbit space of $\mathscr{J}(A_n)$ is a n-dimensional tori in lattice (ω,ω') with fixed symplectic base of the torus homology and with a branching divisor $Y := \{\frac{1}{\omega} = 0\}$. There exist another geometric interpretation of this covering in terms of the flat coordinates of the intersection form $u, v_0, v_1, .., v_n, \tau$. Indeed, in these coordinates the covering is defined to be the quotient of $\mathbb{C}\oplus\mathbb{C}^n\oplus\mathbb{H}$ by the group $\mathscr{J}(A_n)$ without the A_n and $SL_2(\mathbb{Z})$ action, i.e. doing the quotient only by the action of the group $\mathbb{Z}^n\oplus\tau\mathbb{Z}^n$ in the notation of section 5.1.

This covering space of the orbit space of $\mathscr{J}(A_n)$, in the coordinates $u, v_0, v_1, .., v_n, \tau$, is defined by

$$\widetilde{\mathbb{C}\oplus\mathbb{C}^n\oplus\mathbb{H}/\mathscr{J}(A_n)} := \left(\mathbb{C}\oplus\mathbb{C}^n\oplus\mathbb{H}\right)/\left(\mathbb{Z}^n\oplus\tau\mathbb{Z}^n\right). \tag{5.81}$$

Note that

$$E_\tau^n = \mathbb{C}^n/\left(\mathbb{Z}^n\oplus\tau\mathbb{Z}^n\right) \tag{5.82}$$

is a n-dimensional tori with respect the lattice $(1,\tau)$. Then, the covering (5.81) can be thought as line bundle over n-dimensional toric fibration.

Remark 5.6.1. Following the same discussion regarding the correspondence between covering of the orbit spaces and covering of Hurwitz space started in remarks 4.9.1 and 4.9.2, we will consider the correspondence between the covering (5.81) and a suitable covering of the Hurwitz space $H_{1,n}$. Note that, a base in the first homology class of a torus is isomorphic to a lattice, therefore, fixing a $SL_2(\mathbb{Z})$ orbit in the orbit space of $\mathscr{J}(A_n)$ (fixing a lattice) is equivalent to fix a base of homology in the Hurwitz space $H_{1,n}$. Moreover, the action of "forget" the A_n action is equivalent to choice a root of λ (5.21) near ∞ due to the discussions of remarks 4.9.1 and 4.9.2.

In order to manipulate the geometric objects of this covering, it is more convenient to use their ring of functions. Hence, we define:

Definition 5.6.1. The extended ring of Jacobi forms with respect the ring of coefficients is the following ring

$$\widetilde{M}_\bullet[\varphi_0,\varphi_2,..,\varphi_{n+1}], \tag{5.83}$$

where

$$\widetilde{M_\bullet} = M_\bullet \oplus \{g_1(\tau)\}. \tag{5.84}$$

LEMMA 5.6.3. The coefficients of the intersection form g^{ij} on the coordinates $\varphi_0, \varphi_2, .., \varphi_{n+1}, \tau$ belong to the ring $\widetilde{M_\bullet}[\varphi_0, \varphi_2, .., \varphi_{n+1}]$.

PROOF. It is a consequence of the formula (5.29).

□

LEMMA 5.6.4. The coefficients of the intersection form $g^{\alpha\beta}$ on the coordinates $t^1, .., t^{n+1}, \tau$ belong to the ring $\widetilde{M_\bullet}[t^1, .., t^{n+1}, \frac{1}{t^{n+1}}]$.

PROOF. Using the transformation law of $g^{\alpha\beta}$

$$g^{\alpha\beta} = \frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j} g(d\varphi_i, d\varphi_j), \tag{5.85}$$

we realise the term $\frac{\partial t^\alpha}{\partial \varphi_i}$ as polynomial in $t^1, .., t^{n+1}, \frac{1}{t^{n+1}}$ due to the relations (5.56) and (5.60).
□

Recall that the quotient

$$f_i = \frac{\varphi_i}{\varphi_{n+1}} \tag{5.86}$$

is a elliptic function of weight $i - n - 1$ in the variables $(v_0, v_1, ., v_n, \tau)$, and $\varphi_{n+1} = \frac{1}{(2\omega)^{n+1}}$. Then, we can promote the elliptic function f_i in the lattice $(1, \tau)$ to a elliptic function in the lattice ω, ω' by doing

$$f_i(v_0, v_1, ., v_n, \tau) \mapsto \hat{f}_i(z_0, z_1, .., z_n, \omega, \omega') := \frac{1}{(2\omega)^{n+1-i}} f_i(v_0, v_1, ., v_n, \tau). \tag{5.87}$$

The same can be done in the modular forms in $(1, \tau)$ of weight k. Indeed,

$$h(\tau) \mapsto \hat{h}(\omega, \omega') := \frac{1}{(2\omega)^k} h(\tau). \tag{5.88}$$

Therefore, due to the equation (5.56), in the coordinates defined in (5.77) the ring $\widetilde{M_\bullet}[t^1, .., t^{n+1}, \frac{1}{t^{n+1}}]$ takes the following form

$$\tilde{M}_{\bullet \omega,\omega'}\left[\hat{f}_0, \hat{f}_2, .., \hat{f}_n\right] \tag{5.89}$$

where $\hat{f}_i$ is defined in (5.87) and $\tilde{M}_{\bullet \omega,\omega'}$ is the space of modular forms in the lattice ω, ω'.

5.7. Christoffel symbols of the intersection form

In this section, we will focus to prove that the Christoffel symbols of the intersection form belongs to the algebra $\widetilde{M}_{\bullet}[\varphi_0, \varphi_2, .., \varphi_{n+1}]$ on the coordinates $\varphi_0, \varphi_2, .., \varphi_{n+1}, \tau$, and to the algebra $\widetilde{M}_{\bullet}[t^1, .., t^{n+1}, \frac{1}{t^{n+1}}]$ on the coordinates $t^1, .., t^{n+1}, \tau$. Moreover, we will prove that on the coordinates $\varphi_0, \varphi_2, .., \varphi_{n+1}, \tau$ the Christoffel Christoffel symbols of the intersection form is at most linear on φ_0. This fact is necessary to realise the pair g^*, η^* as a flat pencil metric.

Recall that the Christoffel symbols $\Gamma^{ij}_k(\varphi)$ associated with the intersection form g^* is given in terms of the following conditions (4.13).

Lemma 5.7.1. Let $\varphi_0, \varphi_2, .., \varphi_n, \tau$, be defined in (5.21), then Γ^{ii}_j depend at most linear on φ_0.

Proof. Using the first condition of (4.13)

$$\partial_k g^{ii} = 2\Gamma^{ii}_k.$$

Recall that due to the theorem 5.4.2, the metric g^{ij} depend at most linear on φ_0. Then,

$$2\frac{\partial^2 \Gamma^{ii}_k}{\partial \varphi_0^2} = \partial_0^2 \partial_k g^{ii} = \partial_k \partial_0^2 g^{ii} = 0.$$

□

Lemma 5.7.2. Let $\varphi_0, , \varphi_1, \varphi_2, .., \varphi_{n+1}, \tau$, defined in (5.21), then

$$\begin{aligned} \Gamma^{i\tau}_j &= 0, \\ \Gamma^{\tau k}_k &= -2\pi i \frac{\delta_{jk}}{k}. \end{aligned} \tag{5.90}$$

Proof. Let $\Gamma^{ij}_k(x)$, in the coordinates $x_1, .., x_n$, and $\Gamma^{pq}_l(y)$ in the coordinates $y_1, .., y_n$, then the transformation law of the Christoffel symbol in defined in the cotangent bundle is the following

$$\Gamma^{ij}_k(x) = \frac{\partial x^i}{\partial y^p}\frac{\partial x^j}{\partial y^q}\frac{\partial y^l}{\partial x^k}\Gamma^{pq}_l(y) + \frac{\partial x^i}{\partial y^p}\frac{\partial}{\partial x^k}\left(\frac{\partial x^j}{\partial y^q}\right) g^{pq}(y). \tag{5.91}$$

In particular, the $\Gamma^{ij}_k(\varphi)$ in the coordinates $(\varphi_0, \varphi_1, .., \varphi_n, v_{n+1}, \tau)$ could be derived from the Christoffel symbol in the coordinates $v_0, v_1, .., v_{n+1}, \tau$ which is 0. Then,

$$\Gamma^{ij}_k(\varphi) = \frac{\partial \varphi_i}{\partial v_p}\frac{\partial}{\partial \varphi_k}\left(\frac{\partial \varphi_j}{\partial v_q}\right) g^{pq}(v). \tag{5.92}$$

Computing $\Gamma^{i\tau}_j$,

$$\begin{aligned} \Gamma^{i\tau}_k(\varphi) &= \frac{\partial \varphi_i}{\partial v_p}\frac{\partial}{\partial \varphi_k}\left(\frac{\partial \tau}{\partial v_q}\right) g^{pq}(v) \\ &= -2\pi \varphi_i \frac{\partial}{\partial \varphi_k}(1) = 0. \end{aligned} \tag{5.93}$$

Computing $\Gamma^{\tau i}_j$ by using the first condition of (4.13),

$$\begin{aligned}\Gamma^{\tau k}_j(\varphi) &= \partial_j g^{k\tau} - \Gamma^{k\tau}_j \\ &= \partial_j g^{k\tau} = -2\pi i \frac{\delta_{jk}}{k}.\end{aligned} \tag{5.94}$$

□

PROPOSITION 5.7.3. The Christoffel symbols $\Gamma^{ij}_k(\varphi)$ belong to the ring $\widetilde{M}_\bullet[\varphi_0, \varphi_2, .., \varphi_{n+1}]$.

PROOF. Note that the invariance of the Jacobi form φ_i with respect the first two actions of (5.1), and equivariant by the third one implies that the differential $d\varphi_i$ is invariant under the first two actions of (5.1), and behaves as follows under the $SL_2(\mathbb{Z})$

$$d\varphi_i \mapsto \frac{d\varphi_i}{(c\tau+d)^{k_i}} - \frac{c\varphi_i}{(c\tau+d)^{k_i+1}}. \tag{5.95}$$

Therefore the Christoffel symbol Γ^{ij}_k

$$\nabla_{(d\varphi_i)^\#} d\varphi_j = \Gamma^{ij}_k d\varphi_k \tag{5.96}$$

is a Jacobi form if φ_i has weight 0. Hence, doing the change of coordinates

$$\varphi_i \mapsto \hat{\varphi}_i := \eta^{2i}(\tau)\varphi_i, \tag{5.97}$$

we have that the Christoffel symbol $\hat{\Gamma}^{ij}_k$

$$\frac{1}{\eta^{2i+2j}} \nabla_{(d\hat{\varphi}_i)^\#} d\hat{\varphi}_j = \hat{\Gamma}^{ij}_k d\hat{\varphi}_k \tag{5.98}$$

is a Jacobi form.

Comparing $\hat{\Gamma}^{ij}_k$ with Γ^{ij}_k

$$\begin{aligned}\nabla_{(d\hat{\varphi}_j)^\#} d\hat{\varphi}_i &= \nabla_{(2jg_1\eta^{2j}\varphi_j d\tau + \eta^{2j} d\varphi_j)^\#}\left(2ig_1\eta^{2i}\varphi_i d\tau + \eta^{2i} d\varphi_i\right) \\ &= \nabla_{(2jg_1\eta^{2j}\varphi_j d\tau)^\#}\left(2ig_1\eta^{2i}\varphi_i d\tau\right) + \nabla_{(2jg_1\eta^{2j}\varphi_j d\tau)^\#}\left(\eta^{2i} d\varphi_i\right) \\ &+ \nabla_{(\eta^{2j} d\varphi_j)^\#}\left(2ig_1\eta^{2i}\varphi_i d\tau\right) + \nabla_{(\eta^{2j} d\varphi_j)^\#}\left(\eta^{2i} d\varphi_i\right) \\ &= 2jg_1\eta^{2j}\varphi_j g^{l\tau}\nabla_{\frac{\partial}{\partial\varphi_l}}\left(2i\eta^{2i}g_1\varphi_i d\tau\right) + 2jg_1\eta^{2j}\varphi_j g^{l\tau}\nabla_{\frac{\partial}{\partial\varphi_l}}\left(\eta^{2i} d\varphi_i\right) \\ &+ \eta^{2j}g^{lj}\nabla_{\frac{\partial}{\partial\varphi_l}}\left(2ig_1\eta^{2i}\varphi_i d\tau\right) + \eta^{2j}g^{lj}\nabla_{\frac{\partial}{\partial\varphi_l}}\left(\eta^{2i} d\varphi_i\right) \\ &= 4ijg_1'g_1\varphi_i\eta^{2i+2j}\varphi_j g^{\tau\tau}d\tau + 4i^2jg_1^3\eta^{2i+2j}\varphi_j\varphi_i g^{\tau\tau}d\tau + 4ijg_1\eta^{2i+2j}\varphi_j g^{i\tau}d\tau \\ &+ 4ijg_1^2\eta^{2i+2j}\varphi_j g^{\tau\tau}d\varphi_i + 2jg_1\eta^{2i+2j}\varphi_j\Gamma^{\tau i}_k d\varphi_k + 4i^2g_1^2\eta^{2i+2j}\varphi_i g^{\tau j}d\tau \\ &+ 2ig_1'\eta^{2i+2j}\varphi_i g^{lj}d\tau + 2ig_1\eta^{2i+2j}g^{ij}d\tau + 2ig_1\varphi_i\eta^{2i+2j}\Gamma^{i\tau}_k d\varphi_k \\ &+ \eta^{2i+2j}g_1 g^{l\tau}d\varphi_i + \eta^{2i+2j}\Gamma^{ji}_k d\varphi_k.\end{aligned} \tag{5.99}$$

Dividing the equation (5.99) by η^{2i+2j} and isolating $\Gamma^{ji}_k d\varphi_k$, we have

$$
\begin{aligned}
\Gamma^{ji}_k d\varphi_k = & -4ijg_1'g_1\varphi_i\varphi_j g^{\tau\tau}d\tau - 4i^2jg_1^3\varphi_j\varphi_i g^{\tau\tau}d\tau + 4ijg_1\eta^{2i+2j}\varphi_j g^{i\tau}d\tau \\
& - 4ijg_1^2\varphi_j g^{\tau\tau}d\varphi_i - 2jg_1\varphi_j\Gamma^{\tau i}_k d\varphi_k - 4i^2g_1^2\varphi_i g^{\tau j}d\tau \\
& - 2ig_1'\varphi_i g^{lj}d\tau - 2ig_1\eta^{2i+2j}g^{ij}d\tau - 2ig_1\varphi_i\Gamma^{i\tau}_k d\varphi_k \\
& - g_1 g^{l\tau}d\varphi_i + \hat{\Gamma}^{ji}_k d\varphi_k.
\end{aligned} \tag{5.100}
$$

Since the differential forms has a vector space structure and the right hand side of (5.100) depends only on $g^{ij}, g_1(\tau), \varphi_i$, and $\Gamma^{\tau i}_k$ which belongs to the ring $\widetilde{M_\bullet}[\varphi_0, \varphi_2, .., \varphi_{n+1}]$, the desired result is proved.

□

Lemma 5.7.4. The Christoffel symbols $\Gamma^{ij}_k(\varphi)$ depend at most linearly on φ_0.

Proof. The proposition 5.7.3 gives to the space of Christoffel symbols the structure of graded algebra, in particular we can compute the degree m regarding to the algebra of Jacobi forms. Let $\phi \in \widetilde{M_\bullet}[\varphi_0, \varphi_2, .., \varphi_{n+1}]$ with index m_ϕ and weight k_ϕ, then we write

$$
\begin{aligned}
deg_m\phi &= m_\phi, \\
deg_k\phi &= k_\phi.
\end{aligned} \tag{5.101}
$$

If $k \neq \tau$,

$$
deg_m\Gamma^{ij}_k = deg_m\left(\frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial\varphi_k}\left(\frac{\partial\varphi_j}{\partial v_q}\right)g^{pq}(v)\right) = 1. \tag{5.102}
$$

Therefore, Γ^{ij}_k is at most linear on φ_0. If $k = \tau$,

$$
deg_k\Gamma^{ij}_\tau = deg_k\left(\frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial\tau}\left(\frac{\partial\varphi_j}{\partial v_q}\right)g^{pq}(v)\right) = -i - j + 4.
$$

Suppose that Γ^{ij}_τ contains a the term $a(\tau)\varphi_0^2$, where $a(\tau)$, then

$$
deg_k a(v_{n+1}, \tau) = -i - j + 4 > 0.
$$

The possible Christoffel symbols that could depend on φ_0^2 are

$$
\Gamma^{04}_\tau, \Gamma^{40}_\tau, \Gamma^{22}_\tau, \Gamma^{20}_\tau, \Gamma^{02}_\tau, \Gamma^{00}_\tau. \tag{5.103}
$$

But $\Gamma^{22}_\tau, \Gamma^{00}_\tau$ is linear on φ_0 due to lemma 8.8.1.
Computing Γ^{ij}_τ

$$
\begin{aligned}
\Gamma^{ij}_\tau &= \frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial\tau}\left(\frac{\partial\varphi_j}{\partial v_q}\right)g^{pq}(v) \\
&= \frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(\frac{\partial\varphi_j}{\partial\tau}\right)g^{pq}(v)
\end{aligned} \tag{5.104}
$$

Recall that in (5.19), there exist a relation between the Jacobi form $\{\varphi_i\}$ and the he elementary symmetric polynomials $a_2, .., a_{n+1}$ be given by the Taylor expansion of $\{\varphi_i\}$.

Note that the Christoffel symbol depend on φ_0 iff it contains the term constant in its expansion. Our strategy is to show that the Christoffel symbols (5.103) contains only higher order polynomials in its expansions. Computing the lowest degree term in the expansion of (5.104)

$$
\begin{aligned}
\Gamma_{\tau}^{ij} &= \frac{\partial \varphi_i}{\partial v_p} \frac{\partial}{\partial v_q} \left(\frac{\partial \varphi_j}{\partial \tau} \right) g^{pq}(v) \\
&= \frac{\partial a_i}{\partial v_p} \frac{\partial}{\partial v_q} \left(\frac{\partial b_j(\tau)}{\partial \tau} a_{j+1} \right) g^{pq}(v) + .. \\
&= \frac{\partial P_i}{\partial v_p} \frac{\partial b_j(\tau)}{\partial \tau} \frac{\partial a_{j+1}}{\partial v_q} g^{pq}(v) + .. \\
&= \frac{\partial b_j(\tau)}{\partial \tau} a_{i+j+3} + ...
\end{aligned}
\tag{5.105}
$$

Therefore, we have that the associated Christoffel symbol do not depend on φ_0^2. $\square$

5.8. Unit and Euler vector field of the orbit space of $\mathscr{J}(A_n)$

This section is devoted to study the action of the Euler vector field, and Unit vector field in the geometric structure of the orbit space of the group $\mathscr{J}(A_n)$. The action of the Euler vector field is particularly important, because it would give rise to the quasi homogeneous condition to the WDVV solution, which we aim to construct.

Lemma 5.8.1. Let $\lambda, \varphi_0, .., \varphi_n, \varphi_{n+1}, \varphi_{n+2} = \tau$ be defined in (5.21), $(t^1, ., t^{n+1}, \tau)$ the flat coordinates of η defined in (5.50), and the Euler vector field be defined by (5.31). Then,

$$
\begin{aligned}
E(\lambda) &= \lambda, \\
E(\varphi_i) &= d_i \varphi_i, \\
E(t^{\alpha}) &= d_{\alpha} t^{\alpha},
\end{aligned}
\tag{5.106}
$$

where

$$
\begin{aligned}
d_i &= 1, \quad i < n+2, \\
d_i &= 0, \quad i = n+2, \\
d_{\alpha} &= \frac{n+1-\alpha}{n},
\end{aligned}
\tag{5.107}
$$

Proof. Recall that the function λ is given by

$$
\begin{aligned}
\lambda &= e^{-2\pi i u} \frac{\prod_{i=0}^{n} \theta_1(z - v_i, \tau)}{\theta_1^{n+1}(z, \tau)} \\
&= \varphi_{n+1} \wp^{n-1}(z, \tau) + \varphi_n \wp^{n-2}(z, \tau) + ... + \varphi_2 \wp(z, \tau) + \varphi_0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{1}{2\pi i} \frac{\partial}{\partial u}(\lambda) &= \lambda \\
&= E(\varphi_{n+1}) \wp^{n-1}(z, \tau) + E(\varphi_n) \wp^{n-2}(z, \tau) + ... + E(\varphi_2) \wp(z, \tau) + E(\varphi_0),
\end{aligned}
$$

therefore,

$$E(\varphi_i) = \varphi_i,$$

furthermore,

$$E(\tau) = 0.$$

Recall that t^α can written in terms of equation (5.54) or in more convenient way

$$\begin{aligned} t^\alpha &= \frac{n}{n+1-\alpha} \operatorname*{res}_{v=0} \lambda^{\frac{n+1-\alpha}{n}}(v)dv, \quad \alpha \neq 1, \\ t^1 &= \varphi_0 + 4\pi i g_1(\tau)\varphi_2. \end{aligned} \tag{5.108}$$

Applying the Euler vector in (5.108) we get the desired result.

□

COROLLARY 5.8.1.1. The Euler vector field (5.31) in the flat coordinates of η^* has the following form

$$E := \sum_{\alpha=1}^{n+1} d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha}, \tag{5.109}$$

where

$$d_\alpha = \frac{n+2-\alpha}{n+1}. \tag{5.110}$$

PROOF. Recall that

$$\begin{aligned} E &= \frac{1}{2\pi i}\frac{\partial}{\partial u} \\ &= E(t^\alpha)\frac{\partial}{\partial t^\alpha}, \\ &= \sum_{\alpha=1}^{n+1} d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha}. \end{aligned}$$

□

LEMMA 5.8.2. The Euler vector field (5.31) acts in the vector fields $\frac{\partial}{\partial t^\alpha}$, $\frac{\partial}{\partial \varphi_i}$ and differential forms dt^α, $d\varphi_i$ as follows:

$$\begin{aligned} Lie_E d\varphi_i &= d_i d\varphi_i, \\ Lie_E dt^\alpha &= d_\alpha dt^\alpha, \\ Lie_E \frac{\partial}{\partial \varphi_i} &= -d_i \frac{\partial}{\partial \varphi_i}, \\ Lie_E \frac{\partial}{\partial t^\alpha} &= -d_\alpha \frac{\partial}{\partial t^\alpha}. \end{aligned} \tag{5.111}$$

LEMMA 5.8.3. The intersection form g^{ij} be defined in (5.27) and its Christoffel symbol Γ_k^{ij} in the coordinates $\varphi_0, .., \varphi_n, \varphi_{n+1}, \varphi_{n+2} = \tau$ be defined in (5.21) are weighted polynomials in the variables $\varphi_0, \varphi_2, .., \varphi_{n+1}$, with degrees

$$\begin{aligned} deg\left(g^{ij}\right) &= d_i + d_j, \\ deg\left(\Gamma_k^{\alpha\beta}\right) &= d_i + d_j - d_k. \end{aligned} \tag{5.112}$$

PROOF. The function g^{ij} and Γ_k^{ij} belong to the ring $\widetilde{M}_\bullet[\varphi_0, \varphi_2, .., \varphi_{n+1}]$ due to lemma 5.6.3 and 5.7.3. The degrees are computed by using the following formulae

$$\begin{aligned} E\left(g^{ij}(\varphi)\right) &= E\left(\frac{\partial\varphi_i}{\partial v^l}\frac{\partial\varphi_j}{\partial v^m}g^{lm}(v)\right) \\ &= E\left(\frac{\partial\varphi_i}{\partial v^l}\right)\frac{\partial\varphi_j}{\partial v^m}g^{lm}(v) + \frac{\partial\varphi_i}{\partial v^l}E\left(\frac{\partial\varphi_j}{\partial v^m}g^{lm}(v)\right) \\ &= \frac{\partial E(\varphi_i)}{\partial v^l}\frac{\partial\varphi_j}{\partial v^m}g^{lm}(v) + \frac{\partial\varphi_i}{\partial v^l}\frac{\partial E(\varphi_j)}{\partial v^m}g^{lm}(v) \\ &= (d_i + d_j)\frac{\partial\varphi_i}{\partial v^l}\frac{\partial\varphi_j}{\partial v^m}g^{lm}(v). \end{aligned}$$

and

$$\begin{aligned} E\left(\Gamma_k^{ij}(\varphi)\right) &= E\left(\frac{\partial\varphi_i}{\partial v^l}\frac{\partial}{\partial\varphi_k}\left(\frac{\partial\varphi_j}{\partial v^m}\right)g^{lm}(v)\right) \\ &= E\left(\frac{\partial\varphi_i}{\partial v^l}\right)\frac{\partial}{\partial\varphi_k}\left(\frac{\partial\varphi_j}{\partial v^m}\right)g^{lm}(v) + \frac{\partial\varphi_i}{\partial v^l}E\left(\frac{\partial}{\partial\varphi_k}\left(\frac{\partial\varphi_j}{\partial v^m}\right)g^{lm}(v)\right) \\ &= \frac{\partial E(\varphi_i)}{\partial v^l}\frac{\partial}{\partial\varphi_k}\left(\frac{\partial\varphi_j}{\partial v^m}\right)g^{lm}(v) + \frac{\partial\varphi_i}{\partial v^l}\frac{\partial}{\partial\varphi_k}\left(\frac{\partial E(\varphi_j)}{\partial v^m}\right)g^{lm}(v) \\ &- d_k\frac{\partial\varphi_i}{\partial v^l}\frac{\partial}{\partial\varphi_k}\left(\frac{\partial\varphi_j}{\partial v^m}\right)g^{lm}(v) \\ &= (d_i + d_j - d_k)\frac{\partial\varphi_i}{\partial v^l}\frac{\partial}{\partial\varphi_k}\left(\frac{\partial\varphi_j}{\partial v^m}\right)g^{lm}(v). \end{aligned}$$

□

LEMMA 5.8.4. The intersection form $g^{\alpha\beta}$ be defined in (5.27) in the coordinates $(t^1, ., t^{n+1}, \tau)$ be defined in (5.50) and its Christoffel symbol $\Gamma_\gamma^{\alpha\beta}$ are weighted polynomials in the variables $t^1, .., t^{n+1}, \frac{1}{t^{n+1}}$ with degrees

$$\begin{aligned} deg\left(g^{\alpha\beta}\right) &= d_\alpha + d_\beta, \\ deg\left(\Gamma_\gamma^{\alpha\beta}\right) &= d_\alpha + d_\beta - d_\gamma. \end{aligned} \tag{5.113}$$

PROOF. Lemma 5.6.4 guarantee that $g^{\alpha\beta} \in \widetilde{M}_{\bullet}[t^0, t^1, .., t^{n+1}, \frac{1}{t^{n+1}}]$. Using the formula

$$
\begin{aligned}
E(g^{\alpha\beta}) &= E(\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j} g^{ij}(\varphi)) \\
&= E(\frac{\partial t^\alpha}{\partial \varphi_i})\frac{\partial t^\beta}{\partial \varphi_j} g^{ij}(\varphi) + \frac{\partial t^\alpha}{\partial \varphi_i} E(\frac{\partial t^\beta}{\partial \varphi_j}) g^{ij}(\varphi) + \frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j} E(g^{ij}(\varphi)) \\
&= \frac{\partial E(t^\alpha)}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j} g^{ij}(\varphi) - d_i \frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j} g^{ij}(\varphi) + \frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial E(t^\beta)}{\partial \varphi_j} g^{ij}(\varphi) - d_j \frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j} g^{ij}(\varphi) \\
&+ (d_i + d_j)\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j} g^{ij}(\varphi) \\
&= (d_\alpha + d_\beta)\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j} g^{ij}(\varphi).
\end{aligned}
$$

The Christoffel symbol $\Gamma_\gamma^{\alpha\beta}$ is given by the following

$$
\Gamma_\gamma^{\alpha\beta} = \frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j}\frac{\partial \varphi_k}{\partial t^\gamma}\Gamma_k^{ij} + \frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial}{\partial t^\gamma}\left(\frac{\partial t^\beta}{\partial \varphi_j}\right) g^{ij}.
$$

$\Gamma_j^{ij}, \frac{\partial \varphi_k}{\partial t^\gamma} \in \widetilde{M}_{\bullet}[t^1, .., t^{n+1}]$ due to Lemma 5.7.3 and equations (5.65), (5.61). But $\frac{\partial t^\alpha}{\partial \varphi_i} \in \widetilde{M}_{\bullet}[t^1, .., t^{n+1}, \frac{1}{t^{n+1}}]$, see the proof of lemma 5.6.4 for details. Therefore, $\Gamma_\gamma^{\alpha\beta}$ are weighted polynomials in the variables $t^1, .., t^{n+1}, \frac{1}{t^{n+1}}$. Computing the degree of $\Gamma_\gamma^{\alpha\beta}$

$$
\begin{aligned}
E(\Gamma_\gamma^{\alpha\beta}) &= E(\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j}\frac{\partial \varphi_k}{\partial t^\gamma}\Gamma_k^{ij} + \frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial}{\partial t^\gamma}\left(\frac{\partial t^\beta}{\partial \varphi_j}\right) g^{ij}) \\
&= (d_\alpha - d_i)\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j}\frac{\partial \varphi_k}{\partial t^\gamma}\Gamma_k^{ij} + (d_\beta - d_j)\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j}\frac{\partial \varphi_k}{\partial t^\gamma}\Gamma_k^{ij} \\
&+ (d_k - d_\gamma)\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j}\frac{\partial \varphi_k}{\partial t^\gamma}\Gamma_k^{ij} + (d_\alpha - d_i)\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial}{\partial t^\gamma}\left(\frac{\partial t^\beta}{\partial \varphi_j}\right) g^{ij} \\
&+ (d_\beta - d_\gamma)\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial}{\partial t^\gamma}\left(\frac{\partial t^\beta}{\partial \varphi_j}\right) g^{ij} + (d_\alpha - d_i)\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial}{\partial t^\gamma}\left(\frac{\partial t^\beta}{\partial \varphi_j}\right) g^{ij} \\
&+ (d_i + d_j)\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial}{\partial t^\gamma}\left(\frac{\partial t^\beta}{\partial \varphi_j}\right) g^{ij} \\
&= (d_\alpha + d_\beta - d_\gamma)\Gamma_\gamma^{\alpha\beta}.
\end{aligned}
$$

□

LEMMA 5.8.5. The Unit vector field (5.32) in the flat coordinates of η^* has the following form

$$
e = \frac{\partial}{\partial t^1}. \tag{5.114}
$$

PROOF.

$$\frac{\partial}{\partial\varphi_0} = \frac{\partial t^\alpha}{\partial\varphi_0}\frac{\partial}{\partial t^\alpha}$$
$$= \frac{\partial}{\partial t^1}.$$

□

LEMMA 5.8.6. Let the metric η^* defined on (5.36) and the Euler vector field (5.31). Then,

$$Lie_E\eta^{\alpha\beta} = (d_\alpha + d_\beta - d_1)\eta^{\alpha\beta}. \tag{5.115}$$

5.9. Construction of WDVV solution

The main aim of this section is to extract a WDVV equation from the data of the group $\mathscr{J}(A_n)$.

LEMMA 5.9.1. The orbit space of $\mathscr{J}(A_n)$ carries a flat pencil metric

$$g^{\alpha\beta}, \quad \eta^{\alpha\beta} := \frac{\partial g^{\alpha\beta}}{\partial t^1} \tag{5.116}$$

with the correspondent Christoffel symbols.

$$\Gamma^{\alpha\beta}_\gamma, \quad \eta^{\alpha\beta} := \frac{\partial \Gamma^{\alpha\beta}_\gamma}{\partial t^1} \tag{5.117}$$

PROOF. The metric (5.116) satisfies the hypothesis of Lemma 4.8.1 which proves the desired result.

□

The following lemma shows that flat pencil structure is almost the same as Dubrovin Frobenius structure due to lemma 4.8.1.

LEMMA 5.9.2. Let the intersection form (5.27), unit vector field (5.32), and Euler vector field (5.31). Then, there exist a function

$$F(t^1,t^2,..,t^{n+1},\tau) = \frac{(t^1)^2\tau}{4\pi i} + \frac{t^1}{2}\sum_{\alpha,\beta\neq 1,\tau}\eta_{\alpha\beta}t^\alpha t^\beta + G(t^2,..,t^{n+1},\tau), \tag{5.118}$$

such that

$$\begin{aligned} &Lie_E F = 2F + \text{quadratic terms},\\ &Lie_E\left(F^{\alpha\beta}\right) = g^{\alpha\beta},\\ &\frac{\partial^2 G(t^1,t^2,..,t^{n+1},\tau)}{\partial t^\alpha\partial t^\beta} \in \widetilde{M}_\bullet[t^2,..,t^{n+1},\frac{1}{t^{n+1}}], \end{aligned} \tag{5.119}$$

where

$$F^{\alpha\beta} = \eta^{\alpha\alpha'}\eta^{\beta\beta'}\frac{\partial F^2}{\partial t^{\alpha'}\partial t^{\beta'}}. \tag{5.120}$$

Proof. Let $\Gamma_{\gamma}^{\alpha\beta}(t)$ the Christoffel symbol of the intersection form (5.27) in the coordinates the flat coordinates of η^*, i.e $t^1, t^2, .., t^{n+1}, \tau$. According to the lemma 4.8.1, we can represent $\Gamma_{\gamma}^{\alpha\beta}(t)$ as

$$\Gamma_{\gamma}^{\alpha\beta}(t) = \eta^{\alpha\epsilon}\partial_{\epsilon}\partial_{\gamma}f^{\beta}(t). \tag{5.121}$$

Using the relations (5.113), (5.111) and lemma 5.8.6

$$\begin{aligned} Lie_E(\Gamma_{\gamma}^{\alpha\beta}(t)) &= Lie_E(\eta^{\alpha\epsilon})\partial_{\epsilon}\partial_{\gamma}f^{\beta}(t) + \eta^{\alpha\epsilon}Lie_E(\partial_{\epsilon}\partial_{\gamma}f^{\beta}(t)) \\ &= (d_{\alpha} + d_{\epsilon} - d_1)\eta^{\alpha\epsilon}\partial_{\epsilon}\partial_{\gamma}f^{\beta}(t) + (-d_{\epsilon} - d_{\gamma})\eta^{\alpha\epsilon}\partial_{\epsilon}\partial_{\gamma}Lie_E(f^{\beta}(t)) \\ &= (d_{\alpha} + d_{\beta} - d_{\gamma})\eta^{\alpha\epsilon}\partial_{\epsilon}\partial_{\gamma}f^{\beta}(t). \end{aligned}$$

Then, by isolation $Lie_E\left(f^{\beta}(t)\right)$ we get

$$Lie_E\left(f^{\beta}(t)\right) = (d_{\beta} + d_1)f^{\beta} + A_{\sigma}^{\beta}t^{\sigma} + B^{\beta}, \quad A_{\sigma}^{\beta}, B^{\beta} \in \mathbb{C}. \tag{5.122}$$

Considering the second relation of (4.13) for $\alpha = \tau$

$$g^{\tau\sigma}\Gamma_{\sigma}^{\beta\gamma} = g^{\beta\sigma}\Gamma_{\sigma}^{\tau\gamma}, \tag{5.123}$$

and using lemma 5.7.2 and the fact that

$$g(d\varphi_i, d\tau) = 2\pi i\varphi_i,$$

which implies

$$g(dt^{\alpha}, d\tau) = 2\pi i d_{\alpha}t^{\alpha},$$

we have.

$$2\pi i d_{\sigma}t^{\sigma}\eta^{\beta\epsilon}\partial_{\sigma}\partial_{\epsilon}f^{\gamma} = 2\pi i d_{\sigma}\delta_{\sigma}^{\gamma}g^{\beta\sigma}, \tag{5.124}$$

which is equivalent to

$$Lie_E\left(\eta^{\beta\epsilon}\partial_{\epsilon}f^{\gamma}\right) = d_{\gamma}g^{\beta\gamma}. \tag{5.125}$$

Using (5.122) in the equation (5.125), we have

$$(d_{\beta} + d_{\gamma})\eta^{\beta\epsilon}\partial_{\epsilon}f^{\gamma} = d_{\gamma}g^{\beta\gamma}. \tag{5.126}$$

If $\gamma \neq \tau$, we define

$$F^{\gamma} = \frac{f^{\gamma}}{d_{\gamma}}, \tag{5.127}$$

and note that $g^{\beta\gamma}$ is symmetric with respect the indices β, γ. Hence,

$$(d_{\beta} + d_{\gamma})\eta^{\beta\epsilon}\partial_{\epsilon}F^{\gamma} = (d_{\beta} + d_{\gamma})\eta^{\gamma\epsilon}\partial_{\epsilon}F^{\beta}, \tag{5.128}$$

which is the integrability condition for

$$F^{\gamma} = \eta^{\gamma\mu}\partial_{\mu}F. \tag{5.129}$$

In order to extract information from $\gamma = \tau$, take $\beta = \tau$ in equation (5.126)

$$\begin{aligned} d_\gamma \eta^{\tau 0} \partial_0 f^\gamma &= d_\gamma g^{\tau\gamma} \\ 2\pi i d_\gamma \partial_0 f^\gamma &= 2\pi i d_\gamma t^\gamma \end{aligned} \tag{5.130}$$

which is equivalent to

$$\eta^{\gamma\epsilon} \partial_\epsilon \partial_0 F = t^\gamma,$$

inverting $\eta^{\gamma\epsilon}$

$$\partial_\alpha \partial_0 F = \eta_{\alpha\gamma} t^\gamma, \tag{5.131}$$

integrating equation (5.131), we obtain

$$F(t^1, t^2, .., t^{n+1}, \tau) = \frac{(t^1)^2 \tau}{4\pi i} + \frac{t^1}{2} \sum_{\alpha,\beta \neq 1,\tau} \eta_{\alpha\beta} t^\alpha t^\beta + G(t^2, .., t^{n+1}, \tau). \tag{5.132}$$

Substituting the equation (5.132) in the (5.126) for $\gamma \neq \tau$, we get

$$\begin{aligned} g^{\beta\gamma} &= (d_\beta + d_\gamma) \eta^{\beta\epsilon} \eta^{\gamma\mu} \partial_\epsilon \partial_\mu F, \\ &= Lie_E(F^{\beta\gamma}) \end{aligned} \tag{5.133}$$

Since $g^{\beta\gamma}$ is a symmetric matrix, the equation (5.133) is equivalent to the second equation of (5.119) for either β and γ different from τ. Therefore, the missing part of the second equation of (5.119) is only for the case $\beta = \gamma = \tau$. Moreover, the intersection form $g^{\beta\gamma}$ is proportional to the Hessian of the equation (5.132) for for either β and γ different from τ. Recall that from the data of a Hessian, we can reconstruct uniquely a function up to quadratic terms, therefore, by defining

$$Lie_E \left(\frac{\partial^2 F}{\partial {t^1}^2} \right) = g^{\tau\tau}. \tag{5.134}$$

Then substituting (5.132) in (5.134).

$$\begin{aligned} Lie_E \left(\frac{\partial^2 F}{\partial {t^1}^2} \right) &= Lie_E \left(\frac{\tau}{2\pi i} \right) \\ &= 0 = g^{\tau\tau}. \end{aligned}$$

Hence, we proved the second equation (5.119). Substituting the equation (5.132) in the second equation (5.119) for $\alpha, \beta \neq \tau$

$$\begin{aligned} Lie_E \left(F^{\alpha\beta} \right) &= Lie_E \left(\eta^{\alpha\alpha'} \eta^{\beta\beta'} \frac{\partial F^2}{\partial t^{\alpha'} \partial t^{\beta'}} \right) \\ &= Lie_E \left(\eta^{\alpha\alpha'} \eta^{\beta\beta'} \frac{\partial G^2}{\partial t^{\alpha'} \partial t^{\beta'}} \right) \\ &= g^{\alpha\beta} \in \widetilde{M}_\bullet[t^2, .., t^{n+1}, \frac{1}{t^{n+1}}]. \end{aligned}$$

Hence, the second equation (5.119) prove the third equation of (5.119).

Substituting (5.132) in (5.122), and using (5.129), we obtain

$$
\begin{aligned}
Lie_E\left(f^\beta\right) &= Lie_E\left(\eta^{\beta\epsilon}\partial_\epsilon F\right)\\
&= Lie_E\left(\eta^{\beta\epsilon}\partial_\epsilon F\right)\partial_\epsilon F + \eta^{\beta\epsilon} Lie_E\left(\partial_\epsilon F\right)\\
&= (d_\beta + d_\epsilon - d_1)\eta^{\beta\epsilon}\partial_\epsilon F\partial_\epsilon F + \eta^{\beta\epsilon}\partial_\epsilon Lie_E\left(F\right) - d_\epsilon \eta^{\beta\epsilon}\partial_\epsilon F\\
&= (d_\beta - d_1)\eta^{\beta\epsilon}\partial_\epsilon F\partial_\epsilon F + \eta^{\beta\epsilon}\partial_\epsilon Lie_E\left(F\right)\\
&= (d_\beta + d_1)\eta^{\beta\epsilon}\partial_\epsilon F + A^\beta_\sigma t^\sigma + B^\beta
\end{aligned}
$$

Hence, isolating $Lie_E\left(F\right)$

$$\eta^{\beta\epsilon}\partial_\epsilon Lie_E\left(F\right) = 2\eta^{\beta\epsilon}\partial_\epsilon F + A^\beta_\sigma t^\sigma + B^\beta,$$

inverting $\eta^{\beta\epsilon}$

$$\partial_\alpha Lie_E\left(F\right) = 2\partial_\alpha F + \eta_{\alpha\beta}A^\beta_\sigma t^\sigma + \eta_{\alpha\beta}B^\beta,$$

integrating

$$Lie_E\left(F\right) = 2F + \eta_{\alpha\beta}A^\beta_\sigma t^\alpha t^\sigma + \eta_{\alpha\beta}B^\beta t^\alpha,$$

Lemma proved.

$\square$

LEMMA 5.9.3. Let

(5.135) $$c_{\alpha\beta\gamma} = \frac{\partial F^3}{\partial t^\alpha \partial t^\beta \partial t^\gamma},$$

then,

(5.136) $$c^\gamma_{\alpha\beta} = \eta^{\gamma\epsilon}c_{\alpha\beta\epsilon}$$

is a structure constant of a commutative algebra given by the following rule in the flat coordinate of η

(5.137) $$\partial_\alpha \bullet \partial_\beta = c^\gamma_{\alpha\beta}\partial_\gamma$$

such that

(5.138) $$\eta(\partial_\alpha \bullet \partial_\beta, \partial_\gamma) = \eta(\partial_\alpha, \partial_\beta \bullet \partial_\gamma), \quad \textbf{Frobenius condition}.$$

PROOF.

(1) **Commutative**

The product defined in (5.137) is commutative, because its structure constant (5.136) is symmetric with respect its indices α, β, γ due to the commutative behaviour of the partial derivatives $\frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta}, \frac{\partial}{\partial t^\gamma}$.

(2) **Frobenius condition**

$$\begin{aligned}\eta(\partial_\alpha \bullet \partial_\beta, \partial_\gamma) &= c_{\alpha\beta}^{\epsilon}\eta(\partial_\epsilon, \partial_\gamma)\\ &= c_{\alpha\beta}^{\epsilon}\eta_{\epsilon\gamma}\\ &= c_{\alpha\beta\gamma}\\ &= c_{\beta\gamma}^{\epsilon}\eta_{\alpha\epsilon} = \eta(\partial_\alpha, \partial_\beta \bullet \partial_\gamma).\end{aligned}$$

Lemma proved.

□

LEMMA 5.9.4. The unit vector field defined in (5.32) is the unit of the algebra defined in lemma 5.9.3.

PROOF. Substituting (5.129) and (5.127) in (5.121), we obtain

$$\Gamma_{\gamma}^{\alpha\beta} = d_\beta c_{\gamma}^{\alpha\beta}, \tag{5.139}$$

where

$$c_{\gamma}^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\epsilon}c_{\epsilon\mu\gamma}. \tag{5.140}$$

Substituting $\alpha = \tau$ in (5.139) and using lemma 5.7.2

$$\begin{aligned}\Gamma_{\gamma}^{\tau\beta} &= 2\pi i d_\beta \delta_{\gamma}^{\beta},\\ &= d_\beta c_{\gamma}^{\tau\beta}.\end{aligned}$$

Then,

$$c_{0\gamma}^{\beta} = \delta_{\gamma}^{\beta}.$$

Computing

$$\partial_0 \bullet \partial_\gamma = c_{0\gamma}^{\beta}\partial_\beta = \partial_\gamma.$$

Lemma proved. □

LEMMA 5.9.5. The algebra defined in lemma 5.9.3 is associative.

PROOF. Recall that the Christoffel symbol $\Gamma_{\gamma}^{\alpha\beta}$ is proportional to the structure constant of the algebra defined in lemma 5.9.3 for $\beta \neq \tau$

$$\Gamma_{\gamma}^{\alpha\beta} = d_\beta c_{\gamma}^{\alpha\beta}.$$

Then, using (4.59), we obtain

$$\Gamma_{\sigma}^{\alpha\beta}\Gamma_{\epsilon}^{\sigma\gamma} = \Gamma_{\sigma}^{\alpha\gamma}\Gamma_{\epsilon}^{\sigma\beta} \tag{5.141}$$

Substituting (5.139) in (5.141), we have

$$c_{\sigma}^{\alpha\beta}c_{\epsilon}^{\sigma\gamma} = c_{\sigma}^{\alpha\gamma}c_{\epsilon}^{\sigma\beta}, \quad \text{for} \quad \beta, \gamma \neq \tau.$$

If $\beta = \tau$,

$$\begin{aligned} c_{\sigma}^{\alpha\tau}c_{\epsilon}^{\sigma\gamma} &= 2\pi i\delta_{\sigma}^{\alpha}c_{\epsilon}^{\sigma\gamma} \\ &= 2\pi i c_{\epsilon}^{\alpha\gamma} \\ &= 2\pi i\delta_{\epsilon}^{\sigma}c_{\sigma}^{\alpha\gamma} \\ &= c_{\sigma}^{\alpha\gamma}c_{\epsilon}^{\sigma\tau}. \end{aligned}$$

□

Recall of the covering space of the orbit space of $\mathscr{J}(A_n)$ defined in (5.81), see section 5.6 for details.

THEOREM 5.9.6. The orbit space $\widetilde{\mathbb{C}\oplus\mathbb{C}^n}\oplus\mathbb{H}/\mathscr{J}(A_n)$ with the intersection form (5.27), unit vector field (5.32), and Euler vector field (5.31) has a Dubrovin Frobenius manifold structure.

PROOF. The function (5.118) satisfy a WDVV equation due to the lemmas 5.9.2, 5.9.3, 5.9.4, 5.9.5.

□

REMARK 5.9.1. The Dubrovin Frobenius structure associated with the group $\mathscr{J}(A_n)$ does not live in the orbit space of $\mathscr{J}(A_n)$, but in a suitable covering. This covering is described by the space such that the ring of functions is $\widetilde{M_\bullet}\left[t^1, t^2, .., t^{n+1}, \frac{1}{t^{n+1}}\right]$.

REMARK 5.9.2. There are two interpretations of the Dubrovin Frobenius structure on the orbit space of $\mathscr{J}(A_n)$. In the first one, the Dubrovin Frobenius structure in the orbit space of $\mathscr{J}(A_n)$ exist only locally due to the $SL_2(\mathbb{Z})$ action, then the orbit space of $\mathscr{J}(A_n)$ is said to have a twisted Frobenius structure, see details in appendix B of [**12**]. The second one, the Dubrovin Frobenius manifold structure exist truly in the a covering, where we fix the ambiguity, in this case, we fix a symplectic base of homology and a branching of the root of φ_{n+1}.

CHAPTER 6

Differential geometry of orbit space of Extended Jacobi group A_1

This chapter is based in the work done in [**1**], which have the aim to give a gentle introduction to the general theory of orbit space of the group $\mathscr{J}(\tilde{A}_n)$, see section 8 for details. The focus of this chapter is the definition of a new extension of the finite Coxeter group A_1 such that it contains the affine Weyl group $\tilde{A}_1$ and the Jacobi group $\mathscr{J}(A_1)$. This new extension will be denoted by Extended affine Jacobi group $\mathscr{J}(\tilde{A}_1)$. Further, we prove that from the data of the group $\mathscr{J}(\tilde{A}_1)$, we can reconstruct the Dubrovin Frobenius structute of the Hurwitz space $H_{1,0,0}$ on the orbit space of $\mathscr{J}(\tilde{A}_1)$. The advantage of this orbit space construction is the Chevalley theorem 6.2.9, which gives a global interpretation for orbit space of $\mathscr{J}(\tilde{A}_1)$. Furthermore, it attaches the group $\mathscr{J}(\tilde{A}_1)$ to the Hurwitz space $H_{1,0,0}$, and this fact might be useful in the general understanding of WDVV/group correspondence. The results of this chapter is interesting because the Hurwitz space $H_{1,0,0}$ is well know to have a rich Dubrovin Frobenius structure called tri-hamiltonian structure [**26**], [**27**], and [**25**]. This fact realise the orbit space of $\mathscr{J}(\tilde{A}_1)$ as suitable ambient space for Dubrovin Frobenius submanifolds, furthermore, it gives interesting relation between the integrable systems of the ambient space and the integrable systems of its Dubrovin Frobenius submanifolds.

6.1. The Group $\mathscr{J}(\tilde{A}_1)$

The main goal of this section is to motivate and to define the group $\mathscr{J}(\tilde{A}_1)$. In order to do that, it will be necessary to recall the definition of the group A_1, and some of its extensions. Moreover, the goal is to understand how to derive WDDV solution starting from these groups.

Recall the action of the group A_n in $L^{A_n} \otimes \mathbb{C}$ in section 4.1, but let us concentrate on the simplest possible case, i.e $n = 1$. In this case, the action on $\mathbb{C} \cong L^{A_1} \otimes \mathbb{C}$ is just:

$$v_0 \mapsto -v_0. \tag{6.1}$$

The understanding of the orbit space of A_1 requires a Chevalley theorem 4.2.1 for the ring of invariants. In the A_1 case, the ring of invariants is just

$$\mathbb{C}[v_0^2] \cong \mathbb{C}[a_2],$$

then the orbit space of A_1 is just the

$$Spec\left(\mathbb{C}[v_0^2]\right).$$

In the paper [11] and [12], it was demonstrated that $\mathbb{C}/A_1$ has structure of Dubrovin-Frobenius manifold, furthermore, it is isomorphic to the Hurwitz space $H_{0,1}$, i.e. the space of rational functions with a double pole. The isomorphism can be realized by the following map

$$[v_0] \mapsto \lambda^{A_1}(p, v_0) = (p - v_0)(p + v_0) = p^2 + a_2. \tag{6.2}$$

Note that the isomorphism works, because $\lambda^{A_1}(p, v_0)$ is invariant under the A_1 action. Applying the methods developed in [11] and [12], one can show that the WDVV solution associated with this orbit space is

$$F(t^1) = \frac{(t^1)^3}{6}, \tag{6.3}$$

where t^1 is the flat coordinate of the metric η.

In [12], [15] it was also considered the extended affine A_1 that is denoted by $\tilde{A}_1$. The action on

$$\left(L^{A_1} \otimes \mathbb{C}\right) \oplus \mathbb{C} = \{(v_0, v_1, v_2) \in \mathbb{C}^3 : \sum_{i=0}^{1} v_i = 0\}$$

is:

$$\begin{aligned} v_0 &\mapsto \pm v_0 + \mu_0, \\ v_2 &\mapsto v_2 + \mu_2, \end{aligned} \tag{6.4}$$

where $\mu_0, \mu_2 \in \mathbb{Z}$.

A notion of invariant ring for the group extended affine A_n were define in [15] , and Dubrovin and Zhang proved that this invariant ring for the case $\tilde{A}_1$ is isomorphic to

$$\mathbb{C}[e^{2\pi i v_2} cos(2\pi i v_0), e^{2\pi i v_2}].$$

Therefore, the orbit space of $\tilde{A}_1$ is the weight projective variety associated with

$$Spec\left(\mathbb{C}\left[e^{2\pi i v_2}\cos(2\pi i v_0), e^{2\pi i v_2}\right]\right).$$

Further, a Dubrovin Frobenius manifold structure was built on the orbit space of $\tilde{A}_1$ with the following WDVV solution

$$F(t^1, t^2) = \frac{(t^1)^2 t^2}{2} + e^{t^2}. \tag{6.5}$$

The orbit space of $\tilde{A}_1$ is also associated with a Hurwitz space, but the relation is slightly less straightforward. The first step is to consider the following map

$$[v_0, v_2] \mapsto \lambda^{\tilde{A}_1}(p, v_0, v_2) = e^p + e^{2\pi i v_2}\cos(2\pi i v_0) + e^{2\pi i v_2} e^{-p}$$

The second is to consider the Legendre transformation of S_2 type (Appendix B, and Chapter 5 of [**12**]). Consider

$$b = e^{2\pi i v_2}\cos(2\pi i v_0), \quad a = e^{2\pi i v_2},$$

and the following choice of primary differential $d\tilde{p}$ implicity given by

$$dp = \frac{d\tilde{p}}{\tilde{p}-b}.$$

Then, in this new coordinates $\lambda^{\tilde{A}_1}$ is given by

$$\lambda(\tilde{p},a,b) = \tilde{p} + \frac{a}{\tilde{p}-b} \tag{6.6}$$

Hence, the orbit space of $\tilde{A}_1$ is isomorphic to the Hurwitz space $H_{0,0,0}$, i.e. space of fractional functions with two simple poles.

The next example of group to be considered is the Jacobi group $\mathscr{J}(A_1)$ already considered in 5.1, which acts on

$$\Omega^{\mathscr{J}(A_1)} := \left(L^{A_1}\otimes\mathbb{C}\right)\oplus\mathbb{C}\oplus\mathbb{H} = \{(v_0,v_1,u,\tau)\in\mathbb{C}^3\oplus\mathbb{H} : \sum_{i=0}^{1} v_i \in \mathbb{Z}+\tau\mathbb{Z}\}$$

as follows:
A_1 action:

$$v_0 \mapsto -v_0, \quad u\mapsto u, \quad \tau\mapsto\tau. \tag{6.7}$$

Translations:

$$v_0 \mapsto v_0+\mu_0+\lambda_0\tau, \quad u\mapsto u-\lambda_0 v_0 - \frac{\lambda_0^2}{2}\tau, \quad \tau\mapsto\tau, \tag{6.8}$$

where $\mu_0,\lambda_0\in\mathbb{Z}$.
$SL_2(\mathbb{Z})$ action:

$$v_0\mapsto\frac{v_0}{c\tau+d}, \quad u\mapsto u-\frac{cv_0^2}{2(c\tau+d)}, \quad \tau\mapsto\frac{a\tau+b}{c\tau+d}. \tag{6.9}$$

where $a,b,c,d\in\mathbb{Z}$, and $ad-bc=1$.
The notion of invariant ring of $\mathscr{J}(A_1)$ was first defined in [**18**]. However, the definitions stated in [**34**], [**8**], [**9**] are more suitable for this purpose. Then, we use the definition of Jacobi forms 5.2.1 for the case $\mathscr{J}(A_1)$.

