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Black Holes in 4D $\mathcal{N} = 4$ Super-Yang-Mills Field Theory

Francesco Benini[1,2,3,*] and Paolo Milan[1,2,†]

[1]*SISSA, via Bonomea 265, 34136 Trieste, Italy*
[2]*INFN, Sezione di Trieste, via Valerio 2, 34127 Trieste, Italy*
[3]*ICTP, Strada Costiera 11, 34151 Trieste, Italy*

(Received 28 November 2019; revised manuscript received 27 January 2020; accepted 6 April 2020; published 18 May 2020)

Black-hole solutions to general relativity carry a thermodynamic entropy, discovered by Bekenstein and Hawking to be proportional to the area of the event horizon, at leading order in the semiclassical expansion. In a theory of quantum gravity, black holes must constitute ensembles of quantum microstates whose large number accounts for the entropy. We study this issue in the context of gravity with a negative cosmological constant. We exploit the most basic example of the holographic description of gravity (AdS/CFT): type IIB string theory on $\mathrm{AdS}_5 \times S^5$, equivalent to maximally supersymmetric Yang-Mills theory in four dimensions. We thus resolve a long-standing question: Does the four-dimensional $\mathcal{N} = 4$ $\mathrm{SU}(N)$ Super-Yang-Mills theory on S^3 at large N contain enough states to account for the entropy of rotating electrically charged supersymmetric black holes in 5D anti–de Sitter space? Our answer is positive. By reconsidering the large N limit of the superconformal index, using the so-called Bethe-ansatz formulation, we find an exponentially large contribution which exactly reproduces the Bekenstein-Hawking entropy of the black holes. Besides, the large N limit exhibits a complicated structure, with many competing exponential contributions and Stokes lines, hinting at new physics. Our method opens the way toward a quantitative study of quantum properties of black holes in anti–de Sitter space.

DOI: 10.1103/PhysRevX.10.021037 Subject Areas: Gravitation, Particles and Fields, String Theory

I. INTRODUCTION AND RESULTS

Black holes are possibly the most simple and featureless classical solutions to Einstein's theory of gravitation, according to no-hair theorems, but simultaneously the most difficult objects to understand conceptually at the quantum level. One of the fascinating aspects of black-hole physics is its connection with the laws of thermodynamics. Of particular importance is the fact that black holes carry a macroscopic entropy [1], semiclassically determined in terms of the area of the event horizon. In the search for a theory of quantum gravity, explaining the microscopic origin of black-hole thermodynamics—namely, the multitude of classically indistinguishable quantum states responsible for it—is a challenging but fundamental test.

String theory was proposed to be a theory that embeds gravity in a consistent quantum system, hence it should, in particular, explain the black-hole entropy in terms of a degeneracy of string states. This was beautifully shown to be the case in the seminal paper [2] by Strominger and Vafa, where the Bekenstein-Hawking entropy of a class of supersymmetric asymptotically flat black holes was microscopically reproduced.

In the case of asymptotically anti–de Sitter (AdS) black holes—namely, black holes in a gravitational theory with negative cosmological constant—the AdS/CFT or gauge-gravity duality [3,4] constitutes a natural and wonderful framework to study their properties at the quantum level. In fact, the duality provides a fully consistent nonperturbative definition of quantum gravity, in terms of a conformal field theory (CFT) living at the boundary of AdS space. The problem of offering a microscopic account of the black-hole entropy is rephrased into that of counting particular states in the dual CFT. However, despite the very favorable setup, the problem remains challenging, for two reasons. First and most importantly, the dual CFT is strongly coupled, and thus performing computations is daunting. Second, the "phenomenologically" interesting regime of weak-curvature gravity corresponds to a large N limit in the CFT, i.e., a limit in which the central charge goes to infinity. Indeed, this problem in four or more dimensions has remained unsolved for many years, and only recently a concrete example was successfully studied in Refs. [5,6]. There, the Bekenstein-Hawking entropy of a class of static dyonic Bogomol'nyi-Prasad-Sommerfield

*fbenini@sissa.it
†pmilan@sissa.it

(BPS) black holes (i.e., black holes that preserve some supersymmetries and are thus extremal and at zero temperature) in AdS_4 was holographically reproduced in the dual CFT_3, exploiting the nonperturbative computational technique known as supersymmetric localization in the CFT [7–9]. Since then, the matching has been extended to many other classes of magnetically charged BPS black holes in various dimensions [10–22], sometimes including the first quantum corrections [23–28].

When moving to rotating, purely electric black holes, the situation becomes more complicated. Famously, the microstate counting for BPS black holes in AdS_5 has remained a long-standing open problem, which dates back to the work of [29,30]. The context is the first discovered, most basic and best studied example of AdS/CFT: the gravitational theory known as type IIB string theory on the spacetime $AdS_5 \times S^5$, whose dual boundary description is in terms of the four-dimensional Super-Yang-Mills (SYM) theory with $\mathcal{N} = 4$ supersymmetries and gauge group $SU(N)$ [3]. In this context, BPS black holes arise as rotating electrically charged solutions of type IIB supergravity on $AdS_5 \times S^5$ [31–35]. Their holographic description is in terms of $1/16$ BPS states (i.e., states that preserve one complex supercharge) of the boundary 4D $\mathcal{N} = 4$ $SU(N)$ SYM theory on S^3. Those states can be counted (with sign) by computing the superconformal index [30,36], i.e., a supersymmetric grand canonical partition function of the theory with an insertion of the fermion parity operator. One would expect the contribution of the black-hole microstates to the index to dominate the large N (i.e., weak-curvature) expansion. However, the large N computation of the index performed in Ref. [30] showed no rapid-enough growth of the number of states, and thus it could not reproduce the entropy of the dual black holes. Additionally, that result was followed by several studies of BPS operators at weak coupling [37] in which no sign of high degeneracy of states was found.

Recently, the issue received renewed attention pointing toward a different conclusion. First, the authors of Ref. [38] were able to analyze the thermodynamics of extremal black holes by studying complexified solutions to the Einstein equations, and in this way equating the Bekenstein-Hawking entropy to the (complexified) regularized on-shell action of the gravitational black-hole solutions. Since, in AdS/CFT, the bulk on-shell action corresponds to the $S^3 \times S^1$ supersymmetric grand canonical partition function of the boundary theory, this classical computation reinforces the expectation that the field theory index should grow with N. Second, the authors of Refs. [39,40] analyzed the index in a Cardy-like limit in which the fugacities are brought to the unit circle (later developments include Refs. [41]). The Cardy limit, which captures states with very large charges, can be followed by a large N limit which allowed those authors to find evidence that for very large BPS black holes, whose size is much larger than the AdS radius, the index does account for the entropy.

In this paper we offer a direct and conclusive resolution of the issue by revisiting the counting of $1/16$ BPS states in the boundary $\mathcal{N} = 4$ SYM theory at large N, at arbitrary values of the charges. We approach the problem by using a new expression for the superconformal index of the theory, derived in Refs. [42,43] and dubbed Bethe-ansatz (BA) formula, which allows for an easier and very precise analysis of the large N limit, at fixed and generic complex values of the fugacities. We find that the superconformal index, i.e., the grand canonical partition function of $1/16$ BPS states, does in fact grow very rapidly with N—as $e^{\mathcal{O}(N^2)}$—for generic complex values of the fugacities. Although the BA formula of [43] can handle the general case, this is technically difficult and in this paper we restrict to states and black holes with two equal angular momenta, as in Ref. [32].

The BA formulation reveals that the large N limit has a rather complicated structure. There are many exponentially large contributions to the superconformal index, that somehow play the role of saddle points. As we vary the complex fugacities, those contributions compete and in different regions of the parameter space, different contributions dominate. This gives rise to Stokes lines, separating different domains of analyticity of the limit. The presence of Stokes lines could also resolve the apparent tension with the computation of Ref. [30], that was performed with real fugacities. We show that when the fugacities are taken to be real, all exponentially large contributions organize into competing pairs that can conceivably cancel against each other.

Our main result is to identify a particular exponential contribution, such that extracting from it the microcanonical degeneracy of states *exactly* reproduces the Bekenstein-Hawking entropy of BPS black holes in AdS_5 (whose Legendre transform was obtained in Ref. [44]). Along the way, we show that the very same $\mathcal{I}$-extremization principle [5,6] found in AdS_4, is at work in AdS_5 as well. The $\mathcal{I}$-extremization principle is a mechanism which guarantees that, and explains why, the index captures the total number of single-center BPS black-hole states (rather than the mere net number of bosonic minus fermionic states) at leading order in N.

At the same time, we step into many other exponentially large contributions: We expect them to describe very interesting new physics, that we urge to uncover. To that purpose, we study in greater detail the case of BPS black holes with equal charges and angular momenta [31]. We find that while for large black holes their entropy dominates the superconformal index, this is not so for smaller black holes. This seems to suggest [45] that an instability and consequently a phase transition, possibly toward hairy or multicenter black holes, might develop as the charges are decreased. Similar observations were made in Ref. [40]. It would be extremely interesting if there were some connections with the recent works [46], and we leave this issue for future investigations.

The paper is organized as follows. In Sec. II we review the charges and entropy of BPS black holes in AdS_5. In Sec. III we present the BA formula for the superconformal index of $\mathcal{N}=4$ SYM, and in Sec. IV we compute its large N limit. Sections V and VI are devoted to extracting the black-hole entropy from the index.

II. BPS BLACK HOLES IN AdS_5

In this paper we study the entropy of rotating charged BPS black holes in AdS_5 [31–35] that can be embedded in type IIB string theory on $\mathrm{AdS}_5\times S^5$ [47]. In order to set the stage, let us briefly review such gravitational solutions. The black holes are solutions to the equations of motion of type IIB supergravity that preserve one complex supercharge [48], thus being $1/16$ BPS. The metric interpolates between the AdS_5 boundary and a fibration of AdS_2 on S^3 at the horizon. Moreover, the black holes carry three charges $Q_{1,2,3}$ for $\mathrm{U}(1)^3\subset \mathrm{SO}(6)$ acting on S^5, that appear as electric charges in AdS_5, and two angular momenta $J_{1,2}$ associated to the Cartan $\mathrm{U}(1)^2\subset\mathrm{SO}(4)$ (each Cartan generator acts on an $\mathbb{R}^2$ plane inside $\mathbb{R}^4$). Their mass is fixed by the linear BPS constraint

$$M = g(|J_1|+|J_2|+|Q_1|+|Q_2|+|Q_3|), \tag{1}$$

where $g=\ell_5^{-1}$ is the gauge coupling, determined in terms of the curvature radius ℓ_5 of AdS_5 (whereas charges are dimensionless). It turns out that regular BPS black holes with no closed timelike curves only exist when the five charges satisfy certain nonlinear constraints. The first constraint relies on the fact that one parametrizes the solutions by four real parameters $\mu_{1,2,3}$, Ξ [35,49]. The second constraint is

$$g^2\mu_{1,2,3} > \Xi - 1 \geq 0. \tag{2}$$

Alternatively, one can have the same constraint with Ξ substituted by Ξ^{-1} which corresponds to exchanging $J_1\leftrightarrow J_2$. The third constraint is

$$S_{\mathrm{BH}}\in\mathbb{R}, \tag{3}$$

where the Bekenstein-Hawking entropy S_{BH} is defined in Eq. (9) below.

Charges and angular momenta of the black holes are completely determined by these four parameters μ_I, Ξ with $I=1,2,3$. Defining

$$\gamma_1=\sum_I \mu_I,\qquad \gamma_2=\sum_{I<J}\mu_I\mu_J,\qquad \gamma_3=\mu_1\mu_2\mu_3, \tag{4}$$

the electric charges and angular momenta are

$$\begin{aligned} Q_I &= \frac{\pi}{4G_N}\left[\frac{\mu_I}{g}+\frac{g}{2}\left(\gamma_2-\frac{2\gamma_3}{\mu_I}\right)\right],\\ J_{1,2} &= \frac{\pi}{4G_N}\left[\frac{g\gamma_2}{2}+g^3\gamma_3+\frac{\mathcal{J}}{g^3}\left(\Xi^{\pm1}-1\right)\right], \end{aligned} \tag{5}$$

where G_N is the five-dimensional Newton constant and

$$\mathcal{J}=\prod_I(1+g^2\mu_I). \tag{6}$$

It is easy to see that one of the charges Q_I can be zero or negative [50]. There are some combinations, though, that we can bound above zero, for instance

$$\begin{aligned} Q_1+Q_2+Q_3 &= \frac{\pi}{4G_N}\left[\frac{\gamma_1}{g}+\frac{g\gamma_2}{2}\right]>0,\\ Q_I+Q_K &= \frac{\pi}{4G_N}\left[\frac{\mu_I+\mu_K}{g}+g\mu_I\mu_K\right]>0 \quad \text{for } I\neq K. \end{aligned} \tag{7}$$

In particular, at most one charge can be zero or negative. Setting $g=1$ for the sake of clarity, we also have

$$Q_I+J_{1,2}=\frac{\pi}{4G_N}(1+\mu_K)(1+\mu_L)[(1+\mu_I)\Xi^{\pm1}-1]>0 \tag{8}$$

for $I\neq K\neq L\neq I$. The inequality follows from Eq. (2).

The Bekenstein-Hawking entropy is proportional to the horizon area, and can be written as a function of the black-hole charges [51]:

$$S_{\mathrm{BH}}=\frac{\text{area}}{4G_N}=2\pi\sqrt{\sum_{I<J}Q_IQ_J-\frac{\pi}{4G_Ng^3}(J_1+J_2)}. \tag{9}$$

The constraint equation (3) requires the quantity inside the radical to be positive. The BPS solutions have a regular well-defined event horizon only if the angular momenta are nonzero: In other words, there is no static limit in gauged supergravity.

