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Dynamical models for viscoelastic materials in domains with a growing crack

Ph.D. Thesis

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Contents

Introduction

This thesis is devoted to the study of some dynamic viscoelastic models in domains with prescribed growing cracks. From the mathematical point of view, the study of these models, with prescribed growing cracks, is the first step to the study of Fracture Mechanics in which the evolution of the crack is unknown (see for example [6, 13, 16, 38, 48]).

Let T and d be a positive real number and a natural number. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with Lipschitz boundary, which represents the reference configuration of the viscoelastic material, and $\Gamma \subset \overline{\Omega}$ a $(d-1)$-dimensional closed set, which describes the prescribed path of the crack. We consider $\{\Gamma_t\}_{t\in[0,T]}$ a family of closed subsets of Γ increasing in time with respect to the inclusion, which represents the evolution of the crack, and finally let $u(t)\colon \Omega\setminus\Gamma_t \to \mathbb{R}^d$ be the displacement. In this setting, the displacement u solves the following system out of the crack

$$\ddot{u}(t) - \operatorname{div}(\sigma(t)) = f(t) \qquad \text{in } \Omega\setminus\Gamma_t, \quad t \in [0,T] \tag{1}$$

with some prescribed boundary and initial conditions. Here f is the loading term and σ is the stress tensor, which in our models can linearly depend on both the strain $eu := \frac{\nabla u + \nabla u^T}{2}$ and its first derivative in time $e\dot{u}$.

Some materials, or materials under some conditions, exhibit a time-dependent response to a given stress or strain, and this can be caused by a change in the properties of the material and by viscosity. In the literature we can find two different classes of viscoelastic materials: materials with short memory and materials with long memory. The term short memory refers to a material in which the state of the stress at the instant t only depends on the strain at that instant, and in the first chapter we analyze a local model in time whose stress-strain dependence is the following

$$\sigma(t) := \mathbb{A}eu(t) + \Psi^2(t)\mathbb{B}e\dot{u}(t) \qquad \text{in } \Omega\setminus\Gamma_t, \quad t\in[0,T], \tag{2}$$

where Ψ is a suitable function, $\mathbb{A}$ and $\mathbb{B}$ are the elastic and the viscous tensors. On the contrary, the term long memory refers to a material in which the state of the stress at the instant t depends also on the past history of the strain up to time t, and in the other chapters of the thesis we deal with non-local models in time whose stress-strain dependences are the following

$$\sigma(t) := (\mathbb{A}+\mathbb{B})eu(t) - \int_{-\infty}^{t} \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau \qquad \text{in } \Omega\setminus\Gamma_t, \quad t\in(-\infty,T], \tag{3}$$

$$\sigma(t) := \mathbb{A}eu(t) + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathbb{F}(t-\tau)(eu(\tau)-eu(0))\mathrm{d}\tau \qquad \text{in } \Omega\setminus\Gamma_t, \quad t\in[0,T], \tag{4}$$

where

$$\mathbb{F}(t) := \rho(t)\mathbb{B}, \quad \rho(t) := \frac{1}{\boldsymbol{\Gamma}(1-\alpha)t^\alpha} \qquad t\in[0,T], \tag{5}$$

$\boldsymbol{\Gamma}$ is Euler's Gamma Function, $\beta > 0$, and $\alpha \in (0,1)$.

The contents of the thesis are organized into four chapters.

Chapter 1: A dynamic model for viscoelastic materials with growing cracks

In the theory of Dynamic Fracture, the deformation of an elastic material evolves according to the elastodynamics system, while the evolution of the crack follows Griffith's dynamic criterion, see [36]. This principle, originally formulated in [27] for the quasistatic setting, states that there is an exact balance between the energy released during the evolution and the energy used to increase the crack, which is postulated to be proportional to the area increment of the crack itself.

The elastodynamics system leads to (1) with $\sigma(t) = \mathbb{A}eu(t)$. In this case, Griffith's dynamic criterion reads

$$\mathcal{E}(t) + \mathcal{H}^{d-1}(\Gamma_t \setminus \Gamma_0) = \mathcal{E}(0) + \text{work of external forces},$$

where $\mathcal{E}(t)$ is the total energy at time t, given by the sum of the kinetic and the elastic energy, and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure.

When we want to take into account the viscoelastic properties of the material, Kelvin-Voigt's model is the most common local model in time. If no crack is present, this leads to the damped system

$$\ddot{u}(t) - \operatorname{div}(\mathbb{A}eu(t)) - \operatorname{div}(\mathbb{B}e\dot{u}(t)) = f(t) \qquad \text{in } \Omega, \quad t \in [0,T]. \tag{6}$$

As it is well-known, the solutions to (6) satisfy the energy-dissipation balance

$$\mathcal{E}(t) + \int_0^t \int_\Omega \mathbb{B}e\dot{u}(\tau,x) \cdot e\dot{u}(\tau,x) \, \mathrm{d}x \, \mathrm{d}\tau = \mathcal{E}(0) + \text{work of external forces}. \tag{7}$$

When we consider a crack in a viscoelastic material, Griffith's dynamic criterion becomes

$$\mathcal{E}(t) + \mathcal{H}^{d-1}(\Gamma_t \setminus \Gamma_0) + \int_0^t \int_\Omega \mathbb{B}e\dot{u}(\tau,x) \cdot e\dot{u}(\tau,x) \, \mathrm{d}x \, \mathrm{d}\tau = \mathcal{E}(0) + \text{work of external forces}. \tag{8}$$

For a prescribed crack evolution, this model was already considered by [13] in the antiplane case, and more in general by [48] for the vector-valued case. As proved in the quoted papers, the solutions to (6) on a domain with a prescribed time-dependent crack, i.e., with Ω replaced by $\Omega \setminus \Gamma_t$, satisfy (7) for every time. This equality implies that (8) cannot be satisfied unless $\Gamma_t = \Gamma_0$ for every t. This phenomenon was already well-known in Mechanics as the Viscoelastic Paradox, see for instance [47, Chapter 7].

To overcome this problem, in [7] which is a joint work with M. Caponi, we modify Kelvin-Voigt's model by considering a possibly degenerate viscosity term depending on t and x. More precisely, we study system (1) with the stress-strain dependence (2), i.e.

$$\ddot{u}(t) - \operatorname{div}(\mathbb{A}eu(t)) - \operatorname{div}(\Psi^2(t)\mathbb{B}e\dot{u}(t)) = f(t) \qquad \text{in } \Omega \setminus \Gamma_t, \quad t \in [0,T]. \tag{9}$$

On the function Ψ we only require some regularity assumptions (see (1.7)); a particularly interesting case is when $\Psi(t)$ assumes the value zero on some points of Ω, which means that the material is purely elastic in such a zone.

The main result of this chapter is Theorem 1.2.1, in which we show the existence of a weak solution to (9). To this aim, we first perform a time discretization in the same spirit of [13], and then we pass to the limit as the time step goes to zero by relying on energy estimates; as a byproduct, we obtain the energy-dissipation inequality (1.39). By using the change of variables method implemented in [16, 38], we also prove a uniqueness result, but only in dimension $d = 2$ and when $\Psi(t)$ vanishes on a neighborhood of the tip of Γ_t.

We complete our work by providing an example in $d = 2$ of a weak solution to (9) for which the fracture can grow while balancing the energy. More precisely, when the crack Γ_t

moves with constant speed along the x_1-axis and $\Psi(t)$ is zero in a neighborhood of the crack tip, we construct a function u which solves (9) and satisfies

$$\mathcal{E}(t)+\mathcal{H}^1(\Gamma_t\setminus\Gamma_0)+\int_0^t\int_\Omega|\Psi(\tau,x)|^2\mathbb{B}e\dot{u}(\tau,x)\cdot e\dot{u}(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$=\mathcal{E}(0)+\text{work of external forces.} \qquad (10)$$

Notice that this is the natural extension of Griffith's dynamic criterion (8) to this setting.

Chapter 2: A dynamic model with memory for viscoelasticity in domains with time-dependent cracks

In this chapter we study the dynamic evolution of viscoelastic materials with long memory in domains with prescribed growing cracks. When no crack is present, important contributions in the theory of linear viscoelasticity are due to such scientists as Maxwell, Kelvin, and Voigt. Boltzmann was the first to develop a three-dimensional theory of isotropic viscoelasticity in [5], and later Volterra in [49] obtained similar results for anisotropic solids.

As you can find in [24] and [25], in the case of viscoelastic materials with long memory the general stress-strain dependence is the following

$$\sigma(t):=G(0)\nabla u(t)+\int_{-\infty}^t\dot{G}(t-\tau)\nabla u(\tau)\mathrm{d}\tau \qquad \text{in } \Omega,\quad t\in(-\infty,T],$$

for a suitable choice of the memory kernel G, and with some prescribed boundary and initial conditions. In particular, in the case of Maxwell's model the kernel G has an exponential form (see for example [47]), hence the displacement u satisfies (1) with the stress-strain dependence (3), i.e.

$$\ddot{u}(t)-\operatorname{div}((\mathbb{A}+\mathbb{B})eu(t))+\int_{-\infty}^t\frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\operatorname{div}(\mathbb{B}eu(\tau))\mathrm{d}\tau=\ell(t) \qquad \text{in } \Omega,\quad t\in[0,T], \qquad (11)$$

where $\beta>0$ is a material constant, and $\ell(t)$ is the external loading term at time t. As in [12, 24], we suppose that the past history of the displacement up to time 0 is already known, hence we have the following boundary and initial conditions

$$u(t)=z(t) \qquad \text{on } \partial\Omega,\quad t\in[0,T], \qquad (12)$$
$$u(t)=u_{in}(t) \qquad \text{in } \Omega,\quad t\in(-\infty,0], \qquad (13)$$

where z and u_{in} are prescribed functions, the latter representing the history of the displacement for $t\le 0$.

In this chapter, we consider Maxwell's model in the context of fractures and when the crack evolution $t\mapsto\Gamma_t$ is prescribed. In this case, the displacement u satisfies (11) on the cracked domains $\Omega\setminus\Gamma_t$. Thanks to (13) it is convenient to write system (11) as

$$\ddot{u}(t)-\operatorname{div}((\mathbb{A}+\mathbb{B})eu(t))+\int_0^t\frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\operatorname{div}(\mathbb{B}eu(\tau))\mathrm{d}\tau$$
$$=\ell(t)-\int_{-\infty}^0\frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\operatorname{div}(\mathbb{B}eu_{in}(\tau))\mathrm{d}\tau \qquad \text{in } \Omega\setminus\Gamma_t,\quad t\in[0,T]. \qquad (14)$$

The main results of this chapter are Theorems 2.2.1 and 2.3.3, in which we prove, by two different methods, the existence of a solution to (14).

The first method, considered in Theorem 2.2.1, is based on a generalization of Lax-Milgram's Theorem ([33, Chapter 3, Theorem 1.1]). We follow the lines of the proof of

Theorem 2.1 in [11]. In doing so, the main difficulty is given by the fact that the set $\Omega \setminus \Gamma_t$, where the system (14) holds, depends on time. This requires the introduction of suitable function spaces used to adapt the proof in [11].

The second method, provided by Theorem 2.3.3, is based on a time discretization scheme that yields a solution which, in addition, satisfies the energy-dissipation inequality (2.132). This procedure, adopted in [13] for the wave equation in a time-dependent domain, consists of the following steps: time discretization, construction of an approximate solution, discrete energy estimates, and passage to the limit.

The main difficulty in applying this procedure, in the same way it was done in [13], is the identification of the term in the energy-dissipation inequality which corresponds to the non-local in time viscous term $\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau$ appearing in (14).

To fix this issue, given w^0 we introduce the auxiliary variable

$$w(t) := w^0\mathrm{e}^{-\frac{t-\tau}{\beta}} + \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}} eu(\tau)\mathrm{d}\tau \qquad \text{in } \Omega \setminus \Gamma_t, \quad t \in [0,T],$$

and we transform our system (14) into an equivalent coupled system (see Definition 2.3.1) of two equations in the two variables u and w, without long memory terms, which has to be solved on the time-dependent domain $\Omega \setminus \Gamma_t$. The advantage of this strategy lies in the fact that we transform a non-local model (the system in the variable u) into a local one (the coupled system in the two variables u and w).

We discretize the time interval $[0,T]$ by using the time step $\tau_n := \frac{T}{n}$. To define the approximate solution (u_n, w_n) at time $(k+1)\tau_n$, we solve an incremental problem (see (2.94)) depending on the values of (u_n, w_n) at times $(k-1)\tau_n$ and $k\tau_n$. Since the new system has a natural notion of energy, we also obtain a discrete energy estimate for (u_n, w_n). Then, we extend (u_n, w_n) to the whole interval $[0,T]$ by a suitable interpolation, and by using the energy estimates together with a compactness result we pass to the limit, along a subsequence of (u_n, w_n). It is now possible to prove that the limit of this subsequence of (u_n, w_n) is a solution to the coupled system, which is equivalent to our viscoelastic dynamic system (14). As a byproduct, from the discrete energy estimates we obtain the energy-dissipation inequality (2.132).

Chapter 3: An existence result for the fractional Kelvin-Voigt's model on time-dependent cracked domains

This chapter deals with the mathematical analysis of the dynamics of a different kind of viscoelastic materials in the presence of external forces and time-dependent cracks.

In the classical theory of linear viscoelasticity, the constitutive stress-strain dependence of the so called Kelvin-Voigt's model is given by

$$\sigma(t) = \mathbb{A}eu(t) + \mathbb{B}e\dot{u}(t) \qquad \text{in } \Omega \setminus \Gamma_t, \quad t \in [0,T]. \tag{15}$$

The local model associated to (15) has already been widely studied and we can find several existence results in the literature; we refer to [6, 7, 13, 16, 38, 48] for existence and uniqueness results in the pure elastodynamics case ($\mathbb{B} = 0$) and in the classic Kelvin-Voigt's one.

In recent years, materials whose constitutive equations can be described by non-local models are of increasing interest. For solid viscoelastic materials, some experiments are particularly in agreement with models using fractional derivative, see for example [22, 23, 46, 50] and the references therein.

In this chapter, which contains the results of [8] obtained in collaboration with M. Caponi, we focus on the fractional Kelvin-Voigt's model, i.e. we consider the following constitutive stress-strain dependence

$$\sigma(t) = \mathbb{A}eu(t) + \mathbb{B}D_t^\alpha eu(t) \qquad \text{in } \Omega \setminus \Gamma_t, \quad t \in [0,T],$$

where D_t^α denotes a fractional derivative of order $\alpha \in (0,1)$. In the literature we can find several definitions for the fractional derivative of a function $g\colon (a,b) \to \mathbb{R}$; here we focus on the most used ones which are Riemann-Liouville's derivative of order α at starting point a

$$ {}^{RL}_{\ a}D_t^\alpha g(t) := \frac{1}{\mathbf{\Gamma}(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{g(\tau)}{(t-\tau)^\alpha} \, \mathrm{d}\tau, $$

and Caputo's derivative of order α at starting point a

$$ {}^{C}_{a}D_t^\alpha g(t) := \frac{1}{\mathbf{\Gamma}(1-\alpha)} \int_a^t \frac{\dot g(\tau)}{(t-\tau)^\alpha} \, \mathrm{d}\tau, $$

where $\mathbf{\Gamma}$ denotes Euler's Gamma function. Notice that in order to define Caputo's derivative the function g must be differentiable, while this is not necessary for Riemann-Liouville's derivative. Given $g \in AC([a,b])$, and $t \in (a,b)$ we have the following relation between Riemann-Liouville's and Caputo's derivatives (see, e.g., [28]):

$$ {}^{RL}_{\ a}D_t^\alpha g(t) = {}^{C}_{a}D_t^\alpha g(t) + \frac{1}{\mathbf{\Gamma}(1-\alpha)} \frac{g(a)}{(t-a)^\alpha}. \tag{16} $$

In particular, when $g(a) = 0$, these two notions coincide. For more properties regarding these two fractional derivatives, we refer for example to [9, 35, 41, 43] and the references therein.

In this chapter we use Caputo's derivative, which means we consider the dynamic system

$$ \ddot u(t) - \mathrm{div}(\mathbb{A}eu(t)) - \mathrm{div}(\mathbb{B}\, {}^{C}_{0}D_t^\alpha eu(t)) = f(t) \qquad \text{in } \Omega \setminus \Gamma_t, \quad t \in [0,T]. \tag{17} $$

One of the qualities of this definition for the fractional derivative is that the initial conditions can be imposed in the classical sense, see for example [35, 41]. The choice of 0 as a starting point is due to the fact that we want to couple the dynamic system with the initial conditions at time $t = 0$.

Dealing with (17) is very difficult, since in the definition of ${}^{C}_{0}D_t^\alpha eu(t)$ we need that eu is differentiable, which is a very strong request. Hence, we rephrase Caputo's derivative in a more suitable way. Thanks to (16) for $g \in AC([0,T])$ we can write

$$ {}^{C}_{0}D_t^\alpha g(t) = \frac{1}{\mathbf{\Gamma}(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{1}{(t-\tau)^\alpha} (g(\tau) - g(0)) \, \mathrm{d}\tau. \tag{18} $$

This formulation of Caputo's derivative is well-posed in the distributional sense also when the function g is only integrable. We point out that formula (18) can be found in the recent literature on fractional derivatives, where it is used to define the notion of weak Caputo's derivative for less regular functions, see for example [21, 32].

Thanks to formula (18), we can write system (17) in a weaker form (see Definition 3.1.2) as (1) with the stress-strain dependence (4), i.e.

$$ \ddot u(t) - \mathrm{div}\left(\mathbb{A}eu(t) + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathbb{F}(t-\tau)(eu(\tau) - eu(0)) \, \mathrm{d}\tau\right) = f(t) \qquad \text{in } \Omega \setminus \Gamma_t, \quad t \in [0,T], \tag{19} $$

where $\mathbb{F}$ is defined by (5). Notice that the scalar function ρ appearing in $\mathbb{F}$ is positive, decreasing, and convex on $(0,\infty)$. Moreover, $\rho \in L^1(0,T)$ for every $T > 0$, but it is not bounded on $(0,T)$. In particular, we cannot compute the derivative in front of the convolution integral in (19).

In the literature we can find several existence and uniqueness results for fractional type systems related to (19), but only when Ω is a smooth domain without cracks. For example in [10] the authors studied an integral version of (19) with eu replaced by ∇u, and in [4, 29, 40] other fractional viscoelastic models are considered and the existence of solutions is obtained via Laplace's transform. In the case of prescribed fracture there are no existence results for the problem (19), since most of the previous techniques fail because the set $\Omega \setminus \Gamma_t$ is irregular and time-dependent.

To prove the existence of a solution to (19) we proceed into two steps, taking inspiration from [10]. First we consider a regularized version of (19), where we replace the kernel $\mathbb{F}$ by a regular kernel $\mathbb{G} \in C^2([0,T])$. Then we prove the existence of a solution to the more regular system

$$\ddot{u}(t) - \operatorname{div}\Big(\mathbb{A}eu(t) + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathbb{G}(t-\tau)(eu(\tau)-eu(0))\mathrm{d}\tau\Big) = f(t) \qquad \text{in } \Omega\setminus\Gamma_t, \quad t\in[0,T], \tag{20}$$

and we show that this solution satisfies a uniform bound depending on the L^1-norm of $\mathbb{G}$. Finally, we consider a sequence of regular tensors $\mathbb{G}^\varepsilon$ converging to $\mathbb{F}$ in L^1 and we take the solutions to (20) with $\mathbb{G} := \mathbb{G}^\varepsilon$. By a compactness argument, we show that the sequence u^ε converge to a function u^* which solves (19). Moreover, we prove that this solution satisfies an energy-dissipation inequality. We conclude this chapter by showing that, when the crack is not moving, the fractional Kelvin-Voigt's system (19) admits a unique solution.

Chapter 4: Quasistatic limit of a dynamic viscoelastic model with memory

In this chapter we consider a domain without cracks and we study a different problem for the viscoelastic model with memory (11)–(13) of Chapter 2: the quasistatic limit. The results of this chapter are obtained in collaboration with Prof. G. Dal Maso, see [18].

The quasistatic limit of the solutions to problem (11)–(13) means the limit of these solutions when the rate of change of the data tends to zero. More precisely, given a small parameter $\varepsilon > 0$, we consider the solution u^ε of (11)–(13) corresponding to $\ell(\varepsilon t)$, $z(\varepsilon t)$, and $u_{in}(\varepsilon t)$. To study the asymptotic behaviour of u^ε as $\varepsilon \to 0^+$ it is convenient to introduce the rescaled solution $u_\varepsilon(t) := u^\varepsilon(t/\varepsilon)$, which turns out to be the solution of the system

$$\varepsilon^2\ddot{u}_\varepsilon(t) - \operatorname{div}((\mathbb{A}+\mathbb{B})eu_\varepsilon(t)) + \int_{-\infty}^t \frac{1}{\beta\varepsilon}\mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}}\operatorname{div}(\mathbb{B}eu_\varepsilon(\tau))\mathrm{d}\tau = \ell(t) \qquad \text{in } \Omega, \quad t\in[0,T], \tag{21}$$

with boundary and initial conditions (12) and (13).

Under different assumptions on $\ell(t)$, $z(t)$, and $u_{in}(t)$ we prove (Theorems 4.2.6 and 4.2.7) that $u_\varepsilon(t)$ converges, as $\varepsilon \to 0^+$, to the solution $u_0(t)$ of the stationary problem

$$-\operatorname{div}(\mathbb{A}eu_0(t)) = \ell(t) \qquad \text{in } \Omega, \quad t\in[0,T], \tag{22}$$

with boundary condition (12).

By using just the energy-dissipation inequality, it is not difficult to prove a similar result for the Kelvin-Voigt model, in which the viscosity term

$$-\operatorname{div}(\mathbb{B}eu(t)) + \int_{-\infty}^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\operatorname{div}(\mathbb{B}eu(\tau))\mathrm{d}\tau \tag{23}$$

is replaced by $-\operatorname{div}(\mathbb{B}e\dot{u}(t))$. On the other hand, in the case of the equation of elastodynamics without damping terms, i.e., when $\mathbb{B} = 0$, by using the Fourier decomposition with respect to the eigenfunctions of the operator $-\operatorname{div}(\mathbb{A}eu)$, we can easily see that the convergence of u_ε to u_0 does not hold in general. The purpose of this paper is to prove that the non-local damping term (23) is enough to obtain the convergence of the solutions of the evolution problems to the solution of the stationary problem.

Our result can be considered in the framework of the study of the quasistatic limits, i.e. the convergence of the solutions to second order evolution equations with rescaled times towards the solutions to the corresponding stationary equations. Similar problems in finite dimension have been studied in [26, 1, 37, 45]. A special case involving the wave equations on time-dependent intervals in dimension one has been studied in [31, 42]. The main novelty of our problem is the the non-local form of the damping term, given by (23).

The main tools to prove our results are two different estimates (Lemmas 4.2.8 and 4.4.2), related to the energy-dissipation balance (4.24) and to the elliptic system (4.80) obtained from (21) via Laplace Transform. After a precise statement of all assumptions, more details on the line of proof will be given after Theorem 4.2.7.

Notation

Basic notation. The space of $m \times d$ matrices with real entries is denoted by $\mathbb{R}^{m\times d}$, and in the case $m = d$, the subspace of symmetric matrices is denoted by $\mathbb{R}^{d\times d}_{sym}$. We denote by A^T the transpose of $A \in \mathbb{R}^{d\times d}$, and by A^{sym} the symmetric part, namely $A^{sym} := \frac{1}{2}(A + A^T)$; we use Id to denote the identity matrix in $\mathbb{R}^{d\times d}$. The Euclidian scalar product in $\mathbb{R}^d$ is denoted by $\cdot$ and the corresponding Euclidian norm by $|\cdot|$; the same notation is used also for $\mathbb{R}^{m\times d}$. We denote by $a \otimes b \in \mathbb{R}^{d\times d}$ the tensor product between two vectors $a, b \in \mathbb{R}^d$, and by $a \odot b \in \mathbb{R}^{d\times d}_{sym}$ the symmetrized tensor product, namely the symmetric part of $a \otimes b$.

The d-dimensional Lebesgue measure in $\mathbb{R}^d$ is denoted by $\mathcal{L}^d$, and the $(d-1)$-dimensional Hausdorff measure by $\mathcal{H}^{d-1}$. Given a bounded open set Ω with Lipschitz boundary, we denote by ν the outer unit normal vector to $\partial\Omega$, which is defined $\mathcal{H}^{d-1}$-a.e. on the boundary. We use $B_r(x)$ to denote the ball of radius r and center x in $\mathbb{R}^d$, namely $B_r(x) \coloneqq \{y \in \mathbb{R}^d : |y-x| < r\}$, and id to denote the identity function in $\mathbb{R}^d$, possibly restricted to a subset.

The partial derivatives with respect to the variable x_i are denoted by ∂_i or ∂_{x_i}. Given a function $u\colon \mathbb{R}^d \to \mathbb{R}^m$, we denote its Jacobian matrix by ∇u, whose components are $(\nabla u)_{ij} \coloneqq \partial_j u_i$ for $i = 1, \dots, m$ and $j = 1, \dots, d$. When $u\colon \mathbb{R}^d \to \mathbb{R}^d$, we use eu to denote its symmetrized gradient, namely $eu \coloneqq \frac{1}{2}(\nabla u + \nabla u^T)$. Given $u\colon \mathbb{R}^d \to \mathbb{R}$, we use Δu to denote its Laplacian, which is defined as $\Delta u \coloneqq \sum_{i=1}^d \partial_i^2 u$. For a tensor field $T\colon \mathbb{R}^d \to \mathbb{R}^{m\times d}$, by $\operatorname{div} T$ we mean its divergence with respect to rows, namely $(\operatorname{div} T)_i \coloneqq \sum_{j=1}^d \partial_j T_{ij}$ for $i = 1, \dots, m$.

Function spaces. Given two metric spaces X and Y, we use $C^0(X; Y)$ and $\mathrm{Lip}(X; Y)$ to denote, respectively, the space of continuous and Lipschitz functions from X to Y. Given an open set $\Omega \subset \mathbb{R}^d$, we denote by $C^k(\Omega; \mathbb{R}^m)$ the space of $\mathbb{R}^m$-valued functions with k continuous derivatives; we use $C^k_c(\Omega; \mathbb{R}^m)$ and $C^{k,1}(\Omega; \mathbb{R}^m)$ to denote, respectively, the subspace of functions with compact support in Ω, and of functions whose k-derivatives are Lipschitz. For every $1 \le p \le \infty$ we denote by $L^p(\Omega; \mathbb{R}^m)$ the Lebesgue space of p-th power integrable functions, and by $W^{k,p}(\Omega; \mathbb{R}^m)$ the Sobolev space of functions with k derivatives; for $p = 2$ we set $H^k(\Omega; \mathbb{R}^m) := W^{k,2}(\Omega; \mathbb{R}^m)$, and for $m = 1$ we omit $\mathbb{R}^m$ in the previous spaces. The boundary values of a Sobolev function are always intended in the sense of traces. The scalar product in $L^2(\Omega; \mathbb{R}^m)$ is denoted by $(\cdot,\cdot)_{L^2(\Omega)}$ and the norm in $L^p(\Omega; \mathbb{R}^m)$ by $\|\cdot\|_{L^p(\Omega)}$; a similar notation is valid for the Sobolev spaces. For simplicity, we use $\|\cdot\|_{L^\infty(\Omega)}$ to denote also the supremum norm of continuous and bounded functions.

The norm of a generic Banach space X is denoted by $\|\cdot\|_X$; when X is a Hilbert space, we use $(\cdot,\cdot)_X$ to denote its scalar product. We denote by X' the dual of X, and by $\langle\cdot,\cdot\rangle_{X'}$ the duality product between X' and X. Given two Banach spaces X_1 and X_2, the space of linear and continuous maps from X_1 to X_2 is denoted by $\mathscr{L}(X_1; X_2)$; given $\mathbb{A} \in \mathscr{L}(X_1; X_2)$ and $u \in X_1$, we write $\mathbb{A}u \in X_2$ to denote the image of u under $\mathbb{A}$.

Given an open interval $(a, b) \subset \mathbb{R}$ and $1 \le p \le \infty$, we denote by $L^p(a, b; X)$ the space of L^p functions from (a, b) to X; we use $W^{k,p}(a, b; X)$ and $H^k(a, b; X)$ (for $p = 2$) to denote the Sobolev space of functions from (a, b) to X with k derivatives. Given $u \in W^{1,p}(a, b; X)$, we denote by $\dot u \in L^p(a, b; X)$ its derivative in the sense distributions. The set of functions from $[a, b]$ to X with k continuous derivatives is denoted by $C^k([a, b]; X)$; we use $C^k_c(a, b; X)$ to denote the subspace of functions with compact support in (a, b). The space of absolutely continuous functions from $[a, b]$ to X is denoted by $AC([a, b]; X)$; we use $C^0_w([a, b]; X)$ to denote the set of weakly continuous functions from $[a, b]$ to X, namely

$$C^0_w([a, b]; X) \coloneqq \{u\colon [a, b] \to X : t \mapsto \langle x', u(t)\rangle_{X'} \text{ is continuous in } [a, b] \text{ for every } x' \in X'\}.$$

When dealing with an element $u \in H^1(a, b; X)$ we always assume u to be the continuous representative of its class. In particular, it makes sense to consider the pointwise value $u(t)$ for every $t \in [a, b]$.

Chapter 1

A dynamic model for viscoelastic materials with growing cracks

The chapter is organized as follows. In Section 1.1 we fix the notation adopted throughout the chapter, we list the main assumptions on the family of cracks $\{\Gamma_t\}_{t\in[0,T]}$ and on the function Ψ, and we specify the notion of solution to (9). In Section 1.2 we state our main existence result (Theorem 1.2.1), which is obtained by means of a time discretization scheme. We conclude the proof of Theorem 1.2.1 in Section 1.3, where we show the validity of the initial conditions (1.16) and the energy-dissipation inequality (1.39). Section 1.4 deals with the uniqueness problem. Under stronger regularity assumptions on the cracks sets, in Theorem 1.4.5 we prove the uniqueness, but only when the space dimension is $d = 2$. To this aim, we assume also that the function Ψ is zero in a neighborhood of the crack-tip. We conclude with Section 1.5, where, in dimension $d = 2$ and for an antiplane evolution, we show an example of a moving crack which satisfies the dynamic energy-dissipation balance (10).

The results presented here are obtained in collaboration with M. Caponi and are contained in the published paper [7].

1.1 Preliminary results

Let T be a positive real number and let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. Let $\partial_D\Omega$ be a (possibly empty) Borel subset of $\partial\Omega$ and let $\partial_N\Omega$ be its complement. We assume the following hypotheses on the geometry of the cracks:

(E1) $\Gamma \subset \overline{\Omega}$ is a closed set with $\mathcal{L}^d(\Gamma) = 0$ and $\mathcal{H}^{d-1}(\Gamma \cap \partial\Omega) = 0$;

(E2) for every $x \in \Gamma$ there exists an open neighborhood U of x in $\mathbb{R}^d$ such that $(U \cap \Omega) \setminus \Gamma$ is the union of two disjoint open sets U^+ and U^- with Lipschitz boundary;

(E3) $\{\Gamma_t\}_{t\in[0,T]}$ is a family of closed subsets of Γ satisfying $\Gamma_s \subset \Gamma_t$ for every $0 \le s \le t \le T$.

Thanks (E1)–(E3) the space $L^2(\Omega \setminus \Gamma_t; \mathbb{R}^m)$ coincides with $L^2(\Omega; \mathbb{R}^m)$ for every $t \in [0, T]$ and $m \in \mathbb{N}$. In particular, we can extend a function $u \in L^2(\Omega \setminus \Gamma_t; \mathbb{R}^m)$ to a function in $L^2(\Omega; \mathbb{R}^m)$ by setting $u = 0$ on Γ_t. Moreover, the trace of $u \in H^1(\Omega \setminus \Gamma)$ is well defined on $\partial\Omega$. Indeed, we may find a finite number of open sets with Lipschitz boundary $U_j \subset \Omega \setminus \Gamma$, $j = 1, \dots m$, such that $\partial\Omega \setminus (\Gamma \cap \partial\Omega) \subset \cup_{j=1}^m \partial U_j$. Since $\mathcal{H}^{d-1}(\Gamma \cap \partial\Omega) = 0$, there exists a constant $C > 0$, depending only on Ω and Γ, such that

$$\|u\|_{L^2(\partial\Omega)} \le C\|u\|_{H^1(\Omega\setminus\Gamma)} \quad \text{for every } u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d). \tag{1.1}$$

Similarly, we can find a finite number of open sets $U_j \subset \Omega \setminus \Gamma$, $j = 1, \dots m$, with Lipschitz boundary, such that $\Omega \setminus \Gamma = \cup_{j=1}^m U_j$. By using second Korn's inequality in each U_j (see, e.g., [39, Theorem 2.4]) and taking the sum over j we can find a constant C_K, depending only on Ω and Γ, such that

$$\|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} \leq C_K \left(\|u\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|eu\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{sym})}\right) \quad \text{for every } u \in H^1(\Omega\setminus\Gamma;\mathbb{R}^d), \tag{1.2}$$

where eu is the symmetric part of ∇u, i.e., $eu := \frac{1}{2}(\nabla u + \nabla u^T)$.

For every $t \in [0,T]$ we define

$$V_t := \{u \in L^2(\Omega\setminus\Gamma_t;\mathbb{R}^d) : eu \in L^2(\Omega\setminus\Gamma_t;\mathbb{R}^{d\times d}_{sym})\}.$$

Notice that in the definition of V_t we are considering only the distributional gradient of u in $\Omega\setminus\Gamma_t$ and not the one in Ω. The set V_t is a Hilbert space with respect to the following norm

$$\|u\|_{V_t} := (\|u\|^2 + \|eu\|^2)^{\frac{1}{2}} \quad \text{for every } u \in V_t.$$

To simplify our exposition, for every $m \in \mathbb{N}$ we set $H := L^2(\Omega;\mathbb{R}^m)$ and $H_N := L^2(\partial_N\Omega;\mathbb{R}^m)$; we always identify the dual of H by H itself and $L^2(0,T;L^2(\Omega;\mathbb{R}^m))$ by $L^2((0,T)\times\Omega;\mathbb{R}^m)$.

Thanks to (1.2), the space V_t coincides with the usual Sobolev space $H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d)$. Therefore, by (1.1), it makes sense to consider for every $t\in[0,T]$ the set

$$V_t^D := \{u \in V_t : u = 0 \text{ on } \partial_D\Omega\},$$

which is a Hilbert space with respect to $\|\cdot\|_{V_t}$. Moreover, by combining (1.2) with (1.1), we derive also the existence of a constant $C_{tr} > 0$ such that

$$\|u\|_{H_N} \leq C_{tr}\|u\|_V \quad \text{for every } u \in V. \tag{1.3}$$

Let $\mathbb{A},\mathbb{B}\colon \Omega \to \mathscr{L}(\mathbb{R}^{d\times d}_{sym};\mathbb{R}^{d\times d}_{sym})$ be the elastic and viscosity tensors, which are fourth-order tensors such that

$$\mathbb{A},\mathbb{B} \in L^\infty(\Omega;\mathscr{L}(\mathbb{R}^{d\times d}_{sym};\mathbb{R}^{d\times d}_{sym})), \tag{1.4}$$

and which satisfy for a.e. $x\in\Omega$ the following properties:

$$\mathbb{A}(x)\xi_1\cdot\xi_2 = \xi_1\cdot\mathbb{A}(x)\xi_2, \qquad \mathbb{B}(x)\xi_1\cdot\xi_2 = \xi_1\cdot\mathbb{B}(x)\xi_2 \qquad \text{for every } \xi_1,\xi_2\in\mathbb{R}^{d\times d}_{sym}, \tag{1.5}$$

$$c_{\mathbb{A}}|\xi|^2 \leq \mathbb{A}(x)\xi\cdot\xi \leq C_{\mathbb{A}}|\xi|^2, \quad c_{\mathbb{B}}|\xi|^2 \leq \mathbb{B}(x)\xi\cdot\xi \leq C_{\mathbb{B}}|\xi|^2 \quad \text{for every } \xi\in\mathbb{R}^{d\times d}_{sym}, \tag{1.6}$$

for some positive constants $c_{\mathbb{A}}$, $c_{\mathbb{B}}$, $C_{\mathbb{A}}$, and $C_{\mathbb{B}}$ independent of x. Let $\Psi\colon (0,T)\times\Omega\to\mathbb{R}$ be a function satisfying

$$\Psi \in L^\infty((0,T)\times\Omega), \quad \nabla\Psi \in L^\infty((0,T)\times\Omega;\mathbb{R}^d). \tag{1.7}$$

Given $f \in L^2(0,T;H)$, $z \in H^2(0,T;H)\cap H^1(0,T;V_0)$, $N \in H^1(0,T;H_N)$, $u^0 \in V_0$ with $u^0 - z(0) \in V_0^D$, and $u^1 \in H$, we want to find a solution to the viscoelastic dynamic system

$$\ddot u(t) - \operatorname{div}(\mathbb{A}eu(t)) - \operatorname{div}(\Psi^2(t)\mathbb{B}e\dot u(t)) = f(t) \quad \text{in } \Omega\setminus\Gamma_t,\ t\in(0,T), \tag{1.8}$$

satisfying the following boundary and initial conditions

$$u(t) = z(t) \qquad \text{on } \partial_D\Omega,\ t\in(0,T), \tag{1.9}$$

$$(\mathbb{A}eu(t) + \Psi^2(t)\mathbb{B}e\dot u(t))\nu = N(t) \qquad \text{on } \partial_N\Omega,\ t\in(0,T), \tag{1.10}$$

$$(\mathbb{A}eu(t) + \Psi^2(t)\mathbb{B}e\dot u(t))\nu = 0 \qquad \text{on } \Gamma_t,\quad t\in(0,T), \tag{1.11}$$

$$u(0) = u^0, \quad \dot u(0) = u^1. \tag{1.12}$$

As usual, the Neumann boundary conditions are only formal, and their meaning will be specified in Definition 1.1.4.

Throughout the chapter we always assume that the family $\{\Gamma_t\}_{t\in[0,T]}$ satisfies (E1)–(E3), as well as $\mathbb{A}$, $\mathbb{B}$, Ψ, f, z, N, u^0, and u^1 the previous hypotheses. Let us define the following functional spaces:

$$\begin{aligned}
&\mathcal{V} := \{\varphi \in L^2(0,T;V) : \dot\varphi \in L^2(0,T;H), \varphi(t) \in V_t \text{ for a.e. } t \in (0,T)\},\\
&\mathcal{V}^D := \{\varphi \in \mathcal{V} : \varphi(t) \in V_t^D \text{ for a.e. } t \in (0,T)\},\\
&\mathcal{W} := \{u \in \mathcal{V} : \Psi\dot u \in L^2(0,T;V), \Psi(t)\dot u(t) \in V_t \text{ for a.e. } t \in (0,T)\}.
\end{aligned}$$

Remark 1.1.1. In the classical viscoelastic case, namely when Ψ is identically equal to 1, the solution u to system (1.8) has derivative $\dot u(t) \in V_t$ for a.e. $t \in (0,T)$ with $e\dot u \in L^2(0,T;H)$. For a generic Ψ we expect to have $\Psi e\dot u \in L^2(0,T;H)$. Therefore $\mathcal{W}$ is the natural setting where looking for a solution to (1.8). Indeed, from a distributional point of view we have

$$\Psi(t)e\dot u(t) = e(\Psi(t)\dot u(t)) - \nabla\Psi(t) \odot \dot u(t) \quad \text{in } \mathcal{D}'(\Omega\setminus\Gamma_t;\mathbb{R}^{d\times d}_{sym}) \text{ for a.e. } t \in (0,T),$$

and $e(\Psi\dot u), \nabla\Psi \odot \dot u \in L^2(0,T;H)$ if $u \in \mathcal{W}$, thanks to (1.7).

Remark 1.1.2. The set $\mathcal{W}$ coincides with the space of functions $u \in H^1(0,T;H)$ such that $u(t) \in V_t$ and $\Psi(t)\dot u(t) \in V_t$ for a.e. $t \in (0,T)$, and satisfying

$$\int_0^T \|u(t)\|^2_{V_t} + \|\Psi(t)\dot u(t)\|^2_{V_t}\mathrm{d}t < \infty. \tag{1.13}$$

This is a consequence of the strong measurability of the maps $t \mapsto u(t)$ and $t \mapsto \Psi(t)\dot u(t)$ from $(0,T)$ into V, which gives that (1.13) is well defined and $u, \Psi\dot u \in L^2(0,T;V)$. To prove the strong measurability of these two maps, it is enough to observe that V is a separable Hilbert space and that the maps $t \mapsto \dot u(t)$ and $t \mapsto \Psi(t)\dot u(t)$ from $(0,T)$ into V are weakly measurable. Indeed, for every $\varphi \in C_c^\infty(\Omega\setminus\Gamma_T)$ the maps

$$\begin{aligned}
t &\mapsto \int_{\Omega\setminus\Gamma_T} eu(t,x)\varphi(x)\mathrm{d}x = -\int_{\Omega\setminus\Gamma_T} u(t,x) \odot \nabla\varphi(x)\mathrm{d}x,\\
t &\mapsto \int_{\Omega\setminus\Gamma_T} e(\Psi(t,x)\dot u(t,x))\varphi(x)\mathrm{d}x = -\int_{\Omega\setminus\Gamma_T} \Psi(t,x)\dot u(t,x) \odot \nabla\varphi(x)\mathrm{d}x
\end{aligned}$$

are measurable from $(0,T)$ into $\mathbb{R}$, and $C_c^\infty(\Omega\setminus\Gamma_T)$ is dense in $L^2(\Omega)$.

Lemma 1.1.3. *The spaces $\mathcal{V}$ and $\mathcal{W}$ are Hilbert spaces with respect to the following norms:*

$$\begin{aligned}
&\|\varphi\|_{\mathcal{V}} := \left(\|\varphi\|^2_{L^2(0,T;V)} + \|\dot\varphi\|^2_{L^2(0,T;H)}\right)^{\frac12} \quad \textit{for every } \varphi \in \mathcal{V},\\
&\|u\|^2_{\mathcal{W}} := \left(\|u\|_{\mathcal{V}} + \|\Psi\dot u\|^2_{L^2(0,T;V)}\right)^{\frac12} \qquad \textit{for every } u \in \mathcal{W}.
\end{aligned}$$

Moreover, $\mathcal{V}^D$ is a closed subspace of $\mathcal{V}$.

Proof. It is clear that $\|\cdot\|_{\mathcal{V}}$ and $\|\cdot\|_{\mathcal{W}}$ are norms on $\mathcal{V}$ and $\mathcal{W}$ induced by scalar products. We just have to check the completeness of such spaces with respect to these norms.

Let $\{\varphi_k\}_k \subset \mathcal{V}$ be a Cauchy sequence. Then, $\{\varphi_k\}_k$ and $\{\dot\varphi_k\}_k$ are Cauchy sequences, respectively, in $L^2(0,T;V)$ and $L^2(0,T;H)$, which are complete Hilbert spaces. Thus there exists $\varphi \in L^2(0,T;V)$ with $\dot\varphi \in L^2(0,T;H)$ such that $\varphi_k \to \varphi$ in $L^2(0,T;V)$ and $\dot\varphi_k \to \dot\varphi$ in $L^2(0,T;H)$. In particular there exists a subsequence $\{\varphi_{k_j}\}_j$ such that $\varphi_{k_j}(t) \to \varphi(t)$ in V for a.e. $t \in (0,T)$. Since $\varphi_{k_j}(t) \in V_t$ for a.e. $t \in (0,T)$ we deduce that $\varphi(t) \in V_t$ for a.e. $t \in (0,T)$. Hence $\varphi \in \mathcal{V}$ and $\varphi_k \to \varphi$ in $\mathcal{V}$. With a similar argument, we can prove that $\mathcal{V}^D \subset \mathcal{V}$ is a closed subspace.

Let us now consider a Cauchy sequence $\{u_k\}_k \subset \mathcal{W}$. We have that $\{u_k\}_k$ and $\{\Psi \dot{u}_k\}_k$ are Cauchy sequences, respectively, in $\mathcal{V}$ and $L^2(0,T;V)$, which are complete Hilbert spaces. Thus there exist two functions $u \in \mathcal{V}$ and $w \in L^2(0,T;V)$ such that $u_k \to u$ in $\mathcal{V}$ and $\Psi \dot{u}_k \to w$ in $L^2(0,T;V)$. Since $\dot{u}_k \to \dot{u}$ in $L^2(0,T;H)$ and $\Psi \in L^\infty((0,T)\times\Omega)$, we also have that $\Psi\dot{u}_k \to \Psi\dot{u}$ in $L^2(0,T;H)$, which gives that $w = \Psi\dot{u}$. Finally let us prove that $\Psi(t)\dot{u}(t) \in V_t$ for a.e. $t \in (0,T)$. By the fact that $\Psi\dot{u}_k \to \Psi\dot{u}$ in $L^2(0,T;V)$, there exists a subsequence $\{\Psi\dot{u}_{k_j}\}_j$ such that $\Psi(t)\dot{u}_{k_j}(t) \to \Psi(t)\dot{u}(t)$ in V for a.e. $t \in (0,T)$. Since $\Psi(t)\dot{u}_{k_j}(t) \in V_t$ for a.e. $t \in (0,T)$ we deduce that $\Psi(t)\dot{u}(t) \in V_t$ for a.e. $t \in (0,T)$. Hence $u \in \mathcal{W}$ and $u_k \to u$ in $\mathcal{W}$. □

We are now in position to define a weak solution to (1.8)–(1.11).

Definition 1.1.4 (Weak solution)**.** We say that $u \in \mathcal{W}$ is a *weak solution* to system (1.8) with boundary conditions (1.9)–(1.11) if $u - z \in \mathcal{V}^D$ and

$$
\begin{aligned}
&-\int_0^T (\dot{u}(t),\dot{\varphi}(t))\mathrm{d}t + \int_0^T (\mathbb{A}eu(t), e\varphi(t))\mathrm{d}t + \int_0^T (\mathbb{B}e(\Psi(t)\dot{u}(t)), \Psi(t)e\varphi(t))\mathrm{d}t \\
&-\int_0^T (\mathbb{B}\nabla\Psi(t)\odot\dot{u}(t), \Psi(t)e\varphi(t))\mathrm{d}t = \int_0^T (f(t),\varphi(t))\mathrm{d}t + \int_0^T (N(t),\varphi(t))_{H_N}\mathrm{d}t
\end{aligned}
\tag{1.14}
$$

for every $\varphi \in \mathcal{V}^D$ such that $\varphi(0) = \varphi(T) = 0$.

Notice that the Neumann boundary conditions (1.10) and (1.11) can be obtained from (1.14), by using integration by parts in space, only when $u(t)$ and Γ_t are sufficiently regular.

Remark 1.1.5. If $\dot{u}$ is regular enough (for example $\dot{u} \in L^2(0,T;V)$ with $\dot{u}(t) \in V_t$ for a.e. $t \in (0,T)$), then we have $\Psi e\dot{u} = e(\Psi\dot{u}) - \nabla\Psi\odot\dot{u}$. Therefore (1.14) is coherent with the strong formulation (1.8). In particular, for a function $u \in \mathcal{W}$ we can define

$$
\Psi e\dot{u} := e(\Psi\dot{u}) - \nabla\Psi\odot\dot{u} \in L^2(0,T;H),
\tag{1.15}
$$

so that equation (1.14) can be rephrased as

$$
\begin{aligned}
&-\int_0^T (\dot{u}(t),\dot{\varphi}(t))\mathrm{d}t + \int_0^T (\mathbb{A}eu(t), e\varphi(t))\mathrm{d}t + \int_0^T (\mathbb{B}\Psi(t)e\dot{u}(t), \Psi(t)e\varphi(t))\mathrm{d}t \\
&= \int_0^T (f(t),\varphi(t))\mathrm{d}t + \int_0^T (N(t),\varphi(t))_{H_N}\mathrm{d}t
\end{aligned}
$$

for every $\varphi \in \mathcal{V}^D$ such that $\varphi(0) = \varphi(T) = 0$.

Definition 1.1.6 (Initial conditions)**.** We say that $u \in \mathcal{W}$ satisfies the initial conditions (1.12) if

$$
\lim_{h\to 0^+} \frac{1}{h}\int_0^h (\|u(t)-u^0\|^2_{V_t} + \|\dot{u}(t)-u^1\|^2)\mathrm{d}t = 0.
\tag{1.16}
$$

1.2 Existence

We now state our main existence result, whose proof will be given at the end of Section 1.3.

Theorem 1.2.1. *There exists a weak solution $u \in \mathcal{W}$ to (1.8)–(1.11) satisfying the initial conditions $u(0) = u^0$ and $\dot{u}(0) = u^1$ in the sense of (1.16). Moreover $u \in C_w([0,T];V)$, $\dot{u} \in C_w([0,T];H) \cap H^1(0,T;(V_0^D)')$, and*

$$
\lim_{t\to 0^+} u(t) = u^0 \ \textit{in } V, \quad \lim_{t\to 0^+} \dot{u}(t) = u^1 \ \textit{in } H.
$$

To prove the existence of a weak solution to (1.8)–(1.11), we use a time discretization scheme in the same spirit of [13]. Let us fix $n \in \mathbb{N}$ and set

$$\tau_n := \frac{T}{n}, \quad u_n^0 := u^0, \quad u_n^{-1} := u^0 - \tau_n u^1.$$

We define

$$\begin{aligned}
&V_n^k := V_{k\tau_n}^D, && N_n^k := N(k\tau_n) && z_n^k := z(k\tau_n) && \text{for } k = 0,\dots,n,\\
&f_n^k := \fint_{(k-1)\tau_n}^{k\tau_n} f(s)\mathrm{d}s, && \Psi_n^k := \fint_{(k-1)\tau_n}^{k\tau_n} \Psi(s)\mathrm{d}s, && \delta N_n^k := \frac{N_n^k - N_n^{k-1}}{\tau_n} && \text{for } k = 1,\dots,n,\\
&\delta z_n^0 := \dot z(0), && \delta z_n^k := \frac{z_n^k - z_n^{k-1}}{\tau_n}, && \delta^2 z_n^k := \frac{\delta z_n^k - \delta z_n^{k-1}}{\tau_n} && \text{for } k = 1,\dots,n,
\end{aligned}$$

For every $k = 1,\dots,n$ let $u_n^k \in V$, with $u_n^k - z_n^k \in V_n^k$, be the solution to

$$(\delta^2 u_n^k, v) + (\mathbb{A}eu_n^k, ev) + (\mathbb{B}\Psi_n^k e\delta u_n^k, \Psi_n^k ev) = (f_n^k, v) + (N_n^k, v)_{H_N} \quad \text{for every } v \in V_n^k, \tag{1.17}$$

where

$$\delta u_n^k := \frac{u_n^k - u_n^{k-1}}{\tau_n} \quad \text{for } k = 0,\dots,n, \quad \delta^2 u_n^k := \frac{\delta u_n^k - \delta u_n^{k-1}}{\tau_n} \quad \text{for } k = 1,\dots,n.$$

The existence of a unique solution u_n^k to (1.17) is an easy application of Lax-Milgram's theorem.

Remark 1.2.2. Since $\delta u_n^k \in V_{(k-1)\tau_n}$, then $\Psi_n^k e\delta u_n^k = e(\Psi_n^k u_n^k) - \nabla\Psi_n^k \odot u_n^k$, so that the discrete equation (1.17) is coherent with the weak formulation given in (1.14).

In the next lemma, we show a uniform estimate for the family $\{u_n^k\}_{k=1}^n$ with respect to $n \in \mathbb{N}$ that will be used later to pass to the limit in the discrete equation (1.17).