DEFINITION 6.1.1. The weak A_1 -invariant Jacobi forms of weight k, and index m are holomorphic functions on $\Omega = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H} \ni (u, v_0, \tau)$ which satisfy

$$
\begin{aligned}
&\varphi(u, -v_0, \tau) = \varphi(u, v_0, \tau), \quad A_1 \text{ invariant condition} \\
&\varphi\left(u - \lambda_0 v_0 - \frac{{\lambda_0}^2}{2}\tau, v_0 + \lambda_0\tau + \mu, \tau\right) = \varphi(u, v_0, \tau) \\
&\varphi\left(u + \frac{cv_0^2}{2(c\tau+d)}, \frac{v_0}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \varphi(u, v_0, \tau) \\
&E\varphi(u, v_0, \tau) := \frac{1}{2\pi i}\frac{\partial}{\partial u}\varphi(u, v_0, \tau) = m\varphi(u, v_0, \tau)
\end{aligned} \tag{6.10}
$$

Moreover,

(1) φ is locally bounded functions of v_0 as $\Im(\tau) \mapsto +\infty$ (weak condition).

The space of $\tilde{A}_1$-invariant Jacobi forms of weight k, and index m is denoted by $J_{k,m}^{A_1}$, and $J_{\bullet,\bullet}^{\mathscr{J}(A_1)} = \bigoplus_{k,m} J_{k,m}^{A_1}$ is the space of Jacobi forms A_1 invariant.

In [18], it was proved the following a version of the Chevalley theorem, which is a particular case of the Theorem 5.2.2 and corollary 5.2.3.1 .

THEOREM 6.1.1. Let $J_{\bullet,\bullet}^{\mathscr{J}(A_1)}$ the ring of Jacobi forms A_1 invariant, then

$$J_{\bullet,\bullet}^{\mathscr{J}(A_1)} \cong M_\bullet[\varphi_0, \varphi_2] \tag{6.11}$$

where $M_\bullet$ is the ring of holomorphic modular forms, and

$$
\begin{aligned}
\varphi_2 &= e^{2\pi i u}\left(\frac{\theta_1(v_0, \tau)}{\theta_1'(0, \tau)}\right)^2, \\
\varphi_0 &= \varphi_2 \wp(v_0, \tau),
\end{aligned} \tag{6.12}
$$

θ_1 is the Jacobi theta 1 function (3.25), and $\wp$ is the Weierstrass P function, which is defined as

$$\wp(v, \tau) = \frac{1}{v^2} + \sum_{m^2+n^2 \neq 0}^{\infty} \frac{1}{(v-m-n\tau)^2} - \frac{1}{(m+n\tau)^2}, \tag{6.13}$$

Note that this Chevalley theorem is slightly different from the others, the ring of the coefficients is the ring of holomorphic of modular forms, instead of just $\mathbb{C}$. The geometric interpretation of this fact is that the orbit space of $\mathscr{J}(A_1)$ is a line bundle such that its base is family of elliptic curves E_τ quotient by the group A_1 parametrised by $\mathbb{H}/SL_2(\mathbb{Z})$. In [8] and [9], it was proved that orbit space of $\mathscr{J}(A_1)$ has a Dubrovin Frobenius structure, for the convenience of the reader this result was also prove in chapter 5. Furthermore, the orbit space of $\mathscr{J}(A_1)$ is isomorphic to $H_{1,1}$, i.e space of elliptic functions with 1 double pole. The explicit isomorphism is given by the map

$$[(u, v_0, \tau)] \mapsto \lambda^{\mathscr{J}(A_1)}(v, u, v_0, \tau) = e^{2\pi i u}\frac{\theta_1(v - v_0, \tau)\theta_1(v+v_0, \tau)}{\theta_1^2(v, \tau)} \tag{6.14}$$

As in the A_1 case, the isomorphism is only possible, because the map (6.14) is invariant under (6.7), (6.8), (6.9). A WDVV solution for this case is the following

$$F(t^1, t^2, \tau) = \frac{(t^1)^2\tau}{2} + \frac{t^1(t^2)^2}{2} - \frac{\pi i (t^2)^2}{48} E_2(\tau), \tag{6.15}$$

where

$$E_2(\tau) = 1 + \frac{3}{\pi^2} \sum_{m \neq 0} \sum_{n=-\infty}^{\infty} \frac{1}{(m+n\tau)^2}.$$

A remarkable fact in these orbit space constructions are its correspondences with Hurwitz spaces, which can be summarize in the following diagram.

$$\begin{tikzcd}
H_{0,1} \cong \mathbb{C}/A_1 \arrow[r, "1"] \arrow[d, "2"'] & H_{0,0,0} \cong \mathbb{C}^2/\tilde{A}_1 \arrow[d, "4"] \\
H_{1,1} \cong (\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H}) / \mathscr{J}(A_1) \arrow[r, "3"] & H_{1,0,0} \cong ?
\end{tikzcd}$$

The arrows of the diagram above have a double meaning. The first one is simply an extension of the group, the arrow 2 is "Jacobi" extension, and the arrow 1 is "affine" extension. The second meaning is related with the Hurwitz space side: the arrow 2 and 4 increase by one the genus, and the arrow 1 and 3 split a double pole in 2 simple poles. The missing part of the diagram is exactly the orbit space counter part of $H_{1,0,0}$. The diagram suggest that the new group should be an extension of the A_1 group such that combine the groups $\tilde{A}_1$, and $\mathscr{J}(A_1)$, furthermore, it should preserve $H_{1,0,0}$ in a similiar way for what was done in (6.14). To construct the desired group, we start from the group $\mathscr{J}(A_1)$ and make an extension in order to incorporate the $\tilde{A}_1$ group. Concretely, we extend the domain $\Omega^{\mathscr{J}(A_1)}$ to

$$\Omega^{\mathscr{J}(\tilde{A}_1)} := \Omega^{A_1} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H} = \{(v_0, v_1, v_2, u, \tau) \in \mathbb{C}^4 \oplus \mathbb{H} : v_0 + v_1 \in \mathbb{Z} \oplus \tau\mathbb{Z}\},$$

and we extend the group action $\mathscr{J}(A_1)$ to the following action:
A_1 action:

$$\begin{aligned}
v_0 &\mapsto -v_0, \\
v_2 &\mapsto v_2, \\
u &\mapsto u, \\
\tau &\mapsto \tau.
\end{aligned} \tag{6.16}$$

Translations:

$$\begin{aligned}
v_0 &\mapsto v_0 + \mu_0 + \lambda_0\tau, \\
v_2 &\mapsto v_2 + \mu_2 + \lambda_2\tau, \\
u &\mapsto u - 2\lambda_0 v_0 + 2\lambda_2 v_2 - \lambda_0^2\tau + \lambda_2^2\tau + k. \\
\tau &\mapsto \tau
\end{aligned} \tag{6.17}$$

where $(\lambda_0, \lambda_2), (\mu_0, \mu_2) \in \mathbb{Z}^2$, and $k \in \mathbb{Z}$.
$SL_2(\mathbb{Z})$ action:

$$\begin{aligned} v_0 &\mapsto \frac{v_0}{c\tau + d}, \\ v_2 &\mapsto \frac{v_2}{c\tau + d}, \\ u &\mapsto u + \frac{c(v_0^2 - v_2^2)}{(c\tau + d)}, \\ \tau &\mapsto \frac{a\tau + b}{c\tau + d}. \end{aligned} \tag{6.18}$$

where $a, b, c, d \in \mathbb{Z}$, and $ad - bc = 1$.
The group action (6.16), (6.17), and (6.18) is called extended affine Jacobi group A_1, and denoted by $\mathscr{J}(\tilde{A}_1)$.

Remark 6.1.1. The translations of the group $\tilde{A}_1$ is a subgroup of the translations of the group $\mathscr{J}(\tilde{A}_1)$. Therefore, it is in that sense that $\mathscr{J}(\tilde{A}_1)$ is a combination of $\tilde{A}_1$ and $\mathscr{J}(A_1)$.

In order to rewrite the action of $\mathscr{J}(\tilde{A}_1)$ in an intrinsic way, consider the A_1 in the following extended space

$$L^{\tilde{A}_1} = \{(z_0, z_1, z_2) \in \mathbb{Z}^3 : \sum_{i=0}^{3} z_i = 0\}.$$

The action of A_1 on $L^{\tilde{A}_1}$ is given by

$$w(z_0, z_1, z_2) = (z_1, z_0, z_2)$$

permutations in the first 2 variables. Moreover, A_1 also acts on the complexfication of $L^{\tilde{A}_1} \otimes \mathbb{C}$. Let us use the following identification $\mathbb{Z}^2 \cong L^{\tilde{A}_1}, \mathbb{C}^2 \cong L^{\tilde{A}_1} \otimes \mathbb{C}$ that is possible due to the maps

$$(v_0, v_2) \mapsto (v_0, -v_0, v_2),$$
$$(v_0, v_1, v_2) \mapsto (v_0, v_2).$$

The action of A_1 on $\mathbb{C}^2 \ni v = (v_0, v_2)$ is:

$$w(v) = w(v_0, v_2) = (-v_0, v_2).$$

Let the quadratic form $\langle , \rangle_{\tilde{A}_1}$ is given by

$$\begin{aligned} \langle v, v \rangle_{\tilde{A}_1} &= v^T M_{\tilde{A}_1} v \\ &= v^T \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} v \\ &= 2v_0^2 - 2v_2^2. \end{aligned} \tag{6.19}$$

Consider the following group $L^{\tilde{A}_1} \times L^{\tilde{A}_1} \times \mathbb{Z}$ with the following group operation

$$\forall(\lambda, \mu, k), (\tilde{\lambda}, \tilde{\mu}, \tilde{k}) \in L^{\tilde{A}_1} \times L^{\tilde{A}_1} \times \mathbb{Z},$$
$$(\lambda, \mu, k) \bullet (\tilde{\lambda}, \tilde{\mu}, \tilde{k}) = (\lambda + \tilde{\lambda}, \mu + \tilde{\mu}, k + \tilde{k} + \langle \lambda, \tilde{\lambda} \rangle_{\tilde{A}_1}).$$

Note that $\langle , \rangle_{\tilde{A}_1}$ is invariant under A_1 group, then A_1 acts on $L^{\tilde{A}_1} \times L^{\tilde{A}_1} \times \mathbb{Z}$. Hence, we can take the semidirect product $A_1 \ltimes (L^{\tilde{A}_1} \times L^{\tilde{A}_1} \times \mathbb{Z})$ given by the following product.

$$\forall(w, \lambda, \mu, k), (\tilde{w}, \tilde{\lambda}, \tilde{\mu}, \tilde{k}) \in A_1 \times L^{\tilde{A}_1} \times L^{\tilde{A}_1} \times \mathbb{Z},$$
$$(w, \lambda, \mu, k) \bullet (\tilde{w}, \tilde{\lambda}, \tilde{\mu}, \tilde{k}) = (w\tilde{w}, w\lambda + \tilde{\lambda}, w\mu + \tilde{\mu}, k + \tilde{k} + \langle \lambda, \tilde{\lambda} \rangle_{\tilde{A}_1}).$$

Denoting $W(\tilde{A}_1) := A_1 \ltimes (L^{\tilde{A}_1} \times L^{\tilde{A}_1} \times \mathbb{Z})$, we can define

Definition 6.1.2. The Jacobi group $\mathscr{J}(\tilde{A}_1)$ is defined as a semidirect product $W(\tilde{A}_1) \rtimes SL_2(\mathbb{Z})$. The group action of $SL_2(\mathbb{Z})$ on $W(\tilde{A}_1)$ is defined as

$$Ad_\gamma(w) = w,$$
$$Ad_\gamma(\lambda, \mu, k) = (a\mu - b\lambda, -c\mu + d\lambda, k + \frac{ac}{2}\langle \mu, \mu \rangle_{\tilde{A}_1} - bc\langle \mu, \lambda \rangle_{\tilde{A}_1} + \frac{bd}{2}\langle \lambda, \lambda \rangle_{\tilde{A}_1}).$$

for $(w, t = (\lambda, \mu, k)) \in W(\tilde{A}_1), \gamma \in SL_2(\mathbb{Z})$. Then the multiplication rule is given as follows

$$(w, t, \gamma) \bullet (\tilde{w}, \tilde{t}, \tilde{\gamma}) = (w\tilde{w}, tAd_\gamma(w\tilde{t}), \gamma\tilde{\gamma}).$$

Then the action of Jacobi group $\mathscr{J}(\tilde{A}_1)$ on $\Omega^{\mathscr{J}(\tilde{A}_1)} := \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H} \in (u, v, \tau)$ is described by the main three generators

$$\hat{w} = \left(w, 0, I_{SL_2(\mathbb{Z})}\right),$$
$$t = \left(I_{A_1}, \lambda, \mu, k, I_{SL_2(\mathbb{Z})}\right),$$
$$\gamma = \left(I_{A_1}, 0, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

, which acts on $\Omega^{\mathscr{J}(\tilde{A}_1)}$ as follows:

$$\hat{w}(u, v = (v_0, v_2), \tau) = (u, -v_0, v_2, \tau),$$
$$t(u, v = (v_0, v_2), \tau) = \left(u - \langle \lambda, v \rangle_{\tilde{A}_1} - \frac{1}{2}\langle \lambda, \lambda \rangle_{\tilde{A}_1}\tau + k, v_0 + \lambda_0\tau + \mu_0, v_2 + \lambda_2\tau + \mu_2, \tau\right),$$
$$\gamma(u, v = (v_0, v_2), \tau) = \left(u + \frac{c\langle v, v \rangle_{\tilde{A}_1}}{2(c\tau + d)}, \frac{v_0}{c\tau + d}, \frac{v_2}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right),$$

where $\lambda, \mu, k \in L^{\tilde{A}_1} \times L^{\tilde{A}_1} \times \mathbb{Z}$,

$$\lambda = (\lambda_0, \lambda_2), \quad \mu = (\mu_0, \mu_2).$$

Writing in a more condensed way, we have the following proposition:

Proposition 6.1.2. The group $\mathscr{J}(\tilde{A}_1) \ni (\hat{w}, t, \gamma)$ acts on $\Omega := \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H} \ni (u, v, \tau)$ as follows

$$
\begin{aligned}
&\hat{w}(u, v, \tau) = (u, w(v), \tau)\,,\\
&t(u, v, \tau) = \left(u - \langle \lambda, v\rangle_{\tilde{A}_1} - \frac{1}{2}\langle \lambda, \lambda\rangle_{\tilde{A}_1}\tau + k, v + \lambda\tau + \mu, \tau\right),\\
&\gamma(u, v, \tau) = \left(u + \frac{c\langle v, v\rangle_{\tilde{A}_1}}{2(c\tau + d)}, \frac{v}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right).
\end{aligned}
\tag{6.20}
$$

Substituting (6.19) in (6.20), we get the transformation law (6.16),(6.17), and (6.18). The explanation of why (6.20) is an group action for $\mathscr{J}(\tilde{A}_1)$ is just straightforward computations, but a bit long, then, this part of the proof will be omitted.

6.2. Jacobi forms of $\mathscr{J}(\tilde{A}_1)$

In order to understand the differential geometry of orbit space, first we need to study the algebra of the invariant functions. Informally, every time that there is a group W acting on a vector space V, one could think the orbit spaces V/W as V, but you should remember yourself that it is only allowed to use the $W-$invariant sections of V. Hence, motivated by the definition of Jacobi forms of group A_n defined in [**34**], and used in the context of Dubrovin-Frobenius manifold in [**8**],[**9**], and summarised in Chapter 5 we give the following:

Definition 6.2.1. The weak $\mathscr{J}(\tilde{A}_1)$ -invariant Jacobi forms of weight $k \in \mathbb{Z}$, order $l \in \mathbb{N}$, and index $m \in \mathbb{N}$ are functions on $\Omega = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H} \ni (u, v_0, v_2, \tau) = (u, v, \tau)$ which satisfy

$$
\begin{aligned}
&\varphi\left(w\left(u, v, \tau\right)\right) = \varphi(u, v, \tau), \quad A_1 \text{ invariant condition}\\
&\varphi\left(t\left(u, v, \tau\right)\right) = \varphi(u, v, \tau),\\
&\varphi\left(\gamma\left(u, v, \tau\right)\right) = \left(c\tau + d\right)^{-k}\varphi(u, v, \tau),\\
&E\varphi(u, v, \tau) := -\frac{1}{2\pi i}\frac{\partial}{\partial u}\varphi(u, v_0, v_2, \tau) = m\varphi(u, v_0, v_2, \tau).
\end{aligned}
\tag{6.21}
$$

Moreover,

(1) φ is locally bounded functions of v_0 as $\Im(\tau) \mapsto +\infty$ (weak condition).
(2) For fixed u, v_0, τ the function $v_2 \mapsto \varphi(u, v_0, v_2, \tau)$ is meromorphic with poles of order at most $l + 2m$ at in $v_2 = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \mod \mathbb{Z} \oplus \tau\mathbb{Z}$.
(3) For fixed $u, v_2 \neq 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \mod \mathbb{Z} \oplus \tau\mathbb{Z}, \tau$ the function $v_0 \mapsto \varphi(u, v_0, v_2, \tau)$ is holomorphic.
(4) For fixed $u, v_0, v_2 \neq 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \mod \mathbb{Z} \oplus \tau\mathbb{Z}$. the function $\tau \mapsto \varphi(u, v_0, v_2, \tau)$ is holomorphic.

The space of $\tilde{A}_1$-invariant Jacobi forms of weight k, order l, and index m is denoted by $J^{\tilde{A}_1}_{k,l,m}$, and $J^{\mathscr{J}(\tilde{A}_1)}_{\bullet,\bullet,\bullet} = \bigoplus_{k,l,m} J^{\tilde{A}_1}_{k,l,m}$ is the space of Jacobi forms $\tilde{A}_1$ invariant.

Remark 6.2.1. The condition $E\varphi(u, v_0, v_2, \tau) = m\varphi(u, v_0, v_2, \tau)$ implies that $\varphi(u, v_0, v_2, \tau)$ has the following form

$$\varphi(u, v_0, v_2, \tau) = f(v_0, v_2, \tau)e^{2\pi i m u}$$

and the function $f(v_0, v_2, \tau)$ has the following transformation law

$$\begin{aligned} f(v_0, v_2, \tau) &= f(-v_0, v_2, \tau) \\ f(v_0, v_2, \tau) &= e^{-2\pi i m(\langle\lambda, v\rangle + \frac{\langle\lambda,\lambda\rangle}{2}\tau)} f(v_0 + m_0 + n_0\tau, v_2 + m_2 + n_2\tau, \tau) \\ f(v_0, v_2, \tau) &= (c\tau + d)^{-k} e^{2\pi i m(\frac{c\langle v,v\rangle}{(c\tau+d)})} f(\frac{v_0}{c\tau + d}, \frac{v_2}{c\tau + d}, \frac{a\tau + b}{c\tau + d}) \end{aligned} \tag{6.22}$$

The functions $f(v_0, v_2, \tau)$ are more closely related to the definition of Jacobi form of Eichler-Zagier type [18]. The coordinate u works as kind of automorphic correction in this functions $f(v_0, v_2, \tau)$. Further, the coordinate u will be crucial to construct an equivariant metric on the orbit space of $\mathscr{J}(\tilde{A}_1)$, see section 6.3.

Remark 6.2.2. Note that the Jacobi forms in the Definition 6.1.1 are holomorphic, and in the Definition 6.2.1, the Jacobi forms are meromorphic in the variable v_2. This fact will also reflect in the difference between the Chevalley theorems of 6.2.1, and 6.1.1. See Theorem 6.2.9 for details.

The main result of this section is the following.

The ring of $\tilde{A}_1$ invariant Jacobi forms is polynomial over a suitable ring $E_{\bullet,\bullet} := J^{\mathscr{J}(\tilde{A}_1)}_{\bullet,\bullet,0}$ on suitable generators φ_0, φ_1.
Before stating precisely the theorem, I will define the objects $E_{\bullet,\bullet}, \varphi_0, \varphi_1$.

The ring $E_{\bullet,l} := J^{\mathscr{J}(\tilde{A}_1)}_{\bullet,l,0}$ is the space of meromorphic Jacobi forms of index 0 with poles of order at most l at $0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \mod \mathbb{Z} \oplus \tau\mathbb{Z}$, by definition. The sub-ring $J^{\mathscr{J}(\tilde{A}_1)}_{\bullet,0,0} \subset E_{\bullet,\bullet}$ has a nice structure, indeed:

Lemma 6.2.1. The sub-ring $J^{\mathscr{J}(\tilde{A}_1)}_{\bullet,0,0}$ is equal to $M_\bullet := \bigoplus M_k$, where M_k is the space of modular forms of weight k for the full group $SL_2(\mathbb{Z})$.

Proof. Using the Remark 6.2.1, we know that functions $\varphi(u, v_0, v_2, \tau) \in J^{\mathscr{J}(\tilde{A}_1)}_{\bullet,0,0}$ can not depend on u, then $\varphi(u, v_0, v_2, \tau) = \varphi(v_0, v_2, \tau)$. Moreover, for fixed v_2, τ the functions $v_0 \mapsto \varphi(v_0, v_2, \tau)$ are holomorphic elliptic function. Therefore, by Liouville theorem, these function are constant in v_0. Similar argument shows that these function do not depend on v_2, because $l + 2m = 0$, i.e there is no pole. Then, $\varphi = \varphi(\tau)$ are standard holomorphic modular forms. $\square$

LEMMA 6.2.2. If $\varphi \in E_{\bullet,\bullet} = J^{\mathscr{J}(\tilde{A}_1)}_{\bullet,\bullet,0}$, then φ depends only on the variables v_2, τ. Moreover, if $\varphi \in J^{\mathscr{J}(\tilde{A}_1)}_{0,l,0}$ for fixed τ the function $v_2 \mapsto \varphi(v_2, \tau)$ is an elliptic function with poles of order at most l on $0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ mod $\mathbb{Z} \oplus \tau\mathbb{Z}$.

PROOF. The proof is essentially the same of the lemma (6.2.1), the only difference is that now we have poles at $v_2 = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ mod $\mathbb{Z} \oplus \tau\mathbb{Z}$. Then, we have dependence on v_2. □

As a consequence of lemma 6.2.2, the function $\varphi \in E_{k,l} = J^{\mathscr{J}(\tilde{A}_1)}_{k,l,0}$ has the following form

$$\varphi(v_2, \tau) = f(\tau) g(v_2, \tau)$$

where $f(\tau)$ is holomorphic modular form of weight k, and for fixed τ, the function $v_2 \mapsto g(v_2, \tau)$ is an elliptic function of order at most l at the poles $0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ mod $\mathbb{Z} \oplus \tau\mathbb{Z}$.

At this stage, we are able to define φ_0, φ_1. Note that a natural way to produce meromorphic Jacobi forms is by using rational functions of holomorphic Jacobi forms. Starting from now, we will denote the Jacobi forms related with the Jacobi group $\mathscr{J}(A_1)$ with the upper index $\mathscr{J}(A_1)$, for instance

$$\varphi^{\mathscr{J}(A_1)},$$

and the Jacobi forms related with the Jacobi group $\mathscr{J}(\tilde{A}_1)$ with the upper index $\mathscr{J}(\tilde{A}_1)$

$$\varphi^{\mathscr{J}(\tilde{A}_1)}.$$

In [8], Bertola found basis of the generators of the Jacobi form algebra by producing a holomorphic Jacobi form of type A_n as product of theta functions.

$$\varphi^{\mathscr{J}(A_n)} = e^{2\pi i u} \prod_{i=1}^{n+1} \frac{\theta_1(z_i, \tau)}{\theta_1'(0, \tau)}. \tag{6.23}$$

Afterwards, Bertola defined a recursive operator to produce the remaining basic generators. In order to recall the details see section 5.2. Our strategy will follow the same logic of Bertola method, we use theta functions to produce a basic generator and thereafter, we produce a recursive operator to produce the remaining part.

LEMMA 6.2.3. Let be $\varphi_3^{\mathscr{J}(A_2)}(u_1, z_1, z_2, \tau)$ the holomorphic $A_2 - \textit{invariant}$ Jacobi form which correspond to the algebra generator of maximal weight degree, in this case degree 3. More explicitly,

$$\varphi_3^{\mathscr{J}(A_2)} = e^{-2\pi i u_1} \frac{\theta_1(z_1, \tau)\theta_1(z_2, \tau)\theta_1(-z_1 - z_2, \tau)}{\theta_1'(0, \tau)^3}. \tag{6.24}$$

Let be $\varphi_2^{\mathscr{J}(A_1)}(u_2, z_3, \tau)$ the holomorphic $A_1 - \textit{invariant}$ Jacobi form which correspond to the algebra generator of maximal weight degree, in this case degree 2.

$$\varphi_2^{\mathscr{J}(A_1)} = e^{-2\pi i u_2} \frac{\theta_1(z_3, \tau)^2}{\theta_1'(0, \tau)^2}. \tag{6.25}$$

Then, the function

$$\varphi_1^{\mathscr{J}(\tilde{A}_1)} = \frac{\varphi_3^{\mathscr{J}(A_2)}}{\varphi_2^{\mathscr{J}(A_1)}} \tag{6.26}$$

is meromorphic Jacobi form of index 1, weight -1, order 0.

Proof. For our convenience, we change the labels z_1, z_2, z_3 to $v_0 + v_2, -v_0 + v_2, 2v_2$ respectively. Then (6.26) has the following form

$$\varphi_1^{\mathscr{J}(\tilde{A}_1)}(u, v_0, v_2, \tau) = e^{-2\pi i u}\frac{\theta_1(v_0 + v_2, \tau)\theta_1(-v_0 + v_2, \tau)}{\theta_1'(0, \tau)\theta_1(2v_2, \tau)} \tag{6.27}$$

Let us prove each item separated.

(1) A_1 **invariant**
The A_1 group acts on (6.27) by permuting its roots, thus (6.27) remains invariant under this operation.

(2) **Translation invariant**
Recall that under the translation $v \mapsto v + m + n\tau$, the Jacobi theta function transform as [8], [33]:

$$\theta_1(v_i + \mu_i + \lambda_i\tau, \tau) = (-1)^{\lambda_i+\mu_i}e^{-2\pi i(\lambda_i v_i + \frac{\lambda_i^2}{2}\tau)}\theta_1(v_i, \tau). \tag{6.28}$$

Then substituting the transformation (6.42) into (6.27), we conclude that (6.27) remains invariant.

(3) $SL_2(\mathbb{Z})$ **invariant**
Under $SL_2(\mathbb{Z})$ action the following function transform as

$$\frac{\theta_1\left(\frac{v_i}{c\tau+d}, \frac{a\tau+d}{c\tau+d}\right)}{\theta_1'\left(0, \frac{a\tau+d}{c\tau+d}\right)} = (c\tau + d)^{-1}\exp\left(\frac{\pi i c v_i^2}{c\tau + d}\right)\frac{\theta_1(v_i, \tau)}{\theta_1'(0, \tau)}. \tag{6.29}$$

Then, substituting (6.43) in (6.27), we get

$$\varphi_1^{\mathscr{J}(\tilde{A}_1)} \mapsto \frac{\varphi_1^{\mathscr{J}(\tilde{A}_1)}}{c\tau + d}$$

(4) **Index** 1

$$\frac{1}{2\pi i}\frac{\partial}{\partial u}\varphi_1\mathscr{J}(\tilde{A}_1) = \varphi_1^{\mathscr{J}(\tilde{A}_1)}. \tag{6.30}$$

(5) **Analytic behavior**
Note that $\varphi_1^{\mathscr{J}(\tilde{A}_1)}\theta_1^2(2v_2, \tau)$ is holomorphic function in all the variables v_i. Therefore $\varphi_1^{\mathscr{J}(\tilde{A}_1)}$ are holomorphic functions on the variables v_0, and meromorphic function in the variable v_2 with poles on $\frac{j}{2} + \frac{l\tau}{2}, j, l = 0, 1$ of order 2, i.e $l = 0$, since $m = 1$.

□

In order to define the desired recursive operator, it is necessary to enlarge the domain of the Jacobi forms from $\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H} \ni (u, v_0, v_2, \tau)$ to $\mathbb{C} \oplus \mathbb{C}^3 \oplus \mathbb{H} \ni (u, v_0, v_2, p, \tau)$. In addition, we define a lift of Jacobi forms defined in $\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H}$ to $\mathbb{C} \oplus \mathbb{C}^3 \oplus \mathbb{H}$ as

$$\varphi(u, v_0 + v_2, -v_0 + v_2, \tau) \mapsto \hat{\varphi}(p) := \varphi(u, v_0 + v_2 + p, -v_0 + v_2 + p, \tau)$$

A convenient way to do computation in these extended Jacobi forms is by using the following coordinates

$$\begin{aligned} s &= u + g_1(\tau)p^2, \\ z_1 &= v_0 + v_2 + p, \\ z_2 &= -v_0 + v_2 + p, \\ z_3 &= 2v_2 + p, \\ \tau &= \tau. \end{aligned} \tag{6.31}$$

The bilinear form $\langle v, v \rangle_{\tilde{A}_1}$ is extended to

$$\langle (z_1, z_2, z_3), (z_1, z_2, z_3) \rangle_E = z_1^2 + z_2^2 - z_3^2, \tag{6.32}$$

or equivalently,

$$\langle (v_0, v_2, p), (v_0, v_2, p) \rangle_E = 2v_0^2 - 2v_2^2 + p^2. \tag{6.33}$$

The action of the Jacobi group $\tilde{A}_1$ in this extended space is

$$\begin{aligned} \hat{w}_E(u, v, p, \tau) &= (u, w(v), p, \tau) \\ t_E(u, v, p, \tau) &= \left(u - \langle \lambda, v \rangle_E - \frac{1}{2} \langle \lambda, \lambda \rangle_E \tau + k, v + p + \lambda\tau + \mu, \tau \right) \\ \gamma_E(u, v, p, \tau) &= \left(u + \frac{c\langle v, v \rangle_E}{2(c\tau + d)}, \frac{v}{c\tau + d}, \frac{p}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) \end{aligned} \tag{6.34}$$

Proposition 6.2.4. Let be $\varphi \in J_{k,m,\bullet}^{\mathscr{J}(\tilde{A}_1)}$, and $\hat{\varphi}$ the correspondent extended Jacobi form. Then,

$$\frac{\partial}{\partial p} (\hat{\varphi}) \bigg|_{p=0} \in J_{k-1,m,\bullet}^{\mathscr{J}(\tilde{A}_1)}. \tag{6.35}$$

Proof. (1) A_1**-invariant**

The vector field $\frac{\partial}{\partial p}$ in coordinates s, z_1, z_2, z_3, τ reads

$$\frac{\partial}{\partial p} = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3} + 2g_1(\tau)p\frac{\partial}{\partial u}$$

Moreover, in the coordinates s, z_1, z_2, z_3, τ the A_1 group acts by permuting z_1 and z_2. Then

$$\begin{aligned}\frac{\partial}{\partial p}\left(\varphi(s,z_2,z_1,z_3,\tau)\right)\Big|_{p=0} &= \left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}+\frac{\partial}{\partial z_3}\right)\left(\varphi(s,z_2,z_1,z_3,\tau)\right)|_{p=0}\\ &= \left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}+\frac{\partial}{\partial z_3}\right)\left(\varphi(s,z_1,z_2,z_3,\tau)\right)|_{p=0}.\end{aligned}$$

(2) **Translation invariant**

$$\begin{aligned}&\frac{\partial}{\partial p}\left(\varphi(u-\langle\lambda,v\rangle_E-\langle\lambda,\lambda\rangle_E, v+p+\lambda\tau+\mu,\tau)\right)\Big|_{p=0}\\ &= \frac{\partial}{\partial p}\langle\lambda,v\rangle_E\Big|_{p=0}\varphi(u,v,\tau)+\frac{\partial\varphi}{\partial p}\left(u-\langle\lambda,v\rangle_{\tilde A_1}-\frac{1}{2}\langle\lambda,\lambda\rangle_{\tilde A_1}\tau+k, v+\lambda\tau+\mu,\tau\right)\\ &= \frac{\partial\varphi}{\partial p}\left(u-\langle\lambda,v\rangle_{\tilde A_1}-\frac{1}{2}\langle\lambda,\lambda\rangle_{\tilde A_1}\tau+k, v+\lambda\tau+\mu,\tau\right)\\ &= \frac{\partial\varphi}{\partial p}(u,v,\tau).\end{aligned}$$

(3) $SL_2(\mathbb{Z})$ **equivariant of weight** k

$$\begin{aligned}&\frac{\partial}{\partial p}\left(\varphi(u+\frac{c\langle v,v\rangle_E}{2(c\tau+d)},\frac{v}{c\tau+d},\frac{p}{c\tau+d},\frac{a\tau+b}{c\tau+d})\right)\Big|_{p=0}\\ &= \frac{c}{2(c\tau+d)}\frac{\partial}{\partial p}\langle v,v\rangle_E\Big|_{p=0}\varphi(u,v,\tau)+\frac{1}{c\tau+d}\frac{\partial\varphi}{\partial p}\left(u+\frac{c\langle v,v\rangle_E}{2(c\tau+d)},\frac{v}{c\tau+d},\frac{p}{c\tau+d},\frac{a\tau+b}{c\tau+d})\right)\\ &= \frac{1}{c\tau+d}\frac{\partial\varphi}{\partial p}\left(u+\frac{c\langle v,v\rangle_E}{2(c\tau+d)},\frac{v}{c\tau+d},\frac{p}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)\\ &= \frac{1}{(c\tau+d)^k}\frac{\partial\varphi}{\partial p}(u,v,\tau).\end{aligned}$$

Then,

$$\frac{\partial\varphi}{\partial p}\left(u+\frac{c\langle v,v\rangle_E}{2(c\tau+d)},\frac{v}{c\tau+d},\frac{p}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)=\frac{1}{(c\tau+d)^{k-1}}\frac{\partial\varphi}{\partial p}(u,v,\tau)$$

(4) **Index 1**

$$\frac{1}{2\pi i}\frac{\partial}{\partial u}\frac{\partial}{\partial p}\hat\varphi=\frac{1}{2\pi i}\frac{\partial}{\partial p}\frac{\partial}{\partial u}\hat\varphi=\frac{\partial}{\partial p}\hat\varphi.$$

□

Corollary 6.2.4.1. The function

$$\left[e^{z\frac{\partial}{\partial p}}\left(e^{2\pi i u}\frac{\theta_1(v_0+v_2+p)\theta_1(-v_0+v_2+p)}{\theta_1(2v_2+p)\theta_1'(0)}\right)\right]\Big|_{p=0}=\varphi_1^{\mathscr{J}(\tilde A_1)}+\varphi_0^{\mathscr{J}(\tilde A_1)}z+O(z^2), \tag{6.36}$$

generates the Jacobi forms $\varphi_0^{\mathscr{J}(\tilde A_1)}$ and $\varphi_1^{\mathscr{J}(A_1)}$, where

$$\varphi_0^{\mathscr{J}(\tilde A_1)}:=\frac{\partial}{\partial p}\left(\hat\varphi_1^{\mathscr{J}(\tilde A_1)}\right)\Big|_{p=0}. \tag{6.37}$$

PROOF. Acting $\frac{\partial}{\partial p}$ k times in $\varphi_1^{\mathscr{J}(\tilde{A}_1)}$, we have

$$\left[\frac{\partial^k}{\partial^k p}\left(e^{2\pi i u}\frac{\theta_1(v_0+v_2+p)\theta_1(-v_0+v_2+p)}{\theta_1(2v_2+p)\theta_1'(0)}\right)\right]\Big|_{p=0} \in J_{1-k,1,\bullet}^{\mathscr{J}(\tilde{A}_1)}.$$

□

COROLLARY 6.2.4.2. The generating function can be written as

$$\left[e^{z\frac{\partial}{\partial p}}\left(e^{2\pi i u}\frac{\theta_1(v_0+v_2+p)\theta_1(-v_0+v_2+p)}{\theta_1(2v_2+p)\theta_1'(0)}\right)\right]\Big|_{p=0} = e^{-2\pi i(u+ig_1(\tau)z^2)}\frac{\theta_1(z-v_0+v_2,\tau)\theta_1(z+v_0+v_2,\tau)}{\theta_1'(0)\theta_1(z+2v_2)}. \tag{6.38}$$

PROOF.

$$\begin{aligned}
&\left[e^{z\frac{\partial}{\partial p}}\left(e^{2\pi i u}\frac{\theta_1(v_0+v_2+p)\theta_1(-v_0+v_2+p)}{\theta_1'(0)\theta_1(2v_2+p)}\right)\right]\Big|_{p=0} = \\
&= \left[e^{z\frac{\partial}{\partial p}}\left(e^{2\pi i(s+ig_1(\tau)p^2}\frac{\theta_1(v_0+v_2+p)\theta_1(-v_0+v_2+p)}{\theta_1(2v_2+p)\theta_1'(0)}\right)\right]\Big|_{p=0} \\
&= e^{-2\pi i(u+ig_1(\tau)z^2)}\frac{\theta_1(z-v_0+v_2,\tau)\theta_1(z+v_0+v_2,\tau)}{\theta_1'(0)\theta_1(z+2v_2)}.
\end{aligned} \tag{6.39}$$

□

The next lemma is one of the main points of this Chapter, because this lemma identify the orbit space of the group $\mathscr{J}(\tilde{A}_1)$ with the Hurwitz space $H_{1,0,0}$. This relationship is possible due to the construction of the generating function of the Jacobi forms of type $\tilde{A}_1$, which can be completed to be the Landau-Ginzburg superpotential of $H_{1,0,0}$ as follows

$$e^{-2\pi i(u+ig_1(\tau)z^2)}\frac{\theta_1(z-v_0+v_2,\tau)\theta_1(z+v_0+v_2,\tau)}{\theta_1'(0)\theta_1(z+2v_2)} \mapsto e^{-2\pi i u}\frac{\theta_1(v-v_0+v_2,\tau)\theta_1(v+v_0+v_2,\tau)}{\theta_1(v\tau)\theta_1(v+2v_2,\tau)}. \tag{6.40}$$

LEMMA 6.2.5. There exists a local isomorphism between $\Omega/\mathscr{J}(\tilde{A}_1)$ and $H_{1,0,0}$.

PROOF. The correspondence is realized by the map:

$$[(u,v_0,v_2,\tau)] \longleftrightarrow \lambda(v) = e^{-2\pi i u}\frac{\theta_1(v-v_0,\tau)\theta_1(v+v_0,\tau)}{\theta_1(v-v_2,\tau)\theta_1(v+v_2,\tau)} \tag{6.41}$$

where $\theta_1(v,\tau)$ is the Jacobi theta function defined on (3.25).
It is necessary to prove that the map is well defined and one to one.

(1) **Well defined**
Note that the map does not depend on the choice of the representative of $[(u,v_0,v_2,\tau)]$ if the function (6.41) is invariant under the action of $\mathscr{J}(\tilde{A}_1)$. Therefore, let us prove the invariance of the map (6.41).

(2) A_1 **invariant**
The A_1 group acts on (6.41) by permuting its roots, thus (6.41) remains invariant under this operation.

(3) **Translation invariant**

Recall that under the translation $v \mapsto v + m + n\tau$, the Jacobi theta function transform as [**33**]:

$$\theta_1(v_i + \mu_i + \lambda_i\tau, \tau) = (-1)^{\lambda_i+\mu_i} e^{-2\pi i(\lambda_i v_i + \frac{\lambda_i^2}{2}\tau)} \theta_1(v_i, \tau). \tag{6.42}$$

Then substituting the transformation (6.42) into (6.41), we conclude that (6.41) remains invariant.

(4) $SL_2(\mathbb{Z})$ **invariant**

Under $SL_2(\mathbb{Z})$ action the following function transform as

$$\frac{\theta_1\left(\frac{v_i}{c\tau+d}, \frac{a\tau+d}{c\tau+d}\right)}{\theta_1'\left(0, \frac{a\tau+d}{c\tau+d}\right)} = (c\tau+d)^{-1} \exp\left(\frac{\pi i c v_i^2}{c\tau+d}\right) \frac{\theta_1(v_i, \tau)}{\theta_1'(0, \tau)}. \tag{6.43}$$

Then substituting the transformation (6.43) into (6.41), we conclude that (6.41) remains invariant.

(5) **Injectivity**

Note that for fixed v, v_0, v_2, u, the function $\tau \mapsto f(\tau) := \lambda(v, v_0, v_2, u, \tau)$ is a modular form with character [**18**]. This is clear because $\lambda(v, v_0, v_2, u, \tau)$ is rational function of $\theta_1(z, \tau)$, which is modular form with character for special values of z [**18**]. Let $\lambda(v, v_0, v_2, u, \tau) = \lambda(v, \hat{v}_0, \hat{v}_2, \hat{u}, \hat{\tau})$, then for fixed $v, v_0, v_2, u, \hat{v}_0, \hat{v}_2, \hat{u}$, we have $f(\tau) = f(\hat{\tau})$, in particular, $f(\tau), f(\hat{\tau})$ have the same vanishing order, and this implies that $\tau, \hat{\tau}$ belongs to the same $SL_2(\mathbb{Z})$ orbit.

Two elliptic functions are equal if they have the same zeros and poles with multiplicity mod $\mathbb{Z} \oplus \tau\mathbb{Z}$. Then, for a fixed τ in the $SL_2(\mathbb{Z})$ orbit

$$\begin{aligned}
&\hat{v}_0 = v_0 + \lambda_0\tau + \mu_0,\\
&\hat{v}_2 = v_2 + \lambda_2\tau + \mu_2,\\
&(\lambda_i, \mu_i) \in \mathbb{Z}^2.
\end{aligned}$$

Furthermore, for two different representative of the same $SL_2(\mathbb{Z})$ orbit, but considering fixed cells, we have

$$\begin{aligned}
\hat{v}_0 &= \frac{v_0}{c\tau+d},\\
\hat{v}_2 &= \frac{v_2}{c\tau+d},\\
\hat{\tau} &= \frac{a\tau+b}{c\tau+d},
\end{aligned}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

Since, $\lambda(v, v_0, v_2, u, \tau)$ is invariant under translations, and $SL_2(\mathbb{Z})$, for $\hat{\tau} = \tau$, we have

$$\hat{u} = u - \langle \lambda, v \rangle_{\tilde{A}_1} - \langle \lambda, \lambda \rangle_{\tilde{A}_1} \frac{\tau}{2} + k.$$

For $\hat{\tau} = \frac{a\tau+b}{c\tau+d}$,

$$\hat{u} = u - \frac{c\langle v, v \rangle_{\tilde{A}_1}}{2(c\tau + d)} + k,$$

where $k \in \mathbb{Z}$.

(6) **Surjectivity**

Any elliptic function can be written as rational functions of Weierstrass sigma function up to a multiplication factor [**33**]. By using the formula

$$\begin{aligned} \sigma(v_i, \tau) &= \frac{\theta_1(v_i, \tau)}{\theta_1'(0, \tau)} \exp(-2\pi i g_1(\tau) v_i^2) \\ g_1(\tau) &= \frac{\eta'(\tau)}{\eta(\tau)} \end{aligned} \tag{6.44}$$

where $\eta(\tau)$ is the Dedekind η function, we get the desire result.

□

Remark 6.2.3. Lemma 6.2.5 is a local equivalence between $H_{1,0,0}$, and the orbit space of $\mathscr{J}(\tilde{A}_1)$, but it is not a global statement. The Theorem 6.2.9 below characterises the ring of invariants of $\Omega^{\mathscr{J}(\tilde{A}_1)}/\mathscr{J}(\tilde{A}_1)$, therefore, we have the global understanding of $\Omega^{\mathscr{J}(\tilde{A}_1)}/\mathscr{J}(\tilde{A}_1)$ by using the ring of functions/ manifold correspondence. On another hand, the Dubrovin Frobenius structure in a Hurwitz space is based on an open dense domain of a solution of a Darboux-Egorrof system [**12**], [**29**]. Hence, it is a local construction . In this way, the construction of the orbit space of $\mathscr{J}(\tilde{A}_1)$ complements the construction of the Hurwitz space $H_{1,0,0}$, because now, there exist global object where the local Dubrovin Frobenius structure of $H_{1,0,0}$ lives. In addition, lemma 6.2.5 associates a group to $H_{1,0,0}$, and this could be useful for the general understanding of the WDDV solutions/ discrete group correspondence [**12**].

Remark 6.2.4. Lemma 6.2.5 is a local biholomorphism of manifolds, but this does not necessarily means isomorphism of Dubrovin Frobenius structure. On a Hurwitz space may exist several inequivalent Dubrovin Frobenius structure. For instance, in [**27**] Romano constructed two generalised WDDV solution on the Hurwitz space $H_{1,0,0}$, furthermore, in [**8**] and [**9**], Bertola constructed two different Dubrovin Frobenius structures on the orbit space of the Jacobi group G_2. The Dubrovin Frobenius structure of this orbit space will be constructed only on section 6.3.

Corollary 6.2.5.1. The functions $(\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1})$ obtained by the formula

$$\begin{aligned}\lambda^{\tilde{A}_1} &= e^{-2\pi i u}\frac{\theta_1(v-v_0,\tau)\theta_1(v+v_0,\tau)}{\theta_1(v-v_2,\tau)\theta_1(v+v_2,\tau)} \\ &= \varphi_1^{\tilde{A}_1}\left[\zeta(v-v_2,\tau)-\zeta(v+v_2,\tau)+2\zeta(v_2,\tau)\right]+\varphi_0^{\tilde{A}_1}\end{aligned} \tag{6.45}$$

are Jacobi forms of weight $0,-1$ respectively, index 1, and order 0. More explicitly,

$$\begin{aligned}\varphi_1^{\tilde{A}_1} &= \frac{\theta_1(v_0+v_2,\tau)\theta_1(-v_0+v_2,\tau)}{\theta_1'(0,\tau)\theta_1(2v_2,\tau)}e^{-2\pi i u}, \\ \varphi_0^{\tilde{A}_1} &= -\varphi_1^{\tilde{A}_1}\left[\zeta(v_0-v_2,\tau)-\zeta(v_0+v_2,\tau)+2\zeta(v_2,\tau)\right],\end{aligned} \tag{6.46}$$

where $\zeta(v,\tau)$ is the Weierstrass zeta function for the lattice $(1,\tau)$, i.e.

$$\zeta(v,\tau)=\frac{1}{v}+\sum_{m^2+n^2\neq 0}^{\infty}\frac{1}{v-m-n\tau}+\frac{1}{m+n\tau}+\frac{v}{(m+n\tau)^2}. \tag{6.47}$$

Proof. Let us prove each item separated.

(1) A_1 **invariant, translation invariant**

The first line of (6.45) are A_1 invariant, and translation invariant by the lemma (6.2.5). Then, by the Laurent expansion of $\lambda^{\tilde{A}_1}$, we have that $\varphi_i^{\tilde{A}_1}$ are A_1 invariant, and translation invariant.

(2) $SL_2(\mathbb{Z})$ **equivariant**

The first line of (6.45) are $SL_2(\mathbb{Z})$ invariant, but the Weierstrass zeta functions of the second line of (6.45) have the following transformation law

$$\zeta\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)\zeta(z,\tau). \tag{6.48}$$

Then, $\varphi_i^{\tilde{A}_1}$ must have the following transformation law

$$\begin{aligned}\varphi_0^{\tilde{A}_1}\left(u+\frac{c\langle v,v\rangle_{\tilde{A}_1}}{2(c\tau+d)},\frac{v}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right) &= \varphi_0^{\tilde{A}_1}(u,v,\tau), \\ \varphi_1^{\tilde{A}_1}\left(u+\frac{c\langle v,v\rangle_{\tilde{A}_1}}{2(c\tau+d)},\frac{v}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)^{-k}\varphi_1^{\tilde{A}_1}(u,v,\tau).\end{aligned} \tag{6.49}$$

(3) **Index** 1

$$\frac{1}{2\pi i}\frac{\partial}{\partial u}\lambda^{\tilde{A}_1}=\lambda^{\tilde{A}_1}. \tag{6.50}$$

Then

$$\frac{1}{2\pi i}\frac{\partial}{\partial u}\varphi_i^{\tilde{A}_1}=\varphi_i^{\tilde{A}_1}. \tag{6.51}$$

(4) **Analytic behavior**
Note that $\lambda^{\tilde{A}_1}\theta_1^2(2v_2,\tau)$ is holomorphic function in all the variables v_i. Therefore $\varphi_i^{\tilde{A}_1}$ are holomorphic functions on the variables v_0, and meromorphic function in the variable v_2 with poles on $\frac{j}{2}+\frac{l\tau}{2}, j,l=0,1$ of order 2, i.e $l=0$, since $m=1$ for all $\varphi_i^{\tilde{A}_1}$.

To prove the formula (6.46) let us compute the following limit

$$\lim_{z\to v_2}\lambda^{\tilde{A}_1}v_2=\varphi_1^{\tilde{A}_1}=e^{-2\pi i u}\frac{\theta_1(v_0+v_2,\tau)\theta_1(-v_0+v_2,\tau)}{\theta_1'(0,\tau)\theta_1(2v_2,\tau)}.$$

Let us compute the zeros of $\lambda^{\tilde{A}_1}$

$$\lambda^{\tilde{A}_1}(v_0)=0=\varphi_1^{\tilde{A}_1}\left[\zeta(v_0-v_2,\tau)-\zeta(v_0+v_2,\tau)+2\zeta(v_2,\tau)\right]+\varphi_0^{\tilde{A}_1}.$$

□

LEMMA 6.2.6. The functions $\varphi_0^{\tilde{A}_1}$, $\varphi_1^{\tilde{A}_1}$ are algebraically independent over the ring $E_{\bullet,\bullet}$.

PROOF. If $P(X,Y)$ is any polynomial in $E_{\bullet,\bullet}(X,Y)$, such that $P(\varphi_0^{\tilde{A}_1},\varphi_1^{\tilde{A}_1})=0$, then, the fact $\varphi_0^{\tilde{A}_1},\varphi_1^{\tilde{A}_1}$ have index implies that each homogeneous component $P_d(\varphi_0^{\tilde{A}_1},\varphi_1^{\tilde{A}_1})$ has to vanish identically. Defining $p_d\left(\frac{\varphi_0^{\tilde{A}_1}}{\varphi_1^{\tilde{A}_1}}\right):=\frac{P_d\left(\varphi_0^{\tilde{A}_1},\varphi_1^{\tilde{A}_1}\right)}{\left(\varphi_1^{(\tilde{A}_1)}\right)^d}$, we have that $p_d\left(\frac{\varphi_0^{\tilde{A}_1}}{\varphi_1^{\tilde{A}_1}}\right)$ is identically 0 iff $\frac{\varphi_0^{\tilde{A}_1}}{\varphi_1^{\tilde{A}_1}}$ is constant (belongs to $E_{\bullet,\bullet}$), but

$$\frac{\varphi_0^{\tilde{A}_1}}{\varphi_1^{\tilde{A}_1}}=\frac{\wp'(v_2,\tau)}{\wp(v_0,\tau)-\wp(v_2,\tau)}\neq a(v_2,\tau) \tag{6.52}$$

where $a(v_2,\tau)$ is any function belongs to $E_{\bullet,\bullet}$. Then, $\varphi_0^{\tilde{A}_1}$, $\varphi_1^{\tilde{A}_1}$ are algebraically independent over the ring $E_{\bullet,\bullet}$.
Recall that $\wp(v,\tau)$ is the Weierstrass P function (6.13). □

Consider the formula (5.21) for the $\mathcal{J}(A_2)$ case

COROLLARY 6.2.6.1. [8] The ring of A_2 invariant Jacobi forms is free module of rank 3 over the ring of modular forms, moreover there exist a formula for its generators given by

$$\begin{aligned}\lambda^{A_2}&=e^{-2\pi i u_2}\frac{\theta_1(z+v_0+v_2,\tau)\theta_1(z-v_0+v_2,\tau)\theta_1(z-2v_2)}{\theta_1^3(z,\tau)}\\&=-\frac{\varphi_3^{A_2}}{2}\wp'(z,\tau)+\varphi_2^{A_2}\wp(z,\tau)+\varphi_0^{A_2}\end{aligned} \tag{6.53}$$

LEMMA 6.2.7. Let $\{\varphi_0^{\tilde{A}_1},\varphi_1^{\tilde{A}_1}\}$ be set of functions given by the formula (6.45) ,and $\{\varphi_0^{A_2},\varphi_2^{A_2},\varphi_3^{A_2}\}$ given by (6.53), then

$$\begin{aligned}\varphi_3^{A_2}&=\varphi_1^{\tilde{A}_1}\varphi_2^{A_1}\\\varphi_2^{A_2}&=\varphi_0^{\tilde{A}_n}\varphi_2^{A_1}+a_2(v_2,\tau)\varphi_j^{\tilde{A}_n}\varphi_2^{A_1}\\\varphi_0^{A_2}&=a_0(v_2,\tau)\varphi_0^{\tilde{A}_1}\varphi_2^{A_1}+b_0(v_2,\tau)\varphi_2^{\tilde{A}_1}\varphi_2^{A_1}\end{aligned} \tag{6.54}$$

where

$$\varphi_2^{A_1} := \frac{\theta_1^2(2v_2,\tau)}{\theta_1'(0,\tau)^2} e^{2\pi i(-u_2+u_1)}$$

and a_i, b_i are elliptic functions on v_2.

PROOF. Note the following relation

$$\begin{aligned}\frac{\lambda^{A_2}}{\lambda^{\tilde{A}_1}} &= \frac{\theta_1(z-2v_2,\tau)\theta_1(z+2v_2),\tau}{\theta_1^2(z,\tau)} e^{2\pi i(-u_2+u_1)}\\ &= \varphi_2^{A_1}\wp(z,\tau) - \varphi_2^{A_1}\wp(2v_2,\tau)\end{aligned}$$

Hence,

$$\begin{aligned}&-\frac{\varphi_3^{A_2}}{2}\wp'(z,\tau) + \varphi_2^{A_2}\wp(z,\tau) + \varphi_0^{A_2} = \\ &= \left(\varphi_1^{\tilde{A}_1}[\zeta(z,\tau) - \zeta(z+2v_2,\tau) + 2\zeta(v_2,\tau)] + \varphi_0^{\tilde{A}_n}\right)\left(\varphi_2^{A_1}\wp(z,\tau) - \varphi_2^{A_1}\wp(2v_2,\tau)\right).\end{aligned} \tag{6.55}$$

Then, the desired result is obtained by doing a Laurent expansion in the variable z in both side of the equality.

□

COROLLARY 6.2.7.1.

$$E_{\bullet,\bullet}\left[\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1}\right] = E_{\bullet,\bullet}\left[\frac{\varphi_0^{A_2}}{\varphi_2^{A_1}}, \frac{\varphi_2^{A_2}}{\varphi_2^{A_1}}, \frac{\varphi_3^{A_2}}{\varphi_2^{A_1}}\right].$$

Moreover, we have the following lemma

LEMMA 6.2.8. Let be $\varphi \in J^{\tilde{A}_1}_{\bullet,\bullet,m}$, then $\varphi \in E_{\bullet,\bullet}\left[\frac{\varphi_0^{A_2}}{\varphi_2^{A_1}}, \frac{\varphi_2^{A_2}}{\varphi_2^{A_1}}, \frac{\varphi_3^{A_2}}{\varphi_2^{A_1}}\right]$.

