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On critical behaviour in generalized Kadomtsev–Petviashvili equations

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Abstract

An asymptotic description of the formation of dispersive shock waves in solutions to the generalized Kadomtsev–Petviashvili (KP) equation is conjectured. The asymptotic description based on a multiscales expansion is given in terms of a special solution to an ordinary differential equation of the Painlevé I hierarchy. Several examples are discussed numerically to provide strong evidence for the validity of the conjecture. The numerical study of the long time behavior of these examples indicates persistence of dispersive shock waves in solutions to the (subcritical) KP equations, while in the supercritical KP equations a blow-up occurs after the formation of the dispersive shock waves.

1 Introduction

In this manuscript we consider the Cauchy problem for the generalized Kadomtsev–Petviashvili (KP) equations

$$(u_t + u^n u_x + \epsilon^2 u_{xxx})_x = \sigma u_{yy}, \quad n \in \mathbb{N}, \tag{1.1}$$

where ϵ is a small positive parameter and $\sigma = \pm 1$. We are interested in studying the behaviour of solutions $u(x, y, t; \epsilon)$ for $\epsilon \to 0$ when the initial data $u(x, y, t = 0; \epsilon) =$

$u_0(x, y)$ are independent from ϵ. Generically the solution develops oscillations that are called *Dispersive Shock Waves* [26] in the nonlinear wave community or *undular bore* in the fluid dynamics community. The related mathematical problem is now well understood for several dispersive equations in one space dimension, starting with the seminal work of Lax and Levermore [43] (see also [58] and [11]) for the Korteweg–de Vries (KdV) equation (see [23] for a numerical treatment of the problem). However, a rigorous mathematical study of two-dimensional dispersive shock waves remains an open problem even though there are some heuristic arguments for several dispersive equations in two spatial dimension like [2], [27], [18].

Our main result is a description of the solution $u(x, y, t; \epsilon)$ of the generalized KP equation for generic initial data, on the onset of the oscillations in terms of a particular solution of an ordinary differential equation, the so-called Painlevé I2 equation (PI2), the second member of the Painlevé I hierarchy. This description extends the universality results on critical behaviour of Hamiltonian PDEs, first obtained for one-dimensional evolutionary equations [13], [14], [15], [16], to PDEs in two spatial dimensions.

For $n = 1$ the equations (1.1) are known as KP equations (KP I for $\sigma = +1$ and KP II for $\sigma = -1$). They were introduced in [32] to study the transverse stability of the solitary wave solution of the KdV in a $2 + 1$ dimensional setting. Both cases can be derived as models for nonlinear dispersive waves on the surface of fluids [32] (see also [30, 36] for further references). The settings studied in this context are essentially one-dimensional waves with weak transverse modulation. KP I is applicable in the case of strong surface tension, whereas KP II is a model for weak surface tension. Note that KP I has a focusing effect, whereas KP II is defocusing. KP type equations also arise as a model for sound waves in ferromagnetic media [57] and in the description of two-dimensional nonlinear matter-wave pulses in Bose–Einstein condensates, see e.g. [28, 31]. As for the KdV equation, KP equations appear as an asymptotic description in the limit of long wavelengths. In general the dispersion of KP equations is too strong compared to what is found in applications. A way to tilt the balance between dispersion and nonlinearity towards the nonlinearity is to consider generalized KP equations (1.1) with a stronger nonlinearity $n > 1$. Interestingly the generalized KP for $n = 2$ appears as a model for the evolution of sound waves in antiferromagnetic materials, see [57].

In the dimensionless generalized KP equation, i.e., equation (1.1) with $\epsilon = 1$, a parameter ϵ can be introduced in the following way: a possible approach to studying the long time behavior of solutions of this dimensionless generalized KP equation is to consider slowly varying initial data of the form $u_0(\epsilon x, \epsilon y)$ where $0 < \epsilon \ll 1$ is a small parameter and $u_0(x, y)$ is some given initial profile. As $\epsilon \to 0$ the initial data approach a constant value. Hence, in order to see nontrivial effects one has to wait until sufficiently long times of order $t \sim O(1/\epsilon)$, which consequently requires to rescale the spatial variables onto macroscopically large scales $x \sim O(1/\epsilon)$, too. In other words, we consider $x \mapsto \tilde{x} = x\epsilon$, $x \mapsto \tilde{y} = y\epsilon$, $t \mapsto \tilde{t} = t\epsilon$ and put $u^\epsilon(\tilde{t}, \tilde{x}, \tilde{y}) = u(\tilde{t}/\epsilon, \tilde{x}/\epsilon, \tilde{y}/\epsilon)$ to obtain equation (1.1) (we omit the 'tildes' for simplicity). The limit of small ϵ is called the *small dispersion limit*.

We expect that for a reasonable class of smooth initial data $u_0(x,y)$ the solution $u(x,y,t;\epsilon)$ to the Cauchy problem for the generalized KP equation remains smooth and depends continuously on the sufficiently small scaling parameter ϵ on a finite time interval. The solution $u(x,y,t,\epsilon)$ of the generalized KP equation (1.1) is expected to converge in the limit $\epsilon \to 0$ to the solution $u(x,y,t)$ of the generalized dKP equation (1.2)

$$(u_t + u^n u_x)_x = \pm u_{yy}, \quad n \in \mathbb{N}, \tag{1.2}$$

for $0 < t < t_c$ where t_c is the time where the solution of equation (1.2) first develops a singularity (blow-up of gradient) which generically appears in one point (x_c, y_c) of the plane. Equations (1.2) are called generalized dKP equations or generalized dispersionless KP equations, even though the equations (1.2) have dispersion. For $n = 1$ these equations were derived earlier than the KP equation by Lin, Reissner and Tsien [44] and Khokhlov and Zabolotskaya [60] in a $3+1$ dimensional setting. However, in this manuscript we call (1.2) generalized dKP equations.

Local well-posedness of the Cauchy problem for dKP equation ($n = 1$) has been proved in certain Sobolev spaces in [53], while for the generalized dKP equation a similar result is still missing to the best of our knowledge. Furthermore although many strong results about the Cauchy problem for the KP equation in various functional spaces are now available (see, e.g., [5, 46, 50, 54]), these results are still insufficient to rigorously justify the small ϵ behaviour of solutions to KP and its generalizations. Namely, for $t < t_c$ the solution of the generalized KP equation $u(x,y,t;\epsilon)$ is conjectured to be approximated in the limit $\epsilon \to 0$ by the solution $u(x,y,t)$ of the generalized dKP equation with the same initial data. The numerical results of Sections 4, 5 provide a rather convincing motivation for the above conjectural statement. Such conjectural small ϵ behaviour is somehow expected in the subcritical case, where solutions of generalized KP do no have blow-ups, while in the supercritical case, when solutions of generalized KP equations can have a blow-up in finite time $T < \infty$, our conjecture implies that $T > t_c$.

Moreover, our numerical results strongly support the conjectural analytic description of the leading term in the asymptotic expansion of solutions *near* the critical time t_c that we will explain now. Let (x_c, y_c) be the point where the solution of generalized dKP equation (1.2) develops a singularity and let $u_c = u(x_c, y_c, t_c)$. Furthermore let us denote by $\bar{x} := x - x_c$, $\bar{y} := y - y_c$ and $\bar{t} := t = t_c$ the shifted coordinates in the neighbourhood of the critical point and let us introduce the following variables

$$X = \bar{x} - u_c^n \bar{t} + c_1 \bar{t}\bar{y} + c_2 \bar{y} + c_3 \bar{y}^2 + c_4 \bar{y}^3, \quad T = \bar{t} + b\,\bar{y}^2 \tag{1.3}$$

where $c_1, \ldots, c_4$ and b are constants that depend on the equation and the initial data. Then in the double scaling limit $\bar{x} \to 0$, $\bar{y} \to 0$, $\bar{t} \to 0$ and $\epsilon \to 0$ in such a way that $X/\epsilon^{\frac{6}{7}}$ and $T/\epsilon^{\frac{4}{7}}$ remain finite, the solution of the generalized KP equation $u(x,y,t;\epsilon)$ in a neighbourhood of the point (x_c, y_c, t_c) at the onset of the oscillations has the following expansion

$$u(x,y,t;\epsilon) = u_c + \frac{6}{n u_c^{n-1}} \left(\frac{\epsilon^2}{\kappa^2}\right)^{\frac{1}{7}} U\left(\frac{X}{(\kappa\epsilon^6)^{1/7}}, \frac{T}{(\kappa^3\epsilon^4)^{1/7}}\right) + \beta\bar{y} + O(\epsilon^{\frac{4}{7}}), \tag{1.4}$$

where β and κ are constants and X and T have been defined above in (1.3).

The function $U = U(\mathcal{X}, \mathcal{T})$ satisfies the PI2 equation

$$\mathcal{X} = 6\mathcal{T}U - \left(U^3 + \frac{1}{2}U_{\mathcal{X}}^2 + UU_{\mathcal{X}X} + \frac{1}{10}U_{\mathcal{X}XXX} \right). \tag{1.5}$$

The relevant solution is uniquely determined by the asymptotic conditions

$$U(\mathcal{X}, \mathcal{T}) = \mp|\mathcal{X}|^{\frac{1}{3}} \mp \frac{2\mathcal{T}}{|\mathcal{X}|^{\frac{1}{3}}} + O(|\mathcal{X}|^{-1}), \quad |\mathcal{X}| \to \infty. \tag{1.6}$$

The existence for real $\mathcal{X}$ and $\mathcal{T}$ of a smooth real solution of the PI2 equation (1.5) satisfying the boundary conditions (1.6) has been conjectured in [13] and proved in [9]. The solution has an oscillatory region that is formed around the point $\mathcal{X} = 0$ and at about the time $\mathcal{T} = 0$ and it is developing for $\mathcal{T} \gg 1$ into the Gurevich–Pitaevski solution [26], [8], [23], [55]. We remark that from (1.4) for generic initial data the oscillatory front is, approximately, a straight line in the (x, y) plane, while for initial data having a y-symmetry, the oscillatory front has a parabolic shape.

Several numerical examples are presented to provide strong numerical support for our conjecture. Furthermore we will show in section 6 that the formula (1.6) catches some of the qualitative features of the oscillatory regime also for some time considerably bigger then the critical time t_c.

The paper is organised as follows: in section 2 we collect some mathematical facts about KP equations and present the main conjecture of this paper. In section 3 we briefly summarize the used numerical techniques. Solutions to the KP equations are discussed for various initial data in the vicinity of the critical points in section 4 and compared to the asymptotic description. In section 5 we study solutions to the generalized KP equation for $n = 3$ for the same initial data near the critical points. We conclude in section 6 with a preliminary discussion of the longtime behavior of solutions to KP and generalized KP equations, i.e., the formation of dispersive shocks for all times in the former and eventual blow-up in the later, and we outline directions of further research.