Lemma 1.2.3. *There exists a constant $C > 0$, independent of $n \in \mathbb{N}$, such that*

$$\max_{i=1,..,n} \|\delta u_n^i\| + \max_{i=1,..,n} \|eu_n^i\| + \sum_{i=1}^n \tau_n \|\Psi_n^i e\delta u_n^i\|^2 \le C. \tag{1.18}$$

Proof. We fix $n \in \mathbb{N}$. To simplify the notation we set

$$a(u,v) := (\mathbb{A}eu, ev), \quad b_n^k(u,v) := (\mathbb{B}\Psi_n^k eu, \Psi_n^k ev) \quad \text{for every } u, v \in V.$$

By taking as test function $v = \tau_n(\delta u_n^k - \delta z_n^k) \in V_n^k$ in (1.17), for $k = 1,\dots,n$ we obtain

$$\|\delta u_n^k\|^2 - (\delta u_n^{k-1}, \delta u_n^k) + a(u_n^k, u_n^k) - a(u_n^k, u_n^{k-1}) + \tau_n b_n^k(\delta u_n^k, \delta u_n^k) = \tau_n L_n^k,$$

where

$$L_n^k := (f_n^k, \delta u_n^k - \delta z_n^k) + (N_n^k, \delta u_n^k - \delta z_n^k)_{H_N} + (\delta^2 u_n^k, \delta w_n^k) + a(u_n^k, \delta z_n^k) + b_n^k(\delta u_n^k, \delta z_n^k).$$

Thanks to the following identities

$$\begin{aligned}
\|\delta u_n^k\|^2 - (\delta u_n^{k-1}, \delta u_n^k) &= \frac{1}{2}\|\delta u_n^k\|^2 - \frac{1}{2}\|\delta u_n^{k-1}\|^2 + \frac{\tau_n^2}{2}\|\delta^2 u_n^k\|^2,\\
a(u_n^k, u_n^k) - a(u_n^k, u_n^{k-1}) &= \frac{1}{2}a(u_n^k, u_n^k) - \frac{1}{2}a(u_n^{k-1}, u_n^{k-1}) + \frac{\tau_n^2}{2}a(\delta u_n^k, \delta u_n^k),
\end{aligned}$$

and by omitting the terms with τ_n^2, which are non negative, we derive

$$\frac{1}{2}\|\delta u_n^k\|^2 - \frac{1}{2}\|\delta u_n^{k-1}\|^2 + \frac{1}{2}a(u_n^k, u_n^k) - \frac{1}{2}a(u_n^{k-1}, u_n^{k-1}) + \tau_n b_n^k(\delta u_n^k, \delta u_n^k) \le \tau_n L_n^k.$$

We fix $i \in \{1, \dots, n\}$ and sum over $k = 1, \dots, i$ to obtain the following discrete energy inequality

$$\frac{1}{2}\|\delta u_n^i\|^2 + \frac{1}{2}a(u_n^i, u_n^i) + \sum_{k=1}^{i} \tau_n b_n^k(\delta u_n^k, \delta u_n^k) \le \mathcal{E}_0 + \sum_{k=1}^{i} \tau_n L_n^k, \tag{1.19}$$

where $\mathcal{E}_0 := \frac{1}{2}\|u^1\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0)$. Let us now estimate the right-hand side in (1.19) from above. By (1.3) and (1.4) we have

$$\left|\sum_{k=1}^{i} \tau_n (f_n^k, \delta u_n^k - \delta z_n^k)\right| \le \|f\|^2_{L^2(0,T;H)} + \frac{1}{2}\|\dot z\|^2_{L^2(0,T;H)} + \frac{1}{2}\sum_{k=1}^{i} \tau_n \|\delta u_n^k\|_H^2, \tag{1.20}$$

$$\left|\sum_{k=1}^{i} \tau_n a(u_n^k, \delta z_n^k)\right| \le \frac{C_{\mathbb{A}}}{2}\|\dot z\|^2_{L^2(0,T;V_0)} + \frac{C_{\mathbb{A}}}{2}\sum_{k=1}^{i} \tau_n \|eu_n^k\|^2, \tag{1.21}$$

$$\left|\sum_{k=1}^{i} \tau_n (N_n^k, \delta z_n^k)_{H_N}\right| \le \frac{1}{2}\|N\|^2_{L^2(0,T;H_N)} + \frac{C_{tr}^2}{2}\|\dot z\|^2_{L^2(0,T;V_0)}. \tag{1.22}$$

For the other term involving N_n^k, we perform the following discrete integration by parts

$$\sum_{k=1}^{i} \tau_n (N_n^k, \delta u_n^k)_{H_N} = (N_n^i, u_n^i)_{H_N} - (N(0), u^0)_{H_N} - \sum_{k=1}^{i} \tau_n (\delta N_n^k, u_n^{k-1})_{H_N}. \tag{1.23}$$

Hence for every $\varepsilon \in (0,1)$, by using (1.3) and Young's inequality, we get

$$\left|\sum_{k=1}^{i} \tau_n (N_n^k, \delta u_n^k)_{H_N}\right| \tag{1.24}$$

$$\le \frac{\varepsilon}{2}\|u_n^i\|^2_{H_N} + \frac{1}{2\varepsilon}\|N\|^2_{L^\infty(0,T;H_N)} + \|N(0)\|_{H_N}\|u^0\|_{H_N} + \sum_{k=1}^{i} \tau_n \|\delta N_n^k\|_{H_N}\|u_n^{k-1}\|^2_{H_N}$$

$$\le C_\varepsilon + \frac{\varepsilon C_{tr}^2}{2}\|u_n^i\|_V^2 + \frac{C_{tr}^2}{2}\sum_{k=1}^{i} \tau_n \|u_n^k\|_V^2, \tag{1.25}$$

where C_ε is a positive constant depending on ε. Thanks to Jensen's inequality we can write

$$\|u_n^l\|_V^2 \le \|eu_n^l\|^2 + \left(\|u_0\| + \sum_{j=1}^{l} \tau_n \|\delta u_n^j\|\right)^2 \le \|eu_n^l\|^2 + 2\|u^0\|^2 + 2T\sum_{j=1}^{l} \tau_n \|\delta u_n^j\|^2,$$

so that (1.24) can be further estimated as

$$\left|\sum_{k=1}^{i} \tau_n (N_n^k, \delta u_n^k)_{H_N}\right| \le C_\varepsilon + \frac{\varepsilon C_{tr}^2}{2}\left(\|eu_n^i\|^2 + 2\|u^0\|^2 + 2T\sum_{j=1}^{i} \tau_n \|\delta u_n^j\|^2\right)$$

$$+ \frac{C_{tr}^2}{2}\sum_{k=1}^{i} \tau_n \left(\|eu_n^k\|^2 + 2\|u^0\|^2 + 2T\sum_{j=1}^{k} \tau_n \|\delta u_n^j\|^2\right) \tag{1.26}$$

$$\le \tilde C_\varepsilon + \frac{\varepsilon C_{tr}^2}{2}\|eu_n^i\|^2 + \tilde C\sum_{k=1}^{i} \tau_n \left(\|\delta u_n^k\|^2 + \|eu_n^k\|^2\right),$$

for some positive constants $\tilde{C}_\varepsilon$ and $\tilde{C}$, with $\tilde{C}_\varepsilon$ depending on ε. Similarly to (1.23), we can say

$$\sum_{k=1}^{i}\tau_n(\delta^2 u_n^k,\delta z_n^k)=(\delta u_n^i,\delta w_n^i)-(\delta u_n^0,\delta w_n^0)-\sum_{k=1}^{i}\tau_n(\delta u_n^{k-1},\delta^2 w_n^k), \tag{1.27}$$

from which we deduce that for every $\varepsilon>0$

$$\begin{aligned}\Big|\sum_{k=1}^{i}\tau_n(\delta^2 u_n^k,\delta z_n^k)\Big|&\\ &\le\|\delta u_n^i\|\|\delta z_n^i\|+\|u^1\|\|\dot z(0)\|+\sum_{k=1}^{i}\tau_n\|\delta u_n^{k-1}\|\|\delta^2 z_n^k\|\\ &\le\frac{1}{2\varepsilon}\|\delta z_n^i\|^2+\frac{\varepsilon}{2}\|\delta u_n^i\|^2+\|u^1\|\|\dot z(0)\|+\frac12\sum_{k=1}^{i}\tau_n\|\delta u_n^{k-1}\|^2+\frac12\sum_{k=1}^{i}\tau_n\|\delta^2 z_n^k\|^2\\ &\le\bar C_\varepsilon+\frac{\varepsilon}{2}\|\delta u_n^i\|^2+\frac12\sum_{k=1}^{i}\tau_n\|\delta u_n^k\|^2,\end{aligned} \tag{1.28}$$

where $\bar{C}_\varepsilon$ is a positive constant depending on ε. We estimate from above the last term in right-hand side of (1.19) in the following way

$$\begin{aligned}\sum_{k=1}^{i}\tau_n b_n^k(\delta u_n^k,\delta z_n^k)&\le\sum_{k=1}^{i}\tau_n(b_n^k(\delta u_n^k,\delta u_n^k))^{\frac12}(b_n^k(\delta z_n^k,\delta z_n^k))^{\frac12}\\ &\le\frac12\sum_{k=1}^{i}\tau_n b_n^k(\delta u_n^k,\delta u_n^k)+\frac12\|\mathbb{B}\|_\infty\|\Psi\|_\infty^2\|\dot z\|^2_{L^2(0,T;V_0)}.\end{aligned} \tag{1.29}$$

By considering (1.19)–(1.29) and using (1.6) we obtain

$$\Big(\frac{1-\varepsilon}{2}\Big)\|\delta u_n^i\|^2+\frac{c_{\mathbb{A}}-\varepsilon C_{tr}^2}{2}\|eu_n^i\|^2+\frac12\sum_{k=1}^{i}\tau_n b_n^k(\delta u_n^k,\delta u_n^k)\le\hat C_\varepsilon+\hat C\sum_{k=1}^{i}\tau_n\Big(\|\delta u_n^k\|^2+\|eu_n^k\|^2\Big)$$

for two positive constants $\hat{C}_\varepsilon$ and $\hat{C}$, with $\hat{C}_\varepsilon$ depending on ε. We choose $\varepsilon<\frac12\min\Big\{1,\frac{c_{\mathbb{A}}}{C_{tr}^2}\Big\}$ to derive the following estimate

$$\frac14\|\delta u_n^i\|^2+\frac14\|eu_n^i\|^2+\frac12\sum_{k=1}^{i}\tau_n b_n^k(\delta u_n^k,\delta u_n^k)\le C_1+C_2\sum_{k=1}^{i}\tau_n\Big(\|\delta u_n^k\|^2+\|eu_n^k\|^2\Big), \tag{1.30}$$

where C_1 and C_2 are two positive constants depending only on u^0, u^1, f, N, and z. Thanks to a discrete version of Gronwall's lemma (see, e.g., [2, Lemma 3.2.4]) we deduce the existence of a constant $C_3>0$, independent of i and n, such that

$$\|\delta u_n^i\|+\|eu_n^i\|\le C_3\quad\text{for every } i=1,\dots,n \text{ and for every } n\in\mathbb{N}.$$

By combining this last estimate with (1.30) and (1.6) we finally get (1.18) and we conclude. □

We now want to pass to the limit into the discrete equation (1.17) to obtain a weak solution to (1.8)–(1.11). We start by defining the following approximating sequences of our limit solution

$$u_n(t)=u_n^k+(t-k\tau_n)\delta u_n^k\qquad\text{for } t\in[(k-1)\tau_n,k\tau_n] \text{ and } k=1,\dots,n,$$

$$u_n^+(t) = u_n^k \qquad \text{for } t \in ((k-1)\tau_n, k\tau_n] \text{ and } k = 1, \dots, n, \qquad u_n^+(0) = u_n^0,$$
$$u_n^-(t) = u_n^{k-1} \qquad \text{for } t \in [(k-1)\tau_n, k\tau_n) \text{ and } k = 1, \dots, n, \qquad u_n^-(T) = u_n^n.$$

Moreover, we consider also the sequences

$$\tilde{u}_n(t) = \delta u_n^k + (t - k\tau_n)\delta^2 u_n^k \quad \text{for } t \in [(k-1)\tau_n, k\tau_n] \text{ and } k = 1, \dots, n,$$
$$\tilde{u}_n^+(t) = \delta u_n^k \qquad \text{for } t \in ((k-1)\tau_n, k\tau_n] \text{ and } k = 1, \dots, n, \quad \tilde{u}_n^+(0) = \delta u_n^0,$$
$$\tilde{u}_n^-(t) = \delta u_n^{k-1} \qquad \text{for } t \in [(k-1)\tau_n, k\tau_n) \text{ and } k = 1, \dots, n, \quad \tilde{u}_n^-(T) = \delta u_n^n,$$

which approximate the first time derivative of the solution. Notice that $u_n \in H^1(0,T;H)$ with $\dot{u}_n(t) = \delta u_n^k = \tilde{u}_n^+(t)$ for $t \in ((k-1)\tau_n, k\tau_n)$ and $k = 1, \dots, n$. Let us approximate Ψ and z by

$$\Psi_n^+(t) := \Psi_n^k, \qquad z_n^+(t) := z_n^k \qquad t \in ((k-1)\tau_n, k\tau_n], k = 1, \dots, n,$$
$$\Psi_n^-(t) := \Psi_n^{k-1}, \qquad z_n^-(t) := z_n^{k-1} \qquad t \in [(k-1)\tau_n, k\tau_n), k = 1, \dots, n.$$

Lemma 1.2.4. *There exists a function $u \in \mathcal{W}$, with $u - z \in \mathcal{V}^D$, such that, up to a not relabeled subsequence*

$$u_n \xrightarrow[n\to\infty]{H^1(0,T;H)} u, \quad u_n^\pm \xrightarrow[n\to\infty]{L^2(0,T;V)} u, \quad \tilde{u}_n^\pm \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{u}, \tag{1.31}$$
$$\nabla\Psi_n^\pm \odot \tilde{u}_n^\pm \xrightarrow[n\to\infty]{L^2(0,T;H)} \nabla\Psi \odot \dot{u}, \quad e(\Psi_n^\pm \tilde{u}_n^\pm) \xrightarrow[n\to\infty]{L^2(0,T;H)} e(\Psi\dot{u}). \tag{1.32}$$

Proof. Thanks to Lemma 1.2.3 the sequences $\{u_n\}_n \subset H^1(0,T;H) \cap L^\infty(0,T;V)$, $\{u_n^\pm\}_n \subset L^\infty(0,T;V)$, and $\{\tilde{u}_n^\pm\}_n \subset L^\infty(0,T;H)$ are uniformly bounded. By Banach-Alaoglu's theorem there exist $u \in H^1(0,T;H)$ and $v \in L^2(0,T;V)$ such that, up to a not relabeled subsequence

$$u_n \xrightarrow[n\to\infty]{L^2(0,T;V)} u, \quad \dot{u}_n \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{u}, \quad u_n^+ \xrightarrow[n\to\infty]{L^2(0,T;V)} v.$$

Since there exists a constant $C > 0$ such that

$$\|u_n - u_n^+\|_{L^\infty(0,T;H)} \le C\tau_n \xrightarrow[n\to\infty]{} 0,$$

we can conclude that $u = v$. Moreover, given that $u_n^-(t) = u_n^+(t - \tau_n)$ for $t \in (\tau_n, T)$, $\tilde{u}_n^+(t) = \dot{u}_n(t)$ for a.e. $t \in (0,T)$, and $\tilde{u}_n^-(t) = \tilde{u}_n^+(t - \tau_n)$ for $t \in (\tau_n, T)$, we deduce

$$u_n^- \xrightarrow[n\to\infty]{L^2(0,T;V)} u, \quad \tilde{u}_n^\pm \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{u}.$$

By (1.18) we derive that the sequences $\{e(\Psi_n^+ \tilde{u}_n^+)\}_n \subset L^2(0,T;H)$ and $\{\nabla\Psi_n^+ \odot \tilde{u}_n^+\}_n \subset L^2(0,T;H)$ are uniformly bounded. Indeed there exists a constant $C > 0$ independent of n such that

$$\|\nabla\Psi_n^+ \odot \tilde{u}_n^+\|^2_{L^2(0,T;H)} = \sum_{k=1}^n \int_{(k-1)\tau_n}^{k\tau_n} \|\nabla\Psi_n^k \odot \delta u_n^k\|^2 \mathrm{d}t \le \|\nabla\Psi\|^2_\infty \sum_{k=1}^n \tau_n \|\delta u_n^k\|^2_H \le C,$$
$$\|e(\Psi_n^+ \tilde{u}_n^+)\|^2_{L^2(0,T;H)} = \sum_{k=1}^n \int_{(k-1)\tau_n}^{k\tau_n} \|e(\Psi_n^k \delta u_n^k)\|^2 \mathrm{d}t = \sum_{k=1}^n \tau_n \|\Psi_n^k e\delta u_n^k + \nabla\Psi_n^k \odot \delta u_n^k\|^2$$
$$\le 2\sum_{k=1}^n \tau_n \|\Psi_n^k e\delta u_n^k\|^2 + 2\sum_{k=1}^n \tau_n \|\nabla\Psi_n^k \odot \delta u_n^k\|^2 \le C.$$

Therefore, there exists $w_1, w_2 \in L^2(0,T;H)$ such that, up to a further not relabeled subsequence

$$\nabla\Psi_n^+ \odot \tilde{u}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} w_1, \quad e(\Psi_n^+\tilde{u}_n^+) \xrightarrow[n\to\infty]{L^2(0,T;H)} w_2.$$

We want to identify the limit functions w_1 and w_2. Consider $\varphi \in L^2(0,T;H)$, then

$$\int_0^T (\nabla\Psi_n^+ \odot \tilde{u}_n^+, \varphi)\mathrm{d}t = \frac{1}{2}\int_0^T (\tilde{u}_n^+, \varphi\nabla\Psi_n^+)\mathrm{d}t + \frac{1}{2}\int_0^T (\tilde{u}_n^+, \varphi^T\nabla\Psi_n^+)\mathrm{d}t = \int_0^T (\tilde{u}_n^+, \varphi^{sym}\nabla\Psi_n^+)\mathrm{d}t,$$

where $\varphi^{sym} := \frac{\varphi+\varphi^T}{2}$. Since $\tilde{u}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{u}$ and $\varphi^{sym}\nabla\Psi_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \varphi^{sym}\nabla\Psi$ by dominated convergence theorem, we obtain

$$\int_0^T (\nabla\Psi_n^+ \odot \tilde{u}_n^+, \varphi)\mathrm{d}t \xrightarrow[n\to\infty]{} \int_0^T (\dot{u}, \varphi^{sym}\nabla\Psi)\mathrm{d}t = \int_0^T (\nabla\Psi \odot \dot{u}, \varphi)\mathrm{d}t,$$

and so $w_1 = \nabla\Psi \odot \dot{u}$. Moreover for $\phi \in L^2(0,T;H)$ we have

$$\int_0^T (\Psi_n^+\tilde{u}_n^+, \phi)\mathrm{d}t = \int_0^T (\tilde{u}_n^+, \phi\Psi_n^+)\mathrm{d}t \xrightarrow[n\to\infty]{} \int_0^T (\dot{u}, \Psi\phi)\mathrm{d}t = \int_0^T (\Psi\dot{u}, \phi)\mathrm{d}t,$$

thanks to $\tilde{u}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{u}$ and $\Psi_n^+\phi \xrightarrow[n\to\infty]{L^2(0,T;H)} \Psi\phi$, again implied by dominated convergence theorem. Therefore $\Psi_n^+\tilde{u}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \Psi\dot{u}$, from which $e(\Psi_n^+\tilde{u}_n^+) \xrightarrow[n\to\infty]{\mathcal{D}'(0,T;H)} e(\Psi\dot{u})$, that gives $w_2 = e(\Psi\dot{u})$. In particular we have $\Psi\dot{u} \in L^2(0,T;V)$. By arguing in a similar way we also obtain

$$\nabla\Psi_n^- \odot \tilde{u}_n^- \xrightarrow[n\to\infty]{L^2(0,T;H)} \nabla\Psi \odot \dot{u}, \quad e(\Psi_n^-\tilde{u}_n^-) \xrightarrow[n\to\infty]{L^2(0,T;H)} e(\Psi\dot{u}).$$

Let us check that $u \in \mathcal{W}$. To this aim, let us consider the following set

$$\mathcal{U} := \{v \in L^2(0,T;V) : v(t) \in V_t \text{ for a.e. } t \in (0,T)\} \subset L^2(0,T;V).$$

We have that $\mathcal{U}$ is a (strong) closed convex subset of $L^2(0,T;V)$, and so by Hahn-Banach Theorem the set $\mathcal{U}$ is weakly closed. Notice that $\{u_n^-\}_n, \{\Psi_n^-\tilde{u}_n^-\}_n \subset \mathcal{U}$, indeed

$$\begin{aligned} u_n^-(t) &= u_n^{k-1} \in V_{(k-1)\tau_n} \subset V_t && \text{for } t \in [(k-1)\tau_n, k\tau_n),\ k = 1,\dots,n,\\ \Psi_n^-(t)\tilde{u}_n^-(t) &= \Psi_n^{k-1}\delta u_n^{k-1} \in V_{(k-1)\tau_n} \subset V_t && \text{for } t \in [(k-1)\tau_n, k\tau_n),\ k = 1,\dots,n. \end{aligned}$$

Since $u_n^- \xrightarrow[n\to\infty]{L^2(0,T;V)} u$ and $\Psi_n^-\tilde{u}_n^- \xrightarrow[n\to\infty]{L^2(0,T;V)} \Psi\dot{u}$, we conclude that $u, \Psi\dot{u} \in \mathcal{U}$. Finally, to show that $u - z \in \mathcal{V}^D$ we observe

$$u_n^-(t) - z_n^-(t) = u_n^{k-1} - z_n^{k-1} \in V_n^{k-1} \subset V_t^D \quad \text{for } t \in [(k-1)\tau_n, k\tau_n),\ k = 1,\dots,n.$$

Therefore $\{u_n^- - z_n^-\}_n \subset \{v \in L^2(0,T;V) : v(t) \in V_t^D \text{ for a.e. } t \in (0,T)\}$, which is a (strong) closed convex subset of $L^2(0,T;V)$, and so it is weakly closed. Since $u_n^- \xrightarrow[n\to\infty]{L^2(0,T;V)} u$ and $w_n^- \xrightarrow[n\to\infty]{L^2(0,T;V_0)} z$, we get that $u(t) - z(t) \in V_t^D$ for a.e. $t \in (0,T)$, which implies $u - z \in \mathcal{V}^D$. □

We now use Lemma 1.2.4 to pass to the limit in the discrete equation (1.17).

Lemma 1.2.5. *The limit function $u \in \mathcal{W}$ of Lemma 1.2.4 is a weak solution to* (1.8)–(1.11).

Proof. We only need to prove that $u \in \mathcal{W}$ satisfies (1.14). We fix $n \in \mathbb{N}$, $\varphi \in C^1_c(0,T;V)$ such that $\varphi(t) \in V_t^D$ for every $t \in (0,T)$, and we consider

$$\varphi_n^k := \varphi(k\tau_n) \quad \text{for } k = 0, \dots, n, \quad \delta\varphi_n^k := \frac{\varphi_n^k - \varphi_n^{k-1}}{\tau_n} \quad \text{for } k = 1, \dots, n,$$

and the approximating sequences

$$\varphi_n^+(t) := \varphi_n^k, \qquad \tilde{\varphi}_n^+(t) := \delta\varphi_n^k \qquad t \in ((k-1)\tau_n, k\tau_n], k = 1, \dots, n.$$

If we use $\tau_n\varphi_n^k \in V_n^k$ as test function in (1.17), after summing over $k = 1, ..., n$, we get

$$\sum_{k=1}^n \tau_n(\delta^2 u_n^k, \varphi_n^k) + \sum_{k=1}^n \tau_n(\mathbb{A}eu_n^k, e\varphi_n^k) + \sum_{k=1}^n \tau_n(\mathbb{B}\Psi_n^k e\delta u_n^k, \Psi_n^k e\varphi_n^k) = \sum_{k=1}^n \tau_n(f_n^k, \varphi_n^k) + \sum_{k=1}^n \tau_n(N_n^k, \varphi_n^k)_{H_N}. \tag{1.33}$$

By these identities

$$\sum_{k=1}^n \tau_n(\delta^2 u_n^k, \varphi_n^k) = -\sum_{k=1}^n \tau_n(\delta u_n^{k-1}, \delta\varphi_n^k) = -\int_0^T (\tilde{u}_n^-(t), \tilde{\varphi}_n^+(t))\mathrm{d}t,$$

from (1.33) we deduce

$$-\int_0^T (\tilde{u}_n^-, \tilde{\varphi}_n^+)\mathrm{d}t + \int_0^T (\mathbb{A}eu_n^+, e\varphi_n^+)\mathrm{d}t - \int_0^T (\mathbb{B}\nabla\Psi_n^+ \odot \tilde{u}_n^+, e\varphi_n^+)\mathrm{d}t + \int_0^T (\mathbb{B}e(\Psi_n^+\tilde{u}_n^+), e\varphi_n^+)\mathrm{d}t = \int_0^T (f_n^+, \varphi_n^+)\mathrm{d}t + \int_0^T (N_n^+, \varphi_n^+)_{H_N}\mathrm{d}t. \tag{1.34}$$

Thanks to (1.31), (1.32), and the following convergences

$$\varphi_n^+ \xrightarrow[n\to\infty]{L^2(0,T;V)} \varphi, \quad \tilde{\varphi}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{\varphi}, \quad f_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} f, \quad N_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H_N)} N,$$

we can pass to the limit in (1.34), and we get that $u \in \mathcal{W}$ satisfies (1.14) for every $\varphi \in C^1_c(0,T;V)$ such that $\varphi(t) \in V_t^D$ for every $t \in (0,T)$. Finally, by using a density argument (see [17, Remark 2.9]), we conclude that $u \in \mathcal{W}$ is a weak solution to (1.8)–(1.11). □

1.3 Initial conditions and Energy-Dissipation Inequality

To complete our existence result, it remains to prove that the function $u \in \mathcal{W}$ given by Lemma 1.2.5 satisfies the initial conditions (1.12) in the sense of (1.16). Let us start by showing that the second distributional derivative $\ddot{u}$ belongs to $L^2(0,T;(V_0^D)')$. If we consider the discrete equation (1.17), for every $v \in V_0^D \subset V_n^k$, with $\|v\|_{V_0} \le 1$, we have

$$|(\delta^2 u_n^k, v)| \le C_{\mathbb{A}}\|eu_n^k\| + \|\mathbb{B}\|_\infty\|\Psi\|_\infty\|\Psi_n^k e\delta u_n^k\| + \|f_n^k\| + C_{tr}\|N_n^k\|_{H_N}.$$

Therefore, taking the supremum over $v \in V_0^D$ with $\|v\|_{V_0} \le 1$, we obtain the existence of a positive constant C such that

$$\|\delta^2 u_n^k\|^2_{(V_0^D)'} \le C\big(\|eu_n^k\|^2 + \|\Psi_n^k e\delta u_n^k\|^2 + \|f_n^k\|^2 + \|N_n^k\|^2_{H_N}\big).$$

If we multiply this inequality by τ_n and we sum over $k = 1, \dots, n$, we get

$$\sum_{k=1}^n \tau_n\|\delta^2 u_n^k\|^2_{(V_0^D)'} \tag{1.35}$$

$$\leq C\Big(\sum_{k=1}^n \tau_n\|eu_n^k\|^2 + \sum_{k=1}^n \tau_n\|\Psi_n^k e\delta u_n^k\|^2 + \|f\|^2_{L^2(0,T;H)} + \|N\|^2_{L^2(0,T;H_N)}\Big). \tag{1.36}$$

Thanks to (1.35) and Lemma 1.2.3 we conclude that

$$\sum_{k=1}^n \tau_n\|\delta^2 u_n^k\|^2_{(V_0^D)'} \leq \tilde{C} \quad \text{for every } n\in\mathbb{N},$$

for a positive constant $\tilde{C}$ independent on $n \in \mathbb{N}$. In particular the sequence $\{\tilde{u}_n\}_n \subset H^1(0,T;(V_0^D)')$ is uniformly bounded (notice that $\dot{\tilde{u}}_n(t) = \delta^2 u_n^k$ for $t \in ((k-1)\tau_n, k\tau_n)$ and $k = 1,\dots,n$). Hence, up to extract a further (not relabeled) subsequence from the one of Lemma 1.2.4, we get

$$\tilde{u}_n \xrightarrow[n\to\infty]{H^1(0,T;(V_0^D)')} w_3, \tag{1.37}$$

and by using the following estimate

$$\|\tilde{u}_n - \tilde{u}_n^+\|_{L^2(0,T;(V_0^D)')} \leq \tau_n\|\dot{\tilde{u}}_n\|_{L^2(0,T;(V_0^D)')} \leq \tilde{C}\tau_n \xrightarrow[n\to\infty]{} 0$$

we conclude that $w_3 = \dot{u}$.

Let us recall the following result, whose proof can be found for example in [20].

Lemma 1.3.1. *Let X, Y be two reflexive Banach spaces such that $X \hookrightarrow Y$ continuously. Then*

$$L^\infty(0,T;X)\cap C^0_w([0,T];Y) = C^0_w([0,T];X).$$

Since $H^1(0,T;(V_0^D)') \hookrightarrow C^0([0,T],(V_0^D)')$, by using Lemmas 1.2.4 and 1.3.1 we get that our weak solution $u \in \mathcal{W}$ satisfies

$$u \in C^0_w([0,T];V), \quad \dot{u} \in C^0_w([0,T];H), \quad \ddot{u} \in L^2(0,T;(V_0^D)').$$

By (1.31) and (1.37) we hence obtain

$$u_n(t) \xrightarrow[n\to\infty]{H} u(t), \quad \tilde{u}_n(t) \xrightarrow[n\to\infty]{(V_0^D)'} \dot{u}(t) \quad \text{for every } t\in[0,T], \tag{1.38}$$

so that $u(0) = u^0$ and $\dot{u}(0) = u^1$, since $u_n(0) = u^0$ and $\tilde{u}_n(0) = u^1$.

To prove that

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \left(\|u(t)-u^0\|^2_{V_t} + \|\dot{u}(t)-u^1\|^2_H\right) \mathrm{d}t = 0$$

we will actually show

$$\lim_{t\to 0^+} u(t) = u^0 \text{ in } V, \quad \lim_{t\to 0^+} \dot{u}(t) = u^1 \text{ in } H.$$

This is a consequence of following energy-dissipation inequality which holds for the weak solution $u \in \mathcal{W}$ of Lemma 1.2.5. Let us define the total energy as

$$\mathscr{E}(t) := \frac{1}{2}\|\dot{u}(t)\|^2 + \frac{1}{2}(\mathbb{A}eu(t), eu(t)) \quad t\in[0,T].$$

Notice that $\mathscr{E}(t)$ is well defined for every $t \in [0,T]$ since $u \in C^0_w([0,T];V)$, $\dot{u} \in C^0_w([0,T];H)$, and that $\mathscr{E}(0) = \frac{1}{2}\|u^1\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0)$. By defining

$$\mathscr{D}(t) := \int_0^t (\mathbb{B}\Psi(\tau)e\dot{u}(\tau), \Psi(\tau)e\dot{u}(\tau))\mathrm{d}\tau \quad \text{for every } t\in[0,T]$$

we have the following theorem.

Theorem 1.3.2. *The weak solution $u \in \mathcal{W}$ to (1.8)–(1.11), given by Lemma 1.2.5, satisfies for every $t \in [0,T]$ the following energy-dissipation inequality*

$$\mathscr{E}(t) + \mathscr{D}(t) \le \mathscr{E}(0) + \mathscr{W}_{tot}(t), \tag{1.39}$$

where $\Psi e\dot{u}$ is the function defined in (1.15) and $\mathscr{W}_{tot}(t)$ is the total work on the solution u at time $t \in [0,T]$, which is given by

$$\begin{aligned}\mathscr{W}_{tot}(t) := &\int_0^t [(f(\tau), \dot{u}(\tau) - \dot{z}(\tau)) + (\mathbb{A}eu(\tau), e\dot{z}(\tau)) + (\mathbb{B}\Psi(\tau)e\dot{u}(\tau), \Psi(\tau)e\dot{z}(\tau))]\,\mathrm{d}\tau\\ &- \int_0^t (\dot{u}(\tau), \ddot{z}(\tau))\mathrm{d}\tau + (\dot{u}(t), \dot{z}(t)) - (u^1, \dot{z}(0))\\ &- \int_0^t (\dot{N}(\tau), u(\tau) - z(\tau))_{H_N}\mathrm{d}\tau + (N(t), u(t) - z(t))_{H_N} - (N(0), u^0 - z(0))_{H_N}.\end{aligned} \tag{1.40}$$

Remark 1.3.3. From the classical point of view, the total work on the solution u at time $t \in [0,T]$ is given by

$$\mathscr{W}_{tot}(t) := \mathscr{W}_{load}(t) + \mathscr{W}_{bdry}(t), \tag{1.41}$$

where $\mathscr{W}_{load}(t)$ is the work on the solution u at time $t \in [0,T]$ due to the loading term, which is defined as

$$\mathscr{W}_{load}(t) := \int_0^t (f(\tau), \dot{u}(\tau))\mathrm{d}\tau,$$

and $\mathscr{W}_{bdry}(t)$ is the work on the solution u at time $t \in [0,T]$ due to the varying boundary conditions, which one expects to be equal to

$$\mathscr{W}_{bdry}(t) := \int_0^t (N(\tau), \dot{u}(\tau))_{H_N}\mathrm{d}\tau + \int_0^t ((\mathbb{A}eu(\tau) + \Psi^2(\tau)\mathbb{B}e\dot{u}(\tau))\nu, \dot{z}(\tau))_{H_D}\mathrm{d}\tau,$$

being $H_D := L^2(\partial_D\Omega; \mathbb{R}^d)$. Unfortunately, $\mathscr{W}_{bdry}(t)$ is not well defined under our assumptions on u. Notice that when $\Psi \equiv 1$ on a neighborhood U of the closure of $\partial_N\Omega$, then every weak solution u to (1.8)–(1.11) satisfies $u \in H^1(0,T; H^1((\Omega \cap U) \setminus \Gamma; \mathbb{R}^d))$, which gives that $u \in H^1(0,T; H_N)$ by our assumptions on Γ. Hence the first term of $\mathscr{W}_{bdry}(t)$ makes sense and satisfies

$$\int_0^t (N(\tau), \dot{u}(\tau))_{H_N}\mathrm{d}\tau = (N(t), u(t))_{H_N} - (N(0), u(0))_{H_N} - \int_0^t (\dot{N}(\tau), u(\tau))_{H_N}\mathrm{d}\tau.$$

The term involving the Dirichlet datum z is more difficult to handle since the trace of $(\mathbb{A}eu + \Psi^2\mathbb{B}e\dot{u})\nu$ on $\partial_D\Omega$ is not well defined even when $\Psi \equiv 1$ on a neighborhood of the closure of $\partial_D\Omega$. If we assume that $u \in H^1(0,T; H^2(\Omega \setminus \Gamma; \mathbb{R}^d)) \cap H^2(0,T; L^2(\Omega; \mathbb{R}^d))$ and that Γ is a smooth manifold, then we can integrate by part equation (1.14) to deduce that u satisfies (1.8). In this case, $(\mathbb{A}eu + \Psi^2\mathbb{B}e\dot{u})\nu \in L^2(0,T; H_D)$ and by using (1.8), together with the divergence theorem and the integration by parts formula, we deduce

$$\begin{aligned}&\int_0^t ((\mathbb{A}eu(\tau) + \Psi^2(\tau)\mathbb{B}e\dot{u}(\tau))\nu, \dot{z}(\tau))_{H_D}\mathrm{d}\tau\\ &= \int_0^t \left[(\operatorname{div}(\mathbb{A}eu(\tau) + \Psi^2(\tau)\mathbb{B}e\dot{u}(\tau)), \dot{z}(\tau)) + (\mathbb{A}eu(\tau), e\dot{z}(\tau))\right]\mathrm{d}\tau\\ &\quad + \int_0^t \left[(\Psi^2(\tau)\mathbb{B}e\dot{u}(\tau), e\dot{z}(\tau)) - (N(\tau), \dot{z}(\tau))_{H_N}\right]\mathrm{d}\tau\\ &= \int_0^t \left[(\ddot{u}(\tau), \dot{z}(\tau)) - (f(\tau), \dot{z}(\tau)) + (\mathbb{A}eu(\tau) + \Psi^2(\tau)\mathbb{B}e\dot{u}(\tau), e\dot{z}(\tau)) - (N(\tau), \dot{z}(\tau))_{H_N}\right]\mathrm{d}\tau\end{aligned}$$

$$= \int_0^t [(\mathbb{A}eu(\tau), e\dot{z}(\tau)) + (\mathbb{B}\Psi(\tau)e\dot{u}(\tau), \Psi(\tau)e\dot{z}(\tau)) - (f(\tau), \dot{z}(\tau))]\,\mathrm{d}\tau$$
$$+ \int_0^t (\dot{N}(\tau), z(\tau))_{H_N}\mathrm{d}\tau - (N(t), z(t))_{H_N} + (N(0), z(0))_{H_N}$$
$$- \int_0^t (\dot{u}(\tau), \ddot{z}(\tau))\mathrm{d}\tau + (\dot{u}(t), \dot{z}(t)) - (u^1, \dot{z}(0))$$

Hence, the definition of total work given in (1.40) is coherent with the classical one (1.41). Notice that if u is the solution to (1.8)–(1.11) given by Lemma 1.2.5, then (1.40) is well defined for every $t \in [0,T]$, since $N \in C^0([0,T]; H_N)$, $\dot{z} \in C^0([0,T]; H)$, $u \in C^0_w([0,T]; V)$, and $\dot{u} \in C^0_w([0,T]; H)$. In particular, the function $t \mapsto \mathscr{W}_{tot}(t)$ from $[0,T]$ to $\mathbb{R}$ is continuous.

Proof. Fixed $t \in (0,T]$, for every $n \in \mathbb{N}$ there exists a unique $j \in \{1, \dots, n\}$ such that $t \in ((j-1)\tau_n, j\tau_n]$. After setting $t_n := j\tau_n$, we can rewrite (1.19) as

$$\frac{1}{2}\|\tilde{u}_n^+(t)\|^2 + \frac{1}{2}(\mathbb{A}eu_n^+(t), eu_n^+(t)) + \int_0^{t_n} (\mathbb{B}\Psi_n^+(\tau)e\tilde{u}_n^+(\tau), \Psi_n^+(\tau)e\tilde{u}_n^+(\tau))\mathrm{d}\tau \le \mathcal{E}(0) + \mathscr{W}_n^+(t), \tag{1.42}$$

where

$$\mathscr{W}_n^+(t) := \int_0^{t_n} \left[(\mathbb{A}eu_n^+(\tau), e\tilde{z}_n^+(\tau)) + (\mathbb{B}\Psi_n^+(\tau)e\tilde{u}_n^+(\tau), \Psi_n^+(\tau)e\tilde{z}_n^+(\tau))\right]\mathrm{d}\tau$$
$$+ \int_0^{t_n} \left[(f_n^+(\tau), \tilde{u}_n^+(\tau) - \tilde{z}_n^+(\tau)) + (\tilde{u}_n^+(\tau), \tilde{z}_n^+(\tau)) + (N_n^+(\tau), \tilde{u}_n^+(\tau) - \tilde{z}_n^+(\tau))_{H_N}\right]\mathrm{d}\tau.$$

Thanks to (1.18), we have

$$\|u_n(t) - u_n^+(t)\| = \|u_n^j + (t - j\tau_n)\delta u_n^j - u_n^j\| \le \tau_n\|\delta u_n^j\| \le C\tau_n \xrightarrow[n\to\infty]{} 0,$$
$$\|\tilde{u}_n(t) - \tilde{u}_n^+(t)\|^2_{(V_0^D)'} = \|\delta u_n^j + (t - j\tau_n)\delta^2 u_n^j - \delta u_n^j\|^2_{(V_0^D)'} \le \tau_n^2\|\delta^2 u_n^j\|^2_{(V_0^D)'} \le C\tau_n \xrightarrow[n\to\infty]{} 0.$$

The last convergences and (1.38) imply

$$u_n^+(t) \xrightarrow[n\to\infty]{H} u(t), \qquad \tilde{u}_n^+(t) \xrightarrow[n\to\infty]{(V_0^D)'} \dot{u}(t),$$

and since $\|u_n^+(t)\|_V + \|\tilde{u}_n^+(t)\| \le C$ for every $n \in \mathbb{N}$, we get

$$u_n^+(t) \xrightarrow[n\to\infty]{V} u(t), \qquad \tilde{u}_n^+(t) \xrightarrow[n\to\infty]{H} \dot{u}(t). \tag{1.43}$$

By the lower semicontinuity properties of $v \mapsto \|v\|^2$ and $v \mapsto (\mathbb{A}ev, ev)$, we conclude

$$\|\dot{u}(t)\|^2 \le \liminf_{n\to\infty} \|\tilde{u}_n^+(t)\|^2, \tag{1.44}$$
$$(\mathbb{A}eu(t), eu(t)) \le \liminf_{n\to\infty} (\mathbb{A}eu_n^+(t), eu_n^+(t)). \tag{1.45}$$

Thanks to Lemma 1.2.4 and (1.15), we obtain

$$\Psi_n^+ e\tilde{u}_n^+ = e(\Psi_n^+\tilde{u}_n^+) - \nabla\Psi_n^+ \odot \tilde{u}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} e(\Psi\dot{u}) - \nabla\Psi \odot \dot{u} = \Psi e\dot{u},$$

so that

$$\int_0^t (\mathbb{B}\Psi(\tau)e\dot{u}(\tau), \Psi(\tau)e\dot{u}(\tau))\mathrm{d}\tau \le \liminf_{n\to\infty} \int_0^t (\mathbb{B}\Psi_n^+(\tau)e\tilde{u}_n^+(\tau), \Psi_n^+(\tau)e\tilde{u}_n^+(\tau))\mathrm{d}\tau$$

$$\leq \liminf_{n\to\infty} \int_0^{t_n} (\mathbb{B}\Psi_n^+(\tau)e\tilde{u}_n^+(\tau), \Psi_n^+(\tau)e\tilde{u}_n^+(\tau))\mathrm{d}\tau, \tag{1.46}$$

since $t \leq t_n$ and $v \mapsto \int_0^t (\mathbb{B}v(\tau), v(\tau))\mathrm{d}\tau$ is a non negative quadratic form on $L^2(0,T;H)$. Let us study the right-hand side of (1.42). Given that we have

$$\chi_{[0,t_n]} f_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \chi_{[0,t]} f, \quad \tilde{u}_n^+ - \tilde{z}_n^+ \xrightharpoonup[n\to\infty]{L^2(0,T;H)} \dot{u} - \dot{z},$$

we can deduce

$$\int_0^{t_n} (f_n^+(\tau), \tilde{u}_n^+(\tau) - \tilde{z}_n^+(\tau))\mathrm{d}\tau \xrightarrow[n\to\infty]{} \int_0^t (f(\tau), \dot{u}(\tau) - \dot{z}(\tau))\mathrm{d}\tau. \tag{1.47}$$

In a similar way, we can prove

$$\int_0^{t_n} (\mathbb{A}eu_n^+(\tau), e\tilde{z}_n^+(\tau))\mathrm{d}\tau \xrightarrow[n\to\infty]{} \int_0^t (\mathbb{A}eu(\tau), e\dot{z}(\tau))\mathrm{d}\tau, \tag{1.48}$$

$$\int_0^{t_n} (\mathbb{B}\Psi_n^+(\tau)e\tilde{u}_n^+(\tau), \Psi_n^+(\tau)e\tilde{z}_n^+(\tau))\mathrm{d}\tau \xrightarrow[n\to\infty]{} \int_0^t (\mathbb{B}\Psi(\tau)e\dot{u}(\tau), \Psi(\tau)e\dot{z}(\tau))\mathrm{d}\tau, \tag{1.49}$$

since the following convergences hold

$$\chi_{[0,t_n]} e\tilde{z}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \chi_{[0,t]} e\dot{z}, \qquad \mathbb{A}eu_n^+ \xrightharpoonup[n\to\infty]{L^2(0,T;H)} \mathbb{A}eu,$$

$$\chi_{[0,t_n]} \Psi_n^+ e\tilde{z}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \chi_{[0,t]} \Psi e\dot{z}, \qquad \Psi_n^+ e\tilde{u}_n^+ \xrightharpoonup[n\to\infty]{L^2(0,T;H)} \Psi e\dot{u}.$$

It remains to study the behaviour as $n \to \infty$ of the terms

$$\int_0^{t_n} (\dot{\tilde{u}}_n(\tau), \tilde{z}_n^+(\tau))\mathrm{d}\tau, \qquad \int_0^{t_n} (N_n^+(\tau), \tilde{u}_n^+(\tau) - \tilde{z}_n^+(\tau))_{H_N}\mathrm{d}\tau.$$

Thanks to formula (1.27) we have

$$\int_0^{t_n} (\dot{\tilde{u}}_n(\tau), \tilde{z}_n^+(\tau))\mathrm{d}\tau = (\tilde{u}_n^+(t), \tilde{z}_n^+(t)) - (u^1, \dot{z}(0)) - \int_0^{t_n} (\tilde{u}_n^-(\tau), \dot{\tilde{z}}_n(\tau))\mathrm{d}\tau.$$

By arguing as before we hence deduce

$$\int_0^{t_n} (\dot{\tilde{u}}_n(\tau), \tilde{z}_n^+(\tau))\mathrm{d}\tau \xrightarrow[n\to\infty]{} (\dot{u}(t), \dot{z}(t)) - (u^1, \dot{z}(0)) - \int_0^t (\dot{u}(\tau), \ddot{z}(\tau))\mathrm{d}\tau, \tag{1.50}$$

thanks to (1.43) and by these convergences

$$\chi_{[0,t_n]} \dot{\tilde{z}}_n \xrightarrow[n\to\infty]{L^2(0,T;H)} \chi_{[0,t]} \ddot{z}, \quad \tilde{u}_n^- \xrightharpoonup[n\to\infty]{L^2(0,T;H)} \dot{u},$$

$$\|\tilde{z}_n^+(t) - \dot{z}(t)\| = \left\| \frac{z(j\tau_n) - z((j-1)\tau_n)}{\tau_n} - \dot{z}(t) \right\| \leq \fint_{(j-1)\tau_n}^{j\tau_n} \|\dot{z}(\tau) - \dot{z}(t)\|\mathrm{d}\tau \xrightarrow[n\to\infty]{} 0.$$

Notice that in the last convergence we used the continuity of z from $[0,T]$ in H. Similarly we have

$$\int_0^{t_n} (N_n^+(\tau), \tilde{u}_n^+(\tau) - \tilde{z}_n^+(\tau))_{H_N}\mathrm{d}\tau$$
$$= (N_n^+(t), u_n^+(t) - z_n^+(t))_{H_N} - (N(0), u^0 - z(0))_{H_N} - \int_0^{t_n} (\dot{N}_n(\tau), u_n^-(\tau) - z_n^-(\tau))_{H_N}\mathrm{d}\tau$$

so that we get

$$\int_0^{t_n} (N_n^+(\tau), \tilde{u}_n^+(\tau) - \tilde{z}_n^+(\tau))_{H_N} \mathrm{d}\tau$$
$$\xrightarrow[n\to\infty]{} (N(t), u(t) - z(t))_{H_N} - (N(0), u^0 - z(0))_{H_N} - \int_0^t (\dot{N}(\tau), u(\tau) - z(\tau))_{H_N} \mathrm{d}\tau \quad (1.51)$$

thanks to (1.43), the continuity of $s \mapsto N(s)$ in H_N, and the fact that

$$\chi_{[0,t_n]} \dot{N}_n \xrightarrow[n\to\infty]{L^2(0,T;H_N)} \chi_{[0,t]} \dot{N}, \qquad u_n^- - z_n^- \xrightarrow[n\to\infty]{L^2(0,T;H_N)} u - z.$$

By combining (1.44)–(1.51), we deduce the energy-dissipation inequality (1.39) for every $t \in (0, T]$. Finally, for $t = 0$ the inequality trivially holds since $u(0) = u^0$ and $\dot{u}(0) = u^1$. □

We now are in position to prove the validity of the initial conditions.

Lemma 1.3.4. *The weak solution $u \in \mathcal{W}$ to (1.8)–(1.11) of Lemma 1.2.5 satisfies*

$$\lim_{t\to 0^+} u(t) = u^0 \text{ in } V, \quad \lim_{t\to 0^+} \dot{u}(t) = u^1 \text{ in } H. \quad (1.52)$$

In particular u satisfies the initial conditions (1.12) in the sense of (1.16).

Proof. By sending $t \to 0^+$ into the energy-dissipation inequality (1.39) and using that $u \in C_w^0([0, T]; V)$, $\dot{u} \in C_w^0([0, T]; H)$, and the lower semicontinuity of the real valued functions

$$t \mapsto \|\dot{u}(t)\|^2 \quad t \mapsto (\mathbb{A}eu(t), eu(t)),$$

we deduce

$$\mathscr{E}(0) = \frac{1}{2}\|u^1\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0) \le \frac{1}{2}\Big[\liminf_{t\to 0^+}\|\dot{u}(t)\|^2 + \liminf_{t\to 0^+}(\mathbb{A}eu(t), eu(t))\Big]$$
$$\le \liminf_{t\to 0^+}\Big[\frac{1}{2}\|\dot{u}(t)\|^2 + \frac{1}{2}(\mathbb{A}eu(t), eu(t))\Big] = \liminf_{t\to 0^+}\mathscr{E}(t) \le \limsup_{t\to 0^+}\mathscr{E}(t) \le \mathscr{E}(0),$$

since the right-hand side of (1.39) is continuous in t, $u(0) = u^0$, and $\dot{u}(0) = u^1$. Therefore there exists $\lim_{t\to 0^+}\mathscr{E}(t) = \mathscr{E}(0)$. Moreover, we have

$$\mathscr{E}(0) \le \frac{1}{2}\liminf_{t\to 0^+}\|\dot{u}(t)\|^2 + \frac{1}{2}\liminf_{t\to 0^+}(\mathbb{A}eu(t), eu(t))$$
$$\le \frac{1}{2}\limsup_{t\to 0^+}\|\dot{u}(t)\|^2 + \frac{1}{2}\liminf_{t\to 0^+}(\mathbb{A}eu(t), eu(t))$$
$$\le \limsup_{t\to 0^+}\Big[\frac{1}{2}\|\dot{u}(t)\|^2 + \frac{1}{2}(\mathbb{A}eu(t), eu(t))\Big] = \mathscr{E}(0),$$

which gives

$$\lim_{t\to 0^+}\|\dot{u}(t)\|^2 = \|u^1\|^2.$$

Similarly we can dedeuce

$$\lim_{t\to 0^+}(\mathbb{A}eu(t), eu(t)) = (\mathbb{A}eu^0, eu^0).$$

Finally, since we have

$$\dot{u}(t) \xrightarrow[t\to 0^+]{H} u^1, \quad eu(t) \xrightarrow[t\to 0^+]{H} eu^0,$$

we deduce (1.52). In particular the functions $u\colon [0, T] \to V$ and $\dot{u}\colon [0, T] \to H$ are continuous at $t = 0$, which implies (1.16). □

We can finally prove Theorem 1.2.1.

Proof of Theorem 1.2.1. It is enough to combine Lemmas 1.2.5 and 1.3.4. □

Remark 1.3.5. We have proved Theorem 1.2.1 for the d-dimensional linear elastic case, namely when the displacement u is a vector-valued function. The same result is true with identical proofs in the antiplane case, that is when the displacement u is a scalar function and satisfies (9).

1.4 Uniqueness

In this section we investigate the uniqueness properties of system (1.8) with boundary and initial conditions (1.9)–(1.12). To this aim, we need to assume stronger regularity assumptions on the crack sets $\{\Gamma_t\}_{t\in[0,T]}$ and on the function Ψ. Moreover, we have to restrict our problem to the dimensional case $d = 2$, since in our proof we need to construct a suitable family of diffeomorphisms which maps the time-dependent crack Γ_t into a fixed set, and this can be explicitly done only for $d = 2$ (see [16, Example 2.14]).

We proceed in two steps; first, in Lemma 1.4.2 we prove a uniqueness result in every dimension d, but when the cracks are not increasing, that is $\Gamma_T = \Gamma_0$. Next, in Theorem 1.4.5 we combine Lemma 1.4.2 with the finite speed of propagation theorem of [15] and the uniqueness result of [17] to derive the uniqueness of a weak solution to (1.8)–(1.12) in the case $d = 2$.

Let us start with the following lemma, whose proof is similar to that one of [17, Proposition 2.10].

