Derived loop stacks and categorification of orbifold products

DERIVED LOOP STACKS AND CATEGORIFICATION OF ORBIFOLD PRODUCTS

SARAH SCHEROTZKE AND NICOLÒ SIBILLA

Abstract. The existence of interesting multiplicative cohomology theories for orbifolds was initially suggested by string theorists. Orbifold products have been intensely studied by mathematicians for the last fifteen years. In this paper we focus on the *virtual orbifold product* that was first introduced in Lupercio et al. (2007). We construct a categorification of the virtual orbifold product that leverages the geometry of derived loop stacks. After work of Ben-Zvi Francis and Nadler, this reveals connections between virtual orbifold products and Drinfeld centers of monoidal categories, thus answering a question of Hinich.

Contents

1. Introduction

The existence of non-trivial multiplicative cohomology theories for orbifolds was suggested by work of string theorists [Z]. The first mathematical implementation of these ideas is due to Chen and Ruan [CR]. They introduced a cohomology theory for orbifolds which is currently referred to as *Chen-Ruan (CR) cohomology.* If $\mathcal{X}$ is an orbifold, its Chen-Ruan cohomology $H^*_{CR}(\mathcal{X})$ is linearly isomorphic to the cohomology of the inertia orbifold $I\mathcal{X}$, but carries a non-trivial *orbifold product* capturing the degree zero Gromov-Witten theory of $\mathcal{X}$. Although initially developed for differentiable orbifolds, the theory of CR cohomology was later recast within the framework of algebraic geometry [FG, AGV1, AGV2]. In [JKK] Jarvis, Kaufmann

Key words and phrases. orbifold products, categorification, loop space.

and Kimura defined the *orbifold K-theory* of a global quotient DM stack $\mathcal{X}$. Similarly to Chen-Ruan cohomology, orbifold K-theory is linearly isomorphic to the rational algebraic K-theory of the inertia orbifold $I\mathcal{X}$ and is equipped with a non-trivial orbifold product. Further, a multiplicative Chern character map relates orbifold K-theory and CR cohomology.

Chen-Ruan cohomology is not the only possible definition of a multiplicative cohomology theory for orbifolds. In fact orbifold cohomology theories admit a rich web of distinct multiplicative structures (called *inertial products* in [EJK2]) that are governed by a choice of virtual bundle on the double inertia stack $I^2\mathcal{X}$: the orbifold product is only one of them. An especially important variant of the orbifold product is the *virtual orbifold product* introduced in [LUX, LUX+], and investigated in [EJK1, EJK2] from the perspective of algebraic geometry. We denote $*^{virt}$ the virtual orbifold product, and $K^{virt}(\mathcal{X}) := K_0(I\mathcal{X}, *^{virt})$ the virtual orbifold K-theory of $\mathcal{X}$. In this paper we construct a categorification of virtual orbifold K-theory.

Our main theorem, Theorem 1.1, takes the shape of a comparison result equating the virtual orbifold product with a different ring structure on K-theory, which is induced by a tensor product on the triangulated category of coherent sheaves. The virtual orbifold product is defined in the literature under some restrictive assumptions [EJK1, EJK2]. These are therefore mirrored by the assumptions in our Theorem 1.1. As we discuss more fully below, one of the upshots of our work is that in fact the virtual orbifold product can be defined in much greater generality by converting our Theorem 1.1 into a definition.

Let $\mathcal{X}$ be a smooth DM stack which admits a presentation as a global quotient $\mathcal{X} = [X/G]$, where X is an affine scheme and G is a linear group. Under these assumptions Edidin, Jarvis and Kimura define the virtual orbifold K-theory $K_0(I\mathcal{X}, *^{virt})$ [EJK2]. Denote $L\mathcal{X}$ the derived loop stack of $\mathcal{X}$ in the sense of [TV, BFN]. The bounded derived category of $L\mathcal{X}$, which we denote $Coh(L\mathcal{X})$, carries a braided monoidal structure that was introduced in [BFN]: we denote it $\otimes^{str}$, and let $G_0(L\mathcal{X}, \otimes^{str})$ be the Grothendieck group of $Coh(L\mathcal{X})$ together with the commutative product induced by $\otimes^{str}$.

Theorem 1.1. *Let $\iota : I\mathcal{X} \to L\mathcal{X}$ be the natural inclusion. Then ι_* gives rise to an isomorphism of rings:*

$$\iota_* : K^{virt}(\mathcal{X}) = K_0(I\mathcal{X}, *^{virt}) \xrightarrow{\simeq} G_0(L\mathcal{X}, \otimes^{str}).$$

Using results from [BFN] and [BNP] Theorem 1.1 can be reformulated as follows.

Theorem 1.2. *Let $\mathcal{Z}(Coh(\mathcal{X}))$ be the derived Drinfeld center (in the sense of* [BFN]*) of the symmetric monoidal category $Coh(\mathcal{X})$. Then there is a natural isomorphism:*

$$K^{virt}(\mathcal{X}) \simeq K_0(\mathcal{Z}(Coh(\mathcal{X}))).$$

This result has several useful consequences, we list some below:

- The prescription in [EJK2] gives a definition of virtual orbifold cohomology for smooth global quotient DM stacks. By setting $K^{virt}(\mathcal{X}) := G_0(L\mathcal{X}, \otimes^{str})$ we obtain a definition of virtual orbifold cohomology that applies to all smooth DM stacks with finite stabilizers which are *perfect* in the sense of [BFN], and in fact, to a wide class of derived ∞-stacks. Namely, it is sufficient that $\mathcal{X}$ is a regular, locally Noetherian perfect stack with representable diagonal. Under these assumptions Proposition 2.4 applies to $\mathcal{X}$ and therefore $G_0(L\mathcal{X}, \otimes^{str})$ has a well-defined product structure.

- Since $*^{virt}$ lifts to a tensor product on $Coh(L\mathcal{X})$ it induces a multiplicative structure on the full G-theory spectrum of $L\mathcal{X}$, which is equivalent to the K-theory spectrum of $I\mathcal{X}$, $G_*(L\mathcal{X}) \simeq K_*(I\mathcal{X})$.[1] This is a much richer invariant than the virtual orbifold K-theory of $\mathcal{X}$, which can be recovered by taking π_0, $K^{virt}(\mathcal{X}) = \pi_0(K_*(I\mathcal{X}))$. Also in this way we achieve a fully motivic definition of the virtual product, which is therefore not confined to K-theory but extends to any lax monoidal invariant of stable categories: for instance, our result enables the definition of virtual orbifold products on Hochschild homology and negative cyclic homology.
- In [BGNX] Behrend, Ginot, Noohi and Xu develop the theory of string topology for differentiable stacks (expositions of ordinary string topology of manifolds can be found in [CS, CJ]). It is an interesting problem to work out an analogue of string topology for derived ∞-stacks. Our work can be interpreted from this perspective. Our main theorem shows that the product structure on $K^{virt}(\mathcal{X})$ encodes the Chas-Sullivan product on the K-theory of the derived loop stack.

The initial motivation for this project came from a proposal of Hinich. In [Hi] Hinich proves the that abelian category of coherent sheaves over $I\mathcal{X}$ is isomorphic to the (*underived*) Drinfeld center [JS] of the abelian tensor category of coherent sheaves over $\mathcal{X}$, and notes that $Coh(I\mathcal{X})$ inherits from this equivalence an interesting braided tensor product. Hinich asks whether this braided tensor product gives an alternative description of the orbifold product of [JKK] on $K_0(I\mathcal{X})$. Theorem 1.2 implies that the answer to Hinich's question is negative: the tensor product of the Drinfeld center of $Coh(\mathcal{X})$ does not descend to the orbifold product on $K_0(I\mathcal{X})$, but rather to the *virtual orbifold product.* The two are almost always different: notable exceptions include the case of classifying stacks of finite groups and of smooth schemes. For classifying stacks of finite groups a different but related connection between orbifold products and Drinfeld doubles was studied by Kaufmann and Pham [KP]. Our work was also inspired by ideas of Manin and Toën on categorification of quantum cohomology, see [Ma] and [To1] Section 4.4 (6). Some recent work in this direction can be found in preprints of Toën [To2], and of Mann and Robalo [MR].

Acknowledgments: We thank Ralph Kaufmann and Timo Schürg for inspiring conversations in the early stages of this project. We are grateful to Kai Behrend and David Carchedi for useful discussions and for answering our many questions. We would also like to thank the referee for useful comments.

2. Preliminaries

2.1. **∞-categories.** Throughout the paper, we fix a ground field κ of characteristic zero. It is well known that triangulated categories are not well adapted to capture many important functoriality properties of categories of sheaves. Various possible ways to obviate these deficiencies are now available. In the early 90-s, Bondal and Kapranov [BoK] proposed the formalism of pre-triangulated dg categories as a better behaved replacement of ordinary triangulated category theory. We will work with a different enhancement of triangulated

[1]The equivalence $G_*(L\mathcal{X}) \simeq K_*(I\mathcal{X})$ follows from Barwick's Theorem of the heart [Ba2], see Section 2.2.

categories, which is provided by stable ∞-categories. Several different but equivalent formulations of the theory of ∞-categories exist. Our model of choice is given by quasi-categories. Quasi-categories were originally defined by Boardman and Vogt, and have been extensively investigated by Joyal [Jo]. We will rely on the comprehensive treatment of this theory provided by Lurie's work [Lu]. In this paper we will be exclusively concerned with characteristic zero applications: we remark that under this assumption the theory of κ-linear stable ∞-categories is equivalent to the theory of pre-triangulated dg categories [Co]. In the sequel we will always refer to quasi-categories simply as ∞-categories.