2 Asymptotic description of break-up in KP solutions

In this section we present an asymptotic description of the formation of a dispersive shock wave in solutions to the generalized KP equations for initial data in the Schwartz class. We first collect some mathematical facts about generalized KP equations, then summarize previous work on the formation of dispersive shock waves in KdV solutions. Finally the latter results are generalized to the case of KP equations.

2.1 Mathematical preliminaries

The integrability of the KP equations was obtained in 1974 [12] via Lax pair. However, the problem to effectively integrate the equation proved to be quite difficult and the first achievements were obtained in [61, 47, 19, 1]. For initial data $u_0(x, y)$ in the Schwartz class, the solution $u(x, y, t; \epsilon)$ remains a smooth function of x and y. However, the solution $u(x, y, t; \epsilon)$ immediately leaves the Schwartz space of rapidly decreasing functions as soon as $t > 0$ even though it remains in $L^2(\mathbb{R})$ since the quantity

$$\int_{\mathbb{R}^2} u^2(x, y, t; \epsilon) dxdy = \int_{\mathbb{R}^2} u_0^2(x, y) dxdy,$$

is conserved in time.

For $x \to -t\infty$ the solution is still rapidly decreasing while for $x \to t\infty$ one has for KP I [4]

$$u(x, y, t; \epsilon) = \frac{c}{\sqrt{tx}x} \int_{\mathbb{R}^2} dx'dy'u_0(x', y') + o(|x|^{-\frac{3}{2}}) \tag{2.1}$$

with c a constant. Furthermore the solution for $t > 0$ satisfies an infinite number of dynamical constraints, the first two, taking the form [45]

$$\int_{\mathbb{R}} u(x, y, t; \epsilon) dx = 0, \tag{2.2}$$

$$\int_{\mathbb{R}} x u_y(x, y, t; \epsilon) dx = 0. \tag{2.3}$$

The dynamical constraints are satisfied even if the initial data do not satisfy them. This is a manifestation of the infinite speed of propagation inherent to the KP equations. The constraint (2.2) is satisfied also for the generalized KP equation [51]. For initial data in the Schwartz space, the solution $u(x, y, t; \epsilon)$ of the KP equation $u(x, y, t; \epsilon)$ is a C^∞ function in $\mathbb{R}^3_+ = \{(x, y, z)\,;\, t > 0\}$ even if the constraints are not satisfied at $t = 0$. This result has been proved via inverse scattering for the KP I equation for small norm initial data [20] and for the KP II equation in [1]. Global well-posedness for KP II was shown in [5] in the Sobolev space $H^s(\mathbb{R}^2)$, $s \geq 0$, so that for $s \geq 4$ one gets classical solutions while the global well-posedness for KP I was obtained in some subset of the Sobolev space $H^s(\mathbb{R}^2)$ [50]. For the generalized KP equation, local well-posedness results have been established in some weighted Sobolev space, see e.g. [29].

We can conclude that for initial data in the Schwarz space the solution $u(x, y, t; \epsilon)$ of the generalized KP equation is sufficiently regular in space and time for $0 < t < T$, where in general $T < \infty$ for the generalized KP equations, while $T = \infty$ for the KP equations. We also assume that the solution of the generalized KP equation depends continuously on the small parameter ϵ.

2.2 Break-up in dKP solutions

The same concepts can be applied to the solution of the generalized dKP equation (1.2). For generic initial data the solution of (1.2) exists till a time t_c where a gradient catastrophe occurs, however for $t < t_c$ the solution $u(x, y, t)$ of the generalized dKP equation is expected to be smooth in both in time and space for initial data in the Schwartz class. The L^2 norm of the solution is conserved as for the generalized KP equation. Integrating the generalized dKP equation with respect to x one obtains the constraint

$$\int_{\mathbb{R}} u_{yy}(x, y, t)dx = 0.$$

While the Cauchy problem for the generalized KP equation has seen considerable attention, the generalized dKP initial valued problem has been less studied. A Lax pair for dKP equation ($n = 1$) was obtained in [56]. A definition of integrability for the dKP equation was obtained in [21] where the method of hydrodynamic reduction is used. This method was introduced in [40], to obtain particular solutions of the dKP equation while more general solutions have been obtained in [48], [49] using inverse scattering. General solutions have also been obtained [52]. In [53] it is proved that the Cauchy problem of dKP ($n = 1$) is well posed in the Sobolev spaces $H^s(\mathbb{R}^2)$, for $t < t_c$ and $s > 2$, while a similar statement is missing for generalized dKP equation.

In a recent paper, two of the present authors, inspired by the works [48], [49] have studied the Cauchy problem for the dKP equation with smooth initial data using a change of the independent variable x suggested by the method of characteristics for the $1 + 1$ dimensional case. In the following we are using the same idea to obtain the solution of the generalized dKP equation (1.2) in the following form

$$\begin{cases} u(x, y, t) = F(\xi, y, t) \\ x = tF^n(\xi, y, t) + \xi \\ F(x, y, 0) = u_0(x, y) \end{cases} \tag{2.4}$$

where $u_0(x, y)$ is an initial datum in the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ of rapidly decreasing smooth functions. Plugging the above ansatz into the generalized dKP equation one obtains an equation for the function $F(\xi, y, t)$

$$\left(\frac{F_t \pm ntF^{n-1}F_y^2}{1 + ntF^{n-1}F_\xi}\right)_\xi = \pm F_{yy}, \tag{2.5}$$

with initial condition

$$F(x, y, 0) = u_0(x, y).$$

One can easily check that for y-independent initial data equation (2.4) is equivalent to the method of characteristics. The advantage of such a transformation is that while the solution $u(x, y, t)$ of the generalized dKP equation develops a singularity at a certain critical time t_c and at the point (x_c, y_c), the solution $F(\xi, y, t)$ seems to exist for much longer times, at least numerically.

For initial data in the Schwartz class, we assume that the solution of equation (2.5) is sufficiently smooth in time and space for certain time $0 < t < T$ with $T > t_c$. The change of the independent variable from x to ξ has a smoothing effect on the solution $F(\xi, y, t)$ in the sense that, numerically, the solution stays regular for much longer times than the critical time t_c where the derivatives of $u(x, y, t)$ blow up.

The first singularity appears when the change of coordinates $x = tF(\xi, y, t) + \xi$ is not invertible any more. The equations that describe the singularity formation have been considered in [3] using PDE techniques. To study the local behaviour of the function $u(x, y, t)$ (as a multivalued function) around the critical time t_c when the first singularity appears we make a Taylor expansion of (2.4) near the critical point $u_c(x_c, y_c, t_c)$ where $\xi = \xi_c$. Let us consider the gradients

$$u_x = \frac{F_\xi}{\Delta} \quad u_y = \frac{F_y}{\Delta}, \quad \Delta = 1 + ntF^{n-1}F_\xi(\xi, y, t). \tag{2.6}$$

The time of gradient blow-up is the smallest time t_c where the gradient goes to infinity, namely where

$$\Delta = 1 + ntF^{n-1}F_\xi(\xi, y, t) = 0.$$

For simplicity let us introduce the quantity

$$G(\xi, y, t) := F^n(\xi, y, t).$$

Since the quantity $\Delta(\xi, y, t)$, for $t < t_c$, has a definite sign in the ξ and y plane, the first point where it vanishes is a double zero, therefore the point of gradient catastrophe is characterised by the equations

$$\begin{aligned}
&\Delta = \pm 1 + ntF^{n-1}F_\xi(\xi, y, t) = 1 + tG_\xi(\xi, y, t) = 0\\
&\Delta_\xi = ntF^{n-2}(FF_{\xi\xi} + (n-1)F_\xi^2) = tG_{\xi\xi} = 0\\
&\Delta_y = ntF^{n-2}(FF_{\xi y} + (n-1)F_yF_\xi) = tG_{\xi y} = 0\\
&u(x, y, t) = F(\xi, y, t)\\
&x = tF^n(\xi, y, t) + \xi.
\end{aligned} \tag{2.7}$$

The point of gradient catastrophe is generic if

$$\Delta_{\xi\xi}(\xi_c, y_c, t_c) \neq 0, \quad \Delta_{\xi y}(\xi_c, y_c, t_c) \neq 0 \quad \Delta_{yy}(\xi_c, y_c, t_c) \neq 0.$$

Furthermore, at the point of gradient catastrophe it turns out (numerically) that F_{yy} remains bounded. So the solution of the equation (2.5) at the critical time satisfies the necessary conditions

$$\begin{aligned}
&F_t^c \pm t_c G_y^c F_y^c = 0, \quad F_{yt}^c \pm (t_c(G_y^c F_{yy}^c + G_{yy}^c F_y^c)) = 0,\\
&F_{\xi t}^c \pm t_c(G_y^c F_{y\xi}^c + G_{y\xi}^c F_y^c) = 0.
\end{aligned} \tag{2.8}$$

Now we are going to study the analytic behaviour of the solution (1.2) near the point (x_c, y_c, t_c). Introducing the shifted variables

$$x - x_c = \bar{x}, \quad t - t_c = \bar{t}, \quad y - y_c = \bar{y}, \quad \xi - \xi_c = \bar{\xi}$$

we obtain the equation

$$\begin{aligned}
&\bar{x} - \bar{t}(G^c + t_c G^c_t) - \bar{t}\bar{y}(G^c_y + t_c G^c_{yt}) - t_c(G^c_y \bar{y} + \frac{1}{6}G^c_{yyy}\bar{y}^3 + \frac{1}{2}G^c_{yy}\bar{y}^2) \\
&= \frac{t_c}{6}G^c_{\xi\xi\xi}\bar{\xi}^3 + \frac{1}{2}t_c G^c_{\xi\xi y}\bar{y}\bar{\xi}^2 + \frac{1}{2}(t_c\bar{y}^2 G^c_{\xi yy} + (2t_c G^c_{\xi t} + 2G^c_\xi)\bar{t})\bar{\xi} + o(\bar{t}^2, \bar{y}^4, \bar{\xi}^4, \bar{t}(\bar{y}^2 + \bar{\xi}^2))
\end{aligned} \tag{2.9}$$

where the notation $G^c_{\xi\xi\xi}$ stands for $\left.\frac{\partial^3}{\partial\xi^3}G(\xi, y_c, t_c)\right|_{\xi=\xi_c}$ and analogous notations hold for the other quantities. This suggests to introduce shifted variables (using $t_c = -1/G^c_\xi$):