Lemma 1.4.1. *Let $u \in \mathcal{W}$ be a weak solution to (1.8)–(1.11) satisfying the initial condition $\dot u(0) = 0$ in the following sense*

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \|\dot u(t)\|^2 \mathrm{d}t = 0.$$

Then u satisfies

$$-\int_0^T (\dot u(t), \dot\varphi(t))\mathrm{d}t + \int_0^T (\mathbb{A}eu(t), e\varphi(t))\mathrm{d}t + \int_0^T (\mathbb{B}\Psi(t)e\dot u(t), \Psi(t)e\varphi(t))\mathrm{d}t$$
$$= \int_0^T (f(t), \varphi(t))\mathrm{d}t + \int_0^T (N(t), \varphi(t))_{H_N}\mathrm{d}t$$

for every $\varphi \in \mathcal{V}^D$ such that $\varphi(T) = 0$, where $\Psi e\dot u$ is the function defined in (1.15).

Proof. We fix $\varphi \in \mathcal{V}^D$ with $\varphi(T) = 0$ and for every $\varepsilon > 0$ we define the following function

$$\varphi_\varepsilon(t) := \begin{cases} \frac{t}{\varepsilon}\varphi(t) & t \in [0, \varepsilon], \\ \varphi(t) & t \in [\varepsilon, T]. \end{cases}$$

We have that $\varphi_\varepsilon \in \mathcal{V}^D$ and $\varphi_\varepsilon(0) = \varphi_\varepsilon(T) = 0$, so we can use φ_ε as test function in (1.14). By proceeding as in [17, Proposition 2.10] we obtain

$$\lim_{\varepsilon\to 0^+}\int_0^T (\dot u(t), \dot\varphi_\varepsilon(t))\mathrm{d}t = \int_0^T (\dot u(t), \dot\varphi(t))\mathrm{d}t,$$
$$\lim_{\varepsilon\to 0^+}\int_0^T (\mathbb{A}eu(t), e\varphi_\varepsilon(t))\mathrm{d}t = \int_0^T (\mathbb{A}eu(t), e\varphi(t))\mathrm{d}t,$$

$$\lim_{\varepsilon\to 0^+}\int_0^T (f(t),\varphi_\varepsilon(t))\mathrm{d}t = \int_0^T (f(t),\varphi(t))\mathrm{d}t.$$

It remains to consider the terms involving $\mathbb{B}$ and N. We have

$$\begin{aligned}
&\int_0^T (\mathbb{B}\Psi(t)e\dot{u}(t),\Psi(t)e\varphi_\varepsilon(t))\mathrm{d}t\\
&=\int_0^\varepsilon (\mathbb{B}\Psi(t)e\dot{u}(t),\frac{t}{\varepsilon}\Psi(t)e\varphi(t))\mathrm{d}t+\int_\varepsilon^T (\mathbb{B}\Psi(t)e\dot{u}(t),\Psi(t)e\varphi(t))\mathrm{d}t,\\
&\int_0^T (N(t),\varphi_\varepsilon(t))_{H_N}\mathrm{d}t=\int_0^\varepsilon (N(t),\frac{t}{\varepsilon}\varphi(t))_{H_N}\mathrm{d}t+\int_\varepsilon^T (N(t),\varphi(t))_{H_N}\mathrm{d}t,
\end{aligned}$$

hence by the dominated convergence theorem we get

$$\begin{gathered}
\int_\varepsilon^T (\mathbb{B}\Psi(t)e\dot{u}(t),\Psi(t)e\varphi(t))\mathrm{d}t \xrightarrow[\varepsilon\to 0^+]{} \int_0^T (\mathbb{B}\Psi(t)e\dot{u}(t),\Psi(t)e\varphi(t))\mathrm{d}t,\\
\int_\varepsilon^T (N(t),\varphi(t))_{H_N}\mathrm{d}t \xrightarrow[\varepsilon\to 0^+]{} \int_0^T (N(t),\varphi(t))_{H_N}\mathrm{d}t,\\
\Big|\int_0^\varepsilon (\mathbb{B}\Psi(t)e\dot{u}(t),\frac{t}{\varepsilon}\Psi(t)e\varphi(t))\mathrm{d}t\Big| \le \|\mathbb{B}\|_\infty\|\Psi\|_\infty\int_0^\varepsilon \|\Psi(t)e\dot{u}(t)\|\|e\varphi(t)\|\mathrm{d}t \xrightarrow[\varepsilon\to 0^+]{} 0,\\
\Big|\int_0^\varepsilon (N(t),\frac{t}{\varepsilon}\varphi(t))_{H_N}\mathrm{d}t\Big| \le \int_0^\varepsilon \|N(t)\|_{H_N}\|\varphi(t)\|_{H_N}\mathrm{d}t \xrightarrow[\varepsilon\to 0^+]{} 0.
\end{gathered}$$

By combining together all the previous convergences we get the thesis. □

We now state the uniqueness result in the case of a fixed domain, that is $\Gamma_T=\Gamma_0$. We follow the same ideas of [30], and we need to assume

$$\Psi\in \mathrm{Lip}([0,T]\times\overline{\Omega}),\quad \nabla\dot{\Psi}\in L^\infty((0,T)\times\Omega;\mathbb{R}^d), \tag{1.53}$$

while on Γ_0 we do not require any further hypotheses.

Lemma 1.4.2 (Uniqueness in a fixed domain)**.** *Assume* (1.53) *and* $\Gamma_T=\Gamma_0$. *Then the viscoelastic dynamic system* (1.8) *with boundary and initial conditions* (1.9)–(1.12) (*the latter in the sense of* (1.16)) *has a unique weak solution.*

Proof. Let $u_1,u_2\in\mathcal{W}$ be two weak solutions to (1.8)–(1.11) with initial conditions (1.12). The function $u:=u_1-u_2$ satisfies

$$\frac{1}{h}\int_0^h \left(\|u(t)\|_{V_t}^2+\|\dot{u}(t)\|^2\right)\mathrm{d}t \xrightarrow[h\to 0^+]{} 0, \tag{1.54}$$

hence by Lemma 1.4.1 it solves

$$-\int_0^T (\dot{u}(t),\dot{\varphi}(t))\mathrm{d}t+\int_0^T (\mathbb{A}eu(t),e\varphi(t))\mathrm{d}t+\int_0^T (\mathbb{B}\Psi(t)e\dot{u}(t),\Psi(t)e\varphi(t))\mathrm{d}t=0 \tag{1.55}$$

for every $\varphi\in\mathcal{V}^D$ such that $\varphi(T)=0$. We fix $s\in(0,T]$ and consider the function

$$\varphi_s(t):=\begin{cases}-\int_t^s u(\tau)\mathrm{d}\tau & t\in[0,s],\\ 0 & t\in[s,T].\end{cases}$$

Since $\varphi_s\in\mathcal{V}^D$ and $\varphi_s(T)=0$, we can use it as test function in (1.55) to obtain

$$-\int_0^s (\dot{u}(t),u(t))\mathrm{d}t+\int_0^s (\mathbb{A}e\dot{\varphi}_s(t),e\varphi_s(t))\mathrm{d}t+\int_0^s (\mathbb{B}\Psi(t)e\dot{u}(t),\Psi(t)e\varphi_s(t))\mathrm{d}t=0.$$

In particular we deduce

$$-\frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2\mathrm{d}t + \frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{A}e\varphi_s(t), e\varphi_s(t))\mathrm{d}t + \int_0^s (\mathbb{B}\Psi(t)e\dot{u}(t), \Psi(t)e\varphi_s(t))\mathrm{d}t = 0,$$

which implies

$$\frac{1}{2}\|u(s)\|^2 + \frac{1}{2}(\mathbb{A}e\varphi_s(0), e\varphi_s(0)) = \int_0^s (\mathbb{B}\Psi(t)e\dot{u}(t), \Psi(t)e\varphi_s(t))\mathrm{d}t, \tag{1.56}$$

since $u(0) = 0 = \varphi_s(s)$. From the distributional point of view the following equality holds

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Psi eu) = \dot{\Psi}eu + \Psi e\dot{u} \in L^2(0, T; H), \tag{1.57}$$

indeed, for all $v \in C_c^\infty(0, T; H)$ we have

$$\begin{aligned}
&\int_0^T (\frac{\mathrm{d}}{\mathrm{d}t}(\Psi(t)eu(t)), v(t))\mathrm{d}t = -\int_0^T (\Psi(t)eu(t), \dot{v}(t))\,\mathrm{d}t \\
&= -\int_0^T (e(\Psi(t)u(t)) - \nabla\Psi(t) \odot u(t), \dot{v}(t))\,\mathrm{d}t \\
&= \int_0^T (e(\dot{\Psi}(t)u(t)) + e(\Psi(t)\dot{u}(t)), v(t))\mathrm{d}t - \int_0^T (\nabla\dot{\Psi}(t) \odot u(t) + \nabla\Psi(t) \odot \dot{u}(t), v(t))\mathrm{d}t \\
&= \int_0^T (\dot{\Psi}(t)eu(t), v(t))\mathrm{d}t + \int_0^T (\Psi(t)e\dot{u}(t), v(t))\mathrm{d}t.
\end{aligned}$$

In particular $\Psi eu \in H^1(0, T; H) \subset C^0([0, T], H)$, so that by (1.54)

$$\|\Psi(0)eu(0)\|^2 = \lim_{h\to 0}\frac{1}{h}\int_0^h \|\Psi(t)eu(t)\|^2\mathrm{d}t \le C\lim_{h\to 0}\frac{1}{h}\int_0^h \|u(t)\|^2_{V_t}\mathrm{d}t = 0$$

which yields $\Psi(0)eu(0) = 0$. Thanks to (1.57) and to property $\Psi u \in H^1(0, T; H)$, we deduce

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{B}\Psi eu, \Psi e\varphi_s) &= (\mathbb{B}\dot{\Psi}eu, \Psi e\varphi_s) + (\mathbb{B}\Psi e\dot{u}, \Psi e\varphi_s) + (\mathbb{B}\Psi eu, \dot{\Psi}e\varphi_s) + (\mathbb{B}\Psi eu, \Psi e\dot{\varphi}_s) \\
&= 2(\mathbb{B}\Psi eu, \dot{\Psi}e\varphi_s) + (\mathbb{B}\Psi e\dot{u}, \Psi e\varphi_s) + (\mathbb{B}\Psi eu, \Psi e\dot{\varphi}_s),
\end{aligned}$$

and by integrating on $[0, s]$ we get

$$\begin{aligned}
&\int_0^s (\mathbb{B}\Psi(t)e\dot{u}(t), \Psi(t)e\varphi_s(t))\mathrm{d}t = \int_0^s \frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{B}\Psi(t)eu(t), \Psi(t)e\varphi_s(t))\mathrm{d}t \\
&- \int_0^s \left[(\mathbb{B}\Psi(t)e\dot{\varphi}_s(t), \Psi(t)e\dot{\varphi}_s(t)) + 2(\mathbb{B}\Psi(t)eu(t), \dot{\Psi}(t)e\varphi_s(t))\right]\mathrm{d}t \\
&\le (\mathbb{B}\Psi(s)eu(s), \Psi(s)e\varphi_s(s)) - (\mathbb{B}\Psi(0)eu(0), \Psi(0)e\varphi_s(0)) - \int_0^s (\mathbb{B}\Psi(t)e\dot{\varphi}_s(t), \Psi(t)e\dot{\varphi}_s(t))\mathrm{d}t \\
&\quad + \int_0^s \left[2(\mathbb{B}\Psi(t)eu(t), \Psi(t)eu(t))^{\frac{1}{2}}(\mathbb{B}\dot{\Psi}(t)e\varphi_s(t), \dot{\Psi}(t)e\varphi_s(t))^{\frac{1}{2}}\right]\mathrm{d}t \\
&\le \int_0^s \left[(\mathbb{B}\Psi(t)eu(t), \Psi(t)eu(t)) + (\mathbb{B}\dot{\Psi}(t)e\varphi_s(t), \dot{\Psi}(t)e\varphi_s(t)) - (\mathbb{B}\Psi(t)e\dot{\varphi}_s(t), \Psi(t)e\dot{\varphi}_s(t))\right]\mathrm{d}t \\
&\le \|\mathbb{B}\|_\infty\|\dot{\Psi}\|^2_\infty \int_0^s \|e\varphi_s(t)\|^2\mathrm{d}t,
\end{aligned}$$

since $e\varphi_s(s) = 0 = \Psi(0)eu(0)$ and $e\dot{\varphi}_s = eu$ in $(0, s)$. By combining the previous inequality with (1.56) and using the coercivity of the tensor $\mathbb{A}$, we derive

$$\frac{c_{\mathbb{A}}}{2}\|e\varphi_s(0)\|^2 + \frac{1}{2}\|u(s)\|^2 \le \frac{1}{2}(\mathbb{A}e\varphi_s(0), e\varphi_s(0)) + \frac{1}{2}\|u(s)\|^2 \le \|\mathbb{B}\|_\infty\|\dot{\Psi}\|^2_\infty\int_0^s \|e\varphi_s(t)\|^2\mathrm{d}t.$$

Let us set $\xi(t) := \int_0^t u(\tau)\mathrm{d}\tau$, then

$$\|e\varphi_s(0)\|^2 = \|e\xi(s)\|^2, \quad \|e\varphi_s(t)\|^2 = \|e\xi(t) - e\xi(s)\|^2 \leq 2\|e\xi(t)\|^2 + 2\|e\xi(s)\|^2,$$

from which we deduce

$$\frac{c_{\mathbb{A}}}{2}\|e\xi(s)\|^2 + \frac{1}{2}\|u(s)\|^2 \leq C\int_0^s \|e\xi(t)\|^2\mathrm{d}t + Cs\|e\xi(s)\|^2, \tag{1.58}$$

where $C := 2\|\mathbb{B}\|_\infty\|\dot{\Psi}\|^2_\infty$. Therefore, if we set $s_0 := \frac{c_{\mathbb{A}}}{4C}$, for all $s \leq s_0$ we obtain

$$\frac{c_{\mathbb{A}}}{4}\|e\xi(s)\|^2 \leq \Big(\frac{c_{\mathbb{A}}}{2} - Cs\Big)\|e\xi(s)\|^2 \leq C\int_0^s \|e\xi(t)\|^2\mathrm{d}t.$$

By Gronwall's lemma the last inequality implies $e\xi(s) = 0$ for all $s \leq s_0$. Hence, thanks to (1.58) we get $\|u(s)\|^2 \leq 0$ for all $s \leq s_0$, which yields $u(s) = 0$ for all $s \leq s_0$. Since s_0 depends only on $\mathbb{A}$, $\mathbb{B}$, and Ψ, we can repeat this argument starting from s_0, and with a finite number of steps we obtain $u \equiv 0$ on $[0, T]$. □

In order to prove our uniqueness result in the case of a moving crack we need two auxiliary results, which are [14, Theorem 6.1] and [17, Theorem 4.3]. For the sake of the readers, we rewrite below the statements without proof.

The first one ([14, Theorem 6.1]) is a generalization of the well-known result of finite speed of propagation for the wave equation. Given an open bounded set $U \subset \mathbb{R}^d$, we define by $\partial_L U$ the Lipschitz part of the boundary ∂U, which is the collection of points $x \in \partial U$ for which there exist an orthogonal coordinate system $y_1, \dots, y_d$, a neighborhood V of x of the form $A \times I$, with A open in $\mathbb{R}^{d-1}$ and I open interval in $\mathbb{R}$, and a Lipschitz function $g\colon A \to I$, such that $V \cap U := \{(y_1, \dots, y_d) \in V : y_d < g(y_1, \dots, y_{d-1})\}$. Moreover, given a Borel set $S \subset \partial_L U$, we define

$$H_S(U; \mathbb{R}^d) := \{u \in H^1(U; \mathbb{R}^d) : u = 0 \text{ on } S\}.$$

Notice that $H_S(U; \mathbb{R}^d)$ is a Hilbert space, and we denote its dual by $H^{-1}_S(U; \mathbb{R}^d)$.

Theorem 1.4.3 (Finite speed of propagation)**.** *Let $U \subset \mathbb{R}^d$ be an open bounded set and let $\partial_L U$ be the Lipschitz part of ∂U. Let S_0 and S_1 be two Borel sets with $S_0 \subset S_1 \subset \partial_L U$, and let $\mathbb{A}\colon U \to \mathscr{L}(\mathbb{R}^{d\times d}_{sym}; \mathbb{R}^{d\times d}_{sym})$ be a fourth-order tensor satisfying (1.4)–(1.6). Let*

$$u \in L^2(0, T; H^1_{S_0}(U; \mathbb{R}^d)) \cap H^1(0, T; L^2(U; \mathbb{R}^d)) \cap H^2(0, T; H^{-1}_{S_1}(U; \mathbb{R}^d))$$

be a solution to

$$\langle \ddot{u}(t), \psi\rangle_{H^{-1}_{S_1}(U;\mathbb{R}^d)} + (\mathbb{A}eu(t), e\psi)_{L^2(U;\mathbb{R}^{d\times d}_{sym})} = 0 \quad \textit{for every } \psi \in H^1_{S_1}(U; \mathbb{R}^d),$$

with initial conditions $u(0) = 0$ and $\dot{u}(0) = 0$ in the sense of $L^2(U; \mathbb{R}^d)$ and $H^{-1}_{S_1}(U; \mathbb{R}^d)$, respectively. Then

$$u(t) = 0 \quad \textit{a.e. in } U_t := \{x \in U : \operatorname{dist}(x, S_1 \setminus S_0) > t\sqrt{\|\mathbb{A}\|_\infty}\}$$

for every $t \in [0, T]$.

Proof. See [14, Theorem 6.1]. □

The second one ([17, Theorem 4.3]) is a uniqueness result for the weak solutions of the wave equation in a moving domain. Let $\hat{H}$ be a separable Hilbert space, and let $\{\hat{V}_t\}_{t\in[0,T]}$ be a family of separable Hilbert spaces with the following properties:

(*i*) for every $t \in [0,T]$ the space $\hat V_t$ is contained and dense in $\hat H$ with continuous embedding;

(*ii*) for every $s,t \in [0,T]$, with $s<t$, $\hat V_s \subset \hat V_t$ and the Hilbert space structure on $\hat V_s$ is the one induced by $\hat V_t$.

Let $a\colon \hat V \times \hat V \to \mathbb{R}$ be a bilinear symmetric form satisfying the following conditions:

(*iii*) there exists M_0 such that

$$|a(u,v)| \le M_0 \|u\|_{\hat V} \|v\|_{\hat V} \quad \text{for every } u,v \in \hat V;$$

(*iv*) there exist $\lambda_0 > 0$ and $\nu_0 \in \mathbb{R}$ such that

$$a(u,u) \ge \lambda_0 \|u\|^2_{\hat V} - \nu_0 \|u\|^2_{\hat H} \quad \text{for every } u \in \hat V.$$

Assume that

(U1) for every $t \in [0,T]$ there exists a continuous and linear bijective operator $Q_t\colon \hat V_t \to \hat V_0$, with continuous inverse $R_t\colon \hat V_0 \to \hat V_t$;

(U2) Q_0 and R_0 are the identity maps on $\hat V_0$;

(U3) there exists a constant M_1 independent of t such that

$$\begin{aligned} &\|Q_t u\|_{\hat H} \le M_1 \|u\|_{\hat H} \quad \text{for every } u \in \hat V_t, &&\|R_t u\|_{\hat H} \le M_1 \|u\|_{\hat H} \quad \text{for every } u \in \hat V_0,\\ &\|Q_t u\|_{\hat V_0} \le M_1 \|u\|_{\hat V_t} \quad \text{for every } u \in \hat V_t, &&\|R_t u\|_{\hat V_t} \le M_1 \|u\|_{\hat V_0} \quad \text{for every } u \in \hat V_0. \end{aligned}$$

Since $\hat V_t$ is dense in $\hat H$, (U3) implies that R_t and Q_t can be extended to continuous linear operators from $\hat H$ into itself, still denoted by Q_t and R_t. We also require

(U4) for every $v \in \hat V_0$ the function $t \mapsto R_t v$ from $[0,T]$ into $\hat H$ has a derivative, denoted by $\dot R_t v$;

(U5) there exists $\eta \in (0,1)$ such that

$$\|\dot R_t Q_t v\|^2_{\hat H} \le \lambda_0 (1-\eta) \|v\|^2_{\hat V_t} \quad \text{for every } v \in \hat V_t;$$

(U6) there exists a constant M_2 such that

$$\|Q_t v - Q_s v\|_{\hat H} \le M_2 \|v\|_{\hat V_s} (t-s) \quad \text{for every } 0 \le s < t \le T \text{ and every } v \in \hat V_s;$$

(U7) for very $t \in [0,T)$ and for every $v \in \hat V_t$ there exists an element of $\hat H$, denoted by $\dot Q_t v$, such that

$$\lim_{h \to 0^+} \frac{Q_{t+h} v - Q_t v}{h} = \dot Q_t v \text{ in } \hat H.$$

For every $t \in [0,T]$, define

$$\begin{aligned} &\alpha(t)\colon \hat V_0 \times \hat V_0 \to \mathbb{R} \quad \text{as } \alpha(t)(u,v) := a(R_t u, R_t v) \text{ for } u,v \in \hat V_0,\\ &\beta(t)\colon \hat V_0 \times \hat V_0 \to \mathbb{R} \quad \text{as } \beta(t)(u,v) := (\dot R_t u, \dot R_t v) \text{ for } u,v \in \hat V_0,\\ &\gamma(t)\colon \hat V_0 \times \hat H \to \mathbb{R} \quad \text{as } \gamma(t)(u,v) := (\dot R_t u, R_t v) \text{ for } u \in \hat V_0 \text{ and } v \in \hat H,\\ &\delta(t)\colon \hat H \times \hat H \to \mathbb{R} \quad \text{as } \delta(t)(u,v) := (R_t u, R_t v) - (u,v) \text{ for } u,v \in \hat H. \end{aligned}$$

We assume that there exists a constant M_3 such that

(U8) the maps $t \mapsto \alpha(t)(u,v)$, $t \mapsto \beta(t)(u,v)$, $t \mapsto \gamma(t)(u,v)$, and $t \mapsto \delta(t)(u,v)$ are Lipschitz continuous and for a.e. $t \in (0,T)$ their derivatives satisfy

$$\begin{aligned} |\dot\alpha(t)(u,v)| &\le M_3 \|u\|_{\hat V_0} \|v\|_{\hat V_0} \quad \text{for } u,v \in \hat V_0,\\ |\dot\beta(t)(u,v)| &\le M_3 \|u\|_{\hat V_0} \|v\|_{\hat V_0} \quad \text{for } u,v \in \hat V_0,\\ |\dot\gamma(t)(u,v)| &\le M_3 \|u\|_{\hat V_0} \|v\|_{\hat H} \quad \text{for } u \in \hat V_0 \text{ and } v \in \hat H,\\ |\dot\delta(t)(u,v)| &\le M_3 \|u\|_{\hat H} \|v\|_{\hat H} \quad \text{for } u,v \in \hat H. \end{aligned}$$

Theorem 1.4.4 (Uniqueness for the wave equation)**.** *Assume that $\hat H$, $\{\hat V_t\}_{t\in[0,T]}$, and a satisfy (i)–(iv) and that (U1)–(U8) hold. Given $u^0 \in \hat V_0$, $u^1 \in \hat H$, and $f \in L^2(0,T;\hat H)$, there exists a unique solution*

$$u \in \hat{\mathcal V} := \{\varphi \in L^2(0,T;\hat V) : \dot u \in L^2(0,T;\hat H), u(t) \in \hat V_t \text{ for a.e. } t \in (0,T)\}$$

to the wave equation

$$-\int_0^T (\dot u(t), \dot\varphi(t))_{\hat H} \mathrm{d}t + \int_0^T a(u(t),\varphi(t))\mathrm{d}t = \int_0^T (f(t),\varphi(t))_{\hat H}\mathrm{d}t \quad \text{for every } \varphi \in \hat{\mathcal V},$$

satisfying the initial conditions $u(0) = u^0$ and $\dot u(0) = u^1$ in the sense that

$$\lim_{h\to 0^+} \frac{1}{h} \int_0^h \left(\|u(t)-u^0\|^2_{\hat V_t} + \|\dot u(t) - u^1\|^2_{\hat H}\right)\mathrm{d}t = 0.$$

Proof. See [17, Theorem 4.3]. □

We now are in position to prove the uniqueness theorem in the case of a moving domain. We consider the dimensional case $d = 2$, and we require the following assumptions:

(H1) there exists a $C^{2,1}$ simple curve $\Gamma \subset \overline\Omega \subset \mathbb{R}^2$, parametrized by arc-length $\gamma\colon [0,\ell] \to \overline\Omega$, such that $\Gamma \cap \partial\Omega = \gamma(0) \cup \gamma(\ell)$ and $\Omega \setminus \Gamma$ is the union of two disjoint open sets with Lipschitz boundary;

(H2) there exists a non decreasing function $s\colon [0,T] \to (0,\ell)$ of class $C^{1,1}$ such that $\Gamma_t = \gamma([0,s(t)])$;

(H3) $|\dot s(t)|^2 < \frac{c_{\mathbb A}}{C_K}$, where $c_{\mathbb A}$ is the ellipticity constant of $\mathbb A$ and C_K is the constant that appears in Korn's inequality in (1.2).

Notice that hypotheses (H1) and (H2) imply (E1)–(E3). We also assume that Ψ satisfies (1.53) and there exists a constant $\varepsilon > 0$ such that for every $t \in [0,T]$

$$\Psi(t,x) = 0 \quad \text{for every } x \in \{y \in \overline\Omega : |y - \gamma(s(t))| < \varepsilon\}. \tag{1.59}$$

Theorem 1.4.5. *Assume $d = 2$ and (H1)–(H3), (1.53), and (1.59). Then the system (1.8) with boundary conditions (1.9)–(1.11) has a unique weak solution $u \in \mathcal W$ which satisfies $u(0) = u^0$ and $\dot u(0) = u^1$ in the sense of (1.16).*

Proof. As before let $u_1, u_2 \in \mathcal W$ be two weak solutions to (1.8)–(1.11) with initial conditions (1.12). Then $u := u_1 - u_2$ satisfies (1.54) and (1.55) for every $\varphi \in \mathcal V^D$ such that $\varphi(T) = 0$. Let us define

$$t_0 := \sup\{t \in [0,T] : u(s) = 0 \text{ for every } s \in [0,t]\},$$

and assume by contradiction that $t_0 < T$. Consider first the case in which $t_0 > 0$. By (H1), (H2), (1.53), and (1.59) we can find two open sets A_1 and A_2, with $A_1 \subset\subset A_2 \subset\subset \Omega$, and a number $\delta > 0$ such that for every $t \in [t_0 - \delta, t_0 + \delta]$ we have $\gamma(s(t)) \in A_1$, $\Psi(t, x) = 0$ for every $x \in \overline{A}_2$, and $(A_2 \setminus A_1) \setminus \Gamma$ is the union of two disjoint open sets with Lipschitz boundary. Let us define

$$\hat{V}^1 := \{u \in H^1((A_2 \setminus A_1) \setminus \Gamma_{t_0-\delta}; \mathbb{R}^2) : u = 0 \text{ on } \partial A_1 \cup \partial A_2\}, \quad \hat{H}^1 := L^2(A_2 \setminus A_1; \mathbb{R}^2).$$

Since every function in $\hat{V}^1$ can be extended to a function in $V^D_{t_0-\delta}$, by classical results for linear hyperbolic equations (se, e.g., [20]), we deduce $\ddot{u} \in L^2(t_0 - \delta, t_0 + \delta; (\hat{V}^1)')$ and that u satisfies for a.e. $t \in (t_0 - \delta, t_0 + \delta)$

$$\langle \ddot{u}(t), \phi \rangle_{(\hat{V}^1)'} + (\mathbb{A}eu(t), e\phi)_{\hat{H}^1} = 0 \quad \text{for every } \phi \in \hat{V}^1.$$

Moreover, we have $u(t_0) = 0$ as element of $\hat{H}^1$ and $\dot{u}(t_0) = 0$ as element of $(\hat{V}^1)'$, since $u(t) \equiv 0$ in $[t_0 - \delta, t_0)$, $u \in C^0([t_0 - \delta, t_0]; \hat{H}^1)$, and $\dot{u} \in C^0([t_0 - \delta, t_0]; (\hat{V}^1)')$. We are now in position to apply the result of finite speed of propagation of Theorem 1.4.3. This theorem ensures the existence of a third open set A_3, with $A_1 \subset\subset A_3 \subset\subset A_2$, such that, up to choose a smaller δ, we have $u(t) = 0$ on ∂A_3 for every $t \in [t_0, t_0 + \delta]$, and both $(\Omega \setminus A_3) \setminus \Gamma$ and $A_3 \setminus \Gamma$ are union of two disjoint open sets with Lipschitz boundary.

In $\Omega \setminus A_3$ the function u solves

$$-\int_{t_0-\delta}^{t_0+\delta} \int_{\Omega \setminus A_3} \dot{u}(t, x) \cdot \dot{\varphi}(t, x) \mathrm{d}x\mathrm{d}t + \int_{t_0-\delta}^{t_0+\delta} \int_{\Omega \setminus A_3} \mathbb{A}(x)eu(t, x) \cdot e\varphi(t, x) \mathrm{d}x\mathrm{d}t \\ + \int_{t_0-\delta}^{t_0+\delta} \int_{\Omega \setminus A_3} \mathbb{B}(x)\Psi(t, x)e\dot{u}(t, x) \cdot \Psi(t, x)e\varphi(t, x) \mathrm{d}x\mathrm{d}t = 0$$

for every $\varphi \in L^2(t_0 - \delta, t_0 + \delta; \hat{V}^2) \cap H^1(t_0 - \delta, t_0 + \delta; \hat{H}^2)$ such that $\varphi(t_0 - \delta) = \varphi(t_0 + \delta) = 0$, where

$$\hat{V}^2 := \{u \in H^1((\Omega \setminus A_3) \setminus \Gamma_{t_0-\delta}; \mathbb{R}^2) : u = 0 \text{ on } \partial_D\Omega \cup \partial A_3\}, \quad \hat{H}^2 := L^2(\Omega \setminus A_3; \mathbb{R}^2).$$

Since $u(t) = 0$ on $\partial_D\Omega \cup \partial A_3$ for every $t \in [t_0 - \delta, t_0 + \delta]$ and $u(t_0 - \delta) = \dot{u}(t_0 - \delta) = 0$ in the sense of (1.16) (recall that $u \equiv 0$ in $[t_0 - \delta, t_0)$), we can apply Lemma 1.4.2 to deduce $u(t) = 0$ in $\Omega \setminus A_3$ for every $t \in [t_0 - \delta, t_0 + \delta]$.

On the other hand in A_3, by setting

$$\hat{V}^3_t := \{u \in H^1(A_3 \setminus \Gamma_t; \mathbb{R}^2) : u = 0 \text{ on } \partial A_3\}, \quad \hat{H}^3 := L^2(A_3; \mathbb{R}^2),$$

we get that the function u solves

$$-\int_{t_0-\delta}^{t_0+\delta} \int_{A_3} \dot{u}(t, x) \cdot \dot{\varphi}(t, x) \mathrm{d}x\mathrm{d}t + \int_{t_0-\delta}^{t_0+\delta} \int_{A_3} \mathbb{A}(x)eu(t, x) \cdot e\varphi(t, x) \mathrm{d}x\mathrm{d}t = 0$$

for every $\varphi \in L^2(t_0 - \delta, t_0 + \delta; \hat{V}^3_{t_0+\delta}) \cap H^1(t_0 - \delta, t_0 + \delta; \hat{H}^3)$ such that $\varphi(t) \in \hat{V}^3_t$ for a.e. $t \in (t_0 - \delta, t_0 + \delta)$ and $\varphi(t_0 - \delta) = \varphi(t_0 + \delta) = 0$. Here we would like to apply the uniqueness result of Theorem 1.4.4 for the spaces $\{\hat{V}^3_t\}_{t \in [t_0-\delta, t_0+\delta]}$ and $\hat{H}^3$, endowed with the usual norms, and for the bilinear form

$$a(u, v) := \int_{A_3} \mathbb{A}(x)eu(x) \cdot ev(x) \mathrm{d}x \quad \text{for every } u, v \in \hat{V}^3_{t_0+\delta}.$$

As show in [16, Example 2.14] we can construct two maps $\Phi, \Lambda \in C^{1,1}([t_0 - \delta, t_0 + \delta] \times \overline{A}_3; \mathbb{R}^2)$ such that for every $t \in [0, T]$ the function $\Phi(t, \cdot) \colon \overline{A}_3 \to \overline{A}_3$ is a diffeomorfism of A_3 in itself with inverse $\Lambda(t, \cdot) \colon \overline{A}_3 \to \overline{A}_3$. Moreover, $\Phi(0, y) = y$ for every $y \in \overline{A}_3$, $\Phi(t, \Gamma \cap \overline{A}_3) = \Gamma \cap \overline{A}_3$

and $\Phi(t,\Gamma_{t_0-\delta}\cap\overline{A}_3)=\Gamma_t\cap\overline{A}_3$ for every $t\in[t_0-\delta,t_0+\delta]$. For every $t\in[t_0-\delta,t_0+\delta]$, the maps $(Q_tu)(y):=u(\Phi(t,y))$, $u\in\hat V^3_t$ and $y\in A_3$, and $(R_tv)(x):=v(\Lambda(t,x))$, $v\in\hat V^3_{t_0-\delta}$ and $x\in A_3$, provide a family of linear and continuous operators which satisfy the assumptions (U1)–(U8) of Theorem 1.4.4 (see [17, Example 4.2]). The only condition to check is (U5). The bilinear form a satisfies the following ellipticity condition

$$a(u,u)\ge c_{\mathbb{A}}\|eu\|^2_{L^2(A_3;\mathbb{R}^{2\times2}_{sym})}\ge\frac{c_{\mathbb{A}}}{\hat C_k}\|u\|^2_{\hat V^3_{t_0+\delta}}-c_{\mathbb{A}}\|u\|^2_{\hat H^3}\quad\text{for every } u\in\hat V^3_{t_0+\delta},\tag{1.60}$$

where $\hat C_K$ is the constant in Korn's inequality in $\hat V^3_{t_0+\delta}$, namely

$$\|\nabla u\|^2_{L^2(A_3;\mathbb{R}^{2\times2})}\le\hat C_K(\|u\|^2_{L^2(A_3;\mathbb{R}^2)}+\|eu\|^2_{L^2(A_3;\mathbb{R}^{2\times2}_{sym})})\quad\text{for every } u\in\hat V^3_{t_0+\delta}.$$

Notice that for $t\in[t_0-\delta,t_0+\delta]$

$$(\dot R_tv)(x)=\nabla v(\Lambda(t,x))\dot\Lambda(t,x)\quad\text{for a.e. } x\in A_3,$$

from which we obtain

$$\|\dot R_tQ_tu\|^2_{\hat H^3}\le\int_{A_3}|\nabla u(x)|^2|\dot\Phi(t,\Lambda(t,x))|^2\mathrm{d}x.$$

Hence, have to show the property

$$|\dot\Phi(t,y)|^2<\frac{c_{\mathbb{A}}}{\hat C_K}\quad\text{for every } t\in[t_0-\delta,t_0+\delta]\text{ and } y\in\overline{A}_3.$$

This is ensured by (H3). Indeed, as explained in [16, Example 3.1], we can construct the maps Φ and Λ in such a way that

$$|\dot\Phi(t,y)|^2<\frac{c_{\mathbb{A}}}{C_K},$$

since $|\dot s(t)|^2<\frac{c_{\mathbb{A}}}{C_K}$. Moreover, every function in $\hat V^3_{t_0+\delta}$ can be extended to a function in $H^1(\Omega\setminus\Gamma;\mathbb{R}^d)$. Hence, for Korn's inequality in $\hat V^3_{t_0+\delta}$, we can use the same constant C_K of $H^1(\Omega\setminus\Gamma;\mathbb{R}^d)$. This allows us to apply Theorem 1.4.4, which implies $u(t)=0$ in A_3 for every $t\in[t_0,t_0+\delta]$. In the case $t_0=0$, it is enough to argue as before in $[0,\delta]$, by exploiting (1.54). Therefore $u(t)=0$ in Ω for every $t\in[t_0,t_0+\delta]$, which contradicts the maximality of t_0. Hence $t_0=T$, that yields $u(t)=0$ in Ω for every $t\in[0,T]$. □

Remark 1.4.6. Also Theorem 1.4.5 is true in the antiplane case, with essentially the same proof. Notice that, when the displacement is scalar, we do not need to use Korn's inequality in (1.60) to get the coercivity in $\hat V^3_{t_0+\delta}$ of the bilinear form a defined before. Therefore, in this case in (H3) it is enough to assume $|\dot s(t)|^2<c_{\mathbb{A}}$.

1.5 A moving crack satisfying Griffith's Dynamic Energy-Dissipation Balance

We conclude this chapter with an example of a moving crack $\{\Gamma_t\}_{t\in[0,T]}$ and weak solution to (1.8)–(1.12) which satisfy the energy-dissipation balance of Griffith's dynamic criterion, as happens in [14] for the purely elastic case. In dimension $d=2$ we consider an antiplane evolution, which means that the displacement u is scalar, and we take $\Omega:=\{x\in\mathbb{R}^2:|x|<R\}$, with $R>0$. We fix a constant $0<c<1$ such that $cT<R$, and we set

$$\Gamma_t:=\{(\sigma,0)\in\overline{\Omega}:\sigma\le ct\}.$$

Let us define the following function

$$S(x_1, x_2) := Im(\sqrt{x_1 + ix_2}) \quad x \in \mathbb{R}^2 \setminus \{(\sigma, 0) : \sigma \leq 0\},$$

where Im denotes the imaginary part of a complex number. Notice that the function previously defined satisfies $S \in H^1(\Omega \setminus \Gamma_0) \setminus H^2(\Omega \setminus \Gamma_0)$, and it is a weak solution to

$$\begin{cases} \Delta S = 0 & \text{in } \Omega \setminus \Gamma_0, \\ \nabla S \cdot \nu = \partial_2 S = 0 & \text{on } \Gamma_0. \end{cases}$$

Let us consider the function

$$u(t, x) := \frac{2}{\sqrt{\pi}} S\Big(\frac{x_1 - ct}{\sqrt{1 - c^2}}, x_2\Big) \quad t \in [0, T], x \in \Omega \setminus \Gamma_t$$

and let $z(t)$ be its restriction to $\partial\Omega$. Since $u(t)$ has a singularity only at the crack tip $(ct, 0)$, the function $z(t)$ can be seen as the trace on $\partial\Omega$ of a function belonging to $H^2(0, T; L^2(\Omega)) \cap H^1(0, T; H^1(\Omega \setminus \Gamma_0))$, still denoted by $z(t)$. It is easy to see that u solves the wave equation

$$\ddot{u}(t) - \Delta u(t) = 0 \quad \text{in } \Omega \setminus \Gamma_t, t \in (0, T),$$

with boundary conditions

$$\begin{aligned} & u(t) = z(t) && \text{on } \partial\Omega,\ t \in (0, T), \\ & \frac{\partial u}{\partial \nu}(t) = \nabla u(t) \cdot \nu = 0 && \text{on } \Gamma_t,\ t \in (0, T), \end{aligned}$$

and initial data

$$\begin{aligned} u^0(x_1, x_2) &:= \frac{2}{\sqrt{\pi}} S\Big(\frac{x_1}{\sqrt{1 - c^2}}, x_2\Big) \in H^1(\Omega \setminus \Gamma_0), \\ u^1(x_1, x_2) &:= -\frac{2}{\sqrt{\pi}} \frac{c}{\sqrt{1 - c^2}} \partial_1 S\Big(\frac{x_1}{\sqrt{1 - c^2}}, x_2\Big) \in L^2(\Omega). \end{aligned}$$

Let us consider a function Ψ which satisfies the regularity assumptions (1.53) and condition (1.59), namely

$$\Psi(t) = 0 \quad \text{on } B_\varepsilon(t) := \{x \in \mathbb{R}^2 : |x - (ct, 0)| < \varepsilon\} \text{ for every } t \in [0, T],$$

with $0 < \varepsilon < R - cT$. In this case u is a weak solution, in the sense of Definition 1.1.4, to the damped wave equation

$$\ddot{u}(t) - \Delta u(t) - \operatorname{div}(\Psi^2(t) \nabla \dot{u}(t)) = f(t) \quad \text{in } \in \Omega \setminus \Gamma_t, t \in (0, T),$$

with forcing term f given by

$$f := -\operatorname{div}(\Psi^2 \nabla \dot{u}) = -\nabla\Psi \cdot 2\Psi \nabla \dot{u} - \Psi^2 \Delta \dot{u} \in L^2(0, T; L^2(\Omega)),$$

and boundary and initial conditions

$$\begin{aligned} & u(t) = z(t) && \text{on } \partial\Omega,\ t \in (0, T), \\ & \frac{\partial u}{\partial \nu}(t) + \Psi^2(t) \frac{\partial \dot{u}}{\partial \nu}(t) = 0 && \text{on } \Gamma_t,\ t \in (0, T), \\ & u(0) = u^0, \quad \dot{u}(0) = u^1. \end{aligned}$$

Notice that for the homogeneous Neumann boundary conditions on Γ_t we used $\frac{\partial \dot{u}}{\partial \nu}(t) = \nabla \dot{u}(t) \cdot \nu = \partial_2 \dot{u}(t) = 0$ on Γ_t. By the uniqueness result proved in the previous section, the function u coincides with that one found in Theorem 1.2.1. Thanks to the computations done in [14, Section 4], we know that u satisfies for every $t \in [0,T]$ the following energy-dissipation balance for the undamped equation, where ct coincides with the length of $\Gamma_t \setminus \Gamma_0$

$$\begin{aligned}\frac{1}{2}\|\dot u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega;\mathbb{R}^2)} + ct \\ = \frac{1}{2}\|\dot u(0)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(0)\|^2_{L^2(\Omega;\mathbb{R}^2)} + \int_0^t (\frac{\partial u}{\partial \nu}(\tau), \dot z(\tau))_{L^2(\partial\Omega)} \mathrm{d}\tau.\end{aligned} \tag{1.61}$$

Moreover, we have

$$\begin{aligned}\int_0^t (\frac{\partial u}{\partial \nu}(\tau), \dot z(\tau))_{L^2(\partial\Omega)} \mathrm{d}\tau = \int_0^t (\nabla u(\tau), \nabla \dot z(\tau))_{L^2(\Omega;\mathbb{R}^2)} \mathrm{d}\tau - \int_0^t (\dot u(\tau), \ddot z(\tau))_{L^2(\Omega)} \mathrm{d}\tau \\ + (\dot u(t), \dot z(t))_{L^2(\Omega)} - (\dot u(0), \dot z(0))_{L^2(\Omega)}.\end{aligned} \tag{1.62}$$

For every $t \in [0,T]$ we compute

$$\begin{aligned}(f(t), \dot u(t) - \dot z(t))_{L^2(\Omega)} &= -\int_{(\Omega\setminus B_\varepsilon(t))\setminus\Gamma_t} \mathrm{div}[\Psi^2(t,x)\nabla \dot u(t,x)](\dot u(t,x) - \dot z(t,x))\mathrm{d}x \\ &= -\int_{(\Omega\setminus B_\varepsilon(t))\setminus\Gamma_t} \mathrm{div}[\Psi^2(t,x)\nabla \dot u(t,x)(\dot u(t,x) - \dot z(t,x))]\mathrm{d}x \\ &\quad + \int_{(\Omega\setminus B_\varepsilon(t))\setminus\Gamma_t} \Psi^2(t,x)\nabla \dot u(t,x)\cdot(\nabla \dot u(t,x) - \nabla \dot z(t,x))\mathrm{d}x.\end{aligned}$$

If we denote by $\dot u^{\oplus}(t)$ and $\dot z^{\oplus}(t)$ the traces of $\dot u(t)$ and $\dot z(t)$ on Γ_t from above and by $\dot u^{\ominus}(t)$ and $\dot z^{\ominus}(t)$ the trace from below, thanks to the divergence theorem we have

$$\begin{aligned}&\int_{(\Omega\setminus B_\varepsilon(t))\setminus\Gamma_t} \mathrm{div}[\Psi^2(t,x)\nabla \dot u(t,x)(\dot u(t,x) - \dot z(t,x))]\mathrm{d}x \\ &= \int_{\partial\Omega} \Psi^2(t,x)\frac{\partial \dot u}{\partial \nu}(t,x)(\dot u(t,x) - \dot z(t,x))\mathrm{d}x + \int_{\partial B_\varepsilon(t)} \Psi^2(t,x)\frac{\partial \dot u}{\partial \nu}(t,x)(\dot u(t,x) - \dot z(t,x))\mathrm{d}x \\ &\quad - \int_{(\Omega\setminus B_\varepsilon(t))\cap\Gamma_t} \Psi^2(t,x)\partial_2 \dot u^{\oplus}(t,x)(\dot u^{\oplus}(t,x) - \dot z^{\oplus}(t,x))\mathrm{d}\mathcal{H}^1(x) \\ &\quad + \int_{(\Omega\setminus B_\varepsilon(t))\cap\Gamma_t} \Psi^2(t,x)\partial_2 \dot u^{\ominus}(t,x)(\dot u^{\ominus}(t,x) - \dot z^{\ominus}(t,x))\mathrm{d}\mathcal{H}^1(x) = 0,\end{aligned}$$

since $u(t) = z(t)$ on $\partial\Omega$, $\Psi(t) = 0$ on $\partial B_\varepsilon(t)$, and $\partial_2 \dot u(t) = 0$ on Γ_t. Therefore for every $t \in [0,T]$ we get

$$(f(t), \dot u(t) - \dot z(t))_{L^2(\Omega)} = \|\Psi(t)\nabla \dot u(t)\|^2_{L^2(\Omega;\mathbb{R}^2)} - (\Psi(t)\nabla \dot u(t), \Psi(t)\nabla \dot z(t))_{L^2(\Omega;\mathbb{R}^2)}. \tag{1.63}$$

By combining (1.61)–(1.63) we deduce that u satisfies for every $t \in [0,T]$ the following Griffith's energy-dissipation balance for the viscoelastic dynamic equation

$$\begin{aligned}\frac{1}{2}\|\dot u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega;\mathbb{R}^2)} + \int_0^t \|\Psi(\tau)\nabla \dot u(\tau)\|^2_{L^2(\Omega;\mathbb{R}^2)} \mathrm{d}\tau + ct \\ = \frac{1}{2}\|\dot u(0)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(0)\|^2_{L^2(\Omega;\mathbb{R}^2)} + \mathscr{W}_{tot}(t),\end{aligned} \tag{1.64}$$

where in this case the total work takes the form

$$\mathscr{W}_{tot}(t) := \int_0^t \left[(f(\tau), \dot u(\tau) - \dot z(\tau))_{L^2(\Omega)} + (\Psi(\tau)\nabla \dot u(\tau), \Psi(\tau)\nabla \dot z(\tau))_{L^2(\Omega;\mathbb{R}^2)}\right] \mathrm{d}\tau$$

$$
\begin{aligned}
&- \int_0^t \big[(\dot u(\tau), \ddot z(\tau))_{L^2(\Omega)} - (\nabla u(\tau), \nabla \dot z(\tau))_{L^2(\Omega;\mathbb{R}^2)} \big] \, \mathrm{d}\tau \\
&+ (\dot u(t), \dot z(t))_{L^2(\Omega)} - (\dot u(0), \dot z(0))_{L^2(\Omega)}.
\end{aligned}
$$

Notice that equality (1.64) gives (10). This show that in this model Griffith's dynamic energy-dissipation balance can be satisfied by a moving crack, in contrast with the case $\Psi = 1$, which always leads to (7).

Chapter 2

A dynamic model with memory for viscoelasticity in domains with time-dependent cracks

The chapter is organized as follows. In Section 1.2 we fix the notation adopted throughout the chapter. In Section 2.1 we list the standard assumptions on the family of cracks $\{\Gamma_t\}_{t\in[0,T]}$, we state the evolution problem in the general case, and we specify the notion of solution to the problem. In Section 2.2 and 2.3 we deal with the existence of a solution to the viscoelastic dynamic model; in particular in Section 2.2, we provide a solution by means of a generalization of Lax-Milgram's theorem by Lions. After that, in Section 2.3, as previously anticipated, we define a coupled system equivalent to our viscoelastic dynamic system. In particular, in Subsection 2.3.1 we implement the time discretization method on such a system, and we conclude with Subsection 2.3.2 by showing the validity of the energy-dissipation inequality, and of the initial conditions.

The results presented here are contained in [44].

2.1 Formulation of the evolution problem, notion of solution

Let T be a positive real number and $d \in \mathbb{N}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set (which represents the reference configuration of the body) with Lipschitz boundary. Let $\partial_D\Omega$ be a (possibly empty) Borel subset of $\partial\Omega$, on which we prescribe the Dirichlet condition, and let $\partial_N\Omega$ be its complement, on which we give the Neumann condition. Let $\Gamma \subset \overline{\Omega}$ be the prescribed crack path. We assume the following hypotheses on the geometry of the cracks:

(E1) Γ is a closed set with $\mathcal{L}^d(\Gamma) = 0$ and $\mathcal{H}^{d-1}(\Gamma \cap \partial\Omega) = 0$;

(E2) for every $x \in \Gamma$ there exists an open neighborhood U of x in $\mathbb{R}^d$ such that $(U \cap \Omega) \setminus \Gamma$ is the union of two disjoint open sets U^+ and U^- with Lipschitz boundary;

(E3) $\{\Gamma_t\}_{t\in(-\infty,T]}$ is a family of closed subsets of Γ satisfying $\Gamma_s \subset \Gamma_t$ for every $-\infty < s \leq t \leq T$.

Notice that the set Γ_t represents the crack at time t. Thanks to (E1)–(E3) the space $L^2(\Omega \setminus \Gamma_t; \mathbb{R}^d)$ coincides with $L^2(\Omega; \mathbb{R}^d)$ for every $t \in (-\infty, T]$. In particular, we can extend a function $u \in L^2(\Omega \setminus \Gamma_t; \mathbb{R}^d)$ to a function in $L^2(\Omega; \mathbb{R}^d)$ by setting $u = 0$ on Γ_t. Since $\mathcal{H}^{d-1}(\Gamma \cap \partial\Omega) = 0$ the trace of $u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d)$ is well defined on $\partial\Omega$. Indeed, we may find a finite number of open sets with Lipschitz boundary $U_j \subset \Omega \setminus \Gamma$, $j = 1, \dots k$, such that $\partial\Omega \setminus \Gamma \subset \cup_{j=1}^k \partial U_j$. There exists a positive constant C, depending only on Ω and Γ, such that

$$\|u\|_{L^2(\partial\Omega;\mathbb{R}^d)} \leq C\|u\|_{H^1(\Omega\setminus\Gamma;\mathbb{R}^d)} \quad \text{for every } u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d). \tag{2.1}$$

Similarly, we can find a finite number of open sets $V_j \subset \Omega \setminus \Gamma$, $j = 1, \dots l$, with Lipschitz boundary, such that $\Omega \setminus \Gamma = \cup_{j=1}^{l} V_j$. By using the second Korn's inequality in each V_j (see, e.g., [39, Theorem 2.4]) and taking the sum over j we can find a positive constant C_K, depending only on Ω and Γ, such that

$$\|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} \le C_K(\|u\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|eu\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{sym})}) \quad \text{for every } u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d). \tag{2.2}$$

For convenience we set for every $m \in \mathbb{N}$ the space $H := L^2(\Omega; \mathbb{R}^m)$ and we always identify the dual of H with H itself. Moreover, let $H^N := L^2(\partial_N\Omega; \mathbb{R}^d)$ and $H^D := L^2(\partial_D\Omega; \mathbb{R}^d)$; the symbols $(\cdot, \cdot)$ and $\|\cdot\|$ denote the scalar product and the norm in H Moreover, we define the following spaces

$$V := H^1(\Omega \setminus \Gamma; \mathbb{R}^d) \quad \text{and} \quad V_t := H^1(\Omega \setminus \Gamma_t; \mathbb{R}^d) \quad \text{for every } t \in (-\infty, T].$$

Notice that in the definition of V_t and V, we are considering only the distributional gradient of u in $\Omega \setminus \Gamma_t$ and in $\Omega \setminus \Gamma$, respectively, and not the one in Ω. By means of (2.2), we shall use on the set V_t (and also on the set V) the equivalent norm

$$\|u\|_{V_t} := (\|u\|^2 + \|eu\|^2)^{\frac{1}{2}} \quad \text{for every } u \in V_t.$$

Furthermore, by (2.1), we can consider for every $t \in (-\infty, T]$ the set

$$V_t^D := \{u \in V_t : u = 0 \text{ on } \partial_D\Omega\},$$

which is a closed subspace of V_t.

