Stabilized reduced basis methods for parametrized steady Stokes and Navier–Stokes equations

note finali coverpage

(Article begins on next page)

04 May 2024

Stabilized reduced basis methods for parametrized steady Stokes and Navier-Stokes equations

Shafqat Ali*[1, 2], Francesco Ballarin†[1], and Gianluigi Rozza‡[1]

[1]mathLab, Mathematics area, SISSA, via Bonomea 265, I-34136, Trieste, Italy
[2]Current address: Faculty of Engineering Sciences, Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, Topi, 44000 Pakistan

Abstract

It is well known in the Reduced Basis approximation of saddle point problems that the Galerkin projection on the reduced space does not guarantee the inf-sup approximation stability even if a stable high fidelity method was used to generate snapshots. For problems in computational fluid dynamics, the lack of inf-sup stability is reflected by the inability to accurately approximate the pressure field. In this context, inf-sup stability is usually recovered through the enrichment of the velocity space with suitable supremizer functions. The main goal of this work is to propose an alternative approach, which relies on the residual based stabilization techniques customarily employed in the Finite Element literature, such as Brezzi-Pitkaranta, Franca-Hughes, streamline upwind Petrov-Galerkin, Galerkin Least Square. In the spirit of *offline-online* reduced basis computational splitting, two such options are proposed, namely *offline-only stabilization* and *offline-online stabilization*. These approaches are then compared to (and combined with) the state of the art supremizer enrichment approach. Numerical results are discussed, highlighting that the proposed methodology allows to obtain smaller reduced basis spaces (i.e., neglecting supremizer enrichment) for which a modified inf-sup stability is still preserved at the reduced order level.

Keywords: reduced basis method, *offline-online stabilization*, RB inf-sup stability

1   Introduction

Numerical approximation of fluid dynamics problems (modelled by incompressible Stokes and Navier-Stokes equations) is required by several applications in mechanics and engineering, often depending on some parameters. In case of repeated evaluation for different parametric configurations, the Reduced Basis (RB) method [23] offers attractive performance, able to cut down computational costs of standard Finite Element (FE) [41] simulations. In the context of fluid dynamics problems, formulations based on a Galerkin projection, such as FE and RB, require the fulfillment of a discrete inf-sup condition [8]. While several choices of suitable discrete spaces for the velocity and pressure are well known at the FE level [8, 41], the corresponding RB formulation still deserves investigation, even after more than a decade from the original supremizer enrichment proposal [43, 42, 46]. This approach consists in the introduction of the

---

*ali.qau1987@gmail.com, shafqat.ali@giki.edu.pk
†francesco.ballarin@sissa.it
‡gianluigi.rozza@sissa.it (corresponding author)

inner pressure supremizer for the velocity-pressure stability of the RB spaces. Several works on RB method for Stokes problem [35, 45] were developed subsequently using the pressure stabilization via the inner pressure supremizer operator. This approach was then extended to Navier-Stokes problems as well [15, 17, 34, 52, 4].

However, even though nowadays the supremizer enrichment approach is widely used in the RB community, it still carries the side effect of increasing the RB velocity space, resulting in a possible increase in the online computational time. Therefore, in this work we look for an alternative approach which does not entail an enrichment procedure, yet results in a stable RB approximation.

The stabilization approach developed in this work builds upon classical residual based stabilization techniques [25, 12, 26, 11, 18, 53] already known for FE methods, and properly adapted to a reduced order setting in this work. Such stabilization techniques have been employed to handle two sources of instabilities. The first one (which motivates our work) is the incompatibility of velocity and pressure FE pairs. Another possible instability is due to dominating advection for problems characterized by large Reynolds numbers. In this work we will only focus on the inf-sup stability issue, referring to [36, 21, 1, 3, 51] for reduced order methods for advection dominated problems.

The idea of combining stabilization techniques to model reduction methods follows the work of Pacciarini and Rozza [36] and Torlo et al. [51] for advection-dominated problem, where they introduced the concept of *offline-online stabilization* and the *offline-only stabilization*. The main novelty here is the comparison and combination to the supremizer approach by Rozza et al. [46], as both methodologies aim at preserving the inf-sup stability of the reduced order system. As in [36, 38, 37], the *offline-online stabilization* method is based on performing the Galerkin projection with respect to the stabilized formulation in both *offline* and *online* stages. In contrast, the *offline-only stabilization* consists in using the stabilized formulation only during the *offline* stage, while projecting with respect to the standard formulation during the *online* stage.

We refer to some related works in recent past on the stabilization of reduced order models, mostly based on POD, see for instance Baiges et al. [2, 1], Caiazzo et al. [14] and Løvgren et al. [16]. The difference in our approach with respect to previous works is that the stabilization approach we are using is residual based and is a strongly consistent stabilization.

This manuscript is mainly divided into two parts, with the focus on steady Stokes problem in first part and steady Navier-Stokes problem in second part, respectively. After this introduction, in section 2, we present FE and RB approximation for the steady Stokes problem in a parametrized domain, with a particular focus on the introduction of residual based stabilization techniques in the formulation. In section 3 we show some numerical results and error analysis to compare different stabilization options. In the second part, section 4 shows FE and RB discretizations for the steady Navier-Stokes problem. Afterwards, in section 5 we show some numerical results on a benchmark case, addressing the performance (in terms of cost and accuracy) of different stabilization approaches presented. Finally some conclusions and perspectives are provided in section 6.

2 Parametrized steady Stokes problem

The steady Stokes problem in a two-dimensional parametrized domain $\Omega_o(\boldsymbol{\mu}) \subset \mathbb{R}^2$ read as: find $\boldsymbol{u}_o(\boldsymbol{\mu}) : \Omega_o(\boldsymbol{\mu}) \to \mathbb{R}^2$ and $p_o(\boldsymbol{\mu}) : \Omega_o(\boldsymbol{\mu}) \to \mathbb{R}^2$ such that

$$
\begin{cases}
-\nu(\boldsymbol{\mu})\Delta\boldsymbol{u}_o(\boldsymbol{\mu}) + \nabla p_o(\boldsymbol{\mu}) = \boldsymbol{0} & \text{in } \Omega_o(\boldsymbol{\mu}), \\
\text{div } \boldsymbol{u}_\text{o}(\boldsymbol{\mu}) = 0 & \text{in } \Omega_o(\boldsymbol{\mu}), \\
\boldsymbol{u}_o(\boldsymbol{\mu}) = \boldsymbol{g}_D(\boldsymbol{\mu}) & \text{on } \Gamma_{D,o}(\boldsymbol{\mu}), \\
\boldsymbol{u}_o(\boldsymbol{\mu}) = \boldsymbol{0}, & \text{on } \Gamma_{W,o}(\boldsymbol{\mu}),
\end{cases}
\tag{1}
$$

where $\boldsymbol{u}_o(\boldsymbol{\mu})$ is the unknown velocity and $p_o(\boldsymbol{\mu})$ is the unknown pressure, $\nu(\boldsymbol{\mu})$ is the viscosity of fluid. These quantities, as well as the domain $\Omega_o(\boldsymbol{\mu})$, may depend on a parameter $\boldsymbol{\mu} \in \mathbb{P}$ which accounts for parametrized physical properties or geometrical configurations. The boundary $\partial\Omega_o(\boldsymbol{\mu})$ is divided into two parts in such a way that $\partial\Omega_o(\boldsymbol{\mu}) = \Gamma_{D,o}(\boldsymbol{\mu}) \cup \Gamma_{W,o}(\boldsymbol{\mu})$, where $\Gamma_{D,o}(\boldsymbol{\mu})$ is the Dirichlet boundary with non-homogeneous data and $\Gamma_{W,o}(\boldsymbol{\mu})$ denotes the Dirichlet boundary with zero data (i.e., walls).

In order to write the weak formulation of problem (1), we introduce a reference domain, i.e. a $\boldsymbol{\mu}$-independent configuration Ω by assuming that each parametrized domain $\Omega_o(\boldsymbol{\mu})$ can be obtained as the image of $\boldsymbol{\mu}$-independent domain Ω through a parametrized map $\boldsymbol{T}(.; \boldsymbol{\mu}) : \mathbb{R}^2 \to \mathbb{R}^2$, i.e. $\Omega_o(\boldsymbol{\mu}) = \boldsymbol{T}(\Omega; \boldsymbol{\mu})$.

Now the weak formulation of (1) can be obtained by multiplying with the velocity, pressure test functions and using integration by parts; then by tracing everything back onto the reference domain Ω, we have the following parametrized weak formulation of problem (1):

$$
\begin{cases}
\text{Find } \boldsymbol{u}(\boldsymbol{\mu}) \in \boldsymbol{V}, \, p(\boldsymbol{\mu}) \in Q: \\
a(\boldsymbol{u}(\boldsymbol{\mu}), \boldsymbol{v}; \boldsymbol{\mu}) + b(\boldsymbol{v}, p(\boldsymbol{\mu}); \boldsymbol{\mu}) = F(\boldsymbol{v}; \boldsymbol{\mu}) & \forall v \in \boldsymbol{V}, \\
b(\boldsymbol{u}(\boldsymbol{\mu}), q; \boldsymbol{\mu}) = G(q; \boldsymbol{\mu}) & \forall q \in Q,
\end{cases}
\tag{2}
$$

where $\boldsymbol{V} = [H_0^1(\Omega)]^2$ and $Q = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\}$. Bilinear forms related to diffusion and pressure-divergence operators are defined as:

$$
a(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}) = \int_\Omega \frac{\partial\boldsymbol{u}}{\partial x_i}\kappa_{ij}(x; \boldsymbol{\mu})\frac{\partial\boldsymbol{v}}{\partial x_j}d\boldsymbol{x}, \qquad b(\boldsymbol{v}, q; \boldsymbol{\mu}) = -\int_\Omega q\chi_{ij}(x; \boldsymbol{\mu})\frac{\partial v_j}{\partial x_i}d\boldsymbol{x}.
\tag{3}
$$

The transformation tensors for bilinear viscous and pressure divergence terms in (3) are defined as follows [4]:

$$
\begin{aligned}
\boldsymbol{\kappa}(\boldsymbol{x}; \boldsymbol{\mu}) &= \nu(\boldsymbol{\mu})(J_T(\boldsymbol{x}; \boldsymbol{\mu}))^{-1}(J_T(\boldsymbol{x}; \boldsymbol{\mu}))^{-T}|J_T(\boldsymbol{x}; \boldsymbol{\mu})|, \\
\boldsymbol{\chi}(\boldsymbol{x}; \boldsymbol{\mu}) &= (J_T(\boldsymbol{x}; \boldsymbol{\mu}))^{-1}|J_T(\boldsymbol{x}; \boldsymbol{\mu})|,
\end{aligned}
\tag{4}
$$

where $|J_T|$ is the determinant of the Jacobian matrix $J_T \in \mathbb{R}^{2\times2}$ of the map $T(.; \boldsymbol{\mu})$. F and G are terms due to non-homogeneous Dirichlet boundary condition on the boundary are defined as:

$$
\begin{aligned}
F(\boldsymbol{v}; \boldsymbol{\mu}) &= -a(\boldsymbol{l}(\boldsymbol{\mu}), \boldsymbol{v}; \boldsymbol{\mu}), \\
G(q; \boldsymbol{\mu}) &= -b(\boldsymbol{l}(\boldsymbol{\mu}), q; \boldsymbol{\mu}),
\end{aligned}
\tag{5}
$$

where we denote by $\boldsymbol{l}(\boldsymbol{\mu})$ a parametrized lifting function such that $\boldsymbol{l}(\boldsymbol{\mu})|_{\Gamma_{D_g}} = \boldsymbol{g}_D(\boldsymbol{\mu})$.

2.1 Finite Element formulation

For a given parameter $\boldsymbol{\mu} \in \mathbb{P}$, the Galerkin-FE approximation of the parametrized Stokes problem (2) reads as follows:

$$\begin{cases} \text{Find } \boldsymbol{u}_h(\boldsymbol{\mu}) \in \boldsymbol{V}_h, p_h(\boldsymbol{\mu}) \in Q_h : \\ a(\boldsymbol{u}_h(\boldsymbol{\mu}), \boldsymbol{v}_h; \boldsymbol{\mu}) + b(\boldsymbol{v}_h, p_h(\boldsymbol{\mu}); \boldsymbol{\mu}) = F(\boldsymbol{v}_h; \boldsymbol{\mu}) & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b(\boldsymbol{u}_h(\boldsymbol{\mu}), q_h; \boldsymbol{\mu}) = G(q_h; \boldsymbol{\mu}) & \forall q_h \in Q_h, \end{cases} \qquad (6)$$

where $\boldsymbol{V}_h$ and Q_h are finite dimensional subspaces of $\boldsymbol{V}$ and Q of dimension $\mathcal{N}_u$ and $\mathcal{N}_p$, respectively with h related to the computational mesh size. When dealing with saddle point formulations it is well known that $\boldsymbol{V}_h$ and Q_h should be judiciously chosen [41]; we will get back to this point at the end of this subsection.