$$\begin{aligned}
\zeta &= G^c_\xi\left(\bar{\xi} + \frac{G^c_{\xi\xi y}}{G^c_{\xi\xi\xi}}\bar{y}\right) \\
X &= \left[\bar{x} - \bar{t}(G^c + t_c G^c_t) - \bar{t}\bar{y}(G^c_y + t_c G^c_{yt}) - t_c\left(G^c_y\bar{y} + \frac{1}{2}G^c_{yy}\bar{y}^2 + \frac{1}{6}G^c_{yyy}\bar{y}^3\right)\right. \\
&\left. -\frac{1}{3}t_c\frac{(G^c_{\xi\xi y})^3}{(G^c_{\xi\xi\xi})^2}\bar{y}^3 + \frac{1}{2}t_c\frac{G^c_{\xi\xi y}G^c_{\xi yy}}{G^c_{\xi\xi\xi}}\bar{y}^3 + G^c_\xi\frac{G^c_{\xi\xi y}}{G^c_{\xi\xi\xi}}\bar{y}\bar{t}\right] \\
T &= \left[\bar{t} + \frac{t_c^2}{2}\bar{y}^2\left(\frac{(G^c_{\xi\xi y})^2}{G^c_{\xi\xi\xi}} - G^c_{\xi yy}\right)\right],
\end{aligned} \tag{2.10}$$

so that in the variable ζ, (2.9) takes the form

$$-\frac{k}{6}\zeta^3 + T\zeta = X + o(\bar{t}^2, \bar{y}^4, \bar{\xi}^4, \bar{t}(\bar{y}^2 + \bar{\xi}^2)), \tag{2.11}$$

where

$$k = t_c^4 G^c_{\xi\xi\xi}. \tag{2.12}$$

Using the estimates $\bar{\xi} \sim \bar{y} \sim \bar{t}^{1/2}$ and $\bar{x} \sim \bar{t}^{3/2}$ identified previously, the rescaling

$$\begin{aligned}
X &\to \lambda X \\
\bar{t} &\to \lambda^{\frac{2}{3}}\bar{t} \\
\bar{y} &\to \lambda^{\frac{1}{3}}\bar{y} \\
\zeta &\to \lambda^{\frac{1}{3}}\zeta,
\end{aligned} \tag{2.13}$$

in the limit $\lambda \to 0$ reduces (2.11) to the universal cusp singularity

$$-\frac{k}{6}\zeta^3 + T\zeta = X, \tag{2.14}$$

of the solution of a one-dimensional hyperbolic equation.

To leading order in the limit $\lambda \to 0$, it is consistent to expand $u(x, y, t)$ to linear order in $\bar{\xi}, \bar{y}$:

$$u(x, y, t) - u_c = F(\xi, y, t) - F^c \simeq F^c_\xi\bar{\xi} + F^c_y\bar{y} = \frac{F^c_\xi}{G^c_\xi}\zeta(X, T) + \bar{\beta}\bar{y}, \tag{2.15}$$

with

$$\bar{\beta} = F_y^c - \frac{F_\xi^c G_{\xi\xi y}^c}{G_{\xi\xi\xi}^c}. \tag{2.16}$$

2.3 The small dispersion limit of the KdV equation

We conjecture that, before the critical time t_c the solution $u(x, y, t; \epsilon)$ of the generalized KP equation (1.1) in the limit $\epsilon \to 0$ can be approximated by the solution $u(x, y, t)$ of the generalized dKP equation (1.2) with the same ϵ-independent initial data as long as the gradients remain bounded, namely for $t < t_c$ one is expected to have

$$u(x, y, t; \epsilon) = u(x, y, t) + O(\epsilon), \quad t < t_c.$$

We are interested to understand the behaviour of the KP solution $u(x, y, t; \epsilon)$ in a neighborhood of the critical point (x_c, y_c, t_c). For this purpose we first recall some results from the theory of the KdV equation.

We consider the small dispersion limit of the KdV equation

$$u_t + \beta\, uu_x + \epsilon^2 \rho\, u_{xxx} = 0.$$

where β and ρ are constants and ϵ is a small parameter.

For given smooth rapidly decreasing initial data $u_0(x)$, such a limit has been extensively studied in the works [43], [58], [11]. It was pointed out in [13] and numerically shown in [23] that some regions of the (x, t) plane escape the analysis of the small dispersion limit of the KdV equation. In particular one of these regions is a neighbourhood of the critical point (x_c, t_c) where the solution of the Hopf equation

$$u_t + \beta u u_x = 0$$

obtained by setting $\epsilon = 0$ in the KdV equation has a singularity. The solution of this equation for initial data $u_0(x)$ takes the form

$$\begin{cases} u(x, t) = u_0(\xi) \\ x = \beta u_0(\xi) t + \xi. \end{cases}$$

The solution has a point of gradient catastrophe at the time t_c with

$$t_c = \min_\xi \left(-\frac{1}{\beta u_0'(\xi)} \right).$$

The minimum point ξ_c gives $u_c = u_0(\xi_c)$ and the position $x_c = t_c u_0(\xi_c) + \xi_c$. The expansion near the critical points of the Hopf solution gives

$$\bar{x} := x - x_c - \beta u_c (t - t_c) \simeq \beta (t - t_c)(u - u_c) - \frac{t_c^4 \beta^3}{6} u_0'''(\xi_c)(u - u_c)^3.$$

It has been conjectured in [13] and proved in [6] (and for the KdV hierarchy in [7]) that near the point (x_c, t_c) the solution of KdV in the limit $\epsilon \to 0$ is approximated by

$$u(x,t;\epsilon) = u_c + \left(\frac{18\epsilon^2 b}{\gamma^2}\right)^{\frac{1}{7}} U\left(\frac{(48)^{1/7}\bar{x}}{(\epsilon^6 b^3\gamma)^{\frac{1}{7}}}, \beta\frac{t-t_c}{(3^4 2^2\epsilon^4 b^2\gamma^3)^{\frac{1}{7}}}\right) + O(\epsilon^{\frac{4}{7}}),$$

where

$$b = 12\frac{\rho}{\beta} \quad \gamma = -t_c^4\beta^3 u_0'''(\xi_c)$$

and the function $U(\mathcal{X}, \mathcal{T})$ satisfies the ODE (1.5) with the asymptotic conditions (1.6). We remind that this solution of the PI2 equation also satisfies the KdV equation

$$U_{\mathcal{T}} + 6UU_{\mathcal{X}} + U_{\mathcal{X}\mathcal{X}\mathcal{X}} = 0.$$

2.4 The small dispersion limit of the KP equation

Now let us consider the KP equation (1.1). We are looking for a solution of $u(x,y,t;\epsilon)$ near the point of gradient catastrophe (x_c, y_c, t_c) for the dKP equation of the form

$$u(x,y,t;\epsilon) = u_c + \frac{1}{nu_c^{n-1}}h(X,T;\epsilon) + \beta\bar{y}$$

with X and T defined in (2.10) and β is defined in (2.16) and $h(X,Y;\epsilon)$ a function to be determined. We are interested in the multiscale expansion of this function of the form

$$h(X,T;\epsilon) = \lambda^{\frac{1}{3}}H(\mathcal{X},\mathcal{T};\varepsilon) + \mathcal{O}(\lambda)$$
$$X = \lambda\mathcal{X}, \quad T = \lambda^{\frac{2}{3}}\mathcal{T}, \quad \epsilon = \lambda^{\frac{7}{6}}\varepsilon, \quad \bar{y} = \lambda^{\frac{1}{3}}\mathcal{Y}. \tag{2.17}$$

Theorem 2.1 *Let*

$$u(x,y,t;\epsilon) = u_c + \frac{1}{nu_c^{n-1}}h(X,T;\epsilon) + \beta\bar{y} \tag{2.18}$$

be a solution of the generalized KP equation (1.1). Suppose that the limit

$$H(\mathcal{X},\mathcal{T};\varepsilon) = \lim_{\lambda\to 0}\lambda^{-\frac{1}{3}}h(\lambda\mathcal{X},\, \lambda^{\frac{2}{3}}\mathcal{T};\lambda^{\frac{7}{6}}\varepsilon)$$

exists. Then the function $H(\mathcal{X},\mathcal{T};\varepsilon)$ satisfies the KdV equation

$$H_{\mathcal{T}} + HH_{\mathcal{X}} + \varepsilon^2 H_{\mathcal{X}\mathcal{X}\mathcal{X}} = 0. \tag{2.19}$$

Proof Plugging the ansatz into the generalized KP equation one obtains

$$\left(\frac{\partial}{\partial t}h(X,T;\epsilon)+(u_c+\frac{F_\xi^c}{G_\xi^c}h(X,T;\epsilon)+\beta\bar{y})^n\frac{\partial}{\partial x}h(X,T;\epsilon)+\epsilon^2\frac{\partial^3}{\partial x^3}h(X,T;\epsilon)\right)_x=\pm\frac{\partial^2}{\partial y^2}h(X,T;\epsilon)$$

Performing the rescalings (2.17) one obtains

$$\begin{aligned}&(H_{\mathcal{T}}+HH_{\mathcal{X}}+\varepsilon^2H_{\mathcal{X}\mathcal{X}\mathcal{X}})_{\mathcal{X}}+\lambda^{-\frac{1}{3}}H_{\mathcal{X}\mathcal{X}}\left(\frac{\partial X}{\partial t}\mp\left(\frac{\partial X}{\partial y}\right)^2+G_c+\beta nu_c^{n-1}\bar{y}\right)\\&=\pm2H_{\mathcal{T}\mathcal{X}}\frac{\partial T}{\partial y}\frac{\partial X}{\partial y}\pm\lambda^{\frac{1}{3}}H_{\mathcal{T}\mathcal{T}}\left(\frac{\partial T}{\partial y}\right)^2\pm\lambda H_T\frac{\partial^2T}{\partial y^2}\pm\lambda^{\frac{2}{3}}H_{\mathcal{X}}\frac{\partial^2X}{\partial y^2}\\&-\sum_{k=2}^{n}\binom{n}{k}\lambda^{\frac{k-1}{3}}u_c^{n-k}\left((\frac{F_\xi^c}{G_\xi^c}\mathcal{H}+\beta\mathcal{Y})^k\mathcal{H}_{\mathcal{X}}\right)_{\mathcal{X}}\end{aligned}$$

Using (2.10) and taking into account the constraints (2.8) one arrives at the relation

$$(H_{\mathcal{T}}+HH_{\mathcal{X}}+\varepsilon^2H_{\mathcal{X}\mathcal{X}\mathcal{X}})_{\mathcal{X}}=O(\lambda^{\frac{1}{3}}),$$

which in the limit $\lambda\to0$ implies that

$$H_{\mathcal{T}}+HH_{\mathcal{X}}+\varepsilon^2H_{\mathcal{X}\mathcal{X}\mathcal{X}}=const \tag{2.20}$$

In order to show that the constant is equal to zero it is sufficient to observe that the solution $H(\mathcal{X},\mathcal{T};\epsilon)$ of (2.20) has to match the outer solution (2.15) when $\mathcal{X}\to\infty$. □

We observe that choosing $\lambda=\epsilon^{\frac{6}{7}}$ one has $\varepsilon=1$ in (2.19). In the rescaled variables

$$\mathcal{X}=\frac{X}{\epsilon^{\frac{6}{7}}},\quad \mathcal{T}=\frac{T}{\epsilon^{\frac{4}{7}}},$$

the function $H(\mathcal{X},\mathcal{T};\varepsilon)$ satisfies the KdV equation (2.19) with $\varepsilon=1$. Furthermore for $\epsilon\to0$ and fixed X and T the solution has to match the outer solution (2.15), namely

$$H(\mathcal{X},\mathcal{T};\varepsilon)\simeq\mp\left(\frac{6}{k}\right)^{\frac{1}{3}}|\mathcal{X}|^{\frac{1}{3}}\mp2\mathcal{T}\left(\frac{6}{k}\right)^{-\frac{1}{3}}|\mathcal{X}|^{-\frac{1}{3}}+O(\mathcal{X}^{-1}),\quad|\mathcal{X}|\to\infty.$$

Using the results of the previous section on the KdV equation, we arrive at the following conjecture.