We assume that the elasticity and viscosity tensors $\mathbb{A}$ and $\mathbb{B}$ satisfy the following assumptions:

$$\mathbb{A}, \mathbb{B} \in L^\infty(\Omega; \mathscr{L}(\mathbb{R}^{d\times d}_{sym}; \mathbb{R}^{d\times d}_{sym})), , \tag{2.3}$$

and for a.e. $x \in \Omega$

$$\mathbb{A}(x)\xi_1 \cdot \xi_2 = \xi_1 \cdot \mathbb{A}(x)\xi_2, \qquad \mathbb{B}(x)\xi_1 \cdot \xi_2 = \xi_1 \cdot \mathbb{B}(x)\xi_2 \qquad \text{for every } \xi_1, \xi_2 \in \mathbb{R}^{d\times d}_{sym}, \tag{2.4}$$

$$c_{\mathbb{A}}|\xi|^2 \le \mathbb{A}(x)\xi \cdot \xi \le C_{\mathbb{A}}|\xi|^2, \quad c_{\mathbb{B}}|\xi|^2 \le \mathbb{B}(x)\xi \cdot \xi \le C_{\mathbb{B}}|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^{d\times d}_{sym}, \tag{2.5}$$

for some positive constants $c_{\mathbb{A}}$, $c_{\mathbb{B}}$, $C_{\mathbb{A}}$, and $C_{\mathbb{B}}$ independent of x, and the dot denotes the Euclidean scalar product of matrices.

Let β a positive real number. We want to study the following viscoelastic dynamic system

$$\ddot{u}(t) - \operatorname{div}((\mathbb{A}+\mathbb{B})eu(t)) + \int_{-\infty}^{t} \frac{1}{\beta} e^{-\frac{t-\tau}{\beta}} \operatorname{div}(\mathbb{B}eu(\tau))\mathrm{d}\tau = f(t) \quad \text{in } \Omega \setminus \Gamma_t,\, t \in (-\infty, T), \tag{2.6}$$

together with the boundary conditions

$$u(t) = z(t) \qquad \text{on } \partial_D\Omega, \quad t \in (-\infty, T), \tag{2.7}$$

$$\Big[(\mathbb{A}+\mathbb{B})\,eu(t) - \int_{-\infty}^{t} \frac{1}{\beta} e^{-\frac{t-\tau}{\beta}} \mathbb{B}eu(\tau)\mathrm{d}\tau\Big]\nu = N(t) \qquad \text{on } \partial_N\Omega, \quad t \in (-\infty, T), \tag{2.8}$$

$$\Big[(\mathbb{A}+\mathbb{B})\,eu(t) - \int_{-\infty}^{t} \frac{1}{\beta} e^{-\frac{t-\tau}{\beta}} \mathbb{B}eu(\tau)\mathrm{d}\tau\Big]\nu = 0 \qquad \text{on } \Gamma_t, \quad t \in (-\infty, T), \tag{2.9}$$

where the data satisfy

(D1) $f \in L^2_{loc}((-\infty; T]; H)$;

(D2) $N \in L^2_{loc}((-\infty; T]; H^N)$ such that $\dot{N} \in L^2_{loc}((-\infty; T]; H^N)$;

(D3) $z \in L^2_{loc}((-\infty;T];V)$ such that $\dot{z} \in L^2_{loc}((-\infty;T];V)$, $\ddot{z} \in L^2_{loc}((-\infty;T];H)$, and $z(t) \in V_t$ for every $t \in (-\infty, T]$.

Notice that in (2.6)–(2.9) the explicit dependence on x is omitted to enlighten notation.

As usual, the Neumann boundary conditions are only formal, and their meaning will be specified in Definition 2.1.1. To this aim, we define $\mathcal{V}_{loc}(-\infty,T)$ as the space of all function $u \in L^2_{loc}((-\infty,T];V)$ such that $\dot{u} \in L^2_{loc}((-\infty,T];H)$, $u(t) \in V_t$ for a.e. $t \in (-\infty,T)$, and

$$\int_{-\infty}^{T} \mathrm{e}^{\frac{t}{\beta}} \|eu(t)\| \mathrm{d}t < +\infty. \tag{2.10}$$

Now we are in position to explain in which sense we mean that $u \in \mathcal{V}_{loc}(-\infty,T)$ is a solution to the viscoelastic dynamic system (2.6)–(2.9). Roughly speaking, we multiply (2.6) by a test function, we integrate by parts in time and in space, and taking into account (2.7)–(2.9) we obtain the following definition.

Definition 2.1.1 (Weak solution)**.** We say that $u \in \mathcal{V}_{loc}(-\infty,T)$ is a *weak solution* to system (2.6) with boundary conditions (2.7)–(2.9) if $u(t) - z(t) \in V_t^D$ for a.e. $t \in (-\infty,T)$, and

$$\begin{aligned}
&-\int_{-\infty}^{T} (\dot{u}(t), \dot{v}(t))\mathrm{d}t + \int_{-\infty}^{T} ((\mathbb{A}+\mathbb{B})eu(t), ev(t))\mathrm{d}t \\
&-\int_{-\infty}^{T}\int_{-\infty}^{t} \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}} (\mathbb{B}eu(\tau), ev(t))\mathrm{d}\tau\mathrm{d}t = \int_{-\infty}^{T} (f(t), v(t))\mathrm{d}t + \int_{-\infty}^{T} (N(t), v(t))_{H^N}\mathrm{d}t
\end{aligned}$$

for every $v \in C_c^\infty(-\infty,T;V)$ such that $v(t) \in V_t^D$ for every $t \in (-\infty,T]$.

Now, let us consider $a, b \in [0,T]$ such that $a < b$. We define the spaces

$$\begin{aligned}
\mathcal{V}(a,b) &:= \{u \in L^2(a,b;V) \cap H^1(a,b;H) : u(t) \in V_t \text{ for a.e. } t \in (a,b)\},\\
\mathcal{V}^D(a,b) &:= \{v \in \mathcal{V}(a,b) : v(t) \in V_t^D \text{ for a.e. } t \in (a,b)\},\\
\mathcal{D}^D(a,b) &:= \{v \in C_c^\infty(a,b;V) : v(t) \in V_t^D \text{ for every } t \in [a,b]\},
\end{aligned}$$

and we have the following lemma.

Lemma 2.1.2. *The space $\mathcal{V}(a,b)$ is a Hilbert space with respect to the following norm*

$$\|\varphi\|_{\mathcal{V}(a,b)} := \left(\|\varphi\|^2_{L^2(a,b;V)} + \|\dot{\varphi}\|^2_{L^2(a,b;H)}\right)^{\frac{1}{2}} \quad \varphi \in \mathcal{V}(a,b).$$

Moreover, $\mathcal{V}^D(a,b)$ is a closed subspace of $\mathcal{V}(a,b)$, and $\mathcal{D}^D(a,b)$ is a dense subset of the space of functions belonging to $\mathcal{V}^D(a,b)$ which vanish on a and b.

Proof. It is clear that $\|\cdot\|_{\mathcal{V}(a,b)}$ is a norm induced by a scalar product on the set $\mathcal{V}(a,b)$. We just have to check the completeness of this space with respect to this norm. Let $\{\varphi_k\}_k \subset \mathcal{V}(a,b)$ be a Cauchy sequence. Then, $\{\varphi_k\}_k$ and $\{\dot{\varphi}_k\}_k$ are Cauchy sequences in $L^2(a,b;V)$ and $L^2(a,b;H)$, respectively, which are complete Hilbert spaces. Thus, there exists $\varphi \in L^2(a,b;V)$ with $\dot{\varphi} \in L^2(a,b;H)$ such that $\varphi_k \to \varphi$ in $L^2(a,b;V)$ and $\dot{\varphi}_k \to \dot{\varphi}$ in $L^2(a,b;H)$. In particular there exists a subsequence $\{\varphi_{k_j}\}_j$ such that $\varphi_{k_j}(t) \to \varphi(t)$ in V for a.e. $t \in (a,b)$. Since $\varphi_{k_j}(t) \in V_t$ for a.e. $t \in (a,b)$ we deduce that $\varphi(t) \in V_t$ for a.e. $t \in (a,b)$. Hence $\varphi \in \mathcal{V}(a,b)$ and $\varphi_k \to \varphi$ in $\mathcal{V}(a,b)$. With a similar argument, we can prove that $\mathcal{V}^D(a,b) \subset \mathcal{V}(a,b)$ is a closed subspace. For the proof of the last statement we refer to [17, Lemma 2.8]. □

Now, suppose we know the past history of the system up to time 0. In particular, let $u_{in} \in \mathcal{V}_{loc}(-\infty,0)$ be a weak solution to (2.6)–(2.9) on the interval $(-\infty,0)$ in the sense of Definition 2.1.1, in such a way that 0 is a Lebesgue's point for both u_{in} and $\dot{u}_{in}$. This implies that there exist $u^0 \in V_0$, with $u^0 - z(0) \in V_0^D$, and $u^1 \in H$ such that

$$\lim_{h\to 0^+} \frac{1}{h}\int_{-h}^0 \|u_{in}(t)-u^0\|_{V_0}^2 \mathrm{d}t = 0, \qquad \lim_{h\to 0^+} \frac{1}{h}\int_{-h}^0 \|\dot{u}_{in}(t)-u^1\|^2 \mathrm{d}t = 0.$$

From this assumption, by defining

$$F_0(t) := \frac{1}{\beta}\mathrm{e}^{-\frac{t}{\beta}}\int_{-\infty}^0 \mathrm{e}^{\frac{\tau}{\beta}}\mathbb{B}eu_{in}(\tau)\mathrm{d}\tau,$$

we can reformulate (2.6)–(2.9) on the interval $[0,T]$ in the following way: for every $t \in [0,T]$

$$\ddot{u}(t) - \mathrm{div}((\mathbb{A}+\mathbb{B})eu(t)) + \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\,\mathrm{div}(\mathbb{B}eu(\tau))\mathrm{d}\tau = f(t) - \mathrm{div}\,F_0(t), \quad \text{in } \Omega\setminus\Gamma_t, \tag{2.11}$$

with boundary and initial conditions

$$u(t) = z(t) \qquad \text{on } \partial_D\Omega, \tag{2.12}$$

$$\Big[(\mathbb{A}+\mathbb{B})\,eu(t) - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau\Big]\nu = N(t) + F_0(t)\nu \quad \text{on } \partial_N\Omega \tag{2.13}$$

$$\Big[(\mathbb{A}+\mathbb{B})\,eu(t) - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau\Big]\nu = F_0(t)\nu \qquad \text{on } \Gamma_t, \tag{2.14}$$

$$u(0) = u^0, \quad \dot{u}(0) = u^1. \tag{2.15}$$

Thanks to (D1)–(D3) and (2.10) (on the interval $(-\infty,0]$), we have $f \in L^2(0,T;H)$, $F_0 \in C^\infty([0,T];H)$, $N \in H^1(0,T;H^N)$, and $z \in H^2(0,T;H)\cap H^1(0,T;V)$ with $z(t) \in V_t$ for every $t \in [0,T]$.

More in general, given $F \in H^1(0,T;H)$ we will study the following viscoelastic dynamic system: for every $t \in [0,T]$

$$\ddot{u}(t) - \mathrm{div}((\mathbb{A}+\mathbb{B})eu(t)) + \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\,\mathrm{div}(\mathbb{B}eu(\tau))\mathrm{d}\tau = f(t) - \mathrm{div}\,F(t), \quad \text{in } \Omega\setminus\Gamma_t, \tag{2.16}$$

with boundary and initial conditions

$$u(t) = z(t) \qquad \text{on } \partial_D\Omega, \tag{2.17}$$

$$\Big[(\mathbb{A}+\mathbb{B})\,eu(t) - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau\Big]\nu = F(t)\nu \qquad \text{on } \partial_N\Omega, \tag{2.18}$$

$$\Big[(\mathbb{A}+\mathbb{B})\,eu(t) - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau\Big]\nu = F(t)\nu \qquad \text{on } \Gamma_t, \tag{2.19}$$

$$u(0) = u^0, \quad \dot{u}(0) = u^1. \tag{2.20}$$

Notice that system (2.11)–(2.15) is a particular case of system (2.16)–(2.20). As we have already specified for system (2.6)–(2.9), also for (2.16)–(2.20) the Neumann boundary conditions are only formal, and their meaning is clarified by the following definition.

Definition 2.1.3. We say that $u \in \mathcal{V}(0,T)$ is a *weak solution* to the viscoelastic dynamic system (2.16)–(2.20) on the interval $[0,T]$ if $u - z \in \mathcal{V}^D(0,T)$,

$$-\int_0^T (\dot{u}(t),\dot{v}(t))\mathrm{d}t + \int_0^T \Big((\mathbb{A}+\mathbb{B})eu(t) - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau, ev(t)\Big)\mathrm{d}t$$

$$= \int_0^T (f(t), v(t))\mathrm{d}t + \int_0^T (F(t), ev(t))\mathrm{d}t \tag{2.21}$$

for every $v \in \mathcal{D}^D(0,T)$, and

$$\lim_{t\to 0^+} \|u(t) - u^0\| = 0, \quad \lim_{t\to 0^+} \|\dot{u}(t) - u^1\|_{(V_0^D)'} = 0. \tag{2.22}$$

Remark 2.1.4. From Lemma 2.1.2, if a function $u \in \mathcal{V}(0,T)$ satisfies (2.21) for every $v \in \mathcal{D}^D(0,T)$, then it satisfies the same equality for every $v \in \mathcal{V}^D(0,T)$ such that $v(0) = v(T) = 0$.

2.2 Existence by using Dafermos' method

In this section we present an existence result which is to be considered in the framework of functional analysis; in particular it derives from an idea of C. Dafermos (see [11]) based on a generalization of Lax-Milgram's Theorem by J.L. Lions (see [33]). We start by stating the main result of this section.

Theorem 2.2.1. *There exists a weak solution $u \in \mathcal{V}(0,T)$ to the viscoelastic dynamic system (2.16)–(2.20) on the interval $[0,T]$ in the sense of Definition 2.1.3. Moreover, there exists a positive constant $C = C(T, \mathbb{A}, \mathbb{B}, \beta)$ such that*

$$\|u\|_{\mathcal{V}(0,T)} \le C \left(\|f\|_{L^2(0,T;H)} + \|F\|_{H^1(0,T;H)} + \|\ddot{z}\|_{L^2(0,T;H)} + \|z\|_{H^1(0,T;V)} + \|u^0\|_V + \|u^1\|\right). \tag{2.23}$$

Remark 2.2.2. Without loss of generality we may assume that the Dirichlet datum and the initial displacement are identically equal to zero. Indeed, the function u is a weak solution to the viscoelastic dynamic system (2.16)–(2.20) according to Definition 2.1.3 if and only if the function u^*, defined by $u^*(t) := u(t) - u^0 + z(0) - z(t)$, satisfies

$$- \int_0^T (\dot{u}^*(t), \dot{\psi}(t))\mathrm{d}t + \int_0^T ((\mathbb{A}+\mathbb{B})eu^*(t), e\psi(t))\mathrm{d}t$$
$$- \int_0^T \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}} (\mathbb{B}eu^*(\tau), e\psi(t))\mathrm{d}\tau\mathrm{d}t = \int_0^T (f^*(t), \psi(t))\mathrm{d}t + \int_0^T (F^*(t), e\psi(t))\mathrm{d}t,$$

for every $\psi \in \mathcal{D}^D(0,T)$, and

$$\lim_{t\to 0^+} \|u^*(t)\| = 0, \quad \lim_{t\to 0^+} \|\dot{u}^*(t) - u^1_*\|_{(V_0^D)'} = 0,$$

where $f^* := f - \ddot{z}$, $u^1_* := u^1 - \dot{z}(0)$, and for every $t \in [0,T]$

$$F^*(t) := F(t) + \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}ez(\tau)\mathrm{d}\tau - (\mathbb{A}+\mathbb{B})ez(t) - (\mathbb{A} + \mathrm{e}^{-\frac{t}{\beta}}\mathbb{B})(eu^0 - ez(0)).$$

Moreover, if u^* satisfies for some positive constants C^* the following estimate

$$\|u^*\|_{\mathcal{V}(0,T)} \le C^* \left(\|f^*\|_{L^2(0,T;H)} + \|F^*\|_{H^1(0,T;H)} + \|u^1_*\|\right), \tag{2.24}$$

then u satisfies (2.23). Indeed, since

$$\|f^*\|_{L^2(0,T;H)} \le \|f\|_{L^2(0,T;H)} + \|\ddot{z}\|_{L^2(0,T;H)},$$

and for some positive constants $\bar{C} = C(T, \mathbb{A}, \mathbb{B}, \beta)$ we have

$$\|F^*\|_{H^1(0,T;H)} \le \|F\|_{H^1(0,T;H)} + \left(1 + \frac{2^{\frac{1}{2}}}{\beta}\right)\left\| \int_0^{\cdot} \frac{1}{\beta}\mathrm{e}^{-\frac{\cdot-\tau}{\beta}}\mathbb{B}ez(\tau)\mathrm{d}\tau\right\|_{L^2(0,T;H)}$$

$$+ \frac{2^{\frac{1}{2}}}{\beta} C_{\mathbb{B}} \|z\|_{L^2(0,T;V)} + (C_{\mathbb{A}} + C_{\mathbb{B}}) \|z\|_{H^1(0,T;V)}$$
$$+ (C_{\mathbb{A}} + \|\mathrm{e}^{-\frac{\cdot}{\beta}}\|_{H^1(0,T)} C_{\mathbb{B}})(\|u^0\|_V + \|z(0)\|_V)$$
$$\leq \bar{C}(\|F\|_{H^1(0,T;H)} + \|z\|_{H^1(0,T;V)} + \|u^0\|_V),$$

from (2.24) we deduce

$$\begin{aligned} \|u\|_{\mathcal{V}(0,T)} &\leq \|u^*\|_{\mathcal{V}(0,T)} + T^{\frac{1}{2}} (\|u^0\|_V + \|z(0)\|_V) + \|z\|_{\mathcal{V}(0,T)} \\ &\leq C \left(\|f\|_{L^2(0,T;H)} + \|F\|_{H^1(0,T;H)} + \|\ddot{z}\|_{L^2(0,T;H)} + \|z\|_{H^1(0,T;V)} + \|u^0\|_V + \|u^1\| \right), \end{aligned}$$

where $C = C(T, \mathbb{A}, \mathbb{B}, \beta)$ is a positive constant.

Based on Remark 2.2.2, we now assume that the Dirichlet datum and the initial displacement are identically equal to zero. To prove the theorem in this case, we first prove that our weak formulation (2.21) with initial conditions (2.22) is equivalent to another one, which we call Dafermos' Equality. After that, by means of a Lions' theorem we prove that there exists an element which satisfies this equality. Namely, by defining for every $a, b \in [0, T]$ such that $a < b$ the space

$$\mathcal{E}_0^D(a, b) := \{\varphi \in C^\infty([a, b]; V) : \varphi(a) = 0,\ \varphi(t) \in V_t^D \text{ for every } t \in [a, b]\},$$

we can state the following equivalence result.

Proposition 2.2.3. *Suppose that there exists $u \in \mathcal{V}^D(0, T)$ which satisfies the initial condition $u(0) = 0$ in the sense of (2.22), and such that Dafermos' Equality holds:*

$$\begin{aligned} &\int_0^T (\dot{u}(t), \dot{\varphi}(t)) \mathrm{d}t + \int_0^T (t - T) \Big[(\dot{u}(t), \ddot{\varphi}(t)) - ((\mathbb{A} + \mathbb{B}) eu(t), e\dot{\varphi}(t)) \Big] \mathrm{d}t \\ &+ \int_0^T (t - T) \int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} (\mathbb{B} eu(\tau), e\dot{\varphi}(t)) \mathrm{d}\tau \mathrm{d}t = T(u^1, \dot{\varphi}(0)) \\ &- \int_0^T (t - T) \left[(f(t), \dot{\varphi}(t)) + (F(t), e\dot{\varphi}(t)) \right] \mathrm{d}t \quad \textit{for every } \varphi \in \mathcal{E}_0^D(0, T). \end{aligned} \tag{2.25}$$

Then u satisfies (2.21), $u(0) = 0$ and $\dot{u}(0)$ coincides with u^1 in $(V_0^D)'$. Moreover, if $u \in \mathcal{V}^D(0, T)$ is a weak solution in the sense of Definition 2.1.3, then it satisfies (2.25).

At this point, we state and prove some lemmas and propositions needed for the proof of Proposition 2.2.3. In particular, in the following lemma, we highlight a useful relation between $\mathcal{D}^D(0, T)$ and $\mathcal{E}_0^D(0, T)$.

Lemma 2.2.4. *For every $v \in \mathcal{D}^D(0, T)$ the function defined by*

$$\varphi_v(t) = \int_0^t \frac{v(\tau)}{\tau - T} \mathrm{d}\tau$$

is well defined and satisfies $\varphi_v \in \mathcal{E}_0^D(0, T)$.

Proof. Firstly, we can notice that φ_v is well defined because v is a function with compact support, hence it vanishes in a neighborhood of T. Moreover, $\varphi_v(0) = 0$ by definition and $\varphi_v \in C^\infty([0, T]; V)$ because it is a primitive of a function with the same regularity. Now, we can observe that $v(\tau) \in V_\tau^D \subset V_t^D$ for every $\tau \leq t$, therefore we have $\frac{v(\tau)}{\tau - T} \in V_t^D$ for every $\tau \leq t$, and by the properties of Bochner's integral we get $\varphi_v(t) \in V_t^D$. □

In the next proposition we show that the distributional second derivative in time of a weak solution is an element of the space $L^2(0,T;(V_0^D)')$. Therefore, such a solution has an initial velocity in the space $(V_0^D)'$.

Proposition 2.2.5. *Let $u \in \mathcal{V}^D(0,T)$ be a function which satisfies (2.21). Then the distributional derivative of $\dot{u}$ belongs to the space $L^2(0,T;(V_0^D)')$.*

Proof. Let $\Lambda \in L^2(0,T;(V_0^D)')$ be defined in the following way: for a.e. $t \in (0,T)$ and for every $\mathbf{v} \in V_0^D$

$$\langle \Lambda(t), \mathbf{v}\rangle := -((\mathbb{A}+\mathbb{B})eu(t), e\mathbf{v}) + \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), e\mathbf{v})\mathrm{d}\tau + (f(t),\mathbf{v}) + (F(t), e\mathbf{v}) \quad (2.26)$$

where $\langle\cdot,\cdot\rangle$ represents the duality product between $(V_0^D)'$ and V_0^D.
Let us consider a test function $\varphi \in C_c^\infty(0,T)$, then for every $\mathbf{v} \in V_0^D$ the function $\psi(t) := \varphi(t)\mathbf{v}$ belongs to the space $C_c^\infty(0,T;V_0)$, and consequently $\psi \in \mathcal{D}^D(0,T)$. Now we multiply both sides of (2.26) by $\varphi(t)$ and we integrate it on $(0,T)$. Thanks to (2.21) we can write

$$\begin{aligned}\int_0^T \langle \Lambda(t), \mathbf{v}\rangle\varphi(t)\mathrm{d}t = &-\int_0^T ((\mathbb{A}+\mathbb{B})eu(t), e\psi(t))\mathrm{d}t + \int_0^T\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), e\psi(t))\mathrm{d}\tau\mathrm{d}t\\ &+\int_0^T (f(t),\psi(t))\mathrm{d}t + \int_0^T (F(t), e\psi(t))\mathrm{d}t = -\int_0^T (\dot{u}(t),\mathbf{v})\dot{\varphi}(t)\mathrm{d}t,\end{aligned}$$

which implies

$$\langle\int_0^T \Lambda(t)\varphi(t)\mathrm{d}t, \mathbf{v}\rangle = \langle -\int_0^T \dot{u}(t)\dot{\varphi}(t)\mathrm{d}t, \mathbf{v}\rangle \qquad \text{for every } \mathbf{v} \in V_0^D.$$

Hence, we get

$$\int_0^T \Lambda(t)\varphi(t)\mathrm{d}t = -\int_0^T \dot{u}(t)\dot{\varphi}(t)\mathrm{d}t \qquad \text{for every } \varphi \in C_c^\infty(0,T)$$

as elements of $(V_0^D)'$, which concludes the proof. □

Remark 2.2.6. Proposition 2.2.5 implies that $\dot{u} \in H^1(0,T;(V_0^D)')$, hence it admits a continuous representative. Therefore, we can say that there exists $\dot{u}(0) \in (V_0^D)'$ such that

$$\lim_{t\to 0^+} \|\dot{u}(t) - \dot{u}(0)\|_{(V_0^D)'} = 0. \quad (2.27)$$

In the next proposition we show how the weak formulation (2.21) changes if we use test functions which do not vanish at zero. In particular, we use the notation $\eta(T)$ to refer to the family of open neighborhoods of T, and we consider the following spaces

$$\begin{aligned}\mathcal{L}ip^D(0,T) &:= \{\psi \in Lip([0,T];V) : \psi(t) \in V_t^D \text{ for every } t \in [0,T]\},\\ \mathcal{L}ip^D_{0,T}(0,T) &:= \{\psi \in \mathcal{L}ip^D(0,T) : \exists I_\psi \in \eta(T), \text{ s.t. } \psi(t) = 0 \quad \text{for every } t \in I_\psi \cup \{0\}\},\\ \mathcal{L}ip^D_T(0,T) &:= \{\Psi \in \mathcal{L}ip^D(0,T) : \Psi(T) = 0\}.\end{aligned}$$

Proposition 2.2.7. *Let $u \in \mathcal{V}^D(0,T)$ be a function which satisfies (2.21) for every $\psi \in \mathcal{L}ip^D_{0,T}(0,T)$. Then u satisfies the equality*

$$\begin{aligned}-\int_0^T (\dot{u}(t),\dot{\Psi}(t))\mathrm{d}t + \int_0^T ((\mathbb{A}+\mathbb{B})eu(t), e\Psi(t))\mathrm{d}t - \int_0^T\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), e\Psi(t))\mathrm{d}\tau\mathrm{d}t\\ = \int_0^T (f(t),\Psi(t))\mathrm{d}t + \int_0^T (F(t), e\Psi(t))\mathrm{d}t + \langle \dot{u}(0), \Psi(0)\rangle,\end{aligned} \quad (2.28)$$

for every $\Psi \in \mathcal{L}ip^D_T(0,T)$.

Proof. Let us consider $\Psi \in \mathcal{L}ip^D_T(0,T)$ and define for every $\varepsilon \in (0, \frac{T}{3})$ the function

$$\psi_\varepsilon(t) := \begin{cases} \frac{t}{\varepsilon}\Psi(0) & t \in [0,\varepsilon] \\ \Psi(t-\varepsilon) & t \in [\varepsilon, T-2\varepsilon] \\ \left(-\frac{t}{\varepsilon} + \frac{T-\varepsilon}{\varepsilon}\right)\Psi(T-3\varepsilon) & t \in [T-2\varepsilon, T-\varepsilon] \\ 0 & t \in [T-\varepsilon, T]. \end{cases}$$

It is easy to see that $\psi_\varepsilon \in \mathcal{L}ip^D_{0,T}(0,T)$, and by using ψ_ε as test function in (2.21) we get $I_\varepsilon + I^m_\varepsilon + J^m_\varepsilon = 0$, where the three terms I_ε, I^m_ε, and J^m_ε are defined in the following way:

$$\begin{aligned} I_\varepsilon := &-\int_\varepsilon^{T-2\varepsilon} (\dot u(t), \dot\Psi(t-\varepsilon))\mathrm{d}t + \int_\varepsilon^{T-2\varepsilon} ((\mathbb{A}+\mathbb{B})eu(t), e\Psi(t-\varepsilon))\mathrm{d}t \\ &- \int_\varepsilon^{T-2\varepsilon}\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), e\Psi(t-\varepsilon))\mathrm{d}\tau\mathrm{d}t \\ &- \int_\varepsilon^{T-2\varepsilon} (f(t), \Psi(t-\varepsilon))\mathrm{d}t - \int_\varepsilon^{T-2\varepsilon} (F(t), e\Psi(t-\varepsilon))\mathrm{d}t, \end{aligned}$$

$$\begin{aligned} I^m_\varepsilon := &-⨍_0^\varepsilon (\dot u(t), \Psi(0))\mathrm{d}t + ⨍_0^\varepsilon ((\mathbb{A}+\mathbb{B})eu(t), te\Psi(0))\mathrm{d}t \\ &- ⨍_0^\varepsilon \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), te\Psi(0))\mathrm{d}\tau\mathrm{d}t - ⨍_0^\varepsilon (f(t), t\Psi(0))\mathrm{d}t - ⨍_0^\varepsilon (F(t), te\Psi(0))\mathrm{d}t, \end{aligned}$$

and

$$\begin{aligned} J^m_\varepsilon :&= ⨍_{T-2\varepsilon}^{T-\varepsilon} (\dot u(t), \Psi(T-3\varepsilon))\mathrm{d}t + ⨍_{T-2\varepsilon}^{T-\varepsilon} ((\mathbb{A}+\mathbb{B})eu(t), (-t+T-\varepsilon)e\Psi(T-3\varepsilon))\mathrm{d}t \\ &- ⨍_{T-2\varepsilon}^{T-\varepsilon}\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), (-t+T-\varepsilon)e\Psi(T-3\varepsilon))\mathrm{d}\tau\mathrm{d}t \\ &- ⨍_{T-2\varepsilon}^{T-\varepsilon} (f(t), (-t+T-\varepsilon)\Psi(T-3\varepsilon))\mathrm{d}t - ⨍_{T-2\varepsilon}^{T-\varepsilon} (F(t), (-t+T-\varepsilon)e\Psi(T-3\varepsilon))\mathrm{d}t. \end{aligned}$$

Let us study the convergence of I_ε, I^m_ε, and J^m_ε as $\varepsilon \to 0^+$. First of all, we notice that from the definition of ψ_ε and the Lipschitz continuity of Ψ we have

$$\begin{aligned} \|\psi_\varepsilon - \Psi\|^2_{L^2(0,T;V)} &= \int_0^\varepsilon \Big\|\frac{t}{\varepsilon}\Psi(0) - \Psi(t)\Big\|^2_V \mathrm{d}t + \int_\varepsilon^{T-2\varepsilon} \|\Psi(t-\varepsilon) - \Psi(t)\|^2_V \mathrm{d}t \\ &\quad + \int_{T-2\varepsilon}^{T-\varepsilon} \Big\|\Big(-\frac{t}{\varepsilon} + \frac{T-\varepsilon}{\varepsilon}\Big)\Psi(T-3\varepsilon) - \Psi(t)\Big\|^2_V \mathrm{d}t \\ &\le 2\|\Psi(0)\|^2_V \int_0^\varepsilon \frac{t^2}{\varepsilon^2}\mathrm{d}t + 2\int_0^\varepsilon \|\Psi(t)\|^2_V\mathrm{d}t + \int_\varepsilon^{T-2\varepsilon} L^2_\Psi |t-\varepsilon-t|^2\mathrm{d}t \\ &\quad + 2\|\Psi(T-3\varepsilon)\|^2_V \int_{T-2\varepsilon}^{T-\varepsilon}\Big(-\frac{t}{\varepsilon} + \frac{T-\varepsilon}{\varepsilon}\Big)^2\mathrm{d}t + 2\int_{T-2\varepsilon}^{T-\varepsilon} \|\Psi(t)\|^2_V\mathrm{d}t \\ &\le \frac{4}{3}\varepsilon\|\Psi\|^2_{L^\infty(0,T;V)} + 2\int_0^\varepsilon \|\Psi(t)\|^2_V\mathrm{d}t \\ &\quad + 2\int_{T-2\varepsilon}^{T-\varepsilon} \|\Psi(t)\|^2_V\mathrm{d}t + L^2_\Psi\varepsilon^2(T-3\varepsilon) \xrightarrow[\varepsilon\to 0^+]{} 0. \end{aligned} \tag{2.29}$$

From (2.3), (2.29), and to the absolute continuity of Lebesgue's integral, we have

$$\Big|\int_\varepsilon^{T-2\varepsilon} ((\mathbb{A}+\mathbb{B})eu(t), e\Psi(t-\varepsilon))\mathrm{d}t - \int_0^T ((\mathbb{A}+\mathbb{B})eu(t), e\Psi(t))\mathrm{d}t\Big|$$

$$\leq \Big| \int_0^\varepsilon ((\mathbb{A}+\mathbb{B})eu(t), e\Psi(t))\mathrm{d}t \Big| + \Big| \int_{T-2\varepsilon}^T ((\mathbb{A}+\mathbb{B})eu(t), e\Psi(t))\mathrm{d}t \Big|$$
$$+ \Big| \int_\varepsilon^{T-2\varepsilon} ((\mathbb{A}+\mathbb{B})eu(t), e\Psi(t-\varepsilon) - e\Psi(t))\mathrm{d}t \Big|$$
$$\leq (C_\mathbb{A} + C_\mathbb{B}) \Big[\int_0^\varepsilon \|u(t)\|_V \|\Psi(t)\|_V \mathrm{d}t + \int_{T-2\varepsilon}^T \|u(t)\|_V \|\Psi(t))\|_V \mathrm{d}t \Big]$$
$$+ (C_\mathbb{A} + C_\mathbb{B}) \Big[\|u\|_{L^2(0,T;V)} \|\psi_\varepsilon - \Psi\|_{L^2(0,T;V)} \Big] \xrightarrow[\varepsilon\to 0^+]{} 0. \tag{2.30}$$

In the same way we can prove that

$$\int_\varepsilon^{T-2\varepsilon} \int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} (\mathbb{B}eu(\tau), e\Psi(t-\varepsilon))\mathrm{d}\tau\mathrm{d}t \xrightarrow[\varepsilon\to 0^+]{} \int_0^T \int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} (\mathbb{B}eu(\tau), e\Psi(t))\mathrm{d}\tau\mathrm{d}t, \tag{2.31}$$
$$\int_\varepsilon^{T-2\varepsilon} (f(t), \Psi(t-\varepsilon))\mathrm{d}t \xrightarrow[\varepsilon\to 0^+]{} \int_0^T (f(t), \Psi(t))\mathrm{d}t, \tag{2.32}$$
$$\int_\varepsilon^{T-2\varepsilon} (F(t), e\Psi(t-\varepsilon))\mathrm{d}t \xrightarrow[\varepsilon\to 0^+]{} \int_0^T (F(t), e\Psi(t))\mathrm{d}t. \tag{2.33}$$

Notice that, by virtue of the continuity of the translation operator in L^2, and again by the absolute continuity of Lebesgue's integral, we can write

$$\Big| \int_\varepsilon^{T-2\varepsilon} (\dot{u}(t), \dot{\Psi}(t-\varepsilon))\mathrm{d}t - \int_0^T (\dot{u}(t), \dot{\Psi}(t))\mathrm{d}t \Big|$$
$$\leq \Big| \int_0^\varepsilon (\dot{u}(t), \dot{\Psi}(t))\mathrm{d}t \Big| + \Big| \int_\varepsilon^{T-2\varepsilon} (\dot{u}(t), \dot{\Psi}(t-\varepsilon) - \dot{\Psi}(t))\mathrm{d}t \Big| + \Big| \int_{T-2\varepsilon}^T (\dot{u}(t), \dot{\Psi}(t))\mathrm{d}t \Big|$$
$$\leq \int_0^\varepsilon \|\dot{u}(t)\| \|\dot{\Psi}(t)\|\mathrm{d}t + \|\dot{u}\|_{L^2(0,T;H)} \|\dot{\Psi}(\cdot - \varepsilon) - \dot{\Psi}(\cdot)\|_{L^2(0,T;H)}$$
$$+ \int_{T-2\varepsilon}^T \|\dot{u}(t)\| \|\dot{\Psi}(t))\|\mathrm{d}t \xrightarrow[\varepsilon\to 0^+]{} 0. \tag{2.34}$$

Taking into account (2.30)–(2.34) we conclude that

$$I_\varepsilon \xrightarrow[\varepsilon\to 0^+]{} - \int_0^T (\dot{u}(t), \dot{\Psi}(t))\mathrm{d}t + \int_0^T ((\mathbb{A}+\mathbb{B})eu(t), e\Psi(t))\mathrm{d}t$$
$$- \int_0^T \int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} (\mathbb{B}eu(\tau), e\Psi(t))\mathrm{d}\tau\mathrm{d}t - \int_0^T (f(t), \Psi(t))\mathrm{d}t - \int_0^T (F(t), e\Psi(t))\mathrm{d}t.$$

Now we analyze the limit of I_ε^m as $\varepsilon \to 0^+$. By (2.27) we obtain

$$⨍_0^\varepsilon (\dot{u}(t), \Psi(0))\mathrm{d}t = (⨍_0^\varepsilon \dot{u}(t)\mathrm{d}t, \Psi(0)) = \langle ⨍_0^\varepsilon \dot{u}(t)\mathrm{d}t, \Psi(0) \rangle \xrightarrow[\varepsilon\to 0^+]{} \langle \dot{u}(0), \Psi(0) \rangle. \tag{2.35}$$

Moreover

$$\Big| ⨍_0^\varepsilon ((\mathbb{A}+\mathbb{B})eu(t), te\Psi(0))\mathrm{d}t \Big| \leq (C_\mathbb{A} + C_\mathbb{B}) \|\Psi(0)\|_V ⨍_0^\varepsilon t\|u(t)\|_V \mathrm{d}t$$
$$\leq (C_\mathbb{A} + C_\mathbb{B}) \|\Psi\|_{L^\infty(0,T;V)} \Big(\frac{\varepsilon}{3}\Big)^{\frac{1}{2}} \|u\|_{L^2(0,T;V)} \xrightarrow[\varepsilon\to 0^+]{} 0. \tag{2.36}$$

In the same way, we can prove that

$$⨍_0^\varepsilon \int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} (\mathbb{B}eu(\tau), te\Psi(0))\mathrm{d}\tau\mathrm{d}t \xrightarrow[\varepsilon\to 0^+]{} 0, \tag{2.37}$$

$$\fint_0^{\varepsilon}(f(t),t\Psi(0))\mathrm{d}t \xrightarrow[\varepsilon\to0^+]{} 0, \tag{2.38}$$

$$\fint_0^{\varepsilon}(F(t),te\Psi(0))\mathrm{d}t \xrightarrow[\varepsilon\to0^+]{} 0, \tag{2.39}$$

hence, by (2.35)–(2.39) we obtain $I_\varepsilon^m \xrightarrow[\varepsilon\to0^+]{} -\langle\dot u(0),\Psi(0)\rangle$.

Finally, we study the behaviour of J_ε^m as $\varepsilon\to0^+$. Since $\Psi(T)=0$, we can write

$$\begin{aligned}&\Big|\fint_{T-2\varepsilon}^{T-\varepsilon}(\dot u(t),\Psi(T-3\varepsilon))\mathrm{d}t\Big|\\ &\qquad\le \frac{1}{\varepsilon^{\frac12}}\|\dot u\|_{L^2(0,T;H)}\|\Psi(T-3\varepsilon)-\Psi(T)\| \le 3L_\Psi\|\dot u\|_{L^2(0,T;H)}\varepsilon^{\frac12}\xrightarrow[\varepsilon\to0^+]{} 0.\end{aligned} \tag{2.40}$$

Moreover

$$\begin{aligned}&\Big|\fint_{T-2\varepsilon}^{T-\varepsilon}((\mathbb{A}+\mathbb{B})eu(t),(-t+T-\varepsilon)e\Psi(T-3\varepsilon))\mathrm{d}t\Big|\\ &\qquad\le (C_{\mathbb{A}}+C_{\mathbb{B}})\|\Psi(T-3\varepsilon)\|_V\Big(\fint_{T-2\varepsilon}^{T-\varepsilon}(T-t)\|u(t)\|_V\mathrm{d}t+\int_{T-2\varepsilon}^{T-\varepsilon}\|u(t)\|_V\mathrm{d}t\Big)\\ &\qquad\le (C_{\mathbb{A}}+C_{\mathbb{B}})\|\Psi\|_{L^\infty(0,T;V)}\Big(\Big(\frac{7}{3}\Big)^{\frac12}+1\Big)\varepsilon^{\frac12}\|u\|_{L^2(0,T;V)}\xrightarrow[\varepsilon\to0^+]{} 0.\end{aligned} \tag{2.41}$$

By following the same strategy used in (2.41), we can prove that

$$\fint_{T-2\varepsilon}^{T-\varepsilon}\int_0^t\frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau),(-t+T-\varepsilon)e\Psi(T-3\varepsilon))\mathrm{d}\tau\mathrm{d}t\xrightarrow[\varepsilon\to0^+]{} 0, \tag{2.42}$$

$$\fint_{T-2\varepsilon}^{T-\varepsilon}(f(t),(-t+T-\varepsilon)\Psi(T-3\varepsilon))\mathrm{d}t\xrightarrow[\varepsilon\to0^+]{} 0, \tag{2.43}$$

$$\fint_{T-2\varepsilon}^{T-\varepsilon}(F(t),(-t+T-\varepsilon)e\Psi(T-3\varepsilon))\mathrm{d}t\xrightarrow[\varepsilon\to0^+]{} 0. \tag{2.44}$$

Thanks to (2.40)–(2.44) we can say that $J_\varepsilon^m\to0$ as $\varepsilon\to0^+$, and this concludes the proof. □

We are now in position to prove the equivalence result between the viscoelastic dynamic system (2.16)–(2.20) and Dafermos' Equality (2.25), stated in Proposition 2.2.3.

Proof of Proposition 2.2.3. Let $u\in\mathcal{V}^D(0,T)$ be a function with $u(0)=0$, and which satisfies (2.25). Let us consider $v\in\mathcal{D}^D(0,T)$. By Lemma 2.2.4, the function defined by

$$\varphi_v(t)=\int_0^t\frac{v(\tau)}{\tau-T}\mathrm{d}\tau \tag{2.45}$$

is well defined and belongs to the space $\mathcal{E}_0^D(0,T)$. By taking φ_v as a test function in (2.25) we obtain

$$\begin{aligned}&-\int_0^T(\dot u(t),\dot\varphi_v(t)+(t-T)\ddot\varphi_v(t))\mathrm{d}t+\int_0^T((\mathbb{A}+\mathbb{B})eu(t),e((t-T)\dot\varphi_v(t)))\mathrm{d}t\\ &-\int_0^T\int_0^t\frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau,e((t-T)\dot\varphi_v(t)))\mathrm{d}t\\ &=\int_0^T(f(t),(t-T)\dot\varphi_v(t))\mathrm{d}t+\int_0^T(F(t),e((t-T)\dot\varphi_v(t)))\mathrm{d}t,\end{aligned} \tag{2.46}$$

since $\dot{\varphi}_v(0) = \frac{v(0)}{-T} = 0$. Notice that $v(t) = (t-T)\dot{\varphi}_v(t)$ and consequently $\dot{v}(t) = \dot{\varphi}_v(t) + (t-T)\ddot{\varphi}_v(t)$, by the definition of φ_v itself. This, together with (2.46), allows us to conclude that $u \in \mathcal{V}^D(0,T)$ satisfies (2.21) for every $v \in \mathcal{D}^D(0,T)$.

Now we prove that u^1 coincides with $\dot{u}(0)$. Since the function u satisfies (2.21) for every $v \in \mathcal{D}^D(0,T)$, in particular, from Remark 2.1.4, it satisfies the same equality for every $v \in \mathcal{L}ip^D_{0,T}(0,T)$. Thanks to Proposition 2.2.7, the function u satisfies (2.28) for every $v \in \mathcal{L}ip^D_T(0,T)$, and therefore, by defining

$$\mathcal{E}_T(0,T) := \{v \in C^\infty([0,T];V) : \exists I_v \in \eta(T), \text{ s.t. } v(t) = 0 \text{ for every } t \in I_v\},$$

it satisfies (2.28) for every function in the space

$$\mathcal{E}^D_T(0,T) := \{v \in \mathcal{E}_T(0,T) : v(t) \in V^D_t \text{ for every } t \in [0,T]\}.$$

Moreover, if we define φ_v as in (2.45) we have $\varphi_v \in \mathcal{E}^D_0(0,T)$, and we can use it as a test function in (2.25) to deduce

$$\begin{aligned} -\int_0^T (\dot{u}(t), \dot{v}(t))\mathrm{d}t + \int_0^T ((\mathbb{A}+\mathbb{B})eu(t), ev(t))\mathrm{d}t - \int_0^T \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), ev(t))\mathrm{d}\tau\mathrm{d}t \\ = \int_0^T (f(t), v(t))\mathrm{d}t + \int_0^T (F(t), ev(t))\mathrm{d}t + (u^1, v(0)). \quad (2.47) \end{aligned}$$

By taking the difference between (2.28) and (2.47) we get $\langle u^1 - \dot{u}(0), v(0)\rangle = 0$ for every $v \in \mathcal{E}^D_T(0,T)$. Since for every $\mathbf{v} \in V^D_0$ there exists a function $v \in \mathcal{E}^D_T(0,T)$ such that $v(0) = \mathbf{v}$, we can obtain that $\langle u^1 - \dot{u}(0), \mathbf{v}\rangle = 0$ for every $\mathbf{v} \in V^D_0$, and so $u^1 - \dot{u}(0) = 0$ as element of $(V^D_0)'$. This proves the first part of the proposition.

Vice versa, let $u \in \mathcal{V}^D(0,T)$ be a weak solution in the sense of Definition 2.1.3. Therefore, u satisfies (2.21) for every $v \in \mathcal{D}^D(0,T)$, and as we have already shown before, u satisfies (2.28), with u^1 in place of $\dot{u}(0)$, for every function $v \in \mathcal{L}ip^D_T(0,T)$. Let us consider $\varphi \in \mathcal{E}^D_0(0,T)$, then $v_\varphi(t) = (t-T)\dot{\varphi}(t) \in \mathcal{L}ip^D_T(0,T)$, and so it can be used as a test function in (2.28). By noticing that $\dot{v}_\varphi(t) = \dot{\varphi}(t) + (t-T)\ddot{\varphi}(t)$ and $v_\varphi(0) = -T\dot{\varphi}(0)$ we obtain the thesis. □

In view of the previous proposition, it will be enough to prove the existence of a solution to Dafermos' Equality (2.25). In particular, we shall prove the existence of $t_0 \in (0,T]$ and of a function $u \in \mathcal{V}^D(0,t_0)$ such that $u(0) = 0$, and which satisfies Dafermos' Equality on the interval $[0,t_0]$. In order to do this, we use an abstract result due to Lions (see [33, Chapter 3, Theorem 1.1 and Remark 1.2]). We first introduce the necessary setting. Let X be a Hilbert space and $Y \subset X$ be a linear subspace, endowed with the scalar product $(\cdot,\cdot)_Y$ which makes it a pre-Hilbert space. Suppose that the inclusion of Y in X is a continuous map, i.e., there exists a positive constant C such that

$$\|u\|_X \le C\|u\|_Y \quad \text{for every } u \in Y. \quad (2.48)$$

Let us consider a bilinear form $B : X \times Y \to \mathbb{R}$ such that

$$B(\cdot,\varphi) : X \to \mathbb{R} \quad \text{is a linear continuous function on } X \text{ for every } \varphi \in Y, \quad (2.49)$$

$$B(\varphi,\varphi) \ge \alpha\|\varphi\|^2_Y \quad \text{for every } \varphi \in Y, \text{ for some positive constant } \alpha. \quad (2.50)$$

Now, we can state the aforementioned existence theorem.

Theorem 2.2.8 (J.L. Lions)**.** *Suppose that hypotheses (2.48)–(2.50) are satisfied, and let $L : Y \to \mathbb{R}$ be a linear continuous map. Then there exists $u \in X$ such that*

$$B(u,\varphi) = L(\varphi) \qquad \textit{for every } \varphi \in Y.$$

Moreover, the solution u satisfies

$$\|u\|_X \leq \frac{C}{\alpha} \sup\{|L(\varphi)| : \|\varphi\|_Y = 1\}. \tag{2.51}$$

After defining for every $a, b \in [0, T]$ with $a < b$ the space

$$\mathcal{V}_0^D(a, b) := \{u \in \mathcal{V}^D(a, b) : u(a) = 0\},$$

we can state the following proposition.

Proposition 2.2.9. *There exists $t_0 \in (0, T]$ and a function $u \in \mathcal{V}_0^D(0, t_0)$ which satisfies Dafermos' Equality (2.25) on the interval $[0, t_0]$ for every $\varphi \in \mathcal{E}_0^D(0, t_0)$. Moreover, there exists a positive constant $C_0 = C_0(t_0, \mathbb{A})$ such that*

$$\|u\|_{\mathcal{V}(0,t_0)} \leq C_0 \left(\|f\|_{L^2(0,t_0;H)} + \|F\|_{H^1(0,t_0;H)} + \|u^1\|\right). \tag{2.52}$$

Proof. We fix $t_0 \in (0, T]$ such that

$$\begin{cases} t_0 < \frac{1}{2c_{\mathbb{A}}} & \text{if } \frac{1}{2c_{\mathbb{A}}} < T \\ t_0 = T & \text{otherwise.} \end{cases} \tag{2.53}$$

For simplicity of notation, we denote the spaces $\mathcal{V}_0^D(0, t_0)$ and $\mathcal{E}_0^D(0, t_0)$ with the symbols $\mathcal{V}_{t_0}$ and $\mathcal{E}_{t_0}$, respectively. On the space $\mathcal{V}_{t_0}$ we take the usual scalar product, instead on the space $\mathcal{E}_{t_0}$ we consider the following one

$$(\phi, \varphi)_{\mathcal{E}_{t_0}} := \int_0^{t_0} [(\dot\phi(t), \dot\varphi(t)) + (\phi(t), \varphi(t))_V]\mathrm{d}t + t_0(\dot\phi(0), \dot\varphi(0)) \qquad \text{for every } \phi, \varphi \in \mathcal{E}_{t_0},$$

and we denote by $\|\cdot\|_{\mathcal{E}_{t_0}}$ the norm associated.