Let $\{\boldsymbol{\phi}_i^h\}_{i=1}^{\mathcal{N}_u}$ and $\{\psi_j^h\}_{j=1}^{\mathcal{N}_p}$ be basis functions of $\boldsymbol{V}_h$ and Q_h respectively. We introduce the matrices $A(\boldsymbol{\mu}) \in \mathbb{R}^{\mathcal{N}_u \times \mathcal{N}_u}$ and $B(\boldsymbol{\mu}) \in \mathbb{R}^{\mathcal{N}_p \times \mathcal{N}_u}$ whose entries are

$$(A(\boldsymbol{\mu}))_{ij} = a(\boldsymbol{\phi}_j^h, \boldsymbol{\phi}_i^h; \boldsymbol{\mu}), \quad (B(\boldsymbol{\mu}))_{ki} = b(\boldsymbol{\phi}_i^h, \psi_k^h; \boldsymbol{\mu}), \text{ for } 1 \le i, j \le \mathcal{N}_u, 1 \le k \le \mathcal{N}_p, \quad (7)$$

and the algebraic form of discrete problem (6) problem reads

$$\begin{bmatrix} A(\boldsymbol{\mu}) & B^T(\boldsymbol{\mu}) \\ B(\boldsymbol{\mu}) & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}(\boldsymbol{\mu}) \\ \boldsymbol{P}(\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}(\boldsymbol{\mu}) \end{bmatrix} \qquad (8)$$

for the vectors $\boldsymbol{U} = (u_h^{(1)}, ..., u_h^{(\mathcal{N}_u)})^T, \boldsymbol{P} = (p_h^{(1)}, ..., p_h^{(\mathcal{N}_p)})^T$ such that $\boldsymbol{u}_h = \sum_{i=1}^{\mathcal{N}_u} u_h^{(i)} \boldsymbol{\phi}_i^h$ and $p_h = \sum_{j=1}^{\mathcal{N}_p} p_h^{(j)} \psi_j^h$, where for $1 \le i \le \mathcal{N}_u$ and $1 \le k \le \mathcal{N}_p$:

$$(\bar{\boldsymbol{f}}(\boldsymbol{\mu}))_i = -a(\boldsymbol{l}_h, \boldsymbol{\phi}_i^h; \boldsymbol{\mu}), \quad (\bar{\boldsymbol{g}}(\boldsymbol{\mu}))_k = -b(\boldsymbol{l}_h, \psi_k^h; \boldsymbol{\mu}), \qquad (9)$$

with $\boldsymbol{l}_h = \boldsymbol{l}_h(\boldsymbol{\mu}), \boldsymbol{l}(\boldsymbol{\mu})$ in (2) is discretized by the FE interpolant. For the sake of the subsequent reduction procedure, affine parametric dependence is assumed for operators in (8) [23]:

$$\begin{aligned} A(\boldsymbol{\mu}) &= \sum_{q=1}^{Q_a} \Theta_q^a(\boldsymbol{\mu}) A^q, \qquad B(\boldsymbol{\mu}) = \sum_{q=1}^{Q_b} \Theta_q^b(\boldsymbol{\mu}) B^q, \\ \bar{\boldsymbol{f}}(\boldsymbol{\mu}) &= \sum_{q=1}^{Q_f} \Theta_q^f(\boldsymbol{\mu}) \bar{\boldsymbol{f}}^q, \qquad \bar{\boldsymbol{g}}(\boldsymbol{\mu}) = \sum_{q=1}^{Q_g} \Theta_q^g(\boldsymbol{\mu}) \bar{\boldsymbol{g}}^q. \end{aligned} \qquad (10)$$

Finally, coming back to the choice of the FE pairs, the FE spaces $\boldsymbol{V}_h$ and Q_h have to fulfill the following parametrized version of the inf-sup condition [41]:

$$\exists \beta_0(\boldsymbol{\mu}) > 0 : \beta_h(\boldsymbol{\mu}) = \inf_{q_h \in Q_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, q_h; \boldsymbol{\mu})}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h} \|q_h\|_{Q_h}} \ge \beta_0(\boldsymbol{\mu}) \quad \forall \boldsymbol{\mu} \in \mathbb{P}. \qquad (11)$$

This relation holds if, e.g., the Taylor-Hood ($\mathbb{P}_2/\mathbb{P}_1$) FE spaces are chosen. Condition (11) does not hold in case of equal order FE spaces ($\mathbb{P}_k/\mathbb{P}_k$), $k \ge 1$ and for lowest order element ($\mathbb{P}_1/\mathbb{P}_0$). To handle these cases, a stabilized formulation is introduced as in the following.

2.2 Stabilized Finite Element formulation

When the finite dimensional spaces $\boldsymbol{V}_h$ and Q_h do not satisfy the inf-sup stability condition (11), then in order to avoid possible spurious pressure modes the use of a stabilized formulation,

introduced in [25, 12, 26, 11, 18, 53], is nowadays widespread in the FE community. This amounts to modifying (6) as

$$\begin{cases} \text{Find } \boldsymbol{u}_h(\boldsymbol{\mu}) \in \boldsymbol{V}_h,\, p_h(\boldsymbol{\mu}) \in Q_h : \\ a(\boldsymbol{u}_h(\boldsymbol{\mu}), \boldsymbol{v}_h; \boldsymbol{\mu}) + b(\boldsymbol{v}_h, p_h(\boldsymbol{\mu}); \boldsymbol{\mu}) - s_h^{u,v}(\boldsymbol{u}_h, \boldsymbol{v}_h; \boldsymbol{\mu}) - s_h^{p,v}(p_h, \boldsymbol{v}_h; \boldsymbol{\mu}) = F(\boldsymbol{v}_h; \boldsymbol{\mu}) & \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b(\boldsymbol{u}_h(\boldsymbol{\mu}), q_h; \boldsymbol{\mu}) - s_h^{u,q}(\boldsymbol{u}_h, q_h; \boldsymbol{\mu}) - s_h^{p,q}(p_h, q_h; \boldsymbol{\mu}) = G(q_h; \boldsymbol{\mu}) & \forall\, q_h \in Q_h, \end{cases} \tag{12}$$

where $s_h^{u,v}(.,.;\boldsymbol{\mu})$, $s_h^{p,v}(.,.;\boldsymbol{\mu})$, $s_h^{u,q}(.,.;\boldsymbol{\mu})$ and $s_h^{p,q}(.,.;\boldsymbol{\mu})$ are the stabilization terms [41]. Following a strongly consistent residual based approach, these stabilization terms are defined as

$$s_h^{u,v}(\boldsymbol{u}_h, \boldsymbol{v}_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (-\nu\Delta\boldsymbol{u}_h, -\rho\nu\Delta\boldsymbol{v}_h), \tag{13}$$

$$s_h^{p,v}(p_h, \boldsymbol{v}_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\nabla p_h, -\rho\nu\Delta\boldsymbol{v}_h), \tag{14}$$

$$s_h^{u,q}(\boldsymbol{u}_h, q_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (-\nu\Delta\boldsymbol{u}_h, \nabla q_h), \tag{15}$$

and

$$s_h^{p,q}(p_h, q_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\nabla p_h, \nabla q_h), \tag{16}$$

where K is an element of the triangulation of the reference domain Ω, h_K is the diameter of element K, δ is the stabilization coefficient assumed to be constant. For $\rho = 0, 1, -1$, the method (13)-(16) is respectively known as the pressure-poisson stabilized Galerkin (Franca-Hughes) [25], Galerkin least-squares (GALS) [24], Douglas-Wang (DW) [18]. In case of linear interpolation for velocity and pressure ($\mathbb{P}_1/\mathbb{P}_1$) the Laplacian term $-\nu\Delta\boldsymbol{u}_h$ inside the stabilization vanishes and all above choices reduce to Brezzi-Pitkäranta stabilization [10], written as

$$s_h^{u,v}(\boldsymbol{u}_h, \boldsymbol{v}_h; \boldsymbol{\mu}) = s_h^{p,v}(p_h, \boldsymbol{v}_h; \boldsymbol{\mu}) = s_h^{u,q}(\boldsymbol{u}_h, q_h; \boldsymbol{\mu}) := 0, \qquad s_h^{p,q}(p_h, q_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K \nabla p_h \cdot \nabla q_h. \tag{17}$$

Further possible stabilization options are available, see e.g. [41, chapter 9] and references therein. Dependence of these stabilization terms on parameter $\boldsymbol{\mu}$ is motivated by the dependence through the parametrized solution. Further dependence on $\boldsymbol{\mu}$ might be warranted for the stabilization coefficient δ (see e.g. numerical test cases in [51] for the case of geometrical parametrization). In particular, it is often proposed in the FE literature (see e.g. [27]) that the stabilization coefficient depends nonlinearly on the solution. For the sake of guaranteeing affine dependence on the stabilization terms, here we assume instead δ to be a constant, with no noticeable deterioration in the numerical results provided that the stabilization coefficient δ has been properly chosen. It cannot be too small otherwise the stabilization will be poor and spurious modes will not be eliminated, while a large value of parameter δ could result in a poor approximation for the pressure field near to the boundary. Beyond this, if needed, parameter and solution dependent stabilization coefficient could be dealt with the empirical interpolation method [6].

For the sake of exposition, in this work we will only focus on the stabilization option corresponding to $\rho = 0$ (Franca-Hughes) [25], i.e, the stabilization terms (13) and (14) are not taken into account.

After adding the stabilization terms into the system (8), the stabilized algebraic formulation reads

$$\begin{bmatrix} A(\boldsymbol{\mu}) & B^T(\boldsymbol{\mu}) \\ \tilde{B}(\boldsymbol{\mu}) & -S(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \boldsymbol{U}(\boldsymbol{\mu}) \\ \boldsymbol{P}(\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}(\boldsymbol{\mu}) \end{bmatrix}, \tag{18}$$

where $\tilde{B}(\boldsymbol{\mu})$ and $S(\boldsymbol{\mu})$ contains the effects of stabilization and defined as follows:

$$\left(\tilde{B}(\boldsymbol{\mu})\right)_{ki} = b(\boldsymbol{\phi}_i^h, \psi_k^h; \boldsymbol{\mu}) + s_h^{u,q}(\boldsymbol{\phi}_i^h, \psi_k^h; \boldsymbol{\mu}), \quad (S(\boldsymbol{\mu}))_{ij} = s_h^{p,q}(\psi_j^h, \psi_i^h; \boldsymbol{\mu}), \text{ for } 1 \le i,j \le \mathcal{N}_u, 1 \le k \le \mathcal{N}_p, \tag{19}$$

The motivation to choose a *strongly consistent residual based method* is, it improve the stability of Galerkin FE method without compromising the consistency. This choice also allows the stabilized formulation to fulfill a modified version of (11) with a supplementary terms [8, 13, 7]. Indeed, e.g. for the Brezzi-Pitkäranta method, the stabilized formulation requires the FE spaces to fulfill the following modified inf-sup condition

$$\exists \beta_0(\boldsymbol{\mu}) > 0 : \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, q_h; \boldsymbol{\mu})}{\|\nabla \boldsymbol{v}_h\|} + s_h^{p,q}(q_h, q_h; \boldsymbol{\mu})^{1/2} \ge \beta_0(\boldsymbol{\mu}) \|q_h\|, \forall q_h \in Q_h, \tag{20}$$

This condition is a crucial motivation of our work, and will be adapted to the reduced order system in subsection 2.4.

2.3 Reduced Basis formulation

In this section we present the RB formulation of steady Stokes problem. The RB method seeks the approximation of FE solution to (6). In first step we construct a set of global basis functions. Let $\boldsymbol{\mu}^1, ..., \boldsymbol{\mu}^N$ be a set of snapshot parameter values chosen by greedy algorithm [45]. We denote by $\boldsymbol{u}_h(\boldsymbol{\mu}^n)$ and $p_h(\boldsymbol{\mu}^n)$, $n = 1, \ldots, N$, the corresponding snapshot solutions for the velocity and pressure. The snapshot solutions are obtained as solutions of a stable FE formulation of (6). As our focus (especially in the nonlinear case in the second part of the work) is not on the certification of the reduced model, during the greedy iterations we rely on an error indicator based on the residual, rather than a certified error bound [23, 20] which would need to be properly extended to the stabilized case. Nonetheless, the availability of error bounds does not affect the presentation of the rest of the methodology described in this paper. At the end of the greedy procedure, we obtain the reduced velocity space $\boldsymbol{V}_N \subset \boldsymbol{V}_h$ and reduced pressure space $Q_N \subset Q_h$, respectively as:

$$\boldsymbol{V}_N = \text{span}\left\{\boldsymbol{u}_h(\boldsymbol{\mu}^n), 1 \le n \le N_u\right\}, \tag{21}$$

and

$$Q_N = \text{span}\left\{p_h(\boldsymbol{\mu}^n), 1 \le n \le N_p\right\}, \tag{22}$$

where $N_u = N_p = N$ are the dimensions of RB velocity space $\boldsymbol{V}_N$ and RB pressure space Q_N, respectively. Applying the Gram-Schmidt orthogonalization process on the snapshots [23], we denote by $\{\boldsymbol{\xi}_n^u\}_{n=1}^{N_u}$ and $\{\xi_n^p\}_{n=1}^{N_p}$ mutually orthonormal functions obtained from velocity and pressure snapshots, respectively, which we are going to use as basis functions for the reduced spaces instead of the snapshots.

As well known in the RB community, it is important to point out that, even when the spaces $\boldsymbol{V}_N$ and Q_N are obtained collecting snapshots from a stable full order model from section 2.1, a Galerkin projection over the reduced spaces does not guarantee the fulfillment of the following reduced inf-sup condition:

$$\exists \beta_{0,N}(\boldsymbol{\mu}) > 0 : \quad \beta_N(\boldsymbol{\mu}) = \inf_{q_N \in Q_N} \sup_{\boldsymbol{v}_N \in \boldsymbol{V}_N} \frac{b(\boldsymbol{v}_N, q_N; \boldsymbol{\mu})}{\|\boldsymbol{v}_N\|_{\boldsymbol{V}_N} \|q_N\|_{Q_N}} \ge \beta_{0,N}(\boldsymbol{\mu}) \quad \forall \boldsymbol{\mu} \in \mathbb{P}. \tag{23}$$

Indeed, following [46] the usual reduced order methodology relies on the enrichment of the RB velocity space with supremizer solutions. Even though several variants have been discussed in literature [46, 42, 43], for the sake of exposition we will focus only on the so-called *approximate*

supremizer enrichment. This approach requires the introduction of the supremizer operator $T^{\boldsymbol{\mu}}: Q_h \to \boldsymbol{V}_h$ defined as follows:

$$(T^{\boldsymbol{\mu}} q_h, \boldsymbol{v}_h)_{\boldsymbol{V}} = b(\boldsymbol{v}_h, q_h; \boldsymbol{\mu}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{24}$$

which is evaluated for $\boldsymbol{\mu} = \boldsymbol{\mu}^n$ and the corresponding pressure snapshot $q_h := p_h(\boldsymbol{\mu}^n)$, $n = 1, \ldots, N$, to obtain N supremizer snapshots. Afterwards, the RB velocity space $\boldsymbol{V}_N$ is enriched with the supremizer snapshots. We denote the enriched RB velocity space by $\tilde{\boldsymbol{V}}_N$, defined as:

$$\tilde{\boldsymbol{V}}_N = \text{span}\left\{ \boldsymbol{u}_h(\boldsymbol{\mu}^n), 1 \le n \le N_u; T^{\boldsymbol{\mu}^n} p_h(\boldsymbol{\mu}^n), 1 \le n \le N_s \right\}, \tag{25}$$

where $N_s \le N_p$ denotes the number of supremizer snapshots. The dimension of $\tilde{\boldsymbol{V}}_N$ will thus be $N_u + N_s$; in order to keep the notations simple, in the following we will always take $N_u = N_s = N_p = N$. Now the RB formulation corresponding to FE problem (6) can be written as:

$$\begin{cases} \text{Find } \boldsymbol{u}_N(\boldsymbol{\mu}) \in \tilde{\boldsymbol{V}}_N, p_N(\boldsymbol{\mu})) \in Q_N : \\ a(\boldsymbol{u}_N(\boldsymbol{\mu}), \boldsymbol{v}_N; \boldsymbol{\mu}) + b(\boldsymbol{v}_N, p_N(\boldsymbol{\mu})) = F(\boldsymbol{v}_N; \boldsymbol{\mu}) & \forall \boldsymbol{v}_N \in \tilde{\boldsymbol{V}}_N, \\ b(\boldsymbol{u}_N(\boldsymbol{\mu}), q_N; \boldsymbol{\mu}) = G(q_N; \boldsymbol{\mu}) & \forall q_N \in Q_N. \end{cases} \tag{26}$$