In this paper we will focus on the "self-dual" case $J_1=J_2\equiv J$ [32]. Since, in general, $\mathcal{J}>1$ and $\Xi\geq1$, necessarily $\Xi=1$. The constraint Eq. (2) simply becomes

$$\mu_I>0. \tag{10}$$

The charges are

$$\begin{aligned} Q_I &= \frac{\pi}{4G_N}\left[\frac{\mu_I}{g}+\frac{g}{2}\left(\gamma_2-\frac{2\gamma_3}{\mu_I}\right)\right],\\ J &= \frac{\pi}{4G_N}\left[\frac{g\gamma_2}{2}+g^3\gamma_3\right]>0. \end{aligned} \tag{11}$$

The entropy is

$$S_{\mathrm{BH}} = \frac{2\pi^2}{4G_N}\sqrt{(1+g^2\gamma_1)\gamma_3 - \frac{g^2\gamma_2^2}{4}} = 2\pi\sqrt{\sum_{I<J} Q_I Q_J - \frac{\pi}{4G_N g^3} 2J}. \tag{12}$$

Once again, the constraint Eq. (3) requires the quantity inside the radical to be positive [52].

III. THE DUAL FIELD THEORY AND ITS INDEX

A nonperturbative definition of type IIB string theory on $\mathrm{AdS}_5 \times S^5$ is in terms of its boundary dual: 4D $\mathcal{N}=4$ SYM theory with $\mathrm{SU}(N)$ gauge group [3], where

$$N^2 = \frac{\pi \ell_5{}^3}{2G_N} = \frac{\pi}{2G_N g^3}. \tag{13}$$

The weak-curvature limit in gravity corresponds to the large N limit in field theory. Up to the choice of gauge group, SYM is the unique four-dimensional Lagrangian CFT with maximal supersymmetry. The field content, in $\mathcal{N}=1$ notation, consists of a vector multiplet and three chiral multiplets X, Y, Z, all in the adjoint representation of the gauge group. Besides, there is a cubic superpotential $W = \mathrm{Tr}\, X[Y,Z]$. The R-symmetry is $\mathrm{SO}(6)_R$: Going to the Cartan $\mathrm{U}(1)^3$, we choose a basis of generators $R_{1,2,3}$ each giving R-charge 2 to a single chiral multiplet and zero to the other two, in a symmetric way.

Considering the theory in radial quantization on $\mathbb{R}\times S^3$, we are interested in the states that can be dual to the BPS black holes described in Sec. II. These are 1/16 BPS states preserving one complex supercharge $\mathcal{Q}$, and characterized by two angular momenta $J_{1,2}$ on S^3 and three R-charges for $\mathrm{U}(1)^3 \subset \mathrm{SO}(6)_R$. The angular momenta $J_{1,2}$ are semi-integer and each rotates an $\mathbb{R}^2 \subset \mathbb{R}^4$. Indicating with $J_\pm$ the spins under $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \cong \mathrm{SO}(4)$, we set $J_{1,2} = J_+ \pm J_-$. With respect to the $\mathcal{N}=1$ superconformal subalgebra that contains $\mathcal{Q}$, we describe the R-charges in terms of two flavor generators $q_{1,2} = \frac{1}{2}(R_{1,2} - R_3)$ commuting with $\mathcal{Q}$, and the R-charge $r = \frac{1}{3}(R_1+R_2+R_3)$. All fields in the theory have integer charges under $q_{1,2}$. The counting of BPS states is performed by the superconformal index [30,36] defined by the trace

$$\mathcal{I}(p,q,v_1,v_2) = \mathrm{Tr}\,(-1)^F e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} p^{J_1+r/2} q^{J_2+r/2} v_1^{q_1} v_2^{q_2}. \tag{14}$$

Here, p, q, v_a with $a=1, 2$ are complex fugacities associated with the various quantum numbers, while the corresponding chemical potentials τ, σ, ξ_a are defined by

$$p = e^{2\pi i\tau}, \qquad q = e^{2\pi i\sigma}, \qquad v_a = e^{2\pi i\xi_a}. \tag{15}$$

The fermion number is defined as $F = 2(J_+ + J_-) = 2J_1$. The index is well defined for

$$|p|, |q| < 1 \;\Leftrightarrow\; \mathrm{Im}\,\tau, \mathrm{Im}\,\sigma > 0. \tag{16}$$

By standard arguments [53], $\mathcal{I}$ only counts states annihilated by $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ and is thus independent of β.

It will be convenient to redefine the flavor chemical potentials as

$$\Delta_a = \xi_a + \frac{\tau+\sigma}{3} \tag{17}$$

and use

$$y_a = e^{2\pi i\Delta_a}. \tag{18}$$

The index becomes [54]

$$\mathcal{I}(p,q,y_1,y_2) = \mathrm{Tr}\,(-1)^F e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} p^{J_1+R_3/2} q^{J_2+R_3/2} y_1^{q_1} y_2^{q_2}. \tag{19}$$

Notice that J_1, J_2, $\frac{1}{2}F$, $\frac{1}{2}R_3$ are all semi-integer and correlated according to

$$J_1 = J_2 = \frac{F}{2} = \frac{R_3}{2} \pmod 1. \tag{20}$$

It is then manifest from Eq. (19) that the index is a single-valued function of the fugacities.

The index (14) admits an exact integral representation [30,36,55]. In order to evaluate its large N limit, though, we find more convenient to recast it in a different form, called Bethe-ansatz formula [42,43] (see also [56] for a 3D analog, and [57–61] for similar Higgs branch localization formulas). Computing the large N limit with this formula is still challenging, and in this paper we will restrict ourselves to the case of equal fugacities for the two angular momenta

$$\tau = \sigma \;\Rightarrow\; p = q. \tag{21}$$

Hence, let us describe the Bethe-ansatz formula with this restriction [42,62], in the case of $\mathcal{N}=4$ $\mathrm{SU}(N)$ SYM. The superconformal index reads

$$\mathcal{I}(q,y_1,y_2) = \kappa_N \sum_{\hat{u}\in\mathrm{BAEs}} \mathcal{Z}(\hat{u};\Delta,\tau) H(\hat{u};\Delta,\tau)^{-1}. \tag{22}$$

This is a finite sum over the solution set $\{\hat{u}\}$ to a system of transcendental equations, dubbed Bethe-ansatz equations (BAEs), given by

$$1 = Q_i(u;\Delta,\tau) \equiv e^{2\pi i(\lambda+3\sum_j u_{ij})} \times \prod_{j=1}^{N} \frac{\theta_0(u_{ji}+\Delta_1;\tau)\theta_0(u_{ji}+\Delta_2;\tau)\theta_0(u_{ji}-\Delta_1-\Delta_2;\tau)}{\theta_0(u_{ij}+\Delta_1;\tau)\theta_0(u_{ij}+\Delta_2;\tau)\theta_0(u_{ij}-\Delta_1-\Delta_2;\tau)} \tag{23}$$

for $i = 1,\ldots,N$ and where $u_{ij} = u_i - u_j$. We call Q_i the BA operators. The unknowns are the "complexified SU(N) holonomies" u_i subject to the identifications

$$u_i \sim u_i + 1 \sim u_i + \tau, \tag{24}$$

meaning that each one lives on a torus of modular parameter τ, and constrained by

$$\sum_{i=1}^{N} u_i = 0 \quad (\mathrm{mod}\ \mathbb{Z}+\tau\mathbb{Z}), \tag{25}$$

as well as a "Lagrange multiplier" λ. The function θ_0 is defined as

$$\theta_0(u;\tau) = \prod_{k=0}^{\infty}(1-zq^k)(1-z^{-1}q^{k+1}) = (z;q)_\infty (q/z;q)_\infty \tag{26}$$

with $z = e^{2\pi i u}$ and $q = e^{2\pi i\tau}$, where $(z;q)_\infty$ is the so-called q-Pochhammer symbol. The prefactor in Eq. (22) is

$$\kappa_N = \frac{1}{N!}\left(\frac{(q;q)_\infty^2 \tilde{\Gamma}(\Delta_1;\tau,\tau)\tilde{\Gamma}(\Delta_2;\tau,\tau)}{\tilde{\Gamma}(\Delta_1+\Delta_2;\tau,\tau)}\right)^{N-1}, \tag{27}$$

defined in terms of the elliptic gamma function [63]

$$\tilde{\Gamma}(u;\tau,\sigma) = \Gamma(z=e^{2\pi i u}; p=e^{2\pi i\tau}, q=e^{2\pi i\sigma}) = \prod_{m,n=0}^{\infty} \frac{1-p^{m+1}q^{n+1}/z}{1-p^m q^n z}. \tag{28}$$

The function $\mathcal{Z}$ is

$$\mathcal{Z}(u;\Delta,\tau) = \prod_{i\neq j}^{N} \frac{\tilde{\Gamma}(u_{ij}+\Delta_1;\tau,\tau)\tilde{\Gamma}(u_{ij}+\Delta_2;\tau,\tau)}{\tilde{\Gamma}(u_{ij}+\Delta_1+\Delta_2;\tau,\tau)\tilde{\Gamma}(u_{ij};\tau,\tau)}. \tag{29}$$

Finally, the Jacobian H is

$$H\Big|_{\mathrm{BAEs}} = \det\left[\frac{1}{2\pi i}\frac{\partial(Q_1,\ldots,Q_N)}{\partial(u_1,\ldots,u_{N-1},\lambda)}\right] \tag{30}$$

when evaluated on the solutions to the BAEs. Notice that both Q_i, κ_N, $\mathcal{Z}$, and H are invariant under integer shifts of τ, Δ_1, and Δ_2, implying that the superconformal index, Eq. (22), is a single-valued function of the fugacities.

Let us add some comments on how Eqs. (23) and (30) are obtained from the general formalism in [43]. The maximal torus of SU(N) is given by the matrices $\mathrm{diag}(z_1,\ldots,z_{N-1},z_N)$ with $\prod_{j=1}^N z_j = 1$ and, setting $z_j = e^{2\pi i u_j}$, is parametrized by $u_1,\ldots,u_{N-1}$. For general gauge group G, the BA operators Q_i have an index i that runs over the Cartan subalgebra of G. Let us denote the BA operators of SU(N) as $\hat{Q}_1,\ldots,\hat{Q}_{N-1}$, then the BAEs are $\hat{Q}_j = 1$. The BA operators of SU(N) can be written as $\hat{Q}_j = Q_j/Q_N$ in terms of the BA operators $Q_1,\ldots,Q_N$ of U(N). Introducing a "Lagrange multiplier" λ, we can set $Q_N = e^{-2\pi i\lambda}$ and write the BAEs as $e^{2\pi i\lambda}Q_j = 1$ for $j = 1,\ldots,N$ (this includes the definition of λ). Absorbing $e^{2\pi i\lambda}$ into Q_i, we end up with Eq. (23).

The Jacobian H for SU(N) is given by

$$H = \det\left[\frac{1}{2\pi i}\frac{\partial\hat{Q}_i}{\partial u_j}\right]_{i,j=1,\ldots,N-1}. \tag{31}$$

When evaluated on the solutions to the BAEs, we have

$$H\Big|_{\mathrm{BAEs}} = \det\left[\frac{1}{2\pi i}\frac{\partial(Q_i-Q_N)}{\partial u_j}\right]_{i,j=1,\ldots,N-1} = \text{Eq.}(30). \tag{32}$$

To see the last equality, one should notice that $\partial Q_i/\partial\lambda|_{\mathrm{BAEs}} = 2\pi i$.

The chemical potentials u_j are defined modulo 1, and the SU(N) condition implies that they should satisfy $\sum_j u_j \in \mathbb{Z}$. However, it is easy to check that the BAEs, Eq. (23), are invariant under shifts of one of the u_j's by the periods of a complex torus of modular parameter τ, namely $u_k \to u_k + n + m\tau$ for a fixed k. Hence, the BAEs are well defined on $N-1$ copies of the torus. Consistently, both H and $\mathcal{Z}$—when evaluated on the solutions to the BAEs—are invariant under shifts of u_j by the periods of the torus (see Ref. [43] for the general proof).

As one could suspect at this point, the BAEs, Eq. (23), are also invariant under modular transformations of the torus. To see that, it might be convenient to rewrite them in terms of the function $\theta(u;\tau) = e^{-\pi i u + \pi i\tau/6}\theta_0(u;\tau)$ that has simpler modular properties. (See Supplemental Material, Sec. A, at Ref. [64] for a summary of useful properties of special functions.) When doing that, the term $\sum_j u_{ij}$ in the exponential in Eq. (23) disappears. One easily shows that Q_i are invariant under

$$T:\begin{cases}\tau\mapsto\tau+1\\ u\mapsto u,\end{cases}\qquad S:\begin{cases}\tau\mapsto -\dfrac{1}{\tau}\\ u\mapsto \dfrac{u}{\tau},\end{cases}\qquad C:\begin{cases}\tau\mapsto\tau\\ u\mapsto -u,\end{cases} \tag{33}$$

thus showing invariance under the full group SL($2,\mathbb{Z}$). On the other hand, the summand $\kappa_N\mathcal{Z}H^{-1}$ in Eq. (22) is not invariant under modular transformations of τ: This is not a symmetry of the superconformal index.

