Predicting the values of the leptonic CP violation phases in theories with discrete flavour symmetries

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Abstract

Using the fact that the neutrino mixing matrix $U = U_e^\dagger U_\nu$, where U_e and U_ν result from the diagonalisation of the charged lepton and neutrino mass matrices, we consider a number of forms of U_ν associated with a variety of discrete symmetries: i) bimaximal (BM) and ii) tri-bimaximal (TBM) forms, the forms corresponding iii) to the conservation of the lepton charge $L' = L_e - L_\mu - L_\tau$ (LC), iv) to golden ratio type A (GRA) mixing, v) to golden ratio type B (GRB) mixing, and vi) to hexagonal (HG) mixing. Employing the minimal form of U_e, in terms of angles and phases it contains, that can provide the requisite corrections to U_ν so that reactor, atmospheric and solar neutrino mixing angles θ_{13}, θ_{23} and θ_{12} have values compatible with the current data, including a possible sizable deviation of θ_{23} from $\pi/4$, we discuss the possibility to obtain predictions for the CP violation phases in the neutrino mixing matrix. Considering the "standard ordering" of the 12 and the 23 rotations in U_e and following the approach developed in [1] we derive predictions for the Dirac phase δ and the rephasing invariant J_{CP} in the cases of GRA, GRB and HG forms of U_ν (results for the TBM and BM (LC) forms were obtained in [1]). We show also that under rather general conditions within the scheme considered the values of the Majorana phases in the PMNS matrix can be predicted for each of the forms of U_ν discussed. We give examples of these predictions and of their implications for neutrinoless double beta decay. In the GRA, GRB and HG cases, as in the TBM one, relatively large CP violation effects in neutrino oscillations are predicted ($|J_{CP}| \sim (0.031$–$0.034)$). Distinguishing between the TBM, BM (LC), GRA, GRB and HG forms of U_ν requires a measurement of $\cos\delta$ or a relatively high precision measurement of J_{CP}.

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1. Introduction

Determining the status of the CP symmetry in the lepton sector is one of the highest priority principal goals of the program of future research in neutrino physics (see, e.g., [2,3]). As in the case of the quark sector, the CP symmetry can be violated in the lepton sector by the presence of physical phases in the Pontecorvo, Maki, Nakagawa and Sakata (PMNS) neutrino mixing matrix. In the case of 3-neutrino mixing and massive Majorana neutrinos we are going to consider,[2] the 3×3 unitary PMNS matrix $U_{PMNS} \equiv U$ contains, as is well known, one Dirac and two Majorana [4] CP violation (CPV) phases which can be the source of CP violation in the lepton sector. In the widely used standard parametrisation [2] of the PMNS matrix we also are going to employ, U_{PMNS} is expressed in terms of the solar, atmospheric and reactor neutrino mixing angles θ_{12}, θ_{23} and θ_{13}, respectively, and the Dirac and Majorana CPV phases, as follows:

$$U = VQ, \quad Q = \mathrm{diag}\left(1, e^{\mathrm{i}\frac{\alpha_{21}}{2}}, e^{\mathrm{i}\frac{\alpha_{31}}{2}}\right), \tag{1}$$

where $\alpha_{21,31}$ are the two Majorana CPV phases and V is a CKM-like matrix,

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-\mathrm{i}\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{\mathrm{i}\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{\mathrm{i}\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{\mathrm{i}\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{\mathrm{i}\delta} & c_{23}c_{13} \end{pmatrix}. \tag{2}$$

In Eq. (2), δ is the Dirac CPV phase, $0 \leq \delta \leq 2\pi$, we have used the standard notation $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$, and $0 \leq \theta_{ij} \leq \pi/2$. In what concerns the Majorana CPV phases, for the purpose of the present study it is sufficient to consider that they vary in the intervals $0 \leq \alpha_{21,31} \leq 2\pi$.[3] If CP invariance holds, we have $\delta = 0, \pi, 2\pi$, the values 0 and 2π being physically indistinguishable, and [7] $\alpha_{21(31)} = k^{(\prime)}\pi$, $k^{(\prime)} = 0, 1, 2, \ldots$.

The CP symmetry will not hold in the lepton sector if the Dirac and/or Majorana phases possess CP-nonconserving values. If the Dirac phase δ has a CP-nonconserving value, this will induce, as is well known, CP violation effects in neutrino oscillations, i.e., a difference between the 3-flavour neutrino oscillation probabilities $P(\nu_l \to \nu_{l'})$ and $P(\bar{\nu}_l \to \bar{\nu}_{l'})$, $l \neq l' = e, \mu, \tau$.

The flavour neutrino oscillation probabilities $P(\nu_l \to \nu_{l'})$ and $P(\bar{\nu}_l \to \bar{\nu}_{l'})$, $l, l' = e, \mu, \tau$, do not depend on the Majorana phases [4,8]. The Majorana phases can play important role in processes which are characteristic for Majorana neutrinos, in which the total lepton charge L changes by two units, like neutrinoless double beta $((\beta\beta)_{0\nu}$-) decay $(A, Z) \to (A, Z + 2) + e^- + e^-$ (see, e.g., [9–11]), etc. The rates of the processes of emission of two different Majorana

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[2] All compelling data on neutrino masses, mixing and oscillations are compatible with the existence of mixing of three light massive neutrinos ν_i, $i = 1, 2, 3$, in the weak charged lepton current (see, e.g., [2]). It follows also from the data that the masses m_i of the three light neutrinos ν_i do not exceed approximately 1 eV, $m_i \lesssim 1$ eV, i.e., they are significantly smaller than the masses of the charged leptons and quarks.

[3] One should keep in mind, however, that in the case of the seesaw mechanism of neutrino mass generation the Majorana phases α_{21} and α_{31} vary in the interval [5] $0 \leq \alpha_{21,31} \leq 4\pi$. The interval beyond 2π, $2\pi \leq \alpha_{21,31} \leq 4\pi$, is relevant, e.g., in the calculations of the baryon asymmetry within the leptogenesis scenario [5], in the calculation of the neutrinoless double beta decay effective Majorana mass in the TeV scale version of the type I seesaw model of neutrino mass generation [6], etc.

neutrinos, an example of which is the radiative emission of neutrino pair in atomic physics [12], depend in the threshold region on the Majorana phases [13]. The phases $\alpha_{21,31}$ can affect significantly the predictions for the rates of the lepton flavour violating (LFV) decays $\mu \to e + \gamma$, $\tau \to \mu + \gamma$, etc. in a large class of supersymmetric theories incorporating the see-saw mechanism [14].

The existing neutrino oscillation data allow us to determine the two neutrino mass squared differences, Δm_{21}^2 and $|\Delta m_{31(32)}^2|$, and the three angles θ_{12}, θ_{23} and θ_{13}, which drive the neutrino oscillations observed in the experiments with solar, atmospheric, reactor and accelerator neutrinos (see, e.g., [2]) with a relatively good precision [15,16]. The best fit values and the 3σ allowed ranges of the three neutrino mixing parameters which are relevant for our further discussion, $\sin^2 \theta_{12}$, $\sin^2 \theta_{23}$ and $\sin^2 \theta_{13}$, found in the global analysis in Ref. [15] read:

$$\left(\sin^2 \theta_{12}\right)_{\text{BF}} = 0.308, \quad 0.259 \leq \sin^2 \theta_{12} \leq 0.359, \tag{3}$$

$$\left(\sin^2 \theta_{23}\right)_{\text{BF}} = 0.425\,(0.437), \quad 0.357(0.363) \leq \sin^2 \theta_{23} \leq 0.641(0.659), \tag{4}$$

$$\left(\sin^2 \theta_{13}\right)_{\text{BF}} = 0.0234\,(0.0239), \quad 0.0177(0.0178) \leq \sin^2 \theta_{13} \leq 0.0297(0.0300), \tag{5}$$

where the value (the value in brackets) corresponds to $\Delta m_{31(32)}^2 > 0$ ($\Delta m_{31(32)}^2 < 0$). There are also hints from the data about the value of the Dirac phase[4] δ. In both analyses [15,16] the authors find that the best fit value of $\delta \cong 3\pi/2$. The CP conserving values $\delta = 0$ and π ($\delta = 0$) are disfavoured at 1.6σ to 2.0σ (at 2.0σ) for $\Delta m_{31(32)}^2 > 0$ ($\Delta m_{31(32)}^2 < 0$). In the case of $\Delta m_{31(32)}^2 < 0$, the value $\delta = \pi$ is statistically 1σ away from the best fit value $\delta \cong 3\pi/2$ (see, e.g., Fig. 3 in Ref. [15]).

The theoretical predictions for the values of the CPV phases in the neutrino mixing matrix depend on the approach and the type of symmetries one uses in the attempts to understand the pattern of neutrino mixing (see, e.g., [1,19–21] and references quoted therein). In the case of the Dirac phase δ, the predictions vary considerably: they include the values 0, $\pi/2$, π, $3\pi/2$, but not only; in certain cases 0, $\pi/2$, π and $3\pi/2$ are approximate values, the exact predictions being slightly different from these values. Obviously, a sufficiently precise measurement of δ will serve as an additional very useful constraint for identifying the approaches and/or the symmetries, if any, at the origin of the observed pattern of neutrino mixing. Understanding the origin of the patterns of neutrino masses and mixing, emerging from the neutrino oscillation, ^3H β-decay, cosmological, etc. data is one of the most challenging problems in neutrino physics. It is part of the more general fundamental problem in particle physics of understanding the origins of flavour, i.e., of the patterns of the quark, charged lepton and neutrino masses and of the quark and lepton mixing.

Using the fact that the neutrino mixing matrix $U = U_e^\dagger U_\nu$, where U_e and U_ν result from the diagonalisation of the charged lepton and neutrino mass matrices, and assuming that U_ν has a i) tri-bimaximal (TBM) form [22], ii) bimaximal (BM) form [23,24], or else iii) corresponds to the conservation of the lepton charge [23] $L' = L_e - L_\mu - L_\tau$ (LC), that the requisite perturbative

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[4] Using the most recent T2K data on $\nu_\mu \to \nu_e$ oscillations, the T2K collaboration finds for $\delta = 0$, $\sin^2 \theta_{23} = 0.5$ and $|\Delta m_{31(32)}^2| = 2.4 \times 10^{-3}$ eV^2, in the case of $\Delta m_{31(32)}^2 > 0$ ($\Delta m_{31(32)}^2 < 0$) [17]: $\sin^2 2\theta_{13} = 0.140^{+0.038}_{-0.032}$ $(0.170^{+0.045}_{-0.037})$. Thus, the best fit value of $\sin^2 2\theta_{13}$ thus found in the T2K experiment is approximately by a factor of 1.6 (1.9) bigger than that measured in the Daya Bay experiment [18]: $\sin^2 2\theta_{13} = 0.090^{+0.008}_{-0.009}$. The compatibility of the results of the two experiments on $\sin^2 2\theta_{13}$ requires, in particular, that $\delta \neq 0$ (and/or $\sin^2 \theta_{23} \neq 0.5$), which leads to the hints under discussion about the possible value of δ in the global analyses of the neutrino oscillation data.

corrections to the TBM and BM (LC) mixing angles are provided by the matrix U_e, and that U_e has a minimal form in terms of angles and phases it contains that can provide the corrections to U_ν so that the angles θ_{13}, θ_{23} and θ_{12} in the PMNS matrix have values compatible with the current data, we have obtained in [1] predictions for the Dirac phase δ present in the PMNS matrix U.[5] An important requirement is that the corrections due to the matrix U_e should allow sizable deviations of the angle θ_{23} from the BM and TBM value $\pm\pi/4$. These requirements imply that U_e should be a product of two rotations in the 12 and 23 planes, $R_{12}(\theta_{12}^e)$ and $R_{23}(\theta_{23}^e)$, and a diagonal phase matrix which contains, in general, two physical CP violation phases. In the case of "standard" ordering with $U_e \propto R_{23}(\theta_{23}^e) R_{12}(\theta_{12}^e)$, which we are going to consider and which is related to the hierarchy of the charged lepton masses, $m_e^2 \ll m_\mu^2 \ll m_\tau^2$, and is a common feature of the overwhelming majority of the existing models of the charged lepton (and neutrino) masses and the associated mixing, $\cos\delta$ was shown to satisfy in the cases of the TBM and BM (LC) forms of U_ν a new sum rule [1] by which it is expressed in terms of the three angles θ_{12}, θ_{23} and θ_{13} of the PMNS matrix. For the current best fit values of $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$, the following predictions for δ were obtained for the two forms of U_ν [1]: i) $\delta \cong \pi$ in the BM (or LC) case, ii) $\delta \cong 3\pi/2$ or $\pi/2$ in the TBM case,[6] the CP conserving values $\delta = 0, \pi, 2\pi$ being excluded in the TBM case at more than 5σ. A model based on the T' flavour symmetry leading to the TBM form of U_ν, in which the conditions of the general phenomenological approach followed in [1] are realised and thus which predicts, in particular, $\delta \cong 3\pi/2$ or $\pi/2$, was constructed in [19].

In the present article we first generalise in Section 3 the analytic results for the sum rule involving the cosine of the Dirac phase δ, obtained in [1] for the specific BM (LC) and TBM values $\pi/4$ and $\sin^{-1}(1/\sqrt{3})$ of the angle θ_{12}^ν in the matrix U_ν, to the case of arbitrary fixed value of θ_{12}^ν. This allows us to obtain *new* predictions for the phase δ and the J_{CP} factor, which controls the magnitude of CP violation effects due to δ in neutrino oscillations, in the cases of i) golden ratio type A (GRA) mixing [39,40] with $\sin^2\theta_{12}^\nu = (2+r)^{-1} \cong 0.276$, r being the golden ratio, $r = (1+\sqrt{5})/2$, ii) golden ratio type B (GRB) mixing [41] with $\sin^2\theta_{12}^\nu = (3-r)/4 \cong 0.345$, and iii) hexagonal (HG) mixing [42] in which $\theta_{12}^\nu = \pi/6$. As like the TBM and BM forms of U_ν, the GRA form can be obtained from discrete family symmetry in the lepton sector, while the GRB and HG forms are considered on general phenomenological grounds (see, e.g., the reviews [43–45] and [41,42]). In Section 3 we derive also analytic expression for the correction in the new sum rule for $\cos\delta$ due to the possible presence in U_e of the 13 rotation matrix $R_{13}(\theta_{13}^e)$ with angle $\theta_{13}^e \ll 1$ and determine the conditions under which this correction is subdominant. In Section 4 we show that the approximate sum rule for δ proposed in [33] can be obtained in the leading order approximation from the "exact" sum rule for $\cos\delta$ derived in Section 3. We compare the predictions for δ in the cases of the TBM, BM (LC), GRA, GRB and HG forms of the matrix U_ν, obtained using the exact and the leading order sum rules and determine the origin of the difference in the predictions. We next analyse in Section 5 the possibility to obtain predictions for the values of the Majorana phases in the PMNS matrix, α_{21} and α_{31}, using the same approach which allowed us to get predictions for the Dirac phase δ. For the TBM, BM (LC), GRA, GRB and HG forms of U_ν considered by us, we obtain analytic expressions for the contribution to the phases α_{21} and α_{31}, generated by the CPV phases which serve in the approach employed as a "source" for the Dirac phase δ and which are present in the PMNS matrix due to

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[5] The predictions for the Dirac phase δ were obtained in [1] using the framework which was developed in [25–28] for understanding the specific features of the neutrino mixing and in various versions was further exploited by many authors (see, e.g., [29–38]).

