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Painlevé tau-functions
and
Fredholm determinants

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Declaration of Authorship

I, Harini DESIRAJU, declare that this thesis titled, "Painlevé tau-functions and Fredholm determinants" and the work presented in it are my own. Where I have consulted the published work of others, this is always clearly attributed. The thesis is based on the three research papers

- **Harini Desiraju**; The tau-function of the Ablowitz Segur family of solutions to Painlevé II as a Widom constant. In: *Journal of Mathematical Physics 60 (2019), p. 113505*. doi: 10.1063/1.5120357.
- **Harini Desiraju**; Fredholm determinant representation of the Painlevé II tau-function. (*Submitted to Nonlinearity.*) arXiv:2008.01142v3 .
- Fabrizio Del Monte, **Harini Desiraju**, Pavlo Gavrylenko ; Isomonodromic tau functions on a torus as Fredholm determinants, and charged partitions. (*Submitted to Communications in Mathematical Physics.*) arXiv:2011.06292.

I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
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- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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Date: 08/02/2021

Abstract

Harini DESIRAJU

Painlevé tau-functions and Fredholm determinants

It is now known that the tau-functions associated to the generic solutions of the Painlevé equations III, V, VI can be expressed as a Fredholm determinants. The minor expansion of these determinants provide an interesting connection to random partitions. We show that the generic tau-function of the Painlevé II equation can be written as a Fredholm determinant of an integrable (Its-Izergin-Korepin-Slavnov) operator. The tau-function depends on the isomonodromic time t and two Stokes parameters, and the vanishing locus of the tau-function, called the *Malgrange divisor* is determined by the zeros of the Fredholm determinant. As a mid-step, we show that the Fredholm determinant of the Airy kernel which is also the tau-function of the Ablowitz-Segur family of solutions to Painlevé II, can be expressed as the determinant of a combination of Toeplitz operators called the Widom constant. Furthermore, constructing a suitable basis, we obtain the minor expansion of the determinant of the Airy kernel labelled by colourless and chargeless Maya diagrams.

We also generalise the techniques to study the tau-functions of Painlevé III, V, VI to the case of Fuchsian system with generic monodromies in $GL(N,\mathbb{C})$ on a torus, and show that associated the tau-function can be written as a Fredholm determinant of Plemelj operators. We further show that the minor expansion of this Fredholm determinant is described by a series labeled by charged partitions. As an example, we show that in the case of $SL(2,\mathbb{C})$ this combinatorial expression takes the form of a dual Nekrasov-Okounkov partition function.

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Part I

Tau-functions and Painlevé equations

Chapter 1

Introduction

Integrable systems are ubiquitous in several areas of mathematics and physics. There are several compatible notions of integrability and they typically share the following characteristics [62], [2]:

- existence of Lax pairs,
- presence of underlying geometric structure,
- explicit form of the solutions.

Consider a classical mechanical system with the Hamiltonian $H(p_i, q_i)$. The equations of motion read

$$\frac{dp_i}{dT} = -\frac{\P H}{\P q_i}, \qquad \frac{dq_i}{dt} = \frac{\P H}{\P p_i}. \tag{1.0.1}$$

If the equations of motion can be written in terms of a set of matrices L, M, called the Lax matrices, as

$$\frac{dL}{dt} = [L, M]\,, \tag{1.0.2}$$

the Hamiltonian system is said to be integrable. Such systems are also called *isospectral* owing to the fact that the spectrum of the matrix L is conserved, i.e the spectral curve

$$\det\left(L - l\mathbb{1}\right) = 0 \tag{1.0.3}$$

is constant in time. Moreover, the quantities $\mathrm{Tr}\, L^k$ gives the integrals of motion

$$\frac{\P}{\P t}\mathrm{Tr}\, L^k = \mathrm{Tr}\left(k\,[M, L]\, L^{k-1}\right) = 0. \tag{1.0.4}$$

In this thesis, we focus on a particular subclass of integrable systems called the *isomonodromic systems*, which in contrast to the isospectral systems have time dependent Hamiltonians. Moreover, isospectral systems are obtained by an automous reduction of isomonodromic systems [2].

1.1 Isomonodromic deformations

This is a quick review of isomonodromic deformations. Refer to [43], [36], [68] for a complete treatment of the subject, and [100, 101] for reviews covering the subject from a slightly different perspective. Our starting point is a linear system with rational coefficients and $n+1$ singularities

$$\frac{dY}{dI} = A(I)Y, \quad A(I) = \sum_{n=1}^{n}\sum_{k=1}^{r_n+1}\frac{A_{n,k-1}}{(I-a_n)^k} - \sum_{j=-1}^{r_\infty-1} I^j A_{\infty,j+1}, \tag{1.1.1}$$

where

$$A_{n,k-1}, A_{\infty,j+1} \in \mathfrak{sl}_N(\mathbb{C}), \tag{1.1.2}$$

and r_n is called the *Poincaré rank* of the pole at $I = a_n$. We further assume that the highest order matrix coefficients A_{n,r_n} are diagonalizable

$$A_{n,r_n} = G_n \Xi_{n,r_n} G_n^{-1}; \quad \Xi_{n,r_n} = \mathrm{diag}\left(q_{n,1}, \ldots, q_{n,N}\right), \tag{1.1.3}$$

and their eigenvalues are distinct and non-resonant in the sense that:

$$\begin{cases} q_{n,a} \neq q_{n,b}, \quad \text{if} \quad r_n \geq 1, \quad a \neq b, \\ q_{n,a} \neq q_{n,b} \bmod \mathbb{Z}, \quad \text{if} \quad r_n = 0, \quad a \neq b. \end{cases} \tag{1.1.4}$$

Without loss of generality, G_n is chosen to be unimodular $\det G_n = 1$ We also assume that A_{∞,r_∞} can be diagonalized, namely

$$A_{\infty,r_\infty} = G_\infty \Xi_{\infty,r_\infty} G_\infty^{-1}, \tag{1.1.5}$$

where G_∞ can always be set to identity with a suitable transformation. The space of coefficients $\mathbb{A}$ is then defined as

$$\begin{aligned} \mathbb{A} := \{&a_n, A_{n,k-1}, A_{\infty,j+1}, \Xi_{n,r_n}, \Xi_{\infty,r_\infty}, G_n\, ; k = 1\ldots r_n, \\ &j = -1\ldots r_\infty - 2, n = 1\ldots n\} \end{aligned} \tag{1.1.6}$$

and its dimension [100]

$$\dim \mathbb{A} = \left(\sum_{n=1}^{n} r_n + r_\infty\right)(N^2-1) + (N-1)(n+1) + (N^2-1)(n-1) + n - 2. \tag{1.1.7}$$

The formal solution of the linear system (1.1.1) at the singularity a_n is unique, and reads

$$Y^{(n)}_{\text{form}}(I) = G_n \Psi^{(n)}(I) e^{\Xi_n(I)}, \tag{1.1.8}$$

where

$$\Psi^{(n)}(I) = \begin{cases} \mathbb{1} + \sum_{k=1}^{\infty} \Psi_{n,k}(I-a_n)^k, \quad n = 1, \ldots, n, \\ \mathbb{1} + \sum_{k=1}^{\infty} \Psi_{\infty,k} I^{-k}, \quad n = \infty, \end{cases} \tag{1.1.9}$$

and $\Xi_n(l)$ is the matrix valued function

$$\Xi_n(l) := \begin{cases} \sum_{k=1}^{r_n} \frac{\Xi_{n,k}}{k}(l-a_n)^{-k} + \Xi_{n,0}\log(l-a_n), \quad n=1,\dots,n \\ -\sum_{k=1}^{r_\infty} \frac{\Xi_{\infty,k}}{k} l^k - \Xi_{\infty,0}\log l, \quad n=\infty. \end{cases} \tag{1.1.10}$$

The matrix Ξ_{n,r_n} is defined by (1.1.3), $\Xi_{n,k}, k=1,...,r_n$ are diagonal matrices, and along with the matrix valued coefficients of the expression (1.1.9) $\Psi_{n,k}, \Psi_{\infty,k}$, are determined recursively as polynomials of the matrix coefficients $A_{n,k-1}, k=1,...,r_n$ (for further details, refer to [43]),

$$\Xi_{n,r_n-1} + [\Psi_{n,1}, \Xi_{n,r_n}] = G_n^{-1}A_{n,r_n-1}G_n,$$
$$\Xi_{n,r_n-2} + [\Psi_{n,2}, \Xi_{n,r_n}] = G_n^{-1}A_{n,r_n-2}G_n + G_n^{-1}A_{n,r_n-1}G_n\Psi_{n,1} - \Psi_{n,1}\Xi_{n,r_n-1},$$
$$\vdots$$

A system is called Fuchsian if $r_n = 0$, $r_\infty = 0$, i.e the only singularities in the system are simple poles and viceversa.

The Fuchsian case

In this section we consider only the Fuchsian case. The solution $Y(l)$ is multivalued on $\mathbb{C}P^1\backslash\{a_n\}_{n=1}^n$ and its analytic continuation around a closed loop oriented anti-clockwise around a_m, depends solely on the homotopy class of the loop. For each m we define as ℓ_m the close loop around a_m (see Figure 1.1)

$$\ell_m : [0,1] \to \mathbb{C}P^1\backslash\{a_n\}_{n=1}^n, \quad \ell_m(0) = \ell_m(1) = z_0. \tag{1.1.11}$$

The loops $g_1, \dots g_n$ form a set of generators of the fundamental group $p_1\left(\mathbb{C}P^1\backslash\{a_n\}_{n=1}^n; z_0\right)$, and the map

$$M : p_1\left(\mathbb{C}P^1\backslash\{a_n\}_{n=1}^n; z_0\right) \to GL_N(\mathbb{C}). \tag{1.1.12}$$

defines the monodromy representation of the Fuchsian system.

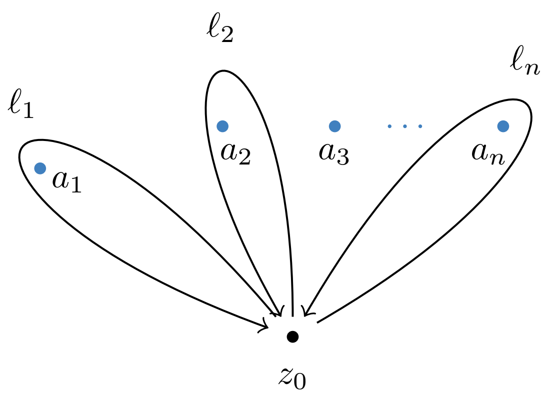


FIGURE 1.1: Fundamental loops ℓ_n.

Any solution in (1.1.8) can be written as $Y^{(n)}(I)C_n$, where C_n is some invertible constant matrix. We choose $Y^{(\infty)}(I)$ as our fundamental matrix solution so that

$$Y(I) = \begin{cases} Y^{(\infty)}(I) = G_\infty \Psi^{(\infty)}(I) e^{\Xi_\infty(I)}, \\ Y^{(n)}(I)C_n = G_n \Psi^{(n)}(I) e^{\Xi_n(I)} C_n, \quad n = 1, \dots, n. \end{cases} \tag{1.1.13}$$

This unambiguously defines the connection matrices C_n. According to the above expansion, it follows that under the transformation $I \to \ell_n I$, where $\ell_n I$ stands for the point I after a counterclocksiwe loop around a_n, we have

$$Y(\ell_n I) = Y(I) M_n,$$

where the monodromy matrix M_n is given by

$$M_n = C_n^{-1} e^{2pi\Xi_{n,0}} C_n. \tag{1.1.14}$$

Note that the monodromy representation $M(g_n) = M_n$ is a group anti-homomorphism:

$$M(\ell_1 \ell_2) = M(\ell_2) M(\ell_1). \tag{1.1.15}$$

Since

$$\ell_n \ell_{n-1} \dots \ell_1 = \ell_\infty^{-1},$$

we obtain the constraint

$$M_1 ... M_n M_\infty = \mathbb{1}. \tag{1.1.16}$$

Remark 1.1. *In the case of a Fuchsian system on a Riemann surface of genus g with n Fuchsian singularities, the non-trivial holonomy around the A-cycles and B-cycles of the surface induces the monodromy $\left(M_{A_j}, M_{B_j}\right)_{j=1}^{g}$ around the respective cycles. The monodromy constraint then reads*

$$M_1 ... M_n \prod_{j=1}^{g} M_{A_j} M_{B_j}^{-1} M_{A_j}^{-1} M_{B_j} = \mathbb{1}. \tag{1.1.17}$$

The case for $g = 1$ will appear in chapter 5.

General case: Irregular singularities

Let us now consider systems with irregular singularities. The asymptotic behaviour of $Y(I)$ in the neighbourhood of an irregular singularity a_n exhibits the *Stokes phenomenon* which is described as follows. Let $e > 0$ be sufficiently small and consider the sector $\Omega_{j,n}$ defined as

$$\Omega_{j,n} = \{ I : 0 < |I - a_n| < e, \frac{p(j-1)}{r_n} - e < \arg(I - a_n) < \frac{pj}{r_n} \} \tag{1.1.18}$$

for $j = 1, \dots, 2r_n + 1$, there exists a unique canonical solution $Y_j^{(n)}$ satisfying the condition

$$Y_j^{(n)}(I) \simeq Y_{\text{form}}^{(n)}(I), \qquad \text{as} \quad I \to a_n, \quad I \in \Omega_{j,n}. \tag{1.1.19}$$

The *Stokes matrices* then relate the canonical solutions in different *Stokes sectors*:

$$S_j^{(n)} := \left(Y_j^{(n)}\right)^{-1} Y_{j+1}^{(n)}, \quad j = 1 \dots 2r_n. \tag{1.1.20}$$

Stokes matrices satisfy a triangularity condition (up to a permutation); if the eigenvalues of the diagonal matrix Θ_{n,r_n} are ordered so that the real parts are strictly decreasing, then the Stokes' matrices are alternatively lower or upper triangular with ones on the diagonal. This structure can be motivated by the fact that both $Y_j^{(n)}$, $Y_{j+1}^{(n)}$ behave as Y_n in the asymptotic limit, implying the following:

- the diagonal entries of $S_j^{(n)}$ are 1,
- the multiplication by $S_j^{(n)}$ on the right can only add a column of $Y_j^{(n)}$ multiplied by a scalar back to itself,

implying that $S_j^{(n)}$ is alternatively lower or upper triangular, we refer to [43, 36, 68] for a complete treatment. Choosing Y_1^{∞} as fundamental matrix solution, the connection matrices are defined as

$$C_n := \left(Y_1^{(n)}\right)^{-1} Y_1^{(\infty)}, \quad n = 1, \dots, n. \tag{1.1.21}$$

and the monodromy matrix M_n around the singularitu a_n reads

$$M_n := C_n^{-1} e^{2pi\Xi_{n,0}} \left(S_{2r_n}^{(n)}\right)^{-1} \dots \left(S_1^{(n)}\right)^{-1} C_n. \tag{1.1.22}$$

Let us now introduce the space of monodromy data $\mathbb{M}$ of the system which consists of formal monodromy exponents $\Xi_{n,0}$, Connection matrices C_n and Stokes matrices $S_j^{(n)}$:

$$\mathbb{M} := \{S_j^{(n)}, \Xi_{n,0}, C_m;\ j = 1, \dots 2r_n, n = 1, \dots, n, \infty, m = 1, \dots, n : M_1 \dots M_n M_\infty = \mathbb{1}\}, \tag{1.1.23}$$

and the dimension [100]

$$\dim \mathbb{M} = \left(\sum_{n=1}^{n} r_n + r_\infty\right)(N^2 - 1) + (N-1)(n+1) + (N^2-1)(n-1). \tag{1.1.24}$$

Note that $\dim \mathbb{M}$ is always even.

Let us now look for the deformations of the linear system (1.1.1) such that the monodromies remain unchanged, i.e *isomonodromic deformations*. For a deformation to be isomonodromic, there should exist a matrix valued, meromorphic one form U such that

$$dY = UY \tag{1.1.25}$$

where $d = da_n \frac{¶}{¶a_n}$. The above fact can be easily verified:

$$dY(g.I) = UY(g.I) \overset{(1.1.12)}{\Rightarrow} d(YM_g) = UYM_g \overset{(1.1.25)}{\Rightarrow} dM_g = 0. \tag{1.1.26}$$

The compatibility condition of (1.1.1), (1.1.25) reads

$$dA - \frac{dU}{dz} = [A, U], \qquad dU = U \wedge U. \tag{1.1.27}$$

We can now introduce the set of isomonodromic times

$$\mathbb{T} = \{a_m, \Xi_{n,k}; k = -r_n, \dots, -1, n = 1, \dots, n, \infty, m = 1, \dots, n\}, \tag{1.1.28}$$

with dimension [100]

$$\dim \mathbb{T} = n + \left(\sum_{n=1}^{n} r_n + r_\infty \right) (N-1) - 2. \tag{1.1.29}$$

Note that $\dim \mathbb{A} = \dim \mathbb{M} + \dim \mathbb{T}$.

1.2 Painlevé equations

Painlevé equations are second order nonlinear ordinary differential equations (ODEs) of the form

$$\frac{d^2u}{dx^2} = F\left(u, \frac{du}{dx}, x\right) \tag{1.2.1}$$

with F rational in $\left(u, \frac{du}{dx}\right)$, analytic in x, and posessing the Painlevé property such that the general solutions cannot be written as contour integrals. Moreover, the solutions are free from branch points and essential singularities that depend on the initial data [36, 43]. For a linear ODE, the singularities of its general solution are uniquely determined by the coefficients of the equation. In contrast, the singularities of the general solutions of nonlinear ODEs depend on the initial data. The only movable singularities of the solutions to the Painlevé equations are poles. The Painlevé equations were first formulated by Fuchs [48], Gambier [51], Painlevé [97], and Picard [99]. The solutions of these equations, called the *Painlevé transcendents*, are transcendental as the name suggests and are often regarded as nonlinear analogues of special functions. Here is the list of the six equations[1]:

$$PI : u'' = 6u^2 + x, \tag{1.2.2}$$

$$PII : u'' = 2u^3 + xu + a, \tag{1.2.3}$$

$$PIII : u'' = \frac{1}{u}(u')^2 - \frac{1}{x}(u') + \frac{au^2}{x} + \frac{b}{x} + gu^3 + \frac{d}{u}, \tag{1.2.4}$$

$$PIV : u'' = \frac{1}{2u}(u')^2 + \frac{3}{2}u^3 + 4xu^2 + 2(x^2 - a)u + \frac{b}{u}, \tag{1.2.5}$$

$$PV : u'' = \left(\frac{1}{2u} + \frac{1}{u-1}\right)(u')^2 - \frac{1}{x}u' + \frac{(u-1)^2}{x^2}\left(au + \frac{b}{u}\right) + \frac{gu}{x} + \frac{du(u+1)}{u-1}, \tag{1.2.6}$$

$$PVI : u'' = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right)(u')^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right)u' + \frac{u(u-1)(u-x)}{x^2(x-1)^2}\left(a + \frac{bx}{u^2} + \frac{g(x-1)}{(u-1)^2} + \frac{dx(x-1)}{(u-x)^2}\right). \tag{1.2.7}$$

The equations (1.2.2)-(1.2.7) have associated Lax pairs with the following singularity structure

$$PI : \quad n = 0, \quad r_\infty = 5, \tag{1.2.8}$$

$$PII : \quad n = 0, \quad r_1 = 0, \quad r_\infty = 3, \tag{1.2.9}$$

$$PIII : \quad n = 1, \quad r_1 = 1, \quad r_\infty = 1, \tag{1.2.10}$$

$$PIV : \quad n = 1, \quad r_1 = 0, \quad r_\infty = 2, \tag{1.2.11}$$

$$PV : \quad n = 2, \quad r_1 = r_2 = 0, \quad r_\infty = 1, \tag{1.2.12}$$

$$PVI : \quad n = 3, \quad r_1 = r_2 = r_3 = r_\infty = 0, \tag{1.2.13}$$

As an example, the Flaschka-Newell Lax pair of the Painlevé II equation (with isomonodromic time x) reads [42]

$$\frac{dY}{dl} = \left[-i\left(4l^2 + x + 2u^2\right)s_3 + 4lus_1 - 2vs_2 - \frac{a}{l}s_1\right]Y, \tag{1.2.14}$$

$$\frac{dY}{dx} = \left[-ils_3 + us_1\right]Y, \tag{1.2.15}$$

where $v = u'$, the Pauli matrices

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.2.16}$$

and the consistency condition (1.1.27) of the Lax pair (1.2.14),(1.2.15)

$$\frac{d}{dx}\frac{dY}{dl} = \frac{d}{dl}\frac{dY}{dx} \tag{1.2.17}$$

[1]Notation: $' = \frac{d}{dx}$

gives Painlevé II (1.2.3). Painlevé equations have several interesting properties out of which, we highlight two aspects relevant to this thesis in this chapter:

- Coalescence of equations,
- Hamiltonian structure.

Coalescence of Painlevé equations

All Painlevé equations can be obtain from an appropriate degeneration of Painlevé VI. As an example, the transformation

$$u(x;a,b) \to \frac{u(x;a)}{e} + \frac{1}{4e^3}, \qquad x \to ex - \frac{1}{4e^3};$$
$$a \to -2a - \frac{1}{32e^6}, \qquad b = -\frac{1}{512e^{12}} \tag{1.2.18}$$

on Painlevé IV (1.2.5) under the limit $e \to 0$ yields Painlevé II (1.2.3) [35].

The degeneration of Painlevé equations is best visualised through the coalescence diagram that was proposed in [30, 31]. Starting from the underlying geometry of linear system of Painlevé VI which is a Riemann sphere with 4 punctures, the geometries of the other Painlevé linear systems are obtained by what are known as *chewing gum* moves that either coalesce two holes to produce a Riemann sphere with one less hole, and two new cusps on the boundary of the coalesced part, or reduce the number of cusps at one hole by 1 by a process called cusp removal. This representation also highlights (the red circle in fig 4.1) the pants decomposition of the associated Riemann surface.

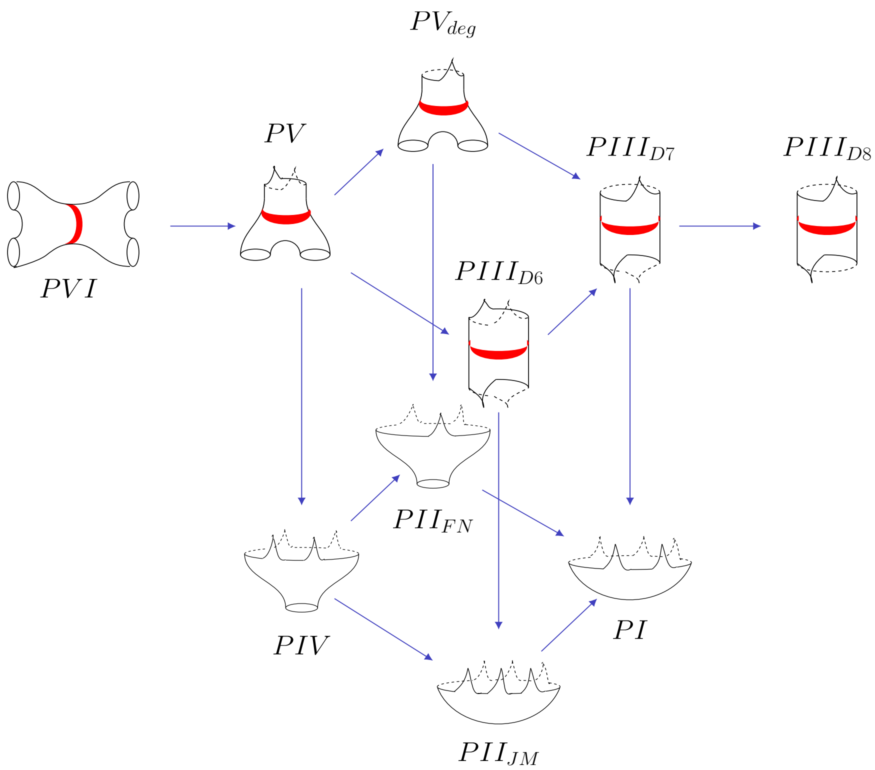


FIGURE 1.2: Coalescence diagram.

The singularity structure of the Painlevé equations can be read off from the above diagram. Each circle on the boundary represents a Fuchsian singularity and the cusps represent Stokes lines. The special solutions of Painlevé equations (obtained through specific constraints on the initial data) are often expressed in terms of classical special functions [88, 91, 82, 86, 117]. For example, the special solutions of Painlevé VI are described by Gauss hypergeometric functions, the one parameter solutions of Painlevé II are related to the Airy functions. Such solutions also have a coalescence diagram similar to the general solutions as in Fig: 1.3.

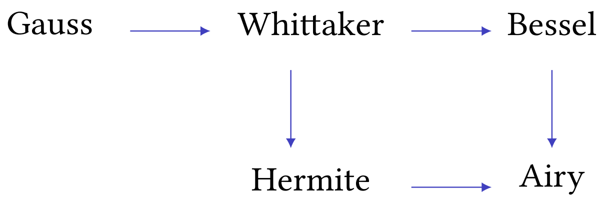


FIGURE 1.3: Coalescence diagram for special functions.

Painlevé-Calogero correspondence

Painlevé equations have the remarkable property of appearing as equations of motion of Hamiltonian systems of time dependent anharmonic oscillators [93]. As an example, Painlevé II arises through the equations of motion

$$p := \frac{\partial u}{\partial x}; \quad \frac{du}{dx} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dx} = -\frac{\partial H}{\partial u}, \tag{1.2.19}$$

of the system with time dependent Hamiltonian $H(p, u, x)$

$$H(p,u,x) = \frac{p^2}{2} + V(u,x), \qquad V(u,x) := -\frac{u^4}{2} - \frac{xu^2}{2} - au. \tag{1.2.20}$$

Moreover, the Hamiltonians associated to Painlevé equations are related to rank-one case of Inozemtsev extension [64] of Calogero-Moser systems, and this correspondence is called the Painlevé-Calogero correspondence [111]. In turn, these Hamiltonians satisfy the so-called *sigma form* of Painlevé equations [94]. In this thesis, the focus will be on Painlevé VI which has a peculiar role as it is associated to a non-autonomous Hamiltonian system with an elliptic potential. The first step to this result [80, 85, 96] was the key observation that Painlevé VI (1.2.7) can be written in terms of elliptic integrals. Below, we outline the computation in Takasaki's paper [111]. Let us introduce the Picard-Fuchs operator $\mathcal{L}_x$

$$\mathcal{L}_x := x(1-x)\frac{d^2}{dx^2} + (1-2x)\frac{d}{dx} - \frac{1}{4}, \tag{1.2.21}$$

that solves the Picard-Fuchs equation for complete elliptic integrals. The Painlevé VI equation for $u = u(x)$ can be equivalently written as

$$x(1-x)\mathcal{L}_x \int_\infty^u \frac{dz}{\sqrt{z(z-1)(z-x)}} = \sqrt{u(u-1)(u-t)} \times \left[a + \frac{bx}{u^2} + \frac{g(x-1)}{(u-1)^2} + \left(d - \frac{1}{2}\right)\frac{x(x-1)}{(u-x)^2}\right]. \quad (1.2.22)$$

Let us consider the change of dependent variable $u \to Q$, and the independent variable $x \to t$

$$Q(x) := \frac{1}{2(e_2-e_1)^{1/2}} \int_\infty^u \frac{dz}{\sqrt{z(z-1)(z-x)}}, \quad (1.2.23)$$

$$x = \frac{e_3(t)-e_1(t)}{e_2(t)-e_1(t)}, \quad (1.2.24)$$

where for $i = 0, 1, 2, 3$,

$$e_i = \wp(w_i), \quad w_0 = 0, \quad w_1 = \frac{1}{2}, \quad w_2 = \frac{1}{2} + \frac{t}{2}, \quad w_3 = \frac{t}{2}, \quad (1.2.25)$$

t being the modular parameter, and the Weirstrass $\wp$-function

$$\wp(z|t) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z+m+nt)^2} - \frac{1}{(m+nt)^2}\right) \quad (1.2.26)$$

with fundamental periods $1, t$. Under the transformation (1.2.23), (1.2.24), the form of Painlevé VI in (1.2.22) reads

$$(2pi)^2 \frac{d^2Q(t)}{dt^2} = \sum_{n=0}^{3} a_n \wp'(Q+w_n), \quad (1.2.27)$$

where the parameters a_i's are related to a, b, g, d in (1.2.22) as

$$a_0 = a, \quad a_1 = -b, \quad a_2 = g, \quad a_3 = -d + \frac{1}{2}. \quad (1.2.28)$$

Setting $a_i = \frac{m^2}{8}$, and using that $\sum_{n=0}^{3} \wp'(Q+w_n) = 8\wp'(2Q)$, (1.2.27) reads

$$(2pi)^2 \frac{d^2Q(t)}{dt^2} = m^2 \wp'(2Q|t), \quad (1.2.29)$$

which is also known as the nonautonomous Calogero-Moser system. This will be the starting point of chapter 5.

1.3 Riemann-Hilbert problems

Let us now introduce an analytic tool that proves critical in studying many aspects of isomonodromic systems, called the Riemann-Hilbert problem [12, 13, 23, 43]. A Riemann-Hilbert problem typically consists of finding an analytic function with prescribed 'jumps' dictated by the monodromy data on a set of contours determined by the singularity data. Finding such function is tantamount to finding a bijective map between the coefficient space $\mathbb{A}$ and the space $\mathbb{M} \times \mathbb{T}$, where $\mathbb{M}$ is the monodromy manifold and $\mathbb{T}$ is the space of isomonodromic times. Let us introduce a minimal setup of a Riemann-Hilbert problem (RHP): If a $N \times N$ matrix valued function $\Psi(l, x)$ depending on the coordinate l and a parameter x with a jump on the smooth contour Σ defined in fig. 1.4 such that

Riemann-Hilbert problem 1.1.

- $\Psi(l, x)$ is analytic on $\mathbb{C} \backslash \Sigma$.
- On the contour Σ, the following boundary value problem is satisfied

$$\Psi_+(l, x) = \Psi_-(l, x) J(l, x), \tag{1.3.1}$$

where $J(l, x) \in Mat(N, \mathbb{C})$, $\det J = 1$, and the jump is identity in the asymptotic limit.

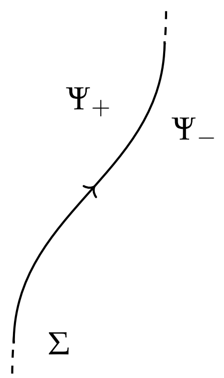


FIGURE 1.4: Contour

- In the limit $l \to \infty$,

$$\Psi(l, x) = \mathbb{1} + \mathcal{O}(l^{-1}). \tag{1.3.2}$$

When $\det J = 1$, the solution of the RHP, if exists, is unique: unimodularity of the jump J implies that $\det \Psi$ is analytic in $l \in \mathbb{C}$ as it has no jump on the contour Σ, and the Liouville theorem then implies that $\det \Psi = 1$. This seemingly obscure construction is vital to provide crucial insights into several aspects of integrable systems such as the asymptotics, and in this thesis will prove essential in constructing what are called the t-functions, that will be introduced in the next chapter. Let us now study the RHP of Painlevé II.

1.3.1 Painlevé II

The equation

$$u'' = 2u^3 + ux \tag{1.3.3}$$

is called the homogeneous Painlevé II equation which is (1.2.3) with $a = 0$, and arises as a consistency condition of the so called Jimbo-Miwa-Garnier Lax pair [71]

$$\frac{dY}{dl} = \left[-i\left(4l^2 + x + 2u^2\right)s_3 + 4lus_1 - 2vs_2\right]Y =: A(l,x)Y, \tag{1.3.4}$$

$$\frac{dY}{dx} = \left[-ils_3 + us_1\right]Y =: U(l,x)Y. \tag{1.3.5}$$

The generalized monodromy data (including the Stokes matrices) [72] for the ODE (1.3.4) defines a RHP depending locally analytically on x and on the Stokes data. Its solvability is the (generalized) inverse monodromy problem [42]. We introduce the matrix $\Psi(l)$ defined as

$$\Psi(l) := Y(l)e^{iq(l,x)s_3}, \quad q(l) = \frac{4}{3}l^3 + xl. \tag{1.3.6}$$

The matrix $\Psi(l,x) \in GL(2,\mathbb{C})$ satisfies the following RHP on the contour in fig:1.5 (see e.g. [43]).

Riemann-Hilbert problem 1.2.

- $\Psi(l,x)$ is piece-wise analytic for $l \in \mathbb{C}\setminus \cup_{k=1}^{6} g_k$, where

  $$g_k = \left\{l \in \mathbb{C} : \arg l = \frac{p}{6} + \frac{p}{3}(k-1)\right\},\ k = 1,...,6, \tag{1.3.7}$$

  such that $\Psi(l,x) \equiv \Psi_k$ in the respective Stokes sector Ω_k defined by

  $$\Omega_k = \left\{l \in \mathbb{C} : \frac{p}{6}(2k-3) < \arg l < \frac{p}{6}(2k-1)\right\},\ k = 1,...,6. \tag{1.3.8}$$

- For $l \in g_k$, the following boundary condition is satisfied

  $$\Psi_{k+1} = \Psi_k S_k, \tag{1.3.9}$$

  where the matrices S_k are alternatively lower or upper triangular

  $$S_k = \begin{pmatrix} 1 & 0 \\ s_k e^{2iq(l,x)} & 1 \end{pmatrix} \text{ for } k \equiv 1 \bmod 2, \quad S_k = \begin{pmatrix} 1 & s_k e^{-2iq(l,x)} \\ 0 & 1 \end{pmatrix} \text{ for } k \equiv 0 \bmod 2, \tag{1.3.10}$$

  The symmetries of the system introduce constraints on Stokes parameters s_k. We begin by noting that the linear system (1.3.4) has the following symmetry

  $$-A(-l,x) = s_2 A(l,x)s_2, \tag{1.3.11}$$

  which in turn implies that

  $$Y(-l,x) = s_2 Y(l,x)s_2, \qquad S_{n+3} = s_2 S_n s_2. \tag{1.3.12}$$

  Therefore the Stokes parameters s_k satisfy the following constraint

  $$s_{k+3} = -s_k\ ,\ s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0. \tag{1.3.13}$$

- In the asymptotic limit $l \to \infty$,

$$\Psi(l,x) = \mathbb{1} + \mathcal{O}(l^{-1}). \tag{1.3.14}$$

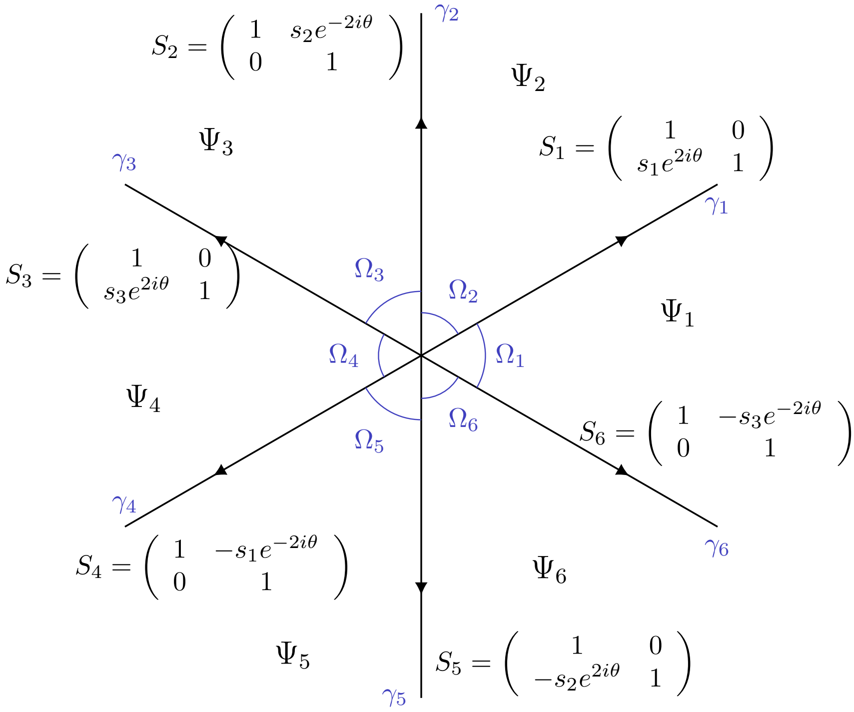


FIGURE 1.5: Stokes rays

The constraint on the Stokes data (1.3.13) implies that the solution $\Psi(z,t)$ depends only on two Stokes parameters. In this thesis, we are concerned with the generic 2-parameter solutions that correspond to the following constraints on the Stokes parameters

$$s_1 s_3 \neq 1; \quad \arg(1 - s_1 s_3) \in (-p, p). \tag{1.3.15}$$

The Painlevé II transcendent can then be obtained from the solution $\Psi(l,x)$ of the RHP 1.2:

$$u(x) = 2(\Psi^{(1)})_{12} = 2\mathrm{Res}_{\,l=\infty}\left(l\,(\Psi)_{12}\right). \tag{1.3.16}$$

Where $\Psi^{(1)}$ is obtained from the asymptotic expansion,

$$\Psi(l) = \mathbb{1} + \Psi^{(1)} l^{-1} + \Psi^{(2)} l^{-2} + \mathcal{O}(l^{-3}). \tag{1.3.17}$$

Indeed, rearranging the terms of the matrix A in (1.3.4) such that

$$A(l) = A_0 + A_1 l + A_2 l^2 \tag{1.3.18}$$

with

$$A_0 := -i\left(x + 2u^2\right)s_3 - 2v s_2, \qquad A_1 := 4u s_1, \qquad A_2 := -4i s_3, \tag{1.3.19}$$

we obtain (1.3.16) by substituting (1.3.6), (1.3.18) in (1.3.4) and comparing powers.

We can thus recast the general solution of Painlevé II in terms of the monodromy data (s_1, s_2, s_3) subject to the constraint (1.3.13)) of the linear system (1.3.4) as opposed to the data of initial values of (1.3.3) as a consequence of the bijection between the spaces $\mathbb{A}$, $\mathbb{M} \times \mathbb{T}$ when the RHP is solvable[2]. Such a construction holds for all the Painlevé equations. Such a formulation of the Painlevé transcendent proves useful to study the asymptotics [24, 70, 27, 38].

Remark 1.2. *The local parametrices of the RHPs of Painlevé equations are usually described by the Wronskians of special functions. For example, the local parametrices of Painlevé VI are described by hypergeometric functions. This property is seen through the pants decomposition of the Riemann surfaces associated to the Painlevé equations fig: 1.2. Painlevé VI is associated to a 4-point sphere which can be cut into two 3-point spheres which in turn are associated to hypergeometric functions. Such decomposition is feasible only for the Painlevé equations sitting on the top tier of the Coalescence diagram as indicated by the red circles in fig: 1.2. We will however see that the local parametrices of the RHP of Painlevé II, after a suitable transformation, are described by parabolic cylinder functions.*

Remark 1.3. *The ratio of the global solutions to the RHPs and their local parametrices often redefine the RHP on a simpler contour. Such ratios will be a recurring theme of this thesis.*

1.3.2 Special solutions of Painlevé II

Special solutions of Painlevé equations are obtained by specific conditions on the initial data, which in turn are equivalent to a constraint on the monodromy data. One such important one-parameter class of special solutions of the homogeneous Painlevé II equation

$$u'' = xu + 2u^3 \tag{1.3.20}$$

are the Ablowitz-Segur family [107] of solutions which are specified uniquely by the boundary condition

$$u(x) \simeq kAi(x); \quad x \to +\infty, \quad k \in \mathbb{C}\,. \tag{1.3.21}$$

or equivalently, setting the Stokes parameter

$$s_2 = 0 \Rightarrow s_1 = -s_3, \quad k := s_1. \tag{1.3.22}$$

Under the constraint (1.3.22), and rotating by $\frac{p}{2}$, fig. 1.5 transforms as

[2]For a detailed study of the coefficient space and monodromy space, refer to Chapters 1.6 and 4.2 of [43].

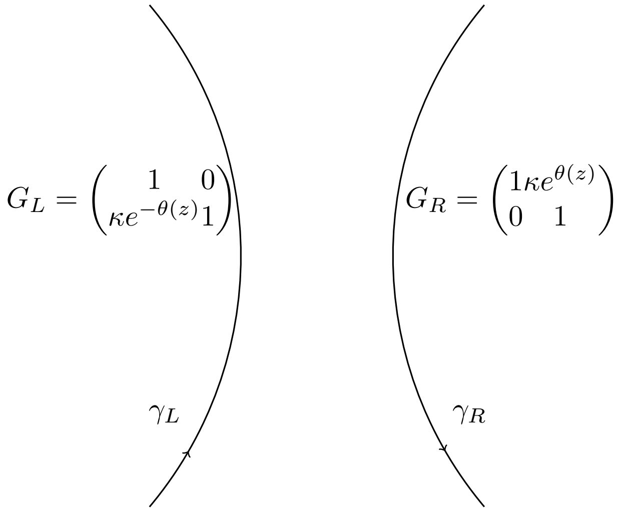


FIGURE 1.6: Contour

and the RHP reads as follows

Riemann-Hilbert problem 1.3.

- $\Gamma(l, x)$ is analytic for $l \in \mathbb{C} \setminus (g_L \cup g_R)$.
- For $l \in (g_L \cup g_R)$, the following bourdary condition is valid

$$\Gamma_-^{-1}\Gamma_+ = G = \begin{cases} G_L, & \text{for } l \in g_L \\ G_R, & \text{for } l \in g_R. \end{cases} \tag{1.3.23}$$

- In the limit $l \to \infty$

$$\Gamma = \mathbb{1} + \mathcal{O}(l^{-1}). \tag{1.3.24}$$

Ablowitz-Segur solutions are well known to be associated to the Airy kernel

$$u(x)^2 = -\frac{d^2}{dx^2} \log\det \left[\mathbb{1} - kK_{Ai}|_{[x,\infty)} \right], \tag{1.3.25}$$

where K_{Ai} is the Airy kernel

$$K_{Ai}(z, w) = \frac{Ai(z)Ai'(w) - Ai'(z)Ai(w)}{z - w}. \tag{1.3.26}$$

In fact special solutions of all Painlevé equations have such determinantal structure and have applications in wide ranging areas of mathematical physics. Another example is a special solution of Painlevé V which is related to the sine kernel. More will be said about these kernels and Airy kernel in particular, in the next chapter.

Chapter 2

Tau-functions and Fredholm determinants

In the theory of isomonodromic deformations, the tau-function (t_{JMU}) was introduced by the Kyoto school [102, 103, 104, 105, 106] and it is constructed starting from a certain 1-form w_{JMU} on the space of the deformation parameters $\mathbb{T}$ [72]. If the parameters are of isomonodromic type, then the form w_{JMU} is closed under differentiation with respect to the parameters. The corresponding t_{JMU} function is defined locally as

$$d \log t_{JMU} = w_{JMU} \tag{2.0.1}$$

where d denotes total differentiation with respect to the parameters.

There are at least two ways of formulating the tau-function.

1. As a generator of isomonodromic Hamiltonian on the space $\mathbb{A} \backslash \{\mathbb{T} = const.\}$ [66] :

$$d \log t_{JMU} := Hdt = \sum_{n=1,\ldots,n,\infty} \operatorname{Res}_{l=a_n} \operatorname{tr}\left(A(l,x)\, dG_n G_n^{-1}\right), \tag{2.0.2}$$

   where $A(l,x)$ is the Lax matrix.

2. For the generic RHP 1.1, the tau-function is defined as a logarithmic potential of a one-form called the *Malgrange form*[1] on $\mathbb{T}$ [84]

$$w_M =: d \log t_{JMU} = \int_{\Sigma} \frac{dz}{2pi} \operatorname{Tr}\left[\Psi_-^{-1} \frac{¶\Psi_-}{¶z} dJJ^{-1}\right]. \tag{2.0.3}$$

   The locus in the parameter space where the problem becomes unsolvable is called the *Malgrange divisor* because (in the language of algebraic geometry) it can be described locally as the zero level set of a local analytic function.

Remark 2.1. *The dependence of the tau-function on the Stokes data is studied by considering an extension of the Malgrange form on $\mathbb{M} \times \mathbb{T}$ [6, 8]. In which case, d depends not only on the isomonodromic times but also the Stokes parameters.*

Remark 2.2. *An important point to be stressed is that the tau-function is not unique and is defined up to multiplicative factors depending only on the monodromy data, owing to the structure of the tau-function as a section of a line bundle [9]. We put this fact to good use throughout this thesis as will become clear.*

Tau-functions play a central role in the study of integrable systems and are the main object of interest in this thesis. Apart from the role tau-functions play in Riemann-Hilbert

[1] Here we are restricting to the $d = \frac{¶}{¶t} dt$

analysis, they also appear in the study of partition functions and correlators of matrix models [45, 43, 46]. Typically, a tau-function also satisfies what is called the Hirota bilinear equation. Please refer to [60] for a comprehensive treatment of the subject. In this thesis, we use the above two notions and show that the tau-functions (2.0.2), (2.0.3) can be expressed as Fredholm determinants. In particular, the Chapters 3, 4 thesis will be mostly concerned with the Riemann-Hilbert approach whereas Chapter 5 will concern the Hamiltonian approach.

For the next two subsections, let us focus on the formulation of the tau-function in terms of the RHP (2.0.3). The solvability of a generic RHP can be recast as invertiblity of an integral operator. The determinants of such operators then contend to be a natural candidate for the tau-function. In fact, when tau-functions are expressed as Fredholm determinants, their zero locus uniquely describes the Malgrange divisor. Furthermore, the asymptotic data of the system such as connection constants become calculable. Let us now present formulation of the isomonodromic tau-function (2.0.3) as a Fredholm determinant of

1. a combination of Toeplitz operators called the Widom constant,
2. an integrable operator in the sense of IIKS.

In the final section, we motivate another formulation of the tau-function as a determinant of Cauchy operators as the construction is case specific.

2.1 "Determinant" of Toeplitz operators

Toeplitz matrices and Toeplitz operators play a significant role in mathematical physics with applications ranging from Statistical physics to non-commutative geometry. For example, the correlation functions of two-dimensional Ising model are described by Toeplitz matrices [120]. For the purposes of this thesis, it is sufficient to understand that a Toeplitz operator is an operator that projects a L^2-space on to its Hardy space.

2.1.1 Symbols on a unit circle

A generic RHP on a unit circle with the jump condition $y_+(z,t) = y_-(z,t)G(z,t)$ (the jump $G(z,t)$ is called the *symbol* in the language of Toeplitz determinants) is solvable if and only if the Toeplitz operator

$$T_G := \Pi_+ G = \int_{S^1} \frac{dw}{2pi} \frac{G(w,t)}{z-w} \tag{2.1.1}$$

is invertible (see [33] for a proof), where Π_+ is the Cauchy projection operator that maps analytic functions on $L^2(S^1)$ to the interior disk of the circle. Moreover, the jump on a unit circle admits *Birkhoff factorization* which means that the matrix valued function G can be factorized as

$$G = f_-(z,t) f_+(z,t)^{-1} = y_-^{-1}(z,t) y_+(z,t) \tag{2.1.2}$$

where the $\pm$ parts represent the positive and negative parts of the Laurent series representation of the function G. The invertibility of the Toeplitz operator $T_{G^{-1}}$ then dictates the solvability of the RHP with the factorization $G = f_-(z,t)\, f_+(z,t)^{-1}$ of the jump matrix. The *Widom* constant is then defined as

$$t\,[G] := \det\left(T_{G^{-1}} \circ T_G\right), \tag{2.1.3}$$

and it is the isomonodromic t function up to a multiplicative factor. The quantity $t\,[G]$ in (2.1.3) was obtained by Widom in the description of the asymptotic behaviour of Toeplitz determinants [118], [119], when the size of the matrix tends to infinity, as a refinement of the strong Szegö theorem. The description of the Widom constant in terms of Toeplitz matrices was obtained by Borodin and Okounkov in [21]. Note that the determinant of a Toeplitz operator is not well defined as the trace is divergent. But, when the symbol is on a unit circle, the Toeplitz operators have the property that

$$T_{G^{-1}} \circ T_G = \mathbb{1} + \text{trace class op.} \tag{2.1.4}$$

and its determinant (2.1.3) is therefore a determinant in the sense of Fredholm. The above construction also holds for RHPs defined on straight line contours with the only difference that the determinant of analogous Toeplitz operators (2.1.3) is usually Hilbert-Schmidt and the determinant is therefore a regularized determinant. Notable examples of tau-function assuming the form of a Widom constant are the Sato-Segal-Wilson tau-function [29], and isomonodromic tau-functions of Painlevé equations III, V, VI [28].

Side note: An operator K belongs to the pth ideal $\mathcal{I}_p$ if $\operatorname{Tr} K^p$ is convergent. The operator K is called trace class if $p = 1$, Hilbert-Schmidt if $p = 2$. The determinant $\det(1 + K)$ is Fredholm if K is trace class, and a regularized determinant otherwise [109].