Let us consider the bilinear form $B : \mathcal{V}_{t_0} \times \mathcal{E}_{t_0} \to \mathbb{R}$ defined by

$$\begin{aligned} B(u, \varphi) := &\int_0^{t_0} [(\dot u(t), \dot\varphi(t)) + (t - t_0)(\dot u(t), \ddot\varphi(t))]\mathrm{d}t \\ &- \int_0^{t_0} (t - t_0)((\mathbb{A} + \mathbb{B})eu(t) - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau, e\dot\varphi(t))\mathrm{d}t, \end{aligned}$$

and the linear operator $L : \mathcal{E}_{t_0} \to \mathbb{R}$ represented by

$$\begin{aligned} L(\varphi) := t_0(u^1, \dot\varphi(0)) &- \int_0^{t_0} (t - t_0)(f(t), \dot\varphi(t))\mathrm{d}t \\ &+ \int_0^{t_0} (t - t_0)(\dot F(t), e\varphi(t))\mathrm{d}t + \int_0^{t_0} (F(t), e\varphi(t))\mathrm{d}t. \end{aligned}$$

Notice that, from these definitions, Dafermos' Equality (2.25) on the interval $[0, t_0]$ can be rephrased as follows

$$B(u, \varphi) = L(\varphi) \qquad \text{for every } \varphi \in \mathcal{E}_{t_0}.$$

Now we are in the framework of Theorem 2.2.8, and we want to show that (2.49) and (2.50) are satisfied. Foremost, we prove the existence of a positive constant α such that

$$B(\varphi, \varphi) \geq \alpha\|\varphi\|^2_{\mathcal{E}_{t_0}} \qquad \text{for every } \varphi \in \mathcal{E}_{t_0}.$$

By definition we have

$$B(\varphi, \varphi) = \int_0^{t_0} [\|\dot\varphi(t)\|^2 + (t - t_0)(\dot\varphi(t), \ddot\varphi(t))]\mathrm{d}t$$

$$- \int_0^{t_0} (t-t_0)\Big[((\mathbb{A}+\mathbb{B})e\varphi(t), e\dot\varphi(t)) - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}e\varphi(\tau), e\dot\varphi(t))\mathrm{d}\tau\Big]\mathrm{d}t. \quad (2.54)$$

Now we define

$$\psi(t) := \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}} e\varphi(\tau)\mathrm{d}\tau \quad \text{hence we have} \quad \dot\psi(t) = \frac{1}{\beta}e\varphi(t) - \int_0^t \frac{1}{\beta^2}\mathrm{e}^{-\frac{t-\tau}{\beta}} e\varphi(\tau)\mathrm{d}\tau;$$

then (2.54) can be reworded as

$$B(\varphi,\varphi) = \int_0^{t_0} \|\dot\varphi(t)\|^2 + (t-t_0)[(\dot\varphi(t),\ddot\varphi(t)) - ((\mathbb{A}+\mathbb{B})e\varphi(t), e\dot\varphi(t)) + (\mathbb{B}\psi(t), e\dot\varphi(t))]\mathrm{d}t. \quad (2.55)$$

Thanks to the chain rule and to the symmetry property (2.4), we can write

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\varphi(t)\|^2 = (\dot\varphi(t),\ddot\varphi(t)), \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}((\mathbb{A}+\mathbb{B})e\varphi(t), e\varphi(t)) = ((\mathbb{A}+\mathbb{B})e\varphi(t), e\dot\varphi(t)),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{B}\psi(t), e\varphi(t)) = (\mathbb{B}\dot\psi(t), e\varphi(t)) + (\mathbb{B}\psi(t), e\dot\varphi(t)).$$

By substituting this information in (2.55), we get after some integration by parts

$$\begin{aligned}
B(\varphi,\varphi) &= \int_0^{t_0}\|\dot\varphi(t)\|^2\mathrm{d}t + \frac{1}{2}\int_0^{t_0}(t-t_0)\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\varphi(t)\|^2\mathrm{d}t - \int_0^{t_0}(t-t_0)(\mathbb{B}\dot\psi(t), e\varphi(t))\mathrm{d}t \\
&\quad + \int_0^{t_0}(t-t_0)\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{B}\psi(t), e\varphi(t))\mathrm{d}t - \frac{1}{2}\int_0^{t_0}(t-t_0)\frac{\mathrm{d}}{\mathrm{d}t}((\mathbb{A}+\mathbb{B})e\varphi(t), e\varphi(t))\mathrm{d}t \\
&= \frac{t_0}{2}\|\dot\varphi(0)\|^2 + \frac{1}{2}\int_0^{t_0}\|\dot\varphi(t)\|^2\mathrm{d}t + \frac{1}{2}\int_0^{t_0}((\mathbb{A}+\mathbb{B})e\varphi(t), e\varphi(t))\mathrm{d}t \\
&\quad - \int_0^{t_0}(\mathbb{B}\psi(t), e\varphi(t))\mathrm{d}t - \int_0^{t_0}(t-t_0)(\mathbb{B}\dot\psi(t), e\varphi(t))\mathrm{d}t \\
&= \frac{t_0}{2}\|\dot\varphi(0)\|^2 + \frac{1}{2}\int_0^{t_0}\|\dot\varphi(t)\|^2\mathrm{d}t + \frac{1}{2}\int_0^{t_0}((\mathbb{A}+\mathbb{B})e\varphi(t), e\varphi(t))\mathrm{d}t \\
&\quad - \int_0^{t_0}(t-t_0)(\beta\mathbb{B}\dot\psi(t), \dot\psi(t))\mathrm{d}t - \int_0^{t_0}(t-t_0)(\mathbb{B}\dot\psi(t), \psi(t)) - \int_0^{t_0}(\mathbb{B}\psi(t), e\varphi(t))\mathrm{d}t \\
&= \frac{t_0}{2}\|\dot\varphi(0)\|^2 + \frac{1}{2}\int_0^{t_0}\|\dot\varphi(t)\|^2\mathrm{d}t + \frac{1}{2}\int_0^{t_0}(\mathbb{A}e\varphi(t), e\varphi(t))\mathrm{d}t \\
&\quad + \frac{1}{2}\int_0^{t_0}(\mathbb{B}(e\varphi(t)-\psi(t)), e\varphi(t)-\psi(t))\mathrm{d}t + \int_0^{t_0}(t_0-t)(\beta\mathbb{B}\dot\psi(t), \dot\psi(t))\mathrm{d}t. \quad (2.56)
\end{aligned}$$

From the coerciveness in (2.5) and to the definition of the V-norm, we have

$$(\mathbb{A}e\varphi(t), e\varphi(t)) \geq c_{\mathbb{A}}\|\varphi(t)\|_V^2 - c_{\mathbb{A}}\|\varphi(t)\|^2 \qquad \text{for every } t\in[0,T]. \quad (2.57)$$

Moreover, since

$$\varphi(t) = \varphi(0) + \int_0^t \dot\varphi(\tau)\mathrm{d}\tau = \int_0^t \dot\varphi(\tau)\mathrm{d}\tau,$$

inequality (2.57) implies

$$\frac{1}{2}\int_0^{t_0}(\mathbb{A}e\varphi(t), e\varphi(t))\mathrm{d}t \geq \frac{c_{\mathbb{A}}}{2}\int_0^{t_0}\|\varphi(t)\|_V^2\mathrm{d}t - \frac{c_{\mathbb{A}}t_0}{2}\int_0^{t_0}\|\dot\varphi(t)\|^2\mathrm{d}t. \quad (2.58)$$

By (2.56), (2.58), and in view of the choice done in (2.53), we can deduce

$$B(\varphi,\varphi) \geq \frac{t_0}{2}\|\dot\varphi(0)\|^2 + \frac{1-c_{\mathbb{A}}t_0}{2}\int_0^{t_0}\|\dot\varphi(t)\|^2\mathrm{d}t + \frac{c_{\mathbb{A}}}{2}\int_0^{t_0}\|\varphi(t)\|_V^2\mathrm{d}t \geq \frac{1}{4}\min\{1,c_{\mathbb{A}}\}\|\varphi\|_{\mathcal{E}_{t_0}}^2,$$

which corresponds to the hypothesis (2.50), with

$$\alpha = \frac{1}{4}\min\{1, c_{\mathbb{A}}\}. \tag{2.59}$$

We now show the validity of assumption (2.49). We have to prove that for every $\varphi \in \mathcal{E}_{t_0}$ the functional $B(\cdot, \varphi)$ is continuous on $\mathcal{V}_{t_0}$, and that $L : \mathcal{E}_{t_0} \to \mathbb{R}$ is a linear continuous operator on the space $\mathcal{E}_{t_0}$. To this aim, we fix $\varphi \in \mathcal{E}_{t_0}$ and we consider $\{u_k\}_k \subset \mathcal{V}_{t_0}$ such that

$$u_k \xrightarrow[k\to\infty]{\mathcal{V}_{t_0}} u.$$

Therefore

$$U_k := u_k - u \xrightarrow[k\to\infty]{L^2(0,t_0;V)} 0 \quad \text{and} \quad \dot{U}_k := \dot{u}_k - \dot{u} \xrightarrow[k\to\infty]{L^2(0,t_0;H)} 0.$$

By using Cauchy-Schwarz's inequality we get

$$\begin{aligned}|B(U_k,\varphi)| &\le \int_0^{t_0} |(\dot{U}_k(t), \dot{\varphi}(t))|\mathrm{d}t + t_0 \int_0^{t_0} |(\dot{U}_k(t), \ddot{\varphi}(t))|\mathrm{d}t \\ &\quad + t_0 \int_0^{t_0} |((\mathbb{A}+\mathbb{B})eU_k(t), e\dot{\varphi}(t))|\mathrm{d}t + t_0 \int_0^{t_0}\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}} |(\mathbb{B}eU_k(\tau), e\dot{\varphi}(t))|\mathrm{d}\tau\mathrm{d}t \\ &\le \|\dot{U}_k\|_{L^2(0,t_0;H)}\|\dot{\varphi}\|_{L^2(0,t_0;H)} + t_0(C_{\mathbb{A}}+C_{\mathbb{B}})\|U_k\|_{L^2(0,t_0;V)}\|\dot{\varphi}\|_{L^2(0,t_0;V)} \\ &\quad + t_0\|\dot{U}_k\|_{L^2(0,t_0;H)}\|\ddot{\varphi}\|_{L^2(0,t_0;H)} + \frac{t_0}{\beta}C_{\mathbb{B}}\int_0^{t_0}\int_0^t |(eU_k(\tau), e\dot{\varphi}(t))|\mathrm{d}\tau\mathrm{d}t. \end{aligned} \tag{2.60}$$

Notice that

$$\begin{aligned}&\int_0^{t_0}\int_0^t |(eU_k(\tau), e\dot{\varphi}(t))|\mathrm{d}\tau\mathrm{d}t \\ &\quad \le \|\dot{\varphi}\|_{L^2(0,t_0;V)}\Big(\int_0^{t_0}\Big(\int_0^t \|U_k(\tau)\|_V\mathrm{d}\tau\Big)^2\mathrm{d}t\Big)^{\frac{1}{2}} \le t_0\|\dot{\varphi}\|_{L^2(0,t_0;V)}\|U_k\|_{L^2(0,t_0;V)},\end{aligned}$$

whence, by considering (2.60), we can say that there exist two positive constants $C_1 = C_1(\varphi, t_0)$ and $C_2 = C_2(\mathbb{A}, \mathbb{B}, t_0, \beta, \varphi)$ such that

$$|B(U_k,\varphi)| \le C_1\|\dot{U}_k\|_{L^2(0,t_0;H)} + C_2\|U_k\|_{L^2(0,t_0,V)} \xrightarrow[k\to\infty]{} 0.$$

Now it remains to show that L is a continuous operator on $\mathcal{E}_{t_0}$, and since it is linear it is enough to show its boundedness. Let $\varphi \in \mathcal{E}_{t_0}$, then

$$|L(\varphi)| \le \Big|\int_0^{t_0}\Big[(t-t_0)(f(t),\dot{\varphi}(t)) - (t-t_0)(\dot{F}(t), e\varphi(t)) - (F(t), e\varphi(t))\Big]\mathrm{d}t\Big| + t_0\|u^1\|\|\dot{\varphi}(0)\|. \tag{2.61}$$

In particular there exists a positive constant $C = C(f, F, t_0)$ such that

$$\begin{aligned}&\int_0^{t_0} |(t-t_0)(f(t),\dot{\varphi}(t)) - (F(t), e\varphi(t)) - (t-t_0)(\dot{F}(t), e\varphi(t))|\mathrm{d}t \\ &\quad \le t_0\|f\|_{L^2(0,t_0;H)}\|\dot{\varphi}\|_{L^2(0,t_0;H)} + \Big(\int_0^{t_0}\|(t-t_0)\dot{F}(t) + F(t)\|^2\mathrm{d}t\Big)^{\frac{1}{2}}\|\varphi\|_{L^2(0,t_0;V)} \\ &\quad \le t_0\|f\|_{L^2(0,t_0;H)}\|\varphi\|_{\mathcal{E}_{t_0}} + 2^{\frac{1}{2}}\max\{t_0,1\}\|F\|_{H^1(0,t_0;H)}\|\varphi\|_{\mathcal{E}_{t_0}} \le C\|\varphi\|_{\mathcal{E}_{t_0}}.\end{aligned} \tag{2.62}$$

Moreover, we have

$$t_0\|u^1\|\|\dot{\varphi}(0)\| \le t_0\|u^1\|t_0^{-\frac{1}{2}}\|\varphi\|_{\mathcal{E}_{t_0}} = t_0^{\frac{1}{2}}\|u^1\|\|\varphi\|_{\mathcal{E}_{t_0}}. \tag{2.63}$$

By applying Theorem 2.2.8 with $X = \mathcal{V}_{t_0}$ and $Y = \mathcal{E}_{t_0}$, we have the existence of a solution to (2.25) on the interval $[0, t_0]$.

Furthermore, we can use (2.51) and (2.59), and by means of (2.61)–(2.63) we obtain (2.52) with

$$C_0 := \frac{\max\{2^{\frac{1}{2}}\max\{t_0, 1\}, t_0^{\frac{1}{2}}\}}{\frac{1}{4}\min\{1, c_{\mathbb{A}}\}}.$$

□

Remark 2.2.10. At this point, from Remark 2.2.2, Propositions 2.2.3 and 2.2.9, we can find a weak solution to the viscoelastic dynamic system (2.16)–(2.20) on the interval $[0, t_0]$.

Now we want to show that it is possible to find a weak solution on the whole interval $[0, T]$. Let $b, c \in [t_0, T)$ be two real numbers such that $b < c$, then we can state the following lemma.

Lemma 2.2.11. *Let $u \in \mathcal{V}^D(0, b)$ be a function which satisfies (2.21) on the interval $[0, b]$, then the following equality holds*

$$\begin{aligned} &\langle \dot u(b), \psi(b)\rangle - \int_0^b (\dot u(t), \dot\psi(t))\mathrm{d}t + \int_0^b ((\mathbb{A}+\mathbb{B})eu(t), e\psi(t))\mathrm{d}t \\ &- \int_0^b \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), e\psi(t))\mathrm{d}\tau\mathrm{d}t = \int_0^b (f(t), \psi(t))\mathrm{d}t + \int_0^b (F(t), e\psi(t))\mathrm{d}t, \end{aligned} \tag{2.64}$$

for every $\psi \in \mathcal{V}^D(0, b)$ such that $\psi(0) = 0$.

Moreover, if $u \in \mathcal{V}^D(b, c)$ is a function which satisfies (2.21) on the interval $[b, c]$, then the following equality holds

$$\begin{aligned} &- \langle \dot u(b), \Psi(b)\rangle - \int_b^c (\dot u(t), \dot\Psi(t))\mathrm{d}t + \int_b^c ((\mathbb{A}+\mathbb{B})eu(t), e\Psi(t))\mathrm{d}t \\ &- \int_b^c \int_b^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), e\Psi(t))\mathrm{d}\tau\mathrm{d}t = \int_b^c (f(t), \Psi(t))\mathrm{d}t + \int_b^c (F(t), e\Psi(t))\mathrm{d}t, \end{aligned} \tag{2.65}$$

for every $\Psi \in \mathcal{V}^D(b, c)$ such that $\Psi(c) = 0$.

Proof. We begin by proving (2.64). We consider $\psi \in \mathcal{V}^D(0, b)$ such that $\psi(0) = 0$, and we define for $\varepsilon \in (0, b)$ the function

$$\psi_\varepsilon(t) = \begin{cases} \psi(t) & t \in [0, b-\varepsilon] \\ \frac{b-t}{\varepsilon}\psi(t) & t \in [b-\varepsilon, b]. \end{cases}$$

Since $\psi_\varepsilon \in \mathcal{V}^D(0, b)$ and $\psi_\varepsilon(0) = \psi_\varepsilon(b) = 0$, we can use it as a test function in (2.21) to obtain $I_\varepsilon + J_\varepsilon = K_\varepsilon$, where

$$\begin{aligned} I_\varepsilon := &- \int_0^{b-\varepsilon} (\dot u(t), \dot\psi(t))\mathrm{d}t + \fint_{b-\varepsilon}^b (\dot u(t), \psi(t))\mathrm{d}t \\ &+ \int_0^{b-\varepsilon} ((\mathbb{A}+\mathbb{B})eu(t) - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau, e\psi(t))\mathrm{d}t, \end{aligned}$$

$$J_\varepsilon := -\fint_{b-\varepsilon}^b (b-t)(\dot u(t), \dot\psi(t))\mathrm{d}t + \fint_{b-\varepsilon}^b (b-t)((\mathbb{A}+\mathbb{B})eu(t) - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau, e\psi(t))\mathrm{d}t,$$

$$K_\varepsilon := \int_0^{b-\varepsilon} (f(t), \psi(t))\mathrm{d}t + \fint_{b-\varepsilon}^b (b-t)(f(t), \psi(t))\mathrm{d}t$$

$$+\int_0^{b-\varepsilon}(F(t),e\psi(t))\mathrm{d}t+\fint_{b-\varepsilon}^{b}(b-t)(F(t),e\psi(t))\mathrm{d}t.$$

Thanks to the absolute continuity of Lebesgue's integral and to Remark 2.2.6 we get

$$\begin{aligned}I_\varepsilon \xrightarrow[\varepsilon\to0^+]{} &-\int_0^b(\dot u(t),\dot\psi(t))\mathrm{d}t+\int_0^b((\mathbb{A}+\mathbb{B})eu(t),e\psi(t))\mathrm{d}t\\ &-\int_0^b\int_0^t\frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau)\mathrm{d}\tau,e\psi(t))\mathrm{d}t+\langle\dot u(b),\psi(b)\rangle,\end{aligned}$$

$$J_\varepsilon\xrightarrow[\varepsilon\to0^+]{}0,\qquad K_\varepsilon\xrightarrow[\varepsilon\to0^+]{}\int_0^b(f(t),\psi(t))\mathrm{d}t+\int_0^b(F(t),e\psi(t))\mathrm{d}t,$$

which concludes the proof of (2.64).

To prove (2.65), it is enough to consider for $\varepsilon\in(0,c-b)$ the function

$$\Psi_\varepsilon(t)=\begin{cases}\frac{t-b}{\varepsilon}\Psi(t) & t\in[b,b+\varepsilon]\\ \Psi(t) & t\in[b+\varepsilon,c]\end{cases}$$

where $\Psi\in\mathcal{V}^D(b,c)$ such that $\Psi(c)=0$, and to repeat similar argument before performed. □

Taking into account the previous lemma we can state and prove the following proposition.

Proposition 2.2.12. *Let $\tilde u\in\mathcal{V}^D(0,b)$ be a weak solution to the viscoelastic dynamic system (2.16)–(2.20) in the sense of Definition 2.1.3 on the interval $[0,b]$ which satisfies for some positive constants $\tilde C$ the following estimate*

$$\|\tilde u\|_{\mathcal{V}(0,b)}\le\tilde C\left(\|f\|_{L^2(0,b;H)}+\|F\|_{H^1(0,b;H)}+\|u^1\|\right).\tag{2.66}$$

Then, for every $l\ge1$ there exists $c\in(b,b+\frac{t_0}{l}]$ such that we can extend $\tilde u$ to a function $u\in\mathcal{V}^D(0,c)$ which is a weak solution on the interval $[0,c]$. Moreover u satisfies for some positive constants C the following estimate

$$\|u\|_{\mathcal{V}(0,c)}\le C\left(\|f\|_{L^2(0,c;H)}+\|F\|_{H^1(0,c;H)}+\|u^1\|\right).\tag{2.67}$$

Proof. We divide the proof into two steps. In the first one, we show how to extend the solution. After this, in the second step, we prove (2.67). We firstly choose $\hat b\in(b-\frac{t_0}{2l},b)$ in such a way that

- $\tilde u(\hat b)\in V$ and
$$\|\tilde u(\hat b)\|_V^2\le\fint_{b-\frac{t_0}{2l}}^{b}\|\tilde u(t)\|_V^2\mathrm{d}t;\tag{2.68}$$
- $\hat b$ is a Lebesgue's point for $\dot{\tilde u}$, that is
$$\lim_{\varepsilon\to0^+}\fint_{\hat b}^{\hat b+\varepsilon}\|\dot{\tilde u}(t)-\dot{\tilde u}(\hat b)\|\mathrm{d}t=0,\tag{2.69}$$
and $\dot{\tilde u}(\hat b)\in H$ satisfies
$$\|\dot{\tilde u}(\hat b)\|^2\le\fint_{b-\frac{t_0}{2l}}^{b}\|\dot{\tilde u}(t)\|^2\mathrm{d}t.\tag{2.70}$$

Notice that (2.68)–(2.70) are possible because $\tilde{u} \in \mathcal{V}(0,b)$.

Step 1. Since $\tilde{u}$ is a weak solution on the interval $[0,b]$, then

$$-\int_0^b (\dot{\tilde{u}}(t), \dot{v}(t))\mathrm{d}t + \int_0^b ((\mathbb{A}+\mathbb{B})e\tilde{u}(t), ev(t))\mathrm{d}t - \int_0^b \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}e\tilde{u}(\tau), ev(t))\mathrm{d}\tau\mathrm{d}t = \int_0^b (f(t), v(t))\mathrm{d}t + \int_0^b (F(t), ev(t))\mathrm{d}t,$$

for every $v \in \mathcal{V}^D(0,b)$ such that $v(0) = v(b) = 0$, and moreover $\tilde{u}$ satisfies

$$\lim_{t\to 0^+} \|\tilde{u}(t)\| = 0 \qquad \text{and} \qquad \lim_{t\to 0^+} \|\dot{\tilde{u}}(t) - u^1\|_{(V_0^D)'} = 0. \tag{2.71}$$

We define the function $G \in H^1(\hat{b}, \hat{b}+\frac{t_0}{l}; H)$ in the following way

$$G(t) := F(t) + \int_0^{\hat{b}} \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}e\tilde{u}(\tau)\mathrm{d}\tau.$$

Since $\frac{t_0}{l} \le t_0$, $\tilde{u}(\hat{b}) \in V$, and $\dot{\tilde{u}}(\hat{b}) \in H$, we can apply Remark 2.2.2, Propositions 2.2.3 and 2.2.9 on the interval $[\hat{b}, \hat{b}+\frac{t_0}{l}]$, to find a function $\bar{u} \in \mathcal{V}^D(\hat{b}, \hat{b}+\frac{t_0}{l})$ which satisfies, for every $v \in \mathcal{V}^D(\hat{b}, \hat{b}+\frac{t_0}{l})$ such that $v(\hat{b}) = v(\hat{b}+\frac{t_0}{l}) = 0$, the following equality

$$-\int_{\hat{b}}^{\hat{b}+\frac{t_0}{l}} (\dot{\bar{u}}(t), \dot{v}(t))\mathrm{d}t + \int_{\hat{b}}^{\hat{b}+\frac{t_0}{l}} ((\mathbb{A}+\mathbb{B})e\bar{u}(t), ev(t))\mathrm{d}t$$
$$-\int_{\hat{b}}^{\hat{b}+\frac{t_0}{l}} \int_{\hat{b}}^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}e\bar{u}(\tau), ev(t))\mathrm{d}\tau\mathrm{d}t = \int_{\hat{b}}^{\hat{b}+\frac{t_0}{l}} (f(t), v(t))\mathrm{d}t + \int_{\hat{b}}^{\hat{b}+\frac{t_0}{l}} (G(t), ev(t))\mathrm{d}t,$$

and also the following limits

$$\lim_{t\to \hat{b}^+} \|\bar{u}(t) - \tilde{u}(\hat{b})\| = 0, \quad \lim_{t\to \hat{b}^+} \|\dot{\bar{u}}(t) - \dot{\tilde{u}}(\hat{b})\|_{(V_0^D)'} = 0. \tag{2.72}$$

Notice that the initial data $\tilde{u}(\hat{b})$ and $\dot{\tilde{u}}(\hat{b})$ are well defined because $\tilde{u} \in C^0([0,b]; H)$ and $\dot{\tilde{u}} \in C^0([0,b]; (V_0^D)')$.

Now we define the function

$$u(t) := \begin{cases} \tilde{u}(t) & t \in [0,\hat{b}] \\ \bar{u}(t) & t \in [\hat{b}, \hat{b}+\frac{t_0}{l}], \end{cases} \tag{2.73}$$

and we claim that it is a weak solution on the interval $[0, \hat{b}+\frac{t_0}{l}]$. Notice that, since $\hat{b} \ge b - \frac{t_0}{2l}$ then $\hat{b}+\frac{t_0}{l} > b$. To prove this, let us fix $\psi \in \mathcal{D}^D(0, \hat{b}+\frac{t_0}{l})$. Clearly $\psi \in \mathcal{V}^D(0,\hat{b})$ and $\psi(0) = 0$, and since $\tilde{u}$ is a weak solution on $[0,\hat{b}]$, we can use (2.64) of Lemma 2.2.11 to get

$$(\dot{\tilde{u}}(\hat{b}), \psi(\hat{b})) - \int_0^{\hat{b}} (\dot{u}(t), \dot{\psi}(t))\mathrm{d}t + \int_0^{\hat{b}} ((\mathbb{A}+\mathbb{B})eu(t), e\psi(t))\mathrm{d}t$$
$$-\int_0^{\hat{b}} \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau)\mathrm{d}\tau, e\psi(t))\mathrm{d}t = \int_0^{\hat{b}} (f(t), \psi(t))\mathrm{d}t + \int_0^{\hat{b}} (F(t), e\psi(t))\mathrm{d}t. \tag{2.74}$$

Moreover, $\psi \in \mathcal{V}^D(\hat{b}, \hat{b}+\frac{t_0}{l})$ and $\psi(\hat{b}+\frac{t_0}{l}) = 0$, and since $\bar{u}$ is a weak solution on $[\hat{b}, \hat{b}+\frac{t_0}{l}]$, by (2.65) of Lemma 2.2.11 we obtain

$$-(\dot{\tilde{u}}(\hat{b}), \psi(\hat{b})) - \int_{\hat{b}}^{\hat{b}+\frac{t_0}{l}} (\dot{u}(t), \dot{\psi}(t))\mathrm{d}t + \int_{\hat{b}}^{\hat{b}+\frac{t_0}{l}} ((\mathbb{A}+\mathbb{B})eu(t), e\psi(t))\mathrm{d}t$$

$$- \int_{\hat{b}}^{\hat{b}+\frac{t_0}{l}} \int_{\hat{b}}^{t} \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} (\mathbb{B}eu(\tau)\mathrm{d}\tau, e\psi(t))\mathrm{d}t = \int_{\hat{b}}^{\hat{b}+\frac{t_0}{l}} (f(t), \psi(t))\mathrm{d}t + \int_{\hat{b}}^{\hat{b}+\frac{t_0}{l}} (G(t), e\psi(t))\mathrm{d}t,$$

that is

$$\begin{aligned} &- (\dot{\tilde{u}}(\hat{b}), \psi(\hat{b})) - \int_{\hat{b}}^{\hat{b}+\frac{t_0}{l}} (\dot{u}(t), \dot{\psi}(t))\mathrm{d}t + \int_{\hat{b}}^{\hat{b}+\frac{t_0}{l}} ((\mathbb{A}+\mathbb{B})eu(t), e\psi(t))\mathrm{d}t \\ &- \int_{\hat{b}}^{\hat{b}+\frac{t_0}{l}} \int_{0}^{t} \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} (\mathbb{B}eu(\tau)\mathrm{d}\tau, e\psi(t))\mathrm{d}t = \int_{\hat{b}}^{\hat{b}+\frac{t_0}{l}} (f(t), \psi(t))\mathrm{d}t + \int_{\hat{b}}^{\hat{b}+\frac{t_0}{l}} (F(t), e\psi(t))\mathrm{d}t. \end{aligned} \tag{2.75}$$

From (2.69) and (2.72), by summing (2.74) and (2.75), we obtain the following equality

$$\begin{aligned} &- \int_{0}^{\hat{b}+\frac{t_0}{l}} (\dot{u}(t), \dot{\psi}(t))\mathrm{d}t + \int_{0}^{\hat{b}+\frac{t_0}{l}} ((\mathbb{A}+\mathbb{B})eu(t), e\psi(t))\mathrm{d}t \\ &- \int_{0}^{\hat{b}+\frac{t_0}{l}} \int_{0}^{t} \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} (\mathbb{B}eu(\tau), e\psi(t))\mathrm{d}\tau\mathrm{d}t = \int_{0}^{\hat{b}+\frac{t_0}{l}} (f(t), \psi(t))\mathrm{d}t + \int_{0}^{\hat{b}+\frac{t_0}{l}} (F(t), e\psi(t))\mathrm{d}t. \end{aligned} \tag{2.76}$$

By setting $c := \hat{b} + \frac{t_0}{l}$ we have that the function u defined in (2.73) is a weak solution to the viscoelastic dynamic system (2.16)–(2.20) in the sense of Definition 2.1.3 on the interval $[0, c]$, since it satisfies (2.71) and (2.76).

Step 2. Now, we want to prove (2.67). We can write

$$\|u\|^2_{\mathcal{V}(0,c)} = \|\tilde{u}\|^2_{\mathcal{V}(0,\hat{b})} + \|\bar{u}\|^2_{\mathcal{V}(\hat{b},c)} \le \|\tilde{u}\|^2_{\mathcal{V}(0,b)} + \|\bar{u}\|^2_{\mathcal{V}(\hat{b},c)}. \tag{2.77}$$

Notice that $\bar{u} - \tilde{u}(\hat{b}) \in \mathcal{V}_0^D(\hat{b}, c)$ is a function which satisfies Dafermos' Equality (2.25) on the interval $[\hat{b}, c]$ with the right-hand side equal to

$$t_0(\dot{\tilde{u}}(\hat{b}), \dot{\varphi}(0)) - \int_{\hat{b}}^{c} (t-t_0) \Big[(f(t), \dot{\varphi}(t)) + (G(t) - \mathbb{A}e\tilde{u}(\hat{b}) - \mathrm{e}^{-\frac{t-\hat{b}}{\beta}} \mathbb{B}e\tilde{u}(\hat{b}), e\dot{\varphi}(t)) \Big] \mathrm{d}t,$$

for every $\varphi \in \mathcal{E}_0^D(\hat{b}, c)$. Therefore, by following the estimates in (2.61)–(2.63), we can apply (2.51) of Theorem 2.2.8, with $X = \mathcal{V}^D(\hat{b}, c)$ and $Y = \mathcal{E}_0^D(\hat{b}, c)$, to obtain the existence of a positive constant $K = K(t_0, \mathbb{A})$ such that

$$\|\bar{u} - \tilde{u}(\hat{b})\|_{\mathcal{V}(\hat{b},c)} \le K \big[\|f\|_{L^2(\hat{b},c;H)} + \|G - \mathbb{A}e\tilde{u}(\hat{b}) - \mathrm{e}^{-\frac{\cdot-\hat{b}}{\beta}} \mathbb{B}e\tilde{u}(\hat{b})\|_{H^1(\hat{b},c;H)} + \|\dot{\tilde{u}}(\hat{b})\| \big]. \tag{2.78}$$

Now notice that

$$\begin{aligned} \|G\|_{H^1(\hat{b},c;H)} &\le \|F\|_{H^1(\hat{b},c;H)} + C_{\mathbb{B}} \Big(\frac{\beta}{2}\Big)^{\frac{1}{2}} \Big(1 + \frac{1}{\beta}\Big) \Big(\int_0^{\hat{b}} \frac{1}{\beta^2} \mathrm{e}^{-\frac{2(\hat{b}-\tau)}{\beta}} \mathrm{d}\tau \Big)^{\frac{1}{2}} \|\tilde{u}\|_{L^2(0,\hat{b};V)} \\ &\le \|F\|_{H^1(\hat{b},c;H)} + \frac{C_{\mathbb{B}}}{2} \Big(1 + \frac{1}{\beta}\Big) \|\tilde{u}\|_{\mathcal{V}(0,\hat{b})}, \end{aligned} \tag{2.79}$$

and

$$\begin{aligned} \|\mathbb{A}e\tilde{u}(\hat{b}) + \mathrm{e}^{-\frac{\cdot-\hat{b}}{\beta}} \mathbb{B}e\tilde{u}(\hat{b})\|_{H^1(\hat{b},c;H)} &\le \Big[C_{\mathbb{A}} \Big(\frac{t_0}{l}\Big)^{\frac{1}{2}} + C_{\mathbb{B}} \|\mathrm{e}^{-\frac{\cdot-\hat{b}}{\beta}}\|_{H^1(\hat{b},c)} \Big] \|\tilde{u}(\hat{b})\|_V \\ &\le \Big[C_{\mathbb{A}} \Big(\frac{t_0}{l}\Big)^{\frac{1}{2}} + C_{\mathbb{B}} \Big(\frac{\beta}{2}\Big)^{\frac{1}{2}} \Big(1 + \frac{1}{\beta}\Big) \Big] \|\tilde{u}(\hat{b})\|_V. \end{aligned} \tag{2.80}$$

Taking into account the information provided by (2.68)–(2.70), we can use estimates (2.78)–(2.80) to deduce the existence of a positive constant $\bar{C} = \bar{C}(t_0, l, \mathbb{A}, \mathbb{B}, \beta)$ such that

$$\|\bar{u}\|_{\mathcal{V}(\hat{b},c)} \le \bar{C} \Big(\|f\|_{L^2(\hat{b},c;H)} + \|F\|_{H^1(\hat{b},c;H)} + \|\tilde{u}\|_{\mathcal{V}(0,b)} \Big). \tag{2.81}$$

By (2.66), (2.77), and (2.81) we obtain the final estimate (2.67). □

Now we are in position to prove the main theorem of this section.

Proof of Theorem 2.2.1. Let us consider $u_0 \in \mathcal{V}^D(0,t_0)$ a weak solution to the viscoelastic dynamic system (2.16)–(2.20) in the sense of Definition 2.1.3 on the interval $[0,t_0]$, whose existence is guaranteed by Remark 2.2.10. Moreover, u_0 satisfies (2.52). By applying a finite number of times Proposition 2.2.12 with $l = 1$ we can extend u_0 to $\tilde{u} \in \mathcal{V}^D(0,b)$ which is a weak solution on the interval $[0,b]$, where $T - b < t_0$. Now we select $\hat{b} \in (T - t_0, b)$ in such a way (2.68)–(2.70) are satisfied on the interval $[T - t_0, b]$. By choosing $l = \frac{t_0}{T-\hat{b}} \geq 1$, since $\hat{b} + \frac{t_0}{l} = T$, thanks to Proposition 2.2.12 we can extend $\tilde{u}$ to a function $u \in \mathcal{V}^D(0,T)$ which is a weak solution to the viscoelastic dynamic system (2.16)–(2.20) on the interval $[0,T]$. Moreover u satisfies (2.67) on $[0,T]$. Finally, by applying Remark 2.2.2 we get the thesis. $\square$

2.3 Existence: A coupled system equivalent to the viscoelastic dynamic system

In this section, we illustrate a second method to find solutions to the viscoelastic dynamic system (2.16)–(2.20) according to Definition 2.1.3. This method is based on a minimizing movement approach deriving from the theory of gradient flows, and it is a classical tool used to prove the existence of solutions in the context of fractures, see, e.g., [7], [13], [17]. By means of this method, we are also able to provide an energy-dissipation inequality satisfied by the solution, and consequently, thanks to this inequality, we prove that such a solution satisfies the initial conditions (2.20) in a stronger sense than the one stated in (2.22).

To this aim, let us define the following *coupled* system

$$\begin{cases} \ddot{u}(t) - \operatorname{div}(\mathbb{A}eu(t)) - \operatorname{div}(\mathbb{B}(eu(t) - w(t))) = f(t) - \operatorname{div} G(t) & \text{in } \Omega \setminus \Gamma_t,\ t \in (0,T), \\ \beta \dot{w}(t) + w(t) = eu(t) \end{cases} \tag{2.82}$$

with the following boundary and initial conditions

$$u(t) = z(t) \qquad \text{on } \partial_D\Omega, \quad t \in (0,T), \tag{2.83}$$

$$[\mathbb{A}eu(t) + \mathbb{B}(eu(t) - w(t))]\nu = G(t)\nu \qquad \text{on } \partial_N\Omega, \quad t \in (0,T), \tag{2.84}$$

$$[\mathbb{A}eu(t) + \mathbb{B}(eu(t) - w(t))]\nu = G(t)\nu \qquad \text{on } \Gamma_t, \quad t \in (0,T), \tag{2.85}$$

$$u(0) = u^0, \quad w(0) = w^0, \quad \dot{u}(0) = u^1, \tag{2.86}$$

where $w^0 \in H$ and $G(t) := F(t) - \mathrm{e}^{-\frac{t}{\beta}}\mathbb{B}w^0$. Also in this case, the strong formulation of the coupled system (2.82)–(2.86) is only formal. By setting

$$\mathcal{V} := \mathcal{V}(0,T), \quad \mathcal{V}^D := \mathcal{V}^D(0,T), \quad \mathcal{D}^D := \mathcal{D}^D(0,T),$$

we give the following definition.

Definition 2.3.1. We say that $(u,w) \in \mathcal{V} \times H^1(0,T;H)$ is a *weak solution* to the coupled system (2.82)–(2.86) if the following conditions hold:

- $u - z \in \mathcal{V}^D$ and

$$-\int_0^T (\dot{u}(t), \dot{\varphi}(t))\mathrm{d}t + \int_0^T (\mathbb{A}eu(t), e\varphi(t))\mathrm{d}t + \int_0^T (\mathbb{B}(eu(t) - w(t)), e\varphi(t))\mathrm{d}t$$
$$= \int_0^T (f(t), \varphi(t))\mathrm{d}t + \int_0^T (F(t), e\varphi(t))\mathrm{d}t - \int_0^T \mathrm{e}^{-\frac{t}{\beta}}(\mathbb{B}w^0, e\varphi(t))\mathrm{d}t, \tag{2.87}$$

  for every $\varphi \in \mathcal{D}^D$;

- for a.e. $t \in (0,T)$
$$\begin{cases} \beta\dot{w}(t) + w(t) = eu(t) \\ w(0) = w^0 \end{cases} \tag{2.88}$$
where the equalities are to be understood in the sense of the Hilbert space H;

- the initial conditions (2.22) are satisfied.

The following result proves that the new problem is equivalent to the first one.

Theorem 2.3.2. *The viscoelastic dynamic system* (2.16)–(2.20) *is equivalent to the coupled system* (2.82)–(2.86)*.*

Proof. Let us consider a weak solution $(u,w) \in \mathcal{V} \times H^1(0,T;H)$ to the coupled system (2.82)–(2.86) according to Definition 2.3.1. In view of the theory of ordinary differential equations valued in Hilbert spaces, by (2.88) we can write

$$w(t) = w^0 \mathrm{e}^{-\frac{t}{\beta}} + \mathrm{e}^{-\frac{t}{\beta}} \int_0^t \frac{1}{\beta} \mathrm{e}^{\frac{\tau}{\beta}} eu(\tau)\mathrm{d}\tau \qquad \text{for every } t \in [0,T]. \tag{2.89}$$

Moreover, by definition $u - z \in \mathcal{V}^D$ and (2.87) holds for every $\varphi \in \mathcal{D}^D$. By substituting (2.89) in (2.87) we obtain

$$\begin{aligned} &- \int_0^T (\dot{u}(t), \dot{\varphi}(t))\mathrm{d}t + \int_0^T ((\mathbb{A}+\mathbb{B})eu(t) - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau, e\varphi(t))\mathrm{d}t \\ &- \int_0^T \mathrm{e}^{-\frac{t}{\beta}}(\mathbb{B}w^0, e\varphi(t))\mathrm{d}t = \int_0^T (f(t),\varphi(t))\mathrm{d}t + \int_0^T (F(t), e\varphi(t))\mathrm{d}t - \int_0^T \mathrm{e}^{-\frac{t}{\beta}}(\mathbb{B}w^0, e\varphi(t))\mathrm{d}t. \end{aligned}$$

Therefore, since, again by definition, (2.22) holds, u is a weak solution to the viscoelastic dynamic system (2.16)–(2.20) in the sense of Definition 2.1.3.

Vice versa, if we consider a solution $u \in \mathcal{V}$ to the viscoelastic dynamic system (2.16)–(2.20), then $u - z \in \mathcal{V}^D$ and

$$\begin{aligned} &- \int_0^T (\dot{u}(t), \dot{\varphi}(t))\mathrm{d}t + \int_0^T ((\mathbb{A}+\mathbb{B})eu(t), e\varphi(t))\mathrm{d}t \\ &- \int_0^T \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), e\varphi(t))\mathrm{d}\tau\mathrm{d}t = \int_0^T (f(t),\varphi(t))\mathrm{d}t + \int_0^T (F(t), e\varphi(t))\mathrm{d}t, \end{aligned} \tag{2.90}$$

for every $\varphi \in \mathcal{D}^D$. Let $w^0 \in H$ and let w be the function defined in (2.89). It is easy to see that $w \in H^1(0,T;H)$ and by summing to both hand sides of (2.90) the term

$$- \int_0^T \mathrm{e}^{-\frac{t}{\beta}}(\mathbb{B}w^0, e\varphi(t))\mathrm{d}t,$$

we get (2.87). This, together with (2.22), shows that $(u,w) \in \mathcal{V} \times H^1(0,T;H)$ is a weak solution to the coupled system (2.82)–(2.86) in the sense of Definition 2.3.1. The proof is then complete. □

Now we are in position to state the main result of this section.

Theorem 2.3.3. *There exists a weak solution* $(u,w) \in \mathcal{V} \times H^1(0,T;H)$ *to the coupled system* (2.82)–(2.86)*. Moreover, we have* $u \in C_w^0([0,T];V)$*,* $\dot{u} \in C_w^0([0,T];H) \cap H^1(0,T;(V_0^D)')$*, and*

$$\lim_{t\to 0^+} u(t) = u^0 \text{ in } V \quad \text{and} \quad \lim_{t\to 0^+} \dot{u}(t) = u^1 \text{ in } H.$$

The proof of this result will be given at the end of this section.

2.3.1 Discretization in time

In this subsection we prove Theorem 2.3.3 by means of a time discretization scheme in the same spirit of [13].

Let us fix $n \in \mathbb{N}$ and set

$$\begin{aligned} &\tau_n := \frac{T}{n}, \quad u_n^0 := u^0, \quad u_n^{-1} := u^0 - \tau_n u^1, \\ &w_n^0 := w^0, \quad F_n^0 := F(0), \quad h_n^0 := \mathbb{B}w^0. \end{aligned} \tag{2.91}$$

We define

$$\begin{aligned} &V_n^k := V_{k\tau_n}^D, \qquad z_n^k := z(k\tau_n) && \text{for } k = 0, \dots, n, \\ &F_n^k := F(k\tau_n), \quad h_n^k := \mathrm{e}^{-\frac{k\tau_n}{\beta}}\mathbb{B}w^0, \quad f_n^k := \fint_{(k-1)\tau_n}^{k\tau_n} f(\tau)\mathrm{d}\tau && \text{for } k = 1, \dots, n. \end{aligned}$$

For $k = 1, ..., n$ let (u_n^k, w_n^k) be the minimizer in $V_n^k \times H$ of the functional

$$\begin{aligned} (u, w) \mapsto &\frac{1}{2\tau_n^2}\|u - 2u_n^{k-1} + u_n^{k-2}\|^2 + \frac{1}{2}(\mathbb{A}eu, eu) + \frac{1}{2}(\mathbb{B}(eu - w), eu - w) \\ &+ \frac{\beta}{2\tau_n^2}(\mathbb{B}(w - w_n^{k-1}), w - w_n^{k-1}) - (f_n^k, u) - (F_n^k - h_n^k, eu). \end{aligned} \tag{2.92}$$

Using the coerciveness (2.5), it is easy to see that the functional in (2.92) is convex and bounded from below by

$$\frac{1}{4}\min\left\{\frac{1}{2\tau_n^2}, C_{\mathbb{A}}, \frac{1}{\tau_n^2}C_{\mathbb{B}}\beta\right\}(\|u\|_V^2 + \|w\|^2) - C_n^k,$$

for a suitable positive constant C_n^k. The existence of a minimizer then follows from the lower semicontinuity of the functional with respect to the strong (and hence to the weak) convergence in $V_n^k \times H$.

To simplify the exposition, for $k = 0, ..., n$ we define

$$\delta u_n^k := \frac{u_n^k - u_n^{k-1}}{\tau_n} \quad \text{and} \quad \delta^2 u_n^k := \frac{\delta u_n^k - \delta u_n^{k-1}}{\tau_n}. \tag{2.93}$$

The Euler equation for (2.92) gives

$$\begin{aligned} &(\delta^2 u_n^k, \varphi) + (\mathbb{A}eu_n^k, e\varphi) + (\mathbb{B}(eu_n^k - w_n^k), e\varphi - \psi) \\ &\qquad + \beta(\mathbb{B}\delta w_n^k, \psi) = (f_n^k, \varphi) + (F_n^k, e\varphi) - (h_n^k, e\varphi), \end{aligned} \tag{2.94}$$

for every $(\varphi, \psi) \in V_n^k \times H$, where δw_n^k is defined for every $k = 1, \dots, n$ as in (2.93), and $\delta u_n^0 = u^1$ by (2.91). Notice that by choosing as a test function the pair $(\varphi, 0)$ with $\varphi \in V_n^k$, we get

$$(\delta^2 u_n^k, \varphi) + ((\mathbb{A} + \mathbb{B})eu_n^k - \mathbb{B}w_n^k, e\varphi) = (f_n^k, \varphi) + (F_n^k, e\varphi) - (h_n^k, e\varphi),$$

which is a discrete-in-time approximation of (2.87). On the other hand, if we use as a test function in (2.94) the pair $(0, \psi)$ with $\psi \in H$, we have

$$(\beta\delta w_n^k + w_n^k - eu_n^k, \psi) = 0,$$

thus $\beta\delta w_n^k + w_n^k - eu_n^k = 0$ (as element of H), which is an approximation in time of (2.88).

In the next lemma we show an estimate for the family $\{(u_n^k, w_n^k)\}_{k=1}^n$, which is uniform with respect to n, and it will be used later to pass to the limit in the discrete equation (2.94).

Lemma 2.3.4. *There exists a positive constant C, independent of n, such that*

$$\max_{i=1,..,n} \|\delta u_n^i\| + \max_{i=1,..,n} \|eu_n^i\| + \max_{i=1,..,n} \|w_n^i\| + \sum_{i=1}^n \tau_n \|\delta w_n^i\|^2 \leq C. \tag{2.95}$$

Proof. To simplify our computations, we define the following two bilinear symmetric forms

$$\begin{aligned} &a : (V \times H) \times (V \times H) \to \mathbb{R} & & b : H \times H \to \mathbb{R} \\ &a((u, w), (\varphi, \psi)) := (\mathbb{A}eu, e\varphi) + (\mathbb{B}(eu - w), e\varphi - \psi), & & b(w, \psi) := \beta(\mathbb{B}w, \psi). \end{aligned}$$

Thanks to (2.5) we have that $a((\varphi, \psi), (\varphi, \psi)) \geq 0$ and $b(\psi, \psi) \geq 0$ for every $\varphi \in V$ and $\psi \in H$. Now we set $\omega_n^k := (u_n^k, w_n^k)$ for $k = 0, \dots, n$, and we take $(\varphi, \psi) = \tau_n(\delta u_n^k - \delta z_n^k, \delta w_n^k) \in V_n^k \times H$ as a test function in (2.94), where $\delta z_n^0 := \dot{z}(0)$ and δz_n^k is defined as in (2.93). Therefore, we obtain

$$\begin{aligned} &\|\delta u_n^k\|^2 - (\delta u_n^{k-1}, \delta u_n^k) - \tau_n(\delta^2 u_n^k, \delta z_n^k) + a(\omega_n^k, \omega_n^k) - a(\omega_n^{k-1}, \omega_n^k) - \tau_n a(\omega_n^k, (\delta z_n^k, 0)) \\ &+ \tau_n b(\delta w_n^k, \delta w_n^k) = \tau_n(f_n^k, \delta u_n^k - \delta z_n^k) + \tau_n(F_n^k, e\delta u_n^k - e\delta z_n^k) - \tau_n(h_n^k, e\delta u_n^k - e\delta z_n^k). \end{aligned} \tag{2.96}$$

By means of the following identities

$$\begin{aligned} &\|\delta u_n^k\|^2 - (\delta u_n^{k-1}, \delta u_n^k) = \frac{1}{2}\|\delta u_n^k\|^2 - \frac{1}{2}\|\delta u_n^{k-1}\|^2 + \frac{\tau_n^2}{2}\|\delta^2 u_n^k\|^2, \\ &a(\omega_n^k, \omega_n^k) - a(\omega_n^{k-1}, \omega_n^k) = \frac{1}{2}a(\omega_n^k, \omega_n^k) - \frac{1}{2}a(\omega_n^{k-1}, \omega_n^{k-1}) + \frac{\tau_n^2}{2}a(\delta\omega_n^k, \delta\omega_n^k), \end{aligned}$$

from (2.96) we infer

$$\frac{1}{2}\|\delta u_n^k\|^2 - \frac{1}{2}\|\delta u_n^{k-1}\|^2 + \frac{1}{2}a(\omega_n^k, \omega_n^k) - \frac{1}{2}a(\omega_n^{k-1}, \omega_n^{k-1}) + \tau_n b(\delta w_n^k, \delta w_n^k) \leq \tau_n W_n^k, \tag{2.97}$$

where

$$\begin{aligned} W_n^k := (f_n^k, \delta u_n^k - \delta z_n^k) &+ (F_n^k, e\delta u_n^k - e\delta z_n^k) \\ &- (h_n^k, e\delta u_n^k - e\delta z_n^k) + (\delta^2 u_n^k, \delta z_n^k) + a(\omega_n^k, (\delta z_n^k, 0)). \end{aligned}$$

We fix $i \in \{1, \dots, n\}$ and we sum in (2.97) over $k = 1, \dots, i$ to obtain the following discrete energy inequality

$$\frac{1}{2}\|\delta u_n^i\|^2 + \frac{1}{2}a(\omega_n^i, \omega_n^i) + \sum_{k=1}^i \tau_n b(\delta w_n^k, \delta w_n^k) \leq \mathscr{E}_0 + \sum_{k=1}^i \tau_n W_n^k, \tag{2.98}$$

where

$$\mathscr{E}_0 := \frac{1}{2}\|u^1\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0) + \frac{1}{2}(\mathbb{B}(eu^0 - w^0), eu^0 - w^0).$$

Let us now estimate the right-hand side of (2.98) from above. We can write

$$\left|\sum_{k=1}^i \tau_n(f_n^k, \delta u_n^k - \delta z_n^k)\right| \leq \|f\|_{L^2(0,T;H)}^2 + \frac{1}{2}\|\dot{z}\|_{L^2(0,T;H)}^2 + \frac{1}{2}\sum_{k=1}^i \tau_n\|\delta u_n^k\|^2, \tag{2.99}$$

$$\begin{aligned} \left|\sum_{k=1}^i \tau_n(h_n^k, \delta z_n^k)\right| &\leq \frac{1}{2}\sum_{k=1}^i \tau_n \mathrm{e}^{-\frac{2k\tau_n}{\beta}}\|\mathbb{B}w^0\|^2 + \frac{1}{2}\sum_{k=1}^i \tau_n\|\delta z_n^k\|^2 \\ &\leq \frac{T}{2}\|\mathbb{B}w^0\|^2 + \frac{1}{2}\|\dot{z}\|_{L^2(0,T;H)}^2, \end{aligned} \tag{2.100}$$

$$\Big|\sum_{k=1}^{i}\tau_n(F_n^k, e\delta z_n^k)\Big| \le \frac{1}{2}\sum_{k=1}^{i}\tau_n\|F_n^k\|^2 + \frac{1}{2}\sum_{k=1}^{i}\tau_n\|e\delta z_n^k\|^2$$
$$\le T\|F(0)\|^2 + T^2\|\dot F\|^2_{L^2(0,T;H)} + \frac{1}{2}\|\dot z\|^2_{L^2(0,T;V)}, \tag{2.101}$$

$$\Big|\sum_{k=1}^{i}\tau_n a(\omega_n^k,(\delta z_n^k,0))\Big| \le \frac{C_{\mathbb{A}}^2}{2}\sum_{k=1}^{i}\tau_n\|eu_n^k\|^2 + \frac{C_{\mathbb{B}}^2}{2}\sum_{k=1}^{i}\tau_n\|eu_n^k - w_n^k\|^2 + \sum_{k=1}^{i}\tau_n\|e\delta z_n^k\|^2$$
$$\le \frac{1}{2}(C_{\mathbb{A}}^2 + C_{\mathbb{B}}^2)\sum_{k=1}^{i}\tau_n\Big[\|eu_n^k\|^2 + \|eu_n^k - w_n^k\|^2\Big] + \|\dot z\|^2_{L^2(0,T;V)}. \tag{2.102}$$

Notice that the following discrete integrations by parts hold

$$\sum_{k=1}^{i}\tau_n(\delta^2 u_n^k, \delta z_n^k) = (\delta u_n^i, \delta z_n^i) - (\delta u_n^0, \delta z_n^0) - \sum_{k=1}^{i}\tau_n(\delta u_n^{k-1}, \delta^2 z_n^k), \tag{2.103}$$

$$\sum_{k=1}^{i}\tau_n(h_n^k, e\delta u_n^k) = (h_n^i, eu_n^i) - (h_n^0, eu_n^0) - \sum_{k=1}^{i}\tau_n(\delta h_n^k, eu_n^{k-1}), \tag{2.104}$$

$$\sum_{k=1}^{i}\tau_n(F_n^k, e\delta u_n^k) = (F_n^i, eu_n^i) - (F_n^0, eu^0) - \sum_{k=1}^{i}\tau_n(\delta F_n^k, eu_n^{k-1}). \tag{2.105}$$

where δh_n^k, δF_n^k, and $\delta^2 z_n^k$ are defined as in (2.93). By (2.103) and

$$\sum_{k=1}^{i}\tau_n\|\delta u_n^{k-1}\|^2 = \sum_{k=0}^{i-1}\tau_n\|\delta u_n^k\|^2 \le T\|u^1\|^2 + \sum_{k=1}^{i}\tau_n\|\delta u_n^k\|^2, \tag{2.106}$$

we can write for every $\varepsilon_1 > 0$

$$\Big|\sum_{k=1}^{i}\tau_n(\delta^2 u_n^k, \delta z_n^k)\Big| \le \frac{1}{2\varepsilon_1}\|\delta z_n^i\|^2 + \frac{\varepsilon_1}{2}\|\delta u_n^i\|^2 + \|u^1\|\|\dot z(0)\| + \sum_{k=1}^{i}\tau_n\|\delta u_n^{k-1}\|\|\delta^2 z_n^k\|$$
$$\le C_{\varepsilon_1} + \|\ddot z\|^2_{L^2(0,T;H)} + \frac{\varepsilon_1}{2}\|\delta u_n^i\|^2 + \frac{1}{2}\sum_{k=1}^{i}\tau_n\|\delta u_n^k\|^2, \tag{2.107}$$

where C_{ε_1} is a positive constant depending on ε_1. Thanks to (2.104) and to (2.106) (applied to eu_n^{k-1} in place of δu_n^{k-1}) we have for every $\varepsilon_2 > 0$

$$\Big|\sum_{k=1}^{i}\tau_n(h_n^k, e\delta u_n^k)\Big| \le \frac{1}{2\varepsilon_2}\|h_n^i\|^2 + \frac{\varepsilon_2}{2}\|eu_n^i\|^2 + \|eu^0\|\|\mathbb{B}w^0\| + \sum_{k=1}^{i}\tau_n\|\delta h_n^k\|\|eu_n^{k-1}\|$$
$$\le C_{\varepsilon_2} + \frac{1}{2\beta}\|\mathbb{B}w^0\|^2 + \frac{\varepsilon_2}{2}\|eu_n^i\|^2 + \frac{1}{2}\sum_{k=1}^{i}\tau_n\|eu_n^k\|^2, \tag{2.108}$$

where C_{ε_2} is a positive constant depending on ε_2. Moreover, notice that

$$u_n^i = \sum_{k=1}^{i}\tau_n\delta u_n^k + u^0,$$

hence by means of the discrete Holder's inequality

$$\|u_n^i\| \le \sum_{k=1}^{i}\tau_n\|\delta u_n^k\| + \|u^0\| \le T^{\frac{1}{2}}\Big(\sum_{k=1}^{i}\tau_n\|\delta u_n^k\|^2\Big)^{\frac{1}{2}} + \|u^0\|. \tag{2.109}$$

By (2.105), (2.106) (applied again to eu_n^{k-1} in place of δu_n^{k-1}), and (2.109) we get for every $\varepsilon_3 > 0$

$$\Big|\sum_{k=1}^{i} \tau_n (F_n^k, e\delta u_n^k)\Big| \leq \frac{1}{2\varepsilon_3}\|F_n^i\|^2 + \frac{\varepsilon_3}{2}\|eu_n^i\|^2 + \|F(0)\|\|eu^0\| + \sum_{k=1}^{i}\tau_n\|\delta F_n^k\|\|eu_n^{k-1}\|$$
$$\leq C_{\varepsilon_3} + \frac{\varepsilon_3}{2}\|eu_n^i\|^2 + \frac{1}{2}\|\dot F\|^2_{L^2(0,T;H)} + \frac{1}{2}\sum_{k=1}^{i}\tau_n\|eu_n^k\|^2, \tag{2.110}$$

where C_{ε_3} is a positive constant depending on ε_3.