[6] More precisely, the predicted values of δ in the TBM case are $\delta \cong 266°$ or $94°$.

the non-trivial form of the charged lepton "correction" matrix U_e. We determine the cases when the phases $\alpha_{21,31}$ can be predicted and give example of prediction of their values. We show in Section 6 that the results obtained on the Majorana phases for the different symmetry forms of the matrix U_ν, can lead, in particular, to specific predictions for the $(\beta\beta)_{0\nu}$-decay effective Majorana mass in the physically important cases of neutrino mass spectrum with inverted ordering or of quasi-degenerate type. The results of the present study are summarised in Section 7.

2. The framework

In what follows we consider 3-neutrino mixing of the three left-handed (LH) flavour neutrinos and antineutrinos, ν_l and $\bar{\nu}_l$, $l = e, \mu, \tau$. The neutrino mixing matrix in this case receives contributions from the diagonalisation of the charged lepton and neutrino Majorana mass terms. Taking into account the contributions from the charged lepton and neutrino sectors, the PMNS neutrino mixing matrix can be written as [26]:

$$U_{\mathrm{PMNS}} = U_e^\dagger U_\nu = (\tilde{U}_e)^\dagger \Psi \tilde{U}_\nu Q_0. \tag{6}$$

Here U_e and U_ν are 3×3 unitary matrices originating from the diagonalisation respectively of the charged lepton[7] and neutrino mass matrices, $\tilde{U}_e$ and $\tilde{U}_\nu$ are CKM-like 3×3 unitary matrices and Ψ and Q_0 are diagonal phase matrices each containing in the general case two physical CPV phases,

$$\Psi = \mathrm{diag}\left(1, e^{-\mathrm{i}\psi}, e^{-\mathrm{i}\omega}\right), \qquad Q_0 = \mathrm{diag}\left(1, e^{\mathrm{i}\frac{\xi_{21}}{2}}, e^{\mathrm{i}\frac{\xi_{31}}{2}}\right). \tag{7}$$

The phase matrix Q_0 contributes to the Majorana phases in the PMNS matrix and can appear in Eq. (6) as a result of the diagonalisation of the neutrino Majorana mass term, while Ψ can originate from the charged lepton sector ($U_e^\dagger = (\tilde{U}_e)^\dagger \Psi$), or from the neutrino sector ($U_\nu = \Psi \tilde{U}_\nu Q_0$), or can receive contributions from both sectors.

Following the results of the analysis performed in [1], we will assume that the matrix $\tilde{U}_e$ is a product of two orthogonal matrices describing rotations in the 12 and 23 planes and that the two rotations in $\tilde{U}_e$ are in the "standard ordering". It proves convenient to adopt for $\tilde{U}_e$ the notation used in [1]:

$$\tilde{U}_e = R_{23}^{-1}\left(\theta_{23}^e\right) R_{12}^{-1}\left(\theta_{12}^e\right), \tag{8}$$

where

$$R_{12}\left(\theta_{12}^e\right) = \begin{pmatrix} \cos\theta_{12}^e & \sin\theta_{12}^e & 0 \\ -\sin\theta_{12}^e & \cos\theta_{12}^e & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_{23}\left(\theta_{23}^e\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23}^e & \sin\theta_{23}^e \\ 0 & -\sin\theta_{23}^e & \cos\theta_{23}^e \end{pmatrix}, \tag{9}$$

and θ_{12}^e and θ_{23}^e are two arbitrary (real) angles.

The fact that $\tilde{U}_e$ does not include the matrix $R_{13}(\theta_{13}^e)$ describing rotation in the 13 plane, i.e., that $\theta_{13}^e \cong 0$, follows from the requirement that U_e has a "minimal" form in terms of angles and phases it contains that can provide the requisite corrections to U_ν, so that the mixing angles θ_{13}, θ_{23} and θ_{12} in U have values compatible with the current data, including the possibility of a sizable deviation of θ_{23} from $\pi/4$. As will be discussed briefly in Section 3, a nonzero

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[7] For charged lepton mass term written in the left-right convention, the matrix U_e diagonalises the hermitian matrix $M_E M_E^\dagger$, $U_e^\dagger M_E M_E^\dagger U_e = \mathrm{diag}(m_e^2, m_\mu^2, m_\tau^2)$, M_E being the charged lepton mass matrix.

$\theta_{13}^e \lesssim 10^{-3}$ generates a correction to $\cos\delta$ derived from the exact sum rule, which does not exceed 11% (4.9%) in the TBM (GRB) cases and is even smaller in the other three cases of symmetry form of $\tilde{U}_\nu$ analysed in the present article. We note that $\theta_{13}^e \cong 0$ is a feature of many theories and models of charged lepton mass generation (see, e.g., [19,35,36,39,43,46]) and was used in a large number of articles dedicated to the problem of understanding the origins of the observed pattern of neutrino mixing (see, e.g., [21,25,26,28,33,34,37,38,47,48]). In large class of GUT inspired models of flavour, for instance, the matrix U_e is directly related to the quark mixing matrix (see, e.g., [30,35,36,43,44]). As a consequence, in this class of models, in particular, θ_{13}^e is negligibly small.

We will assume further that the matrix $\tilde{U}_\nu$ has one of the following symmetry forms: TBM, BM, LC, GRA, GRB and HG. For all symmetry forms of interest, $\tilde{U}_\nu$ is also a product 23 and 12 rotations in the plane:

$$\tilde{U}_\nu = R_{23}(\theta_{23}^\nu) R_{12}(\theta_{12}^\nu). \tag{10}$$

In the case of the TBM, BM, GRA, GRB and HG forms of $\tilde{U}_\nu$ we have $\theta_{23}^\nu = -\pi/4$, while θ_{12}^ν takes the values $\sin^{-1}(1/\sqrt{3})$, $\pi/4$, $\sin^{-1}(1/\sqrt{2+r})$, $\sin^{-1}(\sqrt{3-r}/2)$, and $\pi/6$, respectively. Thus, the matrix $\tilde{U}_\nu$ corresponding to these cases has the form:

$$\tilde{U}_\nu = \begin{pmatrix} \cos\theta_{12}^\nu & \sin\theta_{12}^\nu & 0 \\ -\frac{\sin\theta_{12}^\nu}{\sqrt{2}} & \frac{\cos\theta_{12}^\nu}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{\sin\theta_{12}^\nu}{\sqrt{2}} & \frac{\cos\theta_{12}^\nu}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{11}$$

where θ_{12}^ν takes different fixed values for the different symmetry forms of $\tilde{U}_\nu$. In the case of the LC form of $\tilde{U}_\nu$ we have $\theta_{12}^\nu = \pi/4$, while θ_{23}^ν can have an arbitrary fixed value. Thus, if $U_e = \mathbf{1}$, $\mathbf{1}$ being the unity 3×3 matrix, we have:

i) $\theta_{13} = 0$ in all six cases of interest of $\tilde{U}_\nu$;

ii) $\theta_{23} = -\pi/4$, if $\tilde{U}_\nu$ has any of the forms TBM, BM, GRA, GRB and HG, while θ_{23} can have an arbitrary value if U_ν has the LC form;

iii) $\sin^2\theta_{12} = 0.5$ for the BM and LC forms of $\tilde{U}_\nu$; $\sin^2\theta_{12} = 1/3$ in the TBM case; $\sin^2\theta_{12} \cong 0.276$ and 0.345 for the GRA and GRB mixing and $\sin^2\theta_{12} = 0.25$ for HG mixing. Thus, the matrix U_e has to generate corrections

i) leading to $\theta_{13} \neq 0$ compatible with the observations in all six cases of U_ν considered;

ii) leading to the observed deviation of θ_{12} from a) $\pi/4$, b) from the two golden ratio values[8] and c) from $\pi/6$, in the cases of a) BM and LC, b) GRA and GRB, and c) HG, mixing;

iii) leading to the sizable deviation of θ_{23} from $\pi/4$ for all cases considered except the LC one, if it is confirmed by further data that $\sin^2\theta_{23} \cong 0.40$–$0.44$. The minimal form of U_e in terms of angles and phases it contains, which can produce the requisite corrections discussed above, is the one with $\tilde{U}_e$ given in Eq. (8). The presence of $R_{12}^{-1}(\theta_{12}^e)$ in $\tilde{U}_e$ allows to correct the symmetry values of θ_{12} and θ_{13}, while the presence of $R_{23}^{-1}(\theta_{23}^e)$ allows to have sizable deviations (bigger than $0.5\sin^2\theta_{13}$) of $\sin^2\theta_{23}$ from the symmetry value of 0.5.

In the approach adopted by us following [1] the PMNS neutrino mixing matrix has the form:

$$U_{\text{PMNS}} = U_e^\dagger U_\nu = R_{12}(\theta_{12}^e) R_{23}(\theta_{23}^e) \Psi R_{23}(\theta_{23}^\nu) R_{12}(\theta_{12}^\nu) Q_0, \tag{12}$$

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[8] The GRA and GRB values of $\sin^2\theta_{12} \cong 0.276$ and 0.345 lie at the border of the 2σ allowed range of values of $\sin^2\theta_{12}$ obtained in the global analyses [15,16].

where $\theta_{23}^{\nu} = -\pi/4$ and θ_{12}^{ν} has a known value. As a consequence, the three angles θ_{12}, θ_{23} and θ_{13} and the Dirac CPV phase δ of the PMNS mixing matrix, Eqs. (1)–(2), can be expressed as functions of the two real angles, θ_{12}^{e} and θ_{23}^{e}, and the two phases, ψ and ω, of the phase matrix Ψ. The results will depend on the specific value of the angle θ_{12}^{ν}, i.e., on the assumed symmetry form of $\tilde{U}_{\nu}$. We will discuss how the Majorana phases in the PMNS matrix, α_{21} and α_{31}, are expressed in terms of these parameters later.

As was shown in [1], the product of matrices $R_{23}(\theta_{23}^{e})\Psi R_{23}(\theta_{23}^{\nu} = -\pi/4)$ in the expression (12) for U_{PMNS} can be rearranged as follows:

$$R_{23}(\theta_{23}^{e})\Psi R_{23}(\theta_{23}^{\nu}) = P_1 \Phi R_{23}(\hat{\theta}_{23}) Q_1. \tag{13}$$

Here the angle $\hat{\theta}_{23}$ is determined by

$$\sin^2 \hat{\theta}_{23} = \frac{1}{2}\left(1 - 2 \sin\theta_{23}^{e} \cos\theta_{23}^{e} \cos(\omega - \psi)\right), \tag{14}$$

and

$$P_1 = \mathrm{diag}\left(1, 1, e^{-\mathrm{i}\alpha}\right), \qquad \Phi = \mathrm{diag}\left(1, e^{\mathrm{i}\phi}, 1\right), \qquad Q_1 = \mathrm{diag}\left(1, 1, e^{\mathrm{i}\beta}\right), \tag{15}$$

where

$$\alpha = \gamma + \psi + \omega, \qquad \beta = \gamma - \phi, \tag{16}$$

and

$$\gamma = \arg\left(-e^{-\mathrm{i}\psi} \cos\theta_{23}^{e} + e^{-\mathrm{i}\omega} \sin\theta_{23}^{e}\right), \qquad \phi = \arg\left(e^{-\mathrm{i}\psi} \cos\theta_{23}^{e} + e^{-\mathrm{i}\omega} \sin\theta_{23}^{e}\right). \tag{17}$$

The phase α in the matrix P_1 is unphysical. The phase β contributes to the matrix of physical Majorana phases, which now is equal to $\hat{Q} = Q_1 Q_0$. The PMNS matrix takes the form:

$$U_{\mathrm{PMNS}} = R_{12}(\theta_{12}^{e})\Phi(\phi) R_{23}(\hat{\theta}_{23}) R_{12}(\theta_{12}^{\nu})\hat{Q}, \tag{18}$$

where θ_{12}^{ν} has a fixed value which depends on the symmetry form of $\tilde{U}_{\nu}$ used. Thus, the four observables $\theta_{12}, \theta_{23}, \theta_{13}$ and δ are functions of three parameters $\theta_{12}^{e}, \hat{\theta}_{23}$ and ϕ. As a consequence, the Dirac phase δ can be expressed as a function of the three PMNS angles θ_{12}, θ_{23} and θ_{13}, leading to a new "sum rule" relating δ and θ_{12}, θ_{23} and θ_{13} [1]. Using the measured values of θ_{12}, θ_{23} and θ_{13}, we have obtained in [1] predictions for the values of δ and of the rephasing invariant $J_{\mathrm{CP}} = \mathrm{Im}(U_{e1}^* U_{\mu 3}^* U_{e3} U_{\mu 1})$, which controls the magnitude of CP violating effects in neutrino oscillations [49], in the cases of the TBM, BM (LC) forms of $\tilde{U}_{\nu}$. Here we will first obtain predictions for δ and J_{CP} in the cases of GRA, GRB and HG forms of $\tilde{U}_{\nu}$. After that we will analyse the possibility to obtain predictions for the Majorana phases in the PMNS matrix within the framework described above.

3. The Dirac phase in the PMNS matrix

Using Eq. (18) we get for the angles θ_{12}, θ_{23} and θ_{13} of the standard parametrisation of U_{PMNS} [1]:

$$\sin\theta_{13} = |U_{e3}| = \sin\theta_{12}^{e} \sin\hat{\theta}_{23}, \tag{19}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1 - |U_{e3}|^2} = \sin^2\hat{\theta}_{23}\frac{\cos^2\theta_{12}^{e}}{1 - \sin^2\theta_{12}^{e}\sin^2\hat{\theta}_{23}} = \frac{\sin^2\hat{\theta}_{23} - \sin^2\theta_{13}}{1 - \sin^2\theta_{13}}, \tag{20}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2} = \frac{|\sin\theta_{12}^v \cos\theta_{12}^e + e^{i\phi}\cos\theta_{12}^v \cos\hat{\theta}_{23}\sin\theta_{12}^e|^2}{1 - \sin^2\theta_{12}^e \sin^2\hat{\theta}_{23}}, \tag{21}$$

where Eq. (19) was used in order to obtain the expression for $\sin^2\theta_{23}$ in terms of $\hat{\theta}_{23}$ and θ_{13}. Within the approach employed the expressions in Eqs. (19)–(21) are exact. It follows from Eqs. (19) and (20) that the angle $\hat{\theta}_{23}$ differs little from the angle θ_{23} and that $\sin^2\theta_{12}^e \ll 1$: for, e.g., the best fit values of $\sin^2\theta_{13} = 0.0234$ and $\sin^2\theta_{23} \cong 0.425$ we have $\sin^2\hat{\theta}_{23} \cong 0.438$ and $\sin\theta_{12}^e \cong 0.23$.

We will derive next first a general expression for the cosine of the CPV phase ϕ in terms of the angles θ_{12}, θ_{23}, θ_{13} and θ_{12}^v, then a relation between the phases ϕ and the Dirac phase δ of the standard parametrisation of the PMNS matrix, and finally an expression for δ in terms of θ_{12}, θ_{23}, θ_{13}, θ_{12}^v and θ_{12}^v. This will allow us to obtain new predictions for δ in the cases of GRA, GRB and HG symmetry forms of the matrix $\tilde{U}_v$.