PROOF. Let be $\varphi \in J^{\tilde{A}_1}_{\bullet,\bullet,m}$, then the function $\frac{\varphi}{\left(\varphi_1^{\tilde{A}_1}\right)^m}$ is an elliptic function on the variables (v_0, v_2) with poles on $v_0 - v_2, v_0 + v_2, 2v_2$ due to the zeros of $\varphi_1^{\tilde{A}_1}$ and the poles of φ, which are by definition in $2v_2$. Expanding the function $\frac{\varphi}{\left(\varphi_1^{\tilde{A}_1}\right)^m}$ in the variables v_0, v_2 we get

$$\frac{\varphi}{\left(\varphi_1^{\tilde{A}_1}\right)^m} = \sum_{i=-1}^{m} a^i \wp^{(i)}(v_0+v_2) + \sum_{i=-1}^{m} b^i \wp^{(i)}(-v_0+v_2) + c(v_2,\tau), \tag{6.56}$$

where $\wp^{-1}(v) := \zeta(v)$, and $c(v_2,\tau)$ is an elliptic function in the variable v_2.
But the function $\frac{\varphi}{\left(\varphi_1^{\tilde{A}_1}\right)^m}$ is invariant under the permutations of the variables v_0, then the equation (6.56) is

$$\frac{\varphi}{\left(\varphi_1^{\tilde{A}_1}\right)^m} = \sum_{i=-1}^{m} a^i \left(\wp^{(i)}(v_0+v_2) + \wp^{(i)}(-v_0+v_2)\right) + c(v_2,\tau), \tag{6.57}$$

Now we complete this function to A_2 invariant function by summing and subtracting the following function in e.q (6.57)

$$f(v_2,\tau)=\sum_{i=-1}^{m} a^i \wp^{(i)}(2v_2).$$

Hence,

$$\frac{\varphi}{\left(\varphi_1^{\tilde{A}_1}\right)^m}=\sum_{i=-1}^{m} a^i\left(\wp^{(i)}(v_0+v_2)+\wp^{(i)}(-v_0+v_2)+\wp^{(i)}(2v_2)\right)+g(v_2,\tau), \tag{6.58}$$

Multiplying both side of the equation (6.58) by $\varphi_1^{A_1}$, we get

(6.59)
$$\varphi=\left(\sum_{i=-1}^{m} a^i\left(\wp^{(i)}(v_0+v_2)+\wp^{(i)}(-v_0+v_2)+\wp^{(i)}(2v_2)\right)\right)\left(\varphi_3^{A_2}\right)^m+g(v_2,\tau)\left(\varphi_3^{A_2}\right)^m.$$

To finish the prove, we will show that

$$\left(\sum_{i=-1}^{m} a^i\left(\wp^{(i)}(v_0+v_2)+\wp^{(i)}(-v_0+v_2)+\wp^{(i)}(2v_2)\right)\right)\left(\varphi_3^{A_2}\right)^m$$

is a weak holomorphic Jacobi form of type A_2. To finish the proof note the following

(1) The functions $\left(\varphi_3^{A_2}\right)^m\left(\wp^{(i)}(v_0+v_2)+\wp^{(i)}(-v_0+v_2)+\wp^{(i)}(2v_2)\right)$ are A_2 invariant by construction,
(2) The functions $\left(\varphi_3^{A_2}\right)^m\left(\wp^{(i)}(v_0+v_2)+\wp^{(i)}(-v_0+v_2)+\wp^{(i)}(2v_2)\right)$ are invariant under the action of $(\mathbb{Z}\oplus\tau\mathbb{Z})^2$, because $\varphi_3^{A_2}$ invariant, and $\wp^{(i)}(v_0+v_2)+\wp^{(i)}(-v_0+v_2)+\wp^{(i)}(2v_2)$ are elliptic functions.
(3) The functions $\left(\varphi_3^{A_2}\right)^m\left(\wp^{(i)}(v_0+v_2)+\wp^{(i)}(-v_0+v_2)+\wp^{(i)}(2v_2)\right)$ are equivariant under the action of $SL_2(\mathbb{Z})$, because $\varphi_3^{A_2}$ is equivariant, and $\wp^{(i)}(v_0+v_2)+\wp^{(i)}(-v_0+v_2)+\wp^{(i)}(2v_2)$ are elliptic functions.
(4) The function $\varphi_3^{A_2}$ has zeros on $v_0-v_2, v_0+v_2, 2v_2$ of order m, and $\wp^{(i)}(v_0+v_2)+\wp^{(i)}(-v_0+v_2)+\wp^{(i)}(2v_2)$ has poles on $v_0-v_2, v_0+v_2, 2v_2$ of order $i+2\leq m$. Then, the functions $\left(\varphi_3^{A_2}\right)^m\left(\wp^{(i)}(v_0+v_2)+\wp^{(i)}(-v_0+v_2)+\wp^{(i)}(2v_2)\right)$ are holomorphic.

Hence,

$$\varphi\in E_{\bullet,\bullet}\left[\frac{\varphi_0^{A_2}}{\varphi_2^{A_1}},\frac{\varphi_2^{A_2}}{\varphi_2^{A_1}},\frac{\varphi_3^{A_2}}{\varphi_2^{A_1}}\right]. \tag{6.60}$$

$\square$

At this stage, the principal theorem can be stated in precise way as follows.

Theorem 6.2.9. The trigraded algebra of weak $\mathscr{J}(\tilde{A}_1)$ -invariant Jacobi forms $J_{\bullet,\bullet,\bullet}^{\mathscr{J}(\tilde{A}_1)} = \bigoplus_{k,l,m} J_{k,l,m}^{\tilde{A}_1}$ is freely generated by 2 fundamental Jacobi forms $(\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1})$ over the graded ring $E_{\bullet,\bullet}$

$$J_{\bullet,\bullet,\bullet}^{\mathscr{J}(\tilde{A}_1)} = E_{\bullet,\bullet}\left[\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1}\right]. \tag{6.61}$$

Proof.

$$J_{\bullet,\bullet,\bullet}^{\tilde{A}_1} \subset E_{\bullet,\bullet}\left[\frac{\varphi_0^{A_2}}{\varphi_2^{A_1}}, \frac{\varphi_2^{A_2}}{\varphi_2^{A_1}}, \frac{\varphi_3^{A_2}}{\varphi_2^{A_1}}\right] = E_{\bullet,\bullet}\left[\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1}\right] \subset J_{\bullet,\bullet,\bullet}^{\tilde{A}_1}. \tag{6.62}$$

□

Remark 6.2.5. The structural difference between the Chevalley theorems of the groups $J(A_1)$, and $\mathscr{J}(\tilde{A}_1)$ lies in the ring of coefficients. The ring of coefficients of Jacobi forms with respect $J(A_1)$ are modular forms, and the ring of coefficients of Jacobi forms with respect $\mathscr{J}(\tilde{A}_1)$ are ,for fixed τ, the ring of elliptic functions with poles on $0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \mod \mathbb{Z} \oplus \tau\mathbb{Z}$. See lemma 6.2.2.

Remark 6.2.6. The geometry of $\Omega^{\mathscr{J}(\tilde{A}_1)}/\mathscr{J}(\tilde{A}_1)$ is similar to $\Omega^{\mathscr{J}(A_1)}/\mathscr{J}(A_1)$. Indeed, the orbit space of $\mathscr{J}(\tilde{A}_1)$ is locally a line bundle over a family of two elliptic curves $E_\tau/A_1 \otimes E_\tau$, where the first one is quotient by A_1, and both are parametrised by $\mathbb{H}/SL_2(\mathbb{Z})$.

6.3. Frobenius structure on the Orbit space of $\mathscr{J}(\tilde{A}_1)$

In this section, a Dubrovin-Frobenius manifold structure will be constructed on the orbit space of $\mathscr{J}(\tilde{A}_1)$. More precisely, It will be defined the data $(\Omega^{\mathscr{J}(\tilde{A}_1}/\mathscr{J}(\tilde{A}_1), g^*, e, E)$, with the intersection form g^*, unit vector field e, and Euler vector field E. These data will be written naturally in terms of the invariant functions of $\mathscr{J}(\tilde{A}_1)$. Thereafter, it will be proved that these data are enough to the construction of the Dubrovin-Frobenius structure.
The first step to be done is the construction of the intersection form. It will be shown that such metric can be constructed by using just the data of the group $\mathscr{J}(\tilde{A}_1)$. The strategy is to combine the intersection form of the group $\tilde{A}_1$ and $\mathscr{J}(A_1)$. Recall that the intersection form of the group $\tilde{A}_1$ [**12**], [**15**] is:

$$ds^2 = 2dv_0^2 - 2dv_2^2,$$

and the intersection form of $\mathscr{J}(A_1)$ [**8**], [**9**], [**12**] is:

$$ds^2 = dv_0^2 + 2du d\tau.$$

Therefore, the natural candidate to be the intersection form of $\mathscr{J}(\tilde{A}_1)$ is:

$$ds^2 = 2dv_0^2 - 2dv_2^2 + 2du d\tau. \tag{6.63}$$

The following lemma proves that this metric is invariant metric of the group $\mathscr{J}(\tilde{A}_1)$. To be precise, the metric will be invariant under the action of A_1, and translations, and equivariant under the action of $SL_2(\mathbb{Z})$.

Lemma 6.3.1. The metric

$$ds^2 = 2dv_0^2 - 2dv_2^2 + 2dud\tau \tag{6.64}$$

is invariant under the transformations (6.16),(6.17). Moreover, the transformations (6.18) determine a conformal transformation of the metric ds^2, i.e:

$$2dv_0^2 - 2dv_2^2 + 2dud\tau \mapsto \frac{2dv_0^2 - 2dv_2^2 + 2dud\tau}{(c\tau + d)^2}. \tag{6.65}$$

Proof. Under (6.16),(6.17), the differentials transform as:

$$\begin{aligned}
&dv_0 \mapsto -dv_0,\\
&dv_0 \mapsto dv_0 + \lambda_0 d\tau,\\
&dv_2 \mapsto dv_2 + \lambda_2 d\tau,\\
&du \mapsto du - \lambda_0^2 d\tau - 2\lambda_0 dv_0 + \lambda_2^2 d\tau + 2\lambda_2 dv_2,\\
&d\tau \mapsto d\tau.
\end{aligned} \tag{6.66}$$

Hence:

$$\begin{aligned}
&dv_0^2 \mapsto dv_0^2,\\
&dv_0^2 \mapsto dv_0^2 + 2\lambda_0 dv_0 d\tau + \lambda_0^2 d\tau^2,\\
&dv_2^2 \mapsto dv_2^2 + 2\lambda_2 dv_2 d\tau + \lambda_2^2 d\tau^2,\\
&2dud\tau \mapsto 2dud\tau - 2\lambda_0^2 d\tau^2 - 4\lambda_0 dv_0 d\tau + 2\lambda_2^2 d\tau^2 + 4\lambda_2 dv_2 d\tau.
\end{aligned} \tag{6.67}$$

Then:

$$2dv_0^2 - 2dv_2^2 + 2dud\tau \mapsto 2dv_0^2 - 2dv_2^2 + 2dud\tau. \tag{6.68}$$

Let us show that the metric has conformal transformation under the transformations (6.18):

$$\begin{aligned}
&dv_0 \mapsto \frac{dv_0}{c\tau + d} - \frac{v_0 d\tau}{(c\tau + d)^2},\\
&dv_2 \mapsto \frac{dv_2}{c\tau + d} - \frac{v_2 d\tau}{(c\tau + d)^2},\\
&d\tau \mapsto \frac{d\tau}{(c\tau + d)^2},\\
&du \mapsto du + \frac{c(2v_0 dv_0 - 2v_2 dv_2^2)}{c\tau + d} - \frac{c(v_0^2 - v_2^2)d\tau}{(c\tau + d)^2}.
\end{aligned} \tag{6.69}$$

Then:

$$
\begin{aligned}
dv_0^2 &\mapsto \frac{dv_0^2}{(c\tau+d)^2} - \frac{2v_0dv_0d\tau}{(c\tau+d)^3} + \frac{v_0^2d\tau^2}{(c\tau+d)^4},\\
dv_2^2 &\mapsto \frac{dv_2^2}{(c\tau+d)^2} - \frac{2v_2dv_2d\tau}{(c\tau+d)^3} + \frac{v_2^2d\tau^2}{(c\tau+d)^4},\\
2dud\tau &\mapsto \frac{2dud\tau}{(c\tau+d)^2} + \frac{c(4v_0dv_0-4v_2dv_2)d\tau}{(c\tau+d)^3} - \frac{c(2v_0^2-2v_2^2)d\tau^2}{(c\tau+d)^4}.
\end{aligned} \tag{6.70}
$$

Then,

$$
2dv_0^2 - 2dv_2^2 + 2dud\tau \mapsto \frac{2dv_0^2-2dv_2^2+2dud\tau}{(c\tau+d)^2}. \tag{6.71}
$$

□

The next step is the construction of the Euler vector field. Recall that the coordinates (u, v_0, v_2, τ) are natural coordinates of the orbit space of $\mathscr{J}(\tilde{A}_1)$. The Euler vector field will be defined as:

$$
E = -\frac{1}{2\pi i}\frac{\partial}{\partial u}. \tag{6.72}
$$

The last structure to be defined is the unit vector field

$$
e = \frac{\partial}{\partial \varphi_0}. \tag{6.73}
$$

In order to construct the Dubrovin Frobenius structure, it will be necessary to introduce the coordinates (t^1, t^2, t^3, t^4).

Lemma 6.3.2. There is a change of coordinates in $\Omega^{\mathscr{J}(\tilde{A}_1)}/\mathscr{J}(\tilde{A}_1)$ be given by:

$$
\begin{aligned}
t^1 &= \varphi_0 + 2t^2\frac{\theta_1'(v_2|\tau)}{\theta_1(v_2|\tau)},\\
t^2 &= \varphi_1,\\
t^3 &= v_2,\\
t^4 &= \tau.
\end{aligned} \tag{6.74}
$$

Proof. Note that the function (6.41) can be parametrised by (t^1, t^2, t^3, t^4) as follows

$$
\begin{aligned}
\lambda &= \varphi_0 + \varphi_1[\zeta(v-v_2|\tau) - \zeta(v+v_2|\tau) + 2\zeta(v_2)]\\
&= \varphi_0 + \varphi_1[\frac{\theta_1'(v-v_2|\tau)}{\theta_1(v-v_2|\tau)} - \frac{\theta_1'(v+v_2|\tau)}{\theta_1(v+v_2|\tau)} + 2\frac{\theta_1'(v_2|\tau)}{\theta_1(v_2|\tau)}]\\
&= \varphi_0 + 2\frac{\theta_1'(v_2|\tau)}{\theta_1(v_2|\tau)} + \varphi_1[\frac{\theta_1'(v-v_2|\tau)}{\theta_1(v-v_2|\tau)} - \frac{\theta_1'(v+v_2|\tau)}{\theta_1(v+v_2|\tau)}]\\
&= t^1 + t^2[\frac{\theta_1'(v-t^3|t^4)}{\theta_1(v-t^3|t^4)} - \frac{\theta_1'(v+t^3|t^4)}{\theta_1(v+t^3|t^4)}]
\end{aligned} \tag{6.75}
$$

from the first line to the second line was used the following equation

$$\zeta(v-v_2,\tau)=\frac{\theta_1'(v-v_2|\tau)}{\theta_1(v-v_2|\tau)}+4\pi i g_1(\tau)(v-v_2).$$

In this way, (t^1,t^2,t^3,t^4) are local coordinates of $\Omega^{\mathscr{J}(\tilde{A}_1)}/\mathscr{J}(\tilde{A}_1)$ due to lemma 6.2.5. $\square$

The side back effect of the coordinates (t^1,t^2,t^3,t^4) is the fact that they are not globally single valued functions on the quotient.

Lemma 6.3.3. The coordinates (t^1,t^2,t^3,t^4) have the following transformation laws under the action of the group $\mathscr{J}(\tilde{A}_1)$: they are invariant under (6.16). They transform as follows under (6.17):

$$\begin{aligned}&t^1\mapsto t^1-\lambda_2 t^2\\&t^2\mapsto t^2\\&t^3\mapsto t^3+\mu_2+\lambda_2 t^4\\&t^4\mapsto t^4\end{aligned}\tag{6.76}$$

Moreover, they transform as follows under (6.18)

$$\begin{aligned}&t^1\mapsto t^1+\frac{2ct^2t^3}{ct^4+d}\\&t^2\mapsto \frac{t^2}{ct^4+d}\\&t^3\mapsto \frac{t^3}{ct^4+d}\\&t^4\mapsto \frac{at^4+b}{ct^4+d}\end{aligned}\tag{6.77}$$

Proof. The invariance under (6.16) is clear since only t^1 depend on v_0, and its dependence is given by φ_0 which is invariant under (6.16). Let us check how t^α transform under (6.17), (6.18): Since $t^3=v_2$, $t^4=\tau$, we have the desired transformations law by the definition of $\mathscr{J}(\tilde{A}_1)$. The coordinate $t^2=\varphi_1$ is a invariant under (6.17) and transform as modular form of weight -1 under (6.17). The only non-trivial term is t^1, because it contains the term $\frac{\theta_1'(v_2|\tau)}{\theta_1(v_2|\tau)}$, which transform as follows under (6.17),(6.18) [33].

$$\begin{aligned}&\frac{\theta_1'(v_2|\tau)}{\theta_1(v_2|\tau)}\mapsto\frac{\theta_1'(v_2|\tau)}{\theta_1(v_2|\tau)}-2\pi i n_2\\&\frac{\theta_1'(v_2|\tau)}{\theta_1(v_2|\tau)}\mapsto(c\tau+d)\frac{\theta_1'(v_2|\tau)}{\theta_1(v_2|\tau)}+2\pi i c t^3\end{aligned}\tag{6.78}$$

The proof is completed when we do the rescaling from t^1 to $\frac{t^1}{2\pi i}$.

$\square$

In order to make the coordinates (t^1, t^2, t^3, t^4) being well defined, it will be necessary to define them in a suitable covering over $\Omega^{\mathscr{J}(\tilde{A}_1)}/\mathscr{J}(\tilde{A}_1)$. It is clear that the multivaluedness comes from the coordinates t^3, t^4 essentially. Therefore, the problem is solved by defining a suitable covering over the orbit space of $\mathscr{J}(\tilde{A}_1)$. This can be done by fixing a lattice $(1, t^4)$, and a representative of orbit given by the action

$$t^3 \mapsto t^3 + \mu_2 + \lambda_2 t^4. \tag{6.79}$$

In order to realise also the coordinates $(u, v_0, v_2.\tau)$ as globally well-behaviour in the covering of the orbit space of $\mathscr{J}(\tilde{A}_1)$, we also forget the A_1 action by fixing a representative of each orbit. Therefore in the following covering the problem

$$\widetilde{\Omega^{\mathscr{J}(\tilde{A}_1)}/\mathscr{J}}(\tilde{A}_1) := \Omega^{\mathscr{J}(\tilde{A}_1)}/\mathbb{Z} \oplus \tau\mathbb{Z} \tag{6.80}$$

where $\mathbb{Z} \oplus \tau\mathbb{Z}$ acts on $\Omega^{\mathscr{J}(\tilde{A}_1)}$ as

$$\begin{aligned} v_0 &\mapsto v_0 + \lambda_0\tau + \mu_0, \\ u &\mapsto u - 2\lambda_0 v_0 - n_0^2\tau, \\ v_2 &\mapsto v_2, \\ \tau &\mapsto \tau. \end{aligned} \tag{6.81}$$

This covering is similar to the covering defined in section 5.6 for the orbit space of $\mathscr{J}(A_n)$. In the covering (6.80) the coordinates t^α, and the intersection form g^* are globally single valued. Hence, we have necessary condition to have Dubrovin-Frobenius manifold, since its geometry structure should be globally well defined. Note that, $\Omega^{\mathscr{J}(\tilde{A}_1)}/\mathscr{J}(\tilde{A}_1)$ has the structure of Twisted Frobenius manifold [**12**].

Remark 6.3.1. (t^1, t^2) lives in an enlargement of the algebra of $E_{\bullet,\bullet}[\varphi_0, \varphi_1]$. The extended algebra is the same as $E_{\bullet,\bullet}[\varphi_0, \varphi_1]$, but it is necessary to add the function $\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}$ in the ring of coefficients $E_{\bullet,\bullet}$.

Remark 6.3.2. As it was already discussed in remark 5.6.1, a covering in the orbit space correspond to a covering in the Hurwitz space. The fixation of a lattice in the orbit space of $\mathscr{J}(\tilde{A}_1)$ is equivalent to a choice of homology basis in the Hurwitz space $H_{1,0,0}$. Moreover, a choice of the representative of the action 6.79 in the variable v_2 is a choice of logarithm root in the Hurwitz space $H_{1,0,0}$, furthermore, fixing a representative of the A_1 action is to choice a pole or equivalently to choice a sheet in the Hurwitz space $H_{1,0,0}$.

Theorem 6.3.4. There exists Dubrovin-Frobenius structure on the manifold $\widetilde{\Omega/\mathscr{J}(\tilde{A}_1)}$ with the intersection form (6.64), the Euler vector field (6.72), and the unity vector field (6.73). Moreover, $\widetilde{\Omega/\mathscr{J}(\tilde{A}_1)}$ is isomorphic as Dubrovin-Frobenius manifold to $\widetilde{H}_{1,0,0}$

PROOF. The first step to be done is the computation of the intersection form in coordinates (t^1, t^2, t^3, t^4). Hence, consider the transformation formula of ds^2:

$$g^{\alpha\beta}(t) = \frac{\partial t^\alpha}{\partial x^i}\frac{\partial t^\beta}{\partial x^j} g^{ij}. \tag{6.82}$$

where $x^1 = u, x^2 = v_0, x^3 = v_2, x^4 = \tau$.
From the expression:

$$ds^2 = 2dv_0^2 - 2dv_2^2 + 2dud\tau = g_{ij}dx^i dx^j,$$

we have:

$$(g_{ij}) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Therefore

$$(g^{ij}) = (g_{ij})^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

To compute $g^{\alpha\beta}(t)$, let us write t^α in terms of x^i.

$$\begin{aligned} t^4 &= \tau, \\ t^3 &= v_2, \\ t^2 &= -\frac{\theta_1(v_0 + v_2, \tau)\theta_1(v_0 - v_2, \tau)}{\theta_1(2v_2, \tau)\theta_1'(0, \tau)} e^{-2\pi i u}, \end{aligned} \tag{6.83}$$

using the following formulae [33]

$$\begin{aligned} \frac{\wp'(v_2)}{\wp(v_0) - \wp(v_0)} &= \zeta(v_0 - v_2, \tau) - \zeta(v_0 + v_2, \tau) + 2\zeta(v_2, \tau), \\ \wp(v_0, \tau) - \wp(v_2, \tau) &= -\frac{\sigma(v_0 + v_2, \tau)\sigma(v_0 - v_2, \tau)}{\sigma^2(v_0, \tau)\sigma^2(v_2, \tau)}, \\ \frac{\sigma(2v_2, \tau)}{\sigma^4(v_2, \tau)} &= -\wp'(v_2, \tau), \end{aligned} \tag{6.84}$$

it is possible to rewrite t^1 in a more suitable way:

$$
\begin{aligned}
t^1 &= -t^2[\zeta(v_0-v_2,\tau)-\zeta(v_0+v_2,\tau)+2\zeta(v_2,\tau)]+2t^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\\
&= -t^2\frac{\wp'(v_2,\tau)}{\wp(v_0,\tau)-\wp(v_2,\tau)}+2t^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\\
&= -t^2\frac{\wp'(v_2,\tau)\theta_1^2(v_2,\tau)\theta_1^2(v_0,\tau)}{\theta_1(v_0+v_2,\tau)\theta_1(v_0-v_2,\tau)\theta_1'(0,\tau)^2}+2t^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\\
&= -\frac{\wp'(v_2,\tau)\theta_1^2(v_2,\tau)\theta_1^2(v_0,\tau)}{\theta_1(2v_2,\tau)\theta_1'(0,\tau)^3}e^{-2\pi iu}+2t^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\\
&= \frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi iu}+2t^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}
\end{aligned}
\tag{6.85}
$$

Summarizing:

$$t^1=\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi iu}+2t^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}, \tag{6.86}$$

$$t^2=-\frac{\theta_1(v_0+v_2,\tau)\theta_1(v_0-v_2,\tau)}{\theta_1(2v_2,\tau)\theta_1'(0,\tau)}e^{-2\pi iu}, \tag{6.87}$$

$$t^3=v_2, \tag{6.88}$$

$$t^4=\tau. \tag{6.89}$$

Computing $g^{\alpha\beta}$ according to (6.82):

$$g^{\alpha\beta}=\frac{1}{2}\frac{\partial t^\alpha}{\partial v_0}\frac{\partial t^\beta}{\partial v_0}-\frac{1}{2}\frac{\partial t^\alpha}{\partial v_2}\frac{\partial t^\beta}{\partial v_2}+\frac{\partial t^\alpha}{\partial u}\frac{\partial t^\beta}{\partial \tau}+\frac{\partial t^\alpha}{\partial \tau}\frac{\partial t^\beta}{\partial u}, \tag{6.90}$$

Trivially, we get:

$$g^{44}=g^{34}=0, \tag{6.91}$$

$$g^{33}=-\frac{1}{2}, \tag{6.92}$$

and

$$g^{24}=-2\pi it^2, \tag{6.93}$$

$$g^{14}=-2\pi it^1. \tag{6.94}$$

The following non-trivial terms are computed in Appendix.

$$g^{23}=-\frac{t^1}{2}+t^2\frac{\theta_1'\left(2t^3,\tau\right)}{\theta_1\left(2t^3,\tau\right)}, \tag{6.95}$$

$$g^{13}=-2\pi it^2\frac{\partial}{\partial\tau}\left(\log\frac{\theta_1'\left(0,\tau\right)}{\theta_1\left(2t^3,\tau\right)}\right), \tag{6.96}$$

$$g^{22} = 2\left(t^2\right)^2 \left[\frac{\theta_1''\left(2t^3,\tau\right)}{\theta_1\left(2t^3,\tau\right)} - \frac{\theta_1'^2\left(2t^3,\tau\right)}{\theta_1^2\left(2t^3,\tau\right)}\right], \tag{6.97}$$

$$g^{12} = -2\pi i \left(t^2\right)^2 \left[\frac{\partial^2}{\partial t^3 \partial \tau}\left(\log\left(\frac{\theta_1'(0,\tau)}{\theta_1\left(2t^3,\tau\right)}\right)\right)\right], \tag{6.98}$$

$$\begin{aligned} g^{11} = & -4\left(t^2\right)^2 \frac{\theta_1'\left(t^3,\tau\right)}{\theta_1\left(t^3,\tau\right)} \frac{\partial}{\partial t^3}\left(\frac{\theta_1'\left(t^3,\tau\right)}{\theta_1\left(t^3,\tau\right)}\right)\left[2\frac{\theta_1'\left(t^3,\tau\right)}{\theta_1\left(t^3,\tau\right)} - 2\frac{\theta_1'\left(2t^3,\tau\right)}{\theta_1\left(2t^3,\tau\right)}\right] \\ & + 8\frac{\theta_1'^2\left(t^3,\tau\right)}{\theta_1^2\left(t^3,\tau\right)}\left(t^2\right)^2\left[\frac{\theta_1''\left(2t^3,\tau\right)}{\theta_1\left(2t^3,\tau\right)} - \frac{\theta_1'^2\left(2t^3,\tau\right)}{\theta_1^2\left(2t^3,\tau\right)}\right] \\ & - 2\left(t^2\right)^2\left[\frac{\partial}{\partial t^3}\left(\frac{\theta_1'\left(t^3,\tau\right)}{\theta_1\left(t^3,\tau\right)}\right)\right]^2 - 16\pi i\left(t^2\right)^2 \frac{\theta_1'\left(t^3,\tau\right)}{\theta_1\left(t^3,\tau\right)}\frac{\partial}{\partial \tau}\left(\frac{\theta_1'\left(t^3,\tau\right)}{\theta_1\left(t^3,\tau\right)}\right). \end{aligned} \tag{6.99}$$

Differentiating $g^{\alpha\beta}$ w.r.t. t^1 we obtain a constant matrix η^*:

$$\left(\eta^{\alpha\beta}\right) = \frac{\partial}{\partial t^1}\left(g^{\alpha\beta}\right) = \begin{bmatrix} 0 & 0 & 0 & -2\pi i \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ -2\pi i & 0 & 0 & 0 \end{bmatrix}$$

So t^1, t^2, t^3, t^4 are the flat coordinates.

The next step is to calculate the matrix $F^{\alpha\beta}$ using formula (2.12), namely

$$F^{\alpha\beta} = \frac{g^{\alpha\beta}}{deg\left(g^{\alpha\beta}\right)}. \tag{6.100}$$

We can compute $deg\left(g^{\alpha\beta}\right)$ using the fact that we compute $deg\left(t^\alpha\right)$. Indeed:

$$E = -\frac{1}{2\pi i}\frac{\partial}{\partial u}. \tag{6.101}$$

Implies that:

$$deg\left(t^1\right) = deg\left(t^2\right) = 1, \tag{6.102}$$

$$deg\left(t^3\right) = deg\left(t^4\right) = 0. \tag{6.103}$$

Then the function F is obtained from the equation:

$$\frac{\partial^2 F}{\partial t^\alpha \partial t^\beta} = \eta_{\alpha\alpha'}\eta_{\beta\beta'}F^{\alpha'\beta'} \tag{6.104}$$

Computing

$$F^{\alpha 4} = \frac{g^{\alpha 4}}{deg\left(g^{\alpha 4}\right)}, \tag{6.105}$$

we derive

$$\frac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta}, \tag{6.106}$$

Hence

$$F = \frac{i}{4\pi}\left(t^1\right)^2 t^4 - 2t^1 t^2 t^3 + f\left(t^2, t^3, t^4\right). \tag{6.107}$$

Substituting, F^{23},F^{13} in (6.107)

$$F = \frac{i}{4\pi}\left(t^1\right)^2 t^4 - 2t^1 t^2 t^3 - \left(t^2\right)^2 \log\left(\frac{\theta_1'\left(0, t^4\right)}{\theta_1\left(2t^3, t^4\right)}\right) + h\left(t^2\right) + A_{\alpha\beta} t^\alpha t^\beta + C_\alpha t^\alpha + D, \tag{6.108}$$

where $A_{\alpha\beta}$,C_α,C_α are constants. Note that F^{22},F^{12} contains the same information, furthermore, there is no information in F^{33},F^{34},F^{44} because:

$$deg\left(g^{33}\right) = deg\left(g^{34}\right) = deg\left(g^{44}\right) = 0 \tag{6.109}$$

However, $h(t^2)$ can be computed by using g^{33}

$$g^{33} = -\frac{1}{2} = E^\epsilon \eta^{3\mu} \eta^{3\lambda} c_{\epsilon\mu\lambda} = \frac{t^2}{4} c_{222}. \tag{6.110}$$

Using the formula (2.10), we have:

$$F\left(t^1, t^2, t^3, t^4\right) = \frac{i}{4\pi}\left(t^1\right)^2 t^4 - 2t^1 t^2 t^3 - \left(t^2\right)^2 \log\left(t^2 \frac{\theta_1'\left(0, t^4\right)}{\theta_1\left(2t^3, t^4\right)}\right). \tag{6.111}$$

The remaining part of proof is to show that the equation (6.111) satisfies WDDV equations. Let us prove it step by step

(1) **Commutative of the algebra**
Defining the structure constant of the algebra as

$$c_{\alpha\beta\gamma}(t) = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \tag{6.112}$$

commutative is straightforward.

(2) **Normalization**
Using equation (6.106), we obtain

$$c_{1\alpha\beta}(t) = \frac{\partial^3 F}{\partial t^1 \partial t^\beta \partial t^\gamma} = \eta_{\alpha\beta}. \tag{6.113}$$

(3) **Quasi homogeneity**
Applying the Euler Vector field in the function (6.111), we have

$$E(F) = 2F - 2t^2. \tag{6.114}$$

(4) **Associativity** In order to prove that the algebra is associativity, we will first shown that the algebra is semisimple. First of all note that the multiplication by the Euler vector field is equivalent to the intersection form. Indeed,

$$E \bullet \partial_\alpha = t^\sigma c^\beta_{\sigma\alpha}\partial_\beta = t^\sigma \partial_\sigma \left(\eta^{\beta\mu}\partial_\alpha\partial_\mu F\right)\partial_\beta = \\ = (d_\alpha - d_\beta)\eta^{\beta\mu}\partial_\alpha\partial_\mu F\partial_\beta = \eta_{\alpha\mu}g^{\mu\beta}\partial_\beta \tag{6.115}$$

Therefore, the multiplication by the Euler vector field is semisimple if the following polynomial

$$det(\eta_{\alpha\mu}g^{\mu\beta} - u\delta^\beta_\alpha) = 0, \tag{6.116}$$

has only simple roots. Since $det(\eta_{\alpha\mu}) \neq 0$, the equation (6.116) is equivalent to

$$det(g^{\alpha\beta} - u\eta^{\alpha\beta}) = 0. \tag{6.117}$$

Using that $\eta^{\alpha\beta} = \partial_1 g^{\alpha\beta}$, we have that

$$det(g^{\alpha\beta} - u\eta^{\alpha\beta}) = det\left(g^{\alpha\beta}(t^1 - u, t^2, t^3, t^4)\right) = 0. \tag{6.118}$$

Then, it is enough to compute $detg^{\alpha\beta}$. In particular, computing $detg$ in the coordinates $(\varphi_0, \varphi_1, v_2, \tau)$. Recall that

$$g^{\alpha\beta} = g(dt^\alpha, dt^\beta), \quad \text{in coordinates}(t^1, t^2, t^3, t^4),$$
$$g^{lm} = g(dv_l, dv_m), \quad \text{in coordinates}(u, v_0, v_2, \tau),$$
$$g^{ij} = g(d\varphi_i, d\varphi_j), \quad \text{in coordinates}(\varphi_0, \varphi_1, v_2, \tau).$$

Then,

$$detg^{ij} = det\left(\frac{\partial\varphi_i}{\partial v_l}\right) det\left(\frac{\partial\varphi_j}{\partial v_m}\right) det\left(g^{lm}\right).$$

Remark 6.3.3. The coordinates (u, v_0, v_2, τ) are defined away from the submanifold defined by $\det g = 0$. Then, we have to change coordinates to compute the roots of $\det g = 0$.

Hence, it is enough to compute the $det\left(\frac{\partial\varphi_i}{\partial v_l}\right)$

$$det\left(\frac{\partial\varphi_i}{\partial v_l}\right) = \begin{bmatrix} \frac{\partial\varphi_0}{\partial v_0} & \frac{\partial\varphi_0}{\partial v_2} & \frac{\partial\varphi_0}{\partial \tau} & -2\pi i\varphi_0 \\ \frac{\partial\varphi_1}{\partial v_0} & \frac{\partial\varphi_1}{\partial v_2} & \frac{\partial\varphi_1}{\partial \tau} & -2\pi i\varphi_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = -2\pi i\varphi_0\varphi_1\left[2\frac{\theta_1'(v_0)}{\theta_1(v_0)} - \frac{\theta_1'(-v_0+v_2)}{\theta_1(-v_0+v_2)} + \frac{\theta_1'(v_0+v_2)}{\theta_1(v_0+v_2)}\right] \\ = -2\pi i e^{-4\pi i u}\frac{\theta_1(2v_0)}{\theta_1(2v_2)\theta_1'(0)^2}. \tag{6.119}$$

Then, equation (6.119) has four distinct roots $v_0 = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$. Hence, the following system of equation

$$\begin{aligned} det\left(g^{\alpha\beta}(t^1,t^2,t^3,t^4)\right) &= 0, \\ det\left(\eta^{\alpha\beta}(t^1,t^2,t^3,t^4)\right) &\neq 0, \end{aligned} \tag{6.120}$$

implies in existence of 4 functions $y_i(t^2,t^3,t^4)$ such that

$$t^1 = y_i(t^2,t^3,t^4), \quad i = 1,2,3,4. \tag{6.121}$$

Sending $t^1 \mapsto t^1 - u$ in (6.120), we obtain

$$u^i = t^1 - y^i(t^2,t^3,t^4), \quad i = 1,2,3,4. \tag{6.122}$$

The multiplication by the Euler vector field

$$g^i_j = \eta_{jk}g^{ki}, \quad \text{in canonical coordinates}(u^1,u^2,u^3,u^4)$$

is diagonal, then

$$g^{ij} = u^i\eta^{ij}\delta_{ij}, \tag{6.123}$$

where η^{ij} is the canonical coordinates (u^1,u^2,u^3,u^4), and the unit vector field have the following form

$$\frac{\partial}{\partial t^1} = \sum_{i=1}^{4}\frac{\partial u_i}{\partial t^1}\frac{\partial}{\partial u_i} = \sum_{i=1}^{4}\frac{\partial}{\partial u_i}. \tag{6.124}$$

Moreover, since

$$[E,e] = [t^1\frac{\partial}{\partial t^1} + t^2\frac{\partial}{\partial t^2}, \frac{\partial}{\partial t^1}] = -e, \tag{6.125}$$

the Euler vector field in the coordinates (u^1,u^2,u^3,u^4) takes the following form

$$E = \sum_{i=1}^{4} u^i\frac{\partial}{\partial u_i}. \tag{6.126}$$

Using the relation (6.115) in the coordinates (u^1,u^2,u^3,u^4), we have

$$u^i\eta^{ij}\delta_{ij} = u^l\eta^{im}\eta^{jn}c_{lmn}, \tag{6.127}$$

differentiating both side of the equation (6.127) with respect t^1

$$c^k_{ij} = \delta_{ij}, \tag{6.128}$$

which proves that the algebra is associative and semisimple.

The Function F is exactly the Free energy of the Dubrovin-Frobenius manifold of the Hurwitz space $\widetilde{H}_{1,0,0}$. Therefore, the equation (6.111) solves the WDVV equations by the lemma 2.1.2, then the theorem is proved.

Conclusion

The WDVV solution of $H_{1,0,0}$, which is (6.111), contains the term $\log\left(\frac{\theta_1'(0,t^4)}{\theta_1(2t^3,t^4)}\right)$ on the two exceptional variables (t^3,t^4). This is a reflection of how the ring of invariants affects the WDVV solution. The same pattern is obtained in $\mathscr{J}(A_1)$, and $\tilde{A}_1$. The equation (6.15) contains $E_2(\tau)$ which is a quasi modular form, and the equation (6.5) contains e^{t^2}. These facts could be useful on the understanding of the WDVV/ groups correspondence.

The arrows of the diagram of in section 6.1 may have a third meaning, which is an embedding of Dubron Frobenius submanifolds [**31**], [**32**] in to the ambient space $H_{1,0,0}$. The fact that $H_{1,0,0}$ contains 3 Dubrovin Frobenius submanifolds is not an accident, this comes from the tri hamiltonian structure that $H_{1,0,0}$ has [**25**], [**26**]. In a subsequent publication, we will study the Dubrovin Frobenius manifolds of $H_{1,0,0}$, and its associated integrable systems.

6.4. Appendix

Computing g^{12}:

$$
\begin{aligned}
g^{23} &= -\frac{1}{2}\frac{\partial t^2}{\partial v_2} = -\frac{t^2}{2}\left[-\frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)} + \frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)} - 2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right]\\
&= -\frac{t^2}{2}\left[-\frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)} + \frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)} - 2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right] - t^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\\
&\quad + t^2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\\
&= -\frac{1}{2\wp'(v_2,\tau)}\left[-\zeta(v_0-v_2,\tau)+\zeta(v_0+v_2,\tau)-2\zeta(v_2,\tau)\right] - t^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\\
&\quad + t^2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\\
&=\frac{1}{2}\frac{1}{\wp(z_0,\tau)-\wp(z_2,\tau)} - t^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} + t^2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\\
&= -\frac{t^1}{2} + t^2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}.
\end{aligned}
\tag{6.129}
$$

Computing g^{13}:

$$
\begin{aligned}
g^{13} = & -\frac{1}{2}\frac{\partial t^1}{\partial v_2} = -\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{1}{\wp(z_0)-\wp(z_2)} - \frac{\partial t^2}{\partial v_2}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} \\
& - t^2\left[\frac{\theta_1''(v,\tau)}{\theta_1(v,\tau)} - \frac{\theta_1'^2(v,\tau)}{\theta_1^2(v,\tau)}\right] \\
= & -\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{1}{\wp(z_0)-\wp(z_2)} - t^1\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} - 2t^2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} \\
& - t^2\left[\frac{\theta_1''(v,\tau)}{\theta_1(v,\tau)} - \frac{\theta_1'^2(v,\tau)}{\theta_1^2(v,\tau)}\right] \\
= & -2t^2\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)} - 2t^2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} - t^2\left[\frac{\theta_1''(v,\tau)}{\theta_1(v,\tau)} - \frac{\theta_1'^2(v,\tau)}{\theta_1^2(v,\tau)}\right] \\
= & -t^2\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)} - 2t^2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} - t^2\frac{\theta_1''(v,\tau)}{\theta_1(v,\tau)}.
\end{aligned}
\tag{6.130}
$$

To simplify this expression we need the following lemma:

Lemma 6.4.1. [8] When $x+y+z=0$ holds:

$$
\begin{aligned}
& \frac{\theta_1''(x,\tau)}{\theta_1(x,\tau)} + \frac{\theta_1''(y,\tau)}{\theta_1(y,\tau)} - 2\frac{\theta_1'(x,\tau)}{\theta_1(x,\tau)}\frac{\theta_1'(y,\tau)}{\theta_1(y,\tau)} = \\
& = 4\pi i\frac{\partial}{\partial\tau}\left(\log\left(\frac{\theta_1'(0,\tau)}{\theta(x-y,\tau)}\right)\right) + 2\frac{\theta_1'(x-y,\tau)}{\theta_1(x-y,\tau)}\left[\frac{\theta_1'(x,\tau)}{\theta_1(x,\tau)} - \frac{\theta_1'(y,\tau)}{\theta_1(y,\tau)}\right].
\end{aligned}
\tag{6.131}
$$

Proof. Applying the formulas

$$
\begin{aligned}
\zeta(v,\tau) &= \frac{\theta_1'(v,\tau)}{\theta_1(v,\tau)} + 4\pi i g_1(\tau)\, v, \\
\wp(v,\tau) &= -\frac{\theta_1''(v,\tau)}{\theta_1(v,\tau)} + \left(\frac{\theta_1'(v,\tau)}{\theta_1(v,\tau)}\right)^2 - 4\pi i g_1(\tau),
\end{aligned}
\tag{6.132}
$$

in the identity [33]

$$
[\zeta(x)+\zeta(y)+\zeta(z)]^2 = \wp(x)+\wp(y)+\wp(z),
\tag{6.133}
$$

we get:

$$
\begin{aligned}
& \left(\frac{\theta_1'(x,\tau)}{\theta_1(x,\tau)} + \frac{\theta_1'(y,\tau)}{\theta_1(y,\tau)} + \frac{\theta_1'(z,\tau)}{\theta_1(z,\tau)}\right)^2 = \\
& = -12\pi i g_1(\tau) - \frac{\theta_1''(x,\tau)}{\theta_1(x,\tau)} + \frac{\theta_1'^2(x,\tau)}{\theta_1^2(x,\tau)} - \frac{\theta_1''(y,\tau)}{\theta_1(y,\tau)} + \frac{\theta_1'^2(y,\tau)}{\theta_1^2(y,\tau)} - \frac{\theta_1''(z,\tau)}{\theta_1(z,\tau)} + \frac{\theta_1'^2(z,\tau)}{\theta_1^2(z,\tau)}.
\end{aligned}
\tag{6.134}
$$

Simplifying:

$$
\begin{aligned}
& 2\frac{\theta_1'(x-y,\tau)}{\theta_1(x-y,\tau)}\left[\frac{\theta_1'(x,\tau)}{\theta_1(x,\tau)} - \frac{\theta_1'(y,\tau)}{\theta_1(y,\tau)}\right] + 2\frac{\theta_1'(x,\tau)}{\theta_1(x,\tau)}\frac{\theta_1'(y,\tau)}{\theta_1(y,\tau)} = \\
& = 3\frac{\eta}{\omega} - \frac{\theta_1''(x,\tau)}{\theta_1(x,\tau)} - \frac{\theta_1''(y,\tau)}{\theta_1(y,\tau)} - \frac{\theta_1''(z,\tau)}{\theta_1(z,\tau)},
\end{aligned}
\tag{6.135}
$$

using the fact that

$$4\pi i\frac{\partial_\tau\theta_1'(0,\tau)}{\theta_1'(0,\tau)} = -12\pi i g_1(\tau), \tag{6.136}$$

$$\frac{\partial^2}{\partial^2 v}\theta_1(v,\tau) = 4\pi i\frac{\partial}{\partial\tau}\theta_1(v,\tau), \tag{6.137}$$

and doing the substitution $y \mapsto -y$, $z \mapsto x-y$, we get the desired identity. □

Substituting in the lemma $x = v_2$, $y = -v_2$ we get:

$$2\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)} + 2\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)} = 4\pi i\frac{\partial}{\partial\tau}\left(\log\left(\frac{\theta_1'(0,\tau)}{\theta_1(2v_2,\tau)}\right)\right) + 4\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}. \tag{6.138}$$

Substituting (6.138) in (6.130)

$$g^{13} = -2\pi i t^2\frac{\partial}{\partial\tau}\left(\log\left(\frac{\theta_1'(0,\tau)}{\theta_1(2v_2,\tau)}\right)\right). \tag{6.139}$$

Computing g^{22}:

$$\begin{aligned} g^{22} &= \frac{1}{2}\left(\frac{\partial t^2}{\partial v_0}\right)^2 - \frac{1}{2}\left(\frac{\partial t^2}{\partial v_2}\right)^2 + 2\frac{\partial t^2}{\partial u}\frac{\partial t^2}{\partial\tau} \\ &= \frac{1}{2}\left(\frac{\partial t^2}{\partial v_0}\right)^2 - \frac{1}{2}\left(\frac{\partial t^2}{\partial v_2}\right)^2 - 4\pi i t^2\frac{\partial t^2}{\partial\tau}. \end{aligned} \tag{6.140}$$

First, we separately compute $\frac{\partial t^2}{\partial v_2}$, $\frac{\partial t^2}{\partial v_0}$, $\frac{\partial t^2}{\partial\tau}$

$$\begin{aligned} \frac{1}{2}\left(\frac{\partial t^2}{\partial v_0}\right)^2 &= \frac{\left(t^2\right)^2}{2}\left[\frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)} + \frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}\right]^2, \\ -\frac{1}{2}\left(\frac{\partial t^2}{\partial v_2}\right)^2 &= -\frac{\left(t^2\right)^2}{2}\left[-\frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)} + \frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)} - 2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right]^2, \\ -4\pi i t^2\frac{\partial t^2}{\partial\tau} &= -4\pi i\frac{\left(t^2\right)^2}{2}\left[\frac{\partial_\tau\theta_1(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)} + \frac{\partial_\tau\theta_1(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)} - \frac{\partial_\tau\theta_1(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right] \\ &\quad - 4\pi i\frac{\left(t^2\right)^2}{2}\left[-\frac{\partial_\tau\theta_1'(0,\tau)}{\theta_1'(0,\tau)}\right]. \end{aligned} \tag{6.141}$$

Summing the equations we get:

(6.142)

$$\begin{aligned} g^{22} =& \frac{(t^2)^2}{2}\left[4\frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}\frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}\right] \\ &+ \frac{(t^2)^2}{2}\left[4\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\left[-\frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)} + \frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}\right] - 4\frac{\theta_1'^2(2v_2,\tau)}{\theta_1^2(2v_2,\tau)}\right] \\ &+ \frac{(t^2)^2}{2}\left[-2\frac{\theta_1''(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)} - 2\frac{\theta_1''(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)} - 8\pi i\left[-\frac{\partial_\tau\theta_1(2v_2,\tau)}{\theta_1(2v_2,\tau)} - \frac{\partial_\tau\theta_1'(0,\tau)}{\theta_1'(0,\tau)}\right]\right], \end{aligned}$$

where was used (6.137). Substituting in the lemma 2.3 $x = v_0 + v_2$, $y = v_0 - v_2$ we get:

$$
\begin{aligned}
&\frac{\theta_1''(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)} + \frac{\theta_1''(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} - 2\frac{\theta_1'(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)}\frac{\theta_1'(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} = \\
&= 4\pi i \frac{\partial}{\partial \tau}\left(\log\left(\frac{\theta_1'(0,\tau)}{\theta(2v_2,\tau)}\right)\right) + 2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\left[\frac{\theta_1'(v_0 + v_2, \tau)}{\theta_1(v_0 + v_2, \tau)} - \frac{\theta_1'(v_0 - v_2, \tau)}{\theta_1(v_0 - v_2, \tau)}\right].
\end{aligned} \tag{6.143}
$$

Substituting the last identity in g^{22} we get:

$$
g^{22} = 2\left(t^2\right)^2\left[\frac{\theta_1''(2v_2,\tau)}{\theta_1(2v_2,\tau)} - \frac{\theta_1'^2(2v_2,\tau)}{\theta_1^2(2v_2,\tau)}\right]. \tag{6.144}
$$

Computing g^{12}:

$$
\begin{aligned}
g^{12} &= \frac{1}{2}\frac{\partial t^1}{\partial v_0}\frac{\partial t^2}{\partial v_0} - \frac{1}{2}\frac{\partial t^1}{\partial v_2}\frac{\partial t^2}{\partial v_2} + \frac{\partial t^1}{\partial u}\frac{\partial t^2}{\partial \tau} + \frac{\partial t^2}{\partial u}\frac{\partial t^1}{\partial \tau} \\
&= \frac{1}{2}\frac{\partial t^1}{\partial v_0}\frac{\partial t^2}{\partial v_0} - \frac{1}{2}\frac{\partial t^1}{\partial v_2}\frac{\partial t^2}{\partial v_2} - 2\pi i t^2\frac{\partial t^1}{\partial \tau} - 2\pi i t^1\frac{\partial t^2}{\partial \tau}.
\end{aligned} \tag{6.145}
$$

We have that:

$$
\begin{aligned}
\frac{\partial t^1}{\partial v_0} &= 2\frac{\theta_1'(v_0,\tau)}{\theta_1(v_0,\tau)}\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u} + 2\frac{\partial t^2}{\partial v_0}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}, \\
\frac{\partial t^1}{\partial v_2} &= -2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u} + 2\frac{\partial t^2}{\partial v_2}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} + 2t^2\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)} - \frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right], \\
\frac{\partial t^1}{\partial \tau} &= 2\left[\frac{\partial_\tau\theta_1(v_0,\tau)}{\theta_1(v_0,\tau)} - \frac{\partial_\tau\theta_1(v_2,\tau)}{\theta_1(v_2,\tau)}\right]\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u} + 2\frac{\partial t^2}{\partial \tau}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} \\
&\quad + 2t^2\frac{\partial}{\partial \tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right).
\end{aligned} \tag{6.146}
$$

Therefore:

$$
\begin{aligned}
\frac{1}{2}\frac{\partial t^1}{\partial v_0}\frac{\partial t^2}{\partial v_0} =& t^2\left[\frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}+\frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}\right]\frac{\theta_1'(v_0,\tau)}{\theta_1(v_0,\tau)}\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}\\
&+\left(\frac{\partial t^2}{\partial v_0}\right)^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)},\\
-\frac{1}{2}\frac{\partial t^1}{\partial v_2}\frac{\partial t^2}{\partial v_2} =& -t^2\left[-\frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}+\frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}\right]\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}\\
&-t^2\left[-2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right]\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}\\
&-(\frac{\partial t^2}{\partial v_2})^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}-t^2\frac{\partial t^2}{\partial v_2}\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}-\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right],\\
-2\pi i t^1\frac{\partial t^2}{\partial\tau} =& -2\pi i\left[\frac{\partial_\tau\theta_1(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}+\frac{\partial_\tau\theta_1(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}\right]t^2\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}\\
&-2\pi i\left[-\frac{\partial_\tau\theta_1(2v_2,\tau)}{\theta_1(2v_2,\tau)}-\frac{\partial_\tau\theta_1'(0,\tau)}{\theta_1'(0,\tau)}\right]t^2\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}\\
&-4\pi i t^2\frac{\partial t^2}{\partial\tau}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)},\\
-2\pi i t^2\frac{\partial t^1}{\partial\tau} =& -4\pi i t^2\left[\frac{\partial_\tau\theta_1(v_0,\tau)}{\theta_1(v_0,\tau)}-\frac{\partial_\tau\theta_1(v_2,\tau)}{\theta_1(v_2,\tau)}\right]\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}\\
&-4\pi i t^2\frac{\partial t^2}{\partial\tau}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}-4\pi i(t^2)^2\frac{\partial}{\partial\tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right).
\end{aligned}
\tag{6.147}
$$

Let us separate g^{12} in three terms:

$$
g^{12}=(1)+(2)+(3)
\tag{6.148}
$$

where:

(6.149)

$$
\begin{aligned}
(1) =& t^2\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}\left[\frac{\theta_1'(v_0,\tau)}{\theta_1(v_0,\tau)}\left(\frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}+\frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}\right)\right]\\
&+t^2\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}\left[\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\left(-\frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}+\frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}-2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right)\right]\\
&+t^2\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}\left[-2\pi i\left(\frac{\partial_\tau\theta_1(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}+\frac{\partial_\tau\theta_1(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}\right)\right]\\
&+t^2\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}\left[-2\pi i\left(-\frac{\partial_\tau\theta_1(2v_2,\tau)}{\theta_1(2v_2,\tau)}-\frac{\partial_\tau\theta_1'(0,\tau)}{\theta_1'(0,\tau)}\right)\right]\\
&+t^2\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}\left[-4\pi i\left(\frac{\partial_\tau\theta_1(v_0,\tau)}{\theta_1(v_0,\tau)}-\frac{\partial_\tau\theta_1(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\right],
\end{aligned}
$$

$$(6.150)\quad \begin{aligned}(2) =&\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\left[\left(\frac{\partial t^2}{\partial v_0}\right)^2-\left(\frac{\partial t^2}{\partial v_2}\right)^2-8\pi i t^2\frac{\partial t^2}{\partial \tau}\right]\\ =&4\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}(t^2)^2\left[\frac{\theta_1''(2v_2,\tau)}{\theta_1(2v_2,\tau)}-\frac{\theta_1'^2(2v_2,\tau)}{\theta_1^2(2v_2,\tau)}\right],\end{aligned}$$

where was used the previous computation of g^{22}:

$$(6.151)\quad (3) = -4\pi i(t^2)^2\frac{\partial}{\partial\tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)-t^2\frac{\partial t^2}{\partial v_2}\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}-\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right].$$

To simplify the expression (1) we need to use the lemma 6.4.1 with the following substitutions $x=v_0$,$y=v_2$:

$$(6.152)\quad \begin{aligned}&\frac{\theta_1''(v_0,\tau)}{\theta_1(v_0,\tau)}+\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}-2\frac{\theta_1'(v_0,\tau)}{\theta_1(v_0,\tau)}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}=\\ =&4\pi i\frac{\partial}{\partial\tau}\left(\log\left(\frac{\theta_1'(0,\tau)}{\theta(v_0-v_2,\tau)}\right)\right)+2\frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}\left[\frac{\theta_1'(v_0,\tau)}{\theta_1(v_0,\tau)}-\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right].\end{aligned}$$