2.1.2 Symbols on the imaginary axis

We now start with a RHP with the jump condition

$$\Psi_+(z,t) = \Psi_-(z,t) J(z,t) \tag{2.1.5}$$

on the imaginary axis where the jump $J \in GL(N,\mathbb{C})$. The duplicability of the above exercise for the present choice of contour lies in the fact that both the contours, the unit circle and the straight line, split their respective L^2 spaces orthogonally. The contour $i\mathbb{R}$ divides the complex plane into the right half (negative side) and the left half (positive side). The space $L^2\left(i\mathbb{R}, |dz|\right) \otimes \mathbb{C}^2$ can be split as the direct sum of two closed subspaces (*Hardy spaces*):

$$\mathcal{H} = L^2(i\mathbb{R}, \mathbb{C}^2) = \mathcal{H}_+ \oplus \mathcal{H}_-$$

where the functions of $\mathcal{H}$ are thought of as column vectors. The two subspaces consist of (vector valued) functions in $L^2(i\mathbb{R})$ that are boundary values from the left$(+)$/right$(-)$ of analytic functions that tend to 0 as $\Re z > 0$, $|z| \to \infty$ respectively. On these spaces, one can define projection operators $\Pi_\pm$ such that

$$\Pi_+ : \mathcal{H} \to \mathcal{H}_+ \quad ; \quad \Pi_- : \mathcal{H} \to \mathcal{H}_-.$$

Explicitly, $\Pi_\pm$ are just the Cauchy transforms

$$\Pi_+ f(z) = \int_{i\mathbb{R}} \frac{dw}{2pi} \frac{f(w)}{w-z} \qquad \Re z < 0 \tag{2.1.6}$$

$$\Pi_- f(z) = -\int_{i\mathbb{R}} \frac{dw}{2pi} \frac{f(w)}{w-z} \qquad \Re z > 0 \tag{2.1.7}$$

with the equality $\Pi_+ + \Pi_- \equiv \mathbb{1}$. The Birkhoff factorization of the jump matrix J

$$J(z,t) = \Theta_-(z,t)\Theta_+(z,t)^{-1} = \Psi_-^{-1}(z,t)\Psi_+(z,t) \tag{2.1.8}$$

introduces the *dual* RHP $\Theta_+ = J^{-1}\Theta_-$ in addition to the *direct* RHP (2.1.5). To define the tau-function, we write the Toeplitz operator $T_J : \mathcal{H} \to \mathcal{H}_+$ for the symbol J

$$T_J(f) = \Pi_+(Jf) \tag{2.1.9}$$

where the test function is acted upon by multiplication, followed by the projection, and its inverse reads

$$(T_J)^{-1}(f) = \Pi_+\Psi_+^{-1}(\Psi_- f) : \quad \mathcal{H}_+ \to \mathcal{H}_+. \tag{2.1.10}$$

It can be verified that (2.1.10) is the inverse of (2.1.9), i.e

$$(T_J)^{-1} \circ T_J = \mathbb{1}$$

$$\begin{aligned}(T_J)^{-1} \circ T_J(f) &= \Pi_+\Psi_+^{-1}(\Psi_- T_J(f)) \\ &= \Pi_+\Psi_+^{-1}(\Psi_-\Pi_+ J)(f) \\ &= \Pi_+\Psi_+^{-1}[\Psi_-(1-\Pi_-)(Jf)] \\ &= \Pi_+\Psi_+^{-1}[\Psi_+ f - \Psi_-\Pi_- Jf].\end{aligned} \tag{2.1.11}$$

Since $\Psi_-\Pi_- Jf \in \mathcal{H}_-$, $\Pi_+\Psi_+^{-1}[\Psi_-\Pi_- Jf] = 0$, and $\Psi_+ f \in \mathcal{H}_+$ implying that $\Pi_+\Psi_+^{-1}[\Psi_+ f] = f$. Therefore,

$$(T_J)^{-1} \circ T_J(f) = f. \tag{2.1.12}$$

Definition 2.1. *We define the Widom constant for a symbol on a straight line contour as*

$$t[J] = \det\left(T_{J^{-1}} \circ T_J\right). \tag{2.1.13}$$

Proposition 2.1. *The Widom constant $t[J]$ as defined in (2.1.13) admits an equivalent representation as the determinant*

$$t[J] = \det{}_{\mathcal{H}}[1+U] \tag{2.1.14}$$

where **1** *denotes the identity operator on $\mathcal{H}$, $U : \mathcal{H} \to \mathcal{H}$ is an operator represented in the splitting $\mathcal{H}_\pm$ as $U = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ and $a : \mathcal{H}_- \to \mathcal{H}_+$; $b : \mathcal{H}_+ \to \mathcal{H}_-$ are given by,*

$$a = \Psi_+\Pi_+\Psi_+^{-1} - \Pi_+ \quad ; \quad b = \Pi_- - \Psi_-\Pi_-\Psi_-^{-1}.$$

Proof. Substituting (2.1.9) in (2.1.13) and manipulating the terms gives the familiar form of the determinant representation of tau-function in [28].

$$\begin{aligned} t[J] = \det{}_{\mathcal{H}_+}[T_{J^{-1}} \circ T_J] &= \det{}_{\mathcal{H}_+}[\Pi_+ J^{-1} \Pi_+ J] \\ &= \det{}_{\mathcal{H}_+}[\Pi_+ \Psi_+^{-1} \Psi_- \Pi_+ \Psi_-^{-1} \Psi_+] \\ &= \det{}_{\mathcal{H}_+}[\Psi_+ \Pi_+ \Psi_+^{-1} \Psi_- \Pi_+ \Psi_-^{-1}] \\ &= \det{}_{\mathcal{H}_+}[\Psi_+ \Pi_+ \Psi_+^{-1} \Psi_- (1 - \Pi_-) \Psi_-^{-1}] \\ &= \det{}_{\mathcal{H}_+}[\mathbb{1} - (\Psi_+ \Pi_+ \Psi_+^{-1})(\Psi_- \Pi_- \Psi_-^{-1})] \\ &= \det{}_{\mathcal{H}_+}[\mathbb{1} - (\Psi_+ \Pi_+ \Psi_+^{-1} - \Pi_+)(\Pi_- - \Psi_- \Pi_- \Psi_-^{-1})] \\ &= \det{}_{\mathcal{H}}[\mathbb{1} + U] \end{aligned} \tag{2.1.15}$$

where

$$U = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \tag{2.1.16}$$

and

$$a = \Psi_+ \Pi_+ \Psi_+^{-1} - \Pi_+ \quad ; \quad b = \Pi_- - \Psi_- \Pi_- \Psi_-^{-1}. \tag{2.1.17}$$

Notice that

$$a : \mathcal{H}_- \to \mathcal{H}_+ \quad ; \quad b : \mathcal{H}_+ \to \mathcal{H}_-.$$

□

Now, one can repeat the same computation as above in terms of the dual RHP $\Theta_\pm$ and get the following.

$$a = \Theta_+ \Pi_+ \Theta_+^{-1} - \Pi_+ \quad b = \Pi_- - \Theta_- \Pi_- \Theta_-^{-1}. \tag{2.1.18}$$

Proposition 2.2. *The logarithmic derivative of Widom constant in* (3.2.1) *is related to the isomonodromic tau-function* (2.0.3) *as [28]*

$$\P_s \log t[J] = \int_{i\mathbb{R}} \frac{dz}{2pi} \operatorname{Tr} \left\{ J^{-1} \P_t J \left[\Psi_+^{-1} \P_z \Psi_+ + \P_z \Theta_+ \Theta_+^{-1} \right] \right\} . \tag{2.1.19}$$

Proof. Begin with the Fredholm determinant [2]

$$t[J] = \det \left[T_{J^{-1}} \circ T_J \right] = \det [PQ] \tag{2.1.20}$$

[2]This computation follows from Theorem 2.3 in [28]. The difference being the choice of factorisation.

where $P = T_{J^{-1}} = \Pi_+ J^{-1}$ and $Q = T_J = \Pi_+ J$. Inverses are $P^{-1} = \Theta_- \Pi_+ \Theta_+^{-1}$ and $Q^{-1} = \Psi_+^{-1} \Pi_+ \Psi_-^{-1}$. Computing the logarithmic derivative

$$
\begin{aligned}
\P_t \log\det [PQ] &= \mathrm{Tr}\left[\P_t P P^{-1} + Q^{-1} \P_t Q\right] \\
&= \mathrm{Tr}\left[-\Pi_+ J^{-1} \P_t J J^{-1} \Theta_- \Pi_+ \Theta_+^{-1} + \Psi_+^{-1} \Pi_+ \Psi_- \Pi_+ \P_t J\right] \\
&= \mathrm{Tr}\left[-\Pi_+ J^{-1} \P_t J \Theta_+ \Pi_+ \Theta_+^{-1} + \Psi_+^{-1} \Pi_+ \Psi_+ J^{-1} \left(1 - \Pi_-\right) \P_t J\right] \\
&= \int_{i\mathbb{R}} \frac{dz}{2pi} \mathrm{Tr}\left\{J^{-1} \P_t J \left[\Psi_+^{-1} \P_z \Psi_+ + \P_z \Theta_+ \Theta_+^{-1}\right]\right\}
\end{aligned}
\tag{2.1.21}
$$

to obtain the last expression we use the fact that $J^{-1}\P_t J$ is a multiplication operator and only the diagonal parts of $Y_+ \Pi_+ Y_+^{-1}$ and $\Theta_+^{-1} \Pi_+ \Theta_+$ contribute to the expression. □

2.2 Integrable kernels

Another important notion of tau-functions is in the form of the determinants of 'integrable kernels' which were first introduced by Its, Izergin, Korepin and Slavnov [69] to study certain Fredholm determinants describing quantum correlation functions for Bose gas as the tau-function of a special solution to the Painlevé V equation. Adhering to the theme of this chapter, given an integrable operator $\mathcal{K}$, the invertibility of the operator $(\mathbb{1} - \mathcal{K})$ is equivalent to solving a suitable Riemann-Hilbert problem on the complex plane. The name integrable is due to the fact that any such operator solves some integrable equation (see [37]). The tau-function is then defined as the determinant $\det(\mathbb{1} - \mathcal{K})$ and we will show that it is related to the isomonodromic tau-function (2.0.3) through multiplicative factors. The advantage of the present construction is that the requirement of factorizability of the jump function of the RHP can be lifted.

Let $\Sigma \subset \mathbb{C}$ be a collection of contours and $\Gamma(z)$ solve the following RHP

Riemann-Hilbert problem 2.1.

- $\Gamma(z)$ is a $N \times N$ matrix valued function is analytic for $z \in \mathbb{C} \backslash \Sigma$.
- The following jump condition holds for $z \in \Sigma$

$$
\Gamma_+(z) = \Gamma_-(z) G(z); \quad G(z) := \left(\mathbb{1} - 2pi f(z) g^T(z)\right) \tag{2.2.1}
$$

  where $f(z), g(z)$ are rectangular matrices of suitable dimension under the constraint

$$
f^T(z) g(z) = 0. \tag{2.2.2}
$$

- In the limit $z \to \infty$,

$$
\Gamma(z) = \mathbb{1} + \mathcal{O}(z^{-1}). \tag{2.2.3}
$$

Let us now define the integral operator $\mathcal{K}$ with kernel $K(z,w)$

$$
K(z,w) := \frac{f^T(z) g(w)}{z - w}, \tag{2.2.4}
$$

and $K(z,z) = f'^T(z)g(z)$. Moreover, the Jacobi variational formula reads:

$$d \log \det (\mathbb{1} - \mathcal{K}) = - \mathrm{Tr}_{L^2(\Sigma)} \left((\mathbb{1} + \mathcal{R}) \circ d\mathcal{K} \right) \tag{2.2.5}$$

where $\mathcal{R}$ is the resolvent operator:

$$\mathcal{R} := (\mathbb{1} - \mathcal{K})^{-1} \circ \mathcal{K}. \tag{2.2.6}$$

The important result [37], [69] is contained in the following theorem.

Theorem 2.1. *The RHP 2.1 is solvable iff the operator $(\mathbb{1} - \mathcal{K})$ is invertible and the resolvent operator is also an integrable operator with kernel given by*

$$R(z,w) := (\mathbb{1} - K)^{-1} \circ K(z,w) = \frac{F^T(z)G(w)}{z-w}, \tag{2.2.7}$$

where,

$$F^T(z) := (\mathbb{1} - \mathcal{K})^{-1} f^T \tag{2.2.8}$$

$$G(z) := g\,(\mathbb{1} - \mathcal{K})^{-1}, \tag{2.2.9}$$

Here the inverse is applied entry-wisely to the vectors f, g. Moreover, the solution of the RHP 2.1 determines the resolvent kernel

$$R(z,w) = \frac{f^T(z)\Gamma^T(z)\Gamma^{-T}(w)g(w)}{z-w}. \tag{2.2.10}$$

Proof. The equation for the resolvent is

$$(\mathbb{1} + \mathcal{R}) \circ (\mathbb{1} - \mathcal{K}) = \mathbb{1} \quad \Leftrightarrow \quad \mathcal{R} - \mathcal{K} = \mathcal{R} \circ \mathcal{K}. \tag{2.2.11}$$

Now we rewrite this equation using the expression (2.2.10)

$$R \circ K(z,w) = \int_\Sigma \frac{f^T(z)\Gamma^T(z)\Gamma^{-T}(x)g(x)}{z-x} \frac{f^T(x)g(w)}{x-w} \mathrm{d}x. \tag{2.2.12}$$

To proceed we observe that the matrix Γ^{-T} solves the jump condition

$$\Gamma_+^{-T} = \Gamma_-^{-T} \left(\mathbb{1} + 2ip\, g f^T \right), \qquad z \in \Sigma. \tag{2.2.13}$$

Observe also that for $x \in \Sigma$ we have

$$\Gamma_+^{-T} - \Gamma_-^{-T} = 2ip\Gamma_-^{-T} g f^T \tag{2.2.14}$$

and therefore the right hand side does not depend on the boundary value, namely $\Gamma_-^{-T} g f^T = \Gamma_+^{-T} g f^T$, by using (2.2.14) and recalling that $g^T f \equiv 0$. Thus (2.2.12) yields

$$\int_\Sigma f^T(z)\Gamma^T(z)\left(\Gamma_+^{-T}(x) - \Gamma_-^{-T}(x)\right) g(w) \frac{1}{(z-x)(x-w)} \frac{\mathrm{d}x}{2ip} = \tag{2.2.15}$$

$$= \int_\Sigma f^T(z)\Gamma^T(z)\left(\Gamma_+^{-T}(x) - \Gamma_-^{-T}(x)\right) g(w) \frac{1}{z-w}\left(\frac{1}{z-x} + \frac{1}{x-w}\right)\frac{\mathrm{d}x}{2ip} = \tag{2.2.16}$$

$$= \frac{f^T(z)\Gamma^T(z)}{z-w}\int_\Sigma \left(\Gamma_+^{-T}(x) - \Gamma_-^{-T}(x)\right)\left(\frac{1}{z-x} + \frac{1}{x-w}\right)\frac{\mathrm{d}x}{2ip} g(w) \tag{2.2.17}$$

This expressions splits into two similar integrals; for this purpose we choose $z \notin \Sigma$ and, using Cauchy's theorem together with the fact that $\Gamma(\infty) = \mathbb{1}$, we have

$$\int_\Sigma \frac{\Gamma_+^{-T}(x) - \Gamma_-^{-T}(x)}{z-x}\frac{\mathrm{d}x}{2ip} = \mathbb{1} - \Gamma^{-T}(z) \tag{2.2.18}$$

Continuing then the chain of equalities we find:

$$(2.2.12) = \frac{f^T(z)\Gamma^T(z)}{z-w}\Bigg[\overbrace{\int_\Sigma \frac{\left(\Gamma_-^{-T}(x) - \Gamma_+^{-T}(x)\right)}{z-x}\frac{\mathrm{d}x}{2ip}}^{\mathbb{1}-\Gamma^T(z)} + \tag{2.2.19}$$

$$+ \overbrace{\int_\Sigma \frac{\left(\Gamma_-^{-T}(x) - \Gamma_+^{-T}(x)\right)}{x-w}\frac{\mathrm{d}x}{2ip}}^{\Gamma^{-T}(w)-\mathbb{1}}\Bigg] g(w) = R(z,w) - K(z,w) \tag{2.2.20}$$

and thus we have shown $\mathcal{R} \circ \mathcal{K} = \mathcal{R} - \mathcal{K}$.

We now show the converse statement; supposing that $\mathbb{1} - \mathcal{K}$ is an invertible operator, in terms of the vector $F(z)$ we construct the matrix Γ as a Cauchy–type integral

$$\Gamma(z) := \mathbb{1} - \int_\Sigma \mathrm{d}x \frac{F(x)g^T(x)}{x-z}. \tag{2.2.21}$$

We then observe that, for $z \in \Sigma$, we have $(\Gamma_+(z) - \Gamma_-(z)) f(z) = -2ipg(z)f^t(z) = 0$, so that the two boundary values coincide. We then evaluate directly either of these two boundary values as follows:

$$\Gamma_\pm(z)f(z) = f(z) - \int_\Sigma \mathrm{d}x \frac{F(x)g^T(x)f(z)}{x-z} = f(z) + F\mathcal{K}^T = f + F - F(\mathbb{1}-\mathcal{K})^T \tag{2.2.22}$$

$$= f + F - f(\mathbb{1}-\mathcal{K})^{-T}(\mathbb{1}-\mathcal{K})^T = f + F - f = F. \tag{2.2.23}$$

Using (2.2.23) thus the formula for Γ (2.2.21) reads

$$\Gamma(z) := \mathbb{1} - \int_\Sigma \mathrm{d}x \frac{\Gamma(x)f(x)g^T(x)}{x-z} \tag{2.2.24}$$

from which it follows that Γ solves the RHP (2.1). $\square$

In order to write the tau-function of a RHP as a Fredholm determinant of an integrable operator, it is therefore sufficient to express the associated jump function as (2.2.1) under the constraint (2.2.2).

Relation to the Malgrange form

Theorem 2.2. *The logarithmic derivative of the determinant of the integrable operator with kernel* (2.2.4) *is related to the JMU tau-function* (2.0.3) *as follows*

$$\P_t \log t_{JMU} = \P_t \log \det (\mathbb{1} - K) - H(J) \tag{2.2.25}$$

where

$$H(J) = \int_\Sigma \left(\P_t f'^T g + f'^T \P_t g \right) dz - 2pi \int_\Sigma g^T f' \P_t g^T f dz \tag{2.2.26}$$

See Appendix A of [10] for a proof.

Our goal in this thesis is to study the formulation of Painlevé tau-functions as Fredholm determinants. Many relevant solutions of the Painlevé equations that appear in various branches of mathematics turn out to be expressed as a Fredholm determinant of some IIKS operator [113, 20, 22, 69]. A few notable examples are: the tau-function of the Painlevé II equation which is expressed as a Fredholm determinant of the Airy kernel and describes the Tracy-Widom distribution [113], the gap probability distribution in random matrices that is described by the sine kernel (Painlevé V) [113], the correlation function of stochastic point processes on a one-dimensional lattice that originated from representations of the infinite symmetric group is a Fredholm determinant with hypergeometric kernel (Painlevé VI) [20],[22].

2.3 Determinant of Plemelj operators

A third construction of tau-function as Fredholm determinants is case specific. The following is the overview of the construction in [54] for the case of a Riemann sphere and Fuchsian singularities. Let $Y(z,t)$ be the solution of a linear problem on a Riemann sphere with $n+2$ Fuchsian singularities. Such a surface can then be decomposed to n pairs of pants or trinions $T^{[k]}, k = 1, ..., n$, which are glued together by annuli $A^{[l]}, l = 1, ..., n-1$. Let $\Phi_k(z,t)$ solve the linear problem associated to each pair of pants $T^{[k]}$.

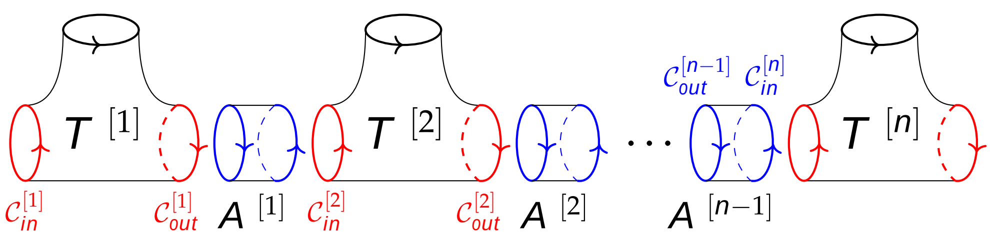


FIGURE 2.1: Pants decomposition for the $n+2$-punctured sphere

A Hilbert space $H^{[k]}$ of vector values functions can be associated to each trinion $T^{[k]}$:

$$H^{[k]} = \bigoplus_{e=in,out} \left(H_{e,+}^{[k]} \oplus H_{e,-}^{[k]} \right), \qquad H_{e,\pm}^{[k]} = \mathbb{C}^N \otimes V_\pm \left(\mathcal{C}_e^{[k]} \right), \tag{2.3.1}$$

where $V_\pm(\mathcal{C})$ denote the space of functions holomorphic inside and outside a neighbourhood of the contour $\mathcal{C}$ respectively, and the total Hilbert space H

$$H := \bigoplus_{k=1}^{n} H^{[k]} = H_+ \oplus H_-, \tag{2.3.2}$$

We then define the following operators

- In terms of $\Phi_k(z,t)$ that solves the linear problem associated to the trinion $T^{[k]}$, we define the operator $\mathcal{P}_\oplus : H \to H$ as

$$\mathcal{P}_\oplus := \bigoplus_{k=1}^{n} \mathcal{P}^{[k]}; \quad \mathcal{P}^{[k]} f^{[k]}(z) := \int_{\mathcal{C}_{in}^{[k]} \cup \mathcal{C}_{out}^{[k]}} \frac{\Phi_k(z)\Phi_k(w)^{-1} f^{[k]}(w)}{z-w} \frac{dw}{2pi} \tag{2.3.3}$$

  where $\mathcal{C}_{in}^{[k]}, \mathcal{C}_{out}^{[k]}$ are the contours shown in the fig. 2.1, and the functions $f^{[k]}(z) \in H^{[k]}$.

- In terms of $Y(z,t)$ that solves the linear problem associated to the n-point sphere, we define the operator $\mathcal{P}_\Sigma : H \to H$

$$\mathcal{P}_\Sigma f(z) := \int_{\mathcal{C}_\Sigma} \frac{Y(z)Y(w)^{-1} f(w)}{z-w} \frac{dw}{2pi}, \qquad \mathcal{C}_\Sigma = \bigcup_{k=1}^{n-1} \mathcal{C}_{out}^{[k]} \cup \mathcal{C}_{in}^{[k+1]} \tag{2.3.4}$$

  where $f(z) \in H$.

The singular factor $\frac{1}{z-w}$ in (2.3.3), (2.3.4) is interpreted by appropriately deforming the contours $\mathcal{C}_{in}, \mathcal{C}_{out}$ in order to avoid the pole at $z = w$. The tau-function is then defined as

$$t := \det_{H_+} \left[\mathcal{P}_\Sigma^{-1} \mathcal{P}_\oplus \right]. \tag{2.3.5}$$

Furthermore, the logarithmic derivative of (2.3.5) relates to the Hamiltonian construction of the tau-function (2.0.2) through multiplicative factors. Refer to [54] for the proof. The case for genus 1 surface with n-punctures is presented in chapter 5.

Remark 2.3. *Note that the constructions in Sections 2.1, 2.2 gave a Fredholm determinant interpretation to the tau-function defined in terms of RHP* (2.0.3) *as opposed to the construction in Section 2.3 which enables us to write the Hamiltonian formulation of the tau-function* (2.0.2) *as a Fredholm determinant.*

2.4 Maya diagrams

One of the advantages of expressing the tau-function as a Fredholm determinant is that the principal minor expansion labelled by Maya diagrams, yields a combinatorial representation of the tau-function. In this subsection, we review the construction [28].

The determinant of an operator $\mathcal{K} \in \mathbb{C}^{m\times m}$ can be expanded in terms of its principal minors. For a finite $m \times m$ matrix $\mathcal{K}$, the minor expansion is

$$\det(\mathbb{1}+\mathcal{K}) = \sum_{n=0}^{\infty} \sum_{i_1<\ldots<i_n} \det(\mathcal{K}_{i_p,i_q})_{p,q=1}^{n}. \tag{2.4.1}$$

This sequence obviously terminates after $n = m$. Now we generalise (2.4.1) if $\mathcal{K}$ is instead an infinite dimensional matrix.

- The matrix $\mathcal{K}$ is now labelled by an infinite discrete set. Define a half-integer lattice $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$. Then, the set of all finite subsets of $\mathbb{Z}'$ is given by $\{0,1\}^{\mathbb{Z}'}$ and $c \subset \{0,1\}^{\mathbb{Z}'}$. For $\mathbb{Z}'_{\mp} = \mathbb{Z}'_{\lessgtr 0}$, we define 'particles' (p_c) to be the positions $p_c = c \cap \mathbb{Z}'_{-}$ and 'holes' (h_c) to be the positions $h_c = c \cap \mathbb{Z}'_{+}$. (p_c, h_c) define point configurations on $\mathbb{Z}'$. Furthermore, for a block matrix, we have two indices.
  - Expansion of the block determinant, given by the particles and holes (p_c, h_c)
  - The index within the block, which is called the colour index.
- Maya diagram m_c is constructed by drawing filled circles at the points $(\mathbb{Z}'_{+} \backslash p_c) \cup h_c$ and empty circles at $p_c \cup (\mathbb{Z}'_{-} \backslash h_c)$. Set of all Maya diagrams is denoted by $\mathbb{M} = \cup_c m_c$.
- For $\det(\mathbb{1}+\mathcal{K})$, the minors can be labelled by the half integer lattice $\mathbb{Z}'$. Rows and coloumns will now be labelled by $c \subset \{0,1\}^{\mathbb{Z}'}$. The minor expansion is given by

$$\det[\mathbb{1}+\mathcal{K}] = \sum_{c\subset\{0,1\}^{\mathbb{Z}'}} \det \mathcal{K}_c \tag{2.4.2}$$

- Maya diagrams can also be written as Young diagrams by playing the following game. Reading the Maya diagram from the left end, draw the arrow ($\searrow$) for every filled circle ($\bullet$) and a vertical line pointing downwards ($\nearrow$) for every empty circle ($\circ$).
- The charge of the Maya diagram m_c is defined as

$$Q_c := \#h_c - \#p_c, \tag{2.4.3}$$

  and the total charge is zero

$$\sum_c Q_c = 0. \tag{2.4.4}$$

Example:

Let $\mathcal{K}$ be a 2×2 block operator acting on a Hilbert space that has an orthogonal decomposition $\mathcal{H} = \mathcal{H}_{+} \oplus \mathcal{H}_{-}$,

$$\mathcal{K} = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \tag{2.4.5}$$

and A, B are 2×2 matrix valued operators acting on $\mathcal{H}_{\mp} \to \mathcal{H}_{\pm}$ respectively. Since A, B are 2×2, the minors will be labelled by a set of 2 colours red and blue. For example, the entries of A are labelled as follows

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{2.4.6}$$

The minor expansion is then obtained by expanding the operator in an appropriate basis of the Hilbert space $\mathcal{H}$ that produces an infinite dimensional matrix $\mathcal{K}$, and summing over all the principal minors. We now explain the labelling for the following choice of a 3×3 principal minor shown in fig. 2.2.

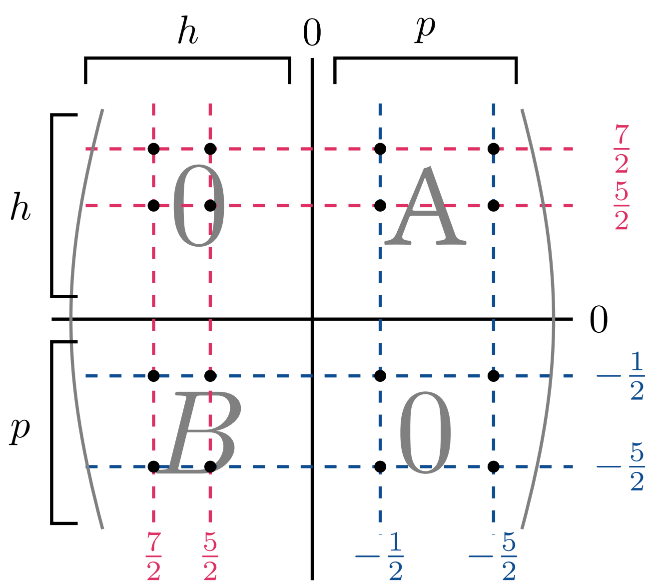


FIGURE 2.2: Minor expansion

From fig. 2.2, let us collect the positive (holes) and negative (particles)[2] of the two colours blue and red:

$$p_c = \left(-\frac{5}{2}, -\frac{1}{2}\right); \quad h_c = \left(\frac{5}{2}, \frac{7}{2}\right) \tag{2.4.7}$$

Recalling that Young diagrams are constructed from the associated Maya diagrams by drawing the arrow ($\searrow$) for every filled circle ($\bullet$) and a vertical line pointing downwards ($\nearrow$) for every empty circle ($\circ$), the configurations of Maya diagrams and Young diagrams are the following

[2]The notation of particles and holes, filled and empty circles can be interchanged in a consistent manner taking care of the definition of the charge (2.4.3)

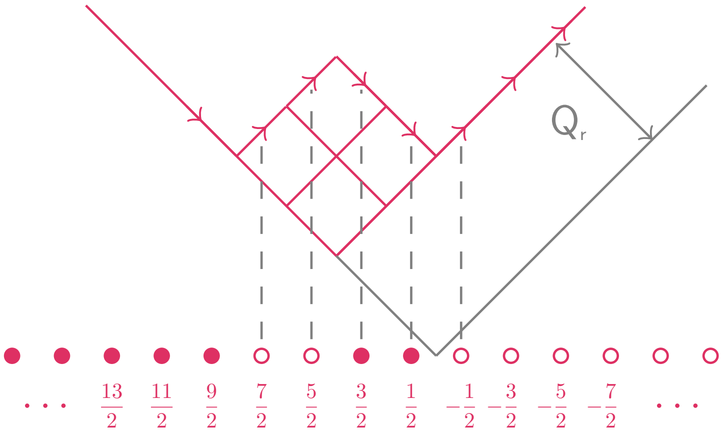


FIGURE 2.3: $m_c = \left(\frac{5}{2}, \frac{7}{2}\right)$, the corresponding charge $Q_r = +2$

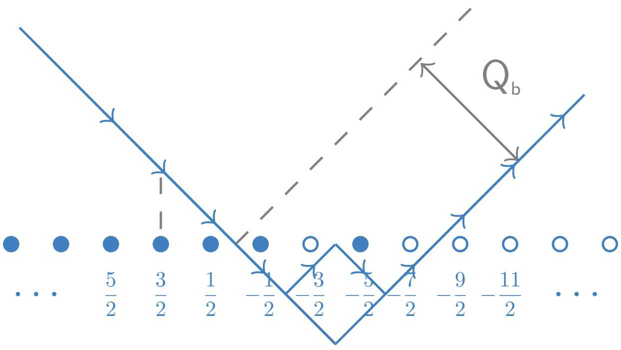


FIGURE 2.4: $m_c = \left(-\frac{5}{2}, -\frac{1}{2}\right)$, the corresponding charge $Q_b = -2$

As is evident from Fig. 2.3, 2.4, the total charge is zero, and since the minor associated to any configuration of Fig: 2.2 that is chosen such that the total charge is non-zero vanishes. Therefore, the labelling of the minors by Maya diagrams automatically retains only the non-zero terms in the expansion.

Remark 2.4. *The zero charge condition depends on the topology of the system. We will see in Chapter 7 that for tau-functions on a torus, the total charge is not zero, but a constant and appears as an overall exponential factor in the minor expansion.*

Remark 2.5. *A tau-function when expressed as a Fredholm determinant could have a combinatorial interpretation through its minor expansion. Such structure however is possible only through the existence of a suitable basis for minor expansion. We will see in this thesis that the tau-function of the general solution of Painlevé II as we construct in Chapter 4 does not have a combinotrial structure due to the lack of an appropriate basis.*

2.5 Aspects of the Airy kernel

In this section we highlight two computations concerning the Airy kernel which are well known but we consider worthy of repetition. The first computation recovers the Painlevé II equation: this exercise is beneficial to understand the working of IIKS kernels better. In the second subsection, we highlight the integral representations of IIKS kernels through the example of the Airy kernel. Similar expressions for Bessel kernel are well known in the literature.

2.5.1 Airy kernel and Painlevé II

Besides constructing the tau-function, given an integrable kernel one can find the associated integrable equation. In this subsection, we discuss the procedure to start form the Airy kernel and obtain the Painlevé II equation. Although the procedure is well known, it is a quick and important exercise that is worth repeating here[3].

Let us recall the Airy kernel

$$K_{Ai} := \frac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x-y} = \frac{f^T(x)g(y)}{x-y}, \tag{2.5.1}$$

where

$$f(x) = \begin{pmatrix} Ai(x) \\ Ai'(x) \end{pmatrix}; \quad g(x) = \begin{pmatrix} Ai'(x) \\ -Ai(x) \end{pmatrix}, \tag{2.5.2}$$

and it is straightforward to see that $f(x)^T g(x) = 0$. A translation by a parameter t gives

$$K_t := \frac{Ai(x+t)Ai'(y+t) - Ai(y+t)Ai'(x+t)}{x-y} = \frac{f^T(x+t)g(y+t)}{x-y}. \tag{2.5.3}$$

Invertibility of the operator $(1 - \mathcal{K}_t)$ informs the solvability of the following RHP defined on the interval $\Sigma \equiv [0, +\infty)$

Riemann-Hilbert problem 2.2.

- $\Psi(x,t)$ is analytic on $\mathbb{C}\backslash\Sigma$
- The following jump condition is valid

$$\Psi_+(x,t) = \Psi_-(x,t)J(x,t), \tag{2.5.4}$$

  where $J(x,t) = \mathbb{1} - 2pi f(x+t)g^T(x+t)$

- Asymptotically,

$$\Psi(x,t) = \mathbb{1} + \frac{M_1}{x} + \frac{M_2}{x^2} + O(x^{-3}). \tag{2.5.5}$$

[3]We change the convention in this subsection with respect to (1.3.26). This subsection is to be studied with the notation fixed by (2.5.1)

Now, we show that the jump can be made constant by conjugation. To see this, we write

$$\begin{aligned} J(x,t) = \mathbb{1} - 2pi f(x+t) g^T(x+t) &= I - 2pi \begin{pmatrix} Ai(x+t) \\ Ai'(x+t) \end{pmatrix} \begin{pmatrix} Ai'(x+t) & -Ai(x+t) \end{pmatrix} \\ &= \mathbb{1} - 2pi \begin{pmatrix} Ai(x+t)Ai'(x+t) & -Ai(x+t)^2 \\ (Ai'(x+t))^2 & -Ai'(x+t)Ai(x+t) \end{pmatrix} \\ &= \begin{pmatrix} 1 - 2pi\,(Ai(x+t)Ai'(x+t)) & -Ai(x+t)^2 \\ (Ai'(x+t))^2 & 1 + 2pi\,(Ai'(x+t)Ai(x+t)) \end{pmatrix}, \end{aligned} \tag{2.5.6}$$

and we know that the Airy functions solve the following equation

$$y''(x) - xy(x) = 0, \tag{2.5.7}$$

with 2 independent solutions: $Ai(x)$, $Bi(x)$, and the Wronskian satisfies the following identity

$$\mathcal{W}\left[Ai(x), Bi(x)\right] = \frac{1}{p}. \tag{2.5.8}$$

An important observation now is that the matrix

$$B := \begin{bmatrix} Ai(x+t) & pBi(x+t) \\ Ai'(x+t) & pBi'(x+t) \end{bmatrix}, \tag{2.5.9}$$

satisfies the ODE

$$\frac{\P B}{\P x} = \begin{bmatrix} 0 & 1 \\ x+t & 0 \end{bmatrix} B, \tag{2.5.10}$$

and brings $J(x,t)$ to the desired constant form by conjugation

$$J_0 = B^{-1}(x,t) J(x,t) B(x,t) = \begin{bmatrix} 1 & 2pi \\ 0 & 1 \end{bmatrix}. \tag{2.5.11}$$

The function $\Gamma(x,t) := \Psi(x,t)B(x,t)$, where Ψ is the solution of (2.5.4), then has a constant jump J_0 on the contour Σ. The Lax pair (A, U) is computed as follows

$$\frac{d\Gamma}{dx}\Gamma^{-1}|_{x\to\infty} + \frac{d\Gamma}{dx}\Gamma^{-1}|_{x\to 0} = A(x,t) = xA_1 + A_0 + \frac{1}{x}A_{-1}, \tag{2.5.12}$$

and

$$\frac{d\Gamma}{dt}\Gamma^{-1}|_{x\to\infty} + \frac{d\Gamma}{dt}\Gamma^{-1}|_{x\to 0} = U(x,t) = xU_1 + U_0. \tag{2.5.13}$$

Let us now study the local behaviour of $\frac{d\Gamma}{dx}\Gamma^{-1}$ at 0 and ∞:

- **Behaviour at 0:** Near $x = 0$, $\Psi(x,t)$ can at most have a logarithmic dependence on x

$$\lim_{x\to 0} \Psi(x,t) = \mathcal{O}(\log x). \tag{2.5.14}$$

The behaviour at 0 of $\Gamma(x,t) = \Psi(x,t)B(x,t)$ is then determined by the jump. Given the jump J_0 on the contour $[0,\infty)$, the local behaviour of the solution at 0 is

$$\lim_{x\to 0}\Gamma(x,t) = g(x)x^{-\frac{1}{2pi}\log J_0} = g(x)\exp\left[-\left(\frac{1}{2pi}\log J_0\right)\log x\right]. \qquad (2.5.15)$$

The term

$$\log J_0 = \log\left(\begin{bmatrix} 1 & 2pi \\ 0 & 1 \end{bmatrix}\right) = \log\left(I + \begin{bmatrix} 0 & 2pi \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 2pi \\ 0 & 0 \end{bmatrix}. \qquad (2.5.16)$$

Using the identity

$$\log(\mathbb{1}+K) = K - \frac{K^2}{2} + \dots \qquad (2.5.17)$$

and noticing that

$$\begin{bmatrix} 0 & 2pi \\ 0 & 0 \end{bmatrix}^2 = 0, \qquad (2.5.18)$$

(2.5.15) reads

$$\begin{aligned}\lim_{x\to 0}\Gamma(x,t) = g(x)\exp\left[-\left(\frac{1}{2pi}\log J_0\right)\log x\right] &= g(x)\exp\left[-\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\log x\right] \\ &= g(x)\left[\mathbb{1} - \log x\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]. \end{aligned} \qquad (2.5.19)$$

To obtain the last step, we expand the exponential and notice that powers of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are all 0. Therefore,

$$\begin{aligned}\frac{d\Gamma}{dx}\Gamma^{-1}|_{x=0} &= g'(x)[1-\log x(.)][1+\log x(.)]g^{-1}(x) \\ &+ g(x)\left[0 - \frac{1}{x}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right][1+\log x(.)]g^{-1}(x) \\ &= -\frac{1}{x}g(x)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}g^{-1}(x) = \frac{1}{x}\begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix}, \end{aligned} \qquad (2.5.20)$$

with

$$g(x) = \begin{bmatrix} u & \widetilde{u} \\ v & \widetilde{v} \end{bmatrix}. \qquad (2.5.21)$$

- **Behaviour at ∞:** A direct substitution of the asymptotics gives

$$\begin{aligned}\frac{d\Gamma}{dx}\Gamma^{-1}|_{x=\infty} &= \left[\left(\mathbb{1}+\frac{M_1}{x}+\frac{M_2}{x^2}\right)B(x,t)\right]' B(x,t)^{-1}\left(\mathbb{1}+\frac{M_1}{x}+\frac{M_2}{x^2}\right)^{-1}\\ &= -\frac{M_1}{x^2}\left(\mathbb{1}-\frac{M_1}{x}\right)+\left(\mathbb{1}+\frac{M_1}{x}+\frac{M_2}{x^2}\right)B(x,t)'B(x,t)^{-1}\left(\mathbb{1}+\frac{M_1}{x}+\frac{M_2}{x^2}\right)^{-1}\\ &= \left(\mathbb{1}+\frac{M_1}{x}\right)\begin{pmatrix}0 & 1\\ x+t & 0\end{pmatrix}\left(\mathbb{1}-\frac{M_1}{x}\right)\\ &= \left(\mathbb{1}+\frac{M_1}{x}\right)\left[x\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}+\begin{pmatrix}0 & 1\\ t & 0\end{pmatrix}\right]\left(\mathbb{1}-\frac{M_1}{x}\right)\\ &= x\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}+\left[M_1,\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right]+\begin{pmatrix}0 & 1\\ t & 0\end{pmatrix}. \end{aligned} \tag{2.5.22}$$

Denoting by

$$M_1 = \begin{pmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}, \tag{2.5.23}$$

then the commutator gives

$$\left[M_1,\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right] = \begin{pmatrix} m_{12} & 0\\ m_{22}-m_{11} & -m_{12}\end{pmatrix}. \tag{2.5.24}$$

Substituting (2.5.24) in (2.5.22),

$$\begin{aligned} x\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}+\left[M_1,\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right]+\begin{pmatrix}0 & 1\\ t & 0\end{pmatrix} &= x\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\\ &+\begin{pmatrix} m_{12} & 0\\ m_{22}-m_{11} & -m_{12}\end{pmatrix}+\begin{pmatrix}0 & 1\\ t & 0\end{pmatrix}\\ &= x\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}+\begin{pmatrix} m_{12} & 1\\ t+m_{22}-m_{11} & -m_{12}\end{pmatrix}\\ &= x\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}+\begin{pmatrix} w & 1\\ p & -w\end{pmatrix}, \end{aligned} \tag{2.5.25}$$

where $w := m_{12}$, $p := t + m_{22} - m_{11}$.

Gathering all the terms,

$$\begin{aligned} A(x,t) &= \frac{d\Gamma}{dx}\Gamma^{-1}|_{x=\infty}+\frac{d\Gamma}{dx}\Gamma^{-1}|_{x=0}\\ &= \frac{1}{x}\begin{pmatrix} uv & -u^2\\ v^2 & -uv\end{pmatrix}+x\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}+\begin{pmatrix} w & 1\\ p & -w\end{pmatrix}. \end{aligned} \tag{2.5.26}$$

Since

$$B(x,t)'B(x,t)^{-1} = \dot{B}(x,t)B(x,t)^{-1}, \tag{2.5.27}$$

$$U(x,t) = A_0 + xA_1 \tag{2.5.28}$$

To obtain the integrable equation, we simply compute the consistency condition of the Lax matrices

$$A_t - U_x = [U, A]. \tag{2.5.29}$$

Starting with the right hand side of the above equation

$$[U, A] = \left[(A_0 + xA_1), \left(\frac{1}{x}A_{-1}\right)\right] = \frac{1}{x}[A_0, A_{-1}] + [A_1, A_{-1}], \tag{2.5.30}$$

and the commutator

$$\begin{aligned}
[A_0, A_{-1}] &= \begin{pmatrix} w & 1 \\ p & -w \end{pmatrix} \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix} - \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix} \begin{pmatrix} w & 1 \\ p & -w \end{pmatrix} \\
&= \begin{pmatrix} uvw + v^2 & -u^2w - uv \\ puv - wv^2 & -pu^2 + wuv \end{pmatrix} - \begin{pmatrix} uvw - pu^2 & uv + wu^2 \\ wv^2 - puv & v^2 + uvw \end{pmatrix} \\
&= \begin{pmatrix} pu^2 + v^2 & -2(u^2w + uv) \\ 2(puv - wv^2) & -pu^2 - v^2 \end{pmatrix}.
\end{aligned} \tag{2.5.31}$$

Similarly,

$$\begin{aligned}
[A_1, A_{-1}] &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix} - \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ uv & -u^2 \end{pmatrix} - \begin{pmatrix} -u^2 & 0 \\ -uv & 0 \end{pmatrix} \\
&= \begin{pmatrix} u^2 & 0 \\ 2uv & -u^2 \end{pmatrix}.
\end{aligned} \tag{2.5.32}$$

Therefore,

$$\begin{aligned}
A_t - U_x &= \frac{1}{x}[A_0, A_{-1}] + [A_1, A_{-1}] \\
\frac{1}{x} \begin{pmatrix} (uv)_t & -2uu_t \\ 2vv_t & -(uv)_t \end{pmatrix} &+ \begin{pmatrix} w_t & 0 \\ p_t & -w_t \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\end{aligned} \tag{2.5.33}$$

Comparing constant coefficients in (2.5.30) and (2.5.33)

$$\begin{aligned}
[A_1, A_{-1}] &= \begin{pmatrix} w_t & 0 \\ p_t - 1 & -w_t \end{pmatrix} \\
\begin{pmatrix} u^2 & 0 \\ 2uv & -u^2 \end{pmatrix} &= \begin{pmatrix} w_t & 0 \\ p_t - 1 & -w_t \end{pmatrix} \\
&\Rightarrow w_t = u^2; \quad p_t = 2uv + 1,
\end{aligned} \tag{2.5.34}$$

and comparing the coefficients of $1/x$

$$[A_0, A_{-1}] = \begin{pmatrix} (uv)_t & -2uu_t \\ 2vv_t & -(uv)_t \end{pmatrix}$$
$$\begin{pmatrix} pu^2+v^2 & -2(u^2w+uv) \\ 2(puv-wv^2) & -pu^2-v^2 \end{pmatrix} = \begin{pmatrix} (uv)_t & -2uu_t \\ 2vv_t & -(uv)_t \end{pmatrix}$$
$$\Rightarrow u_t = uw+v; \quad v_t = pu-wv. \tag{2.5.35}$$

From (2.5.34) and (2.5.35), we see that

$$2uu_t = 2u^2w + 2vu = 2w_tw + p_t - 1$$
$$\Rightarrow u^2 = w^2 + p - t + const. \tag{2.5.36}$$

Painlevé II is now derived from (2.5.34), (2.5.35) and (2.5.36)

$$u_{tt} = u_tw + uw_t + v_t = (uw+v)w + u^3 + pu - wv = uw^2 + pu + u^3$$
$$= u(w^2+p) + u^3 = u(u^2 + t - const.) + u^3$$
$$= 2u^3 + ut \quad . \tag{2.5.37}$$

The $const.$ can be set to zero by a translation ($t \to t + const.$).

Remark 2.6. *The above procedure works only if the jump of the RHP can be conjugated to be a constant and the Wronskian satisfies an ODE.*

2.5.2 Integral representation of the Airy kernel

We now turn to integral representation of integrable kernels, which are better tailored to the formalism of RHPs. In this section, we will represent the Airy kernel in terms of some contour integrals on the Fourier space. Let us recall the kernel

$$K_t = \frac{Ai(x+t)Ai'(y+t) - Ai'(x+t)Ai(y+t)}{x-y}, \tag{2.5.38}$$

and the following equality holds

$$\frac{d^2}{dz^2}Ai(z+t) = \frac{d^2}{dt^2}Ai(z+t) = \frac{d^2}{dz\,dt}Ai(z+t) = (z+t)Ai(z+t). \tag{2.5.39}$$

With (2.5.39), the following statement about (2.5.38) is true:

$$\frac{d}{dt}K_t = \frac{Ai(x+t)Ai''(y+t) - Ai''(x+t)Ai(y+t)}{x-y}$$
$$= \frac{(y+t)Ai(x+t)Ai(y+t) - (x+t)Ai(x+t)Ai(y+t)}{x-y}$$
$$= -Ai(x+t)Ai(y+t). \tag{2.5.40}$$

Therefore,

$$K_t = -\int_0^{+\infty} \frac{dt}{2pi} Ai(x+t)Ai(y+t). \tag{2.5.41}$$

Furthermore, Airy functions have the following integral representation

$$Ai(z) = \int_{\infty e^{-ip/3}}^{\infty e^{+ip/3}} e^{\frac{s^3}{3} - zs} \frac{ds}{2pi}. \tag{2.5.42}$$

Using (2.5.42), (2.5.41) can be rewritten as follows

$$\begin{aligned} K_t &= -\int_0^{+\infty} \frac{dt}{2pi} Ai(x+t)Ai(y+t) \\ &= -\int_0^{+\infty} \frac{dt}{2pi} \int_{\infty e^{-ip/3}}^{\infty e^{+ip/3}} \frac{ds}{2pi} e^{\frac{s^3}{3} - (x+t)s} \int_{\infty e^{-ip/3}}^{\infty e^{+ip/3}} \frac{ds'}{2pi} e^{\frac{(s')^3}{3} - (y+t)s'}. \end{aligned} \tag{2.5.43}$$

transform $s' \to -s'$

$$\begin{aligned} (2.5.43) &= -\int_0^{+\infty} \frac{dt}{2pi} \int_{\infty e^{-ip/3}}^{\infty e^{+ip/3}} \frac{ds}{2pi} e^{\frac{s^3}{3} - (x+t)s} \int_{\infty e^{-ip/3}}^{\infty e^{+ip/3}} \frac{ds'}{2pi} e^{-\frac{(s')^3}{3} + (y+t)s'} \\ &= -\int_{\infty e^{-ip/3}}^{\infty e^{+ip/3}} \frac{ds}{2pi} e^{\frac{s^3}{3} - xs} \int_{\infty e^{-ip/3}}^{\infty e^{+ip/3}} \frac{ds'}{2pi} e^{-\frac{(s')^3}{3} + ys'} \int_0^{+\infty} \frac{dt}{2pi} e^{-t(s-s')} \\ &= \int_{\infty e^{-ip/3}}^{\infty e^{+ip/3}} \frac{ds}{2pi} \int_{\infty e^{-ip/3}}^{\infty e^{+ip/3}} \frac{ds'}{2pi} \frac{1}{s'-s} e^{\frac{s^3}{3} - xs - \frac{(s')^3}{3} + ys'} \\ &= \int_{g_s} \frac{ds}{2pi} \int_{g_{s'}} \frac{ds'}{2pi} \frac{e^{q_x(s) - q_y(s')}}{s'-s}, \end{aligned} \tag{2.5.44}$$

where $q_x(s) = \frac{s^3}{3} - xs$. Therefore, the Airy kernel is

$$K_t = \int_{g_s} \frac{ds}{2pi} \int_{g_{s'}} \frac{ds'}{2pi} \frac{e^{q_x(s) - q_y(s')}}{s'-s}, \quad q_x(s) = \frac{s^3}{3} - xs. \tag{2.5.45}$$

In the next chapter, we reproduce this kernel as the tau-function of the Ablowitz-Segur family of solutions through Riemann-Hilbert techniques.

The story so far...

The recent works of Lisovyy, Cafasso, Gavrylenko [28, 54] provide a method to formulate the isomonodromic tau-functions (related to the 2-parameter solutions) of certain Painlevé equations (PIII, PV, PVI) as Fredholm determinants. Not only does the Fredholm determinant representation of the tau function provide an explicit formulation of the general solution to the Painlevé equations (Painlevé transcendents), but also reveals the combinatorial structure in terms of charged partitions underlying the tau functions, that are organized as a convergent power series in the isomonodromic time: such a representation is especially remarkable given the transcendental nature of these solutions. There are two key aspects to their construction. One is the property that the RHPs of these Painlevé equations can be reduced on to a RHP on the unit circle (see remark:1.3). The second feature is that the jump on the unit circle contour admits Birkhoff factorization enabling the formulation of the tau-function as a determinant of Toeplitz operators called the Widom constant. This approach cannot be directly implemented to the cases where the RHP is formulated on a contour that is not a circle as is the case for the Painlevé equations PI, PII and PIV. However it is expected that the generic RHP for these equations could be reduced to a RHP on the line for a jump matrix G. Considering the Ablowitz-Segur family of solutions of Painlevé II as a toy model, we show in Chapter 3 that the corresponding tau-function, which is the determinant of the Airy kernel (2.5.45), can be expressed as a Widom constant on the imaginary axis 2.1.2. In Chapter 4, we consider the general solutions of Painlevé II (1.3.3) and show that the corresponding tau-function can be written as a Fredholm determinant of an Integrable kernel in the sense of IIKS following the set up in Section 2.2. In Chapter 5, we instead shift the focus to the formulation of Painlevé VI as the 2-particle non-autonomous Calogero-Moser system (1.2.29) and show that the associated tau-function has the form of Fredholm determinant of Plemelj operators 2.3 on a torus . In the final part of the thesis, we present the minor expansion of two of the three tau-functions obtained in Part 2. In particular, we will show that the Airy kernel has a combinatorial representation in terms of colourless, chargeless Maya diagrams, and the minor expansion of the tau-functions on a torus are labelled by coloured and charged partitions.

Part II

Fredholm determinants

Chapter 3

Ablowitz-Segur family of solutions

3.1 Toy model: Ablowitz-Segur solution

The special 1-parameter (Ablowitz-Segur) family of solutions to the Painlevé II equation can be recast as a RHP on the imaginary axis as opposed to the unit circle in [28], hinting at a similar structure for the general RHP of Painlevé II. As a consequence, the corresponding tau-function which is known to be the determinant of the Airy kernel [113], can be formulated as a Widom constant.

From the RHP 1.3, the isomonodromic tau-function of Ablowitz-Segur family of solutions is defined by[1] [72]

$$¶_s \log t_{JMU}(s) = -\mathrm{Res}_{z=\infty} \mathrm{Tr}\left[\Gamma(z,s)'\Gamma^{-1}(z,s)\,(z s_3)\right], \tag{3.1.1}$$

and we have the relation

$$u^2(s) = -\frac{d^2}{ds^2}\log t_{JMU}(s).$$

The RHPs on each of the contours g_L, g_R can be solved locally and let $\Theta_{L,R}(z,s)$ be the respective solutions.

Riemann-Hilbert problem 3.1.

- The 2×2 matrix valued functions Θ_i are analytic for $z \in \mathbb{C}\backslash g_i$, $i = L, R$.
- For $z \in g_i$,

$$\Theta_{i+} = \Theta_{i-} G_i. \tag{3.1.2}$$

- Asymptotically,

$$\lim_{z\to\infty} \Theta_i = \mathbb{1}. \tag{3.1.3}$$

The RHP 3.1 is easy to solve and it is straightforward to see that the solutions Θ_L, Θ_R are given by the Cauchy transforms of the respective jumps G_L, G_R (see Fig. 3.1):

[1] In this chapter, we use the variables (z, s) instead of (l, x).

$$\Theta_L(z,s) = \begin{bmatrix} 1 & 0 \\ k\int_{g_L} \frac{e^{-q(l,s)}}{l-z}\frac{dl}{2pi} & 1 \end{bmatrix} \tag{3.1.4}$$

$$\Theta_R(z,s) = \begin{bmatrix} 1 & k\int_{g_R} \frac{e^{q(l,s)}}{l-z}\frac{dl}{2pi} \\ 0 & 1 \end{bmatrix}. \tag{3.1.5}$$

Define a new function

$$\Psi(z,t) = \begin{cases} \Gamma\Theta_L^{-1}, & \text{for } \Re z < 0, \\ \Gamma\Theta_R^{-1}, & \text{for } \Re z > 0. \end{cases} \tag{3.1.6}$$

Such function Ψ solves the following RHP on the imaginary axis

Riemann-Hilbert problem 3.2.

- $\Psi(z,t)$ is analytic in $z \in \mathbb{C}\backslash i\mathbb{R}$.
- The following jump condition on $i\mathbb{R}$ holds

$$\Psi_+ = \Psi_- J \tag{3.1.7}$$

The jump J is explicitly

$$J(z,s) = \Psi_-^{-1}\Psi_+ = \Theta_R(z,s)\Theta_L(z,s)^{-1}$$
$$= \begin{bmatrix} 1 & k\int_{g_R} \frac{e^{q(l,s)}}{l-z}\frac{dl}{2pi} \\ -k\int_{g_L} \frac{e^{-q(l,s)}}{l-z}\frac{dl}{2pi} & 1 \end{bmatrix}. \tag{3.1.8}$$

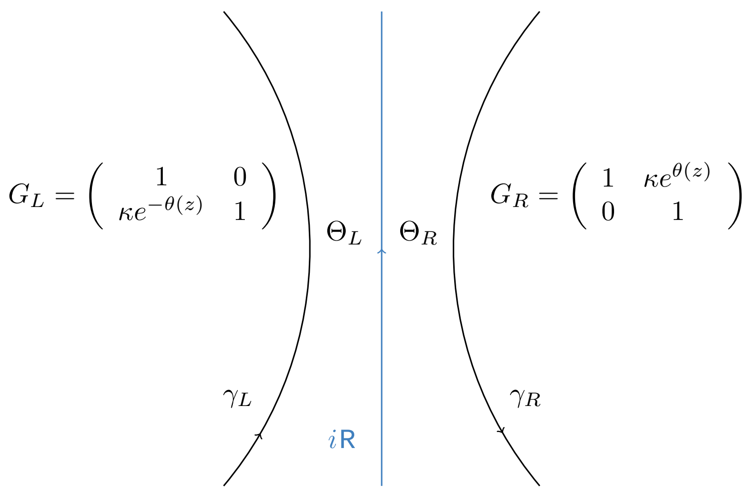


FIGURE 3.1: Contour

With the original RHP now recast as a RHP on the imaginary axis, we can construct the tau-function as a Widom constant (see Section 2.1.2).