Conjecture 2.2 *Let us assume that the solution $u(x,y,t;\epsilon)$ to the Cauchy problem for the generalized KP equation*

$$(u_t+u^nu_x+\epsilon^2u_{xxx})_x=\pm u_{yy}$$

with ϵ-independent initial data

$$u(x,y,t=0;\epsilon)=u_0(x,y),$$

is at least C^4 in $\mathbb{R}^3_+ = \{(x,y,t)\,|\,t>0\}$. Then the solution $u(x,y,t;\epsilon)$ admits the following expansion near the critical point (x_c,y_c,t_c) and $u_c = u(x_c,y_c,t_c)$ for the solution of the dKP equation $(u_t+u^nu_x)_x = \pm u_{yy}$. In the limit $\epsilon\to 0$ and $x\to x_c$, $y\to y_c$ and $t\to t_c$ in such a way that the limits

$$\lim \frac{X}{\epsilon^{\frac{6}{7}}}, \quad \lim \frac{T}{\epsilon^{\frac{4}{7}}}, \quad \lim \frac{y-y_c}{\epsilon^{\frac{2}{7}}}$$

remain finite, with X and T defined in (2.10), the solution of the generalized KP equation (1.1) is approximated by

$$u(x,y,t;\epsilon) = u_c + \frac{6}{nu_c^{n-1}}\left(\frac{\epsilon^2}{\kappa^2}\right)^{\frac{1}{7}} U\left(\frac{X}{(\kappa\epsilon^6)^{1/7}}, \frac{T}{(\kappa^3\epsilon^4)^{1/7}}\right) + \bar{y}(F_y - F_\xi \frac{G_{\xi\xi y}}{G_{\xi\xi\xi}}) + O(\epsilon^{\frac{4}{7}}) \tag{2.21}$$

where

$$\kappa = -36\, G^c_{\xi\xi\xi} t_c^4$$

and $U(\mathcal{X},\mathcal{T})$ is the particular solution of the P1-2 equation (1.6) described in the introduction.

In the particular case in which the initial data is an even function of y, namely $u_0(x,y) = u_0(x,-y)$ one can easily check that this property is preserved by the KP equation, namely $u(x,y,t,\epsilon) = u(x,-y,t,\epsilon)$ and therefore all the odd derivatives with respect to y of the function F vanish. The formula (2.21) can be simplified to the form

$$u(x,y,t;\epsilon) \simeq u_c + \frac{6\left(\frac{\epsilon^2}{\kappa^2}\right)^{\frac{1}{7}}}{nu_c^{n-1}} U\left(\frac{\bar{x} - u_c^n\bar{t} - \frac{t_c}{2}G^c_{yy}\bar{y}^2}{(\kappa\epsilon^6)^{1/7}}, \frac{\bar{t} - \frac{t_c^2}{2}\bar{y}^2 G^c_{\xi yy}}{(\kappa^3\epsilon^4)^{1/7}}\right) + O(\epsilon^{\frac{4}{7}}), \tag{2.22}$$

where $\bar{x} = x - x_c$, $\bar{t} = t - t_c$ and $\bar{y} = y - y_c$ and $G = F^n$.

3 Numerical approaches

The numerical task in this paper is to solve the generalized KP equation (1.1) in *evolutionary form*,

$$u_t + u^n u_x + \epsilon^2 u_{xxx} = \sigma\partial_x^{-1}u_{yy}, \quad \sigma = \pm 1, \tag{3.1}$$

and the generalized dKP equation (equation (3.1) after formally putting $\epsilon = 0$) after the transformation (2.4), i.e., (2.5) in evolutionary form,

$$F_t = \sigma\left((1+tnF^{n-1}F_\xi)\partial_\xi^{-1}F_{yy} - tnF^{n-1}F_y^2\right); \tag{3.2}$$

the nonlocal operators ∂_x^{-1} and ∂_ξ^{-1} are defined in Fourier space by their respective singular Fourier symbols $-i/k_x$ and $-i/k_\xi$ respectively where k_x and k_ξ are the Fourier variables dual to x and ξ respectively. To avoid problems with these singular Fourier symbols, we will always consider initial data with $\partial_x^{-1}u(x,y,0) \in \mathcal{S}(R^2)$, i.e., initial data which are the x-derivative of a function in the Schwartz space of rapidly decreasing smooth functions.

As discussed in [37], for such initial data it is convenient to use Fourier methods. Denoting with $\hat{u}$ the 2-dimensional Fourier transform of u, equations (3.1) and (3.2) can be written in the form

$$\hat{u}_t = \mathcal{L}\hat{u} + \mathcal{N}(\hat{u}), \tag{3.3}$$

where $\mathcal{L}$ is a linear, *diagonal* operator, which is ik_y^2/k_ξ for (3.2), and $ik_y^2/k_x + i\epsilon^2 k_x^3$ for (3.1), and $\mathcal{N}(\hat{u})$ is a nonlinear term. The idea of an *exponential time differencing* (ETD) scheme is to treat the linear part of (3.3) exactly. We use the fourth order ETD method by Cox and Matthews [10], but other schemes offer a very similar performance as discussed in [37].

The Fourier transform will be approximated in standard manner by discrete Fourier transforms. This means the solution will be treated as essentially periodic on the domain $L_x[-\pi,\pi]\times L_y[-\pi,\pi]$, where L_x, L_y are chosen large enough that the function and the discrete Fourier transform decrease to machine precision (here 10^{-16}) if possible. We are working here on serial computers and can access a resolution of $N_xN_y = 2^{15}$. Note that the nonlocal operators in (3.1) and (3.2) imply that solutions for this equation for generic initial data in $\mathcal{S}(\mathbb{R}^2)$ will not stay in this space, but will develop tails with an algebraic fall off towards infinity, see the discussion in [39] and references therein. The resulting loss of regularity at the domain boundaries leads to a slower decrease of the Fourier coefficients than for an exponentially decreasing function. Thus one is forced to use higher resolution or larger domains for KP than for KdV solutions where the solution for initial data in $\mathcal{S}(\mathbb{R})$ stays in this space. As mentioned, the nonlocal terms in (3.1) and (3.2) correspond to singular Fourier symbols. To compute their action numerically, we regularize these symbols by adding some constant of the order of machine precision. In addition we use for (3.2) Krasny filtering [41], i.e., Fourier coefficients with a modulus smaller than 10^{-10} are put equal to zero. This is necessary for (3.2) since the nonlocality there appears not only in the linear part that is treated exactly in ETD schemes.

The decrease of the Fourier coefficients allows in any case to control the spatial resolution in the numerical solution. The accuracy in time is controlled via the L^2 norms of the solutions u and F which are both exactly conserved for solutions of (3.1) and (3.2). Due to unavoidable numerical errors, the L^2 norm of the numerical solutions will depend in general on time. We use the relative computed L^2 norm denoted by Δ_2 as an indicator of the numerical accuracy of the solution. As discussed in [37] this quantity is a valid criterion for sufficient spatial resolution and overestimates the numerical accuracy by one to two orders of magnitude.

Since the focus of this paper is on the critical behavior of the solution to general-

ized KP equations near the critical points (2.7) of the solutions to the corresponding dispersionless equations, it is crucial to obtain these critical points and the solution at this point with high accuracy. To this end we run the code for (3.2) with a resolution of $N_x = 2^9$ and $N_y = 2^{11}$ for the data symmetric with respect to $y \mapsto -y$ and with $N_x = N_y = 2^{10}$ for the non symmetric data with $N_t = 1000$ time steps to some estimated break up time. Once the quantity Δ in (2.7) becomes negative, the code is stopped. Then it is restarted with the same parameters between the last time t_i with positive Δ and the first time t_e with negative Δ with the computed solution at t_i as initial data (this means the time resolution for this step has been increased by a factor 1000). If needed, this procedure is iterated. Note that the Fourier coefficients in all studied examples for (3.2) decrease to the order of the Krasny filter during the whole computation, and the computed L^2 norm of the solution to (3.2) is conserved to better than 10^{-12}.

It turns out that the precision obtained in this way for the critical time t_c is sufficient to assure that equations (2.7) can be satisfied to better than plotting accuracy (10^{-3}). If much higher precision were needed, an interpolation of the solution on the recorded time steps could be used. To identify the critical values x_c and y_c via (2.7), an interpolation in the spatial coordinates is, however, needed (the interpolation is done by using the exact representation of the truncated Fourier series). To solve the equations $G_{\xi\xi} = G_{\xi y} = 0$, we identify the coordinates of the minimum of Δ at the found critical time. These values are used as the starting values for an iterative solution via the algorithm [42] distributed in Matlab as *fminsearch*. The equations are solved to machine precision. At the identified critical point, the derivatives entering formula (2.21) are computed.

The PI2 transcendent appearing in the asymptotic formula (2.21) is computed as detailed in [22] to essentially machine precision.

4 Solutions to the Kadomtsev–Petviashvili equations near the critical points

In this section we study solutions to the KP I and II equations for $\epsilon = 0.01$ near a critical point of the solutions to the corresponding dispersionless equations for the same initial data. The solutions are compared to the asymptotic formula (2.21). Throughout the paper, we consider the initial data

$$u_0(x, y) = -6\partial_x \mathrm{sech}^2(x^2 + y^2), \tag{4.1}$$

which are symmetric with respect to $y \mapsto -y$, and

$$u_0(x, y) = 6\partial_x \exp(-x^2 - 5y^2 - 3xy), \tag{4.2}$$

which do not have such a symmetry. Solutions to the dKP equation for these initial data have been discussed in detail in [25], where also figures can be found. Therefore we will concentrate here on the related KP solutions.