Now we consider (2.98)–(2.110). By choosing $\varepsilon_1 = \frac{1}{2}$, $\varepsilon_2 = \varepsilon_3 = \frac{c_{\mathbb{A}}}{4}$ and using (2.4) and (2.5) we obtain the existence of two positive constants C_1 and C_2 such that

$$\frac{1}{4}\|\delta u_n^i\|^2 + \frac{c_{\mathbb{A}}}{4}\|eu_n^i\|^2 + \frac{c_{\mathbb{B}}}{2}\|eu_n^i - w_n^i\|^2 + \beta c_{\mathbb{B}}\sum_{k=1}^{i}\tau_n\|\delta w_n^k\|^2$$
$$\leq C_1 + C_2\sum_{k=1}^{i}\tau_n\Big[\|\delta u_n^k\|^2 + \|eu_n^k\|^2 + \|eu_n^k - w_n^k\|^2 + \sum_{l=1}^{k}\tau_n\|\delta w_n^l\|^2\Big]. \tag{2.111}$$

By defining

$$a_n^i := \|\delta u_n^i\|^2 + \|eu_n^i\|^2 + \|eu_n^i - w_n^i\|^2 + \sum_{k=1}^{i}\tau_n\|\delta w_n^k\|^2,$$

from (2.111) we can derive

$$a_n^i \leq \tilde C_1 + \tilde C_2 \sum_{k=1}^{i}\tau_n a_n^k,$$

for two positive constants $\tilde C_1$ and $\tilde C_2$. Taking into account a discrete version of Gronwall's lemma (see, e.g., [2, Lemma 3.2.4]) we deduce that a_n^i is bounded by a positive constant C^* independent of i and n; i.e.,

$$\|\delta u_n^i\|^2 + \|eu_n^i\|^2 + \|eu_n^i - w_n^i\|^2 + \sum_{k=1}^{i}\tau_n\|\delta w_n^k\|^2 \leq C^* \quad \text{for every } i = 1,\dots,n \text{ and } n\in\mathbb{N}.$$

Therefore

$$\|\delta u_n^i\|^2 + \|eu_n^i\|^2 + \|w_n^i\|^2 + \sum_{k=1}^{i}\tau_n\|\delta w_n^k\|^2 \leq 3C^* \quad \text{for every } i = 1,\dots,n \text{ and } n\in\mathbb{N},$$

and this concludes the proof. □

We now want to pass to the limit into the discrete equation (2.94) to obtain a solution to the coupled system (2.82)–(2.86) according to Definition 2.3.1. We start by defining the following interpolation sequences of our limit solution

$$\begin{aligned}
&u_n(t) = u_n^k + (t - k\tau_n)\delta u_n^k && \text{for } t\in[(k-1)\tau_n, k\tau_n] \text{ and } k = 1,\dots,n,\\
&u_n^+(t) = u_n^k && \text{for } t\in((k-1)\tau_n, k\tau_n] \text{ and } k = 1,\dots,n, \quad u_n^+(0) = u_n^0,\\
&u_n^-(t) = u_n^{k-1} && \text{for } t\in[(k-1)\tau_n, k\tau_n) \text{ and } k = 1,\dots,n, \quad u_n^-(T) = u_n^n.
\end{aligned}$$

and the same approximations w_n, w_n^+, w_n^- for the function w. Moreover, we consider also the sequences

$$\tilde u_n(t) = \delta u_n^k + (t - k\tau_n)\delta^2 u_n^k \quad \text{for } t\in[(k-1)\tau_n, k\tau_n] \text{ and } k = 1,\dots,n,$$

$$\tilde{u}_n^+(t) = \delta u_n^k \qquad \text{for } t \in ((k-1)\tau_n, k\tau_n] \text{ and } k = 1, \dots, n, \quad \tilde{u}_n^+(0) = \delta u_n^0,$$
$$\tilde{u}_n^-(t) = \delta u_n^{k-1} \qquad \text{for } t \in [(k-1)\tau_n, k\tau_n) \text{ and } k = 1, \dots, n, \quad \tilde{u}_n^-(T) = \delta u_n^n,$$

which approximate the first time derivative of u. By using this notation, we can state the following convergence lemma.

Lemma 2.3.5. *There exists $(u, w) \in \mathcal{V} \times H^1(0,T;H)$, with $u - z \in \mathcal{V}^D$, such that, up to a not relabeled subsequence*

$$u_n \xrightarrow[n\to\infty]{H^1(0,T;H)} u, \quad u_n^\pm \xrightarrow[n\to\infty]{L^2(0,T;V)} u, \quad \tilde{u}_n^\pm \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{u}, \tag{2.112}$$
$$w_n \xrightarrow[n\to\infty]{H^1(0,T;H)} w, \quad w_n^\pm \xrightarrow[n\to\infty]{L^2(0,T;H)} w. \tag{2.113}$$

Proof. Thanks to Lemma 2.3.4 the sequences

$$\{u_n\}_n \subset H^1(0,T;H) \cap L^\infty(0,T;V), \qquad \{w_n\}_n \subset H^1(0,T;H) \cap L^\infty(0,T;H),$$
$$\{u_n^\pm\}_n \subset L^\infty(0,T;V), \qquad \{w_n^\pm\}_n \subset L^\infty(0,T;H),$$
$$\{\tilde{u}_n^\pm\}_n \subset L^\infty(0,T;H),$$

are uniformly bounded. Indeed, by means of (2.95) and (2.109) there exists a positive constant $\bar{C}$ such that $\|u_n^i\|_V \le \bar{C}$ for every $n \in \mathbb{N}$ and $i = 1,..,n$, and therefore

$$\|u_n\|_{L^\infty(0,T;V)} \le \max_{k=1,..,n} \sup_{t\in[(k-1)\tau_n, k\tau_n]} \| \left(1 - k + t\tau_n^{-1}\right) u_n^k + \left(k - t\tau_n^{-1}\right) u_n^{k-1}\|_V \le 2\bar{C}.$$

By Banach-Alaoglu's Theorem there exist some functions

$$u \in H^1(0,T;H), \quad w \in H^1(0,T;H), \quad v_1 \in L^2(0,T;V), \quad v_2 \in L^2(0,T;H)$$

such that, up to a not relabeled subsequence

$$u_n \xrightarrow[n\to\infty]{L^2(0,T;V)} u, \qquad \dot{u}_n \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{u}, \qquad u_n^+ \xrightarrow[n\to\infty]{L^2(0,T;V)} v_1, \tag{2.114}$$
$$w_n \xrightarrow[n\to\infty]{L^2(0,T;H)} w, \qquad \dot{w}_n \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{w}, \qquad w_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} v_2. \tag{2.115}$$

Since there exists a positive constant C such that

$$\|u_n - u_n^+\|_{L^\infty(0,T;H)} \le C\tau_n \xrightarrow[n\to\infty]{} 0, \quad \|w_n - w_n^+\|_{L^\infty(0,T;H)} \le C\tau_n \xrightarrow[n\to\infty]{} 0, \tag{2.116}$$

by using (2.114), (2.115) and triangle inequality, we can conclude that $u = v_1$ and $w = v_2$. Moreover, given that

$$u_n^-(t) = u_n^+(t - \tau_n), \qquad w_n^-(t) = w_n^+(t - \tau_n) \quad \text{for } t \in (\tau_n, T),$$
$$\tilde{u}_n^-(t) = \tilde{u}_n^+(t - \tau_n), \qquad \text{for } t \in (\tau_n, T),$$
$$\tilde{u}_n^+(t) = \dot{u}_n(t), \qquad \text{for a.e. } t \in (0, T),$$

with (2.116) and the continuity of the translations in L^2 we deduce that

$$u_n^- \xrightarrow[n\to\infty]{L^2(0,T;V)} u, \quad \tilde{u}_n^\pm \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{u}, \quad w_n^- \xrightarrow[n\to\infty]{L^2(0,T;H)} w.$$

Now let us check that $u \in \mathcal{V}$. To this aim, we define the following sets

$$\tilde{\mathcal{V}} := \{u \in L^2(0,T;V) : u(t) \in V_t \text{ for a.e. } t \in (0,T)\} \subset L^2(0,T;V),$$

$$\tilde{\mathcal{V}}^D := \{u \in \tilde{\mathcal{V}} : u(t) \in V_t^D \text{ for a.e. } t \in (0,T)\} \subset L^2(0,T;V).$$

Notice that $\tilde{\mathcal{V}}$ is a (strong) closed convex subset of $L^2(0,T;V)$, and so by Hahn-Banach Theorem the set $\tilde{\mathcal{V}}$ is weakly closed. In the same way we can prove that $\tilde{\mathcal{V}}^D$ is also a weakly closed set. Notice that $\{u_n^-\}_n \subset \tilde{\mathcal{V}}$, indeed

$$u_n^-(t) = u_n^{k-1} \in V_{(k-1)\tau_n} \subset V_t \quad \text{for } t \in [(k-1)\tau_n, k\tau_n),\ k = 1,\dots,n.$$

Since $u_n^- \xrightarrow[n\to\infty]{L^2(0,T;V)} u$, we conclude that $u \in \tilde{\mathcal{V}}$. Moreover $\tilde{u}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{u}$ and so $\dot{u} \in L^2(0,T;H)$, from which we have $u \in \mathcal{V}$. Finally, to show that $u - z \in \mathcal{V}^D$ we observe that

$$u_n^-(t) - z_n^-(t) = u_n^{k-1} - z_n^{k-1} \in V_n^{k-1} \subset V_t^D \qquad \text{for } t \in [(k-1)\tau_n, k\tau_n),\ k = 1,\dots,n,$$

therefore $\{u_n^- - z_n^-\}_n \subset \tilde{\mathcal{V}}^D$. Since

$$u_n^- \xrightarrow[n\to\infty]{L^2(0,T;V)} u, \quad z_n^- \xrightarrow[n\to\infty]{L^2(0,T;V)} z,$$

we get $u - z \in \mathcal{V}^D$. This concludes the proof. □

With the next lemma we show that the limit identified by Lemma 2.3.5 is actually a weak solution to the coupled system (2.82)–(2.86).

Lemma 2.3.6. *The limit pair $(u,w) \in \mathcal{V} \times H^1(0,T;H)$ of Lemma 2.3.5 satisfies (2.87) and (2.88).*

Proof. We fix $n \in \mathbb{N}$ and the functions $\varphi \in \mathcal{D}^D$ and $\psi \in C_c^\infty(0,T;H)$. We consider the following piecewise-constant approximating sequences

$$\begin{aligned} \varphi_n^k &:= \varphi(k\tau_n) & \psi_n^k &:= \psi(k\tau_n) & &\text{for } k = 0,\dots,n,\\ \delta\varphi_n^k &:= \frac{\varphi_n^k - \varphi_n^{k-1}}{\tau_n} & \delta\psi_n^k &:= \frac{\psi_n^k - \psi_n^{k-1}}{\tau_n} & &\text{for } k = 1,\dots,n, \end{aligned}$$

and the approximating sequences

$$\begin{aligned} \varphi_n^+(t) &:= \varphi_n^k, & \tilde{\varphi}_n^+(t) &:= \delta\varphi_n^k & &t \in ((k-1)\tau_n, k\tau_n], & &k = 1,\dots,n,\\ \psi_n^+(t) &:= \psi_n^k, & \tilde{\psi}_n^+(t) &:= \delta\psi_n^k & &t \in ((k-1)\tau_n, k\tau_n], & &k = 1,\dots,n. \end{aligned}$$

If we use $\tau_n(\varphi_n^k, 0) \in V_n^k \times H$ as a test function in (2.94), after summing over $k = 1,...,n$, we get

$$\sum_{k=1}^n \tau_n(\delta^2 u_n^k, \varphi_n^k) + \sum_{k=1}^n \tau_n((\mathbb{A}+\mathbb{B})eu_n^k - \mathbb{B}w_n^k, e\varphi_n^k)$$
$$= \sum_{k=1}^n \tau_n(f_n^k, \varphi_n^k) + \sum_{k=1}^n \tau_n(F_n^k, e\varphi_n^k) - \sum_{k=1}^n \tau_n(h_n^k, e\varphi_n^k). \tag{2.117}$$

Since $\varphi_n^0 = \varphi_n^n = 0$ we obtain

$$\sum_{k=1}^n \tau_n(\delta^2 u_n^k, \varphi_n^k) = \sum_{k=1}^n (\delta u_n^k, \varphi_n^k) - \sum_{k=1}^n (\delta u_n^{k-1}, \varphi_n^k) = \sum_{k=0}^{n-1} (\delta u_n^k, \varphi_n^k) - \sum_{k=0}^{n-1} (\delta u_n^k, \varphi_n^{k+1})$$
$$= -\sum_{k=0}^{n-1} (\delta u_n^k, \delta\varphi_n^{k+1}) = -\sum_{k=1}^n \tau_n(\delta u_n^{k-1}, \delta\varphi_n^k) = -\int_0^T (\tilde{u}_n^-(t), \tilde{\varphi}_n^+(t))\mathrm{d}t,$$

and from (2.117) we deduce

$$-\int_0^T (\tilde{u}_n^-(t), \tilde{\varphi}_n^+(t))\mathrm{d}t + \int_0^T ((\mathbb{A}+\mathbb{B})eu_n^+(t) - \mathbb{B}w_n^+(t), e\varphi_n^+(t))\mathrm{d}t$$
$$= \int_0^T (f_n^+(t), \varphi_n^+(t))\mathrm{d}t + \int_0^T (F_n^+(t), e\varphi_n^+(t))\mathrm{d}t - \int_0^T (h_n^+(t), e\varphi_n^+(t))\mathrm{d}t. \qquad (2.118)$$

Thanks to (2.112), (2.113), and to the convergences

$$\varphi_n^+ \xrightarrow[n\to\infty]{L^2(0,T;V)} \varphi, \qquad \tilde{\varphi}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{\varphi}$$

we can pass to the limit in (2.118), and we get that $u \in \mathcal{V}$ satisfies (2.87) for every function $\varphi \in \mathcal{D}^D$.

If we use $\tau_n(0, \psi_n^k) \in V_n^k \times H$ as a test function in (2.94), we have

$$(\beta\delta w_n^k + w_n^k - eu_n^k, \psi_n^k) = 0,$$

which corresponds to

$$(\beta\dot{w}_n(t) + w_n^+(t) - eu_n^+(t), \psi_n^+(t)) = 0 \qquad t \in ((k-1)\tau_n, k\tau_n], \quad k = 1, \dots, n.$$

Therefore, for every $(a, b) \subset (0, T)$, from (2.112) and (2.113), we can write

$$0 = \lim_{n\to\infty} \int_a^b (\beta\dot{w}_n(t) + w_n^+(t) - eu_n^+(t), \psi_n^+(t))\mathrm{d}t = \int_a^b (\beta\dot{w}(t) + w(t) - eu(t), \psi(t))\mathrm{d}t. \quad (2.119)$$

Now we pass to the limit in (2.119) as $a \to b$ and we obtain

$$(\beta\dot{w}(b) + w(b) - eu(b), \psi(b)) = 0 \qquad \text{for every } b \in [0, T].$$

Given that, fixed $b \in (0, T)$ for every $\mathbf{p} \in H$ there exists $\psi_{\mathbf{p}}(t) := (t + 1 - b)\mathbf{p} \in H^1(0, T; H)$ such that $\psi_{\mathbf{p}}(b) = \mathbf{p}$, we can say that for a.e. $t \in (0, T)$ we have $\beta\dot{w}(t) + w(t) - eu(t) = 0$ in H. Finally, since $w_n(0) = w^0$, taking into account (2.113) we can conclude that $w(0) = w^0$. □

It remains to show that the limit previously found assumes the initial data in the sense of (2.22). Before doing this, let us recall the following result, whose proof can be found for example in [20].

Lemma 2.3.7. *Let X, Y be reflexive Banach spaces such that $X \hookrightarrow Y$ continuously. Then*

$$L^\infty(0, T; X) \cap C_w^0([0, T]; Y) = C_w^0([0, T]; X).$$

Proposition 2.3.8. *The limit pair $(u, w) \in \mathcal{V} \times H^1(0, T; H)$ of Lemma 2.3.5 is a weak solution to the coupled system (2.82)–(2.86). Moreover, $u \in C_w^0([0, T]; V)$, $\dot{u} \in C_w^0([0, T]; H)$ and it admits a distributional derivative in the space $L^2(0, T; (V_0^D)')$.*

Proof. From the discrete equation (2.94) we deduce

$$|(\delta^2 u_n^k, \varphi)| \le C_{\mathbb{A}}\|eu_n^k\| + C_{\mathbb{B}}\|eu_n^k - w_n^k\| + \beta C_{\mathbb{B}}\|\delta w_n^k\| + \|f_n^k\| + \|F_n^k\| + \|h_n^k\|,$$

for every $(\varphi, \psi) \in V_0^D \times H \subset V_n^k \times H$ such that $\|(\varphi, \psi)\|_{V\times H} \le 1$. Therefore, taking the supremum over $(\varphi, \psi) \in V_0^D \times H$ with $\|(\varphi, \psi)\|_{V\times H} \le 1$, we obtain the existence of a positive constant C' such that

$$\|\delta^2 u_n^k\|^2_{(V_0^D)'} \le C'(\|eu_n^k\|^2 + \|eu_n^k - w_n^k\|^2 + \|\delta w_n^k\|^2 + \|f_n^k\|^2 + \|F_n^k\|^2 + \|h_n^k\|^2).$$

By multiplying this inequality by τ_n and then by summing over $k = 1, \dots, n$, we get

$$\sum_{k=1}^{n} \tau_n \|\delta^2 u_n^k\|^2_{(V_0^D)'} \le C' \Big(\sum_{k=1}^{n} \tau_n \|eu_n^k\|^2 + \sum_{k=1}^{n} \tau_n \|eu_n^k - w_n^k\|^2 + \sum_{k=1}^{n} \tau_n \|\delta w_n^k\|^2 + C'' \Big), \quad (2.120)$$

where

$$C'' := \|f\|^2_{L^2(0,T;H)} + \|F\|^2_{L^2(0,T;H)} + T\|\mathbb{B}w^0\|^2.$$

Thanks to (2.120) and Lemma 2.3.4 we conclude that there exists a positive constant $\tilde{C}$, which does not depend on n, such that

$$\sum_{k=1}^{n} \tau_n \|\delta^2 u_n^k\|^2_{(V_0^D)'} \le \tilde{C}. \quad (2.121)$$

In particular $\{\tilde{u}_n\}_n \subset H^1(0,T;(V_0^D)')$ is uniformly bounded (notice that $\dot{\tilde{u}}_n(t) = \delta^2 u_n^k$ for $t \in ((k-1)\tau_n, k\tau_n)$ and $k = 1, \dots, n$). Hence, up to extracting a further (not relabeled) subsequence from the one of Lemma 2.3.5, we have

$$\tilde{u}_n \xrightarrow[n\to\infty]{H^1(0,T;(V_0^D)')} v, \quad (2.122)$$

and by using the following estimate

$$\|\tilde{u}_n - \tilde{u}_n^+\|^2_{L^2(0,T;(V_0^D)')} \le \tilde{C}\tau_n^2 \xrightarrow[n\to\infty]{} 0,$$

we conclude that $v = \dot{u}$.

Since $H^1(0,T;(V_0^D)') \hookrightarrow C^0([0,T],(V_0^D)')$, by using Lemma 2.3.5 and Lemma 2.3.7 we deduce that the limit pair $(u,w) \in \mathcal{V} \times H^1(0,T;H)$ satisfies

$$u \in C^0_w([0,T];V) \quad \text{and} \quad \dot{u} \in C^0_w([0,T];H).$$

By (2.112) and (2.122) we then obtain

$$u_n(t) \xrightarrow[n\to\infty]{H} u(t) \quad \text{and} \quad \tilde{u}_n(t) \xrightarrow[n\to\infty]{(V_0^D)'} \dot{u}(t) \qquad \text{for every } t \in [0,T], \quad (2.123)$$

so that $u(0) = u^0$ and $\dot{u}(0) = u^1$, since $u_n(0) = u^0$ and $\tilde{u}_n(0) = u^1$. By Lemma 2.3.6 we get the thesis. □

2.3.2 Energy Estimate

In this subsection, we prove an energy-dissipation inequality which holds for the weak solution $(u,w) \in \mathcal{V} \times H^1(0,T;H)$ to the coupled system (2.82)–(2.86), provided by Lemma 2.3.5. Thanks to this, we are able to show the validity of the initial conditions in a stronger sense. The energy-dissipation inequality give us a relation among the mechanical energy defined by the sum of kinetic and elastic energy, the dissipation and the total work exerted by external forces and by the boundary conditions. Therefore, let us define the total energy as

$$\mathscr{E}_{u,w}(t) := \frac{1}{2}\|\dot{u}(t)\|^2 + \frac{1}{2}(\mathbb{A}eu(t), eu(t)) + \frac{1}{2}(\mathbb{B}(eu(t) - w(t)), eu(t) - w(t)). \quad (2.124)$$

Notice that $\mathscr{E}_{u,w}(t)$ is well defined for every time $t \in [0,T]$ since $u \in C^0_w([0,T];V)$, $\dot{u} \in C^0_w([0,T];H)$ and $w \in C^0([0,T];H)$, and that

$$\mathscr{E}_{u,w}(0) = \frac{1}{2}\|u^1\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0) + \frac{1}{2}(\mathbb{B}(eu^0 - w^0), eu^0 - w^0).$$

The dissipation, on the interval $[0,t]$, is defined by

$$\mathscr{D}_{u,w}(t) := \beta \int_0^t (\mathbb{B}\dot{w}(\tau), \dot{w}(\tau))\mathrm{d}\tau, \tag{2.125}$$

and the total work is given by

$$\begin{aligned}\mathscr{W}_{tot}(t) :=& \int_0^t [(f(\tau), \dot{u}(\tau) - \dot{z}(\tau)) + ((\mathbb{A}+\mathbb{B})eu(\tau) - \mathbb{B}w(\tau), e\dot{z}(\tau))]\mathrm{d}\tau \\ &- \int_0^t (\dot{F}(\tau), eu(\tau) - ez(\tau))\mathrm{d}\tau + (F(t), eu(t) - ez(t)) - (F(0), eu^0 - ez(0)) \\ &- \int_0^t (\dot{u}(\tau), \ddot{z}(\tau))\mathrm{d}\tau + (\dot{u}(t), \dot{z}(t)) - (u^1, \dot{z}(0)) \\ &+ \int_0^t \Big[\mathrm{e}^{-\frac{\tau}{\beta}}(\mathbb{B}w^0, e\dot{z}(\tau)) - \frac{1}{\beta}\mathrm{e}^{-\frac{\tau}{\beta}}(\mathbb{B}w^0, eu(\tau))\Big]\mathrm{d}\tau - \mathrm{e}^{-\frac{t}{\beta}}(\mathbb{B}w^0, eu(t)) + (\mathbb{B}w^0, eu^0).\end{aligned} \tag{2.126}$$

Remark 2.3.9. From the classical point of view, the total work on the solution (u,w) at time $t \in [0,T]$ is given by

$$\mathscr{W}_C(t) := \mathscr{W}_{load}(t) + \mathscr{W}_{bdry}(t), \tag{2.127}$$

where $\mathscr{W}_{load}(t)$ is the work on the solution at time $t \in [0,T]$ due to the loading term, which is defined as

$$\mathscr{W}_{load}(t) := \int_0^t (f(\tau), \dot{u}(\tau))\mathrm{d}\tau + \int_0^t (\mathrm{div}(\mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0 - F(\tau)), \dot{u}(\tau))\mathrm{d}\tau, \tag{2.128}$$

and $\mathscr{W}_{bdry}(t)$ is the work on the solution at time $t \in [0,T]$ due to the varying boundary conditions, which one expects to be equal to

$$\begin{aligned}\mathscr{W}_{bdry}(t) :=& \int_0^t ((F_+(\tau) - \mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0_+)\nu, \dot{u}(\tau))_{L^2(\Gamma_\tau)}\mathrm{d}\tau + \int_0^t ((F_-(\tau) - \mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0_-)\nu, \dot{u}(\tau))_{L^2(\Gamma_\tau)}\mathrm{d}\tau \\ &+ \int_0^t ((F(\tau) - \mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0)\nu, \dot{u}(\tau))_{H^N}\mathrm{d}\tau + \int_0^t (((\mathbb{A}+\mathbb{B})eu(\tau) - \mathbb{B}w(\tau))\nu, \dot{z}(\tau))_{H^D}\mathrm{d}\tau,\end{aligned}$$

where $F_+(t)$, w^0_+ and $F_-(t)$, w^0_- are the traces of $F(t)$ and w^0, respectively, from above and below on Γ_t.

Unfortunately, $\mathscr{W}_{load}(t)$ and $\mathscr{W}_{bdry}(t)$ are not well defined under our assumptions on u, F, and w^0. However, if we suppose more regularity, i.e. $u \in H^1(0,T;H^2(\Omega\setminus\Gamma;\mathbb{R}^d)) \cap H^2(0,T;H)$, $w \in H^1(0,T;H^1(\Omega\setminus\Gamma;\mathbb{R}^{d\times d}_{sym}))$, $F \in H^1(0,T;H^1(\Omega\setminus\Gamma;\mathbb{R}^{d\times d}_{sym}))$, $w^0 \in V_0$, and that Γ is a smooth manifold, then we can deduce from (2.87), (2.88), and (2.22) that the pair (u,w) satisfies

$$\begin{cases}\ddot{u}(t) - \mathrm{div}(\mathbb{A}eu(t)) - \mathrm{div}(\mathbb{B}(eu(t) - w(t))) = f(t) + g(t) & \text{in } \Omega\setminus\Gamma_t, \quad t \in (0,T), \\ \beta\dot{w}(t) + w(t) - eu(t) = 0\end{cases} \tag{2.129}$$

with boundary and initial conditions

$$\begin{aligned}&u(t) = z(t) && \text{on } \partial_D\Omega, \quad t \in (0,T), \\ &[(\mathbb{A}+\mathbb{B})eu(t) - \mathbb{B}w(t)]\nu = [F(t) - \mathrm{e}^{-\frac{t}{\beta}}\mathbb{B}w^0]\nu && \text{on } \partial_N\Omega, \quad t \in (0,T), \\ &[(\mathbb{A}+\mathbb{B})eu_+(t) - \mathbb{B}w_+(t)]\nu = [F_+(t) - \mathrm{e}^{-\frac{t}{\beta}}\mathbb{B}w^0_+]\nu && \text{on } \Gamma_t, \quad t \in (0,T), \\ &[(\mathbb{A}+\mathbb{B})eu_-(t) - \mathbb{B}w_-(t)]\nu = [F_-(t) - \mathrm{e}^{-\frac{t}{\beta}}\mathbb{B}w^0_-]\nu && \text{on } \Gamma_t, \quad t \in (0,T),\end{aligned}$$

$$u(0) = u^0, \quad w(0) = w^0, \quad \dot{u}(0) = u^1,$$

where $g(t) := \operatorname{div}(\mathrm{e}^{-\frac{t}{\beta}}\mathbb{B}w^0 - F(t))$.

In this case, $((\mathbb{A}+\mathbb{B})eu - w)\nu \in L^2(0,T;H^D)$ and by using (2.129), together with the divergence theorem and the integration by parts formula, we deduce

$$\begin{aligned}
&\int_0^t (((\mathbb{A}+\mathbb{B})eu(\tau) - \mathbb{B}w(\tau))\nu, \dot{z}(\tau))_{H^D} \mathrm{d}\tau = \int_0^t ((\mathbb{A}+\mathbb{B})eu(\tau) - \mathbb{B}w(\tau), e\dot{z}(\tau))\mathrm{d}\tau \\
&\quad + \int_0^t \Big[(\operatorname{div}((\mathbb{A}+\mathbb{B})eu(\tau) - \mathbb{B}w(\tau)), \dot{z}(\tau)) + ((\mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0 - F(\tau))\nu, \dot{z}(\tau))_{H^N}\Big] \mathrm{d}\tau \\
&\quad + \int_0^t \Big[((\mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0_+ - F_+(\tau))\nu, \dot{z}(\tau))_{L^2(\Gamma_\tau)} + ((\mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0_- - F_-(\tau))\nu, \dot{z}(\tau))_{L^2(\Gamma_\tau)}\Big] \mathrm{d}\tau \\
&= \int_0^t \Big[((\mathbb{A}+\mathbb{B})eu(\tau) - \mathbb{B}w(\tau), e\dot{z}(\tau)) + ((\mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0 - F(\tau))\nu, \dot{z}(\tau))_{H^N}\Big] \mathrm{d}\tau \\
&\quad + \int_0^t \Big[(\ddot{u}(\tau), \dot{z}(\tau)) - (f(\tau), \dot{z}(\tau)) + (\operatorname{div} F(\tau), \dot{z}(\tau)) - \mathrm{e}^{-\frac{\tau}{\beta}}(\operatorname{div}(\mathbb{B}w^0), \dot{z}(\tau))\Big] \mathrm{d}\tau \\
&\quad + \int_0^t \Big[((\mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0_+ - F_+(\tau))\nu, \dot{z}(\tau))_{L^2(\Gamma_\tau)} + ((\mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0_- - F_-(\tau))\nu, \dot{z}(\tau))_{L^2(\Gamma_\tau)}\Big] \mathrm{d}\tau \\
&= \int_0^t \Big[((\mathbb{A}+\mathbb{B})eu(\tau) - \mathbb{B}w(\tau), e\dot{z}(\tau)) + ((\mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0 - F(\tau))\nu, \dot{z}(\tau))_{H^N}\Big] \mathrm{d}\tau \\
&\quad - \int_0^t (\dot{u}(\tau), \ddot{z}(\tau))\mathrm{d}\tau + (\dot{u}(t), \dot{z}(t)) - (u^1, \dot{z}(0)) \\
&\quad + \int_0^t \Big[-(f(\tau), \dot{z}(\tau)) + (\operatorname{div} F(\tau), \dot{z}(\tau)) - \mathrm{e}^{-\frac{\tau}{\beta}}(\operatorname{div}(\mathbb{B}w^0), \dot{z}(\tau))\Big] \mathrm{d}\tau \\
&\quad + \int_0^t \Big[((\mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0_+ - F_+(\tau))\nu, \dot{z}(\tau))_{L^2(\Gamma_\tau)} + ((\mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0_- - F_-(\tau))\nu, \dot{z}(\tau))_{L^2(\Gamma_\tau)}\Big] \mathrm{d}\tau. \quad (2.130)
\end{aligned}$$

From (2.130) and the definition of $\mathscr{W}_{bdry}$, we have

$$\begin{aligned}
\mathscr{W}_{bdry}(t) &= \int_0^t \Big[((\mathbb{A}+\mathbb{B})eu(\tau) - \mathbb{B}w(\tau), e\dot{z}(\tau)) + ((F(\tau) - \mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0)\nu, \dot{u}(\tau) - \dot{z}(\tau))_{H^N}\Big] \mathrm{d}\tau \\
&\quad + \int_0^t \big[-(f(\tau), \dot{z}(\tau)) + (\operatorname{div}(F(\tau) - \mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0), \dot{z}(\tau))\big]\mathrm{d}\tau \\
&\quad - \int_0^t (\dot{u}(\tau), \ddot{z}(\tau))\mathrm{d}\tau - (u^1, \dot{z}(0)) + (\dot{u}(t), \dot{z}(t)) \\
&\quad + \int_0^t ((F_+(\tau) - \mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0_+)\nu, \dot{u}(\tau) - \dot{z}(\tau))_{L^2(\Gamma_\tau)}\mathrm{d}\tau \\
&\quad + \int_0^t ((F_-(\tau) - \mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0_-)\nu, \dot{u}(\tau) - \dot{z}(\tau))_{L^2(\Gamma_\tau)}\mathrm{d}\tau. \quad (2.131)
\end{aligned}$$

Taking into account (2.128) and (2.131), the classical work (2.127) can be written as

$$\begin{aligned}
\mathscr{W}_C(t) &= \int_0^t [(f(\tau), \dot{u}(\tau) - \dot{z}(\tau)) + ((\mathbb{A}+\mathbb{B})eu(\tau) - \mathbb{B}w(\tau), e\dot{z}(\tau))]\,\mathrm{d}\tau \\
&\quad - \int_0^t (\dot{u}(\tau), \ddot{z}(\tau))\mathrm{d}\tau + (\dot{u}(t), \dot{z}(t)) - (u^1, \dot{z}(0)) \\
&\quad + \int_0^t ((F_+(\tau) - \mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0_+)\nu, \dot{u}(\tau) - \dot{z}(\tau))_{L^2(\Gamma_\tau)}\mathrm{d}\tau \\
&\quad + \int_0^t ((F_-(\tau) - \mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0_-)\nu, \dot{u}(\tau) - \dot{z}(\tau))_{L^2(\Gamma_\tau)}\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^t (\operatorname{div}(F(\tau)-\mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0),\dot u(\tau)-\dot z(\tau))\mathrm{d}\tau\\
&+\int_0^t ((F(\tau)-\mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0)\nu,\dot u(\tau)-\dot z(\tau))_{H^N}\mathrm{d}\tau\\
=&\int_0^t [(f(\tau),\dot u(\tau)-\dot z(\tau))+((\mathbb{A}+\mathbb{B})eu(\tau)-\mathbb{B}w(\tau),e\dot z(\tau))]\,\mathrm{d}\tau+(\dot u(t),\dot z(t))\\
&+\int_0^t \Big[(F(\tau)-\mathrm{e}^{-\frac{\tau}{\beta}}\mathbb{B}w^0,e\dot u(\tau)-e\dot z(\tau))-(\dot u(\tau),\ddot z(\tau))\Big]\,\mathrm{d}\tau-(u^1,\dot z(0))\\
=&\int_0^t \Big[(f(\tau),\dot u(\tau)-\dot z(\tau))+((\mathbb{A}+\mathbb{B})eu(\tau)-\mathbb{B}w(\tau),e\dot z(\tau))+\mathrm{e}^{-\frac{\tau}{\beta}}(\mathbb{B}w^0,e\dot z(\tau))\Big]\,\mathrm{d}\tau\\
&-\int_0^t (\dot u(\tau),\ddot z(\tau))\mathrm{d}\tau+(\dot u(t),\dot z(t))-(u^1,\dot z(0))\\
&-\int_0^t (\dot F(\tau),eu(\tau)-ez(\tau))\mathrm{d}\tau+(F(t),eu(t)-ez(t))-(F(0),eu^0-ez(0))\\
&-\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{\tau}{\beta}}(\mathbb{B}w^0,eu(\tau))\mathrm{d}\tau+(\mathbb{B}w^0,eu^0)-\mathrm{e}^{-\frac{t}{\beta}}(\mathbb{B}w^0,eu(t)).
\end{aligned}
$$

Therefore, the definition of total work given in (2.126) is coherent with the classical one (2.127).

Now we are in position to prove the energy-dissipation inequality before mentioned. For convenience of notation we set $h(t):=\mathrm{e}^{-\frac{t}{\beta}}\mathbb{B}w^0$.

Theorem 2.3.10. *The weak solution $(u,w)\in\mathcal{V}\times H^1(0,T;H)$ to the coupled system (2.82)–(2.86), given by Lemma 2.3.5, satisfies for every $t\in[0,T]$ the following energy-dissipation inequality*

$$
\mathscr{E}_{u,w}(t)+\mathscr{D}_{u,w}(t)\le\mathscr{E}_{u,w}(0)+\mathscr{W}_{tot}(t), \tag{2.132}
$$

where $\mathscr{E}_{u,w}$, $\mathscr{D}_{u,w}$, and $\mathscr{W}_{tot}$ are defined in (2.124), (2.125), and (2.126), respectively.

Proof. Fixed $t\in(0,T]$, for every $n\in\mathbb{N}$ there exists a unique $j\in\{1,\dots,n\}$ such that $t\in((j-1)\tau_n,j\tau_n]$. In particular, denoting by $\lceil x\rceil$ the superior integer part of the number x, it reads as

$$
j(n)=\left\lceil\frac{t}{\tau_n}\right\rceil.
$$

After setting $t_n:=j\tau_n$, we can rewrite (2.98) as follows

$$
\begin{aligned}
\frac12\|\tilde u_n^+(t)\|^2+\frac12(\mathbb{A}eu_n^+(t),eu_n^+(t))+\frac12(\mathbb{B}(eu_n^+(t)-w_n^+(t)),eu_n^+(t)-w_n^+(t))&\\
+\beta\int_0^{t_n}(\mathbb{B}\dot w_n(\tau),\dot w_n(\tau))\mathrm{d}\tau\le\mathscr{E}_{u,w}(0)+\mathscr{W}_n^+(t),&
\end{aligned} \tag{2.133}
$$

where

$$
\begin{aligned}
\mathscr{W}_n^+(t):=&\int_0^{t_n}[(f_n^+(\tau),\tilde u_n^+(\tau)-\tilde z_n^+(\tau))+(F_n^+(\tau),e\tilde u_n^+(\tau)-e\tilde z_n^+(\tau))+(\dot{\tilde u}_n(\tau),\tilde z_n^+(\tau))]\mathrm{d}\tau\\
&+\int_0^{t_n}\left[((\mathbb{A}+\mathbb{B})eu_n^+(\tau)-\mathbb{B}w_n^+(\tau),e\tilde z_n^+(\tau))-(h_n^+(\tau),e\tilde u_n^+(\tau)-e\tilde z_n^+(\tau))\right]\mathrm{d}\tau.
\end{aligned}
$$

Thanks to (2.95) and (2.121), we have

$$
\|w_n(t)-w_n^+(t)\|^2=\|w_n^j+(t-j\tau_n)\delta w_n^j-w_n^j\|^2\le\tau_n^2\|\delta w_n^j\|^2\le C\tau_n\xrightarrow[n\to\infty]{}0,
$$

$$\|u_n(t) - u_n^+(t)\| = \|u_n^j + (t - j\tau_n)\delta u_n^j - u_n^j\| \le \tau_n\|\delta u_n^j\| \le C\tau_n \xrightarrow[n\to\infty]{} 0,$$
$$\|\tilde{u}_n(t) - \tilde{u}_n^+(t)\|^2_{(V_0^D)'} = \|\delta u_n^j + (t - j\tau_n)\delta^2 u_n^j - \delta u_n^j\|^2_{(V_0^D)'} \le \tau_n^2\|\delta^2 u_n^j\|^2_{(V_0^D)'} \le \tilde{C}\tau_n \xrightarrow[n\to\infty]{} 0.$$

The last convergences and (2.123) imply

$$u_n^+(t) \xrightarrow[n\to\infty]{H} u(t), \qquad w_n^+(t) \xrightarrow[n\to\infty]{H} w(t), \qquad \tilde{u}_n^+(t) \xrightarrow[n\to\infty]{(V_0^D)'} \dot{u}(t),$$

and since $\|u_n^+(t)\|_V + \|\tilde{u}_n^+(t)\| \le C$ for every $n \in \mathbb{N}$, we get

$$u_n^+(t) \xrightharpoonup[n\to\infty]{V} u(t), \qquad w_n^+(t) \xrightharpoonup[n\to\infty]{H} w(t), \qquad \tilde{u}_n^+(t) \xrightharpoonup[n\to\infty]{H} \dot{u}(t). \tag{2.134}$$

By (2.134) and the lower semicontinuity property of the maps $v \mapsto \|v\|^2$, $v \mapsto (\mathbb{A}v, v)$, and $v \mapsto (\mathbb{B}v, v)$, we conclude

$$\|\dot{u}(t)\|^2 \le \liminf_{n\to\infty} \|\tilde{u}_n^+(t)\|^2, \tag{2.135}$$
$$(\mathbb{A}eu(t), eu(t)) \le \liminf_{n\to\infty} (\mathbb{A}eu_n^+(t), eu_n^+(t)), \tag{2.136}$$
$$(\mathbb{B}(eu(t) - w(t)), eu(t) - w(t)) \le \liminf_{n\to\infty} (\mathbb{B}(eu_n^+(t) - w_n^+(t)), eu_n^+(t) - w_n^+(t)). \tag{2.137}$$

Moreover, from Lemma 2.3.5, and in particular by (2.113) we get

$$\begin{aligned}\int_0^t (\mathbb{B}\dot{w}(\tau), \dot{w}(\tau))\mathrm{d}\tau &\le \liminf_{n\to\infty} \int_0^t (\mathbb{B}\dot{w}_n(\tau), \dot{w}_n(\tau))\mathrm{d}\tau \\ &\le \liminf_{n\to\infty} \int_0^{t_n} (\mathbb{B}\dot{w}_n(\tau), \dot{w}_n(\tau))\mathrm{d}\tau, \end{aligned} \tag{2.138}$$

since $t \le t_n$ and $v \mapsto \int_0^t (\mathbb{B}v(\tau), v(\tau))\mathrm{d}\tau$ is a non negative quadratic form on $L^2(0, T; H)$.

Now, we study the right-hand side of (2.133). Since we have

$$\chi_{[0,t_n]} f_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \chi_{[0,t]} f \quad \text{and} \quad \tilde{u}_n^+ - \tilde{z}_n^+ \xrightharpoonup[n\to\infty]{L^2(0,T;H)} \dot{u} - \dot{z},$$

we deduce that

$$\int_0^{t_n} (f_n^+(\tau), \tilde{u}_n^+(\tau) - \tilde{z}_n^+(\tau))\mathrm{d}\tau \xrightarrow[n\to\infty]{} \int_0^t (f(\tau), \dot{u}(\tau) - \dot{z}(\tau))\mathrm{d}\tau. \tag{2.139}$$

In a similar way, since the following convergences hold

$$\chi_{[0,t_n]} e\tilde{z}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \chi_{[0,t]} e\dot{z}, \qquad h_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} h,$$
$$(\mathbb{A} + \mathbb{B})eu_n^+ - \mathbb{B}w_n^+ \xrightharpoonup[n\to\infty]{L^2(0,T;H)} (\mathbb{A} + \mathbb{B})eu - \mathbb{B}w,$$

we obtain

$$\int_0^{t_n} (h_n^+(\tau), e\tilde{z}_n^+(\tau))\mathrm{d}\tau \xrightarrow[n\to\infty]{} \int_0^t (h(\tau), e\dot{z}(\tau))\mathrm{d}\tau \tag{2.140}$$
$$\int_0^{t_n} ((\mathbb{A} + \mathbb{B})eu_n^+(\tau) - \mathbb{B}w_n^+(\tau), e\tilde{z}_n^+(\tau))\mathrm{d}\tau \xrightarrow[n\to\infty]{} \int_0^t ((\mathbb{A} + \mathbb{B})eu(\tau) - \mathbb{B}w(\tau), e\dot{z}(\tau))\mathrm{d}\tau. \tag{2.141}$$

By means of the discrete integration by parts formulas (2.103)–(2.105) we can write

$$\int_0^{t_n} (\dot{\tilde{u}}_n(\tau), \tilde{z}_n^+(\tau))\mathrm{d}\tau = (\tilde{u}_n^+(t), \tilde{z}_n^+(t)) - (u^1, \dot{z}(0)) - \int_0^{t_n} (\tilde{u}_n^-(\tau), \dot{\tilde{z}}_n(\tau))\mathrm{d}\tau, \tag{2.142}$$

$$\int_0^{t_n}(h_n^+(\tau),e\tilde{u}_n^+(\tau))\mathrm{d}\tau=(eu_n^+(t),h_n^+(t))-(eu^0,h(0))-\int_0^{t_n}(\tilde{h}_n^+(\tau),eu_n^-(\tau))\mathrm{d}\tau, \tag{2.143}$$

$$\int_0^{t_n}(F_n^+(\tau),e\tilde{u}_n^+(\tau)-e\tilde{z}_n^+(\tau))\mathrm{d}\tau=(F_n^+(t),eu_n^+(t)-ez_n^+(t))-(F(0),eu^0-ez(0)) \\ -\int_0^{t_n}(\tilde{F}_n^+(\tau),eu_n^-(\tau)-ez_n^-(\tau))\mathrm{d}\tau. \tag{2.144}$$

Notice that the following convergences hold

$$\|\tilde{z}_n^+(t)-\dot{z}(t)\|=\left\|\frac{z(j\tau_n)-z((j-1)\tau_n)}{\tau_n}-\dot{z}(t)\right\|\le\int_{(j-1)\tau_n}^{j\tau_n}\|\dot{z}(\tau)-\dot{z}(t)\|\mathrm{d}\tau\xrightarrow[n\to\infty]{}0,$$
$$\|h_n^+(t)-h(t)\|=\|\mathbb{B}w^0\||\mathrm{e}^{-\frac{j\tau_n}{\beta}}-\mathrm{e}^{-\frac{t}{\beta}}|\le\frac{1}{\beta^2}\|\mathbb{B}w^0\||t-j\tau_n|\le\frac{1}{\beta^2}\|\mathbb{B}w^0\|\tau_n\xrightarrow[n\to\infty]{}0,$$
$$\|z_n^+(t)-z(t)\|_V=\|z(j\tau_n)-z(t)\|_V\le(j\tau_n-t)^{\frac12}\|\dot{z}\|_{L^2(0,T;V)}\le\tau_n^{\frac12}\|\dot{z}\|_{L^2(0,T;V)}\xrightarrow[n\to\infty]{}0,$$
$$\|F_n^+(t)-F(t)\|=\|F(j\tau_n)-F(t)\|\le(j\tau_n-t)^{\frac12}\|\dot{F}\|_{L^2(0,T;H)}\le\tau_n^{\frac12}\|\dot{F}\|_{L^2(0,T;H)}\xrightarrow[n\to\infty]{}0,$$

$$\chi_{[0,t_n]}\dot{\tilde{z}}_n\xrightarrow[n\to\infty]{L^2(0,T;H)}\chi_{[0,t]}\ddot{z},\qquad\chi_{[0,t_n]}\tilde{h}_n^+\xrightarrow[n\to\infty]{L^2(0,T;H)}\chi_{[0,t]}\dot{h},$$
$$z_n^-\xrightarrow[n\to\infty]{L^2(0,T;V)}z,\qquad\chi_{[0,t_n]}\tilde{F}_n^+\xrightarrow[n\to\infty]{L^2(0,T;H)}\chi_{[0,t]}\dot{F}.$$

By means of these convergences, (2.134), and Lemma 2.3.5, we can argue as before to deduce from (2.142)–(2.144) that

$$\int_0^{t_n}(\dot{\tilde{u}}_n(\tau),\tilde{z}_n^+(\tau))\mathrm{d}\tau\xrightarrow[n\to\infty]{}(\dot{u}(t),\dot{z}(t))-(u^1,\dot{z}(0))-\int_0^t(\dot{u}(\tau),\ddot{z}(\tau))\mathrm{d}\tau, \tag{2.145}$$
$$\int_0^{t_n}(h_n^+(\tau),e\tilde{u}_n^+(\tau))\mathrm{d}\tau\xrightarrow[n\to\infty]{}(h(t),eu(t))-(h(0),eu^0)-\int_0^t(\dot{h}(\tau),eu(\tau))\mathrm{d}\tau, \tag{2.146}$$

$$\int_0^{t_n}(F_n^+(\tau),e\tilde{u}_n^+(\tau)-e\tilde{z}_n^+(\tau))\mathrm{d}\tau\xrightarrow[n\to\infty]{}(F(t),eu(t)-ez(t))-(F(0),eu^0-ez(0)) \\ -\int_0^t(\dot{F}(\tau),eu(\tau)-ez(\tau))\mathrm{d}\tau. \tag{2.147}$$

By combining (2.133) and (2.135)–(2.147) we obtain the energy-dissipation inequality (2.132) for $t\in(0,T]$. Finally, for $t=0$ the inequality trivially holds since $u(0)=u^0$ and $\dot{u}(0)=u^1$. □

Remark 2.3.11. Thanks to the last theorem and to the equivalence between the viscoelastic dynamic system (2.16)–(2.20) and the coupled system (2.82)–(2.86), we can derive an energy-dissipation inequality for a weak solution to our viscoelastic dynamic system. As can be seen from (2.87) and the proof of Theorem 2.3.2 it is not restrictive to assume $w^0=0$.