3.2 Computing the Widom constant

In this section we want to make sense of the quantity $\det\left(T_{J^{-1}} \circ T_J\right)$ introduced in (2.1.13) when the matrix J admits Birkhoff factorization (3.1.8). In Proposition 2.1 we show that

$$t[J] = \det\left(T_{J^{-1}} \circ T_J\right) = \det{}_{\mathcal{H}}\left[\mathbf{1} + U\right] \tag{3.2.1}$$

where $\mathbf{1}$ denotes the identity operator on $\mathcal{H}$, and $U = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ with $a : \mathcal{H}_- \to \mathcal{H}_+$; $b : \mathcal{H}_+ \to \mathcal{H}_-$ given by,

$$a = \Theta_L \Pi_+ \Theta_L^{-1} - \Pi_+ \quad ; \quad b = \Pi_- - \Theta_R \Pi_- \Theta_R^{-1}.$$

where the explicit form of Θ_L, Θ_R is known (3.1.4), (3.1.5). We now want to show that the quantity (3.2.1) is a Fredholm determinant and coincides with the tau-function defined in [7].

We remind the reader of the result in [7] where the tau-function of the Ablowitz-Segur family of solutions for Painlevé II is given by the following Fredholm determinant[1]

$$t(s) = \det\left[Id_{L^2(g_+ \cup g_-)} - k\begin{bmatrix} 0 & \mathcal{F} \\ \mathcal{G} & 0 \end{bmatrix}\right] = \det\left[Id_{L^2(g_+)} - k^2 \mathcal{F} \circ \mathcal{G}\right] \tag{3.2.2}$$

with

$$\mathcal{F} : L^2(g_-) \to L^2(g_+) \quad \mathcal{G} : L^2(g_+) \to L^2(g_-) \tag{3.2.3}$$

and

$$(\mathcal{F}g)(z) = e^{-\frac{i}{2}q(z,s)} \int_{\mathbb{R}-ic} \frac{dw}{2pi} \frac{e^{\frac{i}{2}q(w,s)} g(w)}{w - z} \tag{3.2.4}$$

$$(\mathcal{G}g)(z) = e^{\frac{i}{2}q(z,s)} \int_{\mathbb{R}+ic} \frac{dw}{2pi} \frac{e^{-\frac{i}{2}q(w,s)} g(w)}{w - z} \tag{3.2.5}$$

Theorem 3.1. *The tau-function (3.2.2) of the Ablowitz-Segur family of solutions of the Painlevé II equation is the Widom constant defined in (3.2.1).*

Proof. The Widom constant can be obtained from (3.2.1) by computing the operators a, b explicity. Let $f(z) \in \mathcal{H}_-$, $h(z) \in \mathcal{H}_+$ be vector valued functions

$$f \equiv \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}; \; h \equiv \begin{pmatrix} h_1 \\ h_2 \end{pmatrix},$$

then

$$af(z) = \int_{i\mathbb{R}} \frac{dw}{2pi} \frac{\Theta_R(z)\Theta_R^{-1}(w) - 1}{w - z} f(w) \tag{3.2.6}$$

$$bh(z) = \int_{i\mathbb{R}} \frac{dw}{2pi} \frac{1 - \Theta_L(z)\Theta_L^{-1}(w)}{w - z} h(w), \tag{3.2.7}$$

[1] $g_\pm$ are $g_{1,2}$ rotated by $p/2$ and this is also the source of the factor of i in the exponential in [7].

with Θ_1 and Θ_2 are as in (3.1.5). We begin by computing (3.2.6)

$$\Theta_R(z)\Theta_R^{-1}(w) - 1 = \begin{bmatrix} 0 & k\int_{g_R}\left(\frac{e^{q(l,s)}}{l-z} - \frac{e^{q(l,s)}}{l-w}\right)\frac{dl}{2pi} \\ 0 & 0 \end{bmatrix} \tag{3.2.8}$$

substituting (3.2.8) in (3.2.7) and focusing on the only non-zero entry a_{12},

$$a_{12}f_2(z) = -k\int_{i\mathbb{R}}\frac{dw}{2pi}\int_{g_R}\frac{dl}{2pi}\frac{e^{q(l,s)}}{(l-z)(l-w)}f_2(w) \tag{3.2.9}$$

integrating over l,

$$a_{12}f_2(z) = -k\int_{i\mathbb{R}-e}\frac{dw}{2pi}\frac{e^{q(w,s)}}{w-z}f_2(w) \tag{3.2.10}$$

A similar computation for b gives that the only non-zero entry is b_{21} that reads

$$b_{21}h_1(z) = k\int_{i\mathbb{R}}\int_{g_L}\frac{e^{-q(l,s)}}{(l-z)(l-w)}h_1(w)\frac{dl}{2pi}\frac{dw}{2pi} \tag{3.2.11}$$

integrating over l

$$b_{21}h_1(z) = k\int_{i\mathbb{R}+e}\frac{e^{-q(w,s)}}{(w-z)}h_1(w)\frac{dw}{2pi} \tag{3.2.12}$$

Substituing a and b back in (3.2.1), we get the following

$$t(s) = \det\left[Id_{L^2(i\mathbb{R})\otimes\mathbb{C}^2} - \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}\right] = \det\left[Id_{L^2(i\mathbb{R})\otimes\mathbb{C}^2} - \begin{bmatrix} 0 & 0 & 0 & a_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_{21} & 0 & 0 & 0 \end{bmatrix}\right] \tag{3.2.13}$$

$$= \det\left[Id_{L^2(i\mathbb{R})} - \begin{bmatrix} 0 & a_{12} \\ b_{21} & 0 \end{bmatrix}\right]. \tag{3.2.14}$$

Further, it is straightforward to see that the operator $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ is Hilbert-Schmidt. In other words, a_{12} and b_{21} are Hilbert-Schmidt

$$|a_{12}|^2 = -k^2\int_{i\mathbb{R}+e}|dz|\int_{i\mathbb{R}-e}|dw|\frac{e^{q(w,s)+q(\bar{w})}}{|w-z|^2} = -k^2\int_{i\mathbb{R}+e}|dz|\int_{i\mathbb{R}-e}|dw|\frac{e^{2Req(w,s)}}{|w-z|^2} < +\infty \tag{3.2.15}$$

$$|b_{21}|^2 = -k^2\int_{i\mathbb{R}-e}|dz|\int_{i\mathbb{R}+e}|dw|\frac{e^{-q(w,s)-q(\bar{w})}}{|w-z|^2} = -k^2\int_{i\mathbb{R}-e}|dz|\int_{i\mathbb{R}+e}|dw|\frac{e^{-2Req(w,s)}}{|w-z|^2} < +\infty \tag{3.2.16}$$

(3.2.16) and (3.2.15) are clearly convergent, implying that a_{21} and b_{12} are Hilbert-Schmidt operators. Therefore, the determinant $\det_{\mathcal{H}}[1 + a\circ b]$ is Fredholm and coincides with the tau-function in (3.2.2). □

Remark 3.1. *Further, it is shown in [10] that*

$$\det\left[Id_{L^2(g_L\cup g_R)} + U\right] = \det\left[Id_{L^2([s,\infty))} - kK_{Ai}|_{[s,\infty)}\right] \tag{3.2.17}$$

where K_{Ai} is the Airy kernel, which implies the tau-function (3.2.14) *is the determinant of the Airy Kernel. It is a well known result [113] that the solution of* (1.3.21) *is related to the Airy Kernel as*

$$u(s)^2 = -\frac{d^2}{ds^2} \log \det \left[\mathbf{1} - kK_{Ai}|_{[s,\infty)} \right] \tag{3.2.18}$$

3.3 Relation to the JMU tau-function

The logarithmic derivative of the Widom constant (3.2.1) can be shown to coincide with the logarithmic derivative of the isomonodromic tau-function (3.1.1).

Notation: $' \equiv \frac{¶}{¶z}$ and $^{\cdot} \equiv \frac{¶}{¶s}$. All functions depend on z,s unless stated otherwise.

Using the above proposition we can identify the Widom constant with the isomonodromic t function.

Proposition 3.1. *The logarithmic derivative of the Widom constant* (2.1.19) *coincides exactly with the logarithmic derivative of the (isomonodromic) JMU t-function* (3.1.1) *for Ablowitz-Segur family of solutions namely :*

$$¶_s \log t[J] = ¶_s \log t_{JMU} = -Res_{z=\infty} \operatorname{Tr} \left[\Gamma^{-1}\Gamma' \left(z s_3 \right) \right] . \tag{3.3.1}$$

Proof. We prove the statement by simplifying the expression of $¶_s \log t[J]$ in (2.1.19):

$$\operatorname{Tr} \left\{ J^{-1} \dot{J} \left[\Psi_+^{-1} \Psi_+' + \Theta_+' \Theta_+^{-1} \right] \right\} . \tag{3.3.2}$$

We first perform algebraic manipulation on $\Psi_+^{-1}\Psi_+'$ with $\Theta_+ \equiv \Theta_L$, and using (3.1.6)

$$\begin{aligned} \Psi_+^{-1}\Psi_+ &= \left(\Theta_L \Gamma^{-1} \right) \left(\Gamma \Theta_L^{-1} \right)' \\ &= \left(\Theta_L \Gamma^{-1} \right) \left(\Gamma' \Theta_L^{-1} - \Gamma \Theta_L^{-1} \Theta_L' \Theta_L^{-1} \right) \\ &= \Theta_L \left(\Gamma^{-1}\Gamma' - \Theta_L^{-1}\Theta_L' \right) \Theta_L^{-1} \end{aligned} \tag{3.3.3}$$

and expressing J in terms of Θ_L and Θ_R we obtain

$$\begin{aligned} J^{-1}\dot{J} &= (\Theta_L \Theta_R^{-1}) ¶_t (\Theta_R \Theta_L^{-1}) \\ &= (\Theta_L \Theta_R^{-1})(\dot{\Theta}_R \Theta_L^{-1} - \Theta_R \Theta_L^{-1} \dot{\Theta}_L \Theta_L^{-1}) \\ &= \Theta_L (\Theta_R^{-1}\dot{\Theta}_R - \Theta_L^{-1}\dot{\Theta}_L) \Theta_L^{-1} \\ &= -\Theta_L \Delta(\Theta^{-1}\dot{\Theta}) \Theta_L^{-1} \end{aligned} \tag{3.3.4}$$

where

$$\Delta(\Theta^{-1}\dot{\Theta}) = \Theta_L^{-1}\dot{\Theta}_L - \Theta_R^{-1}\dot{\Theta}_R.$$

Substituting (3.3.3) and (3.3.4) in (3.3.2) and using cyclicity of trace,

$$\operatorname{Tr} \left\{ J^{-1}\dot{J}\, \Psi_+^{-1}\Psi_+' \right\} = -\operatorname{Tr} \left\{ \Delta \left(\Theta^{-1}\dot{\Theta} \right) \left(\Gamma^{-1}\Gamma' - \Theta_L^{-1}\Theta_L' \right) \right\} \tag{3.3.5}$$

The term $\mathrm{Tr}\left[\Delta\left(\Theta^{-1}\dot{\Theta}\right)\Theta_L^{-1}\Theta_L'\right]$ is explicit and cancels the term $\mathrm{Tr}\left[J^{-1}\dot{J}\Theta_L'\Theta_L^{-1}\right]$. After the simplification, (2.1.19) is

$$-\int_{i\mathbb{R}}\frac{dz}{2pi}\mathrm{Tr}\left\{\Delta\left(\Theta^{-1}\dot{\Theta}\right)\Gamma^{-1}\Gamma'\right\} \tag{3.3.6}$$

since Γ has no jump on $i\mathbb{R}$, (3.3.6) can be further simplified

$$\begin{aligned}-\int_{i\mathbb{R}}\frac{dz}{2pi}\mathrm{Tr}\left\{\Delta\left(\Theta^{-1}\dot{\Theta}\right)\Gamma^{-1}\Gamma'\right\} &= -\int_{i\mathbb{R}}\frac{dz}{2pi}\mathrm{Tr}\,\Delta\left\{\Theta^{-1}\dot{\Theta}\Gamma^{-1}\Gamma'\right\}\\ &= \int_{\Sigma}\frac{dz}{2pi}\mathrm{Tr}\,\Delta\left\{\Theta^{-1}\dot{\Theta}\Gamma^{-1}\Gamma'\right\}\\ = \int_{g_L}\frac{dz}{2pi}\mathrm{Tr}\,\Delta\left\{\Theta^{-1}\dot{\Theta}\Gamma^{-1}\Gamma'\right\} &+ \int_{g_R}\frac{dz}{2pi}\mathrm{Tr}\,\Delta\left\{\Theta^{-1}\dot{\Theta}\Gamma^{-1}\Gamma'\right\}\end{aligned} \tag{3.3.7}$$

Let us begin by computing the integral on g_L in (3.3.7)

$$\mathrm{Tr}\,\Delta\left\{\Theta^{-1}\dot{\Theta}\Gamma^{-1}\Gamma'\right\} = \mathrm{Tr}\left\{\Theta_{L+}^{-1}\dot{\Theta}_{L+}\Gamma_+^{-1}\Gamma_+' - \Theta_{L-}^{-1}\dot{\Theta}_{L-}\Gamma_-^{-1}\Gamma_-'\right\} \tag{3.3.8}$$

computing (3.3.8) term by term by substituting (1.3.23) for Γ_+

$$\begin{aligned}\Gamma_+^{-1}\Gamma_+' &= (G_L^{-1}\Gamma_-^{-1})(\Gamma_- G_L)'\\ &= G_L^{-1}\left[\Gamma_-^{-1}\Gamma_-' + G_L'G_L^{-1}\right]G_L\end{aligned} \tag{3.3.9}$$

and (3.1.2) for Θ_{L+}

$$\begin{aligned}\Theta_{L+}^{-1}\dot{\Theta}_{L+} &= G_L^{-1}\Theta_{L-}^{-1}\P_t(\Theta_{L-}G_L)\\ &= G_L^{-1}\left[\Theta_{L-}^{-1}\dot{\Theta}_{L-} + \dot{G}_L G_L^{-1}\right]G_L.\end{aligned} \tag{3.3.10}$$

Substituting (3.3.9), (3.3.10) in (3.3.7) and using cyclicity

$$\mathrm{Tr}\left\{\left(\Theta_{L+}^{-1}\dot{\Theta}_{L+}\right)\Gamma_+^{-1}\Gamma_+'\right\} = \mathrm{Tr}\left[\left(\Theta_{L-}^{-1}\dot{\Theta}_{L-} + \dot{G}_L G_L^{-1}\right)\left(\Gamma_-^{-1}\Gamma_-' + G_L'G_L^{-1}\right)\right] \tag{3.3.11}$$

In (3.3.11), notice that the term $\left(\Theta_{L-}^{-1}\dot{\Theta}_{L-} + \dot{G}_L G_L^{-1}\right)\left(G_L'G_L^{-1}\right)$ is traceless. Furthermore, we have the following identity $2\dot{G}_L G_L^{-1} = -zG_L s_3 G_L^{-1} + zs_3$. The terms $\Theta_{L-}^{-1}\dot{\Theta}_{L-}\Gamma_-^{-1}\Gamma_-'$ in (3.3.11) and (3.3.8) cancel each other out. So, all that is left to compute on the contour g_L is the following

$$\int_{g_L}\frac{dz}{2pi}\mathrm{Tr}\left[\dot{G}_L G_L^{-1}\Gamma_-^{-1}\Gamma_-'\right] = \frac{1}{2}\int_{g_1}\frac{dz}{2pi}\mathrm{Tr}\left[\left(-zG_L s_3 G_L^{-1} + zs_3\right)\Gamma_-^{-1}\Gamma_-'\right] \tag{3.3.12}$$

We begin by computing the following term

$$\mathrm{Tr}\left(G_L s_3 G_L^{-1}\Gamma_-^{-1}\Gamma_-'\right) \tag{3.3.13}$$

the term $\Gamma_-^{-1}\Gamma'_-$ can be simplified by substituting (1.3.23)

$$\begin{aligned}\Gamma_-^{-1}\Gamma'_- &= \left(G_L\Gamma_+^{-1}\right)\left(\Gamma_+G_L^{-1}\right)' \\ &= G_L\left(\Gamma_+^{-1}\Gamma'_+ - G_L^{-1}G'_L\right)G_L \end{aligned} \tag{3.3.14}$$

substituting (3.3.14) in (3.3.13) and using the cyclic property of the trace

$$\operatorname{Tr}\left(G_L s_3 G_L^{-1}\Gamma_-^{-1}\Gamma'_-\right) = \operatorname{Tr}\left[s_3\Gamma_+^{-1}\Gamma'_+ - G_L^{-1}G'_L s_3\right] \tag{3.3.15}$$

note that $G_L^{-1}G'_L s_3$ is traceless. Substituting (3.3.15) in (3.3.12) we have

$$\frac{1}{2}\int_{g_1}\frac{dz}{2pi}\operatorname{Tr}\left[\left(-zG_L s_3 G_L^{-1} + zs_3\right)\Gamma_-^{-1}\Gamma'_-\right] = \frac{1}{2}\int_{g_1}\frac{dz}{2pi}\operatorname{Tr}\left[-zs_3\left(\Gamma_+^{-1}\Gamma'_+ - \Gamma_-^{-1}\Gamma'_-\right)\right] \tag{3.3.16}$$

repeating this exercise and computing the integral on g_R in (3.3.5), we get exactly the same expression. Putting all together

$$\begin{aligned}¶_s \ln t\,[J] &= \int_\Sigma \frac{dz}{2pi}\operatorname{Tr}\left[-zs_3\left(\Gamma_+^{-1}\Gamma'_+ - \Gamma_-^{-1}\Gamma'_-\right)\right] \\ &= -Res_{z=\infty}\operatorname{Tr}\left(zs_3\Gamma^{-1}\Gamma'\right)\end{aligned} \tag{3.3.17}$$

□

Chapter 4

General solutions of Painlevé II

In the previous chapter we saw that the Ablowitz-Segur family of solutions can be expressed as a Widom constant. Now let us investigate the case of general solutions of the homogeneous Painlevé II equation (1.3.3):

$$u'' = xu + 2u^3 \tag{4.0.1}$$

and relate the solution to an appropriate Fredholm determinant.

4.1 Riemann-Hilbert Problem

In order to modify the Riemann–Hilbert contour of Painlevé II, we restrict to the case $x \in \mathbb{R}$ and perform the change of variables

$$l = (-x)^{1/2} z, \quad t = (-x)^{3/2}. \tag{4.1.1}$$

The characteristic exponent $\exp(q(l))$ in (1.3.10) is then replaced by

$$e^{itq(z)}, \quad q(z) = \frac{4}{3}z^3 - z. \tag{4.1.2}$$

The stationary points are then $z_\pm = \pm 1/2$. Noticing that the product of Stokes' matrices $(S_3S_4S_5)^{-1}$ can be written as a product of lower triangular, diagonal and upper triangular matrices (LDU) for $s_1s_3 \neq 1$

$$(S_3S_4S_5)^{-1} = \begin{pmatrix} 1-s_1s_3 & s_1e^{2itq(z)} \\ s_1e^{-2itq(z)} & 1+s_1s_2 \end{pmatrix} = S_LS_DS_U$$
$$= \begin{pmatrix} 1 & 0 \\ s_1(1-s_1s_3)^{-1}e^{-2itq(z)} & 1 \end{pmatrix} \begin{pmatrix} 1-s_1s_3 & 0 \\ 0 & (1-s_1s_3)^{-1} \end{pmatrix} \begin{pmatrix} 1 & s_1(1-s_1s_3)^{-1}e^{2itq(z)} \\ 0 & 1 \end{pmatrix}, \tag{4.1.3}$$

the contour in fig: 1.5 can be transformed into fig 4.1, we call Σ, and one can easily check the no monodromy condition around the points $z = \pm 1/2$.

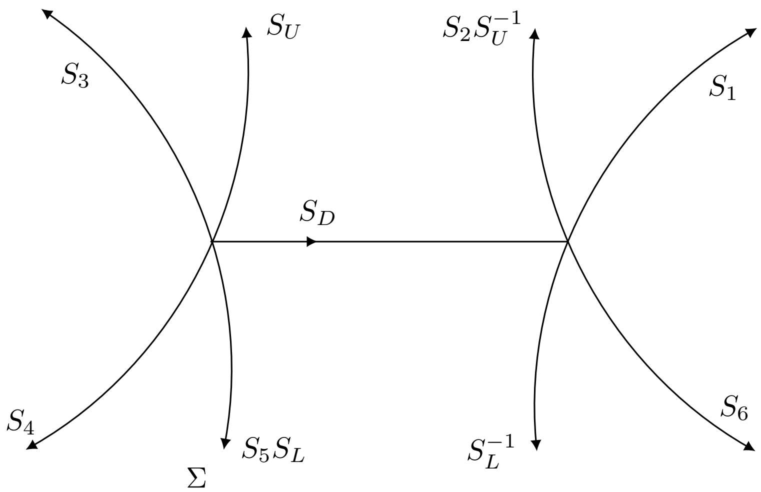


FIGURE 4.1: Deformed Painlevé II Riemann–Hilbert contour Σ.

On the contour Σ in fig:4.1, the function $\Psi(z,t)$ in (1.3.4) solves the following RHP.

Riemann-Hilbert problem 4.1.

- $\Psi(z,t)$ is analytic on $z \in \mathbb{C}\backslash\Sigma$.
- For $z \in \Sigma$, on each of the Stokes' rays

$$\Psi_-^{-1}(z,t)\,\Psi_+(z,t) = G(z,t), \tag{4.1.4}$$

  where $G(z,t)$ is piece-wise defined on each of the rays on the contour Σ.
- $\lim_{z\to\infty}\Psi(z,t) = \mathbb{1}$

In terms of the RHP (4.1.4), the tau-function (2.0.3) is

$$¶_t \log t_{PII} \equiv ¶_t \log t_\Sigma(t) := \int_\Sigma \frac{dz}{2pi}\,\mathrm{Tr}\left[\Psi_-^{-1}\Psi'_-\dot{G}G^{-1}\right], \tag{4.1.5}$$

which we will express as the Fredholm determinant in Theorem 4.1.

4.1.1 Parametrices

To express the tau-function (4.1.5) in terms of a Fredholm determinant we need to construct a "parametrix", namely an "approximate solution" of the RHP in the sense that the actual problem can be recast as the solution of a compact (trace–class) perturbation of the identity.

The effectiveness of the idea relies entirely upon the level of simplicity of this parametrix; the simpler (or rather, more explicit) this reference parametrix is, the more practical the approach is in studying the final problem.

Keeping this in mind, in this section we construct an explicit solution to a Riemann–Hilbert problem to be used as parametrix for the final one. To this end we recall from ([43], Ch.9 pg.318) the construction of the local parametrices of Painlevé II RHP in fig: 4.1(the left and right parametrices around the points $z = \pm 1/2$ respectively), in terms of parabolic cylinder functions [43]. Define the right parametrix around $z_+ = 1/2$ as

$$y_r(z,t) = \left(z(z)\frac{z-z_-}{z-z_+}\right)^{ns_3}\left(-\frac{h_1}{s_3}\right)^{-s_3/2} e^{\frac{it}{3}s_3} 2^{-s_3/2}\begin{pmatrix} z(z) & 1 \\ 1 & 0 \end{pmatrix} Z(z(z))\left(-\frac{h_1}{s_3}\right)^{s_3/2} e^{itq(z)s_3}, \tag{4.1.6}$$

and the left parametrix around $z = -1/2$ is determined through the relation

$$y_l(z,t) = s_2 y_r(-z,t) s_2. \tag{4.1.7}$$

In (4.1.6), the variable

$$z(z) = 2\sqrt{-\frac{4it}{3}z^3 + itz - \frac{it}{3}} \tag{4.1.8}$$

where the branch of the square root is defined by $z(z) = e^{3pi/4} 2\sqrt{2t/3}\left(z - \frac{1}{2}\right)\sqrt{z+1}$, the parameters n and h_1 are defined in terms of the Stokes' parameters s_1, s_3 as follows

$$n = -\frac{1}{2pi}\log(1 - s_1 s_3)\,; \quad h_1 = \frac{\sqrt{2p}}{\Gamma(-n)} e^{ipn}, \tag{4.1.9}$$

and the function $\left(\frac{z-z_-}{z-z_+}\right)^n$ is defined on $\mathbb{C} \setminus [z_-, z_+]$ and the branch is fixed by the following asymptotic condition for $z \to \infty$

$$\left(\frac{z-z_-}{z-z_+}\right)^n \to \mathbb{1}. \tag{4.1.10}$$

Furthermore, the matrix $Z(z(z))$ is determined by parabolic cylinder functions and solves the following RHP.

Riemann-Hilbert problem 4.2.

- $Z(z)$ is a piece-wise holomorphic function defined as follows in each sector shown in fig:4.2

$$Z(z) = \begin{cases} Z_0(z), & \arg z \in \left(-\frac{p}{4}, 0\right) \\ Z_1(z), & \arg z \in \left(0, \frac{p}{2}\right) \\ Z_2(z), & \arg z \in \left(\frac{p}{2}, p\right) \\ Z_3(z), & \arg z \in \left(p, \frac{3p}{2}\right) \\ Z_4(z), & \arg z \in \left(\frac{3p}{2}, \frac{7p}{4}\right). \end{cases} \tag{4.1.11}$$

Under the transformation $z \to -z$ following symmetry relation holds

$$s_3 Z_{k+2}\left(e^{ip} z\right) s_3 = Z_k(z) e^{-ip(n+1)s_3}. \tag{4.1.12}$$

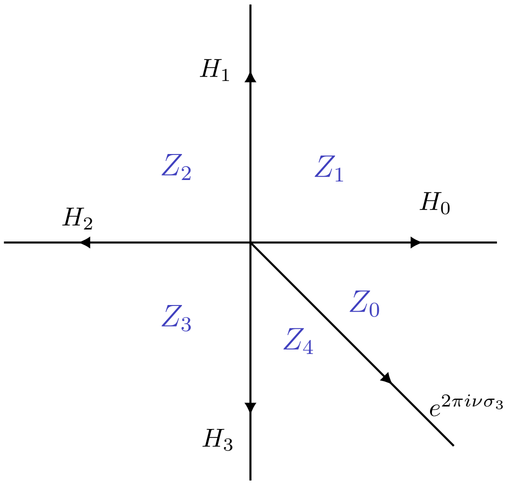


FIGURE 4.2: Riemann–Hilbert contour of parabolic cylinder function.

- In each sector, following jump conditions are satisfied

$$Z_{k+1}(z) = Z_k(z)H_k, \quad \arg z = \frac{p}{2}k, \quad k = 0,1,2,3,4, \tag{4.1.13}$$

and $Z_5 = Z_0$. The jump matrices

$$H_0 = \begin{pmatrix} 1 & 0 \\ h_0 & 1 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 1 & h_1 \\ 0 & 1 \end{pmatrix}, \quad H_4 \equiv H_D = e^{2pins_3}; \\ H_{k+2} = e^{ip\left(n+\frac{1}{2}\right)s_3} H_k e^{-ip\left(n+\frac{1}{2}\right)s_3}, \quad \text{for } k = 0,1. \tag{4.1.14}$$

The Stokes' parameters h_0 and h_1 are defined as follows

$$h_0 = -i\frac{\sqrt{2p}}{\Gamma(n+1)}, \quad h_1 = \frac{\sqrt{2p}}{\Gamma(-n)}e^{ipn}, \quad 1 + h_0h_1 = e^{2pin}, \tag{4.1.15}$$

and the identity $e^{2pins_3}H_0H_1H_2H_3 = I$ implies the triviality of the monodromy at the origin.

- As $z \to \infty$,

$$Z(z) = z^{-s/2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\left(\mathbb{1} + O(z^{-2})\right)e^{\left(\frac{z^2}{4}-(n+\frac{1}{2})\log z\right)s_3}. \tag{4.1.16}$$

In the zeroth sector, $Z(z)$ is expressed in terms of the Wronskian of the parabolic cylinder functions as follows

$$Z_0(z) = 2^{-s_3/2}\begin{pmatrix} D_{-n-1}(iz) & D_n(z) \\ \frac{d}{dz}D_{-n-1}(iz) & \frac{d}{dz}D_n(z) \end{pmatrix}\begin{pmatrix} e^{i\frac{p}{2}(n+1)} & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.1.17}$$

It is straightforward to check that, under the transformation (4.1.8), the parametrix on the right half-plane $y_r(z,t)$ defined in (4.1.6) holds sector-wise as shown in fig:4.3

$$y_r^{(i)}(z,t) = \left(z(z)\frac{z-z_-}{z-z_+}\right)^{ns_3}\left(-\frac{h_1}{s_3}\right)^{-s_3/2} e^{\frac{it}{3}s_3}2^{-s_3/2}\begin{pmatrix} z(z) & 1 \\ 1 & 0 \end{pmatrix} Z_i(z(z))\left(-\frac{h_1}{s_3}\right)^{s_3/2} e^{itq(z)s_3}. \tag{4.1.18}$$

The jumps on Stokes' rays in the right and left half planes are denoted by

$$G_r := G(z,t)|_{\Re(z)>0}\,; \quad G_l := G(z,t)|_{\Re(z)<0}. \tag{4.1.19}$$

As a consequence of (4.1.7),

$$G_l(z,t) = s_2 G_r(-z,t)s_2 \tag{4.1.20}$$

Remark 4.1. *The transformation (4.1.8) is not valid at the point $t = 0$. This implies that $t_{P_{II}}$ in (4.4.1) is valid for $t \in \mathbb{C}\backslash 0$.*

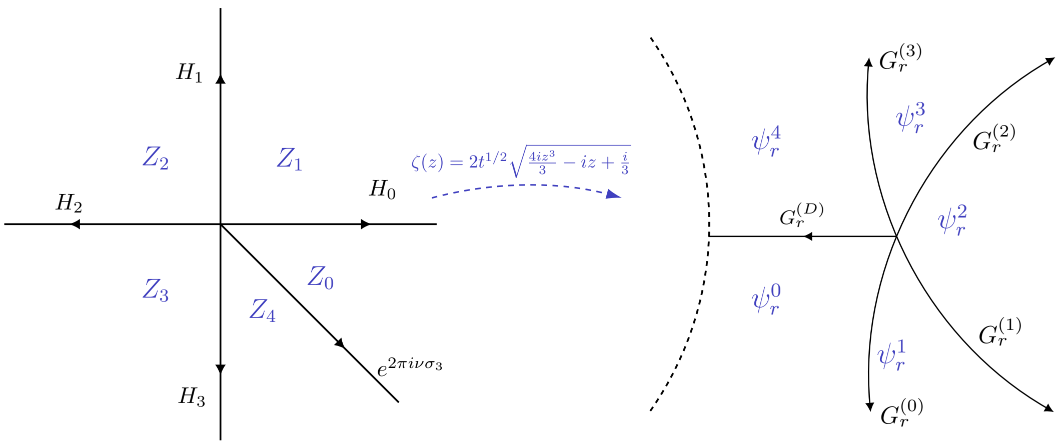


FIGURE 4.3: Mapping the z-plane to the right-half of z-plane

We now establish the relation between the Stokes' matrices of the parabolic cylinder functions H_i in (4.1.14) and $G_r \equiv G_r^{(i)}$ in (4.1.19). Introducing the notation

$$h = \left(-\frac{h_1}{s_3}\right)^{1/2}, \tag{4.1.21}$$

in a sector i on the right half-plane in fig:4.1, y_r satisfies the following jump condition

$$\begin{aligned} y_r^{i+1}(z,t) &= y_r^i(z,t)e^{-itq(z)s_3}\left(-\frac{h_1}{s_3}\right)^{-s_3/2} Z_i^{-1}Z_{i+1}\left(-\frac{h_1}{s_3}\right)^{s_3/2} e^{itq(z)s_3} \\ &= y_r^i(z,t)e^{-itq(z)s_3}h^{-s_3}H_i h^{s_3}e^{itq(z)s_3} \\ &= y_r^i(z,t)G_r^i. \end{aligned} \tag{4.1.22}$$

Note that $Z_5 = Z_0$ implies that $y_r^5(z,t) = y_r^0(z,t)$. Therefore, in terms of H_i, G_r is

$$G_r^{(i)}(z,t) = e^{-itqs_3} h^{-s_3} H_i h^{s_3} e^{itqs_3}. \tag{4.1.23}$$

We define the variable

$$x(z,t) := z(-z,t) \tag{4.1.24}$$

that maps the x-plane to the left half-plane of fig:4.1 and a similar computation follows for the left parametrix due to the symmetry relation (4.1.7). We denote the jump condition in each sector on the respective half-planes in fig:4.1 by

$$y_{r,l;+}(z,t) = y_{r,l;-}(z,t) G_{r,l}(z,t). \tag{4.1.25}$$

4.2 Reduction to a RHP along the imaginary axis

Define a matrix function $\Theta(z,t)$ as a ratio of the global solution Ψ on Σ in (4.1.4) and the local parametrices y_r in (4.1.6), y_l in (4.1.7).

$$\Theta(z,t) := \begin{cases} \Psi(z,t) y_r^{-1}(z,t); \ \Re(z) > 0 \\ \Psi(z,t) y_l^{-1}(z,t); \ \Re(z) < 0. \end{cases} \tag{4.2.1}$$

Note that the local parametrices cancel the jump of the global parametrix on Σ, ensuring that the function $\Theta(z,t)$ has a jump only on the imaginary axis, solving the following RHP.

Riemann-Hilbert problem 4.3.

- $\Theta(z,t)$ is analytic on $z \in \mathbb{C} \backslash i\mathbb{R}$
- For $z \in i\mathbb{R}$,

$$\Theta_+(z,t) = \Theta_-(z,t) J(z,t). \tag{4.2.2}$$

- As $z \to \infty$, $\Theta(z,t) = \mathbb{1}$.

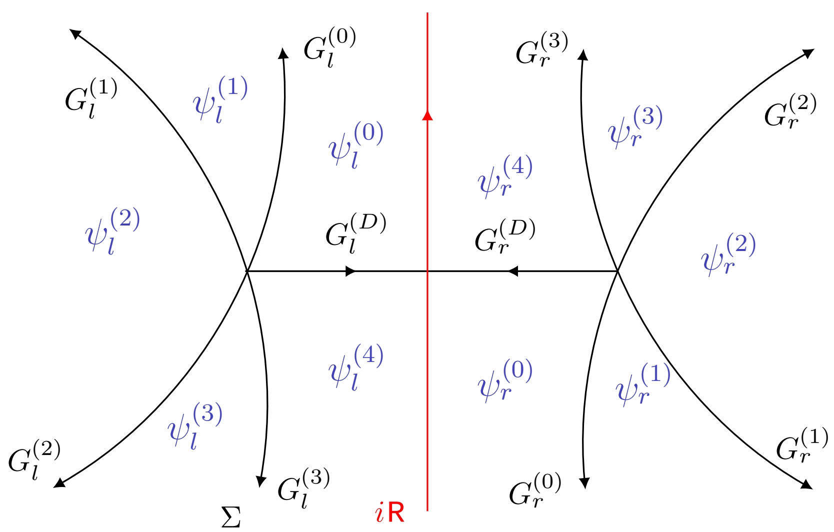


FIGURE 4.4: Reducing the Painlevé II RHP on to the imaginary axis.

Remark 4.2. *The solution of the RHP:4.1 defines, via (4.2.1) a solution of the RHP 4.3. Vice-versa any solution of the RHP 4.3 provides a solution to the RHP:4.1 by means of the inverse of the transformation (4.2.1). Thus we regard these two problems as equivalent in the sense that the solvability of one of them is necessary and sufficient condition for the solvability of the other.*

For later use we compute the expression of the jump matrix J in (4.2.2).

Lemma 4.1. *The jump on the imaginary axis*

$$J(z,t) = \Theta_-(z,t)^{-1}\Theta_+(z,t) = y_r^{(0)}(z,t)\left[y_l^{(4)}(z,t)\right]^{-1} = \begin{pmatrix} \mathcal{A}(z,t) & \mathcal{B}(z,t) \\ \mathcal{C}(z,t) & \mathcal{D}(z,t) \end{pmatrix} \tag{4.2.3}$$

where

$$\mathcal{A}(z,t) = \tfrac{1}{h^4} z^n x^n e^{\frac{2i}{3}t}\left(-e^{-pin}h^4 D_{-n}(iz)D_{-n}(ix) - n^2 e^{2pin} D_{n-1}(z)D_{n-1}(x)\right)$$

$$\mathcal{B}(z,t) = -\tfrac{1}{h^2}\left(\tfrac{z-z_-}{z-z_+}\right)^{2n} z^n x^{-n}\left(-ie^{-pin}h^4 D_{-n}(iz)D_{-n-1}(ix) - ne^{2pin}D_{n-1}(z)D_n(x)\right)$$

$$\mathcal{C}(z,t) = \mathcal{B}(-z,t)\,; \quad \det J = 1.$$

$$\tag{4.2.4}$$

The variables $z \equiv z(z,t)$, $x \equiv x(z,t)$ are defined in (4.1.8), (4.1.24); and h is defined in terms of Stokes' parameters in (4.1.21).

Proof. Since $\Psi(z,t)$ has no jump on $i\mathbb{R}$, $J(z,t)$ can be determined solely in terms of $y_r^{(0)}(z,t)$ and $y_l^{(4)}(z,t)$. One can check the no monodromy condition at the origin,

$$y_r^{(0)}(z,t)\left[y_l^{(4)}(z,t)\right]^{-1} = y_r^{(4)}(z,t)\left[y_l^{(0)}(z,t)\right]^{-1}. \tag{4.2.5}$$

To ease the notation, we define

$$m(z) := \frac{z-z_-}{z-z_+}, \tag{4.2.6}$$

and observe that the following identities hold

$$\begin{gathered} q(z) = \tfrac{4}{3}z^3 - z = i\tfrac{z^2}{4t} - \tfrac{1}{3} = -i\tfrac{x^2}{4t} + \tfrac{1}{3}, \\ (z^2 + x^2) = -\tfrac{8it}{3}. \end{gathered} \tag{4.2.7}$$

The function $y_r^{(0)}(z,t)$ is computed by substituting the zeroth sector solution of the parabolic cylinder function (4.1.17) in (4.1.18),

$$y_r^{(0)}(z) = \left(z(z)\frac{z-z_-}{z-z_+}\right)^{ns_3}\left(-\frac{h_1}{s_3}\right)^{-s_3/2} e^{\frac{it}{3}s_3}2^{-s_3/2}\begin{pmatrix} z(z) & 1 \\ 1 & 0 \end{pmatrix} Z_0(z(z))\left(-\frac{h_1}{s_3}\right)^{s_3/2} \times e^{itq(z)s_3}$$

$$= \begin{bmatrix} e^{ipn/2}e^{-z^2/4}m(z)^n z^n D_{-n}(iz) & \frac{n}{h^2}e^{2it/3}e^{z^2/4}m(z)^n z^n D_{n-1}(z) \\ ih^2 e^{ipn/2}e^{-2it/3}e^{-z^2/4}m(z)^{-n}z^{-n}D_{-n-1}(iz) & e^{z^2/4}m(z)^{-n}z^{-n}D_n(z) \end{bmatrix}. \tag{4.2.8}$$

We use (4.2.7) to obtain the final form of $y_r^{(0)}$. The left parametrix $y_l^{(4)}$ can be obtained in a similar fashion, first by substituting $Z_4 = Z_0 e^{-2ipns_3}$ from (4.1.13) in (4.1.18) to obtain y_r^4 and using the relation (4.1.7) to obtain $y_l^{(4)}$ as follows

$$
\begin{aligned}
y_l^4(z,t) = s_2 y_r^4(-z,t) s_2 &= s_2 \left(z(-z) \frac{z-z_+}{z-z_-} \right)^{ns_3} \left(-\frac{h_1}{s_3} \right)^{-s_3/2} e^{\frac{it}{3}s_3} 2^{-s_3/2} \\
&\times \begin{pmatrix} z(-z) & 1 \\ 1 & 0 \end{pmatrix} Z_4(z(-z)) \left(-\frac{h_1}{s_3} \right)^{s_3/2} e^{itq(z)s_3} s_2 \\
&= s_2 \left(z(-z) \frac{z-z_+}{z-z_-} \right)^{ns_3} \left(-\frac{h_1}{s_3} \right)^{-s_3/2} e^{\frac{it}{3}s_3} 2^{-s_3/2} \\
&\times \begin{pmatrix} z(-z) & 1 \\ 1 & 0 \end{pmatrix} Z_0(z(-z)) H_D^{-1} \left(-\frac{h_1}{s_3} \right)^{s_3/2} e^{itq(z)s_3} s_2 \\
= \begin{bmatrix} e^{2pin} e^{x^2/4} m(z)^n x^{-n} D_n(x) & -ih^2 e^{-3pin/2} e^{-2it/3} e^{x^2/4} m(z)^n x^{-n} D_{-n-1}(ix) \\ -e^{2pin} n h^{-2} e^{2it/3} e^{x^2/4} m(z)^{-n} x^n D_{n-1}(x) & e^{-3pin/2} e^{-y^2/4} m(z)^{-n} x^n D_{-n}(ix) \end{bmatrix}&.
\end{aligned}
\tag{4.2.9}
$$

To obtain the last line, we substitute the expression for Z_0 (4.1.17) and simplify the resulting expression using (4.2.7).Furthermore,

$$
\det\left[y_r^{(0)}(z,t) \right] = 1; \quad \det\left[y_l^{(4)}(z,t) \right] = 1 \tag{4.2.10}
$$

due to the following identity for the Wronskian determinant of parabolic cylinder functions

$$
\mathcal{W}\left[D_{-n-1}(iz), D_n(z) \right] = ie^{-ipn/2}. \tag{4.2.11}
$$

The jump $J(z,t)$ is then obtained by a straightforward substitution of (4.2.8) and (4.2.9) in (4.2.3), and using (4.2.7).

$$
J(z,t) = y_r^{(0)}(z,t) \left[y_l^{(4)}(z,t) \right]^{-1} = \begin{pmatrix} \mathcal{A}(z,t) & \mathcal{B}(z,t) \\ \mathcal{C}(z,t) & \mathcal{D}(z,t) \end{pmatrix} \tag{4.2.12}
$$

where

$$
\begin{aligned}
\mathcal{A}(z,t) &= z^n x^n e^{\frac{2i}{3}t} \left(e^{-pin} D_{-n}(iz) D_{-n}(ix) + n^2 h^{-4} e^{2pin} D_{n-1}(z) D_{n-1}(x) \right) \\
\mathcal{B}(z,t) &= \left(\tfrac{z-z_-}{z-z_+} \right)^{2n} z^n x^{-n} \left(ih^2 e^{-ipn} D_{-n}(iz) D_{-n-1}(ix) + nh^{-2} e^{2pin} D_{n-1}(z) D_n(x) \right) \\
\mathcal{C}(z,t) &= \left(\tfrac{z-z_-}{z-z_+} \right)^{-2n} z^{-n} x^n \left(ih^2 e^{-ipn} D_{-n-1}(iz) D_{-n}(ix) + nh^{-2} e^{2pin} D_n(z) D_{n-1}(x) \right) = \mathcal{B}(-z,t) \\
\mathcal{D}(z,t) &= z^{-n} x^{-n} e^{-\frac{2i}{3}t} \left(-e^{-pin} h^4 D_{-n-1}(iz) D_{-n-1}(ix) + e^{2pin} D_n(z) D_n(x) \right)
\end{aligned}
\tag{4.2.13}
$$

It is obvious that $\det J(z,t) = 1$. Recall that (4.1.24):$x(-z,t) = z(z,t)$ with z defined in (4.1.8), $h = \left(-\frac{h_1}{s_3} \right)^{1/2}$: (4.1.21) where h_1, n are determined by the Stokes' parameters s_1, s_3

as in (4.1.15), (4.1.9) respectively. □

The two equivalent RHPs 4.1 , 4.3 give rise to two corresponding Malgrange forms. Although the two problems are equivalent, the two corresponding tau-function may (and in fact do) differ, but only by a non-vanishing term which we now set up to compute. Recalling the Malgrange form of Painlevé II on Σ in (4.1.5):

$$\P_t \log t_\Sigma = \int_\Sigma \frac{dz}{2pi} \mathrm{Tr}\left[\Psi_-^{-1}\Psi_-' \dot{G} G^{-1}\right]. \tag{4.2.14}$$

Similarly on $i\mathbb{R}$, the RHP:4.3 satisfies the jump condition $\Theta_+ = \Theta_- J$ and the corresponding Malgrange form (2.0.3) is

$$\P_t \log t_{i\mathbb{R}} = \int_{i\mathbb{R}} \frac{dz}{2pi} \mathrm{Tr}\left[\Theta_-^{-1}\Theta_-' \dot{J} J^{-1}\right]. \tag{4.2.15}$$

Proposition 4.1. *The Malgrange forms corresponding to the RHPs on the contours Σ and $i\mathbb{R}$ are related as*

$$\P_t \log t_\Sigma = \P_t \log t_{i\mathbb{R}} - \int_{i\mathbb{R}} \frac{dz}{2pi} \mathcal{F}(z,t;n,h) - \left[\frac{4in}{3} + \frac{2n^2}{t}\right], \tag{4.2.16}$$

where $\mathcal{F}(z,t;n,h)$ is a regular function explicit in terms of parabolic cylinder functions.

Proof. [4] We begin by computing the expression

$$\mathrm{Tr}\left\{\Theta_-^{-1}\Theta_-' \dot{J} J^{-1}\right\}. \tag{4.2.17}$$

Computing (4.2.17) term by term using (4.2.1): $\Theta_- = \Psi y_r^{-1}$,

$$\begin{aligned}\Theta_-^{-1}\Theta_-' &= (\Psi y_r^{-1})^{-1}(\Psi y_r^{-1})' = y_r \Psi^{-1}\left(\Psi' y_r^{-1} - \Psi y_r^{-1} y_r' y_r^{-1}\right) \\ &= y_r\left(\Psi^{-1}\Psi' - y_r^{-1} y_r'\right) y_r^{-1}.\end{aligned} \tag{4.2.18}$$

Since (4.2.3): $J = y_r y_l^{-1}$,

$$\begin{aligned}\dot{J} J^{-1} &= \frac{\P}{\P t}(y_r y_l^{-1})(y_r y_l^{-1})^{-1} = \left(\dot{y}_r y_l^{-1} - y_r y_l^{-1} \dot{y}_l y_l^{-1}\right) y_l y_r^{-1} \\ &= -y_r \Delta\left(y^{-1}\dot{y}\right) y_r^{-1},\end{aligned} \tag{4.2.19}$$

where

$$\Delta(y^{-1}\dot{y}) = y_l^{-1}\dot{y}_l - y_r^{-1}\dot{y}_r.$$

Substituting (4.2.18) and (4.2.19) in (4.2.17) and using cyclicity of trace,

$$\mathrm{Tr}\left\{\Theta_-^{-1}\Theta_-' \dot{J} J^{-1}\right\} = \mathrm{Tr}\left\{\left(-\Psi^{-1}\Psi' + y_r^{-1} y_r'\right)\Delta\left(y^{-1}\dot{y}\right)\right\}. \tag{4.2.20}$$

[4] In the proof, we drop the z, t dependence for the ease of writing. All the functions here on depend on z, t unless specified.

Since the term $y_r^{-1}y_r'\Delta\left(y^{-1}\dot{y}\right)$ is integrated on $i\mathbb{R}$ in (4.2.15),

$$\int_{i\mathbb{R}}\frac{dz}{2pi}\left[y_r^{-1}y_r'\Delta\left(y^{-1}\dot{y}\right)\right]=\int_{i\mathbb{R}}\frac{dz}{2pi}\operatorname{Tr}\left[\left(y_r^{(0)}\right)^{-1}\left(y_r^{(0)}\right)'\left\{\left(y_l^{(4)}\right)^{-1}\dot{y}_l{}^{(4)}-\left(y_r^{(0)}\right)^{-1}\dot{y}_r{}^{(0)}\right\}\right] \tag{4.2.21}$$

with $y_r^{(0)}$ defined in (4.2.8), $y_l^{(4)}$ in (4.2.9). We collect the explicit terms and compute them in the end. Since Ψ has no jump on $i\mathbb{R}$, using Cauchy theorem

$$\begin{aligned}-\int_{i\mathbb{R}}\frac{dz}{2pi}\operatorname{Tr}\left\{\Psi^{-1}\Psi'\Delta\left(y^{-1}\dot{y}\right)\right\}&=-\int_{i\mathbb{R}}\frac{dz}{2pi}\operatorname{Tr}\Delta\left\{\Psi^{-1}\Psi'\left(y^{-1}\dot{y}\right)\right\}\\&=\int_{\Sigma}\frac{dz}{2pi}\operatorname{Tr}\Delta\left\{\Psi^{-1}\Psi'\left(y^{-1}\dot{y}\right)\right\}\\&=\int_{\Sigma_L}\frac{dz}{2pi}\operatorname{Tr}\Delta\left\{\Psi^{-1}\Psi'\left(y^{-1}\dot{y}\right)\right\}+\int_{\Sigma_R}\frac{dz}{2pi}\operatorname{Tr}\Delta\left\{\Psi^{-1}\Psi'\left(y^{-1}\dot{y}\right)\right\}.\end{aligned} \tag{4.2.22}$$

In order to estimate (4.2.22), we begin by computing the integrand on Σ_L:

$$\operatorname{Tr}\Delta\left\{\Psi^{-1}\Psi'\left(y^{-1}\dot{y}\right)\right\}=\operatorname{Tr}\left\{\Psi_+^{-1}\Psi_+'\left(y_{l+}^{-1}\dot{y_{l+}}\right)-\Psi_-^{-1}\Psi_-'\left(y_{l-}^{-1}\dot{y_{l-}}\right)\right\}. \tag{4.2.23}$$

Computing (4.2.23) term by term using (4.1.4): $\Psi_+=\Psi_-G_l$,

$$\begin{aligned}\Psi_+^{-1}\Psi_+'=(\Psi_-G_l)^{-1}(\Psi_-G_l)'&=G_l^{-1}\Psi_-^{-1}(\Psi_-'G_l+\Psi_-G_l')\\&=G_l^{-1}(\Psi_-^{-1}\Psi_-'+G_l'G_l^{-1})G_l.\end{aligned} \tag{4.2.24}$$

Since (4.1.25): $y_{l+}=y_{l-}G_l$,

$$\begin{aligned}y_{l+}^{-1}\dot{y_{l+}}&=G_l^{-1}y_{l-}^{-1}\left(\dot{y_{l-}}G_l+y_{l-}\dot{G}_l\right)\\&=G_l^{-1}\left(y_{l-}^{-1}\dot{y_{l-}}+\dot{G}_lG_l^{-1}\right)G_l.\end{aligned} \tag{4.2.25}$$

The product of (4.2.24) and (4.2.25) under the trace reads

$$\operatorname{Tr}\left\{\Psi_+^{-1}\Psi_+'\left(\dot{y_{l+}}y_{l+}^{-1}\right)\right\}=\operatorname{Tr}\left[\left(\Psi_-^{-1}\Psi_-'+G_l'G_l^{-1}\right)\left(y_{l-}^{-1}\dot{y_{l-}}+\dot{G}_lG_l^{-1}\right)\right]. \tag{4.2.26}$$

Substituting (4.2.26) in (4.2.23),

$$\operatorname{Tr}\Delta\left\{\Psi^{-1}\Psi'\left(\dot{y}y^{-1}\right)\right\}=\operatorname{Tr}\left[\Psi_-^{-1}\Psi_-'\dot{G}_lG_l^{-1}+G_l'G_l^{-1}\left(y_{l-}^{-1}\dot{y_{l-}}+\dot{G}_lG_l^{-1}\right)\right] \tag{4.2.27}$$

A parallel computation for Σ_R gives

$$\operatorname{Tr}\Delta\left\{\Psi^{-1}\Psi'\left(\dot{y}y^{-1}\right)\right\}=\operatorname{Tr}\left[\Psi_-^{-1}\Psi_-'\dot{G}_rG_r^{-1}+G_r'G_r^{-1}\left(y_{r-}^{-1}\dot{y_{r-}}+\dot{G}_rG_r^{-1}\right)\right]. \tag{4.2.28}$$

Summing the terms (4.2.27) and (4.2.28), we obtain that

$$
\begin{aligned}
\P_t \log t_{i\mathbb{R}} = \int_{i\mathbb{R}} \frac{dz}{2pi} \operatorname{Tr}\left[\Theta_-^{-1}\Theta'_- \dot{J} J^{-1}\right] = \int_{\Sigma_R} \frac{dz}{2pi} \operatorname{Tr}\left[\Psi_-^{-1}\Psi'_- \dot{G}_r G_r^{-1}\right] + \int_{\Sigma_L} \frac{dz}{2pi} \operatorname{Tr}\left[\Psi_-^{-1}\Psi'_- \dot{G}_l G_l^{-1}\right] \\
+ \int_{\Sigma_L} \frac{dz}{2pi} \operatorname{Tr}\left[G_l' G_l^{-1}\left(y_{l-}^{-1} \dot{y_{l-}} + \dot{G}_l G_l^{-1}\right)\right] + \int_{\Sigma_R} \frac{dz}{2pi} \operatorname{Tr}\left[G_r' G_r^{-1}\left(y_{r-}^{-1} \dot{y_{r-}} + \dot{G}_r G_r^{-1}\right)\right] \\
+ \int_{i\mathbb{R}} \frac{dz}{2pi} \operatorname{Tr}\left[y_r^{-1} y_r' \Delta\left(y^{-1}\dot{y}\right)\right] \\
= \int_{\Sigma} \frac{dz}{2pi} \operatorname{Tr}\left[\Psi_-^{-1}\Psi'_- \dot{G} G^{-1}\right] + \int_{i\mathbb{R}} \frac{dz}{2pi} \operatorname{Tr}\left[y_r^{-1} y_r' \Delta\left(y^{-1}\dot{y}\right)\right] \\
+ \int_{\Sigma_L} \frac{dz}{2pi} \operatorname{Tr}\left[G_l' G_l^{-1}\left(y_{l-}^{-1} \dot{y_{l-}} + \dot{G}_l G_l^{-1}\right)\right] + \int_{\Sigma_R} \frac{dz}{2pi} \operatorname{Tr}\left[G_r' G_r^{-1}\left(y_{r-}^{-1} \dot{y_{r-}} + \dot{G}_r G_r^{-1}\right)\right] \\
= \P_t \log t_{\Sigma} + \int_{i\mathbb{R}} \frac{dz}{2pi} \operatorname{Tr}\left[y_r^{-1} y_r' \Delta\left(y^{-1}\dot{y}\right)\right] \\
+ \int_{\Sigma_L} \frac{dz}{2pi} \operatorname{Tr}\left[G_l' G_l^{-1}\left(y_{l-}^{-1} \dot{y_{l-}} + \dot{G}_l G_l^{-1}\right)\right] + \int_{\Sigma_R} \frac{dz}{2pi} \operatorname{Tr}\left[G_r' G_r^{-1}\left(y_{r-}^{-1} \dot{y_{r-}} + \dot{G}_r G_r^{-1}\right)\right].
\end{aligned}
\tag{4.2.29}
$$

Notice that $y_{r,l}$ and $G_{r,l}$ are completely determined in terms of parabolic cylinder functions. The final expression is

$$
\begin{aligned}
\P_t \log t_{\Sigma} = \P_t \log t_{i\mathbb{R}} - \int_{i\mathbb{R}} \frac{dz}{2pi} \operatorname{Tr}\left[y_r^{-1} y_r' \Delta\left(y^{-1}\dot{y}\right)\right] \\
- \int_{\Sigma_L} \frac{dz}{2pi} \operatorname{Tr}\left[G_l' G_l^{-1}\left(y_{l-}^{-1} \dot{y_{l-}} + \dot{G}_l G_l^{-1}\right)\right] - \int_{\Sigma_R} \frac{dz}{2pi} \operatorname{Tr}\left[G_r' G_r^{-1}\left(y_{r-}^{-1} \dot{y_{r-}} + \dot{G}_r G_r^{-1}\right)\right].
\end{aligned}
\tag{4.2.30}
$$

The following can be said about the explicit terms in (4.2.30).