A. Exact solutions to the BAEs

When evaluating the BA formula (22), the hardest task is to solve the BAEs, Eq. (23). The very same equations appear in the $T^2 \times S^2$ topologically twisted index [7], and one exact solution was found in Refs. [11,65]:

$$u_{ij} = \frac{\tau}{N}(j-i), \quad u_j = \frac{\tau(N-j)}{N} + \bar{u}, \quad \lambda = \frac{N-1}{2}. \quad (34)$$

Here, $\bar{u}$ is a suitable constant that solves the $\mathrm{SU}(N)$ constraint, Eq. (25); since all expressions depend solely on u_{ij}, we will not specify that constant. Notice that the solution does not depend on the chemical potentials Δ_a. To prove that it is a solution, we compute

$$\begin{aligned}\prod_{j=1}^{N}\frac{\theta_0(u_{ji}+\Delta)}{\theta_0(u_{ij}+\Delta)} &= \frac{\prod_{k=0}^{i-1}\theta_0(\frac{\tau}{N}k+\Delta)\times\prod_{k=i-N}^{-1}\theta_0(\frac{\tau}{N}k+\Delta)}{\prod_{k=0}^{N-i}\theta_0(\frac{\tau}{N}k+\Delta)\times\prod_{k=1-i}^{-1}\theta_0(\frac{\tau}{N}k+\Delta)} \\ &= \frac{\prod_{k=0}^{N-1}\theta_0(\frac{\tau}{N}k+\Delta)\times\prod_{k=i-N}^{-1}(-q^{k/N}y)}{\prod_{k=1}^{N-1}\theta_0(\frac{\tau}{N}k+\Delta)\times\prod_{k=1-i}^{-1}(-q^{k/N}y)} \\ &= (-1)^{N-1}y^{N-2i+1}q^{i-(N+1)/2}. \end{aligned} \quad (35)$$

Taking the product over $\Delta = \{\Delta_1, \Delta_2, -\Delta_1-\Delta_2\}$ we precisely reproduce the inverse of the prefactor of Eq. (23), for every i. Furthermore, notice that the shift $\bar{u} \to \bar{u} + 1/N$ generates a new inequivalent solution that solves the $\mathrm{SU}(N)$ constraint. Repeating the shift N times, because of the torus periodicities, we go back to the original solution. Therefore, Eq. (34) actually represents N inequivalent solutions.

Because the BAEs are modular invariant, we could transform τ to $\tau' = (a\tau+b)/(c\tau+d)$, then write the solution $u'_{ij} = \tau'(j-i)/N$, and finally go back to $\tau = (d\tau'-b)/(a-c\tau')$. This gives, for any $a, b \in \mathbb{Z}$ with $\gcd(a,b) = 1$, an $\mathrm{SL}(2,\mathbb{Z})$-transformed solution

$$u_{ij} = \frac{a\tau+b}{N}(j-i). \quad (36)$$

However, one should only keep the solutions that are not equivalent—either because of periodicities on the torus or because of Weyl group transformations.

On the other hand, a larger class of inequivalent solutions has been found in Ref. [65] (we do not know if this is the full set or other solutions exist). For given N, every choice of three nonnegative integers $\{m,n,r\}$ that decompose $N = m\,n$ and with $0 \le r < n$ leads to an exact solution

$$u_{\hat{j}\hat{k}} = \frac{\hat{j}}{m} + \frac{\hat{k}}{n}\left(\tau + \frac{r}{m}\right) + \bar{u}, \quad (37)$$

where $\hat{j} = 0,\ldots,m-1$ and $\hat{k} = 0,\ldots,n-1$ are an alternative parametrization of the index $j = 0,\ldots,N-1$. As we show below, the first class is contained into the second class.

The solutions (37) organize into orbits of $\mathrm{PSL}(2,\mathbb{Z})$ with the following action:

$$\begin{aligned} T&: \{m,n,r\} \mapsto \{m,n,r+m\} \\ S&: \{m,n,r\} \mapsto \left\{p, \frac{mn}{p}, -m(r/p)^{-1}_{\mathrm{mod}\,n/p}\right\}, \end{aligned} \quad (38)$$

where $p = \gcd(n,r)$, we indicated as $(x)^{-1}_{\mathrm{mod}\,y}$ the modular inverse of x mod y (which exists whenever x, y are coprime), and the last entry of $\{m,n,r\}$ is understood mod n. One can check that $S^2 = (TS)^3 = \mathbb{1}$. If $\{m,n,r\}$ have a common divisor, then one can see that also the images $\{m',n',r'\}$ under T, S have that common divisor, and since T, S are invertible, it follows that $\gcd(m,n,r) \equiv d$ is an invariant along $\mathrm{PSL}(2,\mathbb{Z})$ orbits.

We can prove that if $\{m,n,r\}$ have $\gcd(m,n,r) = 1$, then they are in the orbit of $\{1,mn,0\}$, i.e., there exists a $\mathrm{PSL}(2,\mathbb{Z})$ transformation that maps them to $\{1,mn,0\}$. Indeed, let $\tilde{r} = \gcd(m,r)$. We can perform a number of T transformations to reach $\{m,n,\tilde{r}\}$. Necessarily $\gcd(n,\tilde{r}) = 1$, therefore an S transformation gives $\{1,mn,-m(\tilde{r})^{-1}_{\mathrm{mod}\,n}\}$. Now a number of T transformations gives $\{1,mn,0\}$. On the other hand, we observe that if $\gcd(m,n,r) = d > 1$, then the orbit under $\mathrm{PSL}(2,\mathbb{Z})$ is in one-to-one correspondence with the one of $\{m/d,n/d,r/d\}$, which is generated by $\{1,mn/d^2,0\}$. This shows that the number of orbits is equal to the number of divisors d^2 of N which are also squares. Each orbit is generated by $\{d,N/d,0\}$, and is in one-to-one correspondence with the orbit generated by $\{1,N/d^2,0\}$, which we can regard as the "canonical form."

At this point we recognize that the set of inequivalent solutions in the first class (36) is precisely the $\mathrm{PSL}(2,\mathbb{Z})$ orbit with $\gcd(m,n,r) = 1$ in the second class (37). Indeed, start with a solution of type (36) for some N and some coprime integers a, b. Let $m = \gcd(a,N)$ and $n = N/m$. We can write the solution as

$$u_j = -\frac{(a/m)j}{n}\tau - \frac{bj}{N} + \bar{u} \quad (\mathrm{mod}\ \mathbb{Z}+\tau\mathbb{Z}). \quad (39)$$

We can identify $\hat{k} = (a/m)j$ mod n. Since (a/m) and n are coprime, as j runs from 0 to $n-1$, $\hat{k}$ takes all values in the same range once. Moreover, there exists $s = (a/m)^{-1}$ mod n, such that $j = s\hat{k}$ mod n. In other words, (a/m) is invertible mod n and its inverse s is coprime with n. We can write

$$j = s\hat{k} + n\hat{j} \quad (40)$$

and as j runs from 0 to $N-1$, $\hat{\jmath}$ covers a range of length m. Substituting the expression for j we obtain

$$u_j = -\frac{b}{m}\hat{\jmath} - \frac{\hat{k}}{n}\left(\tau + \frac{bs}{m}\right) + \bar{u} \quad (\mathrm{mod}\ \mathbb{Z}+\tau\mathbb{Z}). \tag{41}$$

Notice that $\gcd(b,m)=1$. Indeed, suppose that b and m have a common factor, then this must also be a factor of a, which is a contradiction. Therefore, we have the equality of sets $\{b\hat{\jmath} \bmod m\} = \{\hat{\jmath} \bmod m\}$. Finally, we set $r = bs \bmod n$ and we reproduce the expression in Eq. (37). The values $\{m,n,r\}$ obtained this way have $\gcd(m,n,r)=1$. Indeed, suppose they have a common factor, then this must also be a factor of a but not of (a/m), and thus it must also be a factor of b, which is a contradiction.

On the contrary, start with a solution $\{m,n,r\}$ of type (37) with $\gcd(m,n,r)=1$. It is easy to see, by repeating the procedure, that it is equivalent to a solution of type (36) with $a=m$ and $b=r$ (which imply $s=1$).

IV. THE LARGE N LIMIT

In this section we take the large N limit of the BA formula (22) for the superconformal index. The first part of the section is technical, and the uninterested reader could directly jump to Sec. IV C where the final result is presented.

In the related context of the $T^2\times S^2$ topologically twisted index [7,66], it was shown in Ref. [11] that the basic solution (34) leads to the dominant contribution in the high temperature limit. Assuming that such a solution gives an important contribution in our setup as well, we will start evaluating its large N limit. We will find that it scales as $e^{\mathcal{O}(N^2)}$, therefore in the following we will systematically neglect any factor whose logarithm is subleading with respect to $\mathcal{O}(N^2)$. We will also find that the solution (34) is not necessarily dominant in our setup, rather other solutions can compete, and we will thus have to include the contributions of some of the solutions (36).

First of all, consider the prefactor κ_N in Eq. (22) and the multiplicity of the BA solutions, whose contribution does not depend on the particular solution. Each BA solution (37) has multiplicity $N(N!)$, where the first factor comes from shifts of $\bar{u}$ while the second factor from the Weyl group action. Thus, from Eq. (27) and applying Stirling's approximation, we find

$$N(N!)\,\kappa_N = e^{\mathcal{O}(N)}. \tag{42}$$

This contribution can be neglected at leading order.

A. Contribution of the basic solution

Here, we consider only the contribution of the basic solution (34) to the sum in Eq. (22).

1. *The Jacobian.*—We use the expression in Eq. (30). The derivative of Q_i with respect to u_j can be computed and it gives

$$\frac{\partial \log Q_i(u;\Delta,\tau)}{\partial u_j} = \sum_{k=1}^{N} \partial_{u_j} u_{ik}\left(6\pi i + \sum_{\Delta}\frac{\mathcal{G}'(u_{ik};\Delta,\tau)}{\mathcal{G}(u_{ik};\Delta,\tau)}\right), \tag{43}$$

where the second sum is over $\Delta\in\{\Delta_1,\Delta_2,-\Delta_1-\Delta_2\}$, we defined the function

$$\mathcal{G}(u;\Delta,\tau) = \frac{\theta_0(-u+\Delta;\tau)}{\theta_0(u+\Delta;\tau)}, \tag{44}$$

and $\partial_{u_j}u_{ik} = \delta_{ij}-\delta_{kj}-\delta_{iN}+\delta_{kN}$. This relation holds because we take $u_1,\ldots,u_{N-1}$ as the independent variables, and fix u_N using Eq. (25). Substituting, we get

$$\frac{\partial \log Q_i}{\partial u_j} = (\delta_{ij}-\delta_{iN})\left(6\pi i N + \sum_{k=1}^{N}\sum_{\Delta}\frac{\mathcal{G}'(u_{ik};\Delta,\tau)}{\mathcal{G}(u_{ik};\Delta,\tau)}\right) + \sum_{\Delta}\left(\frac{\mathcal{G}'(u_{iN};\Delta,\tau)}{\mathcal{G}(u_{iN};\Delta,\tau)} - \frac{\mathcal{G}'(u_{ij};\Delta,\tau)}{\mathcal{G}(u_{ij};\Delta,\tau)}\right). \tag{45}$$

When we evaluate this expression on $u_{ij}=\tau(j-i)/N$, we notice that—for generic values of Δ_a—the terms in the second line are of order $\mathcal{O}(1)$. Indeed, the distribution of points u_{ij} generically does not hit any zeros or poles of $\mathcal{G}$. Retaining only the terms in the first line, the Jacobian reads

$$H = \det\begin{pmatrix} A_1 & \mathcal{O}(1) & \cdots & \mathcal{O}(1) & 1 \\ \mathcal{O}(1) & A_2 & & \vdots & 1 \\ \vdots & & \ddots & \vdots & \vdots \\ \mathcal{O}(1) & \cdots & \cdots & A_{N-1} & 1 \\ -A_N & -A_N & \cdots & -A_N & 1 \end{pmatrix} \tag{46}$$

where the diagonal entries are

$$A_i = 3N + \frac{1}{2\pi i}\sum_{k=1}^{N}\sum_{\Delta}\frac{\mathcal{G}'(u_{ik};\Delta,\tau)}{\mathcal{G}(u_{ik};\Delta,\tau)}. \tag{47}$$

Let us estimate the behavior of A_i with N. By the same argument as above, A_i contains the sum of N elements of order $\mathcal{O}(1)$ and thus it scales like $\mathcal{O}(N)$ (or smaller). The determinant can be computed at leading order and it gives

$$H = \sum_{k=1}^{N}\prod_{j(\neq k)=1}^{N} A_j + \text{subleading}. \tag{48}$$

This scales as $\mathcal{O}(N^N)$, therefore $\log H = \mathcal{O}(N\log N)$ and can be neglected.

2. *Functions* $\tilde{\Gamma}$.—The dominant contribution comes from the function $\mathcal{Z}$ defined in Eq. (29). To evaluate it, let us analyze $\sum_{i\neq j}^{N}\log\tilde{\Gamma}(u_{ij}+\Delta;\tau,\tau)$ with $\Delta\in\{\Delta_1,\Delta_2,\Delta_1+\Delta_2\}$ separately. We make use of a relation proven in [63], and write

$$\tilde{\Gamma}(u_{ij}+\Delta;\tau,\tau) = \frac{e^{-\pi i \mathcal{Q}(u_{ij}+\Delta;\tau,\tau)}}{\theta_0\left(\frac{u_{ij}+\Delta}{\tau};-\frac{1}{\tau}\right)} \prod_{k=0}^{\infty} \frac{\psi\left(\frac{k+1+u_{ij}+\Delta}{\tau}\right)}{\psi\left(\frac{k-u_{ij}-\Delta}{\tau}\right)}. \quad (49)$$

Here,

$$\begin{aligned}\mathcal{Q}(u;\tau,\sigma) = \frac{u^3}{3\tau\sigma} - \frac{\tau+\sigma-1}{2\tau\sigma}u^2 \\ + \frac{(\tau+\sigma)^2+\tau\sigma-3(\tau+\sigma)+1}{6\tau\sigma}u \\ + \frac{(\tau+\sigma-1)(\tau+\sigma-\tau\sigma)}{12\tau\sigma}\end{aligned} \quad (50)$$

is a cubic polynomial in u. The function ψ is defined as

$$\psi(t) = \exp\left[t\log(1-e^{-2\pi i t}) - \frac{1}{2\pi i}\mathrm{Li}_2(e^{-2\pi i t})\right]. \quad (51)$$

The branch of the logarithm is determined by the series expansion $\log(1-z) = -\sum_{k=1}^{\infty} z^k/k$, whereas we have $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} z^k/k^2$. The branch cut discontinuities cancel out so that $\psi(t)$ is a meromorphic function on the complex plane. It holds $\psi(t)\psi(-t) = e^{-\pi i (t^2-1/6)}$.