Using the substitutions $x=v_0$,$y=-v_2$

$$(6.153)\quad \begin{aligned}&\frac{\theta_1''(v_0,\tau)}{\theta_1(v_0,\tau)}+\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}+2\frac{\theta_1'(v_0,\tau)}{\theta_1(v_0,\tau)}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}=\\ =&4\pi i\frac{\partial}{\partial\tau}\left(\log\left(\frac{\theta_1'(0,\tau)}{\theta(v_0+v_2,\tau)}\right)\right)+2\frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}\left[\frac{\theta_1'(v_0,\tau)}{\theta_1(v_0,\tau)}+\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right].\end{aligned}$$

Summing (6.152) with (6.153):

$$(6.154)\quad \begin{aligned}&2\frac{\theta_1''(v_0,\tau)}{\theta_1(v_0,\tau)}+2\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}-4\pi i\frac{\partial}{\partial\tau}\left(\log\left(\frac{\theta_1'(0,\tau)}{\theta(v_0-v_2,\tau)}\right)\right)\\ &-4\pi i\frac{\partial}{\partial\tau}\left(\log\left(\frac{\theta_1'(0,\tau)}{\theta(v_0+v_2,\tau)}\right)\right)\\ =&2\frac{\theta_1'(v_0,\tau)}{\theta_1(v_0,\tau)}\left(\frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}+\frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}\right)\\ &+2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\left(-\frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}+\frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}\right).\end{aligned}$$

Substituting in (1) we get :

(6.155)

$$\begin{aligned}(1) =&t^2\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}\left[-2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right]\\ &+t^2\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}\left[-2\pi i(-\frac{\partial_\tau\theta_1(2v_2,\tau)}{\theta_1(2v_2,\tau)}+\frac{\partial_\tau\theta_1'(0,\tau)}{\theta_1'(0,\tau)})+8\pi i(\frac{\partial_\tau\theta_1(v_2,\tau)}{\theta_1(v_2,\tau)})\right]\\ =&t^2\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}\left[-2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}-2\pi i\frac{\partial}{\partial\tau}\left(\log\frac{\theta_1'(0,\tau)}{\theta_1(2v_2,\tau)}\right)+2\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}\right].\end{aligned}$$

Using the identity (6.131), We get:

$$(6.156)\quad (1) = t^2\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}-\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right].$$

We compute (3)

$$\begin{aligned}(3) =& -4\pi i(t^2)^2\frac{\partial}{\partial\tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right) - t^2\frac{\partial t^2}{\partial v_2}\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)} - \frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right]\\
=& -4\pi i(t^2)^2\frac{\partial}{\partial\tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right) - t^2\left(t^1 - 2t^2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right)\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)} - \frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right]\\
=& -4\pi i(t^2)^2\frac{\partial}{\partial\tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right) + 2(t^2)^2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)})\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)} - \frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right]\\
&- t^2\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi iu}\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)} - \frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right]\\
&- 2(t^2)^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)})\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)} - \frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right].\end{aligned} \tag{6.157}$$

The result implies:

$$\begin{aligned}(1)+(3) =& -4\pi i(t^2)^2\frac{\partial}{\partial\tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\\
&- 2(t^2)^2\left[\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} - \frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)})\right]\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)} - \frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right].\end{aligned} \tag{6.158}$$

Computing g^{12}:

$$\begin{aligned}g^{12} =& -4\pi i(t^2)^2\frac{\partial}{\partial\tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\\
&- 2(t^2)^2\left[\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} - \frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)})\right]\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)} - \frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right]\\
&+ 4\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}(t^2)^2\left[\frac{\theta_1''(2v_2,\tau)}{\theta_1(2v_2,\tau)} - \frac{\theta_1'^2(2v_2,\tau)}{\theta_1^2(2v_2,\tau)}\right].\end{aligned} \tag{6.159}$$

To simplify this expression we need to prove one more lemma:

Lemma 6.4.2.

$$\begin{aligned}&2\frac{\theta_1'''(v_2,\tau)}{\theta_1(v_2,\tau)} + 2\frac{\theta_1''(v_2,\tau)\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} - 4\frac{\theta_1'^3(v_2,\tau)}{\theta_1^3(v_2,\tau)} = 4\pi i\frac{\partial^2}{\partial v_2\partial\tau}\left(\log\left(\frac{\theta_1'(0,\tau)}{\theta_1(2v_2,\tau)}\right)\right)\\
&+ 8\frac{\theta_1''(2v_2,\tau)}{\theta_1(2v_2,\tau)}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} - 8\frac{\theta_1'^2(2v_2,\tau)}{\theta_1^2(2v_2,\tau)}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} + 4\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}\\
&- 4\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}.\end{aligned} \tag{6.160}$$

Proof. Differentiating the identity with respect to v_2 we obtain (6.160). □

Computing g^{12}:

$$\begin{aligned} g^{12} =&(t^2)^2\left[-\frac{\theta_1'''(v_2,\tau)}{\theta_1(v_2,\tau)}+\frac{\theta_1'(v_2,\tau)\theta_1''(v_2,\tau)}{\theta_1^2(v_2,\tau)}-2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}+2\frac{\theta_1'^3(v_2,\tau)}{\theta_1^3(v_2,\tau)}\right] \\ &+(t^2)^2\left[2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)})\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}-2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}+4\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\theta_1''(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right] \\ &+(t^2)^2\left[-4\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\theta_1'^2(2v_2,\tau)}{\theta_1^2(2v_2,\tau)}\right]. \end{aligned} \tag{6.161}$$

Applying (6.160), we get:

$$g^{12}=-2\pi i(t^2)^2\left[\frac{\partial^2}{\partial v_2\partial\tau}\left(\log\left(\frac{\theta_1'(0,\tau)}{\theta_1(2v_2,\tau)}\right)\right)\right]. \tag{6.162}$$

Computing g^{11}:

$$\begin{aligned} g^{11}&=\frac{1}{2}\left(\frac{\partial t^1}{\partial v_0}\right)^2-\frac{1}{2}\left(\frac{\partial t^1}{\partial v_2}\right)^2+2\frac{\partial t^1}{\partial u}\frac{\partial t^1}{\partial\tau} \\ &=\frac{1}{2}\left(\frac{\partial t^1}{\partial v_0}\right)^2-\frac{1}{2}\left(\frac{\partial t^1}{\partial v_2}\right)^2-4\pi i t^1\frac{\partial t^1}{\partial\tau} \end{aligned} \tag{6.163}$$

Computing $\frac{1}{2}\left(\frac{\partial t^1}{\partial v_0}\right)^2$, $\frac{1}{2}\left(\frac{\partial t^1}{\partial v_2}\right)^2$ and $-4\pi i t^1\frac{\partial t^1}{\partial\tau}$:
To simplify the computation let us define:

$$A:=\frac{\theta_1^2(v_0,\tau)}{\theta_1^2(v_2,\tau)}e^{-2\pi i u}. \tag{6.164}$$

Then,

$$\frac{1}{2}\left(\frac{\partial t^1}{\partial v_0}\right)^2=2\frac{\theta_1'^2(v_0,\tau)}{\theta_1^2(v_0,\tau)}A^2+4A\frac{\theta_1'(v_0,\tau)}{\theta_1(v_0,\tau)}\frac{\partial t^2}{\partial v_0}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}+2\left(\frac{\partial t^2}{\partial v_0}\right)^2\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}, \tag{6.165}$$

$$\begin{aligned} -\frac{1}{2}\left(\frac{\partial t^1}{\partial v_2}\right)^2=&-2\frac{\theta_1'^2(v_2\tau)}{\theta_1^2(v_2,\tau)}A^2 \\ &+2A\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\left[2\frac{\partial t^2}{\partial v_2}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}+2t^2\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}-\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right]\right] \\ &-2\left(\frac{\partial t^2}{\partial v_2}\right)^2\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}-4t^2\frac{\partial t^2}{\partial v_2}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}-\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right] \\ &-2(t^2)^2\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}-\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right]^2, \end{aligned} \tag{6.166}$$

(6.167)
$$-4\pi i t^1\frac{\partial t^1}{\partial\tau} = -8\pi i A^2\left[\frac{\partial_\tau\theta_1(v_0,\tau)}{\theta_1(v_0,\tau)}-\frac{\partial_\tau\theta_1(v_2,\tau)}{\theta_1(v_2,\tau)}\right]-8\pi i A\frac{\partial t^2}{\partial\tau}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}$$
$$-8\pi i A t^2\frac{\partial}{\partial\tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)-16\pi i A t^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\left[\frac{\partial_\tau\theta_1(v_0,\tau)}{\theta_1(v_0,\tau)}-\frac{\partial_\tau\theta_1(v_2,\tau)}{\theta_1(v_2,\tau)}\right]$$
$$-16\pi i t^2\frac{\partial t^2}{\partial\tau}\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}-16\pi i(t^2)^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial\tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right).$$

Then, we have:

$$g^{11}=(1)+(2)+(3)+(4)+(5), \tag{6.168}$$

where:

$$\begin{aligned}(1)=&A^2\left[2\frac{\theta_1'^2(v_0,\tau)}{\theta_1^2(v_0,\tau)}-2\frac{\theta_1'^2(v_2\tau)}{\theta_1^2(v_2,\tau)}-8\pi i\left[\frac{\partial_\tau\theta_1(v_0,\tau)}{\theta_1(v_0,\tau)}-\frac{\partial_\tau\theta_1(v_2,\tau)}{\theta_1(v_2,\tau)}\right]\right]\\ =&A^2\left[2\frac{\theta_1'^2(v_0,\tau)}{\theta_1^2(v_0,\tau)}-2\frac{\theta_1'^2(v_2\tau)}{\theta_1^2(v_2,\tau)}-2\frac{\theta_1''(v_0,\tau)}{\theta_1(v_0,\tau)}+2\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}\right]\\ =&2A^2\left[\wp(v_0)-\wp(v_2)\right]=2\frac{16\omega^4}{\left[\wp(v_0)-\wp(v_2)\right]^2}\left[\wp(v_0)-\wp(v_2)\right]\\ =&32\frac{\omega^4}{\wp(v_0)-\wp(v_2)},\end{aligned}\tag{6.169}$$

(6.170)
$$(2)=-8\pi i t^2 A\frac{\partial}{\partial\tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)$$
$$+2At^2\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\left[2\frac{\theta_1'(v_0,\tau)}{\theta_1(v_0,\tau)}\left[\frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}+\frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}\right]\right]$$
$$+2At^2\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\left[2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\left[\frac{-\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}+\frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}-2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right]\right]$$
$$+2At^2\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\left[2\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}-\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right]-8\pi i\left[\frac{\partial_\tau\theta_1(v_0,\tau)}{\theta_1(v_0,\tau)}-\frac{\partial_\tau\theta_1(v_2,\tau)}{\theta_1(v_2,\tau)}\right]\right]$$
$$+2At^2\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\left[-4\pi i\left[\frac{\partial_\tau\theta_1(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}+\frac{\partial_\tau\theta_1(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}-\frac{\partial_\tau\theta_1(2v_2,\tau)}{\theta_1(2v_2,\tau)}-\frac{\partial_\tau\theta_1'(0,\tau)}{\theta_1'(0,\tau)}\right]\right].$$

Using (6.131),

(6.171)
$$2\frac{\theta_1''(v_0,\tau)}{\theta_1(v_0,\tau)}+2\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}-4\pi i\frac{\partial}{\partial\tau}\left(\log\left(\frac{\theta_1'(0,\tau)}{\theta(v_0-v_2,\tau)}\right)\right)-4\pi i\frac{\partial}{\partial\tau}\left(\log\left(\frac{\theta_1'(0,\tau)}{\theta(v_0+v_2,\tau)}\right)\right)=$$
$$=2\frac{\theta_1'(v_0,\tau)}{\theta_1(v_0,\tau)}\left(\frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}+\frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}\right)+2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\left(-\frac{\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)}+\frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)}\right),$$

$$
\begin{aligned}
(2) =& -8\pi i t^2 A\frac{\partial}{\partial\tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right) + 2At^2\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\left[-4\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right] \\
&+ 2At^2\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\left[2\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)} - \frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right] + 4\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)}\right] \\
&+ 2At^2\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\left[-4\pi i\frac{\partial_\tau\theta_1'(0,\tau)}{\theta_1'(0,\tau)} + 4\pi i\frac{\partial_\tau\theta_1(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right].
\end{aligned}
\tag{6.172}
$$

Using again (6.131):

$$
(2) = -8\pi i t^2 A\frac{\partial}{\partial\tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right) + 8At^2\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)} - \frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right],
\tag{6.173}
$$

$$
\begin{aligned}
(3) &= 4\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\left[\frac{1}{2}\left(\frac{\partial t^2}{\partial v_0}\right)^2 - \frac{1}{2}\left(\frac{\partial t^2}{\partial v_2}\right)^2 - 4\pi i t^2\frac{\partial t^2}{\partial\tau}\right] \\
&= 8\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}(t^2)^2\left[\frac{\theta_1''(2v_2,\tau)}{\theta_1(2v_2,\tau)} - \frac{\theta_1'^2(2v_2,\tau)}{\theta_1^2(2v_2,\tau)}\right],
\end{aligned}
\tag{6.174}
$$

$$
(4) = -2(t^2)^2\left[\frac{\partial}{\partial v_2}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\right]^2 - 16\pi i(t^2)^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial\tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right),
\tag{6.175}
$$

$$
\begin{aligned}
(5) =& -4(t^2)\frac{\partial t^2}{\partial v_2}\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial v_2}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right) \\
=& -4(t^2)^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial v_2}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\left[\frac{-\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)} + \frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)} - 2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right] \\
=& -4(t^2)^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial v_2}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\left[\frac{-\theta_1'(v_0-v_2,\tau)}{\theta_1(v_0-v_2,\tau)} + \frac{\theta_1'(v_0+v_2,\tau)}{\theta_1(v_0+v_2,\tau)} - 2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right] \\
&- 4(t^2)^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial v_2}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\left[2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} - 2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right] \\
=& -4(t^2)A\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial v_2}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right) \\
&- 4(t^2)^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial v_2}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\left[2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} - 2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right].
\end{aligned}
\tag{6.176}
$$

Summing (2) and (5):

$$
\begin{aligned}
(2)+(5) =& -8\pi i t^2 A \frac{\partial}{\partial\tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right) + 8At^2\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\left[\frac{\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)} - \frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}\right]\\
&- 4(t^2)A\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial v_2}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\\
&- 4(t^2)^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial v_2}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\left[2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} - 2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right]\\
&= -4(t^2)^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial v_2}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\left[2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} - 2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right]\\
&+ At^2\left[-2\frac{\theta_1'''(v_2,\tau)}{\theta_1(v_2,\tau)} + 6\frac{\theta_1'(v_2,\tau)\theta_1''(v_2,\tau)}{\theta_1(v_2,\tau)} - 4\frac{\theta_1'^3(v_2,\tau)}{\theta_1^3(v_2,\tau)}\right]\\
&= -4(t^2)^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial v_2}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\left[2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} - 2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right]\\
&+ 2At^2\wp'(v_2)\\
&= -4(t^2)^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial v_2}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\left[2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} - 2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right]\\
&- 32\frac{\omega^4}{\wp(v_0)-\wp(v_2)}.
\end{aligned}
\tag{6.177}
$$

Summing (1) and (2) + (5):

$$
(1)+(2)+(5) = -4(t^2)^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial v_2}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\left[2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} - 2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right]. \tag{6.178}
$$

From the above results, we find:

$$
\begin{aligned}
g^{11} =&(1)+(2)+(5)+(3)+(4)\\
=& -4(t^2)^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial v_2}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\left[2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)} - 2\frac{\theta_1'(2v_2,\tau)}{\theta_1(2v_2,\tau)}\right]\\
&+ 8\frac{\theta_1'^2(v_2,\tau)}{\theta_1^2(v_2,\tau)}(t^2)^2\left[\frac{\theta_1''(2v_2,\tau)}{\theta_1(2v_2,\tau)} - \frac{\theta_1'^2(2v_2,\tau)}{\theta_1^2(2v_2,\tau)}\right]\\
&- 2(t^2)^2\left[\frac{\partial}{\partial v_2}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right)\right]^2 - 16\pi i(t^2)^2\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\frac{\partial}{\partial\tau}\left(\frac{\theta_1'(v_2,\tau)}{\theta_1(v_2,\tau)}\right).
\end{aligned}
\tag{6.179}
$$

Summarizing, we have proved the identities (6.95)-(6.99).

□

CHAPTER 7

Coalescence phenomenon and Dubrovin Frobenius submanifold of the orbit space of $\mathscr{J}(\tilde{A}_1)$

In differential geometry, one of the most common problem is the study of the submanifolds. Dubrovin Frobenius manifolds consists in a manifold with a large number of geometric conditions, therefore, quite often some of these conditions are not satisfied in a submanifold, for instance flatness. In [**31**] and [**32**] , Strachan investigates the geometric structure of the discriminant locus and the caustic of Hurwitz spaces, which is described as

$$\begin{aligned} &u_i = 0, \quad \text{discriminant locus,} \\ &u_i = u_j, \quad \text{for} \quad i \neq j, \quad \text{caustic.} \end{aligned}$$

In these spaces, there exist a very rich geometric structure, which is almost a Dubrovin Frobenius structure, but, the induced metric of the ambient space is quite often curved. The aim of this section is to point out the rich geometric structure that the orbit space of $\mathscr{J}(\tilde{A}_1)$ has. Indeed, in [**26**], Romano proved that the Hurwitz space $H_{1,0,0}$ have a tri-hamiltonian structure. This fact realise the Hurwitz space $H_{1,0,0}$ as suitable ambient space to support the Hurwitz space $H_{1,1}$ as Dubrovin Frobenius submanifold. This was done supported in the argument that the Darboux-Egoroff systems of both Hurwitz spaces are parametrized by the same Painleve VI transcendents. In addition, in [**30**] Shramchenko shows that the Stokes matrices of the Hurwitz space $H_{1,0,0}$ depends on the Stokes matrices of the Hurwitz space $H_{0,0,0}$. This fact suggests that there exist some submanifold of $H_{1,0,0}$ which contains some geometric information regarding the Hurwitz space $H_{0,0,0}$. Therefore, this section will investigates the submanifolds of $H_{1,0,0}$ by coalescing the canonical coordinates.

7.1. Review of Tri-hamiltonian structure

In [**26**], Romano inspired by the work done in [**25**] introduced the a notion of tri-hamiltonian structure. This new structure implies in the existence of third flat metric compatible with the flat pencil structure of the Dubrovin Frobenius manifold.

DEFINITION 7.1.1. [**26**] A 2n-dimensional Dubrovin Frobenius manifold has a tri-hamiltonian structure if its Euler vector field has the following form

$$E = \sum_{i=1}^{n} t^{\alpha} \frac{\partial}{\partial t^{\alpha}} + (1 + 2\mu) \sum_{i=n+1}^{2n} \frac{\partial}{\partial t^{\alpha}}, \tag{7.1}$$

for some non zero constant μ.

The name tri-hamiltonian is motivated by the following

PROPOSITION 7.1.1. [26] Let be a 2n-dimensional Dubrovin Frobenius manifold with tri-hamiltonian structure, then the metric

$$\tilde{\eta}^{\alpha\beta} = \eta^{\epsilon\alpha}\left(\mathscr{U}^2\right)^{\beta}_{\epsilon}, \quad \mathscr{U}^{\alpha}_{\beta} = E^{\mu}c^{\beta}_{\mu\alpha} \tag{7.2}$$

is flat.

COROLLARY 7.1.1.1. [26] Let a 2n-dimensional Dubrovin Frobenius manifold with tri-hamiltonian structure, then the metrics $\eta^*, g^*, \tilde{\eta}$ form a 2-parameter flat pencil metric

$$g_{\epsilon_1,\epsilon_2} = \tilde{\eta} - \epsilon_1 g - \epsilon_2 \eta. \tag{7.3}$$

If the a 2n-dimensional Dubrovin Frobenius manifold with a tri-hamiltonian structure is semisimple, the metrics $\eta, g, \tilde{\eta}$, in canonical coordinates, have the following form

$$\begin{aligned} \eta &= \sum_{i=1}^{2n} \eta_{ii}(du_i)^2, \\ g &= \sum_{i=1}^{2n} \frac{\eta_{ii}}{u_i}(du_i)^2, \\ \tilde{\eta} &= \sum_{i=1}^{2n} \frac{\eta_{ii}}{u_i^2}(du_i)^2, \end{aligned} \tag{7.4}$$

7.2. Review of Dubrovin Frobenius submanifolds

The aim of this section is to introduce the definition of Dubrovin Frobenius submanifolds, for this purpose, it will be necessary to introduce the notion of induced structure first. Dubrovin Frobenius manifolds is manifold together a large amount of conditions, therefore, we will gradually introduce the induced structure once by time.

7.2.1. Induced structure.

DEFINITION 7.2.1. [32] An F-manifold is a pair $(M, \bullet)$ where M is a manifold and $\bullet$ is a commutative, associative multiplication $\bullet : TM \times TM \mapsto TM$ satisfying the following conditions

$$Lie_{X\bullet Y}(\bullet) = X \bullet Lie_Y(\bullet) + Y \bullet Lie_X(\bullet), \quad \forall X, Y \in TM. \tag{7.5}$$

DEFINITION 7.2.2. [32]

(1) An F_E manifold is an F-manifold with an Euler vector field of weight d. This is a global vector field satisfying the conditions

$$Lie_E(\bullet) = d.\bullet\,. \tag{7.6}$$

(2) An F_η is an F-manifold with a metric $\eta(,)$ compatible with the multiplication:

$$\eta(X \bullet Y, Z) = \eta(X, Y \bullet Z), \quad X, Y, Z \in TM. \tag{7.7}$$

(3) An $\mathscr{F}$ is both F_E and F_η manifold with E and η related by the relation

$$Lie_E \eta(,) = D\eta(,) \tag{7.8}$$

for some constant D.

At this stage, we can define the notion of natural submanifold.

Definition 7.2.3. [**32**] A natural submanifold N of an F_E manifold $(M, \bullet, E)$ is a submanifold $N \subset M$ such that

$$\begin{aligned} &TN \bullet TN \subset TN, \\ &E_x \in TN, \quad \forall x \in N. \end{aligned} \tag{7.9}$$

Then, we say that the vector field $E|_N$ is the induced Euler vector field.

Definition 7.2.4. [**32**] Consider a F_η manifold $(M, \bullet, \eta)$ with a submanifold $N \subset M$, then we define an induced metric on N by $\eta|_N$, and an induced product $\star$ by

$$X \star Y = pr(X \bullet Y), \quad X, Y \in T_xN \subset T_xM, \tag{7.10}$$

where pr denotes the projection with respect the metric η. Moreover, if we also have a unit vector field e, we defined an induced unit vector field by $e|_N$.

At this point, we have collected the minimal definitions to give a notion of Dubrovin Frobenius submanifolds.

Definition 7.2.5. [**32**] Let $(M, \bullet, \eta, e, E)$ be a Dubrovin Frobenius manifold with Frobenius product $\bullet$, metric η, unit vector field e, and Euler vector field E. Then, we say that a submanifold $N \subset M$ is a Dubrovin Frobenius submanifold if $(N, \star, \eta|_N, e|_N, E|_N)$ is a Dubrovin Frobenius manifold. i.e. N is a Dubrovin Frobenius manifold with respect the induced structure.

In [**31**], Strachan proved an important Theorem, which could give a source of Dubrovin Frobenius submanifolds.

Theorem 7.2.1. [**31**] Let N be a natural flat submanifold of a Dubrovin Frobenius manifold M. If the unit vector field e and the Euler vector field E are both tangential to N at all $t \in N$, then N is a Dubrovin Frobenius submanifold.

For two-dimensional submanifolds we have an even stronger result.

Theorem 7.2.2. [**32**] Let M a Dubrovin Frobenius manifold of dimension n and N a two-dimensional submanifold. If the unity vector field e is tangential to the submanifold N at all points $t \in N$, then N is a Dubrovin Frobenius submanifold.

7.2.2. Semisimple Dubrovin Frobenius submanifold.

The goal of this subsection is to show that the discriminant locus and the caustic of a semisimple Dubrovin Frobenius manifolds is a promising source of Dubrovin Frobenius submanifolds, since the caustic for instance fails to be a Dubrovin Frobenius manifold only due the non-flatness of the induced metric η.

Consider a semisimple $\mathscr{F}$ manifold, then there exists coordinates $(u_1, u_2, u_3, .., u_n)$ such that the Frobenius product and the Euler vector field assume the following form

$$\begin{aligned} &\frac{\partial}{\partial u_i} \bullet \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i}, \\ &E = \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i}. \end{aligned} \tag{7.11}$$

Definition 7.2.6. [32]

(1) A submanifold defined by the condition $u_i = 0$ for one or more values of i is a discriminant hypersurface, and will be denoted by $\mathscr{D}$.
(2) A submanifold defined by the condition $u_i = u_j$ for some pair u_i and u_j for $i \neq j$ is known as caustic, and will be denoted by $\mathscr{C}$.

The next Theorem proved by Strachan in [32] shows that discriminant and caustic hypersurfaces are the only source of possible Dubrovin Frobenius submanifolds.

Theorem 7.2.3. [32] Let $(M, \bullet, E, \eta)$ be a semisimple $\mathscr{F}$ manifold. Then

(1) The only natural submanifolds are the caustic and the discriminant hypersurfaces.
(2) The identity is tangential to a natural submanifold if and only if it is pure caustic. i.e. it does not have intersection with any discriminant hypersurface.

Thereore, the Theorems 7.2.1, 7.2.2, and 7.2.3 proves the following corollary, which is our main source of Dubrovin Frobenius submanifolds.

Corollary 7.2.3.1. [32] Any flat caustic of semisimple Dubrovin Frobenius manifold is itself a Dubrovin Frobenius manifold, i.e. Dubrovin Frobenius submanifold. All two-dimensional caustic are Dubrovin Frobenius submanifolds.

7.3. Discriminant of $\mathscr{J}(\tilde{A}_1)$

We start this section by comparing the Landau-Ginzburg superpotential of $H_{1,0,0}$ and $H_{1,1}$.

In the appendix J of [12], Dubrovin proved the following lemma relating the coordinates (z, ω, ω') with the coordinates (u, v_0, τ).

Lemma 7.3.1. [12] The map

$$
\begin{aligned}
u &= \frac{-1}{2\pi i}\left[2\log\sigma(z_0,\omega,\omega') - \frac{\eta}{\omega}z_0^2\right] \\
v_0 &= \frac{z_0}{2\omega}, \\
\tau &= \frac{\omega'}{\omega}.
\end{aligned} \tag{7.12}
$$

determine local coordinates in $H_{1,1}$, where $\eta = \zeta(\omega,\omega,\omega')$ is Weiestrass Zeta function evaluated in ω.

Lemma 7.3.2. The canonical coordinates of the orbit space $\mathscr{J}(A_1)$ are given by

$$
\begin{aligned}
u_1 &= -t^1 + t^2\frac{\theta_2''(0,\tau)}{\theta_2(0,\tau)}, \\
u_2 &= -t^1 + t^2\frac{\theta_3''(0,\tau)}{\theta_3(0,\tau)}, \\
u_3 &= -t^1 + t^2\frac{\theta_4''(0,\tau)}{\theta_4(0,\tau)}.
\end{aligned} \tag{7.13}
$$

where (t^1, t^2, τ) are the flat coordinates of η with respect the orbit space $\mathscr{J}(A_1)$.

Proof. Consider the Landau-Ginzburg superpotential (5.21) for $n = 1$

$$\lambda^{\mathscr{J}(A_1)}(v) = \varphi_2\wp(v,\tau) + \varphi_0. \tag{7.14}$$

where

$$\varphi_0 = -\varphi_2\wp(v_0,\tau). \tag{7.15}$$

Computing the critical points of (7.14),

$$\lambda^{\mathscr{J}(A_1)'}(v) = \varphi_2\wp'(v,\tau) = 0. \tag{7.16}$$

We obtain $v = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ as roots of (7.16). Then, by writing (7.14) in terms of the flat coordinates of η

$$\lambda^{\mathscr{J}(A_1)}(v) = \varphi_2\frac{d^2\log\theta_1(v,\tau)}{dv^2} - t^1. \tag{7.17}$$

We obtain the desired result by substituting $v = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ in(7.17), and by using the following relations between the Jacobi theta functions [33]

$$
\begin{aligned}
\theta_2(v,\tau) &= \theta_1\left(v + \frac{1}{2}, \tau\right), \\
\theta_3(v,\tau) &= e^{\left(iv + \frac{i\pi\tau}{4}\right)}\theta_1\left(v + \frac{1+\tau}{2}, \tau\right), \\
\theta_4(v,\tau) &= ie^{\left(iv + \frac{i\pi\tau}{4}\right)}\theta_1\left(v + \frac{\tau}{2}, \tau\right).
\end{aligned} \tag{7.18}
$$

$\square$

LEMMA 7.3.3. The canonical coordinates of the orbit space $\mathscr{J}(\tilde{A}_1)$ are given by

$$v_1 = t^1 - 2t^2 \frac{\theta_1'(t^3,\tau)}{\theta_1(t^3,\tau)},$$
$$v_2 = t^1 - 2t^2 \frac{\theta_2'(t^3,\tau)}{\theta_2(t^3,\tau)},$$
$$v_3 = t^1 - 2t^2 \frac{\theta_3'(t^3,\tau)}{\theta_3(t^3,\tau)}, \tag{7.19}$$
$$v_4 = t^1 - 2t^2 \frac{\theta_4'(t^3,\tau)}{\theta_4(t^3,\tau)}.$$

PROOF. Consider the critical points of (6.75),

$$\begin{aligned}\lambda'(p) &= t^2 \left[-\wp(p-t^3,\tau) + \wp(p+t^3,\tau)\right] \\ &= t^2 \frac{\sigma(2p,\tau)\sigma(2t^3,\tau)}{\sigma^2(p-t^3,\tau)\sigma^2(p-t^3,\tau)} = 0.\end{aligned} \tag{7.20}$$

Then, $p = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ solves (7.20). Consequently, the canonical coordinates of the orbit space of $\mathscr{J}(\tilde{A}_1)$ read

$$v_1 = t^1 - 2t^2 \frac{\theta_1'(t^3,\tau)}{\theta_1(t^3,\tau)},$$
$$v_2 = t^1 - 2t^2 \frac{\theta_2'(t^3,\tau)}{\theta_2(t^3,\tau)},$$
$$v_3 = t^1 - 2t^2 \frac{\theta_3'(t^3,\tau)}{\theta_3(t^3,\tau)},$$
$$v_4 = t^1 - 2t^2 \frac{\theta_4'(t^3,\tau)}{\theta_4(t^3,\tau)},$$

due to (7.18). □

LEMMA 7.3.4. The equations

$$\begin{aligned}u &= \frac{-1}{2\pi i}\left[4\log\sigma(z_2,\omega,\omega') - \log\sigma(z_0-z_2,\omega,\omega') - \log\sigma(z_0+z_2,\omega,\omega') - \frac{\eta}{\omega}\left(z_0^2 - z_2^2\right)\right] \\ v_0 &= \frac{z_0}{2\omega}, \\ v_2 &= \frac{z_2}{2\omega}, \\ \tau &= \frac{\omega'}{\omega},\end{aligned} \tag{7.21}$$

determine local coordinates in $H_{1,0,0}$, where $\eta = \zeta(\omega,\omega,\omega')$ is Weiestrass Zeta function evaluated in ω.

PROOF. The σ function for the lattice generated by ω, ω' in terms of Jacobi theta 1 is

$$\sigma(z,\omega,\omega') = 2\omega \frac{\theta_1(\frac{z}{2\omega},\frac{\omega'}{\omega})}{\theta_1'(0,\frac{\omega'}{\omega})} e^{\frac{\eta}{2\omega}z^2}. \tag{7.22}$$

Substituting (7.22) into (7.21), we obtain

$$-2\pi i u = \log\left((2\omega)^2 \frac{\theta_1^4(v_2,\tau)}{\theta_1(v_0-v_2,\tau)\theta_1(v_0+v_2,\tau)\theta_1'(0,\tau)^2}\right), \tag{7.23}$$

Then, the equations (7.21) determine the inverse map of (7.24) below

$$\begin{aligned} 4\omega^2 &= \frac{\theta_1(v_0-v_2,\tau)\theta_1(v_0+v_2,\tau)\theta_1'(0,\tau)^2}{\theta_1^4(v_2,\tau)} e^{-2\pi i u}, \\ z_0 &= 2\omega v_0, \\ z_2 &= 2\omega v_2, \\ \omega' &= \tau\omega. \end{aligned} \tag{7.24}$$

□

Consider the Landau-Ginzburg superpotential of $H_{1,0,0}$ (6.45) written in the coordinates $(z_0, z_2, \omega, \omega')$

LEMMA 7.3.5. The Landau-Ginzburg superpotential of $H_{1,0,0}$ (6.45) in the coordinates $(z_0, z_2, \omega, \omega')$ have the following form

$$\lambda^{\mathscr{I}(\tilde{A}_1)} = \frac{1}{\wp(z_0,\omega,\omega') - \wp(z_2,\omega,\omega')} - \frac{1}{\wp(z,\omega,\omega') - \wp(z_2,\omega,\omega')}. \tag{7.25}$$

PROOF.

$$\begin{aligned} \lambda^{\mathscr{I}(\tilde{A}_1)} &= e^{-2\pi i u} \frac{\theta_1(v-v_0,\tau)\theta_1(v+v_0,\tau)}{\theta_1(v-v_2,\tau)\theta_1(v+v_2,\tau)} \\ &= (2\omega)^2 \frac{\theta_1^4(v_2,\tau)}{\theta_1(v_0-v_2,\tau)\theta_1(v_0+v_2,\tau)\theta_1'(0,\tau)^2} \frac{\theta_1(v-v_0,\tau)\theta_1(v+v_0,\tau)}{\theta_1(v-v_2,\tau)\theta_1(v+v_2,\tau)} \\ &= \frac{\sigma^4(z_2,\omega,\omega')}{\sigma(z_0-z_2,\omega,\omega'\sigma(z_0-z_2,\omega,\omega')} \frac{\sigma(z-z_0,\omega,\omega')\sigma(z+z_0,\omega,\omega')}{\sigma(z-z_2,\omega,\omega')\sigma(z+z_2,\omega,\omega')} \\ &= \frac{\sigma^2(z_2)\sigma(z_0)}{\sigma(z_0-z_2)\sigma(z_0+z_2)} \frac{\sigma^2(z)\sigma(z_0)}{\sigma(z-z_2)\sigma(z+z_2)} \frac{\sigma(z-z_0)\sigma(z+z_0)}{\sigma^2(z)\sigma(z_0)} \\ &= \frac{\wp(z_0,\omega,\omega') - \wp(z,\omega,\omega')}{\left(\wp(z_0,\omega,\omega') - \wp(z_2,\omega,\omega')\right)\left(\wp(z,\omega,\omega') - \wp(z_2,\omega,\omega')\right)} \\ &= \frac{1}{\wp(z_0,\omega,\omega') - \wp(z_2,\omega,\omega')} - \frac{1}{\wp(z,\omega,\omega') - \wp(z_2,\omega,\omega')}. \end{aligned} \tag{7.26}$$

□

A convenient way to write the Landau-Ginzburg superpotential (7.14) is by taking the new coordinates

$$\begin{aligned} 4\omega^2 &= \varphi_2, \\ v_0 &= \frac{z_0}{\omega}, \\ \tau &= \frac{\omega'}{\omega}. \end{aligned} \tag{7.27}$$

Then, substituting (7.27) in (7.14)

$$\lambda^{\mathscr{J}(A_1)}(z) = \wp(z,\omega,\omega') - \wp(z_0,\omega,\omega'). \tag{7.28}$$

Hence, identifying the ω, ω' of the Landau-Ginzburg superpotential of $\mathscr{J}(A_1)$ (7.28) and of $\mathscr{J}(\tilde{A}_1)$ (7.25), we obtain

$$\lambda^{\mathscr{J}(\tilde{A}_1)}(v) = \frac{1}{\wp(z_0,\omega,\omega') - \wp(z_2,\omega,\omega')} - \frac{1}{\lambda^{\mathscr{J}(A_1)}(v)}. \tag{7.29}$$

Consequently, we receive the following corollary.

Corollary 7.3.5.1. Let (v_1, v_2, v_3, v_4) the canonical coordinates of the orbit space of $\mathscr{J}(\tilde{A}_1)$ be given by (7.19), and (u_1, u_2, u_3) the canonical coordinates of the orbit space of $\mathscr{J}(A_1)$. Then, the following relation holds

$$\begin{aligned} v_1 &= v_1, \\ v_2 &= v_1 - \frac{1}{u_1}, \\ v_3 &= v_1 - \frac{1}{u_2}, \\ v_4 &= v_1 - \frac{1}{u_3}. \end{aligned} \tag{7.30}$$

Proof. From the equations (7.19) and (6.74), we have

$$v_1 = \varphi_0. \tag{7.31}$$

Hence, recalling that $0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ are the critical points of (6.45) and $\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ are the critical points of (7.14), we obtain the desired result by using the equation (7.29).

□

Note that the Hurwitz space $H_{1,0,0}$ has a tri-hamiltonian structure due its Euler vector field (6.72) in flat coordinates of η. i.e.

$$E = t^1 \frac{\partial}{\partial t^1} + t^2 \frac{\partial}{\partial t^2}. \tag{7.32}$$

This fact implies that $H_{1,0,0}$ has three flat metrics $\eta^*, g^*, \tilde{\eta}^*$. The next proposition will realise a discriminant hypersurface of the Hurwitz space $H_{1,0,0}$ as Dubrovin Frobenius submanifold. However, we should consider the induced vector field E^2, instead of the induced unit vector field e of the orbit space of $H_{1,0,0}$.

Proposition 7.3.6. Consider the orbit space of the group $\mathscr{J}(\tilde{A}_1)$, and let (u, v_0, v_2, τ) be the flat coordinates of its intersection form g^* (6.64). Then, the submanifold $v_0 = 0$ is a Dubrovin Frobenius submanifold with respect the induced structure $(g^*|_{v_0=0}, E|_{v_0=0}, E^2|_{v_0=0})$. Moreover, the hyperplane $v_0 = 0$ is isomorphic as Dubrovin Frobenius manifold to the orbit space of $\mathscr{J}(A_1)$.

PROOF. The intersection form (6.64) induces the metric

$$-2dv_2^2 + 2dud\tau, \tag{7.33}$$

in the submanifold $v_0 = 0$. Note that the induced intersecition form (7.33) is flat and is equal to the intersection form of the orbit space of $\mathscr{J}(A_1)$ (once, we send $u \mapsto -u$) . Then, it remains to show that the induced Euler vector field and unit vector field are the same of those of the orbit space of $\mathscr{J}(A_1)$, because due to 2.1.2, we can reconstruct the Dubrovin Frobenius structure from this data.

The equations (7.30) in the submanifold $v_0 = 0$ have the following form

$$\begin{aligned} v_1 &= 0, \\ v_2 &= \frac{1}{u_1}, \\ v_3 &= \frac{1}{u_2}, \\ v_4 &= \frac{1}{u_3}. \end{aligned} \tag{7.34}$$

The unit vector field e , Euler vector field E and the square of the Euler vector field E^2 in canonical coordinates read

$$\begin{aligned} e &= \sum_{i=1}^{4} \frac{\partial}{\partial v_i}, \\ E &= \sum_{i=1}^{4} v_i \frac{\partial}{\partial v_i}, \\ E^2 &= \sum_{i=1}^{4} v_i^2 \frac{\partial}{\partial v_i}. \end{aligned} \tag{7.35}$$

In the submanifold $v_0 = 0$, the vector fields (7.35) have the following form

$$\begin{aligned} e|_{v_0=0} &= \sum_{i=1}^{3} -u_i^3 \frac{\partial}{\partial u_i}, \\ E|_{v_0=0} &= \sum_{i=1}^{3} u_i \frac{\partial}{\partial u_i}, \\ E^2\big|_{v_0} &= \sum_{i=1}^{3} -\frac{\partial}{\partial u_i}. \end{aligned} \tag{7.36}$$

Therefore, the Euler vector field on the ambient space induces the correct Euler vector in the submanifold, but the unit vector does not induces the unit vector field of $H_{1,1}$. Fortunately, E^2 induces the correct unit vector field of $H_{1,1}$. Hence, we have that $(g^*|_{v_0=0}, E|_{v_0=0}, E^2\big|_{v_0=0})$ are the same data of the Hurwitz space $H_{1,1}$ and using the Theorem 5.9.6, we can reconstruct the Dubrovin Frobenius manifold of $H_{1,1}$.

□

Remark 7.3.1. The metric η of $H_{1,0,0}$ does not induce the metric η of $H_{1,1}$, but, $H_{1,0,0}$ has the tri-hamiltonian structure with three flat metric. The third metric $\tilde{\eta}$ of $H_{1,0,0}$ induce the metric η of $H_{1,1}$. The vector field E^2 inducing the correct unit vector field in the submanifold $v_0=0$ is the realisation of this fact.

Corollary 7.3.6.1. The Dubrovin Frobenius submanifold described in proposition 7.3.6 lives in the discriminant locus of the orbit space of $\mathscr{J}(\tilde{A}_1)$.

Proof. It is a direct consequence of the equation (7.34).

□

7.4. Nilpotent caustic of the orbit space $\mathscr{J}(\tilde{A}_1)$

Lemma 7.4.1. Consider the orbit space of the group $\mathscr{J}(\tilde{A}_1)$, and let u, v_0, v_2, τ be the flat coordinates of its intersection form g^*. Then, the submanifold defined by

$$N=\{(u,v_0,v_2,\tau)\in\left(\mathbb{C}\oplus\mathbb{C}^2\oplus\mathbb{H}\right)/\mathscr{J}(\tilde{A}_1): u=0,\quad Im(\tau)\mapsto\infty\} \tag{7.37}$$

lives in the caustic of the orbit space of the group $\mathscr{J}(\tilde{A}_1)$.

Proof. Recall the following relation of the log derivatives of Jacobi theta functions [**33**]

$$\begin{aligned}
\frac{\theta_1'(v,\tau)}{\theta_1(v,\tau)}&=\cot v+4\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin(2nv),\\
\frac{\theta_2'(v,\tau)}{\theta_2(v,\tau)}&=\tan v+4\sum_{n=1}^{\infty}(-1)^n\frac{q^{2n}}{1-q^{2n}}\sin(2nv),\\
\frac{\theta_3'(v,\tau)}{\theta_3(v,\tau)}&=4\sum_{n=1}^{\infty}(-1)^n\frac{q^{2n}}{1-q^{2n}}\sin(2nv),\\
\frac{\theta_4'(v,\tau)}{\theta_4(v,\tau)}&=+4\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin(2nv).
\end{aligned} \tag{7.38}$$

Doing the limit $Im(\tau)\mapsto\infty$ in (7.38), we obtain

$$\begin{aligned}
\frac{\theta_1'(v,\tau)}{\theta_1(v,\tau)}&=\cot v,\\
\frac{\theta_2'(v,\tau)}{\theta_2(v,\tau)}&=\tan v,\\
\frac{\theta_3'(v,\tau)}{\theta_3(v,\tau)}&=0,\\
\frac{\theta_4'(v,\tau)}{\theta_4(v,\tau)}&=0.
\end{aligned} \tag{7.39}$$

Substituting (7.39) in (7.19), we obtain

$$v_3=v_4. \tag{7.40}$$

Then, N is the caustic of the orbit space of $\mathscr{J}(\tilde{A}_1)$. $\square$

PROPOSITION 7.4.2. Consider the orbit space of the group $\mathscr{J}(\tilde{A}_1)$, and let u, v_0, v_2, τ be the flat coordinates of its intersection form g^*. Then, the submanifold defined by

$$N = \{(u, v_0, v_2, \tau) \in \left(\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H}\right) / \mathscr{J}(\tilde{A}_1) : u = 0, \quad Im(\tau) \mapsto \infty\} \tag{7.41}$$

is Dubrovin Frobenius submanifold with respect the induced structure $(\star, \tilde{\eta}^*|_N, E|_N, e|_N)$. Moreover, N is isomorphic as Dubrovin Frobenius manifold to the orbit space of $\tilde{A}_1$.

PROOF. From lemma 7.4.1, and corollary 7.2.3.1, we derive that N is Dubrovin Frobenius submanifold of the orbit space of $\mathscr{J}(\tilde{A}_1)$. In order to proof the remaining part. Note that the induced intersection form in $u = 0, Im(\tau) \mapsto \infty$, is given by

$$2dv_0^2 - 2dv_2^2,$$

which is the intersection form of the orbit space of $\tilde{A}_1$. Taking the limit $u \mapsto 0, Im\tau \mapsto \infty$ in (6.45), we have

$$\lim_{Im\tau \mapsto \infty, u \mapsto 0} \lambda^{\mathscr{J}(\tilde{A}_1)} = \frac{\sin(v - v_0)\sin(v + v_0)}{\sin(v - v_2)\sin(v + v_2)}, \tag{7.42}$$

after doing some Moebius transformation in v and some change of coordinates the (7.42) became the superpotential

$$\lambda^{\tilde{A}_1} = e^p + a + be^{-p}. \tag{7.43}$$

From the data of (7.43) we can derive the Euler vector field and unit of the orbit space of $\tilde{A}_1$ [**15**]. Hence, we can reconstruct the Dubrovin Frobenius structure of the orbit space of $\tilde{A}_1$ by the arguments of section 2.1.2. $\square$

PROPOSITION 7.4.3. Consider the orbit space of the group $\mathscr{J}(A_1)$, and let u, v_0, τ be the flat coordinates of its the intersection form g^*. Then, the submanifold defined by

$$N = \{(u, v_0, \tau) \in (\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H}) / \mathscr{J}(A_1) : u = 0, \quad Im(\tau) \mapsto \infty\} \tag{7.44}$$

is a Dubrovin Frobenius submanifold of the orbit space of $\mathscr{J}(A_1)$ with respect the induced structure $(\star, \tilde{\eta}^*|_N, E|_N, e|_N)$. Moreover, N is isomorphic as a Dubrovin Frobenius manifold to the orbit space of A_1.

PROOF. The induced intersection form in $u = 0, Im(\tau) \mapsto \infty$, is given by

$$2dv_0^2,$$

which is the intersection form of the orbit space of A_1. Recall that the canonical coordinates of the orbit space $\mathscr{J}(A_1)$ are given by

$$
\begin{aligned}
u_1 &= t^1 - t^2 \frac{\theta_2''(0,\tau)}{\theta_2(0,\tau)},\\
u_2 &= t^1 - t^2 \frac{\theta_3''(0,\tau)}{\theta_3(0,\tau)},\\
u_3 &= t^1 - t^2 \frac{\theta_4''(0,\tau)}{\theta_4(0,\tau)}.
\end{aligned}
\tag{7.45}
$$

In the limit $u = 0, Im(\tau) \mapsto \infty$,, the canonical coordinates take the form

$$u_1 = u_2 = u_3 = t^1. \tag{7.46}$$

Hence, the unit vector field of $H_{1,1}$ takes the form

$$e = 3\frac{\partial}{\partial t^1}, \tag{7.47}$$

which is the unit and the Euler of the orbit space of A_1. Lemma is proved due to the discussion of section 2.1.2.

□

Corollary 7.4.3.1. Consider the orbit space of the group $\mathscr{J}(A_1)$, and let u, v_0, τ be the flat coordinates of its intersection form g^*. Then, the submanifold defined by

$$N = \{(u, v_0, \tau) \in (\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H}) / \mathscr{J}(A_1) : u = 0, \quad Im(\tau) \mapsto \infty\} \tag{7.48}$$

lives in the caustic of the orbit space of the group $\mathscr{J}(A_1)$.

Proof. It is a direct consequence of (7.46).

□

Proposition 7.4.4. The Dubrovin Frobenius submanifolds described in proposition 7.4.2 and 7.4.3 live in the nilpotent locus of the orbit space of $\mathscr{J}(\tilde{A}_1)$ and $\mathscr{J}(A_1)$ respectively.

Proof. Consider the following identity

$$\log\left(\frac{\theta_1(v|\tau)}{\theta_1'(0|\tau)}\right) = \log(\sin(\pi v)) + 4\sum_{m=1}^{\infty} \frac{q^{2m}}{1-q^{2m}} \frac{\sin^2(m\pi v)}{m} \tag{7.49}$$

where $q = e^{i\pi\tau}$. Differentiating with respect τ and then computing the limit $\Im\tau \mapsto \infty$, we receive

$$\lim_{\Im\tau\mapsto\infty} \frac{\partial}{\partial\tau}\left(\log\left(\frac{\theta_1(v,\tau)}{\theta_1'(0,\tau)}\right)\right) = 4\lim_{\Im\tau\mapsto\infty}\sum_{m=1}^{\infty}\left[\frac{2mq^{2m}}{1-q^{2m}} - \frac{2mq^{4m}}{(1-q^{2m})^2}\right]\frac{\sin^2(m\pi v)}{m} = 0. \tag{7.50}$$

Therefore, considering the WDVV solution given by (6.111), we have that $c_{4\alpha\beta}$ evaluated in the Dubrovin Frobenius submanifold $u = 0, \Im\tau \mapsto \infty$ is 0. Therefore

$$\partial_4 \bullet \partial_4 = c_{444}\partial_1 + c_{443}\partial_2 + c_{442}\partial_3 = 0. \tag{7.51}$$

Then, the vector field ∂_4 is nilpotent in the submanifold $u = 0, \Im\tau \mapsto \infty$.

Consider the identity

$$\frac{\theta_1'''(0,\tau)}{\theta_1'(0,\tau)} = -1 + 24\sum_{i=0}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2}. \tag{7.52}$$

Differentiating with respect τ and then computing the limit $\Im\tau \mapsto \infty$, we receive

$$\lim_{Im\tau\mapsto\infty} \frac{\partial}{\partial\tau}\left(\frac{\theta_1'''(0,\tau)}{\theta_1'(0,\tau)}\right) = -24 \lim_{Im\tau\mapsto\infty} \sum_{i=0}^{\infty} \frac{2nq^{2n}}{(1-q^{2n})^2} - \frac{2nq^{4n}}{(1-q^{2n})^3} = 0. \tag{7.53}$$

From appendix C of [**12**], we have that the WDVV solution of the orbit space of $\mathscr{J}(A_1)$ is

$$F = \frac{(t^1)^2\tau}{2} + \frac{t^1(t^2)^2}{2} - \frac{i\pi}{48}(t^2)^4 E_2(\tau). \tag{7.54}$$

Using the fact that $E_2(\tau)$ is proportional to $\frac{\theta_1'''(0,\tau)}{\theta_1'(0,\tau)}$, we have that ∂_τ is nilpotent in the submanifold $u = 0, \Im\tau \mapsto \infty$ in the orbit space of $\mathscr{J}(A_1)$, by the same reason of the case $u = 0, \Im\tau \mapsto \infty$ in the orbit space of $\mathscr{J}(\tilde{A}_1)$.

$\square$

CHAPTER 8

Differential geometry of the orbit space of extended Jacobi group A_n

This Chapter is dedicated to generalise the group $\mathscr{J}(\tilde{A}_1)$ defined in Chapter 6 for arbitrary n, this new class of groups will be denoted by $\mathscr{J}(\tilde{A}_n)$. From the data of the group $\mathscr{J}(\tilde{A}_n)$, we will construct the Dubrovin Frobenius manifold in the orbit space of $\mathscr{J}(\tilde{A}_n)$. Furthermore, this Dubrovin Frobenius manifold will be locally isomorphic to the Hurwitz space $H_{1,n-1,0}$. In the section 1.4, there is a scheme of the technical steps that we should take to built the desired Dubrovin Frobenius manifold.

8.1. The Group $\mathscr{J}(\tilde{A}_n)$

In this section, we define the group $\mathscr{J}(\tilde{A}_n)$. In order to understand the motivation of this group see 1.4.

Consider the A_n in the following extended space

$$L^{\tilde{A}_n} = \{(z_0, z_1, .., z_n, z_{n+1}) \in \mathbb{Z}^{n+2} : \sum_{i=0}^{n} z_i = 0\}.$$

The action of A_n on $L^{\tilde{A}_n}$ is given by

$$w(z_0, z_1, z_2, .., z_{n-1}, z_n, z_{n+1}) = (z_{i_0}, z_{i_1}, z_{i_2}, .., z_{i_{n-1}}, z_{i_n}, z_{n+1})$$

permutations in the first $n+1$ variables. Moreover, A_n also acts on the complexfication of $L^{\tilde{A}_n} \otimes \mathbb{C}$. Let the quadratic form $\langle , \rangle_{\tilde{A}_n}$ be given by

$$\begin{aligned}
\langle v, v\rangle_{\tilde{A}_n} &= v^T M_{\tilde{A}_n} v \\
&= v^T \begin{pmatrix} 2 & 1 & 1 & ...1 & 1 & 0 \\ 1 & 2 & 1 & ....1 & 1 & 0 \\ 1 & 1 & 2 & ....1 & 1 & 0 \\ . & . & . & ... & . & 0 \\ . & . & . & ... & . & 0 \\ 1 & 1 & 1 & ...2 & 1 & 0 \\ 1 & 1 & 1 & .... & 2 & 0 \\ 0 & 0 & 0 & .... & 0 & -n(n+1) \end{pmatrix} v \\
&= 2\sum_{i=0}^{n-1} v_i^2 + 2\sum_{i>j} v_i v_j - n(n+1)v_{n+1}^2.
\end{aligned}$$

Consider the following group $L^{\tilde{A}_n} \times L^{\tilde{A}_n} \times \mathbb{Z}$ with the following group operation

$$\begin{aligned}
&\forall(\lambda, \mu, k), (\tilde{\lambda}, \tilde{\mu}, \tilde{k}) \in L^{\tilde{A}_n} \times L^{\tilde{A}_n} \times \mathbb{Z} \\
&(\lambda, \mu, k) \bullet (\tilde{\lambda}, \tilde{\mu}, \tilde{k}) = (\lambda + \tilde{\lambda}, \mu + \tilde{\mu}, k + \tilde{k} + \langle \lambda, \tilde{\lambda}\rangle_{\tilde{A}_n})
\end{aligned}$$

Note that $\langle , \rangle_{\tilde{A}_n}$ is invariant under A_n group, then A_n acts on $L^{\tilde{A}_n} \times L^{\tilde{A}_n} \times \mathbb{Z}$. Hence, we can take the semidirect product $A_n \ltimes (L^{\tilde{A}_n} \times L^{\tilde{A}_n} \times \mathbb{Z})$ given by the following product.

$$\begin{aligned}
&\forall(w, \lambda, \mu, k), (\tilde{w}, \tilde{\lambda}, \tilde{\mu}, \tilde{k}) \in A_n \times L^{\tilde{A}_n} \times L^{\tilde{A}_n} \times \mathbb{Z}, \\
&(w, \lambda, \mu, k) \bullet (\tilde{w}, \tilde{\lambda}, \tilde{\mu}, \tilde{k}) = (w\tilde{w}, w\lambda + \tilde{\lambda}, w\mu + \tilde{\mu}, k + \tilde{k} + \langle \lambda, \tilde{\lambda}\rangle_{\tilde{A}_n}).
\end{aligned}$$

Denoting $W(\tilde{A}_n) := A_n \ltimes (L^{\tilde{A}_n} \times L^{\tilde{A}_n} \times \mathbb{Z})$, we can define

Definition 8.1.1. The Jacobi group $\mathscr{J}(\tilde{A}_n)$ is defined as a semidirect product $W(\tilde{A}_n) \rtimes SL_2(\mathbb{Z})$. The group action of $SL_2(\mathbb{Z})$ on $W(\tilde{A}_n)$ is defined as

$$\begin{aligned}
&Ad_\gamma(w) = w, \\
&Ad_\gamma(\lambda, \mu, k) = (a\mu - b\lambda, -c\mu + d\lambda, k + \frac{ac}{2}\langle \mu, \mu\rangle_{\tilde{A}_n} - bc\langle \mu, \lambda\rangle_{\tilde{A}_n} + \frac{bd}{2}\langle \lambda, \lambda\rangle_{\tilde{A}_n}),
\end{aligned}$$

for $(w, t = (\lambda, \mu, k)) \in W(\tilde{A}_n), \gamma \in SL_2(\mathbb{Z})$. Then the multiplication rule is given as follows

$$(w, t, \gamma) \bullet (\tilde{w}, \tilde{t}, \tilde{\gamma}) = (w\tilde{w}, t \bullet Ad_\gamma(w\tilde{t}), \gamma\tilde{\gamma}).$$

Let us use the following identification $\mathbb{Z}^{n+1} \cong L^{\tilde{A}_n}, \mathbb{C}^{n+1} \cong L^{\tilde{A}_n} \otimes \mathbb{C}$ that is possible due to maps

$$(v_0, .., v_{n-1}, v_{n+1}) \mapsto (v_0, .., v_{n-1}, -\sum_{i=0}^{n} v_i, v_{n+1}),$$
$$(v_0, .., v_{n-1}, v_n, v_{n+1}) \mapsto (v_0, .., v_{n-1}, v_{n+1}).$$

Then the action of Jacobi group $\mathscr{J}(\tilde{A}_n)$ on $\Omega := \mathbb{C} \oplus \mathbb{C}^{n+1} \oplus \mathbb{H}$ is given as follows

PROPOSITION 8.1.1. The group $\mathscr{J}(\tilde{A}_n) \ni (w, t, \gamma)$ acts on $\Omega := \mathbb{C} \oplus \mathbb{C}^{n+1} \oplus \mathbb{H} \ni (u, v, \tau)$ as follows

$$\begin{aligned} w(u, v, \tau) &= (u, wv, \tau) \\ t(u, v, \tau) &= (u - \langle \lambda, v \rangle_{\tilde{A}_n} - \frac{1}{2}\langle \lambda, \lambda \rangle_{\tilde{A}_n} \tau, v + \lambda\tau + \mu, \tau) \\ \gamma(u, v, \tau) &= (\phi + \frac{c\langle v, v \rangle_{\tilde{A}_n}}{2(c\tau + d)}, \frac{v}{c\tau + d}, \frac{a\tau + b}{c\tau + d}) \end{aligned} \tag{8.1}$$

The proof of the proposition 8.1.1 follows from the proposition 5.1.1 with small adaptation with the extra trivial action in the exceptional variable v_{n+1} with respect the A_n action.