The solution $(\boldsymbol{u}_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu})) \in \tilde{\boldsymbol{V}}_N \times Q_N$ of (26) can be expressed as a linear combination of the basis functions:

$$\boldsymbol{u}_N(\boldsymbol{\mu}) = \sum_{n=1}^{2N} U_{N,n}(\boldsymbol{\mu}) \boldsymbol{\xi}_n^u, \qquad p_N(\boldsymbol{\mu}) = \sum_{n=1}^{N} P_{N,n}(\boldsymbol{\mu}) \xi_n^p, \tag{27}$$

where $\boldsymbol{U}_N(\boldsymbol{\mu}) = [U_{N,n}(\boldsymbol{\mu})]_{n=1}^{N_u}$ and $\boldsymbol{P}_N(\boldsymbol{\mu}) = [P_{N,n}(\boldsymbol{\mu})]_{n=1}^{N_p}$ denote the vector of coefficients of the reduced basis approximation for velocity and pressure. Finally, we write the system in compact form as

$$\begin{bmatrix} A_N(\boldsymbol{\mu}) & B_N^T(\boldsymbol{\mu}) \\ B_N(\boldsymbol{\mu}) & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_N(\boldsymbol{\mu}) \\ \boldsymbol{P}_N(\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}_N(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}_N(\boldsymbol{\mu}) \end{bmatrix}, \tag{28}$$

where the RB tensors are computed as

$$\begin{aligned} A_N(\boldsymbol{\mu}) &= Z_{u,s}^T A(\boldsymbol{\mu}) Z_{u,s}, \quad B_N(\boldsymbol{\mu}) = Z_p^T B(\boldsymbol{\mu}) Z_{u,s}, \\ \bar{\boldsymbol{f}}_N(\boldsymbol{\mu}) &= Z_{u,s}^T \bar{\boldsymbol{f}}(\boldsymbol{\mu}), \qquad \bar{\boldsymbol{g}}_N(\boldsymbol{\mu}) = Z_p^T \bar{\boldsymbol{g}}(\boldsymbol{\mu}), \end{aligned} \tag{29}$$

being $Z_{u,s} \in \mathbb{R}^{\mathcal{N}_u \times N_{u,s}}$ and $Z_p \in \mathbb{R}^{\mathcal{N}_p \times N_p}$ rectangular matrices that contain the FE degrees of freedom of the basis of $\tilde{\boldsymbol{V}}_N$ and Q_N, respectively.

2.4 Stabilized Reduced Basis formulation

We introduce now the stabilized Reduced Basis model derived from the stabilized FE problem (12). During the offline phase, the computation of the reduced spaces is done in section 2.3, resulting in a reduced velocity space $\boldsymbol{V}_N$, enriched reduced velocity space $\tilde{\boldsymbol{V}}_N$, and reduced pressure space Q_N. During the online stage, the stabilized RB problem reads

$$\begin{cases} \text{Find } \boldsymbol{u}_N(\boldsymbol{\mu}) \in \boldsymbol{V}_N, p_N(\boldsymbol{\mu}) \in Q_N : \\ a(\boldsymbol{u}_N(\boldsymbol{\mu}), \boldsymbol{v}_N; \boldsymbol{\mu}) + b(\boldsymbol{v}_N, p_N(\boldsymbol{\mu}); \boldsymbol{\mu}) - s_N^{u,v}(\boldsymbol{u}_N, \boldsymbol{v}_N; \boldsymbol{\mu}) - s_N^{p,v}(p_N, \boldsymbol{v}_N; \boldsymbol{\mu}) = F(\boldsymbol{v}_N; \boldsymbol{\mu}) & \forall \boldsymbol{v}_N \in \boldsymbol{V}_N, \\ b(\boldsymbol{u}_N(\boldsymbol{\mu}), q_N; \boldsymbol{\mu}) - s_N^{u,q}(\boldsymbol{u}_N, q_N; \boldsymbol{\mu}) - s_N^{p,q}(p_N, q_N; \boldsymbol{\mu}) = G(q_N; \boldsymbol{\mu}) & \forall q_N \in Q_N, \end{cases} \tag{30}$$

where $s_N^{u,v}(.,.;\boldsymbol{\mu})$, $s_N^{p,v}(.,.;\boldsymbol{\mu})$, $s_N^{u,q}(.,.;\boldsymbol{\mu})$ and $s_N^{p,q}(.,.;\boldsymbol{\mu})$ are the reduced order stabilization terms defined as

$$s_N^{u,v}(\boldsymbol{u}_N, \boldsymbol{v}_N; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (-\nu \Delta \boldsymbol{u}_N, -\rho \nu \Delta \boldsymbol{v}_N), \tag{31}$$

$$s_N^{p,v}(p_h, \boldsymbol{v}_N; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\nabla p_N, -\rho \nu \Delta \boldsymbol{v}_N), \tag{32}$$

$$s_N^{u,q}(\boldsymbol{u}_N, q_N; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (-\nu \Delta \boldsymbol{u}_N, \nabla q_N), \tag{33}$$

and

$$s_N^{p,q}(p_N, q_N; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\nabla p_N, \nabla q_N), \tag{34}$$

In a similar way, one could seek solutions in the enriched velocity space by replacing $\boldsymbol{V}_N$ with $\tilde{\boldsymbol{V}}_N$ in (30). With slight abuse of notation, we will keep denoting by $(\boldsymbol{u}_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu}))$ solutions to either (26) and (30), as it will be clear from the context to which RB formulation we will refer to. The corresponding algebraic formulation reads

$$\begin{bmatrix} A_N(\boldsymbol{\mu}) & B_N^T(\boldsymbol{\mu}) \\ \tilde{B}_N(\boldsymbol{\mu}) & -S_N(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_N(\boldsymbol{\mu}) \\ \boldsymbol{P}_N(\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}_N(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}_N(\boldsymbol{\mu}) \end{bmatrix}, \tag{35}$$

where $\tilde{B}_N(\boldsymbol{\mu})$ and $S_N(\boldsymbol{\mu})$ are RB stabilization matrices defined as:

$$\tilde{B}_N(\boldsymbol{\mu}) = Z_p^T \tilde{B}(\boldsymbol{\mu}) Z_{u,s}, \quad S_N(\boldsymbol{\mu}) = Z_p^T S(\boldsymbol{\mu}) Z_p, \tag{36}$$

To further present our methodology, we also define the reduced order version of modified inf-sup condition (20) as:

$$\exists \beta_{0,N}(\boldsymbol{\mu}) > 0 : \sup_{\boldsymbol{v}_N \in \boldsymbol{V}_N} \frac{b(\boldsymbol{v}_N, q_N; \boldsymbol{\mu})}{\|\nabla \boldsymbol{v}_N\|} + s_N^{p,q}(q_N, q_N; \boldsymbol{\mu})^{1/2} \geq \beta_{0,N}(\boldsymbol{\mu}) \|q_N\|, \forall q_N \in Q_N, \tag{37}$$

where $s_N^{p,q}(.,.;\boldsymbol{\mu})$ is due to the addition of stabilization terms in RB formulation. It is now clear that the two addend on the left-hand side can contribute in different ways to the overall inf-sup stability of the reduced problem. In particular, one could either increase the first term by replacing $\boldsymbol{V}_N$ with $\tilde{\boldsymbol{V}}_N$, thus exploiting the existing supremizer enrichment procedure, or rely on the contribution of the second term due to the underlying FE residual based stabilization, or both. Overall, this results in the following four different combinations:

(i) the first option is to increase both the first addend, through RB velocity space enrichment with supremizer solutions, and the second addend as well, through residual based stabilization. Thus, during the online stage we will solve (30), upon replacing $\boldsymbol{V}_N$ with $\tilde{\boldsymbol{V}}_N$. As the residual based stabilization terms are present both *offline* and *online*, we will call this methodology *offline-online stabilization* in agreement with the denomination introduced in [36]. Furthermore, as supremizer enrichment is performed, we will denote this option by *offline-online stabilization* with supremizer;

(ii) as a second option, we do not enrich the RB velocity space with supremizer solutions, yet we still rely on the residual based stabilization during the online phase. Thus, during the online stage, we solve (30). When compared to (i), this option seems attractive in terms of CPU time because it decreases the dimensionality of the velocity space, resulting in possible larger speedups. We call this option *offline-online stabilization* without supremizer;

8

(iii) the third option is to enrich the RB velocity space with supremizer solutions and neglect stabilization during the online phase. Thus, online we end up solving (26). If compared to (i), this option seems attractive in perspective for cases where the stabilization coefficient δ depends nonlinearly on the solution, and an accurate approximation of the stabilization terms would require preprocessing through EIM. We denote this case with *offline-only stabilization* with supremizer, the first part of the name underlining the fact that stabilization is applied only offline;

(iv) a fourth (fictitious) option could be to completely avoid both supremizer enrichment and residual based stabilization. We would call this option as *offline-only stabilization* without supremizer. It is clear, however, that this methodology would hardly grant a positive reduced inf-sup constant, as neither stabilizing contributions are enabled. For this reason, results for this case are not reported in numerical results presented in this paper.

In the following, we employ equal order FE spaces $(\mathbb{P}_k/\mathbb{P}_k), k \geq 1$ during the offline stage. We remark that, during this stage, reduced solutions required by the greedy algorithm are computed via option (i). Indeed, the consistency of the *offline-online stabilization* is necessary to ensure that the same parameter will not be selected twice by the greedy algorithm. In contrast, during the online stage we study the performance of options (i)-(iii).

3 Numerical results and discussion for parametrized Stokes problems

In this section, we present some numerical results for stabilized reduced order model for steady Stokes problem developed in section 2. Numerical simulations are carried out in FreeFem++ [22], and also with RBniCS [5] for comparison.

As a test case we consider the parametrized cavity flow problem [25]. We set the parametrized domain $\Omega_o(\boldsymbol{\mu}) = (0, 1 + \mu_2) \times (0, 1)$, where we define $\boldsymbol{\mu} = (\mu_1, \mu_2)$ such that μ_1 is a physical parameter (kinematic viscosity of fluid) and μ_2 is a geometrical parameter (length of domain). The parametrized domain is shown in Fig. 1, along with the partition $\Gamma_{D,o}(\boldsymbol{\mu}) \cup \Gamma_{W,o}(\boldsymbol{\mu})$ of its boundary $\partial\Omega$; unit horizontal velocity is imposed on the side $\Gamma_{D,o}(\boldsymbol{\mu})$.
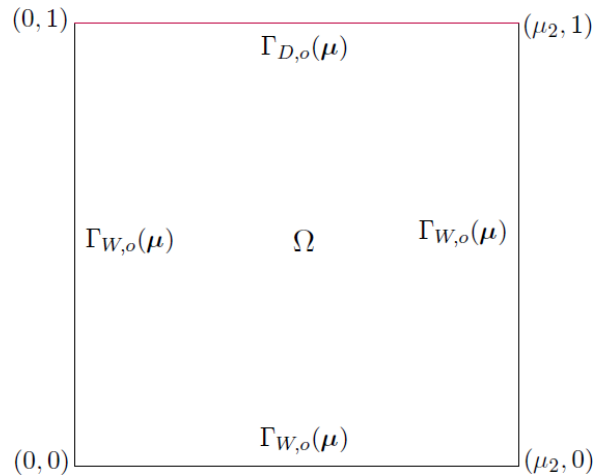

Figure 1: Parametrized domain.

3.1 Numerical results for $\mathbb{P}_k/\mathbb{P}_k$ for $k = 1, 2$

Since a stabilized FE formulation is used, we need not to employ Taylor-Hood elements, rather we can rely on equal order velocity-pressure FE pairs. The results in this section are obtained for $\mathbb{P}_k/\mathbb{P}_k$ for $k = 1, 2$. The case $k = 1$ is attractive in the FE context as it allows to employ the lowest order continuous FE spaces for both velocity and pressure. However, unless suitable gradient recovery techniques are used, all the various residual based stabilization techniques (13)-(15) coincide with the simplest Brezzi-Pitkäranta stabilization (17) when using lowest order elements. Therefore, in order to test the applicability of the Franca-Hughes stabilization we also provide results for $k = 2$. Details of the problem and computational costs are summarized in Table 1.

In Fig. 2 we show a comparison between the FE velocity solution and the RB solutions obtained for three different options (i)-(iii), for $(\mu_1, \mu_2) = (0.6, 2)$. These plots are shown in the $\mathbb{P}_2/\mathbb{P}_2$ case, and the corresponding plots for the $\mathbb{P}_1/\mathbb{P}_1$ pair hardly look any different. From Fig. 2, we see that the RB velocity and pressure solutions obtained by using the *offline-online stabilization* with/without supremizer looks similar to the FE solution. However, the RB solutions obtained by the *offline-only stabilization* is poor, in particular pressure solution is highly oscillatory.

In order to see a more quantitive comparison between the *offline-online stabilization* with/without supremizer and *offline-only stabilization* with supremizer we report next the results of an error analysis, starting from the $\mathbb{P}_1/\mathbb{P}_1$ FE pair with $\delta = 0.05$. These results show that *offline-online stabilization* result in the most accurate methods, while *offline-only stabilization* is inaccurate. We also see that the enrichment of supremizer together with *offline-online stabilization* is beneficial for the pressure approximation, improving results by more than an order of magnitude, while has negigible effect on the velocity.