To make progress, we perform a series expansion of $\log\theta_0$ and $\log\psi$, evaluate this expansion on the basic solution for u_{ij} in Eq. (34), and perform the sum $\sum_{i\neq j}^N$. We define the modular transformed variables

$$\tilde{z} = e^{2\pi i u/\tau}, \qquad \tilde{y} = e^{2\pi i y/\tau}, \qquad \tilde{q} = e^{-2\pi i/\tau}. \quad (52)$$

We have

$$\begin{aligned}&\sum_{i\neq j}^{N} \log\theta_0\left(\frac{u_{ij}+\Delta}{\tau};-\frac{1}{\tau}\right) \\ &= \sum_{n=0}^{\infty}\sum_{i\neq j}^{N} \log\left[\left(1-\frac{\tilde{z}_i}{\tilde{z}_j}\tilde{y}\tilde{q}^n\right)\left(1-\frac{\tilde{z}_j}{\tilde{z}_i}\tilde{y}^{-1}\tilde{q}^{n+1}\right)\right] \\ &= -\sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty}\sum_{i\neq j}^{N}\frac{1}{\ell}\left[\left(\frac{\tilde{z}_i}{\tilde{z}_j}\tilde{y}\tilde{q}^n\right)^{\ell} + \left(\frac{\tilde{z}_j}{\tilde{z}_i}\tilde{y}^{-1}\tilde{q}^{n+1}\right)^{\ell}\right] \\ &= -\sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\ell}[A_\ell \tilde{y}^\ell \tilde{q}^{n\ell} + A_\ell \tilde{y}^{-\ell}\tilde{q}^{(n+1)\ell}] \\ &= -\sum_{\ell=1}^{\infty}\frac{1}{\ell}A_\ell \frac{\tilde{y}^\ell + \tilde{y}^{-\ell}\tilde{q}^\ell}{1-\tilde{q}^\ell},\end{aligned} \quad (53)$$

where we introduced A_ℓ which denotes the following sum over i, j:

$$\begin{aligned}A_\ell &\equiv \sum_{i\neq j}^{N}\left(\frac{\tilde{z}_i}{\tilde{z}_j}\right)^{\ell} = \sum_{i\neq j}^{N} e^{2\pi i (j-i)\ell/N} \\ &= \begin{cases} N^2 - N & \text{for } \ell = 0 \mod N \\ -N & \text{for } \ell \neq 0 \mod N. \end{cases}\end{aligned} \quad (54)$$

The series could be analytically resummed to the expression $N\log[\theta_0(N\Delta/\tau;-N/\tau)/\theta_0(\Delta/\tau;-1/\tau)]$, however, we do not need that. We collect the terms into two groups:

$$\text{Eq.}(53) = N\sum_{\ell=1}^{\infty}\frac{1}{\ell}\frac{\tilde{y}^\ell + \tilde{y}^{-\ell}\tilde{q}^\ell}{1-\tilde{q}^\ell} - N\sum_{j=1}^{\infty}\frac{1}{j}\frac{\tilde{y}^{Nj}+\tilde{y}^{-Nj}\tilde{q}^{Nj}}{1-\tilde{q}^{Nj}}, \quad (55)$$

where the second term comes from the cases $\ell = Nj$. For $|\tilde{q}| < |\tilde{y}| < 1$, namely for

$$\mathrm{Im}\left(-\frac{1}{\tau}\right) > \mathrm{Im}\left(\frac{\Delta}{\tau}\right) > 0, \quad (56)$$

the series converges. The second term is suppressed at large N, whereas the first term is of order $\mathcal{O}(N)$ and can be neglected.

We then perform a similar analysis of $\log\psi$, using the series expansions of the functions log and Li_2. We find

$$\begin{aligned}&\sum_{i\neq j}^{N}\sum_{k=0}^{\infty}\log\frac{\psi\left(\frac{k+1+u_{ij}+\Delta}{\tau}\right)}{\psi\left(\frac{k-u_{ij}-\Delta}{\tau}\right)} \\ &= \sum_{i\neq j}^{N}\sum_{k=0}^{\infty}\sum_{\ell=1}^{\infty}\left[-\frac{1}{\ell}\left(\frac{k+1+\Delta}{\tau}\tilde{y}^{-\ell}\tilde{q}^{\ell} - \frac{k-\Delta}{\tau}\tilde{y}^{\ell}\right) + \right. \\ &\left. -\frac{u_{ij}}{\ell\tau}(\tilde{y}^\ell + \tilde{y}^{-\ell}\tilde{q}^\ell) + \frac{1}{2\pi i \ell^2}(\tilde{y}^\ell - \tilde{y}^{-\ell}\tilde{q}^\ell)\right]\left(\frac{\tilde{z}_i}{\tilde{z}_j}\tilde{q}^k\right)^{\ell} \\ &= \sum_{k=0}^{\infty}\sum_{\ell=1}^{\infty}\left[-\frac{A_\ell}{\ell}\left(\frac{k+1+\Delta}{\tau}\tilde{y}^{-\ell}\tilde{q}^\ell - \frac{k-\Delta}{\tau}\tilde{y}^\ell\right)\tilde{q}^{k\ell} + \right. \\ &\left. +\frac{1}{2\pi i}\frac{A_\ell}{\ell^2}(\tilde{y}^\ell - \tilde{y}^{-\ell}\tilde{q}^\ell)\tilde{q}^{k\ell}\right],\end{aligned} \quad (57)$$

where we used that the following sum vanishes:

$$B_\ell \equiv \sum_{i\neq j}^{N}\tilde{u}_{ij}\left(\frac{\tilde{z}_i}{\tilde{z}_j}\right)^{\ell} = \frac{1}{N}\sum_{i\neq j}^{N}(j-i)e^{2\pi i (j-i)\ell/N} = 0. \quad (58)$$

Once again, the expression can be resummed by breaking the sum into two groups (corresponding to generic ℓ and $\ell = Nj$):

$$\text{Eq.}(57) = \sum_{k=0}^{\infty}\left[-N\log\frac{\psi\left(\frac{k+1+\Delta}{\tau}\right)}{\psi\left(\frac{k-\Delta}{\tau}\right)} + \log\frac{\psi\left(\frac{N(k+1-\Delta)}{\tau}\right)}{\psi\left(\frac{N(k-\Delta)}{\tau}\right)}\right]. \quad (59)$$

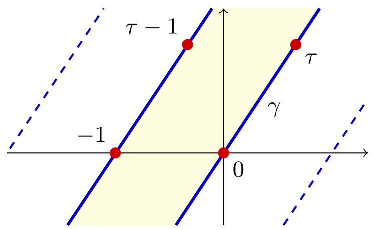

FIG. 1. In yellow is highlighted the domain, Eq. (56), in the complex Δ plane. The right boundary is the line γ, passing through 0 and τ. The left boundary is the line $\gamma-1$, passing through -1 and $\tau-1$. The dashed lines are other elements of $\gamma+\mathbb{Z}$.

The first term (that comes from setting $A_\ell \to N$) is of order $\mathcal{O}(N)$ and can be neglected. The second term is

$$\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\bigg[-\frac{N}{j}\left(\frac{k+1+\Delta}{\tau}(\tilde{q}/\tilde{y})^{Nj}-\frac{k-\Delta}{\tau}\tilde{y}^{Nj}\right)\tilde{q}^{Nkj} + \frac{1}{2\pi i j^2}(\tilde{y}^{Nj}-(\tilde{q}/\tilde{y})^{Nj})\tilde{q}^{Nkj}\bigg]. \tag{60}$$

In the regime of convergence (56) this series goes to zero as $N\to\infty$. We conclude that the only contribution at leading order in N is from the polynomial $\mathcal{Q}$ in Eq. (50).

The limit we computed is valid as long as Δ satisfies Eq. (56). That inequality has the interpretation that Δ should lie inside an infinite strip, bounded on the left by the line through -1 and $\tau-1$, and on the right by the line (that we dub γ) through 0 and τ (see Fig. 1). On the other hand, $\tilde{\Gamma}(u_{ij}+\Delta;\tau,\tau)$ is a periodic function invariant under shifts $\Delta\to\Delta+1$. Therefore, unless Δ sits exactly on one image of the line γ under periodic integer shifts, there always exists a shift that brings Δ inside the strip. This means that we can use our computation to extract the limit for all $\Delta\in\mathbb{C}\backslash\{\gamma+\mathbb{Z}\}$.

Let us define the periodic discontinuous function

$$[\Delta]_\tau \equiv \left[\Delta+n \;\middle|\; n\in\mathbb{Z},\ \mathrm{Im}\left(-\frac{1}{\tau}\right) > \mathrm{Im}\left(\frac{\Delta+n}{\tau}\right) > 0\right] \quad \text{for } \mathrm{Im}\left(\frac{\Delta}{\tau}\right)\notin\mathbb{Z}\times\mathrm{Im}\left(\frac{1}{\tau}\right). \tag{61}$$

The function is not defined for $\mathrm{Im}(\Delta/\tau)\in\mathbb{Z}\times\mathrm{Im}(1/\tau)$. Essentially, this function is constructed in such a way that $[\Delta]_\tau = \Delta \bmod 1$, and $[\Delta]_\tau$ satisfies Eq. (56) when it is defined. It also satisfies

$$[\Delta+1]_\tau = [\Delta]_\tau, \qquad [-\Delta]_\tau = -[\Delta]_\tau - 1, \qquad [\Delta+\tau]_\tau = [\Delta]_\tau + \tau. \tag{62}$$

We use such a function to express the limit as

$$\begin{aligned}\lim_{N\to\infty}\sum_{i\neq j}^{N}\log\tilde{\Gamma}(u_{ij}+\Delta;\tau,\tau)\Big|_{(34)} &= -\pi i\sum_{i\neq j}^{N}\mathcal{Q}(u_{ij}+[\Delta]_\tau;\tau,\tau)+\mathcal{O}(N)\\ &= -\pi i N^2\frac{([\Delta]_\tau-\tau)([\Delta]_\tau-\tau+\frac{1}{2})([\Delta]_\tau-\tau+1)}{3\tau^2}\end{aligned} \tag{63}$$

up to terms of order N. This expression is, by construction, invariant under integer shifts $\Delta\to\Delta+1$. The lines

$$\mathrm{Im}(\Delta/\tau)\in\mathbb{Z}\times\mathrm{Im}(1/\tau) \tag{64}$$

that we have dubbed $\gamma+\mathbb{Z}$ are Stokes lines: They represent transitions between regions in the complex Δ plane in which different exponential contributions dominate the large N limit, and along which the limit is discontinuous [67]. We do not know what is the limit along the lines, because different contributions compete and a more precise estimate would be necessary to evaluate their sum. We will elaborate on Stokes lines in Sec. IV C.

The term with $\Delta=0$ requires a special treatment, because it does not satisfy Eq. (56). We can still use the expansion Eq. (49). The term $\log\theta_0$ is evaluated as

$$\begin{aligned}\sum_{i\neq j}^{N}\log\theta_0\left(\frac{u_{ij}}{\tau};-\frac{1}{\tau}\right) &= \sum_{i\neq j}^{N}\log\left(1-\frac{\tilde{z}_i}{\tilde{z}_j}\right)+2\sum_{i\neq j}^{N}\sum_{k=1}^{\infty}\log\left(1-\frac{\tilde{z}_i}{\tilde{z}_j}\tilde{q}^k\right)\\ &= N\log N + 2N\log\frac{(\tilde{q}^N;\tilde{q}^N)_\infty}{(\tilde{q};\tilde{q})_\infty}.\end{aligned} \tag{65}$$

To calculate the first term in the second line, we notice that $x^N-1=\prod_{j=1}^{N}(x-e^{2\pi i j/N})$. Factoring $(x-1)$ on both sides we get $x^{N-1}+\cdots+x+1=\prod_{j=1}^{N-1}(x-e^{2\pi i j/N})$, and substituting $x=1$ we get $N=\prod_{j=1}^{N-1}(1-e^{2\pi i j/N})$. At this point we can shift j by k and multiply over k:

$$N^N=\prod_{k=1}^{N}\prod_{j(\neq k)=1}^{N}(1-e^{2\pi i(j-k)/N}). \tag{66}$$

To compute the second term we use the series expansion as before. We see that $\log\theta_0$ contributes at order $\mathcal{O}(N\log N)$ and can be neglected. The product of terms $\log\psi$ gives

$$\begin{aligned}\sum_{i\neq j}^{N}\log\prod_{k=0}^{\infty}\frac{\psi\left(\frac{k+1+u_{ij}}{\tau}\right)}{\psi\left(\frac{k-u_{ij}}{\tau}\right)} &= -\sum_{i\neq j}^{N}\log\psi\left(\frac{u_{ij}}{\tau}\right)\\ &= \sum_{i<j}^{N}\pi i\left(\frac{(j-i)^2}{N^2}-\frac{1}{6}\right)\\ &= \frac{i\pi}{12}(N-1).\end{aligned} \tag{67}$$

In the first equality we changed sign to u_{ij} because it is summed over i, j. This term is of order $\mathcal{O}(N)$ and can be neglected. We conclude that

$$\lim_{N\to\infty}\sum_{i\neq j}^{N}\log\tilde{\Gamma}(u_{ij};\tau,\tau)\Big|_{(34)} = \pi i N^2\frac{\tau(\tau-\frac{1}{2})(\tau-1)}{3\tau^2} + \mathcal{O}(N\log N). \tag{68}$$