- We can completely determine the integrals on $\Sigma_{R,L}$. The symmetry relations (4.1.7), (4.1.20) imply that

$$
\int_{\Sigma_L} \frac{dz}{2pi} \operatorname{Tr}\left[G_l' G_l^{-1}\left(y_{l-}^{-1} \dot{y_{l-}} + \dot{G}_l G_l^{-1}\right)\right] = \int_{\Sigma_R} \frac{dz}{2pi} \operatorname{Tr}\left[G_r' G_r^{-1}\left(y_{r-}^{-1} \dot{y_{r-}} + \dot{G}_r G_r^{-1}\right)\right].
\tag{4.2.31}
$$

Furthermore, (4.1.15) implies that the jump $G_r^{(i)}$ in (4.1.23) is lower triangular for $i = 0, 2$; upper triangular for $i = 1, 3$; diagonal and constant for $i = 4$. Therefore,

$$
\operatorname{Tr}\left[G_r' G_r^{-1} \dot{G}_r G_r^{-1}\right] = \operatorname{Tr}\left[G_l' G_l^{-1} \dot{G}_l G_l^{-1}\right] = 0.
\tag{4.2.32}
$$

We now proceed to compute the following term in (4.2.30)

$$
\int_{\Sigma_R} \frac{dz}{2pi} \operatorname{Tr}\left[G_r' G_r^{-1} y_{r-}^{-1} \dot{y_{r-}}\right] = \sum_{i=1}^{5} \int_{\Sigma_i} \frac{dz}{2pi} \operatorname{Tr}\left[(G_r^{(i)})'(G_r^{(i)})^{-1}(y_{r-}^{(i-1)})^{-1} \dot{y_{r-}}^{(i-1)}\right].
\tag{4.2.33}
$$

In each sector, y_r and G_r can be computed starting from $y_r^{(0)}$ in (4.2.8), and the jumps in (4.1.23). A lengthy but straightforward computation yields

$$\int_{\Sigma_R} \frac{dz}{2pi} \operatorname{Tr}\left[G_r' G_r^{-1}\left(y_{r-}^{-1} \dot{y_{r-}} + \dot{G}_r G_r^{-1}\right)\right] = \left[\frac{2in}{3} + \frac{n^2}{t}\right]. \tag{4.2.34}$$

The relation (4.2.31) then implies,

$$\int_{\Sigma_L} \frac{dz}{2pi} \operatorname{Tr}\left[G_l' G_l^{-1}\left(y_{l-}^{-1} \dot{y_{l-}} + \dot{G}_l G_l^{-1}\right)\right] + \int_{\Sigma_R} \frac{dz}{2pi} \operatorname{Tr}\left[G_r' G_r^{-1}\left(y_{r-}^{-1} \dot{y_{r-}} + \dot{G}_r G_r^{-1}\right)\right] = \left[\frac{4in}{3} + \frac{2n^2}{t}\right]. \tag{4.2.35}$$

- The remaining explicit term in (4.2.30)

$$\int_{i\mathbb{R}} \frac{dz}{2pi} \operatorname{Tr}\left[y_r^{-1} y_r' \Delta\left(y^{-1} \dot{y}\right)\right] \equiv \int_{i\mathbb{R}} \frac{dz}{2pi} \operatorname{Tr}\left[\left(y_r^{(0)}\right)^{-1}\left(y_r^{(0)}\right)'\left\{\left(y_l^{(4)}\right)^{-1} \dot{y_l}^{(4)} - \left(y_r^{(0)}\right)^{-1} \dot{y_r}^{(0)}\right\}\right]. \tag{4.2.36}$$

The functions $y_r^{(0)}$ and $y_l^{(4)}$ depend on z through $z(z,t)$ as in (4.1.8) and $x(z,t)$ as in (4.1.24) respectively. In order to solve the integral, we need to compute integrals of the form

$$\int \frac{dz}{2pi} D_n(z) D_m(x) D_{-r}(iz) D_{-s}(ix), \tag{4.2.37}$$

which is not exactly solvable. The expression (4.2.36) is however, explicit. Defining a function $\mathcal{F}$ as

$$\mathcal{F}(z,t;n,h) := \operatorname{Tr}\left[y_r^{-1} y_r'\left(y_l^{-1} \dot{y}_l - y_r^{-1} \dot{y}_r\right)\right], \tag{4.2.38}$$

The final expression in (4.2.30) reads

$$\P_t \log t_\Sigma = \P_t \log t_{i\mathbb{R}} - \int_{i\mathbb{R}} \frac{dz}{2pi} \mathcal{F}(z,t;n,h) - \left[\frac{4in}{3} + \frac{2n^2}{t}\right]. \tag{4.2.39}$$

□

4.3 Integrable kernel and Fredholm determinant

Up to this point, we started with the RHP of Painlevé II in fig:4.1, used the description of the local parametrices in terms of parabolic cylinder functions in the subsection 4.1.1 to define a RHP on $i\mathbb{R}$ (4.2.2) in section 4.2. We then showed that the corresponding Malgrange forms are related in proposition: 4.1. Our goal now reduces to expressing $t_{i\mathbb{R}}$ as a Fredholm determinant.

It is known that a jump $J(z,t) \in SL(2,\mathbb{C})$ on non-intersecting contours can be expressed in terms of lower and upper triangular matrices called the LULU decomposition and the corresponding tau-function can then be written as a Fredholm determinant of an integrable operator [9]. Here, we modify the construction in [9] by using LDU decomposition instead, which then gives us a simpler kernel. In this section, we

1. transform RHP:4.3 on to a set of two parallel lines with lower and upper triangular jumps using the LDU decomposition,
2. formulate the tau-function on the set of parallel lines, call it t_{LU} as a Fredholm determinant of an integrable operator, and
3. prove that the Malgrange forms on the contours LU and $i\mathbb{R}$ coincide.

4.3.1 LU decomposition

The RHP on $i\mathbb{R}$ can be transformed on to a set of two parallel lines with jumps that are upper and lower triangular respectively. We decompose the jump $J(z,t)$ (4.2.3) into lower, diagonal and upper triangular matrices, called the LDU decomposition [19], which recasts the RHP:4.3 on to a set of three parallel lines.

$$J(z,t) = \begin{pmatrix} \mathcal{A}(z,t) & \mathcal{B}(z,t) \\ \mathcal{C}(z,t) & \mathcal{D}(z,t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\mathcal{C}(z,t)}{\mathcal{A}(z,t)} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A}(z,t) & 0 \\ 0 & \frac{1}{\mathcal{A}(z,t)} \end{pmatrix} \begin{pmatrix} 1 & \frac{\mathcal{B}(z,t)}{\mathcal{A}(z,t)} \\ 0 & 1 \end{pmatrix}$$
$$:= F_1(z,t)F_2(z,t)F_3(z,t). \tag{4.3.1}$$

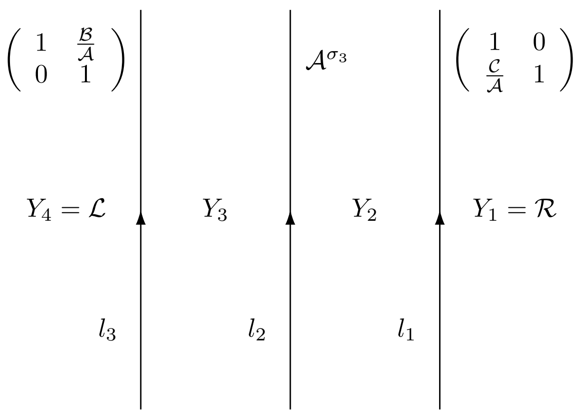


FIGURE 4.5: LDU decomposition

The function $Y(z,t)$ then solves the following RHP.

Riemann-Hilbert problem 4.4.

- $Y(z,t)$ is a piecewise analytic in $\mathbb{C}\setminus\left(\cup_{i=1}^{3} l_i\right)$.
- On each line l_i in fig:4.5, the following jump condition holds

$$Y_{i+1}(z,t) = Y_i(z,t)F_i(z,t), \tag{4.3.2}$$

with the identification

$$Y_4(z,t) = \Theta_+(z,t)\,; \quad Y_1(z,t) = \Theta_-(z,t). \tag{4.3.3}$$

[5]The author thanks A.Its for suggesting LDU decomposition.

$\Theta_\pm$ are defined in (4.2.1).

- $\lim_{z\to\infty} Y(z,t) = \mathbb{1}$.

The RHP:4.4 can be further transformed with the observation that the function $j(z,t)^{s_3}$ defined as

$$j(z,t) := \exp\left[\int_{i\mathbb{R}} \frac{dw}{2pi} \frac{\log \mathcal{A}(w,t)}{z-w}\right], \tag{4.3.4}$$

solves RHP on l_2 with the diagonal jump $\mathcal{A}^{s_3}$ locally with $\mathcal{A}$ defined in (4.2.4). The ratio of $Y(z,t)$, $j(z,t)^{s_3}$

$$\widetilde{Y}_i(z,t) := Y_i(z,t)\, j(z,t)^{-s_3} \tag{4.3.5}$$

is such that $\widetilde{Y}(z,t)$ jumps only on $l_1 \cup l_3$ and solves the following RHP.

Riemann-Hilbert problem 4.5.

- $\widetilde{Y}(z,t)$ is piece-wise analytic in $\mathbb{C}\backslash(l_1 \cup l_3)$.

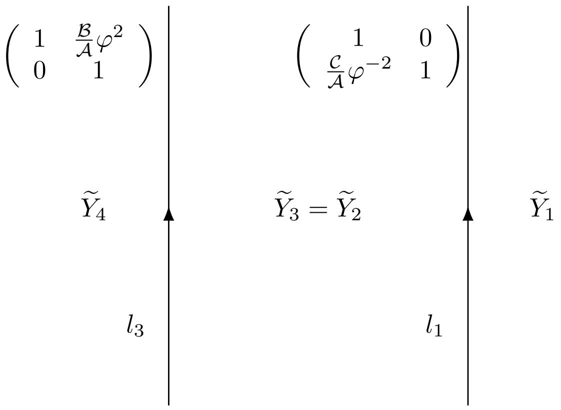


FIGURE 4.6: RHP with lower and upper triangular jumps.

- The following jump conditions are valid on the contours l_i, $i = 1,3$

$$\widetilde{Y}_{i+1}(z,t) = \widetilde{Y}_i(z,t)\widetilde{F}_i(z,t). \tag{4.3.6}$$

  Note that $\widetilde{Y}(z,t)$ has no jump on l_2, implying that $\widetilde{Y}_3(z,t) = \widetilde{Y}_2(z,t)$.

The RHP:4.5 is of the 'integrable' type and its solvability is determined by the invertibility of an integrable operator i.e, its tau-function is the Fredholm determinant of an integrable operator.

4.3.2 Integrable kernel

Proposition 4.2. *The tau-function on $l_1 \cup l_3$ denoted by t_{LU} is a Fredholm determinant of an integrable operator*

$$t_{LU} = \det\left[\mathbb{1}_{L^2(i\mathbb{R})} - \widetilde{\mathcal{K}}\right] \tag{4.3.7}$$

where

$$\left(\widetilde{\mathcal{K}}g\right)(z) = \frac{\mathcal{C}(z,t)}{\mathcal{A}(z,t)} \int_{i\mathbb{R}} \frac{dw}{2pi} \int_{i\mathbb{R}+e} \frac{d\widetilde{w}}{2pi} \frac{j_+^2(w)\, j_+^{-2}(\widetilde{w})}{(z-\widetilde{w})(\widetilde{w}-w)} \mathcal{A}(\widetilde{w},t)\mathcal{B}(\widetilde{w},t)g(\widetilde{w}). \tag{4.3.8}$$

The functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are defined in (4.2.4) and j_+ is the positive (left of the imaginary axis) boundary value of (4.3.4).

Proof. Let us start with the jumps in (4.3.6)

$$\widetilde{F}(z,t) = \begin{cases} \widetilde{F}_1(z,t) = \begin{pmatrix} 1 & 0 \\ \frac{\mathcal{C}(z,t)}{\mathcal{A}(z,t)} j(z,t)^2 & 1 \end{pmatrix} ; \text{on } l_1 \\ \widetilde{F}_3(z,t) = \begin{pmatrix} 1 & \frac{\mathcal{B}(z,t)}{\mathcal{A}(z,t)} j(z,t)^{-2} \\ 0 & 1 \end{pmatrix} ; \text{on } l_3 \end{cases} . \tag{4.3.9}$$

We define the functions

$$f(z,t) = \frac{1}{2pi} \begin{pmatrix} \frac{\mathcal{B}(z,t)}{\mathcal{A}(z,t)} c_3(z) \\ \\ \frac{\mathcal{C}(z,t)}{\mathcal{A}(z,t)} c_1(z) \end{pmatrix} \quad ; \quad g(z,t) = \begin{pmatrix} j(z,t)^2 c_1(z) \\ \\ j(z,t)^{-2} c_3(z) \end{pmatrix} \tag{4.3.10}$$

where $c_1(z)$, $c_3(z)$ denote the characteristic functions on the contours l_1, l_3 respectively. The jump $\widetilde{F}(z,t)$ can be written in terms of (4.3.10) as

$$\widetilde{F} = 1 - 2pi f(z) g^T(z), \tag{4.3.11}$$

and clearly $f^T(z)g(z) = 0$. The associated integrable kernel is then

$$\begin{aligned} K(z,w) &= \frac{f^T(z)g(w)}{z-w} = \\ &= \frac{1}{(2pi)(z-w)} \begin{pmatrix} c_1(z) & c_3(z) \end{pmatrix} \begin{pmatrix} 0 & \frac{\mathcal{C}(z,t)}{\mathcal{A}(z,t)} j^{-2}(w,t) \\ \frac{\mathcal{B}(z,t)}{\mathcal{A}(z,t)} j^2(w,t) & 0 \end{pmatrix} \begin{pmatrix} c_1(w) \\ c_3(w) \end{pmatrix} \\ &\equiv \begin{pmatrix} c_1(z) & c_3(z) \end{pmatrix} \begin{pmatrix} 0 & K_{31}(z,w) \\ K_{13}(z,w) & 0 \end{pmatrix} \begin{pmatrix} c_1(w) \\ c_3(w) \end{pmatrix}. \end{aligned} \tag{4.3.12}$$

The kernels $K_{13}(z,w)$ and $K_{31}(z,w)$ in (5.5) take the form

$$K_{13}(z,w) = \frac{\mathcal{B}(z,t)\, j^2(w,t)}{(2pi)\mathcal{A}(z,t)(z-w)}$$

$$K_{31}(z,w) = \frac{\mathcal{C}(z,t)\, j^{-2}(w,t)}{(2pi)\mathcal{A}(z,t)(z-w)}.$$

We introduce the operators

$$\begin{aligned} \mathcal{K}_{31} &: L^2(l_3) \to L^2(l_1) \\ \mathcal{K}_{13} &: L^2(l_1) \to L^2(l_3), \end{aligned} \tag{4.3.13}$$

defined as

$$\begin{aligned} \left(\mathcal{K}_{31}h\right)(z) &= \int_{I_3} K_{31}(z,w)h(w)dw, \\ \left(\mathcal{K}_{13}g\right)(z) &= \int_{I_1} K_{13}(z,w)g(w)dw. \end{aligned} \tag{4.3.14}$$

The tau-function corresponding to the RHP:4.5 is then

$$t_{LU}(t) = \det\left[\mathbb{1}_{L^2(I_1\cup I_3)} - \begin{pmatrix} 0 & \mathcal{K}_{31} \\ \mathcal{K}_{13} & 0 \end{pmatrix}\right]. \tag{4.3.15}$$

Since $j^2(w,t)$ is analytic in $\Re(w) > 0$ and $\lim_{w\to\infty} j(w,t) = 1$, $\mathcal{K}_{13}$, $\mathcal{K}_{31}$ are Trace–class. Therefore we can write $t_{LU}(t)$ in the form

$$t_{LU}(t) = \det\left[\mathbb{1}_{L^2(I_3)} - \mathcal{K}_{13}\circ\mathcal{K}_{31}\right]. \tag{4.3.16}$$

The form of the tau-function (4.3.16) can be further modified such that the operator acts on $L^2(i\mathbb{R})$ instead of $L^2(I_3)$. We begin by splitting the function $h(z)$ as

$$h(z) = h_L(z) + h_R(z) \tag{4.3.17}$$

where $h_{L,R}(z)$ are analytic to the left and right of I_3 respectively, and $h_{L,R}(z) = \mathcal{O}(z^{-1})$ as $z\to\infty$. The integrable operator (4.3.14) acts on $h(z)$ as

$$\left(\mathcal{K}_{13}\mathcal{K}_{31}h_R\right)(z) \equiv 0 \Rightarrow \left(\mathcal{K}_{13}\mathcal{K}_{31}h\right)(z) = \left(\mathcal{K}_{13}\mathcal{K}_{31}h_L\right)(z). \tag{4.3.18}$$

We can therefore move the integration in w from I_3 to $i\mathbb{R}$ in (4.3.14) and identify the space of functions $\left(\mathcal{K}_{31}h\right)(z)$ with $H_R(i\mathbb{R})$, the Hardy space on the right half–plane. So, the operator

$$\left(\widetilde{\mathcal{K}}g\right)(z) := \left(\mathcal{K}_{13}\mathcal{K}_{31}g\right)(z) = \frac{\mathcal{C}(z,t)}{\mathcal{A}(z,t)}\int_{I_1}\frac{d\widetilde{w}}{2pi}\int_{i\mathbb{R}}\frac{dw}{2pi}\frac{j^{-2}(\widetilde{w})}{z-\widetilde{w}}\frac{\mathcal{B}(\widetilde{w},t)}{\mathcal{A}(\widetilde{w},t)}\frac{j_+^2(w,t)}{\widetilde{w}-w}g(w) \tag{4.3.19}$$

The kernel, $\widetilde{K}(z,w)$ is

$$\widetilde{K}(z,w) = \frac{\mathcal{C}(z,t)}{\mathcal{A}(z,t)}j_+^2(w)\int_{I_1}\frac{d\widetilde{w}}{2pi}\frac{j^{-2}(\widetilde{w})}{(z-\widetilde{w})(\widetilde{w}-w)}\frac{\mathcal{B}(\widetilde{w},t)}{\mathcal{A}(\widetilde{w},t)}. \tag{4.3.20}$$

We can now move I_1 to $i\mathbb{R}+e$ from the right without changing the kernel $\widetilde{\mathcal{K}}$

$$\begin{aligned} \widetilde{K}(z,w) &= \frac{\mathcal{C}(z,t)}{\mathcal{A}(z,t)}j_+^2(w)\int_{i\mathbb{R}+e}\frac{d\widetilde{w}}{2pi}\frac{j_-^{-2}(\widetilde{w})}{(z-\widetilde{w})(\widetilde{w}-w)}\frac{\mathcal{B}(\widetilde{w},t)}{\mathcal{A}(\widetilde{w},t)} && (4.3.21) \\ &= \frac{\mathcal{C}(z,t)}{\mathcal{A}(z,t)}j_+^2(w)\int_{i\mathbb{R}+e}\frac{d\widetilde{w}}{2pi}\frac{j_+^{-2}(\widetilde{w})}{(z-\widetilde{w})(\widetilde{w}-w)}\mathcal{A}(\widetilde{w},t)\mathcal{B}(\widetilde{w},t), && (4.3.22) \end{aligned}$$

where in the last identity we use the relation from (4.3.2), (4.3.4): $j_+(\widetilde{w}) = j_-(\widetilde{w})\mathcal{A}(\widetilde{w},t)$. Therefore we conclude from (4.3.16) and the above discussion that

$$t_{LU}(t) = \det\left[\mathbb{1}_{L^2(I_3)} - \mathcal{K}_{13}\circ\mathcal{K}_{31}\right] = \det\left[\mathbb{1}_{L^2(i\mathbb{R})} - \widetilde{\mathcal{K}}\right]. \tag{4.3.23}$$

Moreover, it is immediate to see from Theorem 2.2 that $t_{LU}(t)$ is indeed the JMU tau-function. □

4.3.3 Malgrange forms

In (4.3.23), we expressed the tau-function on LU as a Fredholm determinant. To relate t_{LU} to t_Σ in (4.2.14), we will first prove that the tau-function corresponding to the RHP:4.4, call it t_{LDU}, is equal to $t_{i\mathbb{R}}$ plus non-vanishing explicit factors as in proposition:4.1, and then show that t_{LU} is related to t_{LDU} up to explicit terms. We know that the Malgrange form for the RHP on $i\mathbb{R}$ (4.2.15) is :

$$\P_t \log t_{i\mathbb{R}} = \int_{i\mathbb{R}} \frac{dz}{2pi}\,\mathrm{Tr}\left[\Theta_-^{-1}\Theta_-'\,\dot{J}J^{-1}\right]. \tag{4.3.24}$$

Similarly, the Malgrange form of the RHP on LDU (4.3.2): $Y_{i+1} = Y_i F_i$ is

$$\P_t \log t_{LDU} = \sum_{i=1}^{3}\int_{I_i} \frac{dz}{2pi}\,\mathrm{Tr}\left[Y_i^{-1}Y_i'\dot{F}_i F_i^{-1}\right]. \tag{4.3.25}$$

Proposition 4.3. *The Malgrange forms for the RHPs on the contours $i\mathbb{R}$ (RHP:4.3) and on LDU (RHP:4.4) are related as*

$$\P_t \log t_{i\mathbb{R}} = \P_t \log t_{LDU} - \int_{i\mathbb{R}} \frac{dz}{2pi}\left(\frac{\dot{\mathcal{B}}}{\mathcal{A}}\right)\left(\mathcal{A}\mathcal{C}' - \mathcal{A}'\mathcal{C}\right). \tag{4.3.26}$$

the functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are defined in (4.2.4).

Proof. We begin by substituting (4.3.1): $J = F_1F_2F_3$ in the term

$$\begin{aligned}\dot{J}J^{-1} &= \left(\dot{F}_1F_2F_3 + F_1\dot{F}_2F_3 + F_1F_2\dot{F}_3\right)\left(F_3^{-1}F_2^{-1}F_1^{-1}\right)\\ &= \left(\dot{F}_1F_1^{-1} + F_1\dot{F}_2F_2^{-1}F_1^{-1} + F_1F_2\dot{F}_3F_3^{-1}F_2^{-1}F_1^{-1}\right).\end{aligned} \tag{4.3.27}$$

Substituting in (4.3.27) in the integrand of (4.3.24),

$$\mathrm{Tr}\left[\Theta_-^{-1}\Theta_-'\,\dot{J}J^{-1}\right] = \mathrm{Tr}\left[\Theta_-^{-1}\Theta_-'\left(\dot{F}_1F_1^{-1} + F_1\dot{F}_2F_2^{-1}F_1^{-1} + F_1F_2\dot{F}_3F_3^{-1}F_2^{-1}F_1^{-1}\right)\right]. \tag{4.3.28}$$

The equivalence (4.3.3) along with the jump condition (4.3.2) imply that

$$\Theta_- = Y_1, \quad \Theta_-F_1 = Y_2, \quad \Theta_-F_1F_2 = Y_3. \tag{4.3.29}$$

Substituting (4.3.29) in (4.3.28),

$$\begin{aligned}
&\mathrm{Tr}\left[\Theta_-^{-1}\Theta_-'\left(\dot{F}_1F_1^{-1}+F_1\dot{F}_2F_2^{-1}F_1^{-1}+F_1F_2\dot{F}_3F_3^{-1}F_2^{-1}F_1^{-1}\right)\right]\\
&=\mathrm{Tr}\left[\Theta_-^{-1}\Theta_-'\dot{F}_1F_1^{-1}+F_1^{-1}\Theta_-^{-1}\Theta_-'F_1\dot{F}_2F_2^{-1}+F_2^{-1}F_1^{-1}\Theta_-^{-1}\Theta_-'F_1F_2\dot{F}_3F_3^{-1}\right]\\
&=\mathrm{Tr}\left[Y_1^{-1}Y_1'\dot{F}_1F_1^{-1}+Y_2^{-1}Y_2'\dot{F}_2F_2^{-1}-F_1^{-1}F_1'\dot{F}_2F_2^{-1}+Y_3^{-1}Y_3'\dot{F}_3F_3^{-1}-(F_1F_2)^{-1}(F_1F_2)'\dot{F}_3F_3^{-1}\right]\\
&=\sum_{i=1}^{3}\mathrm{Tr}\left[Y_i^{-1}Y_i'\dot{F}_iF_i^{-1}\right]-\mathrm{Tr}\left[F_1^{-1}F_1'\dot{F}_2F_2^{-1}+(F_1F_2)^{-1}(F_1F_2)'\dot{F}_3F_3^{-1}\right]. \qquad (4.3.30)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{i\mathbb{R}}\frac{dz}{2pi}\mathrm{Tr}\left[\Theta_-^{-1}\Theta_-'\dot{J}J^{-1}\right]&=\sum_{i=1}^{3}\int_{l_i}\frac{dz}{2pi}\mathrm{Tr}\left[Y_i^{-1}Y_i'\dot{F}_iF_i^{-1}\right]-\int_{i\mathbb{R}}\frac{dz}{2pi}\mathrm{Tr}\left[F_1^{-1}F_1'\dot{F}_2F_2^{-1}\right]\\
&\quad-\int_{i\mathbb{R}}\frac{dz}{2pi}\mathrm{Tr}\left[(F_1F_2)^{-1}(F_1F_2)'\dot{F}_3F_3^{-1}\right]. \qquad (4.3.31)
\end{aligned}$$

Let us analyze the explicit terms.

- Since F_1 is upper triangular and F_2 is diagonal as defined in (4.3.1),

$$\mathrm{Tr}\left[F_1^{-1}F_1'\dot{F}_2F_2^{-1}\right]=0. \qquad (4.3.32)$$

- Substituting $F_{1,2,3}$ in the last term in (4.3.31),

$$\mathrm{Tr}\left[(F_1F_2)^{-1}(F_1F_2)'\dot{F}_3F_3^{-1}\right]=\left(\frac{\dot{\mathcal{B}}}{\mathcal{A}}\right)\left(\mathcal{A}\mathcal{C}'-\mathcal{A}'\mathcal{C}\right) \qquad (4.3.33)$$

  where $\mathcal{A},\mathcal{B},\mathcal{C}$ are explicit in terms of parabolic cylinder functions (4.2.4).

  Therefore,

$$¶_t\log t_{i\mathbb{R}}=¶_t\log t_{LDU}-\int_{i\mathbb{R}}\frac{dz}{2pi}\left(\frac{\dot{\mathcal{B}}}{\mathcal{A}}\right)\left(\mathcal{A}\mathcal{C}'-\mathcal{A}'\mathcal{C}\right). \qquad (4.3.34)$$

□

Recall from proposition:4.3, the Malgrange form of the RHP on LDU (4.3.25):

$$¶_t\log t_{LDU}=\sum_{i=1}^{3}\int_{l_i}\frac{dz}{2pi}\mathrm{Tr}\left[Y_i^{-1}Y_i'\dot{F}_iF_i^{-1}\right]. \qquad (4.3.35)$$

For the RHP on LU (RHP:4.5) with the jump condition (4.3.6): $\widetilde{Y}_{i+1}=\widetilde{Y}_i\widetilde{F}_i$ where $i=1,3$, the Malgrange form reads

$$¶_t\log t_{LU}=\sum_{i=1,3}\int_{l_i}\frac{dz}{2pi}\mathrm{Tr}\left[\widetilde{Y}_i^{-1}\widetilde{Y}_i'\dot{\widetilde{F}}_i\widetilde{F}_i^{-1}\right]. \qquad (4.3.36)$$

Proposition 4.4. *The Malgrange forms of the RHPs on contours LDU (RHP:4.4) and LU (RHP:4.5) are related as*

$$\P_t \log t_{LDU} = \P_t \log t_{LU} + 2\int_{i\mathbb{R}} \frac{dz}{2pi}\frac{\dot{\mathcal{A}}(z,t)}{\mathcal{A}(z,t)}\int_{i\mathbb{R}_-}\frac{dw}{2pi}\frac{\mathcal{A}'(w,t)}{\mathcal{A}(w,t)(z-w)}. \tag{4.3.37}$$

Proof. We will first simplify the integrals on l_1 and l_3 in (4.3.35). Given that (4.3.5):$Y_i = \widetilde{Y}_i j^{s_3}$ and (4.3.9):$F_i = j^{-s_3}\widetilde{F}_i j^{s_3}$,

$$\begin{aligned}
&\sum_{i=1,3}\int_{l_i}\frac{dz}{2pi}\operatorname{Tr}\left[Y_i^{-1}Y_i'\dot{F}_iF_i^{-1}\right]\\
&= \sum_{i=1,3}\int_{l_i}\frac{dz}{2pi}\operatorname{Tr}\left[\left(\widetilde{Y}_i j^{s_3}\right)^{-1}\left(\widetilde{Y}_i j^{s_3}\right)'\left(j^{-s_3}\dot{\widetilde{F}}_i j^{s_3}\right)\left(j^{-s_3}\widetilde{F}_i^{-1} j^{s_3}\right)\right]\\
&= \sum_{i=1,3}\int_{l_i}\frac{dz}{2pi}\operatorname{Tr}\left[\left(\widetilde{Y}_i j^{s_3}\right)^{-1}\left(\widetilde{Y}_i j^{s_3}\right)'\left(j^{-s_3}\dot{\widetilde{F}}_i j^{s_3}\right)\left(j^{-s_3}\widetilde{F}_i^{-1} j^{s_3}\right)\right]\\
&= \sum_{i=1,3}\int_{l_i}\frac{dz}{2pi}\operatorname{Tr}\left[\left(\widetilde{Y}_i^{-1}\widetilde{Y}_i' + s_3 j' j^{-1}\right)\left(-s_3 \dot{j} j^{-1} + \dot{M}_i\widetilde{F}_i^{-1} + \widetilde{F}_i s_3 \dot{j} j^{-1}\widetilde{F}_i^{-1}\right)\right]\\
&= \sum_{i=1,3}\int_{l_1}\frac{dz}{2pi}\operatorname{Tr}\left[\widetilde{Y}_i^{-1}\widetilde{Y}_i'\dot{\widetilde{F}}_i\widetilde{F}_i^{-1}\right] + \sum_{i=1,3}\int_{l_i}\frac{dz}{2pi}\operatorname{Tr}\left[\left(\widetilde{Y}_i^{-1}\widetilde{Y}_i'\right)\left(-s_3\dot{j} j^{-1} + \widetilde{F}_i s_3 \dot{j} j^{-1}\widetilde{F}_i^{-1}\right)\right]\\
&\quad + \sum_{i=1,3}\int_{l_i}\frac{dz}{2pi}\operatorname{Tr}\left[s_3 j' j^{-1}\left(-s_3\dot{j} j^{-1} + \dot{\widetilde{F}}_i\widetilde{F}_i^{-1} + \widetilde{F}_i s_3 \dot{j} j^{-1}\widetilde{F}_i^{-1}\right)\right].
\end{aligned} \tag{4.3.38}$$

In (4.3.38), $\widetilde{F}_i$ are either lower or upper triangular (4.3.9). Therefore,

$$\int_{l_1\cup l_3}\frac{dz}{2pi}\operatorname{Tr}\left[s_3 j' j^{-1}\left(-s_3\dot{j} j^{-1} + \dot{M}_i\widetilde{F}_i^{-1} + \widetilde{F}_i s_3 \dot{j} j^{-1}\widetilde{F}_i^{-1}\right)\right] = 0. \tag{4.3.39}$$

Therefore, given (4.3.36), (4.3.38) reads

$$\begin{aligned}
\sum_{i=1,3}\int_{l_i}\frac{dz}{2pi}\operatorname{Tr}\left[Y_i^{-1}Y_i'\dot{F}_iF_i^{-1}\right] &= \sum_{i=1,3}\int_{l_i}\frac{dz}{2pi}\operatorname{Tr}\left[\widetilde{Y}_i^{-1}\widetilde{Y}_i'\dot{\widetilde{F}}_i\widetilde{F}_i^{-1}\right]\\
&\quad + \sum_{i=1,3}\int_{l_i}\frac{dz}{2pi}\operatorname{Tr}\left[\left(\widetilde{Y}_i^{-1}\widetilde{Y}_i'\right)\left(-s_3\dot{j} j^{-1} + \widetilde{F}_i s_3 \dot{j} j^{-1}\widetilde{F}_i^{-1}\right)\right]\\
&= \P_t \log t_{LU} + \sum_{i=1,3}\int_{l_i}\frac{dz}{2pi}\operatorname{Tr}\left[\left(\widetilde{Y}_i^{-1}\widetilde{Y}_i'\right)\left(-s_3\dot{j} j^{-1} + \widetilde{F}_i s_3 \dot{j} j^{-1}\widetilde{F}_i^{-1}\right)\right].
\end{aligned} \tag{4.3.40}$$

Recalling (4.3.6): $\widetilde{Y}_{i+1} = \widetilde{Y}_i \widetilde{F}_i$, the second term in (4.3.40) can be further simplified

$$\begin{aligned}
&\sum_{i=1,3} \int_{l_i} \frac{dz}{2pi} \operatorname{Tr}\left[\left(\widetilde{Y}_i^{-1}\widetilde{Y}_i'\right)\left(-s_3 j j^{-1} + \widetilde{F}_i s_3 j j^{-1} \widetilde{F}_i^{-1}\right)\right] \\
&= \sum_{i=1,3} \int_{l_i} \frac{dz}{2pi} \operatorname{Tr}\left[-\widetilde{Y}_i^{-1}\widetilde{Y}_i' s_3 j j^{-1} + \widetilde{F}_i^{-1}\widetilde{Y}_i^{-1}\widetilde{Y}_i'\widetilde{F}_i s_3 j j^{-1}\right] \\
&= \sum_{i=1,3} \int_{l_i} \frac{dz}{2pi} \operatorname{Tr}\left[-\widetilde{Y}_i^{-1}\widetilde{Y}_i' s_3 j j^{-1} + \widetilde{Y}_{i+1}^{-1}\widetilde{Y}_{i+1}' s_3 j j^{-1} - \widetilde{F}_i^{-1}\widetilde{F}_i' s_3 j j^{-1}\right] \\
&= \sum_{i=1,3} \int_{l_i} \frac{dz}{2pi} \operatorname{Tr}\left[\Delta\left(\widetilde{Y}_i^{-1}\widetilde{Y}_i'\right) s_3 j j^{-1} - \widetilde{F}_i^{-1}\widetilde{F}_i' s_3 j j^{-1}\right] \\
&= \sum_{i=1,3} \int_{l_i} \frac{dz}{2pi} \operatorname{Tr}\left[\Delta\left(\widetilde{Y}_i^{-1}\widetilde{Y}_i'\right) s_3 j j^{-1}\right],
\end{aligned} \tag{4.3.41}$$

where $\Delta\left(\widetilde{Y}_i^{-1}\widetilde{Y}_i'\right) = \widetilde{Y}_{i+1}^{-1}\widetilde{Y}_{i+1}' - \widetilde{Y}_i^{-1}\widetilde{Y}_i'$. The last line is obtained using the fact that $\operatorname{Tr}\left[\widetilde{F}_i^{-1}\widetilde{F}_i' s_3 j j^{-1}\right] = 0$ since $\widetilde{F}_i$ is either lower or upper triangular and j is scalar.

The final expression in (4.3.41) can be further simplified by noting that the function j has no jumps on l_1 and l_3. Beginning with the integral on l_1,

$$\begin{aligned}
\int_{l_1} \frac{dz}{2pi} \operatorname{Tr}\left[\Delta\left(\widetilde{Y}_1^{-1}\widetilde{Y}_1'\right) s_3 j j^{-1}\right] &= \int_{l_1} \frac{dz}{2pi} \operatorname{Tr}\left[\left(\widetilde{Y}_2^{-1}\widetilde{Y}_2' - \widetilde{Y}_1^{-1}\widetilde{Y}_1'\right) s_3 j j^{-1}\right] \\
&= \int_{l_1} \frac{dz}{2pi} \operatorname{Tr}\left[\widetilde{Y}_2^{-1}\widetilde{Y}_2' s_3 j j^{-1}\right].
\end{aligned} \tag{4.3.42}$$

To obtain the last line, we notice that $\int_{l_1} \frac{dz}{2pi} \operatorname{Tr}\left[\widetilde{Y}_1^{-1}\widetilde{Y}_1' s_3 j j^{-1}\right] = 0$ by closing the contour on the right. A similar computation follows for the integral on l_3 in (4.3.41)

$$\begin{aligned}
\int_{l_3} \frac{dz}{2pi} \operatorname{Tr}\left[\Delta\left(\widetilde{Y}_3^{-1}\widetilde{Y}_3'\right) s_3 j j^{-1}\right] &= \int_{l_3} \frac{dz}{2pi} \operatorname{Tr}\left[\left(\widetilde{Y}_4^{-1}\widetilde{Y}_4' - \widetilde{Y}_3^{-1}\widetilde{Y}_3'\right) s_3 j j^{-1}\right] \\
&= -\int_{l_3} \frac{dz}{2pi} \operatorname{Tr}\left[\widetilde{Y}_3^{-1}\widetilde{Y}_3' s_3 j j^{-1}\right].
\end{aligned} \tag{4.3.43}$$

To obtain the last line, we note that $\int_{l_3} \frac{dz}{2pi} \operatorname{Tr}\left[\widetilde{Y}_4^{-1}\widetilde{Y}_4' s_3 j j^{-1}\right] = 0$ by closing the contour on the left.

Gathering the terms (4.3.42), (4.3.43), and using (4.3.6):$\widetilde{Y}_2 = \widetilde{Y}_3$, (4.3.41) reads

$$\begin{aligned}
\sum_{i=1,3} \int_{l_i} \frac{dz}{2pi} \operatorname{Tr}\left[\Delta\left(\widetilde{Y}_i^{-1}\widetilde{Y}_i'\right) s_3 j j^{-1}\right] &= \int_{l_1} \frac{dz}{2pi} \operatorname{Tr}\left[\widetilde{Y}_2^{-1}\widetilde{Y}_2' s_3 j j^{-1}\right] - \int_{l_3} \frac{dz}{2pi} \operatorname{Tr}\left[\widetilde{Y}_3^{-1}\widetilde{Y}_3' s_3 j j^{-1}\right] \\
&= -\int_{l_2} \frac{dz}{2pi} \operatorname{Tr}\left[\widetilde{Y}_2^{-1}\widetilde{Y}_2' s_3 j j^{-1}\right] + \int_{l_2} \frac{dz}{2pi} \operatorname{Tr}\left[\widetilde{Y}_3^{-1}\widetilde{Y}_3' s_3 j j^{-1}\right] = 0.
\end{aligned} \tag{4.3.44}$$

Substituting (4.3.44) in (4.3.40),

$$\sum_{i=1,3} \int_{l_i} \frac{dz}{2pi} \operatorname{Tr}\left[Y_i^{-1} Y_i' \dot{F}_i F_i^{-1}\right] = \sum_{i=1,3} \int_{l_i} \frac{dz}{2pi} \operatorname{Tr}\left[\widetilde{Y}_i^{-1}\widetilde{Y}_i' \dot{\widetilde{F}}_i \widetilde{F}_i^{-1}\right] = \P_t \log t_{LU}. \tag{4.3.45}$$

We now compute the integral on l_2 in (4.3.38)

$$\int_{l_2} \frac{dz}{2pi} \mathrm{Tr}\left[Y_2^{-1} Y_2' \dot{F}_2 F_2^{-1}\right] = \int_{l_2} \frac{dz}{2pi} \mathrm{Tr}\left[\left(\widetilde{Y}_2 j_-^{s_3}\right)^{-1}\left(\widetilde{Y}_2' j_-^{s_3} + \widetilde{Y}_2\left(j_-^{s_3}\right)'\right) \dot{F}_2 F_2^{-1}\right]$$
$$= \int_{l_2} \frac{dz}{2pi} \mathrm{Tr}\left[j_-^{-s_3} \widetilde{Y}_2^{-1}\left(\widetilde{Y}_2' j_-^{s_3} + \widetilde{Y}_2\left(j_-^{s_3}\right)'\right) \dot{F}_2 F_2^{-1}\right]$$
$$= \int_{l_2} \frac{dz}{2pi} \mathrm{Tr}\left[\widetilde{Y}_2^{-1} \widetilde{Y}_2' \dot{F}_2 F_2^{-1} + s_3 j_-^{-1} j'_- \dot{F}_2 F_2^{-1}\right]. \tag{4.3.46}$$

Since $\widetilde{Y}_2$ does not jump on l_2, Liouville theorem implies that

$$\mathrm{Tr}\left[\widetilde{Y}_2^{-1} \widetilde{Y}_2' \dot{F}_2 F_2^{-1}\right] = 0. \tag{4.3.47}$$

The term

$$\mathrm{Tr}\left[s_3 j_-^{-1} j'_- \dot{F}_2 F_2^{-1}\right] \tag{4.3.48}$$

in (4.3.46) is an explicit function of $\mathcal{A}(z,w)$ in (4.2.4). From (4.3.1),

$$F_2 = \mathcal{A}^{s_3} \Rightarrow \dot{F}_2 F_2^{-1} = \frac{\dot{\mathcal{A}}}{\mathcal{A}} s_3. \tag{4.3.49}$$

The function j_- is the boundary value of j defined in (4.3.4)

$$j_- = \int_{i\mathbb{R}-e} \frac{dw}{2pi} \frac{\log \mathcal{A}(w,t)}{z-w} \Rightarrow j_-^{-1} j'_- = \int_{i\mathbb{R}-e} \frac{dw}{2pi} \frac{\mathcal{A}'(w,t)}{\mathcal{A}(w,t)(z-w)}. \tag{4.3.50}$$

The expression (4.3.46) simplifies as follows due to (4.3.49), (4.3.50)

$$\int_{l_2} \frac{dz}{2pi} \mathrm{Tr}\left[Y_2^{-1} Y_2' \dot{F}_2 F_2^{-1}\right] = \int_{l_2} \frac{dz}{2pi} \mathrm{Tr}\left[s_3 j_-^{-1} j'_- \dot{F}_2 F_2^{-1}\right]$$
$$= \int_{i\mathbb{R}} \frac{dz}{2pi} \left(2\frac{\dot{\mathcal{A}}(z,t)}{\mathcal{A}(z,t)}\right) \int_{i\mathbb{R}-e} \frac{dw}{2pi} \frac{\mathcal{A}'(w,t)}{\mathcal{A}(w,t)(z-w)}. \tag{4.3.51}$$

Substituting (4.3.51) and (4.3.45), in (4.3.35)

$$\P_t \log t_{LDU} = \sum_{i=1}^{3} \int_{l_i} \frac{dz}{2pi} \mathrm{Tr}\left[Y_i^{-1} Y_i' \dot{F}_i F_i^{-1}\right]$$
$$= \sum_{i=1,3} \int_{l_i} \frac{dz}{2pi} \mathrm{Tr}\left[Y_i^{-1} Y_i' \dot{F}_i F_i^{-1}\right] + \int_{i\mathbb{R}} \frac{dz}{2pi} \mathrm{Tr}\left[s_3 j_-^{-1} j'_- \dot{F}_2 F_2^{-1}\right]$$
$$= \P_t \log t_{LU} + +2 \int_{i\mathbb{R}} \frac{dz}{2pi} \frac{\dot{\mathcal{A}}(z,t)}{\mathcal{A}(z,t)} \int_{i\mathbb{R}_-} \frac{dw}{2pi} \frac{\mathcal{A}'(w,t)}{\mathcal{A}(w,t)(z-w)} \tag{4.3.52}$$

□

4.4 Fredholm determinant representation of Painlevé II tau-function

Theorem 4.1. *The tau-function of Painlevé II equation can be expressed in terms of a Fredholm determinant of an integrable operator as follows*

$$\P_t \log t_{PII} = \P_t \log\det\left[\mathbb{1}_{L^2(i\mathbb{R})} - \widetilde{\mathcal{K}}\right] - \left[\frac{4in}{3} + \frac{2n^2}{t}\right] + \mathcal{F}(t,n,h). \tag{4.4.1}$$

The kernel

$$\widetilde{\mathcal{K}}(z,w) = \frac{\mathcal{C}(z,t)}{\mathcal{A}(z,t)} j_-^2(w) \int_{i\mathbb{R}-e} \frac{d\widetilde{w}}{2pi} \frac{j_-^{-2}(\widetilde{w})}{(z-\widetilde{w})(\widetilde{w}-w)} \mathcal{A}(\widetilde{w},t)\mathcal{B}(\widetilde{w},t) \tag{4.4.2}$$

with

$$z \equiv z(z,t) = 2t^{1/2}\sqrt{-\frac{4i}{3}z^3 + iz - \frac{i}{3}}; \quad x = z(-z,t)$$

$$\mathcal{A}(z,t) = \frac{e^{-2pin}}{h^4} e^{\frac{2i}{3}t} z^n x^n \left(-e^{ipn}h^4 D_{-n}(iz)D_{-n}(ix) - n^2 e^{4pin} D_{n-1}(z) D_{n-1}(x)\right)$$

$$\mathcal{B}(z,t) = -\frac{e^{-2pin}}{h^2}\left(\frac{z+1/2}{z-1/2}\right)^{2n} z^n x^{-n}\left(-ie^{ipn}h^4 D_{-n}(iz)D_{-n-1}(ix) - ne^{4pin}D_{n-1}(z)D_n(x)\right)$$

$$\mathcal{C}(z,t) = \mathcal{B}(-z,t)\,; \quad j_-(w,t) = \int_{i\mathbb{R}-e} \frac{dw'}{2pi} \frac{\log\mathcal{A}(w',t)}{w-w'}. \tag{4.4.3}$$

The coordinate z, isomonodromic time t are related to l, x in (1.3.4) as

$$l = (-x)^{1/2} z, \quad t = (-x)^{3/2}. \tag{4.4.4}$$

$D_n(x)$ is the parabolic cylinder function, and the constants n, h are determined by the Stokes' parameters s_1, s_3.

$$n = -\frac{1}{2pi}\log(1 - s_1 s_3), \tag{4.4.5}$$

$$h = -\frac{\sqrt{2p}}{\Gamma(-n)s_3} e^{ipn}. \tag{4.4.6}$$

The term $\mathcal{F}(t,n,h)$ is a regular function of t and the Stokes' data and is defined in (4.4.9).

Proof. The Propositions 4.1, 4.3, 4.4 imply that the tau-functions t_Σ and t_{LU} are related through explicit factors, and Proposition 4.2 expresses t_{LU} as a Fredholm determinant. Therefore, the tau-function of Painlevé II equation defined in (4.1.5)

$$\P_t \log t_{P_{II}} \equiv \P_t \log t_{\Sigma} \overset{(4.2.39)}{=} \P_t \log t_{i\mathbb{R}} - \int_{i\mathbb{R}} \frac{dz}{2pi} \widetilde{\mathcal{F}}(z,t;n,h) - \left[\frac{4in}{3} + \frac{2n^2}{t}\right]$$

$$\overset{(4.3.34)}{=} \P_t \log t_{LDU} - \int_{i\mathbb{R}} \frac{dz}{2pi} \left\{ \left(\frac{\dot{\mathcal{B}}}{\mathcal{A}}\right) (\mathcal{A}\mathcal{C}' - \mathcal{A}'\mathcal{C}) + \widetilde{\mathcal{F}}(z,x,t;n,h) \right\} - \left[\frac{4in}{3} + \frac{2n^2}{t}\right]$$

$$\overset{(4.3.52)}{=} \P_t \log t_{LU} + \int_{i\mathbb{R}} \frac{dz}{2pi} \left\{ \frac{2\dot{\mathcal{A}}(z,t)}{\mathcal{A}(z,t)} \left(\int_{i\mathbb{R}-e} \frac{dw}{2pi} \frac{\mathcal{A}'(w,t)}{\mathcal{A}(w,t)(z-w)} \right) - \left(\frac{\dot{\mathcal{B}}}{\mathcal{A}}\right) (\mathcal{A}\mathcal{C}' - \mathcal{A}'\mathcal{C}) - \widetilde{\mathcal{F}}(z,t;n,h) \right\}$$
$$- \left[\frac{4in}{3} + \frac{2n^2}{t}\right]$$

$$\overset{(4.3.23)}{=} \P_t \log \det \left[\mathbb{1}_{L^2(i\mathbb{R})} - \widetilde{\mathcal{K}}\right] - \left[\frac{4in}{3} + \frac{2n^2}{t}\right]$$
$$+ \int_{i\mathbb{R}} \frac{dz}{2pi} \left\{ \frac{2\dot{\mathcal{A}}(z,t)}{\mathcal{A}(z,t)} \left(\int_{i\mathbb{R}-e} \frac{dw}{2pi} \frac{\mathcal{A}'(w,t)}{\mathcal{A}(w,t)(z-w)} \right) - \left(\frac{\dot{\mathcal{B}}}{\mathcal{A}}\right) (\mathcal{A}\mathcal{C}' - \mathcal{A}'\mathcal{C}) - \widetilde{\mathcal{F}}(z,t;n,h) \right\}. \quad (4.4.7)$$

In (4.4.7), the functions $\mathcal{A}, \mathcal{B}, \mathcal{C}$ defined in (4.2.4) are explicit in terms of parabolic cylinder functions, $\widetilde{\mathcal{F}}$ is defined in (4.2.38), and the term

$$\int_{i\mathbb{R}} \frac{dz}{2pi} \left\{ \frac{2\dot{\mathcal{A}}(z,t)}{\mathcal{A}(z,t)} \left(\int_{i\mathbb{R}-e} \frac{dw}{2pi} \frac{\mathcal{A}'(w,t)}{\mathcal{A}(w,t)(z-w)} \right) - \left(\frac{\dot{\mathcal{B}}}{\mathcal{A}}\right) (\mathcal{A}\mathcal{C}' - \mathcal{A}'\mathcal{C}) - \widetilde{\mathcal{F}}(z,t;n,h) \right\} \quad (4.4.8)$$

depends only on h, n and t. We then define

$$\mathcal{F}(t,n,h) := \int_{i\mathbb{R}} \frac{dz}{2pi} \left\{ \frac{2\dot{\mathcal{A}}(z,t)}{\mathcal{A}(z,t)} \left(\int_{i\mathbb{R}-e} \frac{dw}{2pi} \frac{\mathcal{A}'(w,t)}{\mathcal{A}(w,t)(z-w)} \right) - \left(\frac{\dot{\mathcal{B}}}{\mathcal{A}}\right) (\mathcal{A}\mathcal{C}' - \mathcal{A}'\mathcal{C}) - \widetilde{\mathcal{F}}(z,t;n,h) \right\}. \quad (4.4.9)$$

In terms of $\mathcal{F}(t,n,h)$, (4.4.7) reads

$$\P_t \log t_{P_{II}} = \P_t \log \det \left[\mathbb{1}_{L^2(i\mathbb{R})} - \widetilde{\mathcal{K}}\right] + F(t,n,h) - \left[\frac{4in}{3} + \frac{2n^2}{t}\right]. \quad (4.4.10)$$

Therefore, the tau-function of Painlevé II can be expressed as a Fredholm determinant of an integrable operator up to explicit factors. Furthermore, solving the RHP:4.5 is equivalent to solving the RHP:4.3, which in turn is tantamount to solving the RHP:4.1. Therefore, the zeros of $t_{P_{II}}$ (solvability condition of 4.1) are completely determined by the zeros of the Fredholm determinant (4.3.23). □

Some further comments are in order.