Let (u,w) be the weak solution to the coupled system (2.82)–(2.86) provided by Lemma 2.3.5. Then, it satisfies the energy-dissipation inequality (2.132). Moreover, from Theorem 2.3.2 the function u is a solution to the viscoelastic dynamic system (2.16)–(2.20) in the sense of Definition 2.1.3. Therefore, by substituting (2.89) in (2.132) we get for the conservative part

$$\mathscr{E}_{u,w}(t)=\frac12\|\dot{u}(t)\|^2+\frac12(\mathbb{A}eu(t),eu(t))+\frac12(\mathbb{B}(eu(t)-w(t)),eu(t)-w(t))$$

$$
\begin{aligned}
&= \frac{1}{2}\|\dot u(t)\|^2 + \frac{1}{2}((\mathbb{A}+\mathbb{B})eu(t), eu(t)) - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), eu(t))\mathrm{d}\tau \\
&\quad + \frac{1}{2\beta^2}\int_0^t\int_0^t \mathrm{e}^{-\frac{2t-r-\tau}{\beta}}(\mathbb{B}eu(r), eu(\tau))\mathrm{d}r\mathrm{d}\tau \qquad (2.148)
\end{aligned}
$$

and for the dissipation

$$
\begin{aligned}
\mathscr{D}_{u,w}(t) &= \int_0^t (\mathbb{B}\dot w(\tau), eu(\tau) - w(\tau))\mathrm{d}\tau = \int_0^t (\mathbb{B}\dot w(\tau), eu(\tau))\mathrm{d}\tau - \int_0^t (\mathbb{B}\dot w(\tau), w(\tau))\mathrm{d}\tau \\
&= \frac{1}{\beta}\int_0^t \Big(\mathbb{B}eu(\tau) - \int_0^\tau \frac{1}{\beta}\mathrm{e}^{-\frac{\tau-r}{\beta}}\mathbb{B}eu(r)\mathrm{d}r, eu(\tau)\Big)\mathrm{d}\tau - \frac{1}{2}(\mathbb{B}w(t), w(t)) \\
&= \frac{1}{\beta}\int_0^t (\mathbb{B}eu(\tau), eu(\tau))\mathrm{d}\tau - \frac{1}{\beta^2}\int_0^t\int_0^\tau \mathrm{e}^{-\frac{\tau-r}{\beta}}(\mathbb{B}eu(r), eu(\tau))\mathrm{d}r\mathrm{d}\tau \\
&\quad - \frac{1}{2\beta^2}\int_0^t\int_0^t \mathrm{e}^{-\frac{2t-r-\tau}{\beta}}(\mathbb{B}eu(r), eu(\tau))\mathrm{d}r\mathrm{d}\tau. \qquad (2.149)
\end{aligned}
$$

By substituting the same information in the total work, we obtain

$$
\begin{aligned}
\mathscr{W}_{tot}(t) &= \int_0^t \Big[(f(\tau), \dot u(\tau) - \dot z(\tau)) + ((\mathbb{A}+\mathbb{B})eu(\tau), e\dot z(\tau)) - \int_0^\tau \frac{1}{\beta}\mathrm{e}^{-\frac{\tau-r}{\beta}}(\mathbb{B}eu(r), e\dot z(\tau))\mathrm{d}r\Big]\mathrm{d}\tau \\
&\quad - \int_0^t (\dot F(\tau), eu(\tau) - ez(\tau))\mathrm{d}\tau + (F(t), eu(t) - ez(t)) - (F(0), eu^0 - ez(0)) \\
&\quad - \int_0^t (\dot u(\tau), \ddot z(\tau))\mathrm{d}\tau + (\dot u(t), \dot z(t)) - (u^1, \dot z(0)). \qquad (2.150)
\end{aligned}
$$

After defining the elastic energy as

$$
\begin{aligned}
\mathscr{E}(t) &:= \frac{1}{2}\|\dot u(t)\|^2 + \frac{1}{2}((\mathbb{A}+\mathbb{B})eu(t), eu(t)) \\
&\quad - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), eu(t))\mathrm{d}\tau + \frac{1}{2\beta^2}\int_0^t\int_0^t \mathrm{e}^{-\frac{2t-r-\tau}{\beta}}(\mathbb{B}eu(r), eu(\tau))\mathrm{d}r\mathrm{d}\tau,
\end{aligned}
$$

and the dissipative term

$$
\begin{aligned}
\mathscr{D}(t) &:= \frac{1}{\beta}\int_0^t (\mathbb{B}eu(\tau), eu(\tau))\mathrm{d}\tau - \frac{1}{\beta^2}\int_0^t\int_0^\tau \mathrm{e}^{-\frac{\tau-r}{\beta}}(\mathbb{B}eu(r), eu(\tau))\mathrm{d}r\mathrm{d}\tau \\
&\quad - \frac{1}{2\beta^2}\int_0^t\int_0^t \mathrm{e}^{-\frac{2t-r-\tau}{\beta}}(\mathbb{B}eu(r), eu(\tau))\mathrm{d}r\mathrm{d}\tau,
\end{aligned}
$$

taking into account (2.148), (2.149), and (2.150) we can rephrase the energy-dissipation inequality (2.132) as

$$
\mathscr{E}(t) + \mathscr{D}(t) \le \mathscr{E}(0) + \mathscr{W}_{tot}(t),
$$

where the total work $\mathscr{W}_{tot}$ now depends just on the function u.

Finally, in view of Theorem 2.3.10 we are ready to show that our weak solution satisfies the initial conditions in a stronger sense than the one stated in (2.22), that is the content of the following lemma.

Lemma 2.3.12. *The weak solution $(u, w) \in \mathcal{V} \times H^1(0, T; H)$ to the coupled system (2.82)–(2.86), provided by Lemma 2.3.5, satisfies the initial conditions in the following sense:*

$$
\lim_{t\to 0^+} u(t) = u^0 \text{ in } V, \qquad \lim_{t\to 0^+} \dot u(t) = u^1 \text{ in } H, \qquad \lim_{t\to 0^+} w(t) = w^0 \text{ in } H. \qquad (2.151)
$$

Proof. Since $u \in C^0_w([0,T];V)$, $\dot u \in C^0_w([0,T];H)$, $w \in C^0([0,T];H)$, from the lower semicontinuity of the real valued functions

$$t \mapsto \|\dot u(t)\|^2, \quad t \mapsto (\mathbb{A}eu(t), eu(t)), \quad t \mapsto (\mathbb{B}(eu(t)-w(t)), eu(t)-w(t)),$$

we can let $t \to 0^+$ into the energy-dissipation inequality (2.132) to deduce that

$$\begin{aligned}
\mathscr{E}_{u,w}(0) &= \frac{1}{2}\|u^1\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0) + \frac{1}{2}(\mathbb{B}(eu^0 - w^0), eu^0 - w^0)\\
&\le \frac{1}{2}\Big[\liminf_{t\to0^+}\|\dot u(t)\|^2 + \liminf_{t\to0^+}(\mathbb{A}eu(t), eu(t)) + \liminf_{t\to0^+}(\mathbb{B}(eu(t)-w(t)), eu(t)-w(t))\Big]\\
&\le \liminf_{t\to0^+}\Big[\frac{1}{2}\|\dot u(t)\|^2 + \frac{1}{2}(\mathbb{A}eu(t), eu(t)) + \frac{1}{2}(\mathbb{B}(eu(t)-w(t)), eu(t)-w(t))\Big]\\
&= \liminf_{t\to0^+}\mathscr{E}_{u,w}(t) \le \limsup_{t\to0^+}\mathscr{E}_{u,w}(t) \le \mathscr{E}_{u,w}(0).
\end{aligned} \tag{2.152}$$

Notice that the last inequality in (2.152) holds because the right-hand side of (2.132) is continuous in t, and $u(0) = u^0$, $\dot u(0) = u^1$, and $w(0) = w^0$. Therefore, there exists $\lim_{t\to0^+}\mathscr{E}_{u,w}(t) = \mathscr{E}_{u,w}(0)$. Moreover, we have

$$\begin{aligned}
\mathscr{E}_{u,w}(0) &\le \frac{1}{2}\liminf_{t\to0^+}\|\dot u(t)\|^2 + \frac{1}{2}\liminf_{t\to0^+}\Big[(\mathbb{A}eu(t), eu(t)) + (\mathbb{B}(eu(t)-w(t)), eu(t)-w(t))\Big]\\
&\le \frac{1}{2}\limsup_{t\to0^+}\|\dot u(t)\|^2 + \frac{1}{2}\liminf_{t\to0^+}\Big[(\mathbb{A}eu(t), eu(t)) + (\mathbb{B}(eu(t)-w(t)), eu(t)-w(t))\Big]\\
&\le \limsup_{t\to0^+}\Big[\frac{1}{2}\|\dot u(t)\|^2 + \frac{1}{2}(\mathbb{A}eu(t), eu(t)) + \frac{1}{2}(\mathbb{B}(eu(t)-w(t)), eu(t)-w(t))\Big] = \mathscr{E}_{u,w}(0),
\end{aligned}$$

which gives

$$\lim_{t\to0^+}\|\dot u(t)\|^2 = \|u^1\|^2.$$

In a similar way, we can also show that

$$\lim_{t\to0^+}(\mathbb{A}eu(t), eu(t)) = (\mathbb{A}eu^0, eu^0).$$

Finally, since we have

$$\dot u(t) \xrightharpoonup[t\to0^+]{H} u^1, \quad eu(t) \xrightharpoonup[t\to0^+]{H} eu^0$$

and $u \in C^0([0,T];H)$, we deduce (2.151). In particular the functions $u\colon [0,T] \to V$ and $\dot u\colon [0,T] \to H$ are continuous at $t = 0$. □

We can finally prove the main theorem of Section 2.3.

Proof of Theorem 2.3.3. It is enough to combine Proposition 2.3.8 and Lemma 2.3.12. □

Remark 2.3.13. We have proved Theorem 2.3.3 for the d-dimensional linear viscoelastic case, namely when the displacement u is a vector-valued function. The same result is true with identical proof in the *antiplane* case, that is when the displacement u is a scalar function and satisfies (9).

Chapter 3

An existence result for the fractional Kelvin-Voigt's model on time-dependent cracked domains

The chapter is organized as follows. In Section 3.1 we fix the notation and the framework of our problem. Moreover, we give the notion of solution to the fractional Kelvin-Voigt's system involving Caputo's derivative (19) and we state our main existence result (see Theorem 3.1.4). Section 3.2 deals with the regularized system (20). First, by a time-discretization procedure in Theorem 3.2.13 we prove the existence of a solution to (20). Then, in Lemma 3.2.14 we derive the uniform energy estimate which depends on the L^1-norm of $\mathbb{G}$. In Section 3.3 we consider Kelvin-Voigt's system (19): we prove the existence of a generalized solution to system (19) and in Theorem 3.3.2 we show that such a solution satisfies an energy-dissipation inequality. Finally, in Section 3.4 we prove that, for a not moving crack, the solution to (19) is unique.

The results presented here are obtained in collaboration with M. Caponi and are contained in the published paper [8].

3.1 Framework of the problem

Let T be a positive real number and let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. Let $\partial_D\Omega$ be a (possibly empty) Borel subset of $\partial\Omega$ and let $\partial_N\Omega$ be its complement. Throughout the chapter we assume the following hypotheses on the geometry of the cracks:

(H1) $\Gamma \subset \overline{\Omega}$ is a closed set with $\mathcal{L}^d(\Gamma) = 0$ and $\mathcal{H}^{d-1}(\Gamma \cap \partial\Omega) = 0$;

(H2) for every $x \in \Gamma$ there exists an open neighborhood U of x in $\mathbb{R}^d$ such that $(U \cap \Omega) \setminus \Gamma$ is the union of two disjoint open sets U^+ and U^- with Lipschitz boundary;

(H3) $\{\Gamma_t\}_{t\in[0,T]}$ is an increasing family in time of closed subsets of Γ, i.e. $\Gamma_s \subset \Gamma_t$ for every $0 \le s \le t \le T$.

Thanks (H1)–(H3) the space $L^2(\Omega \setminus \Gamma_t; \mathbb{R}^m)$ coincides with $L^2(\Omega; \mathbb{R}^m)$ for every $t \in [0, T]$ and $m \in \mathbb{N}$. In particular, we can extend a function $u \in L^2(\Omega \setminus \Gamma_t; \mathbb{R}^m)$ to a function in $L^2(\Omega; \mathbb{R}^m)$ by setting $u = 0$ on Γ_t. To simplify our exposition, for every $m \in \mathbb{N}$ we define the spaces $H := L^2(\Omega; \mathbb{R}^m)$, $H_N := L^2(\partial_N\Omega; \mathbb{R}^m)$ and $H_D := L^2(\partial_D\Omega; \mathbb{R}^m)$; we always identify the dual of H by H itself, and $L^2((0, T) \times \Omega; \mathbb{R}^m)$ by the space $L^2(0, T; H)$. We define

$$V_t := H^1(\Omega \setminus \Gamma_t; \mathbb{R}^d) \quad \text{for every } t \in [0, T].$$

Notice that in the definition of V_t we are considering only the distributional gradient of u in $\Omega \setminus \Gamma_t$ and not the one in Ω. By (H2) we can find a finite number of open sets $U_j \subset \Omega \setminus \Gamma$, $j = 1, \dots m$, with Lipschitz boundary, such that $\Omega \setminus \Gamma = \cup_{j=1}^m U_j$. By using second Korn's inequality in each U_j (see, e.g., [39, Theorem 2.4]) and taking the sum over j we can find a constant C_K, depending only on Ω and Γ, such that

$$\|\nabla u\|^2 \le C_K \left(\|u\|^2 + \|eu\|^2\right) \quad \text{for every } u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d),$$

where eu is the symmetric part of ∇u. Therefore, we can use on the space V_t the equivalent norm

$$\|u\|_{V_t} := (\|u\|^2 + \|eu\|^2)^{\frac{1}{2}} \quad \text{for every } u \in V_t.$$

Furthermore, the trace of $u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d)$ is well defined on $\partial\Omega$. Indeed, we may find a finite number of open sets with Lipschitz boundary $V_k \subset \Omega \setminus \Gamma$, $k = 1, \dots l$, such that $\partial\Omega \setminus (\Gamma \cap \partial\Omega) \subset \cup_{k=1}^l \partial V_k$. Since $\mathcal{H}^{d-1}(\Gamma \cap \partial\Omega) = 0$, there exists a constant C, depending only on Ω and Γ, such that

$$\|u\|_{L^2(\partial\Omega;\mathbb{R}^d)} \le C\|u\|_{H^1(\Omega\setminus\Gamma;\mathbb{R}^d)} \quad \text{for every } u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d).$$

Hence, we can consider the set

$$V_t^D := \{u \in V_t : u = 0 \text{ on } \partial_D\Omega\} \quad \text{for every } t \in [0, T],$$

which is a closed subspace of V_t. Moreover, there exists a positive constant C_{tr} such that

$$\|u\|_{H_N} \le C_{tr}\|u\|_V \quad \text{for every } u \in V.$$

Now, we define the following sets of functions

$$\mathcal{C}_w := \{u \in C_w^0([0, T]; V) : \dot u \in C_w^0([0, T]; H),\ u(t) \in V_t \text{ for every } t \in [0, T]\},$$
$$\mathcal{C}_c^1 := \{\varphi \in C_c^1(0, T; V) : \varphi(t) \in V_t^D \text{ for every } t \in [0, T]\},$$

in which we develop our theory. Moreover, we consider the Banach space

$$B := L^\infty(\Omega; \mathcal{L}_{sym}(\mathbb{R}^{d\times d}_{sym}, \mathbb{R}^{d\times d}_{sym})),$$

where $\mathcal{L}_{sym}(\mathbb{R}^{d\times d}_{sym}, \mathbb{R}^{d\times d}_{sym})$ represents the space of symmetric tensor fields, i.e. the collections of linear and continuous maps $\mathbb{A}\colon \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}^{d\times d}_{sym}$ satisfying

$$\mathbb{A}\xi_1 \cdot \xi_2 = \xi_1 \cdot \mathbb{A}\xi_2 \quad \text{for every } \xi_1, \xi_2 \in \mathbb{R}^{d\times d}_{sym}.$$

We consider a tensor $\mathbb{A} \in B$ such that

$$c_{\mathbb{A}}|\xi|^2 \le \mathbb{A}(x)\xi \cdot \xi \le C_{\mathbb{A}}|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^d \text{ and a.e. } x \in \Omega, \tag{3.1}$$

where $c_{\mathbb{A}}$ and $C_{\mathbb{A}}$ are to positive constants independent of x.

We assume that the Dirichlet datum z, the Neumann datum N, the forcing term f, the initial displacement u^0, and the initial velocity u^1 satisfy

$$z \in W^{2,1}(0, T; V_0), \tag{3.2}$$
$$N \in W^{1,1}(0, T; H_N), \quad f \in L^2(0, T; H), \tag{3.3}$$
$$u^0 \in V_0 \text{ with } u^0 - z(0) \in V_0^D, \quad u^1 \in H. \tag{3.4}$$

Moreover, let us take a time-dependent tensor $\mathbb{F}\colon (0, T_0) \to B$, where $T_0 := T + \delta_0$ with $\delta_0 > 0$, satisfying

$$\mathbb{F} \in C^2(0, T_0; B) \cap L^1(0, T_0; B), \tag{3.5}$$
$$\mathbb{F}(t, x)\xi \cdot \xi \ge 0 \quad \text{for every } \xi \in \mathbb{R}^d,\ t \in (0, T_0), \text{ and a.e. } x \in \Omega, \tag{3.6}$$
$$\dot{\mathbb{F}}(t, x)\xi \cdot \xi \le 0 \quad \text{for every } \xi \in \mathbb{R}^d,\ t \in (0, T_0), \text{ and a.e. } x \in \Omega, \tag{3.7}$$
$$\ddot{\mathbb{F}}(t, x)\xi \cdot \xi \ge 0 \quad \text{for every } \xi \in \mathbb{R}^d,\ t \in (0, T_0), \text{ and a.e. } x \in \Omega. \tag{3.8}$$

Remark 3.1.1. The tensor $\mathbb{F}$ may be not defined at $t=0$ and unbounded on $(0,T_0)$. In the case of (5), the function $\mathbb{F}$ associated to the fractional Kelvin-Voigt's model involving Caputo's derivative, satisfies (3.5)–(3.8) provided that $\mathbb{B}\in B$ is non-negative, that is

$$\mathbb{B}(x)\xi\cdot\xi\geq 0\quad\text{for every } \xi\in\mathbb{R}^d \text{ and a.e. } x\in\Omega.$$

Since in our existence result we first regularize the tensor $\mathbb{F}$ by means of translations (see Section 3.3) we need that $\mathbb{F}$ is defined also on the right of T. This is not a problem, because our standard example for $\mathbb{F}$, which is (5), is defined on the whole $(0,\infty)$.

In this chapter we want to study the following problem

$$\begin{cases}\ddot u(t)-\operatorname{div}(\mathbb{A}eu(t))-\operatorname{div}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\mathbb{F}(t-\tau)(eu(\tau)-eu^0)\mathrm{d}\tau\right)=f(t) & \text{in } \Omega\setminus\Gamma_t,\quad t\in(0,T),\\ u(t)=z(t) & \text{on } \partial_D\Omega,\quad t\in(0,T),\\ \mathbb{A}eu(t)\nu+\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\mathbb{F}(t-\tau)(eu(\tau)-eu^0)\mathrm{d}\tau\right)\nu=N(t) & \text{on } \partial_N\Omega,\quad t\in(0,T),\\ \mathbb{A}eu(t)\nu+\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\mathbb{F}(t-\tau)(eu(\tau)-eu^0)\mathrm{d}\tau\right)\nu=0 & \text{on } \Gamma_t,\quad t\in(0,T),\\ u(0)=u^0,\quad \dot u(0)=u^1 & \text{in } \Omega\setminus\Gamma_0.\end{cases}\tag{3.9}$$

We give the following notion of solution to system (3.9):

Definition 3.1.2 (Generalized solution)**.** Assume (3.2)–(3.8). A function $u\in\mathcal{C}_w$ is a *generalized solution* to system (3.9) if $u(t)-z(t)\in V_t^D$ for every $t\in[0,T]$, $u(0)=u^0$ in V_0, $\dot u(0)=u^1$ in H, and for every $\varphi\in\mathcal{C}_c^1$ the following equality holds

$$\begin{aligned}&-\int_0^T(\dot u(t),\dot\varphi(t))\mathrm{d}t+\int_0^T(\mathbb{A}eu(t),e\varphi(t))\mathrm{d}t-\int_0^T\int_0^t(\mathbb{F}(t-\tau)(eu(\tau)-eu^0),e\dot\varphi(t))\mathrm{d}\tau\mathrm{d}t\\&=\int_0^T(f(t),\varphi(t))\mathrm{d}t+\int_0^T(N(t),\varphi(t))_{H_N}\mathrm{d}t.\end{aligned}\tag{3.10}$$

Remark 3.1.3. The Neumann conditions appearing in (3.9) are only formal; they are used to pass from the strong formulation in (3.9) to the weak one (3.10).

The main existence result of this chapter is the following theorem:

Theorem 3.1.4. *Assume (3.2)–(3.8). Then there exists a generalized solution $u\in\mathcal{C}_w$ to system (3.9).*

The proof of this theorem requires several preliminary results. First, in the next section, we prove the existence of a generalized solution when the tensor $\mathbb{F}$ is replaced by a tensor $\mathbb{G}\in C^2([0,T];B)$. Then, we show that such a solution satisfies an energy estimate, which depends via $\mathbb{G}$ only by its L^1-norm. In Section 3.3 we combine these two results to prove Theorem 3.1.4.

3.2 The regularized model

In this section we deal with a regularized version of the system (3.9), where the tensor $\mathbb{F}$ is replaced by a tensor $\mathbb{G}$ which is bounded at $t=0$. More precisely, we consider the following system

$$\begin{cases} \ddot{u}(t) - \operatorname{div}(\mathbb{A}eu(t)) - \operatorname{div}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathbb{G}(t-\tau)(eu(\tau)-eu^0)\mathrm{d}\tau\right) = f(t) & \text{in } \Omega\setminus\Gamma_t, \quad t\in(0,T),\\ u(t) = z(t) & \text{on } \partial_D\Omega, \quad t\in(0,T),\\ \mathbb{A}eu(t)\nu + \left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathbb{G}(t-\tau)(eu(\tau)-eu^0)\mathrm{d}\tau\right)\nu = N(t) & \text{on } \partial_N\Omega, \quad t\in(0,T),\\ \mathbb{A}eu(t)\nu + \left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathbb{G}(t-\tau)(eu(\tau)-eu^0)\mathrm{d}\tau\right)\nu = 0 & \text{on } \Gamma_t, \quad t\in(0,T),\\ u(0)=u^0, \quad \dot{u}(0)=u^1 & \text{in } \Omega\setminus\Gamma_0, \end{cases} \tag{3.11}$$

and we assume that $\mathbb{G}\colon [0,T]\to B$ satisfies

$$\mathbb{G}\in C^2([0,T];B), \tag{3.12}$$

$$\mathbb{G}(t,x)\xi\cdot\xi\geq 0 \qquad \text{for every } \xi\in\mathbb{R}^d,\ t\in[0,T], \text{ and a.e. } x\in\Omega, \tag{3.13}$$

$$\dot{\mathbb{G}}(t,x)\xi\cdot\xi\leq 0 \qquad \text{for every } \xi\in\mathbb{R}^d,\ t\in[0,T], \text{ and a.e. } x\in\Omega, \tag{3.14}$$

$$\ddot{\mathbb{G}}(t,x)\xi\cdot\xi\geq 0 \qquad \text{for every } \xi\in\mathbb{R}^d,\ t\in[0,T], \text{ and a.e. } x\in\Omega. \tag{3.15}$$

As before, on N, u^0, u^1, and $\mathbb{A}$ we assume (3.3)–(3.1), while for the Dirichlet datum z we can require the weaker assumption

$$z\in W^{2,1}(0,T;H)\cap W^{1,1}(0,T;V_0). \tag{3.16}$$

The notion of generalized solution to (3.11) is the same as before.

Definition 3.2.1 (Generalized solution)**.** Assume (3.3)–(3.1) and (3.12)–(3.16). A function $u\in\mathcal{C}_w$ is a *generalized solution* to system (3.11) if $u(t)-z(t)\in V_t^D$ for every $t\in[0,T]$, $u(0)=u^0$ in V_0, $\dot{u}(0)=u^1$ in H, and for every $\varphi\in\mathcal{C}_c^1$ the following equality holds

$$\begin{aligned} &-\int_0^T(\dot{u}(t),\dot{\varphi}(t))\mathrm{d}t + \int_0^T(\mathbb{A}eu(t),e\varphi(t))\mathrm{d}t - \int_0^T\int_0^t(\mathbb{G}(t-\tau)(eu(\tau)-eu^0),e\dot{\varphi}(t))\mathrm{d}\tau\mathrm{d}t\\ &=\int_0^T(f(t),\varphi(t))\mathrm{d}t + \int_0^T(N(t),\varphi(t))_{H_N}\mathrm{d}t. \end{aligned} \tag{3.17}$$

Since the time-dependent tensor $\mathbb{G}$ is well defined in $t=0$, we can give another notion of solution. In particular, the convolution integral is now differentiable, and we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\mathbb{G}(t-\tau)(eu(\tau)-eu^0)\mathrm{d}\tau = \mathbb{G}(0)(eu(t)-eu^0)+\int_0^t\dot{\mathbb{G}}(t-\tau)(eu(\tau)-eu^0)\mathrm{d}\tau.$$

Definition 3.2.2 (Weak solution)**.** Assume (3.3)–(3.1) and (3.12)–(3.16). A function $u\in\mathcal{C}_w$ is a *weak solution* to system (3.11) if $u(t)-z(t)\in V_t^D$ for every $t\in[0,T]$, $u(0)=u^0$ in V_0, $\dot{u}(0)=u^1$ in H, and for every $\varphi\in\mathcal{C}_c^1$ the following equality holds

$$\begin{aligned} &-\int_0^T(\dot{u}(t),\dot{\varphi}(t))\mathrm{d}t + \int_0^T(\mathbb{A}eu(t),e\varphi(t))\mathrm{d}t + \int_0^T(\mathbb{G}(0)(eu(t)-eu^0),e\varphi(t))\mathrm{d}t\\ &+\int_0^T\int_0^t(\dot{\mathbb{G}}(t-\tau)(eu(\tau)-eu^0),e\varphi(t))\mathrm{d}\tau\mathrm{d}t = \int_0^T(f(t),\varphi(t))\mathrm{d}t + \int_0^T(N(t),\varphi(t))_{H_N}\mathrm{d}t. \end{aligned} \tag{3.18}$$

In this framework the two previous definitions are equivalent.

Proposition 3.2.3. *Assume* (3.3)*–*(3.1) *and* (3.12)*–*(3.16)*. Then $u\in\mathcal{C}_w$ is a generalized solution to* (3.11) *if and only if u is a weak solution.*

Proof. We only need to prove that (3.18) is equivalent to (3.17). This is true if and only if the function $u \in \mathcal{C}_w$ satisfies for every $\varphi \in \mathcal{C}_c^1$ the following equality

$$\int_0^T (\mathbb{G}(0)(eu(t) - eu^0), e\varphi(t))\mathrm{d}t + \int_0^T \int_0^t (\dot{\mathbb{G}}(t-\tau)(eu(\tau) - eu^0), e\varphi(t))\mathrm{d}\tau\mathrm{d}t \\ = -\int_0^T \int_0^t (\mathbb{G}(t-\tau)(eu(\tau) - eu^0), e\dot{\varphi}(t))\mathrm{d}\tau\mathrm{d}t. \tag{3.19}$$

Let us consider for $t \in [0, T]$ the function

$$p(t) := \int_0^t (\mathbb{G}(t-\tau)(eu(\tau) - eu^0), e\varphi(t))\mathrm{d}\tau.$$

We claim that $p \in \mathrm{Lip}([0, T])$. Indeed, for every $s, t \in [0, T]$ with $s < t$ we have

$$\begin{aligned} |p(s) - p(t)| \le &\Big|\int_s^t (\mathbb{G}(t-\tau)(eu(\tau) - eu^0), e\varphi(t))\mathrm{d}\tau\Big| \\ &+ \Big|\int_0^s (\mathbb{G}(s-\tau)(eu(\tau) - eu^0), e\varphi(t) - e\varphi(s))\mathrm{d}\tau\Big| \\ &+ \Big|\int_0^s ((\mathbb{G}(t-\tau) - \mathbb{G}(s-\tau))(eu(\tau) - eu^0), e\varphi(t))\mathrm{d}\tau\Big|. \end{aligned}$$

Since

$$\begin{aligned} &\Big|\int_s^t (\mathbb{G}(t-\tau)(eu(\tau) - eu^0), e\varphi(t))\mathrm{d}\tau\Big| \le 2(t-s)\|\mathbb{G}\|_{C^0([0,T];B)}\|e\varphi\|_{C^0([0,T];H)}\|eu\|_{L^\infty(0,T;H)}, \\ &\Big|\int_0^s (\mathbb{G}(s-\tau)(eu(\tau) - eu^0), e\varphi(t) - e\varphi(s))\mathrm{d}\tau\Big| \\ &\qquad \le 2(t-s)\|\mathbb{G}\|_{C^0([0,T];B)}\|e\dot{\varphi}\|_{C^0([0,T];H)}T\|eu\|_{L^\infty(0,T;H)}, \\ &\Big|\int_0^s ((\mathbb{G}(t-\tau) - \mathbb{G}(s-\tau))(eu(\tau) - eu^0), e\varphi(t))\mathrm{d}\tau\Big| \\ &\qquad \le 2(t-s)\|\dot{\mathbb{G}}\|_{C^0([0,T];B)}\|e\varphi\|_{C^0([0,T];H)}T\|eu\|_{L^\infty(0,T;H)}, \end{aligned}$$

we deduce that $p \in \mathrm{Lip}([0, T])$. In particular, there exists $\dot{p}(t)$ for a.e. $t \in (0, T)$. Given $t \in (0, T)$ and $h > 0$ we can write

$$\begin{aligned} \frac{p(t+h) - p(t)}{h} = &\int_0^t (\frac{\mathbb{G}(t+h-\tau) - \mathbb{G}(t-\tau)}{h}(eu(\tau) - eu^0), e\varphi(t+h))\mathrm{d}\tau \\ &+ \fint_t^{t+h} (\mathbb{G}(t+h-\tau)(eu(\tau) - eu^0), e\varphi(t+h))\mathrm{d}\tau \\ &+ \int_0^t (\mathbb{G}(t-\tau)(eu(\tau) - eu^0), \frac{e\varphi(t+h) - e\varphi(t)}{h})\mathrm{d}\tau. \end{aligned}$$

Let us compute these three limits separately. We claim that for a.e. $t \in (0, T)$ we have

$$\lim_{h\to 0^+} \fint_t^{t+h} (\mathbb{G}(t+h-\tau)(eu(\tau) - eu^0), e\varphi(t+h))\mathrm{d}\tau = (\mathbb{G}(0)(eu(t) - eu^0), e\varphi(t)).$$

Indeed, by the Lebesgue's differentiation theorem, for a.e. $t \in (0, T)$ we get

$$\begin{aligned} &\Big|\fint_t^{t+h} (\mathbb{G}(t+h-\tau)(eu(\tau) - eu^0), e\varphi(t+h))\mathrm{d}\tau - (\mathbb{G}(0)(eu(t) - eu^0), e\varphi(t))\Big| \\ &\qquad \le \|\mathbb{G}(0)\|_B\|e\varphi(t)\|\fint_t^{t+h} \|eu(t) - eu(\tau)\|\mathrm{d}\tau \end{aligned}$$

$$+\|\mathbb{G}(0)\|_B\|e\varphi(t+h)-e\varphi(t)\|\int_t^{t+h}\|eu(\tau)-eu^0\|\mathrm{d}\tau$$
$$+\|e\varphi(t+h)\|\int_t^{t+h}\|\mathbb{G}(t+h-\tau)-\mathbb{G}(0)\|_B\|eu(\tau)-eu^0\|\mathrm{d}\tau\xrightarrow[h\to0^+]{}0.$$

Moreover, for every $t\in(0,T)$ we have

$$\lim_{h\to0^+}\int_0^t\Big(\frac{\mathbb{G}(t+h-\tau)-\mathbb{G}(t-\tau)}{h}(eu(\tau)-eu^0),e\varphi(t+h)\Big)\mathrm{d}\tau$$
$$=\int_0^t(\dot{\mathbb{G}}(t-\tau)(eu(\tau)-eu^0),e\varphi(t))\mathrm{d}\tau$$

since

$$e\varphi(t+h)\xrightarrow[h\to0^+]{H}e\varphi(t),\quad\frac{\mathbb{G}(t+h-\cdot)-\mathbb{G}(t-\cdot)}{h}(eu(\cdot)-eu^0)\xrightarrow[h\to0^+]{L^1(0,t;H)}\dot{\mathbb{G}}(t-\cdot)(eu(\cdot)-eu^0).$$

Finally, for every $t\in(0,T)$ we get

$$\lim_{h\to0^+}\int_0^t\Big(\mathbb{G}(t-\tau)(eu(\tau)-eu^0),\frac{e\varphi(t+h)-e\varphi(t)}{h}\Big)\mathrm{d}\tau=\int_0^t(\mathbb{G}(t-\tau)(eu(\tau)-eu^0),e\dot\varphi(t))\mathrm{d}\tau$$

because

$$\frac{e\varphi(t+h)-e\varphi(t)}{h}\xrightarrow[h\to0^+]{H}e\dot\varphi(t).$$

Therefore, by the identity

$$0=p(T)-p(0)=\int_0^T\dot p(t)\mathrm{d}t$$

and the previous computations we deduce (3.19). □

In the particular case in which the tensor $\mathbb{G}$ appearing in (3.11) is the one associated to the Standard viscoelastic model, i.e.

$$\mathbb{G}(t)=\frac{1}{\beta}\mathrm{e}^{-\frac{t}{\beta}}\mathbb{B}\quad\text{for } t\in[0,T]$$

with $\beta>0$ and $\mathbb{B}\in B$ non-negative tensor, then the existence of weak solutions (and so generalized solutions) was proved in [44]. Here we adapt the techniques of [44] to a general tensor $\mathbb{G}$ satisfying (3.12)–(3.15).

3.2.1 Existence and Energy-Dissipation Inequality

In this subsection we prove the existence of a generalized solution to system (3.11), by means of a time discretization scheme in the same spirit of [13]. Moreover, we show that such a solution satisfies the energy-dissipation inequality (3.49).

We fix $n\in\mathbb{N}$ and we set

$$\tau_n:=\frac{T}{n},\quad u_n^0:=u^0,\quad u_n^{-1}:=u^0-\tau_nu^1,\quad \delta z_n^0:=\dot z(0),\quad \delta\mathbb{G}_n^0:=0.$$

Let us define for $j=0,\dots,n$

$$V_n^j:=V_{j\tau_n}^D,\quad z_n^j:=z(j\tau_n),\quad \mathbb{G}_n^j:=\mathbb{G}(j\tau_n),$$

and for $j=1,\dots,n$

$$\delta z_n^j:=\frac{z_n^j-z_n^{j-1}}{\tau_n},\quad \delta^2z_n^j:=\frac{\delta z_n^j-\delta z_n^{j-1}}{\tau_n},\quad \delta\mathbb{G}_n^j:=\frac{\mathbb{G}_n^j-\mathbb{G}_n^{j-1}}{\tau_n},\quad \delta^2\mathbb{G}_n^j:=\frac{\delta\mathbb{G}_n^j-\delta\mathbb{G}_n^{j-1}}{\tau_n}.$$

Regarding the forcing term and the Neumann datum we pose

$$N_n^j := N(j\tau_n) \qquad\qquad \text{for } j = 0,\dots,n,$$
$$f_n^j := \fint_{(j-1)\tau_n}^{j\tau_n} f(\tau)\mathrm{d}\tau, \quad \delta N_n^j := \frac{N_n^j - N_n^{j-1}}{\tau_n} \qquad \text{for } j = 1,\dots,n.$$

For every $j = 1,\dots,n$ let us consider the unique $u_n^j \in V$ with $u_n^j - z_n^j \in V_n^j$, which satisfies

$$(\delta^2 u_n^j, v) + (\mathbb{A}eu_n^j, ev) + (\mathbb{G}_n^0(eu_n^j - eu^0), ev) + \sum_{k=1}^{j} \tau_n(\delta\mathbb{G}_n^{j-k}(eu_n^k - eu^0), ev) = (f_n^j, v) + (N_n^j, v)_{H_N} \tag{3.20}$$

for every $v \in V_n^j$, where

$$\delta u_n^j := \frac{u_n^j - u_n^{j-1}}{\tau_n} \quad \text{for } j = 0,\dots,n, \quad \delta^2 u_n^j := \frac{\delta u_n^j - \delta u_n^{j-1}}{\tau_n} \quad \text{for } j = 1,\dots,n.$$

The existence and uniqueness of u_n^j is a consequence of Lax-Milgram's lemma. Notice that equation (3.20) is a sort of discrete version of (3.18), which we already know that is equivalent to (3.17).

We now use equation (3.20) to derive an energy estimate for the family $\{u_n^j\}_{j=1}^n$, which is uniform with respect to $n \in \mathbb{N}$.

Lemma 3.2.4. *Assume (3.3)–(3.1) and (3.12)–(3.16). Then there exists a constant C, independent of $n \in \mathbb{N}$, such that*

$$\max_{j=0,\dots,n} \|\delta u_n^j\| + \max_{j=0,\dots,n} \|eu_n^j\| \le C. \tag{3.21}$$

Proof. First, since

$$\mathbb{G}_n^{j-1} - \mathbb{G}_n^0 = \sum_{k=0}^{j-1} \tau_n \delta\mathbb{G}_n^k = \sum_{k=1}^{j} \tau_n \delta\mathbb{G}_n^{j-k} \quad \text{for } j = 1,\dots,n,$$

we have for $j = 1,\dots,n$ that

$$\mathbb{G}_n^0(eu_n^j - eu^0) + \sum_{k=1}^{j} \tau_n\delta\mathbb{G}_n^{j-k}(eu_n^k - eu^0) = \mathbb{G}_n^{j-1}(eu_n^j - eu^0) + \sum_{k=1}^{j} \tau_n\delta\mathbb{G}_n^{j-k}(eu_n^k - eu_n^j).$$

Therefore, equation (3.20) can be written as

$$(\delta^2 u_n^j, v) + (\mathbb{A}eu_n^j, ev) + (\mathbb{G}_n^{j-1}(eu_n^j - eu^0), ev) + \sum_{k=1}^{j} \tau_n(\delta\mathbb{G}_n^{j-k}(eu_n^k - eu_n^j), ev) = (f_n^j, v) + (N_n^j, v)$$

for every $v \in V_n^j$. We fix $i \in \{1,\dots,n\}$. By taking $v := \tau_n(\delta u_n^j - \delta z_n^j) \in V_n^j$ and summing over $j = 1,\dots,i$, we get the following identity

$$\sum_{j=1}^{i} \tau_n(\delta^2 u_n^j, \delta u_n^j) + \sum_{j=1}^{i} \tau_n(\mathbb{A}eu_n^j, e\delta u_n^j) + \sum_{j=1}^{i} \tau_n(\mathbb{G}_n^{j-1}(eu_n^j - eu^0), e\delta u_n^j) + \sum_{j=1}^{i}\sum_{k=1}^{j} \tau_n^2(\delta\mathbb{G}_n^{j-k}(eu_n^k - eu_n^j), e\delta u_n^j) = \sum_{j=1}^{i} \tau_n L_n^j, \tag{3.22}$$

where

$$L_n^j := (f_n^j, \delta u_n^j - \delta z_n^j) + (N_n^j, \delta u_n^j - \delta z_n^j)_{H_N} + (\delta^2 u_n^j, \delta z_n^j) \\ + (\mathbb{A} e u_n^j, e\delta z_n^j) + (\mathbb{G}_n^{j-1}(eu_n^j - eu^0), e\delta z_n^j) + \sum_{k=1}^{j} \tau_n (\delta \mathbb{G}_n^{j-k}(eu_n^k - eu_n^j), e\delta z_n^j).$$

By using the identity

$$|a|^2 - a \cdot b = \frac{1}{2}|a|^2 - \frac{1}{2}|b|^2 + \frac{1}{2}|a-b|^2 \quad \text{for every } a, b \in \mathbb{R}^d$$

we deduce

$$\tau_n(\delta^2 u_n^j, \delta u_n^j) = \|\delta u_n^j\|^2 - (\delta u_n^j, \delta u_n^{j-1}) = \frac{1}{2}\|\delta u_n^j\|^2 - \frac{1}{2}\|\delta u_n^{j-1}\|^2 + \frac{1}{2}\tau_n^2 \|\delta^2 u_n^j\|^2.$$

Therefore

$$\sum_{j=1}^{i} \tau_n(\delta^2 u_n^j, \delta u_n^j) = \frac{1}{2}\sum_{j=1}^{i}\|\delta u_n^j\|^2 - \frac{1}{2}\sum_{j=1}^{i}\|\delta u_n^{j-1}\|^2 + \frac{1}{2}\sum_{j=1}^{i}\tau_n^2\|\delta^2 u_n^j\|^2 \\ = \frac{1}{2}\|\delta u_n^i\|^2 - \frac{1}{2}\|u^1\|^2 + \frac{1}{2}\sum_{j=1}^{i}\tau_n^2\|\delta^2 u_n^j\|^2. \tag{3.23}$$

Similarly, we have

$$\sum_{j=1}^{i} \tau_n(\mathbb{A} e u_n^j, e\delta u_n^j) = \frac{1}{2}(\mathbb{A} e u_n^i, e u_n^i) - \frac{1}{2}(\mathbb{A} e u^0, e u^0) + \frac{1}{2}\sum_{j=1}^{i}\tau_n^2(\mathbb{A} e\delta u_n^j, e\delta u_n^j). \tag{3.24}$$

Moreover, we can write

$$\tau_n(\mathbb{G}_n^{j-1}(eu_n^j - eu^0), e\delta u_n^j) = (\mathbb{G}_n^{j-1}(eu_n^j - eu^0), eu_n^j - eu^0) - (\mathbb{G}_n^{j-1}(eu_n^j - eu^0), eu_n^{j-1} - eu^0) \\ = \frac{1}{2}(\mathbb{G}_n^{j-1}(eu_n^j - eu^0), eu_n^j - eu^0) - \frac{1}{2}(\mathbb{G}_n^{j-1}(eu_n^{j-1} - eu^0), eu_n^{j-1} - eu^0) \\ + \frac{\tau_n^2}{2}(\mathbb{G}_n^{j-1} e\delta u_n^j, e\delta u_n^j) \\ = \frac{1}{2}(\mathbb{G}_n^{j}(eu_n^j - eu^0), eu_n^j - eu^0) - \frac{1}{2}(\mathbb{G}_n^{j-1}(eu_n^{j-1} - eu^0), eu_n^{j-1} - eu^0) \\ - \frac{1}{2}\tau_n(\delta\mathbb{G}_n^{j}(eu_n^j - eu^0), eu_n^j - eu^0) + \frac{1}{2}\tau_n^2(\mathbb{G}_n^{j-1} e\delta u_n^j, e\delta u_n^j).$$

As consequence of this we obtain

$$\sum_{j=1}^{i} \tau_n(\mathbb{G}_n^{j-1}(eu_n^j - eu^0), e\delta u_n^j) \\ = \frac{1}{2}\sum_{j=1}^{i}(\mathbb{G}_n^{j}(eu_n^j - eu^0), eu_n^j - eu^0) - \frac{1}{2}\sum_{j=1}^{i}(\mathbb{G}_n^{j-1}(eu_n^{j-1} - eu^0), eu_n^{j-1} - eu^0) \\ - \frac{1}{2}\sum_{j=1}^{i}\tau_n(\delta\mathbb{G}_n^{j}(eu_n^j - eu^0), eu_n^j - eu^0) + \frac{1}{2}\sum_{j=1}^{i}\tau_n^2(\mathbb{G}_n^{j-1} e\delta u_n^j, e\delta u_n^j) \\ = \frac{1}{2}(\mathbb{G}_n^{i}(eu_n^i - eu^0), eu_n^i - eu^0) + \frac{1}{2}\sum_{j=1}^{i}\tau_n^2(\mathbb{G}_n^{j-1} e\delta u_n^j, e\delta u_n^j) \\ - \frac{1}{2}\sum_{j=1}^{i}\tau_n(\delta\mathbb{G}_n^{j}(eu_n^j - eu^0), eu_n^j - eu^0)$$

Finally, let us consider the term

$$\sum_{j=1}^{i}\sum_{k=1}^{j}\tau_n^2(\delta\mathbb{G}_n^{j-k}(eu_n^k-eu_n^j),e\delta u_n^j)=\sum_{k=1}^{i}\sum_{j=k}^{i}\tau_n^2(\delta\mathbb{G}_n^{j-k}(eu_n^k-eu_n^j),e\delta u_n^j).$$

We can write

$$\begin{aligned}
&\sum_{j=k}^{i}\tau_n^2(\delta\mathbb{G}_n^{j-k}(eu_n^k-eu_n^j),e\delta u_n^j)=-\sum_{j=k}^{i}\tau_n(\delta\mathbb{G}_n^{j-k}(eu_n^j-eu_n^k),eu_n^j-eu_n^{j-1})\\
&=-\sum_{j=k}^{i}\tau_n(\delta\mathbb{G}_n^{j-k}(eu_n^j-eu_n^k),eu_n^j-eu_n^k)+\sum_{j=k}^{i}\tau_n(\delta\mathbb{G}_n^{j-k}(eu_n^j-eu_n^k),eu_n^{j-1}-eu_n^k)\\
&=-\frac{1}{2}\sum_{j=k}^{i}\tau_n(\delta\mathbb{G}_n^{j-k}(eu_n^j-eu_n^k),eu_n^j-eu_n^k)+\frac{1}{2}\sum_{j=k}^{i}\tau_n(\delta\mathbb{G}_n^{j-k}(eu_n^{j-1}-eu_n^k),eu_n^{j-1}-eu_n^k)\\
&\qquad\qquad-\frac{1}{2}\sum_{j=k}^{i}\tau_n^3(\delta\mathbb{G}_n^{j-k}e\delta u_n^j,e\delta u_n^j)\\
&=-\frac{1}{2}\sum_{j=k}^{i}\tau_n(\delta\mathbb{G}_n^{j-k+1}(eu_n^j-eu_n^k),eu_n^j-eu_n^k)+\frac{1}{2}\sum_{j=k}^{i}\tau_n(\delta\mathbb{G}_n^{j-k}(eu_n^{j-1}-eu_n^k),eu_n^{j-1}-eu_n^k)\\
&\quad+\frac{1}{2}\sum_{j=k}^{i}\tau_n^2(\delta^2\mathbb{G}_n^{j-k+1}(eu_n^j-eu_n^k),eu_n^j-eu_n^k)-\frac{1}{2}\sum_{j=k}^{i}\tau_n^3(\delta\mathbb{G}_n^{j-k}e\delta u_n^j,e\delta u_n^j)\\
&=\frac{1}{2}\sum_{j=k}^{i}\tau_n^2(\delta^2\mathbb{G}_n^{j-k+1}(eu_n^j-eu_n^k),eu_n^j-eu_n^k)-\frac{1}{2}\sum_{j=k}^{i}\tau_n^3(\delta\mathbb{G}_n^{j-k}e\delta u_n^j,e\delta u_n^j)\\
&\qquad\qquad-\frac{1}{2}\tau_n(\delta\mathbb{G}_n^{i-k+1}(eu_n^i-eu_n^k),eu_n^i-eu_n^k)
\end{aligned}$$

because $\delta\mathbb{G}_n^0=0$. Therefore, we deduce

$$\begin{aligned}
&\sum_{j=1}^{i}\sum_{k=1}^{j}\tau_n^2(\delta\mathbb{G}_n^{j-k}(eu_n^k-eu_n^j),e\delta u_n^j)\\
&=\frac{1}{2}\sum_{k=1}^{i}\sum_{j=k}^{i}\tau_n^2(\delta^2\mathbb{G}_n^{j-k+1}(eu_n^j-eu_n^k),eu_n^j-eu_n^k)-\frac{1}{2}\sum_{k=1}^{i}\sum_{j=k}^{i}\tau_n^3(\delta\mathbb{G}_n^{j-k}e\delta u_n^j,e\delta u_n^j)\\
&\quad-\frac{1}{2}\sum_{k=1}^{i}\tau_n(\delta\mathbb{G}_n^{i-k+1}(eu_n^i-eu_n^k),eu_n^i-eu_n^k)\\
&=\frac{1}{2}\sum_{j=1}^{i}\sum_{k=1}^{j}\tau_n^2(\delta^2\mathbb{G}_n^{j-k+1}(eu_n^j-eu_n^k),eu_n^j-eu_n^k)-\frac{1}{2}\sum_{j=1}^{i}\sum_{k=1}^{j}\tau_n^3(\delta\mathbb{G}_n^{j-k}e\delta u_n^j,e\delta u_n^j)\\
&\quad-\frac{1}{2}\sum_{j=1}^{i}\tau_n(\delta\mathbb{G}_n^{i-j+1}(eu_n^i-eu_n^j),eu_n^i-eu_n^j).
\end{aligned}\tag{3.25}$$

By combining together (3.22)–(3.25), we obtain for $i=1,\dots,n$ the following discrete energy equality

$$\frac{1}{2}\|\delta u_n^i\|^2+\frac{1}{2}(\mathbb{A}eu_n^i,eu_n^i)+\frac{1}{2}(\mathbb{G}_n^i(eu_n^i-eu^0),eu_n^i-eu^0)$$

$$
\begin{aligned}
&- \frac{1}{2}\sum_{j=1}^{i}\tau_n(\delta\mathbb{G}_n^{i-j+1}(eu_n^i - eu_n^j), eu_n^i - eu_n^j) - \frac{1}{2}\sum_{j=1}^{i}\tau_n(\delta\mathbb{G}_n^{j}(eu_n^j - eu^0), eu_n^j - eu^0) \\
&+ \frac{1}{2}\sum_{j=1}^{i}\sum_{k=1}^{j}\tau_n^2(\delta^2\mathbb{G}_n^{j-k+1}(eu_n^j - eu_n^k), eu_n^j - eu_n^k) + \frac{\tau_n^2}{2}\left(\sum_{j=1}^{i}\|\delta^2 u_n^j\|^2 + \sum_{j=1}^{i}(\mathbb{A}e\delta u_n^j, e\delta u_n^j)\right) \\
&+ \frac{\tau_n^2}{2}\left(\sum_{j=1}^{i}(\mathbb{G}_n^{j-1}e\delta u_n^j, e\delta u_n^j) - \sum_{j=1}^{i}\sum_{k=1}^{j}\tau_n(\delta\mathbb{G}^{j-k}e\delta u_n^j, e\delta u_n^j)\right) \\
&\qquad\qquad = \frac{1}{2}\|u^1\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0) + \sum_{j=1}^{i}\tau_n L_n^j. \qquad (3.26)
\end{aligned}
$$

By our assumptions on $\mathbb{G}$ we deduce for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^d$ that

$$
\begin{aligned}
&\mathbb{G}_n^j(x)\xi\cdot\xi \geq 0 && j = 0,\dots,n,\\
&\delta\mathbb{G}_n^j(x)\xi\cdot\xi = \fint_{(j-1)\tau_n}^{j\tau_n}\dot{\mathbb{G}}(\tau,x)\xi\cdot\xi\,\mathrm{d}\tau \leq 0 && j = 1,\dots,n,\\
&\delta^2\mathbb{G}_n^j(x)\xi\cdot\xi = \fint_{(j-1)\tau_n}^{j\tau_n}\fint_{\tau-\tau_n}^{\tau}\ddot{\mathbb{G}}(r,x)\xi\cdot\xi\,\mathrm{d}r\mathrm{d}\tau \geq 0 && j = 2,\dots,n.
\end{aligned}
$$

Hence, thanks to (3.26), for every $i = 1,\dots,n$ we can write

$$
\frac{1}{2}\|\delta u_n^i\|^2 + \frac{1}{2}(\mathbb{A}eu_n^i, eu_n^i) \leq \frac{1}{2}\|u^1\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0) + \sum_{j=1}^{i}\tau_n L_n^j. \qquad (3.27)
$$

Let us estimate the right-hand side in (3.27) from above. We set

$$
K_n := \max_{j=0,..,n}\|\delta u_n^j\|, \quad E_n := \max_{j=0,..,n}\|eu_n^j\|.
$$

Therefore, we have the following bounds

$$
\left|\sum_{j=1}^{i}\tau_n(f_n^j, \delta u_n^j)\right| \leq \sqrt{T}\|f\|_{L^2(0,T;H)}K_n, \qquad (3.28)
$$

$$
\left|\sum_{j=1}^{i}\tau_n(f_n^j, \delta z_n^j)\right| \leq \|f\|_{L^2(0,T;H)}\|\dot{z}\|_{L^2(0,T;H)}, \qquad (3.29)
$$

$$
\left|\sum_{j=1}^{i}\tau_n(\mathbb{A}eu_n^j, e\delta z_n^j)\right| \leq C_{\mathbb{A}}\|e\dot{z}\|_{L^1(0,T;H)}E_n, \qquad (3.30)
$$

$$
\left|\sum_{j=1}^{i}\tau_n(\mathbb{G}_n^{j-1}(eu_n^j - eu^0), e\delta z_n^j)\right| \leq 2\|\mathbb{G}\|_{C^0([0,T];B)}\|e\dot{z}\|_{L^1(0,T;H)}E_n. \qquad (3.31)
$$

Notice that the following discrete integrations by parts hold

$$
\sum_{j=1}^{i}\tau_n(\delta^2 u_n^j, \delta z_n^j) = (\delta u_n^i, \delta z_n^i) - (\delta u_n^0, \delta z_n^0) - \sum_{j=1}^{i}\tau_n(\delta u_n^{j-1}, \delta^2 z_n^j), \qquad (3.32)
$$

$$
\sum_{j=1}^{i}\tau_n(N_n^j, \delta u_n^j)_{H_N} = (N_n^i, u_n^i)_{H_N} - (N_n^0, u_n^0)_{H_N} - \sum_{j=1}^{i}\tau_n(\delta N_n^j, u_n^{j-1})_{H_N}, \qquad (3.33)
$$

$$\sum_{j=1}^{i} \tau_n (N_n^j, \delta z_n^j)_{H_N} = (N_n^i, z_n^i)_{H_N} - (N_n^0, z_n^0)_{H_N} - \sum_{j=1}^{i} \tau_n (\delta N_n^j, z_n^{j-1})_{H_N}. \tag{3.34}$$

By means of (3.32) we can write

$$\begin{aligned} \Big| \sum_{j=1}^{i} (\delta^2 u_n^j, \delta z_n^j) \Big| &\le \|\delta u_n^i\| \|\delta z_n^i\| + \|\delta u_n^0\| \|\delta z_n^0\| + \sum_{j=1}^{i} \tau_n \|\delta u_n^{j-1}\| \|\delta^2 z_n^j\| \\ &\le (2\|\dot z\|_{C^0([0,T];H)} + \|\ddot z\|_{L^1(0,T;H)}) K_n. \end{aligned} \tag{3.35}$$

Moreover, thanks to

$$\|u_n^i\|_V \le \|u_n^i\| + E_n \le \sum_{j=1}^{i} \tau_n \|\delta u_n^j\| + \|u^0\| + E_n \le TK_n + E_n + \|u^0\| \quad \text{for } i = 0, \dots, n \tag{3.36}$$

and to (3.33) we obtain

$$\begin{aligned} \Big| \sum_{j=1}^{i} \tau_n (N_n^j, \delta u_n^j)_{H_N} \Big| &\le \|N_n^i\|_{H_N} \|u_n^i\|_{H_N} + \|N_n^0\|_{H_N} \|u_n^0\|_{H_N} + \sum_{j=1}^{i} \tau_n \|\delta N_n^j\|_{H_N} \|u_n^{j-1}\|_{H_N} \\ &\le C_{tr} \|N\|_{C^0([0,T];H_N)} (\|u_n^i\|_V + \|u_n^0\|_V) + C_{tr} \sum_{j=1}^{i} \tau_n \|\delta N_n^j\|_{H_N} \|u_n^{j-1}\|_V \\ &\le C_{tr} \big(2\|N\|_{C^0([0,T];H_N)} + \|\dot N\|_{L^1(0,T;H_N)}\big)(E_n + TK_n + \|u^0\|). \end{aligned} \tag{3.37}$$

Similarly, by (3.34) we obtain

$$\Big| \sum_{j=1}^{i} \tau_n (N_n^j, \delta z_n^j)_{H_N} \Big| \le C_{tr} \big(2\|N\|_{C^0([0,T];H_N)} + \|\dot N\|_{L^1(0,T;H_N)}\big) \|z\|_{C^0([0,T];V_0)}. \tag{3.38}$$

Finally, we have

$$\begin{aligned} \Big| \sum_{j=1}^{i} \sum_{k=1}^{j} \tau_n^2 (\delta \mathbb{G}_n^{j-k} (eu_n^k - eu_n^j), e\delta z_n^j) \Big| &\le \sum_{j=1}^{i} \sum_{k=1}^{j} \tau_n^2 \|\delta \mathbb{G}_n^{j-k}\|_B \|eu_n^k - eu_n^j\| \|e\delta z_n^j\| \\ &\le 2T \|\dot{\mathbb{G}}\|_{C^0([0,T];B)} \|e\dot z\|_{L^1(0,T;H)} E_n. \end{aligned} \tag{3.39}$$

By considering (3.27)–(3.39) and using (3.1), we obtain the existence of a positive constant $C_1 = C_1(z, N, f, u^0, \mathbb{A}, \mathbb{G})$ such that

$$\|\delta u_n^i\|^2 + c_{\mathbb{A}} \|eu_n^i\|^2 \le \|u^1\|^2 + C_{\mathbb{A}} \|eu^0\|^2 + C_1 (1 + K_n + E_n) \quad \text{for } i = 1, \dots, n.$$

In particular, since the right-hand side is independent of i, $u_n^0 = u^0$ and $\delta u_n^0 = u^1$, there exists another constant $C_2 = C_2(z, N, f, u^0, u^1, \mathbb{A}, \mathbb{G})$ for which we have

$$K_n^2 + E_n^2 \le C_2(1 + K_n + E_n) \quad \text{for every } n \in \mathbb{N}.$$

This implies the existence of a constant $C = C(z, N, f, u^0, u^1, \mathbb{A}, \mathbb{G})$ independent of $n \in \mathbb{N}$ such that

$$\|\delta u_n^j\| + \|eu_n^j\| \le K_n + E_n \le C \qquad \text{for every } j = 1, \dots, n \text{ and } n \in \mathbb{N},$$

which gives (3.21). □

A first consequence of Lemma 3.2.4 is the following uniform estimate on the family $\{\delta^2 u_n^j\}_{j=1}^n$.