Figures 4 and 5 are plotted to see the error comparison for velocity and pressure, respectivley using Franca-Hughes stabilization (15) for $\mathbb{P}_2/\mathbb{P}_2$ FE pair and by varying the stabilization coefficient $\delta = 0.5, 0.05$. If we compare these results with previous results obtained for the $\mathbb{P}_1/\mathbb{P}_1$ case, we see that Franca-Hughes stabilization is able to perform comparably between the cases of *offline-online stabilization* with and without supremizer, even for the pressure approximation. Therefore, we conclude that when we use Franca-Hughes stabilization with $\mathbb{P}_2/\mathbb{P}_2$ FE pair, there is no need to enrich the RB velocity space with supremizer solutions. In this way we can reduce also the online computational cost by decreasing the dimension of reduced velocity space (see Table 1). Furthermore, the results for different values of δ show the robustness of the methodology when varying the stabilization coefficient.

Physical parameter	μ_1 (fluid viscosity)
Geometrical parameter	μ_2 (horizontal length of domain)
Range of μ_1	[0.25,0.75]
Range of μ_2	[1,3]
μ_1 *online*	0.6
μ_2 *online*	2
FE degrees of freedom	6222 ($\mathbb{P}_1/\mathbb{P}_1$) 10935 ($\mathbb{P}_2/\mathbb{P}_2$)
RB dimension	$N_u = N_s = N_p = 20$
Computation time ($\mathbb{P}_2/\mathbb{P}_1$)	260s (*offline*), 12s (*online*) with supremizer
Offline time ($\mathbb{P}_1/\mathbb{P}_1$)	180s (*offline-online stabilization* with supremizer) 130s (*offline-online stabilization* without supremizer) 105s (*offline-only stabilization* with supremizer)
Offline time ($\mathbb{P}_2/\mathbb{P}_2$)	348s (*offline-online stabilization* with supremizer) 309s (*offline-online stabilization* without supremizer) 280s (*offline-only stabilization* with supremizer)
Online time ($\mathbb{P}_1/\mathbb{P}_1$)	10s (*offline-online stabilization* with supremizer) 8s (*offline-online stabilization* without supremizer) 7s (*offline-only stabilization* with supremizer)
Online time ($\mathbb{P}_2/\mathbb{P}_2$)	15s (*offline-online stabilization* with supremizer) 13s (*offline-online stabilization* without supremizer) 10s (*offline-only stabilization* with supremizer)

**Table 1:** Stokes problem: Computational details for physical and geometrical parameters.

(a) FE Velocity

(b) FE Pressure

(c) RB Velocity: *offline-online stabilization* with supremizer

(d) RB Pressure: *offline-online stabilization* with supremizer

(e) RB Velocity: *offline-online stabilization* without supremizer

(f) RB Pressure: *offline-online stabilization* without supremizer

(g) RB Velocity: *offline-only* supremizer

(h) RB Pressure: *offline-only* supremizer
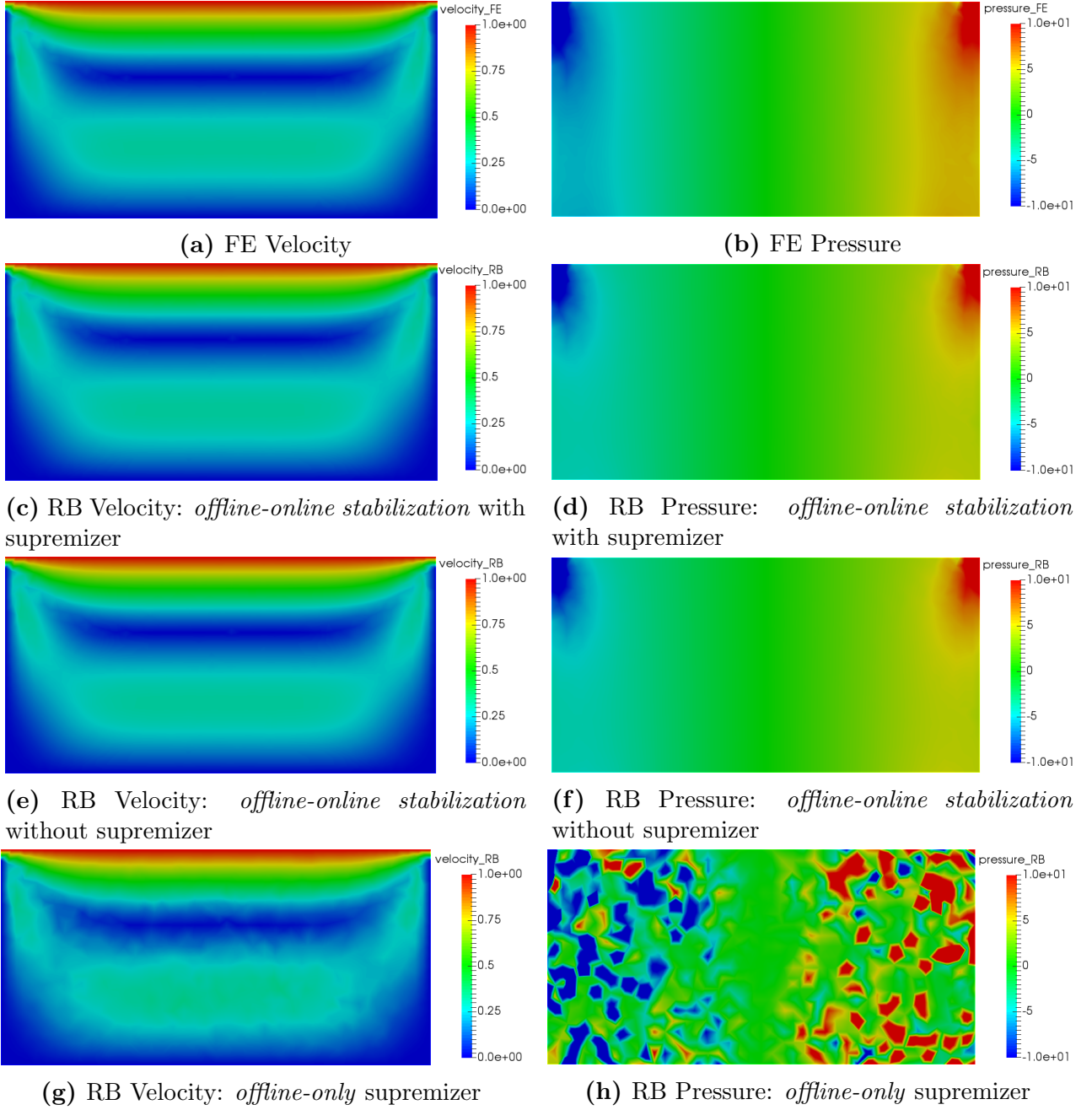
Figure 2: Stokes problem: FE and RB solutions for velocity and pressure at $(\mu_1, \mu_2) = (0.6, 2)$; $N_u = N_p = 20$ and using $\mathbb{P}_2/\mathbb{P}_2$.

From these results it is clear that there is a trade-off between increasing by supremizers reduced basis velocity spaces and improving accuracy of results.
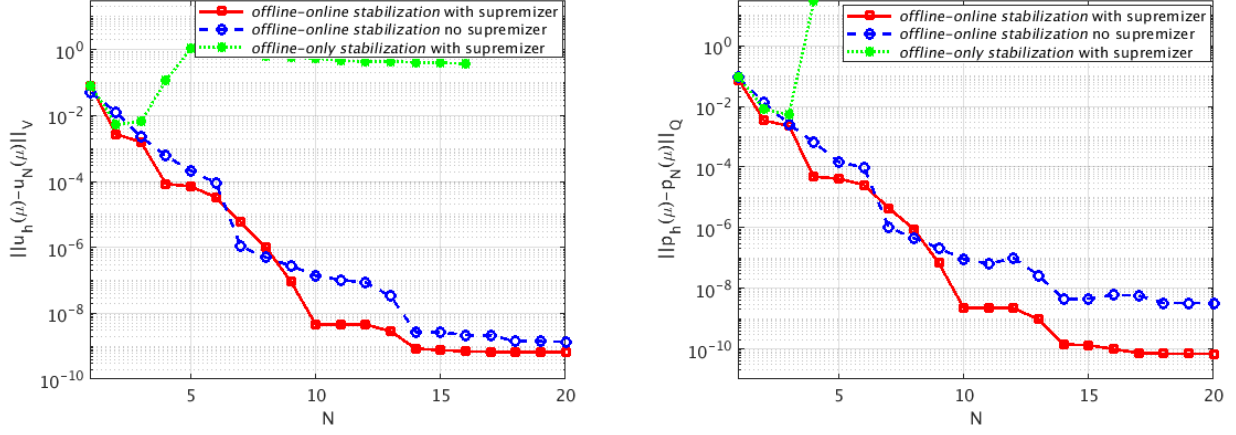
Figure 3: Stokes problem: Brezzi-Pitkaranta stabilization on cavity flow; Velocity (left) and pressure (right) error comparison between the *offline-online stabilization* with/without supremizer and *offline-only stabilization* with supremizer using $\mathbb{P}_1/\mathbb{P}_1$; stabilization coefficient $\delta = 0.05$; $N_u = N_p = N_s = 20$.
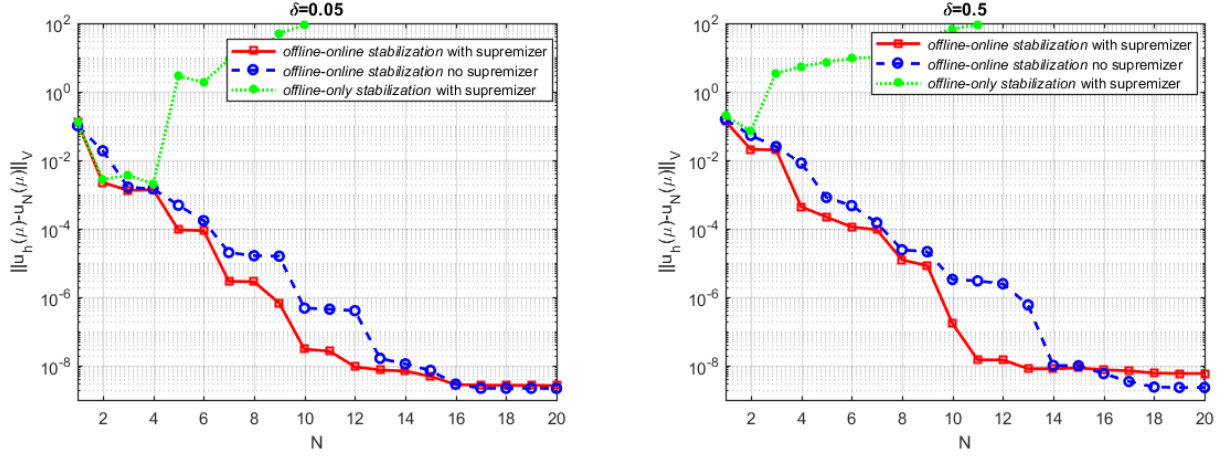

Figure 4: Stokes problem: Franca-Hughes stabilization on cavity flow; Velocity error comparison between the *offline-online stabilization* with/without supremizer and *offline-only stabilization* with supremizer using $\mathbb{P}_2/\mathbb{P}_2$; stabilization coefficient $\delta = 0.05, 0.5$; $N_u = N_p = N_s = 20$.
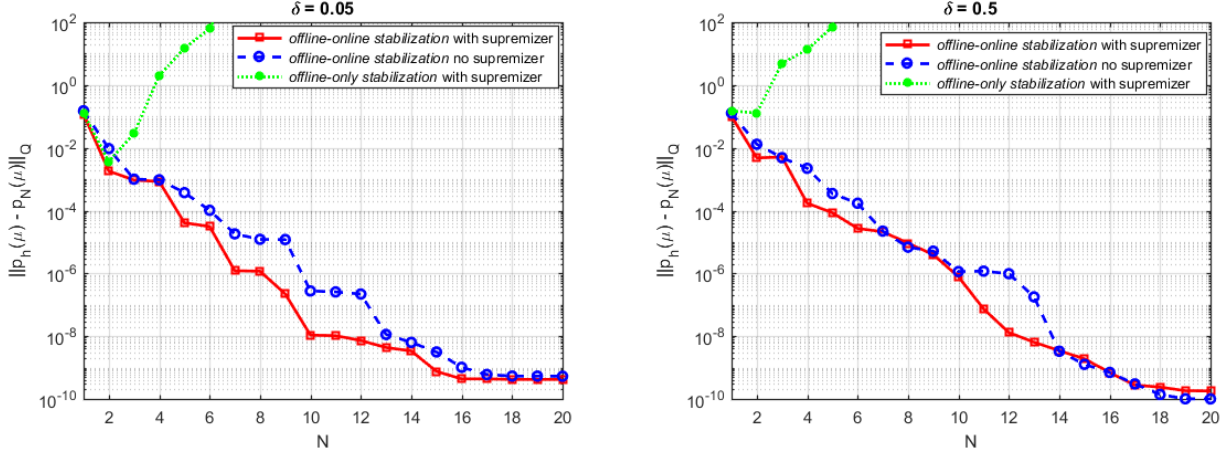
Figure 5: Stokes problem: Franca-Hughes stabilization on cavity flow; Pressure error comparison between the *offline-online stabilization* with/without supremizer and *offline-only stabilization* with supremizer using $\mathbb{P}_2/\mathbb{P}_2$; stabilization coefficient $\delta = 0.05, 0.5$; $N_u = N_p = N_s = 20$.

3.2   Numerical results for $\mathbb{P}_1/\mathbb{P}_0$

In this section we discuss the solution of steady parametrized Stokes problem using the FE pair given by $\mathbb{P}_1$ approximation for velocity and discontinuous $\mathbb{P}_0$ discretization for pressure. We present this case to show that the proposed methodology is applicable to any type of stabilization, and not necessarily residual based. Indeed, for $\mathbb{P}_1/\mathbb{P}_0$, both (13)-(15) vanish; one can alternatively resort to [41]:

$$ s_h^{p,q}(q_h; \boldsymbol{\mu}) := \delta \sum_{\sigma \in \Gamma_h} h_\sigma \int_\sigma [p_h]_\sigma [q_h]_\sigma \tag{38} $$

where Γ_h is the set of all edges σ of the triangulation except for those belonging to the boundary $\partial\Omega$, h_σ is the length of σ and $[q_h]_\sigma$ denotes its jump across σ.