From Eq. (21) using Eqs. (19) and (20) we find:

$$\cos\phi = 2\frac{\sin^2\theta_{12}(1 - \cos^2\theta_{23}\cos^2\theta_{13}) - (\sin^2\theta_{23}\sin^2\theta_{12}^v + \cos^2\theta_{23}\cos^2\theta_{12}^v\sin^2\theta_{13})}{\sin 2\theta_{12}^v \sin 2\theta_{23}\sin\theta_{13}}. \tag{22}$$

As it follows from Eqs. (13), (15) and (16), the phase ϕ contributes to the Majorana phase α_{31}, in particular, via the phase β. Thus, we will give next the values of $\cos\phi$ and $|\sin\phi|$ for the different symmetry forms of the matrix $\tilde{U}_v$ we are considering, TBM, BM (LC), GRA, GRB and HG.[9] These values will be relevant in the discussion of the Majorana phases determination. Using the best fit values of the neutrino mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ quoted in Eqs. (3)–(5), for $\Delta m_{31}^2 > 0$ and the specific value of θ_{12}^v characterising a given case of $\tilde{U}_v$, we get:

TBM: $\cos\phi \cong -0.219,$ $|\sin\phi| \cong 0.976,$ (23)

GRA: $\cos\phi \cong +0.116,$ $|\sin\phi| \cong 0.993,$ (24)

GRB: $\cos\phi \cong -0.286,$ $|\sin\phi| \cong 0.958,$ (25)

HG: $\cos\phi \cong +0.286,$ $|\sin\phi| \cong 0.958.$ (26)

The same procedure leads in the BM (LC) case to the unphysical value of $\cos\phi \cong -1.13$. This reflects the fact that the scheme under discussion with BM (LC) form of the matrix $\tilde{U}_v$ does not provide a good description of the current data on θ_{12}, θ_{23} and θ_{13} [1]. Thus, we will calculate $\cos\phi$ using the best values of $\sin^2\theta_{12} = 0.32$, $\sin^2\theta_{23} = 0.41\ (0.42)$ and $\sin\theta_{13} = 0.158$, determined for $\Delta m_{31}^2 > 0\ (\Delta m_{31}^2 < 0)$ in the statistical analysis performed in [1]. For these values of $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin\theta_{13}$ in the case of $\Delta m_{31}^2 > 0$ we get:

BM (LC): $\cos\phi \cong -0.981,$ $|\sin\phi| \cong 0.193.$ (27)

We do not give the results on $\cos\phi$ for $\Delta m_{31}^2 < 0$ since they differ little from those shown.

Comparing the imaginary and real parts of $U_{e1}^* U_{\mu3}^* U_{e3} U_{\mu1}$, obtained using Eq. (18) and the standard parametrisation of U_{PMNS}, one gets the following relation between ϕ and δ:

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[9] Using the current data one can determine directly only $\cos\phi$ but not $\sin\phi$, and therefore the sign of $\sin\phi$ is undetermined. The measurement of $\sin\delta$ will allow to determine $\sin\phi$ as well.

$$\sin \delta = -\frac{\sin 2\theta_{12}^{\nu}}{\sin 2\theta_{12}} \sin \phi, \tag{28}$$

$$\cos \delta = \frac{\sin 2\theta_{12}^{\nu}}{\sin 2\theta_{12}} \cos \phi \left(-1 + \frac{2 \sin^2 \theta_{23}}{\sin^2 \theta_{23} \cos^2 \theta_{13} + \sin^2 \theta_{13}} \right)$$

$$+ \frac{\cos 2\theta_{12}^{\nu}}{\sin 2\theta_{12}} \frac{\sin 2\theta_{23} \sin \theta_{13}}{\sin^2 \theta_{23} \cos^2 \theta_{13} + \sin^2 \theta_{13}}. \tag{29}$$

Within the scheme considered the results quoted above, including those for $\sin \delta$ and $\cos \delta$, are exact and are valid for arbitrary fixed θ_{12}^{ν}. As can be shown, in particular, we have: $\sin^2 \delta + \cos^2 \delta = 1$. In Section 5 we will derive an exact relation between the CPV phases δ and ϕ (see Eq. (94)).

Substituting the expression (22) for $\cos \phi$ in Eqs. (28) and (29), we get a general expressions for $\sin \delta$ and $\cos \delta$ in terms of $\theta_{12}, \theta_{23}, \theta_{13}$ and θ_{12}^{ν}. We give below the result for $\cos \delta$:

$$\cos \delta = \frac{\tan \theta_{23}}{\sin 2\theta_{12} \sin \theta_{13}} \left[\cos 2\theta_{12}^{\nu} + \left(\sin^2 \theta_{12} - \cos^2 \theta_{12}^{\nu} \right) \left(1 - \cot^2 \theta_{23} \sin^2 \theta_{13} \right) \right]. \tag{30}$$

For $\theta_{12}^{\nu} = \pi/4$ and $\theta_{12}^{\nu} = \sin^{-1}(1/\sqrt{3})$ the expression (30) for $\cos \delta$ we have derived reduces to those found in [1] in the BM (LC) and TBM cases, respectively.

From Eq. (30) we find in the cases of TBM and BM (LC) forms of $\tilde{U}_{\nu}$ [10]:

> TBM: $\quad \cos \delta = -0.0851, \quad \delta = 265.1° \text{ or } 94.9°,$ $\tag{31}$

> BM (LC): $\quad \cos \delta = -0.978, \quad \delta = 191.9° \text{ or } 168.1°.$ $\tag{32}$

The value of $\cos \delta$ corresponds in the TBM case to the b.f.v. of $\sin^2 \theta_{ij}$ given in Eqs. (3)–(5); in the BM (LC) case it is obtained for [1] $\sin^2 \theta_{12} = 0.32$, $\sin^2 \theta_{23} = 0.41$ and $\sin \theta_{13} = 0.158$.

For the new cases considered by us we get using the best fit values of $\sin^2 \theta_{ij}$ quoted in Eqs. (3)–(5) for $\Delta m_{31}^2 > 0$ ($\Delta m_{31}^2 < 0$) and the value of θ_{12}^{ν} characterising a given case:

> GRA: $\quad \cos \delta \cong 0.273 \ (0.274), \quad \delta \cong 285.8° \ (285.9°) \text{ or } 74.2° \ (74.1°),$ $\tag{33}$

> GRB: $\quad \cos \delta \cong -0.161 \ (-0.165), \quad \delta \cong 260.7° \ (260.5°) \text{ or } 99.3° \ (99.5°),$ $\tag{34}$

> HG: $\quad \cos \delta \cong 0.438 \ (0.442), \quad \delta \cong 296.0° \ (296.2°) \text{ or } 64.0° \ (63.8°).$ $\tag{35}$

It follows from the results derived and quoted above that, in general, the predicted values of $\cos \delta$ and δ vary significantly with the assumed symmetry form of the matrix $\tilde{U}_{\nu}$. One exception are the predictions of δ in the cases of TBM and GRB forms of $\tilde{U}_{\nu}$: they differ only by approximately 5°. We note also that, except for the BM (LC) case, the values of $\cos \delta$ and $\cos \phi$ differ significantly for a given assumed form of the symmetry mixing, TBM, GRA, etc.

If we consider the indications obtained in [15,16] that $\delta \cong 3\pi/2$, only the case of BM (LC) mixing is weakly disfavoured for $\Delta m_{31}^2 > 0$ at approximately 1.4σ, while for $\Delta m_{31}^2 < 0$ all cases of the form of $\tilde{U}_{\nu}$ considered by us are statistically compatible with the results on δ found in [15, 16] (see, e.g., Fig. 3 in [15]).

As was mentioned in Section 2, a nonzero $|\sin \theta_{13}^{e}| \ll 1$, θ_{13}^{e} being the angle of rotation in the 13 plane, generates a correction to the value of $\cos \delta$ derived from the exact sum rule. In this case we have: $\cos \delta(\theta_{13}^{e}) = \cos \delta - \Delta(\cos \delta)$, where $\cos \delta$ is the value obtained from the exact sum rule

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[10] There is a small difference between the values of $\cos \delta$ and δ obtained for $\Delta m_{31}^2 > 0$ and $\Delta m_{31}^2 < 0$. We report here the values corresponding to $\Delta m_{31}^2 > 0$.

and $\Delta(\cos\delta)$ is the correction due to $|\sin\theta_{13}^e| \neq 0$. As can be shown using the parametrisation $\tilde{U}_e = R_{23}^{-1}(\theta_{23}^e) R_{13}^{-1}(\theta_{13}^e) R_{12}^{-1}(\theta_{12}^e)$, to leading order in $|\sin\theta_{13}^e| \ll 1$ (i.e., neglecting terms of order of, or smaller than, $\sin^2\theta_{13}^e$, $\sin\theta_{13}^e \sin\theta_{13}$) we have:

$$\Delta(\cos\delta) \cong \frac{\sin\theta_{13}^e}{\sin\theta_{13}} \frac{\cos\kappa}{\sin\hat{\theta}_{23}} \tan\theta_{12}^v \cot\theta_{12} \tan\theta_{23}, \tag{36}$$

where $\kappa = \arg(c_{23}^e e^{-i\omega} - s_{23}^e e^{-i\psi})$. The result (36) for $\Delta(\cos\delta)$ can be derived by taking into account, in particular, that $|\sin\theta_{13}^e| \ll 1$ and that in the approximation employed by us $\cos\delta(\theta_{13}^e) \sin\theta_{13} \cong \cos\delta \sin\theta_{13}$. It is not difficult to convince oneself that for the best fit values of the neutrino mixing parameters and the symmetry forms of $\tilde{U}_v$ considered, the correction satisfies the inequality: $|\Delta(\cos\delta)| \lesssim C |\sin\theta_{13}^e|$, where the constant $C = 9.0$, 12.7, 7.9, 9.2, and 7.3 for the TBM, BM, GRA, GRB and HG forms of $\tilde{U}_v$, respectively. Thus, for $|\sin\theta_{13}^e| \lesssim 10^{-3}$, the correction $|\Delta(\cos\delta)|$ to the exact sum rule result for $\cos\delta$ does not exceed 11% (4.9%) in the case of the TBM (GRB) form and is even smaller for the BM, GRA and HG forms of $\tilde{U}_v$. In what follows we concentrate on the case of negligibly small $\sin\theta_{13}^e \cong 0$.

The fact that the value of the Dirac CPV phase δ is determined (up to an ambiguity of the sign of $\sin\delta$) by the values of the three mixing angles θ_{12}, θ_{23} and θ_{13} of the PMNS matrix and the value of θ_{12}^v of the matrix $\tilde{U}_v$, Eq. (11), is the most striking prediction of the model considered. This result implies also that in the scheme under discussion, the rephasing invariant J_{CP} associated with the Dirac phase δ, which determines the magnitude of CP violation effects in neutrino oscillations [49] and in the standard parametrisation of the PMNS matrix has the well known form,

$$J_{\text{CP}} = \text{Im}\{U_{e1}^* U_{\mu3}^* U_{e3} U_{\mu1}\} = \frac{1}{8} \sin\delta \sin2\theta_{13} \sin2\theta_{23} \sin2\theta_{12} \cos\theta_{13}, \tag{37}$$

is also a function of the three angles θ_{12}, θ_{23} and θ_{13} of the PMNS matrix and of θ_{12}^v:

$$J_{\text{CP}} = J_{\text{CP}}\big(\theta_{12}, \theta_{23}, \theta_{13}, \delta(\theta_{12}, \theta_{23}, \theta_{13}, \theta_{12}^v)\big) = J_{\text{CP}}\big(\theta_{12}, \theta_{23}, \theta_{13}, \theta_{12}^v\big). \tag{38}$$

This allows us to obtain predictions for the range of possible values of J_{CP} in the cases of different symmetry forms of $\tilde{U}_v$, which are specified by the value of θ_{12}^v, using the current data on $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$. Using the best fit values of the neutrino mixing angles, we have for $\Delta m_{31}^2 > 0$ ($\Delta m_{31}^2 < 0$):

TBM: $J_{\text{CP}} \cong \mp 0.034$, (39)

BM (LC): $J_{\text{CP}} \cong \mp 0.008$ (∓ 0.003), (40)

GRA: $J_{\text{CP}} \cong \mp 0.0328$ (∓ 0.0332), (41)

GRB: $J_{\text{CP}} \cong \mp 0.0336$ (∓ 0.0341), (42)

HG: $J_{\text{CP}} \cong \mp 0.0306$ (∓ 0.0310), (43)

where the results in the TBM and BM cases were obtained in [1].[11] It follows from Eqs. (39)–(43) that, apart from the BM (LC) case, the $|J_{\text{CP}}|$ factor has rather similar values in the TBM, GRA,

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[11] The statistical analyses performed in [1] showed, in particular, that, given the indication for $\delta \cong 3\pi/2$ found in the global analyses of the current neutrino oscillation data, in the TBM case the value of $J_{\text{CP}} \cong +0.034$ is statistically disfavoured with respect to the value $J_{\text{CP}} \cong -0.034$.

GRB and HG mixing cases. As our results show, distinguishing between these cases requires a measurement of $\cos\delta$ or a very high precision measurement of $|J_{CP}|$.

4. The case of $|\sin\theta_{23}^e| \ll 1$

4.1. Negligible θ_{23}^e

The case of negligible $\theta_{23}^e \cong 0$ was analysed by many authors (see, e.g., [25–31,33] as well as [21]). It corresponds to a large number of theories and models of charged lepton and neutrino mass generation (see, e.g., [30,31,35,36,38,43]). In the limit of negligibly small θ_{23}^e we find from Eqs. (14), (16) and (17):

$$\sin^2\hat{\theta}_{23} = \frac{1}{2}, \qquad \gamma = -\psi + \pi, \qquad \phi = -\psi, \qquad \beta = \gamma - \phi = \pi. \tag{44}$$

The phase ω is unphysical. All results obtained in the previous section are valid also in the case of negligibly small θ_{23}^e: one has to set $\sin^2\hat{\theta}_{23} = 0.5$ in the expressions derived for arbitrary $\sin^2\hat{\theta}_{23}$ in the preceding Section. From Eqs. (19)–(21), using the fact that $\sin^2\hat{\theta}_{23} = 0.5$, we get the well known results for $\sin\theta_{13}$ and $\sin^2\theta_{23}$,

$$\sin\theta_{13} = \frac{1}{\sqrt{2}}\sin\theta_{12}^e, \qquad \sin^2\theta_{23} = \frac{1 - 2\sin^2\theta_{13}}{2(1 - \sin^2\theta_{13})} \cong \frac{1}{2}(1 - \sin^2\theta_{13}), \tag{45}$$

and the following new exact expression for $\sin^2\theta_{12}$:

$$\sin^2\theta_{12} = \sin^2\theta_{12}^\nu + \cos 2\theta_{12}^\nu \frac{\sin^2\theta_{13}}{1 - \sin^2\theta_{13}} + \sin 2\theta_{12}^\nu \sin\theta_{13}\cos\phi \frac{(1 - 2\sin^2\theta_{13})^{\frac{1}{2}}}{1 - \sin^2\theta_{13}}. \tag{46}$$

In the case of $\theta_{23}^e = 0$, as is well known, $\sin^2\theta_{23}$ can deviate only by $0.5\sin^2\theta_{13}$ from 0.5. Let us emphasise that the exact sum rules in Eqs. (22) and (30) correspond to $\sin\theta_{23}^e \neq 0$, including the case of a relatively small but non-negligible $\sin\theta_{23}^e$.