3. *Total contribution from the basic solution.*—At this point we can collect the various contributions and obtain the large N limit of $\log\mathcal{Z}$ in Eq. (29) evaluated on the solution Eq. (34). It is also useful to define the primed bracket

$$[\Delta]'_\tau = [\Delta]_\tau + 1 \;\Rightarrow\; 0 > \mathrm{Im}\left(\frac{[\Delta]'_\tau}{\tau}\right) > \mathrm{Im}\left(\frac{1}{\tau}\right). \tag{69}$$

The primed bracket selects the image of Δ, under integer shifts, that sits inside the strip on the right of the line γ through zero and τ, as opposed to the strip on the left. The expression of $\log\mathcal{Z}$ depends on $[\Delta_1]_\tau$, $[\Delta_2]_\tau$, and $[\Delta_1+\Delta_2]_\tau$. We notice the following relation:

$$[\Delta_1+\Delta_2]_\tau = \begin{cases} [\Delta_1]_\tau + [\Delta_2]_\tau & \text{if } \mathrm{Im}\left(-\frac{1}{\tau}\right) > \mathrm{Im}\left(\frac{[\Delta_1]_\tau+[\Delta_2]_\tau}{\tau}\right) > 0, \quad \text{first case} \\ [\Delta_1]_\tau + [\Delta_2]_\tau + 1 & \text{if } \mathrm{Im}\left(-\frac{2}{\tau}\right) > \mathrm{Im}\left(\frac{[\Delta_1]_\tau+[\Delta_2]_\tau}{\tau}\right) > \mathrm{Im}\left(-\frac{1}{\tau}\right), \quad \text{second case.} \end{cases} \tag{70}$$

The second case can also be rewritten as

$$[\Delta_1+\Delta_2]_\tau = [\Delta_1]'_\tau + [\Delta_2]'_\tau - 1 \quad \text{if } 0 > \mathrm{Im}\left(\frac{[\Delta_1]'_\tau+[\Delta_2]'_\tau}{\tau}\right) > \mathrm{Im}\left(\frac{1}{\tau}\right). \tag{71}$$

The large N limit of the summand is then

$$\lim_{N\to\infty}\log\mathcal{Z}\Big|_{(34)} = -\pi i N^2\Theta(\Delta_1,\Delta_2;\tau), \tag{72}$$

where we have introduced the following function for compactness:

$$\Theta(\Delta_1,\Delta_2;\tau) = \begin{cases} \frac{[\Delta_1]_\tau[\Delta_2]_\tau(2\tau-1-[\Delta_1]_\tau-[\Delta_2]_\tau)}{\tau^2}, & \text{first case} \\ \frac{[\Delta_1]'_\tau[\Delta_2]'_\tau(2\tau+1-[\Delta_1]'_\tau-[\Delta_2]'_\tau)}{\tau^2} - 1, & \text{second case.} \end{cases} \tag{73}$$

The two cases were defined in Eqs. (70) and (71).

We can rewrite the function Θ in a way that will be useful in Sec. VI. Define an auxiliary chemical potential Δ_3, modulo 1, such that

$$\Delta_1+\Delta_2+\Delta_3-2\tau\in\mathbb{Z}. \tag{74}$$

It follows that $[\Delta_3]_\tau = 2\tau - [\Delta_1+\Delta_2]_\tau - 1$. Hence,

$$\Theta(\Delta_1,\Delta_2;\tau) = \begin{cases} \frac{[\Delta_1]_\tau[\Delta_2]_\tau[\Delta_3]_\tau}{\tau^2}, & \text{first case} \\ \frac{[\Delta_1]'_\tau[\Delta_2]'_\tau[\Delta_3]'_\tau}{\tau^2} - 1, & \text{second case.} \end{cases} \tag{75}$$

Irrespective of the integer appearing in Eq. (74), the bracketed potentials satisfy the following constraints:

$$[\Delta_1]_\tau + [\Delta_2]_\tau + [\Delta_3]_\tau - 2\tau + 1 = 0, \quad \text{first case}$$
$$[\Delta_1]'_\tau + [\Delta_2]'_\tau + [\Delta_3]'_\tau - 2\tau - 1 = 0, \quad \text{second case.} \tag{76}$$

Such constraints have already appeared in Refs. [38,39,44].

B. Contribution of SL$(2,\mathbb{Z})$-transformed solutions

As discussed in Sec. III A, Eq. (34) is not the only solution to the BAEs: Each inequivalent SL$(2,\mathbb{Z})$ transformation of it, given in Eq. (36), is another solution—and even more generally there are the $\{m,n,r\}$ solutions (37) found in [65]. Some of those solutions might contribute at the same leading order in N.

A class of inequivalent solutions—particularly simple to study—that contribute at leading order in N is obtained through T transformations:

$$u_{ij} = \frac{\tau+r}{N}(j-i) \quad \text{for } r=0,\ldots,N-1. \tag{77}$$

These are the solutions $\{1,N,r\}$ in the notation of Sec. III A. To evaluate their contribution, simply notice that both $\mathcal{Z}$ in Eq. (29) and H are invariant under $\tau\to\tau+r$, thus the contribution of Eq. (77) is the same as in Eq. (72) but with $\tau\to\tau+r$. In the large N limit, r runs over $\mathbb{Z}$.

We have not evaluated the contribution of all other $\{m,n,r\}$ solutions, which is a difficult task. However, in order to have an idea of what their contribution could be, let us estimate the contribution from the S-transformed solution

$$u_{ij} = \frac{j-i}{N}, \tag{78}$$

which is $\{N,1,0\}$ in the notation of Eq. (37). The large N limit of κ_N does not depend on the solution, and is subleading. The large N limit of $\log H$ is computed in the same way as in Sec. IV A, and it gives $\mathcal{O}(N\log N)$ or smaller. Let us then analyze $\mathcal{Z}$. In the regime $|q|^2 < |y| < 1$ we can directly expand $\log\tilde{\Gamma}$ in its plethystic form:

$$\begin{aligned}\sum_{i\neq j}^{N}&\log\tilde{\Gamma}(u_{ij}+\Delta;\tau,\tau)\\ &= \sum_{i\neq j}^{N}\sum_{\ell=1}^{\infty}\sum_{m=0}^{\infty}\frac{m+1}{\ell}\left[\left(\frac{z_i}{z_j}\right)^{\ell}y^{\ell} - \left(\frac{z_j}{z_i}\right)^{\ell}\frac{q^{2\ell}}{y^{\ell}}\right]q^{m\ell}\\ &= N\log\frac{\tilde{\Gamma}(N\Delta;N\tau,N\tau)}{\tilde{\Gamma}(\Delta;\tau,\tau)} = \mathcal{O}(N).\end{aligned} \tag{79}$$

If $|y|$ is outside the range of convergence of the plethystic expansion, either above or below, we can simply shift $\Delta\to\Delta\pm\tau$. This gives a shift by [68]

$$\pm\sum_{i\neq j}^{N}\log\theta_0(u_{ij}+\Delta;\tau) = \mathcal{O}(N), \tag{80}$$

which can be treated in a similar way. This allows us to use the estimate above also when the Δ's are outside the original regime of convergence. The case $\Delta=0$ requires a special treatment. We have

$$\begin{aligned}\sum_{i\neq j}^{N}&\log\tilde{\Gamma}(u_{ij};\tau,\tau)\\ &= \sum_{i\neq j}^{N}\left[-\log\left(1-\frac{z_i}{z_j}\right) + 2\sum_{\ell,m=1}^{\infty}\left(\frac{z_i}{z_j}\right)^{\ell}\frac{q^{m\ell}}{\ell}\right]\\ &= -N\log N + 2N\log\frac{(q;q)_\infty}{(q^N;q^N)_\infty} = \mathcal{O}(N\log N).\end{aligned} \tag{81}$$

Thus, there is no contribution from $\log\mathcal{Z}$ at leading order in N.

In the following, we will assume that the only solutions contributing at leading order, namely $\mathcal{O}(N^2)$, are the T-transformed solutions.

C. Final result and Stokes lines

Since we end up with competing exponentials, the one with the largest real part dominates the large N limit. Assuming that solutions other than the T-transformed ones are of subleading order in N, we find the final formula

$$\begin{aligned}\lim_{N\to\infty}\log\mathcal{I}(q,y_1,y_2) &= \widetilde{\max_{r\in\mathbb{Z}}}[-\pi i N^2\Theta(\Delta_1,\Delta_2;\tau+r)]\\ &\equiv \log\mathcal{I}_\infty.\end{aligned} \tag{82}$$

The function Θ is defined in Eqs. (73) or (75). The meaning of $\widetilde{\max}$ is that we should choose the value of $r\in\mathbb{Z}$ such that the real part of the argument is maximized. One of the good features of Eq. (82) is that it is periodic under integer shifts of τ, Δ_1, Δ_2. We already observed that Θ is periodic in $\Delta_{1,2}$ because the functions $[\Delta_{1,2}]_\tau$ are. Taking the $\widetilde{\max}$ over $\tau\to\tau+r$ gives periodicity in τ as well. This implies that the right hand side of Eq. (82) is actually a single-valued function of the fugacities q, y_1, y_2. This is a property of the index at finite N, as manifest in Eqs. (19) and (22), and it is reassuring that the large N expression we found respects the same property.

The function $\mathcal{I}_\infty$ has a complicated structure. The full range of allowed fugacities q, y_1, y_2 gets divided into multiple domains of analyticity, separated by *Stokes lines*. In each domain of analyticity, only one exponential contribution (for some value of r) dominates the large N limit: The function $\log\mathcal{I}_\infty$ takes the form of a simple rational function given by $\Theta(\Delta_1,\Delta_2;\tau+r)$. The Stokes lines are real-codimension-one surfaces, in the space of fugacities, that separate the different domains. When crossing a Stokes line, a different exponential contribution dominates, and $\log\mathcal{I}_\infty$ takes the form of a different rational function. In particular, on top of a Stokes line there are two (or more) exponential contributions that compete: Their exponents have equal real part. This characterizes the locations of Stokes lines. In terms of the function Θ:

$$\operatorname{Im}\Theta(\Delta_1,\Delta_2;\tau+r_1) = \operatorname{Im}\Theta(\Delta_1,\Delta_2;\tau+r_2) \tag{83}$$

for some $r_{1,2}\in\mathbb{Z}$.

In fact, also the values of Δ_1 and Δ_2 such that $\Theta(\Delta_1,\Delta_2;\tau+r)$ is discontinuous (for the value of r picked up by $\widetilde{\max}$) should be regarded as forming a Stokes line. In this case, the two competing exponents correspond to the values of Θ on the two sides of the discontinuity. There are two possible sources of discontinuity. First, one of the bracket functions, say $[\Delta_1]_\tau$, could be discontinuous. This happens when $\operatorname{Im}(\Delta_1/\tau)\in\mathbb{Z}\times\operatorname{Im}(1/\tau)$, namely when $\alpha\equiv\lim_{\epsilon\to0^+}[\Delta_1-\epsilon]_\tau/\tau\in\mathbb{R}$. Taking into account that on the left of the discontinuity we are in the first case, while on the right we are in the second case—in the terminology of Eq. (70)—and assuming that Δ_2 is generic, we find

$$\lim_{\epsilon\to0^+}[\Theta(\Delta_1-\epsilon,\Delta_2;\tau)-\Theta(\Delta_1+\epsilon,\Delta_2;\tau)] = (\alpha-1)^2\in\mathbb{R}, \tag{84}$$

where the limit is taken with ϵ real positive. Second, we could pass from the first to the second case of the definition (73). This happens when $[\Delta_1]_\tau+[\Delta_2]_\tau+1=\alpha\tau$ for some $\alpha\in\mathbb{R}$. Assuming that $\Delta_{1,2}$ are otherwise generic, we find

$$\Delta\Theta = (\alpha-1)^2\in\mathbb{R}. \tag{85}$$

In both cases we confirm that the codimension-one surface of discontinuity is a Stokes line, because $\mathrm{Im}\,\Theta$ is equal on the two sides.

When we sit exactly on a Stokes line, two (or more) exponential contributions compete, and in order to compute the large N limit we should sum them. However, we do not know the relative phases, because they are affected by all subleading terms and a more accurate analysis would be required. Therefore, we cannot determine the large N limit of the index along Stokes lines.

It turns out that a value of r that maximizes the real part of the argument of $\widetilde{\max}$ may or may not exist. We can estimate the behavior of the real part at large r by noticing that

$$\lim_{r\to\pm\infty} \frac{[\Delta]_{\tau+r}}{\tau+r} = \frac{\mathrm{Im}\,\Delta}{\mathrm{Im}\,\tau}. \tag{86}$$

This implies that

$$\lim_{r\to\pm\infty} \mathrm{Im}\,\Theta(\Delta_1,\Delta_2;\tau+r) = \frac{\mathrm{Im}\,\Delta_1 \mathrm{Im}\,\Delta_2 \mathrm{Im}(2\tau-\Delta_1-\Delta_2)}{(\mathrm{Im}\,\tau)^2}. \tag{87}$$

Thus, the real part of the argument of $\widetilde{\max}$ approaches a constant value. If there is no maximum but rather the constant value is a supremum, then our computation is not finished: All contributions from the T-transformed solutions should be summed, however, for large $|r|$ they form an infinite number of competing exponentials, whose sum crucially depends on how they interfere. In order to determine such a sum we would need more accurate information.

We conclude by stressing that—even though only the dominant exponential determines the large N limit of the index—we expect that all exponential contributions, including the subdominant ones, have some physical meaning. Each of them plays the role of a "saddle point," although our treatment is not the standard saddle-point approximation. We will make this comment more concrete in Sec. VI, when comparing the large N limit of the index with BPS black-hole solutions in supergravity.