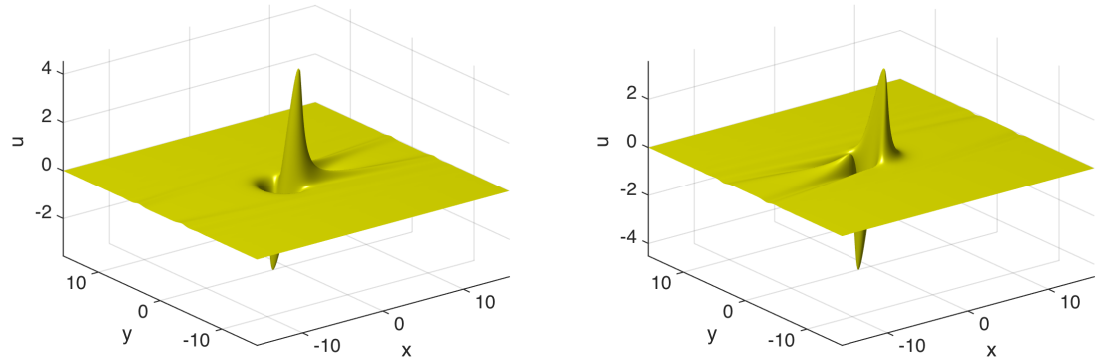

Figure 1: Solutions to the KP equation with $\epsilon = 0.01$ for the initial data (4.1) at the critical time $t_c \approx 0.222$; on the left for KP I, on the right for KP II.

4.1 Symmetric initial data

In Fig. 1 we show the solution to the KP I and II equation with $\epsilon = 0.01$ for the symmetric initial data (4.1) at the critical time $t_c \approx 0.222$. We use $N_x = 2^{13}$ and $N_y = 2^{10}$ Fourier modes and $N_t = 1000$ time steps. Note that while a solution to the dKP I equation (1.2) gives under the simultaneous transformation $x \mapsto -x$, $u \mapsto -u$ a solution to the dKP II equation, this is not the case for the full KP equation. As can be seen in Fig. 1, the KP I solution for localized initial data develops a tail with algebraic decrease to the right, whereas such a tail goes to the left for KP II solutions. Due to the imposed periodicity, these tails reenter the computational domain on the opposing side. Since we are interested here in the formation of dispersive shocks that are not affected by this reentering of the tails, we can use the shown domain size (the Fourier coefficients for both cases decrease to at least 10^{-7}). The numerically computed L^2 norm of the solution is conserved to better than 10^{-10}. The oscillations are hardly visible on these plots which is why we show only close-ups in the following.

The mass transfer to infinity via these tails implies stronger gradients on the respective side and leads to a break-up exactly on this side. Consequently also the first oscillation of the KP solutions are observed on the side of the tails. For dKP I the first critical point is $x_c \approx 1.79$ and $y_c = 0$. First oscillations appear around the critical time near this point as can be seen in Fig. 2. Near the critical point, the KP I solution is well approximated by the asymptotic solution (2.21) in terms of the PI2 transcendent. The approximation is local (for small $|x - x_c|$ and small $|y - y_c|$), but it can be seen that even the first oscillation will be approximately captured by small enough ϵ. Also the y-dependence is well reproduced, but gets worse for larger values of $|y - y_c|$.

The asymptotic description (2.21) via the PI2 transcendent of the KP solutions is local near the critical point (x_c, y_c, t_c). For the spatial dependence, this can be seen in

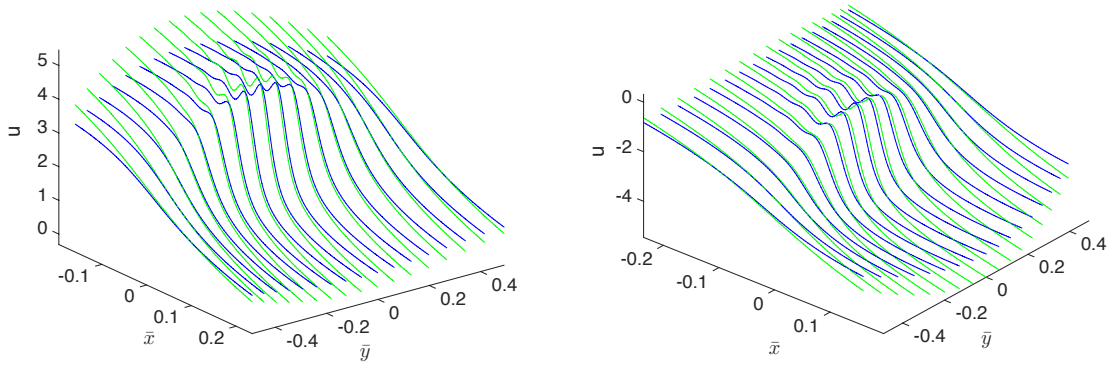

Figure 2: Solutions to the KP equations with $\epsilon = 0.01$ for the symmetric initial data (4.1) at the critical time $t_c \approx 0.222$ near the critical points $x_c \approx \pm 1.79$ and $y_c = 0$ in blue and the corresponding PI2 asymptotic solution (2.21) in green; on the left the KP I solution, on the right the KP II solution.

Fig. 2 at the critical time. However the description is also valid for small $|t - t_c|$ as is visible in Fig. 3 for times before and after the critical time. It can be seen that the PI2 asymptotics is shifted slightly to the left for KP I as before and moves with higher speed than the KP I solution to right. Thus the approximation becomes close the inflection point better with time, but the oscillations are less well reproduced. Note, however, that there appears to be the same number of oscillations of the KP and PI2 solution.

The oscillatory zone of the corresponding KP II solution can be seen in Fig. 2 on the right. It appears near $x_c \approx -1.79$, and due to the symmetry between dKP I and dKP II solutions, it is again decreasing with x. Note, however, that the asymptotic solution (2.21) is slightly shifted towards more positive values of x with respect to the KP II solution, whereas it is slightly shifted to more negative x-values for KP I. The quality of the approximation is the same in both cases. The time dependence of the asymptotic description can be seen in Fig. 3 on the y-axis. The PI2 asymptotics is slightly shifted to the right for $t \leq t_c$ and travels with somewhat higher speed than the KP II dispersive shock front to the left. Thus the approximation of the shock front becomes again better with time, and this time also the oscillations are better approximated.

4.2 Second critical point for symmetric initial data

The initial data (4.1) are odd functions in x. For the 1+1 dimensional Hopf equation, a possible break up would occur at the same time at the points $\pm x_c$. For dKP solutions for such initial data, this is no longer true. But the method detailed in [25] to numerically integrate the dKP equation also allows to reach the second breaking of the dKP solution

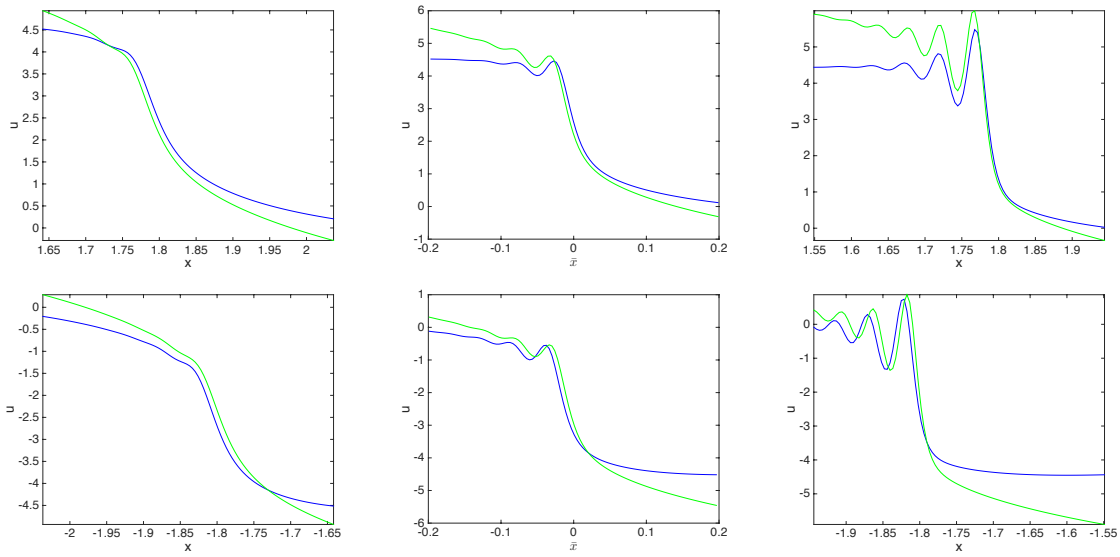

Figure 3: Solutions to the KP I equation in the upper row and KP II equation in the lower row with $\epsilon = 0.01$ for the symmetric initial data (4.1) on the left for $t = 0.204 < t_c$, in the center for $t = t_c$ and on the right for $t = 0.24 > t_c$ on the y-axis near the critical point $x_c \approx -1.79$ in blue and the corresponding asymptotic solution (2.21) in terms of the PI2 transcendent in green.

for the data (4.1). It occurs at the time $t_c \approx 0.3001$ at $x_c \approx -2.033$ for KP I and $-x_c$ for KP II. The behaviour of the solution of KP I near the critical point (x_c, t_c, y_c) is shown in Fig. 4 in the upper row. It can be seen that the PI2 asymptotic description (2.21) is slightly worse than at the first breaking, it is again somewhat shifted towards the left.

The same situation for KP II can be see in the lower row of Fig. 4. Here the asymptotic solution is as for the first breaking in KP II shifted towards the right.

4.3 Nonsymmetric initial data

Both the numerical approach to dKP in [25] and the asymptotic formula (2.21) are applicable to non symmetric initial data as for example in (4.2). For such data, we could not reach the second breaking, but the first break-up is well resolved. The KP I solution for these data can be seen in the upper row of Fig. 5 at the critical time $t_c \approx 0.0855$ in the vicinity of the critical point $x_c \approx 0.1045$ and $y_c \approx -0.2566$ on the left. Visibly the PI2 asymptotic description (2.21) captures well the onset of the oscillations, also in dependence of $y - y_c$. The corresponding situation for KP II can be seen in upper row of Fig. 5 on the right. The PI2 asymptotic solution (2.21) matches also well with the KP II numerical solution.