In Fig. 6 we show some snapshots for velocity and pressure fields using *stabilized* FE method and *stabilized* RB method. From these solution plots we conclude that we are able to recover a good qualitative approximation of FE solution at reduced order level using the *offline-online stabilization* without supremizers, whereas the *offline-only stabilization* is not enough to recover FE approximation.
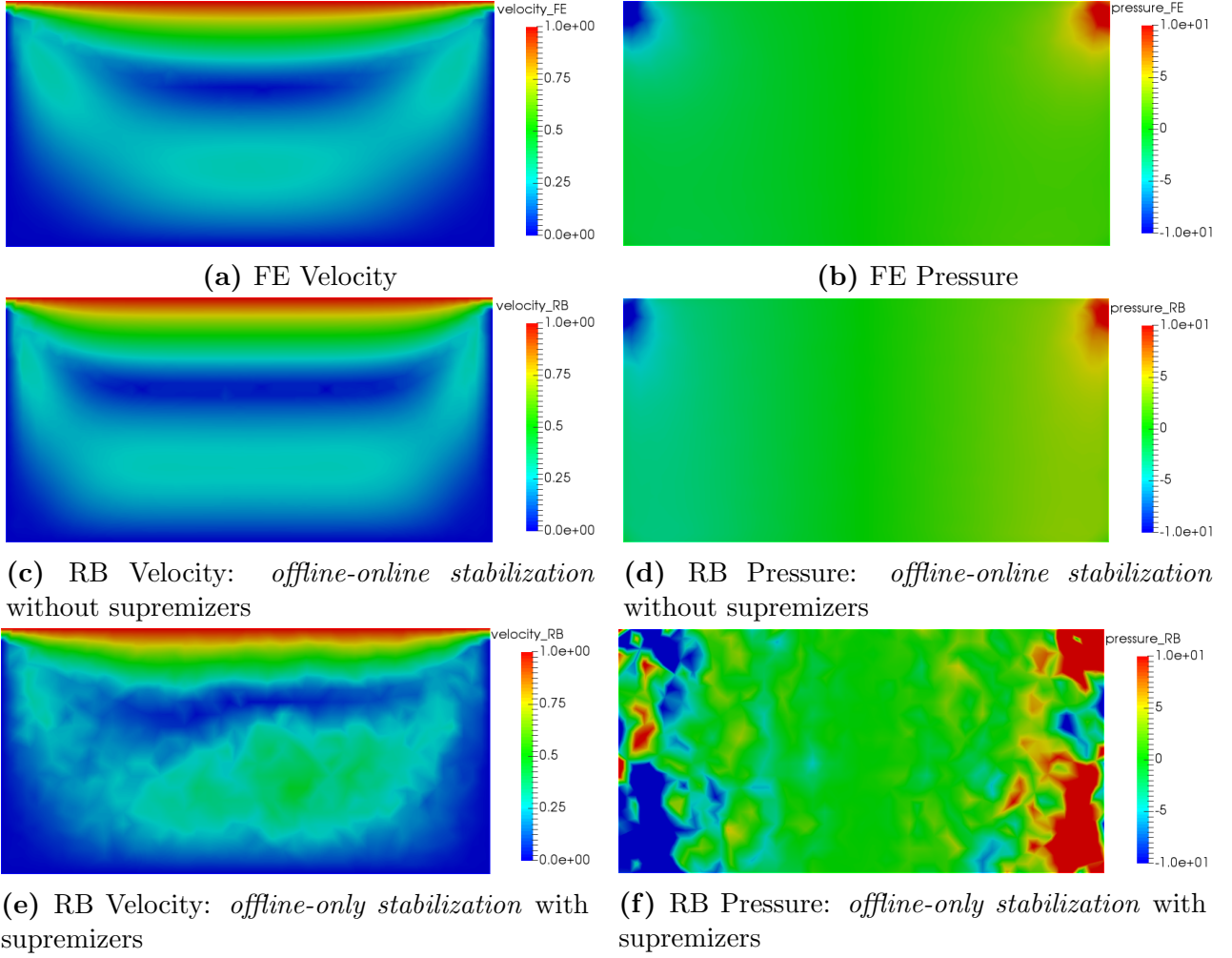
(a) FE Velocity

(b) FE Pressure

(c) RB Velocity: *offline-online stabilization* without supremizers

(d) RB Pressure: *offline-online stabilization* without supremizers

(e) RB Velocity: *offline-only stabilization* with supremizers

(f) RB Pressure: *offline-only stabilization* with supremizers

Figure 6: Stokes problem: FE and RB solutions for velocity and pressure at $(\mu_1, \mu_2) = (0.6, 2)$; $N_u = N_p = 20$ and using $\mathbb{P}_1/\mathbb{P}_0$.

In Figs. 7 we plot the comparison between *offline-online stabilization* with/without supremizer and *offline-only stabilization* with supremizer for velocity and pressure, respectively for $\mathbb{P}_1/\mathbb{P}_0$. These comparison shows that the *offline-online stabilization* results in the most accurate method, and that the addition of supremizer to velocity space is not necessary for pressure recovery. Indeed, a good approximation of pressure is obtained even without the supremizer, and the enrichment accounts for a further improvement of only one order of magnitude.
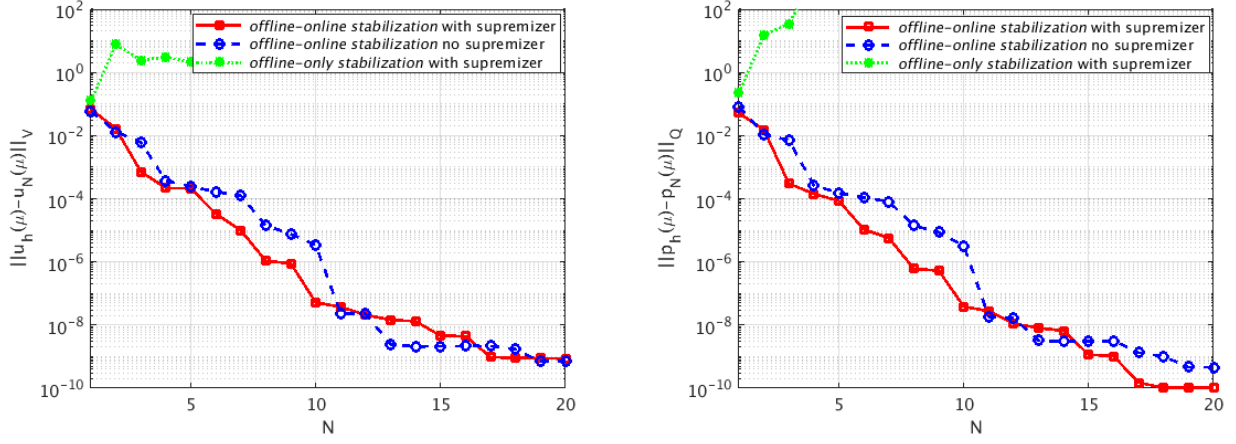
Figure 7: Stokes problem: Stabilization with $\mathbb{P}_1/\mathbb{P}_0$ on cavity flow: Velocity (left) and pressure (right) error between FE solution and RB solution for different possible options.

3.3 Numerical results for stable $\mathbb{P}_2/\mathbb{P}_1$

Just for completeness, here we plot a comparison between the FE solution and RB solution for velocity and pressure in Fig. 8 using stable FE pair $\mathbb{P}_2/\mathbb{P}_1$ [46]. We compare results with and without supremizer and conclude that supremizer is necessary to enrich the RB velocity space. In this case we are not using any *stabilization* method, therefore we need supremizer to fulfill reduced inf-sup condition.
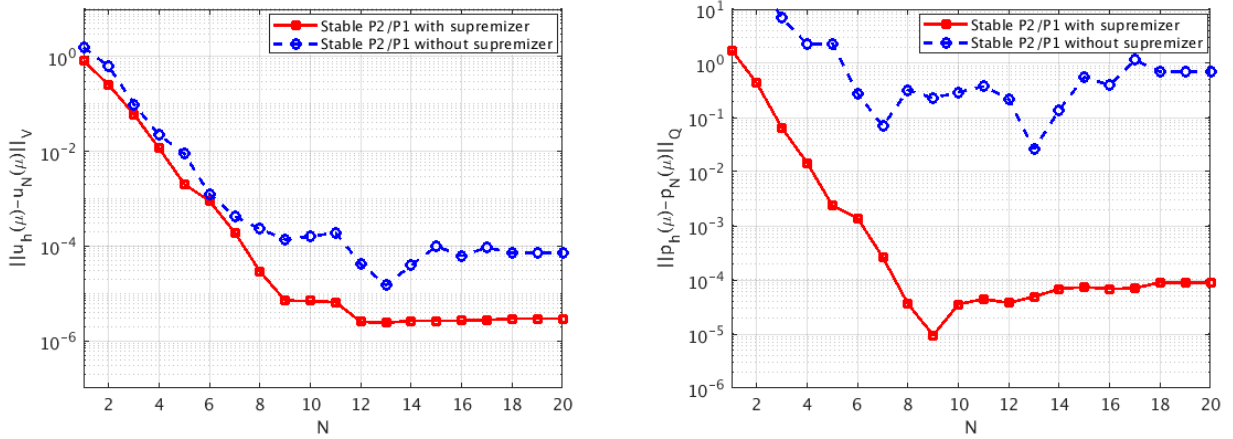

Figure 8: Stokes problem: Error between FE and RB solutions for velocity (left) and pressure (right) with/without supremizer using $\mathbb{P}_2/\mathbb{P}_1$.

4   Parametrized Steady Navier-Stokes problem

Let us now consider the steady incompressible Navier-Stokes equations in a parametrized domain $\Omega_o(\boldsymbol{\mu}) \subset \mathbb{R}^2$. The continuous parametrized formulation read as follows: find $\boldsymbol{u}_o(\boldsymbol{\mu})$: $\Omega_o(\boldsymbol{\mu}) \to \mathbb{R}^2$ and $p_o(\boldsymbol{\mu}) : \Omega_o(\boldsymbol{\mu}) \to \mathbb{R}^2$ such that

$$
\begin{cases}
-\nu(\boldsymbol{\mu})\Delta\boldsymbol{u}_o(\boldsymbol{\mu}) + (\boldsymbol{u}_o(\boldsymbol{\mu}) \cdot \nabla)\,\boldsymbol{u}_o(\boldsymbol{\mu}) + \nabla p_o(\boldsymbol{\mu}) = \boldsymbol{0} & \text{in } \Omega_o(\boldsymbol{\mu}), \\
\text{div } \boldsymbol{u}_o(\boldsymbol{\mu}) = 0 & \text{in } \Omega_o(\boldsymbol{\mu}), \\
\boldsymbol{u}_o(\boldsymbol{\mu}) = \boldsymbol{g}_D(\boldsymbol{\mu}) & \text{on } \Gamma_{D,o}(\boldsymbol{\mu}), \\
\boldsymbol{u}_o(\boldsymbol{\mu}) = \boldsymbol{0}, & \text{on } \Gamma_{W,o}(\boldsymbol{\mu}),
\end{cases}
\tag{39}
$$

where the parameter, unknowns, coefficients and boundaries have the same meaning as in the Stokes case (1). For the sake of the numerical test cases, we introduced the Reynolds number defined as $Re(\boldsymbol{\mu}) = L(\boldsymbol{\mu})|\bar{\boldsymbol{u}}(\boldsymbol{\mu})|/\nu(\boldsymbol{\mu})$, being $L(\boldsymbol{\mu})$ a characteristic length of the parametrized domain, $\bar{\boldsymbol{u}}(\boldsymbol{\mu})$ a typical flow velocity and $\nu(\boldsymbol{\mu})$ the kinematic viscosity.

The weak formulation of problem (39) can be obtained, as in section 2, with the aid of a reference domain Ω, on which function spaces $\boldsymbol{V}$ and Q are defined, as in section 2. Proceeding as in the Stokes case, we have the following parametrized weak formulation of problem (39):

$$
\begin{cases}
\text{Find } \boldsymbol{u}(\boldsymbol{\mu}) \in \boldsymbol{V}, p(\boldsymbol{\mu}) \in Q : \\
a(\boldsymbol{u}(\boldsymbol{\mu}), \boldsymbol{v}; \boldsymbol{\mu}) + b(\boldsymbol{v}, p(\boldsymbol{\mu}); \boldsymbol{\mu}) + c(\boldsymbol{u}(\boldsymbol{\mu}), \boldsymbol{u}(\boldsymbol{\mu}), \boldsymbol{v}; \boldsymbol{\mu}) + d(\boldsymbol{u}(\boldsymbol{\mu}), \boldsymbol{v}; \boldsymbol{\mu}) = F(\boldsymbol{v}; \boldsymbol{\mu}) & \forall\, v \in \boldsymbol{V}, \\
b(\boldsymbol{u}(\boldsymbol{\mu}), q; \boldsymbol{\mu}) = G(q; \boldsymbol{\mu}) & \forall\, q \in Q,
\end{cases}
\tag{40}
$$

where $a(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu})$, $b(\boldsymbol{v}, q; \boldsymbol{\mu})$ are the bilinear forms related to diffusion and pressure-divergence operators, respectively defined in (3), whereas the trilinear form related to the convective term is defined as:

$$
c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}; \boldsymbol{\mu}) = \int_\Omega u_i \chi_{ji}(x; \boldsymbol{\mu}) \frac{\partial v_m}{\partial x_j} w_m d\boldsymbol{x}.
\tag{41}
$$

Other terms appearing due to the lifting of Dirichlet boundary conditions are defined as

$$
\begin{aligned}
d(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}) &= c(\boldsymbol{l}(\boldsymbol{\mu}), \boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}) + c(\boldsymbol{u}, \boldsymbol{l}(\boldsymbol{\mu}), \boldsymbol{v}; \boldsymbol{\mu}), \\
F(\boldsymbol{v}; \boldsymbol{\mu}) &= -a(\boldsymbol{l}(\boldsymbol{\mu}), \boldsymbol{v}, \boldsymbol{\mu}) - c(\boldsymbol{l}(\boldsymbol{\mu}), \boldsymbol{l}(\boldsymbol{\mu}), \boldsymbol{v}; \boldsymbol{\mu}), \\
G(q; \boldsymbol{\mu}) &= -b(\boldsymbol{l}(\boldsymbol{\mu}), q; \boldsymbol{\mu}),
\end{aligned}
\tag{42}
$$

being $\boldsymbol{l}(\boldsymbol{\mu})$ a parametrized lifting function as in section 2.