Equation (46) represents an exact sum rule connecting the value of the CPV phase ϕ with the values of the angles θ_{13} and θ_{12} for $\theta_{23}^e = 0$. From Eq. (46) we can get approximate sum rules taking into account that $\sin\theta_{13} \cong 0.15$:

$$\sin^2\theta_{12} = \sin^2\theta_{12}^\nu + \sin 2\theta_{12}^\nu \cos\phi \sin\theta_{13} + \cos 2\theta_{12}^\nu \sin^2\theta_{13} + O(\sin^4\theta_{13}), \tag{47}$$

$$\sin^2\theta_{12} = \sin^2\theta_{12}^\nu + \sin 2\theta_{12}^\nu \cos\phi \sin\theta_{13} + O(\sin^2\theta_{13}). \tag{48}$$

We have given the sum rules up to corrections of order $O(\sin^4\theta_{13})$ and of order $O(\sin^2\theta_{13})$ because both will serve our further discussion. By adding and subtracting the negligible (within the approximation used) term $(\cos\theta_{12}^\nu \cos\phi \sin\theta_{13})^2$ to the r.h.s. of Eq. (48), and by using $\sin\theta_{13} \cong \theta_{13}$, we get $\sin^2\theta_{12} \cong \sin^2(\theta_{12}^\nu + \theta_{13}\cos\phi)$, which leads to

$$\theta_{12} \cong \theta_{12}^\nu + \theta_{13}\cos\phi + O(\theta_{13}^2). \tag{49}$$

In what concerns the phase δ, in the limit of negligible θ_{23}^e we find from Eqs. (30) and (29) the following exact expressions for $\cos\delta$ and the relation between $\cos\delta$ and $\cos\phi$:

$$\cos\delta = \frac{(1 - 2\sin^2\theta_{13})^{\frac{1}{2}}}{\sin 2\theta_{12}\sin\theta_{13}}\left[\cos 2\theta_{12}^\nu + (\sin^2\theta_{12} - \cos^2\theta_{12}^\nu)\frac{1 - 3\sin^2\theta_{13}}{1 - 2\sin^2\theta_{13}}\right], \tag{50}$$

$$\cos\delta = \frac{\sin 2\theta_{12}^\nu}{\sin 2\theta_{12}}\cos\phi\frac{1 - 3\sin^2\theta_{13}}{1 - \sin^2\theta_{13}} + 2\frac{\cos 2\theta_{12}^\nu}{\sin 2\theta_{12}}\frac{(1 - 2\sin^2\theta_{13})^{\frac{1}{2}}}{1 - \sin^2\theta_{13}}\sin\theta_{13}. \tag{51}$$

Eq. (50) can also be cast in the form of Eqs. (46), (47) and (48):

$$\sin^2\theta_{12} = \sin^2\theta_{12}^\nu + \frac{(1 - 2\sin^2\theta_{13})^{\frac{1}{2}}}{1 - 3\sin^2\theta_{13}}\sin 2\theta_{12}\sin\theta_{13}\cos\delta - \frac{\sin^2\theta_{13}}{1 - 3\sin^2\theta_{13}}\cos 2\theta_{12}^\nu \tag{52}$$

$$= \sin^2\theta_{12}^\nu + (1 + 2\sin^2\theta_{13})\sin 2\theta_{12}\cos\delta\sin\theta_{13} - \cos 2\theta_{12}^\nu\sin^2\theta_{13} + O(\sin^4\theta_{13}) \tag{53}$$

$$= \sin^2\theta_{12}^\nu + \sin 2\theta_{12}\cos\delta\sin\theta_{13} + O(\sin^2\theta_{13}). \tag{54}$$

We note that Eqs. (50)–(52) are exact. We have given the approximate sum rules involving $\cos\delta$ up to corrections of order $O(\sin^4\theta_{13})$ and of order $O(\sin^2\theta_{13})$ in Eqs. (53) and (54) because both sum rules will be used in the analysis which follows.

It is not difficult to show, using the same steps which allowed us to get Eq. (49) from Eq. (48) that, to leading order in θ_{13}, the sum rule in Eq. (54) leads to

$$\theta_{12} \cong \theta_{12}^\nu + \theta_{13}\cos\delta + O(\theta_{13}^2). \tag{55}$$

This implies that, to leading order in $\sin\theta_{13}$, the sum rule in Eq. (48) is equivalent to the sum rule in Eq. (54), and thus, to leading order in $\sin\theta_{13}$, we have $\cos\phi \cong \cos\delta$.[12] The different expressions in Eqs. (48) and (54) lead to the same leading order sum rules (49) and (55) as a consequence of the fact that the neglected corrections in the two cases differ.

The approximate sum rules given in Eqs. (48), (49) and (55) and similar relations, were considered or found in specific models, for different fixed symmetry forms of $\tilde{U}_\nu$ (BM, TBM, etc.), e.g., in [25–27,29–31,33–35].[13] For arbitrary fixed value of θ_{12}^ν the sum rule in Eq. (49) was proposed in [33], where the approximate relation $\cos\phi \cong \cos\delta$, which holds to leading order in $\sin\theta_{13}$, was implicitly used (see further). The approximation $|\cos\phi| \cong |\cos\delta|$ is employed also, e.g., in Refs. [29,30,35]. It was suggested in Ref. [34] that the sum rule (55) should be used to obtain the value of $\cos\delta$ using the experimentally determined values of $\sin^2\theta_{12}$ and $\sin\theta_{13}$, e.g., in the case of the TBM form of $\tilde{U}_\nu$. The same sum rule (55) is given also, e.g., in the review articles [43,50].

The derivation of the sum rule of interest, given in Ref. [33], is based on the following expression for $\sin^2\theta_{12}$:

$$\sin^2\theta_{12} \cong \left|\sin\theta_{12}^\nu + \cos\theta_{12}^\nu\sin\theta_{13}e^{i(\delta_{12}^\nu - \delta_{12}^e + \pi)}\right|^2 \tag{56}$$

$$\cong \sin^2\theta_{12}^\nu + \sin 2\theta_{12}^\nu\sin\theta_{13}\cos(\delta_{12}^\nu - \delta_{12}^e + \pi), \tag{57}$$

where[14] δ_{12}^ν and δ_{12}^e are two of the phases introduced in [33]. The expression for $\sin^2\theta_{12}$ in Eq. (57) is obtained in [33] by keeping the leading order corrections in $\sin\theta_{13}$ and $\sin\theta_{23}^e \neq 0$, $\sin\theta_{23}^e \ll 1$, and neglecting terms of order of, or smaller than, $\sin^2\theta_{13}$, $\sin^2\theta_{23}^e$ and $\sin\theta_{13}\sin\theta_{23}^e$.

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[12] The change $\sin\theta_{12}^e \to -\sin\theta_{12}^e$ in Eq. (9) would lead to the relation $\cos\phi \cong -\cos\delta$, which appears in a number of articles (see, e.g., [33,30,35]).

[13] In the BM case, for instance, Eq. (48) can be obtained i) from Eq. (32) in [26] by setting the parameters $A = B = 0$, ii) from Eqs. (31)–(32) in the first article quoted in [31] by setting the parameter $s_{13}^e = 0$.

[14] Expression (57) follows from Eqs. (15c) and (18) in Ref. [33] after, following Ref. [33], one neglects the term $\propto \theta_{13}^e$ in Eq. (15c) and uses $c_{23}^\nu = s_{23}^\nu$.

The presence of $\sin 2\theta_{12}^\nu$ (rather than $\sin 2\theta_{12}$) in Eq. (57) suggests a similarity between this equation and Eq. (48) in which the phase ϕ is present. It can be shown that the following exact relation holds between the phase ψ, defined in Eq. (7), and the phases δ_{12}^ν and δ_{12}^e, introduced[15] in [33]:

$$\psi = -\left(\delta_{12}^\nu - \delta_{12}^e + \pi\right). \tag{58}$$

Further, it follows from Eq. (17) that the phases ψ and ϕ are related in the following way:

$$\psi = -(\phi - \tilde{\phi}), \tag{59}$$

where

$$\sin\tilde{\phi} = \frac{\sin\theta_{23}^e \sin(\psi - \omega)}{\sqrt{1 + \sin 2\theta_{23}^e \cos(\psi - \omega)}}. \tag{60}$$

We note that in the absence of 1–3 rotations in $\tilde{U}_e$ and $\tilde{U}_\nu$, the relations (58)–(60) are exact. It follows from Eqs. (58)–(60) that for the phase $(\delta_{12}^\nu - \delta_{12}^e + \pi)$ in Eq. (57) we get:

$$\delta_{12}^\nu - \delta_{12}^e + \pi = \phi - \tilde{\phi}. \tag{61}$$

This implies that, in the approximation employed in Ref. [33] in which terms of order $\sin\theta_{13}\sin\theta_{23}^e$ are neglected, the contribution of $\tilde{\phi}$ in Eq. (57) should also be neglected and we get:

$$\sin^2\theta_{12} \cong \sin^2\theta_{12}^\nu + \sin 2\theta_{12}^\nu \sin\theta_{13}\cos\phi, \tag{62}$$

which coincides with Eq. (48) in which the phase ϕ, rather than the phase δ, is present.

In the case of $\theta_{23}^e = 0$ we get from Eqs. (60) and (61) the exact relation:

$$\delta_{12}^\nu - \delta_{12}^e + \pi = \phi. \tag{63}$$

We find the same relation comparing the expressions for the rephasing invariant J_{cP}, Eq. (37), in the standard parametrisation of the PMNS matrix and in the parametrisation employed in Ref. [33]. This allows us to obtain a relation between the phase δ and the phase $(\delta_{12}^\nu - \delta_{12}^e)$, which in turn, via Eq. (28), leads to a relation between $(\delta_{12}^\nu - \delta_{12}^e)$ and ϕ. Indeed, taking into account that in the case of $\theta_{23}^e = 0$, $\sin^2\theta_{23}$ is given in Eq. (45), and that in the parametrisation used in [33] one has $\theta_{23}^\nu = \pi/4$, $\sin\theta_{13} = \sin\theta_{12}^e/\sqrt{2}$, we get equating the two expressions of interest for the J_{cP} factor:

$$\sin\delta = -\frac{\sin 2\theta_{12}^\nu}{\sin 2\theta_{12}}\sin\left(\delta_{12}^\nu - \delta_{12}^e + \pi\right). \tag{64}$$

This result is exact. Comparing the above equation with Eq. (28) we can conclude that

$$\sin\left(\delta_{12}^\nu - \delta_{12}^e + \pi\right) = \sin\phi, \tag{65}$$

which leads to Eq. (63)

As we have already noted, in the derivation of the sum rule under discussion proposed in [33], terms of order $\sin^2\theta_{13}$, $\sin^2\theta_{23}^e$ and $\sin\theta_{13}\sin\theta_{23}^e$ and higher order corrections are neglected. In the next subsection we will consider the corrections due to $\sin\theta_{23}^e \neq 0$. Here we would like

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[15] The relations between the phases $(\delta_{12}^\nu - \delta_{12}^e)$ and ψ or ϕ we are going to derive are valid, obviously, modulo 2π.

to note that for the TBM, GRA, GRB and HG forms of the matrix $\tilde{U}_\nu$ of interest, we have $|\sin^2\theta_{12} - \sin^2\theta_{12}^\nu| \sim \sin^2\theta_{13}$. Indeed, for the best fit value of $\sin^2\theta_{12} = 0.308$, this difference in the TBM, GRA, GRB and HG cases reads, respectively: 0.032; 0.025; 0.037; 0.058. Therefore in all four cases under discussion we have $\sin^2\theta_{12} = \sin^2\theta_{12}^\nu + a\sin^2\theta_{13}$, with $|a| \cong (1.1 - 2.5)$. The last relation implies:

$$\theta_{12} = \theta_{12}^\nu + \frac{a\,\theta_{13}^2}{\sin 2\theta_{12}^\nu} + O\left(a^2\theta_{13}^4\right), \quad 1.1 \lesssim |a| \lesssim 2.5, \tag{66}$$

where $\sin 2\theta_{12}^\nu \cong 0.94$, 0.89, 0.95 and 0.87 for the TBM, GRA, GRB and HG forms of $\tilde{U}_\nu$, respectively. Thus, if one should be consistent, working in the leading order approximation in $\sin\theta_{13}$, i.e., neglecting terms $\sim \sin^n\theta_{13}$ for $n \geq 2$, for the TBM, GRA, GRB and HG forms of $\tilde{U}_\nu$, one should also neglect the difference between $\sin^2\theta_{12}$ and $\sin^2\theta_{12}^\nu$ in Eqs. (48) and (54), or equivalently, the difference between θ_{12} and θ_{12}^ν in Eqs. (49) and (55). In this case we get $\cos\phi = \cos\delta$, but also $\cos\phi = 0$ and $\cos\delta = 0$, for the indicated symmetry forms of $\tilde{U}_\nu$. If the sum rules are derived in the TBM, GRA, GRB and HG cases taking into account the difference $|\theta_{12} - \theta_{12}^\nu| \sim \theta_{13}^2 \neq 0$, (or $|\sin^2\theta_{12} - \sin^2\theta_{12}^\nu| \sim \sin^2\theta_{13} \neq 0$), a consistent application of the approximations used requires in these cases to take also terms of order $\sin^2\theta_{13}$ into account, i.e., to use the sum rules given in Eqs. (47) and (53), rather than the sum rules (48) and (54) (or (49) and (55)). We will use quotation marks in the term "leading order sum rules" to denote the inconsistency of the approximations used to derive the sum rules in Eqs. (48) and (54), and, correspondingly, in Eqs. (49) and (55), in the TBM, GRA, GRB and HG cases. We will return to the problem of correct implementation of the approximations employed to derive the sum rules in Eqs. (49) and (55) in the next subsection, where we will analyse in detail the corrections due to $\sin\theta_{23}^e \neq 0$.

The above considerations do not apply to the case of BM (LC) form of the matrix $\tilde{U}_\nu$ since in this case we have $|\sin^2\theta_{12} - \sin^2\theta_{12}^\nu| \sim \sin\theta_{13}$. Thus, for the BM (LC) form of $\tilde{U}_\nu$, the leading order approximation in $\sin\theta_{13}$ is consistent with taking into account the difference between θ_{12} and θ_{12}^ν in the sum rules given in Eqs. (49) and (55), and in Eqs. (48) and (54).

We will show next that the sum rules in Eqs. (48) and (54), and the equivalent "leading order sum rules" in Eqs. (49) and (55), give imprecise, and in some cases – largely incorrect, results for both $\cos\phi$ and $\cos\delta$ in the cases of TBM, GRA, GRB and HG forms of $\tilde{U}_\nu$.

Indeed, using the "leading order sum rules" in Eqs. (48) and (54), we get for the best fit values of $\sin^2\theta_{12} = 0.308$ and $\sin^2\theta_{13} = 0.0234$ in the TBM, GRA, GRB and HG cases[16]:

TBM, Eqs. (54) and (48):	$\cos\delta = -0.179$;	$\cos\phi \cong -0.176$;	(67)
GRA, Eqs. (54) and (48):	$\cos\delta \cong 0.227$;	$\cos\phi \cong 0.234$;	(68)
GRB, Eqs. (54) and (48):	$\cos\delta \cong -0.262$;	$\cos\phi \cong -0.254$;	(69)
HG, Eqs. (54) and (48):	$\cos\delta \cong 0.411$;	$\cos\phi \cong 0.438$.	(70)

Clearly, in all these cases we have $\cos\delta \cong \cos\phi$. The slight differences in the values of $\cos\delta$ and $\cos\phi$ are caused by the differences between the factors $\sin 2\theta_{12}^\nu$ and $\sin 2\theta_{12}$ in Eqs. (48) and (54). In the approximation in which Eqs. (49) and (55) are derived, these differences should be neglected and we would have $\sin 2\theta_{12}^\nu = \sin 2\theta_{12}$. But in this case, as we have already have noticed, we would have also $\theta_{12}^\nu = \theta_{12}$, and thus $\cos\delta = \cos\phi = 0$.

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[16] Practically the same results are obtained employing the equivalent "leading order sum rules" in Eqs. (49) and (55).