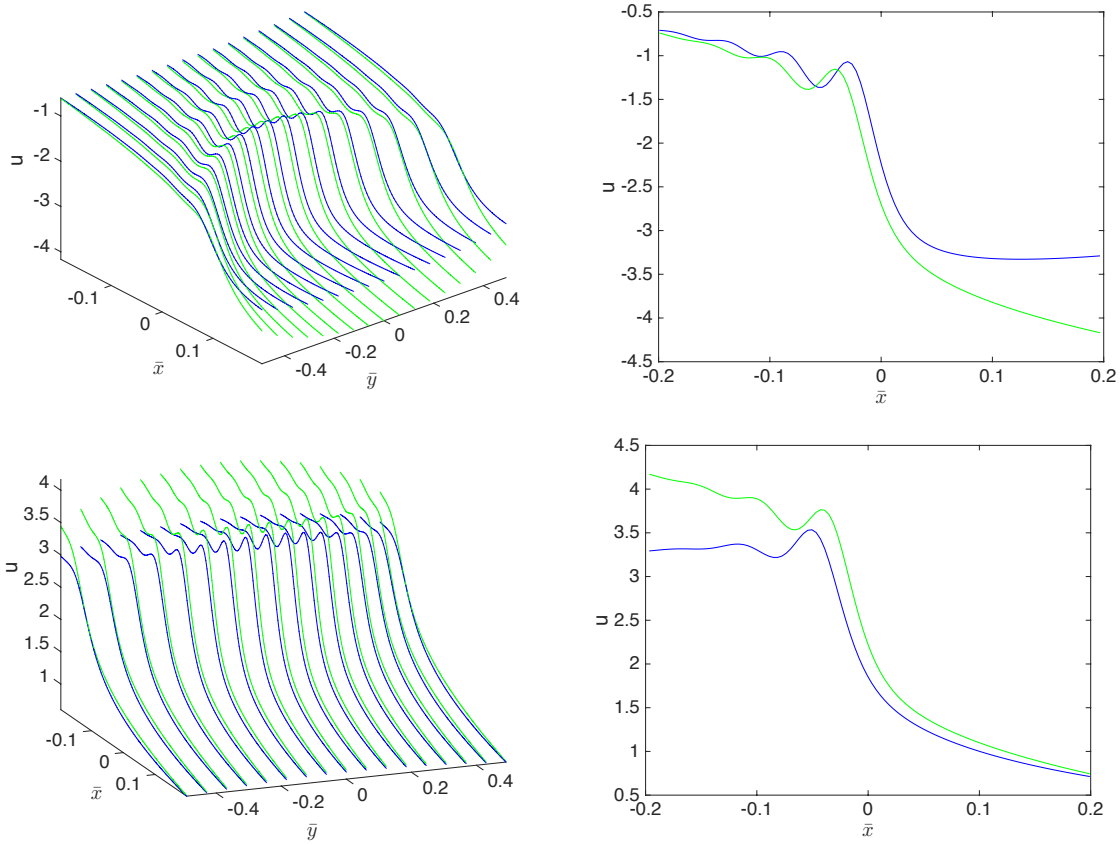

Figure 4: Solutions to the KP equation with $\epsilon = 0.01$ for the symmetric initial data (4.1) at the critical time $t \approx 0.3001$ of the second break-up in blue and the corresponding asymptotic solution (2.21) in green; on the left in the vicinity of the critical point, on the right on the y-axis, in the upper row for KP I, in the lower row for KP II.

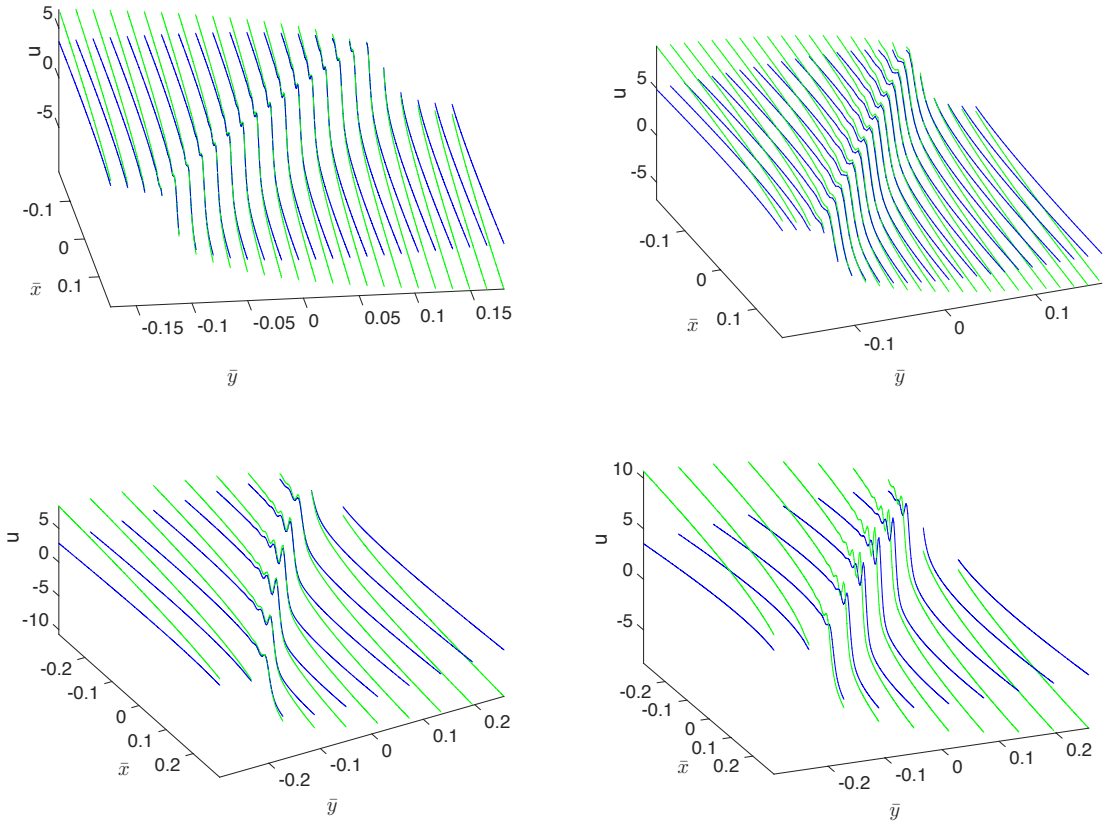

Figure 5: Solutions to the KP equations with $\epsilon = 0.01$ for the asymmetric initial data (4.2) in the vicinity of the critical point in blue and the corresponding PI2 asymptotic solution (2.21) in green; on the left KP I, on the right KP II, in the upper row at the critical time $t \approx 0.0855$, in the lower row at the time $t = 0.09 > t_c$.

It can be seen in the upper row of Fig. 5 that the situation is not the same for KP I and KP II, the asymptotic solution (2.21) approximates the critical behavior better for KP I than for KP II. This persists for larger values of t as can be seen in the lower row of Fig. 5. Still it is remarkable that the asymptotic solution also catches at least qualitatively the behavior of the KP solution in larger distance from the critical points, especially the y-dependence.

5 Solutions to the generalized Kadomtsev–Petviashvili equations near the critical point

Generalized KP equations allow to study the competing influence of dispersion and nonlinearity in a $2+1$ dimensional model. It is known that solutions to generalized KP I equations with $n \geq 4/3$ can have blow-up. There is no theoretical description of this blow-up, it is just known that the L^2 norm of u_y can explode in finite time, see [54] and [46]. There are no theoretical predictions for generalized KP II so far. It is just expected that the defocusing effect in generalized KP II should make blow-up less likely than in generalized KP I situations (there is an analogy between generalized KP and nonlinear Schrödinger (NLS) equations in this respect). Numerical studies in [36] and in more detail in [34] have indicated that a self-similar blow-up can be expected for generalized KP I for $n \geq 4/3$ (see also the following section). No blow-up was found for generalized KP II for $n < 3$. Therefore we consider here only the case $n = 3$ where also for generalized KP II blow-up was observed. In addition an odd exponent n has the advantage that the breaking is similar to the case $n = 1$ studied in the previous section. We always use $N_x = N_y = 2^{12}$ Fourier modes and $N_t = 2000$ time steps.

We consider the same initial data as in the KP case. It is expected that the balance between dispersion and nonlinearity is here tilted towards the nonlinearity. This means that dispersive shocks will be less pronounced (less oscillations confined to smaller domains), and that the break-up will happen at earlier times. In fact we find for the symmetric initial data (4.1)that a first break-up in the generalized dKP solution occurs at $t_c \approx 0.0059$ (we did not study a second breaking) and $x_c \approx 1.33$ and $y_c = 0$. The corresponding solution to the generalized KP I equation can be seen for the critical time in the vicinity of the critical point in Fig. 6 in the upper row. Due to the reduced dispersive effects, the PI2 asymptotic description (2.21) is almost more accurate than for KP I, but in a smaller domain.

In Fig. 6 in the lower row we show the corresponding solution for the generalized KP II equation. The PI2 asymptotic description (2.21) again matches the generalized KP II solution well in the vicinity of the critical point and catches qualitatively the first oscillation. The domain of applicability of the approximation is more confined than for KP II.