2.2. **Derived algebraic geometry.** It is often useful to consider spaces of maps from simplicial sets to algebraic geometric objects such as schemes and stacks. Derived algebraic geometry provides a language in which to make sense of these constructions. A careful definition of derived stacks can be found in [To1]. For an agile exposition of this material see [BFN] Section 2.3, which employs as we do the language of ∞-categories. Recall that we work over a fixed ground field κ of characteristic 0. Let $d\mathcal{A}lg_\kappa$ be the ∞-category of simplicial commutative κ-algebras. The opposite category of $d\mathcal{A}lg_\kappa$, which we denote $d\mathcal{A}ff_\kappa$, is a site with the étale topology (see Section 2.3 of [BFN]): we denote it $(d\mathcal{A}ff_\kappa)_{ét}$. Derived stacks are sheaves over $(d\mathcal{A}ff_\kappa)_{ét}$ with values in the ∞-category of topological spaces, $\mathcal{T}op$.

Derived stacks form the ∞-category $d\mathcal{S}t_\kappa$. Some important examples of derived stacks are:

- ordinary schemes and stacks of groupoids (in the following, we will refer to these simply as schemes and stacks)
- topological spaces (that are viewed as constant sheaves of spaces), and more generally underived *higher* stacks [To1]
- derived affine schemes, that is objects of $d\mathcal{A}ff_\kappa$

There exists a truncation functor $t_0(-)$ that maps derived stacks to underived stacks: if F is a derived stack, there is a canonical closed embedding $t_0(F) \to F$. All limits and colimits of derived stacks are taken in the ∞-category $d\mathcal{S}t_\kappa$, that is, they are always *derived*. This also applies to limits and colimits of schemes: for instance, if $X \to Y \leftarrow Z$ is a diagram of schemes, $X \times_Y Z$ denotes the *derived* fiber product of X and Z, which in general differs from the ordinary fiber product. The ordinary fiber product can be recovered as $t_0(X \times_Y Z)$.

If K is in $\mathcal{T}op$ and $\mathcal{X}$ is in $d\mathcal{S}t_\kappa$ the space of maps from K to $\mathcal{X}$ is also a derived stack, which we denote $\mathcal{X}^K$. The derived loop stack of $\mathcal{X}$ is the space of maps from S^1 into $\mathcal{X}$. We will often denote the loop stack $L\mathcal{X}$. Recall that S^1 can be realized as the colimit of the diagram $* \leftarrow (* \coprod *) \to *$ in $\mathcal{T}op$: this captures the fact that a circle can be obtained by joining two intervals at their endpoints. Since the mapping stack functor is right exact, $L\mathcal{X}$ is equivalent to the fiber product of the diagonal $\mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times \mathcal{X}$ with itself,

$$L\mathcal{X} \simeq \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}.$$

Note that even for ordinary schemes, loop stacks have a non-trivial derived structure. In fact by the Hochschild-Konstant-Rosenberg isomorphism, if $\mathcal{X}$ is a smooth scheme, $L\mathcal{X}$ is equivalent to the total space of the shifted tangent bundle $T\mathcal{X}[-1]$.

Proposition 2.1 ([ACH], Theorem 4.9)**.** *Let $\mathcal{X}$ be a DM stack, then $t_0(L\mathcal{X})$ is isomorphic to the inertia stack $I\mathcal{X}$. Further, if $\mathcal{X}$ is the global quotient of a smooth scheme by a finite group, there is an equivalence $L\mathcal{X} \simeq TI\mathcal{X}[-1]$.*

We can attach to derived stacks various categories of sheaves. Quasi-coherent sheaves on a derived stack $\mathcal{X}$ form a presentable and stable symmetric monoidal ∞-category, that we denote $\mathcal{Q}Coh(\mathcal{X})$. The category of perfect complexes on $\mathcal{X}$, $\mathcal{P}\mathrm{erf}(\mathcal{X})$, is the subcategory of compact objects in $\mathcal{Q}Coh(\mathcal{X})$. If $\mathcal{X}$ satisfies some additional assumptions (e.g. if it is a derived DM stack) $\mathcal{Q}Coh(\mathcal{X})$ can be equipped with a canonical t-structure, we denote its heart $qcoh(\mathcal{X})$. Coherent sheaves are quasi-perfect and quasi-truncated objects in $\mathcal{Q}Coh(\mathcal{X})$, see [Lu8] Definition 2.6.20. They form a full stable subcategory $\mathcal{C}oh(\mathcal{X})$ of $\mathcal{Q}Coh(\mathcal{X})$. The canonical t-structure on $\mathcal{Q}Coh(\mathcal{X})$ restricts to a bounded t-structure on $\mathcal{C}oh(\mathcal{X})$ and we denote its heart $coh(\mathcal{X})$.

The *K-theory* of a derived DM stack $\mathcal{X}$ is the K-theory of its category of perfect complexes, while its *G-theory* is by definition the K-theory of $\mathcal{C}oh(\mathcal{X})$, $G_*(\mathcal{X}) := K_*(\mathcal{C}oh(\mathcal{X}))$. We refer the reader to [Ba1, BGT] for foundations on the K-theory of ∞-categories. Recall also that if $\mathcal{X}$ is a smooth ordinary DM stack there is an equivalence $\mathcal{C}oh(\mathcal{X}) \simeq \mathcal{P}\mathrm{erf}(\mathcal{X})$, and therefore the G-theory and K-theory of $\mathcal{X}$ are naturally identified.

By Barwick's "theorem of the heart" [Ba2] (the analogous statement for triangulated categories had been originally proved by Neeman [Ne]) we know that the G-theory of $\mathcal{X}$ and of $t_0(\mathcal{X})$ are equivalent.

Proposition 2.2 ([Ba2] Proposition 9.2)**.** *Let $\mathcal{X}$ be a derived DM stack. Then there is an equivalence of spectra*

$$\iota_* : G_*(t_0(\mathcal{X})) = K_*(\mathcal{C}oh(t_0(\mathcal{X}))) \xrightarrow{\simeq} G_*(\mathcal{X}) = K_*(\mathcal{C}oh(\mathcal{X})).$$

In particular $\iota_* : G_0(t_0(\mathcal{X})) \to G_0(\mathcal{X})$ is an isomorphism of groups that sends the class $\sum_{i=0}^{\infty}(-1)^i \pi_i(\mathcal{O}_{\mathcal{X}}) \in G_0(t_0(\mathcal{X}))$ to the class of $\mathcal{O}_{\mathcal{X}}$ in $G_0(\mathcal{X})$.

2.3. **Derived Drinfeld center and convolution tensor product.** Here we review some results from [BFN] and [BNP] that will play a key role in the following. Denote $\mathcal{P}r^L$ the closed symmetric monoidal ∞-category of presentable ∞-categories (and left adjoint functors between them). Let $\mathcal{X}$ be a *perfect* derived stack in the sense of [BFN] Definition 3.2. Note that, in characteristic zero, quotients of quasi-projective derived schemes by a linear action of an algebraic group are perfect (see [BFN] Corollary 3.22). As in ordinary homological algebra, we can define the Hochschild homology and cohomology of $\mathcal{Q}Coh(\mathcal{X})$ as an associative algebra object in $\mathcal{P}r^L$. Following [BFN] we call these respectively the derived trace and derived center of $\mathcal{Q}Coh(\mathcal{X})$, and denote them $\mathcal{T}r(\mathcal{Q}Coh(\mathcal{X}))$ and $\mathcal{Z}(\mathcal{Q}Coh(\mathcal{X}))$. The derived center $\mathcal{Z}(\mathcal{Q}Coh(\mathcal{X}))$ is an $\mathcal{E}_2$*-category* with the convolution tensor product $- \otimes^{conv} -$. Recall that $\mathcal{E}_2$-categories are the analogue in ∞-category theory of braided monoidal categories.