Using the exact sum rules for $\cos\phi$ and $\cos\delta$, given in Eqs. (50) and (46), we find:

TBM exact: $\cos\delta = -0.114;$ $\cos\phi \cong -0.230;$ (71)

GRA exact: $\cos\delta \cong 0.289;$ $\cos\phi \cong 0.153;$ (72)

GRB exact: $\cos\delta \cong -0.200;$ $\cos\phi \cong -0.307;$ (73)

HG exact: $\cos\delta \cong 0.476;$ $\cos\phi \cong 0.347.$ (74)

As we see comparing Eqs. (67)–(70) with Eqs. (71)–(74), the values of $\cos\delta$, obtained using the exact sum rule (50) in the TBM, GRA, GRB and HG cases differ from those calculated using the "leading order sum rule" (54), by the factors 1.57, 0.78, 1.31 and 0.86, respectively. In the case of $\cos\phi$, the corresponding factors are 0.76, 1.53, 0.83 and 1.26. The higher order corrections have opposite effect on the leading order results for $|\cos\delta|$ and $|\cos\phi|$: if the exact sum rule value of $|\cos\delta|$ is smaller (larger) than the "leading order sum rule" value, as in the TBM and GRB (GRA and HG) cases, the corresponding exact sum rule value of $|\cos\phi|$ is larger (smaller) than the "leading order sum rule" value. We see also from Eqs. (71)–(74) that the values of $\cos\delta$ and $\cos\phi$, derived from the exact sum rules in the cases of TBM, GRA, GRB and HG forms of the matrix $\tilde{U}_\nu$ indeed differ approximately by factors (1.5–2.0). As we have seen, for finite values of θ_{23}^e, for which we have $\sin^2\theta_{23} \cong (0.43$–$0.44)$, $\cos\phi$ and $\cos\delta$ in all cases we are considering with the exception of the BM (LC) one, differ approximately by the same factor of (1.5–2.0).

The origin of these significant differences between the results derived using the exact and the "leading order sum rules" for $\cos\delta$ and $\cos\phi$ for the TBM, GRA, GRB and HG forms of the matrix $\tilde{U}_\nu$ can be traced to the importance of the next-to-leading order corrections $\propto \sin^2\theta_{13}$ in the "leading order sum rules" for[17] $\cos\delta$ and $\cos\phi$. For arbitrary fixed θ_{12}^ν these corrections are given in Eqs. (47) and (53). In the specific cases of TBM GRA, GRB and HG forms of $\tilde{U}_\nu$, up to corrections $O(\sin^4\theta_{13})$ the sum rules for $\cos\delta$ read:

TBM: $\sin^2\theta_{12} \cong \dfrac{1}{3}(1 - \sin^2\theta_{13}) + (1 + 2\sin^2\theta_{13})\sin 2\theta_{12}\sin\theta_{13}\cos\delta,$ (75)

GRA: $\sin^2\theta_{12} \cong 0.276(1 + 2\sin^2\theta_{13}) - \sin^2\theta_{13}$
$$+ (1 + 2\sin^2\theta_{13})\sin 2\theta_{12}\sin\theta_{13}\cos\delta,$$ (76)

GRB: $\sin^2\theta_{12} \cong 0.345(1 + 2\sin^2\theta_{13}) - \sin^2\theta_{13}$
$$+ (1 + 2\sin^2\theta_{13})\sin 2\theta_{12}\sin\theta_{13}\cos\delta,$$ (77)

HG: $\sin^2\theta_{12} \cong \dfrac{1}{4}(1 - 2\sin^2\theta_{13}) + (1 + 2\sin^2\theta_{13})\sin 2\theta_{12}\sin\theta_{13}\cos\delta,$ (78)

where we have used $\sin^2\theta_{12}^\nu = (2 + r)^{-1} \cong 0.276$, $\sin^2\theta_{12}^\nu = (3 - r)/4 \cong 0.345$ and $\sin^2\theta_{12}^\nu = 1/4$ in the GRA, GRB and HG cases, respectively (we recall that $r = (1 + \sqrt{5})/2$ is the golden ratio). Similarly, for the sum rules involving the phase ϕ we find:

TBM: $\sin^2\theta_{12} = \dfrac{1}{3}(1 + \sin^2\theta_{13}) + \dfrac{2\sqrt{2}}{3}\sin\theta_{13}\cos\phi + O(\sin^4\theta_{13}),$ (79)

GRA: $\sin^2\theta_{12} \cong 0.276\cos 2\theta_{13} + \sin^2\theta_{13} + 0.894\sin\theta_{13}\cos\phi + O(\sin^4\theta_{13}),$ (80)

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[17] Note that since in the sum rules of interest $\cos\delta$ and $\cos\phi$ are always multiplied by $\sin\theta_{13}$, the corrections $\sim\sin^2\theta_{13}$ in the sum rules lead effectively to corrections $\sim\sin\theta_{13} \cong 0.16$ in the values of $\cos\delta$ and $\cos\phi$.

GRB: $\sin^2 \theta_{12} = 0.345 \cos 2\theta_{13} + \sin^2 \theta_{13} + 0.951 \sin \theta_{13} \cos \phi + O(\sin^4 \theta_{13})$,   (81)

HG: $\sin^2 \theta_{12} = \dfrac{1}{4}(1 + 2 \sin^2 \theta_{13}) + \dfrac{\sqrt{3}}{2} \sin \theta_{13} \cos \phi + O(\sin^4 \theta_{13})$.   (82)

As can be easily checked, the approximate sum rules given in Eqs. (75)–(77),(82), lead to results for $\cos \delta$ and $\cos \phi$, which practically coincide with those quoted in Eqs. (71)–(74) and obtained using the exact sum rules given in Eqs. (50) and (46). It follows from Eqs. (47) and (53) that the important corrections $\propto \sin^2 \theta_{13}$ to the "leading order sum rules" Eqs. (48) and (54), are given respectively by $(+\cos 2\theta_{12}^{\nu} \sin^2 \theta_{13})$ and by $(-\cos 2\theta_{12}^{\nu} \sin^2 \theta_{13})$, i.e., they coincide in absolute value but have opposite signs. This explains the effect of these corrections on the values of $|\cos \delta|$ and $|\cos \phi|$ derived from the "leading order sum rules" (48) and (54): given the value of $|\cos 2\theta_{12}^{\nu} \sin^2 \theta_{13}|$, the corrections make maximal the difference between $|\cos \delta|$ and $|\cos \phi|$. The fact that the correction $\propto \sin^2 \theta_{13}$ of interest is given by the term $\pm \cos 2\theta_{12}^{\nu} \sin^2 \theta_{13}$ explains also why the results for $\cos \delta$ and $\cos \phi$ obtained using the exact sum rules (50) and (46) and leading order sum rules (48) and (54) do not differ significantly for BM (LC) form of the matrix $\tilde{U}_{\nu}$: in the BM (LC) case these correction is zero since $\cos 2\theta_{12}^{\nu} = 0$. Thus, the corrections to the leading order sum rule are $O(\sin^3 \theta)$ and $O(\sin^4 \theta)$ and have minor effect on the determination of $\cos \delta$ and $\cos \phi$ in the BM (LC) case.

We would like to emphasise once again that the corrections $\propto \sin^2 \theta_{13}$ to the "leading order sum rules" for $\cos \delta$ and $\cos \phi$ (55) and (49), as well as, (54) and (48), are significant and have to be taken into account when the difference $|\sin^2 \theta_{12} - \sin^2 \theta_{12}^{\nu}| \sim \sin^2 \theta_{13}$, and thus is of the order of the correction. For the current best fit value of $\sin^2 \theta_{12} = 0.308$ this is the case of the TBM, GRA, GRB and HG forms of the matrix $\tilde{U}_{\nu}$ considered in the present article.

4.2. The corrections generated by non-negligible $\sin \theta_{23}^{e} \ll 1$

The sum rule (56), which leads to the "leading order sum rules" (49) and (55) of interest, was derived in [33] assuming that $\theta_{12}^{e} \neq 0$, $\theta_{23}^{e} \neq 0$ and $|\sin \theta_{23}^{e}| \ll 1$, and keeping terms $\sim \sin \theta_{13}$ and $\sim \sin \theta_{23}^{e}$ in the relation between $\sin \theta_{12}$, $\sin \theta_{12}^{\nu}$ and $\cos \delta$. The corrections of the order of, or smaller than, $\sin^2 \theta_{13}$, $\sin^2 \theta_{23}^{e}$ and $\sin \theta_{13} \sin \theta_{23}^{e}$ were neglected. The exact sum rules for $\cos \phi$ and $\cos \delta$ given in Eqs. (22) and (30), were derived for any $\theta_{12}^{e} \neq 0$, $\sin \theta_{13}$ and $\sin \theta_{23}^{e}$. Thus, the sum rule (55) is an approximate version of the exact sum rule (30): Eq. (55) can be obtained from Eq. (30) in the leading order approximation by treating not only $\sin \theta_{13}$, but also $\sin \theta_{23}^{e}$ as a small parameter. In this subsection, from the exact sum rules (22) and (30), we will derive the corrections due to both $\sin \theta_{13}$ and $\sin \theta_{23}^{e} \neq 0$ in the "leading order sum rules" in Eqs. (48) and (54), and in Eqs. (49) and (55).

It follows from Eq. (14) that

$$\sin 2\theta_{23}^{e} \cos(\omega - \psi) \equiv X = 1 - 2 \sin^2 \hat{\theta}_{23} \cong 0.124,$$ (83)

The relation between $\sin^2 \theta_{23}$ and $\sin^2 \hat{\theta}_{23}$ is given in Eq. (20). The numerical value quoted in Eq. (83) is for $\sin^2 \hat{\theta}_{23} \cong 0.438$, which corresponds to $\sin^2 \theta_{23} = 0.425$ and $\sin^2 \theta_{13} = 0.0234$.

Equation (83) implies that $|\sin 2\theta_{23}^{e}| \gtrsim 0.124$. Following the analysis performed in [33], we will assume that $0 < \sin \theta_{23}^{e} \ll 1$, and thus $X \ll 1$. From the exact sum rules for $\cos \phi$ and $\cos \delta$ given in Eqs. (22) and (30), we will derive approximate sum rules for the two CPV phases, in which, in contrast to the approximation employed in Ref. [33] leading to Eq. (55), the next-to-leading order corrections $\sim \sin^2 \theta_{13}$, $\sim \sin^2 \theta_{23}^{e}$ and $\sim \sin \theta_{13} \sin \theta_{23}^{e}$ are included. This means

that, in addition to keeping terms $\sim \sin\theta_{13}$ and $\sim \sin^2\theta_{13}$ in the sum rules, we will keep also terms $\sim X$, $\sim X^2$ and $\sim X\sin\theta_{13}$. It is not difficult to show that in this next-to-leading order approximation we get from Eqs. (30) and (22):

$$\sin^2\theta_{12} = \sin^2\theta_{12}^{\nu} + (1+X)\sin 2\theta_{12}\sin\theta_{13}\cos\delta - \cos 2\theta_{12}^{\nu}\sin^2\theta_{13}$$
$$+ O\left(X\sin^2\theta_{13}, X^2\sin\theta_{13}, \sin^3\theta_{13}, X^3\right), \tag{84}$$

$$\sin^2\theta_{12} = \sin^2\theta_{12}^{\nu} + (1+X)\sin 2\theta_{12}^{\nu}\cos\phi\sin\theta_{13} + \cos 2\theta_{12}^{\nu}\sin^2\theta_{13}$$
$$+ O\left(X\sin^2\theta_{13}, X^2\sin\theta_{13}, \sin^3\theta_{13}, X^3\right). \tag{85}$$

Comparing Eqs. (84) and (85) respectively with Eqs. (53) and (47), we see that the next-to-leading order correction due to $X \sim \sin\theta_{23}^e \neq 0$ amounts formally to multiplying the terms $\propto \sin\theta_{13}\cos\delta$ and $\propto \sin\theta_{13}\cos\phi$ by the factor $(1+X)$. The "leading order sum rules" in Eqs. (48) and (54), and in Eqs. (49) and (55), do not depend on $\sin\theta_{23}^e$ because in the sum rules (84) and (85) there are no terms of the order of $\sin\theta_{23}^e$: the small parameter $\sin\theta_{23}^e$ appears only in the next-to-leading order correction $\sim \sin\theta_{23}^e\sin\theta_{13}$.

It follows from Eqs. (83) and (20) that we have: $1+X = 2\cos^2\hat{\theta}_{23} = 2\cos^2\theta_{23}(1-\sin^2\theta_{13})$. Thus, in the approximation of interest the sum rules for $\cos\delta$ and $\cos\phi$ take the form:

$$\sin^2\theta_{12} = \sin^2\theta_{12}^{\nu} + 2\cos^2\theta_{23}\sin 2\theta_{12}\sin\theta_{13}\cos\delta - \cos 2\theta_{12}^{\nu}\sin^2\theta_{13}$$
$$+ O\left(X\sin^2\theta_{13}, X^2\sin\theta_{13}, \sin^3\theta_{13}, X^3\right), \tag{86}$$

$$\sin^2\theta_{12} = \sin^2\theta_{12}^{\nu} + 2\cos^2\theta_{23}\sin 2\theta_{12}^{\nu}\cos\phi\sin\theta_{13} + \cos 2\theta_{12}^{\nu}\sin^2\theta_{13}$$
$$+ O\left(X\sin^2\theta_{13}, X^2\sin\theta_{13}, \sin^3\theta_{13}, X^3\right). \tag{87}$$

For $\theta_{23}^e = 0$, we have $2\cos^2\theta_{23} = (1-\sin^2\theta_{13})^{-1}$, and, within the approximation employed, Eqs. (86) and (87) reduce to Eqs. (53) and (47). In the case of non-negligible θ_{23}^e, however, $\sin^2\theta_{23}$ can deviate sizably from 0.5. In this case, as it follows from Eqs. (30), (22), (86) and (87), the exact and the approximate (next-to-leading order) sum rules for $\cos\delta$ and $\cos\phi$ depend not only on θ_{12} and θ_{13}, but also on θ_{23}. If, for instance, $\cos^2\theta_{23} = 0.6\ (0.4)$, the effect of the factor $2\cos^2\theta_{23}$, e.g., in the approximate sum rules (86) and (87) is to decrease (increase) the values of $\cos\delta$ and $\cos\phi$, evaluated without taking into account the correction due to $\theta_{23}^e \neq 0$, by a factor of 1.2 (1.25). This dependence, as well as the variation of the predictions for $\cos\delta$ and $\cos\phi$ with the variation of the values of $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ in their experimentally allowed ranges, will be investigated elsewhere [51].

5. The Majorana phases

We will analyse next the possibility to obtain predictions for the values of Majorana phases α_{21} and α_{31} in the PMNS matrix using the approach described above, We will show in what follows that in many cases of interest it is possible to determine the phases α_{21} and α_{31} if the values of the phase ϕ, or δ, and of the phases ξ_{21} and ξ_{31} in the diagonal matrix Q_0 in Eq. (6) are known. The matrix $\tilde{U}_{\nu}Q_0$, as we have already briefly discussed, originates from the diagonalisation of the flavour neutrino Majorana mass term. In many theories and models of neutrino mixing the values of the phases ξ_{21} and ξ_{31} are fixed by the form of flavour neutrino Majorana mass term, which is dictated by the chosen discrete (or continuous) flavour symmetry (see, e.g., [36,19]), or on phenomenological grounds (see, e.g., [37]). Typical values of the phases ξ_{21} and ξ_{31} are 0,

$\pi/2$ and π. In the model with T' flavour symmetry in the lepton sector constructed in [19], for instance, ξ_{21} and ξ_{31} can take two sets of values: $(\xi_{21}, \xi_{31}) = (0, 0)$ and $(0, \pi)$.

In what follows we will assume that the phases ξ_{21} and ξ_{31} are known. Under this condition the Majorana phases α_{21} and α_{31} can be determined, as we will discuss in greater detail below, i) if the angles θ^e_{12} and θ^e_{23}, or the angle θ^e_{12} and the phase $(\psi - \omega)$, are known, or ii) if the angle θ^e_{12} is known and the phase ψ or ω takes one of the specific values 0, $\pi/2$, π and $3\pi/2$. In processes like the $(\beta\beta)_{0\nu}$-decay, which are characteristic of the Majorana nature of the light massive neutrinos ν_j, the phase α_{31} can play under certain conditions a subdominant role (see further), while the rate of the processes depends strongly on the phase α_{21}. As we will see, the phase α_{21} can be determined (given the phase ξ_{21}) knowing only the values of the phase ϕ (or δ) and of the angle θ^e_{12}.