4.1   Finite Element formulation

In order to write the Galerkin-FE formulation for (40), we first need to introduce two finite-dimensional subspaces $\boldsymbol{V}_h \subset \boldsymbol{V}$, $Q_h \subset Q$ of dimension $\mathcal{N}_u$ and $\mathcal{N}_p$, respectively, as in section 2.1. The Galerkin-FE approximation of the parametrized problem (40) reads as follows:

$$
\begin{cases}
\text{Find } \boldsymbol{u}_h(\boldsymbol{\mu}) \in \boldsymbol{V}_h, p_h(\boldsymbol{\mu}) \in Q_h : \\
a(\boldsymbol{u}_h(\boldsymbol{\mu}), \boldsymbol{v}_h; \boldsymbol{\mu}) + b(\boldsymbol{v}_h, p_h(\boldsymbol{\mu}); \boldsymbol{\mu}) + c(\boldsymbol{u}_h(\boldsymbol{\mu}), \boldsymbol{u}_h(\boldsymbol{\mu}), \boldsymbol{v}_h; \boldsymbol{\mu}) \\
+ d(\boldsymbol{u}_h(\boldsymbol{\mu}), \boldsymbol{v}_h; \boldsymbol{\mu}) = F(\boldsymbol{v}_h; \boldsymbol{\mu}) & \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \\
b(\boldsymbol{u}_h(\boldsymbol{\mu}), q_h; \boldsymbol{\mu}) = G(q_h; \boldsymbol{\mu}) & \forall\, q_h \in Q_h,
\end{cases}
\tag{43}
$$

where

$$
\begin{aligned}
d(\boldsymbol{u}_h(\boldsymbol{\mu}), \boldsymbol{v}_h; \boldsymbol{\mu}) &= c(\boldsymbol{l}_h(\boldsymbol{\mu}), \boldsymbol{u}_h, \boldsymbol{v}_h; \boldsymbol{\mu}) + c(\boldsymbol{u}_h, \boldsymbol{l}_h(\boldsymbol{\mu}), \boldsymbol{v}_h; \boldsymbol{\mu}), \\
F(\boldsymbol{v}_h; \boldsymbol{\mu}) &= -a(\boldsymbol{l}_h(\boldsymbol{\mu}), \boldsymbol{v}_h, \boldsymbol{\mu}) - c(\boldsymbol{l}_h(\boldsymbol{\mu}), \boldsymbol{l}_h(\boldsymbol{\mu}), \boldsymbol{v}_h; \boldsymbol{\mu}), \\
G(q_h; \boldsymbol{\mu}) &= -b(\boldsymbol{l}_h(\boldsymbol{\mu}), q_h; \boldsymbol{\mu}).
\end{aligned}
\tag{44}
$$

Let $\{\phi_i^h\}_{i=1}^{\mathcal{N}_u}$ and $\{\psi_j^h\}_{j=1}^{\mathcal{N}_p}$ be basis functions of $\boldsymbol{V}_h$ and Q_h respectively. We introduce the matrices $A(\boldsymbol{\mu}) \in \mathbb{R}^{\mathcal{N}_u \times \mathcal{N}_u}$, $C(\boldsymbol{u}(\boldsymbol{\mu}); \boldsymbol{\mu}) \in \mathbb{R}^{\mathcal{N}_u \times \mathcal{N}_u}$, and $B(\boldsymbol{\mu}) \in \mathbb{R}^{\mathcal{N}_p \times \mathcal{N}_u}$ whose entries are

$$(A(\boldsymbol{\mu}))_{ij} = a(\phi_j^h, \phi_i^h; \boldsymbol{\mu}) + d(\phi_j^h, \phi_i^h; \boldsymbol{\mu}), \qquad (B(\boldsymbol{\mu}))_{ki} = b(\phi_i^h, \psi_k^h; \boldsymbol{\mu}),$$

$$(C(\boldsymbol{u}(\boldsymbol{\mu}); \boldsymbol{\mu}))_{ij} = \sum_{m=1}^{\mathcal{N}_u} \boldsymbol{u}_h^m(\boldsymbol{\mu}) c(\phi_m^h, \phi_j^h, \phi_i^h; \boldsymbol{\mu}), \quad \text{for } 1 \le i, j \le \mathcal{N}_u, 1 \le k \le \mathcal{N}_p, \tag{45}$$

and therefore the nonlinear algebraic system is:

$$\begin{bmatrix} A(\boldsymbol{\mu}) + C(\boldsymbol{u}_h(\boldsymbol{\mu}); \boldsymbol{\mu}) & B^T(\boldsymbol{\mu}) \\ B(\boldsymbol{\mu}) & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}(\boldsymbol{\mu}) \\ \boldsymbol{P}(\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}(\boldsymbol{\mu}) \end{bmatrix}, \tag{46}$$

for the vectors of coefficients $\boldsymbol{U} = (u_h^{(1)}, ..., u_h^{(\mathcal{N}_u)})^T$, $\boldsymbol{P} = (p_h^{(1)}, ..., p_h^{(\mathcal{N}_p)})^T$ such that $\boldsymbol{u}_h = \sum_{i=1}^{\mathcal{N}_u} u_h^{(i)} \phi_i^h$ and $p_h = \sum_{j=1}^{\mathcal{N}_p} p_h^{(j)} \psi_j^h$, where for $1 \le i \le \mathcal{N}_u$ and $1 \le k \le \mathcal{N}_p$:

$$(\bar{\boldsymbol{f}}(\boldsymbol{\mu}))_i = -a(\boldsymbol{l}_h, \phi_i^h; \boldsymbol{\mu}) - c(\boldsymbol{l}_h, \boldsymbol{l}_h, \phi_i^h; \boldsymbol{\mu}), \quad (\bar{\boldsymbol{g}}(\boldsymbol{\mu}))_k = -b(\boldsymbol{l}_h, \psi_k^h; \boldsymbol{\mu}), \tag{47}$$

with $\boldsymbol{l}_h = \boldsymbol{l}_h(\boldsymbol{\mu})$, the FE interpolant of lifting function. To solve this nonlinear system we use the Newton method [40].

For an efficient RB method, we need to ensure the assumption of affine parametric dependence on operators (45) and (47), i.e, these operators can be written as:

$$A(\boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \Theta_q^a(\boldsymbol{\mu}) A^q, \qquad C(.; \boldsymbol{\mu}) = \sum_{q=1}^{Q_c} \Theta_q^c(\boldsymbol{\mu}) C^q(.), \qquad B(\boldsymbol{\mu}) = \sum_{q=1}^{Q_b} \Theta_q^b(\boldsymbol{\mu}) B^q,$$

$$\bar{\boldsymbol{f}}(\boldsymbol{\mu}) = \sum_{q=1}^{Q_f} \Theta_q^f(\boldsymbol{\mu}) \bar{\boldsymbol{f}}^q, \qquad \bar{\boldsymbol{g}}(\boldsymbol{\mu}) = \sum_{q=1}^{Q_g} \Theta_q^g(\boldsymbol{\mu}) \bar{\boldsymbol{g}}^q. \tag{48}$$

4.2   Stabilized Finite Element formulation

In this section we introduce the stabilization terms into the FE formulation of (43). The stabilized FE formulation of (43) read as:

$$\begin{cases} \text{Find } \boldsymbol{u}_h(\boldsymbol{\mu}) \in \boldsymbol{V}_h, p_h(\boldsymbol{\mu}) \in Q_h : \\ a(\boldsymbol{u}_h(\boldsymbol{\mu}), \boldsymbol{v}_h; \boldsymbol{\mu}) + b(\boldsymbol{v}_h, p_h(\boldsymbol{\mu}); \boldsymbol{\mu}) + c(\boldsymbol{u}_h(\boldsymbol{\mu}), \boldsymbol{u}_h(\boldsymbol{\mu}), \boldsymbol{v}_h; \boldsymbol{\mu}) \\ + d(\boldsymbol{u}_h(\boldsymbol{\mu}), \boldsymbol{v}_h; \boldsymbol{\mu}) - s_h^{u,v}(\boldsymbol{u}_h, \boldsymbol{v}_h; \boldsymbol{\mu}) - s_h^{p,v}(p_h, \boldsymbol{v}_h; \boldsymbol{\mu}) = F(\boldsymbol{v}_h; \boldsymbol{\mu}) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b(\boldsymbol{u}_h(\boldsymbol{\mu}), q_h; \boldsymbol{\mu}) - s_h^{u,q}(\boldsymbol{u}_h, q_h; \boldsymbol{\mu}) - s_h^{p,q}(p_h, q_h; \boldsymbol{\mu}) = G(q_h; \boldsymbol{\mu}) \qquad \forall q_h \in Q_h, \end{cases} \tag{49}$$

where $s_h^{u,v}(., .; \boldsymbol{\mu})$, $s_h^{p,v}(., .; \boldsymbol{\mu})$, $s_h^{u,q}(., .; \boldsymbol{\mu})$ and $s_h^{p,q}(., .; \boldsymbol{\mu})$ are the stabilization terms [41] defined as:

$$s_h^{u,v}(\boldsymbol{u}_h, \boldsymbol{v}_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (-\nu \Delta \boldsymbol{u}_h + \boldsymbol{u}_h \cdot \nabla \boldsymbol{u}_h, -\rho \nu \Delta \boldsymbol{v}_h),$$

$$s_h^{p,v}(p_h, \boldsymbol{v}_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\nabla p_h, -\rho \nu \Delta \boldsymbol{v}_h),$$

$$s_h^{u,q}(\boldsymbol{u}_h, q_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (-\nu \Delta \boldsymbol{u}_h + \boldsymbol{u}_h \cdot \nabla \boldsymbol{u}_h, \nabla q_h), \tag{50}$$

$$s_h^{p,q}(p_h, q_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\nabla p_h, \nabla q_h),$$

where δ is the stabilization coefficient needs to be chosen properly [31, 9, 32]. For $\rho = 0, 1$, the stabilization (50) is respectively known as Streamline Upwind Petrov Galerkin (SUPG) [12], Galerkin least-squares (GLS) [26]. The case $\rho = -1$ was studied by Franca and Frey [19]. Several other works on these kind of stabilization techniques can be found in [30, 50, 53, 33] and references therein.

In this paper we discuss only the SUPG stabilization. Therefore, after adding the SUPG stabilization terms, the stabilized version of nonlinear system reads as (46):

$$\begin{bmatrix} A(\boldsymbol{\mu}) + \tilde{C}(\boldsymbol{u}_h(\boldsymbol{\mu}); \boldsymbol{\mu}) & \tilde{B}^T(\boldsymbol{\mu}) \\ \tilde{B}(\boldsymbol{\mu}) & -S_N(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \boldsymbol{U}(\boldsymbol{\mu}) \\ \boldsymbol{P}(\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}(\boldsymbol{\mu}) \end{bmatrix}, \tag{51}$$

where $\tilde{B}$, includes the the effects of stabilization in mass equation on divergence term, $\tilde{B}^T$ includes the effects on pressure gradient and $\tilde{C}$ contains the nonlinear stabilization terms [41].

4.3 Reduced Basis formulation

In this section we present the RB formulation of parametrized Navier-Stokes problem (39) [39, 29] in a similar way as we have done for the Stokes case. The RB spaces for velocity and pressure are defined in section 2.3.

Once, we have built the RB for velocity and pressure fields during the *offline* stage, the RB formulation corresponding to FE formulation (43) reads as:

$$\begin{cases} \text{Find } \boldsymbol{u}_N(\boldsymbol{\mu}) \in \tilde{\boldsymbol{V}}_N, p_N(\boldsymbol{\mu})) \in Q_N : \\ a(\boldsymbol{u}_N(\boldsymbol{\mu}), \boldsymbol{v}_N; \boldsymbol{\mu}) + b(\boldsymbol{v}_N, p_N(\boldsymbol{\mu}); \boldsymbol{\mu}) + c(\boldsymbol{u}_N(\boldsymbol{\mu}), \boldsymbol{u}_N(\boldsymbol{\mu}), \boldsymbol{v}_N; \boldsymbol{\mu}) \\ + d(\boldsymbol{u}_N(\boldsymbol{\mu}), \boldsymbol{v}_N; \boldsymbol{\mu}) = F(\boldsymbol{v}_N; \boldsymbol{\mu}) & \forall \boldsymbol{v}_N \in \tilde{\boldsymbol{V}}_N, \\ b(\boldsymbol{u}_N(\boldsymbol{\mu}), q_N; \boldsymbol{\mu}) = G(q_N; \boldsymbol{\mu}) & \forall q_N \in Q_N. \end{cases} \tag{52}$$

Owing to the representation (27), the corresponding nonlinear reduced system

$$\begin{bmatrix} A_N(\boldsymbol{\mu}) + C_N(\boldsymbol{u}(\boldsymbol{\mu}); \boldsymbol{\mu}) & B_N^T(\boldsymbol{\mu}) \\ B_N(\boldsymbol{\mu}) & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_N(\boldsymbol{\mu}) \\ \boldsymbol{P}_N(\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}_N(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}_N(\boldsymbol{\mu}) \end{bmatrix}, \tag{53}$$

where the reduced order matrices are defined as:

$$\begin{aligned} A_N(\boldsymbol{\mu}) &= Z_{u,s}^T A(\boldsymbol{\mu}) Z_{u,s}, \quad B_N(\boldsymbol{\mu}) = Z_p^T B(\boldsymbol{\mu}) Z_{u,s}, \quad C_N(.; \boldsymbol{\mu}) = Z_{u,s}^T C(.; \boldsymbol{\mu}) Z_{u,s}, \\ \bar{\boldsymbol{f}}_N(\boldsymbol{\mu}) &= Z_{u,s}^T \bar{\boldsymbol{f}}(\boldsymbol{\mu}), \quad \bar{\boldsymbol{g}}_N(\boldsymbol{\mu}) = Z_p^T \bar{\boldsymbol{g}}(\boldsymbol{\mu}), \end{aligned} \tag{54}$$

with $Z_{u,s}$, the velocity basis matrix and Z_p denotes the pressure basis matrix.