8.2. Jacobi forms of $\mathscr{J}(\tilde{A}_n)$

This section is the generalisation of the section 5.2 and 6.2 for the case of the group $\mathscr{J}(\tilde{A}_n)$.

DEFINITION 8.2.1. The weak $\mathscr{J}(\tilde{A}_n)$ -invariant Jacobi forms of weight $k \in \mathbb{Z}$, order $l \in \mathbb{N}$, and index $m \in \mathbb{N}$ are functions on $\Omega = \mathbb{C} \oplus \mathbb{C}^{n+1} \oplus \mathbb{H} \ni (u, v', v_{n+1}, \tau) = (u, v, \tau)$ which satisfy

$$\begin{aligned} &\varphi(w(u, v, \tau)) = \varphi(u, v, \tau), \quad A_n \text{ invariant condition} \\ &\varphi(t(u, v, \tau)) = \varphi(u, v, \tau) \\ &\varphi(\gamma(u, v, \tau)) = (c\tau + d)^{-k}\varphi(u, v, \tau) \\ &E\varphi(u, v, \tau) := -\frac{1}{2\pi i}\frac{\partial}{\partial u}\varphi(u, v, \tau) = m\varphi(u, v, \tau) \end{aligned} \tag{8.2}$$

Moreover,

(1) φ is locally bounded functions on v' as $\Im(\tau) \mapsto +\infty$ (weak condition).
(2) For fixed u, v', τ the function $v_{n+1} \mapsto \varphi(u, v', v_{n+1}, \tau)$ is meromorphic with poles of order at most $l + 2m$ on $v_{n+1} = \frac{j}{n} + \frac{l\tau}{n} \mod \mathbb{Z} \oplus \tau\mathbb{Z}$, $0 \leq l, j \leq n - 1$.
(3) For fixed $u, \tau, v_{n+1} = \frac{j}{n} + \frac{l\tau}{n} \mod \mathbb{Z} \oplus \tau\mathbb{Z}$, $0 \leq l, j \leq n - 1$ the function $(i \neq n + 1)$ $v_i \mapsto \varphi(u, v', v_{n+1}, \tau)$ is holomorphic.
(4) For fixed $u, v', v_{n+1} = \frac{j}{n} + \frac{l\tau}{n} \mod \mathbb{Z} \oplus \tau\mathbb{Z}$, $0 \leq l, j \leq n - 1$. the function $\tau \mapsto \varphi(u, v', v_{n+1}, \tau)$ is holomorphic.

The space of $\mathscr{J}(\tilde{A}_n)$-invariant Jacobi forms of weight k, order l, and index m is denoted by $J^{\tilde{A}_n}_{k,l,m}$, and $J^{\mathscr{J}(\tilde{A}_n)}_{\bullet,\bullet,\bullet} = \bigoplus_{k,l,m} J^{\tilde{A}_n}_{k,l,m}$ is the space of $\tilde{A}_n$ invariant Jacobi forms .

REMARK 8.2.1. The condition $E\varphi(u,v',v_{n+1},\tau)=m\varphi(u,v',v_{n+1},\tau)$ implies that $\varphi(u,v',v_{n+1},\tau)$ has the following form

$$\varphi(u,v',v_{n+1},\tau)=f(v',v_{n+1},\tau)e^{2\pi imu}.$$

See also remarks 6.2.1.

The main result of section is the following.

The ring of $\tilde{A}_n$ invariant Jacobi forms is polynomial over a suitable ring $E_{\bullet,\bullet}:=J^{\mathscr{J}(\tilde{A}_n)}_{\bullet,\bullet,0}$ on suitable generators $\varphi_0,\varphi_1,\varphi_2,..,\varphi_n$.
Before state precisely the theorem, I will define the objects $E_{\bullet,\bullet},\varphi_0,\varphi_1,\varphi_2,..,\varphi_n$.

The ring $E_{\bullet,l}:=J^{\mathscr{J}(\tilde{A}_n)}_{\bullet,l,0}$ is the space of meromorphic Jacobi forms of index 0 with poles of order at most l on $v_{n+1}=\frac{j}{n}+\frac{l\tau}{n},0\leq l,j\leq n-1 \text{mod } \mathbb{Z}\oplus\tau\mathbb{Z}$, by definition. The sub-ring $J^{\mathscr{J}(\tilde{A}_n)}_{\bullet,0,0}\subset E_{\bullet,\bullet}$ has a nice structure, indeed:

LEMMA 8.2.1. The sub-ring $J^{\mathscr{J}(\tilde{A}_n)}_{\bullet,0,0}$ is equal to $M_\bullet:=\bigoplus M_k$, where M_k is the space of modular forms of weight k for the full group $SL_2(\mathbb{Z})$.

PROOF. Using the Remark 8.2.1, we know that functions $\varphi(u,v',v_{n+1},\tau)\in J^{\mathscr{J}(\tilde{A}_n)}_{\bullet,0,0}$ can not depend on u, then $\varphi(u,v',v_{n+1},\tau)=\varphi(v',v_{n+1},\tau)$. Moreover, for fixed v_{n+1},τ the functions $v_i\mapsto\varphi(v',v_{n+1},\tau))$ are holomorphic elliptic function for any $i\neq n+1$. Therefore, by Liouville theorem, these function are constant in v'. Similar argument shows that these function do not depend on v_{n+1}, because $l+2m=0$, i.e there is no pole. Then, $\varphi=\varphi(\tau)$ are standard holomorphic modular forms. □

LEMMA 8.2.2. If $\varphi\in E_{\bullet,\bullet}=J^{\mathscr{J}(\tilde{A}_n)}_{\bullet,\bullet,0}$, then φ depends only on the variables v_{n+1},τ. Moreover, if $\varphi\in J^{\mathscr{J}(\tilde{A}_n)}_{0,l,0}$ for fixed τ the function $\tau\mapsto\varphi(v_{n+1},\tau)$ is a elliptic function with poles of order at most l $v_{n+1}=\frac{j}{n}+\frac{l\tau}{n},0\leq l,j\leq n-1\text{mod } \mathbb{Z}\oplus\tau\mathbb{Z}$.

PROOF. The proof follows essentially in the same way of the lemma 8.2.1, the only difference is that now we have poles on $v_{n+1}=\frac{j}{n}+\frac{l\tau}{n},0\leq l,j\leq n-1\text{mod } \mathbb{Z}\oplus\tau\mathbb{Z}$. Then, we have depedence in v_{n+1}. □

As a consequence of lemma 8.2.2, the function $\varphi\in E_{k,l}=J^{\mathscr{J}(\tilde{A}_n)}_{k,l,0}$ has the following form

$$\varphi(v_{n+1},\tau)=f(\tau)g(v_{n+1},\tau)$$

where $f(\tau)$ is holomorphic modular form of weight k, and for fixed τ, the function $v_{n+1}\mapsto g(v_{n+1},\tau)$ is an elliptic function of order at most l on the poles $v_{n+1}=\frac{j}{n}+\frac{l\tau}{n},0\leq l,j\leq n-1$ mod $\mathbb{Z}\oplus\tau\mathbb{Z}$.

At this point, we will generalise the construction in sections 5.2 and 6.2 regarding a generating functions of the basic generators of the algebra of Jacobi forms.

Note that a natural way to produce meromorphic Jacobi form is by using rational functions of holomorphic Jacobi forms. Starting from now, we will denote the Jacobi forms related with the Jacobi group $\mathscr{J}(A_{n+1})$ with the upper index $\mathscr{J}(A_{n+1})$, for instance

$$\varphi^{\mathscr{J}(A_{n+1})},$$

and the Jacobi forms related with the Jacobi group $\mathscr{J}(\tilde{A}_n)$ with the with the upper index $\mathscr{J}(\tilde{A}_n)$

$$\varphi^{\mathscr{J}(\tilde{A}_n)}.$$

In [**8**], Bertola found basis of the algebra of Jacobi form by producing a holomorphic Jacobi form of type A_n as product of theta functions.

$$\varphi_{n+2}^{\mathscr{J}(A_{n+1})} = e^{-2\pi i u} \prod_{i=1}^{n+2} \frac{\theta_1(z_i,\tau)}{\theta_1'(0,\tau)}. \tag{8.3}$$

Afterwards, Bertola defined a recursive operator to produce the remaining basic generators. In order to recall the details see section 5.2. Our strategy will follow the same logic of Bertola method, we use theta functions to produce a basic generator and thereafter, we produce a recursive operator to produce the remaining part.

Lemma 8.2.3. Let be $\varphi_{n+2}^{\mathscr{J}(A_{n+1})}(u_1, z_1, z_2, .., z_n, \tau)$ the holomorphic $A_{n+1}-\textit{invariant}$ Jacobi forms which correspond to the algebra generator of maximal weight degree, in this case degree n+2. More explicitly,

$$\varphi_{n+2}^{\mathscr{J}(A_2)} = e^{-2\pi i u_1} \prod_{i=1}^{n+2} \frac{\theta_1(z_i,\tau)}{\theta_1'(0,\tau)}. \tag{8.4}$$

Let be $\varphi_2^{\mathscr{J}(A_1)}(u_2, z_{n+1}, \tau)$ the holomorphic $A_1 - \textit{invariant}$ Jacobi form which correspond to the algebra generator of maximal weight degree, in this case degree 2.

$$\varphi_2^{\mathscr{J}(A_1)} = e^{-2\pi i u_2} \frac{\theta_1(z_{n+2},\tau)^2}{\theta_1'(0,\tau)^2}. \tag{8.5}$$

Then, the function

$$\varphi_n^{\mathscr{J}(\tilde{A}_n)} = \frac{\varphi_{n+2}^{\mathscr{J}(A_{n+1})}}{\varphi_2^{\mathscr{J}(A_1)}} \tag{8.6}$$

is meromorphic Jacobi form of index 1, weight -n, order 0.

PROOF. For our convenience, we change the labels $u_2 - u_1, z_1, z_2, ..., z_{n+2}$ to

$$
\begin{aligned}
u &= u_2 - u_1, \\
z_1 &= v_0 - v_{n+1}, \\
z_2 &= v_1 - v_{n+1}, \\
&. \\
&. \\
z_{n+1} &= -\sum_{i=0}^{n} v_i - v_{n+1}, \\
z_{n+2} &= (n+1)v_{n+1}.
\end{aligned}
\tag{8.7}
$$

Then (8.6) has the following form

$$
\varphi_n^{\mathscr{J}(\tilde{A}_n)}(u, v_0, v_1, .., v_{n+1}, \tau) = e^{-2\pi i u} \frac{\prod_{i=0}^{n} \theta_1(v_i - v_{n+1}, \tau)}{\theta_1'(0,\tau)^n \theta_1((n+1)v_{n+1}, \tau)}
\tag{8.8}
$$

Let us prove each item separated.

(1) A_n **invariant**

The A_n group acts on (8.8) by permuting its roots, thus (8.8) remains invariant under this operation.

(2) **Translation invariant**

Recall that under the translation $v \mapsto v + m + n\tau$, the Jacobi theta function transform as [8], [33]:

$$
\theta_1(v_i + \mu_i + \lambda_i \tau, \tau) = (-1)^{\lambda_i + \mu_i} e^{-2\pi i (\lambda_i v_i + \frac{\lambda_i^2}{2}\tau)} \theta_1(v_i, \tau).
\tag{8.9}
$$

Then substituting the transformation (8.9) into (8.8), we conclude that (8.8) remains invariant.

(3) $SL_2(\mathbb{Z})$ **invariant**

Under $SL_2(\mathbb{Z})$ action the following function transform as

$$
\frac{\theta_1(\frac{v_i}{c\tau+d}, \frac{a\tau+d}{c\tau+d})}{\theta_1'(0, \frac{a\tau+d}{c\tau+d})} = (c\tau+d)^{-1} \exp\left(\frac{\pi i c v_i^2}{c\tau+d}\right) \frac{\theta_1(v_i,\tau)}{\theta_1'(0,\tau)}.
\tag{8.10}
$$

Then, substituting (8.10) in (8.8), we get

$$
\varphi_n^{\mathscr{J}(\tilde{A}_n)} \mapsto \frac{\varphi_n^{\mathscr{J}(\tilde{A}_n)}}{(c\tau+d)^n}
$$

(4) **Index** 1

$$
\frac{1}{2\pi i} \frac{\partial}{\partial u} \varphi_n^{\mathscr{J}(\tilde{A}_n)} = \varphi_n^{\mathscr{J}(\tilde{A}_n)}.
\tag{8.11}
$$

(5) **Analytic behavior**
Note that $\varphi_n^{\mathscr{J}(\tilde{A}_n)}\theta_1^2((n+1)v_{n+1},\tau)$ is holomorphic function in all the variables v_i. Therefore $\varphi_n^{\mathscr{J}(\tilde{A}_n)}$ are holomorphic functions on the variables $\{v_i\}$, and meromorphic function in the variable v_{n+1} with poles on $\frac{j}{n}+\frac{l\tau}{n}, j,l=0,1,..,n-1$ of order 2, i.e $l=0$, since $m=1$.

$\square$

In order to define the desired recursive operator, it is necessary to enlarge the domain of the Jacobi forms from $\mathbb{C}\oplus\mathbb{C}^n\oplus\mathbb{H}\ni(u,v_0,v_1,..,v_{n+1},\tau)$ to $\mathbb{C}\oplus\mathbb{C}^{n+1}\oplus\mathbb{H}\ni(u,v_0,v_1,..,v_{n+1},p,\tau)$. In addition, we define lift a of Jacobi forms defined in $\mathbb{C}\oplus\mathbb{C}^2\oplus\mathbb{H}$ to $\mathbb{C}\oplus\mathbb{C}^3\oplus\mathbb{H}$ as

$$\varphi(u,v_0-v_{n+1},v_1-v_{n+1},..,(n+1)v_{n+1},\tau)\mapsto\hat{\varphi}(p):=\varphi(u,v_0-v_{n+1}+p,v_1-v_{n+1}+p,..,(n+1)v_{n+1}+p,\tau)$$

A convenient way to do computation in these extended Jacobi forms is by using the following coordinates

$$\begin{aligned}
s&=u+ng_1(\tau)p^2,\\
z_1&=v_0-v_{n+1}+p,\\
z_2&=v_1-v_{n+1}+p,\\
&.\\
&.\\
z_{n+1}&=-\sum_{i=0}^{n}v_i-v_{n+1}+p,\\
z_{n+2}&=(n+1)v_{n+1}+p,\\
\tau&=\tau.
\end{aligned}\tag{8.12}$$

The bilinear form $\langle v,v\rangle_{\tilde{A}_1}$ is extended to

$$\langle(z_1,z_2,..,z_{n+2}),(z_1,z_2,..,z_{n+2})\rangle_E=\sum_{i=1}^{n+1}z_i^2-z_{n+2}^2,\tag{8.13}$$

The action of the Jacobi group $\tilde{A}_n$ in this extended space is

$$\begin{aligned}
\hat{w}_E(u,v,p,\tau)&=(u,w(v),p,\tau)\\
t_E(u,v,p,\tau)&=\left(u-\langle\lambda,v\rangle_E-\frac{1}{2}\langle\lambda,\lambda\rangle_E\tau+k,v+p+\lambda\tau+\mu,\tau\right)\\
\gamma_E(u,v,p,\tau)&=\left(u+\frac{c\langle v,v\rangle_E}{2(c\tau+d)},\frac{v}{c\tau+d},\frac{p}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)
\end{aligned}\tag{8.14}$$

Proposition 8.2.4. Let be $\varphi \in J^{\mathscr{J}(\tilde{A}_n)}_{k,m,\bullet}$, and $\hat{\varphi}$ the correspondent extended Jacobi form. Then,

$$\frac{\partial}{\partial p}(\hat{\varphi})\Big|_{p=0} \in J^{\mathscr{J}(\tilde{A}_n)}_{k-1,m,\bullet}. \tag{8.15}$$

Proof. (1) A_n**-invariant**

The vector field $\frac{\partial}{\partial p}$ in coordinates $s, z_1, z_2, .., z_{n+2}, \tau$ reads

$$\frac{\partial}{\partial p} = \sum_{i=1}^{n+2} \frac{\partial}{\partial z_i} + 2n g_1(\tau) p \frac{\partial}{\partial u}$$

Moreover, in the coordinates $s, z_1, z_2, ., z_{n+1}, z_{n+2}, \tau$ the A_n group acts by permuting $z_1, z_2, ..., z_{n+1}$. Then

$$\begin{aligned}\frac{\partial}{\partial p}\left(\varphi(s, z_2, z_1, z_3, \tau)\right)\Big|_{p=0} &= \left(\sum_{i=1}^{n+2} \frac{\partial}{\partial z_i}\right)\left(\varphi(s, z_{i_1}, z_{i_2} .., z_{i_{n+1}}, z_{n+2}, \tau)\right)\big|_{p=0} \\ &= \left(\sum_{i=1}^{n+2} \frac{\partial}{\partial z_i}\right)\left(\varphi(s, z_1, z_2, , .., z_{n+1}, z_{n+2}, \tau)\right)\big|_{p=0}.\end{aligned}$$

(2) **Translation invariant**

$$\begin{aligned}&\frac{\partial}{\partial p}\left(\varphi(u - \langle \lambda, v\rangle_E - \langle \lambda, \lambda\rangle_E, v + p + \lambda\tau + \mu, \tau)\right)\Big|_{p=0} \\ &= \frac{\partial}{\partial p}\langle \lambda, v\rangle_E\Big|_{p=0} \varphi(u, v, \tau) + \frac{\partial \varphi}{\partial p}\left(u - \langle \lambda, v\rangle_{\tilde{A}_n} - \frac{1}{2}\langle \lambda, \lambda\rangle_{\tilde{A}_n}\tau + k, v + \lambda\tau + \mu, \tau\right) \\ &= \frac{\partial \varphi}{\partial p}\left(u - \langle \lambda, v\rangle_{\tilde{A}_n} - \frac{1}{2}\langle \lambda, \lambda\rangle_{\tilde{A}_n}\tau + k, v + \lambda\tau + \mu, \tau\right) \\ &= \frac{\partial \varphi}{\partial p}(u, v, \tau)\Big|_{p=0}.\end{aligned}$$

(3) $SL_2(\mathbb{Z})$ **equivariant**

$$\begin{aligned}&\frac{\partial}{\partial p}\left(\varphi(u + \frac{c\langle v, v\rangle_E}{2(c\tau + d)}, \frac{v}{c\tau + d}, \frac{p}{c\tau + d}, \frac{a\tau + b}{c\tau + d})\right)\Big|_{p=0} \\ &= \frac{c}{2(c\tau + d)}\frac{\partial}{\partial p}\langle v, v\rangle_E\Big|_{p=0} \varphi(u, v, \tau) + \frac{1}{c\tau + d}\frac{\partial \varphi}{\partial p}\left(u + \frac{c\langle v, v\rangle_{\tilde{A}_n}}{2(c\tau + d)}, \frac{v}{c\tau + d}, \frac{p}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) \\ &= \frac{1}{c\tau + d}\frac{\partial \varphi}{\partial p}\left(u + \frac{c\langle v, v\rangle_{\tilde{A}_n}}{2(c\tau + d)}, \frac{v}{c\tau + d}, \frac{p}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) \\ &= \frac{1}{(c\tau + d)^k}\,\frac{\partial \varphi}{\partial p}(u, v, \tau)\Big|_{p=0}.\end{aligned}$$

Then,

$$\frac{\partial \varphi}{\partial p}\left(u + \frac{c\langle v, v\rangle_E}{2(c\tau + d)}, \frac{v}{c\tau + d}, \frac{p}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right)\Big|_{p=0} = \frac{1}{(c\tau + d)^{k-1}}\,\frac{\partial \varphi}{\partial p}(u, v, \tau)\Big|_{p=0}.$$

(4) **Index 1**

$$\frac{1}{2\pi i}\frac{\partial}{\partial u}\frac{\partial}{\partial p}\hat{\varphi}=\frac{1}{2\pi i}\frac{\partial}{\partial p}\frac{\partial}{\partial u}\hat{\varphi}=\frac{\partial}{\partial p}\hat{\varphi}.$$

□

COROLLARY 8.2.4.1. The function

$$\begin{aligned}&\left[e^{z\frac{\partial}{\partial p}}\left(e^{2\pi iu}\frac{\prod_{i=0}^{n}\theta_1(p+v_i-v_{n+1},\tau)}{\theta_1'(0,\tau)^n\theta_1(p+(n+1)v_{n+1},\tau)}\right)\right]\Bigg|_{p=0}\\&=\varphi_n^{\mathscr{J}(\tilde{A}_n)}+\varphi_{n-1}^{\mathscr{J}(\tilde{A}_n)}z+\varphi_{n-2}^{\mathscr{J}(\tilde{A}_n)}z^2+...+\varphi_0^{\mathscr{J}(\tilde{A}_n)}z^n+O(z^{n+1}),\end{aligned}\tag{8.16}$$

is a generating function for the Jacobi forms $\varphi_n^{\mathscr{J}(\tilde{A}_n)},\varphi_{n-1}^{\mathscr{J}(\tilde{A}_n)},\varphi_{n-2}^{\mathscr{J}(\tilde{A}_n)},..,\varphi_0^{\mathscr{J}(\tilde{A}_n)}$, where

$$\varphi_k^{\mathscr{J}(\tilde{A}_n)}:=\frac{\partial^{n-k}}{\partial p^{n-k}}\left(\hat{\varphi}_n^{\mathscr{J}(\tilde{A}_n)}\right)\Bigg|_{p=0}.\tag{8.17}$$

PROOF. Acting $\frac{\partial}{\partial p}$ k times in $\varphi_n^{\mathscr{J}(\tilde{A}_n)}$, we have

$$\left[\frac{\partial^k}{\partial^k p}\left(e^{2\pi iu}\cdot\frac{\prod_{i=0}^{n}\theta_1(p+v_i-v_{n+1},\tau)}{\theta_1'(0,\tau)^n\theta_1(p+(n+1)v_{n+1},\tau)}\right)\right]\Bigg|_{p=0}\in J_{-n+k,1,\bullet}^{\mathscr{J}(\tilde{A}_n)}.$$

□

COROLLARY 8.2.4.2. The generating function can be written as

$$\left[e^{z\frac{\partial}{\partial p}}\left(e^{2\pi iu}\frac{\prod_{i=0}^{n}\theta_1(p+v_i-v_{n+1},\tau)}{\theta_1'(0,\tau)^n\theta_1(p+(n+1)v_{n+1},\tau)}\right)\right]\Bigg|_{p=0}=e^{-2\pi i(u+ng_1(\tau)z^2)}\frac{\prod_{i=0}^{n}\theta_1(z+v_i-v_{n+1},\tau)}{\theta_1'(0,\tau)^n\theta_1(z+(n+1)v_{n+1},\tau)}.\tag{8.18}$$

PROOF.

$$\begin{aligned}&\left[e^{z\frac{\partial}{\partial p}}\left(e^{2\pi iu}\frac{\prod_{i=0}^{n}\theta_1(p+v_i-v_{n+1},\tau)}{\theta_1'(0,\tau)^n\theta_1(p+(n+1)v_{n+1},\tau)}\right)\right]\Bigg|_{p=0}=\\&=\left[e^{z\frac{\partial}{\partial p}}\left(e^{2\pi i(s+ng_1(\tau)p^2}\frac{\prod_{i=0}^{n}\theta_1(p+v_i-v_{n+1},\tau)}{\theta_1'(0,\tau)^n\theta_1(p+(n+1)v_{n+1},\tau)}\right)\right]\Bigg|_{p=0}\\&=e^{-2\pi i(u+ng_1(\tau)z^2)}\frac{\prod_{i=0}^{n}\theta_1(z+v_i-v_{n+1},\tau)}{\theta_1'(0,\tau)^n\theta_1(z+(n+1)v_{n+1},\tau)}.\end{aligned}\tag{8.19}$$

□

The next lemma is one of the main points of this section, because this lemma identify the orbit space of the group $\mathscr{J}(\tilde{A}_n)$ with the Hurwitz space $H_{1,n-1,0}$. This relationship is possible due to the construction of the generating function of the Jacobi forms of type $\tilde{A}_n$, which can be completed to be the Landau-Ginzburg superpotential of $H_{1,n-1,0}$ as follows

LEMMA 8.2.5. There is local biholomorphism between the orbit space $\mathscr{J}(\tilde{A}_n)$ and $H_{1,n-1,0}$, i.e the space of elliptic functions with 1 pole of order n, and one simple pole.

PROOF. The correspondence is realized by the map:

$$[(u, v_0, v_1, .., v_{n-1}, v_{n+1}, \tau)] \longleftrightarrow \lambda(v) = e^{-2\pi iu} \frac{\prod_{i=0}^{n} \theta_1(z - v_i, \tau)}{\theta_1^n(v,\tau)\theta_1(v + (n+1)v_{n+1}, \tau)} \tag{8.20}$$

Note that this map is well defined and one to one. Indeed:

(1) **Well defined**

Note that proof that the map does not depend on the choice of the representant of $[(\phi, v_0, v_1, .., v_{n-1}, v_{n+1}, \tau)]$ is equivalent to prove that the function (8.20) is invariant under the action of $\mathscr{J}(\tilde{A}_n)$. Indeed

(2) A_n **invariant**

The A_n group acts on (8.20) by permuting its roots, thus (8.20) remais invariant under this operation.

(3) **Translation invariant**

Recall that under the translation $v \mapsto v + m + n\tau$, the Jacobi theta function transform as [**8**], [**33**]:

$$\theta_1(v_i + \mu_i + \lambda_i\tau, \tau) = (-1)^{\lambda_i + \mu_i} e^{-2\pi i(\lambda_i v_i + \frac{\lambda_i^2}{2}\tau)} \theta_1(v_i, \tau) \tag{8.21}$$

Then substituting the transformation (8.21) into (8.20), we conclude that (8.20) remains invariant.

(4) $SL_2(\mathbb{Z})$ **invariant**

Under $SL_2(\mathbb{Z})$ action the following function transform as

$$\frac{\theta_1(\frac{v_i}{c\tau+d}, \frac{a\tau+d}{c\tau+d})}{\theta_1'(0, \frac{a\tau+d}{c\tau+d})} = \exp(\frac{\pi i c v_i^2}{c\tau + d}) \frac{\theta_1(v_i, \tau)}{\theta_1'(0, \tau)} \tag{8.22}$$

Then substituting the transformation (8.22) into (8.20), we conclude that (8.20) remains invariant.

(5) **Injectivity**

Two elliptic functions are equal if they have the same zeros and poles with multiplicity.

(6) **Surjectivity**

Any elliptic function can be written as rational functions of Weierstrass sigma function up to a multiplication factor [**33**]. By using the formula to relate Weierstrass sigma function and Jacobi theta function

$$\sigma(v_i, \tau) = \frac{\theta_1(v_i, \tau)}{\theta_1'(0, \tau)} \exp(-2\pi i g_1(\tau) v_i^2) \tag{8.23}$$

□

Corollary 8.2.5.1. The functions $(\varphi_0^{\tilde{A}_n}, \varphi_1^{\tilde{A}_n}, .., \varphi_n^{\tilde{A}_n})$ obtained by the formula

$$
\begin{aligned}
\lambda^{\tilde{A}_n} &= e^{2\pi i u} \frac{\prod_{i=0}^{n} \theta_1(z - v_i + v_{n+1}, \tau)}{\theta_1^n(z,\tau)\theta_1(z + (n+1)v_{n+1})} \\
&= \varphi_n^{\tilde{A}_n} \wp^{n-2}(z,\tau) + \varphi_{n-1}^{\tilde{A}_n} \wp^{n-3}(z,\tau) + ... + \varphi_2^{\tilde{A}_n} \wp(z,\tau) \\
&+ \varphi_1^{\tilde{A}_n} [\zeta(z,\tau) - \zeta(z + (n+1)v_{n+1}, \tau) + \varphi_0^{\tilde{A}_n}
\end{aligned} \tag{8.24}
$$

are Jacobi forms of weight $0, -1, -2, .., -n$ respectively, index 1, and order 0.

Proof. Let us prove each item separated.

(1) A_n **invariant, translation invariant**
The l.h.s of (8.24) are A_n invariant, and translation invariant by the lemma (8.2.5). Then, by the uniqueness of Laurent expansion of $\lambda^{\tilde{A}_n}$, we have that $\varphi_i^{\tilde{A}_n}$ are A_n invariant, and translation invariant.

(2) $SL_2(\mathbb{Z})$ **equivariant**
The l.h.s of (8.24) are $SL_2(\mathbb{Z})$ invariant, but the Weierstrass functions of the r.h.s have the following transformation law

$$
\wp^{(k-2)}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \wp^{(k-2)}(z,\tau). \tag{8.25}
$$

Then, $\varphi_k^{\tilde{A}_n}$ must have the following transformation law

$$
\varphi_k^{\tilde{A}_n}\left(u + \frac{c\langle v, v\rangle_{\tilde{A}_n}}{2(c\tau+d)}, \frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-k} \varphi_k^{\tilde{A}_n}(u,v,\tau). \tag{8.26}
$$

(3) **Index** 1

$$
\frac{1}{2\pi i}\frac{\partial}{\partial u}\lambda^{\tilde{A}_n} = \lambda^{\tilde{A}_n}. \tag{8.27}
$$

Then

$$
\frac{1}{2\pi i}\frac{\partial}{\partial u}\varphi_i^{\tilde{A}_n} = \varphi_i^{\tilde{A}_n}. \tag{8.28}
$$

(4) **Analytic behavior**
Note that $\lambda^{\tilde{A}_n}\theta_1^2((n+1)v_{n+1}, \tau)$ is holomorphic function in all the variables v_i. Therefore $\varphi_i^{\tilde{A}_n}$ are holomorphic functions on the variables $v_0, v_1, .., v_{n-1}$, and meromorphic function in the variable $(n+1)v_{n+1}$ with poles on $\frac{j}{n} + \frac{l\tau}{n}, j, l = 0, .., n-1$ of order 2, i.e $l = 0$, since $m = 1$ for all $\varphi_i^{\tilde{A}_n}$.

□

8.3. Proof of the Chevalley theorem

At this stage, the principal theorem can be state in precise way as follows.

THEOREM 8.3.1. The trigraded algebra of weak $\mathscr{I}(\tilde{A}_n)$ -invariant Jacobi forms $J^{\mathscr{I}(\tilde{A}_n)}_{\bullet,\bullet,\bullet} = \bigoplus_{k,l,m} J^{\tilde{A}_n}_{k,l,m}$ is freely generated by $n+1$ fundamental Jacobi forms $(\varphi_0, \varphi_1, , \varphi_2, .., , \varphi_n)$ over the graded ring $E_{\bullet,\bullet}$

$$J^{\mathscr{I}(\tilde{A}_n)}_{\bullet,\bullet,\bullet} = E_{\bullet,\bullet}\left[\varphi_0, \varphi_1, , \varphi_2, .., , \varphi_n\right], \tag{8.29}$$

where

$$E_{\bullet,\bullet} = J_{\bullet,\bullet,0} \quad \text{is the ring of coefficients.}$$

More specifically, the ring of function $E_{\bullet,\bullet}$ is the space of coefficients $f(v_{n+1}, \tau)$ such that for fixed τ, the functions $v_{n+1} \mapsto f(v_{n+1}, \tau)$ is an elliptic function.

Before proving this Theorem, some auxiliary lemmas will be necessary.

LEMMA 8.3.2. Let $\{\varphi_i^{\tilde{A}_n}\}$ be set of functions given by the formula (8.24) ,and $\{\varphi_j^{A_{n+1}}\}$ given by (5.21) , then

$$\begin{aligned}
\varphi_{n+2}^{\mathscr{I}(A_{n+1})} &= \varphi_n^{\tilde{A}_n} \varphi_2^{\mathscr{I}(A_1)}, \\
\varphi_{n+1}^{\mathscr{I}(A_{n+1})} &= \varphi_{n-1}^{\mathscr{I}(\tilde{A}_n)} \varphi_2^{\mathscr{I}(A_1)} + a_{n-1}^n \varphi_n^{\mathscr{I}(\tilde{A}_n)} \varphi_2^{\mathscr{I}(A_1)}, \\
\varphi_{n+2}^{\mathscr{I}(A_{n+1})} &= \varphi_{n-2}^{\mathscr{I}(\tilde{A}_n)} \varphi_2^{\mathscr{I}(A_1)} + a_{n-2}^{n-1} \varphi_{n-1}^{\mathscr{I}(\tilde{A}_n)} \varphi_2^{\mathscr{I}(A_1)} + a_{n-2}^n \varphi_n^{\mathscr{I}(\tilde{A}_n)} \varphi_2^{\mathscr{I}(A_1)}, \\
&. \\
&. \\
\varphi_2^{\mathscr{I}(A_{n+1})} &= \varphi_0^{\mathscr{I}(\tilde{A}_n)} \varphi_2^{\mathscr{I}(A_1)} + \sum_{j=1}^{n} a_0^j \varphi_j^{\mathscr{I}(\tilde{A}_n)} \varphi_2^{\mathscr{I}(A_1)}, \\
\varphi_0^{\mathscr{I}(A_{n+1})} &= \sum_{j=0}^{n} a_{-1}^j \varphi_j^{\mathscr{I}(\tilde{A}_n)} \varphi_2^{\mathscr{I}(A_1)}.
\end{aligned} \tag{8.30}$$

where $\varphi_2^{\mathscr{I}(A_1)}$ is defined on (8.5) for $z_{n+2} = (n+1)v_{n+1}$, and $a_i^j = a_i^j(v_{n+1}, \tau)$ are elliptic functions on v_{n+1}.

PROOF. Note the following relation

$$\begin{aligned}
\frac{\lambda^{\mathscr{I}(A_{n+1})}}{\lambda^{\mathscr{I}(\tilde{A}_n)}} &= \frac{\theta_1(z-(n+1)v_{n+1}, \tau)\theta_1(z+(n+1)v_{n+1}), \tau}{\theta_1^2(z, \tau)} e^{-2\pi i u_2} \\
&= \varphi_2^{\mathscr{I}(A_1)} \wp(z, \tau) - \varphi_2^{\mathscr{I}(A_1)} \wp((n+1)v_{n+1}, \tau)
\end{aligned}$$

Hence,

(8.31)

$$\varphi_{n+2}^{\mathscr{I}(A_{n+1})}\wp^{n-2}(z,\tau)+\varphi_{n+1}^{\mathscr{I}(A_{n+1})}\wp^{n-3}(z,\tau)+...+\varphi_{2}^{\mathscr{I}(A_{n+1})}\wp(z,\tau)+\varphi_{0}^{\mathscr{I}(A_{n+1})}$$
$$=(\varphi_{n}^{\mathscr{I}(\tilde{A}_{n})}\wp^{n-2}(z,\tau)+\varphi_{n-1}^{\mathscr{I}(\tilde{A}_{n})}\wp^{n-3}(z,\tau)+...+\varphi_{2}^{\mathscr{I}(\tilde{A}_{n})}\wp(z,\tau)$$
$$+\varphi_{1}^{\mathscr{I}(\tilde{A}_{n})}[\zeta(z,\tau)-\zeta(z+(n+1)v_{n+1},\tau)+\varphi_{0}^{\mathscr{I}(\tilde{A}_{n})})(\varphi_{2}^{\mathscr{I}(A_{1})}\wp(z,\tau)-\varphi_{2}^{\mathscr{I}(A_{1})}\wp((n+1)v_{n+1},\tau)).$$

Then, the desired result is obtained by doing a Laurent expansion in the variable z in both side of the equality.

□

As a consequence of the previous lemma, we have

Corollary 8.3.2.1. The Jacobi forms $\{\varphi_{i}^{\mathscr{I}(\tilde{A}_{n})}\}$ are algebraically independent.

Proof. Suppose that there exist polynomial $h(x_0,x_1,..,x_n)$ not identically 0, such that

$$h(\varphi_{0}^{\mathscr{I}(\tilde{A}_{n})},\varphi_{1}^{\mathscr{I}(\tilde{A}_{n})},\varphi_{2}^{\mathscr{I}(\tilde{A}_{n})},..,\varphi_{n}^{\mathscr{I}(\tilde{A}_{n})})=0$$

then, because $J_{\bullet,\bullet,\bullet}^{\mathscr{I}(\tilde{A}_{n})}$ is graded ring $h(x_0,x_1,..,x_n)$ should be 0 in each homogeneous component $h_m(x_0,x_1,..,x_n)$ of index m. Let $\tilde{h}_m:=(\varphi_{2}^{\mathscr{I}(A_{1})})^{m}h_m(\varphi_{0}^{\tilde{\mathscr{I}}(A_{n})},\varphi_{1}^{\mathscr{I}(\tilde{A}_{n})},\varphi_{2}^{\mathscr{I}(\tilde{A}_{n})},..,\varphi_{n}^{\mathscr{I}(\tilde{A}_{n})})$. Let us expand the functions $\varphi_{i}^{\mathscr{I}(\tilde{A}_{n})}$ in the variables v_i, then $\tilde{h}_m$ vanishes iff its vanishes in each order of this expansion.

From equations (5.19), we know that the lowest term of the taylor expansion of $\varphi_{n+2}^{\mathscr{I}(A_{n+1})}$ are the elementary symmetric polynomials. Using lemma 8.3.2, we conclude that the lowest term of $\varphi_{j}^{\mathscr{I}(\tilde{A}_{n})}$ is the same as the lowest term of $\varphi_{j+2}^{\mathscr{I}(A_{n+1})}$, but those terms are exactly the elementary symmetric polynomials. But the elementary symmetric polynomials are algebraically independent, then they can not solve any polynomial equation. Lemma proved. □

Corollary 8.3.2.2.

$$E_{\bullet,\bullet}\left[\varphi_{0}^{\mathscr{I}(\tilde{A}_{n})},\varphi_{1}^{\mathscr{I}(\tilde{A}_{n})},\varphi_{2}^{\mathscr{I}(\tilde{A}_{n})},..,\varphi_{n}^{\mathscr{I}(\tilde{A}_{n})}\right]=E_{\bullet,\bullet}\left[\frac{\varphi_{0}^{\mathscr{I}(A_{n+1})}}{\varphi_{2}^{\mathscr{I}(A_{1})}},\frac{\varphi_{2}^{\mathscr{I}(A_{n+1})}}{\varphi_{2}^{\mathscr{I}(A_{1})}},...,\frac{\varphi_{n}^{\mathscr{I}(A_{n+1})}}{\varphi_{2}^{\mathscr{I}(A_{1})}}\right]$$

Moreover, we have the following lemma

Lemma 8.3.3. Let $\varphi\in J_{\bullet,\bullet,m}^{\mathscr{I}(\tilde{A}_{n})}$, then $\varphi\in E_{\bullet,\bullet}\left[\frac{\varphi_{0}^{\mathscr{I}(A_{n+1})}}{\varphi_{2}^{\mathscr{I}(A_{1})}},\frac{\varphi_{2}^{\mathscr{I}(A_{n+1})}}{\varphi_{2}^{\mathscr{I}(A_{1})}},...,\frac{\varphi_{n}^{\mathscr{I}(A_{n+1})}}{\varphi_{2}^{\mathscr{I}(A_{1})}}\right]$.

Proof. Let $\varphi\in J_{\bullet,\bullet,m}^{\mathscr{I}(\tilde{A}_{n})}$, then the function $\frac{\varphi}{\varphi_{n}^{\mathscr{I}(A_{n})}}$ is an elliptic function on the variables $(v_0,v_1,..,v_{n-1},v_{n+1})$ with poles on $v_i-v_{n+1},(n+1)v_{n+1}$. Expanding the function $\frac{\varphi}{\varphi_{n}^{\mathscr{I}(A_{n})}}$ in the variables $v_0, v_1, .., v_{n-1}$ we get

$$\frac{\varphi}{\varphi_n^{\mathscr{I}(A_n)}} = \sum_{i=0}^{n-1} a_m^i \wp^{(m-2)}(v_i - v_{n+1}) + \sum_{i=0}^{n-1} a_{m-1}^i \wp^{(m-3)}(v_i - v_{n+1}) + .. + \sum_{i=0}^{n-1} a_1^i \zeta^{(m-2)}(v_i - v_{n+1}) + b(v_{n+1}, \tau)$$

But the function $\frac{\varphi}{\varphi_n^{\mathscr{I}(A_n)}}$ is invariant under the permutations of the variables v_i, then

$$\frac{\varphi}{\varphi_n^{\mathscr{I}(A_n)}} = a_m \sum_{i=0}^{n-1} \wp^{(m-2)}(v_i - v_{n+1}) + a_{m-1} \sum_{i=0}^{n-1} \wp^{(m-3)}(v_i - v_{n+1}) + .. + a_1 \sum_{i=0}^{n-1} \zeta^{(m-2)}(v_i - v_{n+1}) + b(v_{n+1}, \tau) \tag{8.32}$$

Now we complete this function to A_{n+1} invariant function by summing and subtracting the following function in e.q (8.32)

$$f(v_{n+1}, \tau) = a_m \sum_{i=0}^{n-1} \wp^{(m-2)}((n+1)v_{n+1}) + a_{m-1} \sum_{i=0}^{n-1} \wp^{(m-3)}((n+1)v_{n+1}) + .. + a_1 \sum_{i=0}^{n-1} \zeta^{(m-2)}((n+1)v_{n+1})$$

Hence,

$$\begin{aligned} \frac{\varphi}{\varphi_n^{\mathscr{I}(A_n)}} &= a_m(\sum_{i=0}^{n-1} \wp^{(m-2)}(v_i - v_{n+1}) + \wp^{(m-2)}(n+1)v_{n+1})) \\ &+ a_{m-1} \sum_{i=0}^{n-1} (\wp^{(m-3)}(v_i - v_{n+1}) + \wp^{(m-3)}((n+1)v_{n+1})) + .. \\ &+ a_1 \sum_{i=0}^{n-1} (\zeta^{(m-2)}(v_i - v_{n+1}) + \zeta^{(m-2)}((n+1)v_{n+1})) + \tilde{b}(v_{n+1}, \tau) \end{aligned} \tag{8.33}$$

To finish the proof note the following

(1) The functions $\varphi_{n+2}^{\mathscr{I}(A_{n+1})}[\wp^{(j)}(v_i - v_{n+1}) + \wp^{(j)}(n+1)v_{n+1})]$ are A_{n+1} by construction,

(2) The functions $\varphi_{n+2}^{\mathscr{I}(A_{n+1})}[\wp^{(j)}(v_i - v_{n+1}) + \wp^{(j)}(n+1)v_{n+1})]$ are invariant under the action of $(\mathbb{Z} \oplus \tau\mathbb{Z})^{2n+2}$, because $\varphi_{n+2}^{\mathscr{I}(A_{n+1})}$ invariant, and $\wp^{(j)}(v_i - v_{n+1}) + \wp^{(j)}(n+1)v_{n+1})]$ are elliptic functions.

(3) The functions $\varphi_{n+2}^{\mathscr{I}(A_{n+1})}[\wp^{(j)}(v_i - v_{n+1}) + \wp^{(j)}(n+1)v_{n+1})]$ are equivariant under the action of $SL_2(\mathbb{Z})$, because $\varphi_{n+2}^{\mathscr{I}(A_{n+1})}$ is equivariant, and $\wp^{(j)}(v_i - v_{n+1}) + \wp^{(j)}(n+1)v_{n+1})]$ are elliptic functions.

(4) The function $\varphi_{n+2}^{\mathscr{I}(A_{n+1})}$ has zeros on $v_i - v_{n+1}, (n+1)v_{n+1}$ of order m, and $\wp^{(j)}(v_i - v_{n+1}) + \wp^{(j)}(n+1)v_{n+1})]$ has poles on $v_i - v_{n+1}, (n+1)v_{n+1}$ of order $j + 2 \leq m$. Then, the functions $\varphi_{n+2}^{\mathscr{I}(A_{n+1})}[\wp^{(j)}(v_i - v_{n+1}) + \wp^{(j)}(n+1)v_{n+1})]$ are holomorphic.

(5) We conclude that $g_j := \varphi_{n+2}^{\mathscr{J}(A_{n+1})}[\wp^{(j)}(v_i - v_{n+1}) + \wp^{(j)}(n+1)v_{n+1})] \in J_{\bullet,\bullet}^{\mathscr{J}(A_{n+1})}$. Hence,

(8.34)

$$\varphi = \sum_{i=1}^{m} a_i \frac{g_i}{(\varphi_2^{\mathscr{J}(A_1)})^m} + \tilde{b}(v_{n+1},\tau)(\frac{\varphi_{n+2}^{\mathscr{J}(A_{n+1})}}{\varphi_2^{\mathscr{J}(A_1)}})^m \in E_{\bullet,\bullet}[\frac{\varphi_0^{\mathscr{J}(A_{n+1})}}{\varphi_2^{\mathscr{J}(A_1)}}, \frac{\varphi_2^{\mathscr{J}(A_{n+1})}}{\varphi_2^{\mathscr{J}(A_1)}}, ..., \frac{\varphi_n^{\mathscr{J}(A_{n+1})}}{\varphi_2^{\mathscr{J}(A_1)}}].$$

□

PROOF.

(8.35)

$$J_{\bullet,\bullet,\bullet}^{\mathscr{J}(\tilde{A}_n)} \subset E_{\bullet,\bullet}\left[\frac{\varphi_0^{\mathscr{J}(A_{n+1})}}{\varphi_2^{\mathscr{J}(A_1)}}, \frac{\varphi_2^{\mathscr{J}(A_{n+1})}}{\varphi_2^{\mathscr{J}(A_1)}}, ..., \frac{\varphi_n^{\mathscr{J}(A_{n+1})}}{\varphi_2^{\mathscr{J}(A_1)}}\right] = E_{\bullet,\bullet}\left[\varphi_0^{\mathscr{J}(\tilde{A}_n)}, \varphi_1^{\mathscr{J}(\tilde{A}_n)}, \varphi_2^{\mathscr{J}(\tilde{A}_n)}, .., \varphi_n^{\mathscr{J}(\tilde{A}_n)}\right] \subset J_{\bullet,\bullet,\bullet}^{\mathscr{J}(\tilde{A}_n)}.$$

□

8.4. Intersection form

This section generalise the definitions and results of the section 5.3. In addition, we generalise the formula (5.35).

REMARK 8.4.1. From this point, we will use $(\varphi_0, \varphi_1, .., \varphi_n)$ to denote the Jacobi forms of the group $\mathscr{J}(\tilde{A}_n)$ again, since there will not be anymore Jacobi form from others Jacobi groups.

DEFINITION 8.4.1. Let

$$\begin{aligned} g &= \sum_{i=0}^{n} dv_i^2|_{\sum_{i=0}^n v_i = 0} - n(n+1)dv_{n+1}^2 + 2dud\tau, \\ &= \sum_{i,j=0}^{n-1} A_{ij} dv_i dv_j - n(n+1)dv_{n+1}^2 + 2dud\tau, \\ &= \sum_{i,j=0}^{n+1} g_{ij} dv_i dv_j + 2dud\tau. \end{aligned} \tag{8.36}$$

where A_{ij} is same as g_{ij} of the Coxeter case (4.5). The intersection form is given by

$$g^* = \sum_{i,j=0}^{n-1} A_{ij}^{-1} \frac{\partial}{\partial v_i} \otimes \frac{\partial}{\partial v_j} - \frac{1}{n(n+1)} \frac{\partial}{\partial v_{n+1}} \otimes \frac{\partial}{\partial v_{n+1}} + \frac{\partial}{\partial u} \otimes \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \tau} \otimes \frac{\partial}{\partial u}. \tag{8.37}$$

PROPOSITION 8.4.1. The intersection form (8.36) is invariant under the first two actions of (8.1), and behaves a modular form of weight 2 under the last action of (8.1).

PROOF. The first action of (8.1) only acts on the variables v_i, the intersection form is invariant under this action because $A_{ij} dv_i dv_j$ is invariant under permutation.

Under the second action of (8.1), the differentials transform as:

$$
\begin{aligned}
&dv_i \mapsto dv_i + \lambda_i d\tau,\\
&du \mapsto du - \langle\lambda,\lambda\rangle d\tau - 2\sum_{i=0}^{n+1} g_{ij}\lambda_j dv_i,\\
&d\tau \mapsto d\tau.
\end{aligned}
\tag{8.38}
$$

Hence:

$$
\begin{aligned}
&g_{ij}dv_i dv_j \mapsto g_{ij}dv_i dv_j + 2g_{ij}\lambda_i dv_i d\tau + g_{ij}\lambda_i\lambda_j d\tau^2,\\
&2du d\tau \mapsto 2du d\tau - \langle\lambda,\lambda\rangle d^2\tau - 2\sum_{i=0}^{n-1} g^{ij}\lambda_j dv_i d\tau.
\end{aligned}
\tag{8.39}
$$

Then:

$$
\sum_{i,j=0}^{n+1} g_{ij}dv_i dv_j + 2du d\tau \mapsto \sum_{i,j=0}^{n+1} g_{ij}dv_i dv_j + 2du d\tau.
\tag{8.40}
$$

Let us show that the metric has conformal transformation under the third action of transformations (8.1):

$$
\begin{aligned}
&dv_i \mapsto \frac{dv_i}{c\tau+d} - \frac{cv_i d\tau}{(c\tau+d)^2},\\
&d\tau \mapsto \frac{d\tau}{(c\tau+d)^2},\\
&du \mapsto du + \frac{cg_{ij}v_i dv_j}{c\tau+d} - \frac{c\langle v,v\rangle d\tau}{2(c\tau+d)^2}.
\end{aligned}
\tag{8.41}
$$

Then:

$$
\begin{aligned}
&g_{ij}dv_i dv_j \mapsto \frac{g_{ij}dv_i dv_j}{(c\tau+d)^2} - \frac{2cg_{ij}v_i dv_j d\tau}{(c\tau+d)^3} + \frac{g_{ij}v_i v_j d\tau^2}{(c\tau+d)^4},\\
&2du d\tau \mapsto \frac{2du d\tau}{(c\tau+d)^2} + \frac{2cg_{ij}v_i dv_j d\tau}{(c\tau+d)^3} - \frac{c\langle v,v\rangle d\tau^2}{(c\tau+d)^4}.
\end{aligned}
\tag{8.42}
$$

Then,

$$
\sum_{i,j=0}^{n+1} g_{ij}dv_i dv_j + 2du d\tau \mapsto \frac{\sum_{i,j=0}^{n+1} g_{ij}dv_i dv_j + 2du d\tau}{(c\tau+d)^2}.
\tag{8.43}
$$

□

An efficient way to compute all $g^*(d\varphi_i, d\varphi_j)$ is by collecting all of them in a generating function. Note that $(d\varphi_i, d\varphi_j)$ is not a Jacobi form, and this fact makes the computation more difficult. Hence, in order to circle around this problem, we define the following coefficients in (8.45).

LEMMA 8.4.2. Let $\varphi_i \in J^{A_n}_{k_i,m_i}$, then the metric given by

$$\frac{1}{\eta^{2i+2j}} g^*(d\eta^{2i}\varphi_i, d\eta^{2j}\varphi_j) \frac{\partial}{\partial\varphi_i}\left(\eta^{2i}.\right) \otimes \frac{\partial}{\partial\varphi_j}\left(\eta^{2j}.\right) \tag{8.44}$$

is invariant under the first two actions of (8.1), and behaves a modular form of weight 2 under the last action of (8.1).
Moreover, the coeffiecients of the metric (8.44)

$$\begin{aligned} M\left(d\varphi_i, d\varphi_j\right) &:= \frac{1}{\eta^{2i+2j}} g^*(d\eta^{2i}\varphi_i, d\eta^{2j}\varphi_j) \\ &= g^*\left(d\varphi_i, d\varphi_j\right) - 4\pi i g_1(\tau)\left(k_i m_j + k_j m_i\right)\varphi_i\varphi_j, \end{aligned} \tag{8.45}$$

belong to $J^{A_n}_{k_i+k_j-2,m_i+m_j}$.

PROOF. The metric (8.44) is invariant under the first two actions of (8.1) due to proposition 8.4.1, and because η do not change under this action. The equivariance with respect the $SL_2(\mathbb{Z})$ follows again from proposition 8.4.1, and from the fact that the transformation laws of η get canceled.