The PMNS matrix we obtain from Eq. (18) in the scheme we consider has the form:

$$U_{\text{PMNS}} = \begin{pmatrix} c^e_{12}c^\nu_{12} - s^e_{12}\hat{c}_{23}s^\nu_{12}e^{i\phi} & c^e_{12}s^\nu_{12} + s^e_{12}\hat{c}_{23}c^\nu_{12}e^{i\phi} & s^e_{12}\hat{s}_{23}e^{i\phi} \\ -s^e_{12}c^\nu_{12} - c^e_{12}\hat{c}_{23}s^\nu_{12}e^{i\phi} & -s^e_{12}s^\nu_{12} + c^e_{12}\hat{c}_{23}c^\nu_{12}e^{i\phi} & c^e_{12}\hat{s}_{23}e^{i\phi} \\ \hat{s}_{23}s^\nu_{12} & -\hat{s}_{23}c^\nu_{12} & \hat{c}_{23} \end{pmatrix} Q_1 Q_0, \quad (88)$$

where we have used the standard notations $c^e_{12} \equiv \cos\theta^e_{12}$, $c^\nu_{12} \equiv \cos\theta^\nu_{12}$, $\hat{c}_{23} \equiv \cos\hat{\theta}_{23}$, etc. Obviously, the matrix (88) does not have the form of the standard parametrisation of the PMNS matrix. As we will show below, bringing the matrix (88) to the standard parametrisation form leads to contributions to the Majorana phases α_{21} and α_{31}, which are associated with the phase ϕ. Thus, the phase ϕ not only generates the Dirac phase δ, but also contributes to the values of the Majorana phases α_{21} and α_{31}.

The first thing to notice is that using Eqs. (19)–(21) it can be shown that the absolute values of the elements of the matrix given in Eq. (88) coincide with the absolute values of the elements of the PMNS matrix in the standard parametrisation, defined in Eqs. (1)–(2): $|c^e_{12}c^\nu_{12} - s^e_{12}\hat{c}_{23}s^\nu_{12}e^{i\phi}| = c_{12}c_{13} = |U_{e1}|$, $|c^e_{12}s^\nu_{12} + s^e_{12}\hat{c}_{23}c^\nu_{12}e^{i\phi}| = s_{12}c_{13} = |U_{e2}|$, $s^e_{12}\hat{s}_{23} = s_{13} = |U_{e3}|$, $c^e_{12}\hat{s}_{23} = s_{23}c_{13} = |U_{\mu3}|$, $\hat{c}_{23} = c_{23}c_{13} = |U_{\tau3}|$, etc. It is more difficult technically to demonstrate that for the elements $U_{\mu1}$, $U_{\mu2}$, $U_{\tau1}$, $U_{\tau2}$, but it can be easily checked numerically using, e.g., the best fit values of the angles $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ to determine numerically θ^e_{12}, $\hat{\theta}_{23}$ and ϕ, and correspondingly, δ, for each given value of θ^ν_{12}, and then using these "data" to calculate the absolute values of the indicated elements of the PMNS matrices given in Eqs. (1)–(2) and in Eq. (88). As a consequence, the PMNS matrix in Eq. (88) can be written as

$$U_{\text{PMNS}} = \begin{pmatrix} |U_{e1}|e^{i\beta_{e1}} & |U_{e2}|e^{i\beta_{e2}} & |U_{e3}|e^{i\phi} \\ |U_{\mu1}|e^{i\beta_{\mu1}} & |U_{\mu2}|e^{i\beta_{\mu2}} & |U_{\mu3}|e^{i\phi} \\ |U_{\tau1}| & |U_{\tau2}|e^{-i\pi} & |U_{\tau3}| \end{pmatrix} Q_1 Q_0, \quad (89)$$

where

$$\beta_{e1} = \arg(c^e_{12}c^\nu_{12} - s^e_{12}\hat{c}_{23}s^\nu_{12}e^{i\phi}), \quad (90)$$

$$\beta_{e2} = \arg(c^e_{12}s^\nu_{12} + s^e_{12}\hat{c}_{23}c^\nu_{12}e^{i\phi}), \quad (91)$$

$$\beta_{\mu1} = \arg(-s^e_{12}c^\nu_{12} - c^e_{12}\hat{c}_{23}s^\nu_{12}e^{i\phi}), \quad (92)$$

$$\beta_{\mu2} = \arg(-s^e_{12}s^\nu_{12} + c^e_{12}\hat{c}_{23}c^\nu_{12}e^{i\phi}). \quad (93)$$

The phases β_{e1}, β_{e2}, $\beta_{\mu1}$ and $\beta_{\mu2}$ can be calculated for any of the specific values of θ^ν_{12} of interest since, for a given θ^ν_{12}, the angles θ^e_{12}, $\hat{\theta}_{23}$ and the phase ϕ can be determined from the values of

the neutrino mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$. One has to remember that although $\cos\phi$ is uniquely determined, the sign of $\sin\phi$ cannot be determined using the current data. Thus, two values of ϕ, and correspondingly of the phases β_{e1}, β_{e2}, $\beta_{\mu1}$, $\beta_{\mu2}$ and δ, are compatible with the data and have to be considered.

As we know from the analysis in Section 3, the phase ϕ does not coincide with the Dirac phase δ. It is not difficult to convince oneself that we have:

$$\delta = -\phi + \beta_{e1} + \beta_{e2}. \tag{94}$$

Using Eqs. (19)–(21), (90) and (91), it is rather straightforward to demonstrate, for instance, that $\sin(-\phi + \beta_{e1} + \beta_{e2}) = -\sin\phi \sin 2\theta_{12}^\nu / \sin 2\theta_{12} = \sin\delta$, where the last equality follows from Eq. (28). The result given in Eq. (94) indicates what rearrangement of the phases in the PMNS matrix in Eq. (89) we have to perform in order to bring it to the standard parametrisation form:

$$U_{\text{PMNS}} = P_2 \begin{pmatrix} |U_{e1}| & |U_{e2}| & |U_{e3}|e^{-i(-\phi+\beta_{e1}+\beta_{e2})} \\ |U_{\mu1}|e^{i(\beta_{\mu1}+\beta_{e2}-\phi)} & |U_{\mu2}|e^{i(\beta_{\mu2}+\beta_{e1}-\phi)} & |U_{\mu3}| \\ |U_{\tau1}|e^{i\beta_{e2}} & |U_{\tau2}|e^{i(\beta_{e1}-\pi)} & |U_{\tau3}| \end{pmatrix}$$
$$\times Q_2 Q_1 Q_0, \tag{95}$$

where, as we have shown, $(-\phi + \beta_{e1} + \beta_{e2}) = \delta$ and

$$P_2 = \text{diag}\left(e^{i(\beta_{e1}+\beta_{e2})}, e^{i\phi}, 1\right), \tag{96}$$

$$Q_2 = \text{diag}\left(e^{-i\beta_{e2}}, e^{-i\beta_{e1}}, 1\right) = e^{-i\beta_{e2}} \text{diag}\left(1, e^{i(\beta_{e2}-\beta_{e1})}, e^{i\beta_{e2}}\right). \tag{97}$$

The phases in the diagonal matrix P_2 are unphysical – they can be absorbed by the electron and muon fields in the weak charged lepton current. The phases $(\beta_{e2} - \beta_{e1})$ and β_{e2} in the diagonal matrix Q_2 give contribution to the Majorana phases $\alpha_{21}/2$ and $\alpha_{31}/2$, respectively, while the common phase $(-\beta_{e2})$ in Q_2 is also unphysical and we will not keep it in our further analysis. One can show further (analytically or numerically) that we have:

$$\beta_{\mu1} + \beta_{e2} - \phi = \arg(U_{\mu1}) = \arg\left(-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta}\right), \tag{98}$$

$$\beta_{\mu2} + \beta_{e1} - \phi = \arg(U_{\mu2}) = \arg\left(c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta}\right), \tag{99}$$

$$\beta_{e2} = \arg(U_{\tau1}) = \arg\left(s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta}\right), \tag{100}$$

$$\beta_{e1} = \arg(U_{\tau2}) + \pi = \arg\left[\left(-c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta}\right)e^{i\pi}\right]. \tag{101}$$

This implies that the matrix in Eq. (95) is in the standard parametrisation form. Correspondingly, the Majorana phases $\alpha_{21}/2$ and $\alpha_{31}/2$ in the matrix Q in Eq. (1) are determined by the phases in the matrix $\overline{Q} = Q_2 Q_1 Q_0$: $Q = \overline{Q}$ and

$$\frac{\alpha_{21}}{2} = \beta_{e2} - \beta_{e1} + \frac{\xi_{21}}{2}, \qquad \frac{\alpha_{31}}{2} = \beta_{e2} + \beta + \frac{\xi_{31}}{2}. \tag{102}$$

The expressions we have obtained for the phases β_{e1} and β_{e2}, Eqs. (90) and (101) and (91), (100), are exact. It follows from these expressions that the phases β_{e1}, β_{e2} can be determined knowing the values of θ_{12}^e and ϕ, or, alternatively, of δ and of θ_{12}, θ_{23} and θ_{13}. In what concerns the phases ξ_{21} and ξ_{31} in Eq. (102), they are assumed to be fixed by the symmetry which determines the TBM, BM, GRA, etc. form of the matrix $\tilde{U}_\nu$.

More specifically, the phases β_{e1}, β_{e2} can be calculated either using Eqs. (90) and (91), or from Eqs. (100) and (101). It follows from Eqs. (100) and (101), in particular, that

we have approximately $|\sin\beta_{e1}| \cong \tan\theta_{12}\cot\theta_{23}\sin\theta_{13}|\sin\delta| \cong 0.12|\sin\delta|$, and $|\sin\beta_{e2}| \cong \cot\theta_{12}\cot\theta_{23}\sin\theta_{13}|\sin\delta| \cong 0.27|\sin\delta|$, where we have used the b.f.v. of $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ quoted in Eqs. (3)–(5). These estimates imply that $\cos\beta_{e1}$ and $\cos\beta_{e2}$ will have values close to 1. Indeed, we get, e.g., utilising the values of $\cos\delta$ given in Eqs. (31)–(35) and the corresponding b.f.v. of $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$:

$$\text{TBM:} \qquad \cos\beta_{e1} \cong 0.9929, \qquad \beta_{e1} \cong \pm 6.81°, \tag{103}$$

$$\text{BM (LC):} \qquad \cos\beta_{e1} \cong 0.999, \qquad \beta_{e1} \cong \pm 1.77°, \tag{104}$$

$$\text{GRA:} \qquad \cos\beta_{e1} \cong 0.994, \qquad \beta_{e1} \cong \pm 6.31°, \tag{105}$$

$$\text{GRB:} \qquad \cos\beta_{e1} \cong 0.9927, \qquad \beta_{e1} \cong \pm 6.92°, \tag{106}$$

$$\text{HG:} \qquad \cos\beta_{e1} \cong 0.995, \qquad \beta_{e1} \cong \pm 5.79°, \tag{107}$$

$$\text{TBM:} \qquad \cos\beta_{e2} \cong 0.976, \qquad \beta_{e2} \cong \pm 12.64°, \tag{108}$$

$$\text{BM (LC):} \qquad \cos\beta_{e2} \cong 0.9989, \qquad \beta_{e2} \cong \pm 2.58°, \tag{109}$$

$$\text{GRA:} \qquad \cos\beta_{e2} \cong 0.973, \qquad \beta_{e2} \cong \pm 13.24°, \tag{110}$$

$$\text{GRB:} \qquad \cos\beta_{e2} \cong 0.977, \qquad \beta_{e2} \cong \pm 12.31°, \tag{111}$$

$$\text{HG:} \qquad \cos\beta_{e2} \cong 0.975, \qquad \beta_{e2} \cong \pm 12.91°. \tag{112}$$

We see that the phases β_{e1} and β_{e2}, with the exception of the BM (LC) case, take values approximately in the intervals $\pm(6°–7°)$ and $\pm(12°–13°)$, respectively. For the phase difference $(\beta_{e2} - \beta_{e1})$, which contributes to the Majorana phase $\alpha_{21}/2$, we get taking into account that $\sin\beta_{e1}\sin\beta_{e2} < 0$:

$$\text{TBM:} \qquad \cos(\beta_{e2} - \beta_{e1}) \cong 0.943, \qquad \beta_{e2} - \beta_{e1} \cong \pm 19.45°, \tag{113}$$

$$\text{BM (LC):} \qquad \cos(\beta_{e2} - \beta_{e1}) \cong 0.997, \qquad \beta_{e2} - \beta_{e1} \cong \pm 4.35°, \tag{114}$$

$$\text{GRA:} \qquad \cos(\beta_{e2} - \beta_{e1}) \cong 0.942, \qquad \beta_{e2} - \beta_{e1} \cong \pm 19.55°, \tag{115}$$

$$\text{GRB:} \qquad \cos(\beta_{e2} - \beta_{e1}) \cong 0.944, \qquad \beta_{e2} - \beta_{e1} \cong \pm 19.23°, \tag{116}$$

$$\text{HG:} \qquad \cos(\beta_{e2} - \beta_{e1}) \cong 0.947, \qquad \beta_{e2} - \beta_{e1} \cong \pm 18.70°. \tag{117}$$

It follows from the results we have obtained that the contributions of the phases $2(\beta_{e2} - \beta_{e1})$ and $2\beta_{e2}$ to the Majorana phases α_{21} and α_{31} are practically negligible in the BM (LC) case. In all other cases of the form of the matrix $\tilde{U}_\nu$ considered by us, TBM, GRA, GRB and HG, these contributions have to be taken into account. If the sign of $\sin\delta$ will be determined experimentally, the ambiguity in the signs of $\sin\beta_{e1}$, $\sin\beta_{e2}$ and $\sin(\beta_{e2} - \beta_{e1})$ will be removed and β_{e1}, β_{e2} and $(\beta_{e2} - \beta_{e1})$ will be uniquely determined.

We note that by writing, $2\beta_{e2} = \pm r_2°$ and $2(\beta_{e2} - \beta_{e1}) = \pm r_{21}°$ we imply, in the convention used by us for the intervals in which the phases α_{21} and α_{31} vary, $2\beta_{e2} =^+_{(-)} 2r_2° + 360°k_2(\prime)$ and $2(\beta_{e2} - \beta_{e1}) =^+_{(-)} 2r_{21}° + 360°k_{21}(\prime)$, $k_2, k_{21} = 0, 1$ $(k'_2, k'_{21} = 1, 2)$, where $k_2 = 1$ $(k'_2 = 2)$ and $k_{21} = 1$ $(k'_{21} = 2)$ has to be taken into account in certain cases [5] when the flavour neutrino Majorana mass term is generated by the type I seesaw mechanism [52].

We will consider next the possibility to calculate also the phase $\beta = \gamma - \phi$ determined in Eqs. (16) and (17). We note first that the phase β enters only in the expression for the Majorana phase α_{31}. The latter plays a subdominant role in a number of cases of processes, characteristic of the Majorana nature of massive neutrinos ν_j. More specifically, the term involving the

Majorana phase α_{31} gives a subdominant contribution in the $(\beta\beta)_{0\nu}$-decay rate in the cases of neutrino mass spectrum i) with inverted ordering (IO), corresponding to $\Delta m^2_{31(32)} < 0$, and ii) of quasi-degenerate (QD) type (see, e.g., [2,11]), the reason being that the term of interest involves the suppression factor $\sin^2\theta_{13} \cong (0.023\text{--}0.024)$. For the same reason the rate of the process of radiative emission of two different Majorana neutrinos in atomic physics depends weakly on the Majorana phase α_{31} [13]. The value of the phase α_{31} plays important role, for example, for the prediction of the $(\beta\beta)_{0\nu}$-decay rate if neutrino mass spectrum is with normal ordering (NO) but is not quasi-degenerate, i.e., if $\Delta m^2_{31(32)} > 0$, $m_1 < (\ll)m_{2,3}$ and $m_1 \lesssim \sqrt{\Delta m^2_{31}} \cong 0.05$ eV (see, e.g., [2]).