D. Comparison with previous literature

The large N limit of the superconformal index of $\mathcal{N}=4$ SYM was already computed in [30]. There, it was found that the large N limit does not depend on N, and therefore it does not show a rapid enough growth of the number of states to reproduce the black-hole entropy. In this section, we would like to explain how the results here and there can be compatible.

The authors of Ref. [30] took the large N limit of the index, for *real* fugacities. Their result, in our notation and restricted to the case $p=q$, is

$$\lim_{N\to\infty} \mathcal{I}(q,y_1,y_2) = \prod_{n=1}^{\infty} \frac{1}{1-f(q^n,y_1^n,y_2^n)} \tag{88}$$

with

$$1-f(q,y_1,y_2) = \frac{(1-y_1)(1-y_2)(1-q^2/y_1y_2)}{(1-q)^2}. \tag{89}$$

In particular, $\log\mathcal{I}$ is of order $\mathcal{O}(1)$. On the contrary, we computed the large N limit for generic *complex* fugacities, and found that $\log\mathcal{I}$ is of order $\mathcal{O}(N^2)$.

The resolution we propose relies on the fact that, for complex fugacities, the limit shows Stokes lines. As we described, along those codimension-one surfaces multiple exponentials compete. In order to know what the limit is there, we would need to sum those competing exponentials, but this requires a more accurate knowledge of the subleading terms.

What we notice, though, is that the codimension-three subspace of real fugacities is precisely within a Stokes line. Therefore, although we cannot prove it, it is conceivable that the competing terms cancel exactly, leaving the $\mathcal{O}(1)$ result, Eq. (88). Indeed, in the Supplemental Material, Sec. B in Ref. [64], we prove the following result, which is stronger than the statement that we sit on a Stokes line. Take the angular fugacity q to be real positive, namely $0<q<1$ and set $\tau\in i\mathbb{R}_{\geq 0}$ for concreteness, and take the flavor fugacities $y_{1,2}$ to be real. Then $\Theta(\Delta_1,\Delta_2;\tau)$ is along a Stokes line and is not defined, while

$$\Theta(\Delta_1,\Delta_2;\tau-r) = -\overline{\Theta(\Delta_1,\Delta_2;\tau+r)} - 1 \tag{90}$$

for $r>0$. On the other hand, take the angular fugacity real negative, namely $-1<q<0$ and set $\tau\in -\frac{1}{2}+i\mathbb{R}_{\geq 0}$, and take again the flavor fugacities to be real. Then

$$\Theta(\Delta_1,\Delta_2;\tau-r) = -\overline{\Theta(\Delta_1,\Delta_2;\tau+r+1)} - 1 \tag{91}$$

for $r\geq 0$. Therefore, among the various contributions from T-transformed solutions parametrized by $r\in\mathbb{Z}$, there is an exact pairing of all well-defined terms where, in each pair, two terms have the same real part and can conceivably cancel. In other words, not only the term with maximal real part can cancel, but also all other terms we computed at order $\mathcal{O}(N^2)$. This scenario is a strong check of our result, that possibly makes it compatible with [30].

V. STATISTICAL INTERPRETATION AND $\mathcal{I}$-EXTREMIZATION

We wish to extract the number of BPS states, for given electric charges and angular momenta, from the large N limit of the exact expression Eq. (22) of the superconformal index. Since the latter counts states weighted by the fermion number $(-1)^F$, one may worry that strong cancelations take place and that the total number of states is not accessible. However, one can argue [5,6] that the index (14) or (19) is equal to

$$\mathcal{I}(p,q,y_1,y_2) = \mathrm{Tr}\, e^{i\pi R_{\mathrm{trial}}(\tau,\sigma,\Delta_1,\Delta_2)} e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} \times e^{-2\pi\,\mathrm{Im}[\tau C_1+\sigma C_2+\Delta_1 C_3+\Delta_2 C_4]}, \tag{92}$$

where the trace is taken in the IR $\mathcal{N}=2$ super quantum mechanics (QM) obtained by reducing the 4D theory on S^3, R_{trial} is a trial R-symmetry, and $C_{1,2,3,4}$ are the charges appearing in Eq. (19):

$$C_1 = J_1 + \frac{1}{2}R_3, \qquad C_3 = q_1,$$
$$C_2 = J_2 + \frac{1}{2}R_3, \qquad C_4 = q_2. \tag{93}$$

Indeed, because of the relations (20), we can represent the fermion number as $(-1)^F = e^{i\pi R_3}$. Substituting in Eq. (19) and separating the chemical potentials into real and imaginary part, we obtain the expression (92) with

$$R_{\mathrm{trial}}(\tau,\sigma,\Delta_1,\Delta_2) = R_3 + 2(\mathrm{Re}\,\tau)C_1 + 2(\mathrm{Re}\,\sigma)C_2 + 2(\mathrm{Re}\,\Delta_1)C_3 + 2(\mathrm{Re}\,\Delta_2)C_4. \tag{94}$$

From the point of view of the super QM, R_3 is an R-symmetry while the other four operators are flavor charges, hence R_{trial} is an R-symmetry. We see in Eq. (92) that only the first exponential can produce possibly dangerous phases, while the other two are real positive.

Now, for a single-center black hole in the microcanonical ensemble, the near-horizon AdS_2 region is dual to an $\mathcal{N}=2$ superconformal QM. The black-hole states are vacua of the $\mathfrak{su}(1,1|1)$ 1D superconformal algebra. Since we are in the microcanonical ensemble, each of those states is invariant under the global conformal algebra $\mathfrak{su}(1,1)\cong\mathfrak{so}(2,1)$ (because AdS_2 is) as well as under the fermionic generators (because the black hole is supersymmetric). This necessarily implies that those states are invariant under the exact superconformal R-symmetry $\mathfrak{u}(1)_{\mathrm{sc}}\subset\mathfrak{su}(1,1|1)$, i.e., that they have vanishing IR superconformal R-charge R_{sc}. Thus, when R_{trial} is tuned to R_{sc}, the index counts the black-hole states with no extra signs or phases (this is similar to Refs. [69]). Of course, in a given charge sector there will be more BPS states than just the single-center black hole, but assuming that the single-center black hole dominates, the index captures its entropy. It remains to understand how to identify R_{sc}. At large N, the entropy is extracted from the index with a Legendre transform, and this operation can be argued to effectively select R_{sc} among the R_{trial}'s (this large N principle was dubbed $\mathcal{I}$-extremization in Refs. [5,6]).

Let us elaborate on this point. The index is the grand canonical partition function of BPS states. Introducing an auxiliary variable Δ_3 and the corresponding fugacity $y_3 = e^{2\pi i\Delta_3}$ such that $\Delta_1+\Delta_2+\Delta_3-\tau-\sigma+1\in 2\mathbb{Z}$, we can rewrite Eq. (92) as

$$\mathcal{I}(p,q,y_1,y_2) = \mathrm{Tr}_{\mathrm{BPS}}\, p^{J_1} q^{J_2} y_1^{Q_1} y_2^{Q_2} y_3^{Q_3}. \tag{95}$$

Here, the trace is over states with $\{\mathcal{Q},\mathcal{Q}^\dagger\}=0$, and we have identified $Q_I = R_I/2$ (for $I=1,2,3$) with the electric charges in supergravity. We recognize that the black-hole angular momenta $J_{1,2}$ are associated with the chemical potentials τ, σ and the charges $Q_{1,2,3}$ with $\Delta_{1,2,3}$. The microcanonical degeneracies at fixed quantum numbers are extracted by computing the Fourier transform of Eq. (95). However, since Δ_3 is not an independent variable, what we obtain are the degeneracies for fixed values of the four charge operators appearing in Eq. (93), summed over Q_3. Using the supergravity notation, those four fixed charge operators are

$$C_1 = J_1 + Q_3, \qquad C_3 = Q_1 - Q_3,$$
$$C_2 = J_2 + Q_3, \qquad C_4 = Q_2 - Q_3. \tag{96}$$

Thus, what we can compute is

$$\sum_{Q_3} d(J,Q)\Big|_{C_{1,2,3,4}} = \int d\tau\, d\sigma\, d\Delta_1 d\Delta_2\, \mathcal{I}(p,q,y_1,y_2)\, p^{-J_1} q^{-J_2} \prod_{I=1}^{3} y_I^{-Q_I}, \tag{97}$$

where $d(J,Q)$ are the (weighted) degeneracies with all charges $J_{1,2}$ and $Q_{1,2,3}$ fixed.

Nevertheless, we can take advantage of the fact, reviewed in Sec. II, that the charges of BPS back holes are constrained, and for fixed $C_{1,2,3,4}$ there is at most one black hole—for a certain value of the fifth charge Q_3. We can then use Eq. (97) to extract its degeneracy $d(J,Q) = \exp S_{\mathrm{BH}}(J,Q)$ at leading order because the latter will dominate the sum over Q_3.

In the large N limit, the integral (97) reduces by saddle-point approximation to a Legendre transform with respect to the independent variables $\{\tau,\sigma,\Delta_1,\Delta_2\}$:

$$\begin{aligned} S_{\mathrm{BH}}(J,Q) &= \log\mathcal{I}(\hat\tau,\hat\sigma,\hat\Delta_1,\hat\Delta_2) - 2\pi i\Big(\hat\tau J_1 + \hat\sigma J_2 + \sum_{I=1}^{3}\hat\Delta_I Q_I\Big) \\ &= \log\mathcal{I} - 2\pi i(\hat\tau C_1 + \hat\sigma C_2 + \hat\Delta_1 C_3 + \hat\Delta_2 C_4) + 2\pi i Q_3 \end{aligned} \tag{98}$$

where hatted variables denote the critical point. In this approach, Q_3 can be determined as the unique value that makes the entropy $S_{\mathrm{BH}}(J,Q)$ real [5,6].

In the particular case of 4D $\mathcal{N}=4$ SYM, the large N limit of the index is a function with multiple domains of analyticity, separated by Stokes lines. This makes things more interesting. In each domain we should perform the

Legendre transform, and whenever the critical point falls inside the domain itself, we obtain a self-consistent contribution to the total entropy. Even more generally, we have written the index as a sum of competing exponentials (one for each Bethe-ansatz solution) and we can compute the Legendre transform of each of those exponentials—irrespective of which one dominates. We expect each contribution to represent the entropy of some classical solution—very similarly to a standard saddle point—even when the entropy is smaller than that of the dominant solution.

VI. BLACK-HOLE ENTROPY FROM THE INDEX

In this section we show that the contribution of the basic solution Eq. (34) to the superconformal index at large N, in the domain of analyticity that we called "first case" in Eq. (70), given by

$$\begin{aligned} &-\pi i N^2 \Theta(\Delta_1,\Delta_2;\tau)\Big|_{\text{first case}} \\ &\quad = -\pi i N^2 \frac{[\Delta_1]_\tau [\Delta_2]_\tau (2\tau - 1 - [\Delta_1]_\tau - [\Delta_2]_\tau)}{\tau^2} \end{aligned} \tag{99}$$

precisely reproduces the Bekenstein-Hawking entropy (12) of single-center black holes in AdS_5 (this is in line with the result of [39] in a double-scaling Cardy-like limit). It amounts to show that the Legendre transform of Eq. (99) is the black-hole entropy (this will be reviewed below), and that the critical point involved in the Legendre transform consistently lies within the domain of analyticity in which Eq. (99) holds.

Recall that the contribution of the basic solution corresponds to the $r = 0$ sector in Eq. (82). For black holes with large charges, i.e., for black holes that are large compared with the AdS_5 scale, that is indeed the dominant contribution to the index. However, intriguingly enough, as we reduce the charges the contribution of the single-center black hole may cease to dominate. We will highlight this phenomenon in Sec. VI A in the very special case of black holes with equal charges. This seems to suggest that, below a certain threshold, the BPS black holes may develop instabilities, possibly toward hairy or multicenter black holes. Indications that this is the case have also been given in Refs. [39,40]. It would be nice if there were a connection between this observation and recently constructed hairy black holes in AdS_5 [46].

a. *The entropy function.*—The Legendre transform of the black-hole entropy (9) in the general case, also called entropy function, was obtained in Ref. [44]. Let us review it, following the detailed discussion in Appendix B of Ref. [38]. The entropy function is

$$\mathcal{S} = -2\pi i \nu \frac{X_1 X_2 X_3}{\omega_1 \omega_2} \quad \text{with} \quad \nu = \frac{N^2}{2} = \frac{\pi}{4 G_N g^3} \tag{100}$$

and with the constraint

$$\sum_{a=1,2,3} X_a - \sum_{i=1,2} \omega_i + 1 = 0. \tag{101}$$

Because of the constraint, $\mathcal{S}$ is really a function of four variables. The entropy S_{BH} is the Legendre transform of $\mathcal{S}$ with its constraint. We can compute it as the critical point of

$$\begin{aligned} \hat{\mathcal{S}} = \mathcal{S} &- 2\pi i \left(\sum_a Q_a X_a + \sum_i J_i \omega_i \right) \\ &- 2\pi i \Lambda \left(\sum_a X_a - \sum_i \omega_i + 1 \right) \end{aligned} \tag{102}$$

in which the constraint is imposed with a Lagrange multiplier Λ. The equations for the critical point are

$$Q_a + \Lambda = \frac{1}{2\pi i} \frac{\partial \mathcal{S}}{\partial X_a}, \qquad J_i - \Lambda = \frac{1}{2\pi i} \frac{\partial \mathcal{S}}{\partial \omega_i}, \tag{103}$$

and the constraint Eq. (101). In detail,

$$\begin{aligned} &Q_1 + \Lambda = -\nu \frac{X_2 X_3}{\omega_1 \omega_2}, \qquad J_1 - \Lambda = \nu \frac{X_1 X_2 X_3}{\omega_1^2 \omega_2}, \\ &Q_2 + \Lambda = -\nu \frac{X_1 X_3}{\omega_1 \omega_2}, \qquad J_2 - \Lambda = \nu \frac{X_1 X_2 X_3}{\omega_1 \omega_2^2}, \\ &Q_3 + \Lambda = -\nu \frac{X_1 X_2}{\omega_1 \omega_2}. \end{aligned} \tag{104}$$