The solutions to the generalized KP I equation with $\epsilon = 0.01$ and $n = 3$ break for

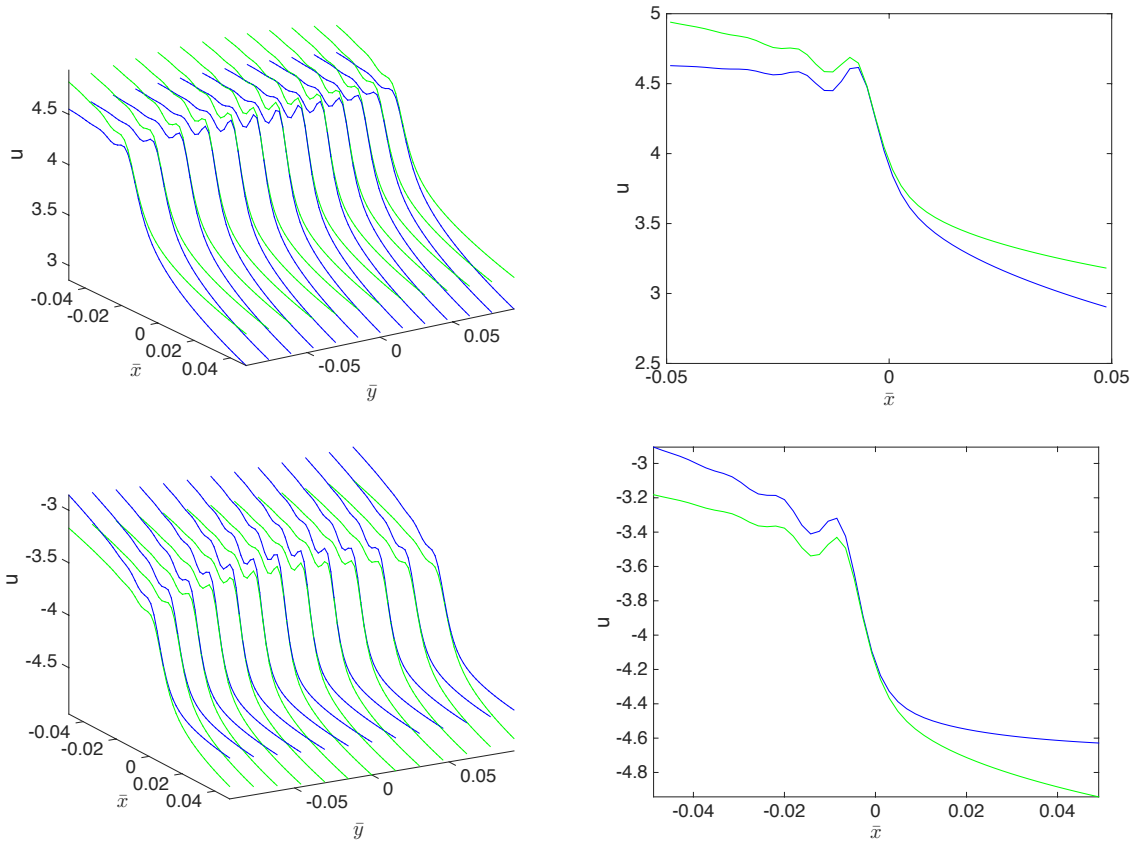

Figure 6: Solutions to the generalized KP equations with $n = 3$ and $\epsilon = 0.01$ for the symmetric initial data (4.1) at the critical time $t_c \approx 0.0059$ near the critical points x_c and $y_c = 0$ in blue and the corresponding PI2 asymptotic solution (2.21) in green; on the left in dependence on x and y, on the right on the y-axis, in the upper row for KP I, in the lower row for KP II.

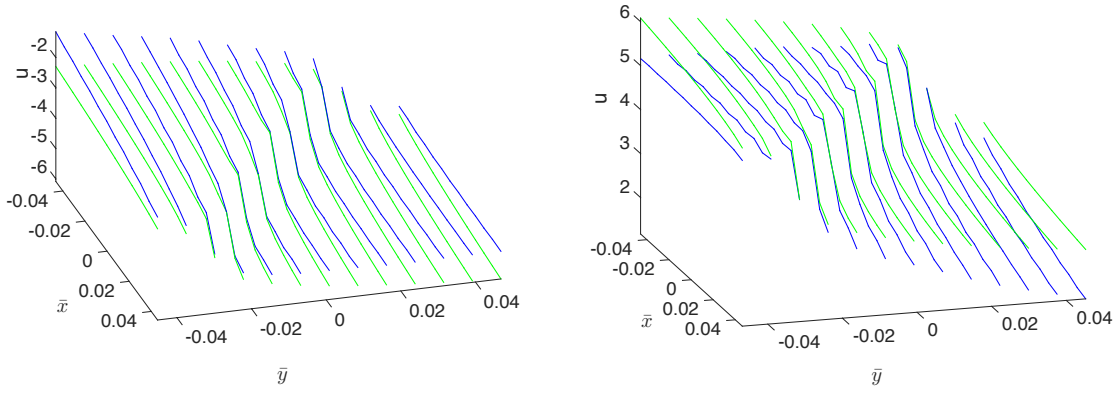

Figure 7: Solutions to the generalized KP equations with $n = 3$ and $\epsilon = 0.01$ for the asymmetric initial data (4.2) at the critical time $t \approx 0.0028$ in the vicinity of the critical point in blue and the corresponding PI2 asymptotic solution (2.21) in green; on the left generalized KP I, on the right generalized KP II.

the asymmetric initial data (4.2) at $t_c \approx 0.0028$ at $x_c \approx 0.2074$ and $y_c \approx -0.0085$. The resulting solution at the critical time can be seen in the vicinity of the critical point in Fig. 7 on the left. The PI2 asymptotic solution (2.21) describes the breaking well. The corresponding generalized dKP II solution breaks at the same t_c at $-x_c$, y_c. It can be seen in Fig. 7 on the right.

6 Outlook: Longtime behavior

In this section we study the solutions of the previous sections for times $t \gg t_c$. The solution of the KP equations develops in the (x, y) plane a region of fast oscillations after the critical time t_c, the dispersive shock waves.

The generalized KP equation with $n \geq 4/3$ is, however, critical or supercritical and its solution may develop blow-up in finite time. In [54] and [46], it was shown that in generalized KP I solutions, a blow-up of the L^2 norm of u_y can be observed for certain initial data. There are no rigorous analytic results on blow-up in the generalized KP II solutions so far. The first numerical studies of the generalized KP solutions appear to have been presented in [59], a more detailed study was performed in [34]. The numerical results in [34] led to the conjecture that there is an L^∞ blow-up with a self similar blow-up profile for $n \geq 4/3$ for generalized KP I and for $n \geq 3$ for generalized KP II. In [33], blow-up in dispersive shocks for solutions to generalized KdV equations were studied numerically in the limit of small dispersion. These results are compared with the blow-up found below in solutions to generalized KP equations in the small dispersion limit.

Since we did not have access to parallel computers for this work, we lack the nec-

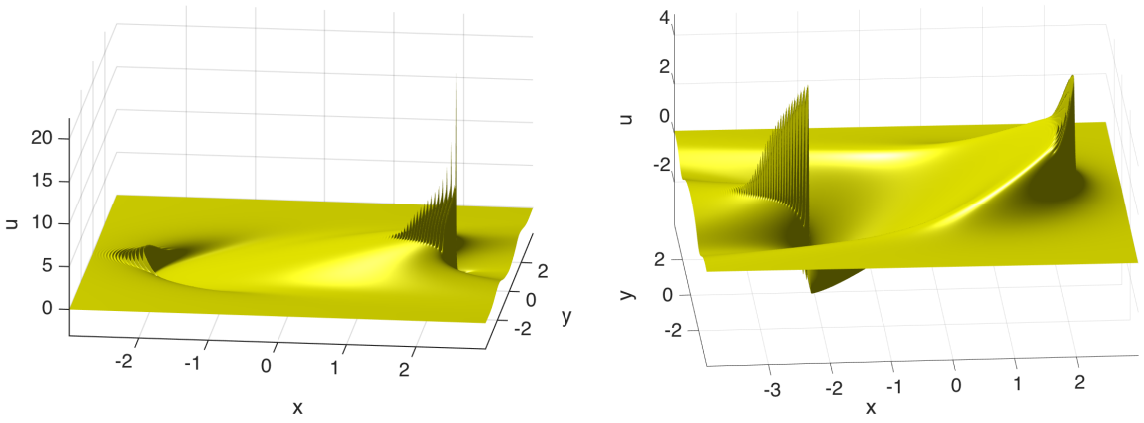

Figure 8: Solution to the KP I (left) and KPII (right) equations with $\epsilon = 0.01$ for the symmetric initial data (4.1) for $t = 0.4 > t_c \simeq 0.22$. The KP I focusing effect can be observed from the different scales in the u-axis of the two plots. A zoom in of these plots is shown in Fig. 9 and Fig.10 respectively.

essary resolution to address these questions with the needed resolution (we use here $N_x = 2^{13}$ and $N_y = 2^{12}$ with $N_t = 5000$ up to $N_t = 20000$). The figures shown below have thus to be seen as indicative with the goal to outline directions of future research.

6.1 Solution to the KP equations for symmetric initial data

In Fig. 8 we show the dispersive shock waves that are formed in the KP I and KP II solutions for the initial data (4.1) at $t = 0.4 > t_c$ i.e. well after both break-ups. In Fig. 9 we zoom in the oscillatory zones in the KP I solution. The oscillations appear near the two critical points discussed in section 4. It can be seen that the oscillations follow a parabolic pattern initially as suggested by the asymptotic formula (2.21). But the focusing effect of the KP I equation appears to lead to the formation of a cusp. This is presumably a real effect and not related to a lack of resolution since a similar behavior was seen in [35] where considerably higher resolution could be used. The analogy between KP and NLS suggests that this cusp could be related to higher genus solutions appearing in the asymptotic description of the oscillations. For times closer to the critical time, the PI2 asymptotic (2.21) in the lower row of Fig. 9 gives a qualitative approximation to the oscillations both for the parabolic shape and the number of oscillations. For larger times, the PI2 asymptotic has considerably more oscillations.

The corresponding KP II solution can be seen in Fig. 10. Due to the defocusing character of KP II, the oscillations have a parabolic pattern at least in the range of times we considered. It is unclear whether some sort of cusp will appear at later times