1. The tau-function (4.4.1) is defined on $\mathbb{C} \setminus \{0\}$ and is analytic in t. Refer to Remark:4.1 for the details.
2. The degeneration limit from the general tau-function of Painlevé II in (4.4.1) to the tau-function of the Ablowitz-Segur family of solutions (determinant of the Airy kernel) in [40] is singular.

Chapter 5

Tau-function of a n-point torus

In this chapter, we extend the determinant formalism of [54] that was summarized in Section 2.3 to isomonodromic tau-functions on a torus with an arbitrary number of Fuchsian singularities [76, 77, 79, 81, 110], providing both the Fredholm determinant representation of the general case, and its minor expansion in terms of Nekrasov-Okounkov functions [90, 89] as will be seen in Chapter 7. This extends and completes the analysis of [15, 14], where these cases were studied by using CFT methods.

5.1 The 2-particle nonautonomous Calogero-Moser system: a toy model

Our starting point is the elliptic form of the Painlevé VI equation with an arbitrary parameter $m \in \mathbb{C}$ (1.2.29)

$$(2pi)^2 \frac{d^2 Q(t)}{dt^2} = m^2 \wp'(2Q(t)|t), \tag{5.1.1}$$

which describes the equation of motion of the nonautonomous 2-particle Calogero-Moser system [110]. The Weierstrass $\wp$ function is defined in terms of the theta function q_1 by

$$\wp(z|t) := -\frac{\P^2}{\P z^2} \log q_1(z|t) - \frac{1}{6}\frac{q_1'''(0|t)}{q_1'(0|t)} \equiv -\frac{\P^2}{\P z^2} \log q_1(z|t) - \frac{1}{6}\frac{q_1'''}{q_1'}, \tag{5.1.2}$$

$$q_1(z|t) := \sum_{n\in\mathbb{Z}} (-1)^{n-\frac{1}{2}} e^{ip\left(n+\frac{1}{2}\right)^2} e^{2pi\left(n+\frac{1}{2}\right)z}, \tag{5.1.3}$$

with the theta function satisfying the following periodicity properties:

$$q_1(z+1|t) = -q_1(z|t), \qquad q_1(z+t|t) = -e^{-2pi(z+\frac{t}{2})} q_1(z|t). \tag{5.1.4}$$

The Weierstrass $\wp$ function also has a series representation:

$$\wp(z|t) = \frac{1}{z^2} + \sum_{(p,q)\neq(0,0)} \left(\frac{1}{(z+p+qt)^2} - \frac{1}{(p+qt)^2} \right) \tag{5.1.5}$$

with fundamental periods $1, t$. The modular parameter of the torus t lies in the upper-half plane $\mathbb{H}$ and assumes the role of the isomonodromic time. The equation (5.1.1) arises as the compatibility condition of the following linear system[1] on a torus with one puncture set at zero [77, 81, 110],

$$\begin{aligned} \P_z \mathcal{Y}_{CM}(z,t) &= \mathcal{Y}_{CM}(z,t) L_{CM}(z,t), \\ 2pi \P_t \mathcal{Y}_{CM}(z,t) &= \mathcal{Y}_{CM}(z,t) M_{CM}(z,t), \end{aligned} \tag{5.1.6}$$

where (L_{CM}, M_{CM}) is the Lax pair of 2-particle non-autonomous Calogero-Moser system

$$\begin{aligned} L_{CM}(z,t) &= \begin{pmatrix} P(t) & mx(-2Q(t),z) \\ mx(2Q(t),z) & -P(t) \end{pmatrix}, \\ M_{CM}(z,t) &= m \begin{pmatrix} 0 & y(-2Q(t),z) \\ y(2Q(t),z) & 0 \end{pmatrix}. \end{aligned} \tag{5.1.7}$$

The functions $x(x,z)$, $y(x,z)$ and P in (5.1.7) are respectively,

$$x(x,z) = \frac{q_1(z-x|t)q_1'(0|t)}{q_1(z|t)q_1(x|t)}, \qquad y(x,z) = \P_x x(x,z), \qquad P(t) = 2pi\frac{dQ(t)}{dt}. \tag{5.1.8}$$

As opposed the behaviour of the Lax matrices on the sphere, the Lax matrix L_{CM} in (5.1.7) is not single-valued, and satisfies the relations

$$L_{CM}(z+1,t) = L_{CM}(z,t), \qquad L_{CM}(z+t,t) = e^{-2piQ(t)s_3} L_{CM}(z,t) e^{2piQ(t)s_3}. \tag{5.1.9}$$

Subsequently, the solution of the linear system (5.1.7) has the following monodromy properties around A,B cycles of the torus and around the puncture:

$$\begin{gathered} \mathcal{Y}_{CM}(z+1,t) = M_A \mathcal{Y}_{CM}(z,t), \qquad \mathcal{Y}_{CM}(z+t,t) = M_B \mathcal{Y}_{CM}(z,t) e^{2piQ(t)s_3}, \\ \mathcal{Y}_{CM}(e^{2pi}z,t) = M_0 \mathcal{Y}_{CM}(z,t), \end{gathered} \tag{5.1.10}$$

under the constraint

$$M_0 = M_A^{-1} M_B^{-1} M_A M_B, \tag{5.1.11}$$

and without loss of generality, it is always possible to set M_A to be diagonal by conjugation, so that

$$M_A = e^{2pias_3}, \qquad\qquad M_0 \sim e^{2pims_3}, \tag{5.1.12}$$

where $\sim$ means "in the same conjugacy class of", s_3 is the Pauli sigma matrix, and the arbitrary constants $a, m \in \mathbb{C}$.

5.2 Tau-function

The Hamiltonian of the system (5.1.7) is the A-cycle contour integral [77, 79]

$$H_{CM}(t) = \oint_A dz \frac{1}{2} \operatorname{tr} L_{CM}^2(z,t) = P(t)^2 - m^2 \wp(2Q(t)|t) + 4pim^2 \P_t \log h(t), \tag{5.2.1}$$

[1]Note that in this chapter we multiply the linear system from the left and the monodromies from the right. This choice makes the computations that follow easier.

where $h(t)$ is Dedekind's eta function

$$h(t) := \left(\frac{q_1'(0|t)}{2p}\right)^{1/3}. \tag{5.2.2}$$

The generator of the Hamiltonian H_{CM} is called the (isomonodromic) tau-function $\mathcal{T}_{CM}$ of the 2-particle non-autonomous Calogero-Moser system, and is defined by

$$2pi\P_t \log \mathcal{T}_{CM}(t) := H_{CM}(t). \tag{5.2.3}$$

Another notion of a tau-function describes it as a Fredholm determinant (if it exists) of an operator whose vanishing locus, called the *Malgrange divisor* [84], defines the non-solvability of some linear problem [8, 6]. In this spirit, following the construction in [54], we define a tau-function as the Fredholm determinant of certain Plemelj operators. The overview of the construction for the one-punctured torus is as follows:

- The pants decomposition [61] of the one-punctured torus consists of a trinion with two legs identified [57], whose boundaries become the A-cycle of the torus;
- A linear system with 3 Fuchsian singularities, whose solution is explicitly described by hypergeometric functions, is associated to the trinion;
- Boundary (Hilbert) spaces are defined on the two legs of the trinion;
- Two Plemelj operators, P_Σ and $P_\oplus$, are defined in terms of the solutions to the linear systems on the torus and on the trinion respectively. The Plemelj operators project one boundary space on to the other, effectively 'gluing' the cut along the A-cycle and giving us the one-punctured torus.
- A tau-function is then defined in (5.2.32) as a determinant of some combination of (restrictions of) the operators $\mathcal{P}_\Sigma$ and $\mathcal{P}_\oplus$.

5.2.1 Pants decomposition and Plemelj operators

Let us introduce the 2×2 matrix-valued function $\widetilde{Y}(z)$ that solves the following auxiliary linear system on a cylinder with 3 punctures at $-i\infty, 0, +i\infty$:

$$\P_z \widetilde{\mathcal{Y}}(z) = \widetilde{\mathcal{Y}}(z) L_{3pt}(z), \qquad L_{3pt}(z) = -2piA_0 - 2pi\frac{A_1}{1 - e^{2piz}}, \tag{5.2.4}$$

where A_0, A_1 are constant 2×2 matrices, and the fundamental solution $\widetilde{\mathcal{Y}}(z)$ of the linear system is described by hypergeometric functions, see [54, 15]. The local monodromy exponents of the Lax matrix in (5.2.4) are chosen so that they coincide with those on the torus (5.1.12):

$$A_0 \sim as_3, \qquad A_1 \sim ms_3, \tag{5.2.5}$$

and $\widetilde{\mathcal{Y}}(z)$ itself is chosen in such a way that

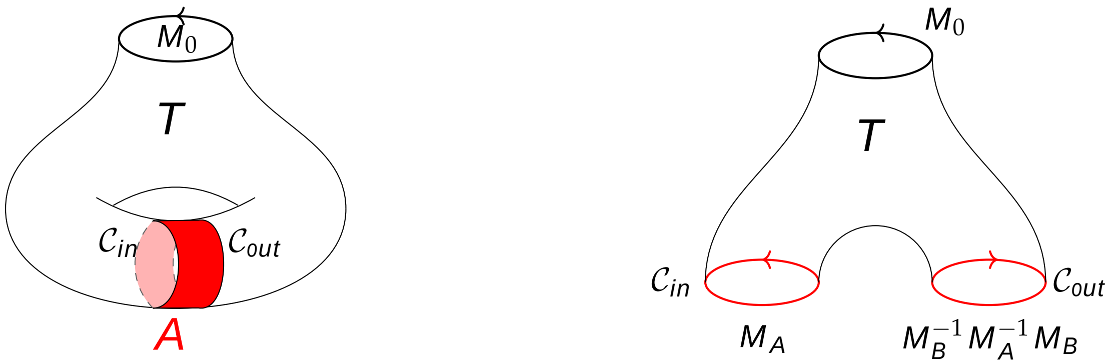
$$\widetilde{\mathcal{Y}}(z)^{-1}\mathcal{Y}_{CM}(z)$$

is regular and single-valued around $z = 0$ and has no monodromy around the A-cycles closest to the annulus A . In other words, $\widetilde{\mathcal{Y}}(z)$ "approximates" analytic behavior of $\mathcal{Y}(z)$ in the fundamental domain having the same monodromies around puncture and around two closest A-cycles.

The trinion T can then be viewed as being obtained by cutting the torus along its A-cycle, see Figure 5.1, inducing a homomorphism of monodromy groups [2] $p_1(C_{3,0}) \to p_1(C_{1,1})$

$$M_A M_0 M_B^{-1} M_A^{-1} M_B = 1 = (M_A) M_0 (M_B^{-1} M_A M_B)^{-1} := M_{-i\infty}^{3pt} M_0^{3pt} M_{i\infty}^{3pt}, \tag{5.2.6}$$

that defines the monodromies of the three-punctured cylinder around $-i\infty, 0, +i\infty$ in terms of the monodromy representation of the torus as in Figure 5.1b.



(A) Pants decomposition of $C_{1,1}$

(B) Trinion

FIGURE 5.1

Remark 5.1. *The linear system (5.2.4) is simply the usual three-point Fuchsian problem on the sphere, having mapped the sphere to a cylinder by $z \to e^{-2piz}$. The punctures at $0, 1, \infty$ become punctures at $-i\infty, 0, i\infty$ respectively.*

Definition 5.1. *Out of the solutions $\mathcal{Y}_{CM}(z)$, $\widetilde{\mathcal{Y}}(z)$ of the linear problems (5.1.6), (5.2.4) respectively, we define two matrix-valued functions $Y_{CM}(z)$, $\widetilde{Y}(z)$ with diagonal monodromies around the boundary circles $\mathcal{C}_{in}$ and $\mathcal{C}_{out}$ in Figure 5.1, by the following equations:*

$$Y_{CM}(z)|_{\mathcal{C}_{in}} := \mathcal{Y}_{CM}(z)|_{\mathcal{C}_{in}} \in \mathcal{H}_{in}, \qquad Y_{CM}(z)|_{\mathcal{C}_{out}} := M_B^{-1} \mathcal{Y}_{CM}(z)|_{\mathcal{C}_{out}} \in \mathcal{H}_{out}, \tag{5.2.7}$$

$$\widetilde{Y}(z)|_{\mathcal{C}_{in}} \equiv \widetilde{Y}_{in}(z) := \widetilde{\mathcal{Y}}(z)|_{\mathcal{C}_{in}} \in \mathcal{H}_{in}, \quad \widetilde{Y}(z)|_{\mathcal{C}_{out}} \equiv \widetilde{Y}_{out}(z) := M_B^{-1} \widetilde{\mathcal{Y}}(z)|_{\mathcal{C}_{out}} \in \mathcal{H}_{out}. \tag{5.2.8}$$

Notice that $Y_{CM}(z)$ and $\widetilde{Y}(z)$ also solve (5.1.6), (5.2.4) respectively. Moreover,

$$\widetilde{Y}(z)^{-1} Y_{CM}(z) = \widetilde{\mathcal{Y}}(z)^{-1} \mathcal{Y}_{CM}(z), \tag{5.2.9}$$

so effectively they can be exchanged in the formulas where they appear in the form of such ratios. Notice also that under such definition

$$Y_{CM}(z + t) = Y_{CM}(z) e^{2piQ}, \quad z \in \mathcal{C}_{in}. \tag{5.2.10}$$

[2] $C_{g,n}$ represents a Riemann surface of genus g with n punctures.

The Hilbert spaces $\mathcal{H}_{in}$, $\mathcal{H}_{out}$ on the boundaries of the pants $\mathcal{C}_{in}, \mathcal{C}_{out}$ respectively (see Figure 5.1) have an orthogonal decomposition into spaces of positive and negative Fourier modes. A Hilbert space $\mathcal{H}$ defined as the direct sum of $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ is then associated to the trinion T :

$$\mathcal{H} := \mathcal{H}_{in} \oplus \mathcal{H}_{out} = (\mathcal{H}_{in,-} \oplus \mathcal{H}_{out,+}) \oplus (\mathcal{H}_{in,+} \oplus \mathcal{H}_{out,-}) := \mathcal{H}_+ \oplus \mathcal{H}_-, \tag{5.2.11}$$

where

$$\mathcal{H}_+ = \mathcal{H}_{in,-} \oplus \mathcal{H}_{out,+}, \quad \mathcal{H}_- = \mathcal{H}_{in,+} \oplus \mathcal{H}_{out,-}. \tag{5.2.12}$$

The functions $f(z) \in \mathcal{H}$ then have the decomposition

$$f(z) = \begin{pmatrix} f_{in} \\ f_{out} \end{pmatrix} = \begin{pmatrix} f_{in,-} \\ f_{out,+} \end{pmatrix} \oplus \begin{pmatrix} f_{in,+} \\ f_{out,-} \end{pmatrix} \equiv f_+ \oplus f_-, \tag{5.2.13}$$

where

$$f_+ = \begin{pmatrix} f_{in,-} \\ f_{out,+} \end{pmatrix} \in \mathcal{H}_+, \quad f_- = \begin{pmatrix} f_{in,+} \\ f_{out,-} \end{pmatrix} \in \mathcal{H}_-, \tag{5.2.14}$$

and the $\pm$ parts of the function are defined by their Fourier expansions:

$$\begin{aligned} f_{in,+} &= e^{2piazs_3} \sum_{n=0}^{\infty} f_{in,n} e^{-2pinz}, \qquad & f_{in,-} &= e^{2piazs_3} \sum_{n=1}^{\infty} f_{in,-n} e^{2pinz}, \\ f_{out,+} &= e^{2piazs_3} \sum_{n=0}^{\infty} f_{out,n} e^{-2pinz}, \qquad & f_{out,-} &= e^{2piazs_3} \sum_{n=1}^{\infty} f_{out,n} e^{2pinz}. \end{aligned} \tag{5.2.15}$$

On the space $\mathcal{H}$ we introduce two Plemelj projectors in terms of the solutions to the linear systems (5.1.7), (5.2.4) respectively.

Definition 5.2. *The Plemelj operator $\mathcal{P}_{\Sigma_{1,1}} : \mathcal{H} \to \mathcal{H}$ is defined in terms of the solution to the linear system on the torus* (5.1.7) *as*

$$\begin{aligned} \left(\mathcal{P}_{\Sigma_{1,1}} f\right)(z) &= \int_{\mathcal{C}_{in} \cup \mathcal{C}_{out}} \frac{dw}{2pi} Y_{CM}(z;t) \Xi_2(z,w;t) Y_{CM}(w;t)^{-1} f(w) \\ &\equiv \int_{\mathcal{C}} \frac{dw}{2pi} Y_{CM}(z;t) \Xi_2(z,w;t) Y_{CM}(w;t)^{-1} f(w), \end{aligned} \tag{5.2.16}$$

where

$$\Xi_2(z,w;t) = \begin{pmatrix} \frac{q_1(z-w+Q-r)q_1'(0)}{q_1(z-w)q_1(Q-r)} & 0 \\ 0 & -\frac{q_1(z-w-Q-r)q_1'(0)}{q_1(z-w)q_1(Q+r)} \end{pmatrix}. \tag{5.2.17}$$

The function $\Xi_2(z,w;t)dw$ in (5.2.17) is a twisted Cauchy kernel, with the properties

$$\Xi_2(z+t,w;t) = e^{-2piQs_3+2pir} \Xi_2(z,w;t), \quad \Xi_2(x,w+t;t) = \Xi_2(z,w;t) e^{2piQs_3-2pir}, \tag{5.2.18}$$

The variable[3] $Q \equiv Q(t)$ is the solution of the non-autonomous Calogero-Moser system (5.1.1), and r is a parameter encoding a $U(1)$ B-cycle monodromy of the twisted Cauchy

[3]Here on, we drop the t dependence of Q for brevity.

kernel as can be seen in (5.2.18). It does not appear in the linear problem (5.1.6), but rather it is an arbitrary parameter whose role will become clear later (see remark 5.2). The expansion of $\Xi_2(z,w;t)$ for $z \sim w$ reads

$$\Xi_2(z,w;t) = \frac{\mathbb{I}}{z-w} + \operatorname{diag}\left[\frac{q_1'(Q-r)}{q_1(Q-r)}, -\frac{q_1'(Q+r)}{q_1(Q+r)}\right] + \frac{1}{2}(z-w)\operatorname{diag}\left[\frac{q_1''(Q-r)}{q_1(Q-r)}, \frac{q_1''(Q+r)}{q_1(Q+r)}\right] - \frac{\mathbb{I}}{6}(z-w)\frac{q_1'''}{q_1'} + \mathcal{O}\left((z-w)^2\right). \tag{5.2.19}$$

Definition 5.3. *Since the integrand in (5.2.16) has a singularity at $w = z$, we define the following rule: each time w approaches z, we go around the singularity in clockwise direction. Sometimes it is also useful to use the notation $\mathcal{C} = \mathcal{C}_{in} \cup \mathcal{C}_{out}$, and $\underline{\mathcal{C}}$, $\overline{\mathcal{C}}$ for the shifted contours as in Figure 5.2.*

One can verify that $\mathcal{P}^2_{\Sigma_{1,1}} = \mathcal{P}_{\Sigma_{1,1}}$, and that the space of functions on the annulus A, which is defined by the equation (5.2.39) (see also Figure 5.1a), is

$$\mathcal{H}_A \subseteq \ker \mathcal{P}_{\Sigma_{1,1}}. \tag{5.2.20}$$

Definition 5.4. *The Plemelj operator $\mathcal{P}_\oplus : \mathcal{H} \to \mathcal{H}$ is defined in terms of the solution of the 3–point linear system (5.2.4) as*

$$\begin{aligned}\left(\mathcal{P}_\oplus f\right)(z) &= \int_{\mathcal{C}_{in}\cup\mathcal{C}_{out}} dw\, \frac{\widetilde{Y}(z)\widetilde{Y}(w)^{-1}}{1-e^{-2pi(z-w)}} f(w) \\ &= \int_{\mathcal{C}} dw\, \frac{\widetilde{Y}(z)\widetilde{Y}(w)^{-1}}{1-e^{-2pi(z-w)}} f(w).\end{aligned} \tag{5.2.21}$$

For $z \sim w$,

$$\frac{1}{1-e^{-2pi(z-w)}} = \frac{1}{2pi(z-w)} + \frac{1}{2} + \frac{2pi}{12}(z-w) + \mathcal{O}\left((z-w)^2\right). \tag{5.2.22}$$

It can be verified that $\mathcal{P}^2_\oplus = \mathcal{P}_\oplus$, and

$$\ker \mathcal{P}_\oplus = \mathcal{H}_-. \tag{5.2.23}$$

Furthermore, one can prove that

$$\mathcal{P}_\oplus \mathcal{P}_{\Sigma_{1,1}} = \mathcal{P}_{\Sigma_{1,1}}, \quad \mathcal{P}_{\Sigma_{1,1}} \mathcal{P}_\oplus = \mathcal{P}_\oplus, \tag{5.2.24}$$

and therefore, the space of functions on the trinion T in Figure 5.1a is defined as

$$\mathcal{H}_T := \operatorname{im} \mathcal{P}_\oplus = \operatorname{im} \mathcal{P}_{\Sigma_{1,1}}. \tag{5.2.25}$$

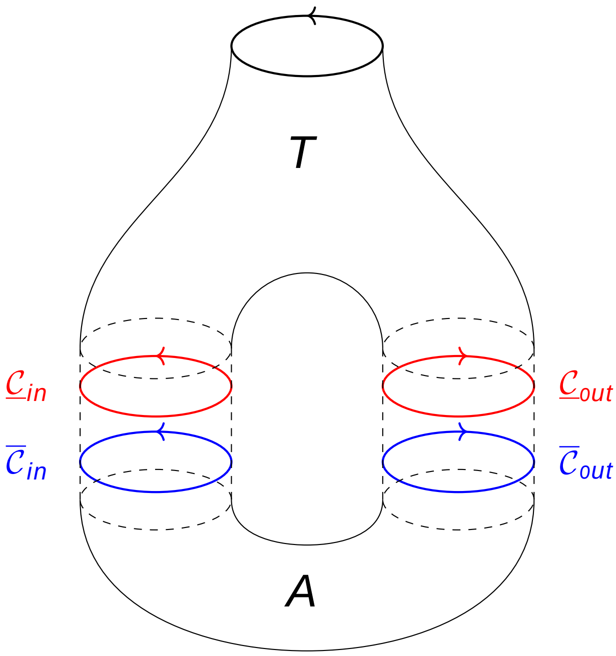


FIGURE 5.2: Contours

The components of $\mathcal{P}_\oplus$ under the orthogonal decomposition are obtained by computing its action on the function $f(z) \in \mathcal{H}$:

$$\begin{aligned}(\mathcal{P}_\oplus f)(z)_{in} = &\oint_{\mathcal{C}_{in}} dw \frac{1}{1-e^{-2pi(z-w)}} f_{in}(w) \\ &+ \oint_{\mathcal{C}_{in}} dw \frac{\widetilde{Y}_{in}(z)\widetilde{Y}_{in}(w)^{-1} - 1}{1-e^{-2pi(z-w)}} f_{in}(w) \\ &+ \oint_{\mathcal{C}_{out}} dw \frac{\widetilde{Y}_{in}(z)\widetilde{Y}_{out}(w)^{-1}}{1-e^{-2pi(z-w)}} f_{out}(w),\end{aligned} \tag{5.2.26}$$

$$\begin{aligned}(\mathcal{P}_\oplus f)(z)_{out} = &\oint_{\mathcal{C}_{out}} dw \frac{1}{1-e^{-2pi(z-w)}} f_{out}(w) \\ &+ \oint_{\mathcal{C}_{out}} dw \frac{\widetilde{Y}_{out}(z)\widetilde{Y}_{out}(w)^{-1} - 1}{1-e^{-2pi(z-w)}} f_{out}(w) \\ &+ \oint_{\mathcal{C}_{in}} dw \frac{\widetilde{Y}_{out}(z)\widetilde{Y}_{in}(w)^{-1}}{1-e^{-2pi(z-w)}} f_{in}(w).\end{aligned} \tag{5.2.27}$$

To analyze the formulas above we notice that

$$\widetilde{Y}_{out}(z) \in \mathbb{C}[[e^{2piz}]] \otimes e^{2piazs_3} End(\mathbb{C}^2), \quad \widetilde{Y}_{in}(z) \in \mathbb{C}[[e^{-2piz}]] \otimes e^{2piazs_3} End(\mathbb{C}^2), \tag{5.2.28}$$

and

$$\int_{-\frac{1}{2}+ic}^{\frac{1}{2}+ic} dw \frac{1}{1-e^{-2pi(z-w)}} f(w) = \begin{cases} f_-(z), & \mathrm{Im}\, z < c \\ -f_+(z), & \mathrm{Im}\, z > c. \end{cases} \tag{5.2.29}$$

Because of (5.2.28), (5.2.29), the action of $\mathcal{P}_\oplus$ on $f(z)$ in (5.2.26), (5.2.27) can be rewritten as

$$\begin{aligned}(\mathcal{P}_\oplus f)(z) &= \begin{pmatrix}(\mathcal{P}_\oplus f)_{in,-}\\(\mathcal{P}_\oplus f)_{out,+}\end{pmatrix} \oplus \begin{pmatrix}(\mathcal{P}_\oplus f)_{in,+}\\(\mathcal{P}_\oplus f)_{out,-}\end{pmatrix} = \\ &= \begin{pmatrix} f_{in,-}\\ f_{out,+}\end{pmatrix} \oplus \begin{pmatrix}\mathsf{a} & \mathsf{b}\\ \mathsf{c} & \mathsf{d}\end{pmatrix}\begin{pmatrix} f_{in,-}\\ f_{out,+}\end{pmatrix},\end{aligned} \tag{5.2.30}$$

where $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ are the components of $\mathcal{P}_\oplus$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_{in} \oplus \mathcal{H}_{out}$:

$$\begin{aligned}(\mathsf{a}f)(z) &= \oint_{\mathcal{C}_{in}} dw \frac{\widetilde{Y}_{in}(z)\widetilde{Y}_{in}(w)^{-1} - 1}{1 - e^{-2pi(z-w)}} f_{in}(w), && z \in \mathcal{C}_{in},\\ (\mathsf{b}f)(z) &= \oint_{\mathcal{C}_{out}} dw \frac{\widetilde{Y}_{in}(z)\widetilde{Y}_{out}(w)^{-1}}{1 - e^{-2pi(z-w)}} f_{out}(w), && z \in \mathcal{C}_{in},\\ (\mathsf{c}f)(z) &= \oint_{\mathcal{C}_{in}} dw \frac{\widetilde{Y}_{out}(z)\widetilde{Y}_{in}(w)^{-1}}{1 - e^{-2pi(z-w)}} f_{in}(w), && z \in \mathcal{C}_{out},\\ (\mathsf{d}f)(z) &= \oint_{\mathcal{C}_{out}} dw \frac{\widetilde{Y}_{out}(z)\widetilde{Y}_{out}(w)^{-1} - 1}{1 - e^{-2pi(z-w)}} f_{out}(w), && z \in \mathcal{C}_{out}.\end{aligned} \tag{5.2.31}$$

The functions $\widetilde{Y}_{in}, \widetilde{Y}_{out}$ are the local solutions of the three-point problem (5.2.4) around $\mp i\infty$, defined in Definition 5.1. They are given by, respectively (5.2.50), which is well-defined as a series in e^{-2piz}, convergent for $|e^{-2piz}| < 1$, and (5.2.51), which is well-defined as a series in e^{2piz}.

Definition 5.5. *The tau-function $\mathcal{T}^{(1,1)}$ is defined, in terms of the Plemelj operators $\mathcal{P}_\oplus, \mathcal{P}_{\Sigma_{1,1}}$ in definitions 5.4 and 5.2, as:*

$$\mathcal{T}^{(1,1)}(t) := \det_{\mathcal{H}_+}\left[\mathcal{P}^{-1}_{\Sigma_{1,1},+}\mathcal{P}_{\oplus,+}\right], \tag{5.2.32}$$

where

$$\mathcal{P}_{\cdot,+} := \mathcal{P}_\cdot|_{\mathcal{H}_+}. \tag{5.2.33}$$

In general, it is useful to introduce the following notation:

Notation 5.1. *$\mathcal{T}^{(g,n)}$ denotes the determinant tau-function on genus g Riemann Surfaces with n Fuchsian singularities.*

5.2.2 Constructing the Fredholm determinant

As a stepping stone to theorem 5.1, that links the determinant tau-function (5.2.32) to the isomonodromic tau-function (5.2.3), in the following proposition we show that the tau-function $\mathcal{T}^{(1,1)}$ of Definition 5.5 depends solely on the operators $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ defined by the three-point problem.

Proposition 5.1. *The tau-function $\mathcal{T}^{(1,1)}(t)$ is the Fredholm determinant of an operator acting on $L^2(S^1) \otimes \mathbb{C}^2$, explicitly determined by hypergeometric functions*

$$\mathcal{T}^{(1,1)}(t) = \det\left[\mathbb{I} - K_{1,1}\right], \tag{5.2.34}$$

where

$$K_{1,1}(z,w) = \begin{pmatrix} -e^{-2pir} \frac{\widetilde{Y}_{out}(z+t)\widetilde{Y}_{in}(w)^{-1}}{1-e^{-2pi(z-w+t)}} & \frac{\widetilde{Y}_{out}(z+t)\widetilde{Y}_{out}(w+t)^{-1}-\mathbb{I}}{1-e^{-2pi(z-w)}} \\ \frac{\mathbb{I}-\widetilde{Y}_{in}(z)\widetilde{Y}_{in}(w)^{-1}}{1-e^{-2pi(z-w)}} & e^{2pir} \frac{\widetilde{Y}_{in}(z)\widetilde{Y}_{out}(w+t)^{-1}}{1-e^{-2pi(z-w-t)}} \end{pmatrix}, \tag{5.2.35}$$

$\widetilde{Y}_{in}$ and $\widetilde{Y}_{out}$ are the solutions of the three-point problem on the cylinder (5.2.4), given by (5.2.50) and (5.2.51) respectively, r parametrizes the $U(1)$ shift of the B-cycle monodromy of $\mathcal{P}_\Sigma$, and t is the modular parameter of the torus.

Proof. Starting from the definition (5.2.32) of $\mathcal{T}^{(1,1)}$, we compute the action of $\mathcal{P}^{-1}_{\Sigma_{1,1},+}\mathcal{P}_{\oplus,+}$ on a function $f \in \mathcal{H}_+$:

$$F := \mathcal{P}^{-1}_{\Sigma_{1,1},+}\mathcal{P}_{\oplus,+} f \quad \Rightarrow \quad \mathcal{P}_{\Sigma_{1,1}} F = \mathcal{P}_\oplus f\,, \qquad F \in \mathcal{H}_+. \tag{5.2.36}$$

Noting that for any projector $\mathcal{P}$ acting on a vector x, one has $x - \mathcal{P}x \in \ker \mathcal{P}$, and that[4] $\ker \mathcal{P}_{\Sigma_{1,1}} = \mathcal{H}_A$:

$$F = (F - \mathcal{P}_{\Sigma_{1,1}}F) + \mathcal{P}_{\Sigma_{1,1}}F = A + \mathcal{P}_\oplus F, \qquad A := F - \mathcal{P}_{\Sigma_{1,1}}F \in \mathcal{H}_A\,. \tag{5.2.37}$$

In components, A reads

$$A = \begin{pmatrix} A_{in,-}(z) \\ A_{out,+}(z) \end{pmatrix} \oplus \begin{pmatrix} A_{in,+}(z) \\ A_{out,-}(z) \end{pmatrix}. \tag{5.2.38}$$

The identification of $\mathcal{C}_{in}$ with $\mathcal{C}_{out}$, that produces the torus from the trinion as in Figure 5.1, is implemented at the level of functional spaces by setting

$$A_{in,\pm} = \nabla^{-1} A_{out,\pm}, \tag{5.2.39}$$

where $\nabla : \mathcal{H}_{in} \to \mathcal{H}_{out}$ is a translation operator acting on an arbitrary function $g(z) \in \mathcal{H}_{in}$ as

$$\nabla g(z) = e^{2pir} g(z-t). \tag{5.2.40}$$

The factor e^{2pir} takes into account the $U(1)$ B-cycle monodromy of the Cauchy kernel in (5.2.18). Using the explicit form of $\mathcal{P}_\oplus$ in (5.2.30), together with the fact that $F \in \mathcal{H}_+$, equation (5.2.37) reads:

$$\begin{pmatrix} F_{in,-} \\ F_{out,+} \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A_{in,-} \\ A_{out,+} \end{pmatrix} \oplus \begin{pmatrix} A_{in,+} \\ A_{out,-} \end{pmatrix} + \begin{pmatrix} f_{in,-} \\ f_{out,+} \end{pmatrix} \oplus \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & \mathsf{d} \end{pmatrix} \begin{pmatrix} f_{in,-} \\ f_{out,+}. \end{pmatrix} \tag{5.2.41}$$

The $\mathcal{H}_-$ components of (5.2.41) are solved by

$$\begin{aligned} A_{out,-} &= -\mathsf{c}\, f_{in,-} - \mathsf{d}\, f_{out,+} = \nabla A_{in,-}\,, \\ A_{in,+} &= -\mathsf{a}\, f_{in,-} - \mathsf{b}\, f_{out,+} = \nabla^{-1} A_{out,+}\,, \end{aligned} \tag{5.2.42}$$

[4] When $(\mathbb{I} - K_{1,1})$ is invertible, $\mathcal{H} = \mathcal{H}_T \oplus \mathcal{H}_A$, and therefore $\ker \mathcal{P}_{\Sigma_{1,1}} = \mathcal{H}_A$.

and substituting (5.2.42) into (5.2.41) gives

$$F = \begin{pmatrix} F_{in,-} \\ F_{out,+} \end{pmatrix} = \begin{pmatrix} f_{in,-} \\ f_{out,+} \end{pmatrix} - \begin{pmatrix} \nabla^{-1}\mathsf{c} & \nabla^{-1}\mathsf{d} \\ \nabla\mathsf{a} & \nabla\mathsf{b} \end{pmatrix} \begin{pmatrix} f_{in,-} \\ f_{out,+} \end{pmatrix} := \left(\mathbb{I} - \widehat{K}_{1,1}\right) f. \tag{5.2.43}$$

We note that the kernel $\widehat{K}$ in (5.2.43), when expressed in spherical coordinates, becomes the one appearing in Section 4 of [15]. It is however more natural to conjugate the kernel $\widehat{K}_{1,1}$ by the operator $\mathrm{diag}(1, \nabla^{-1})$:

$$K_{1,1} := \mathrm{diag}(1, \nabla^{-1})\, \widehat{K}_{1,1}\, \mathrm{diag}(1, \nabla) = \begin{pmatrix} \nabla^{-1}\mathsf{c} & \nabla^{-1}\mathsf{d}\nabla \\ \mathsf{a} & \mathsf{b}\nabla \end{pmatrix} \tag{5.2.44}$$

The advantage of such a conjugation is the following: recall that we identify $\mathcal{C}_{in}$ and $\mathcal{C}_{out}$ with two copies of the A-cycle obtained by cutting the B-cycle of the torus. They are given by the segments in figure (5.3) with endpoints identified.

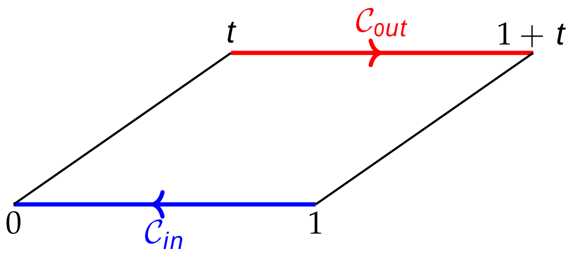


FIGURE 5.3: $\mathcal{C}_{in}, \mathcal{C}_{out}$ in coordinates on the torus

After the conjugation, $\widehat{K}_{1,1}$ is defined on a single circle, since all the functions on $\mathcal{C}_{out}$ are translated by t, as is clear from the explicit expression

$$K_{1,1}(z,w) = \begin{pmatrix} -e^{-2pir} \frac{\widetilde{Y}_{out}(z+t)\widetilde{Y}_{in}(w)^{-1}}{1-e^{-2pi(z-w+t)}} & \frac{\widetilde{Y}_{out}(z+t)\widetilde{Y}_{out}(w+t)^{-1} - \mathbb{I}}{1-e^{-2pi(z-w)}} \\ \frac{\mathbb{I} - \widetilde{Y}_{in}(z)\widetilde{Y}_{in}(w)^{-1}}{1-e^{-2pi(z-w)}} & e^{2pir} \frac{\widetilde{Y}_{in}(z)\widetilde{Y}_{out}(w+t)^{-1}}{1-e^{-2pi(z-w-t)}} \end{pmatrix}. \tag{5.2.45}$$

The tau-function $\mathcal{T}^{(1,1)}$ in (5.2.32) is therefore

$$\mathcal{T}^{(1,1)}(t) = \det_{\mathcal{H}_+} \left[\mathcal{P}^{-1}_{\Sigma_{1,1},+} \mathcal{P}_{\oplus,+}\right] = \det[\mathbb{I} - K_{1,1}]. \tag{5.2.46}$$

□

Let us highlight the block determinant structure of the tau-function

$$\mathcal{T}^{(1,1)}(t) = \det_{\mathcal{H}_+} \left[\mathcal{P}^{-1}_{\Sigma_{1,1},+} \mathcal{P}_{\oplus,+}\right] = \det\left[\mathbb{I} - \begin{pmatrix} \nabla^{-1}\mathsf{c} & \nabla^{-1}\mathsf{d}\nabla \\ \mathsf{a} & \mathsf{b}\nabla \end{pmatrix}\right], \tag{5.2.47}$$

which will prove important in theorem 5.2, that generalizes proposition 5.1 to the case of a genus 1 surface with n punctures, with tau-function $\mathcal{T}^{(1,n)}$.

5.2.3 Relation to the Hamiltonian

In this section we prove that the logarithmic derivative of the tau-function (5.2.32) differs from the Hamiltonian (5.2.1) by a factor that we compute.

Theorem 5.1. *The isomonodromic tau-function $\mathcal{T}_{CM}$ for the one-punctured torus is given by the following expression:*

$$\mathcal{T}_{CM}(t) = \det\left[\mathbb{I} - K_{1,1}\right]\left(\frac{e^{-2\pi i r} h(t)^2}{q_1(Q(t)-r)q_1(Q(t)+r)}\right) e^{2\pi i t\left(a^2+\frac{1}{6}\right)} \Upsilon_{1,1}(a,m), \tag{5.2.48}$$

where r is an arbitrary constant, $Q(t)$ is the solution of the equation of motion for the 2-particle nonautonomous Calogero-Moser system (5.1.1). The kernel $K_{1,1}(z,w;t)$ reads

$$K_{1,1}(z,w;t) = \begin{pmatrix} -e^{-2\pi i r}\frac{\widetilde{Y}_{out}(z+t)\widetilde{Y}_{in}(w)^{-1}}{e^{2\pi i(z-w+t)}-1} & \frac{\widetilde{Y}_{out}(z+t)\widetilde{Y}_{out}(w+t)^{-1}-\mathbb{I}}{1-e^{-2\pi i(z-w)}} \\ \frac{\mathbb{I}-\widetilde{Y}_{in}(z)\widetilde{Y}_{in}(w)^{-1}}{1-e^{-2\pi i(z-w)}} & e^{2\pi i r}\frac{\widetilde{Y}_{in}(z)\widetilde{Y}_{out}(w+t)^{-1}}{e^{2\pi i(z-w-t)}-1} \end{pmatrix}. \tag{5.2.49}$$

and the corresponding operator acts on $L^2(S^1)\otimes\mathbb{C}^2$. The function

$$\begin{aligned}\widetilde{Y}_{in}(z) &= (1-e^{-2\pi i z})^m \times \operatorname{diag}(e^{2\pi i a z}, e^{-2\pi i a z})\times \\ &\times \begin{pmatrix} {}_2F_1(m, m-2a, -2a, e^{-2\pi i z}) & -\frac{m}{2a}{}_2F_1(1+m, m-2a, 1-2a, e^{-2\pi i z}) \\ \frac{m e^{-2\pi i z}}{2a+1}{}_2F_1(1+m, 1+m+2a, 2+2a, e^{-2\pi i z}) & {}_2F_1(m, 1+m+2a, 1+2a, e^{-2\pi i z}) \end{pmatrix},\end{aligned} \tag{5.2.50}$$

is the local behavior of the solution to the associated three-point spherical problem for $z\to -i\infty$, normalized in such a way that the monodromy around $-i\infty$ is diagonal and equal to $e^{2\pi i a s_3}$, well-defined as a series in $e^{-2\pi i z}$, convergent for $|e^{-2\pi i z}|<1$, ${}_2F_1$ are hypergeometric functions, and the function $\widetilde{Y}_{out}$ is defined by

$$\widetilde{Y}_{out}(z) := e^{2\pi i(n+dn(a,m))s_3} s_1 \widetilde{Y}_{in}(-z) s_1, \qquad e^{2\pi i dn(a,m)} = \frac{\Gamma(-2a)\Gamma(1+2a-m)}{\Gamma(1+2a)\Gamma(-2a-m)}. \tag{5.2.51}$$

This expression for $\widetilde{Y}_{out}$, which is well-defined as a series in $e^{2\pi i z}$, was obtained in [16], where n parametrizes the B-cycle monodromy M_B and $dn(a,m)$ is a shift depending on a, m. s_1 is a Pauli sigma matrix, m is the monodromy exponent around the puncture, a is the monodromy exponent around the A-cycle of the torus, and $\Upsilon_{1,1}$ is an arbitrary function of the monodromy data.

Proof. Recall from (5.2.16), (5.2.21) that

$$\mathcal{P}_{\Sigma_{1,1}} f(z) = \int_{\mathcal{C}} \frac{dw}{2\pi i} Y_{CM}(z,t)\Xi_2(z,w;t)Y_{CM}(w,t)^{-1} f(w) \tag{5.2.52}$$

$$\mathcal{P}_{\oplus} f(z) = \int_{\mathcal{C}} dw \frac{\widetilde{Y}(z)\widetilde{Y}(w)^{-1}}{1-e^{-2\pi i(z-w)}} f(w), \tag{5.2.53}$$

and since $\mathcal{P}_{\oplus}$ does not depend on t, the logarithmic derivative of $\mathcal{T}^{(1,1)}$ in (5.2.32) is (see also pg. 20 in [54])

$$\partial_t \log \mathcal{T}^{(1,1)}(t) = -\operatorname{tr}_{\mathcal{H}} \mathcal{P}_{\oplus}\partial_t \mathcal{P}_{\Sigma_{1,1}}. \tag{5.2.54}$$

The computation of the t-derivative of $\mathcal{P}_\Sigma$ needs careful analysis. In principle, the operator $\mathcal{P}_\Sigma$ acts on different spaces for different values of the complex moduli: to define its derivative we need a local identification of these spaces (connection). In the spherical case such an identification is absolutely natural, because we can keep the system of contours $\mathcal{C}_{in,out}$ untouched while varying the complex moduli; which is no longer true in the torus case, since the position of $\mathcal{C}_{out}$ depends on t, see Figure 5.3. In order to make the space $\mathcal{H}_{out}$ t-independent we identify it with $\mathcal{H}_{in}$ using the shift operator ∇ defined in (5.2.40), by setting $\mathcal{H}_{out} = \nabla\mathcal{H}'_{in}$, where the space $\mathcal{H}'_{in}$ is isomorphic to $\mathcal{H}_{in}$. This identification gives us a new operator $\mathcal{P}'_{\Sigma_{1,1}}$ acting on "time-independent" spaces: $\mathcal{P}'_{\Sigma_{1,1}} : \mathcal{H}_{in} \oplus \mathcal{H}'_{in} \to \mathcal{H}_{in} \oplus \mathcal{H}'_{in}$.

$$\mathcal{P}'_{\Sigma_{1,1}} := \operatorname{diag}(1, \nabla^{-1})\mathcal{P}_{\Sigma_{1,1}}\operatorname{diag}(1, \nabla). \tag{5.2.55}$$

We identify $\mathcal{H}'_{in}$ with the space of functions on $\mathcal{C}'_{in}$, which is just another copy of $\mathcal{C}_{in}$, introduced for convenience to describe the block structure of $\mathcal{P}'_\Sigma$ by indicating the positions of the arguments of the kernel. Using these notations, the kernel of $\mathcal{P}'_\Sigma$ is given by the following expressions:

$$\begin{aligned}
&\mathcal{P}'_{\Sigma_{1,1}}(w,z) = \mathcal{P}_{\Sigma_{1,1}}(w,z), \quad \text{for } w,z \in \mathcal{C}_{in},\\
&\mathcal{P}'_{\Sigma_{1,1}}(w,z) = e^{-2pir}\mathcal{P}_{\Sigma_{1,1}}(w+t,z), \quad \text{for } w \in \mathcal{C}'_{in}, z \in \mathcal{C}_{in},\\
&\mathcal{P}'_{\Sigma_{1,1}}(w,z) = e^{2pir}\mathcal{P}_{\Sigma_{1,1}}(w,z+t), \quad \text{for } w \in \mathcal{C}_{in}, z \in \mathcal{C}'_{in},\\
&\mathcal{P}'_{\Sigma_{1,1}}(w,z) = \mathcal{P}_{\Sigma_{1,1}}(w+t,z+t), \quad \text{for } w,z \in \mathcal{C}'_{in}.
\end{aligned} \tag{5.2.56}$$

Now we define the t-derivative of $\mathcal{P}_{\Sigma_{1,1}}$ simply as

$$\P_t\mathcal{P}_{\Sigma_{1,1}} := \operatorname{diag}(1, \nabla)\P_t\mathcal{P}'_{\Sigma_{1,1}}\operatorname{diag}(1, \nabla^{-1}). \tag{5.2.57}$$

Using (5.2.56) we get the kernel of $\P_t\mathcal{P}_{\Sigma_{1,1}}$ explicitly:

$$\begin{aligned}
&\left(\P_t\mathcal{P}_{\Sigma_{1,1}}\right)(w,z) = \P_t\mathcal{P}_{\Sigma_{1,1}}(w,z) \quad \text{for} \quad w,z \in \mathcal{C}_{in},\\
&\left(\P_t\mathcal{P}_{\Sigma_{1,1}}\right)(w,z) = (\P_t + \P_w)\mathcal{P}_{\Sigma_{1,1}}(w,z) \quad \text{for} \quad w \in \mathcal{C}_{out},\ z \in \mathcal{C}_{in},\\
&\left(\P_t\mathcal{P}_{\Sigma_{1,1}}\right)(w,z) = (\P_t + \P_z)\mathcal{P}_{\Sigma_{1,1}}(w,z) \quad \text{for} \quad w \in \mathcal{C}_{in},\ z \in \mathcal{C}_{out},\\
&\left(\P_t\mathcal{P}_{\Sigma_{1,1}}\right)(w,z) = (\P_t + \P_w + \P_z)\mathcal{P}_{\Sigma_{1,1}}(w,z) \quad w,z \in \mathcal{C}_{out}.
\end{aligned} \tag{5.2.58}$$

Therefore[5],

$$\begin{aligned}
&-\operatorname{tr}_{\mathcal{H}}(\mathcal{P}_\oplus\P_t\mathcal{P}_{\Sigma_{1,1}})\\
&= -\oint_{\overline{\mathcal{C}}} dw \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi}\frac{1}{1-e^{-2pi(z-w)}}\operatorname{tr}\left\{\widetilde{Y}(z)\widetilde{Y}(w)^{-1}\P_t\left(Y_{CM}(w)\Xi_2(w,z)Y_{CM}(z)^{-1}\right)\right\}\\
&-\oint_{\overline{\mathcal{C}}} dw \oint_{\underline{\mathcal{C}}_{out}} \frac{dz}{2pi}\frac{1}{1-e^{-2pi(z-w)}}\operatorname{tr}\left\{\widetilde{Y}(z)\widetilde{Y}(w)^{-1}\P_z\left(Y_{CM}(w)\Xi_2(w,z)Y_{CM}(z)^{-1}\right)\right\}\\
&-\oint_{\overline{\mathcal{C}}_{out}} dw \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi}\frac{1}{1-e^{-2pi(z-w)}}\operatorname{tr}\left\{\widetilde{Y}(z)\widetilde{Y}(w)^{-1}\P_w\left(Y_{CM}(w)\Xi_2(w,z)Y_{CM}(z)^{-1}\right)\right\}\\
&= -I_t - I_z - I_w,
\end{aligned} \tag{5.2.59}$$

[5]We drop the t dependence of Y_{CM}, L_{CM} and M_{CM} in this proof for brevity

where

$$I_t := \oint_{\overline{\mathcal{C}}} dw \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \frac{1}{1-e^{-2pi(z-w)}} \mathrm{tr}\left\{ \widetilde{Y}(z)\widetilde{Y}(w)^{-1} ¶_t \left(Y_{CM}(w)\Xi_2(w,z)Y_{CM}(z)^{-1} \right\}, \quad (5.2.60)$$

$$I_z := \oint_{\overline{\mathcal{C}}} dw \oint_{\underline{\mathcal{C}}_{out}} \frac{dz}{2pi} \frac{1}{1-e^{-2pi(z-w)}} \mathrm{tr}\left\{ \widetilde{Y}(z)\widetilde{Y}(w)^{-1} ¶_z \left(Y_{CM}(w)\Xi_2(w,z)Y_{CM}(z)^{-1} \right) \right\}, \quad (5.2.61)$$

$$I_w := \oint_{\overline{\mathcal{C}}_{out}} dw \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \frac{1}{1-e^{-2pi(z-w)}} \mathrm{tr}\left\{ \widetilde{Y}(z)\widetilde{Y}(w)^{-1} ¶_w \left(Y_{CM}(w)\Xi_2(w,z)Y_{CM}(z)^{-1} \right) \right\}. \quad (5.2.62)$$

In the multiple integrals we always use the convention that z is inside w (recall that the notation $\overline{\mathcal{C}}$, $\underline{\mathcal{C}}$ is explained in Figure 5.2) and we close the contours in the direction of A . The reason for such choice of the contour is the following: the kernel $\left(¶_t \mathcal{P}_{\Sigma_{1,1}}\right)(w,z)$ is regular at $z = w$ since $¶_t \frac{1}{w-z} = 0$ and $(¶_t + ¶_z + ¶_w) \frac{1}{w-z} = 0$, which means that the relative positions of the arguments of $¶_t \mathcal{P}_{\Sigma_{1,1}}$ can be arbitrary. Keeping this in mind we first act on $\left(\mathcal{P}_{\Sigma_{1,1}}\right)(w,z_0)$, viewed as a function of w, by $\mathcal{P}_{\oplus}(z,w)$: the action results in an integral over w, whose contour should be chosen according to Definition 5.3. Namely, since $\mathcal{P}_{\oplus}(z,w)$ has pole along the diagonal, we deform the contour for w to $\overline{\mathcal{C}}$, and also move z to $\underline{\mathcal{C}}$ for convenience. After this, we set $z_0 = z$ and integrate over z on $\underline{\mathcal{C}}$ to take trace.