Let P be the two-dimensional pair of pants, that is, P is a genus 0 compact surface with three boundary components. Set $P\mathcal{X} := \mathcal{X}^P$, and note that restriction to the boundary components gives maps,

$$\begin{tikzcd} & P\mathcal{X} \arrow[dl, "p_1"'] \arrow[d, "p_3"] \arrow[dr, "p_2"] & \\ L\mathcal{X} & L\mathcal{X} & L\mathcal{X}. \end{tikzcd}$$

This diagram induces a non trivial tensor product on $\mathcal{Q}Coh(L\mathcal{X})$, that we denote $\otimes^{str}$:

$$\mathcal{F} \otimes^{str} \mathcal{G} = p_{3*}(p_1^*(\mathcal{F}) \otimes p_2^*(\mathcal{G})).^2$$

Theorem 2.3. *$(\mathcal{Q}Coh(L\mathcal{X}), \otimes^{str})$ is an $\mathcal{E}_2$-category, and there is an equivalence of $\mathcal{E}_2$-categories:*

$$(\mathcal{Z}(\mathcal{Q}Coh(\mathcal{X})), \otimes^{conv}) \simeq (\mathcal{Q}Coh(L\mathcal{X}), \otimes^{str}).$$

Proof. The equivalence of $\mathcal{Q}Coh(L\mathcal{X}) \simeq \mathcal{Z}(\mathcal{Q}Coh(\mathcal{X}))$ as symmetric monoidal categories is given in [BFN] Proposition 5.2. The fact that under this equivalence the convolution tensor product is send to the string tensor product follows from Section 5.2 of [BFN], inducing an equivalence of $\mathcal{E}_2$-categories by Corollary 6.7 of [BFN]. □

Now assume that $\mathcal{X}$ is a proper and perfect, underived DM stack. As we mentioned earlier examples of stacks $\mathcal{X}$ satisfying these properties are given, for instance, by proper DM stacks obtained as quotients of a quasi-projective scheme by a linear group. The category $Coh(\mathcal{X}) \simeq \mathcal{P}\mathrm{erf}(\mathcal{X})$ is an associative algebra object in the closed symmetric monoidal ∞-category of small, stable and idempotent complete ∞-categories.

Proposition 2.4. *There is an equivalence of $\mathcal{E}_2$-categories:*

$$(\mathcal{Z}(\mathcal{P}\mathrm{erf}(\mathcal{X})), \otimes^{conv}) \simeq (Coh(L\mathcal{X}), \otimes^{str}).$$

Proof. By definition, the derived center $\mathcal{Z}(\mathcal{P}\mathrm{erf}(\mathcal{X}))$ is the category

$$Fun_{\mathcal{P}\mathrm{erf}(\mathcal{X}\times\mathcal{X})}(\mathcal{P}\mathrm{erf}(\mathcal{X}), \mathcal{P}\mathrm{erf}(\mathcal{X}))$$

with the convolution tensor product. By Theorem 1.1.3 of [BNP], under our assumptions on $\mathcal{X}$, there is an equivalence

$$Fun_{\mathcal{P}\mathrm{erf}(\mathcal{X}\times\mathcal{X})}(\mathcal{P}\mathrm{erf}(\mathcal{X}), \mathcal{P}\mathrm{erf}(\mathcal{X})) \simeq Coh(L\mathcal{X}).$$

This equivalence fits in a commutative diagram of ∞-categories:

$$\begin{tikzcd} \mathcal{Z}(\mathcal{Q}Coh(\mathcal{X})) \arrow[r, "\simeq"] & Fun_{\mathcal{Q}Coh(\mathcal{X}\times\mathcal{X})}(\mathcal{Q}Coh(\mathcal{X}), \mathcal{Q}Coh(\mathcal{X})) \arrow[r, "\simeq"] & \mathcal{Q}Coh(L\mathcal{X}) \\ \mathcal{Z}(\mathcal{P}\mathrm{erf}(\mathcal{X})) \arrow[r, "\simeq"] & Fun_{\mathcal{P}\mathrm{erf}(\mathcal{X}\times\mathcal{X})}(\mathcal{P}\mathrm{erf}(\mathcal{X}), \mathcal{P}\mathrm{erf}(\mathcal{X})) \arrow[r, "\simeq"] \arrow[u] & Coh(L\mathcal{X}) \arrow[u] \end{tikzcd}$$

where the vertical arrows are fully-faithful functors given respectively by extension of scalars, and by the inclusion of coherent shaves into quasi-coherent sheaves. By Theorem 2.3 the top right horizontal arrow is an equivalence of $\mathcal{E}_2$-categories. Further, $Coh(L\mathcal{X})$ is an $\mathcal{E}_2$-category with the restriction of $\otimes^{str}$, this is proved exactly as in Section 5.2 of [BFN]. Thus

[2]The notation $\otimes^{str}$ is motivated by the relation with string topology [CS], [CJ]. See the introduction and Section 6.1 of [BFN] for a discussion of these aspects.

the bottom right horizontal arrow is also an equivalence of $\mathcal{E}_2$-categories, and this concludes the proof. □

3. A proof of the main theorem for classifying stacks and schemes

In this section we work out the two simplest examples of our main theorem: we prove it for DM stacks of the form $BG = [*/G]$, where G is a finite group, and for smooth schemes. In the case of BG, a direct proof follows from results scattered in the literature but it will be useful to sketch it here. Although the proof for schemes does not differ in any essential way from the general argument, it has the advantage that it can be entirely carried out leveraging simple geometric properties of mapping spaces. These geometric ideas motivate the complete proof of Theorem 1.1 that we will give in Section 5 and contribute to clarify it.

3.1. **Classifiying stacks of finite groups.** Let G be a finite group and let $\mathcal{X} = [*/G]$ be the classifying stack of G.

Remark 3.1. For classifying stacks of finite groups $K^{virt}(\mathcal{X})$ is equal to $K^{orb}(\mathcal{X})$ the *orbifold K-theory* of $\mathcal{X}$ defined in [JKK]. This is an immediate consequence of the definitions, see [EJK2] Section 4.3.

Proposition 3.2. *There is an isomorphism* $K^{virt}(\mathcal{X}) \simeq G_0(L\mathcal{X}, \otimes^{str})$.

Proof. Since $\mathcal{X}$ is isomorphic to $[*/G]$ and G is finite, the diagonal map $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is flat. As a consequence there in an equivalence $\mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X} \simeq t_0(\mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X})$: that is, the loop stack $L\mathcal{X}$ is equivalent to the inertia stack $I\mathcal{X}$. Further the convolution tensor product $\otimes^{str}$ on $\mathcal{Z}(Coh(\mathcal{X})) \simeq Coh(I\mathcal{X})$, restricts to an exact tensor product on the abelian heart $coh(I\mathcal{X})$.

Recall that the *underived* Drinfeld center of an ordinary monoidal category was introduced in [JS]. Its relation with the derived Drinfeld center is discussed in Remark 1.8 of [BFN]: in order to avoid confusion between the two, for the rest of the proof we refer to the underived Drinfeld center as the *JS center*. The JS center of a monoidal category carries a braided monoidal structure. As noted in [JS] and in [Hi], $coh(I\mathcal{X})$ is equivalent to the JS center of $coh(\mathcal{X})$. Further the induced braided tensor product on $coh(I\mathcal{X})$ coincides with the convolution tensor product $\otimes^{str}$.

Let $Rep(G)$ be the monoidal abelian category of G-representations. Let $D(\kappa[G])$ be the Drinfeld double of the group algebra of G. The abelian category of representations of $D(\kappa[G])$, $Rep(D(\kappa[G]))$, is equipped with a braided monoidal structure. There are natural monoidal equivalences:

(1) between $Rep(G)$ and $coh(\mathcal{X})$,
(2) and between $Rep(D(\kappa[G]))$ and the JS center of $coh(\mathcal{X})$.

Kaufmann and Pham prove in [KP] Theorem 3.13 that there is an isomorphism of rings

$$K_0(Rep(D(\kappa[G]))) \simeq K^{orb}(\mathcal{X}).$$

As a consequence we obtain a chain of ring isomorphisms

$$G_0(L\mathcal{X}, \otimes^{str}) \simeq K_0(coh(I\mathcal{X}), \otimes^{str}) \simeq K_0(Rep(D(\kappa[G]))) \simeq K^{orb}(\mathcal{X}) = K^{virt}(\mathcal{X}),$$

where the last identity is given by Remark 3.1. □

3.2. **Smooth schemes.** Let X be a smooth scheme. Then $IX = X$ and both the orbifold and the virtual orbifold product on $K_0(X)$ coincide with the ordinary product in the K-theory of X. Denote $\otimes$ the ordinary symmetric tensor product on $\mathcal{Q}Coh(X)$. In this section we prove that $\iota_* : K^{virt}(X) = K_0(X, \otimes) \to G_0(LX, \otimes^{str})$ is an isomorphism of rings. Note that by Proposition 2.2 the map ι_* is a group isomorphism. The following proposition shows that ι_* is also compatible with the product structures. This proves Theorem 1.1 for smooth schemes.

Proposition 3.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be in $\mathcal{Q}Coh(X)$. Then there is a natural equivalence:*

$$(\iota_*\mathcal{F}) \otimes^{str} (\iota_*\mathcal{G}) \simeq \iota_*(\mathcal{F} \otimes \mathcal{G}).$$

Corollary 3.4. *There is an isomorphism $\iota_* : K^{virt}(X) = K_0(X, \otimes) \to G_0(LX, \otimes^{str})$.*

Before proceeding with the proof of Proposition 3.3 we make some preliminary observations. Denote D the closed disc, and let P be the pair of pants. It is useful to model P as the complement of three non-intersecting open discs, D_1 D_2 and D_3, in the 2-sphere S^2. Denote $b_1, b_2, b_3 : S^1 \to P$ the inclusions given by the identification $S^1 = \partial D_i$. Let $\{i, j, k\} = \{1, 2, 3\}$, and denote $P_i = S^2 - (D_j \cup D_k)$, and $P_{i,j} = S^2 - D_k$. Note that in $\mathcal{T}op$ we have equivalences $P_i \simeq S^1$, $P_{i,j} \simeq D$.