In the case of negligibly small θ^e_{23}, as we have seen, $\gamma = -\psi + \pi$, $\phi = -\psi$, and $\beta = \pi$. In the "counter-intuitive" case [26] of $|\sin\theta^e_{23}| = 1$ we have $\gamma = \phi = -\omega$, and $\beta = 0$. In these cases we get, e.g., for $(\xi_{21}, \xi_{31}) = (0, 0)$ using Eqs. (108)–(117):

TBM: $\alpha_{21} \cong \pm 38.90°,\qquad \alpha_{31} \cong \pm 25.28° + 180°\ (0°),$

BM (LC): $\alpha_{21} \cong \pm 8.70°,\qquad\ \alpha_{31} \cong \pm 5.16° + 180°\ (0°),$

GRA: $\alpha_{21} \cong \pm 39.10°,\qquad \alpha_{31} \cong \pm 26.48° + 180°\ (0°),$

GRB: $\alpha_{21} \cong \pm 38.46°,\qquad \alpha_{31} \cong \pm 24.62° + 180°\ (0°),$

HG: $\alpha_{21} \cong \pm 37.40°,\qquad \alpha_{31} \cong \pm 25.82° + 180°\ (0°),$

where the values (values in parentheses) correspond to $\beta = \pi$ ($\beta = 0$).

In the general case of non-negligible θ^e_{23} we get from Eq. (17), using Eq. (14):

$$\cos\gamma = \frac{-\cos\theta^e_{23}\cos\psi + \sin\theta^e_{23}\cos\omega}{\sqrt{2}\sin\hat\theta_{23}}, \qquad \sin\gamma = \frac{\cos\theta^e_{23}\sin\psi - \sin\theta^e_{23}\sin\omega}{\sqrt{2}\sin\hat\theta_{23}}, \quad (118)$$

$$\cos\phi = \frac{\cos\theta^e_{23}\cos\psi + \sin\theta^e_{23}\cos\omega}{\sqrt{2}\cos\hat\theta_{23}}, \qquad \sin\phi = \frac{-\cos\theta^e_{23}\sin\psi - \sin\theta^e_{23}\sin\omega}{\sqrt{2}\cos\hat\theta_{23}}. \quad (119)$$

As it is not difficult to show using Eqs. (118)–(119), the phase β depends on the phases ψ and ω only via their difference $(\psi - \omega)$. Indeed, we have:

$$\cos\beta = -\frac{\cos 2\theta^e_{23}}{\sin 2\hat\theta_{23}}, \qquad \sin\beta = \frac{\sin 2\theta^e_{23}}{\sin 2\hat\theta_{23}}\sin(\psi - \omega), \quad (120)$$

where

$$\sin 2\hat\theta_{23} = \left(1 - \sin^2 2\theta^e_{23}\cos^2(\psi - \omega)\right)^{\frac{1}{2}}. \quad (121)$$

Thus, we have two undetermined parameters θ^e_{23} and $(\psi - \omega)$, which are constrained by their relation to, e.g., $\sin^2\hat\theta_{23}$, whose value is known:

$$2\sin\theta^e_{23}\cos\theta^e_{23}\cos(\psi - \omega) = 1 - 2\sin^2\hat\theta_{23}. \quad (122)$$

This constraint reduces the number of the unknown parameters in terms of which the phase β is expressed to one. The sign of $\sin(\psi - \omega)$ is also undetermined. Obviously, it is impossible to determine the phase β without some additional input. In what follows we will exploit several possibilities.

The first possibility corresponds to the phase ψ or the phase ω having one of the following specific values: 0, $\pi/2$, π and $3\pi/2$. In any of these cases the phase γ is determined (up to a possible sign ambiguity either of $\sin\gamma$ or of $\cos\gamma$) by the phase ϕ, which allows to determine

also the phase β (again up to a possible sign ambiguity of $\cos\beta$ or of $\sin\beta$). This possibility is realised in certain models of neutrino mixing based on discrete flavour symmetries.

To be more specific, assume first that $\psi = 0$. In this case we get from Eqs. (118)–(119):

$$\sin\gamma = \sin\phi \frac{\cos\hat\theta_{23}}{\sin\hat\theta_{23}}, \quad |\sin\gamma| \le 1, \tag{123}$$

$$\cos\phi\cos\hat\theta_{23} + \cos\gamma\sin\hat\theta_{23} = \sqrt{2}\sin\theta^e_{23}\cos\omega. \tag{124}$$

It is clear from Eq. (123) that the value of $\sin\gamma$ can be determined knowing the values of $\sin\phi$ and of $\cot\hat\theta_{23}$, independently of the values of θ^e_{23} and ω. This, obviously, allows to find also $|\cos\gamma|$, but not the sign of $\cos\gamma$. If, however, the inequality $\sqrt{2}|\sin\theta^e_{23}\cos\omega| < |\cos\phi\cos\hat\theta_{23}|$ is fulfilled, Eq. (124) would allow to correlate the sign of $\cos\gamma$ with the sign of $\cos\phi$ and thus to determine γ for a given ϕ: we would have $\cos\gamma < 0$ if $\cos\phi > 0$, and $\cos\gamma > 0$ for $\cos\phi < 0$. In the case of $\sqrt{2}|\sin\theta^e_{23}\cos\omega| > |\cos\phi\cos\hat\theta_{23}|$, the sign of $\cos\gamma$ will coincide with the sign of $\sin\theta^e_{23}\cos\omega$, and if the latter cannot be fixed, the two possible signs of $\cos\gamma$ have to be considered.

In a similar way we find that if $\psi = \pi/2$ we have:

$$\cos\gamma = \cos\phi \frac{\cos\hat\theta_{23}}{\sin\hat\theta_{23}}, \tag{125}$$

$$\sin\phi\cos\hat\theta_{23} + \sin\gamma\sin\hat\theta_{23} = -\sqrt{2}\sin\theta^e_{23}\sin\omega. \tag{126}$$

These relations hold in the model with T' family symmetry proposed in [19].[18] Now the value of $\cos\gamma$ can be determined knowing the values of $\cos\phi$ and of $\cot\hat\theta_{23}$, independently of the values of θ^e_{23} and ω. This allows to find also $|\sin\gamma|$, leaving the sign of $\sin\gamma$ undetermined. Depending on the relative magnitude of the terms $|\sin\phi\cos\hat\theta_{23}|$ and $|\sqrt{2}\sin\theta^e_{23}\sin\omega|$, the sign of $\sin\gamma$ will be anti-correlated either with the sign of $\sin\phi$, or with the sign of $\sin\theta^e_{23}\sin\omega$. In the latter case both signs of $\sin\gamma$ have to be considered if the sign of $\sin\theta^e_{23}\sin\omega$ is undetermined.

Similar results can be obtained if $\psi = \pi$ or $3\pi/2$, or if ω has one of the four values 0, $\pi/2$, π and $3\pi/2$.

One finds $\beta = \pi + 2\pi k$, $k = 0, 1$, if the equality $\psi = \omega$ holds. This possibility is realised in a scheme considered in [37], in which also the phases ξ_{21} and ξ_{31} are fixed: $\xi_{21} = 0$ and $\xi_{31}/2 = -\psi = -\arg[(c^e_{23} - s^e_{23})e^{-i\phi}/(c^e_{23} + s^e_{23})]$, where θ^e_{23} is determined from Eq. (122) in which one has to set $\cos(\psi - \omega) = 1$.

Further, it follows from Eq. (120) that if $|\sin\theta^e_{23}|$ (or $|\cos\theta^e_{23}|$) is known, that will allow to determine $\cos\beta$ and, correspondingly, $|\sin\beta|$. If, for instance, $|\sin\theta^e_{23}| = 0.2$, for the "best fit" value of $\sin^2\hat\theta_{23} = 0.438$ we find: $\cos\beta \cong -0.919$, and thus $\beta = 156.8°$ or $203.2°$.

In a general analysis in which one attempts to reproduce the values of the three neutrino mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ in the cases of the TBM, BM, GRA, etc. forms of the matrix $\tilde U_\nu$ with the help of the "correcting" matrix $(\tilde U_e)^\dagger\Psi = R_{12}(\theta^e_{12})R_{23}(\theta^e_{23})\Psi$, the four parameters θ^e_{12}, θ^e_{23}, ψ and ω will have to satisfy three constraints. This implies that the values of any two parameters, say, θ^e_{23} and $(\psi - \omega)$, will have to be correlated.[19] In addition, θ^e_{23} and

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[18] We correct two typos Eqs. (3.71) and (3.72) in [19]: i) the factor $\sin\hat\theta_{23}/\cos\hat\theta_{23}$ in the r.h.s. of Eq. (3.71) should be replaced by the inverse one, $\cos\hat\theta_{23}/\sin\hat\theta_{23}$, and ii) the factor $1/\sqrt{2}$ in the r.h.s. of Eq. (3.72) should be replaced by $\sqrt{2}$, i.e., Eqs. (3.71) and (3.72) in [19] should coincide respectively with Eqs. (125) and (126) given above.

[19] The author would like to thank W. Rodejohann and He Zhang for useful discussions of this point.

$(\psi - \omega)$ have to satisfy the constraint given in Eq. (14). This can allow to limit significantly the range of possible values of, or even to determine, $|\sin\theta^e_{23}|$. As a consequence, $\cos\beta$ (and therefore $|\sin\beta|$) will either be constrained to lie in a relatively narrow interval, or its value will be determined, which will lead to a similar information about the phase β (up to the possible ambiguity related to the two possible signs of $\sin\beta$). Such an analysis, however, is outside the scope of the present investigation; we intend to perform it elsewhere.

6. Implications for $(\beta\beta)_{0\nu}$-decay

We will discuss next briefly the implications of the results we have obtained on the Dirac and Majorana phases for the predictions of the effective Majorana mass in $(\beta\beta)_{0\nu}$-decay (see, e.g., [9,11]):

$$\left|\langle m\rangle\right| = \left|m_1(c_{12}c_{13})^2 + m_2(s_{12}c_{13})^2 e^{i\alpha_{21}} + m_3 s^2_{13} e^{i(\alpha_{31}-2\delta)}\right|, \tag{127}$$

where $m_j \geq 0$, $j = 1, 2, 3$, are the masses of the three light Majorana neutrinos. As is well known, the existing data do not allow one to determine the sign of $\Delta m^2_{31(32)}$ and the two possible signs of $\Delta m^2_{31(2)}$ correspond to two types of neutrino mass spectrum. In the widely used convention of numbering the neutrinos ν_j with definite mass in the two cases (see, e.g., [2]) we shall also employ, the two spectra read:

i) spectrum with normal ordering (NO): $0 \leq m_1 < m_2 < m_3$, $\Delta m^2_{31(32)} > 0$, $\Delta m^2_{21} > 0$, $m_{2(3)} = (m^2_1 + \Delta m^2_{21(31)})^{\frac{1}{2}}$;

ii) spectrum with inverted ordering (IO): $0 \leq m_3 < m_1 < m_2$, $\Delta m^2_{32(31)} < 0$, $\Delta m^2_{21} > 0$, $m_2 = (m^2_3 + \Delta m^2_{23})^{\frac{1}{2}}$, $m_1 = (m^2_3 + \Delta m^2_{23} - \Delta m^2_{21})^{\frac{1}{2}}$.

The values of $\Delta m^2_{21} > 0$ and $\Delta m^2_{31} > 0$ ($\Delta m^2_{32} < 0$) in the NO (IO) case were determined with relatively high precision in the global analyses of the neutrino oscillation data and read [15]:

$$\left(\Delta m^2_{21}\right)_{\mathrm{BF}} = 7.54 \times 10^{-5} \ \mathrm{eV}^2, \qquad \Delta m^2_{21} = (6.99 - 8.18) \times 10^{-5} \ \mathrm{eV}^2; \tag{128}$$

$$\left(\left|\Delta m^2_{31(32)}\right|\right)_{\mathrm{BF}} = 2.48 \ (2.44) \times 10^{-3} \ \mathrm{eV}^2,$$
$$\left|\Delta m^2_{31(32)}\right| = \left(2.26 \ (2.21) - 2.70 \ (2.65)\right) \times 10^{-3} \ \mathrm{eV}^2, \tag{129}$$

where we have given the best fit values and the 3σ allowed ranges of Δm^2_{21} and $|\Delta m^2_{31}|$ ($|\Delta m^2_{32}|$). Thus, we have, in particular, $\Delta m^2_{21}/|\Delta m^2_{31(32)}| \cong 0.03$.

Consider the case of IO neutrino mass spectrum. Expressing $m_{1,2}$ in terms of m_3, Δm^2_{21} and $\Delta m^2_{23} > 0$ in Eq. (127), and taking into account the fact that $\Delta m^2_{21} \ll \Delta m^2_{23}$, we get:

$$\left|\langle m\rangle\right| \cong \sqrt{m^2_3 + \Delta m^2_{23}} \left| c^2_{13}\left(c^2_{12} + s^2_{12} e^{i\alpha_{21}}\right) - \frac{1}{2}\frac{\Delta m^2_{21}(c_{12}c_{13})^2}{m^2_3 + \Delta m^2_{23}} + s^2_{13}\frac{m_3 e^{i(\alpha_{31}-2\delta)}}{\sqrt{m^2_3 + \Delta m^2_{23}}} \right|. \tag{130}$$

It follows from Eq. (3) that at 3σ we have: $|c^2_{12} + s^2_{12} e^{i\alpha_{21}}| \geq 0.28$. Taking into account the result on $\sin^2\theta_{13}$ quoted in Eq. (5), it is clear that the term $\propto s^2_{13}m_3$ in Eq. (130) is at least by a factor of 10 smaller in absolute value than $|c^2_{12} + s^2_{12} e^{i\alpha_{21}}|$. The term $\propto \Delta m^2_{21}$ in Eq. (130) does not

exceed approximately 0.01. Thus, up to corrections which are not larger than 10%, $|\langle m \rangle|$ in the case of IO spectrum is given by [10]:

$$|\langle m \rangle| \cong \sqrt{m_3^2 + \Delta m_{23}^2} |c_{12}^2 + s_{12}^2 e^{i\alpha_{21}}| = \sqrt{m_3^2 + \Delta m_{23}^2} \left(1 - \sin^2 2\theta_{12} \sin^2 \frac{\alpha_{21}}{2}\right)^{\frac{1}{2}}. \quad (131)$$

The expression for $|\langle m \rangle|$ in the case of QD neutrino mass spectrum (see, e.g., [2]), $m_1 \cong m_2 \cong m_3$, $m_{1,2,3}^2 \gg |\Delta m_{31(32)}^2|$, implying $m_0 \equiv \min(m_j) \gtrsim 0.1$ eV, has a similar form up to corrections $\sim |\Delta m_{31(32)}^2|/m_0^2$ [10]:

$$|\langle m \rangle| \cong m_0 |c_{12}^2 + s_{12}^2 e^{i\alpha_{21}}| = m_0 \left(1 - \sin^2 2\theta_{12} \sin^2 \frac{\alpha_{21}}{2}\right)^{\frac{1}{2}}. \quad (132)$$

It follows from Eqs. (131), (102) and (113),(115)–(117) that for $\xi_{21} = 0$ and the best fit values of the neutrino mixing angles, $|\langle m \rangle|$ will deviate little from the maximal possible value corresponding to the IO spectrum, $|\langle m \rangle| \cong \sqrt{m_3^2 + \Delta m_{23}^2}$, since for all cases considered $\sin^2(\alpha_{21}/2) = \sin^2(\beta_{e2} - \beta_{e1}) \lesssim 0.11$. If, however, $\xi_{21} = \pi$, then $\sin^2(\alpha_{21}/2) = \cos^2(\beta_{e2} - \beta_{e1})$ and $|\langle m \rangle|$ can be expected to be closer to its minimal possible value of $|\langle m \rangle| \cong \sqrt{m_3^2 + \Delta m_{23}^2} \cos 2\theta_{12}$. Using $\sin^2 2\theta_{12} = 0.85$ (corresponding to $\sin^2 \theta_{12} = 0.308$ for which $\cos 2\theta_{12} = 0.39$) and the values of $\cos(\beta_{e2} - \beta_{e1})$ given in Eqs. (114)–(117), we get: $|\langle m \rangle| \cong C_a \sqrt{m_3^2 + \Delta m_{23}^2}$, $a = TBM, BM(LC), GRA, GRB, HG$, where $C_{TBM} \cong 0.49$, $C_{BM(LC)} \cong 0.39$, $C_{GRA} \cong 0.49$, $C_{GRB} \cong 0.49$ and $C_{HG} \cong 0.48$. Thus, in the BM (LC) case $|\langle m \rangle|$ is minimal, while in the other cases $|\langle m \rangle|$ is approximately half of its maximal value. For any other value of ξ_{21}, the prediction for $|\langle m \rangle|$ for a given symmetry case will lie between those quoted for $\xi_{21} = 0$ and $\xi_{21} = \pi$. For the TBM, GRA, GRB and HG symmetry mixing, this implies that $0.49\sqrt{m_3^2 + \Delta m_{23}^2} \lesssim |\langle m \rangle| \leq \sqrt{m_3^2 + \Delta m_{23}^2}$, while for the BM (LC) mixing case, $0.39\sqrt{m_3^2 + \Delta m_{23}^2} \lesssim |\langle m \rangle| \leq \sqrt{m_3^2 + \Delta m_{23}^2}$, where the numerical factors correspond to the best fit values of $\sin^2 \theta_{12}$, $\sin^2 \theta_{23}$ and $\sin^2 \theta_{13}$. Similar results are valid for the QD neutrino mass spectrum.