The integration of w over $\overline{\mathcal{C}}$ then picks up the residue at $w = z$. Let us begin with the integral I_z:

$$\begin{aligned} I_z &= \oint_{\overline{\mathcal{C}}} dw \oint_{\underline{\mathcal{C}}_{out}} \frac{dz}{2pi} \frac{1}{1-e^{-2pi(z-w)}} \mathrm{tr}\left\{ \widetilde{Y}(z)\widetilde{Y}(w)^{-1} ¶_z \left(Y_{CM}(w)\Xi_2(w,z)Y_{CM}(z)^{-1} \right) \right\} \\ &= I_z^{(1)} + I_z^{(2)}, \end{aligned} \quad (5.2.63)$$

where

$$I_z^{(1)} := \oint_{\overline{\mathcal{C}}} dw \oint_{\underline{\mathcal{C}}_{out}} \frac{dz}{2pi} \frac{1}{1-e^{-2pi(z-w)}} \mathrm{tr}\left\{ \widetilde{Y}(w)^{-1} Y_{CM}(w) ¶_z \Xi_2(w,z) Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\}, \quad (5.2.64)$$

$$I_z^{(2)} := \oint_{\overline{\mathcal{C}}} dw \oint_{\underline{\mathcal{C}}_{out}} dz \frac{1}{1-e^{-2pi(z-w)}} \mathrm{tr}\left\{ \widetilde{Y}(w)^{-1} Y_{CM}(w) \Xi_2(w,z) ¶_z Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\}. \quad (5.2.65)$$

To compute $I_z^{(1)}$, we expand $\Xi_2(w,z)$ as in (5.2.19), and use (5.2.22)

$$
\begin{aligned}
I_z^{(1)} &= \oint_{\overline{\mathcal{C}}} dw \oint_{\underline{\mathcal{C}}_{out}} \frac{dz}{2pi} \frac{1}{1-e^{-2pi(z-w)}} \mathrm{tr}\left\{ \widetilde{Y}(w)^{-1} Y_{CM}(w) \P_z \Xi_2(w,z) Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\} \\
&= \oint_{\overline{\mathcal{C}}} \frac{dw}{2pi} \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \mathrm{tr}\left\{ \widetilde{Y}(w)^{-1} Y_{CM}(w) \left[-\frac{1}{(w-z)^3} + \frac{ip}{(w-z)^2} \right] Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\} \\
&+ \oint_{\overline{\mathcal{C}}} \frac{dw}{2pi} \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \mathrm{tr}\left\{ \frac{\widetilde{Y}(w)^{-1} Y_{CM}(w)}{2(w-z)} \left[\begin{pmatrix} \frac{q_1''(Q-r)}{q_1(Q-r)} & 0 \\ 0 & \frac{q_1''(Q+r)}{q_1(Q+r)} \end{pmatrix} - \frac{1}{3} \frac{q_1'''}{q_1'} - \frac{1}{6}(2pi)^2 \right] Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\} \\
&= \frac{1}{2} \oint_{\underline{\mathcal{C}}_{out}} \frac{dz}{2pi} \mathrm{tr}\left\{ \P_z^2 \left(\widetilde{Y}(z)^{-1} Y_{CM}(z) \right) Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\} - \frac{1}{2} \oint_{\underline{\mathcal{C}}_{out}} dz \mathrm{tr}\left\{ L_{CM} - L_{3pt} \right\} \\
&\quad - \oint_{\underline{\mathcal{C}}_{out}} \frac{dz}{2pi} \mathrm{tr}\left\{ \frac{1}{2} \begin{pmatrix} \frac{q_1''(Q-r)}{q_1(Q-r)} & 0 \\ 0 & \frac{q_1''(Q+r)}{q_1(Q+r)} \end{pmatrix} - \frac{\mathbb{I}}{6} \frac{q_1'''}{q_1'} - \frac{(2pi)^2 \mathbb{I}}{12} \right\},
\end{aligned}
\tag{5.2.66}
$$

with $L_{CM}(z)$, $L_{3pt}(z)$ given in (5.1.7), (5.2.4) respectively. Similarly, $I_z^{(2)}$ reads

$$
\begin{aligned}
I_z^{(2)} &= \oint_{\overline{\mathcal{C}}} dw \oint_{\underline{\mathcal{C}}_{out}} \frac{dz}{2pi} \frac{1}{1-e^{-2pi(z-w)}} \mathrm{tr}\left\{ \widetilde{Y}(w)^{-1} Y_{CM}(w) \Xi_2(w,z) \P_z Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\} \\
&= -\oint_{\overline{\mathcal{C}}} \frac{dw}{2pi} \oint_{\underline{\mathcal{C}}_{out}} \frac{dz}{2pi} \mathrm{tr}\left\{ \widetilde{Y}(w)^{-1} Y_{CM}(w) \left[\frac{1}{(w-z)^2} \right] \P_z Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\} \\
&- \oint_{\overline{\mathcal{C}}} \frac{dw}{2pi} \oint_{\underline{\mathcal{C}}_{out}} \frac{dz}{2pi} \mathrm{tr}\left\{ \frac{\widetilde{Y}(w)^{-1} Y_{CM}(w)}{w-z} \left[\begin{pmatrix} \frac{q_1'(Q-r)}{q_1(Q-r)} & 0 \\ 0 & -\frac{q_1'(Q+r)}{q_1(Q+r)} \end{pmatrix} - ip \right] \P_z Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\} \\
&= \oint_{\underline{\mathcal{C}}_{out}} \frac{dz}{2pi} \mathrm{tr}\left\{ \P_z \left(\widetilde{Y}(z)^{-1} Y_{CM}(z) \right) \P_z Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\} \\
&\quad - \oint_{\underline{\mathcal{C}}_{out}} \frac{dz}{2pi} \mathrm{tr}\left\{ \begin{pmatrix} \frac{q_1'(Q-r)}{q_1(Q-r)} & 0 \\ 0 & -\frac{q_1'(Q+r)}{q_1(Q+r)} \end{pmatrix} L_{CM}(z) \right\} + \frac{1}{2} \oint_{\underline{\mathcal{C}}_{out}} dz \mathrm{tr}\, L_{CM}(z).
\end{aligned}
\tag{5.2.67}
$$

Plugging the expressions for $I_z^{(1)}$ in (5.2.66) and $I_z^{(2)}$ in (5.2.67) into (5.2.63), observing that $\mathrm{tr}\, L_{CM} = \mathrm{tr}\, L_{3pt} = 0$, and rearranging the terms we find:

$$
\begin{aligned}
I_z &= \frac{1}{2} \oint_{\underline{\mathcal{C}}_{out}} \frac{dz}{2pi} \mathrm{tr}\left\{ \P_z^2 \left(\widetilde{Y}(z)^{-1} Y_{CM}(z) \right) Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\} \\
&+ \oint_{\underline{\mathcal{C}}_{out}} \frac{dz}{2pi} \mathrm{tr}\left\{ \P_z \left(\widetilde{Y}(z)^{-1} Y_{CM}(z) \right) \P_z Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\} \\
&- \oint_{\underline{\mathcal{C}}_{out}} \frac{dz}{2pi} \mathrm{tr}\left\{ \frac{1}{2} \begin{pmatrix} \frac{q_1''(Q-r)}{q_1(Q-r)} & 0 \\ 0 & \frac{q_1''(Q+r)}{q_1(Q+r)} \end{pmatrix} - \frac{\mathbb{I}}{6} \frac{q_1'''}{q_1'} - \frac{(2pi)^2 \mathbb{I}}{12} \right\}.
\end{aligned}
\tag{5.2.68}
$$

Let us integrate by parts the first two terms in (5.2.68):

$$
\begin{aligned}
&\frac{1}{2}\oint_{\mathcal{C}_{out}}\frac{dz}{2pi}\mathrm{tr}\left\{\P_z^2\left(\widetilde{Y}(z)^{-1}Y_{CM}(z)\right)Y_{CM}(z)^{-1}\widetilde{Y}(z)\right\}\\
&+\oint_{\mathcal{C}_{out}}\frac{dz}{2pi}\mathrm{tr}\left\{\P_z\left(\widetilde{Y}(z)^{-1}Y_{CM}(z)\right)\P_z Y_{CM}(z)^{-1}\widetilde{Y}(z)\right\}\\
=&-\frac{1}{2}\oint_{\mathcal{C}_{out}}\frac{dz}{2pi}\mathrm{tr}\left\{\P_z\left(\widetilde{Y}(z)^{-1}Y_{CM}(z)\right)\P_z\left(Y_{CM}(z)^{-1}\widetilde{Y}(z)\right)\right\}\\
&+\oint_{\mathcal{C}_{out}}\frac{dz}{2pi}\mathrm{tr}\left\{\P_z\left(\widetilde{Y}(z)^{-1}Y_{CM}(z)\right)\P_z Y_{CM}(z)^{-1}\widetilde{Y}(z)\right\}\\
=&-\frac{1}{2}\oint_{\mathcal{C}_{out}}\frac{dz}{2pi}\mathrm{tr}\left\{-L_{3pt}(z)^2+L_{CM}(z)^2\right\}.
\end{aligned}
\tag{5.2.69}
$$

Therefore,

$$
\begin{aligned}
-I_z=&\frac{1}{2}\oint_{\mathcal{C}_{out}}\frac{dz}{2pi}\mathrm{tr}\left\{-L_{3pt}(z)^2+L_{CM}(z)^2+2\begin{pmatrix}\frac{q_1'(Q-r)}{q_1(Q-r)} & 0\\ 0 & -\frac{q_1'(Q+r)}{q_1(Q+r)}\end{pmatrix}L_{CM}(z)\right\}\\
&+\frac{1}{2}\oint_{\mathcal{C}_{out}}\frac{dz}{2pi}\mathrm{tr}\left\{\begin{pmatrix}\frac{q_1''(Q-r)}{q_1(Q-r)} & 0\\ 0 & \frac{q_1''(Q+r)}{q_1(Q+r)}\end{pmatrix}-\mathbb{I}\left(\frac{1}{3}\frac{q_1'''}{q_1'}+\frac{(2pi)^2}{6}\right)\right\}\\
\overset{(5.2.1)}{=}&-2pia^2+\frac{1}{2pi}H_{CM}+\frac{P}{2pi}\left(\frac{q_1'(Q-r)}{q_1(Q-r)}+\frac{q_1'(Q+r)}{q_1(Q+r)}\right)+\frac{1}{4pi}\left(\frac{q_1''(Q-r)}{q_1(Q-r)}+\frac{q_1''(Q+r)}{q_1(Q+r)}\right)\\
&-\left(\frac{1}{6pi}\frac{q_1'''}{q_1'}+\frac{(2pi)}{6}\right).
\end{aligned}
\tag{5.2.70}
$$

To compute the first term in (5.2.70), we use the explicit form (5.2.4), (5.2.5) and recall that the contour $\mathcal{C}_{out}$ is simply the interval $[t, t+1]$:

$$
\oint_{\mathcal{C}_{out}}\frac{dz}{4pi}\mathrm{tr}\,L_{3pt}^2(z)=2pia^2. \tag{5.2.71}
$$

The second term of (5.2.70) is simply the isomonodromic Hamiltonian, while all the other terms are constants, that are unaffected by the integration. The term I_w in (5.2.62) vanishes because the z-loop is contractible.

$$
I_w=0. \tag{5.2.72}
$$

Finally, we compute I_t:

$$
\begin{aligned}
I_t=&\oint_{\mathcal{C}}dw\oint_{\mathcal{C}}\frac{dz}{2pi}\frac{1}{1-e^{-2pi(z-w)}}\mathrm{tr}\left\{\widetilde{Y}(z)\widetilde{Y}(w)^{-1}\P_t\left(Y_{CM}(w)\Xi_2(w,z)Y_{CM}(z)^{-1}\right)\right\}\\
=&I_t^{(1)}+I_t^{(2)}+I_t^{(3)},
\end{aligned}
\tag{5.2.73}
$$

where

$$I_t^{(1)} := \oint_{\overline{\mathcal{C}}} dw \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \frac{1}{1-e^{-2pi(z-w)}} \mathrm{tr} \left\{ \widetilde{Y}(w)^{-1} \P_t \left(Y_{CM}(w) \right) \Xi_2(w,z) Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\}, \tag{5.2.74}$$

$$I_t^{(2)} := \oint_{\overline{\mathcal{C}}} dw \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \frac{1}{1-e^{-2pi(z-w)}} \mathrm{tr} \left\{ \widetilde{Y}(w)^{-1} Y_{CM}(w) \P_t \left(\Xi_2(w,z) \right) Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\} \tag{5.2.75}$$

$$I_t^{(3)} := \oint_{\overline{\mathcal{C}}} dw \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \frac{1}{1-e^{-2pi(z-w)}} \mathrm{tr} \left\{ \widetilde{Y}(w)^{-1} Y_{CM}(w) \Xi_2(w,z) \P_t \left(Y_{CM}(z)^{-1} \right) \widetilde{Y}(z) \right\}. \tag{5.2.76}$$

Expanding $\Xi_2(z,w)$ as in (5.2.19) and using (5.2.22), $I_t^{(1)}$ reads,

$$\begin{aligned} I_t^{(1)} &= \oint_{\overline{\mathcal{C}}} dw \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \frac{1}{1-e^{-2pi(z-w)}} \mathrm{tr} \left\{ \widetilde{Y}(w)^{-1} \P_t \left(Y_{CM}(w) \right) \Xi_2(w,z) Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\} \\ &= - \oint_{\overline{\mathcal{C}}} \frac{dw}{2pi} \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \mathrm{tr} \left\{ \frac{\widetilde{Y}(w)^{-1} \P_t \left(Y_{CM}(w) \right) Y_{CM}(z)^{-1} \widetilde{Y}(z)}{(w-z)^2} \right\} \\ &+ \oint_{\overline{\mathcal{C}}} \frac{dw}{2pi} \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \mathrm{tr} \left\{ \frac{\widetilde{Y}(w)^{-1} \P_t \left(Y_{CM}(w) \right)}{w-z} \left[ip\mathbb{I} - \begin{pmatrix} \frac{q_1'(Q-r)}{q_1(Q-r)} & 0 \\ 0 & \frac{q_1'(-Q-r)}{q_1(-Q-r)} \end{pmatrix} \right] Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\} \\ &= - \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \mathrm{tr} \left\{ \left[ip\mathbb{I} - \begin{pmatrix} \frac{q_1'(Q-r)}{q_1(Q-r)} & 0 \\ 0 & \frac{q_1'(-Q-r)}{q_1(-Q-r)} \end{pmatrix} \right] M_{CM} \right\} = 0, \end{aligned} \tag{5.2.77}$$

where M_{CM} is the matrix in equation (5.1.6). In the last line we use the fact that z lies inside the contour of w, and M_{CM} has no residue at the puncture $z = 0$. Now computing the integral $I_t^{(2)}$,

$$I_t^{(2)} = \oint_{\overline{\mathcal{C}}} dw \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \frac{1}{1-e^{-2pi(z-w)}} \mathrm{tr} \left\{ \widetilde{Y}(z) \widetilde{Y}(w)^{-1} Y_{CM}(w) \P_t \left(\Xi_2(w,z) \right) Y_{CM}(z)^{-1} \right\} = 0, \tag{5.2.78}$$

because $\P_t \Xi_2(z,w)$ is regular at $w = z$. Since $I_t^{(1)}$ and $I_t^{(2)}$ vanish, $I_t = I_t^{(3)}$. Finally, we compute the integral $I_t^{(3)}$ by expanding Ξ as before:

$$
\begin{aligned}
I_t = I_t^{(3)} &= \oint_{\overline{\mathcal{C}}} dw \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \frac{1}{1-e^{-2pi(z-w)}} \mathrm{tr}\left\{ \widetilde{Y}(w)^{-1} Y_{CM}(w) \Xi_2(w,z) \P_t Y_{CM}(z)^{-1} \widetilde{Y}(z) \right\} \\
&= -\oint_{\overline{\mathcal{C}}} \frac{dw}{2pi} \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \mathrm{tr}\left\{ \frac{\widetilde{Y}(w)^{-1} Y_{CM}(w) \P_t \left(Y_{CM}(z)^{-1}\right) \widetilde{Y}(z)}{(w-z)^2} \right\} \\
- \oint_{\overline{\mathcal{C}}} \frac{dw}{2pi} \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \mathrm{tr} & \left\{ \frac{\widetilde{Y}(w)^{-1} Y_{CM}(w)}{w-z} \left[ip\mathbb{I} - \begin{pmatrix} \frac{q_1'(Q-r)}{q_1(Q-r)} & 0 \\ 0 & \frac{q_1'(-Q-r)}{q_1(-Q-r)} \end{pmatrix} \right] \P_t \left(Y_{CM}(z)^{-1} \right) \widetilde{Y}(z) \right\} \\
&= -\oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \mathrm{tr}\left\{ \P_z \left(\widetilde{Y}(z)^{-1} Y_{CM}(z) \right) \P_t \left(Y_{CM}(z)^{-1} \right) \widetilde{Y}(z) \right\} \\
&\quad + \oint_{\underline{\mathcal{C}}} dz \mathrm{tr} \left\{ \left[ip\mathbb{I} - \begin{pmatrix} \frac{q_1'(Q-r)}{q_1(Q-r)} & 0 \\ 0 & \frac{q_1'(-Q-r)}{q_1(-Q-r)} \end{pmatrix} \right] M_{CM} \right\} \\
&= \oint_{\underline{\mathcal{C}}} \frac{dz}{2pi} \mathrm{tr}\left\{ \P_z \left(\widetilde{Y}(z)^{-1} Y_{CM}(z) \right) \P_t \left(Y_{CM}(z)^{-1} \right) \widetilde{Y}(z) \right\}.
\end{aligned}
\tag{5.2.79}
$$

Again, in the last line of (5.2.79) we used the fact that M_{CM} in (5.1.7), is regular at the puncture. Therefore,

$$
\begin{aligned}
-I_t &= -\oint_{\underline{\mathcal{C}}} dz \mathrm{tr}\left\{ \P_z \left(\widetilde{Y}(z)^{-1} Y_{CM}(z) \right) \P_t \left(Y_{CM}(z)^{-1} \right) \widetilde{Y}(z) \right\} \\
&= -\oint_{\underline{\mathcal{C}}} dz \mathrm{tr}\left\{ \left(\P_z \widetilde{Y}(z) \widetilde{Y}(z)^{-1} - \P_z Y_{CM}(z) Y_{CM}(z)^{-1} \right) \P_t Y_{CM}(z) Y_{CM}(z)^{-1} \right\} \\
&= -\oint_{\underline{\mathcal{C}}} dz \left\{ \left(Y_{CM}(z)^{-1} \widetilde{Y}(z) L_{3pt}(z) (Y_{CM}(z)^{-1} \widetilde{Y}(z))^{-1} - L_{CM} \right) M_{CM} \right\}.
\end{aligned}
\tag{5.2.80}
$$

To compute the above expression, we study the behavior of L_{CM}, M_{CM}, L_{3pt} in (5.1.7) and (5.2.4) respectively, at $z = 0$, using the expansions

$$
x(2Q,z) = \frac{1}{z} + \frac{q_1'(2Q)}{q_1(2Q)} + \mathcal{O}(z), \tag{5.2.81}
$$

and

$$
\begin{aligned}
y(2Q,z) &= \left[\frac{q_1''(2Q)}{q_1(2Q)} - \left(\frac{q_1'(2Q)}{q_1(2Q)} \right)^2 \right] + \mathcal{O}(z) \\
&= \wp(2Q) + \mathcal{O}(z).
\end{aligned}
\tag{5.2.82}
$$

Substituting (5.2.81) and (5.2.82) in the Lax matrices one finds (the solutions $Y_{CM}(z)$, $\widetilde{Y}(z)$ can be simultaneously re-normalized in such a way that their monodromy around $z = 0$ is

$e^{2pims_3})$

$$\begin{aligned} L_{CM} &= \frac{Y_{CM}(0)^{-1} m s_3 Y_{CM}(0)}{z} - i\frac{m q_1'(2Q)}{q_1(2Q)} s_2 + P s_3 + \mathcal{O}(z), \\ M_{CM} &= \wp(2Q) s_1 + \mathcal{O}(z), \\ L_{3pt} &= \frac{\widetilde{Y}(0)^{-1} m s_3 \widetilde{Y}(0)}{z} - 2pi\left(A_0 + \frac{A_1}{2}\right) + \mathcal{O}(z). \end{aligned} \tag{5.2.83}$$

From equation (5.2.83), it follows that

$$\begin{aligned} & Y_{CM}(z)^{-1}\widetilde{Y}(z) L_{3pt}(z) (Y_{CM}(z)^{-1}\widetilde{Y}(z))^{-1} - L_{CM} \\ &= -2pi Y_{CM}(0)^{-1}\widetilde{Y}(0)\left(A_0 + \frac{A_1}{2}\right) Y_{CM}(0)^{-1}\widetilde{Y}(0) + i\frac{m q_1'(2Q)}{q_1(2Q)} s_2 - P s_3, \end{aligned} \tag{5.2.84}$$

so, the integrand in equation (5.2.80) has no pole, and

$$I_t = 0. \tag{5.2.85}$$

We have thus shown that the logarithmic derivative of the tau-function $\mathcal{T}^{(1,1)}$ in (5.2.54) is

$$\begin{aligned} 2pi\P_t \log\det[\mathbb{I} - K_{1,1}] &\overset{(5.2.34)}{=} 2pi\P_t \log \mathcal{T}^{(1,1)} \overset{(5.2.54)}{=} -2pi \operatorname{tr}_{\mathcal{H}} \mathcal{P}_{\oplus} \P_t \mathcal{P}_{\Sigma} \\ &\overset{(5.2.59)}{=} 2pi(-I_t - I_z - I_w) \overset{(5.2.72),(5.2.85)}{=} -2pi I_z \\ &\overset{(5.2.70)}{=} -(2pi)^2 a^2 + H_{CM} - \left(\frac{1}{3}\frac{q_1'''}{q_1'} + \frac{(2pi)^2}{6}\right) \\ &+ P\left(\frac{q_1'(Q-r)}{q_1(Q-r)} + \frac{q_1'(Q+r)}{q_1(Q+r)}\right) + \frac{1}{2}\left(\frac{q_1''(Q-r)}{q_1(Q-r)} + \frac{q_1''(Q+r)}{q_1(Q+r)}\right) \\ &\overset{(5.2.3)}{=} 2pi\P_t \log \mathcal{T}_{CM} - (2pi)^2 a^2 - \left(\frac{1}{3}\frac{q_1'''}{q_1'} + \frac{(2pi)^2}{6}\right) \\ &+ P\left(\frac{q_1'(Q-r)}{q_1(Q-r)} + \frac{q_1'(Q+r)}{q_1(Q+r)}\right) + \frac{1}{2}\left(\frac{q_1''(Q-r)}{q_1(Q-r)} + \frac{q_1''(Q+r)}{q_1(Q+r)}\right) \\ &= 2pi\P_t \log \mathcal{T}_{CM} - (2pi)^2 a^2 - \frac{(2pi)^2}{6} + 2pi\frac{d}{dt}\log\left(\frac{q_1(Q-r)q_1(Q+r)}{h(t)^2}\right). \end{aligned} \tag{5.2.86}$$

In the last line we used the heat equation for q_1

$$4pi\P_t q_1(Q \pm r) = q_1''(Q \pm r), \tag{5.2.87}$$

as well as the fact that $P = 2pi\frac{dQ}{dt}$, leading to

$$\begin{aligned}4pi\frac{d}{dt}\log q_1(Q\pm r) &= \frac{1}{q_1(Q\pm r)}4pi\frac{d}{dt}q_1(Q\pm r)\\ &= 4pi\frac{dQ}{dt}\frac{q_1'(Q\pm r)}{q_1(Q\pm r)} + \frac{1}{q_1(Q\pm r)}\P_t q_1(Q\pm r)\\ &= 2P\frac{q_1'(Q\pm r)}{q_1(Q\pm r)} + \frac{q_1''(Q\pm r)}{q_1(Q\pm r)},\end{aligned} \tag{5.2.88}$$

and the expression for Dedekind eta function

$$h(t) = \left(\frac{q_1'}{2p}\right)^{1/3}, \qquad -4pi\frac{d}{dt}\log h(t) = -\frac{1}{3q_1'}4pi\frac{dq_1'}{dt} = -\frac{1}{3}\frac{q_1'''}{q_1'}. \tag{5.2.89}$$

Integrating (5.2.86) on both sides, we obtain (5.2.48) where the explicit form of the kernel $K_{1,1}$ is in (5.2.45). □

Remark 5.2. *Due to the factor $\frac{q_1(Q+r)q_1(Q-r)}{h(t)^2}$ in (5.2.48) between the isomonodromic tau-function $\mathcal{T}_{CM}$ and the determinant tau-function $\mathcal{T}^{(1,1)}$, we have the following statement:*

$$\mathcal{T}^{(1,1)}|_{r=\pm Q} = 0, \tag{5.2.90}$$

i.e. the zero locus of the Fredholm determinant in r computes the solution to the equation (5.1.1). This is an isomonodromic version of Krichever's solution of the isospectral elliptic Calogero-Moser model [78, 15, 14], and justifies the introduction of the extra parameter r.

Remark 5.3. *We see that, in contrast to the spherical case, now there are two different tau-functions, $\mathcal{T}_{CM}$ and $\mathcal{T}^{(1,1)}$. It is usually supposed that the object called 'tau-function' is related to free fermions, has a determinant representation, and satisfies some bilinear relations. It turns out that only $\mathcal{T}^{(1,1)}$ has such properties, in particular, it was shown in [3] that equation (5.1.1) is equivalent to some bilinear relations on the two r-independent parts of $\mathcal{T}^{(1,1)}$. These bilinear relations are the consequences of the blow-up relations for the theory with adjoint matter (for other examples of such equations see [59]). The free-fermionic nature of $\mathcal{T}^{(1,1)}$ was shown in [15]. Instead, $\mathcal{T}_{CM}$ has one property, which $\mathcal{T}^{(1,1)}$ does not have: its derivative gives the Hamiltonian.*

5.3 Generalization to the n-punctured torus

The results for the 2-particle nonautonomous Calogero-Moser system are further generalized to the isomonodromic problem on an n-punctured torus $C_{1,n}$ which is characterised by the following $N \times N$ system of linear differential equations [112, 79]

$$\begin{aligned} &\frac{\P}{\P z}\mathcal{Y}\left(z;t,\{z_i\}_1^n\right) = \mathcal{Y}\left(z;t,\{z_i\}_1^n\right) L\left(z;t,\{z_i\}_1^n\right), \\ &(2pi)\frac{\P}{\P t}\mathcal{Y}\left(z;t,\{z_i\}_1^n\right) = \mathcal{Y}\left(z;t,\{z_i\}_1^n\right) M_t\left(z;t,\{z_i\}_1^n\right), \qquad \begin{matrix} z,z_1,\ldots,z_n \in C_{1,n};\, t \in \mathbb{H}, \\ k=1,\ldots,n \end{matrix} \\ &\frac{\P}{\P z_k}\mathcal{Y}\left(z;t,\{z_i\}_1^n\right) = \mathcal{Y}\left(z;t,\{z_i\}_1^n\right) M_{z_k}\left(z;t,\{z_i\}_1^n\right), \end{aligned} \tag{5.3.1}$$

where[6] $\mathcal{Y}(z) \in GL(N)$, and $L, M_t, M_{z_k} \in \mathfrak{gl}_N$ are the Lax matrices. The isomonodromic time evolution in this case is generated by $n+1$ Poisson commuting Hamiltonians, that can be obtained as before from contour integrals of $\frac{1}{2}\mathrm{tr}\, L^2$, and are generated by the isomonodromic tau-function $\mathcal{T}_H$:

$$2pi\P_t \log \mathcal{T}_H := H_t = \frac{1}{2}\oint_A \mathrm{tr}\, L^2(z)dz, \qquad \P_{z_k}\log\mathcal{T}_H := H_k = \mathrm{Res}_{\,z=z_k}\frac{1}{2}\mathrm{tr}\, L^2(z). \tag{5.3.2}$$

In theorem 5.2 we show that the isomonodromic tau-function for the linear system (5.3.1) is also described by a Fredholm determinant (5.3.51). Furthermore, theorem 7.2 generalizes theorem 7.1, describing the tau-function of the elliptic Garnier system in terms of Nekrasov partition functions.

We now generalize the discussion of the previous section to the $GL(N)$ linear system (5.3.1) on a torus with n punctures, using the expressions derived in [79] for the matrices L, M_{z_k}, M_t. In this case the matrix elements L_{ij} of the Lax matrix $L(z)$ are

$$\begin{aligned} L_{ij}(z) = d_{ij}&\left[P_i + \sum_{k=1}^{n}\frac{q_1'(z-z_k)}{q_1(z-z_k)}\left(S_{ii}^{(k)} + \Lambda_k\right)\right] \\ &+ (1-d_{ij})\sum_{k=1}^{n}\frac{q_1(z-z_k+Q_i-Q_j)q_1'(0)}{q_1(z-z_k)q_1(Q_i-Q_j)}S_{ij}^{(k)} \end{aligned} \tag{5.3.3}$$

while the matrix elements of the M-matrices (5.3.1) are

$$(M_{z_k})_{ij}(z) = -d_{ij}\frac{q_1'(z-z_k)}{q_1(z-z_k)}\left(S_{ii}^{(k)} + \Lambda_k\right) - (1-d_{ij})\frac{q_1(z-z_k+Q_i-Q_j)q_1'(0)}{q_1(z-z_k)q_1(Q_i-Q_j)}S_{ij}^{(k)}, \tag{5.3.4}$$

$$(M_t)_{ij}(z) = \frac{1}{2}d_{ij}\sum_{k=1}^{n}\frac{q_1''(z-z_k)}{q_1(z-z_k)}\left(S_{ii}^{(k)} + \Lambda_k\right) + \sum_{k=1}^{n} y(Q_j-Q_i, z-z_k)S_{ij}^{(k)}, \tag{5.3.5}$$

where the function $y(u,z)$ is defined in (5.1.8). The dynamical variables[7] $Q_1,\ldots,Q_N,P_1,\ldots,P_N$ satisfy $\sum_i Q_i = \sum_i P_i = 0$ and are canonically conjugated, and the matrices $S^{(k)}$ satisfy the

[6] The dependence on the variables t, $z_1,\ldots,z_n$ of the functions $\mathcal{Y}(z)$, $L(z)$, $M(z)$, H_k, H_t, $\mathcal{T}_H$ is dropped henceforth for brevity.

[7] In the interest of brevity, we omit writing the $t, z_1 \ldots z_n$ dependence of the functions $L(z), M(z), Y(z)$ and the dynamical variables Q_i's, P_i's.

Kirillov-Kostant Poisson bracket

$$\{Q_i, P_j\} = d_{ij}, \qquad \left\{S_a^{(k)}, S_b^{(m)}\right\} = d^{km} f_{ab}{}^c S_c^{(k)}, \tag{5.3.6}$$

where we defined $S^{(k)} := S_a^{(k)} t^a$ in terms of a set of generators t_a of $\mathfrak{sl}_N$, and $f_{ab}{}^c$ are the $\mathfrak{sl}_N$ structure constants. The residues take value in $\mathfrak{gl}(N)$ due to the $U(1)$ factors Λ_k.

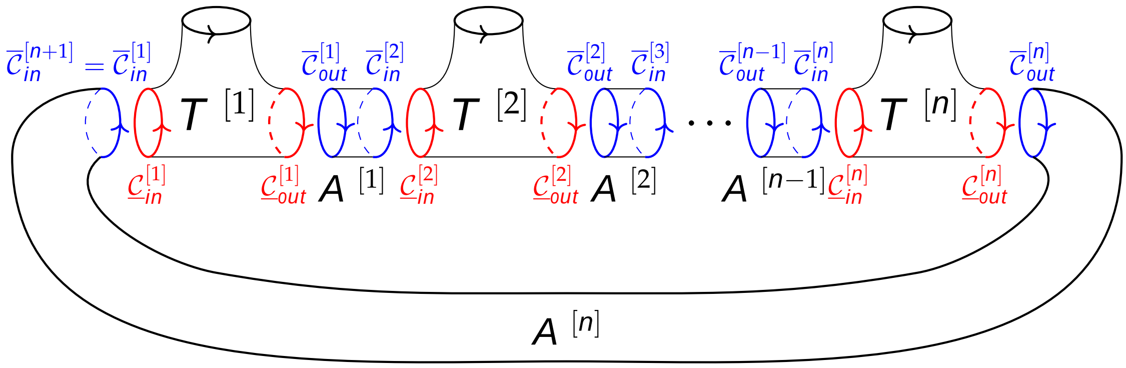


FIGURE 5.4: Pants decomposition for the n-punctured torus

Notation 5.2. *Given an N-tuple of parameters $(x_1, \ldots x_N)$, and a function $g(x_i)$, $i = 1, \ldots, N$ of these parameters, we define*

$$g(\xi) := diag\,(f(x_1), \ldots, g(x_N))\,. \tag{5.3.7}$$

In particular, when $g(x_i) = x_i$, this is

$$\xi = diag\,(x_1, \ldots, x_N)\,. \tag{5.3.8}$$

Remark 5.4. *The generic isomonodromic problem on genus one surfaces is formulated in [79] under the requirement that the matrices $S^{(k)}$, parametrizing the $\mathfrak{sl}_N$ residues at the punctures z_k, satisfy*

$$\sum_{k=1}^{n} S_{ii}^{(k)} = 0. \tag{5.3.9}$$

For consistency of the construction, (5.3.9) will be imposed on the $\mathfrak{sl}_N$ component of the residues, $S^{(k)}$.

The matrices L, M_{z_k}, M_t are not single-valued on the torus, but rather under the shift $z \to z + t$ behave as (using notation 5.2)

$$\begin{aligned} L(z+t) &= e^{-2piQ} L(z) e^{2piQ} - 2pi \sum_{k=1}^{n} \Lambda_k \\ M_{z_k}(z+t) &= e^{-2piQ} M_{z_k}(z) e^{2piQ} + 2pi \sum_{k=1}^{n} \Lambda_k, \\ M_t(z+t) &= e^{-2piQ} \left(M_t(z) + L(z)\right) e^{2pQ} - 2piP - (2pi)^2 \frac{1}{2} \sum_{k=1}^{n} \Lambda_k, \end{aligned} \tag{5.3.10}$$

so that the solution of the linear system (5.1.7) will transform as follows:

$$\mathcal{Y}(z+t) = M_B e^{-2pi\sum_{j=1}^{n}\left(z-z_j+\frac{t}{2}+\frac{1}{2}\right)\Lambda_j}\mathcal{Y}(z)e^{2piQ}, \tag{5.3.11}$$

where $M_B \in SL(N)$. The pants decomposition corresponding to the n-punctured torus consists of n trinions, as shown in Figure 5.4, with each trinion $T^{[k]}$ associated to its own three-point problem.

$$\begin{aligned} &\P_z\widetilde{\mathcal{Y}}^{[k]}(z) = \widetilde{\mathcal{Y}}^{[k]}(z)L_{3pt}^{[k]}(z), \\ &L_{3pt}^{[k]}(z) = -2piA_0^{[k]} - 2pi\frac{A_1^{[k]}}{1-e^{2piz}}, \end{aligned} \tag{5.3.12}$$

where

$$A_0^{[k]} \sim \sigma_k, \qquad\qquad A_1^{[k]} \sim \mu_k, \tag{5.3.13}$$

$$\sigma_k = a_k - \sum_{j=0}^{k-1}\Lambda_j\mathbb{I}, \qquad\qquad \mu_k = m_k + \Lambda_k\mathbb{I} \tag{5.3.14}$$

for $k = 1, \dots n$. As in the 1-point case, we choose $\widetilde{\mathcal{Y}}^{[k]}(z)$ in such a way that

$$\widetilde{\mathcal{Y}}^{[k]}(z)^{-1}\mathcal{Y}(z)$$

is regular and single-valued around $z = z_k$ and has no monodromies around two closest A-cycles.

In (5.3.14) we introduced a $U(1)$ parameter Λ_0 shifting the monodromy exponent σ_1 around $\mathcal{C}_{in}^{[1]}$, whose significance will become apparent in sections 7.1.2 and 7.1.3[8]. The monodromy exponents m_k, a_k parametrize the $SL(N)$ component of the monodromy, and the a_k's satisfy $a_{n+1} = a_1$. In terms of the original problem on the torus, the monodromy exponents σ_k, μ_k in equation (5.3.12) are defined by the conjugacy class of the monodromies around the punctures $\{z_k\}_{k=1}^{n}$, and around the circles $\mathcal{C}_{in}$, $\mathcal{C}_{out}$ being glued in the pants decomposition (see Figure 5.4), which are respectively

$$M_k \sim e^{2pi\mu_k}, \qquad\qquad M_{\mathcal{C}_{in}^{[k]}} = M^{-1}_{\mathcal{C}_{out}^{[k-1]}} = G_k^{-1}e^{2pi\sigma_k}G_k, \tag{5.3.15}$$

for $k = 1, \dots n$, and

$$M_{\mathcal{C}_{out}^{[n]}} = M_B^{-1}e^{-2pi(\sigma_1 - \sum_{j=1}^{n}\Lambda_j\mathbb{I})}M_B. \tag{5.3.16}$$

The matrix G_k is the matrix that diagonalizes $M_{\mathcal{C}_{in}^{[k]}} = M_{\mathcal{C}_{out}}^{[k-1]}$, while $G_{n+1} := M_B$ is the matrix that diagonalizes $M_{\mathcal{C}_{out}}^{[n]}$ as in the one-punctured case, and we fixed $G_1 = \mathbb{I}$. The total Hilbert space $\mathcal{H}$ is decomposed into a direct sum of spaces $\mathcal{H}^{[k]}$ corresponding to each pair

[8]From the point of view of the dynamical system, the monodromy exponents on $\mathcal{C}_{in}^{[1]}$ have the role of initial conditions, so that it is natural that Λ_0 doesn't appear in the Lax matrix, contrary to $\Lambda_1, \dots, \Lambda_N$, which are residues at the punctures.

of pants:

$$\mathcal{H} := \bigoplus_{k=1}^{n} \mathcal{H}^{[k]} = \mathcal{H}_+ \oplus \mathcal{H}_-, \tag{5.3.17}$$

where

$$\mathcal{H}_\pm := \bigoplus_{k=1}^{n} \left(\mathcal{H}^{[k]}_{in,\mp} \oplus \mathcal{H}^{[k]}_{out,\pm} \right), \tag{5.3.18}$$

Definition 5.6. *Corresponding to the solutions $\mathcal{Y}(z)$, $\widetilde{\mathcal{Y}}^{[k]}(z)$ of the linear problems (5.3.1), (5.3.12) respectively, we define two matrix-valued functions: $Y(z)$ with diagonal monodromies around the boundary circles $\mathcal{C}^{[k]}_{in}$ and $\mathcal{C}^{[k]}_{out}$, and $\widetilde{Y}^{[k]}(z)$ with diagonal monodromies around $\mathcal{C}^{[k]}_{in}$ and $\mathcal{C}^{[k]}_{out}$ (see Figure 5.4), by the following equations:*

$$Y(z)|_{\mathcal{C}^{[1]}_{in}} := \mathcal{Y}(z)|_{\mathcal{C}^{[1]}_{in}} \in \mathcal{H}^{[1]}_{in}, \qquad Y(z)|_{\mathcal{C}^{[n]}_{out}} := M_B^{-1}\mathcal{Y}(z)|_{\mathcal{C}^{[n]}_{out}} \in \mathcal{H}^{[n]}_{out}. \tag{5.3.19}$$

$$\widetilde{Y}^{[k]}(z)|_{\mathcal{C}^{[k]}_{in}} \equiv \widetilde{Y}^{[k]}_{in}(z) := G_k^{-1}\widetilde{\mathcal{Y}}^{[k]}(z)|_{\mathcal{C}^{[k]}_{in}} \in \mathcal{H}^{[k]}_{in}, \tag{5.3.20}$$

$$\widetilde{Y}^{[k]}(z)|_{\mathcal{C}^{[k]}_{out}} \equiv \widetilde{Y}^{[k]}_{out}(z) := G_{k+1}^{-1}\widetilde{\mathcal{Y}}^{[k]}(z)|_{\mathcal{C}^{[k]}_{out}} \in \mathcal{H}^{[k]}_{out}, \tag{5.3.21}$$

with $G_1 = \mathbb{I}$, and $G_{n+1} = M_B$.

The functions $f^{[k]}(z) \in \mathcal{H}^{[k]}$ are decomposed as

$$f^{[k]}(z) = \begin{pmatrix} f^{[k]}_{in,-} \\ f^{[k]}_{out,+} \end{pmatrix} \oplus \begin{pmatrix} f^{[k]}_{in,+} \\ f^{[k]}_{out,-} \end{pmatrix}. \tag{5.3.22}$$

The generalization of definition 5.4 to the n-punctured case is as follows:

$$\mathcal{P}_\oplus := \bigoplus_{k=1}^{n} \mathcal{P}^{[k]}_\oplus \tag{5.3.23}$$

where $\mathcal{P}^{[k]}$ is the Plemelj operator given by the solution to the three-point problem (5.3.12) in the pants decomposition,

$$\begin{aligned} \left(\mathcal{P}^{[k]}_\oplus f^{[k]}\right)(z) &= \int_{\mathcal{C}^{[k]}_{in}\cup\mathcal{C}^{[k]}_{out}} dw \frac{\widetilde{Y}^{[k]}(z)\widetilde{Y}^{[k]}(w)^{-1}}{1-e^{-2pi(z-w)}} f^{[k]}(w) \\ &:= \int_{\mathcal{C}^{[k]}} dw \frac{\widetilde{Y}^{[k]}(z)\widetilde{Y}^{[k]}(w)^{-1}}{1-e^{-2pi(z-w)}} f^{[k]}(w). \end{aligned} \tag{5.3.24}$$

Equivalently,

$$\mathcal{P}^{[k]}_\oplus : \begin{pmatrix} f^{[k]}_{in,-} \\ f^{[k]}_{out,+} \end{pmatrix} \oplus \begin{pmatrix} f^{[k]}_{in,+} \\ f^{[k]}_{out,-} \end{pmatrix} \mapsto \begin{pmatrix} f^{[k]}_{in,-} \\ f^{[k]}_{out,+} \end{pmatrix} \oplus \begin{pmatrix} \mathsf{a}^{[k]} & \mathsf{b}^{[k]} \\ \mathsf{c}^{[k]} & \mathsf{d}^{[k]} \end{pmatrix} \begin{pmatrix} f^{[k]}_{in,-} \\ f^{[k]}_{out,+} \end{pmatrix}, \tag{5.3.25}$$

where

$$\begin{aligned}
(\mathsf{a}^{[k]}g)(z) &= \oint_{\mathcal{C}_{in}^{[k]}} dw \frac{\widetilde{Y}_{in}^{[k]}(z)\widetilde{Y}_{in}^{[k]}(w)^{-1}-\mathbb{I}}{1-e^{-2pi(z-w)}} g_{in}(w), && z \in \mathcal{C}_{in}^{[k]}, \\
(\mathsf{b}^{[k]}g)(z) &= \oint_{\mathcal{C}_{out}^{[k]}} dw \frac{\widetilde{Y}_{in}^{[k]}(z)\widetilde{Y}_{out}^{[k]}(w)^{-1}}{1-e^{-2pi(z-w)}} g_{out}(w), && z \in \mathcal{C}_{in}^{[k]}, \\
(\mathsf{c}^{[k]}g)(z) &= \oint_{\mathcal{C}_{in}^{[k]}} dw \frac{\widetilde{Y}_{out}^{[k]}(z)\widetilde{Y}_{in}^{[k]}(w)^{-1}}{1-e^{-2pi(z-w)}} g_{in}(w), && z \in \mathcal{C}_{out}^{[k]}, \\
(\mathsf{d}^{[k]}g)(z) &= \oint_{\mathcal{C}_{out}^{[k]}} dw \frac{\widetilde{Y}_{out}^{[k]}(z)\widetilde{Y}_{out}^{[k]}(w)^{-1}-\mathbb{I}}{1-e^{-2pi(z-w)}} g_{out}(w), && z \in \mathcal{C}_{out}^{[k]}.
\end{aligned} \tag{5.3.26}$$

The functions $\widetilde{Y}_{in}^{[k]}, \widetilde{Y}_{out}^{[k]}$ are the local solutions of the k-th three-point problem around $\mp i\infty$, respectively, defined in Definition 5.6. In the case of a semi-degenerate system (i.e. a linear system with a single independent local monodromy exponent at $z = 0$ instead of N) these solutions are described by generalized hypergeometric functions ${}_NF_{N-1}$ (see eq. 19 in [55]). Similar to (5.2.23), $\mathcal{P}_\oplus^2 = \mathcal{P}_\oplus$, and

$$\ker \mathcal{P}_\oplus = \mathcal{H}_-. \tag{5.3.27}$$

Generalizing definition 5.2, we now introduce the Plemelj operator described by the solution to the n-point linear system (5.3.1),

$$\left(\mathcal{P}_{\Sigma_{1,n}} f\right)(z) = \oint_{\mathcal{C}_\Sigma} \frac{dw}{2pi} Y(z)\Xi_N(z,w)Y(w)^{-1} f(w), \tag{5.3.28}$$

where

$$\mathcal{C}_\Sigma := \bigcup_{k=1}^{n} \mathcal{C}_{out}^{[k]} \cup \mathcal{C}_{in}^{[k+1]}, \quad \mathcal{C}_{in}^{[n+1]} := \mathcal{C}_{in}^{[1]}, \tag{5.3.29}$$

and

$$\Xi_N(z,w) = \operatorname{diag}\left(\frac{q_1(z-w+Q_1-\widetilde{r})q_1'(0)}{q_1(z-w)q_1(Q_1-\widetilde{r})}, \ldots, \frac{q_1(z-w+Q_N-\widetilde{r})q_1'(0)}{q_1(z-w)q_1(Q_N-\widetilde{r})}\right), \tag{5.3.30}$$

where

$$\widetilde{r} := r - \sum_{j=1}^{n} \Lambda_j \left(z_j - \frac{t}{2} - \frac{1}{2}\right), \tag{5.3.31}$$

and as before r is an arbitrary parameter, and Ξ_N transforms as

$$\Xi_N(z+t,w) = e^{-2piQ+2pi\widetilde{r}}\,\Xi_N(z,w), \qquad \Xi_N(x,w+t) = \Xi_N(z,w)e^{2piQ-2pi\widetilde{r}}. \tag{5.3.32}$$

The shift of the parameter r in (5.3.31) makes the monodromies of the Cauchy kernel time-independent (see equation (5.3.11)), and the following is true:

$$\mathcal{H}_A \subseteq \ker \mathcal{P}_{\Sigma_{1,n}}, \tag{5.3.33}$$

where $A := \bigcup_{k=1}^{n} A^{[k]}$ in Figure 5.4, and the space of functions are defined by the equation (5.3.47). It is straightforward to check that $\mathcal{P}^2_{\Sigma_{1,n}} = \mathcal{P}_{\Sigma_{1,n}}$, and one can further prove:

$$\mathcal{P}_{\oplus}\mathcal{P}_{\Sigma_{1,n}} = \mathcal{P}_{\Sigma_{1,n}}, \quad \mathcal{P}_{\Sigma_{1,n}}\mathcal{P}_{\oplus} = \mathcal{P}_{\oplus}. \tag{5.3.34}$$

The space of functions on $T := \bigcup_{k=1}^{n} T^{[k]}$ in Figure 5.4 is

$$\mathcal{H}_T := \text{im}\,\mathcal{P}_{\oplus} = \text{im}\,\mathcal{P}_{\Sigma_{1,n}}. \tag{5.3.35}$$

Definition 5.7. *The determinant tau-function $\mathcal{T}^{(1,n)}$ is defined in terms of the Plemelj operators in equations (5.3.23), (5.3.28), as*

$$\mathcal{T}^{(1,n)} := \det_{\mathcal{H}_+}\left[\mathcal{P}^{-1}_{\Sigma_{1,n}}\mathcal{P}_{\oplus}\right]. \tag{5.3.36}$$

We now proceed to formulate the generalization of proposition 5.1 to the present case.

5.3.1 Block-determinant representation of the tau-function

Proposition 5.2. *The tau-function $\mathcal{T}^{(1,n)}$ in (5.3.36) is the Fredholm determinant of a block operator acting on $L^2(S^1)\otimes\mathbb{C}^N$:*

$$\mathcal{T}^{(1,n)}(t, z_1, ..., z_n) = \det\left[\mathbb{I} - K_{1,n}\right], \tag{5.3.37}$$

where

$$K_{1,n} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \nabla^{-1}\mathsf{c}^{[n]} & \nabla^{-1}\mathsf{d}^{[n]}\nabla \\ 0 & & & & & & 0 \\ \mathsf{c}^{[1]} & U_1 & V_1 & & \mathbf{0} & & 0 \\ 0 & W_1 & U_2 & V_2 & & & \\ 0 & & W_2 & & & & \vdots \\ & & & \ddots & & & \\ \vdots & & & \ddots & & & \\ & & & \ddots & V_{n-1} & & 0 \\ & \mathbf{0} & & W_{n-1} & U_n & & \mathsf{b}^{[n]}\nabla \\ 0 & & & & & & 0 \\ \mathsf{a}^{[1]} & \mathsf{b}^{[1]} & 0 & 0 & \cdots & & 0 \end{pmatrix}, \tag{5.3.38}$$

and

$$U_k = \begin{pmatrix} 0 & \mathsf{a}^{[k+1]} \\ \mathsf{d}^{[k]} & 0 \end{pmatrix}, \quad V_k = \begin{pmatrix} \mathsf{b}^{[k+1]} & 0 \\ 0 & 0 \end{pmatrix}, \quad W_k = \begin{pmatrix} 0 & 0 \\ 0 & \mathsf{c}^{[k+1]} \end{pmatrix}. \tag{5.3.39}$$

The operators $\mathsf{a}^{[k]}, \mathsf{b}^{[k]}, \mathsf{c}^{[k]}, \mathsf{d}^{[k]}$ defined in (5.3.26), and ∇ is the shift operator defined in (5.3.43).

Proof. The proof goes along the same lines as that of Proposition 5.1. Recalling the definition of the tau-function in (5.3.36) and of the Plemelj operators in (5.3.24), (5.3.28), we compute the action of $\mathcal{P}^{-1}_{\Sigma_{1,n},+}\mathcal{P}_{\oplus,+}$ on a function $f \in \mathcal{H}_+$:

$$F := \mathcal{P}^{-1}_{\Sigma_{1,n},+}\mathcal{P}_{\oplus,+} f \quad \Rightarrow \quad \mathcal{P}_{\Sigma_{1,n}} F = \mathcal{P}_{\oplus} f \,, \qquad F \in \mathcal{H}_+. \tag{5.3.40}$$

Now we use that for any projector $\mathcal{P}$ acting on a vector x, one has $x - \mathcal{P}x \in \ker \mathcal{P}$, and that[9] $\ker \mathcal{P}_{\Sigma_{1,n}} = \mathcal{H}_A$:

$$F = (F - \mathcal{P}_{\Sigma_{1,n}}F) + \mathcal{P}_{\Sigma_{1,n}}F = A + \mathcal{P}_{\oplus}F, \qquad A := F - \mathcal{P}_{\Sigma_{1,n}}F \in \mathcal{H}_A \,. \tag{5.3.41}$$

The orthogonal decomposition of A is

$$A = \bigoplus_{k=1}^{n} A^{[k]} = \begin{pmatrix} A^{[k]}_{in,-} \\ A^{[k]}_{out,+} \end{pmatrix} \oplus \begin{pmatrix} A^{[k]}_{in,+} \\ A^{[k]}_{out,-} \end{pmatrix}. \tag{5.3.42}$$

The z-dependent B-cycle monodromy in (5.3.11) implies that the monodromies around $\mathcal{C}^{[1]}_{in}$ and $\mathcal{C}^{[n]}_{out}$ (see Figure 5.4) are given by (5.3.16), prompting the following expression for the shift operator $\nabla : \mathcal{H}^{[1]}_{in} \to \mathcal{H}^{[n]}_{out}$

$$\nabla g(z) = e^{2pi\left(r-(z-t)\sum_{j=1}^{n}\Lambda_j\right)} g(z-t), \tag{5.3.43}$$

in order to 'glue' the boundary spaces on $\mathcal{C}_{in}$, $\mathcal{C}_{out}$. The factor $e^{2piz\sum_{j=1}^{n}\Lambda_j}$ in the above definition of ∇ leads to the following action of ∇^{-1}:

$$\nabla^{-1} h(z) = e^{-2pi\left(r-z\sum_{j=1}^{n}\Lambda_j\right)} h(z+t). \tag{5.3.44}$$

Identifying the boundaries $\mathcal{C}^{[n]}_{out}$ with $\mathcal{C}^{[1]}_{in}$, and $\mathcal{C}^{[k]}_{out}$ with $\mathcal{C}^{[k+1]}_{in}$ for $k = 1...n-1$, produces the torus from the pants decomposition as in Figure 5.4, and translates to the following constraints on A in (5.3.42):

$$A^{[1]}_{in,\pm} = \nabla^{-1} A^{[n]}_{out,\pm}; \qquad A^{[k]}_{out,\pm} = A^{[k+1]}_{in,\pm}, \qquad k = 1, \ldots, n-1, \tag{5.3.45}$$

where the translation operator $\nabla : \mathcal{H}^{[1]}_{in} \to \mathcal{H}^{[n]}_{out}$ is defined as in (5.3.43). Component-wise, equation (5.3.41) reads

$$\begin{aligned} F &= \begin{pmatrix} F_{in,-} \\ F_{out,+} \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \bigoplus_{k=1}^{n} \begin{pmatrix} A^{[k]}_{in,-} \\ A^{[k]}_{out,+} \end{pmatrix} \oplus \begin{pmatrix} A^{[k]}_{in,+} \\ A^{[k]}_{out,-} \end{pmatrix} + \begin{pmatrix} f^{[k]}_{in,-} \\ f^{[k]}_{out,+} \end{pmatrix} \oplus \begin{pmatrix} \mathsf{a}^{[k]} & \mathsf{b}^{[k]} \\ \mathsf{c}^{[k]} & \mathsf{d}^{[k]} \end{pmatrix} \begin{pmatrix} f^{[k]}_{in,-} \\ f^{[k]}_{out,+} \end{pmatrix}. \end{aligned} \tag{5.3.46}$$

[9] As in the previous section, when $(\mathbb{I} - K_{1,n})$ is invertible, $\mathcal{H} = \mathcal{H}_T \oplus \mathcal{H}_A$, and therefore $\ker \mathcal{P}_{\Sigma_{1,n}} = \mathcal{H}_A$.