Lemma 3.5. *The following diagrams of inclusions*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & P_i \arrow[d] \\ P_j \arrow[r] & P_{i,j}, \end{tikzcd} \qquad \begin{tikzcd} S^1 \arrow[r] \arrow[d, "b_i"'] & D_i \arrow[d] \\ P \arrow[r] & P_i, \end{tikzcd}$$

are push-outs in $\mathcal{T}op$.

Lemma 3.6. *Let $X \xrightarrow{i} Z \xleftarrow{j} Y$ be maps of quasi-compact and quasi-separated derived DM stacks. Denote $l_X : X \times_Z Y \to X$ and $l_Y : X \times_Z Y \to Y$ the projections, and set $l_Z = i \circ l_X \simeq j \circ l_Y$. Let $\mathcal{F}$ be in $\mathcal{Q}Coh(X)$ and let $\mathcal{G}$ be in $\mathcal{Q}Coh(Y)$. Then there is a natural equivalence $i_*\mathcal{F} \otimes j_*G \simeq l_{Z*}(l_X^*\mathcal{F} \otimes l_Y^*\mathcal{G})$ in $\mathcal{Q}Coh(Z)$.*

Proof. There is a chain of natural equivalences:

$$i_*\mathcal{F} \otimes j_*\mathcal{G} \simeq i_*(\mathcal{F} \otimes i^*j_*\mathcal{G}) \simeq i_*(\mathcal{F} \otimes l_{X*}l_Y^*\mathcal{G}) \simeq i_*l_{X*}(l_X^*\mathcal{F} \otimes l_Y^*\mathcal{G}) \simeq l_{Z*}(l_X^*\mathcal{F} \otimes l_Y^*\mathcal{G}).$$

The first and third equivalences follow from the projection formula, and the second follows from the base change formula of [To3] Proposition 1.4. □

Proof of Proposition 3.3. Denote $\iota : X = t_0(LX) \to LX$ the natural embedding. Note that ι can be described as the restriction map $X \simeq X^D \to X^{S^1}$. Consider the diagram

$$\begin{tikzcd}[column sep=small] & & & X^{P_{12}} \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & & & \\ & & X^{P_1} \arrow[r, "u_1"] \arrow[dll, "n_1"'] & X^P \arrow[dl, "p_1"] \arrow[d, "p_3"] \arrow[dr, "p_2"'] & X^{P_2} \arrow[l, "u_2"'] \arrow[drr, "n_2"] & & \\ X^D \simeq X \arrow[r, "i"] & X^{S^1} & & X^{S^1} & & X^{S^1} & X^D \simeq X \arrow[l] \end{tikzcd}$$

By Lemma 3.5 the top triangle and the right and left squares are all fiber products. The base change formula (see [To3] Proposition 1.4) implies that we have equivalences $p_1^*\iota_*\mathcal{F} \simeq u_{1*}n_1^*\mathcal{F}$ and $p_2^*\iota_*\mathcal{G} \simeq u_{2*}n_2^*\mathcal{F}$. Using Lemma 3.6 we can write

$$p_1^*\iota_*\mathcal{F} \otimes p_2^*\iota_*\mathcal{G} \simeq u_{1*}n_1^*\mathcal{F} \otimes u_{2*}n_2^*\mathcal{G} \simeq u_{1*}s_{1*}(s_1^*n_1^*\mathcal{F} \otimes s_2^*n_2^*\mathcal{G}) \simeq u_{1*}s_{1*}(\mathcal{F} \otimes \mathcal{G}),$$

where the last equivalence follows from the fact that, since $P_{12} \simeq D \simeq *$, $X^{P_{12}} \simeq X$ and $(n_i \circ s_i)^* \simeq Id$. Thus, $\iota_*\mathcal{F} \otimes^{str} \iota_*\mathcal{G} = p_{3*}(p_1^*\iota_*\mathcal{F} \otimes p_2^*\iota_*\mathcal{G}) \simeq p_{3*}u_{1*}s_{1*}(\mathcal{F} \otimes \mathcal{G}) \simeq \iota_*(\mathcal{F} \otimes \mathcal{G})$, and this concludes the proof. $\square$

4. Some remarks on mapping stacks

Let $\mathcal{X}$ be a derived stack. In this section we collect some facts about the mapping stacks $L\mathcal{X}$ and $P\mathcal{X}$. Consider the evaluation map $S^1 \times L\mathcal{X} \to \mathcal{X}$. We fix a point on S^1, $* \to S^1$. We denote $ev : L\mathcal{X} \to \mathcal{X}$ the map obtained as the composition:

$$L\mathcal{X} \simeq * \times L\mathcal{X} \to S^1 \times L\mathcal{X} \to L\mathcal{X}.$$

Lemma 4.1. *The following diagrams are fiber products in $d\mathcal{S}t_\kappa$:*

$$\begin{tikzcd} P\mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X} \times \mathcal{X}, \end{tikzcd} \qquad \begin{tikzcd} P\mathcal{X} \arrow[r] \arrow[d] & L\mathcal{X} \arrow[d, "ev"] \\ L\mathcal{X} \arrow[r, "ev"] & \mathcal{X}. \end{tikzcd}$$

Proof. P is equivalent in $\mathcal{T}op$ to a wedge of two circles. A wedge of two circles can be obtained by gluing two arcs along their middle and end points, or by gluing two circles along a common point. Thus $\mathcal{P}$ is equivalent in $\mathcal{T}op$ to the push-out of both of the following diagrams: $* \leftarrow * \amalg * \amalg * \to *$, and $S^1 \leftarrow * \to S^1$. The claim follows from the right exactness of the mapping stack functor. $\square$

Corollary 4.2. *Let $\mathcal{X}$ be a DM stack, and denote $I^2\mathcal{X}$ the* double inertia stack *of $\mathcal{X}$. Then $t_0(P\mathcal{X}) \simeq I^2\mathcal{X}$.*

Proof. Recall that $I^2\mathcal{X}$ is the ordinary fiber product of the diagram $I\mathcal{X} \to \mathcal{X} \leftarrow I\mathcal{X}$ (that is, the fiber product in the category of *underived* stacks). Since t_0 is a right adjoint it preserves limits. In particular, if $\mathcal{X} \to \mathcal{Z} \leftarrow \mathcal{Y}$ is a diagram in $d\mathcal{S}t_\kappa$, $t_0(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$ is naturally equivalent to $t_0(t_0(\mathcal{X}) \times_{t_0(\mathcal{Z})} t_0(\mathcal{Y}))$. Thus we obtain a chain of equivalences:

$$t_0(P\mathcal{X}) \simeq t_0(L\mathcal{X} \times_{\mathcal{X}} L\mathcal{X}) \simeq t_0(t_0(L\mathcal{X}) \times_{\mathcal{X}} t_0(L\mathcal{X})) = t_0(I\mathcal{X} \times_{\mathcal{X}} I\mathcal{X}) \simeq I^2\mathcal{X}.$$

$\square$

Remark 4.3. Recall that $I\mathcal{X}$ is a $\mathcal{X}$-group (for a reference see e.g. Remark 79.5.2 [SP]). The multiplication is encoded in a map $\mu : I^2\mathcal{X} \to I\mathcal{X}$. We also have two projection maps $q_1, q_2 : I^2\mathcal{X} \to \mathcal{X}$. We set $q_3 := \mu$. By Lemma 4.1 the derived stack $P\mathcal{X}$ carries two projections $p_1, p_2 : P\mathcal{X} \to L\mathcal{X}$. These maps coincide with the restriction to two of the boundary components of P. The restriction to the third boundary component gives a morphism $p_3 : P\mathcal{X} \to L\mathcal{X}$ (see Section 2.3). We have that $q_i = t_0(p_i)$. Hence these maps fit in a commutative diagram:

$$\begin{tikzcd}
& I^2\mathcal{X} \arrow[dl, "q_1"'] \arrow[d] \arrow[dr, "q_3"] \arrow[drr, "q_2"] & & \\
I\mathcal{X} \arrow[d] & P\mathcal{X} \arrow[dl, "p_1"'] \arrow[dr, "p_3"] \arrow[drr, "p_2"] & I\mathcal{X} \arrow[d] & I\mathcal{X} \arrow[d] \\
L\mathcal{X} & & L\mathcal{X} & L\mathcal{X},
\end{tikzcd}$$

where the vertical arrows are given by the natural embedding $I\mathcal{X} = t_0(L\mathcal{X}) \to L\mathcal{X}$ and $I^2\mathcal{X} = t_0(P\mathcal{X}) \to P\mathcal{X}$.