The equation (8.45) follows from the chain rule, from the identity

$$\frac{\eta'}{\eta}(\tau) = g_1(\tau) \tag{8.46}$$

□

PROPOSITION 8.4.3. [8] Let E^k the space of elliptic function of weight k. The elliptic connection $D_\tau : E^K \mapsto E^k$ is linear map defined by

$$D_\tau F(v,\tau) = \partial_\tau F(v,\tau) - 2kg_1(\tau)F(v,\tau) - \frac{1}{2\pi i}\frac{\theta_1'(v,\tau)}{\theta_1(v,\tau)}F'(v,\tau), \tag{8.47}$$

where $F(v,\tau) \in E^k$.

In order to compute the coefficient of $M^*(d\varphi_i, d\varphi_j)$ it will be necessary to define an extended intersection form g.

DEFINITION 8.4.2. The extended metric g is defined by

$$\widetilde{g} = \sum_{i=0}^{n} dv_i^2\big|_{\sum_{i=0}^{n} v_i=0} - n(n+1)dv_{n+1}^2 + 2dud\tau + ndp^2 + 4ng_1(\tau)pdpd\tau + 2ng_1'(\tau)p^2d\tau^2, \tag{8.48}$$

or alternatively,

$$\widetilde{g} = \sum_{i=1}^{n+1} dz_i^2 - dz_{n+2}^2 + 2dsd\tau, \tag{8.49}$$

where $(s, z_1, ..., z_{n+2}, \tau)$ is given by (8.12). The extended intersection form read as

(8.50)
$$\widetilde{g}^* = \sum_{i,j} A_{ij}^{-1} \frac{\partial}{\partial v_i} \otimes \frac{\partial}{\partial v_j} - \frac{1}{n(n+1)} \frac{\partial}{\partial v_{n+1}} \otimes \frac{\partial}{\partial v_{n+1}} + \frac{1}{n} \frac{\partial}{\partial p} \otimes \frac{\partial}{\partial p} + \frac{\partial}{\partial s} \otimes \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \tau} \otimes \frac{\partial}{\partial s}$$

The following technical result proved by Bertola in [8] will be useful to prove the subsequent results.

Proposition 8.4.4. [8] The following formula holds

$$\left(\frac{\alpha''(x)}{\alpha(x)} + \frac{\alpha''(y)}{\alpha(y)} - \frac{\alpha'(x)\alpha'(y)}{\alpha(x)\alpha(y)} \right) = -4\pi i \frac{\partial_\tau \alpha(x-y)}{\alpha(x-y)} + 2\frac{\alpha'(x-y)}{\alpha(x-y)} \left[\frac{\alpha'(x)}{\alpha(x)} - \frac{\alpha'(y)}{\alpha(y)} \right]. \tag{8.51}$$

where $\alpha(p)$ is given by the second equation (8.55).

The desired generating function is a consequence of the following lemmas.

Lemma 8.4.5. Let $\Phi(p)$ be given by

$$\Phi(p) = e^{-2\pi i u - 2\pi i p^2 n g_1(\tau)} \prod_{i=1}^{n+1} \frac{\theta_1(z_i,\tau)}{\theta_1'(0,\tau)} \frac{\theta_1'(0,\tau)}{\theta_1(z_{n+2},\tau)}, \tag{8.52}$$

and $\widetilde{M}$ the extended modified intersection form

$$\widetilde{M}(d\Phi(p), d\Phi(p')) = \frac{1}{\eta^{4n+4}} \widetilde{g}^* \left(d\left(\eta^{2n+2}\Phi(p)\right), d\left(\eta^{2n+2}\Phi(p')\right)\right), \tag{8.53}$$

then,

$$\begin{aligned} & e^{2\pi i n(p^2+p'^2)} \widetilde{M}(d\Phi(p), d\Phi(p')) = \\ & = 2\pi i n \frac{\nabla_\tau \alpha(p-p')}{\alpha(p-p')} + \frac{\alpha'(p-p')}{\alpha(p-p')} \left[P(p)\frac{dP(p')}{dp'} - P(p')\frac{dP(p)}{dp} \right], \end{aligned} \tag{8.54}$$

where

$$\begin{aligned} \nabla_\tau F(v,\tau) &= \frac{1}{\eta^{2k}} \frac{\partial\left(\eta^{2k} F\right)}{\partial \tau}, \quad F \in E^k, \\ \alpha(p) &= \frac{\theta_1(p,\tau)}{\theta_1'(0,\tau)}, \\ P(p) &= e^{-2\pi i u} \prod_{i=1}^{n+1} \frac{\theta_1(z_i,\tau)}{\theta_1'(0,\tau)} \frac{\theta_1'(0,\tau)}{\theta_1(z_{n+2},\tau)}. \end{aligned} \tag{8.55}$$

Proof.

$$\begin{aligned} & e^{2\pi i n g_1(\tau)(p^2+p'^2)} \widetilde{M}(d\Phi(p), d\Phi(p')) \\ & = e^{2\pi i g_1(\tau) n(p^2+p'^2)} \widetilde{M}\left(d\left(e^{-2\pi i n g_1(\tau)p^2} P(p)\right), d\left(e^{-2\pi i n g_1(\tau)p'^2} P(p')\right)\right) \end{aligned} \tag{8.56}$$

Note that

$$
\begin{aligned}
\frac{\partial}{\partial p} &= \sum_{i=1}^{n+2} \frac{\partial}{\partial z_i} + 2npg_1(\tau)\frac{\partial}{\partial u}, \\
\frac{\partial}{\partial v_{n+1}} &= \sum_{i=1}^{n+1} \frac{\partial}{\partial z_i} + (n+1)\frac{\partial}{\partial z_{n+2}}, \\
\frac{\partial}{\partial \tau} &= \frac{\partial}{\partial \tilde{\tau}} + np^2 g_1'(\tau)\frac{\partial}{\partial u}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
e^{2\pi i n g_1(\tau)p^2} \frac{\partial}{\partial p}\left(e^{-2\pi i n g_1(\tau)p^2} P(p)\right) &= \sum_{i=1}^{n+1} \frac{\partial P(p)}{\partial z_i} - \frac{\partial P(p)}{\partial z_{n+2}}, \\
e^{2\pi i n g_1(\tau)p^2} \frac{\partial}{\partial \tau}\left(e^{-2\pi i n g_1(\tau)p^2} P(p)\right) &= \frac{\partial P(p)}{\partial \tilde{\tau}}.
\end{aligned} \tag{8.57}
$$

Substituting (8.57) in (8.56) we get

$$
\begin{aligned}
&e^{2\pi i n g_1(\tau)(p^2+p'^2)} \widetilde{M}(d\Phi(p), d\Phi(p')) = \\
&= \sum_{i,j=0}^{n-1} A_{ij}^{-1} \frac{\partial P(p)}{\partial v_i}\frac{\partial P(p')}{\partial v_j} + \frac{1}{n}\sum_{i=1}^{n+1}\frac{\partial P(p)}{\partial z_i}\sum_{i=1}^{n+1}\frac{\partial P(p')}{\partial z_i} - \frac{1}{n(n+1)}\frac{\partial P(p)}{\partial v_{n+1}}\frac{\partial P(p')}{\partial v_{n+1}} \\
&- \frac{1}{n}\frac{\partial P(p)}{\partial z_{n+2}}\sum_{i=1}^{n+1}\frac{\partial P(p')}{\partial z_i} - \frac{1}{n}\frac{\partial P(p')}{\partial z_{n+2}}\sum_{i=1}^{n+1}\frac{\partial P(p)}{\partial z_i} + \frac{1}{n}\frac{\partial P(p)}{\partial z_{n+2}}\frac{\partial P(p')}{\partial z_{n+2}} \\
&- 2\pi i P(p)\nabla_\tau P(p') - 2\pi i P(p')\nabla_\tau P(p) \\
&= \sum_{i=1}^{n+1}\frac{\partial P(p)}{\partial z_i}\frac{\partial P(p')}{\partial z_i} - 2\pi i P(p)\nabla_\tau P(p') - 2\pi i P(p')\nabla_\tau P(p).
\end{aligned} \tag{8.58}
$$

Using the following identity in (8.58)

$$
\begin{aligned}
&\sum_{i,j=0}^{n-1} A_{ij}^{-1} \frac{\partial P(p)}{\partial v_i}\frac{\partial P(p')}{\partial v_j} + \frac{1}{n}\sum_{i=1}^{n+1}\frac{\partial P(p)}{\partial z_i}\sum_{i=1}^{n+1}\frac{\partial P(p')}{\partial z_i} \\
&= \sum_{i=1}^{n+1}\frac{\partial P(p)}{\partial z_i}\frac{\partial P(p')}{\partial z_i} + \frac{1}{n(n+1)}\sum_{i=1}^{n+1}\frac{\partial P(p)}{\partial z_i}\sum_{i=1}^{n+1}\frac{\partial P(p')}{\partial z_i},
\end{aligned} \tag{8.59}
$$

we get

$$
\begin{aligned}
&e^{2\pi i n g_1(\tau)(p^2+p'^2)} \widetilde{M}(d\Phi(p), d\Phi(p')) = \\
&= \sum_{i=1}^{n+1}\frac{\partial P(p)}{\partial z_i}\frac{\partial P(p')}{\partial z_i} - \frac{\partial P(p)}{\partial z_{n+2}}\frac{\partial P(p')}{\partial z_{n+2}} - 2\pi i P(p)\nabla_\tau P(p') - 2\pi i P(p')\nabla_\tau P(p),
\end{aligned} \tag{8.60}
$$

we now compute

$$
\begin{aligned}
&\sum_{i=1}^{n+1}\frac{\partial P(p)}{\partial z_i}\frac{\partial P(p')}{\partial z_i}-2\pi i P(p)\nabla_\tau P(p')-2\pi i P(p')\nabla_\tau P(p)=\\
&=\sum_{i=1}^{n+1}\frac{\partial P(p)}{\partial z_i}\frac{\partial P(p')}{\partial z_i}-\frac{2\pi i}{\eta^{2n}}\left[P(p)\frac{\partial\left(\eta^{2n}P(p')\right)}{\partial\tau}+P(p')\frac{\partial\left(\eta^{2n}P(p)\right)}{\partial\tau}\right]\\
&=-\frac{1}{2}\left(\sum_{i=1}^{n+1}\frac{\alpha''(z_i)}{\alpha(z_i)}+\frac{\alpha''(w_i)}{\alpha(w_i)}-2\frac{\alpha'(z_i)\alpha'(w_i)}{\alpha(z_i)\alpha(w_i)}\right)P(p)P(p')+4\pi i g_1 n P(p)P(p')\\
&+\frac{1}{2}\left(\frac{\alpha''(z_{n+2})}{\alpha(z_{n+2})}+\frac{\alpha''(w_{n+2})}{\alpha(w_{n+2})}-2\frac{\alpha'(z_{n+2})\alpha'(w_{n+2})}{\alpha(z_{n+2})\alpha(w_{n+2})}\right)P(p)P(p'),
\end{aligned}
\tag{8.61}
$$

where $z_i:=z_i(v_i,p)$, $w_i:=z_i(v_i,p')$. Substituting (8.51) in (8.61)

$$
\begin{aligned}
&=-\frac{1}{2}\sum_{i=1}^{n+1}\left(\frac{\alpha''(z_i)}{\alpha(z_i)}+\frac{\alpha''(w_i)}{\alpha(w_i)}-2\frac{\alpha'(z_i)\alpha'(w_i)}{\alpha(z_i)\alpha(w_i)}\right)P(p)P(p')+4\pi i g_1 n P(p)P(p')\\
&+\frac{1}{2}\left(\frac{\alpha''(z_{n+2})}{\alpha(z_{n+2})}+\frac{\alpha''(w_{n+2})}{\alpha(w_{n+2})}-2\frac{\alpha'(z_{n+2})\alpha'(w_{n+2})}{\alpha(z_{n+2})\alpha(w_{n+2})}\right)P(p)P(p')\\
&=2\pi i n\frac{\nabla_\tau\alpha(p-p')}{\alpha(p-p')}+\frac{\alpha'(p-p')}{\alpha(p-p')}\left[P(p)\frac{dP(p')}{dp'}-P(p')\frac{dP(p)}{dp}\right].
\end{aligned}
\tag{8.62}
$$

□

Lemma 8.4.6. For the coefficients $\tilde{M}$ of intersection form, we have the following formula

$$
\begin{aligned}
&\sum_{k,j}C_k(p)C_j(p')\widetilde{M}(d\varphi_k,d\varphi_j)=\\
&=2\pi i n\frac{\nabla_\tau\alpha(p-p')}{\alpha(p-p')}\lambda(p)\lambda(p')+\frac{\alpha'(p-p')}{\alpha(p-p')}\left[P(p)\frac{dP(p')}{dp'}-P(p')\frac{dP(p)}{dp}\right]\\
&-\sum_{k,j}\widetilde{M}\left(dC_k(p),dC_j(p')\right)\varphi_{n+1-k}\varphi_{n+1-j}
\end{aligned}
\tag{8.63}
$$

where $C_k(p)$ is given by

$$
\begin{aligned}
&C_k(p)=\frac{(-1)^k}{(k-1)!}\alpha^n(p)\wp^{(k-2)}(p),\\
&\wp^{-1}(p)=\zeta(p)-\zeta(p+(n+1)v_{n+1})+(n+1)\zeta(v_{n+1}),\\
&\wp^{-2}(p)=1,
\end{aligned}
\tag{8.64}
$$

and

$$
\begin{aligned}
\widetilde{M}\left(dC_k(p),dC_j(p')\right):=&-2\pi i\left(C_k(p)\nabla_\tau C_j(p')+C_j(p')\nabla_\tau C_k(p)\right)+\frac{1}{n}\frac{\partial C_k(p)}{\partial p}\frac{\partial C_j(p')}{\partial p'}\\
&-\frac{1}{n(n+1)}\frac{\partial C_k(p)}{\partial v_{n+1}}\frac{\partial C_j(p')}{\partial v_{n+1}}+\sum_{k,j=0}^{n}C_k(p)C_j(p')\frac{\partial\varphi_j}{\partial v_{n+1}}\frac{\partial\varphi_k}{\partial v_{n+1}}.
\end{aligned}
\tag{8.65}
$$

PROOF. Note that $P(p) = \sum_{k=0}^{n+1} C_k(p)\varphi_k$, then

$$\begin{aligned}
&e^{2\pi in(p^2+p'^2)}\widetilde{M}(d\Phi(p), d\Phi(p')) = \\
&= \sum_{j,k} \widetilde{M}\left(d\left(C_k(p)\varphi_k\right), d\left(C_j(p')\varphi_j\right)\right) \\
&= \sum_{j,k} C_k(p)C_j(p')\widetilde{M}\left(d\varphi_k, d\varphi_j\right) + \sum_{j,k} C_j(p')\varphi_k\widetilde{M}\left(dC_k(p), d\varphi_j\right) \\
&+ \sum_{j,k} C_k(p)\varphi_j\widetilde{M}\left(d\varphi_k, dC_j(p')\right) + \sum_{j,k} \varphi_k\varphi_j\widetilde{M}\left(dC_k(p), dC_j(p')\right) \\
&= \sum_{j,k} C_k(p)C_j(p')\widetilde{M}\left(d\varphi_k, d\varphi_j\right) - 2\pi i\sum_{j,k} C_j(p')\varphi_k\varphi_j\nabla_\tau C_k(p) \\
&- 2\pi i\sum_{j,k} C_k(p)\varphi_j\varphi_k\nabla_\tau C_j(p') + \frac{1}{n}\sum_{j,k} \varphi_k\varphi_j\frac{\partial C_k(p)}{\partial p}\frac{\partial C_j(p')}{\partial p'} \\
&- \frac{1}{n(n+1)}\sum_{j,k} \varphi_k\varphi_j\frac{\partial C_k(p)}{\partial v_{n+1}}\frac{\partial C_j(p')}{\partial v_{n+1}} - \frac{1}{n(n+1)}\sum_{j,k} C_k(p)\varphi_j\frac{\partial \varphi_k}{\partial v_{n+1}}\frac{\partial C_j(p')}{\partial v_{n+1}} \\
&- \frac{1}{n(n+1)}\sum_{j,k} C_j(p')\varphi_k\frac{\partial \varphi_j}{\partial v_{n+1}}\frac{\partial C_k(p)}{\partial v_{n+1}}.
\end{aligned} \tag{8.66}$$

Then, isolating $\sum_{k,j} C_k(p)C_j(p')\widetilde{M}(d\varphi_k, d\varphi_j)$

$$\begin{aligned}
&\sum_{k,j} C_k(p)C_j(p')\widetilde{M}(d\varphi_k, d\varphi_j) = \\
&= 2\pi in\frac{\nabla_\tau\alpha(p-p')}{\alpha(p-p')}\lambda(p)\lambda(p') + \frac{\alpha'(p-p')}{\alpha(p-p')}\left[P(p)\frac{dP(p')}{dp'} - P(p')\frac{dP(p)}{dp}\right] \\
&- \sum_{k,j} \widetilde{M}\left(dC_k(p), dC_j(p')\right)\varphi_{n+1-k}\varphi_{n+1-j}.
\end{aligned} \tag{8.67}$$

□

THEOREM 8.4.7. The coefficient of $M^*(d\varphi_i, d\varphi_j)$ is recovered by the generating formula

$$\begin{aligned}
&\sum_{k,j=0}^{n} \frac{(-1)^{k+j}}{(k-1)!(j-1)!}M^*(d\varphi_i, d\varphi_j)\wp(v)^{(k-2)}\wp(v')^{(j-2)} = \\
&= 2\pi i(\lambda(v')D_\tau\lambda(v) + \lambda(v)D_\tau\lambda(v')) - \frac{1}{n+1}\frac{d\lambda(v)}{dv}\frac{d\lambda(v')}{dv'} \\
&+ \frac{1}{2}\frac{\wp'(v)+\wp'(v')}{\wp(v)-\wp(v')}[\lambda(v)\frac{d\lambda(v')}{dv'} - \frac{d\lambda(v)}{dv}\lambda(v')] - \frac{1}{n}\lambda'(p)\lambda'(p') \\
&- \frac{1}{n(n+1)}\frac{\partial\lambda(p)}{\partial v_{n+1}}\frac{\partial\lambda(p')}{\partial v_{n+1}} + \frac{1}{n(n+1)}\sum_{k,j=0}^{n}\frac{(-1)^{k+j}}{(k-1)!(j-1)!}\wp(v)^{(k-2)}\wp(v')^{(j-2)}\frac{\partial\varphi_j}{\partial v_{n+1}}\frac{\partial\varphi_k}{\partial v_{n+1}}.
\end{aligned} \tag{8.68}$$

PROOF. We start by dividing the expression (8.67) by $\alpha^n(p)\alpha^n(p')$

$$
\begin{aligned}
&\sum_{k,j=0}^{n} \frac{(-1)^{k+j}}{(k-1)!(j-1)!} M^*(d\varphi_i, d\varphi_j)\wp(v)^{(k-2)}\wp(v')^{(j-2)} = \\
&= 2\pi i n \frac{\nabla_\tau \alpha(p-p')}{\alpha(p-p')}\lambda(p)\lambda(p') + \frac{\alpha'(p-p')}{\alpha(p-p')}\left[\lambda(p)\frac{dP(p')}{dp'}\frac{1}{\alpha^n(p')} - \lambda(p')\frac{dP(p)}{dp}\frac{1}{\alpha^n(p)}\right] \\
&- \sum_{k,j} \frac{\widetilde{M}\,(dC_k(p), dC_j(p'))}{\alpha^n(p)\alpha^n(p')}\varphi_{n+1-k}\varphi_{n+1-j} \\
&= (1) - (2).
\end{aligned} \tag{8.69}
$$

Computing separately

$$
\begin{aligned}
&(1) := 2\pi i n \frac{\nabla_\tau \alpha(p-p')}{\alpha(p-p')}\lambda(p)\lambda(p') + \frac{\alpha'(p-p')}{\alpha(p-p')}\left[\lambda(p)\frac{dP(p')}{dp'}\frac{1}{\alpha^n(p')} - \lambda(p')\frac{dP(p)}{dp}\frac{1}{\alpha^n(p)}\right] = \\
&= n\left(2\pi i \frac{\nabla_\tau \alpha(p-p')}{\alpha(p-p')} - \frac{\alpha'(p-p')}{\alpha(p-p')}\left[\frac{\alpha'(p)}{\alpha(p)} - \frac{\alpha'(p')}{\alpha(p')}\right]\right)\lambda(p)\lambda(p') \\
&+ \frac{\alpha'(p-p')}{\alpha(p-p')}\left[\lambda(p)\frac{d\lambda(p')}{dp'} - \lambda(p')\frac{d\lambda(p)}{dp}\right].
\end{aligned} \tag{8.70}
$$

and

$$
\begin{aligned}
(2) &:= \sum_{k,j} \frac{\widetilde{M}\,(dC_k(p), dC_j(p'))}{\alpha^n(p)\alpha^n(p')}\varphi_{n+1-k}\varphi_{n+1-j} = \\
&= -2\pi i \left(\lambda(p)\nabla_\tau\lambda(p') + \lambda(p')\nabla_\tau\lambda(p)\right) - 2\pi i n\left[4g_1 + \frac{\partial_\tau\alpha(p)}{\alpha(p)} + \frac{\partial_\tau\alpha(p)}{\alpha(p)}\right]\lambda(p)\lambda(p') \\
&+ \frac{1}{n}\left[\frac{\partial\lambda(p)}{\partial p} + n\frac{\alpha'(p)}{\alpha(p)}\lambda(p)\right]\left[\frac{\partial\lambda(p')}{\partial p'} + n\frac{\alpha'(p')}{\alpha(p')}\lambda(p')\right] \\
&- \frac{1}{n(n+1)}\frac{\partial\lambda(p)}{\partial v_{n+1}}\frac{\partial\lambda(p')}{\partial v_{n+1}} + \sum_{k,j=0}^{n}\frac{(-1)^{k+j}}{(k-1)!(j-1)!}\wp(v)^{(k-2)}\wp(v')^{(j-2)}\frac{\partial\varphi_j}{\partial v_{n+1}}\frac{\partial\varphi_k}{\partial v_{n+1}} \\
&= -2\pi i \left(\lambda(p)\nabla_\tau\lambda(p') + \lambda(p')\nabla_\tau\lambda(p)\right) \\
&- n\left[8\pi i g_1 + 2\pi i\left(\frac{\partial_\tau\alpha(p)}{\alpha(p)} + \frac{\partial_\tau\alpha(p)}{\alpha(p)}\right) - \frac{\alpha'(p)\alpha'(p')}{\alpha(p)\alpha(p')}\right]\lambda(p)\lambda(p') \\
&+ \frac{1}{n}\frac{\partial\lambda(p)}{\partial p}\frac{\partial\lambda(p')}{\partial p'} + \frac{\alpha'(p)}{\alpha(p)}\lambda(p)\frac{\partial\lambda(p')}{\partial p'} + \frac{\alpha'(p')}{\alpha(p')}\lambda(p')\frac{\partial\lambda(p')}{\partial p'} \\
&- \frac{1}{n(n+1)}\frac{\partial\lambda(p)}{\partial v_{n+1}}\frac{\partial\lambda(p')}{\partial v_{n+1}} + \sum_{k,j=0}^{n}\frac{(-1)^{k+j}}{(k-1)!(j-1)!}\wp(v)^{(k-2)}\wp(v')^{(j-2)}\frac{\partial\varphi_j}{\partial v_{n+1}}\frac{\partial\varphi_k}{\partial v_{n+1}} \\
&= -2\pi i \left(\lambda(p)D_\tau\lambda(p') + \lambda(p')D_\tau\lambda(p)\right) \\
&- n\left[8\pi i g_1 + 2\pi i\left(\frac{\partial_\tau\alpha(p)}{\alpha(p)} + \frac{\partial_\tau\alpha(p)}{\alpha(p)}\right) - \frac{\alpha'(p)\alpha'(p')}{\alpha(p)\alpha(p')}\right]\lambda(p)\lambda(p') \\
&+ \frac{1}{n}\frac{\partial\lambda(p)}{\partial p}\frac{\partial\lambda(p')}{\partial p'} + \left(\frac{\alpha'(p)}{\alpha(p)} - \frac{\alpha'(p')}{\alpha(p')}\right)\left[\lambda(p)\frac{d\lambda(p')}{dp'} - \lambda(p')\frac{d\lambda(p)}{dp}\right] \\
&- \frac{1}{n(n+1)}\frac{\partial\lambda(p)}{\partial v_{n+1}}\frac{\partial\lambda(p')}{\partial v_{n+1}} + \sum_{k,j=0}^{n}\frac{(-1)^{k+j}}{(k-1)!(j-1)!}\wp(v)^{(k-2)}\wp(v')^{(j-2)}\frac{\partial\varphi_j}{\partial v_{n+1}}\frac{\partial\varphi_k}{\partial v_{n+1}}.
\end{aligned}
\tag{8.71}
$$

Computing (1)-(2), and using equation (8.51), we obtain

(8.72)

$$
\begin{aligned}
&\sum_{k,j=0}^{n}\frac{(-1)^{k+j}}{(k-1)!(j-1)!}M^*(d\varphi_i, d\varphi_j)\wp(v)^{(k-2)}\wp(v')^{(j-2)} = \\
&= 2\pi i(\lambda(v')D_\tau\lambda(v) + \lambda(v)D_\tau\lambda(v')) - \frac{1}{n+1}\frac{d\lambda(v)}{dv}\frac{d\lambda(v')}{dv'} \\
&+ \frac{1}{2}\frac{\wp'(v)+\wp'(v')}{\wp(v)-\wp(v')}[\lambda(v)\frac{d\lambda(v')}{dv'} - \frac{d\lambda(v)}{dv}\lambda(v')] - \frac{1}{n}\lambda'(p)\lambda'(p') \\
&\frac{1}{n(n+1)}\frac{\partial\lambda(p)}{\partial v_{n+1}}\frac{\partial\lambda(p')}{\partial v_{n+1}} - \frac{1}{n(n+1)}\sum_{k,j=0}^{n}\frac{(-1)^{k+j}}{(k-1)!(j-1)!}\wp(v)^{(k-2)}\wp(v')^{(j-2)}\frac{\partial\varphi_j}{\partial v_{n+1}}\frac{\partial\varphi_k}{\partial v_{n+1}}.
\end{aligned}
$$

□

COROLLARY 8.4.7.1. Let $\tilde{\eta}^*(d\varphi_i, d\varphi_j)$ and $\eta^*(d\varphi_i, d\varphi_j)$ be given by

$$
\begin{aligned}
\tilde{\eta}^*(d\varphi_i, d\varphi_j) &:= \frac{\partial M^*(d\varphi_i, d\varphi_j)}{\partial \varphi_0}, \\
\eta^*(d\varphi_i, d\varphi_j) &:= \frac{\partial g^*(d\varphi_i, d\varphi_j)}{\partial \varphi_0}.
\end{aligned}
\tag{8.73}
$$

The coefficient of $\tilde{\eta}^*(d\varphi_i, d\varphi_j)$ is recovered by the generating formula

$$
\begin{aligned}
&\sum_{k,j=0}^{n+1} \frac{(-1)^{k+j}}{(k-1)!(j-1)!}\tilde{\eta}^*(d\varphi_i, d\varphi_j)\wp(v)^{(k-2)}\wp(v')^{(j-2)} = \\
&= 2\pi i(D_\tau\lambda(v) + D_\tau\lambda(v')) + \frac{1}{2}\frac{\wp'(v)+\wp'(v')}{\wp(v)-\wp(v')}[\frac{d\lambda(v')}{dv'} - \frac{d\lambda(v)}{dv}] \\
&- \frac{1}{n(n+1)}\frac{\partial}{\partial \varphi_0}\left(\frac{\partial \lambda(p)}{\partial v_{n+1}}\right)\frac{\partial \lambda(p')}{\partial v_{n+1}} - \frac{1}{n(n+1)}\frac{\partial \lambda(p)}{\partial v_{n+1}}\frac{\partial}{\partial \varphi_0}\left(\frac{\partial \lambda(p')}{\partial v_{n+1}}\right) \\
&+ \frac{1}{n(n+1)}\sum_{k,j=0}^{n} \frac{(-1)^{k+j}}{(k-1)!(j-1)!}\frac{\partial}{\partial \varphi_0}\left(\frac{\partial \varphi_j}{\partial v_{n+1}}\right)\frac{\partial \varphi_k}{\partial v_{n+1}} \\
&+ \frac{1}{n(n+1)}\sum_{k,j=0}^{n} \frac{(-1)^{k+j}}{(k-1)!(j-1)!}\frac{\partial \varphi_j}{\partial v_{n+1}}\frac{\partial}{\partial \varphi_0}\left(\frac{\partial \varphi_k}{\partial v_{n+1}}\right)
\end{aligned}
\tag{8.74}
$$

Moreover,

$$
\begin{aligned}
\tilde{\eta}^*(d\varphi_i, d\varphi_j) &= \eta^*(d\varphi_i, d\varphi_j), \quad i, j \neq 0, \\
\tilde{\eta}^*(d\varphi_0, d\varphi_j) &= \eta^*(d\varphi_i, d\varphi_j) + 4\pi i k_j\varphi_j.
\end{aligned}
\tag{8.75}
$$

PROOF. Just differentiate equation (8.68) with respect φ_0, and use the equation (8.24). □

COROLLARY 8.4.7.2. The metric $\tilde{\eta}^*$ and η^* defined in (8.73)is invariant under the second action of (8.1), furthermore, behave as modular form of weight 2 under the last action of (8.1).

PROOF. The metric $\tilde{\eta}^*$ and η^* are given by

$$
\begin{aligned}
\tilde{\eta}^* &= Lie_{\frac{\partial}{\partial \varphi_0}} M \\
\eta^* &= Lie_{\frac{\partial}{\partial \varphi_0}} g^*
\end{aligned}
\tag{8.76}
$$

The fact that $\frac{\partial}{\partial \varphi_0}, g^*, M^*$ is invariant under the second action of (8.1),furthermore, behave as modular form of weight 2 under the last action of (8.1) give the desired result.

□

8.5. The second metric of the pencil

In the (8.73), it was defined the second metric η, furthermore, it was derived a generating function for it. The main goal of this section is to extract the coefficients $\eta(d\varphi_i, d\varphi_j)$ from its generating function. In order to do this extraction, some auxiliaries lemmas are needed.

LEMMA 8.5.1. Let φ_1, φ_n be defined on (8.24), then

$$\varphi_1 = e^{2\pi i u} \frac{\prod_{i=0}^{n} \theta_1(v_i + n v_{n+1})}{\theta_1^n((n+1)v_{n+1})\theta_1'(0)},$$
$$\varphi_n = e^{2\pi i u} \frac{\prod_{i=0}^{n} \theta_1(-v_i + v_{n+1})}{\theta_1(-(n+1)v_{n+1})\theta_1'(0)^n}. \tag{8.77}$$

PROOF.

$$\varphi_1 = \lim_{p \mapsto -(n+1)v_{n+1}} (p + (n+1)v_{n+1})\,\lambda(p) = -e^{2\pi i u} \frac{\prod_{i=0}^{n} \theta_1^n(-v_i - n v_{n+1})}{\theta_1^n(-(n+1)v_{n+1})\theta_1'(0)} = e^{2\pi i u} \frac{\prod_{i=0}^{n} \theta_1(v_i + n v_{n+1})}{\theta_1^n((n+1)v_{n+1})\theta_1'(0)},$$
$$\varphi_n = \lim_{p \mapsto 0} p^n \lambda(p) = e^{2\pi i u} \frac{\prod_{i=0}^{n} \theta_1(-v_i + v_{n+1})}{\theta_1(-(n+1)v_{n+1})\theta_1'(0)^n}.$$

□

LEMMA 8.5.2. Let $\varphi_0, \varphi_1, \varphi_2, .., \varphi_n$ be defined on (8.24), then

$$\frac{\partial}{\partial \varphi_0}\left(\frac{\partial \varphi_i}{\partial v_{n+1}}\right) = 0, \quad i > 1,$$
$$\frac{\partial}{\partial \varphi_0}\left(\frac{\partial \varphi_1}{\partial v_{n+1}}\right) = n, \tag{8.78}$$
$$\frac{\partial}{\partial \varphi_0}\left(\frac{\partial \varphi_0}{\partial v_{n+1}}\right) = -n \frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}.$$

PROOF. Computing $\frac{\partial \varphi_n}{\partial v_{n+1}}$ by using the first equation of (8.77)

$$\frac{\partial \varphi_n}{\partial v_{n+1}} = \left(\sum_{i=0}^{n} \frac{\theta_1'(-v_i + v_{n+1})}{\theta_1(-v_i + v_{n+1})} - (n+1)\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\right)\varphi_n. \tag{8.79}$$

Recall the recursive relation between $\{\varphi_i\}$ in (8.17)

$$\varphi_i = \left.\frac{\partial^{(n-i)} \varphi_n(p)}{\partial p^{n-i}}\right|_{p=0}. \tag{8.80}$$

In particular,

$$\varphi_{n-1} = \left.\frac{\partial \varphi_n(p)}{\partial p}\right|_{p=0} = \left(\sum_{i=0}^{n} \frac{\theta_1'(-v_i + v_{n+1})}{\theta_1(-v_i + v_{n+1})} - \frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\right)\varphi_n, \tag{8.81}$$

Then,

$$\frac{\partial \varphi_n}{\partial v_{n+1}} = \varphi_{n-1} - n\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\varphi_n, \tag{8.82}$$

consequently

$$\frac{\partial}{\partial \varphi_0}\left(\frac{\partial \varphi_n}{\partial v_{n+1}}\right) = 0.$$

Suppose that for $i > 1$, we have

$$\varphi_i = f(-v_0 + v_{n+1}, -v_1 + v_{n+1}, .., (n+1)v_{n+1}, \tau)\varphi_n, \tag{8.83}$$

where $f(-v_0+v_{n+1},-v_1+v_{n+1},..,(n+1)v_{n+1},\tau)$ is an elliptic function on the variables $v_0,v_1,..,v_{n+1}$ with zeros on $-v_i+v_{n+1}$ and poles on $(n+1)v_{n+1}$. Consider the extended $\varphi_i(p)$ as

$$\varphi_i(p)=f(p-v_0+v_{n+1},p-v_1+v_{n+1},..,p+(n+1)v_{n+1},\tau)\varphi_n.$$

The action of the vector fields $\frac{\partial}{\partial p}$ and $\frac{\partial}{\partial v_{n+1}}$ in $\varphi_i(p)$ and φ_i are given by

$$\begin{aligned}\left.\frac{\partial\varphi_i(p)}{\partial p}\right|_{p=0}&=\left.\frac{\partial f}{\partial p}\right|_{p=0}\varphi_n+f\left.\frac{\partial\varphi_n(p)}{\partial p}\right|_{p=0},\\ \frac{\partial\varphi_i}{\partial v_{n+1}}&=\frac{\partial f}{\partial v_{n+1}}\varphi_n+f\frac{\partial\varphi_n}{\partial v_{n+1}}.\end{aligned}$$

Note that

$$\left.\frac{\partial f}{\partial p}\right|_{p=0}-\frac{\partial f}{\partial v_{n+1}}=b(v_{n+1},\tau),$$

where $b((n+1)v_{n+1},\tau)$ is an elliptic function on $(n+1)v_{n+1}$, because $\left.\frac{\partial f}{\partial p}\right|_{p=0}$ and $\frac{\partial f}{\partial v_{n+1}}$ are elliptic functions with the same Laurent tail in the variables $-v_i+v_{n+1}$. Hence, using equation (8.80)

$$\begin{aligned}\frac{\partial\varphi_i}{\partial v_{n+1}}&=\left.\frac{\partial\varphi_i(p)}{\partial p}\right|_{p=0}+h(v_{n+1},\tau)\varphi_n,\\ &=\varphi_{i-1}+h(v_{n+1},\tau)\varphi_n.\end{aligned}$$

Then,

$$\frac{\partial}{\partial\varphi_0}\left(\frac{\partial\varphi_i}{\partial v_{n+1}}\right)=0,\quad i>1,$$

Computing φ_0

$$\begin{aligned}\varphi_0&=\left.\frac{\partial\varphi_1}{\partial p}\right|_{p=0}\\ &=\left[\sum_{i=0}^{n}\frac{\theta_1'(v_i+nv_{n+1})}{\theta_1(v_i+nv_{n+1})}-n\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\right]\varphi_1.\end{aligned}$$

Computing $\frac{\partial\varphi_1}{\partial v_{n+1}}$ in terms of φ_0

$$\begin{aligned}\frac{\partial\varphi_1}{\partial v_{n+1}}&=\left[\sum_{i=0}^{n}n\frac{\theta_1'(v_i+nv_{n+1})}{\theta_1(v_i+nv_{n+1})}-n(n+1)\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\right]\varphi_1\\ &=n\varphi_0-n\frac{\theta_1((n+1)v_{n+1}}{\theta_1'((n+1)v_{n+1}}\varphi_1.\end{aligned}\tag{8.84}$$

Then,

$$\frac{\partial}{\partial\varphi_0}\left(\frac{\partial\varphi_1}{\partial v_{n+1}}\right)=n.\tag{8.85}$$

Computing $\frac{\partial}{\partial \varphi_0}\left(\frac{\partial \varphi_0}{\partial v_{n+1}}\right)$

$$
\begin{aligned}
\frac{\partial \varphi_0}{\partial v_{n+1}} &= \left[n\sum_{i=0}^{n}\frac{\partial^2 \log\theta_1(v_i+nv_{n+1})}{\partial v_{n+1}^2} - n(n+1)\frac{\partial^2\log\theta_1((n+1)v_{n+1})}{\partial v_{n+1}^2}\right]\varphi_1\\
&+\left[\sum_{i=0}^{n}\frac{\theta_1'(v_i+nv_{n+1})}{\theta_1(v_i+nv_{n+1})} - n\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\right]\frac{\partial\varphi_1}{\partial v_{n+1}}\\
&= \left[n\sum_{i=0}^{n}\frac{\partial^2 \log\theta_1(v_i+nv_{n+1})}{\partial v_{n+1}^2} - n(n+1)\frac{\partial^2\log\theta_1((n+1)v_{n+1})}{\partial v_{n+1}^2}\right]\varphi_1\\
&+\left[\sum_{i=0}^{n}\frac{\theta_1'(v_i+nv_{n+1})}{\theta_1(v_i+nv_{n+1})} - n\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\right]\left[-n\varphi_0 - n\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\varphi_1\right]\\
&= \left[n\sum_{i=0}^{n}\frac{\partial^2 \log\theta_1(v_i+nv_{n+1})}{\partial v_{n+1}^2} - n(n+1)\frac{\partial^2\log\theta_1((n+1)v_{n+1})}{\partial v_{n+1}^2}\right]\varphi_1\\
&- n\left[\sum_{i=0}^{n}\frac{\theta_1'(v_i+nv_{n+1})}{\theta_1(v_i+nv_{n+1})} - \frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\right]^2\varphi_1 - n\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\varphi_0\\
&= -\frac{\partial^2\log\theta_1((n+1)v_{n+1})}{\partial v_{n+1}^2}\varphi_1 - n\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\varphi_0 + na\varphi_1.
\end{aligned}
\tag{8.86}
$$

where a is defined by

(8.87)

$$
a = \sum_{i=0}^{n}\frac{\partial^2 \log\theta_1(v_i+nv_{n+1})}{\partial v_{n+1}^2} - \frac{\partial^2\log\theta_1((n+1)v_{n+1})}{\partial v_{n+1}^2} - \left[\sum_{i=0}^{n}\frac{\theta_1'(v_i+nv_{n+1})}{\theta_1(v_i+nv_{n+1})} - \frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\right]^2.
$$

Note that $na\varphi_1$ can not be proportional to $b\varphi_0$, for any $b = b(v_{n+1},\tau)$ elliptic function in the variable v_{n+1}. Indeed, if

$$
na\varphi_1 = b\left[\sum_{i=0}^{n}\frac{\theta_1'(v_i+nv_{n+1})}{\theta_1(v_i+nv_{n+1})} - n\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\right]\varphi_1,
$$

we obtain,

$$
na = b\left[\sum_{i=0}^{n}\frac{\theta_1'(v_i+nv_{n+1})}{\theta_1(v_i+nv_{n+1})} - n\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\right].
\tag{8.88}
$$

Analysing the Laurent tail in v_i+nv_{n+1} of (8.87)

(8.89)

$$
a = -2\sum_{i\le j}\frac{\theta_1'(v_i+nv_{n+1})}{\theta_1(v_i+nv_{n+1})}\frac{\theta_1'(v_j+nv_{n+1})}{\theta_1(v_j+nv_{n+1})} + 2\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\sum_{i=0}^{n}\frac{\theta_1'(v_i+nv_{n+1})}{\theta_1(v_i+nv_{n+1})} + \text{regular terms},
$$

then, the first term of the equation (8.89) implies that the left-hand side and right hand side of the equation (8.88) have a different Laurent tail which is an absurd. Hence,

$$
\frac{\partial}{\partial\varphi_0}\left(\frac{\partial\varphi_0}{\partial v_{n+1}}\right) = -n\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}.
$$

$\square$

COROLLARY 8.5.2.1. Let $\varphi_0, \varphi_1, \varphi_2, .., \varphi_n$ be defined on (8.24) and the metric η^* defined in (8.73), then

$$\begin{aligned} &\eta^*(d\varphi_i, dv_{n+1}) = 0, \quad i > 1, \\ &\eta^*(d\varphi_1, dv_{n+1}) = -\frac{1}{n+1}, \\ &\eta^*(d\varphi_0, dv_{n+1}) = -\frac{1}{n+1}\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}. \end{aligned} \tag{8.90}$$

PROOF.

$$\begin{aligned} \eta^*(d\varphi_i, dv_{n+1}) &= \frac{\partial}{\partial\varphi_0}\left(\frac{\partial\varphi_i}{\partial v_{n+1}}\right) = 0, \quad i > 1, \\ \eta^*(d\varphi_1, dv_{n+1}) &= -\frac{1}{n(n+1)}\frac{\partial}{\partial\varphi_0}\left(\frac{\partial\varphi_1}{\partial v_{n+1}}\right) = -\frac{1}{n+1}, \\ \eta^*(d\varphi_0, dv_{n+1}) &= -\frac{1}{n(n+1)}\frac{\partial}{\partial\varphi_0}\left(\frac{\partial\varphi_0}{\partial v_{n+1}}\right) = -\frac{1}{n+1}\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}. \end{aligned}$$

$\square$

LEMMA 8.5.3. Let $\varphi_0, \varphi_1, \varphi_2, .., \varphi_n$ be defined on (8.24) and the metric η^* defined in (8.73), then

$$\begin{aligned} &\eta^*(d\varphi_i, d\tau) = 0, \quad i \neq 0, \\ &\eta^*(d\varphi_0, d\tau) = -2\pi i. \end{aligned} \tag{8.91}$$

PROOF.

$$\eta^*(d\varphi_i, d\tau) = -2\pi i\frac{\partial\varphi_i}{\partial\varphi_0} = \delta_{i0}.$$

$\square$

THEOREM 8.5.4. Let $\eta^*(d\varphi_i, d\varphi_j)$ be defined in (8.73), then its coefficients can be obtained by the formula

$$\begin{aligned} &\tilde{\eta}^*(d\varphi_i, d\varphi_j) = (i+j-2)\varphi_{i+j-2}, \quad i, j \neq 0 \\ &\tilde{\eta}^*(d\varphi_i, d\varphi_0) = 0, \quad i \neq 0, \quad i \neq 1, \\ &\tilde{\eta}^*(d\varphi_1, d\varphi_j) = 0. \quad j \neq 0, \\ &\tilde{\eta}^*(d\varphi_1, d\varphi_0) = \wp((n+1)v_{n+1})\varphi_1. \end{aligned} \tag{8.92}$$

PROOF. We start by dividing the right hand side of the expression (8.74) in two parts:

$$
\begin{aligned}
(a) &:= 2\pi i(D_\tau\lambda(v) + D_\tau\lambda(v')) + \frac{1}{2}\frac{\wp'(v)+\wp'(v')}{\wp(v)-\wp(v')}[\frac{d\lambda(v')}{dv'} - \frac{d\lambda(v)}{dv}],\\
(b) &:= \frac{1}{n(n+1)}\frac{\partial}{\partial\varphi_0}\left(\frac{\partial\lambda(p)}{\partial v_{n+1}}\right)\frac{\partial\lambda(p')}{\partial v_{n+1}} + \frac{1}{n(n+1)}\frac{\partial\lambda(p)}{\partial v_{n+1}}\frac{\partial}{\partial\varphi_0}\left(\frac{\partial\lambda(p')}{\partial v_{n+1}}\right)\\
&\quad - \frac{1}{n(n+1)}\sum_{k,j=0}^{n}\frac{(-1)^{k+j}}{(k-1)!(j-1)!}\frac{\partial}{\partial\varphi_0}\left(\frac{\partial\varphi_j}{\partial v_{n+1}}\right)\frac{\partial\varphi_k}{\partial v_{n+1}}\\
&\quad - \frac{1}{n(n+1)}\sum_{k,j=0}^{n}\frac{(-1)^{k+j}}{(k-1)!(j-1)!}\frac{\partial\varphi_j}{\partial v_{n+1}}\frac{\partial}{\partial\varphi_0}\left(\frac{\partial\varphi_k}{\partial v_{n+1}}\right).
\end{aligned}
\tag{8.93}
$$

Consider the equation (8.24) written in a concise way as follows

$$
\lambda(v) = \sum_{k=0}^{n}\frac{(-1)^k}{(k-1)!}\varphi_k\wp^{k-2}(v). \tag{8.94}
$$

Substituting (8.94) in the first equation of (8.93)

$$
\begin{aligned}
(a) &= \sum_{k=0}^{n}\frac{(-1)^k}{(k-1)!}\varphi_k\left[2\pi i\left(D_\tau\wp^{n-1-k}(v) + D_\tau\wp^{n-1-k}(v')\right)\right]\\
&\quad + \sum_{k=0}^{n}\frac{(-1)^k}{(k-1)!}\varphi_k\left[\left(\zeta(v-v')+\zeta(v')-\zeta(v)\right)\left(\wp^{n-k}(v')-\wp^{n-k}(v')\right)\right].
\end{aligned}
\tag{8.95}
$$

Dividing the expression once more

$$
\begin{aligned}
(a)_1 &= \sum_{k=2}^{n}\frac{(-1)^k}{(k-1)!}\varphi_k\left[2\pi i\left(D_\tau\wp^{n-1-k}(v) + D_\tau\wp^{n-1-k}(v')\right)\right]\\
&\quad + \sum_{k=2}^{n}\frac{(-1)^k}{(k-1)!}\varphi_k\left[\left(\zeta(v-v')+\zeta(v')-\zeta(v)\right)\left(\wp^{n-k}(v')-\wp^{n-k}(v')\right)\right],\\
(a)_2 &:= \varphi_1\left[2\pi i\left(D_\tau\wp^{-1}(v) + D_\tau\wp^{-1}(v')\right)\right]\\
&\quad + \varphi_1\left[\left(\zeta(v-v')+\zeta(v')-\zeta(v)\right)\left(\wp^{-1}(v')-\wp^{-1}(v')\right)\right].
\end{aligned}
\tag{8.96}
$$

Expanding the left-hand side of (8.74), we get

$$
\begin{aligned}
&\sum_{k,j=0}^{n}\frac{(-1)^{k+j}}{(k-1)!(j-1)!}\frac{\partial M^*(d\varphi_i,d\varphi_j)}{\partial\varphi_0}\wp(v)^{(k-2)}\wp(v')^{(j-2)}\\
&= \sum_{k,j=0}^{n}\frac{\partial M^*(d\varphi_i,d\varphi_j)}{\partial\varphi_0}\frac{1}{v^k(v')^j} + \text{Other terms},
\end{aligned}
\tag{8.97}
$$

where "Other terms" in the equation (8.97) means positive powers of either v or v'. For convenience, define

$$
\begin{aligned}
(1) &: = 2\pi i\left(D_\tau\wp^{(k-2)}(v) + D_\tau\wp^{(j-2)}(v')\right)\\
&\quad + \left(\zeta(v-v')+\zeta(v')-\zeta(v)\right)\left(\wp^{(j-1)}(v')-\wp^{(k-1)}(v)\right)
\end{aligned}
\tag{8.98}
$$

In order to better to compute (8.98), consider the analytical behaviour of the term

$$D_\tau \wp^{k-2}(v) = \partial_\tau \wp^{k-2}(v) - 2(k-2)g_1(\tau)\wp^{k-2}(v) - \frac{1}{2\pi i}\frac{\theta_1'(v,\tau)}{\theta_1(v,\tau)}\wp^{k-1}(v). \tag{8.99}$$

The term $\partial_\tau \wp^{k-2}(v)$ in (8.99) is holomorphic, therefore, it does not contribute for the Laurent tail. The term

$$2(k-2)g_1(\tau)\wp^{k-2}(v) \tag{8.100}$$

also does not contribute, because the full expression (8.99) behaves as modular form under the $SL_2(\mathbb{Z})$, but (8.100) is clear a quasi-modular form, since it contains $g_1(\tau)$. Hence, (8.100) is canceled with the Laurent tail of

$$\frac{1}{2\pi i}\frac{\theta_1'(v,\tau)}{\theta_1(v,\tau)}\wp^{k-1}(v). \tag{8.101}$$

To sum up, the analytical behaviour avior of (8.99) is essentially given by (8.101), under this consideration, and by using the equation

$$\begin{aligned} \zeta(v,\tau) &= \frac{\theta_1'(v,\tau)}{\theta_1(v,\tau)} - 4\pi i g_1(\tau) v \\ &= \frac{1}{v} + O(v^3), \end{aligned} \tag{8.102}$$

the equation (8.98) became

$$\begin{aligned} (1) &= -\zeta(v)\wp^{k-1}(v) - \zeta(v')\wp^{k-1}(v') \\ &\quad + \left(\zeta(v-v') + \zeta(v') - \zeta(v)\right)\left(\wp^{k-1}(v') - \wp^{k-1}(v)\right) + \text{Other terms} \\ &= \zeta(v-v')\left(\wp^{k-1}(v') - \wp^{k-1}(v')\right) - \zeta(v)\wp^{k-1}(v') - \zeta(v')\wp^{k-1}(v) + \text{Other terms} \\ &= \frac{1}{v-v'}\left(\frac{(-1)^k k!}{v'^{k+1}} - \frac{(-1)^k k!}{v^{k+1}}\right) - \frac{1}{v}\frac{(-1)^k k!}{v'^{k+1}} - \frac{1}{v'}\frac{(-1)^k k!}{v^{k+1}} + \text{Other terms} \\ &= (-1)^k k!\left(\frac{1}{v-v'}\frac{v^{k+1} - v'^{k+1}}{(v'v)^{k+1}}\right) - \frac{1}{v}\frac{(-1)^k k!}{v'^{k+1}} - \frac{1}{v'}\frac{(-1)^k k!}{v^{k+1}} + \text{Other terms} \\ &= (-1)^k k!\left(\sum_{j=0}^{k}\frac{v^{k-j}v'^j}{(v'v)^{k+1}}\right) - \frac{1}{v}\frac{(-1)^k k!}{v'^{k+1}} - \frac{1}{v'}\frac{(-1)^k k!}{v^{k+1}} + \text{Other terms} \\ &= (-1)^k k!\left(\sum_{j=0}^{k}\frac{1}{v^{1+j}v'^{k+1-j}}\right) - \frac{1}{v}\frac{(-1)^k k!}{v'^{k+1}} - \frac{1}{v'}\frac{(-1)^k k!}{v^{k+1}} + \text{Other terms} \\ &= (-1)^k k!\left(\sum_{j=1}^{k}\frac{1}{v^{1+j}v'^{k+1-j}}\right) + \text{Other terms} \\ &= (-1)^k k!\left(\sum_{j=2}^{k+1}\frac{1}{v^{j}v'^{k+2-j}}\right) + \text{Other terms.} \end{aligned} \tag{8.103}$$

Substituting (8.103) in right-hand side of (8.95)

$$
\begin{aligned}
&\sum_{k=0}^{n}\frac{(-1)^k}{(k-1)!}\varphi_k\left[2\pi i\left(D_\tau\wp^{k-2}(v)+D_\tau\wp^{k-1}(v')\right)\right]\\
&+\sum_{k=0}^{n}\frac{(-1)^k}{(k-1)!}\varphi_k\left[\left(\zeta(v-v')+\zeta(v')-\zeta(v)\right)\left(\wp^{k-1}(v')-\wp^{k-1}(v')\right)\right]\\
&=\sum_{k=0}^{n}\sum_{j=2}^{k+1}\frac{(k)\,\varphi_k}{v^j v'^{k+2-j}}+\text{Other terms}\\
&=\sum_{k=0}^{n}\sum_{j=2}^{n+1}\frac{(k+j)\,\varphi_{k+j}}{v^j v'^{k+2}}+\text{Other terms}\\
&=\sum_{k=2}^{n+2}\sum_{j=2}^{n+1}\frac{(k+j-2)\,\varphi_{k+j-2}}{v^j v'^{k}}+\text{Other terms.}
\end{aligned}
\tag{8.104}
$$

Computing the second expression of (8.96)

(8.105)

$$
\begin{aligned}
(a)_2 &:= \varphi_1\left[2\pi i\left(D_\tau\wp^{-1}(v)+D_\tau\wp^{-1}(v')\right)\right]\\
&+\varphi_1\left[\left(\zeta(v-v')+\zeta(v')-\zeta(v)\right)\left(\wp^{-1\prime}(v')-\wp^{-1\prime}(v)\right)\right]\\
&=\varphi_1\left[\zeta(v)\left(-\wp(v)+\wp(v+(n+1)v_{n+1})\right)+\zeta(v')\left(-\wp(v')+\wp(v'+(n+1)v_{n+1})\right)\right]\\
&+\varphi_1\left(\zeta(v-v')+\zeta(v')-\zeta(v)\right)\left(\wp(v)-\wp(v+(n+1)v_{n+1})\right)\\
&+\varphi_1\left(\zeta(v-v')+\zeta(v')-\zeta(v)\right)\left(-\wp(v')+\wp(v'+(n+1)v_{n+1})\right)+\text{Other terms}\\
&=\varphi_1\zeta(v-v')\left(\wp(v)-\wp(v+(n+1)v_{n+1})-\wp(v')+\wp(v'+(n+1)v_{n+1})\right)\\
&+\varphi_1\left[\zeta(v)\left(-\wp(v')+\wp(v'+(n+1)v_{n+1})\right)+\zeta(v')\left(-\wp(v)+\wp(v+(n+1)v_{n+1})\right)\right]+\text{Other terms}\\
&=\varphi_1\zeta(v-v')\left(-\wp(v+(n+1)v_{n+1})+\wp(v'+(n+1)v_{n+1})\right)\\
&+\varphi_1\left[\zeta(v)\wp(v'+(n+1)v_{n+1})+\zeta(v')\wp(v+(n+1)v_{n+1})\right]+\widetilde{(a)_2}+\text{Other terms,}
\end{aligned}
$$

where $\widetilde{(a)_2}$ are terms that were already computed in (8.104). To compute the second expression in (8.93), consider

(8.106)

$$
\frac{\partial\lambda(v)}{\partial v_{n+1}}=\sum_{k=0}^{n}\frac{(-1)^k}{(k-1)!}\frac{\partial\varphi_k}{\partial v_{n+1}}\wp^{(k-2)}(v)+(n+1)\varphi_1\left[\wp(v+(n+1)v_{n+1})-\wp((n+1)v_{n+1})\right],
$$

and

(8.107)

$$\frac{\partial}{\partial \varphi_0}\left(\frac{\partial \lambda(v)}{\partial v_{n+1}}\right) = \frac{\partial}{\partial \varphi_0}\left(\frac{\partial \varphi_1}{\partial v_{n+1}}\right)[\zeta(v) - \zeta(v+(n+1)v_{n+1}) + \zeta((n+1)v_{n+1})] + \frac{\partial}{\partial \varphi_0}\left(\frac{\partial \varphi_0}{\partial v_{n+1}}\right)$$
$$= n\,[\zeta(v) - \zeta(v+(n+1)v_{n+1}) + \zeta((n+1)v_{n+1})] - n\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}$$
$$= n\,[\zeta(v) - \zeta(v+(n+1)v_{n+1})].$$

Substituting (8.106) and (8.107) in the the second expression in (8.93)

$$\begin{aligned}(b) :=& \frac{1}{n(n+1)}\frac{\partial}{\partial \varphi_0}\left(\frac{\partial \lambda(p)}{\partial v_{n+1}}\right)\frac{\partial \lambda(p')}{\partial v_{n+1}} + \frac{1}{n(n+1)}\frac{\partial \lambda(p)}{\partial v_{n+1}}\frac{\partial}{\partial \varphi_0}\left(\frac{\partial \lambda(p')}{\partial v_{n+1}}\right)\\ &- \frac{1}{n(n+1)}\sum_{k,j=0}^{n}\frac{(-1)^{k+j}}{(k-1)!(j-1)!}\frac{\partial}{\partial \varphi_0}\left(\frac{\partial \varphi_j}{\partial v_{n+1}}\right)\frac{\partial \varphi_k}{\partial v_{n+1}}\\ &- \frac{1}{n(n+1)}\sum_{k,j=0}^{n}\frac{(-1)^{k+j}}{(k-1)!(j-1)!}\frac{\partial \varphi_j}{\partial v_{n+1}}\frac{\partial}{\partial \varphi_0}\left(\frac{\partial \varphi_k}{\partial v_{n+1}}\right)\\ &= \varphi_1\left[\zeta(v') - \zeta(v'+(n+1)v_{n+1})\right][\wp(v+(n+1)v_{n+1}) - \wp((n+1)v_{n+1})]\\ &+ \varphi_1[\zeta(v) - \zeta(v+(n+1)v_{n+1})]\left[\wp(v'+(n+1)v_{n+1}) - \wp((n+1)v_{n+1})\right].\end{aligned} \tag{8.108}$$

Subtracting (8.105) and (8.108)

(8.109)

$$\begin{aligned}(a)_2 - (b) =& -\varphi_1\left[\zeta(v') - \zeta(v'+(n+1)v_{n+1})\right][\wp(v+(n+1)v_{n+1}) - \wp((n+1)v_{n+1})]\\ &- \varphi_1[\zeta(v) - \zeta(v+(n+1)v_{n+1})]\left[\wp(v'+(n+1)v_{n+1}) - \wp((n+1)v_{n+1})\right]\\ &+ \varphi_1\zeta(v-v')\left(-\wp(v+(n+1)v_{n+1}) + \wp(v'+(n+1)v_{n+1})\right)\\ &+ \varphi_1\left[\zeta(v)\wp(v'+(n+1)v_{n+1}) + \zeta(v')\wp(v+(n+1)v_{n+1})\right] + \widetilde{(a)}_2 + \text{Other terms}\\ &= \varphi_1\wp((n+1)v_{n+1})\left[\zeta(v') - \zeta(v'+(n+1)v_{n+1})\right] + \varphi_1\wp((n+1)v_{n+1})[\zeta(v) - \zeta(v+(n+1)v_{n+1})]\\ &+ \varphi_1\left[\zeta(v-v')\right]\left(\wp(v+(n+1)v_{n+1}) + \wp(v'+(n+1)v_{n+1})\right)\\ &+ \varphi_1\left[\zeta(v'+(n+1)v_{n+1})\right](\wp(v+(n+1)v_{n+1}))\\ &+ \varphi_1[\zeta(v+(n+1)v_{n+1})]\left(\wp(v'+(n+1)v_{n+1})\right)\\ &+ \widetilde{(a)}_2 + \text{Other terms}\\ &= \varphi_1\wp((n+1)v_{n+1})\left[\zeta(v') - \zeta(v'+(n+1)v_{n+1})\right] + \varphi_1\wp((n+1)v_{n+1})[\zeta(v) - \zeta(v+(n+1)v_{n+1})]\\ &+ \widetilde{(a)}_2 + \text{Other terms.}\end{aligned}$$

Summing (8.109) and (8.104), we have

$$\begin{aligned}
&\sum_{k,j=0}^{n} \frac{(-1)^{k+j}}{(k-1)!(j-1)!} \frac{\partial M^*(d\varphi_i, d\varphi_j)}{\partial \varphi_0} \wp(v)^{(k-2)} \wp(v')^{(j-2)} \\
&= \sum_{k,j=0}^{n} \frac{\partial M^*(d\varphi_i, d\varphi_j)}{\partial \varphi_0} \frac{1}{v^k (v')^j} + \text{Other terms}, \\
&= \sum_{k=2}^{n+2} \sum_{j=2}^{n+1} \frac{(k+j-2)\,\varphi_{k+j-2}}{v^j {v'}^k} \\
&+ \varphi_1 \wp((n+1)v_{n+1}) \left[\zeta(v') - \zeta(v' + (n+1)v_{n+1})\right] \\
&+ \varphi_1 \wp((n+1)v_{n+1}) \left[\zeta(v) - \zeta(v + (n+1)v_{n+1})\right] \\
&+ \text{Other terms}.
\end{aligned}$$

Hence, we get the desired result.