4.4 Stabilized Reduced Basis formulation

In this section, we present the stabilized RB method for the model problem (39). The stabilized RB formulation corresponding to stabilized FE formulation (49) reads as:

$$\begin{cases} \text{Find } \boldsymbol{u}_N(\boldsymbol{\mu}) \in \boldsymbol{V}_N, p_N(\boldsymbol{\mu}) \in Q_N : \\ a(\boldsymbol{u}_N(\boldsymbol{\mu}), \boldsymbol{v}_N; \boldsymbol{\mu}) + b(\boldsymbol{v}_N, p_N(\boldsymbol{\mu}); \boldsymbol{\mu}) + c(\boldsymbol{u}_N(\boldsymbol{\mu}), \boldsymbol{u}_N(\boldsymbol{\mu}), \boldsymbol{v}_N; \boldsymbol{\mu}) \\ + d(\boldsymbol{u}_N(\boldsymbol{\mu}), \boldsymbol{v}_N; \boldsymbol{\mu}) - s_N^{u,v}(\boldsymbol{u}_N, \boldsymbol{v}_N; \boldsymbol{\mu}) - s_N^{p,v}(p_N, \boldsymbol{v}_N; \boldsymbol{\mu}) = F(\boldsymbol{v}_N; \boldsymbol{\mu}) & \forall \boldsymbol{v}_N \in \boldsymbol{V}_N, \\ b(\boldsymbol{u}_N(\boldsymbol{\mu}), q_N; \boldsymbol{\mu}) - s_N^{u,q}(\boldsymbol{u}_N, q_N; \boldsymbol{\mu}) - s_N^{p,q}(p_N, q_N; \boldsymbol{\mu}) = G(q_N; \boldsymbol{\mu}) & \forall q_N \in Q_N, \end{cases} \tag{55}$$

where $s_N^{u,v}(.,.;\boldsymbol{\mu})$, $s_N^{p,v}(.,.;\boldsymbol{\mu})$, $s_N^{u,q}(.,.;\boldsymbol{\mu})$ and $s_N^{p,q}(.,.;\boldsymbol{\mu})$ are the reduced order stabilization terms defined as:

$$
\begin{aligned}
s_N^{u,v}(\boldsymbol{u}_N, \boldsymbol{v}_N; \boldsymbol{\mu}) &:= \delta \sum_K h_K^2 \int_K (-\nu\Delta\boldsymbol{u}_{N+\boldsymbol{u}_N\cdot\nabla\boldsymbol{u}_N}, -\rho\nu\Delta\boldsymbol{v}_N), \\
s_N^{p,v}(p_h, \boldsymbol{v}_N; \boldsymbol{\mu}) &:= \delta \sum_K h_K^2 \int_K (\nabla p_N, -\rho\nu\Delta\boldsymbol{v}_N), \\
s_N^{u,q}(\boldsymbol{u}_N, q_N; \boldsymbol{\mu}) &:= \delta \sum_K h_K^2 \int_K (-\nu\Delta\boldsymbol{u}_N + \boldsymbol{u}_N \cdot \nabla\boldsymbol{u}_N, \nabla q_N), \\
s_N^{p,q}(p_N, q_N; \boldsymbol{\mu}) &:= \delta \sum_K h_K^2 \int_K (\nabla p_N, \nabla q_N),
\end{aligned}
\tag{56}
$$

The corresponding nonlinear system results in

$$
\begin{bmatrix} A_N(\boldsymbol{\mu}) + \tilde{C}_N(\boldsymbol{u}(\boldsymbol{\mu}); \boldsymbol{\mu}) & \tilde{B}_N^T(\boldsymbol{\mu}) \\ \tilde{B}_N(\boldsymbol{\mu}) & -S_N(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_N(\boldsymbol{\mu}) \\ \boldsymbol{P}_N(\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}_N(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}_N(\boldsymbol{\mu}) \end{bmatrix},
\tag{57}
$$

where $\tilde{B}_N$, $\tilde{B}_N^T$ and $\tilde{C}_N$ are RB stabilization matrices [41] defined as:

$$
\tilde{B}_N(\boldsymbol{\mu}) = Z_p^T \tilde{B}(\boldsymbol{\mu}) Z_{u,s}, \quad \tilde{B}_N^T(\boldsymbol{\mu}) = Z_p \tilde{B}(\boldsymbol{\mu}) Z_{u,s}, \quad S_N(\boldsymbol{\mu}) = Z_p^T S(\boldsymbol{\mu}) Z_p.
\tag{58}
$$

5 Numerical results and discussion for parametrized Navier-Stokes problems

In this section we present some numerical results for the RB approximation of steady parametrized Navier-Stokes problem developed in section 4 and subsections therein. As in section 3, we compare three possible options (*i*) *offline-online stabilization* with supremizer, (*ii*) *offline-online stabilization* without supremizer, (*iii*) *offline-only stabilization* with supremizer. The numerical test case is the same parametrized cavity problem as in section 3. Table 2 illustrates the computation details and the cost of stabilization options using both $\mathbb{P}_1/\mathbb{P}_1$ and $\mathbb{P}_2/\mathbb{P}_2$ FE pairs.

In Fig. 9, we show the FE solution (top) for velocity and pressure, RB solution for velocity and pressure using the *offline-online stabilization* with/without supremizer (center) and the *offline-only stabilization* with supremizer (bottom), for $(\mu_1, \mu_2) = (120, 2)$ and $\delta = 1.0$. These results are carried out using $\mathbb{P}_2/\mathbb{P}_2$ FE pair (results are similar for $\mathbb{P}_1/\mathbb{P}_1$). From these results, we see that the FE solution and RB solution obtained by *offline-online stabilization* are same, but the RB pressure obtained by *offline-only stabilization* is not accurate.

Figures 10 and 11 represents the error between FE and RB velocity obtained by three possible stabilization options using $\mathbb{P}_1/\mathbb{P}_1$ and $\mathbb{P}_2/\mathbb{P}_2$ FE pair, respectively. From these results we conclude that the *offline-online stabilization* without supremizer has better performance in case of velocity, when compared to other two options and is the most consistent stabilization option. However, in case of pressure, supremizer is improving the error upto an order of magnitude. Moreover, when comparing the computational costs of the two *offline-online stabilization options, the one without supremizer has less computation time than the case with supremizer.* Therefore, we conclude that the *offline-online stabilization* is sufficient to guarantee a stable RB solution and we can avoid the supremizer enrichment to increase the computational performance of *online* stage.
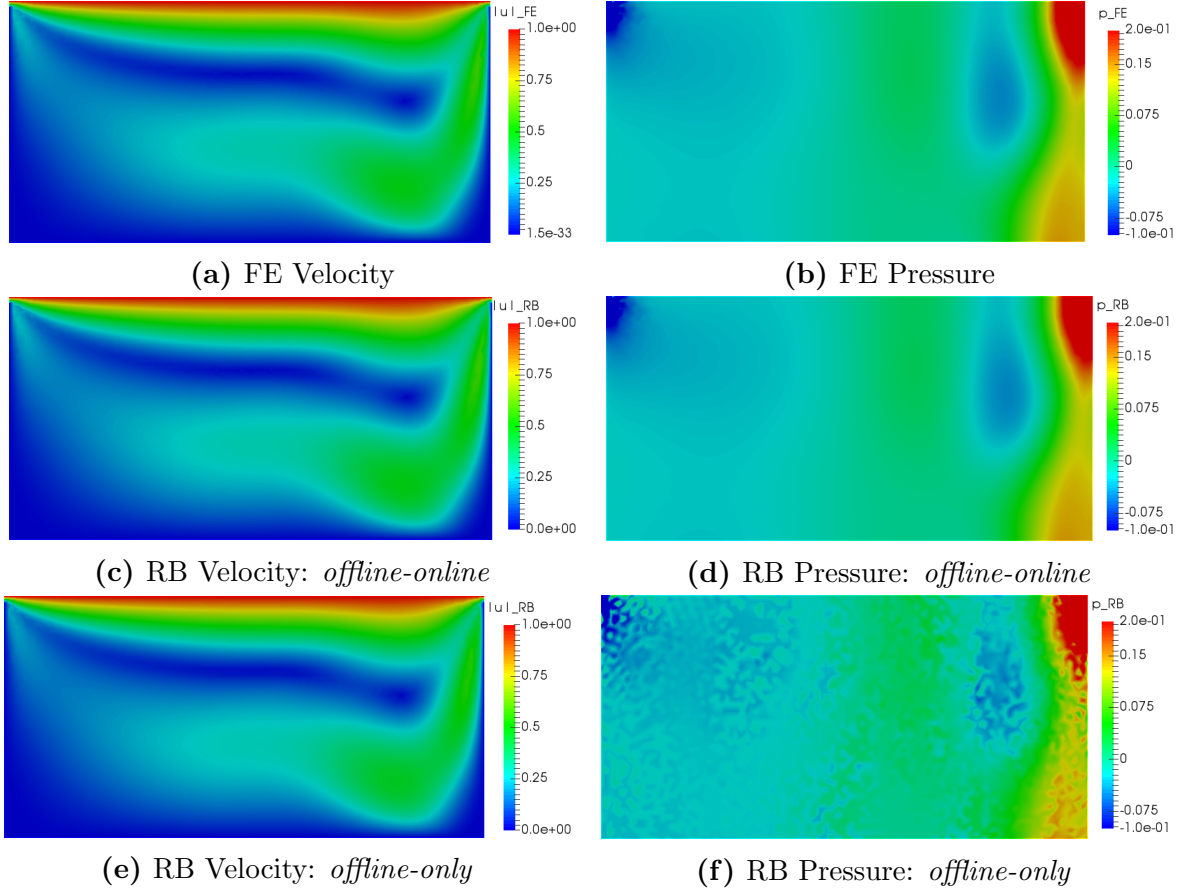
(a) FE Velocity

(b) FE Pressure

(c) RB Velocity: *offline-online*

(d) RB Pressure: *offline-online*

(e) RB Velocity: *offline-only*

(f) RB Pressure: *offline-only*

Figure 9: Navier-Stokes problem with SUPG stabilization: From top to bottom; FE solution (top), RB solution using *offline-online stabilization* (center), RB solution using *offline-only stabilization* (bottom) for $(\mu_1, \mu_2) = (120, 2)$.
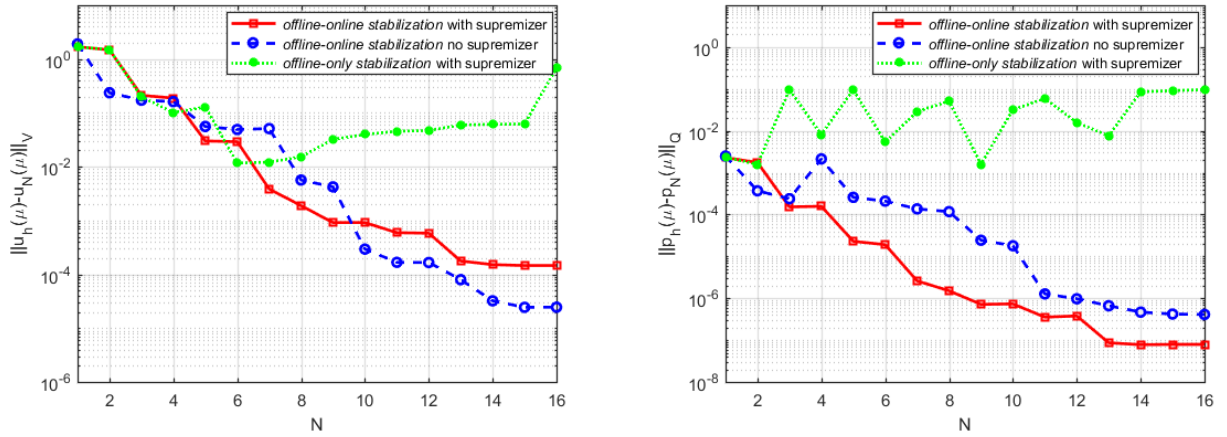

Figure 10: Navier-Stokes problem: SUPG stabilization with geometrical and physical parametrization on cavity flow; Error between FE and RB solution for velocity (left) and pressure (right) using $\mathbb{P}_1/\mathbb{P}_1$.
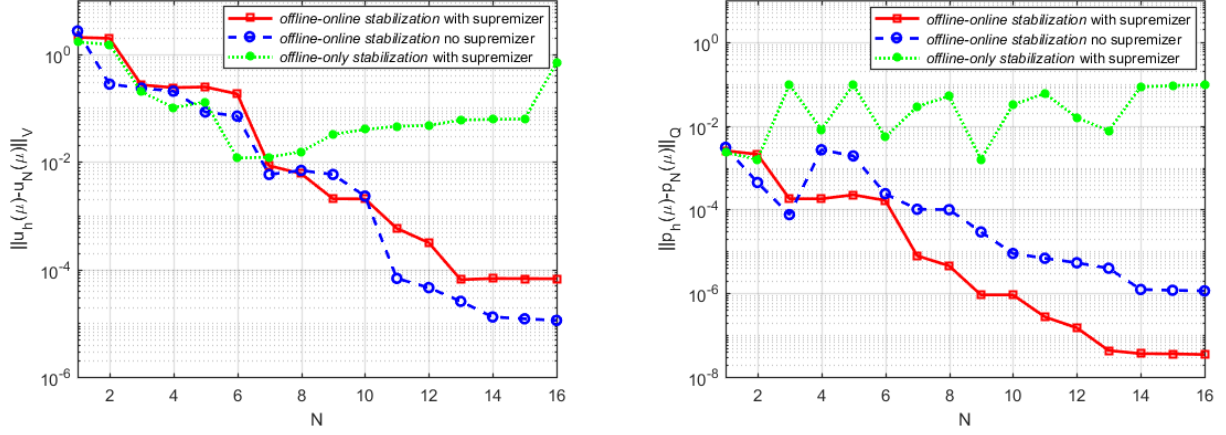
Figure 11: Navier-Stokes problem: SUPG stabilization with geometrical and physical parametrization on cavity flow; Error between FE and RB solution for velocity (left) and pressure (right) using $\mathbb{P}_2/\mathbb{P}_2$.

Physical parameter	μ_1 (Reynolds number)
Geometrical parameter	μ_2 (horizontal length of domain)
Range of μ_1	[100,200]
Range of μ_2	[1.5,3]
μ_1 *online*	120
μ_2 *online*	2
FE degrees of freedom	11160 ($\mathbb{P}_1/\mathbb{P}_1$) 44091 ($\mathbb{P}_2/\mathbb{P}_2$)
RB dimension	$N_u = N_s = N_p = 16$
Computation time ($\mathbb{P}_2/\mathbb{P}_1$)	3909s (*offline*), 195s (*online*) with supremizers
Offline time ($\mathbb{P}_1/\mathbb{P}_1$)	2034s (*offline-online stabilization* with supremizers) 1920s (*offline-online stabilization* without supremizers) 649 (*offline-only stabilization* with supremizers)
Offline time ($\mathbb{P}_2/\mathbb{P}_2$)	4885s (*offline-online stabilization* with supremizers) 4387ss (*offline-online stabilization* without supremizers) 1650s (*offline-only stabilization* with supremizers)
Online time ($\mathbb{P}_1/\mathbb{P}_1$)	110s (*offline-online stabilization* with supremizers) 87s (*offline-online stabilization* without supremizers) 35s (*offline-only stabilization* with supremizers)
Online time ($\mathbb{P}_2/\mathbb{P}_2$)	242s (*offline-online stabilization* with supremizers) 180s (*offline-online stabilization* without supremizers) 90s (*offline-only stabilization* with supremizers)

Table 2: Navier-Stokes problem: computational details for physical and geometrical parameters.