Imposing the condition that the $\mathcal{H}_-$ component of F is zero, and using the constraints in (5.3.45) we get

$$\begin{aligned} A^{[1]}_{in,+} &= -\mathsf{a}^{[1]} f^{[1]}_{in,-} - \mathsf{b}^{[1]} f^{[1]}_{out,+} = \nabla^{-1} A^{[n]}_{out,+}, \\ A^{[k]}_{in,+} &= -\mathsf{a}^{[k]} f^{[k]}_{in,-} - \mathsf{b}^{[k]} f^{[k]}_{out,+} = A^{[k-1]}_{out,+}, \quad \text{for } k = 2 \ldots n, \\ A^{[k]}_{out,-} &= -\mathsf{c}^{[k]} f^{[k]}_{in,-} - \mathsf{d}^{[k]} f^{[k]}_{out,+} = A^{[k+1]}_{in,-}, \quad \text{for } k = 1 \ldots n-1, \\ A^{[n]}_{out,-} &= -\mathsf{c}^{[n]} f^{[n]}_{in,-} - \mathsf{d}^{[n]} f^{[n]}_{out,+} = \nabla A^{[1]}_{in,-}. \end{aligned} \tag{5.3.47}$$

Substituting (5.3.47) in (5.3.46),

$$F = \bigoplus_{k=1}^{n} \begin{pmatrix} A^{[k]}_{in,-} \\ A^{[k]}_{out,+} \end{pmatrix} + \begin{pmatrix} f^{[k]}_{in,-} \\ f^{[k]}_{out,+} \end{pmatrix} = \bigoplus_{k=1}^{n} \begin{pmatrix} f^{[k]}_{in,-} \\ f^{[k]}_{out,+} \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \nabla^{-1}\mathsf{c}^{[n]} & \nabla^{-1}\mathsf{d}^{[n]} \\ 0 & 0 & \mathsf{a}^{[2]} & \mathsf{b}^{[2]} & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \mathsf{c}^{[1]} & \mathsf{d}^{[1]} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathsf{a}^{[3]} & \mathsf{b}^{[3]} & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & \mathsf{c}^{[2]} & \mathsf{d}^{[2]} & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathsf{a}^{[4]} & \mathsf{b}^{[4]} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & & & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \mathsf{a}^{[n]} & \mathsf{b}^{[n]} \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \mathsf{c}^{[n-1]} & \mathsf{d}^{[n-1]} & 0 & 0 \\ \nabla\mathsf{a}^{[1]} & \nabla\mathsf{b}^{[1]} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f^{[1]}_{in,-} \\ f^{[1]}_{out,+} \\ f^{[2]}_{in,-} \\ f^{[2]}_{out,+} \\ f^{[3]}_{in,-} \\ f^{[3]}_{out,+} \\ \vdots \\ f^{[n]}_{in,-} \\ f^{[n]}_{out,+} \end{pmatrix}$$

$$:= \left(\mathbb{I} - \widehat{K}_{1,n} \right) f. \tag{5.3.48}$$

Similar to (5.2.44), we conjugate $\widehat{K}_{1,n}$ with the diagonal operator $\operatorname{diag}\left(1, 1, \ldots, \nabla^{-1}\right)$

$$K_{1,n} := \operatorname{diag}(1, \ldots, \nabla^{-1})\, \widehat{K}_{1,n} \operatorname{diag}(1, \ldots, \nabla) \tag{5.3.49}$$

obtaining equation (5.3.38). □

Remark 5.5. *It is straightforward to recover* (5.2.32) *from*

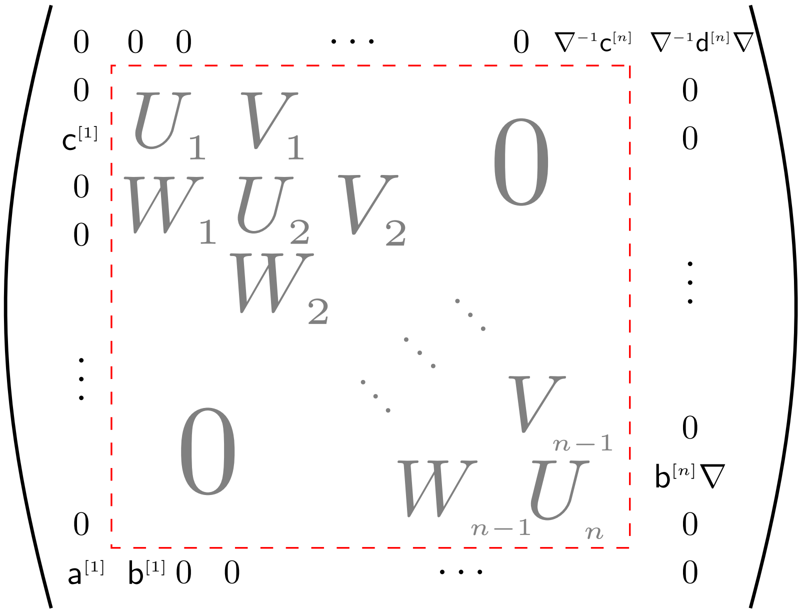


FIGURE 5.5: Kernel $K_{1,n}$ in (5.3.49).

Moreover, the block form of the tau-function $\mathcal{T}^{(1,n)}$ includes naturally an $n \times n$ sub-block identical to the tau-function appearing in pg. 18 of [54] for the $n+2$-punctured sphere, as emphasised in Figure 5.5. This is a consequence of the fact that if we cut the tube that joins the first and last trinion in Figure 5.4, (i.e. if we take the limit $t \to +i\infty$), we obtain a Fuchsian problem for an $n+2$-punctured sphere:

$$\lim_{t \to +i\infty} \mathcal{T}^{(1,n)} \propto \mathcal{T}^{(0,n+2)}. \tag{5.3.50}$$

5.3.2 Relation to the Hamiltonians

Theorem 5.2. *The isomonodromic tau-function $\mathcal{T}_H$ in (5.3.2) is related to the Fredholm determinant of the operator $K_{1,n}$ in (5.3.38) as*

$$\mathcal{T}_H(t) = \det\left[\mathbb{I} - K_{1,n}\right] e^{ip\,t\,tr\left(\sigma_1^2 + \frac{\mathbb{I}}{6}\right)} e^{-ipN\tilde{r}} \prod_{i=0}^{N} \frac{h(t)}{q_1\left(Q_i - \tilde{r}\right)} \prod_{k=1}^{n} e^{-ipz_k\left(tr\,\sigma_{k+1}^2 - tr\,\sigma_k^2\right)} \Upsilon_{1,n}, \tag{5.3.51}$$

where $\Upsilon_{1,n}$ is an arbitrary function of the monodromy data of the linear system (5.3.1), $Q_i \equiv Q_i(t, z_1, ..., z_n)$ are the Calogero-like dynamical variables in the linear system (5.3.3), $\sigma_k = a_k + \sum_{j<k} \Lambda_j$ are the monodromy exponents defined in (5.3.15), and $a_{n+1} = a_1$,

$$\tilde{r} = r + \sum_{k=1}^{n} \Lambda_k \left(z_k - \frac{t}{2} - \frac{1}{2}\right), \tag{5.3.52}$$

and r is an arbitrary parameter.

Proof. Let us recall equation (5.3.36):

$$\mathcal{T}^{(1,n)} = \det_{\mathcal{H}_+}\left[\mathcal{P}_{\Sigma_{1,n}}^{-1}\mathcal{P}_{\oplus}\right], \tag{5.3.53}$$

where the operators $\mathcal{P}_{\oplus}, \mathcal{P}_{\Sigma_{1,n}}$ are defined in (5.3.24) and (5.3.28) respectively. The logarithmic derivative of the tau-function $\mathcal{T}^{(1,n)}$, has two main components: the derivatives with respect to the modular parameter t, and the position of the singularities $\{z_k\}_{k=1}^n$:

$$d\log\mathcal{T}^{(1,n)} = 2pidt\,\P_t\log\mathcal{T}^{(1,n)} + \sum_{k=1}^n dz_k\,\P_{z_k}\log\mathcal{T}^{(1,n)}. \tag{5.3.54}$$

Computation of this derivative can be done exactly in the same way as in [54, page 20]:

$$d\log\mathcal{T}^{(1,n)} = -2pi\mathrm{tr}_{\mathcal{H}}\left[\mathcal{P}_{\oplus}\P_t\mathcal{P}_{\Sigma_{1,n}}\right]dt - \sum_{l=1}^n \mathrm{tr}_{\mathcal{H}^{[l]}}\left[\mathcal{P}_{\oplus}^{[l]}\P_{z_k}\mathcal{P}_{\Sigma_{1,n}}\cdot\right]dz_k \tag{5.3.55}$$

The computation for the first term in (5.3.55) is the same as in the proof for Theorem 5.1 in section 5.2.3: the t-derivative is given by

$$-\mathrm{tr}_{\mathcal{H}}\left[\mathcal{P}_{\oplus}\P_t\mathcal{P}_{\Sigma_{1,n}}\right] = \sum_{l=1}^n -I_t^{(l)} - I_w - I_z, \tag{5.3.56}$$

where

$$\begin{aligned} I_t^{(l)} &= \oint_{\overline{\mathcal{C}}^{[l]}} dw \oint_{\underline{\mathcal{C}}^{[l]}} \frac{dz}{2pi}\frac{1}{1-e^{-2pi(z-w)}}\mathrm{tr}\left[\widetilde{Y}^{[l]}(z)\widetilde{Y}^{[l]}(w)^{-1}\P_t\left(Y(w)\Xi_N(w,z)Y(z)^{-1}\right)\right], \\ I_z &= \oint_{\overline{\mathcal{C}}^{[n]}} dw \oint_{\underline{\mathcal{C}}^{[n]}_{out}} \frac{dz}{2pi}\frac{1}{1-e^{-2pi(z-w)}}\mathrm{tr}\left[\widetilde{Y}^{[n]}(z)\widetilde{Y}^{[n]}(w)^{-1}\P_z\left(Y(w)\Xi_N(w,z)Y(z)^{-1}\right)\right], \\ I_w &= \oint_{\overline{\mathcal{C}}^{[n]}_{out}} dw \oint_{\underline{\mathcal{C}}^{[n]}} \frac{dz}{2pi}\frac{1}{1-e^{-2pi(z-w)}}\mathrm{tr}\left[\widetilde{Y}^{[n]}(z)\widetilde{Y}^{[n]}(w)^{-1}\P_w\left(Y(w)\Xi_N(w,z)Y(z)^{-1}\right)\right]. \end{aligned} \tag{5.3.57}$$

Note that the contours of I_z, I_w involve only the final trinion, because the identification $z \sim z + t$ glues $\mathcal{C}_{out}^{[n]}$ to $\mathcal{C}_{in}^{[1]}$, as in Figure 5.4. Like in the case of one puncture, $I_t = \sum_l I_t^{(l)}$ vanishes because M_t in (5.3.5) has zero residue at the punctures, while I_w vanishes because the z-loop is contractible. Using the notation 5.2,

$$\sum_i \frac{q_1'(Q_i-\tilde{r})}{q_1(Q_i-\tilde{r})}(L)_{ii} \equiv \mathrm{tr}\left[\frac{q_1'(Q-\tilde{r})}{q_1(Q-\tilde{r})}L\right], \tag{5.3.58}$$

we are then left with the following expression (see (5.2.70) for comparison) for the first term in (5.3.55):

$$-\mathrm{tr}_{\mathcal{H}}\left[\mathcal{P}_{\oplus}\P_t\mathcal{P}_{\Sigma_{1,n}}\right] = -I_z = \frac{1}{2}\oint_{\underline{\mathcal{C}}^{[n]}_{out}}\frac{dz}{2pi}\mathrm{tr}\left[-L^{[n]}_{3pt}(z)^2 + L^2(z) + 2ip\,\mathrm{tr}\,L^{[n]}_{3pt}\right]$$
$$+\frac{1}{2}\oint_{\underline{\mathcal{C}}^{[n]}_{out}}\frac{dz}{2pi}\left[2\frac{q_1'(Q-\widetilde{r})}{q_1(Q-\widetilde{r})}L(z) + \frac{q_1''(Q-\widetilde{r})}{q_1(Q-\widetilde{r})} - \mathbb{I}\left(\frac{1}{3}\frac{q_1'''}{q_1} + \frac{(2pi)^2}{6}\right)\right]$$
$$= \frac{H_t}{2pi} + \frac{d}{dt}\log\left(e^{-ip\,t\,\mathrm{tr}\left(\sigma_1^2+\mathbb{I}/6\right)}e^{ip\,t\,N\sum_{j=0}^{n}\Lambda_j}\prod_{i=1}^{N}\frac{q_1(Q_i-\widetilde{r})}{h(t)}\right) \tag{5.3.59}$$
$$= \frac{H_t}{2pi} + \frac{d}{dt}\log\left(e^{-ip\,t\,\mathrm{tr}\left(\sigma_1^2+\mathbb{I}/6\right)}e^{ipN\tilde{r}}\prod_{i=1}^{N}\frac{q_1(Q_i-\widetilde{r})}{h(t)}\right). \tag{5.3.60}$$

In the last line we used

$$\sum_{j=0}^{N}\Lambda_j = \frac{1}{2}\sum_{j=1}^{N}\Lambda_j, \tag{5.3.61}$$

so that

$$ip\,\mathrm{tr}\,L^{[n]}_{3pt} = \frac{N}{2}(2pi)^2\sum_{j=0}^{n}\Lambda_j \overset{(5.3.31)}{=} 2pi\P_t\log\left(e^{ipN\tilde{r}}\right). \tag{5.3.62}$$

Let us now compute the second term in (5.3.55):

$$-\sum_{l=1}^{n}\mathrm{tr}_{\mathcal{H}^{[l]}}\left[\mathcal{P}^{[l]}_{\oplus}\P_{z_k}\mathcal{P}_{\Sigma_{1,n}}\right]$$
$$= -\sum_{l=1}^{n}\oint_{\overline{\mathcal{C}}^{[l]}}dw\oint_{\underline{\mathcal{C}}^{[l]}}\frac{dz}{2pi}\frac{1}{1-e^{-2pi(z-w)}}\mathrm{tr}\left[\widetilde{Y}^{[l]}(z)\widetilde{Y}^{[l]}(w)^{-1}\P_{z_k}\left(Y(w)\Xi_N(w,z)Y(z)^{-1}\right)\right]$$
$$= \sum_{l=1}^{n}\left(I^{(l,1)}_{z_k} + I^{(l,2)}_{z_k} + I^{(l,3)}_{z_k}\right), \tag{5.3.63}$$

where

$$I^{(l,1)}_{z_k} := -\oint_{\overline{\mathcal{C}}^{[l]}}dw\oint_{\underline{\mathcal{C}}^{[l]}}\frac{dz}{2pi}\frac{1}{1-e^{-2pi(z-w)}}\mathrm{tr}\left[\widetilde{Y}^{[l]}(z)\widetilde{Y}^{[l]}(w)^{-1}\P_{z_k}Y(w)\Xi_N(w,z)Y(z)^{-1}\right] = 0 \tag{5.3.64}$$

since the z-loop is contractible, and

$$I^{(l,2)}_{z_k} := -\oint_{\overline{\mathcal{C}}^{[l]}}dw\oint_{\underline{\mathcal{C}}^{[l]}}\frac{dz}{2pi}\frac{1}{1-e^{-2pi(z-w)}}\mathrm{tr}\left[\widetilde{Y}^{[l]}(z)\widetilde{Y}^{[l]}(w)^{-1}Y(w)\P_{z_k}\Xi_N(w,z)Y(z)^{-1}\right] = 0 \tag{5.3.65}$$

since $\P_{z_k}\Xi_N(w,z)$ is regular in $w \sim z$. The term $I_{z_k}^{(l,3)}$ is computed by expanding Ξ_N for $w \sim z$ as in (5.3.30), and using (5.2.22) :

$$\begin{aligned}
I_{z_k}^{(l,3)} &:= -\oint_{\overline{\mathcal{C}}^{[l]}} \frac{dw}{2pi} \oint_{\underline{\mathcal{C}}^{[l]}} \frac{dz}{2pi} \frac{1}{1-e^{-2pi(z-w)}} \mathrm{tr}\left[\widetilde{Y}^{[l]}(z)\widetilde{Y}^{[l]}(w)^{-1}Y(w)\Xi_N(w,z)\P_{z_k}Y(z)^{-1}\right] \\
&= \oint_{\overline{\mathcal{C}}^{[l]}} \frac{dw}{2pi} \oint_{\underline{\mathcal{C}}^{[l]}} \frac{dz}{2pi} \mathrm{tr}\left[\widetilde{Y}^{[l]}(w)^{-1}Y(w)\frac{1}{(w-z)^2}\P_{z_k}Y(z)^{-1}\widetilde{Y}^{[l]}(z)\right] \\
&\oint_{\overline{\mathcal{C}}^{[l]}} \frac{dw}{2pi} \oint_{\underline{\mathcal{C}}^{[l]}} \frac{dz}{2pi} \mathrm{tr}\left[\frac{\widetilde{Y}^{[l]}(w)^{-1}Y(w)}{w-z}\left(\frac{q_1'(Q-\widetilde{r})}{q_1(Q-\widetilde{r})} - ip\right)\P_{z_k}Y(z)^{-1}\widetilde{Y}^{[l]}(z)\right] \\
&= \oint_{\underline{\mathcal{C}}^{[l]}} \frac{dz}{2pi} \mathrm{tr}\left\{\left(Y(z)^{-1}\widetilde{Y}^{[l]}(z)L_{3pt}^{[l]}(z)(Y(z)^{-1}\widetilde{Y}^{[l]}(z))^{-1} - L(z)\right)M_{z_k}(z)\right\} \\
&\quad - \oint_{\underline{\mathcal{C}}^{[k]}} \frac{dz}{2pi} \mathrm{tr}\left\{\left(\frac{q_1'(Q-\widetilde{r})}{q_1(Q-\widetilde{r})} - ip\right)M_{z_k}(z)\right\}.
\end{aligned} \tag{5.3.66}$$

Note that (5.3.66) can be different from zero only for $l = k$, because the integrand is regular for $l \neq k$. To compute the first and second term, we use the regular parts $L(z)_{reg}$ and $L_{3pt}^{[k]}(z)_{reg}$ of $L(z)$ (eq. (5.3.3)) and $L_{3pt}^{[k]}$ (eq. (5.3.12)), as well as the explicit expression (5.3.4): for M_{z_k}:

$$\begin{aligned}
&\oint_{\underline{\mathcal{C}}^{[k]}} \frac{dz}{2pi} \mathrm{tr}\left\{\left(Y(z)^{-1}\widetilde{Y}^{[k]}(z)L_{3pt}^{[k]}(z)_{reg}(Y(z)^{-1}\widetilde{Y}^{[k]}(z))^{-1}\right)M_{z_k}(z)\right\} \\
&= 2pi\,\mathrm{tr}\left(A_1^{[k]}A_0^{[k]}\right) + ip\,\mathrm{tr}\,A_1^{[k]^2} = ip\left(\mathrm{tr}\,\sigma_{k+1}^2 - \mathrm{tr}\,\sigma_k^2\right) = \frac{d}{dz_k}\log\left(e^{ipz_k(\mathrm{tr}\,\sigma_{k+1}^2 - \mathrm{tr}\,\sigma_k^2)}\right),
\end{aligned} \tag{5.3.67}$$

where we used the identity

$$\mathrm{tr}\left(A_0^{[k]}A_1^{[k]}\right) = \frac{1}{2}\mathrm{tr}\left(A_\infty^{[k]^2} - A_0^{[k]^2} - A_1^{[k]^2}\right) = \frac{1}{2}\left(\mathrm{tr}\,\sigma_{k+1}^2 - \mathrm{tr}\,\sigma_k^2 - \mathrm{tr}\,\mu_k^2\right). \tag{5.3.68}$$

To compute the second term in (5.3.66), we note that M_{z_k} in (5.3.4) is simply the singular part at z_k of L in (5.3.3) with a negative sign, so that

$$-\oint_{\underline{\mathcal{C}}^{[k]}} \mathrm{tr}\,L(z)M_{z_k}(z)\frac{dz}{2pi} = \oint_{\underline{\mathcal{C}}^{[k]}} \frac{1}{2}\mathrm{tr}\,L^2(z)\frac{dz}{2pi} = \mathrm{tr}\left(S_kL(z_k)_{reg}\right) = H_{z_k} \tag{5.3.69}$$

The last term in (5.3.66):

$$\begin{aligned}
-\oint_{\underline{\mathcal{C}}^{[k]}} \frac{dz}{2pi} \mathrm{tr}\left\{\left(\frac{q_1'(Q-\widetilde{r})}{q_1(Q-\widetilde{r})} - ip\right)M_{z_k}(z)\right\} &= \sum_{i=1}^N\left(S_{ii}^{(k)} + \Lambda_k\right)\left(\frac{q_1'(Q_i-\widetilde{r})}{q_1(Q_i-\widetilde{r})} - ip\right) \\
&= \sum_{i=1}^N\left(S_{ii}^{(k)} + \Lambda_k\right)\frac{q_1'(Q_i-\widetilde{r})}{q_1(Q_i-\widetilde{r})} - ipN\Lambda_k
\end{aligned} \tag{5.3.70}$$

since $\operatorname{tr} S^{(k)} = 0$. To simplify (5.3.70) further, let us first substitute (5.3.3), (5.3.4) in (5.3.69):

$$H_{z_k} = \sum_{i=1}^{N} S_{ii}^{(k)} P_i + (P\text{-independent part})\,, \tag{5.3.71}$$

which implies, together with the canonical Poisson bracket $\{Q_i, P_j\} = d_{ij}$, that

$$\frac{dQ_i}{dz_k} = \frac{\P H_{z_k}}{\P P_i} = S_{ii}^{(k)} \tag{5.3.72}$$

Then (5.3.70) becomes

$$\begin{aligned}
\oint_{\underline{\mathcal{C}}^{[k]}} \frac{dz}{2pi} \operatorname{tr}\left\{\frac{q_1'(Q-\tilde{r})}{q_1(Q-\tilde{r})} M_{z_k}(z)\right\} &= \sum_{i=1}^{N}\left(S_{ii}^{(k)} + \Lambda_k\right)\frac{q_1'(Q_i-\tilde{r})}{q_1(Q_i-\tilde{r})} - ipN\Lambda_k \\
&\overset{(5.3.31)}{=} \sum_{i=1}^{N}\left(\frac{dQ_i}{dz_k} - \frac{d\tilde{r}}{dz_k}\right)\frac{q_1'(Q_i-\tilde{r})}{q_1(Q_i-\tilde{r})} - ipN\Lambda_k \\
&= \frac{d}{dz_k}\log\left(e^{-ipNz_k\Lambda_k}\prod_{i=1}^{N} q_1(Q_i-\tilde{r})\right) \\
&\overset{(5.3.31)}{=} \frac{d}{dz_k}\log\left(e^{ipN\tilde{r}}\prod_{i=1}^{N} q_1(Q_i-\tilde{r})\right).
\end{aligned} \tag{5.3.73}$$

Substituting (5.3.67), (5.3.69), (5.3.73) back in (5.3.66),

$$I_{z_k}^{(l,3)} = d_k^l\left[\frac{d}{dz_k}\log\left(e^{ipz_k\left(\operatorname{tr}\sigma_{k+1}^2 - \operatorname{tr}\sigma_k^2\right)} e^{ipN\tilde{r}}\right) + H_{z_k} + \frac{d}{dz_k}\log\prod_{i=1}^{N} q_1(Q_i-\tilde{r})\right]. \tag{5.3.74}$$

Putting it all together (5.3.55):

$$\begin{aligned}
d\log\det\left[\mathbb{I} - K_{1,n}\right] &\overset{(5.3.37)}{=} d\log\mathcal{T}^{(1,n)} = -2pi\operatorname{tr}_{\mathcal{H}}\left[\mathcal{P}_{\oplus}\P_t\mathcal{P}_{\Sigma_{1,n}}\right] dt - \sum_{l=1}^{n}\operatorname{tr}_{\mathcal{H}^{[l]}}\left[\mathcal{P}_{\oplus}^{[l]}\P_{z_k}\mathcal{P}_{\Sigma_{1,n}}\right] dz_k \\
&\overset{(5.3.56),(5.3.63)}{=} -2pi\left(\sum_{l=1}^{n} I_t^{(l)} + I_w + I_z\right) dt + \sum_{l=1}^{n}\left(I_{z_k}^{(l,1)} + I_{z_k}^{(l,2)} + I_{z_k}^{(l,3)}\right) dz_k \\
&\overset{(5.3.60),(5.3.64),(5.3.74)}{=} H_t dt + \sum_{k=1}^{n} H_{z_k} dz_k \\
&+ 2pi\frac{d}{dt}\log\left(e^{-ipt\operatorname{tr}\left(\sigma_1^2 + \frac{\mathbb{I}}{6}\right)} e^{ipN\tilde{r}}\prod_{i=1}^{N}\frac{q_1(Q_i-\tilde{r})}{h(t)}\right) dt \\
&+ \sum_{k=1}^{n}\left[\frac{d}{dz_k}\log\left(e^{ipz_k\left(\operatorname{tr}\sigma_{k+1}^2 - \operatorname{tr}\sigma_k^2\right)} e^{ipN\tilde{r}}\right) + \frac{d}{dz_k}\log\prod_{i=1}^{N} q_1(Q_i-\tilde{r})\right] dz_k \\
&\overset{(5.3.2)}{=} d\log\mathcal{T}_H \\
&+ d\log\left[e^{-ipt\operatorname{tr}\left(\sigma_1^2 + \frac{\mathbb{I}}{6}\right)} e^{ipN\tilde{r}}\prod_{i=1}^{N}\frac{q_1(Q_i-\tilde{r})}{h(t)}\prod_{k=1}^{n} e^{ipz_k\left(\operatorname{tr}\sigma_{k+1}^2 - \operatorname{tr}\sigma_k^2\right)}\right]
\end{aligned} \tag{5.3.75}$$

Integrating (5.3.75) and substituting (5.3.31), we obtain (5.3.51). □

Part III

Combinatorics

Chapter 6

Combinatorics of the Airy kernel

6.1 Minor expansion

The Hilbert space $L^2(S^1)$ admits a natural orthonormal basis of Fourier modes (i.e. the monomials $z^n,\ n \in \mathbb{Z}$). The minor expansion of the Fredholm determinant (2.1.3) in this particular basis gives rise to interesting combinatorics. In the case of Painlevé VI, V, III the combinatorics correspond to certain Nekrasov Partition functions of certain Gauge theories [54].

In this spirit, we would like to propose, at least, a reasonable expansion of the Fredholm determinant of our operator in a similar guise. In our case the underlying Hilbert space $L^2(i\mathbb{R})$ does not immediately suggest a natural discrete orthonormal basis. Here below we want to propose a very natural such basis: the main guiding principle is that of identifying the Hardy space $\mathcal{H}_+$ with the Hardy space of the interior of the disk, and pulling back the monomial basis.

Proposition 6.1. *The Fredholm determinant of the tau-function in (3.2.1) can be expanded, on an appropriate basis, in terms of minors, that can be labelled by Maya diagrams (m_X)*

$$t\,[s] = \sum_{m_X \in \mathbb{M};\ |p|=|h|} a_{p_X}^{h_X} b_{h_X}^{p_X} \tag{6.1.1}$$

where the coefficients a_m^n, b_n^m are as follows

$$b_0^0 = a_0^0 = \left(4\frac{\P^2}{\P s^2} - 1\right)\left(1 - \frac{\P}{\P s}\right)^{-1} Ai(s) \tag{6.1.2}$$

where $Ai(s)$ is the Airy function, and

$$a_m^n = \frac{(-1)^{m+n}}{(m!)^2 n!(m+n+1)!}\left(\widetilde{D}\right)^{m+n} a_0^0 \tag{6.1.3}$$

$$b_n^m = \frac{(-1)^{m+n}}{(n!)^2 m!(m+n+1)!}\left(\widetilde{D}\right)^{m+n} b_0^0 \tag{6.1.4}$$

with $\widetilde{D} = 2\left(\frac{\P}{\P s} - 1\right)^2\left(4\frac{\P^2}{\P s^2} - s\right)$

Proof. Recall that,

$$t(s) = \det\left[Id_{L^2(i\mathbb{R})} - \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}\right] = \det\left[Id_{L^2(i\mathbb{R})\otimes\mathbb{C}^2} - \begin{bmatrix} 0 & 0 & 0 & a_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_{21} & 0 & 0 & 0 \end{bmatrix}\right] \quad (6.1.5)$$

of which a_{12} and b_{21} are the only non zero entries. Therefore, the determinant of the 4×4 block operator can be reduced to a determinant of a 2×2 block operator. Let us denote $a_{12} \equiv a$, $b_{21} \equiv b$

$$t[s] = \det\left[Id_{L^2(i\mathbb{R})} - \begin{pmatrix} 0 & a_{12} \\ b_{21} & 0 \end{pmatrix}\right] = \det\left[Id_{L^2(i\mathbb{R})} - \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\right] \quad (6.1.6)$$

We remind the reader here that the block decomposition is due to the splitting $L^2(i\mathbb{R}) = \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

The first step to obtain the minor expansion is constructing a suitable basis to expand $a(z,w)$ and $b(z,w)$,

$$a(z,w) : \mathcal{H}_- \to \mathcal{H}_+ \quad ; \quad b(z,w) : \mathcal{H}_+ \to \mathcal{H}_-. \quad (6.1.7)$$

Basis construction

The spaces $\mathcal{H}_\pm$ are Hardy spaces of functions analytic on the left and right half of the complex planes respectively. To construct the bases of $\mathcal{H}_\pm$, we employ the Paley-Weiner theorem which identifies $\mathcal{H}_+$ as the image under Fourier transform of functions supported on a half–line. Specifically, let $\mathbb{C}_+ = \{z : z = x+iy, y>0\}$,

$$H^2(\mathbb{C}_+) = \left\{ f : f \text{ is analytic in } \mathbb{C}_+ \text{ and } \sup_{0<y<+\infty} \int_{-\infty}^{+\infty} |f(z)|^2 dx < \infty \right\}. \quad (6.1.8)$$

By definition, the boundary values of $f \in H^2(\mathbb{C}_+)$ on $\mathbb{R}$ define a function in $L^2(\mathbb{R})$ and we can think of $H^2(\mathbb{C}_+)$ as a (closed) subspace of $L^2(\mathbb{R})$. With this understanding, the Paley–Wiener theorem can be stated as the following identity:

$$\mathcal{F}H^2(\mathbb{C}_+) = L^2[0,\infty). \quad (6.1.9)$$

The space $\mathcal{H}_+$ can be isometrically mapped to $H^2(\mathbb{C}_+)$ by a variable change $z \to iz$. We have that Laguerre functions $(L_n^l(z)z^l e^{-z})$ provide a basis of $L^2(\mathbb{R}_+)$. Using the Paley–Wiener theorem, upon inverse Fourier transform, they yield a basis for $H^2(\mathbb{C}_+)$ and an innocent change of variable $z \to iz$ gives a basis on $\mathcal{H}_+$. We can comfortably restrict ourselves to $l = 0$. Following [108], the Fourier transform $(\hat{\ell}_n^l(t))$ of the Laguerre functions $L_n^l(x)e^{-\frac{x}{2}}x^{\frac{l}{2}}$ for $l = 0$, and using the notation and $L_n^0 \equiv L_n$, is

$$L_n(x)e^{-\frac{x}{2}} = \frac{e^{-\frac{x}{2}}}{n!}\sum_{k=0}^{n}\frac{(-n)_k x^k}{k!} \quad (6.1.10)$$

$$\hat{\ell}_n(t) = \frac{-2}{n!}\left(\frac{1+2it}{2it-1}\right)^n \frac{1}{2it-1} \quad (6.1.11)$$

$\hat{\ell}_n$ forms a complete basis on $H^2(\mathbb{C}_+, dt)$. With the change of variable $2it = z$, (6.1.7) reads

$$\hat{\ell}_n(z) = \frac{-2}{n!}\left(\frac{1+z}{z-1}\right)^n \frac{1}{z-1} \tag{6.1.12}$$

and will form a basis on $H^2(\mathcal{H}_+, \frac{-i}{2}dz)$. In conclusion

$$e^n_{\mathcal{H}_+} = \frac{i}{n!}\left(\frac{1+z}{z-1}\right)^n \frac{1}{z-1} \tag{6.1.13}$$

is a basis on $H^2(\mathcal{H}_+, dz)$. Similarly,

$$e^n_{\mathcal{H}_-} = \frac{i}{n!}\left(\frac{z-1}{z+1}\right)^n \frac{1}{z+1} \tag{6.1.14}$$

is a basis on $H^2(\mathcal{H}_-, dz)$

Minor expansion

Expanding $a(z,w)$ and $b(z,w)$ in the basis $e_{\mathcal{H}_+}$ and $e_{\mathcal{H}_-}$, the tau-function (3.2.1) can be expressed as a minor expansion. Starting with $a(z,w)$,

$$\begin{aligned} a^n_m &= \langle a(z,w)e^n_{\mathcal{H}_-}, e^m_{\mathcal{H}_+}(z)\rangle \\ &= \int_{i\mathbb{R}} \frac{dz}{2pi}\bar{e}^m_{\mathcal{H}_+}(z)\int_{i\mathbb{R}-e}\frac{dw}{2pi}a(z,w)e^n_{\mathcal{H}_-}(w) \\ &= \frac{-k}{m!n!}\int_{i\mathbb{R}}\frac{dz}{2pi}\left(\frac{\bar{z}+1}{\bar{z}-1}\right)^m\frac{1}{(\bar{z}-1)}\int_{i\mathbb{R}-e}\frac{dw}{2pi}\frac{e^{q(w,s)}}{w-z}\left(\frac{w-1}{w+1}\right)^n\frac{1}{(w+1)} \\ &= \frac{k}{m!n!}\int_{i\mathbb{R}-e}\frac{dw}{2pi}\frac{e^{q(w,s)}(w-1)^{m+n}}{(w+1)^{m+n+2}} \end{aligned} \tag{6.1.15}$$

Similarly for $b(w,z)$

$$\begin{aligned} b^n_m &= \langle b(z,w)e^n_{\mathcal{H}_+}, e^m_{\mathcal{H}_-}(z)\rangle \\ &= \int_{i\mathbb{R}} \frac{dz}{2pi}\bar{e}^m_{\mathcal{H}_-}(z)\int_{i\mathbb{R}+e}\frac{dw}{2pi}b(z,w)e^n_{\mathcal{H}_+}(w) \\ &= \frac{k}{m!n!}\int_{i\mathbb{R}}\frac{dz}{2pi}\left(\frac{\bar{z}-1}{\bar{z}+1}\right)^m\frac{1}{(\bar{z}+1)}\int_{i\mathbb{R}+e}\frac{dw}{2pi}\frac{e^{-q(w,s)}}{w-z}\left(\frac{w+1}{w-1}\right)^n\frac{1}{(w-1)} \\ &= \frac{-k}{m!n!}\int_{i\mathbb{R}+e}\frac{dw}{2pi}\frac{e^{-q(w,s)}(w+1)^{m+n}}{(w-1)^{m+n+2}} \end{aligned} \tag{6.1.16}$$

Recurrence relations

a^n_m, b^n_m can be made explicit by noticing that the functions such as $\int \frac{dw}{2pi}(w+1)^m e^{-q(w)}$ can be written as some derivatives of the Airy function. Define the function c_{m+n} as

$$c_{m+n} = \frac{(w-1)^{m+n}}{(w+1)^{m+n+2}}e^{q(w,s)}. \tag{6.1.17}$$

Then a_m^n in terms of c_{m+n} is simply

$$a_m^n = \frac{k}{m!n!}\int_{i\mathbb{R}-e}\frac{dw}{2pi}\frac{e^{q(w,s)}(w-1)^{m+n}}{(w+1)^{m+n+2}} = \frac{k}{m!n!}\int_{i\mathbb{R}-e}\frac{dw}{2pi}c_{m+n} \tag{6.1.18}$$

and b_n^m in terms of c_{m+n} is

$$\begin{aligned} b_n^m &= \frac{-k}{m!n!}\int_{i\mathbb{R}+e}\frac{dw}{2pi}\frac{e^{-q(w,s)}(w+1)^{m+n}}{(w-1)^{m+n+2}} \\ &= \frac{k}{m!n!}\int_{i\mathbb{R}-e}\frac{dw}{2pi}\frac{e^{q(w,s)}(w-1)^{m+n}}{(w+1)^{m+n+2}} = \frac{k}{m!n!}\int_{i\mathbb{R}-e}\frac{dw}{2pi}c_{m+n} \end{aligned} \tag{6.1.19}$$

The function c_{m+n} obeys a recursion relation that can be derived as follows

$$\begin{aligned} \int\frac{dw}{2pi}c_{m+n}(w,s) &= \int\frac{dw}{2pi}\frac{(w-1)^{m+n}}{(w+1)^{m+n+2}}e^{q(w,s)} \\ &= \frac{2}{(m+n+1)}\int\frac{dw}{2pi}e^{q(w,s)}\P_w\left[\frac{(w-1)^{m+n+1}}{(w+1)^{m+n+1}}\right] \\ &= -\frac{2}{(m+n+1)}\int\frac{dw}{2pi}\left[\frac{(w-1)^{m+n+1}}{(w+1)^{m+n+1}}\right]\P_w e^{q(w,s)} \\ &= -\frac{2}{(m+n+1)}\int\frac{dw}{2pi}\frac{(w-1)^{m+n-1}}{(w+1)^{m+n+1}}(w-1)^2(4w^2-s)e^{q(w,s)} \\ &= -\frac{2}{(m+n+1)}\int\frac{dw}{2pi}\left(\frac{\P}{\P s}-1\right)^2\left(4\frac{\P^2}{\P s^2}-s\right)c_{m+n-1}(w,s) \end{aligned} \tag{6.1.20}$$

which gives the following equation

$$\int_{i\mathbb{R}}\frac{dw}{2pi}\left[c_{m+n}+\frac{2}{(m+n+1)}\left(\frac{\P}{\P s}-1\right)^2\left(4\frac{\P^2}{\P s^2}-s\right)c_{m+n-1}\right] = 0 \tag{6.1.21}$$

We define a function $Ci(t)$ as follows:

$$Ci(t) := \int\frac{dw}{2pi}\frac{1}{(w+1)}e^{q(w)} = \left(1-\frac{\P}{\P s}\right)^{-1}Ai(t) \tag{6.1.22}$$

Then the function c_0 can be computed in terms of $Ci(t)$

$$\begin{aligned} \int_{i\mathbb{R}}\frac{dw}{2pi}c_0 &= \int\frac{dw}{2pi}\frac{1}{(w+1)^2}e^{q(w,s)} = -\int\frac{dw}{2pi}\P_w\left(\frac{1}{w+1}\right)e^{q(w,s)} \\ &= \int\frac{dw}{2pi}\left(\frac{1}{w+1}\right)\P_w e^{q(w,s)} = \int\frac{dw}{2pi}\left(\frac{1}{w+1}\right)(4w^2-s)e^{q(w,s)} \\ &= \int\frac{dw}{2pi}\left(4\frac{\P^2}{\P s^2}-s\right)\left(\frac{1}{w+1}\right)e^{q(w,s)} \\ &= \left(4\frac{\P^2}{\P s^2}-s\right)Ci(s) \end{aligned} \tag{6.1.23}$$

Since both a_m^n and b_n^m are integrals of c_{m+n}(6.1.18), (6.1.19), the recurrsion relation (6.1.21) and the expression for c_0 (6.1.23) give the explicit expressions for a_m^n and b_n^m. Starting with the recursion relation for a_m^n and b_n^m

$$n(m+n+1)a_m^n + 2\left(\frac{\P}{\P s} - 1\right)^2 \left(4\frac{\P^2}{\P s^2} - s\right) a_m^{n-1} = 0 \tag{6.1.24}$$

$$n(m+n+1)b_m^n + 2\left(\frac{\P}{\P s} - 1\right)^2 \left(4\frac{\P^2}{\P s^2} - s\right) b_m^{n-1} = 0 \tag{6.1.25}$$

with

$$a_0^0 = \left(4\frac{\P^2}{\P s^2} - 1\right)\left(1 - \frac{\P}{\P s}\right)^{-1} Ai(s) \tag{6.1.26}$$

Further,

$$na_{m-1}^n = ma_m^{n-1} \quad ; \quad nb_{m-1}^n = mb_m^{n-1} \tag{6.1.27}$$

Notice that (6.1.24) and (6.1.25) are recursive relations in n. Using (6.1.27) similar recursion relations in m can be obtained

$$m(m+n+1)a_m^n + 2\left(\frac{\P}{\P s} - 1\right)^2 \left(4\frac{\P^2}{\P s^2} - s\right) a_{m-1}^n = 0 \tag{6.1.28}$$

$$m(m+n+1)b_m^n + 2\left(\frac{\P}{\P s} - 1\right)^2 \left(4\frac{\P^2}{\P s^2} - s\right) b_{m-1}^n = 0 \tag{6.1.29}$$

Define $2\left(\frac{\P}{\P s} - 1\right)^2 \left(4\frac{\P^2}{\P s^2} - s\right) = \widetilde{D}$. From (6.1.24)

$$a_m^n = (-1)^{m+n}\frac{(1+m)}{n!(m+n+1)!}\widetilde{D}^n a_m^0 \tag{6.1.30}$$

now using (6.1.28)

$$a_m^0 = (-1)^m \frac{1}{m!(m+1)!}\widetilde{D}^m a_0^0 \tag{6.1.31}$$

In terms of a_0^0, a_m^n is explicit

$$a_m^n = \frac{(-1)^{m+n}}{(m!)^2 n!(m+n+1)!}\widetilde{D}^{m+n} a_0^0 \tag{6.1.32}$$

Repeating the same computation for b_n^m

$$b_n^m = \frac{(-1)^{m+n}}{(n!)^2 m!(m+n+1)!}\widetilde{D}^{m+n} b_0^0 \tag{6.1.33}$$

Now for the Ablowitz-Segur tau-function,

$$t[s] = \det\left[1 - \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\right] \tag{6.1.34}$$

a and b can be expanded on a discrete basis $e_{\mathcal{H}_\pm}(z)$

$$a(z,w) = \sum_{m,n\in\mathbb{Z}_+} a^n_m e_{\mathcal{H}_+}(z)^m e_{\mathcal{H}_-}(w)^n; \quad b(z,w) = \sum_{m,n\in\mathbb{Z}_+} b^n_m e_{\mathcal{H}_-}(z)^m e_{\mathcal{H}_+}(w)^n \tag{6.1.35}$$

Since a^n_m and b^m_n in (6.1.35) are not matrices themselves, the corresponding Maya diagrams are "colourless". If a^n_m and b^m_n were $N \times N$ matrices themselves, the corresponding entries in the expansion would be $a^{n;a}_{m;b}$ and $b^{m;b}_{n;a}$ where $a, b = \{1, ..., N\}$ would be the colour indices. Furthermore, given the off-diagonal structure of the matrix U, the minors with $|p| \neq |h|$ vanish. Therefore, the minor expansion reads,

$$t\,[s] = \sum_{m_c\in\mathbb{M};\ |p|=|h|} a^{h_c}_{p_c} b^{p_c}_{h_c}. \tag{6.1.36}$$

The proof is now complete. □

It would be extremely interesting to interpret the terms in this minor expansion in a similar way to the case of Painlevé VI, V, III. However, to our knowledge, in the case of the second Painlevé transcendent, there is no direct analog connection with some field theory. Nonetheless the computation proceeds in a rather natural way and may prove of use in future applications.

Chapter 7

Combinatorial representation of tau-functions on a torus

7.1 Charged partitions and Nekrasov functions

In this section, we expand the Fredholm determinant (5.3.37) in terms of its principal minors labeled by random partitions, and show that the resulting combinatorial expression takes the form of a Fourier series of Nekrasov functions, known as dual Nekrasov-Okounkov partition function [89] in the self-dual Omega-background, for a class of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories called circular quiver gauge theories. These are gauge theories with multiple $SU(N)$ gauge groups, each of which is coupled to two matter hypermultiplets in the bifundamental representation, and their partition functions are equal to free fermion conformal blocks on the torus.

7.1.1 Minor expansion

The Hilbert space $L^2(S^1)$ admits a natural orthonormal basis of Fourier modes. We now compute the minor expansion of the Fredholm determinant (5.3.37) in this particular basis. The kernels of the operators $\mathsf{a}^{[k]}, \mathsf{b}^{[k]}, \mathsf{c}^{[k]}, \mathsf{d}^{[k]}$ in (5.3.26) read:

$$\begin{aligned}
\mathsf{a}^{[k]}(z,w) &= \frac{\mathbb{I} - \widetilde{Y}^{[k]}(z)\widetilde{Y}^{[k]}(w)^{-1}}{1-e^{-2pi(z-w)}}, && z,w \in \mathcal{C}_{in}^{[k]}, \\
\mathsf{b}^{[k]}(z,w) &= \frac{\widetilde{Y}^{[k]}(z)\widetilde{Y}^{[k]}(w)^{-1}}{1-e^{-2pi(z-w)}}, && z \in \mathcal{C}_{in}^{[k]},\ w \in \mathcal{C}_{out}^{[k]} \\
\mathsf{c}^{[k]}(z,w) &= -\frac{\widetilde{Y}^{[k]}(z)\widetilde{Y}^{[k]}(w)^{-1}}{1-e^{-2pi(z-w)}}, && z \in \mathcal{C}_{out}^{[k]},\ w \in \mathcal{C}_{in}^{[k]} \\
\mathsf{d}^{[k]}(z,w) &= \frac{\widetilde{Y}^{[k]}(z)\widetilde{Y}^{[k]}(w)^{-1} - \mathbb{I}}{1-e^{-2pi(z-w)}}, && z,w \in \mathcal{C}_{out}^{[k]}.
\end{aligned} \tag{7.1.1}$$

Since the solution $\widetilde{Y}^{[k]}$ to the k-th three-point problem defined in (5.3.12) is multivalued on $\mathcal{C}_{in}, \mathcal{C}_{out}$, with monodromy determined by σ_k, σ_{k+1} respectively as in equation (5.3.15), the matrix elements of the kernels in (7.1.1) have the following (twisted) Fourier series representation:

$$\begin{aligned}
\mathsf{a}^{[k]}_{a,b}(z,w) &= \sum_{-r,s\in\mathbb{Z}'_+} \mathsf{a}^{-r;a}_{s;b}\, e^{2piz\left(\frac{1}{2}-r+s^{(a)}_k\right)} e^{2piw\left(-\frac{1}{2}-s-s^{(b)}_k\right)},\\
\mathsf{b}^{[k]}_{a,b}(z,w) &= \sum_{r,s\in\mathbb{Z}'_+} \mathsf{b}^{-r;a}_{-s;b}\, e^{2piz\left(\frac{1}{2}-r+s^{(a)}_k\right)} e^{2piw\left(-\frac{1}{2}+s-s^{(b)}_{k+1}\right)},\\
\mathsf{c}^{[k]}_{a,b}(z,w) &= \sum_{r,s\in\mathbb{Z}'_+} \mathsf{c}^{r;a}_{s;b}\, e^{2piz\left(\frac{1}{2}+r+s^{(a)}_{k+1}\right)} e^{2piw\left(-\frac{1}{2}-s-s^{(b)}_k\right)},\\
\mathsf{d}^{[k]}_{a,b}(z,w) &= \sum_{r,s\in\mathbb{Z}'_+} \mathsf{d}^{r;a}_{-s;b}\, e^{2piz\left(\frac{1}{2}+r+s^{(a)}_{k+1}\right)} e^{2piw\left(-\frac{1}{2}+s-s^{(b)}_{k+1}\right)},
\end{aligned} \tag{7.1.2}$$

with $a,b = 1,\dots,N$, and $\mathbb{Z}'_+$ denoting the set of positive half-integers. The Fourier coefficients $\mathsf{a}^{-r;a}_{s;b}, \mathsf{b}^{-r;a}_{-s;b}, \mathsf{c}^{r;a}_{s;b}, \mathsf{d}^{r;a}_{-s;b}$ were computed in [54], but we will not need their explicit form. A submatrix of either $\mathsf{a}^{[k]}, \mathsf{b}^{[k]}, \mathsf{c}^{[k]}, \mathsf{d}^{[k]}$, of size $i\times j$, is denoted by two unordered sets $\{(r,a)_1,\dots,(r,a)_i\} \in 2^{\mathbb{Z}'_+\times\{1,\dots,N\}}$ and $\{(s,b)_1,\dots,(s,b)_j\} \in 2^{\mathbb{Z}'_+\times\{1,\dots,N\}}$ where r,s are the Fourier indices in the expansion (7.1.2), and a,b are the matrix ("color") indices. Such sets comprised of positive (negative) Fourier modes will be denoted by I (J). Minors of K will then be denoted by collections of such sets $\vec{I} := \{I_1,\dots,I_n\}$, $\vec{J} := \{J_1,\dots,J_n\}$, and a generic minor $K_{\vec{I},\vec{J}}$ has the form:

$$K_{\vec{I},\vec{J}} = \begin{pmatrix}
0 & 0 & 0 & 0 & \dots & 0 & \left(\nabla^{-1}\mathsf{c}^{[n]}\right)^{J_1}_{J_n} & \left(\nabla^{-1}\mathsf{d}^{[n]}\nabla\right)^{J_1}_{I_1}\\
0 & 0 & \left(\mathsf{a}^{[2]}\right)^{I_2}_{J_2} & \left(\mathsf{b}^{[2]}\right)^{I_2}_{I_3} & \dots & 0 & 0 & 0\\
\left(\mathsf{c}^{[1]}\right)^{J_2}_{J_1} & \left(\mathsf{d}^{[1]}\right)^{J_2}_{I_2} & 0 & 0 & \dots & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\
0 & 0 & \left(\mathsf{c}^{[2]}\right)^{J_3}_{J_2} & \left(\mathsf{d}^{[2]}\right)^{J_3}_{I_3} & \dots & 0 & 0 & 0\\
\vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots\\
0 & 0 & \dots & 0 & 0 & 0 & \left(\mathsf{a}^{[n]}\right)^{I_n}_{J_n} & \left(\mathsf{b}^{[n]}\nabla\right)^{I_n}_{I_1}\\
0 & 0 & \dots & 0 & \left(\mathsf{c}^{[n-1]}\right)^{J_n}_{J_{n-1}} & \left(\mathsf{d}^{[n-1]}\right)^{J_n}_{I_n} & 0 & 0\\
\left(\mathsf{a}^{[1]}\right)^{I_1}_{J_1} & \left(\mathsf{b}^{[1]}\right)^{I_1}_{I_2} & \dots & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{7.1.3}$$

A combinatorial interpretation in terms of Maya diagrams and charged partitions proves vital in expressing the minors as Nekrasov functions: the multi-indices (I,J) can be viewed as the positions $\mathsf{h}(\mathsf{m}^{(a)})$ and $\mathsf{p}(\mathsf{m}^{(a)})$ of 'holes' and 'particles' respectively, of a coloured Maya diagram $\mathsf{m}^{(a)}$, where $a = 1,...,N$, see figure 7.1. Each particle (hole) carries a positive (negative) unit charge, so that the total charge associated to every Maya diagram is

$$\mathsf{Q}(\mathsf{m}^{(a)}) := |p(\mathsf{m}^{(a)})| - |h(\mathsf{m}^{(a)})|. \tag{7.1.4}$$

Using the notation

$$\vec{\mathsf{m}} := \left(\mathsf{m}^{(1)}, \ldots, \mathsf{m}^{(N)}\right), \qquad \vec{\mathsf{Q}} := \left(\mathsf{Q}^{(1)}, \ldots, \mathsf{Q}^{(N)}\right), \tag{7.1.5}$$

the total charge is

$$\mathsf{Q} := \sum_{a=1}^{N} \mathsf{Q}^{(a)}, \tag{7.1.6}$$

and it is the same for every N-tuple of coloured Maya diagrams appearing in our expansions. Each Maya diagram determines uniquely a charged Young diagram (Y, Q) as exemplified in figure 7.1. Consequently, the minors can be labeled by N-tuples of charged partitions $\left(\vec{\mathsf{Y}}, \vec{\mathsf{Q}}\right)$.