It follows that

$$\begin{aligned} 0 &= (Q_1 + \Lambda)(Q_2 + \Lambda)(Q_3 + \Lambda) + \nu (J_1 - \Lambda)(J_2 - \Lambda) \\ &= \Lambda^3 + p_2 \Lambda^2 + p_1 \Lambda + p_0, \end{aligned} \tag{105}$$

with

$$\begin{aligned} p_2 &= Q_1 + Q_2 + Q_3 + \nu, \\ p_1 &= Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3 - \nu (J_1 + J_2), \\ p_0 &= Q_1 Q_2 Q_3 + \nu J_1 J_2. \end{aligned} \tag{106}$$

It turns out that we can find the value of $\hat{\mathcal{S}}$ at the critical point without knowing the exact solution for the critical point. We use the fact that $\mathcal{S}$ is homogeneous of degree 1 (it is a monomial), and thus

$$\sum_a X_a \frac{\partial \mathcal{S}}{\partial X_a} + \sum_i \omega_i \frac{\partial \mathcal{S}}{\partial \omega_i} = \mathcal{S}. \tag{107}$$

Substituting into Eq. (102) we find

$$S_{BH} = \hat{\mathcal{S}}\big|_{\text{crit}} = -2\pi i \Lambda. \tag{108}$$

Since Λ is the solution to the cubic equation (105), it looks like there are three possible values for the entropy. However, since for real charges the cubic equation has real coefficients, we either find three real roots or one real and two complex conjugate roots for Λ. Imposing that the entropy be *real positive*, we require that there is one real and two imaginary conjugate roots, then only one of them—the one along the positive imaginary axis—leads to an acceptable value for the entropy. Since $(\Lambda-\beta)(\Lambda-i\alpha)(\Lambda+i\alpha)=\Lambda^3-\beta\Lambda^2+\alpha^2\Lambda-\beta\alpha^2$, we obtain the following constraint on the charges:

$$p_0=p_1p_2 \quad \text{and} \quad p_1>0. \tag{109}$$

One can check that the parametrization (5) automatically solves the first equation. Then the roots of Eq. (105) are $\Lambda\in\{-p_2,\pm i\sqrt{p_1}\}$. The physical solution is

$$\Lambda=i\sqrt{p_1} \;\Rightarrow\; S_{\mathrm{BH}}=2\pi\sqrt{p_1}, \tag{110}$$

which is precisely Eq. (9). We stress that the conditions Eq. (109) are necessary, but not sufficient, to guarantee that the supergravity solution is well-defined [70].

It is not difficult to write the values of the chemical potentials at the critical point. To simplify the notation, let us define

$$\begin{aligned} P_{1,2,3}&=Q_{1,2,3}+\Lambda, \qquad & \Phi_{1,2,3}&=X_{1,2,3},\\ P_{4,5}&=J_{1,2}-\Lambda, \qquad & \Phi_{4,5}&=-\omega_{1,2}, \end{aligned} \tag{111}$$

and use an index $A=1,\dots,5$. Equations (104) imply that

$$\Phi_AP_A \quad \text{are all equal for } A=1,\dots,5. \tag{112}$$

Implementing the constraint (101), the solution is

$$\Phi_A=-\frac{1}{P_A}\left(\sum_{B=1}^{5}\frac{1}{P_B}\right)^{-1}. \tag{113}$$

Since, even for real charges, the P_A's are complex, the solutions Φ_A are in general complex.

b. *Equal angular momenta.*—Let us specialize the formulas to the case $J_1=J_2\equiv J$, and determine useful inequalities satisfied by the chemical potentials at the critical point. First of all, from the constraint (101) it immediately follows

$$-\frac{1}{\omega}=\frac{X_1}{\omega}+\frac{X_2}{\omega}+\frac{X_3}{\omega}-2. \tag{114}$$

At the critical point (113) one finds

$$\frac{X_a}{\omega}=-\frac{J-\Lambda}{Q_a+\Lambda}, \qquad \operatorname{Im}\left(\frac{X_a}{\omega}\right)=\sqrt{p_1}\,\frac{Q_a+J}{Q_a^2+p_1}>0. \tag{115}$$

To obtain the last inequality we used that $Q_a+J>0$ for the BPS black holes, as we showed in Eq. (8). This implies that

$$\operatorname{Im}\left(-\frac{1}{\omega}\right)>\operatorname{Im}\left(\frac{X_a}{\omega}\right)>0 \quad \text{for } a=1,2,3. \tag{116}$$

Using the explicit parametrization (11) presented in Sec. II (and setting $g=1$ for the sake of clarity), one can also show that

$$\operatorname{Re}(\omega)=\frac{1}{2(1+\gamma_1)}, \qquad \operatorname{Im}(\omega)=\frac{\nu\gamma_2}{4(1+\gamma_1)\sqrt{p_1}}, \tag{117}$$

where $p_1=\nu^2\big((1+\gamma_1)\gamma_3-\frac{1}{4}\gamma_2^2\big)$. In particular, the first equation shows that

$$0<\operatorname{Re}(\omega)<\frac{1}{2}. \tag{118}$$

c. *Entropy from the index.*—Finally, we compare the contribution to the index from the basic solution in the first case, given in Eq. (99), with the entropy function $\mathcal{S}$ in Eq. (100). The latter, after eliminating X_3 with the constraint (101) and restricting to equal angular fugacities, reads

$$\mathcal{S}=-\pi iN^2\,\frac{X_1X_2(2\omega-1-X_1-X_2)}{\omega^2}. \tag{119}$$

We see that it is exactly equal to Eq. (99), as long as we can identify

$$\tau=\omega, \;\; [\Delta_a]_\tau=X_a \quad \text{for } a=1,2,3. \tag{120}$$

This is not obvious, but we can check that it is indeed possible. First of all, X_1 and X_2 should satisfy the strip inequalities that $[\cdot]_\tau$ does, at least in a neighborhood of the critical point. This is precisely what we proved in Eq. (116). Second, the fugacities at the critical point should also satisfy the inequalities Eq. (70) that define the first case. Because of the constraint, this is the same as requiring that also X_3 satisfies Eq. (116), which is true. Thus, this concludes our proof. Let us stress that, in our approach, the constraint Eq. (101) with the correct constant term simply comes out of the large N limit.

One could wonder what is the physics described by the domain of analyticity named second case in Eq. (73). It appears that it reproduces the very same black-hole entropy as the first case. Indeed, as is apparent from Eq. (75), in the two cases Θ takes almost the same form, the only difference being that $[\cdot]_\tau$ and $[\cdot]'_\tau$ satisfy opposite strip inequalities and a constraint with an opposite constant term. It was already observed in Ref. [38] that the entropy function $\mathcal{S}$ reproduces the black-hole entropy with either one of the two constraints imposed. We leave for future work to understand what is the role of such a twin contribution.

A. Example: Equal charges and angular momenta

In order to make some of the previous statements more concrete, we now study in detail a very special case in which the index counts states with equal charges $Q_a \equiv Q$ and angular momenta $J_i \equiv J$. This will be instructive to elucidate the structure of Stokes lines.

Let us first quickly summarize the properties of black holes and their entropy in this case [31]. We set $\nu = 1$ (all charges are in "units" of ν) so that

$$p_0 = Q^3 + J^2, \quad p_1 = 3Q^2 - 2J, \quad p_2 = 3Q + 1, \tag{121}$$

and the charge constraint is

$$p_1 p_2 - p_0 = 8Q^3 + 3Q^2 - 2(3Q+1)J - J^2 = 0. \tag{122}$$

This is quadratic in J and potentially leads to two branches of solutions. However, only one of them satisfies Eq. (2) when parametrized in terms of μ (we also set $g = 1$):

$$\begin{aligned} &Q = \mu + \frac{1}{2}\mu^2, \qquad \Lambda = i\sqrt{p_1}, \qquad S = 2\pi\sqrt{p_1}, \\ &J = (2Q+1)^{3/2} - 3Q - 1 = \frac{3}{2}\mu^2 + \mu^3, \\ &p_1 = 3Q^2 + 6Q + 2 - 2(2Q+1)^{3/2} = \mu^3 + \frac{3}{4}\mu^4. \end{aligned} \tag{123}$$

The entropy is positive for $Q > 0$, and in this range $J > 0$.

The extremization problem (102) simplifies because we have only two chemical potentials, $X \equiv X_{1,2,3}$ and ω, with the constraint (101). The critical point is

$$\omega = \frac{Q + \Lambda}{2Q + 3J - \Lambda}, \qquad X = -\frac{J - \Lambda}{2Q + 3J + \Lambda}. \tag{124}$$

Let us mention that in the alternative extremization problem in which the constraint Eq. (101) is modified by changing $+1$ into -1, the critical values of X and ω are given by the same expressions, however, the critical value of the Lagrange multiplier becomes $\Lambda = -i\sqrt{p_1}$.

We now turn to the index. Given the identifications $X_a = [\Delta_a]_\tau$ and $\omega = \tau$, we can restrict to chemical potentials such that $[\Delta_1]_\tau = [\Delta_2]_\tau = [\Delta_3]_\tau \equiv [\Delta]_\tau$, where Δ_3 is defined through the general constraint equation (74). Up to integer shifts, this amounts to

$$\Delta_1 = \Delta_2 = \Delta_3 \equiv \Delta = \frac{2\tau - 1}{3}. \tag{125}$$

The critical points, Eq. (124), indeed satisfy this relation. We have thus reduced to a single independent chemical potential τ. Notice that the function

$$\mathcal{I}\Big(\Delta(\tau); \tau\Big) = \mathcal{I}\left(\frac{2\tau - 1}{3}; \tau\right) \tag{126}$$

is periodic under $\tau \to \tau + 3$, therefore we will restrict to $0 \leq \operatorname{Re}\tau < 3$.

We study the large N formula (82) for the index, in particular we want to determine the structure of the leading contributions as τ is varied, and where the Stokes lines are. To do so, we need the values of the bracketed potentials $[\Delta]_{\tau+r}$ for $r \in \mathbb{Z}$. We find

$$[\Delta]_{\tau+r} = \left[\frac{2\tau - 1}{3}\right]_{\tau+r} = \begin{cases} \Delta + \frac{2r}{3} & \text{if } r = 0 \mod 3 \\ \text{undefined} & \text{if } r = 1 \mod 3 \\ \Delta + \frac{2r-1}{3} & \text{if } r = 2 \mod 3. \end{cases} \tag{127}$$

In the second case the bracket is not defined because $\operatorname{Im}[\Delta/(\tau + r)] \in \mathbb{Z} \times \operatorname{Im}[1/(\tau + r)]$, i.e., because Δ sits exactly on the boundary of a strip defined by $\tau + r$. We can, however, consider $[\Delta]_{\tau+r}$ for values of Δ that are a bit off the boundary of the strip in the real direction. We consider the values $\Delta_{(\pm)} = \Delta \pm \epsilon$ with infinitesimal $\epsilon > 0$ and find

$$\begin{aligned} [\Delta_{(+)}]_{\tau+r} &\underset{\epsilon\to 0}{\longrightarrow} \Delta + \tfrac{2r-2}{3} \\ [\Delta_{(-)}]_{\tau+r} &\underset{\epsilon\to 0}{\longrightarrow} \Delta + \tfrac{2r+1}{3} \end{aligned} \qquad \text{if } r = 1 \mod 3. \tag{128}$$

Using these formulas, the values of $\Theta(\Delta, \tau + r)$ are easily computed [71]. In particular, the imaginary parts of Θ computed on $\Delta_{(\pm)}$ are the same.

The dominant contribution to the index is found by comparing the absolute values of $\exp[-\pi i N^2 \Theta(\Delta; \tau + r)]$—or equivalently the imaginary parts of Θ—as we vary r. When there is a particular value $\hat{r}$ for which $\operatorname{Im}\Theta(\Delta; \tau + \hat{r})$ is maximum, there is one dominant contribution which leads to a concrete estimate of the leading behavior of the index. When, instead, there is no maximum, we are left with an infinite number of competing contributions and more detailed information would be needed to resum them. We obtain the following values for the imaginary part of Θ:

$$\operatorname{Im}\Theta(\Delta; \tau + r) = \begin{cases} \dfrac{2\operatorname{Im}\tau}{27}\left(4 + \dfrac{\operatorname{Re}\tau + r}{|\tau + r|^4} - \dfrac{3}{|\tau + r|^2}\right), & \text{if } r = 0 \mod 3 \\ \dfrac{8\operatorname{Im}\tau}{27}, & \text{if } r = 1 \mod 3 \\ \dfrac{2\operatorname{Im}\tau}{27}\left(4 - \dfrac{\operatorname{Re}\tau + r}{|\tau + r|^4} - \dfrac{3}{|\tau + r|^2}\right), & \text{if } r = 2 \mod 3. \end{cases} \tag{129}$$

Notice that the limiting value for large $|r|$ (equal to the value for $r = 1 \mod 3$) is as in Eq. (87). If there is a value of r that maximizes $\operatorname{Im}\Theta$, it must come from the first or third case. In particular, there exists $\hat{r}$ with $\hat{r} = 0 \mod 3$ if τ satisfies the following relation:

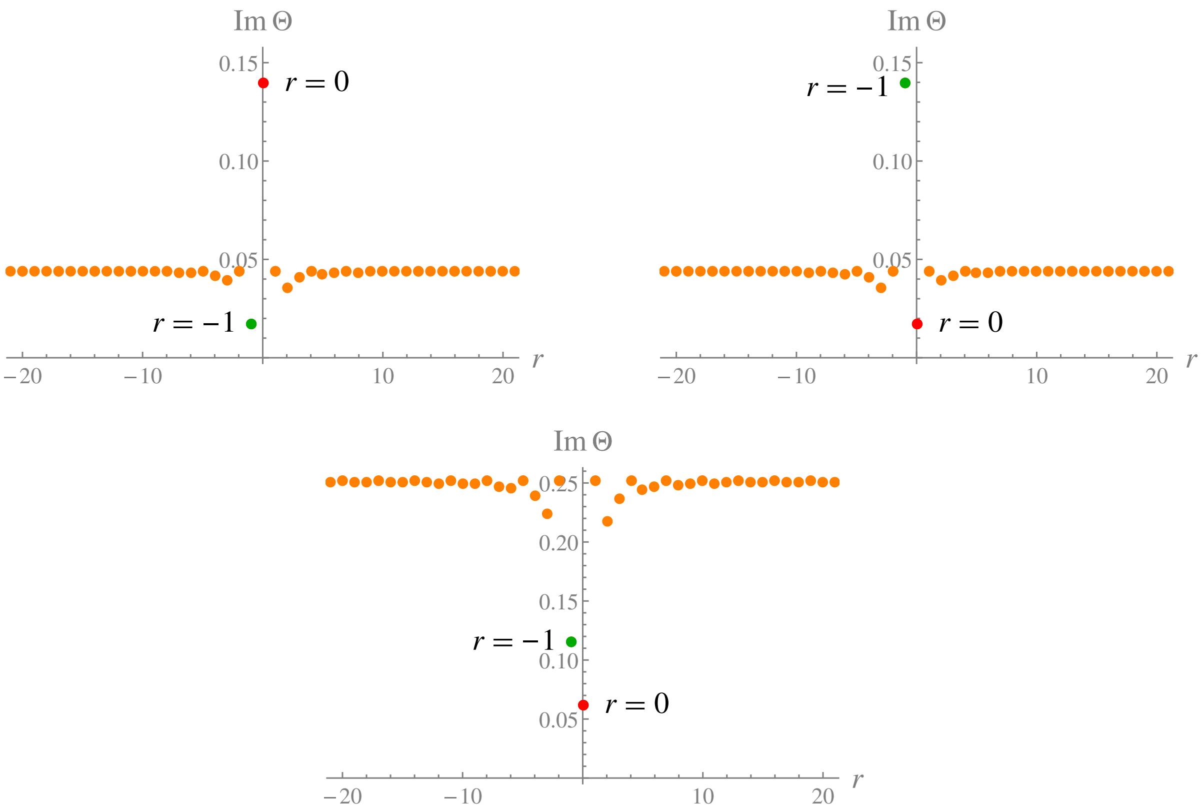

FIG. 2. The upper-left plot shows the values of $\mathrm{Im}\,\Theta(\Delta;\tau+r)$ as a function of r, for a sample value of τ inside the semicircle (130). The red dot corresponds to $r=0$, which is the dominant contribution in this case. The upper-right plot shows $\mathrm{Im}\,\Theta(\Delta;\tau+r)$ for τ inside the semicircle (131). The green dot corresponds to $r=-1$, which is the dominant contribution in this case. The lower plot shows the values of $\mathrm{Im}\,\Theta(\Delta;\tau+r)$ for τ outside the two semicircles, where there is no dominant contribution.

$$\mathrm{Re}\,\tau+\hat{r}>3|\tau+\hat{r}|^2 \quad \text{with} \quad \hat{r}=0 \text{ mod } 3. \tag{130}$$

This corresponds to the interior of a semicircle in the upper half τ plane, centered at the boundary point $\tau=1/6-\hat{r}$ and with radius $1/6$. Similarly, there exists $\hat{r}$ with $\hat{r}=2$ mod 3 if τ satisfies

$$-\mathrm{Re}\,\tau-\hat{r}>3|\tau+\hat{r}|^2 \quad \text{with} \quad \hat{r}=2 \text{ mod } 3. \tag{131}$$

This corresponds to the interior of another semicircle of radius $1/6$, centered at $\tau=-1/6-\hat{r}$. The two inequalities (130) and (131) define two semicircles in the fundamental range $0\leq\mathrm{Re}\,\tau<3$, for $\hat{r}=0$ and $\hat{r}=-1$ respectively, as well as all their images under the periodicity $\tau\to\tau+3$. On the other hand, outside the two regions there is no dominant contribution because, for all values of r, $\mathrm{Im}\,\Theta$ is smaller than the limiting value. In Fig. 2 we provide plots of $\mathrm{Im}\,\Theta(\Delta;\tau+r)$ as r is varied, both for τ inside the semicircle (130), inside the semicircle (131), and outside those two.

In Fig. 3 we represent the fundamental domain with range $0\leq\mathrm{Re}\,\tau<3$ in the upper half τ plane, dividing it into regions according to the dominant contribution. In Fig. 4 we represent the same information in the q plane, using $q^{1/3}$ as the variable. The red semicircle (130) corresponds to the values of τ in which $\hat{r}=0$, while the green semicircle (131) corresponds to $\hat{r}=-1$. These are two different domains of analyticity. The remaining "Max?" region, in blue, corresponds to values of τ for which there is no dominant contribution within the ones we studied in this paper. The three regions are separated by Stokes lines (in black).

Inside the red semicircle (130) the large N limit of the superconformal index is

$$\log\mathcal{I}_\infty(\Delta;\tau)=-\pi iN^2\Theta(\Delta;\tau)$$
$$=-\pi iN^2\frac{[\Delta]_\tau^3}{\tau^2}=-\pi iN^2\frac{(2\tau-1)^3}{27\tau^2}. \tag{132}$$

This expression exactly matches the entropy function (100) of black holes with equal charges Q and angular momenta J, and its Legendre transform selects the critical points (124). We represent the line of critical points, as $\mu>0$ is varied, by a blue solid line in Figs. 3 and 4. As we see from there, for $\mu>\mu_*$ the blue line lies inside the red semicircle, meaning that the entropy of the single-center black hole is the dominant contribution to the index. This seems to confirm that "large" BPS black holes, with $Q>Q_*$ or

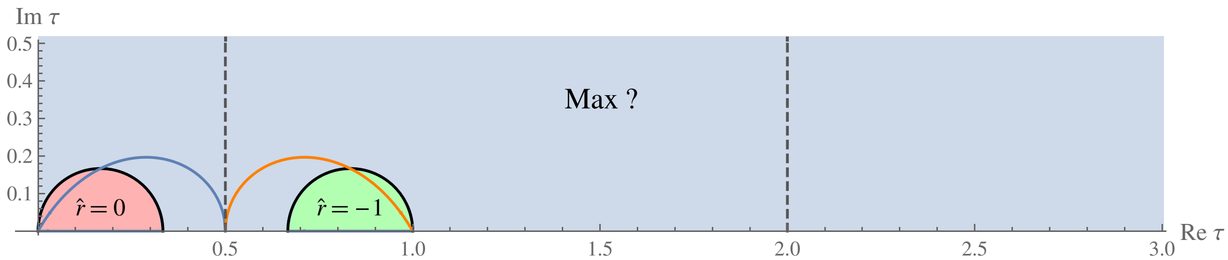

FIG. 3. Stokes lines in the τ plane. The red semicircle corresponds to the domain of analyticity where the $r = 0$ contribution dominates the index. The green semicircle instead corresponds to the domain where the $r = -1$ contribution dominates. In the remaining region we do not have a dominant contribution. The blue and orange lines are the critical points of the entropy function for BPS black holes, in the two possible formulations.

equivalently $J > J_*$, are stable. On the contrary, for $0 < \mu < \mu_*$ the blue line plunges into the "Max?" region. We can still identify the black-hole entropy with the contribution of the basic solution Eq. (34) to the index, however, such a contribution is no longer dominant. This suggests that "small" BPS black holes with $Q < Q_*$ might be unstable toward other supergravity configurations. We find the following values at the transition point:

$$\mu_* = \frac{2}{3}, \quad \tau_* = \frac{1+i}{6}, \quad Q_* = \frac{8}{9}, \quad J_* = \frac{26}{27}, \quad S_* = \frac{4\pi}{3}, \tag{133}$$

where Q, J, S are in units of ν. It would be nice to derive these values from supergravity.

The green circle in Figs. 3 and 4 corresponds to values of τ for which the $r = -1$ contribution dominates. In this domain we find

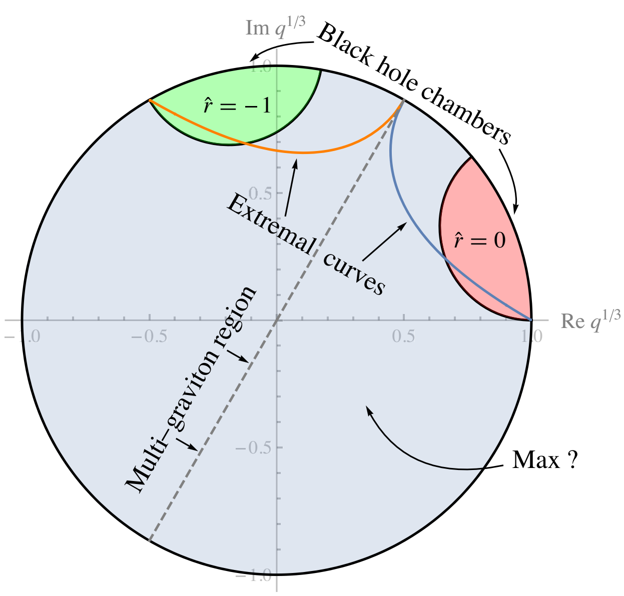

FIG. 4. Stokes lines in the q plane, where the variable is $q^{1/3}$. The notation is the same as in Fig. 3. The dashed line indicates the subspace where both fugacities q and y are real and the computation of Ref. [30], which reproduced the index of multigraviton states, applies.

$$\log \mathcal{I}_\infty(\Delta;\tau) = -\pi i N^2 \Theta(\Delta; \tau - 1) = -\pi i N^2 \left(\frac{(2\tau-1)^3}{27(\tau-1)^2} - 1 \right). \tag{134}$$

This also reproduces the entropy of single-center black holes: This expression matches the entropy function (100) with the alternative constraint among the chemical potentials, given by Eq. (101) with $+1$ substituted with -1. In the figures we have indicated with a solid orange line the critical points obtained with the alternative extremization principle.

It is interesting to draw the subspace where both fugacities q and y are real and the computation of Ref. [30], which reproduced the index of multigraviton states in AdS_5, applies. We include this subspace both in Fig. 3 and, in terms of $q^{1/3}$, in Fig. 4. We see that the real subspace does not intercept the black-hole lines: It only asymptotically reaches them, at the tail that describes black holes much smaller than the AdS radius.

ACKNOWLEDGMENTS

We thank A. Arabi Ardehali, A. Dabholkar, J. Maldacena, S. Minwalla, S. Murthy and A. Zaffaroni for useful discussions on this and related matters. F. B. is supported in part by the MIUR-SIR Grant No. RBSI1471GJ "Quantum Field Theories at Strong Coupling: Exact Computations and Applications."

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$$\sqrt{(1-a^2g^2)(1-b^2g^2)}=\frac{2ab+2g^{-1}(a+b)+3g^{-2}}{\mu_1+\mu_2+\mu_3+3g^{-2}}$$

fixes a, b in terms of $\mu_{1,2,3}$ and Ξ.

[50] For instance, take μ_1 that goes to zero with $\mu_{2,3}$ fixed, then Q_1 becomes negative. One may wonder whether the extra condition that the entropy be real could force the charges to be positive. This is not the case. For instance, setting $\mu_1=\mu_2^2/3(1+\mu_2)$ and $\mu_3=\mu_2$ as well as $\Xi=1$, one finds (up to constant factors and setting $g=1$) that $Q_1\sim-\mu_2^2/6<0$, $Q_2=Q_3\sim\mu_2(\mu_2+2)/2>0$ and $S_{\mathrm{BH}}^2\sim\mu_2^4/12>0$.

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[67] Stokes lines divide the complex plane into regions in which the limit gives different analytic functions. Because of their origin, Stokes lines have the property that only the imaginary part of the function can jump, while the real part must be continuous. One can indeed check that Eq. (63) satisfies this property.

[68] More precisely, $\log\tilde\Gamma(u+k\tau;\tau,\tau)=-2\pi ik(k-1)(u+\frac{1}{2})/2-2\pi ik(k-1)(k-2)\tau/6+k\log\theta_0(u;\tau)+\log\tilde\Gamma(u;\tau,\tau)$. Thus, when summing over $i\neq j$ up to N, one can generate terms of order $\mathcal{O}(N^2)$ linear in the chemical potentials. These terms only shift the charges in the Legendre transform, but do not alter the extremization problem and the entropy discussed in Secs. V and VI.

[69] A. Sen, *Arithmetic of Quantum Entropy Function*, J. High Energy Phys. 08 (2009) 068; A. Dabholkar, J. Gomes,

S. Murthy, and A. Sen, *Supersymmetric Index from Black Hole Entropy*, J. High Energy Phys. 04 (2011) 034.

[70] For example, take $Q_1 = Q_2 = Q_3 \equiv Q$ and $J_1 = J_2 \equiv J$. We find that Eq. (109) is solved by $J = -3Q - 1 \pm (2Q+1)^{3/2}$, and both branches are covered by the parametrization, Eq. (5), as $Q = \mu + \mu^2/2$ and $J = 3\mu^2/2 + \mu^3$. Then we have $p_1 = 3Q^2 + 6Q + 2 \mp 2(2Q+1)^{3/2}$ and one can check that, for $Q > 0$, both branches have $p_1 > 0$. However, only the branch with the upper sign satisfies also Eq. (2)—here $\mu > 0$—and corresponds to well-defined supergravity solutions, while the branch with the lower sign does not.

[71] For $r = 0$ mod 3 one has to use the first case of Θ, while for $r = 2$ mod 3 the second case.