6 Conclusion and perspectives

In this work we have used classical residual based stabilization techniques [25, 12] to develop stabilized RB methods for parametrized steady Stokes and Navier-Stokes problem. While such stabilizations have been used to handle advection dominated problems both at FE [11] and RB [36] level, in this work we have focused solely on the issue of inf-sup stability and pressure

recovery. Indeed, construction of stable reduced basis during the offline stage does not guarantee a stable system in the online stage. The most widespread method in the RB community to tackle this issue is instead to perform a supremizer enrichment of the reduced velocity space [46]. The goal of the work has been to combine and compare the two resulting RB inf-sup stabilization options, i.e. supremizer enrichment and residual based stabilization; this gave rise to four different reduced order methods.

The first conclusion of this work is that, whenever FE stabilization is used, the *offline-online stabilization* is the most appropriate way to perform the online phase. Indeed, the other option, i.e. *offline-only stabilized* method, resulted in sensibly larger errors, for both velocity and pressure. We claim that this is because of the lack of consistency between the offline and online phases. Similar conclusions were reached by [51, 36], although the difference is exacerbated in our case, due to the role of the pressure.

The second conclusion is on the role of supremizer enrichment. On one hand, addition of supremizers to *offline-only stabilized* case does not help in improving the results, which are affected in a greater way by the lack of consistency. One the other hand, *offline-online stabilization* with and without supremizers frequently showed comparable results in the error analysis, with values lower than 10^{-4} in all numerical examples. The case with supremizer often results in better pressure, while the case without supremizer yields better velocity in the nonlinear case. However, the differences in terms of error are often limited to one order of magnitude, still resulting in a reduced solution which would be sufficiently accurate for many practical applications. Therefore, when compared to the existing methodology based on supremizer enrichment alone, our current work allows one to make a tradeoff between accuracy and performance. Better accuracy is obtained combining residual based stabilization and supremizer enrichment, while better performance could have been obtained neglecting the enrichment step.

We still have some open questions and perspectives to improve this work in future, in order to make this approach applicable to more and more complex problems. For instance:

(i) in order to develop a certified stabilized RB method, an a *posteriori* error analysis [23, 52] is needed for residual based stabilization in a reduced order setting, for which we suggest to have a look into the error analysis of stabilized FE methods [48, 28];

(ii) the computational cost of stabilized RB method in case of nonlinear problems can be decreased by using the Empirical Interpolation method (EIM) [6];

(iii) one can extend this work to develop a Variational MultiScale (VMS) method for turbulent flows with moderate-higher Reynolds number [47];

(iv) this work is applicable to optimal control problems, see for instance environmental applications in marine sciences [49];

(v) time-dependent problems are of interest for some applications of the stabilized RB methodology;

Acknowledgements

References

[1] J. Baiges, R. Codina, and S. Idelsohn. Explicit reduced-order models for the stabilized finite element approximation of the incompressible Navier-Stokes equations. *International Journal for Numerical Methods in Fluids*, 72:1219–1243, 2013.

[2] J. Baiges, R. Codina, and S. R. Idelsohn. *Reduced-Order Modelling Strategies for the Finite Element Approximation of the Incompressible Navier-Stokes Equations*, pages 189–216. Springer International Publishing, 2014.

[3] M. Balajewicz and E. H. Dowell. Stabilization of projection-based reduced order models of the Navier-Stokes. *Nonlinear Dynamics*, 70(2):1619–1632, 2012.

[4] F. Ballarin, A. Manzoni, A. Quarteroni, and G. Rozza. Supremizer stabilization of POD-Galerkin approximation of parametrized steady incompressible Navier-Stokes equations. *International Journal for Numerical Methods in Engineering*, 102(5):1136–1161, 2015.

[5] F. Ballarin, A. Sartori, and G. Rozza. RBniCS - reduced order modelling in FEniCS. *http://mathlab.sissa.it/rbnics*, 2016.

[6] M. Barrault, Y. Maday, N. C. Nguyen, and A. T. Patera. An empirical interpolation method: application to efficient reduced-basis discretization of partial differential equations. *Comptes Rendus Mathematique*, 339(9):667 – 672, 2004.

[7] R. Becker and M. Braack. A finite element pressure gradient stabilization for the stokes equations based on local projections. *Calcolo*, 38(4):173–199, 2001.

[8] D. Boffi, F. Brezzi, and M. Fortin. *Mixed finite element methods and applications*, volume 44. Springer, 2013.

[9] M. Braack, E. Burman, V. John, and G. Lube. Stabilized finite element methods for the generalized Oseen problem. *Computer Methods in Applied Mechanics and Engineering*, 196(46):853 – 866, 2007.

[10] F. Brezzi and J. Pitkäranta. On the stabilization of finite element approximations of the Stokes equations. In *Efficient Solutions of Elliptic Systems*, volume 10 of *Notes on Numerical Fluid Mechanics*, pages 11–19. 1984.

[11] A. Brooks and T. Hughes. Streamline Upwind/Petrov-Galerkin methods for advection dominated flows. *Third International Conference on Finite Element Methods in Fluid Flow*, 2 , Calgary, Canada, Calgary Univ., 1980.

[12] A. N. Brooks and T. J. Hughes. Streamline Upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. *Computer Methods in Applied Mechanics and Engineering*, 32(1-3):199 – 259, 1982.

[13] E. Burman and M. Fernández. Galerkin finite element methods with symmetric pressure stabilization for the transient stokes equations: Stability and convergence analysis. *SIAM Journal on Numerical Analysis*, 47(1):409–439, 2009.

[14] A. Caiazzo, T. Iliescu, V. John, and S. Schyschlowa. A numerical investigation of velocity-pressure reduced order models for incompressible flows. *Journal of Computational Physics*, 259:598 – 616, 2014.

[15] S. Deparis. Reduced basis error bound computation of parameter-dependent Navier-Stokes equations by the natural norm approach. *SIAM Journal on Numerical Analysis*, 46(4):2039–2067, 2008.

[16] S. Deparis and A. E. Løvgren. Stabilized reduced basis approximation of incompressible three-dimensional Navier-Stokes equations in parametrized deformed domains. *Journal of Scientific Computing*, 50(1):198–212, 2012.

[17] S. Deparis and G. Rozza. Reduced basis method for multi-parameter-dependent steady Navier-Stokes equations: Applications to natural convection in a cavity. *Journal of Computational Physics*, 228(12):4359 – 4378, 2009.

[18] J. J. Douglas and J. Wang. An absolutely stabilized finite element formulation for the Stokes problem. *Mathematics of Computations*, 52(186):495 – 508, 1989.

[19] L. P. Franca and S. L. Frey. Stabilized finite element methods: II. The incompressible Navier-Stokes equations. *Computer Methods in Applied Mechanics and Engineering*, 99(2):209 – 233, 1992.

[20] A.-L. Gerner and K. Veroy. Certified reduced basis methods for parametrized saddle point problems. *SIAM Journal on Scientific Computing*, 34(5):A2812–A2836, 2012.

[21] S. Giere, T. Iliescu, V. John, and D. Wells. SUPG reduced order models for convection-dominated convection-diffusion-reaction equations. *Computer Methods in Applied Mechanics and Engineering*, 289:454 – 474, 2015.

[22] F. Hecht. New development in FreeFem++. *Journal of Numerical Mathematics*, 20(3-4):251–266, 2013.

[23] J. S. Hesthaven, G. Rozza, and B. Stamm. *Certified Reduced Basis Methods for Parametrized Partial Differential Equations*. Springer Briefs in Mathematics. 2015.

[24] T. J. Hughes and L. P. Franca. A new finite element formulation for computational fluid dynamics: VII. The Stokes problem with various well-posed boundary conditions: Symmetric formulations that converge for all velocity/pressure spaces. *Computer Methods in Applied Mechanics and Engineering*, 65(1):85 – 96, 1987.

[25] T. J. Hughes, L. P. Franca, and M. Balestra. A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuska-Brezzi condition: a stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations. *Computer Methods in Applied Mechanics and Engineering*, 59(1):85 – 99, 1986.

[26] T. J. Hughes, L. P. Franca, and G. M. Hulbert. A new finite element formulation for computational fluid dynamics: VIII. The Galerkin/least-squares method for advective-diffusive equations. *Computer Methods in Applied Mechanics and Engineering*, 73(2):173 – 189, 1989.

[27] E. Jenkins, V. John, A. Linke, and L. Rebholz. On the parameter choice in grad-div stabilization for the Stokes equations. *Advances in Computational Mathematics*, pages 1–26, 2013.

[28] D. Kay and D. Silvester. A Posteriori Error Estimation for Stabilized Mixed Approximations of the Stokes Equations. *SIAM Journal on Scientific Computing*, 21(4):1321–1336, 1999.

[29] A. E. Løvgren. *A reduced basis method for the steady Navier-Stokes problem, in "Reduced basis modelling of hierarchical flow system"*. PhD thesis, Norwegian University of Science and Technology, 2006.

[30] G. Lube and M. A. Olshanskii. Stable finite element calculation of incompressible flows using the rotation form of convection. *IMA Journal of Numerical Analysis*, 22(3):437–461, 2002.

[31] G. Lube and G. Rapin. Residual-based stabilized higher-order FEM for a generalized oseen problem. *Mathematical Models and Methods in Applied Sciences*, 16(07):949–966, 2006.

[32] G. Lube and G. Rapin. Residual-based stabilized higher-order FEM for advection-dominated problems. *Computer Methods in Applied Mechanics and Engineering*, 195(3336):4124 – 4138, 2006.

[33] G. Lube and L. Tobiska. A nonconforming finite element method of streamline diffusion type for the incompressible Navier-Stokes equations. *Journal of Computational Mathematics*, 8(2):147–158, 1990.

[34] A. Manzoni. An efficient computational framework for reduced basis approximation and a posteriori error estimation of parametrized Navier-Stokes flows. *ESAIM: Mathematical Modelling and Numerical Analysis*, 48(4):1199–1226, 2014.

[35] F. Negri, A. Manzoni, and G. Rozza. Reduced basis approximation of parametrized optimal flow control problems for the Stokes equations. *Computers and Mathematics with Applications*, 69(4):319–336, 2015.

[36] P. Pacciarini and G. Rozza. Stabilized reduced basis method for parametrized advection-diffusion PDEs. *Computer Methods in Applied Mechanics and Engineering*, 274:1–18, 2014.

[37] P. Pacciarini and G. Rozza. Stabilized reduced basis method for parametrized scalar advection-diffusion problems at higher Peclet number: roles of the boundary layers and inner fronts. In *Proceedings of the jointly organized 11th World Congress on Computational Mechanics-WCCM XI, 5th European Congress on Computational Mechanics-ECCM V, 6th European Congress on Computational Fluid Dynamics-ECFD VI*, pages 5614–5624, 2014.

[38] P. Pacciarini and G. Rozza. Reduced basis approximation of parametrized advection-diffusion PDEs with high Péclet number. *Lecture Notes in Computational Science and Engineering*, 103:419–426, 2015.

[39] A. Quarteroni and G. Rozza. Numerical solution of parametrized Navier-Stokes equations by reduced basis methods. *Numerical Methods for Partial Differential Equations*, 23(4):923–948, 2007.

[40] A. Quarteroni, R. Sacco, and F. Saleri. *Numerical Mathematics*. Springer, New York, 2000.

[41] A. Quarteroni and A. Valli. *Numerical approximation of partial differential equations*, volume 23. Springer Science & Business Media, 2008.

[42] D. Rovas. *Reduced-basis output bound methods for parametrized partial differential equations*. PhD thesis, Massachusetts Institute of Technology, 2003.

[43] G. Rozza. *Shape design by optimal flow control and reduced basis techniques: applications to bypass configurations in haemodynamics*. PhD thesis, École Polytechnique Fédérale de Lausanne, N. 3400, 2005.

[44] G. Rozza. Reduced basis methods for Stokes equations in domains with non-affine parameter dependence. *Computing and Visualization in Science*, 12(1):23–35, 2009.

[45] G. Rozza, D. B. P. Huynh, and A. Manzoni. Reduced basis approximation and a posteriori error estimation for Stokes flows in parametrized geometries: roles of the inf-sup stability constants. *Numerische Mathematik*, 125(1):115–152, 2013.

[46] G. Rozza and K. Veroy. On the stability of the reduced basis method for Stokes equations in parametrized domains. *Computer Methods in Applied Mechanics and Engineering*, 196:1244–1260, 2007.

[47] G. Stabile, F. Ballarin, G. Zuccarino, and G. Rozza. A reduced order variational multiscale approach for turbulent flows. *Advances in Computational Mathematics*, Nov 2019.

[48] R. Stenberg and J. Videman. On the Error Analysis of Stabilized Finite Element Methods for the Stokes Problem. *SIAM Journal on Numerical Analysis*, 53(6):2626–2633, 2015.

[49] M. Strazzullo, F. Ballarin, R. Mosetti, and G. Rozza. Model reduction for parametrized optimal control problems in environmental marine sciences and engineering. *SIAM Journal on Scientific Computing*, 40(4):B1055–B1079, 2018.

[50] L. Tobiska and R. Verfurth. Analysis of a streamline diffusion finite element method for the Stokes and Navier-stokes equations. *SIAM Journal on Numerical Analysis*, 33(1):107–127, 1996.

[51] D. Torlo, F. Ballarin, and G. Rozza. Stabilized weighted reduced basis methods for parametrized advection dominated problems with random inputs. *SIAM/ASA Journal on Uncertainty Quantification*, 6(4):1475–1502, 2017.

[52] K. Veroy and A. T. Patera. Certified real-time solution of the parametrized steady incompressible Navier-Stokes equations: rigorous reduced-basis a posteriori error bounds. *International Journal for Numerical Methods in Fluids*, 47(8-9):773–788, 2005.

[53] T. X. Zhou and M. F. Feng. A least squares Petrov-Galerkin finite element method for the stationary Navier-Stokes equations. *Mathematics of Computation*, 60(202):531–543, 1993.