Let G be an algebraic group acting on a scheme X, and let $\mathcal{X} = [X/G]$ be the quotient stack. We let $X \times G \to X \times X$ and $X \times G \times G \to X \times X \times X$ be the maps defined on closed points by the assignment $(x, g) \mapsto (x, gx)$ and $(x, g, h) \mapsto (x, gx, hx)$. The next Lemma gives an explicit construction of $L\mathcal{X}$ and $P\mathcal{X}$ as global quotients of derived schemes.

Lemma 4.4.
- *Let L_GX be the derived scheme obtained as the following fiber product:*

$$\begin{tikzcd}
L_GX \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\
X \times G \arrow[r] & X \times X.
\end{tikzcd}$$

Then there is a natural action of G on L_GX and $L\mathcal{X}$ is isomorphic to $[L_GX/G]$.
- *Let P_GX be the derived scheme obtained as the following fiber product:*

$$\begin{tikzcd}
P_GX \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\
X \times G \times G \arrow[r] & X \times X \times X.
\end{tikzcd}$$

Then there is a natural action of G on P_GX and $P\mathcal{X}$ is isomorphic to $[P_GX/G]$.

Proof. The first part of the Lemma is stated without proof in Section 4.4 of [To4]. We include a proof for completeness. Consider the diagram,

$$\begin{tikzcd}
L_GX \arrow[r] \arrow[d] & G \times X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
X \arrow[r] & X \times X \arrow[r] & \mathcal{X} \times \mathcal{X}.
\end{tikzcd}$$

There is an equivalence $G \times X \simeq X \times_{\mathcal{X}} X$. Standard properties of fiber products imply that there is an equivalence $G \times X \simeq (X \times X) \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$, and therefore that the right square is a fiber product. The left square is a fiber product by the definition of L_GX. Thus the exterior square is a fiber product as well.

Next note that the left square in the diagram

$$\begin{tikzcd} L_GX \arrow[r] \arrow[d] & L\mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ X \arrow[r] & \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X}, \end{tikzcd}$$

is a fiber product. Indeed, both the right and the exterior squares are fiber products: the right square is a fiber product by the discussion in Section 2.2, and the fact that the exterior square is also a fiber product was proved in the previous paragraph. This and the fact that $\mathcal{X}$ is equivalent to $[X/G]$ prove that both the right and the left squares in

$$\begin{tikzcd} L_GX \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & * \arrow[d] \\ L\mathcal{X} \arrow[r] & \mathcal{X} \arrow[r] & [*/G], \end{tikzcd}$$

are fiber products, and that therefore the exterior square is as well. Thus $L\mathcal{X}$ is equivalent to $[L_GX/G]$, as we needed to show.

The second part of the Lemma is proved in a very similar way. The fiber product of the diagram $X \times X \times X \to \mathcal{X} \times \mathcal{X} \times \mathcal{X} \leftarrow \mathcal{X}$ is equivalent to $X \times_{\mathcal{X}} X \times_{\mathcal{X}} X \simeq X \times G \times G$. Next consider the diagram,

$$\begin{tikzcd} P_GX \arrow[r] \arrow[d] & G \times G \times X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ X \arrow[r] & X \times X \times X \arrow[r] & \mathcal{X} \times \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

The exterior square is a fiber product, as both right and left squares are. This together with Lemma 4.1 implies that the left square in the diagram

$$\begin{tikzcd} P_GX \arrow[r] \arrow[d] & P\mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ X \arrow[r] & \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X} \times \mathcal{X}, \end{tikzcd}$$

is a fiber product. This implies that the exterior square in

$$\begin{tikzcd} P_GX \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & * \arrow[d] \\ P\mathcal{X} \arrow[r] & \mathcal{X} \arrow[r] & [*/G], \end{tikzcd}$$

is also a fiber product. That is, $P\mathcal{X} \simeq [P_GX/G]$, and this concludes the proof. □

Remark 4.5. Assume now that X is affine, G acts linearly and that $\mathcal{X} = [X/G]$ is a DM stack with finite stabilizers (that is, such that the map $I\mathcal{X} \to \mathcal{X}$ is finite). Under these assumptions we can give a more explicit description of $P\mathcal{X}$. If g, h are in G let $\Gamma_{g,h}$ be the image of X in $X \times X \times X$ under the assignment: $x \mapsto (x, gx, hx)$. Let $\Delta \subset X \times X \times X$ be the diagonal subscheme. Then the derived scheme $P_G X$ decomposes as the disjoint union

$$P_G X = \coprod_{g,h \in G} \Gamma_{g,h} \times_{X \times X \times X} \Delta.$$

5. Virtual orbifold K-theory and the proof of the main Theorem

5.1. **Virtual orbifold K-theory.** The virtual orbifold cohomology of differential orbifolds was introduced in [LUX, LUX+]. Virtual orbifold cohomology is closely related to Chen-Ruan cohomology, and a precise comparison between the two was obtained in Theorem 1.1 of [LUX+]. In the setting of algebraic geometry, the study of virtual orbifold cohomology and virtual orbifold K-theory was pursued in [EJK2, EJK3]. We start by recalling briefly the setting of [EJK2, EJK3].

Let $\mathcal{X}$ be a smooth Deligne-Mumford stack with finite stabilizers. Assume that $\mathcal{X}$ admits a presentation as a global quotient of a smooth affine scheme by a linear algebraic group, $\mathcal{X} = [X/G]$.

Definition 5.1.

- Denote $I_G X$ the *inertia scheme* of $\mathcal{X}$,

$$I_G X := \{(x, g) | gx = x\} \subset X \times G.$$

- Denote $I^2_G X$ the *double inertia scheme* of $\mathcal{X}$,

$$I^2_G X := \{(x, g, h) | gx = hx = x\} \subset X \times G \times G.$$

Remark 5.2. If $g \in G$ denote X^g the *underived* fixed locus of g: that is, if $\Gamma_g \subset X \times X$ is the graph of g, set $X^g := \Gamma_g \cap \Delta$. Similarly if $g, h \in G$, set $X^{g,h} := X^g \cap X^h$. We can decompose $I_G X$ and $I^2_G X$ as the following disjoint unions:

$$I_G X = \coprod_{g \in G} X^g, \ I^2_G X = \coprod_{g,h \in G} X^{g,h}.$$

Remark 5.3. Let $L_G X$ and $P_G X$ be as in the statement of Lemma 4.4. Then we have isomorphisms $I_G X \simeq t_0(L_G X)$ and $I^2_G X \simeq t_0(P_G X)$. In particular, $I^2_G X$ is the *underived* fiber product of $I_G X \to X \leftarrow I_G X$, and we denote the projections $q_1, q_2 : I^2_G X \to I_G X$. Further $I_G X$ is a X-group. We let $\mu : I^2_G X \to I_G X$ be the multiplication. We set $q_3 := \mu$. Note that in Remark 4.3 we used these same notations to denote the projections $I^2\mathcal{X} \to I\mathcal{X}$, and the multiplication on $I\mathcal{X}$: this should cause no confusion as it will be clear from the context whether we are referring to the inertia scheme or to the inertia stack.

Note that $I_G X$ and $I^2_G X$ carry a natural action of G. This gives presentations of the inertia and double inertia stack as global quotients: $I\mathcal{X} \simeq [I_G X/G]$ and $I^2\mathcal{X} \simeq [I^2_G X/G]$. We can describe the sheaf theory of $I\mathcal{X}$ and $I^2\mathcal{X}$ in terms of the equivariant sheaf theory of $I_G X$ and $I^2_G X$. In particular, $K_0(I\mathcal{X})$ and $K_0(I^2\mathcal{X})$ are naturally identified with the equivariant Grothendieck groups $K_G(I_G X)$ and $K_G(I^2_G X)$.

Definition 5.4 ([EJK1] Definition 3.1)**.** Let $\mathcal{R}$ be a class on $K_G(I^2_G X)$. Then we define a product[3] $*_{\mathcal{R}}$ on $K_G(I_G X) \simeq K_0(\mathcal{X})$ by the assignment:

$$x, y \in K_G(I_G X), \quad x *_{\mathcal{R}} y = q_{3*}(q_1^* x \cdot q_2^* y \cdot \lambda_{-1}(\mathcal{R})).$$

Definition 5.5 ([EJK3] Definition 2.16, [LUX] Section 5)**.** Let $u : I^2_G X \subset X \times G \times G \to X$ be the projection on the first factor. Set $\mathcal{B} := u^* T_X + T_{I^2_G X} - q_1^* T_{I_G X} - q_2^* T_{I_G X} \in K_G(I^2\mathcal{X})$, and $\mathcal{R} := \mathcal{B}^\vee$. Then the *virtual orbifold product* $*^{virt}$ is defined by:

$$*^{virt} := *_{\mathcal{R}}.$$

Many properties of the virtual orbifold product have been investigated in [EJK2, EJK3, LUX] and [LUX+]. We list two of the most important here:

- The product $*^{virt}$ is unital, associative and commutative ([EJK2] Proposition 4.3.2).
- $K^{virt}(\mathcal{X})$ is a Frobenius algebra (see [LUX+] Theorem 2.3 and [EJK1] Proposition 3.5).