One can use the same method to obtain predictions for $|\langle m \rangle|$ in the case of non-QD neutrino mass spectrum with normal ordering in the cases when the phase β is known.

7. Summary and conclusions

We have applied the approach developed in Ref. [1] to obtaining predictions for the Dirac and Majorana CP violation phases in the neutrino mixing (PMNS) matrix. The approach is based on the fact that the PMNS matrix $U_{PMNS} = U_e^\dagger U_\nu = (\tilde{U}_e)^\dagger \Psi \tilde{U}_\nu Q_0$, where U_e $(\tilde{U}_e)$ and U_ν $(\tilde{U}_\nu Q_0)$ result respectively from the diagonalisation of the charged lepton and neutrino mass matrices, $\tilde{U}_e$ and $\tilde{U}_\nu$ are 3×3 unitary (CKM-like) matrices, and Ψ and Q_0 are diagonal phase matrices containing, in general, two physical CP violation phases each. The phases in Q_0, $\xi_{21}/2$ and $\xi_{31}/2$, contribute to the Majorana phases $\alpha_{21}/2$ and $\alpha_{31}/2$, present in the standard parametrisation of the PMNS matrix (see Eq. (1)). The CPV phases in Ψ can originate from the charged lepton sector $(U_e^\dagger = (\tilde{U}_e)^\dagger \Psi)$, or from the neutrino sector $(U_\nu = \Psi \tilde{U}_\nu Q_0)$, or can receive contributions from both sectors. We have considered a number of different forms of $\tilde{U}_\nu = \tilde{U}_\nu(\theta_{12}^\nu, \theta_{23}^\nu, \theta_{13}^\nu, \delta^\nu)$ associated with a variety of discrete symmetries, for which $\theta_{13}^\nu = 0$

(and thus one can set $\delta^\nu = 0$) and $\theta_{23}^\nu = -\pi/4$: i) bimaximal (BM) ($\theta_{12}^\nu = \pi/4$), and ii) tri-bimaximal (TBM) ($\theta_{12}^\nu = \sin^{-1}(1/\sqrt{3})$) forms, the forms corresponding iii) to the conservation of the lepton charge $L' = L_e - L_\mu - L_\tau$ (LC) ($\theta_{12}^\nu = \pi/4$), iv) to golden ratio type A (GRA) mixing with $\sin^2\theta_{12}^\nu = (2+r)^{-1} \cong 0.276$, r being the golden ratio, $r = (1+\sqrt{5})/2$, v) golden ratio type B (GRB) mixing, with $\sin^2\theta_{12}^\nu = (3-r)/4 \cong 0.345$, and vi) to hexagonal (HG) mixing, in which $\theta_{12}^\nu = \pi/6$. The TBM, BM and GRA special forms of $\tilde{U}_\nu$, for instance, can be obtained from specific discrete family symmetries in the lepton sector (see, e.g., [39–45]). In the cases of symmetry forms of $\tilde{U}_\nu$ considered, the phases in the matrix $\Psi = \text{diag}(1, e^{-i\psi}, e^{-i\omega})$, generate the Dirac phase δ in the (standard parametrisation of the) PMNS matrix and, as we have shown, give rise to contributions to the Majorana phases $\alpha_{21}/2$ and $\alpha_{31}/2$. The minimal form of $\tilde{U}_e$, in terms of angles it contains, that can provide the requisite corrections to $\tilde{U}_\nu$ so that reactor, atmospheric and solar neutrino mixing angles θ_{13}, θ_{23} and θ_{12} have values compatible with the current data, including a possible sizable deviation of θ_{23} from $\pi/4$, is a product of two orthogonal matrices describing rotations in the 12 and 23 planes, $R_{12}(\theta_{12}^e)$ and $R_{23}(\theta_{23}^e)$. Two orderings of the 12 and the 23 rotations in $\tilde{U}_e$ are possible: "standard" with $\tilde{U}_e = R_{23}(\theta_{23}^e)R_{12}(\theta_{12}^e)$, and "inverse" with $\tilde{U}_e = R_{12}(\theta_{12}^e)R_{23}(\theta_{23}^e)$. The "standard" ordering is related to the hierarchy of the charged lepton masses, $m_e^2 \ll m_\mu^2 \ll m_\tau^2$, and is a common feature of the overwhelming majority of the existing models of the charged lepton masses and the associated mixing. In the present article we have analysed only the more interesting case of "standard" ordering. In this case the Dirac CP violation phase δ, present in the PMNS matrix U, is shown to satisfy a new sum rule, Eq. (30), by which $\cos\delta$ is expressed in terms of the angle θ_{12}^ν of $\tilde{U}_\nu$ and the three angles θ_{12}, θ_{23} and θ_{13} of the PMNS matrix. The sum rule we have derived is exact within the approach employed and is a generalisation of the sum rule found in [1] for the TBM and BM (LC) forms of $\tilde{U}_\nu$. This allowed us to obtain predictions for δ and the J_{CP} factor, which controls the magnitude of the CP violation effects in neutrino oscillations, in the cases of GRA, GRB and HG forms of $\tilde{U}_\nu$; predictions for δ and J_{CP} for the TBM and BM (LC) forms of $\tilde{U}_\nu$ were obtained in [1]. Although the $\cos\delta$ is determined without sign ambiguity, the sign of $\sin\delta$ cannot be fixed using the current data, which leads to a two-fold (sign) ambiguity in the value of δ. The indicated results on δ and the J_{CP} factor are given in Eqs. (31)–(35) and Eqs. (39)–(43). They have been derived for the best fit values of the neutrino mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$. It follows from these results that:

i) $\delta \cong 1.59\pi$ or 0.41π in the GRA case; ii) $\delta \cong 1.45\pi$ or 0.55π in the GRB case; and iii) $\delta \cong 1.64\pi$ or 0.36π in the HG case.

In the TBM and BM (LC) cases we have [1]: iv) TBM: $\delta \cong 1.47\pi$ or 0.53π; v) BM (LC): $\delta \cong 1.07\pi$ or 0.93π. Thus, in the TBM, GRA, GRB and HG cases, relatively large CP violation effects in neutrino oscillations are predicted ($|J_{CP}| \cong (0.031$–$0.034)$), while in the BM (LC) case the indicated CP violation effects are suppressed. Distinguishing between the TBM, BM (LC), GRA, GRB and HG forms of $\tilde{U}_\nu$ requires a measurement of $\cos\delta$ or a relatively high precision measurement of J_{CP}.

We have considered also the case of $|\sin\theta_{23}^e| \ll 1$ (Section 4), analysing first the possibility of negligibly small $|\sin\theta_{23}^e|$ (Subsection 4.1). For $\theta_{23}^e = 0$ we have $\tilde{U}_e = R_{12}(\theta_{12}^e)$. This case has been analysed by many authors in the past (see, e.g., Refs. [25–37]). If $\theta_{23}^e = 0$, as is well known, $\sin^2\theta_{23}$ can deviate only by $0.5\sin^2\theta_{13}$ from 0.5. The phase ω is unphysical. Now the exact sum rule of interest involves the cosine of the Dirac phase δ and the angles θ_{12}^ν, θ_{12} and θ_{13} (Eq. (50)). A similar sum rule can be obtained for the cosine of the phase $\phi = -\psi$, Eq. (46), which is related to, but does not coincide with, δ (the exact relation between $\cos\delta$ and $\cos\phi$ for arbitrary

θ_{12}^{ν} is given in[20] Eq. (51)). We have derived exact, leading order and next-to-leading order sum rules for both $\cos\phi$ and $\cos\delta$, Eqs. (46), (48), (47) and Eqs. (52), (54), (53), respectively. The leading order sum rules (48) and (54) are shown to be equivalent to the sum rules for $\cos\delta$ and $\cos\phi$ given in Eqs. (49) and (55). For arbitrary θ_{12}^{ν}, the leading order sum rule (55) was proposed in [33]. In Ref. [34] it was suggested that the sum rule (55) should be used to obtain the value of $\cos\delta$ using the experimentally determined values of $\sin^2\theta_{12}$ and $\sin\theta_{13}$, e.g., in the case of the TBM form of $\tilde{U}_{\nu}$. We have shown that the sum rule (55) is the leading order approximation of the exact sum rule (30), derived in Section 3. We have also shown that in the cases of TBM, GRA, GRB and HG forms of $\tilde{U}_{\nu}$, and for the current best fit value of $\sin^2\theta_{12}$, the leading order sum rule (55) is not consistent with the approximation employed to derive it. A consistent application of the corrections in the indicated cases leads to $\cos\delta = \cos\phi = 0$. As a consequence, the next-to-leading order corrections to (55), or to the equivalent sum rule (54), derived in Eq. (53) (and in Eq. (47) for $\cos\phi$), are significant and should be taken into account. For the TBM GRA, GRB and HG forms of $\tilde{U}_{\nu}$, the predictions for $\cos\delta$ (and $\cos\phi$) derived using the exact sum rule Eq. (50) (Eq. (46)), or the next to leading order sum rule Eq. (53) (Eq. (47)), differ by factors of (1.2–1.6) from the predictions obtained from the leading order sum rule Eq. (55) (Eq. (49)), or the equivalent one Eq. (54) (Eq. (48)). As we have shown in Subsection 4.2, this difference can be further amplified by an additional factor of 1.2 by the next-to-leading order correction due to $\theta_{23}^{e} \neq 0$, $\sin\theta_{23}^{e} \ll 1$, if $\sin^2\theta_{23} \cong 0.4$. Using the exact sum rules Eqs. (30) and (22) leads for $\theta_{23}^{e} \neq 0$ to practically the same results respectively for $\cos\delta$ and $\cos\phi$ as the next-to-leading order sum rules Eq. (86) and Eq. (87). We have shown also that the leading order sum rule (55) provides a rather accurate prediction for $\cos\delta$ only in the case of BM (LC) form of the matrix $\tilde{U}_{\nu}$.

In Section 5 we have analysed the possibility to obtain predictions for the values of the Majorana phases $\alpha_{21}/2$ and $\alpha_{31}/2$ in the PMNS matrix. We have shown that $\alpha_{21}/2 = \beta_{e2} - \beta_{e1} + \xi_{21}/2$ and $\alpha_{31}/2 = \beta_{e2} + \beta + \xi_{31}/2$, where ξ_{21} and ξ_{31} are the phases of the matrix Q_0, and β_{e1}, β_{e2} and β are real calculable phases. In many theories and models of neutrino mixing the values of the phases ξ_{21} and ξ_{31} are fixed by the form of the neutrino Majorana mass term, which is dictated by the chosen discrete (or continuous) flavour symmetry or on phenomenological grounds. Typical values of $\xi_{21}/2$ and $\xi_{31}/2$ are 0, $\pi/2$ and π.[21] Within the approach adopted in the present article, the phases β_{e1} and β_{e2} can be calculated exactly for each of the five symmetry forms of $\tilde{U}_{\nu}$ considered by us. We have first derived exact analytic expressions for β_{e1} and β_{e2} in terms of the three neutrino mixing angles, θ_{12}, θ_{23}, θ_{13}, and the Dirac phases δ (Eqs. (101) and (100)). Given θ_{12}, θ_{23}, θ_{13} and θ_{12}^{ν} (i.e., the symmetry form of $\tilde{U}_{\nu}$), these expressions allow to get predictions for the values of β_{e1} and β_{e2}. We give such predictions for β_{e1}, β_{e2} and $(\beta_{e2} - \beta_{e1})$ for each of the five symmetry forms of $\tilde{U}_{\nu}$ considered using the best fit values of $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ (Eqs. (103)–(117)). In what concerns the phase β entering into the expression for the Majorana phase $\alpha_{31}/2$, we have discussed a number of cases in which it can be calculated exactly.

Finally, in Section 6 we have analysed the implications of the results obtained on the leptonic CPV phases for the predictions of the effective Majorana mass in $(\beta\beta)_{0\nu}$-decay. This was done

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[20] The exact relation between δ ($\cos\delta$) and ϕ ($\cos\phi$) in the case of $\theta_{23}^{e} \neq 0$, in which $\phi \neq -\psi$ and ϕ is defined in Eq. (17), is given in Eq. (94) (Eq. (29)).

[21] In the model with T' flavour symmetry in the lepton sector constructed in [19], for instance, ξ_{21} and ξ_{31} can take two sets of values: $(\xi_{21}, \xi_{31}) = (0, 0)$ and $(0, \pi)$.

on the examples of the neutrino mass spectra with inverted ordering and of quasi-degenerate type.

The predictions for the leptonic CP violation phases in the PMNS neutrino mixing matrix derived in the present article will be tested in the experiments on CP violation in neutrino oscillations and possibly in the neutrinoless double beta decay experiments.

8. Note added

After this study was completed, results of an updated global analysis of the neutrino oscillation data were published in [53], in which the latest T2K data on $\sin^2 \theta_{23}$ [54], $\sin^2 \theta_{23} = 0.514 + 0.055 / - 0.056$ (0.511 ± 0.055) for the NO (IO) neutrino mass spectrum, were taken into account. As a consequence, the authors of [53] find a somewhat larger central value of $\sin^2 \theta_{23}$ than the one used by us in the numerical predictions for the Dirac and Majorana phases, namely $\sin^2 \theta_{23} = 0.437$ (0.455) in the NO (IO) case. At the same time, the MINOS collaboration finds for the best fit value of $\sin^2 \theta_{23} = 0.41$, performing a 3-neutrino oscillation analysis of their data [55]. Obviously, high precision measurement of $\sin^2 \theta_{23}$ is lacking at present. Our numerical predictions for the values of the Dirac and Majorana phases should be updated when a sufficiently precise determination of $\sin^2 \theta_{23}$ will be available. However, if $\sin^2 \theta_{23}$ is found to lie in the interval (0.40–0.50), the numerical predictions obtained in this study will not change significantly.

Acknowledgements

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