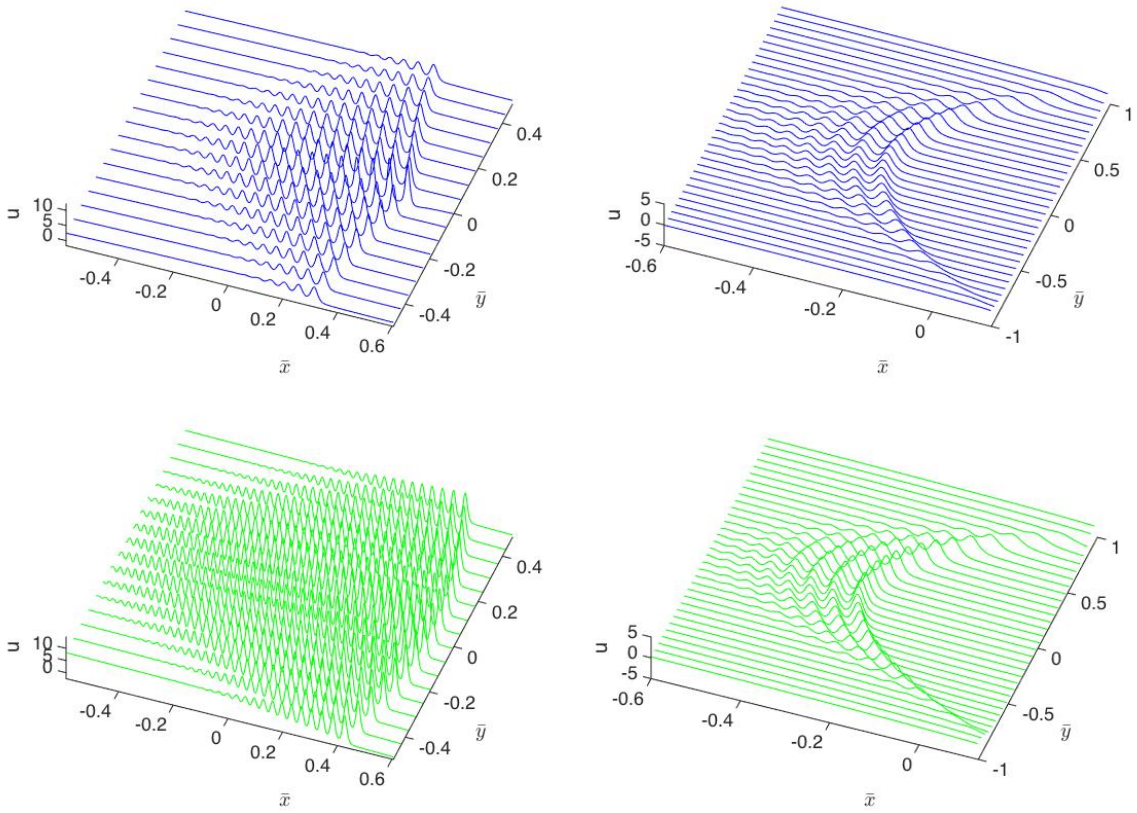

Figure 9: Solution to the KP I equations with $\epsilon = 0.01$ for the symmetric initial data (4.1) for $t = 0.4 > t_c \simeq 0.22$; in the upper row on the left the dispersive shock wave appearing first, on the right the dispersive shock wave appearing at a later time ($t_c \simeq 0.3$), in the lower row the corresponding PI2 asymptotic solutions (2.21).

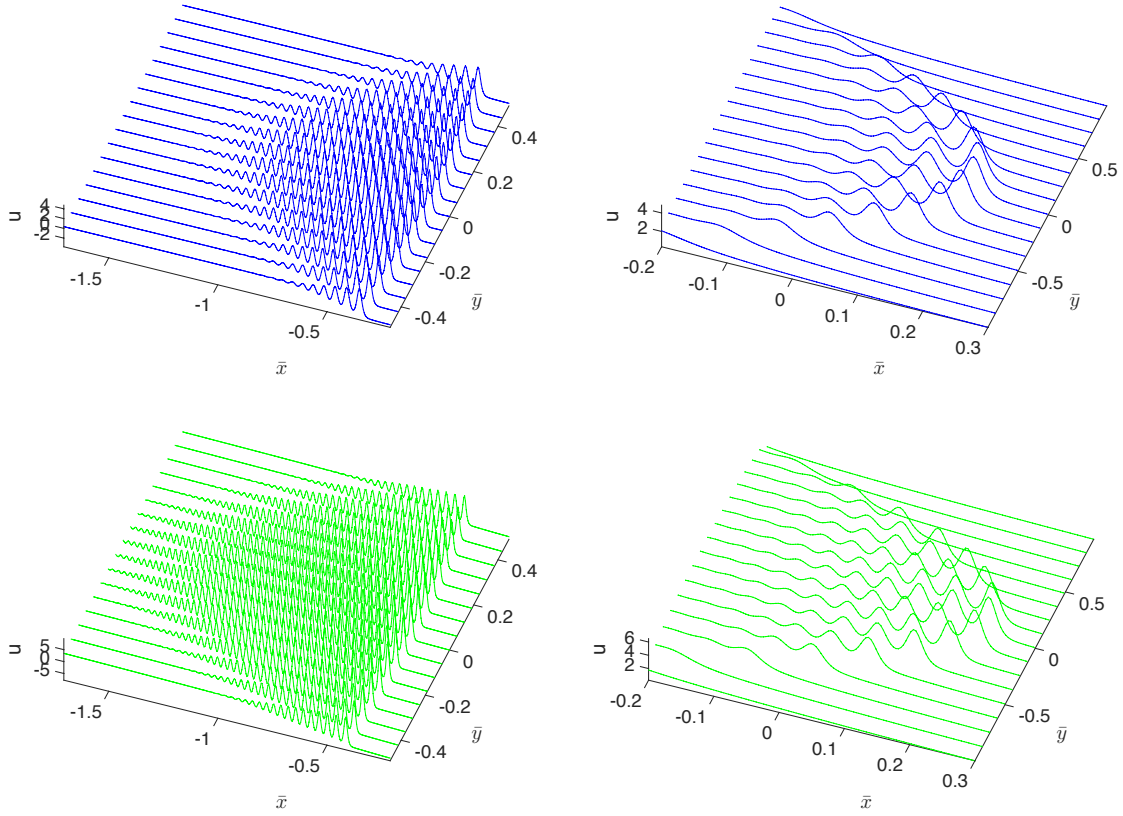

Figure 10: Solution to the KP II equation with $\epsilon = 0.01$ for the symmetric initial data (4.1) for $t = 0.4 > t_c \simeq 0.22$; in the upper row on the left the dispersive shock wave appearing first, on the right the dispersive shock wave appearing at a later time ($t_c \simeq 0.3$); in the lower row the corresponding PI2 asymptotic solutions (2.21.

in the envelope of the oscillatory profile. In the lower row of Fig. 10 we show the corresponding PI2 asymptotic solution (2.21) which shows a similar behavior as for KP I.

The asymptotic description of these oscillation in the small dispersion limit will be the subject of further research. The task is to derive and solve the Whitham equations which asymptotically describe the oscillations and to find the needed phase information in the asymptotic solutions.

6.2 Solution to generalized KP equations for symmetric initial data

In [34] blow-up in generalized KP solutions was studied numerically. It was conjectured that a self similar L^∞ blow-up is observed.

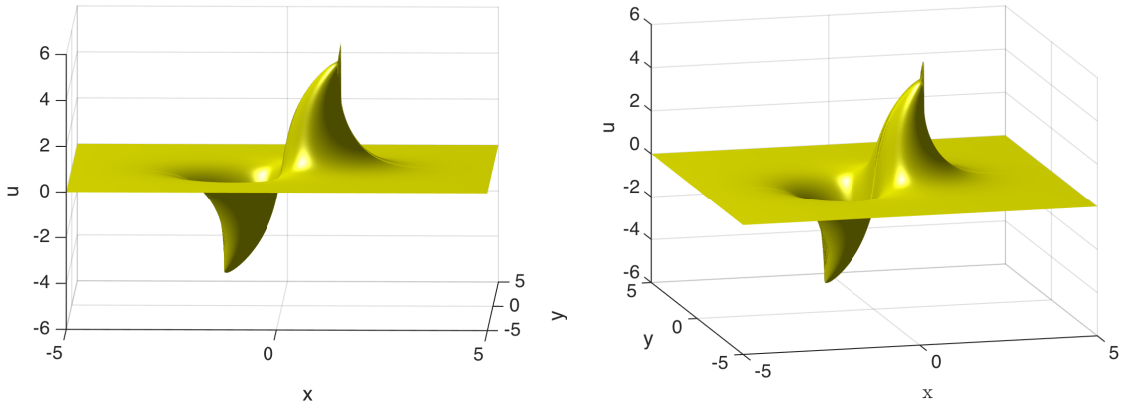

Figure 11: Solution to generalized KP equations with $\epsilon = 0.01$ and $n = 3$ for the initial data (4.1); on the left the generalized KP I solution for $t = 0.00695$ L^∞ norm of u, on the right the generalized KP II solution for $t = 0.0071$.

We consider again the symmetric initial data (4.1) for generalized KP I with $\epsilon = 0.01$ and let the code run with $N_t = 20000$ and $N_x = 2^{13}$, $N_y = 2^{12}$. High resolution in x is needed to resolve the dispersive shock as before, but an even higher resolution in y would be necessary since the blow-up in y according to [34] is stronger in y than in x direction. Thus on the used computers, just indication for future work on larger computers can be obtained. For KP, the relative mass conservation drops below 10^{-3} at $t \approx 0.0085$, but the Fourier coefficients deteriorate even earlier. We show the generalized KP I solution for $t = 0.00695$, i.e., at a time where the solution is still resolved, on the left in Fig. 11. It seems that the cusped structure in the oscillatory zone in Fig. 9 blows up in the gKP I solution.

In Fig. 12 we show the L^∞ norm of the solution which appears to explode. The fuzzy structure of the L^∞ norm indicates a lack of resolution. In fact we did not manage to fit the norms to the asymptotic formulae of [34] which means we do not get close enough to the actual blow-up.

The corresponding generalized KP II solution can be seen in Fig. 11 on the right. There appears to be some blow-up at a slightly later time which is in accordance with the defocusing character of generalized KP II. The blow-up is again in the region of positive values of u, not on the side of the algebraic tails as for generalized KP I. The L^∞ norm of the solution shown in Fig. 12on the right also indicates a blow-up. Again we do mot manage to fit the norms to the formulae of [34] for lack of spatial resolution. It will be the subject of future work to study the details of the blow-up in the presence of a dispersive shock. For instance for generalized KdV in [33] the dependence of the blow-up time as a function of ϵ was obtained.

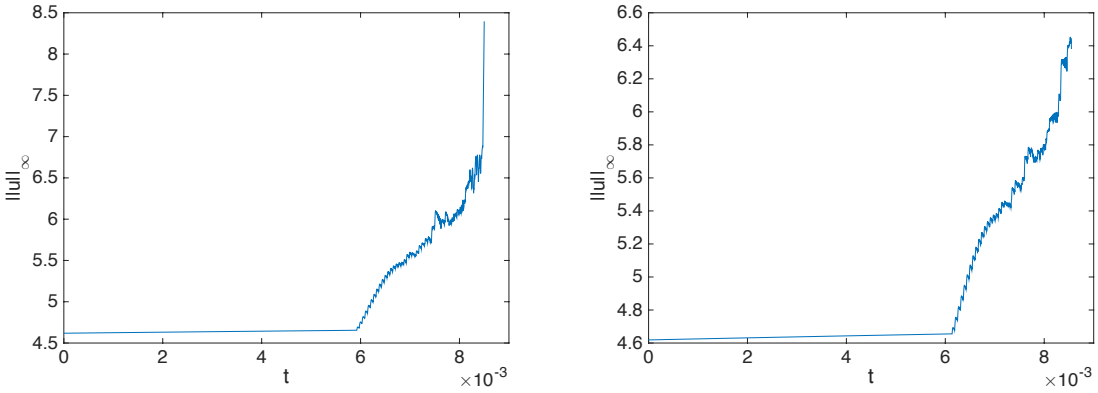

Figure 12: L^∞ norm of the solution to generalized KP equations with $\epsilon = 0.01$ and $n = 3$ for the initial data (4.1) in dependence of time; on the left for KP I, on the right for KP II.

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