□

COROLLARY 8.5.4.1. Let $\eta^*(d\varphi_i, d\varphi_j)$ be defined in (8.73), then its coefficients can be obtained by the formula

$$\begin{aligned}
&\eta^*(d\varphi_i, d\varphi_j) = (i+j-2)\varphi_{i+j-2}, \quad i, j \neq 0 \\
&\eta^*(d\varphi_i, d\varphi_0) = 0, \quad i \neq 0, \quad i \neq 1, \\
&\eta^*(d\varphi_1, d\varphi_j) = 0. \quad j \neq 0, \\
&\eta^*(d\varphi_1, d\varphi_0) = \frac{\partial^2 \log(\theta_1(n+1)v_{n+1})}{\partial v_{n+1}^2} \varphi_1.
\end{aligned} \tag{8.110}$$

8.6. Flat coordinates of η

This section is dedicated to construct the flat coordinates of η and its relationship with the invariant coordinates $\varphi_0, \varphi_1, .., \varphi_n, v_{n+1}, \tau$. Our strategy will be an adaptation of the work done in [**28**]. See section 4.6 for the summary of this approach, and section 5.5 to see this techniques applied in ordinary Jacobi group. The flatness of the Saito metric η is proved in the Theorem 8.6.12.

Let $t^1, t^2, .., t^n$ be given by the following generating function

$$v(z) = \frac{-1}{n} \left(t^n z + t^{n-1} z^2 + ... + t^2 z^{n-1} + t^1 z^n + O(z^{n+1})\right), \tag{8.111}$$

be defined by the following condition

$$\lambda(v) = \frac{1}{z^n}.$$

Moreover,

$$t^0 = \varphi_0 - \frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})} \varphi_1 + 4\pi i g_1(\tau) \varphi_2. \tag{8.112}$$

Lemma 8.6.1. The following identity holds

$$\frac{n}{n+1-\alpha}\operatorname*{res}_{p=\infty}\left(\lambda^{\frac{n+1-\alpha}{n}}(p)dp\right) = -\operatorname*{res}_{\lambda=\infty}\left(v(z)\lambda^{\frac{1-\alpha}{n}}d\lambda\right) \tag{8.113}$$

Proof. Consider the integration by parts

$$\frac{n}{n+1-\alpha}\int\left(\lambda^{\frac{n+1-\alpha}{n}}(p)dp\right) = p\lambda^{\frac{n+1-\alpha}{n}} - \int p\lambda^{\frac{1-\alpha}{n}}d\lambda \tag{8.114}$$

Lemma proved.

□

Lemma 8.6.2. The functions $t^1, t^2, .., t^n$ defined in (8.111) can be obtained by the formula

$$t^\alpha = -\operatorname*{res}_{\lambda=\infty}\left(v(z)\lambda^{\frac{1-\alpha}{n}}d\lambda\right). \tag{8.115}$$

Proof. Let

$$z = \left(\frac{1}{\lambda}\right)^{\frac{1}{n}},$$

then,

$$\begin{aligned}
v(z)\lambda^{\frac{1-\alpha}{n}}d\lambda &= \left(\frac{1}{n}\right)\left(t^n z + t^{n-1}z^2 + ... + t^2 z^{n-1} + t^1 z^n + O(z^{n+1})\right) z^{\alpha-1} n z^{-n-1} dz \\
&= \left(\sum_{\beta=1}^{n} t^\beta z^{n+1-\beta} + O(z^{n+1})\right) z^{\alpha-n-2}dz \\
&= \left(\sum_{\beta=1}^{n} t^\beta z^{\alpha-\beta-1} + O(z^{\alpha-1})\right) dz.
\end{aligned}$$

Hence, the residue is different from 0, when $\alpha = \beta$, resulting in this way the desired result.

□

Corollary 8.6.2.1. The coordinate t^n can be written in terms of the coordinates $\varphi_0, \varphi_1, \varphi_2, .., \varphi_{n+1}$ as

$$t^n = n\left(\varphi_n\right)^{\frac{1}{n}}. \tag{8.116}$$

Proof.

$$\begin{aligned}
t^n &= n\operatorname*{res}_{v=0}\lambda^{\frac{1}{n}}(v)dv \\
&= n\operatorname*{res}_{v=0}\left(\frac{\varphi_n}{v^n} + \frac{\varphi_{n-1}}{v^{n-1}} + .. + \frac{\varphi_2}{v^2} + \frac{\varphi_1}{v} + O(1)\right)^{\frac{1}{n}} dv \\
&= n\operatorname*{res}_{v=0}\frac{(\varphi_n)^{\frac{1}{n}}}{v}\left(1 + O(v)\right)^{\frac{1}{n}} dv = n\left(\varphi_n\right)^{\frac{1}{n}}.
\end{aligned}$$

□

LEMMA 8.6.3. Let $t^1, t^2, .., t^n$ be defined in (8.111), then

$$t^\alpha = \frac{n}{n+1-\alpha} (\varphi_n)^{\frac{n+1-\alpha}{n}} (1+\Phi_{n-\alpha})^{\frac{n+1-\alpha}{n}}, \tag{8.117}$$

where

$$\begin{aligned} (1+\Phi_i)^{\frac{n+1-\alpha}{n}} &= \sum_{d=0}^{\infty} \binom{\frac{n+1-\alpha}{n}}{d} \Phi_i^d, \\ \Phi_i^d &= \sum_{i_1+i_2+..+i_d=i} \frac{\varphi_{(n-i_1)}}{\varphi_n} .... \frac{\varphi_{(n-i_d)}}{\varphi_n}. \end{aligned} \tag{8.118}$$

PROOF.

$$\begin{aligned} t^\alpha &= \frac{n}{n+1-\alpha} \operatorname*{res}_{v=0} \left(\lambda^{\frac{n+1-\alpha}{n}}(v) dv \right) \\ &= \frac{n}{n+1-\alpha} \operatorname*{res}_{v=0} \left(\frac{\varphi_n}{v^n} + \frac{\varphi_{n-1}}{v^{n-1}} + .. + \frac{\varphi_2}{v^2} + \frac{\varphi_1}{v} + O(1) \right)^{\frac{n+1-\alpha}{n}} dv \\ &= \frac{n}{n+1-\alpha} \operatorname*{res}_{v=0} \left(\frac{\varphi_n}{v^n} \right)^{\frac{n+1-\alpha}{n}} \left(1 + \frac{\varphi_{n-1}}{\varphi_n} v + \frac{\varphi_{n-2}}{\varphi_n} v^2 + .. + \frac{\varphi_2}{\varphi_n} v^{n-2} + \frac{\varphi_1}{\varphi_n} v^{n-1} + O\left(v^{n+1}\right) \right)^{\frac{n+1-\alpha}{n}} dv \\ &= \frac{n}{n+1-\alpha} \operatorname*{res}_{v=0} \left(\frac{\varphi_n}{v^n} \right)^{\frac{n+1-\alpha}{n}} \sum_{d=0}^{\infty} \binom{\frac{n+1-\alpha}{n}}{d} \left(1 + \frac{\varphi_{n-1}}{\varphi_n} v + .. + \frac{\varphi_1}{\varphi_n} v^{n-1} + O\left(v^{n+1}\right) \right)^d dv \\ &= \frac{n}{n+1-\alpha} \operatorname*{res}_{v=0} \left(\frac{\varphi_n}{v^n} \right)^{\frac{n+1-\alpha}{n}} \sum_{d=0}^{\infty} \binom{\frac{n+1-\alpha}{n}}{d} \sum_{j_1+..+j_n=d} \frac{d!}{j_1! j_2! .. j_n!} \prod_{i=1}^{n-1} \left(\frac{\varphi_{n-i} v^i}{\varphi_n} \right)^{j_i} (O(v^n))^{j_n} dv \\ &= \frac{n}{n+1-\alpha} \operatorname*{res}_{v=0} (\varphi_n)^{\frac{n+1-\alpha}{n}} \sum_{d=0}^{\infty} \binom{\frac{n+1-\alpha}{n}}{d} \sum_{j_1+..+j_n=d} \frac{d!}{j_1! j_2! .. j_n!} \prod_{i=1}^{n-1} \left(\frac{\varphi_{n-i}}{\varphi_n} \right)^{j_i} v^{\left(\sum_{i=1}^{n-1} i j_i - n - 1 + \alpha\right)} dv \\ &\quad + O(1) \\ &= \frac{n}{n+1-\alpha} (\varphi_n)^{\frac{n+1-\alpha}{n}} \sum_{d=0}^{\infty} \binom{\frac{n+1-\alpha}{n}}{d} \sum_{\substack{j_1+..+j_n=d \\ j_1+2j_2+3j_3+..+(n-1)j_{n-1}=n-\alpha}} \frac{d!}{j_1! j_2! .. j_n!} \prod_{i=1}^{n-1} \left(\frac{\varphi_{n-i}}{\varphi_n} \right)^{j_i} \\ &= \frac{n}{n+1-\alpha} (\varphi_n)^{\frac{n+1-\alpha}{n+1}} \sum_{d=0}^{\infty} \binom{\frac{n+1-\alpha}{n}}{d} \sum_{i_1+..+i_d=n-\alpha} \frac{\varphi_{(n-i_1)}}{\varphi_{n+1}} .... \frac{\varphi_{(n-i_d)}}{\varphi_n} \\ &= \frac{n}{n+1-\alpha} (\varphi_n)^{\frac{n+1-\alpha}{n}} \sum_{d=0}^{\infty} \binom{\frac{n+1-\alpha}{n}}{d} \Phi_{n-\alpha}^d \\ &= \frac{n}{n+1-\alpha} (\varphi_n)^{\frac{n+1-\alpha}{n}} (1+\Phi_{n-\alpha})^{\frac{n+1-\alpha}{n}}. \end{aligned}$$

□

COROLLARY 8.6.3.1. The coordinate t^1 can be written in terms of the coordinates $\varphi_0, \varphi_1, \varphi_2, .., \varphi_n$ as

$$t^1 = \varphi_1 \tag{8.119}$$

Proof. Using the relation (8.117) for $\alpha = 1$

$$t^1 = \varphi_n \left(1 + \Phi_{n-1}\right) = \varphi_n \sum_{i_1 = n-1} \frac{\varphi_{n-i_1}}{\varphi_n} = \varphi_1. \tag{8.120}$$

□

Lemma 8.6.4. Let $\varphi_0, \varphi_1, \varphi_2, .., \varphi_n$ and $\lambda(v)$ be defined in (8.24), then

$$k\varphi_k = \operatorname*{res}_{v=0} k\lambda v^{k-1} dv. \tag{8.121}$$

Proof.

$$\begin{aligned}\operatorname*{res}_{v=0} k\lambda v^{k-1} dv &= \operatorname*{res}_{v=0} k\left(\frac{\varphi_n}{v^n} + \frac{\varphi_{n-1}}{v^{n-1}} + .. + \frac{\varphi_k}{v^k} + .. + \frac{\varphi_2}{v^2} + \frac{\varphi_1}{v} + O(1)\right) v^{k-1} dv \\ &= k\varphi_k.\end{aligned}$$

□

Lemma 8.6.5. Let $\varphi_0, \varphi_1, \varphi_2, .., \varphi_n$ and $\lambda(v)$ be defined in (8.24), then

$$k\varphi_k = -\operatorname*{res}_{\lambda=\infty} v^k d\lambda. \tag{8.122}$$

Proof. Using formula (8.122) and integration by parts

$$k\varphi_k = \operatorname*{res}_{v=0} k\lambda v^{k-1} dv = -\operatorname*{res}_{\lambda=\infty} v^k d\lambda.$$

□

Lemma 8.6.6. Let $\varphi_0, , \varphi_1, \varphi_2, .., \varphi_n$, $\lambda(v)$ be defined in (8.24) and $(t^1, .., t^n)$ be defined in (8.111) , then

$$k\varphi_k = \frac{(-1)^k}{n^{k-1}} T_n^k, \tag{8.123}$$

where

$$T_n^k = \sum_{i_1 + .. + i_k = n} t^{(n+1-i_1)} ... t^{(n+1-i_k)}. \tag{8.124}$$

Proof. Let $z := \left(\frac{1}{\lambda}\right)^{\frac{1}{n}}$, then by using equation (8.122):

$$
\begin{aligned}
-\operatorname*{res}_{\lambda=\infty} v^k d\lambda &= \operatorname*{res}_{z=0} \frac{nv^k(z)dz}{z^{n+1}} \\
&= \operatorname*{res}_{z=0} \frac{(-1)^k}{n^{k-1}} \left(t^n z + t^{n-1}z^2 + .. + t^2 z^{n-1} + O(z^{n+1})\right)^k \frac{dz}{z^{n+1}} \\
&= \operatorname*{res}_{z=0} \frac{(-1)^k}{n^{k-1}} \sum_{j_1+j_2+..+j_n+j_{n+1}=k} (t^n z)^{j_1} \left(t^{n-1}z^2\right)^{j_2} .. \left(t^2 z^{n-1}\right)^{j_n} \left(O(z^{n+1})\right)^{j_{n+2}} \frac{dz}{z^{n+1}} \\
&= \frac{(-1)^k}{n^{k-1}} \sum_{\substack{j_1+j_2+..+j_n=k \\ j_1+2j_2+3j_3+..+(n)j_n=n}} \frac{k!}{j_1!j_2!..j_n!} (t^n)^{j_1} \left(t^{n-1}\right)^{j_2} .. \left(t^2\right)^{j_n} \\
&= \frac{(-1)^k}{n^{k-1}} \sum_{i_1+..i_k=n} t^{(n+1-i_1)}...t^{(n+1-i_k)} \\
&= \frac{(-1)^k}{n^{k-1}} T_n^k.
\end{aligned}
$$

□

LEMMA 8.6.7. Let T_n^k be defined in (8.124), then

$$
\frac{\partial T_n^k}{\partial t^\alpha} = kT_{\alpha-1}^{k-1}. \tag{8.125}
$$

PROOF.

$$
\begin{aligned}
\frac{\partial T_n^k}{\partial t^\alpha} &= \frac{\partial}{\partial t^\alpha}\left(\sum_{i_1+..i_k=n} t^{(n+1-i_1)}...t^{(n+1-i_k)}\right) \\
&= \sum_{i_1+..i_k=n} k\delta_{n+1-i_k,\alpha} t^{(n+1-i_1)}...t^{(n+1-i_{k-1})} \\
&= k \sum_{i_1+..i_{k-1}=\alpha-1} t^{(n+1-i_1)}...t^{(n+1-i_{k-1})} \\
&= kT_{\alpha-1}^{k-1}.
\end{aligned}
$$

□

At this stage, we are able to compute the coefficients of $\eta(dt^\alpha, dt^\beta)$.

THEOREM 8.6.8. Let $(t^1, .., t^n)$ defined in (8.111), and η^* defined in (8.73). Then,

$$
\eta^*(dt^\alpha, dt^{n+3-\beta}) = n\delta_{\alpha\beta}. \tag{8.126}
$$

PROOF. If $i, j \neq 0$

$$
\eta^*(d\varphi_i, d\varphi_j) = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \frac{\partial \varphi_i}{\partial t^\alpha}\frac{\partial \varphi_j}{\partial t^\beta}\eta^*(dt^\alpha, dt^\beta) \tag{8.127}
$$

Using (8.125) and (8.123), we get

$$
\frac{\partial \varphi_k}{\partial t^\alpha} = \frac{(-1)^k}{n^{k-1}} kT_{\alpha-1}^{k-1} \tag{8.128}
$$

$$\sum_{\alpha=1}^{n}\frac{\partial\varphi_i}{\partial t^\alpha}\frac{\partial\varphi_{n+3-j}}{\partial t^{n+3-\alpha}}=\sum_{\alpha=1}^{n}\frac{(-1)^{i-j+n+1}}{n^{i-j+n+1}}T^{i-1}_{\alpha-1}T^{n+2-j}_{n+2-\alpha} \tag{8.129}$$

Using the second of the equation (4.46) in (8.129)

$$\begin{aligned}\sum_{\alpha=1}^{n}\frac{\partial\varphi_i}{\partial t^\alpha}\frac{\partial\varphi_{n+3-j}}{\partial t^{n+3-\alpha}}&=\frac{T^{n+1+i-j}_{n+1}}{n}\\&=\frac{(n+1+i-j)}{n}\varphi_{(n+1+i-j)}\end{aligned} \tag{8.130}$$

Note that the following identity holds

$$\sum_{\alpha=2}^{n+1}\frac{\partial\varphi_i}{\partial t^\alpha}\frac{\partial\varphi_{n+3-j}}{\partial t^{n+3-\alpha}}=\sum_{\alpha=2}^{n+1}\sum_{\alpha=2}^{n+1}\frac{\partial\varphi_i}{\partial t^\alpha}\frac{\partial\varphi_{n+3-j}}{\partial t^{n+3-\beta}}\delta_{\alpha\beta}. \tag{8.131}$$

On another hand, using the first equation of (8.92), we have

$$\begin{aligned}\eta^*(d\varphi_i,d\varphi_{n+3-j})&=(n+1+i-j)\varphi_{n+1+i-j}\\&=n\sum_{\alpha=2}^{n+1}\sum_{\alpha=2}^{n+1}\frac{\partial\varphi_i}{\partial t^\alpha}\frac{\partial\varphi_{n+3-j}}{\partial t^{n+3-\beta}}\delta_{\alpha\beta}\\&=\sum_{\alpha=2}^{n+1}\sum_{\alpha=2}^{n+1}\frac{\partial\varphi_i}{\partial t^\alpha}\frac{\partial\varphi_{n+3-j}}{\partial t^{n+3-\beta}}\eta^*(dt^\alpha,dt^{n+3-\beta})\end{aligned}$$

Then, we obtain

$$\eta^*(dt^\alpha,dt^{n+3-\beta})=n\delta_{\alpha\beta}.$$

□

Lemma 8.6.9. Let $(t^1,..,t^n)$ be defined in (8.111), and η^* be defined in (8.73). Then,

$$\begin{aligned}\eta^*(dt^i,d\tau)&=-2\pi i\delta_{i0},\\\eta^*(dt^i,dv_{n+1})&=-\frac{\delta_{i1}}{n+1}.\end{aligned} \tag{8.132}$$

Proof. Using corollary 8.5.2.1 and lemma 8.5.3, we have

$$\eta^*(dt^i,d\tau)=\frac{\partial t^i}{\partial\varphi_j}\eta^*(d\varphi_j,d\tau)=-2\pi i\delta_{i0}.$$

In addition, if $i\neq 0$

$$\begin{aligned}\eta^*(dt^i,dv_{n+1})&=\frac{\partial t^i}{\partial\varphi_j}\eta^*(d\varphi_j,dv_{n+1})=0,\quad i\neq 1,\\\eta^*(dt^1,dv_{n+1})&=\eta^*(d\varphi_1,dv_{n+1})=-\frac{1}{n+1}.\end{aligned}$$

Computing dt^0 with respect the variables φ_i by using (8.112),

$$dt^0 = d\varphi_0 - (n+1)\frac{\partial^2 \log(\theta_1((n+1)v_{n+1})}{\partial v_{n+1}^2}\varphi_1 dv_{n+1} - \frac{\partial \log(\theta_1((n+1)v_{n+1})}{\partial v_{n+1}}d\varphi_1 \\ + 4\pi i g_1'(\tau)\varphi_2 d\tau + 4\pi i g_1(\tau)d\varphi_2. \tag{8.133}$$

Hence,

$$\begin{aligned}\eta^*(dt^0, dv_{n+1}) &= \eta^*(d\varphi_0, dv_{n+1}) - \frac{\partial \log(\theta_1((n+1)v_{n+1})}{\partial v_{n+1}}\eta^*(d\varphi_1, dv_{n+1}) \\ &= 0.\end{aligned}$$

□

LEMMA 8.6.10. Let t^0 be defined in (8.112), and η^* be defined in (8.73). Then,

$$\eta^*(dt^0, dt^\alpha) = 0, \quad \alpha \neq 0. \tag{8.134}$$

PROOF. Using the definition of η^* in equation (8.73), formula (8.92), and (8.75) If $i > 1$,

$$\begin{aligned}\eta^*(dt^0, d\varphi_i) &= \eta^*(d\varphi_0, d\varphi_i) + 4\pi i g_1(\tau)\eta^*(d\varphi_2, d\varphi_i) + 4\pi i g_1'(\tau)\eta^*(d\tau, d\varphi_i) \\ &= \eta^*(d\varphi_0, d\varphi_i) + 4\pi i g_1(\tau)\eta^*(d\varphi_2, d\varphi_i) \\ &= \tilde{\eta}^*(d\varphi_0, d\varphi_i) - 4\pi i g_1(\tau)k_i\varphi_i + 4\pi i g_1(\tau)k_i\varphi_i = 0 \\ &= -4\pi i g_1(\tau)k_i\varphi_i + 4\pi i g_1(\tau)k_i\varphi_i = 0\end{aligned}$$

Then, if $\alpha > 1$

$$\eta^*(dt^0, dt^\alpha) = \sum_{\alpha=2}^{n}\frac{\partial t^\alpha}{\partial \varphi_i}\eta^*(dt^0, d\varphi_i) = 0.$$

Computing $\eta^*(dt^0, dt^1)$ by using (8.133)

$$\begin{aligned}\eta^*(dt^0, dt^1) &= \eta^*(d\varphi_0, d\varphi_1) - (n+1)\frac{\partial^2 \log(\theta_1((n+1)v_{n+1})}{\partial v_{n+1}^2}\varphi_1\eta^*(dv_{n+1}, d\varphi_1) \\ &= 0.\end{aligned}$$

□

Instead of considering the coefficients of the metric η^*, let us investigate the flatness of its inverse η in order to make the computations shorter.

LEMMA 8.6.11. The metric

$$\sum_{\alpha=2}^{n}\eta(dt^\alpha, dt^{n+3-\alpha})dt^\alpha dt^{n+3-\alpha} - 2(n+1)dt^1 dv_{n+1} - \frac{1}{\pi i}dt^0 d\tau \tag{8.135}$$

is invariant under the second action of (8.1).

PROOF. Under the second action of (8.1), we have that

$$\sum_{\alpha=2}^{n} \eta(dt^{\alpha}, dt^{n+3-\alpha}) dt^{\alpha} dt^{n+3-\alpha}$$

are invariant under the second action of (8.1), because the relationship between t^i and φ_i be given by (8.117), and the fact that the Jacobi forms $\{\varphi_i\}$ are invariant under the second action of (8.1). t^0 and v_{n+1} have the following transformation law

$$\begin{aligned} t^0 &\mapsto t^0 - 2\pi i(n+1)\lambda_{n+1} t^1 \\ v_{n+1} &\mapsto v_{n+1} + \lambda_{n+1}\tau + \mu_{n+1} \end{aligned} \tag{8.136}$$

Hence, its differentials are

$$\begin{aligned} dt^0 &\mapsto dt^0 - 2\pi i(n+1)\lambda_{n+1} dt^1 \\ dv_{n+1} &\mapsto dv_{n+1} + \lambda_{n+1} d\tau \end{aligned} \tag{8.137}$$

Substituting (8.137) in (8.135) we get the desired result. □

THEOREM 8.6.12. Let $(t^0, t^1, t^2, .., t^n)$ defined in (8.111), and η^* defined in (8.73). Then,

$$\begin{aligned} &\eta^*(dt^{\alpha}, dt^{n+3-\beta}) = -(n+1)\delta_{\alpha\beta}, \quad 2 \leq \alpha, \beta \leq n \\ &\eta^*(dt^1, dt^{\alpha}) = 0, \\ &\eta^*(dt^0, dt^{\alpha}) = 0, \\ &\eta^*(dt^i, d\tau) = -2\pi i \delta_{i0}, \\ &\eta^*(dt^i, dv_{n+1}) = -\frac{\delta_{i1}}{n+1}. \end{aligned} \tag{8.138}$$

Moreover, the coordinates $t^0, t^1, t^2, .., t^n, v_{n+1}, \tau$ are the flat coordinates of η^*.

PROOF. The theorem is already proved for $\alpha, \beta \in \{2, .., n\}$ in theorem 8.6.8, and for the rest in the lemma 8.6.10 and 8.6.9. The only missing part is to prove

$$\eta^*(dt^0, dt^0) = 0. \tag{8.139}$$

Recall that from corollary 8.4.7.2, the metric η^* is invariant under the second action of (8.1). Moreover, the same statement is valid for (8.135), because of lemma 8.6.11. However, the tensor $dt^0 \otimes dt^0$ have a non-trivial transformation law under this action. Hence, if the coefficient of the component $dt^0 \otimes dt^0$ is different from 0, we have a contradiction with corollary 8.4.7.2.

□

COROLLARY 8.6.12.1. The metric $\eta^*(d\varphi_i, d\varphi_j) := \frac{\partial g^*(d\varphi_i, d\varphi_j)}{\partial \varphi_0}$ is triangular, and non degenerate.

DEFINITION 8.6.1. Let $\eta^{*} = \eta^{\alpha\beta}\frac{\partial}{\partial t^{\alpha}} \otimes \frac{\partial}{\partial t^{\beta}}$ defined in (8.73). The metric defined by

$$\eta = \eta_{\alpha\beta} dt^{\alpha} \otimes dt^{\beta} \tag{8.140}$$

is denoted by η.

8.7. The extended ring of Jacobi forms

This section generalise the results of section 5.6 and lemma 6.3.3. The flat coordinates of the Saito metric η of the orbit space of $\mathscr{J}(\tilde{A}_n)$ does not live in the orbit space of $\mathscr{J}(\tilde{A}_n)$, but live in a suitable covering of this orbit space. The main goal of this section is in describing this covering as the space such that the ring of functions is a suitable extension of the ring of Jacobi forms.

LEMMA 8.7.1. The coordinates $(t^0, t^1, t^2, .., v_{n+1}, \tau)$ defined on (8.111) have the following transformation laws under the action of the group $\mathscr{J}(\tilde{A}_n)$: They transform as follows under the second action of (8.1):

$$\begin{aligned}
&t^0 \mapsto t^0 - 2\pi i(n+1)\lambda_{n+1} t^1\\
&t^{\alpha} \mapsto t^{\alpha}, \quad \alpha \neq 0\\
&v_{n+1} \mapsto v_{n+1} + \mu_{n+1} + \lambda_{n+1}\tau\\
&\tau \mapsto \tau
\end{aligned} \tag{8.141}$$

Moreover, they transform as follows under the third action (8.1)

$$\begin{aligned}
&t^0 \mapsto t^0 + \frac{2c\sum_{\alpha,\beta\neq 0,\tau} \eta_{\alpha\beta} t^{\alpha} t^{\beta}}{c\tau + d}\\
&t^{\alpha} \mapsto \frac{t^{\alpha}}{c\tau + d}, \quad \alpha \neq 0\\
&v_{n+1} \mapsto \frac{v_{n+1}}{c\tau + d}\\
&\tau \mapsto \frac{a\tau + b}{c\tau + d}
\end{aligned} \tag{8.142}$$

PROOF. Note that the term Φ_i^d equation (8.118) has weight $+i$, then using that φ_n has weight $-n$, we have that the weight of t^{α} for $\alpha \neq 1$ must have weight -1 due to (8.117). The transformation law of t^1 follows from the transformation law of $\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}$ and $g_1(\tau)$

$$\begin{aligned}
\frac{\theta_1'((n+1)v_{n+1} + (n+1)\lambda_{n+1}\tau + \mu_{n+1}, \tau)}{\theta_1((n+1)v_{n+1} + (n+1)\lambda_{n+1}\tau + \mu_{n+1}, \tau)} &= \frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})} - 2\pi i(n+1)\lambda_{n+1},\\
\frac{\theta_1'(\frac{(n+1)v_{n+1}}{c\tau+d}, \frac{a\tau+b}{c\tau+d})}{\theta_1(\frac{(n+1)v_{n+1}}{c\tau+d}, \frac{a\tau+b)}{c\tau+d})} &= \frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})} + 2\pi i c(n+1)v_{n+1},\\
g_1(\frac{a\tau+b}{c\tau+d}) &= (c\tau+d)^2 g_1(\tau) + 2c(c\tau+d),
\end{aligned} \tag{8.143}$$

and by using equation (8.123) for $k = 2$.

$\square$

In addition , from the formula (8.118) it is clear that the multivaluedness of $(t^1, .., t^n)$ comes from $(\varphi_n)^{\frac{1}{n}}$. Therefore, the coordinates lives in a suitable covering over the orbit space of the group $\mathscr{J}(\tilde{A}_n)$. This covering is obtained by forgetting to act the Coxeter group A_n and the $SL_2(\mathbb{Z})$ action, and the translation action $v_{n+1} \mapsto v_{n+1} + \lambda_{n+1}\tau + \mu_{n+1}$ of $\mathscr{J}(\tilde{A}_n)$ on $\mathbb{C} \oplus \mathbb{C}^{n+1} \oplus \mathbb{H}$. The only remaining part of the $\mathscr{J}(\tilde{A}_n)$ action are the translations

$$v_i \mapsto v_i + \lambda_i\tau + \mu_i, \quad i \neq n+1.$$

Hence, the coordinates $(t^1, .., t^n)$ live in n-dimensional tori with fixed symplectic base of the torus homology, a fixed chamber in the tori parametrised by (v_{n+1}, τ), and with a branching divisor $Y := \{\varphi_n = 0\}$. Another way to describe this covering is using the flat coordinates of the intersection form $(u, v_0, v_1, .., v_{n+1}, \tau)$, and to fix a lattice τ, a representative of the action

$$v_{n+1} \mapsto v_{n+1} + \lambda_{n+1}\tau + \mu_{n+1}, \tag{8.144}$$

and a representative of the A_n action. Then, the desired covering of the orbit space of the group $\mathscr{J}(\tilde{A}_n)$ is defined by

$$\mathbb{C} \oplus \widetilde{\mathbb{C}^{n+1}} \oplus \mathbb{H}/\mathscr{J}(\tilde{A}_n) := \mathbb{C} \oplus \mathbb{C}^{n+1} \oplus \mathbb{H}/(\mathbb{Z}^n \oplus \tau\mathbb{Z}^n), \tag{8.145}$$

where $\mathbb{Z}^n \oplus \tau\mathbb{Z}^n$ acts on $\mathbb{C} \oplus \mathbb{C}^{n+1} \oplus \mathbb{H}$ by

$$\begin{aligned} & v_i \mapsto v_i + \lambda_i\tau + \mu_i, \quad i \neq n+1, \\ & u \mapsto u - 2A_{ij}\lambda_i v_j - A_{ij}\lambda_i\lambda_j\tau, \\ & v_{n+1} \mapsto v_{n+1}, \\ & \tau \mapsto \tau. \end{aligned} \tag{8.146}$$

where A_{ij} is given by (4.5).

Remark 8.7.1. As it was already discussed in the remarks 4.9.1, 4.9.2 , 5.6.1, and 6.3.2, we expect that the covering space covering space for the tilde an case is isomorphic to a suitable covering over the Hurwitz space $H_{1,n-1,0}$. The covering over $H_{1,n-1,0}$ is given by a fixation of base of the homology in the tori generated by the lattice $(1, \tau)$, fixation of root λ (8.24) near ∞, and a fixation of a logarithm root.

In order to manipulate the geometric objects of this covering, it is more convenient to use their ring of functions. Hence, we define:

Definition 8.7.1. The extended ring of Jacobi forms with respect the ring of coefficients is the following ring

$$\widetilde{E}_{\bullet,\bullet}[\varphi_0, \varphi_1, .., \varphi_n], \tag{8.147}$$

where

$$\widetilde{E}_{\bullet,\bullet} = E_{\bullet,\bullet} \oplus \{g_1(\tau)\} \oplus \{\frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\}. \tag{8.148}$$

LEMMA 8.7.2. The coefficients of the intersection form g^{ij} on the coordinates $\varphi_0, \varphi_1, .., \varphi_n, v_{n+1}, \tau$ belong to the ring $\widetilde{E}_{\bullet,\bullet}[\varphi_0, \varphi_1, .., \varphi_n]$.

PROOF. It is a consequence of the formula (8.45).

□

LEMMA 8.7.3. The coefficients of the intersection form $g^{\alpha\beta}$ on the coordinates $t^0, t^1, .., t^n, v_{n+1}, \tau$ belong to the ring $\widetilde{E}_{\bullet,\bullet}[t^0, t^1, .., t^n, \frac{1}{t^n}]$.

PROOF. Using the transformation law of $g^{\alpha\beta}$

$$g^{\alpha\beta} = \frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j} g(d\varphi_i, d\varphi_j), \tag{8.149}$$

we realise the term $\frac{\partial t^\alpha}{\partial \varphi_i}$ as polynomial in $t^0, t^1, .., t^n, \frac{1}{t^n}$ due to the relations (8.117) and (8.123).

□

8.8. Christoffel symbols of the intersection form

In this section, we will generalise the results done in 5.7. Roughly speaking, we consider the Christoffel symbols of the intersection form as object living in $\widetilde{E}_{\bullet,\bullet}[\varphi_0, \varphi_1, .., \varphi_n]$ in coordinates $\varphi_0, , \varphi_1, \varphi_2, .., \varphi_n, v_{n+1}, \tau$, in addition, we will show that the Christoffel symbols depend at most linear in φ_0 in this coordinates.

Recall that the Christoffel symbols $\Gamma_k^{ij}(\varphi)$ associated with the intersection form g^* is given in terms of the conditions (4.13).

LEMMA 8.8.1. Let $\varphi_0, , \varphi_1, \varphi_2, .., \varphi_n, v_{n+1}, \tau$ be defined in (8.24), then Γ_j^{ii} depend at most linear on φ_0.

PROOF. Using the first condition of (4.13)

$$\partial_k g^{ii} = 2\Gamma_k^{ii}$$

Recall that due to the corollary 8.5.4.1, the metric g^{ij} depend at most linear on φ_0. Then,

$$2\frac{\partial^2 \Gamma_k^{ii}}{\partial \varphi_0^2} = \partial_0^2 \partial_k g^{ii} = \partial_k \partial_0^2 g^{ii} = 0.$$

□

LEMMA 8.8.2. Let $\varphi_0, , \varphi_1, \varphi_2, .., \varphi_n, v_{n+1}, \tau$ be defined in (8.24), then

$$\begin{aligned} \Gamma_j^{i\tau} &= 0, \\ \Gamma_k^{\tau k} &= -2\pi i \frac{\delta_{jk}}{k}. \end{aligned} \tag{8.150}$$

PROOF. Let $\Gamma^{ij}_k(x)$, in the coordinates $x_1, .., x_n$, and $\Gamma^{pq}_l(y)$ in the coordinates $y_1, .., y_n$, then the transformation law of the Christoffel symbol defined in the cotangent bundle is the following

$$\Gamma^{ij}_k(x) = \frac{\partial x^i}{\partial y^p}\frac{\partial x^j}{\partial y^q}\frac{\partial y^l}{\partial x^k}\Gamma^{pq}_l(y) + \frac{\partial x^i}{\partial y^p}\frac{\partial}{\partial x^k}\left(\frac{\partial x^j}{\partial y^q}\right) g^{pq}(y). \tag{8.151}$$

In particular, the $\Gamma^{ij}_k(\varphi)$ in the coordinates $(\varphi_0, \varphi_1, .., \varphi_n, v_{n+1,\tau})$ could be derived from the Christoffel symbol in the coordinates $v_0, v_1, .., v_{n+1}, \tau$ which is 0. Then,

$$\Gamma^{ij}_k(\varphi) = \frac{\partial \varphi_i}{\partial v_p}\frac{\partial}{\partial \varphi_k}\left(\frac{\partial \varphi_j}{\partial v_q}\right) g^{pq}(v). \tag{8.152}$$

Computing $\Gamma^{iv_{n+1}}_j$,

$$\begin{aligned}\Gamma^{i\tau}_k(\varphi) &= \frac{\partial \varphi_i}{\partial v_p}\frac{\partial}{\partial \varphi_k}\left(\frac{\partial \tau}{\partial v_q}\right) g^{pq}(v)\\ &= -2\pi\varphi_i\frac{\partial}{\partial \varphi_k}(1) = 0.\end{aligned} \tag{8.153}$$

Computing $\Gamma^{v_{n+1}i}_j$ by using the first condition of (4.13),

$$\begin{aligned}\Gamma^{\tau k}_j(\varphi) &= \partial_j g^{k\tau} - \Gamma^{k\tau}_j\\ &= \partial_j g^{k\tau} = -2\pi i\frac{\delta_{jk}}{k}.\end{aligned} \tag{8.154}$$

□

LEMMA 8.8.3. Let $\varphi_0, , \varphi_1, \varphi_2, .., \varphi_n, v_{n+1}, \tau$ be defined in (8.24), then

$$\begin{aligned}&\Gamma^{iv_{n+1}}_j = 0,\\ &\Gamma^{v_{n+1}i}_j = \frac{\partial g^{iv_{n+1}}}{\partial \varphi_j} \in \widetilde{E}_{\bullet,\bullet}[\varphi_0, \varphi_1, ., \varphi_n],\\ &\Gamma^{ij}_{v_{n+1}} \in \widetilde{E}_{\bullet,\bullet}[\varphi_0, \varphi_1, ., \varphi_n].\end{aligned} \tag{8.155}$$

Moreover, these Christoffel symbols are at most linear on φ_0.

PROOF. Let $\Gamma^{ij}_k(x)$, in the coordinates $x_1, .., x_n$, and $\Gamma^{pq}_l(y)$ in the coordinates $y_1, .., y_n$, then the transformation law of the Christoffel symbol defined in the cotangent bundle is the following

$$\Gamma^{ij}_k(x) = \frac{\partial x^i}{\partial y^p}\frac{\partial x^j}{\partial y^q}\frac{\partial y^l}{\partial x^k}\Gamma^{pq}_l(y) + \frac{\partial x^i}{\partial y^p}\frac{\partial}{\partial x^k}\left(\frac{\partial x^j}{\partial y^q}\right) g^{pq}(y). \tag{8.156}$$

In particular, the $\Gamma^{ij}_k(\varphi)$ in the coordinates $(\varphi_0, \varphi_1, .., \varphi_n, v_{n+1,\tau})$ could be derived from the Christoffel symbol in the coordinates $v_0, v_1, .., v_{n+1}, \tau$ which is 0. Then,

$$\Gamma^{ij}_k(\varphi) = \frac{\partial \varphi_i}{\partial v_p}\frac{\partial}{\partial \varphi_k}\left(\frac{\partial \varphi_j}{\partial v_q}\right) g^{pq}(v). \tag{8.157}$$

Computing $\Gamma_k^{iv_{n+1}}$,

$$\Gamma_k^{iv_{n+1}}(\varphi) = \frac{\partial \varphi_i}{\partial v_p}\frac{\partial}{\partial \varphi_k}\left(\frac{\partial v_{n+1}}{\partial v_q}\right) g^{pq}(v) \tag{8.158}$$
$$= \frac{\partial \varphi_i}{\partial v_{n+1}}\frac{\partial}{\partial \varphi_k}(1)\, g^{v_{n+1}v_{n+1}}(v) = 0.$$

Computing $\Gamma_k^{v_{n+1}i}$ by using the first condition of (4.13),

$$\Gamma_k^{v_{n+1}i}(\varphi) = \partial_k g^{iv_{n+1}} - \Gamma_j^{iv_{n+1}} \tag{8.159}$$
$$= \partial_k g^{iv_{n+1}}.$$

Since, $g^{iv_{n+1}} \in \widetilde{E}_{\bullet,\bullet}[\varphi_0, \varphi_1, ., \varphi_n]$, we have that $\Gamma_k^{v_{n+1}i}(\varphi) \in \widetilde{E}_{\bullet,\bullet}[\varphi_0, \varphi_1, ., \varphi_n]$. In addition, since the metric $\eta^{iv_{n+1}}$ is independent of φ_0 due to the corollary 8.5.4.1, we have that $\partial_k g^{iv_{n+1}}$ is at most linear on φ_0

Computing $\Gamma^{ij}_{v_{n+1}}$,

$$\Gamma^{ij}_{v_{n+1}}(\varphi) = \frac{\partial \varphi_i}{\partial v_p}\frac{\partial}{\partial v_{n+1}}\left(\frac{\partial \varphi_j}{\partial v_q}\right) g^{pq}(v) \tag{8.160}$$
$$= \frac{\partial \varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(\frac{\partial \varphi_j}{\partial v_{n+1}}\right) g^{pq}(v)$$

In the subsequent computation, whenever appears a function depend only on v_{n+1}, τ, we will call it by $h(v_{n+1}, \tau)$, because for our purpose, it is enough to prove that the subsequent function belong to the ring $\widetilde{E}_{\bullet,\bullet}[\varphi_0, \varphi_1, ., \varphi_n]$.

If $j > 1$, using equation (8.82),

$$\begin{aligned}
\Gamma^{ij}_{v_{n+1}}(\varphi) &= \frac{\partial \varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(\frac{\partial \varphi_j}{\partial v_{n+1}}\right) g^{pq}(v) \\
&= \frac{\partial \varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(\varphi_{j-1} + h(v_{n+1},\tau)\varphi_n\right) g^{pq}(v) \\
&= \frac{\partial \varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(\varphi_{j-1} + h(v_{n+1},\tau)\varphi_n\right) g^{pq}(v) \\
&= \frac{\partial \varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(\varphi_{j-1}\right) g^{pq}(v) + h(v_{n+1},\tau)\frac{\partial \varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(\varphi_n\right) g^{pq}(v) \\
&+ \varphi_n \frac{\partial \varphi_i}{\partial v_{n+1}}\frac{\partial}{\partial v_{n+1}}\left(h(v_{n+1},\tau)\right) g^{v_{n+1}v_{n+1}}(v) \\
&= g^{i(j-1)}(\varphi) + h(v_{n+1},\tau) g^{in}(\varphi_n) + \frac{h'(v_{n+1},\tau)}{n(n+1)}\left(\varphi_i + g(v_{n+1},\tau)\varphi_n\right)\varphi_n.
\end{aligned} \tag{8.161}$$

If $j=1$, using equation (8.84)

$$(8.162)\quad \begin{aligned}\Gamma^{i1}_{v_{n+1}}(\varphi) &= \frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(\frac{\partial\varphi_1}{\partial v_{n+1}}\right)g^{pq}(v)\\ &= \frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(n\varphi_0 + h(v_{n+1},\tau)\varphi_1\right)g^{pq}(v)\\ &= \frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(n\varphi_0 + h(v_{n+1},\tau)\varphi_1\right)g^{pq}(v)\\ &= \frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(n\varphi_0\right)g^{pq}(v) + h(v_{n+1},\tau)\frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(\varphi_1\right)g^{pq}(v)\\ &+ \varphi_1\frac{\partial\varphi_i}{\partial v_{n+1}}\frac{\partial}{\partial v_{n+1}}\left(h(v_{n+1},\tau)\right)g^{v_{n+1}v_{n+1}}(v)\\ &= ng^{i0}(\varphi) + h(v_{n+1},\tau)g^{i1}(\varphi_1) + \frac{h'(v_{n+1},\tau)}{n(n+1)}\left(\varphi_i + h_3(v_{n+1},\tau)\varphi_n\right)\varphi_1.\end{aligned}$$

If $j=0$, using equation (8.86)

(8.163)

$$\begin{aligned}\Gamma^{i0}_{v_{n+1}}(\varphi) &= \frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(\frac{\partial\varphi_0}{\partial v_{n+1}}\right)g^{pq}(v)\\ &= \frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(-nh_1(v_{n+1},\tau)\varphi_0 + h_2(v_{n+1},\tau)\varphi_1\right)g^{pq}(v)\\ &= \frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(-nh_1(v_{n+1},\tau)\varphi_0 + h_2(v_{n+1},\tau)\varphi_1\right)g^{pq}(v)\\ &= \frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(-nh_1(v_{n+1},\tau)\varphi_0\right)g^{pq}(v) + h_2(v_{n+1},\tau)\frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(\varphi_1\right)g^{pq}(v)\\ &+ \varphi_1\frac{\partial\varphi_i}{\partial v_{n+1}}\frac{\partial}{\partial v_{n+1}}\left(h_2(v_{n+1},\tau)\right)g^{v_{n+1}v_{n+1}}(v)\\ &= -nh_1(v_{n+1},\tau)g^{i0}(\varphi) + h_2(v_{n+1},\tau)g^{i1}(\varphi_n) + \frac{h_2'(v_{n+1},\tau)}{n(n+1)}\left(\varphi_i + h_3(v_{n+1},\tau)\varphi_n\right)\varphi_1\\ &- n\frac{h_1'(v_{n+1},\tau)}{n(n+1)}\left(\varphi_i + h_3(v_{n+1},\tau)\varphi_n\right)\varphi_0.\end{aligned}$$

Hence $\Gamma^{ij}_{v_{n+1}}(\varphi) \in \widetilde{E}_{\bullet,\bullet}[\varphi_0,\varphi_1,.,\varphi_n]$, furthermore, it is at most linear on φ_0.

□

Proposition 8.8.4. The Christoffel symbols $\Gamma^{ij}_k(\varphi)$ belong to the ring $\widetilde{E}_{\bullet,\bullet}[\varphi_0,\varphi_1,..,\varphi_n]$.

Proof. Note that the invariance of the Jacobi form φ_i with respect the first two actions of (8.1), and equivariance by the third one implies that the differential $d\varphi_i$ is invariant under the first two actions of (8.1), and behaves as follows under the $SL_2(\mathbb{Z})$

$$(8.164)\quad d\varphi_i \mapsto \frac{d\varphi_i}{(c\tau+d)^{k_i}} - \frac{c\varphi_i}{(c\tau+d)^{k_i+1}}$$

Therefore the Christoffel symbol Γ^{ij}_k

$$(8.165)\quad \nabla_{(d\varphi_i)^{\#}}d\varphi_j = \Gamma^{ij}_k d\varphi_k$$

is a Jacobi form if φ_i has weight 0. Hence, doing the change of coordinates

$$\varphi_i \mapsto \hat{\varphi}_i := \eta^{2i}(\tau)\varphi_i, \tag{8.166}$$

we have that the Christoffel symbol $\hat{\Gamma}^{ij}_k$

$$\frac{1}{\eta^{2i+2j}}\nabla_{(d\hat{\varphi}_i)^\#} d\hat{\varphi}_j = \hat{\Gamma}^{ij}_k d\hat{\varphi}_k \tag{8.167}$$

is a Jacobi form.

Comparing $\hat{\Gamma}^{ij}_k$ with Γ^{ij}_k

$$\begin{aligned}
\nabla_{(d\hat{\varphi}_j)^\#} d\hat{\varphi}_i &= \nabla_{(2jg_1\eta^{2j}\varphi_j d\tau + \eta^{2j}d\varphi_j)^\#}\left(2ig_1\eta^{2i}\varphi_i d\tau + \eta^{2i}d\varphi_i\right)\\
&= \nabla_{(2jg_1\eta^{2j}\varphi_j d\tau)^\#}\left(2ig_1\eta^{2i}\varphi_i d\tau\right) + \nabla_{(2jg_1\eta^{2j}\varphi_j d\tau)^\#}\left(\eta^{2i}d\varphi_i\right)\\
&+ \nabla_{(\eta^{2j}d\varphi_j)^\#}\left(2ig_1\eta^{2i}\varphi_i d\tau\right) + \nabla_{(\eta^{2j}d\varphi_j)^\#}\left(\eta^{2i}d\varphi_i\right)\\
&= 2jg_1\eta^{2j}\varphi_j g^{l\tau}\nabla_{\frac{\partial}{\partial\varphi_l}}\left(2i\eta^{2i}g_1\varphi_i d\tau\right) + 2jg_1\eta^{2j}\varphi_j g^{l\tau}\nabla_{\frac{\partial}{\partial\varphi_l}}\left(\eta^{2i}d\varphi_i\right)\\
&+ \eta^{2j}g^{lj}\nabla_{\frac{\partial}{\partial\varphi_l}}\left(2ig_1\eta^{2i}\varphi_i d\tau\right) + \eta^{2j}g^{lj}\nabla_{\frac{\partial}{\partial\varphi_l}}\left(\eta^{2i}d\varphi_i\right)\\
&= 4ijg_1'g_1\varphi_i\eta^{2i+2j}\varphi_j g^{\tau\tau}d\tau + 4i^2jg_1^3\eta^{2i+2j}\varphi_j\varphi_i g^{\tau\tau}d\tau + 4ijg_1\eta^{2i+2j}\varphi_j g^{i\tau}d\tau\\
&+ 4ijg_1^2\eta^{2i+2j}\varphi_j g^{\tau\tau}d\varphi_i + 2jg_1\eta^{2i+2j}\varphi_j\Gamma^{\tau i}_k d\varphi_k + 4i^2g_1^2\eta^{2i+2j}\varphi_i g^{\tau j}d\tau\\
&+ 2ig_1'\eta^{2i+2j}\varphi_i g^{lj}d\tau + 2ig_1\eta^{2i+2j}g^{ij}d\tau + 2ig_1\varphi_i\eta^{2i+2j}\Gamma^{i\tau}_k d\varphi_k\\
&+ \eta^{2i+2j}g_1 g^{l\tau}d\varphi_i + \eta^{2i+2j}\Gamma^{ji}_k d\varphi_k.
\end{aligned} \tag{8.168}$$

Dividing the equation (8.168) by η^{2i+2j} and isolating $\Gamma^{ji}_k d\varphi_k$, we have

$$\begin{aligned}
\Gamma^{ji}_k d\varphi_k &= -4ijg_1'g_1\varphi_i\varphi_j g^{\tau\tau}d\tau - 4i^2jg_1^3\varphi_j\varphi_i g^{\tau\tau}d\tau + 4ijg_1\eta^{2i+2j}\varphi_j g^{i\tau}d\tau\\
&- 4ijg_1^2\varphi_j g^{\tau\tau}d\varphi_i - 2jg_1\varphi_j\Gamma^{\tau i}_k d\varphi_k - 4i^2g_1^2\varphi_i g^{\tau j}d\tau\\
&- 2ig_1'\varphi_i g^{lj}d\tau - 2ig_1\eta^{2i+2j}g^{ij}d\tau - 2ig_1\varphi_i\Gamma^{i\tau}_k d\varphi_k\\
&- g_1 g^{l\tau}d\varphi_i + \hat{\Gamma}^{ji}_k d\varphi_k.
\end{aligned} \tag{8.169}$$

Since the differential forms has a vector space structure and the right hand side of (8.169) depends only on $g^{ij}, g_1(\tau), \varphi_i$, and $\Gamma^{\tau i}_k$ which belongs to the ring $\widetilde{E}_{\bullet,\bullet}[\varphi_0,\varphi_1,..,\varphi_n]$, the desired result is proved.