We remark that from the vantage point of Theorem 1.1, the first property is a consequence of the fact that $K^{virt}(\mathcal{X})$ is the Grothendieck group of the $\mathcal{E}_2$-category $\mathcal{C}oh(L\mathcal{X})$. As for the second point, $\mathcal{C}oh(L\mathcal{X}, \otimes^{str})$ is a Frobenius algebra object in the closed symmetric monoidal ∞-category of small, stable and idempotent complete ∞-categories (this is a consequence of [BFN] Proposition 6.3). Thus the same is true of its Grothendieck group, which is isomorphic to $K^{virt}(\mathcal{X})$.

5.2. **The derived double inertia stack and excess intersection.** Let $\mathcal{X}$ be a DM stack satisfying the same assumptions as in the previous section: that is, $\mathcal{X}$ is smooth, has finite stabilizers and can be presented as the global quotient $[X/G]$ of an affine scheme by a linear group. It will be useful to introduce a derived stack, $\mathcal{I}^2\mathcal{X}$, that in a precise sense interpolates between $P\mathcal{X}$ and $I^2\mathcal{X}$. It is possible to describe explicitly $\mathcal{O}_{P\mathcal{X}}$ and $\mathcal{O}_{\mathcal{I}^2\mathcal{X}}$ as classes in $K_0(I^2\mathcal{X})$, and we will do this next: the calculation of the class of $\mathcal{O}_{\mathcal{I}^2\mathcal{X}}$ will be especially important in the proof of Theorem 1.1.

Definition 5.6. The *derived double inertia stack* of $\mathcal{X}$, denoted $\mathcal{I}^2\mathcal{X}$, is the *derived* fiber product of $I\mathcal{X} \to \mathcal{X} \leftarrow I\mathcal{X}$, that is $\mathcal{I}^2\mathcal{X} = I\mathcal{X} \times_{\mathcal{X}} I\mathcal{X}$.

Remark 5.7. There is an equivalence $t_0(\mathcal{I}^2\mathcal{X}) \simeq I^2\mathcal{X}$. The derived double inertia stack $\mathcal{I}^2\mathcal{X}$ can be realized as the quotient of the derived scheme $\mathcal{I}^2_G X := I_G X \times_X I_G X$ by the action of G: we have $\mathcal{I}^2\mathcal{X} = [\mathcal{I}^2_G X/G]$. Note that $\mathcal{I}^2_G X$ decomposes as the following disjoint union:

$$\mathcal{I}^2_G X \simeq \coprod_{g,h \in G} X^g \times_X X^h.$$

Lemma 5.8. *Denote $\iota : I\mathcal{X} \to L\mathcal{X}$ the natural embedding. Let Y be the fiber product of the diagram $L\mathcal{X} \stackrel{ev}{\to} \mathcal{X} \stackrel{\iota}{\leftarrow} I\mathcal{X}$. As in Remark 4.3 denote $p_i : P\mathcal{X} \to L\mathcal{X}$, $i = 1, 2$, the projections.*

[3] It is important to note that $*_{\mathcal{R}}$ in general will not be neither unital nor associative. The works [EJK1, EJK2] contain a careful study of the conditions on $\mathcal{R}$ under which $*_{\mathcal{R}}$ is unital, associative, and has various additional properties.

Then the following diagrams are fiber products:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & P\mathcal{X} \arrow[d, "p_i"] \\ I\mathcal{X} \arrow[r] & L\mathcal{X}, \end{tikzcd} \qquad \begin{tikzcd} \mathcal{I}^2\mathcal{X} \arrow[r] \arrow[d] & Y \arrow[d] \\ Y \arrow[r] & P\mathcal{X}. \end{tikzcd}$$

Proof. Consider the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & P\mathcal{X} \arrow[r, "p_2"] \arrow[d, "p_1"] & L\mathcal{X} \arrow[d] \\ I\mathcal{X} \arrow[r] & L\mathcal{X} \arrow[r] & \mathcal{X}. \end{tikzcd}$$

The right square is a fiber product by Lemma 4.1, and the exterior one is a fiber product by the definition of Y. Thus the left square is also a fiber product.

In order to prove that the second diagram is a fiber product, we write it as the upper-left square in the diagram

$$\begin{tikzcd} \mathcal{I}^2\mathcal{X} \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & I\mathcal{X} \arrow[d, "\iota"] \\ Y \arrow[r] \arrow[d] & P\mathcal{X} \arrow[r, "p_2"] \arrow[d, "p_1"] & L\mathcal{X} \arrow[d] \\ I\mathcal{X} \arrow[r, "\iota"] & L\mathcal{X} \arrow[r] & \mathcal{X}. \end{tikzcd}$$

Note that the top right, bottom right, and bottom left squares are all fiber products by the first part of the Lemma and by Lemma 4.1. Also the derived double inertia stack $\mathcal{I}^2\mathcal{X}$ is the fiber product of $I\mathcal{X} \to \mathcal{X} \leftarrow I\mathcal{X}$: thus, also the exterior square is a fiber product. Standard properties of fiber products imply that the top left square is also a fiber product, and this concludes the proof.

□

Remark 5.9. The derived stack $\mathcal{I}^2\mathcal{X}$ carries two projections $r_1, r_2 : \mathcal{I}^2\mathcal{X} \to I\mathcal{X}$, and a multiplication map $r_3 : \mathcal{I}^2\mathcal{X} \to I\mathcal{X}$. Further we have maps $I^2\mathcal{X} \stackrel{j}{\to} \mathcal{I}^2\mathcal{X} \stackrel{l}{\to} P\mathcal{X}$, where l comes from the second fiber product of Lemma 5.8, and j and $i := l \circ j$ are the canonical embeddings of $I^2\mathcal{X} = t_0(\mathcal{I}^2\mathcal{X}) = t_0(P\mathcal{X})$ into $\mathcal{I}^2\mathcal{X}$ and $P\mathcal{X}$. It is important to clarify the relationship between these maps and the various maps to and from $I^2\mathcal{X}$ and $L^2\mathcal{X}$ that we considered in Remark 4.3. We use the notations of Remark 4.3: for all $i = 1, 2, 3$, the following is a commutative diagram

$$\begin{tikzcd} I^2\mathcal{X} \arrow[r, "j"] \arrow[dr, "q_i"'] & \mathcal{I}^2\mathcal{X} \arrow[r, "l"] \arrow[d, "r_i"'] & P\mathcal{X} \arrow[d, "p_i"] \\ & I\mathcal{X} \arrow[r] & L\mathcal{X}. \end{tikzcd}$$

Lemma 5.10 ([CKS] Proposition A.3)**.** *Let X, Y, Z be smooth schemes. Suppose that there are embeddings $X \to Z \leftarrow Y$, such that $W = X \cap Y$ is smooth, and denote E the* excess (intersection) bundle, $E = T_Z|_W/(T_X|_W + T_Y|_W)$. *Then, in* $K_0(W)$, *we have the identity:*

$$\sum (-1)^i \pi_i \mathcal{O}_{X\times_Z Y} = \sum (-1)^i \Lambda^i E^\vee = \lambda_{-1}(E^\vee).$$

Lemma 5.11. *Let* g, h *be in* G, *and set* $W := X^{g,h}$. *Then:*

- *The class in* $K_0(W)$ *of the excess intersection bundle* $E^{g,h}$ *of* X^g *and* X^h *in* X *is given by* $E^{g,h} = TX - TX^g - TX^h + TX^{g,h}$.
- *The class in* $K_0(W)$ *of the excess intersection bundle* $F^{g,h}$ *of* $\Gamma^{g,h}$ *and* Δ *in* $X\times X\times X$ *is given by* $F^{g,h} = TX + TX^{g,h}$.

Proof. Both in the statement of the Proposition and in the proof all bundles are always implicitly assumed to be restricted to W: TX^g, TX^h and $TX^{g,h}$ denote respectively the g-invariant, h-invariant and $< g, h >$-invariant sub-bundles of $TX|_W$. We start from the first statement: $E^{g,h}$ is by definition the cokernel of the embedding $TX^g \times TX^h \xrightarrow{+} TX$. Thus the class in $K_0(W)$ of $E^{g,h}$ is given by $E^{g,h} = TX - TX^g \oplus TX^h + TX^{g,h} = TX - TX^g - TX^h + TX^{g,h}$. As for the second statement, the excess bundle $F^{g,h}$ is isomorphic to the cokernel of the map

$$TX \times TX \to TX \times TX \times TX,\ (u, v) \mapsto (u + v, gv, hv),$$

and therefore $F^{g,h} = 3TX - TX - TX + TX^{g,h} = TX + TX^{g,h}$ in $K_0(W)$. □

The next proposition gives a global description of the excess intersection bundles $E^{g,h}$ and $F^{g,h}$ from Lemma 5.11 as g and h vary.

Proposition 5.12.