Definition 7.1. *With the labels in terms of partitions* Y *and charges* Q*, let us define the trinion partition function by the following expression:*

$$Z^{\mathsf{Y}_k, \mathsf{Q}_k}_{\mathsf{Y}_{k+1}, \mathsf{Q}_{k+1}}\left(\mathcal{T}^{[k]}\right) = Z^{I_k, J_k}_{I_{k+1}, J_{k+1}}\left(\mathcal{T}^{[k]}\right) := (-1)^{|I_{k+1}|} \det \begin{pmatrix} \left(\mathsf{a}^{[k]}\right)^{I_k}_{J_k} & \left(\mathsf{b}^{[k]}\right)^{I_k}_{I_{k+1}} \\ \left(\mathsf{c}^{[k]}\right)^{J_{k+1}}_{J_k} & \left(\mathsf{d}^{[k]}\right)^{J_{k+1}}_{I_{k+1}} \end{pmatrix}, \tag{7.1.7}$$

where $k = 1, \ldots, n$*, and* $I_{n+1} = I_1$*,* $J_{n+1} = J_1$*.* $\mathcal{T}^{[k]}$ *is the k-th trinion in the pants decomposition in figure 5.4.*

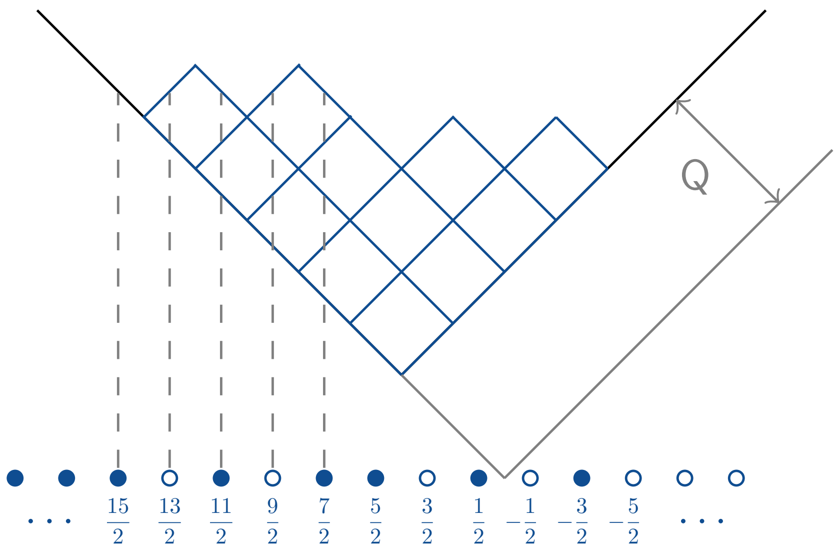


FIGURE 7.1: Young and Maya diagrams

Note that the determinant in (7.1.7) is non zero for $|I_{k+1}| + |J_k| = |I_k| + |J_{k+1}|$, which in turn implies that all the Maya diagrams carry the same charge Q.

Proposition 7.1. *The determinant tau-function $\mathcal{T}^{(1,n)}$ in (5.3.75) has the following minor expansion in terms of the trinion partition functions in (7.1.7):*

$$\mathcal{T}^{(1,n)} = \sum_{\vec{Q}_1,\dots\vec{Q}_n} \sum_{\vec{Y}_1,\dots\vec{Y}_n\in\mathbb{Y}^N} e^{2pit\left[\frac{1}{2}\left(\vec{Q}+\vec{s}_1\right)^2-\frac{1}{2}\vec{s}_1^2+|\vec{Y}|\right]-2pi\left(r-\frac{t}{2}+t\sum_{j=1}^n\Lambda_j\right)Q} \prod_{k=1}^n Z^{\vec{Y}_k,\vec{Q}_k}_{\vec{Y}_{k+1},\vec{Q}_{k+1}}\left(T^{[k]}\right) \tag{7.1.8}$$

where $\vec{s}_1 := \left(s_1^{(1)},\dots,s_1^{(N)}\right)$[1] *is the vector of monodromy exponents along the A-cycle of the torus, with modular parameter t.*

Proof. From (7.1.3), we can read off the minor expansion of the tau-function (5.3.37) in terms of the trinion partition functions in (7.1.7):

$$\mathcal{T}^{(1,n)} = \sum_{(\vec{I},\vec{J})}\prod_{k=1}^{n-1} Z^{I_k,J_k}_{I_{k+1},J_{k+1}}\left(T^{[k]}\right)\times(-1)^{|I_1|}\det\begin{pmatrix}\left(a^{[n]}\right)^{I_n}_{J_n} & \left(b^{[n]}\right)^{I_n}_{I_1}\\ \left(c^{[n]}\right)^{J_1}_{J_n} & \left(d^{[n]}\right)^{J_1}_{I_1}\end{pmatrix}. \tag{7.1.9}$$

[1]Note that here, differently from (5.3.15) where we collected the monodromy exponents $s_i^{(1)},\dots,s_i^{(N)}$ into diagonal matrices denoted by σ_i, we organize them into vectors $\vec{a}_i$, since they are summed with the charges $\vec{Q}_i$, that are vectors in the root lattice $\mathbb{Z}^N$ of $\mathfrak{gl}_N$.

Additionally, we can write the last factor in (7.1.9) as follows

$$\det\begin{pmatrix} \left(\mathsf{a}^{[n]}\right)^{I_n}_{J_n} & \left(\mathsf{b}^{[n]}\nabla\right)^{I_n}_{I_1} \\ \left(\nabla^{-1}\mathsf{c}^{[n]}\right)^{J_1}_{J_n} & \left(\nabla^{-1}\mathsf{d}^{[n]}\nabla\right)^{J_1}_{I_1} \end{pmatrix}$$

$$\overset{(5.3.43),(5.3.44)}{=} \left(\prod_{(r,a)\in J_1} e^{-2pir} e^{2pit\left(\frac{1}{2}+r+\mathsf{s}_1^{(a)}-\sum_{j=1}^n \Lambda_j\right)}\right) \det\begin{pmatrix} \left(\mathsf{a}^{[n]}\right)^{I_n}_{J_n} & \left(\mathsf{b}^{[n]}\right)^{I_n}_{I_1} \\ \left(\mathsf{c}^{[n]}\right)^{J_1}_{J_n} & \left(\mathsf{d}^{[n]}\right)^{J_1}_{I_1} \end{pmatrix}$$

$$\times\left(\prod_{(s,b)\in I_1} e^{2pir} e^{2pit\left(-\frac{1}{2}+s-\mathsf{s}_1^{(b)}+\sum_{j=1}^n \Lambda_j\right)}\right)$$

$$= \det\begin{pmatrix} \left(\mathsf{a}^{[n]}\right)^{I_n}_{J_n} & \left(\mathsf{b}^{[n]}\right)^{I_n}_{I_1} \\ \left(\mathsf{c}^{[n]}\right)^{J_1}_{J_n} & \left(\mathsf{d}^{[n]}\right)^{J_1}_{I_1} \end{pmatrix} \exp\left\{\sum_{(r,a)\in J_1}\left[-2pi\left(r-\frac{t}{2}+t\sum_{j=1}^n \Lambda_j\right)+2pit\left(r+\mathsf{s}_1^{(a)}\right)\right]\right\}$$

$$\times\exp\left\{\sum_{(s,b)\in I_1}\left[2pi\left(r-\frac{t}{2}+t\sum_{j=1}^n \Lambda_j\right)+2pit\left(s-\mathsf{s}_1^{(b)}\right)\right]\right\}$$

$$= \det\begin{pmatrix} \left(\mathsf{a}^{[n]}\right)^{I_n}_{J_n} & \left(\mathsf{b}^{[n]}\right)^{I_n}_{I_1} \\ \left(\mathsf{c}^{[n]}\right)^{J_1}_{J_n} & \left(\mathsf{d}^{[n]}\right)^{J_1}_{I_1} \end{pmatrix} e^{2pit\left[\frac{1}{2}\left(\vec{\mathsf{Q}}+\vec{\mathsf{s}}_1\right)^2-\frac{1}{2}\vec{\mathsf{s}}_1^2+|\vec{\mathsf{Y}}|\right]-2pi\left(r-\frac{t}{2}+t\sum_{j=1}^n \Lambda_j\right)\mathsf{Q}}. \tag{7.1.10}$$

In the second line of (7.1.10), we used the fact that if σ_1 is the monodromy exponent on $\mathcal{C}_{in}^{[1]}$, then the monodromy exponent on $\mathcal{C}_{out}$ is $\sigma_1 - \sum_{j=1}^n \Lambda_j$ Since $\mathsf{s} \in I_1$, the hole positions in the corresponding Maya diagram m are $\mathsf{h}(\mathsf{m}) = \{-\mathsf{s}_1, ..., -\mathsf{s}_k\}$, and since $r \in \mathsf{J}_1$, the particle positions are $\mathsf{p}(\mathsf{m}) = \{r_1, ..., r_l\}$.

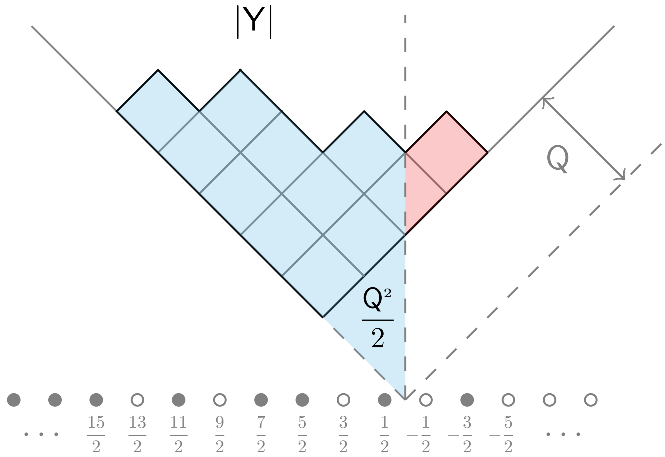


FIGURE 7.2: Pictorial proof of (7.1.11)

To obtain the last line in (7.1.10), we use the following equalities:

$$\sum_l r_l + \sum_k s_k = \frac{\mathsf{Q}(\mathsf{m})^2}{2} + |\vec{\mathsf{Y}}|, \qquad \#r - \#s = \mathsf{Q}(\mathsf{m}), \tag{7.1.11}$$

which can be read off from Figure 7.2 noting that the r's and s's are to the left and right sides of the axis respectively. As an example, in the Figure 7.2, $\mathsf{p}(\mathsf{m}) = \left\{\frac{13}{2}, \frac{9}{2}, \frac{3}{2}\right\}$, $\mathsf{h}(\mathsf{m}) = \left\{\frac{-3}{2}\right\}$. $|Y|$ is the #boxes in the Young diagram which in the present example is 12. The charge $\mathsf{Q}(\mathsf{m}) = 2$. $\sum r$ is the blue area and $\sum s$ is the red area in the Figure 7.2. Equations (7.1.9), (7.1.10) imply (7.1.8). □

Although the determinant tau-function $\mathcal{T}^{(1,n)}$ in (5.3.37) admits the expansion (7.1.8), the trinion partition functions (7.1.7) are known explicitly in terms of Nekrasov functions only in the case where the Lax matrix residues are of rank-1. We denote the determinant tau-function for a generic Fuchsian system on the torus with rank-1 residues,i.e. residues of the form $\boldsymbol{\mu}_k = (m_1, 0, \ldots, 0)$, and monodromy exponents around $\mathcal{C}_{in}^{[k]}$ given by $\vec{\mathsf{s}}_k$ by $\widetilde{\mathcal{T}}^{(1,n)} := \det\left[\mathbb{I} - \widetilde{K}_{1,n}\right]$. Using the expressions for $Z^{\mathsf{Y}_k,\mathsf{Q}_k}_{\mathsf{Y}_{k+1},\mathsf{Q}_{k+1}}(T^{[k]})$ computed in [54, 53] for the rank-1 case, we obtain[2]

$$\begin{aligned}\widetilde{\mathcal{T}}^{(1,n)} &= \sum_{\vec{Q}_1,\ldots\vec{Q}_n}\sum_{\vec{Y}_1,\ldots\vec{Y}_n} e^{-2piQ\left(r-\frac{t}{2}+t\sum_{j=1}^{n}\Lambda_j+\frac{1}{2}\sum_{j=1}^{n}\Lambda_j\right)} e^{2pit\left[\frac{1}{2}(\vec{Q}_1+\vec{s}_1)^2-\frac{1}{2}\vec{s}_1^2+|\vec{Y}_n|\right]} \\ &\times\prod_{j=1}^{n} e^{-2pi(z_j-z_{j-1})\left[\frac{1}{2}(\vec{s}_j+\vec{Q}_j)^2-\frac{1}{2}\vec{s}_j^2+|\vec{Y}_j|\right]} \\ &\times\prod_{k=1}^{n} e^{2pi\vec{Q}_i\cdot\vec{n}_i}\frac{Z_{pert}\left(\vec{s}_k+\vec{Q}_k,\vec{s}_{k+1}+\vec{Q}_{k+1}\right)}{Z_{pert}\left(\vec{s}_k,\vec{s}_{k+1}\right)} Z_{inst}\left(\vec{s}_k+\vec{Q}_k,\vec{s}_{k+1}+\vec{Q}_{k+1}|\vec{Y}_k,\vec{Y}_{k+1}\right) \\ &= \det\left[\mathbb{I}-\widetilde{K}_{1,n}\right]\end{aligned} \tag{7.1.12}$$

where we set $z_0 := z_n$, the Fourier series parameters $\vec{n}_i$ were defined in [54, 53] in terms of the normalization of the three-point solution, and we have used introduced the functions

$$Z_{pert}\left(\vec{s},\vec{m}\right) := \prod_{a,b=1}^{N}\frac{G\left(1+s^{(a)}-m^{(b)}\right)}{G\left(1+s^{(a)}-s^{(b)}\right)}, \tag{7.1.13}$$

$G(x)$ being the Barnes' G-function, and

$$Z_{inst}\left(\vec{s},\vec{m}|\vec{Y},\vec{W}\right) := \prod_{a,b=1}^{N}\frac{Z_{\rm bif}\left(s^{(a)}-m^{(b)}|Y^{(a)},W^{(\rm fi)}\right)}{Z_{\rm bif}\left(s^{(a)}-s^{(b)}|Y^{(a)},Y^{(b)}\right)}, \tag{7.1.14}$$

with

$$Z_{\rm bif}\left(x|Y',Y\right) := \prod_{a,b=1}^{N}\prod_{\square\in Y'}\left(x+1+a_{Y'}(\square)+l_{Y}(\square)\right)\left(x-1-a_{Y}(\square)-l_{Y'}(\square)\right). \tag{7.1.15}$$

In the above equations, $\vec{s},\vec{m}\in\mathbb{C}^N$, $\vec{Y},\vec{W}\in\mathbb{Y}^N$, ere, $a_Y(\square)$ and $l_Y(\square)$ denote respectively the arm and leg length of the box $\square$ in the Young diagram Y, as in figure 7.3.

Remark 7.1. *In (7.1.12), the expression*

$$Z^D := e^{2pit\vec{s}_1^2}\det\left[\mathbb{I}-\widetilde{K}_{1,n}\right] \tag{7.1.16}$$

is the Nekrasov-Okounkov dual partition function [89] of a circular quiver $\mathcal{N}=2$, $SU(N)$ gauge theory. By the AGT correspondence [1], Z^D is equal to a conformal block of N free fermions on the torus, as in [14]. Consequently, we expect Z^D in (7.1.16) to satisfy appropriate bilinear equations, along the lines of [5, 4].

Our next goal is to relate the explicit expression (7.1.12) for the tau-function $\widetilde{\mathcal{T}}^{(1,n)}$ of a linear system on the torus with rank-1 residues, to the tau-function $\mathcal{T}_H$ of an isomonodromic problem, where the residues are generic and satisfy the constraint (5.3.9). With the observation that any $SL(2)$ matrix can be reduced to rank-1 by a scalar transformation, we will

[2]Time-independent term $\frac{1}{2}\sum_{j=1}^{n}\Lambda_j$ comes from the ratios of the asymptotics of $U(1)$ corrections to solutions of the 3-pt problems, given explicitly by $(\sin p(z-z_k))^{\Lambda_k}$.

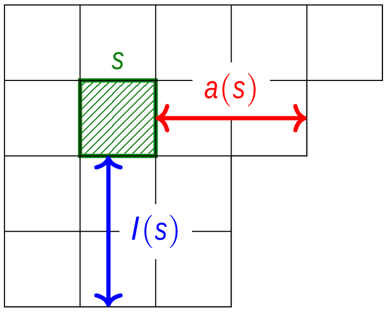


FIGURE 7.3: Arm and leg length

do this for the cases of the 2-particle nonautonomous Calogero-Moser system and of the elliptic Garnier system, which is the restriction to $N=2$, $\Lambda^{(j)}=0$, $j=1,\dots,n$ of the linear system (5.3.1).

7.1.2 Reduction to rank-1 residues: the case of 2-particle nonautonomous Calogero-Moser system

With the above considerations in mind, we formulate the tau-function of the equation (5.1.1) in terms of the dual Nekrasov-Okounkov partition function (7.1.12) for the $\mathcal{N}=2^*$ gauge theory[3]: the Lax matrix L_{CM} in (5.1.6) behaves as follows around the puncture $z=0$

$$L_{CM}(z)=\begin{pmatrix} P & mx(2Q,z) \\ mx(-2Q,z) & -P \end{pmatrix}=\frac{ms_1}{z}+\mathcal{O}(1), \tag{7.1.17}$$

so that it has rank-2 residue. To make it rank-1, we perform the scalar gauge transformation

$$L_{CM}(z)\to \widetilde{L}_{CM}:=L_{CM}(z)-l_{CM}(z)^{-1}\P_z l_{CM}(z)\mathbb{I}_2, \qquad l_{CM}(z)=q_1(z)^m, \tag{7.1.18}$$

after which the Lax matrix and its behavior around the puncture become

$$\widetilde{L}_{CM}(z)=\begin{pmatrix} P-m\frac{q_1'(z)}{q_1(z)} & mx(2Q,z) \\ mx(-2Q,z) & -P-m\frac{q_1'(z)}{q_1(z)} \end{pmatrix}=\frac{m}{z}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}+\mathcal{O}(1). \tag{7.1.19}$$

As a consequence of (7.1.18), the monodromies will be dressed by additional scalar factors that we denote by $g_B(z), g_1$ for the B-cycle and for the monodromy around the puncture respectively. The absence of a factor g_A for the A-cycle, as well as the expression for $g_B(z)$, are determined by the periodicity of theta functions:

$$l_{CM}(z+t)=q_1(z+t)^m=e^{-2pi\left(z+\frac{t+1}{2}\right)m}l_{CM}(z):=g_B(z)\,l_{CM}(z), \tag{7.1.20}$$

$$l_{CM}(z+1)=q_1(z+1)^m=e^{ipm}l_{CM}(z):=g_A\,l_{CM}(z). \tag{7.1.21}$$

The z-dependence of the factor $g_B(z)$ leads to a nontrivial factor g_1 for the monodromy around $z=0$:

$$g_1=e^{-2pim}. \tag{7.1.22}$$

[3] This is the $SU(2)$, $\mathcal{N}=2$ Super Yang-Mills theory with one massive adjoint hypermultiplet.

The Hamiltonian tau-function $\widetilde{\mathcal{T}}_{CM}$ associated to the gauge-transformed Lax matrix (7.1.19) is:

$$2pi\P_t \log \widetilde{\mathcal{T}}_{CM} := \frac{1}{2}\oint_A dz \mathrm{tr}\, \widetilde{L}^2_{CM}. \tag{7.1.23}$$

Proposition 7.2. *The tau-function (5.2.48) of the equation (5.1.1) is related to the tau-function (7.1.23) of the rank-1 Lax matrix in (7.1.18) as*

$$\mathcal{T}_{CM}(t) = \widetilde{\mathcal{T}}_{CM}(t)\left(h(t)e^{\frac{ipt}{6}}\right)^{-2m^2}, \tag{7.1.24}$$

where m is the monodromy exponent at the puncture and t is the isomonodromic time.

Proof. We begin with the equation (7.1.23):

$$\begin{aligned} 2pi\P_t \log \widetilde{\mathcal{T}}_{CM} &= \frac{1}{2}\oint_A dz \mathrm{tr}\, \widetilde{L}^2_{CM} \overset{(7.1.18)}{=} \frac{1}{2}\oint_A dz \mathrm{tr}\, L^2_{CM} + \oint_A dz \left(I^{-1}_{CM}(z)\P_z I_{CM}(z)\right)^2 \\ &= 2pi\P_t \log \mathcal{T}_{CM} + m^2 \int_0^1 dz \left(\frac{q_1'(z)}{q_1(z)}\right)^2. \end{aligned} \tag{7.1.25}$$

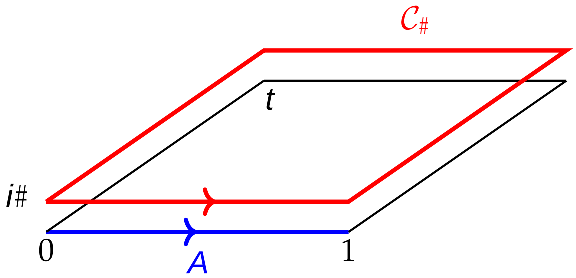


FIGURE 7.4: Contour of integration

To compute the last term in (7.1.25), consider the following integral over the deformed contour $\mathcal{C}_\#$ as in Figure 7.4

$$\begin{aligned} 2pi\,\mathrm{Res}_{z=t}\left(\frac{q_1'(z)}{q_1(z)}\right)^3 &= \oint_{\mathcal{C}_\#}\left(\frac{q_1'(z)}{q_1(z)}\right)^3 dz \\ &= \left[\int_{ie}^{1+ie} + \int_{1+ie}^{1+ie+t} + \int_{1+ie+t}^{ie+t} + \int_{ie+t}^{ie}\right]\left(\frac{q_1'(z)}{q_1(z)}\right)^3 dz \\ &= \int_{ie}^{1+ie}\left[\left(\frac{q_1'(z)}{q_1(z)}\right)^3 - \left(\frac{q_1'(z)}{q_1(z)} - 2pi\right)^3\right] dz \\ &= 6pi\int_{ie}^{1+ie} dz \left(\frac{q_1'(z)}{q_1(z)}\right)^2 - 3(2pi)^2\int_{ie}^{1+ie}\frac{q_1'(z)}{q_1(z)} + (2pi)^3 \\ &= 6pi\int_{ie}^{1+ie} dz\left(\frac{q_1'(z)}{q_1(z)}\right)^2 + \frac{5}{2}(2pi)^3. \end{aligned} \tag{7.1.26}$$

To obtain the last line we use that

$$\int_{ie}^{1+ie}\frac{q_1'(z)}{q_1(z)}dz = \left[\log q_1(z)\right]_{ie}^{1+ie} = -ip. \tag{7.1.27}$$

The residue on the left hand side of (7.1.26) is computed shifting z by t and expanding around 0:

$$\text{Res}_{z=t}\left(\frac{q_1'(z)}{q_1(z)}\right)^3 = \text{Res}_{z=0}\left(\frac{q_1'(z+t)}{q_1(z+t)}\right)^3, \tag{7.1.28}$$

and

$$\left(\frac{q_1'(z+t)}{q_1(z+t)}\right)^3 = \left(\frac{q_1'(z)}{q_1(z)} - 2pi\right)^3 = \frac{1}{z^3} - \frac{6pi}{z^2} + \frac{1}{z}\left(\frac{q_1'''}{q_1'} + 3(2pi)^2\right) + \mathcal{O}(1). \tag{7.1.29}$$

Therefore,

$$\text{Res}_{z=t}\left(\frac{q_1'(z)}{q_1(z)}\right)^3 = \frac{q_1'''}{q_1} + 3(2pi)^2. \tag{7.1.30}$$

Substituting (7.1.30) in (7.1.26), and taking the limit $e \to 0$ we get

$$\begin{aligned} \frac{2pi}{m^2}\P_t \log\left(\frac{\widetilde{\mathcal{T}}_{CM}}{\mathcal{T}_{CM}}\right) &\overset{(7.1.25)}{=} \lim_{e\to 0}\int_{ie}^{1+ie} dz \left(\frac{q_1'(z)}{q_1(z)}\right)^2 \\ &= \frac{1}{3}\frac{q_1'''}{q_1'} + \frac{1}{6}(2pi)^2 \\ &\overset{(5.2.89)}{=} 2pi\P_t \log h(t)^2 e^{ipt/3}. \end{aligned} \tag{7.1.31}$$

Therefore,

$$h(t)^{-2m^2} e^{-iptm^2/3}\widetilde{\mathcal{T}}_{CM} = \mathcal{T}_{CM}, \tag{7.1.32}$$

having set the integration constant to 1 without any loss of generality. □

Theorem 7.1. *The isomonodromic tau-function $\mathcal{T}_{CM}$ admits the following combinatorial expansion:*

$$\begin{aligned} \mathcal{T}_{CM}(t) = & \frac{\left(h(t)e^{-ipt/12}\right)^{2(1-m^2)}}{q_1\left(Q(t) + r - \frac{m(t+1)}{2}\right) q_1\left(Q(t) - r + \frac{m(t+1)}{2}\right)} e^{-2pi\left[r - \frac{t}{2}\left(m+\frac{1}{2}\right) - \frac{m}{2}\right]} \\ & \times \sum_{\vec{Q}} \sum_{\vec{Y}\in\mathbb{Y}^2} e^{2pit\left[\frac{1}{2}(\vec{Q}+\vec{a})^2 + |\vec{Y}|\right]} e^{2pi\left[\vec{Q}\cdot\vec{n} - Q\left(r - \frac{m(t+1)}{2} - \frac{t}{2}\right)\right]} \\ & \times \frac{Z_{pert}\left(\vec{a}+\vec{Q}, \vec{a}+\vec{Q}+m\right)}{Z_{pert}\left(\vec{a}, \vec{a}+m\right)} Z_{inst}\left(\vec{a}+\vec{Q}, \vec{a}+\vec{Q}+m|\vec{Y}, \vec{Y}\right)\widetilde{\Upsilon}_{1,1}, \end{aligned} \tag{7.1.33}$$

where $\vec{a} = (a, -a)$, with a the local monodromy exponent around the A-cycle of the torus, m is the monodromy exponent at the puncture $z = 0$, r is an arbitrary parameter, Q is the solution of the equations of motion of the 2-particle nonautonomous Calogero-Moser system (5.1.1), *$\vec{Q}$ is the vector of charges* (7.1.5), *Q is the total $U(1)$ charge* (7.1.6), *$\widetilde{\Upsilon}_{1,1}$ is an integration constant depending on monodromy data, and Z_{inst}, Z_{pert} are defined in* (7.1.14),(7.1.13).

Proof. The linear system (7.1.19) is the specialisation of (5.3.3) to the case $n = 1$ (with the puncture at 0), $N = 2$. The corresponding monodromy exponents $\vec{s}_1, \vec{s}_2$, and the $U(1)$ shifts

Λ_0, Λ_1 in (5.3.14), and the parameter $\tilde{r}$ in (5.3.31), for the present case are

$$\vec{s}_1 = \left(a - \frac{m}{2}, -a - \frac{m}{2}\right), \qquad \vec{s}_2 = \left(a + \frac{m}{2}, -a + \frac{m}{2}\right), \tag{7.1.34}$$

$$\Lambda_0 = \frac{m}{2}, \qquad \Lambda_1 = -m, \qquad \tilde{r} = r - \frac{m(t+1)}{2}. \tag{7.1.35}$$

Theorem 5.2 then implies that the tau-function $\widetilde{\mathcal{T}}_{CM}$ in (7.1.24) can be written as a Fredholm determinant of an operator we call $\widetilde{K}_{1,1}$ whose minor expansion has an interpretation through Nekrasov functions as in (7.1.12), of the tau-function in (5.3.51). Therefore,

$$\begin{aligned}
\mathcal{T}_{CM} &\overset{(7.1.24)}{=} h(t)^{-2m^2} e^{-ip\,tm^2/3} \widetilde{\mathcal{T}}_{CM} \\
&\overset{(5.3.51)}{=} h(t)^{-2m^2} e^{-ip\,tm^2/3} e^{ip\,t\,\mathrm{tr}\left(\sigma_1^2+\frac{\mathbb{I}}{6}\right)} e^{-2pi\tilde{r}} \frac{h(t)^2}{q_1(Q+\tilde{r})q_1(Q-\tilde{r})} \det\left[\mathbb{I} - \widetilde{K}_{1,1}\right] \widetilde{\Upsilon}_{1,1}, \\
&\overset{(7.1.34),(7.1.35)}{=} \frac{\left(h(t)e^{-\frac{ipt}{12}}\right)^{2-2m^2} e^{2pita^2}}{q_1(Q+r-\frac{m(t+1)}{2})q_1(Q-r+\frac{m(t+1)}{2})} e^{-2pi\left[r-\frac{t}{2}\left(m+\frac{1}{2}\right)-\frac{m}{2}\right]} \det\left[\mathbb{I} - \widetilde{K}_{1,1}\right] \widetilde{\Upsilon}_{1,1} \\
&\overset{(7.1.12)}{=} \frac{\left(h(t)e^{-\frac{ipt}{12}}\right)^{2-2m^2} e^{-2pi\left[r-\frac{t}{2}\left(m+\frac{1}{2}\right)-\frac{m}{2}\right]}}{q_1\left(Q+r-\frac{m(t+1)}{2}\right) q_1\left(Q-r+\frac{m(t+1)}{2}\right)} \sum_{\vec{Q}} \sum_{\vec{Y}\in\mathbb{Y}^2} e^{2pit\left[\frac{1}{2}(\vec{Q}+\vec{a})^2+|\vec{Y}|\right]} \\
&\quad \times e^{2pi\left[\vec{Q}\cdot\vec{n}-Q\left(r-\frac{m(t+1)}{2}-\frac{t}{2}\right)\right]} \frac{Z_{pert}\left(\vec{a}+\vec{Q}, \vec{a}+\vec{Q}+m\right)}{Z_{pert}\left(\vec{a},\vec{a}+m\right)} Z_{inst}\left(\vec{a}+\vec{Q}, \vec{a}+\vec{Q}+m|\vec{Y},\vec{Y}\right) \widetilde{\Upsilon}_{1,1}.
\end{aligned} \tag{7.1.36}$$

□

Remark 7.2. *Equation (7.1.33) coincides with equations (3.48) (4.10) in [15], obtained by CFT methods. To compare the two expressions, one has to set* $s = 0$ *and send* $r + \frac{1}{2} + \frac{t}{2} \to -r + \frac{m(t+1)}{2}$ *in the expressions of [15].*

7.1.3 Elliptic Garnier system and Nekrasov functions

For the $N \times N$ case, it is in general only possible, with a scalar gauge transformation, to reduce the rank of the residues to $N-1$, which means that the minors can be written in terms of Nekrasov functions only in the case of semi-degenerate residues, as in [52, 53].[4] Therefore, we restrict the Lax matrix in (5.3.3) to $N = 2$, which can always be reduced to rank-1 by the scalar gauge transformation

$$l(z) = \prod_{k=1}^{n} q_1(z-z_k)^{m_k} \tag{7.1.37}$$

[4]In the context of class S theories [49, 50] these are called *minimal punctures*. The A_{N-1} six-dimensional theory compactified on a torus with n minimal punctures gives rise to a four-dimensional circular quiver gauge theory.

where m_k is the local monodromy exponent at the puncture z_k. The new Lax matrix is

$$\widetilde{L} := L - l(z)^{-1}\P_z l(z)\mathbb{I}_2 = L(z) - \sum_{k=1}^{n} m_k \frac{q_1'(z-z_k)}{q_1(z-z_k)}. \tag{7.1.38}$$

The $U(1)$ factors around the punctures are given by

$$g_k = e^{-2pim_k}, \tag{7.1.39}$$

while $g_A, g_B(z)$, are induced as before by the periodicity of theta functions:

$$\begin{aligned} l(z+t) &= \prod_{k=1}^{n} q_1(z-z_k+t)^{m_k} \\ &= e^{-2pi\sum_{k=1}^{n}\left(z-z_k+\frac{t+1}{2}\right)m_k} l(z) := g_B(z)\, l(z), \end{aligned} \tag{7.1.40}$$

$$l(z+1) = \prod_{k=1}^{n} q_1(z-z_k+1)^{m_k} = e^{ipm} l(z) := g_A\, l(z), \tag{7.1.41}$$

where we defined

$$m := \sum_{j=1}^{n} m_j. \tag{7.1.42}$$

Again, we want to find the relation between the isomonodromic tau-function $\mathcal{T}_H$ of the $SL(2)$ elliptic Garnier system [76, 112, 79], and the $GL(2)$ tau-function $\widetilde{\mathcal{T}}_H$ for the system with rank-1 residues obtained from the scalar gauge transformation (7.1.37), defined by

$$2pi\P_t \log \widetilde{\mathcal{T}}_H = \oint_A dz \frac{1}{2} \mathrm{tr}\, \widetilde{L}(z)^2, \qquad \P_{z_k} \log \widetilde{\mathcal{T}}_H = \mathrm{Res}_{\,z_k} \frac{1}{2} \mathrm{tr}\, \widetilde{L}(z)^2. \tag{7.1.43}$$

Proposition 7.3. *The tau-function $\widetilde{\mathcal{T}}_H$ (7.1.43) of the rank-1 system is related to the tau-function $\mathcal{T}_H$ (5.3.2) of the Garnier system (whose Lax matrix is (5.3.3) restricted to $N = 2$, $\Lambda_i = 0$ for $i = 1, \dots, n$) as*

$$\mathcal{T}_H(t) = \widetilde{\mathcal{T}}_H(t) \prod_{k=1}^{n} \left(h(t) e^{\frac{ipt}{12}} \right)^{-2m_k^2} \prod_{l \neq k} \left(\frac{q_1(z_k - z_l)}{h(t) e^{-\frac{ipt}{3}}} \right)^{-m_k m_l}, \tag{7.1.44}$$

where m_k is the local monodromy exponent at the puncture z_k, $k = 1, \dots, n$, and t is the modular parameter of the torus.

Proof. Under the transformation (7.1.37), the z_k-derivative of $\widetilde{\mathcal{T}}_H$ is

$$\P_{z_k} \log \widetilde{\mathcal{T}}_H = \mathrm{Res}_{\,z_k} \frac{1}{2} \mathrm{tr}\, \widetilde{L}^2 = \mathrm{Res}_{\,z_k} \frac{1}{2} \mathrm{tr}\, L^2 + \mathrm{Res}_{\,z_k} \left(l^{-1}(z) \P_z l(z) \right)^2$$
$$= \P_{z_k} \log \mathcal{T}_H + \sum_{j=1}^{n} \mathrm{Res}_{\,z_k} \left[m_j^2 \left(\frac{q_1'(z-z_j)}{q_1(z-z_j)} \right)^2 + \sum_{l \neq k} m_j m_l \frac{q_1'(z-z_j)}{q_1(z-z_j)} \frac{q_1'(z-z_l)}{q_1(z-z_l)} \right]$$
$$= \P_{z_k} \log \mathcal{T}_H + \P_{z_k} \log \left[\prod_{l \neq k} (q_1(z_k - z_l))^{2 m_k m_l} \right]. \tag{7.1.45}$$

In the last line we use that

$$\sum_{j=1}^{n} \sum_{l \neq k} m_j m_l \mathrm{Res}_{\,z_k} \left(\frac{q_1'(z-z_k)}{q_1(z-z_k)} \frac{q_1'(z-z_l)}{q_1(z-z_l)} \right) = \P_{z_k} \log \left(\prod_{l \neq k} q_1(z_k - z_l)^{2 m_k m_l} \right), \tag{7.1.46}$$

and

$$\mathrm{Res}_{\,z_k} \left(\frac{q_1'(z-z_k)}{q_1(z-z_k)} \right)^2 = 0. \tag{7.1.47}$$

We now turn to the computation of the t-derivative:

$$2pi \P_t \log \widetilde{\mathcal{T}}_H = \oint_A dz \frac{1}{2} \mathrm{tr}\, \widetilde{L}^2 = \oint_A dz \frac{1}{2} \mathrm{tr}\, L^2 + \oint_A dz \left(l^{-1}(z) \P_z l(z) \right)^2$$
$$= 2pi \P_t \log \mathcal{T}_H + \sum_{k=1}^{n} \int_0^1 dz \left[m_k^2 \left(\frac{q_1'(z-z_k)}{q_1(z-z_k)} \right)^2 + \sum_{l \neq k} m_k m_l \frac{q_1'(z-z_k)}{q_1(z-z_k)} \frac{q_1'(z-z_l)}{q_1(z-z_l)} \right] \tag{7.1.48}$$

Let us consider the A-cycle integral of the first term in equation (7.1.48).

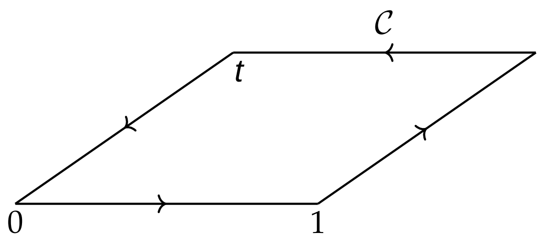


FIGURE 7.5: Contour of integration

The computation goes along the same lines as in the $n = 1$ case (7.1.26), but in the present case we do not shift the contour by ie, since the singularity z_l is now in the interior of the contour $\mathcal{C}$ in figure 7.5:

$$\begin{aligned}2pi\,\mathrm{Res}_{z=z_l}\left(\frac{q_1'(z-z_l)}{q_1(z-z_l)}\right)^3 &= \left[\int_0^1+\int_1^{t+1}+\int_{t+1}^{t}+\int_t^0\right]\left(\frac{q_1'(z-z_l)}{q_1(z-z_l)}\right)^3 dz\\ &= 6pi\int_0^1\left(\frac{q_1'(z-z_l)}{q_1(z-z_l)}\right)^2 dz-3(2pi)^2\int_0^1\frac{q_1'(z-z_l)}{q_1(z-z_l)}dz+(2pi)^3\\ &= 6pi\int_0^1\left(\frac{q_1'(z-z_l)}{q_1(z-z_l)}\right)^2 dz-\frac{1}{2}(2pi)^3,\end{aligned} \tag{7.1.49}$$

while

$$2pi\,\mathrm{Res}_{z=z_l}\left(\frac{q_1'(z-z_l)}{q_1(z-z_l)}\right)^3 = 2pi\frac{q_1'''}{q_1'}\overset{(5.2.89)}{=}3(2pi)^2\P_t\log h(t)^2. \tag{7.1.50}$$

Equating (7.1.49), (7.1.50), we see that the first term of (7.1.48) simply consists of n copies of the 1-point computation (7.1.31):

$$\oint_A dz\sum_{k=1}^n m_k^2\left(\frac{q_1'(z-z_k)}{q_1(z-z_k)}\right)^2 = 2pi\P_t\log\left[\prod_{k=1}^n\left(h(t)e^{\frac{ipt}{6}}\right)^{2m_k^2}\right]. \tag{7.1.51}$$

We then turn to the computation of the second term of (7.1.48):

$$I_{kl} := \oint_A dz\frac{q_1'(z-z_k)}{q_1(z-z_k)}\frac{q_1'(z-z_l)}{q_1(z-z_l)}. \tag{7.1.52}$$

To compute I_{kl}, we consider the following integral over the deformed contour $\mathcal{C}$ in figure 7.5:

$$\begin{aligned}&\left[\int_0^1+\int_1^{1+t}+\int_{1+t}^{t}+\int_t^0\right]dz\frac{q_1'(z-z_k)}{q_1(z-z_k)}\left(\frac{q_1'(z-z_l)}{q_1(z-z_l)}\right)^2\\ &= 2pi\left(\mathrm{Res}_{z=z_k}+\mathrm{Res}_{z=z_l}\right)\frac{q_1'(z-z_k)}{q_1(z-z_k)}\left(\frac{q_1'(z-z_l)}{q_1(z-z_l)}\right)^2.\\ &= 2pi\left[\left(\frac{q_1'(z_k-z_l)}{q_1(z_k-z_l)}\right)^2+\frac{d}{dz_k}\left(\frac{q_1'(z_k-z_l)}{q_1(z_k-z_l)}\right)\right]\overset{(5.2.87)}{=}2pi\frac{q_1''(z_k-z_l)}{q_1(z_k-z_l)}\\ &= (2pi)^2\P_t\log q_1(z_k-z_l)^2.\end{aligned} \tag{7.1.53}$$

The left-hand side of (7.1.53) is

$$\begin{aligned}
&\left[\int_0^1 + \int_1^{1+t} + \int_{1+t}^{t} + \int_t^0\right] dz \frac{q_1'(z-z_k)}{q_1(z-z_k)} \left(\frac{q_1'(z-z_l)}{q_1(z-z_l)}\right)^2 \\
=& \int_0^1 dz \frac{q_1'(z-z_k)}{q_1(z-z_k)} \left(\frac{q_1'(z-z_l)}{q_1(z-z_l)}\right)^2 - \int_0^1 dz \left(\frac{q_1'(z-z_k)}{q_1(z-z_k)} - 2pi\right)\left(\frac{q_1'(z-z_l)}{q_1(z-z_l)} - 2pi\right)^2 \\
&= 4pi I_{kl} - \frac{1}{2}(2pi)^3 + 2pi \int_0^1 dz \left(\frac{q_1'(z-z_l)}{q_1(z-z_l)}\right)^2 \\
&\overset{(7.1.51)}{=} 4pi I_{kl} - \frac{1}{2}(2pi)^3 + (2pi)^2 \P_t \log\left(h(t)^2 e^{\frac{ipt}{3}}\right).
\end{aligned} \tag{7.1.54}$$

Equating (7.1.54) and (7.1.53), we find

$$I_{kl} = 2pi\P_t \log\left(\frac{q_1(z_k-z_l)}{h(t)e^{-\frac{ipt}{3}}}\right). \tag{7.1.55}$$

Therefore, the second term of (7.1.48) reads

$$\begin{aligned}
\sum_{k=1}^n \sum_{l\neq k} m_k m_l \int_0^1 dz \frac{q_1'(z-z_k)q_1'(z-z_l)}{q_1(z-z_k)q_1(z-z_l)} &= 2pi \sum_{k=1}^n \sum_{l\neq k} m_k m_l \P_t \log\left(\frac{q_1(z_k-z_l)}{h(t)e^{-\frac{ipt}{3}}}\right) \\
&= 2pi\P_t \log\left[\prod_{k=1}^n \prod_{l\neq k}\left(\frac{q_1(z_k-z_l)}{h(t)e^{-\frac{ipt}{3}}}\right)^{m_k m_l}\right].
\end{aligned} \tag{7.1.56}$$

Substituting (7.1.51) and (7.1.56) in (7.1.48),

$$2pi\P_t \log\widetilde{\mathcal{T}}_H = 2pi\P_t \log\mathcal{T}_H + 2pi\P_t \log\left[\prod_{k=1}^n \left(h(t)e^{\frac{ipt}{6}}\right)^{2m_k^2} \prod_{l\neq k}\left(\frac{q_1(z_k-z_l)}{h(t)e^{-\frac{ipt}{3}}}\right)^{m_k m_l}\right]. \tag{7.1.57}$$

Combining (7.1.45) and (7.1.57) we find

$$\begin{aligned}
2pi\P_t \log\widetilde{\mathcal{T}}_H + \sum_{k=1}^n \P_{z_k}\log\widetilde{\mathcal{T}}_H &= 2pi\P_t\log\mathcal{T}_H + \sum_{k=1}^n \P_{z_k}\log\mathcal{T}_H \\
2pi\P_t \log&\left[\prod_{k=1}^n \left(h(t)e^{\frac{ipt}{6}}\right)^{2m_k^2} \prod_{l\neq k}\left(\frac{q_1(z_k-z_l)}{h(t)e^{\frac{-ipt}{3}}}\right)^{m_k m_l}\right] \\
&+ \sum_{k=1}^n \P_{z_k}\log\left[\prod_{l\neq k} q_1(z_k-z_l)^{2m_k m_l}\right]
\end{aligned} \tag{7.1.58}$$

Integrating the above equation on both sides and setting the integration constant to 1, we obtain

$$\frac{\widetilde{\mathcal{T}}_H}{\mathcal{T}_H} = \prod_{k=1}^n \left(h(t)e^{\frac{ipt}{6}}\right)^{2m_k^2} \prod_{l\neq k}\left(\frac{q_1(z_k-z_l)}{h(t)e^{-\frac{ipt}{3}}}\right)^{m_k m_l}. \tag{7.1.59}$$

□

Remark 7.3. *Note that (7.1.44) takes the form of the partition function for a Coulomb gas on a torus, with the first term encoding the self-interaction of the particles, while the second term encodes the pairwise interactions.*

Using Proposition 7.3, it is possible to write the tau-function of the elliptic Garnier system as a Fourier series of Nekrasov partition functions.

Theorem 7.2. *The isomonodromic tau-function of the elliptic Garnier system (see (5.3.51) restricted to $N = 2$) admits the following combinatorial expression:*

$$\mathcal{T}_H(t) = \widetilde{\Upsilon}_{1,n}\frac{e^{-2pi\left(\widetilde{r}-\frac{t}{4}\right)}}{q_1\left(Q-\widetilde{r}\right)q_1\left(Q+\widetilde{r}\right)}\prod_{k=1}^{n}\left(h(t)e^{\frac{-ipt}{12}}\right)^{2-2m_k^2}e^{-2piz_km_k^2}\prod_{l\neq k}\left(\frac{q_1(z_k-z_l)}{h(t)e^{-\frac{ipt}{6}}}e^{-ip(z_k-z_l)}\right)^{-m_km_l}$$
$$\times\sum_{\vec{Q}_1,\ldots\vec{Q}_n}\sum_{\vec{Y}_1,\ldots\vec{Y}_n}e^{-2piQ\left(\widetilde{r}-\frac{t}{2}\right)}e^{2pit\left[\frac{1}{2}(\vec{Q}_1+\vec{a}_1)^2+|\vec{Y}_n|\right]}\prod_{j=1}^{n}e^{-2pi(z_j-z_{j-1})\left[\frac{1}{2}(\vec{a}_j+\vec{Q}_j)^2+|\vec{Y}_j|\right]}$$
$$\prod_{k=1}^{n}e^{2pi\vec{Q}_i\cdot\vec{n}_i}\frac{Z_{pert}\left(\vec{a}_k+\vec{Q}_k,\vec{a}_{k+1}+m_k+\vec{Q}_{k+1}\right)}{Z_{pert}\left(\vec{a}_k,\vec{a}_{k+1}+m_k\right)}Z_{inst}\left(\vec{a}_k+\vec{Q}_k,\vec{a}_{k+1}+m_k+\vec{Q}_{k+1}|\vec{Y}_k,\vec{Y}_{k+1}\right),\tag{7.1.60}$$

where $\vec{a}_k = (a_k, -a_k)$, a_k being the $\mathfrak{sl}_2$ local monodromy exponent on the circle $\mathcal{C}_{in}^{[k]}$ in figure 5.4, m_k is the $\mathfrak{sl}_2$ monodromy exponent at the puncture z_k, $Q \equiv Q(t; z_1, ..., z_n)$ is the Calogero-like variable in the Lax matrix (5.3.3) specialized to $N = 2$, t is the modular parameter, Z_{inst}, Z_{pert} are defined in (7.1.14),(7.1.13), $\widetilde{\Upsilon}_{1,n}$ is an integration constant that depends on the monodromy data, $(\vec{Y}, \vec{Q})$ are charged partitions,

$$\widetilde{r} = r - \sum_{j=1}^{n}\Lambda_j\left(z_j - \frac{(t+1)}{2}\right),\tag{7.1.61}$$

and r is an arbitrary parameter.

Proof. The Lax matrix (7.1.38) is the same as (5.3.3) specialised to n-punctures, $N = 2$. The monodromy exponents $\vec{s}_k$, the $U(1)$ shifts Λ_k in (5.3.14), and the parameter $\widetilde{r}$ defined in (5.3.31) read as follows for the present case:

$$\Lambda_j = -m_j \quad \text{for} \quad j = 1...n, \qquad \Lambda_0 = \frac{m}{2},\tag{7.1.62}$$

with m defined in (7.1.42), and

$$\vec{s}_k = \left(a_k - \sum_{j=0}^{k-1}\Lambda_j, -a_k - \sum_{j=0}^{k-1}\Lambda_j\right), \qquad \widetilde{r} = r - \sum_{j=1}^{n}\Lambda_j\left(z_j - \frac{(t+1)}{2}\right).\tag{7.1.63}$$

Theorem 5.2 then implies that the tau-function $\widetilde{\mathcal{T}}_H$ in (7.1.44) can be written in terms of a Fredholm determinant of an operator we call $\widetilde{K}_{1,n}$ which in turn can be written in terms of

Nekrasov functions as in (7.1.12).

$$
\begin{aligned}
\mathcal{T}_H &\overset{(7.1.44)}{=} \left(\prod_{k=1}^{n}\left(h(t)e^{\frac{ipt}{6}}\right)^{-2m_k^2}\prod_{l\neq k}\left(\frac{q_1(z_k-z_l)}{h(t)e^{-\frac{ipt}{3}}}\right)^{-m_k m_l}\right)\widetilde{\mathcal{T}}_H \\
&\overset{(5.3.51)}{=} \left(\prod_{k=1}^{n}\left(h(t)e^{\frac{ipt}{6}}\right)^{-2m_k^2}\prod_{l\neq k}\left(\frac{q_1(z_k-z_l)}{h(t)e^{-\frac{ipt}{3}}}\right)^{-m_k m_l}\right)e^{ipt\,\mathrm{tr}\left(\sigma_1^2+\frac{\mathbb{I}}{6}\right)}e^{-2ip\tilde{r}} \\
&\times\frac{h(t)^2}{q_1(Q-\tilde{r})q_1(Q+\tilde{r})}\left(\prod_{k=1}^{n}e^{-ipz_k\left(\mathrm{tr}\,\sigma_{k+1}^2-\mathrm{tr}\,\sigma_k^2\right)}\right)\det\left[\mathbb{I}-\widetilde{K}_{1,n}\right]\widetilde{\Upsilon}_{1,n}, \\
&\overset{(7.1.62),(7.1.63)}{=} \frac{e^{2pita_1^2}e^{-2pi\left(\tilde{r}-\frac{t}{4}\right)}}{q_1(Q-\tilde{r})q_1(Q+\tilde{r})}\prod_{k=1}^{n}\left(h(t)e^{\frac{-ipt}{12}}\right)^{2-2m_k^2}\prod_{l\neq k}\left(\frac{q_1(z_k-z_l)}{h(t)e^{-\frac{ipt}{6}}}\right)^{-m_k m_l} \\
&\times\prod_{k=1}^{n}e^{-2piz_k\left(a_{k+1}^2-a_k^2+m_k^2+m_k\sum_{j=1}^{k-1}m_j+m_k\right)}\det\left[\mathbb{I}-\widetilde{K}_{1,n}\right]\widetilde{\Upsilon}_{1,n} \\
&\overset{(7.1.12)}{=} \frac{e^{-2pi\left(\tilde{r}-\frac{t}{4}\right)}}{q_1(Q-\tilde{r})q_1(Q+\tilde{r})}\prod_{k=1}^{n}\left(h(t)e^{\frac{-ipt}{12}}\right)^{2-2m_k^2}e^{-2piz_k m_k^2}\prod_{l\neq k}\left(\frac{q_1(z_k-z_l)}{h(t)e^{-\frac{ipt}{6}}}e^{-ip(z_k-z_l)}\right)^{-m_k m_l} \\
&\times\prod_{k=1}^{n}e^{-2piz_k\left(a_{k+1}^2-a_k^2+m_k\right)}\widetilde{\Upsilon}_{1,n}\times\sum_{\vec{Q}_1,\ldots\vec{Q}_n}\sum_{\vec{Y}_1,\ldots\vec{Y}_n}e^{-2piQ\left(\tilde{r}-\frac{t}{2}\right)} \\
&\times e^{2pit\left[\frac{1}{2}(\vec{Q}_1+\vec{a}_1)^2+|\vec{Y}_n|\right]}\prod_{j=1}^{n}e^{-2pi(z_j-z_{j-1})\left[\frac{1}{2}(\vec{a}_j+\vec{Q}_j)^2-\frac{1}{2}\vec{a}_j^2+|\vec{Y}_j|\right]} \\
&\prod_{k=1}^{n}e^{2pi\vec{Q}_i\cdot\vec{n}_i}\frac{Z_{pert}\left(\vec{a}_k+\vec{Q}_k,\vec{a}_{k+1}+m_k+\vec{Q}_{k+1}\right)}{Z_{pert}\left(\vec{a}_k,\vec{a}_{k+1}+m_k\right)}Z_{inst}\left(\vec{a}_k+\vec{Q}_k,\vec{a}_{k+1}+m_k+\vec{Q}_{k+1}|\vec{Y}_k,\vec{Y}_{k+1}\right).
\end{aligned}
\tag{7.1.64}
$$

□

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