□

LEMMA 8.8.5. *The Christoffel symbols $\Gamma^{ij}_k(\varphi)$ depend linearly on φ_0.*

PROOF. The result was already proved, when $k = v_{n+1}$ in the lemma 8.8.3. The proposition 8.8.4 gives to the space of Christoffel symbols the structure of graded algebra, in particular we can compute the degree m regarding to the algebra of Jacobi forms. Let $\phi \in \widetilde{E}_{\bullet,\bullet}[\varphi_0,\varphi_1,..,\varphi_n]$

with index m_ϕ and weight k_ϕ, then we write

$$deg_m\phi = m_\phi, \qquad deg_k\phi = k_\phi. \tag{8.170}$$

If $k \neq \tau, v_{n+1}$,

$$deg_m\Gamma_k^{ij} = deg_m\left(\frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial\varphi_k}\left(\frac{\partial\varphi_j}{\partial v_q}\right)g^{pq}(v)\right) = 1. \tag{8.171}$$

Therefore, Γ_k^{ij} is at most linear on φ_0. If $k = \tau$,

$$deg_k\Gamma_\tau^{ij} = deg_k\left(\frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial\tau}\left(\frac{\partial\varphi_j}{\partial v_q}\right)g^{pq}(v)\right) = -i-j+4.$$

Suppose that Γ_τ^{ij} contains a the term $a(v_{n+1},\tau)\varphi_0^2$, where $a(v_{n+1},\tau)$ is an elliptic function in v_{n+1}, then

$$deg_k a(v_{n+1},\tau) = -i-j+4 > 0.$$

The possible Christoffel symbols that could depend on φ_0^2 are

$$\Gamma_\tau^{04}, \Gamma_\tau^{40}, \Gamma_\tau^{13}, \Gamma_\tau^{31}, \Gamma_\tau^{22}, \Gamma_\tau^{21}, \Gamma_\tau^{12}, \Gamma_\tau^{20}, \Gamma_\tau^{02}, \Gamma_\tau^{11}, \Gamma_\tau^{10}, \Gamma_\tau^{01}, \Gamma_\tau^{00}. \tag{8.172}$$

But $\Gamma_\tau^{22}, \Gamma_\tau^{11}, \Gamma_\tau^{00}$ is linear on φ_0 due to lemma 8.8.1.
Computing Γ_τ^{ij}

$$\Gamma_\tau^{ij} = \frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial\tau}\left(\frac{\partial\varphi_j}{\partial v_q}\right)g^{pq}(v) = \frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(\frac{\partial\varphi_j}{\partial\tau}\right)g^{pq}(v) \tag{8.173}$$

In order to compute it, recall there exist a relationship between the holomorphic Jacobi forms of A_{n+1} type and the meromorphic Jacobi forms of $\tilde{A}_n$ type given by (8.30). Moreover, in (5.19) it was demonstrated that the lowest degree term in the Taylor expansion of $\varphi_i^{\mathscr{J}(A_{n+1})}$ with respect the variables $v_0, v_1, .., v_{n+1}$ are the elementary symmetric polynomials $a_i(v_0, v_1, .., v_{n+1})$ of degree i. Hence, using the equations (5.19) and (8.30), we can estimate the degree of the lowest degree term of the meromorphic Jacobi forms of $\tilde{A}_n$, more specifically,

(8.174)

$$\varphi_n^{\mathscr{J}(\tilde{A}_n)}\varphi_2^{\mathscr{J}(A_1)} = a_{n+2}(v) + b_{n+2}(v_{n+1},\tau)a_{n+3}(v) + O(||v||^{n+4}),$$

$$\varphi_{n-1}^{\mathscr{J}(\tilde{A}_n)}\varphi_2^{\mathscr{J}(A_1)} + b_{n-1}^n\varphi_n^{\mathscr{J}(\tilde{A}_n)}\varphi_2^{A_1} = a_{n+1}(v) + b_{n+1}(v_{n+1},\tau)a_{n+2}(v) + O(||v||^{n+3}),$$

$$\varphi_{n-2}^{\mathscr{J}(\tilde{A}_n)}\varphi_2^{\mathscr{J}(A_1)} + b_{n-2}^{n-1}\varphi_{n-1}^{\mathscr{J}(\tilde{A}_n)}\varphi_2^{\mathscr{J}(A_1)} + a_{n-2}^n\varphi_n^{\mathscr{J}(\tilde{A}_n)}\varphi_2^{\mathscr{J}(A_1)} = a_n(v) + b_{n+2}(v_{n+1},\tau)a_{n+1}(v) + O(||v||^{n+2}),$$

.

.

$$\varphi_0^{\mathscr{J}(\tilde{A}_n)}\varphi_2^{\mathscr{J}(A_1)} + \sum_{j=1}^n b_0^j\varphi_j^{\mathscr{J}(\tilde{A}_n)}\varphi_2^{\mathscr{J}(A_1)} = a_2(v) + b_3(v_{n+1},\tau)a_3(v) + O(||v||^4).$$

Note that the Christoffel symbol depend on φ_0 iff it contains the term $a_2^2(v)$ in its expansion. Our strategy is to show that the Christoffel symbols (8.172) contains only higher order polynomials in its expansions. Computing the lowest degree term in the expansion of (8.173)

$$\begin{aligned}
\Gamma_\tau^{01} &= \frac{\partial\varphi_i}{\partial v_p}\frac{\partial}{\partial v_q}\left(\frac{\partial\varphi_j}{\partial\tau}\right)g^{pq}(v) \\
&= \frac{\partial a_{i+2}}{\partial v_p}\frac{\partial}{\partial v_q}\left(\frac{\partial b_{j+1}(v_{n+1},\tau)}{\partial\tau}a_{j+3}\right)g^{pq}(v)+.. \\
&= \frac{\partial a_{i+2}}{\partial v_p}\frac{\partial b_{j+1}(v_{n+1},\tau)}{\partial\tau}\frac{\partial a_{j+3}}{\partial v_q}g^{pq}(v)+.. \\
&= \frac{\partial b_{j+1}(v_{n+1},\tau)}{\partial\tau}a_{i+j+3}+...
\end{aligned} \tag{8.175}$$

Therefore, for $i+j>1$, we have that the associated Christoffel symbol do not depend on φ_0^2. It remains to check only Γ_τ^{01} and Γ_τ^{10}. Computing Γ_τ^{10} by using the second equation of (4.13) for $i=1, j=0, k=0$

$$\begin{aligned}
\sum_{l=0}^{n+2} g^{1l}\Gamma_l^{00} &= \sum_{l=0}^{n+2} g^{0l}\Gamma_l^{10} \\
&= \sum_{l=0}^{n+1} g^{0l}\Gamma_l^{10} + g^{0\tau}\Gamma_\tau^{10}.
\end{aligned}$$

Isolating Γ_τ^{10},

$$\Gamma_\tau^{10} = \frac{1}{2\pi i\varphi_0}\left[\sum_{l=0}^{n+2} g^{1l}\Gamma_l^{00} - \sum_{l=0}^{n+1} g^{0l}\Gamma_l^{10}\right], \tag{8.176}$$

we have that the right hand side of (8.176) depend at most linear on φ_0. Moreover, using the first equation (4.13), we have

$$\begin{aligned}
\partial_0^2\Gamma_\tau^{10} &= \partial_\tau\partial_0^2 g^{10} - \partial_0^2\Gamma_\tau^{10} \\
&= \partial_\tau\partial_0^2 g^{10} = 0.
\end{aligned}$$

Lemma proved.

□

8.9. Unit and Euler vector field of the orbit space of $\mathscr{J}(\tilde{A}_n)$

The aim of this section is to define the Unit and Euler vector field and its respective actions on the geometric data of the orbit space of $\mathscr{J}(\tilde{A}_n)$. Further, these objects will be fundamental to define the unit of the Frobenius algebra of the desired Dubrovin Frobenius structure, and to give a quasi homogeneous property to the desired WDVV solution.

DEFINITION 8.9.1. The Euler vector field with respect the orbit space $\mathscr{J}(\tilde{A}_n)$ is defined by the last equation of (8.2), i.e

$$E := -\frac{1}{2\pi i}\frac{\partial}{\partial u} \tag{8.177}$$

DEFINITION 8.9.2. A f is quasi homogeneous of degree d if it is an eigenfunction of the Euler vector field (8.177) with eigenvalue d, i.e.

$$E(f) = df.$$

LEMMA 8.9.1. Let $\lambda, \varphi_0, .., \varphi_n, \varphi_{n+1} = v_{n+1}, \varphi_{n+2} = \tau$ be defined in (8.24) and $(t^0, ., t^n, v_{n+1}, \tau)$ the flat coordinates of eta defined in (8.111). Then,

$$\begin{aligned} E(\lambda) &= \lambda, \\ E(\varphi_i) &= d_i \varphi_i, \\ E(t^\alpha) &= d_\alpha t^\alpha, \end{aligned} \tag{8.178}$$

where

$$\begin{aligned} &d_i = 1, \quad i < n+1, \\ &d_i = 0, \quad i \geq n+1, \\ &d_\alpha = \frac{n+1-\alpha}{n}, \quad \alpha \neq 0, \\ &d_0 = 1. \end{aligned} \tag{8.179}$$

PROOF. Recall that the function λ is given by

$$\begin{aligned} \lambda &= e^{-2\pi i u}\frac{\prod_{i=0}^{n}\theta_1(z - v_i + v_{n+1}, \tau)}{\theta_1^n(z,\tau)\theta_1(z + (n+1)v_{n+1})} \\ &= \varphi_n \wp^{n-2}(z,\tau) + \varphi_{n-1}\wp^{n-3}(z,\tau) + ... + \varphi_2\wp(z,\tau) \\ &+ \varphi_1[\zeta(z,\tau) - \zeta(z + (n+1)v_{n+1}, \tau) + \varphi_0. \end{aligned}$$

Hence,

$$\begin{aligned} \lambda &= \frac{1}{2\pi i}\frac{\partial}{\partial u}(\lambda) \\ &= E(\varphi_n)\wp^{n-2}(z,\tau) + E(\varphi_{n-1})\wp^{n-3}(z,\tau) + ... + E(\varphi_2)\wp(z,\tau) \\ &+ E(\varphi_1)[\zeta(z,\tau) - \zeta(z + (n+1)v_{n+1}, \tau) + E(\varphi_0), \end{aligned}$$

therefore, $E(\varphi_i) = \varphi_i$, and $E(v_{n+1}) = E(\tau) = 0$.

Recall that t^α can written in terms of equation (8.115) or in more convenient way

$$\begin{aligned} t^\alpha &= \frac{n}{n+1-\alpha}\operatorname*{res}_{v=0}\lambda^{\frac{n+1-\alpha}{n}}(v)dv, \quad \alpha \neq 0, \\ t^0 &= \varphi_0 - \frac{\theta_1'((n+1)v_{n+1})}{\theta_1((n+1)v_{n+1})}\varphi_1 + 4\pi i g_1(\tau)\varphi_2. \end{aligned} \tag{8.180}$$

Applying the Euler vector in (8.180) we get the desired result.

□

Corollary 8.9.1.1. The Euler vector field (8.177) in the flat coordinates of η^* has the following form

$$E := \sum_{\alpha=0}^{n} d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha}, \tag{8.181}$$

where

$$\begin{aligned} d_\alpha &= \frac{n+1-\alpha}{n}, \quad \alpha \neq 0, \\ d_0 &= 1. \end{aligned} \tag{8.182}$$

Proof. Recall that

$$E = \frac{1}{2\pi i}\frac{\partial}{\partial u} = E(t^\alpha)\frac{\partial}{\partial t^\alpha} = \sum_{\alpha=0}^{n} d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha}.$$

□

Lemma 8.9.2. The Euler vector field (8.177) acts on the vector fields $\frac{\partial}{\partial t^\alpha}$, $\frac{\partial}{\partial \varphi_i}$ and differential forms dt^α, $d\varphi_i$ as follows:

$$\begin{aligned} Lie_E d\varphi_i &= d_i d\varphi_i, \\ Lie_E dt^\alpha &= d_\alpha dt^\alpha, \\ Lie_E \frac{\partial}{\partial \varphi_i} &= -d_i \frac{\partial}{\partial \varphi_i}, \\ Lie_E \frac{\partial}{\partial t^\alpha} &= -d_\alpha \frac{\partial}{\partial t^\alpha}. \end{aligned} \tag{8.183}$$

Proof. Recall that the Lie derivative acts in vector fields by using the Lie bracket and in differential forms by the use of Cartan's magic formula

$$\begin{aligned} Lie_E \frac{\partial}{\partial t^\alpha} &= \left[E, \frac{\partial}{\partial t^\alpha}\right], \\ Lie_E dt^\alpha &= dE(dt^\alpha) + E(d^2 t^\alpha) = dE(dt^\alpha). \end{aligned} \tag{8.184}$$

Using (8.184) and (8.178), we obtain the desired result.

□

Lemma 8.9.3. The intersection form g^{ij} defined in (8.36) and its Christoffel symbol Γ_k^{ij} in the coordinates $\varphi_0, .., \varphi_n, \varphi_{n+1} = v_{n+1}, \varphi_{n+2} = \tau$ defined in (8.24) are weighted polynomials in the variables $\varphi_0, .., \varphi_n$, with degrees

$$deg\left(g^{ij}\right) = d_i + d_j, \quad deg\left(\Gamma_k^{\alpha\beta}\right) = d_i + d_j - d_k. \tag{8.185}$$

PROOF. The function g^{ij} and Γ_k^{ij} belong to the ring $\widetilde{E}_{\bullet,\bullet}[\varphi_0, \varphi_1, .., \varphi_n]$ due to 8.4.2 and 8.8.4. The degrees are computed by using the following formulae

$$\begin{aligned}
E\left(g^{ij}(\varphi)\right) &= E\left(\frac{\partial \varphi_i}{\partial v^l}\frac{\partial \varphi_j}{\partial v^m}g^{lm}(v)\right)\\
&= E\left(\frac{\partial \varphi_i}{\partial v^l}\right)\frac{\partial \varphi_j}{\partial v^m}g^{lm}(v) + \frac{\partial \varphi_i}{\partial v^l}E\left(\frac{\partial \varphi_j}{\partial v^m}g^{lm}(v)\right)\\
&= \frac{\partial E(\varphi_i)}{\partial v^l}\frac{\partial \varphi_j}{\partial v^m}g^{lm}(v) + \frac{\partial \varphi_i}{\partial v^l}\frac{\partial E(\varphi_j)}{\partial v^m}g^{lm}(v)\\
&= (d_i + d_j)\frac{\partial \varphi_i}{\partial v^l}\frac{\partial \varphi_j}{\partial v^m}g^{lm}(v).
\end{aligned}$$

and

$$\begin{aligned}
E\left(\Gamma_k^{ij}(\varphi)\right) &= E\left(\frac{\partial \varphi_i}{\partial v^l}\frac{\partial}{\partial \varphi_k}\left(\frac{\partial \varphi_j}{\partial v^m}\right)g^{lm}(v)\right)\\
&= E\left(\frac{\partial \varphi_i}{\partial v^l}\right)\frac{\partial}{\partial \varphi_k}\left(\frac{\partial \varphi_j}{\partial v^m}\right)g^{lm}(v) + \frac{\partial \varphi_i}{\partial v^l}E\left(\frac{\partial}{\partial \varphi_k}\left(\frac{\partial \varphi_j}{\partial v^m}\right)g^{lm}(v)\right)\\
&= \frac{\partial E(\varphi_i)}{\partial v^l}\frac{\partial}{\partial \varphi_k}\left(\frac{\partial \varphi_j}{\partial v^m}\right)g^{lm}(v) + \frac{\partial \varphi_i}{\partial v^l}\frac{\partial}{\partial \varphi_k}\left(\frac{\partial E(\varphi_j)}{\partial v^m}\right)g^{lm}(v)\\
&- d_k\frac{\partial \varphi_i}{\partial v^l}\frac{\partial}{\partial \varphi_k}\left(\frac{\partial \varphi_j}{\partial v^m}\right)g^{lm}(v)\\
&= (d_i + d_j - d_k)\frac{\partial \varphi_i}{\partial v^l}\frac{\partial}{\partial \varphi_k}\left(\frac{\partial \varphi_j}{\partial v^m}\right)g^{lm}(v).
\end{aligned}$$

$\square$

LEMMA 8.9.4. The intersection form $g^{\alpha\beta}$ defined in (8.36) in the coordinates $(t^0, ., t^n, v_{n+1}, \tau)$ defined in (8.111) and its Christoffel symbol $\Gamma_\gamma^{\alpha\beta}$ are weighted polynomials in the variables $t^0, t^1, .., t^n, \frac{1}{t^n}$ with degrees

$$\begin{aligned}
deg\left(g^{\alpha\beta}\right) &= d_\alpha + d_\beta,\\
deg\left(\Gamma_\gamma^{\alpha\beta}\right) &= d_\alpha + d_\beta - d_\gamma.
\end{aligned} \tag{8.186}$$

PROOF. Lemma 8.7.3 guarantee that $g^{\alpha\beta} \in \widetilde{E}_{\bullet,\bullet}[t^0, t^1, .., t^n, \frac{1}{n}]$. Using the formula

$$\begin{aligned}
E(g^{\alpha\beta}) &= E(\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j}g^{ij}(\varphi))\\
&= E(\frac{\partial t^\alpha}{\partial \varphi_i})\frac{\partial t^\beta}{\partial \varphi_j}g^{ij}(\varphi) + \frac{\partial t^\alpha}{\partial \varphi_i}E(\frac{\partial t^\beta}{\partial \varphi_j})g^{ij}(\varphi) + \frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j}E(g^{ij}(\varphi))\\
&= \frac{\partial E(t^\alpha)}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j}g^{ij}(\varphi) - d_i\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j}g^{ij}(\varphi) + \frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial E(t^\beta)}{\partial \varphi_j}g^{ij}(\varphi) - d_j\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j}g^{ij}(\varphi)\\
&+ (d_i + d_j)\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j}g^{ij}(\varphi)\\
&= (d_\alpha + d_\beta)\frac{\partial t^\alpha}{\partial \varphi_i}\frac{\partial t^\beta}{\partial \varphi_j}g^{ij}(\varphi).
\end{aligned}$$

The Christoffel symbol $\Gamma_{\gamma}^{\alpha\beta}$ is given by the following

$$\Gamma_{\gamma}^{\alpha\beta} = \frac{\partial t^{\alpha}}{\partial \varphi_i}\frac{\partial t^{\beta}}{\partial \varphi_j}\frac{\partial \varphi_k}{\partial t^{\gamma}}\Gamma_{k}^{ij} + \frac{\partial t^{\alpha}}{\partial \varphi_i}\frac{\partial}{\partial t^{\gamma}}\left(\frac{\partial t^{\beta}}{\partial \varphi_j}\right) g^{ij}.$$

$\Gamma_{j}^{ij}, \frac{\partial \varphi_k}{\partial t^{\gamma}} \in \widetilde{E}_{\bullet,\bullet}[t^0, t^1, .., t^n]$ due to Lemma 8.8.4 and equations (8.128), (8.125). But using (8.117), we realise that $\frac{\partial t^{\alpha}}{\partial \varphi_i}$ is polynomial in $t^1, t^2, .., t^n, \frac{1}{t^n}$ due to due to equations (8.123) and (8.118). Therefore, $\Gamma_{\gamma}^{\alpha\beta}$ are weighted polynomials in the variables $t^0, t^1, .., t^n, \frac{1}{t^n}$. Computing the degree of $\Gamma_{\gamma}^{\alpha\beta}$

$$\begin{aligned}
E(\Gamma_{\gamma}^{\alpha\beta}) &= E(\frac{\partial t^{\alpha}}{\partial \varphi_i}\frac{\partial t^{\beta}}{\partial \varphi_j}\frac{\partial \varphi_k}{\partial t^{\gamma}}\Gamma_{k}^{ij} + \frac{\partial t^{\alpha}}{\partial \varphi_i}\frac{\partial}{\partial t^{\gamma}}\left(\frac{\partial t^{\beta}}{\partial \varphi_j}\right) g^{ij})\\
&= (d_{\alpha} - d_i)\frac{\partial t^{\alpha}}{\partial \varphi_i}\frac{\partial t^{\beta}}{\partial \varphi_j}\frac{\partial \varphi_k}{\partial t^{\gamma}}\Gamma_{k}^{ij} + (d_{\beta} - d_j)\frac{\partial t^{\alpha}}{\partial \varphi_i}\frac{\partial t^{\beta}}{\partial \varphi_j}\frac{\partial \varphi_k}{\partial t^{\gamma}}\Gamma_{k}^{ij}\\
&+ (d_k - d_{\gamma})\frac{\partial t^{\alpha}}{\partial \varphi_i}\frac{\partial t^{\beta}}{\partial \varphi_j}\frac{\partial \varphi_k}{\partial t^{\gamma}}\Gamma_{k}^{ij} + (d_{\alpha} - d_i)\frac{\partial t^{\alpha}}{\partial \varphi_i}\frac{\partial}{\partial t^{\gamma}}\left(\frac{\partial t^{\beta}}{\partial \varphi_j}\right) g^{ij}\\
&+ (d_{\beta} - d_{\gamma})\frac{\partial t^{\alpha}}{\partial \varphi_i}\frac{\partial}{\partial t^{\gamma}}\left(\frac{\partial t^{\beta}}{\partial \varphi_j}\right) g^{ij} + (d_{\alpha} - d_i)\frac{\partial t^{\alpha}}{\partial \varphi_i}\frac{\partial}{\partial t^{\gamma}}\left(\frac{\partial t^{\beta}}{\partial \varphi_j}\right) g^{ij}\\
&+ (d_i + d_j)\frac{\partial t^{\alpha}}{\partial \varphi_i}\frac{\partial}{\partial t^{\gamma}}\left(\frac{\partial t^{\beta}}{\partial \varphi_j}\right) g^{ij}\\
&= (d_{\alpha} + d_{\beta} - d_{\gamma})\Gamma_{\gamma}^{\alpha\beta}.
\end{aligned}$$

□

Definition 8.9.3. The Unit vector field with respect the orbit space $\mathscr{J}(\tilde{A}_n)$ is the vector associated to the invariant coordinate φ_0 defined in (8.24) , i.e

(8.187)
$$e := \frac{\partial}{\partial \varphi_0}.$$

Lemma 8.9.5. The Unit vector field (8.177) in the flat coordinates of η^* has the following form

(8.188)
$$e = \frac{\partial}{\partial t^0}.$$

Proof.

$$\frac{\partial}{\partial \varphi_0} = \frac{\partial t^{\alpha}}{\partial \varphi_0}\frac{\partial}{\partial t^{\alpha}} = \frac{\partial}{\partial t^0}.$$

□

Lemma 8.9.6. Let the metric η^* be defined on (8.73) and the Euler vector field (8.177). Then,

(8.189)
$$Lie_E\eta^{\alpha\beta} = (d_{\alpha} + d_{\beta} - d_1)\eta^{\alpha\beta}.$$

8.10. Discriminant locus and the monodromy of the orbit space of $\mathscr{J}(\tilde{A}_n)$

This section relates the critical points of the (8.24) with the zeros of the determinant of the intersection form g^* (8.37). Further, in section 8.11, it will be built a Frobenius algebra in the sections of the orbit space of $\mathscr{J}(\tilde{A}_n)$, furthermore, the intersection form g^* (8.37) can be realised as the multiplication by the Euler vector field (8.177). The results of this section will imply that the intersection form g^* (8.37) is diagonalisable with eigenvalues generically different from 0, and this is equivalent to the Frobenius algebra be semisimple. Moreover, we can realise the isomorphism of orbit space of $\mathscr{J}(\tilde{A}_n)$ with the Hurwitz space $H_{1,n-1,0}$ as a Dubrovin Frobenius manifolds, see Theorem 8.11.7 for details.

Definition 8.10.1. Let g^* the metric defined on (8.37). The discriminant locus of the orbit space of $\mathscr{J}(\tilde{A}_n)$ $\mathbb{C}\oplus\mathbb{C}^{n+1}\oplus\mathbb{H}/\mathscr{J}(\tilde{A}_n)$ is defined by

$$\Sigma = \{x \in \mathbb{C}\oplus\mathbb{C}^{n+1}\oplus\mathbb{H}/\mathscr{J}(\tilde{A}_n) : \det(g^*) = 0\}. \tag{8.190}$$

Lemma 8.10.1. The fixed points of the action $\mathscr{J}(\tilde{A}_n)$ belong to the discriminant locus (8.190).

Proof. Note that the fixed points of the action $\mathscr{J}(\tilde{A}_n)$ on $\mathbb{C}\oplus\mathbb{C}^{n+1}\oplus\mathbb{H} \ni (u, v_0, v_1, .., v_{n-1}, v_{n+1}, \tau)$ are the fixed points of the action A_n on $\mathbb{C}^n \ni (v_0, v_1, .., v_{n_1})$. Therefore, the fixed points are the hyperplanes

$$v_i = v_j \quad i, j \in \{0, 1, .., n-1\}. \tag{8.191}$$

The intersection form (8.37) is given by

$$\begin{aligned} g &= \sum_{i=0}^{n} dv_i^2|_{\sum_{i=0}^{n} v_i=0} - n(n+1)dv_{n+1}^2 + 2dud\tau \\ &= \sum_{i,j=0}^{n-1} A_{ij}dv_i dv_j - n(n+1)dv_{n+1}^2 + 2dud\tau \end{aligned} \tag{8.192}$$

The intersection form is given by

$$g^* = \sum_{i,j=0}^{n-1} A_{ij}^{-1}\frac{\partial}{\partial v_i}\otimes\frac{\partial}{\partial v_j} - \frac{1}{n(n+1)}\frac{\partial}{\partial v_{n+1}}\otimes\frac{\partial}{\partial v_{n+1}} + \frac{\partial}{\partial u}\otimes\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \tau}\otimes\frac{\partial}{\partial u} \tag{8.193}$$

became degenerate on the hyperplanes (8.191), because two columns of the matrix A_{ij}^{-1} became proportional.

□

Lemma 8.10.2. The function $\lambda(p, u, v_0, v_1, .., v_{n+1}, \tau)$ defined on (8.24) has simple critical points if and only if $(u, v_0, v_1, .., v_{n+1}, \tau)$ is a fixed point of the action $\mathscr{J}(\tilde{A}_n)$.

PROOF. Using the local isomorphism given by 8.20

$$[(u, v_0, v_1, .., v_{n-1}, v_{n+1}, \tau)] \longleftrightarrow \lambda(v) = e^{-2\pi i u} \frac{\prod_{i=0}^{n} \theta_1(z - v_i, \tau)}{\theta_1^n(v, \tau)\theta_1(v + (n+1)v_{n+1}, \tau)}, \tag{8.194}$$

we can realise the discriminant locus as the space of parameters of $\lambda(p, u, v_0, v_1, .., v_{n+1}, \tau)$ such that $\lambda(p, u, v_0, v_1, .., v_{n+1}, \tau)$ has repeated roots. In these cases $\lambda(p, u, v_0, v_1, .., v_{n+1}, \tau)$ has non simple critical points.

□

DEFINITION 8.10.2. The canonical coordinates $(u_1, u_2, .., u_{n+2})$ of the orbit space $\mathscr{J}(\tilde{A}_n)$ is given by the following relation

$$\begin{aligned} \lambda(q_i) &= u_i, \\ \lambda'(q_i) &= 0. \end{aligned} \tag{8.195}$$

LEMMA 8.10.3. The determinant of the intersection form g^* defined on (8.36) is proportional to $\prod_{i=1}^{n+2} u_i$.

PROOF. If $u_i = \lambda(q_i, u, v_0, .., v_{n+1}, \tau) = 0$, we have that $\det g^* = 0$ due to lemma (8.10.2), then u_i are zeros of the equation $\det g^* = 0$.

□

PROPOSITION 8.10.4. In the canonical coordinates $(u_1, u_2, .., u_{n+2})$ the unit vector field (8.187), the Euler vector field (8.177), and the intersection form (8.36) have the following form

$$\begin{aligned} g^{ii} &= u^i \eta^{ii} \delta_{ij}, \\ e &= \sum_{i=1}^{n+2} \frac{\partial}{\partial u_i}, \\ E &= \sum_{i=1}^{n+2} u_i \frac{\partial}{\partial u_i}. \end{aligned} \tag{8.196}$$

where η^{ii} are the coefficients of second metric η^* in canonical coordinates.

PROOF. Note that g^* is diagonalisable with distinct eigenvalues if the following equation

$$det(\eta_{\alpha\mu} g^{\mu\beta} - u\delta_\alpha^\beta) = 0, \tag{8.197}$$

has only simple roots. Since $det(\eta_{\alpha\mu}) \neq 0$, the equation (8.197) is equivalent to

$$det(g^{\alpha\beta} - u\eta^{\alpha\beta}) = 0. \tag{8.198}$$

Using that $\eta^{\alpha\beta} = \partial_0 g^{\alpha\beta}$, we have that

$$det(g^{\alpha\beta} - u\eta^{\alpha\beta}) = det\left(g^{\alpha\beta}(t^0 - u, t^1, t^2, t^3, .., t^n, v_{n+1}, \tau\right) = 0. \tag{8.199}$$

Due to the lemma 8.10.3 the equation (8.198) has $n + 2$ distinct roots

$$u^i = t^1 - y^i(t^1, t^2, t^3, .., t^n, v_{n+1}, \tau). \tag{8.200}$$

In the coordinates $(u^1, u^2, .., u^{n+2})$ the matrix g^i_j is diagonal, then

$$g^{ij} = u^i \eta^{ij} \delta_{ij} \tag{8.201}$$

and the unit vector field have the following form

$$\frac{\partial}{\partial t^1} = \sum_{i=1}^{n+2} \frac{\partial u_i}{\partial t^1} \frac{\partial}{\partial u_i} = \sum_{i=1}^{n+2} \frac{\partial}{\partial u_i} \tag{8.202}$$

Moreover, since

$$[E, e] = [\sum_{\alpha=0}^{n} d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^1}] = -e, \tag{8.203}$$

the Euler vector field in the coordinates $(u^1, u^2, .., u^{n+2})$ takes the following form

$$E = \sum_{i=1}^{n+2} u^i \frac{\partial}{\partial u_i}. \tag{8.204}$$

Lemma proved.

□

8.11. Construction of WDVV solution

The main aim of this section is to extract a WDVV equation from the data of the group $\mathscr{J}(\tilde{A}_n)$.

LEMMA 8.11.1. The orbit space of $\mathscr{J}(\tilde{A}_n)$ carries a flat pencil of metrics

$$g^{\alpha\beta}, \quad \eta^{\alpha\beta} := \frac{\partial g^{\alpha\beta}}{\partial t^0} \tag{8.205}$$

with the correspondent Christoffel symbols.

$$\Gamma^{\alpha\beta}_{\gamma}, \quad \eta^{\alpha\beta} := \frac{\partial \Gamma^{\alpha\beta}_{\gamma}}{\partial t^0} \tag{8.206}$$

PROOF. The metric (8.205) satisfies the hypothesis of Lemma 4.8.1 which proves the desired result.

□

LEMMA 8.11.2. Let the intersection form be (8.36), unit vector field be (8.187), and Euler vector field be (8.177). Then, there exist a function

$$F(t^0, t^1, t^2, .., t^n.v_{n+1}, \tau) = -\frac{(t^0)^2\tau}{4\pi i} + \frac{t^0}{2} \sum_{\alpha,\beta\neq 0,\tau} \eta_{\alpha\beta} t^\alpha t^\beta + G(t^1, t^2, .., t^n, v_{n+1}, \tau), \tag{8.207}$$

such that

$$\begin{aligned} &Lie_E F = 2F + \text{quadratic terms}, \\ &Lie_E \left(F^{\alpha\beta}\right) = g^{\alpha\beta}, \\ &\frac{\partial^2 G(t^1, t^2, .., t^n, v_{n+1}, \tau)}{\partial t^\alpha \partial t^\beta} \in \widetilde{E}_{\bullet,\bullet}[t^1, t^2, .., t^n, \frac{1}{t^n}], \end{aligned} \tag{8.208}$$

where

$$F^{\alpha\beta} = \eta^{\alpha\alpha'}\eta^{\beta\beta'}\frac{\partial F^2}{\partial t^{\alpha'}\partial t^{\beta'}}. \tag{8.209}$$

PROOF. Let $\Gamma^{\alpha\beta}_{\gamma}(t)$ the Christoffel symbol of the intersection form (8.36) in the coordinates the flat coordinates of η^*, i.e $t^0, t^1, t^2, .., t^n.v_{n+1}, \tau$. According to the lemma 4.8.1, we can represent $\Gamma^{\alpha\beta}_{\gamma}(t)$ as

$$\Gamma^{\alpha\beta}_{\gamma}(t) = \eta^{\alpha\epsilon}\partial_\epsilon\partial_\gamma f^\beta(t). \tag{8.210}$$

Using the relations (8.186), (8.183) and lemma 8.9.6

$$\begin{aligned} Lie_E(\Gamma^{\alpha\beta}_{\gamma}(t)) &= Lie_E(\eta^{\alpha\epsilon})\partial_\epsilon\partial_\gamma f^\beta(t) + \eta^{\alpha\epsilon}Lie_E(\partial_\epsilon\partial_\gamma f^\beta(t)) \\ &= (d_\alpha + d_\epsilon - d_1)\eta^{\alpha\epsilon}\partial_\epsilon\partial_\gamma f^\beta(t) + (-d_\epsilon - d_\gamma)\eta^{\alpha\epsilon}\partial_\epsilon\partial_\gamma Lie_E(f^\beta(t)) \\ &= (d_\alpha + d_\beta - d_\gamma)\eta^{\alpha\epsilon}\partial_\epsilon\partial_\gamma f^\beta(t). \end{aligned}$$

Then, by isolation $Lie_E\left(f^\beta(t)\right)$ we get

$$Lie_E\left(f^\beta(t)\right) = (d_\beta + d_1)f^\beta + A^\beta_\sigma t^\sigma + B^\beta, \quad A^\beta_\sigma, B^\beta \in \mathbb{C}. \tag{8.211}$$

Considering the second relation of (4.13) for $\alpha = \tau$

$$g^{\tau\sigma}\Gamma^{\beta\gamma}_{\sigma} = g^{\beta\sigma}\Gamma^{\tau\gamma}_{\sigma}, \tag{8.212}$$

and using lemma 8.5.3 and 8.8.2, we have.

$$-2\pi i d_\sigma t^\sigma\eta^{\beta\epsilon}\partial_\sigma\partial_\epsilon f^\gamma = -2\pi i d_\sigma\delta^\gamma_\sigma g^{\beta\sigma}, \tag{8.213}$$

which is equivalent to

$$Lie_E\left(\eta^{\beta\epsilon}\partial_\epsilon f^\gamma\right) = d_\gamma g^{\beta\gamma}. \tag{8.214}$$

Using (8.211) in the equation (8.214), we have

$$(d_\beta + d_\gamma)\eta^{\beta\epsilon}\partial_\epsilon f^\gamma = d_\gamma g^{\beta\gamma}. \tag{8.215}$$

If $\gamma \neq v_{n+1}, \tau$, we define

$$F^\gamma = \frac{f^\gamma}{d_\gamma}, \tag{8.216}$$

and note that $g^{\beta\gamma}$ is symmetric with respect the indices β, γ. Hence,

$$(d_\beta + d_\gamma)\eta^{\beta\epsilon}\partial_\epsilon F^\gamma = (d_\beta + d_\gamma)\eta^{\gamma\epsilon}\partial_\epsilon F^\beta, \tag{8.217}$$

which is the integrability condition for

$$F^\gamma = \eta^{\gamma\mu}\partial_\mu F. \tag{8.218}$$

In order to extract information from $\gamma = \tau$, take $\beta = \tau$ in equation (8.215)

$$(8.219)\qquad \begin{aligned} d_\gamma \eta^{\tau 0}\partial_0 f^\gamma &= d_\gamma g^{\tau\gamma} \\ -2\pi i d_\gamma \partial_0 f^\gamma &= -2\pi i d_\gamma t^\gamma \end{aligned}$$

which is equivalent to

$$\eta^{\gamma\epsilon}\partial_\epsilon\partial_0 F = t^\gamma,$$

inverting $\eta^{\gamma\epsilon}$

$$(8.220)\qquad \partial_\alpha\partial_0 F = \eta_{\alpha\gamma}t^\gamma,$$

integrating equation (8.220), we obtain

$$(8.221)\qquad F(t^0, t^1, t^2, .., t^n, v_{n+1}, \tau) = -\frac{(t^0)^2\tau}{4\pi i} + \frac{t^0}{2}\sum_{\alpha,\beta\neq 0,\tau}\eta_{\alpha\beta}t^\alpha t^\beta + G(t^1, t^2, .., t^n, v_{n+1}, \tau).$$

Substituting the equation (8.221) in the (8.215) for $\gamma \neq v_{n+1}, \tau$, we get

$$(8.222)\qquad \begin{aligned} g^{\beta\gamma} &= (d_\beta + d_\gamma)\eta^{\beta\epsilon}\eta^{\gamma\mu}\partial_\epsilon\partial_\mu F, \\ &= Lie_E(F^{\beta\gamma}) \end{aligned}$$

Since $g^{\beta\gamma}$ is a symmetric matrix the equation (8.222) is equivalent to the second equation of (8.208) for either β and γ different from v_{n+1}, τ. Therefore, the missing part of the second equation of (8.208) is only for the cases $\beta = \gamma = v_{n+1}$ and $\beta = \gamma = \tau$. Moreover, the intersection form $g^{\beta\gamma}$ is proportional to the Hessian of the equation (8.221) for for either β and γ different from v_{n+1}, τ. Recall that from the data of a Hessian, we can reconstruct uniquely a function up to quadratic terms, therefore, by defining

$$(8.223)\qquad \begin{aligned} Lie_E\left(\frac{\partial^2 F}{\partial t^{1^2}}\right) &= g^{v_{n+1}v_{n+1}}, \\ Lie_E\left(\frac{\partial^2 F}{\partial t^{1^2}}\right) &= g^{\tau\tau}. \end{aligned}$$

Just the second equation of (8.223) needs to be proved, since the first equation defines the coefficients of the Hessian $\frac{\partial^2 F}{\partial t^{1^2}}$, in another words, it defines $\frac{\partial^2 G}{\partial t^{1^2}}$. The second equation must be compatible with the equation (8.221), then substituting (8.221) in the second equation of (8.223).

$$\begin{aligned} Lie_E\left(\frac{\partial^2 F}{\partial t^{1^2}}\right) &= Lie_E\left(\frac{\tau}{2\pi i}\right) \\ &= 0 = g^{\tau\tau}. \end{aligned}$$

Hence, we proved the second equation (8.208). Substituting the equation (8.221) in the second equation (8.208) for $\alpha, \beta \neq \tau$

$$Lie_E\left(F^{\alpha\beta}\right) = Lie_E\left(\eta^{\alpha\alpha'}\eta^{\beta\beta'}\frac{\partial F^2}{\partial t^{\alpha'}\partial t^{\beta'}}\right)$$
$$= Lie_E\left(\eta^{\alpha\alpha'}\eta^{\beta\beta'}\frac{\partial G^2}{\partial t^{\alpha'}\partial t^{\beta'}}\right)$$
$$= g^{\alpha\beta} \in \widetilde{E}_{\bullet,\bullet}[t^1, t^2, .., t^n, \frac{1}{t^n}]$$

Hence, the second equation (8.208 prove the third equation of (8.208).

Substituting (8.221) in (8.211), (8.218)

$$Lie_E\left(f^{\beta}\right) = Lie_E\left(\eta^{\beta\epsilon}\partial_\epsilon F\right)$$
$$= Lie_E\left(\eta^{\beta\epsilon}\partial_\epsilon F\right)\partial_\epsilon F + \eta^{\beta\epsilon} Lie_E\left(\partial_\epsilon F\right)$$
$$= (d_\beta + d_\epsilon - d_1)\eta^{\beta\epsilon}\partial_\epsilon F\partial_\epsilon F + \eta^{\beta\epsilon}\partial_\epsilon Lie_E\left(F\right) - d_\epsilon\eta^{\beta\epsilon}\partial_\epsilon F$$
$$= (d_\beta - d_1)\eta^{\beta\epsilon}\partial_\epsilon F\partial_\epsilon F + \eta^{\beta\epsilon}\partial_\epsilon Lie_E\left(F\right)$$
$$= (d_\beta + d_1)\eta^{\beta\epsilon}\partial_\epsilon F + A^{\beta}_{\sigma}t^{\sigma} + B^{\beta}$$

Hence, isolating $Lie_E\left(F\right)$

$$\eta^{\beta\epsilon}\partial_\epsilon Lie_E\left(F\right) = 2\eta^{\beta\epsilon}\partial_\epsilon F + A^{\beta}_{\sigma}t^{\sigma} + B^{\beta},$$

inverting $\eta^{\beta\epsilon}$

$$\partial_\alpha Lie_E\left(F\right) = 2\partial_\alpha F + \eta_{\alpha\beta}A^{\beta}_{\sigma}t^{\sigma} + \eta_{\alpha\beta}B^{\beta},$$

integrating

$$Lie_E\left(F\right) = 2F + \eta_{\alpha\beta}A^{\beta}_{\sigma}t^{\alpha}t^{\sigma} + \eta_{\alpha\beta}B^{\beta}t^{\alpha},$$

Lemma proved.

□

LEMMA 8.11.3. Let be

$$c_{\alpha\beta\gamma} = \frac{\partial F^3}{\partial t^\alpha\partial t^\beta\partial t^\gamma}, \tag{8.224}$$

then,

$$c^{\gamma}_{\alpha\beta} = \eta^{\gamma\epsilon}c_{\alpha\beta\epsilon} \tag{8.225}$$

is a structure constant of a commutative algebra given by the following rule in the flat coordinate of η

$$\partial_\alpha \bullet \partial_\beta = c^{\gamma}_{\alpha\beta}\partial_\gamma \tag{8.226}$$

such that

$$\eta(\partial_\alpha \bullet \partial_\beta, \partial_\gamma) = \eta(\partial_\alpha, \partial_\beta \bullet \partial_\gamma), \quad \textbf{Frobenius condition}. \tag{8.227}$$

Proof.

(1) **Commutative**
The product defined in (8.226) is commutative, because its structure constant (8.225) is symmetric with respect its indices α, β, γ due to the commutative behaviour of the partial derivatives $\frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta}, \frac{\partial}{\partial t^\gamma}$.

(2) **Frobenius condition**

$$\begin{aligned}
\eta(\partial_\alpha \bullet \partial_\beta, \partial_\gamma) &= c^\epsilon_{\alpha\beta}\eta(\partial_\epsilon, \partial_\gamma)\\
&= c^\epsilon_{\alpha\beta}\eta_{\epsilon\gamma}\\
&= c_{\alpha\beta\gamma}\\
&= c^\epsilon_{\beta\gamma}\eta_{\alpha\epsilon} = \eta(\partial_\alpha, \partial_\beta \bullet \partial_\gamma).
\end{aligned}$$

Lemma proved.

□

Lemma 8.11.4. The unit vector field be defined in (8.187) is the unit of the algebra defined in lemma 8.11.3.

Proof. Substituting (8.218) and (8.216) in (8.210), we obtain

$$\Gamma^{\alpha\beta}_\gamma = d_\beta c^{\alpha\beta}_\gamma, \tag{8.228}$$

where

$$c^{\alpha\beta}_\gamma = \eta^{\alpha\mu}\eta^{\beta\epsilon}c_{\epsilon\mu\gamma}. \tag{8.229}$$

Substituting $\alpha = \tau$ in (8.228) and using lemma 8.8.2

$$\begin{aligned}
\Gamma^{\tau\beta}_\gamma &= -2\pi i d_\beta \delta^\beta_\gamma,\\
&= d_\beta c^{\tau\beta}_\gamma.
\end{aligned}$$

Then,

$$c^\beta_{0\gamma} = \delta^\beta_\gamma.$$

Computing

$$\partial_0 \bullet \partial_\gamma = c^\beta_{0\gamma}\partial_\beta = \partial_\gamma.$$

Lemma proved. □

Lemma 8.11.5. The algebra defined in lemma 8.11.3 is associative and semisimple.

PROOF. Recall that the Christoffel symbol $\Gamma_{\gamma}^{\alpha\beta}$ is proportional to the structure constant of the algebra defined in lemma 8.11.3 for $\beta \neq v_{n+1}, \tau$

$$\Gamma_{\gamma}^{\alpha\beta} = d_{\beta} c_{\gamma}^{\alpha\beta}.$$

Then, using (4.59), we obtain

$$\Gamma_{\sigma}^{\alpha\beta}\Gamma_{\epsilon}^{\sigma\gamma} = \Gamma_{\sigma}^{\alpha\gamma}\Gamma_{\epsilon}^{\sigma\beta} \tag{8.230}$$

Substituting (8.228) in (8.230), we have

$$c_{\sigma}^{\alpha\beta} c_{\epsilon}^{\sigma\gamma} = c_{\sigma}^{\alpha\gamma} c_{\epsilon}^{\sigma\beta}, \quad \text{for} \quad \beta, \gamma \neq v_{n+1}, \tau.$$

If $\beta = \tau$,

$$\begin{aligned} c_{\sigma}^{\alpha\tau} c_{\epsilon}^{\sigma\gamma} &= -2\pi i \delta_{\sigma}^{\alpha} c_{\epsilon}^{\sigma\gamma} \\ &= -2\pi i c_{\epsilon}^{\alpha\gamma} \\ &= -2\pi i \delta_{\epsilon}^{\sigma} c_{\sigma}^{\alpha\gamma} \\ &= c_{\sigma}^{\alpha\gamma} c_{\epsilon}^{\sigma\tau}. \end{aligned}$$

In order to prove the associativity for $\beta = v_{n+1}$, note that the multiplication by the Euler vector field is almost the same of the intersection form g^*. Indeed,

$$\begin{aligned} E \bullet \partial_{\alpha} &= t^{\sigma} c_{\sigma\alpha}^{\beta} \partial_{\beta} = t^{\sigma} \partial_{\sigma} \left(\eta^{\beta\mu} \partial_{\alpha} \partial_{\mu} F \right) \partial_{\beta} = \\ &= (d_{\alpha} - d_{\beta}) \eta^{\beta\mu} \partial_{\alpha} \partial_{\mu} F \partial_{\beta} = \eta_{\alpha\mu} g^{\mu\beta} \partial_{\beta} \end{aligned} \tag{8.231}$$

Then,

$$E^{\sigma} c_{\sigma}^{\alpha\beta} = g^{\alpha\beta}. \tag{8.232}$$

Using the relation (8.232) in the coordinates $(u^1, u^2, ..., u^{n+2})$, we have

$$u^i \eta^{ij} \delta_{ij} = u^l \eta^{im} \eta^{jn} c_{lmn}, \tag{8.233}$$

differentiating both side of the equation (8.233) with respect t^1

$$c_{ij}^{k} = \delta_{ij}, \tag{8.234}$$

which proves that the algebra is associative and semisimple.

□

Recall of the covering space of the orbit space of $\mathscr{J}(\tilde{A}_n)$ defined in (8.145), see section 8.7 for details.

THEOREM 8.11.6. The covering $\widetilde{\mathbb{C} \oplus \mathbb{C}^n} \oplus \mathbb{H} / \mathscr{J}(\tilde{A}_n)$ with the intersection form (8.36), unit vector field (8.187), and Euler vector field (8.177) has a Dubrovin Frobenius manifold structure.

PROOF. The function (8.207) satisfy a WDVV equation due to the lemmas 8.11.2, 8.11.3, 8.11.4, 8.11.5.

□

Taking the same covering taking in the orbit space of $\mathscr{J}(\tilde{A}_n)$ in the Hurwitz space $H_{1,n-1,0}$, fixing a symplectic base of cycle, a chamber in the tori where the variable v_{n+1} lives, and branching root of φ_n, denoting this covering by

$$\widetilde{H}_{1,n-1,0}, \tag{8.235}$$

we obtain

THEOREM 8.11.7. The Dubrovin Frobenius structure of the covering space $\mathbb{C} \oplus \widetilde{\mathbb{C}^n} \oplus \mathbb{H}/\mathscr{J}(\tilde{A}_n)$ is isomorphic as Dubrovin Frobenius manifold to the covering $\widetilde{H}_{1,n-1,0}$.

PROOF. Both the orbit space $\mathscr{J}(\tilde{A}_n)$ and the Hurwitz space $H_{1,n-1,0}$ has the same intersection form, Euler vector, unit vector field due to proposition 8.10.4, lemma 8.10.2 and 8.10.3 From this data, one can reconstruct the WDVV solution by using the relation

$$F^{\alpha\beta} = \eta^{\alpha\alpha'}\eta^{\beta\beta'}\frac{\partial^2 F}{\partial t^{\alpha'}\partial t^{\beta'}} = \frac{g^{\alpha\beta}}{\deg g^{\alpha\beta}}. \tag{8.236}$$

Theorem proved. □

Bibliography

[1] G. F. Almeida, *Differential Geometry of Orbit space of Extended Affine Jacobi Group A_1* preprint arxiv:1907.01436 (2020).

[2] G. F. Almeida, *Differential Geometry of Orbit space of Extended Affine Jacobi Group A_n; Parts I,* preprint arxiv:2004.01780(2020).

[3] G. F. Almeida, *Differential Geometry of Orbit space of Extended Affine Jacobi Group A_n; Parts II,* in preparation.

[4] A. Belavin, V. Belavin, *Flat structures on the deformations of Gepner chiral rings,* J. High Energy Phys., 10, 128 (2016)

[5] A. Belavin, V. Belavin, *Frobenius manifolds, integrable hierarchies and minimal Liouville* gravity, JHEP 09 (2014) 151.

[6] A. Belavin, D. Gepner and Y. Kononov, *Flat coordinates for Saito Frobenius manifolds and String theory,* arXiv:1510.06970

[7] V. Belavin, *Unitary minimal Liouville gravity and Frobenius manifolds,* JHEP 07 (2014) 129.

[8] M. Bertola, *Frobenius manifold structure on orbit space of Jacobi groups; Parts I,* Diff. Geom. Appl. 13, (2000), 19-41.

[9] M. Bertola, *Frobenius manifold structure on orbit space of Jacobi groups; Parts II,* Diff. Geom. Appl. 13, (2000), 213-23.

[10] Bourbaki N., *Groupes et Algebres de Lie, Chapitres 4, 5 et 6,* Masson, Paris-New York-Barcelone-Milan-Mexico-Rio de Janeiro, 1981.

[11] B. Dubrovin, *Differential geometry of the space of orbits of a Coxeter group,* Preprint SISSA–29/93/FM (February 1993), hep–th/9303152.

[12] B. Dubrovin, *Geometry of 2D topological field theories,* in: M. Francaviglia and S. Greco, eds., Integrable Systems and Quantum Groups, Montecatini Terme, 1993, Lecture Notes in Math. 1620 (Springer, Berlin, 1996) 120–384.

[13] B. Dubrovin, *Hamiltonian perturbations of hyperbolic PDEs: from classification results to the properties of solutions,* New Trends in Mathematical Physics (Selected Contributions of the XVth Int. Congress on Mathematical Physics) ed V Sidoravicius (Netherlands: Springer) pp 231–76.

[14] B. Dubrovin, I. A. B. Strachan, Y. Zhang, D. Zuo, *Extended affine Weyl groups of BCD type, Frobenius manifolds and their Landau-Ginzburg superpotentials,* Advances in Mathematics, Volume 351, 2019, Pages 897-946.

[15] B. Dubrovin and Y. Zhang, *Extended affine Weyl groups and Frobenius manifolds,* Comp. Math. 111 (1998) 167–219.

[16] B. Dubrovin and Y. Zhang, *Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants,* preprint, available at http://arxiv.org/math.DG/0108160.

[17] M. Cutimanco, and V. Shramchenko, *Explicit examples of Hurwitz Frobenius manifolds in genus one,* J. Math. Phys. 61, 013501 (2020).

[18] M. Eichler and D. Zagier, *The Theory of Jacobi Forms,* (Birkhauser, Boston, 1985).

[19] E. V. Ferapontov, and M. V. Pavlov, L. Xue, *Second-order integrable Lagrangians and WDVV equations,* arXiv:2007.03768, 2020.

[20] C. Hertling, *Multiplication on the tangent bundle,* math.AG/9910116.

[21] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function,* Comm. Math. Phys. 147 (1992), 1-23.

[22] Krichever I.M, *The τ -function of the universal Whitham hierarchy, matrix models and topological field theories,* Comm. Pure Appl. Math. 47 (1994) 437 - 475.

[23] E. K. Morrison, and I. A. B. Strachan, *Modular Frobenius manifolds and their invariant flows,* International Mathematics Research Notices, Volume 2011, Issue 17, 2011, Pages 3957–3982.

[24] E. K. Morrison, and I. A. B. Strachan, *Polynomial modular Frobenius manifolds,* Phys. D 241 (2012), 2145–2155, arXiv:1110.4021.

[25] M. V. Pavlov, S. P. Tsarev,, *Tri-Hamiltonian structures of the Egorov systems,* Funct. Anal. and Appl. 37:1 (2003), 32-45.

[26] S. Romano, *4-dimensional Frobenius manifolds and Painleve' VI,* preprint arxiv 1209.3959 (2012).

[27] S. Romano, *Frobenius structures on double Hurwitz spaces,* Int. Math. Res. Notices, doi:10.1093/imrn/rnt215 (2013), available at 1210.2312.

[28] Saito K., Yano T., and Sekeguchi J., *On a certain generator system of the ring of invariants of a finite reflection group,* Comm. in Algebra 8(4) (1980) 373 - 408.

[29] V. Shramchenko, *Deformations of Frobenius structures on Hurwitz spaces,* International Mathematics Research Notices, Volume 2005, Issue 6, 2005, Pages 339–387.

[30] V. Shramchenko, *Riemann-Hilbert problem associated to Frobenius manifold structures on Hurwitz spaces: irregular singularity,* Duke Math. J. 144, no. 1, 1-52 (2008)

[31] I. A. B. Strachan, *Frobenius submanifolds,* J. Geom. Phys. 38:3-4 (2001), 285-307.

[32] I. A. B. Strachan, *Frobenius manifolds: natural submanifolds and induced bi-hamiltonian structures,* Diff. Geom. Appl. 20:1 (2004), 67-99.

[33] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis,* IV ed. (Cambridge University Press, Cambridge, 1980).

[34] K. Wirthmüller, *Root systems and Jacobi forms,* Comp. Math. 82 (1992) 293–354.

[35] D. Zuo, *Frobenius manifolds and a new class of extended affine Weyl groups of A-type,* Lett. Math. Phys. 110 (2020), no. 7.

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