- *The excess intersection bundles* $E^{g,h}$, *whose class in* K*-theory is given by*

$$E^{g,h} = TX|_{X^{g,h}} - TX|^g_{X^{g,h}} - TX|^h_{X^{g,h}} + TX|^{g,h}_{X^{g,h}},$$

assemble to a bundle $\mathcal{E}_{\mathcal{I}^2}$ *on* $I^2_G X$. *Further we have that*

$$\sum (-1)^i \pi_i \mathcal{O}_{\mathcal{I}^2_G X} = \lambda_{-1}(\mathcal{E}^\vee_{\mathcal{I}^2})$$

in $K_0(I^2_G X)$.
- *The excess intersection bundles* $F^{g,h}$, *whose class in K-theory is given by*

$$F^{g,h} = TX|_{X^{g,h}} + TX|^{g,h}_{X^{g,h}},$$

assemble to a bundle $\mathcal{E}_P$ *on* $I^2_G X$. *Further we have that*

$$\sum (-1)^i \pi_i \mathcal{O}_{P_G X} = \lambda_{-1}(\mathcal{E}^\vee_P)$$

in $K_0(I^2_G X)$.

Proof. By Remark 4.5, Remark 5.2 and Remark 5.7, we have decompositions:

$$P_G X = \coprod_{g,h\in G} \Gamma_{g,h} \times_{X\times X\times X} \Delta,\ \mathcal{I}^2_G X = \coprod_{g,h\in G} X^g \times_X X^h \text{ and } I^2_G X = \coprod_{g,h\in G} X^{g,h}.$$

Thus, by Lemma 5.10, the classes in K-theory of $\mathcal{O}_{\mathcal{I}^2_G X}$ and $\mathcal{O}_{P_G X}$ can be described in terms of the excess intersection bundles on each component $X^{g,h}$ of $I^2_G X$. These have been calculated in Lemma 5.11 and coincide with the classes appearing in the claim. □

The bundles $\mathcal{E}_{\mathcal{I}^2}$ and $\mathcal{E}_P$ carry a canonical G-equivariant structure. With slight abuse of notation we keep denoting these bundles $\mathcal{E}_{\mathcal{I}^2}$, $\mathcal{E}_P$ also when we regard them as objects of the G-equivariant category $\mathcal{C}oh_G(I^2_G X)$ or of $\mathcal{C}oh(I^2\mathcal{X})$. In the following Corollary the notations $\mathcal{E}^2_{\mathcal{I}}$ and $\mathcal{E}_P$ are used precisely in this way: that is, they refer to the corresponding bundles on $I^2\mathcal{X}$.

Corollary 5.13.
- *The class* $\sum(-1)^i\pi_i\mathcal{O}_{\mathcal{I}^2\mathcal{X}}$ *is equal to* $\lambda_{-1}(\mathcal{E}^\vee_{\mathcal{I}^2})$ *in* $K_0(I^2\mathcal{X})$.
- *The class* $\sum(-1)^i\pi_i\mathcal{O}_{P\mathcal{X}}$ *is equal to* $\lambda_{-1}(\mathcal{E}^\vee_P)$ *in* $K_0(I^2\mathcal{X})$.

Proof. The claim follows from Proposition 5.12, and the fact that $P\mathcal{X}$ and $\mathcal{I}^2\mathcal{X}$ are isomorphic to the quotients $P\mathcal{X} = [P_G X/G]$, $\mathcal{I}^2\mathcal{X} = [\mathcal{I}^2_G X/G]$ (see Lemma 4.4 and Remark 5.7). $\square$

Remark 5.14. Note that the class of $\mathcal{E}_{\mathcal{I}^2}$ in $K_0(I^2\mathcal{X})$ coincides with the class $\mathcal{B}$ that appears in the definition of the virtual orbifold product, see Definition 5.5. This observation is a key ingredient in the proof of Theorem 1.1.

5.3. **The proof of the main Theorem.** In this Section we prove that the string tensor product on $\mathcal{C}oh(\mathcal{X})$ categorifies the virtual orbifold product. As before we assume that $\mathcal{X}$ is a smooth DM stack with finite stabilizers that admits a presentation as the global quotient of an affine scheme by a linear group, $\mathcal{X} \simeq [X/G]$. It will be important to refer to various maps to and from $\mathcal{I}^2\mathcal{X}$, $P\mathcal{X}$, $I^2\mathcal{X}$, $L\mathcal{X}$ and $I\mathcal{X}$: we let $\iota : I\mathcal{X} \to L\mathcal{X}$ be the natural embedding, and for the rest use the same notations as in Remark 5.9.

Lemma 5.15. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be in* $\mathcal{C}oh(\mathcal{I}\mathcal{X})$. *Then* $\iota_*\mathcal{F} \otimes^{str} \iota_*\mathcal{G} \simeq \iota_* r_{3*}(r_1^*\mathcal{F} \otimes r_2^*\mathcal{G})$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
 & & \mathcal{I}^2\mathcal{X} \arrow[dl, "s_1"'] \arrow[dr, "s_2"] \arrow[ddll, bend right, "r_1"'] \arrow[ddrr, bend left, "r_2"] & & \\
 & Y \arrow[r, "u_1"] \arrow[dl, "n_1"'] & P\mathcal{X} \arrow[dl, "p_1"'] \arrow[d, "p_3"] \arrow[dr, "p_2"] & Y \arrow[l, "u_2"'] \arrow[dr, "n_2"] & \\
I\mathcal{X} \arrow[r, "\iota"] & L\mathcal{X} & L\mathcal{X} & L\mathcal{X} & I\mathcal{X}. \arrow[l, "\iota"']
\end{tikzcd}$$

Note that the right, left and top squares are all fiber products: we proved that $\mathcal{I}^2\mathcal{X}$ is the fiber product of the top square in Lemma 5.8. As in Remark 5.9, we denote l the composition $u_1 \circ s_1 \simeq u_2 \circ s_2$. The base change formula [To3] Proposition 1.4 gives equivalences

$$p_1^*\iota_*\mathcal{F} \simeq u_{1*}n_1^*\mathcal{F}, \quad p_2^*\iota_*\mathcal{G} \simeq u_{2*}n_2^*\mathcal{F}.$$

Using Lemma 3.6 we can rewrite

$$p_1^*\iota_*\mathcal{F} \otimes p_2^*\iota_*\mathcal{G} \simeq u_{1*}n_1^*\mathcal{F} \otimes u_{2*}n_2^*\mathcal{G} \simeq u_{1*}s_{1*}(s_1^*n_1^*\mathcal{F} \otimes s_2^*n_2^*\mathcal{G}) \simeq l_*(r_1^*\mathcal{F} \otimes r_2^*\mathcal{G}).$$

Recall that $p_3 \circ l \simeq \iota \circ r_3$ (see Remark 5.9), and thus

$$\iota_*\mathcal{F} \otimes^{str} \iota_*\mathcal{G} = p_{3*}(p_1^*\iota_*\mathcal{F} \otimes p_2^*\iota_*\mathcal{G}) \simeq p_{3*}l_*(r_1^*\mathcal{F} \otimes r_2^*\mathcal{G}) \simeq \iota_* r_{3*}(r_1^*\mathcal{F} \otimes r_2^*\mathcal{G}).$$

$\square$

Lemma 5.16. *We denote $-\otimes-$ the product on $K_0(I^2\mathcal{X})$ induced by the ordinary tensor product of sheaves on $I^2\mathcal{X}$. Denote $\mathcal{D}$ the class $\Sigma_i(-1)^i\pi_i\mathcal{O}_{\mathcal{I}^2\mathcal{X}}$ in $K_0(I^2\mathcal{X})$. If $\mathcal{F},\mathcal{G}$ are in $\mathcal{C}oh(I\mathcal{X})$, then $q_{3*}(q_1^*\mathcal{F}\otimes q_2^*\mathcal{G}\otimes\mathcal{D}) = r_{3*}(r_1^*\mathcal{F}\otimes r_2^*\mathcal{G})$ in $K_0(I\mathcal{X})$.*

Proof. As in Remark 5.9 let $j: I^2\mathcal{X}\to\mathcal{I}^2\mathcal{X}$ be the natural embedding. Recall by Proposition 2.2 that the class of $\mathcal{O}_{\mathcal{I}^2\mathcal{X}}$ in $G_0(\mathcal{I}^2\mathcal{X})$ is equal to $j_*\mathcal{D}$. We have the following equalities in $K_0(I\mathcal{X})$:

$$r_{3*}(r_1^*\mathcal{F}\otimes r_2^*\mathcal{G}) = r_{3*}(r_1^*\mathcal{F}\otimes r_2^*\mathcal{G}\otimes\mathcal{O}_{\mathcal{I}^2\mathcal{X}}) = r_{3*}(r_1^*\mathcal{F}\otimes r_2^*\mathcal{G}\otimes j_*\mathcal{D}) = r_{3*}j_*(j^*(r_1^*\mathcal{F}\otimes r_2^*\mathcal{G})\otimes\mathcal{D}),$$

where the last one is a consequence of the projection formula. Further we can write

$$r_{3*}j_*(j^*(r_1^*\mathcal{F}\otimes r_2^*\mathcal{G})\otimes\mathcal{D}) = r_{3*}j_*((j^*r_1^*\mathcal{F}\otimes j^*r_2^*\mathcal{G})\otimes\mathcal{D}) = q_{3*}(q_1^*\mathcal{F}\otimes q_2^*\mathcal{G}\otimes\mathcal{D}),$$

as $q_i = r_i\circ j$ for all $i=1,2,3$ (see Remark 5.9) and this concludes the proof. □

Theorem 5.17. *Let $\iota: I\mathcal{X}\to L\mathcal{X}$ be the natural embedding. Then ι_* is an isomorphism of rings: $\iota_*: K^{virt}(\mathcal{X}) = K_0(I\mathcal{X}, *^{virt}) \xrightarrow{\cong} G_0(L\mathcal{X},\otimes^{str})$.*

Proof. Recall that by Proposition 2.2 the map ι_* is an isomorphism of groups. We need to prove that ι_* is also compatible with the product structures. Let $\mathcal{F}$ and $\mathcal{G}$ be in $\mathcal{C}oh(I\mathcal{X})$. By Lemma 5.15 and Lemma 5.16,

$$\iota_*\mathcal{F}\otimes^{str}\iota_*\mathcal{G} = \iota_* r_{3*}(r_1^*\mathcal{F}\otimes r_2^*\mathcal{G}) = \iota_* q_{3*}(q_1^*\mathcal{F}\otimes q_2^*\mathcal{G}\otimes\mathcal{D}),$$

where $\mathcal{D} = \Sigma_i(-1)^i\pi_i\mathcal{O}_{\mathcal{I}^2\mathcal{X}}$. By Corollary 5.13 there is an identity $\mathcal{D} = \lambda_{-1}(\mathcal{E}_{\mathcal{I}^2}^\vee)$. We pointed out in Remark 5.14 that the class of $\mathcal{E}_{\mathcal{I}^2}$ in $K_0(I\mathcal{X})$ is equal to the class $\mathcal{B}$ from Definition 5.5. As a consequence we can rewrite

$$q_{3*}(q_1^*\mathcal{F}\otimes q_2^*\mathcal{G}\otimes\mathcal{D}) = q_{3*}(q_1^*\mathcal{F}\otimes q_2^*\mathcal{G}\otimes\lambda_{-1}(\mathcal{E}_{\mathcal{I}^2}^\vee)) = \mathcal{F}*^{virt}\mathcal{G}.$$

Applying ι_*, we obtain an identity $\iota_*(\mathcal{F})\otimes^{str}\iota_*(\mathcal{G}) = \iota_*(\mathcal{F}*^{virt}\mathcal{G})$ in $G_0(L\mathcal{X})$, and this concludes the proof. □

References

[ACH] D. Arinkin, A. Caldararu, M. Hablicsek, "Formality of derived intersections and the orbifold HKR isomorphism," `arxiv:1412.5233`

[AGV1] D. Abramovich, T. Graber, A, Vistoli, "Algebraic orbifold quantum products," Orbifolds in mathematics and physics (Madison, WI, 2001), 1–24, Contemp. Math., 310, Amer. Math. Soc., Providence, RI, 2002.

[AGV2] D. Abramovich, T. Graber, A, Vistoli, "Gromov-Witten theory of Deligne-Mumford stacks," Am. J. Math., **130** (5), 1337–1398, (2008).

[Ba1] C. Barwick, "On the algebraic K-theory of higher categories,"Journal of Topology, vol. 9 (2016), 245–347.

[Ba2] C. Barwick, "On exact infinity-categories and the Theorem of the Heart," Compositio Mathematica, vol. 151, no. 11, pp. 2160–2186.

[BGNX] K. Behrend, G. Ginot, B. Noohi and P. Xu, "String topology for stacks," Astérisque **343** (2012).

[BGT] A. Blumberg, D. Gepner, G. Tabuada, "A universal characterization of higher algebraic K-theory," Geom. Topol. 17 (2013), no. 2, 733–838.

[BFN] D. Ben-Zvi, J. Francis, D. Nadler, "Integral transforms and Drinfeld centers in derived algebraic geometry," J. Amer. Math. Soc. 23 909–966 (2010).

[BoK] A. Bondal, M. Kapranov, "Enhanced triangulated categories," USSR Math. USSR Sbornik, vol. 70, No. 1, pp. 93–107 (1991).

[BNP] D. Ben-Zvi, D. Nadler, Anatoly Preygel "Integral transforms for coherent sheaves," ?Journal of the European Mathematical Society 19(12).

[CKS] A. Caldararu, S. Katz, E. Sharpe, "D-branes, B-fields and Ext groups," Adv. Theor. Math. Phys. **7** no. 3, 381–404, (2003).

[Co] L. Cohn, "Differential graded categories are k-linear stable ∞-categories," `arxiv:1308.2587`

[CJ] R.L. Cohen, J.D.S. Jones, "A homotopy theoretic realization of string topology," Math. Ann. 324 no. 4, 773–798, (2002).

[CR] W. Chen, Y. Ruan, "A new cohomology theory of orbifold," Comm. Math. Phys. **248**, no. 1, 1–31(2004).

[CS] M. Chas, D. Sullivan, "String topology," `arxiv:9911159`

[EJK1] D. Edidin, T. J. Jarvis, T. Kimura, "Logarithmic trace and orbifold products," Duke Math. J., **153**, (3), 427-473, (2010).

[EJK2] D. Edidin, T. J. Jarvis, T. Kimura, "A plethora of inertial products," A plethora of inertial products. Ann. K-Theory 1 (2016), no. 1, 85–108.

[EJK3] D. Edidin, T. J. Jarvis, T. Kimura, "Chern classes and compatible power operations in inertial K-theory," Ann. K-Theory 2 (2017), no. 1, 73–130.

[FG] B. Fantechi, L. Göttsche, "Orbifold cohomology for global quotients," Duke Math. J. **117**, 197–227 (2003).

[Hi] V. Hinich, "Drinfeld double for orbifolds," Quantum groups. Contemp. Math. 433, Amer. Math. Soc., 251–265 (2007).

[Jo] A. Joyal, "Quasi-categories and Kan complexes," J. Pure Appl. Algebra 175, 207–222, (2002).

[JS] A. Joyal, R. Street, "Tortile Yang-Baxter operators in tensor categories," J. Pure Appl. Algebra 71, no. 1, 43–51, (1991).

[KP] R. Kaufmann, D. Pham, "The Drinfeld double and twisting in stringy orbifold theories," Internat. J. Math. 20 (2009), no. 5, 623–657.

[JKK] T. Jarvis, R. Kaufmann and T. Kimura, "Stringy K-theory and the Chern character," Inv. Math. **168**, 23–81 (2007).

[Lu] J. Lurie, "Higher topos theory," Annals of Mathematics Studies, 170, Princeton University Press 2009.

[Lu8] J. Lurie, "Derived Algebraic Geometry VIII: Quasi-Coherent Sheaves and Tannaka Duality Theorems," `http://www.math.harvard.edu/ lurie`

[LUX] E. Lupercio, B. Uribe, M. A. Xicoténcatl, "The loop orbifold of the symmetric product," J. Pure and Appl. Alg. 211, 293–306, (2007).

[LUX+] A. González, E. Lupercio, C. Segovia, B. Uribe, M. A. Xicoténcatl, "Chen-Ruan cohomology of cotangent orbifolds and Chas-Sullivan string topology," Math. Res. Lett., **14** (3),491–501, (2007).

[Ma] Y. Manin, "Quantum cohomology, motives, derived categories I,II," IHP, January 2013, lecture notes avaible at
`http://www.math.jussieu.fr/yuy/rega/pdf/1213/16012013a.pdf`

[MR] E. Mann, M. Robalo, "Gromov-Witten invariants: brane actions, path spaces and categorification," available at `http://www.math.univ-montp2.fr/ mann/catGW.pdf`.

[Ne] A. Neeman "The K-theory of triangulated categories," Handbook of K-theory, Vol. 1, 2, Springer, Berlin, 2005, 1011–1078.

[SP] the Stacks Project, an open source reference work available at `http://stacks.math.columbia.edu`.

[To1] B. Toën, "Higher and derived stacks: a global overview," Proc. Sympos. Pure Math. 80, Part 1, Amer. Math. Soc., Providence, RI, (2009).

[To2] B. Toën, "Operations on derived moduli spaces of branes," `arxiv:1307.0405`.

[To3] B. Toën, "Proper local complete intersection morphisms preserve perfect complexes," `arxiv:1210.2827`.

[To4] B. Toën, "Derived algebraic geometry," `arxiv:1401.1044`.

[TV] B. Toën, G. Vezzosi, "Chern character, loop spaces and derived algebraic geometry," Algebraic topology, 331–354, Abel Symp., 4, Springer, Berlin, 2009.

[Vi] A. Vistoli, "Intersection theory on algebraic stacks and their moduli spaces," Invent. Math., **97** 613–670 (1987).

[Z] E. Zaslow, "Topological orbifold models and quantum cohomology rings," Comm. Math. Phys. 156 301–331(1993).

Sarah Scherotzke, Mathematical Institute of the University of Münster, Einsteinstrasse 62, 48149 Münster, Germany
E-mail address: scherotz@math.uni-muenster.de

Nicolò Sibilla, SMSAS, University of Kent, Canterbury, Kent CT2 7NF, UK
E-mail address: N.Sibilla@kent.ac.uk