Corollary 3.2.5. *Assume (3.3)–(3.1) and (3.12)–(3.16). Then there exists a constant $\tilde{C}$, independent of $n \in \mathbb{N}$, such that*

$$\sum_{j=1}^{n} \tau_n \|\delta^2 u_n^j\|^2_{(V_0^D)'} \leq \tilde{C}. \tag{3.40}$$

Proof. Thanks to equation (3.20) and to Lemma 3.2.4, for every $j = 1, \dots, n$ and $v \in V_0^D \subset V_n^j$ with $\|v\|_{V_0} \leq 1$ we have

$$|(\delta^2 u_n^j, v)| \leq C\big(C_{\mathbb{A}} + 2\|\mathbb{G}\|_{C^0([0,T];B)} + 2T\|\dot{\mathbb{G}}\|_{C^0([0,T];B)}\big) + \|f_n^j\| + C_{tr}\|N\|_{C^0([0,T];H_N)}.$$

By taking the supremum over $v \in V_0^D$ with $\|v\|_{V_0} \leq 1$ we obtain

$$\|\delta^2 u_n^j\|^2_{(V_0^D)'} \leq 3C^2\big(C_{\mathbb{A}} + 2\|\mathbb{G}\|_{C^0([0,T];B)} + 2T\|\dot{\mathbb{G}}\|_{C^0([0,T];B)}\big)^2 + 3\|f_n^j\|^2 + 3C_{tr}^2\|N\|^2_{C^0([0,T];H_N)}.$$

We multiply this inequality by τ_n and we sum over $j = 1, \dots, n$ to get (3.40). □

We now want to pass to the limit into equation (3.20) to obtain a generalized solution to system (3.11). Let us recall the following result, whose proof can be found for example in [19].

Lemma 3.2.6. *Let X, Y be two reflexive Banach spaces such that $X \hookrightarrow Y$ continuously. Then*

$$L^\infty(0,T;X) \cap C_w^0([0,T];Y) = C_w^0([0,T];X).$$

Let us define the following sequences of functions which are an approximation of the generalized solution:

$$\begin{aligned}
&u_n(t) = u_n^i + (t - i\tau_n)\delta u_n^i && \text{for } t \in [(i-1)\tau_n, i\tau_n] \text{ and } i = 1, \dots, n, && \\
&u_n^+(t) = u_n^i && \text{for } t \in ((i-1)\tau_n, i\tau_n] \text{ and } i = 1, \dots, n, && u_n^+(0) = u_n^0, \\
&u_n^-(t) = u_n^{i-1} && \text{for } t \in [(i-1)\tau_n, i\tau_n) \text{ and } i = 1, \dots, n, && u_n^-(T) = u_n^n.
\end{aligned}$$

Moreover, we consider also the sequences

$$\begin{aligned}
&\tilde{u}_n(t) = \delta u_n^i + (t - i\tau_n)\delta^2 u_n^i && \text{for } t \in [(i-1)\tau_n, i\tau_n] \text{ and } i = 1, \dots, n, && \\
&\tilde{u}_n^+(t) = \delta u_n^i && \text{for } t \in ((i-1)\tau_n, i\tau_n] \text{ and } i = 1, \dots, n, && \tilde{u}_n^+(0) = \delta u_n^0, \\
&\tilde{u}_n^-(t) = \delta u_n^{i-1} && \text{for } t \in [(i-1)\tau_n, i\tau_n) \text{ and } i = 1, \dots, n, && \tilde{u}_n^-(T) = \delta u_n^n,
\end{aligned}$$

which approximate the first time derivative of the generalized solution. In a similar way, we define also f_n^+, N_n^+, $\tilde{N}_n^+$, $z_n^\pm$, $\tilde{z}_n$, $\tilde{z}_n^+$, $\mathbb{G}_n^\pm$, $\tilde{\mathbb{G}}_n$, $\tilde{\mathbb{G}}_n^+$. Thanks to the uniform estimates of Lemma 3.2.4 we derive the following compactness result:

Lemma 3.2.7. *Assume (3.3)–(3.1) and (3.12)–(3.16). There exists a function $u \in \mathcal{C}_w \cap H^2(0,T;(V_0^D)')$ such that, up to a not relabeled subsequence*

$$u_n \xrightarrow[n\to\infty]{H^1(0,T;H)} u, \quad u_n^\pm \xrightharpoonup[n\to\infty]{L^\infty(0,T;V)}{}^* u, \quad \tilde{u}_n \xrightharpoonup[n\to\infty]{H^1(0,T;(V_0^D)')} \dot{u}, \quad \tilde{u}_n^\pm \xrightharpoonup[n\to\infty]{L^\infty(0,T;H)}{}^* \dot{u}, \tag{3.41}$$

and for every $t \in [0,T]$

$$u_n^\pm(t) \xrightharpoonup[n\to\infty]{V} u(t), \quad \tilde{u}_n^\pm(t) \xrightharpoonup[n\to\infty]{H} \dot{u}(t). \tag{3.42}$$

Proof. Thanks to Lemma 3.2.4 and the estimate (3.40), the sequences

$$\{u_n\}_n \subset L^\infty(0,T;V)\cap H^1(0,T;H), \qquad \{\tilde{u}_n\}_n \subset L^\infty(0,T;H)\cap H^1(0,T;(V_0^D)'),$$
$$\{u_n^\pm\}_n \subset L^\infty(0,T;V), \qquad \{\tilde{u}_n^\pm\}_n \subset L^\infty(0,T;H),$$

are uniformly bounded with respect to $n\in\mathbb{N}$. By Banach-Alaoglu's theorem and Lemma 3.2.6 there exist two functions $u \in C_w^0([0,T];V)\cap H^1(0,T;H)$ and $v\in C_w^0([0,T];H)\cap H^1(0,T;(V_0^D)')$, such that, up to a not relabeled subsequence

$$u_n \xrightarrow[n\to\infty]{H^1(0,T;H)} u, \quad u_n \xrightarrow[n\to\infty]{L^\infty(0,T;V)}{}^{*}\, u, \quad \tilde{u}_n \xrightarrow[n\to\infty]{H^1(0,T;(V_0^D)')} v, \quad \tilde{u}_n \xrightarrow[n\to\infty]{L^\infty(0,T;H)}{}^{*}\, v. \tag{3.43}$$

Thanks to (3.40) we get

$$\|\dot{u}_n-\tilde{u}_n\|^2_{L^2(0,T;(V_0^D)')}\le \tilde{C}\tau_n^2 \xrightarrow[n\to\infty]{} 0,$$

therefore we deduce that $v=\dot{u}$. Moreover, by using (3.21) and (3.40) we have

$$\|u_n^\pm-u_n\|_{L^\infty(0,T;H)}\le C\tau_n \xrightarrow[n\to\infty]{} 0, \qquad \|\tilde{u}_n^\pm-\tilde{u}_n\|^2_{L^2(0,T;(V_0^D)')}\le \tilde{C}\tau_n^2 \xrightarrow[n\to\infty]{} 0.$$

We combine the previous convergences with (3.43) to derive

$$u_n^\pm \xrightarrow[n\to\infty]{L^\infty(0,T;V)}{}^{*}\, u, \qquad \tilde{u}_n^\pm \xrightarrow[n\to\infty]{L^\infty(0,T;H)}{}^{*}\, \dot{u}.$$

By (3.43) for every $t\in[0,T]$ we have

$$u_n(t) \xrightarrow[n\to\infty]{V} u(t), \qquad \tilde{u}_n(t) \xrightarrow[n\to\infty]{H} \dot{u}(t).$$

Again, thanks to (3.21) and (3.40), for every $t\in[0,T]$ we get

$$\|u_n^\pm(t)\|_V\le C, \quad \|u_n^\pm(t)-u_n(t)\|\le C\tau_n \xrightarrow[n\to\infty]{} 0,$$
$$\|\tilde{u}_n^\pm(t)\|\le C, \quad \|\tilde{u}_n^\pm(t)-\tilde{u}_n(t)\|^2_{(V_0^D)'}\le \tilde{C}\tau_n \xrightarrow[n\to\infty]{} 0,$$

which imply (3.42). Finally, observe that for every $t\in[0,T]$

$$u_n^-(t)\in V_t, \qquad u_n^-(t) \xrightarrow[n\to\infty]{V} u(t).$$

Therefore, $u(t)\in V_t$ for every $t\in[0,T]$ since V_t is a closed subspace of V. Hence, $u\in\mathcal{C}_w$. □

Let us check that the limit function u defined before satisfies the boundary and initial conditions.

Corollary 3.2.8. *Assume (3.3)–(3.1) and (3.12)–(3.16). Then the function $u\in\mathcal{C}_w$ of Lemma 3.2.7 satisfies for every $t\in[0,T]$ the condition $u(t)-z(t)\in V_t^D$, and it assumes the initial conditions $u(0)=u^0$ in V_0 and $\dot{u}(0)=u^1$ in H.*

Proof. By (3.41) we have

$$u^0=u_n(0) \xrightarrow[n\to\infty]{V} u(0), \qquad u^1=\tilde{u}_n(0) \xrightarrow[n\to\infty]{H} \dot{u}(0).$$

Hence, $u\in\mathcal{C}_w$ satisfies $u(0)=u^0$ in V_0 and $\dot{u}(0)=u^1$ in H. Moreover, since $z\in C^0([0,T];V_0)$ and thanks to (3.42), we have for every $t\in[0,T]$

$$u_n^-(t)-z_n^-(t)\in V_t^D, \qquad u_n^-(t)-z_n^-(t) \xrightarrow[n\to\infty]{V} u(t)-z(t).$$

Thus, $u(t)-z(t)\in V_t^D$ for every $t\in[0,T]$ because V_t^D is a closed subspace of V. □

Lemma 3.2.9. *Assume (3.3)–(3.1) and (3.12)–(3.16). Then the function $u \in \mathcal{C}_w$ of Lemma 3.2.7 is a generalized solution to system (3.11).*

Proof. We only need to prove that the function $u \in \mathcal{C}_w$ satisfies (3.17). We fix $n \in \mathbb{N}$ and a function $\varphi \in \mathcal{C}^1_c$. Let us consider

$$\varphi_n^j := \varphi(j\tau_n) \qquad \text{for } j = 0, \dots, n, \qquad \delta\varphi_n^j := \frac{\varphi_n^j - \varphi_n^{j-1}}{\tau_n} \qquad \text{for } j = 1, \dots, n,$$

and, as we did before for the family $\{u_n^j\}_{j=1}^n$, we define the approximating sequences $\{\varphi_n^+\}_n$ and $\{\tilde{\varphi}_n^+\}_n$. If we use $\tau_n\varphi_n^j \in V_n^j$ as a test function in (3.20), after summing over $j = 1, ..., n$, we get

$$\sum_{j=1}^n \tau_n(\delta^2 u_n^j, \varphi_n^j) + \sum_{j=1}^n \tau_n(\mathbb{A}eu_n^j, e\varphi_n^j) + \sum_{j=1}^n \tau_n(\mathbb{G}_n^0(eu_n^j - eu^0), e\varphi_n^j)$$
$$+ \sum_{j=1}^n \sum_{k=1}^j \tau_n^2(\delta\mathbb{G}_n^{j-k}(eu_n^k - eu^0), e\varphi_n^j) = \sum_{j=1}^n \tau_n(f_n^j, \varphi_n^j) + \sum_{j=1}^n \tau_n(N_n^j, \varphi_n^j)_{H_N}. \tag{3.44}$$

By means of a time discrete integration by parts we obtain

$$\sum_{j=1}^n \tau_n(\delta^2 u_n^j, \varphi_n^j) = -\sum_{j=1}^n \tau_n(\delta u_n^{j-1}, \delta\varphi_n^j) = -\int_0^T (\tilde{u}_n^-(t), \tilde{\varphi}_n^+(t))\mathrm{d}t,$$

and since $\delta\mathbb{G}_n^0 = 0$ and $\varphi_n^0 = \varphi_n^n = 0$ we get

$$\sum_{j=1}^n \tau_n(\mathbb{G}_n^0(eu_n^j - eu^0), e\varphi_n^j) + \sum_{j=1}^n \sum_{k=1}^j \tau_n^2(\delta\mathbb{G}_n^{j-k}(eu_n^k - eu^0), e\varphi_n^j)$$
$$= -\sum_{j=1}^{n-1} \sum_{k=1}^j \tau_n^2(\mathbb{G}_n^{j-k}(eu_n^k - eu^0), e\delta\varphi_n^{j+1})$$
$$= -\int_0^{T-\tau_n} \int_0^{t_n} (\mathbb{G}_n^-(t_n - \tau)(eu_n^+(\tau) - eu^0), e\tilde{\varphi}_n^+(t + \tau_n))\mathrm{d}\tau\mathrm{d}t,$$

where $t_n := \left\lceil \frac{t}{\tau_n} \right\rceil \tau_n$ for $t \in (0,T)$ and $\lceil x \rceil$ is the superior integer part of the number x. Thanks to (3.44) we deduce

$$-\int_0^T (\tilde{u}_n^-(t), \tilde{\varphi}_n^+(t))\mathrm{d}t - \int_0^{T-\tau_n} \int_0^{t_n} (\mathbb{G}_n^-(t_n - \tau)(eu_n^+(\tau) - eu^0), e\tilde{\varphi}_n^+(t + \tau_n))\mathrm{d}\tau\mathrm{d}t$$
$$+ \int_0^T (\mathbb{A}eu_n^+(t), e\varphi_n^+(t))\mathrm{d}t = \int_0^T (f_n^+(t), \varphi_n^+(t))\mathrm{d}t + \int_0^T (N_n^+(t), \varphi_n^+(t))_{H_N}\mathrm{d}t. \tag{3.45}$$

We use (3.41) and the following convergences

$$\varphi_n^+ \xrightarrow[n\to\infty]{L^2(0,T;V)} \varphi, \quad \tilde{\varphi}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{\varphi}, \quad f_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} f, \quad N_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H_N)} N,$$

to derive

$$\int_0^T (\tilde{u}_n^-(t), \tilde{\varphi}_n^+(t))\mathrm{d}t \xrightarrow[n\to\infty]{} \int_0^T (\dot{u}(t), \dot{\varphi}(t))\mathrm{d}t,$$
$$\int_0^T (\mathbb{A}eu_n^+(t), e\varphi_n^+(t))\mathrm{d}t \xrightarrow[n\to\infty]{} \int_0^T (\mathbb{A}eu(t), e\varphi(t))\mathrm{d}t,$$

$$\int_0^T (f_n^+(t), \varphi_n^+(t))\mathrm{d}t \xrightarrow[n\to\infty]{} \int_0^T (f(t), \varphi(t))\mathrm{d}t,$$
$$\int_0^T (N_n^+(t), \varphi_n^+(t))_{H_N}\mathrm{d}t \xrightarrow[n\to\infty]{} \int_0^T (N(t), \varphi(t))_{H_N}\mathrm{d}t.$$

Moreover, for every fixed $t \in (0, T)$

$$\chi_{[0,T-\tau_n]}(t)\chi_{[0,t_n]}(\cdot)\mathbb{G}_n^-(t_n - \cdot)e\tilde{\varphi}_n^+(t + \tau_n) \xrightarrow[n\to\infty]{L^2(0,T;H)} \chi_{[0,T]}(t)\chi_{[0,t]}(\cdot)\mathbb{G}(t - \cdot)e\dot{\varphi}(t), \tag{3.46}$$

which together with (3.41) gives

$$\chi_{[0,T-\tau_n]}(t)\int_0^{t_n} (\mathbb{G}_n^-(t_n - \tau)(eu_n^+(\tau) - eu^0), e\tilde{\varphi}_n^+(t + \tau_n))\mathrm{d}\tau$$
$$\xrightarrow[n\to\infty]{} \chi_{[0,T]}(t)\int_0^t (\mathbb{G}(t - \tau)(eu(\tau) - eu^0), e\dot{\varphi}(t))\mathrm{d}\tau. \tag{3.47}$$

By (3.21) for every $t \in (0, T)$ we deduce

$$\left|\chi_{[0,T-\tau_n]}(t)\int_0^{t_n} (\mathbb{G}_n^-(t_n - \tau)(eu_n^+(\tau) - eu^0), e\tilde{\varphi}_n^+(t + \tau_n))\mathrm{d}\tau\right|$$
$$\le 2T\|\mathbb{G}\|_{C^0([0,T];B)}C\|e\dot{\varphi}\|_{C^0([0,T];H)}. \tag{3.48}$$

Therefore, we can use the dominated convergence theorem to pass to the limit in the double integral of (3.45), and we obtain that u satisfies (3.17) for every function $\varphi \in \mathcal{C}_c^1$. □

Now we want to deduce an energy-dissipation inequality for the generalized solution $u \in \mathcal{C}_w$ of Lemma 3.2.7. Let us define for every $t \in [0, T]$ the total energy $\mathscr{E}(t)$ and the dissipation $\mathscr{D}(t)$ as

$$\mathscr{E}(t) := \frac{1}{2}\|\dot{u}(t)\|^2 + \frac{1}{2}(\mathbb{A}eu(t), eu(t)) + \frac{1}{2}(\mathbb{G}(t)(eu(t) - eu^0), eu(t) - eu^0)$$
$$- \frac{1}{2}\int_0^t (\dot{\mathbb{G}}(t - \tau)(eu(t) - eu(\tau)), eu(t) - eu(\tau))\mathrm{d}\tau,$$
$$\mathscr{D}(t) := -\frac{1}{2}\int_0^t (\dot{\mathbb{G}}(\tau)(eu(\tau) - eu^0), eu(\tau) - eu^0)\mathrm{d}\tau$$
$$+ \frac{1}{2}\int_0^t\int_0^\tau (\ddot{\mathbb{G}}(\tau - r)(eu(\tau) - eu(r)), eu(\tau) - eu(r))\mathrm{d}r\mathrm{d}\tau.$$

Notice that $\mathscr{E}(t)$ is well defined for every time $t \in [0, T]$ since $u \in C_w^0([0, T]; V)$ and $\dot{u} \in C_w^0([0, T]; H)$. Moreover, by the initial conditions we have

$$\mathscr{E}(0) = \frac{1}{2}\|u^1\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0).$$

Proposition 3.2.10. *Assume (3.3)–(3.1) and (3.12)–(3.16). Then the generalized solution $u \in \mathcal{C}_w$ to system (3.11) of Lemma 3.2.7 satisfies for every $t \in [0, T]$ the following energy-dissipation inequality*

$$\mathscr{E}(t) + \mathscr{D}(t) \le \mathscr{E}(0) + \mathscr{W}_{tot}(t), \tag{3.49}$$

where the total work is defined as

$$\mathscr{W}_{tot}(t) := \int_0^t [(f(\tau), \dot{u}(\tau) - \dot{z}(\tau)) + (\mathbb{A}eu(\tau), e\dot{z}(\tau))]\mathrm{d}\tau$$

$$
\begin{aligned}
&-\int_0^t (\dot N(\tau), u(\tau)-z(\tau))_{H_N}\mathrm{d}\tau + (N(t), u(t)-z(t))_{H_N} - (N(0), u^0 - z(0))_{H_N}\\
&-\int_0^t (\dot u(\tau), \ddot z(\tau))\mathrm{d}\tau + (\dot u(t), \dot z(t)) - (u^1, \dot z(0))\\
&+\int_0^t (\mathbb{G}(\tau)(eu(\tau)-eu^0), e\dot z(\tau))\mathrm{d}\tau + \int_0^t\int_0^\tau (\dot{\mathbb{G}}(\tau-r)(eu(r)-eu(\tau)), e\dot z(\tau))\mathrm{d}r\mathrm{d}\tau.
\end{aligned}
\tag{3.50}
$$

Proof. Fixed $t \in (0,T]$ and $n \in \mathbb{N}$ there exists a unique $i = i(n) \in \{1,\dots,n\}$ such that $t \in ((i-1)\tau_n, i\tau_n]$. In particular, $i(n) = \left\lceil \frac{t}{\tau_n}\right\rceil$. After setting $t_n := i\tau_n$ and using that $\delta\mathbb{G}_n^0 = 0$, we rewrite (3.26) as

$$
\begin{aligned}
&\frac12\|\tilde u_n^+(t)\|^2 + \frac12(\mathbb{A}eu_n^+(t), eu_n^+(t)) + \frac12(\mathbb{G}_n^+(t)(eu_n^+(t)-eu^0), eu_n^+(t)-eu^0)\\
&\quad-\frac12\int_0^{t_n}(\tilde{\mathbb{G}}_n^+(t_n-\tau)(eu_n^+(t)-eu_n^+(\tau)), eu_n^+(t)-eu_n^+(\tau))\mathrm{d}\tau\\
&\quad+\frac12\int_{\tau_n}^{t_n}\int_0^{\bar\tau_n-\tau_n}(\dot{\tilde{\mathbb{G}}}_n(\bar\tau_n-r)(eu_n^+(\tau)-eu_n^+(r)), eu_n^+(\tau)-eu_n^+(r))\mathrm{d}r\mathrm{d}\tau\\
&\quad-\frac12\int_0^{t_n}(\tilde{\mathbb{G}}_n^+(\tau)(eu_n^+(\tau)-eu^0), eu_n^+(\tau)-eu^0)\mathrm{d}\tau \le \frac12\|u^1\|^2 + \frac12(\mathbb{A}eu^0, eu^0) + \mathcal{W}_n^+(t),
\end{aligned}
\tag{3.51}
$$

where $\bar\tau_n := \left\lceil\frac{\tau}{\tau_n}\right\rceil\tau_n$ for $\tau \in (\tau_n, t_n)$, and the approximate total work $\mathscr{W}_n^+(t)$ is given by

$$
\begin{aligned}
\mathscr{W}_n^+(t) &:= \int_0^{t_n}[(f_n^+(\tau), \tilde u_n^+(\tau)-\tilde z_n^+(\tau)) + (N_n^+(\tau), \tilde u_n^+(\tau)-\tilde z_n^+(\tau))_{H_N} + (\dot{\tilde u}_n(\tau), \tilde z_n^+(\tau))]\mathrm{d}\tau\\
&\quad+\int_0^{t_n}[(\mathbb{C}eu_n^+(\tau), e\tilde z_n^+(\tau)) + (\mathbb{G}_n^-(\tau)(eu_n^+(\tau)-eu^0), e\tilde z_n^+(\tau))]\mathrm{d}\tau\\
&\quad+\int_{\tau_n}^{t_n}\int_0^{\bar\tau_n-\tau_n}(\tilde{\mathbb{G}}_n^-(\bar\tau_n-r)(eu_n^+(r)-eu_n^+(\tau)), e\tilde z_n^+(\tau))\mathrm{d}r\mathrm{d}\tau.
\end{aligned}
$$

By (3.1), (3.13), and (3.42) we derive

$$\|\dot u(t)\|^2 \le \liminf_{n\to\infty}\|\tilde u_n^+(t)\|^2, \tag{3.52}$$

$$(\mathbb{A}eu(t), eu(t)) \le \liminf_{n\to\infty}(\mathbb{A}eu_n^+(t), eu_n^+(t)), \tag{3.53}$$

$$(\mathbb{G}(t)(eu(t)-eu^0), eu(t)-eu^0) \le \liminf_{n\to\infty}(\mathbb{G}(t)(eu_n^+(t)-eu^0), eu_n^+(t)-eu^0). \tag{3.54}$$

Moreover, the estimate (3.21) imply

$$\left|((\mathbb{G}(t)-\mathbb{G}_n^+(t))(eu_n^+(t)-eu^0), eu_n^+(t)-eu^0)\right| \le 4C^2\|\dot{\mathbb{G}}\|_{C^0([0,T];B)}\tau_n \xrightarrow[n\to\infty]{} 0,$$

which together with inequality (3.54) gives

$$(\mathbb{G}(t)(eu(t)-eu^0), eu(t)-eu^0) \le \liminf_{n\to\infty}(\mathbb{G}_n^+(t)(eu_n^+(t)-eu^0), eu_n^+(t)-eu^0). \tag{3.55}$$

By (3.14) and (3.42), for every $\tau \in (0,t)$ we have

$$
\begin{aligned}
&(-\dot{\mathbb{G}}(t-\tau)(eu(t)-eu(\tau)), eu(t)-eu(\tau))\\
&\qquad\le \liminf_{n\to\infty}(-\dot{\mathbb{G}}(t-\tau)(eu_n^+(t)-eu_n^+(\tau)), eu_n^+(t)-eu_n^+(\tau)).
\end{aligned}
$$

Moreover

$$\|\tilde{\mathbb{G}}_n^+(t_n-\tau)-\dot{\mathbb{G}}(t-\tau)\|_B\le \fint_{t_n-\bar\tau_n}^{t_n-\bar\tau_n+\tau_n}\|\dot{\mathbb{G}}(r)-\dot{\mathbb{G}}(t-\tau)\|_B\mathrm{d}r\xrightarrow[n\to\infty]{}0$$

because $t_n-\bar\tau_n\to t-\tau$. Hence, we can argue as before to deduce

$$(-\dot{\mathbb{G}}(t-\tau)(eu(t)-eu(\tau)),eu(t)-eu(\tau))\\ \le\liminf_{n\to\infty}(-\tilde{\mathbb{G}}_n^+(t_n-\tau)(eu_n^+(t)-eu_n^+(\tau)),eu_n^+(t)-eu_n^+(\tau)).$$

In particular, we can use Fatou's lemma and the fact that $t\le t_n$ to obtain

$$\int_0^t(-\dot{\mathbb{G}}(t-\tau)(eu(t)-eu(\tau)),eu(t)-eu(\tau))\mathrm{d}\tau\\ \le\liminf_{n\to\infty}\int_0^{t_n}(-\tilde{\mathbb{G}}_n^+(t_n-\tau)(eu_n^+(t)-eu_n^+(\tau)),eu_n^+(t)-eu_n^+(\tau))\mathrm{d}\tau.$$

By arguing in a similar way, we can derive

$$\int_0^t(-\dot{\mathbb{G}}(\tau)(eu(\tau)-eu^0),eu(\tau)-eu^0)\mathrm{d}\tau\\ \le\liminf_{n\to\infty}\int_0^{t_n}(-\tilde{\mathbb{G}}_n^+(\tau)(eu_n^+(\tau)-eu^0),eu_n^+(\tau)-eu^0)\mathrm{d}\tau.$$

Let us consider the double integral in the left-hand side. We fix $\tau\in(0,t)$ and by (3.15) for every $r\in(0,\tau)$ we have

$$(\ddot{\mathbb{G}}(\tau-r)(eu(\tau)-eu(r)),eu(\tau)-eu(r))\\ \le\liminf_{n\to\infty}(\ddot{\mathbb{G}}(\tau-r)(eu_n^+(\tau)-eu_n^+(r)),eu_n^+(\tau)-eu_n^+(r)).$$

Moreover, for a.e. $r\in(0,\bar\tau_n-\tau_n)$ by defining $r_n:=\left\lceil\frac{r}{\tau_n}\right\rceil\tau_n$ we deduce

$$\|\dot{\tilde{\mathbb{G}}}_n(\bar\tau_n-r)-\ddot{\mathbb{G}}(\tau-r)\|_B\le\fint_{\bar\tau_n-r_n}^{\bar\tau_n-r_n+\tau_n}\fint_{\lambda-\tau_n}^{\lambda}\|\ddot{\mathbb{G}}(\theta)-\ddot{\mathbb{G}}(\tau-r)\|_B\mathrm{d}\theta\mathrm{d}\lambda\xrightarrow[n\to\infty]{}0.$$

Therefore, for a.e. $r\in(0,\tau)$ we get

$$(\ddot{\mathbb{G}}(\tau-r)(eu(\tau)-eu(r)),eu(\tau)-eu(r))\\ \le\liminf_{n\to\infty}(\dot{\tilde{\mathbb{G}}}_n(\bar\tau_n-r)(eu_n^+(\tau)-eu_n^+(r)),eu_n^+(\tau)-eu_n^+(r)),$$

since $r\in(0,\bar\tau_n-\tau_n)$ for n large enough. If we apply again Fatou's lemma we conclude

$$\int_0^\tau(\ddot{\mathbb{G}}(\tau-r)(eu(\tau)-eu(r)),eu(\tau)-eu(r))\mathrm{d}r\\ \le\liminf_{n\to\infty}\int_0^\tau(\dot{\tilde{\mathbb{G}}}_n(\bar\tau_n-r)(eu_n^+(\tau)-eu_n^+(r)),eu_n^+(\tau)-eu_n^+(r))\mathrm{d}r.$$

By (3.21) we get

$$\left|\int_{\bar\tau_n-\tau_n}^\tau(\dot{\tilde{\mathbb{G}}}_n(\bar\tau_n-r)(eu_n^+(\tau)-eu_n^+(r)),eu_n^+(\tau)-eu_n^+(r))\mathrm{d}r\right|\\ \le 4C^2\|\ddot{\mathbb{G}}\|_{C^0([0,T];B)}(\tau-\bar\tau_n+\tau_n)\xrightarrow[n\to\infty]{}0,$$

from which we derive

$$\int_0^\tau (\ddot{\mathbb{G}}(\tau-r)(eu(\tau)-eu(r)),eu(\tau)-eu(r))\mathrm{d}r$$
$$\leq \liminf_{n\to\infty}\int_0^{\bar\tau_n-\tau_n}(\dot{\tilde{\mathbb{G}}}_n(\bar\tau_n-r)(eu_n^+(\tau)-eu_n^+(r)),eu_n^+(\tau)-eu_n^+(r))\mathrm{d}r.$$

Since this is true for every $\tau\in(0,t)$, arguing as before we obtain

$$\int_0^t\int_0^\tau (\ddot{\mathbb{G}}(\tau-r)(eu(\tau)-eu(r)),eu(\tau)-eu(r))\mathrm{d}r\mathrm{d}\tau$$
$$\leq \liminf_{n\to\infty}\int_{\tau_n}^{t_n}\int_0^{\bar\tau_n-\tau_n}(\dot{\tilde{\mathbb{G}}}_n(\bar\tau_n-r)(eu_n^+(\tau)-eu_n^+(r)),eu_n^+(\tau)-eu_n^+(r))\mathrm{d}r\mathrm{d}\tau.$$

Let us study the right-hand side of (3.51). Given that

$$\chi_{[0,t_n]}f_n^+\xrightarrow[n\to\infty]{L^2(0,T;H)}\chi_{[0,t]}f,\qquad \tilde u_n^+-\tilde z_n^+\xrightarrow[n\to\infty]{L^2(0,T;H)}\dot u-\dot z,$$
$$\chi_{[0,t_n]}\mathbb{G}_n^-e\tilde z_n^+\xrightarrow[n\to\infty]{L^1(0,T;H)}\chi_{[0,t]}\mathbb{G}e\dot z,\qquad u_n^+\xrightarrow[n\to\infty]{L^\infty(0,T;V)}{}^{*}\,u,$$

we can deduce

$$\int_0^{t_n}(f_n^+(\tau),\tilde u_n^+(\tau)-\tilde z_n^+(\tau))\mathrm{d}\tau\xrightarrow[n\to\infty]{}\int_0^t(f(\tau),\dot u(\tau)-\dot z(\tau))\mathrm{d}\tau, \tag{3.56}$$
$$\int_0^{t_n}(\mathbb{A}eu_n^+(\tau),e\tilde z_n^+(\tau))\mathrm{d}\tau\xrightarrow[n\to\infty]{}\int_0^t(\mathbb{A}eu(\tau),e\dot z(\tau))\mathrm{d}\tau, \tag{3.57}$$
$$\int_0^{t_n}(\mathbb{G}_n^-(\tau)(eu_n^+(\tau)-eu^0),e\tilde z_n^+(\tau))\mathrm{d}\tau\xrightarrow[n\to\infty]{}\int_0^t(\mathbb{G}(\tau)(eu(\tau)-eu^0),e\dot z(\tau))\mathrm{d}\tau. \tag{3.58}$$

By using the same argumentations of (3.46)–(3.48), together with the dominate convergence theorem, we can write

$$\int_{\tau_n}^{t_n}\int_0^{\bar\tau_n-\tau_n}(\tilde{\mathbb{G}}_n^-(\bar\tau_n-r)(eu_n^+(r)-eu_n^+(\tau)),e\tilde z_n^+(\tau))\mathrm{d}r\mathrm{d}\tau$$
$$\xrightarrow[n\to\infty]{}\int_0^t\int_0^\tau(\dot{\mathbb{G}}(\tau-r)(eu(r)-eu(\tau)),e\dot z(\tau))\mathrm{d}r\mathrm{d}\tau. \tag{3.59}$$

Thanks to the discrete integration by parts formulas (3.32)–(3.34) we have

$$\int_0^{t_n}(\dot{\tilde u}_n(\tau),\tilde z_n^+(\tau))\mathrm{d}\tau=(\tilde u_n^+(t),\tilde z_n^+(t))-(u^1,\dot z(0))-\int_0^{t_n}(\tilde u_n^-(\tau),\dot{\tilde z}_n(\tau))\mathrm{d}\tau,$$
$$\int_0^{t_n}(N_n^+(\tau),\tilde u_n^+(\tau)-\tilde z_n^+(\tau))_{H_N}\mathrm{d}\tau=(N_n^+(t),u_n^+(t)-z_n^+(t))_{H_N}-(N(0),u^0-z(0))_{H_N}$$
$$-\int_0^{t_n}(\tilde N_n^+(\tau),u_n^-(\tau)-z_n^-(\tau))_{H_N}\mathrm{d}\tau.$$

By arguing as before we deduce

$$\int_0^{t_n}(\dot{\tilde u}_n(\tau),\tilde z_n^+(\tau))\mathrm{d}\tau\xrightarrow[n\to\infty]{}(\dot u(t),\dot z(t))-(u^1,\dot z(0))-\int_0^t(\dot u(\tau),\ddot z(\tau))\mathrm{d}\tau, \tag{3.60}$$
$$\int_0^{t_n}(N_n^+(\tau),\tilde u_n^+(\tau)-\tilde z_n^+(\tau))_{H_N}\mathrm{d}\tau$$

$$\xrightarrow[n\to\infty]{} (N(t),u(t)-z(t))_{H_N}-(N(0),u^0-z(0))_{H_N}-\int_0^t(\dot N(\tau),u(\tau)-z(\tau))_{H_N}\mathrm{d}\tau, \quad (3.61)$$

thanks to Lemma 3.2.7 and to the following convergences:

$$\|\tilde{\dot z}_n^+(t)-\dot z(t)\|\le \fint_{t_n-\tau_n}^{t_n}\|\dot z(\tau)-\dot z(t)\|\mathrm{d}\tau \xrightarrow[n\to\infty]{} 0,$$
$$\|z_n^+(t)-z(t)\|_{H_N}\le C_{tr}\sqrt{\tau_n}\|\dot z\|_{L^2(0,T;V_0)} \xrightarrow[n\to\infty]{} 0,$$
$$\|N_n^+(t)-N(t)\|_{H_N}\le \int_t^{t_n}\|\dot N(\tau)\|_{H_N}\mathrm{d}\tau \xrightarrow[n\to\infty]{} 0,$$

and

$$\chi_{[0,t_n]}\dot{\tilde{\dot z}}_n \xrightarrow[n\to\infty]{L^1(0,T;H)} \chi_{[0,t]}\ddot z, \qquad \tilde u_n^- \xrightarrow[n\to\infty]{L^\infty(0,T;H)\,*} \dot u,$$
$$\chi_{[0,t_n]}\tilde{\dot N}_n^+ \xrightarrow[n\to\infty]{L^1(0,T;H_N)} \chi_{[0,t]}\dot N, \qquad u_n^- - z_n^- \xrightarrow[n\to\infty]{L^\infty(0,T;V)\,*} u-z.$$

By combining (3.51) with (3.52)–(3.61) we deduce the energy-dissipation inequality (3.49) for every $t\in(0,T]$. Finally, for $t=0$ the inequality trivially holds since $u(0)=u^0$ in V_0 and $\dot u(0)=u^1$ in H. □

Remark 3.2.11. From the classical point of view, the total work on the solution u at time $t\in[0,T]$ is given by

$$\mathscr{W}_{tot}^C(t):=\mathscr{W}_{load}(t)+\mathscr{W}_{bdry}(t), \quad (3.62)$$

where $\mathscr{W}_{load}(t)$ is the work on the solution u at time $t\in[0,T]$ due to the loading term, which is defined as

$$\mathscr{W}_{load}(t):=\int_0^t(f(\tau),\dot u(\tau))\mathrm{d}\tau,$$

and $\mathscr{W}_{bdry}(t)$ is the work on the solution u at time $t\in[0,T]$ due to the varying boundary conditions, which one expects to be equal to

$$\begin{aligned}\mathscr{W}_{bdry}(t):=\int_0^t(N(\tau),\dot u(\tau))_{H_N}\mathrm{d}\tau+\int_0^t(\mathbb{A}eu(\tau)\nu,\dot z(\tau))_{H_D}\mathrm{d}\tau\\ +\int_0^t\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau\mathbb{G}(\tau-r)(eu(r)-eu^0)\mathrm{d}r\right)\nu,\dot z(\tau))_{H_D}\mathrm{d}\tau\end{aligned}$$

Unfortunately, $\mathscr{W}_{bdry}(t)$ is not well defined under our assumptions on u. In particular, the term involving the Dirichlet datum z is difficult to handle since the trace of the function $\mathbb{A}eu(\tau)\nu+\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int_0^\tau\mathbb{G}(\tau-r)eu(r)\mathrm{d}r\right)\nu$ on $\partial_D\Omega$ is not well defined. If we assume that $u\in L^2(0,T;H^2(\Omega\setminus\Gamma;\mathbb{R}^d))\cap H^2(0,T;L^2(\Omega\setminus\Gamma;\mathbb{R}^d))$ and that Γ is a smooth manifold, then the first term of $\mathscr{W}_{bdry}(t)$ makes sense and satisfies

$$\int_0^t(N(\tau),\dot u(\tau))_{H_N}\mathrm{d}\tau=(N(t),u(t))_{H_N}-(N(0),u(0))_{H_N}-\int_0^t(\dot N(\tau),u(\tau))_{H_N}\mathrm{d}\tau.$$

Moreover, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau\mathbb{G}(\tau-r)(eu(r)-eu^0)\mathrm{d}r&=\mathbb{G}(0)(eu(\tau)-eu^0)+\int_0^\tau\dot{\mathbb{G}}(\tau-r)(eu(r)-eu^0)\mathrm{d}r\\&=\mathbb{G}(\tau)(eu(\tau)-eu^0)+\int_0^\tau\dot{\mathbb{G}}(\tau-r)(eu(r)-eu(\tau))\mathrm{d}r,\end{aligned} \quad (3.63)$$

therefore $\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \mathbb{G}(\tau-r)(eu(r)-eu^0)\mathrm{d}r\right)\nu \in L^2(0,T;H_D)$. By using (3.11), together with the divergence theorem and the integration by parts formula, we derive

$$\begin{aligned}
&\int_0^t \Big(\mathbb{A}eu(\tau)\nu + \Big(\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \mathbb{G}(\tau-r)(eu(r)-eu^0)\mathrm{d}r\Big)\nu, \dot z(\tau)\Big)_{H_D}\mathrm{d}\tau\\
&= \int_0^t \Big(\mathbb{A}eu(\tau) + \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \mathbb{G}(\tau-r)(eu(r)-eu^0)\mathrm{d}r, e\dot z(\tau)\Big)\mathrm{d}\tau\\
&\quad + \int_0^t \Big[\Big(\operatorname{div}\Big(\mathbb{A}eu(\tau) + \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \mathbb{G}(\tau-r)(eu(r)-eu^0)\mathrm{d}r\Big), \dot z(\tau)\Big) - (N(\tau), \dot z(\tau))_{H_N}\Big]\mathrm{d}\tau\\
&= \int_0^t \Big(\mathbb{A}eu(\tau) + \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \mathbb{G}(\tau-r)(eu(r)-eu^0)\mathrm{d}r, e\dot z(\tau)\Big)\mathrm{d}\tau\\
&\quad + \int_0^t \Big[(\ddot u(\tau) - f(\tau), \dot z(\tau)) - (N(\tau), \dot z(\tau))_{H_N}\Big]\mathrm{d}\tau\\
&= \int_0^t \Big[\Big(\mathbb{A}eu(\tau) + \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \mathbb{G}(\tau-r)(eu(r)-eu^0)\mathrm{d}r, e\dot z(\tau)\Big) - (f(\tau), \dot z(\tau))\Big]\mathrm{d}\tau\\
&\quad + \int_0^t (\dot N(\tau), z(\tau))_{H_N}\mathrm{d}\tau - (N(t), z(t))_{H_N} + (N(0), z(0))_{H_N}\\
&\quad - \int_0^t (\dot u(\tau), \ddot z(\tau))\mathrm{d}\tau + (\dot u(t), \dot z(t)) - (u^1, \dot z(0)).
\end{aligned} \tag{3.64}$$

Therefore, by (3.63) and (3.64) we deduce the definition of total work given in (3.50) is coherent with the classical one (3.62).

We conclude this subsection by showing that the generalized solution of Lemma 3.2.7 satisfies the initial conditions in a stronger sense than the ones stated in Definition 3.1.2.

Lemma 3.2.12. *Assume* (3.3)–(3.1) *and* (3.12)–(3.16). *Then the generalized solution* $u \in \mathcal{C}_w$ *to system* (3.11) *of Lemma 3.2.7 satisfies*

$$\lim_{t\to 0^+} u(t) = u^0 \quad \text{in } V, \qquad \lim_{t\to 0^+} \dot u(t) = u^1 \quad \text{in } H. \tag{3.65}$$

In particular, the functions $u\colon [0,T]\to V$ *and* $\dot u\colon [0,T]\to H$ *are continuous at* $t=0$.

Proof. The proof is the same of Lemma (1.3.4) □

By combining the previous results together we obtain the following existence result for the system (3.11).

Theorem 3.2.13. *Assume* (3.3)–(3.1) *and* (3.12)–(3.16). *Then there exists a generalized solution* $u \in \mathcal{C}_w$ *to system* (3.11). *Moreover, we have* $u \in H^2(0,T;(V_0^D)')$ *and it satisfies the energy-dissipation inequality* (3.49) *and*

$$\lim_{t\to 0^+} u(t) = u^0 \quad \text{in } V, \qquad \lim_{t\to 0^+} \dot u(t) = u^1 \quad \text{in } H.$$

Proof. It is enough to combine Lemma 3.2.7, Corollary 3.2.8, Lemma 3.2.9, Proposition 3.2.10, and Lemma 3.2.12. □

3.2.2 Uniform energy estimates

In this subsection we show that, under the stronger assumption (3.2) on z, the generalized solution to (3.11) of Theorem 3.2.13 satisfies some uniform estimates which depends on $\mathbb{G}$ only via $\|\mathbb{G}\|_{L^1(0,T;B)}$.

Lemma 3.2.14. *Assume (3.2)–(3.1) and (3.12)–(3.15). Let u be the generalized solution to system (3.11) of Theorem 3.2.13. Then the following estimate holds*

$$\|\dot{u}(t)\| + \|eu(t)\| \leq M \quad \text{for every } t \in [0,T], \tag{3.66}$$

where $M = M(z, N, f, u^0, u^1, \mathbb{A}, \|\mathbb{G}\|_{L^1(0,T;B)})$ is a positive constant.

Proof. We define

$$K := \sup_{t\in[0,T]} \|\dot{u}(t)\| = \|\dot{u}\|_{L^\infty(0,T;H)}, \quad E := \sup_{t\in[0,T]} \|eu(t)\| = \|eu\|_{L^\infty(0,T;H)}.$$

Notice that K and E are well-posed since $u \in C^0_w([0,T];V)$ and $\dot{u} \in C^0_w([0,T];H)$. Let us estimate the total work $\mathscr{W}_{tot}(t)$ in (3.49) by means of K and E. Since

$$\|u(t)\|_V \leq \|u^0\| + TK + E \quad \text{for every } t \in [0,T],$$

we have

$$\begin{aligned}
\left|\int_0^t (f(\tau), \dot{u}(\tau))\mathrm{d}\tau\right| &\leq \sqrt{T}\|f\|_{L^2(0,T;H)}K,\\
\left|\int_0^t (\dot{N}(\tau), u(\tau))_{H_N}\mathrm{d}\tau\right| &\leq C_{tr}\|\dot{N}\|_{L^2(0,T;H_N)}\left(\|u^0\| + TK + E\right),\\
|(N(t), u(t))_{H_N}| &\leq C_{tr}\|N\|_{C^0([0,T];H_N)}\left(\|u^0\| + TK + E\right),\\
|(N(0), u^0)_{H_N}| &\leq C_{tr}\|N\|_{C^0([0,T];H_N)}\left(\|u^0\| + TK + E\right),\\
\left|\int_0^t (f(\tau), \dot{z}(\tau))\mathrm{d}\tau\right| &\leq \sqrt{T}\|f\|_{L^2(0,T;H)}\|\dot{z}\|_{C^0([0,T];H)},\\
\left|\int_0^t (N(\tau), \dot{z}(\tau))_{H_N}\mathrm{d}\tau\right| &\leq C_{tr}\|N\|_{C^0([0,T];H_N)}\|\dot{z}\|_{L^1(0,T;V_0)},\\
\left|\int_0^t (\mathbb{A}eu(\tau), e\dot{z}(\tau))\mathrm{d}\tau\right| &\leq C_{\mathbb{A}}\|e\dot{z}\|_{L^1(0,T;H)}E,\\
\left|\int_0^t (\dot{u}(\tau), \ddot{z}(\tau))\mathrm{d}\tau\right| &\leq \|\ddot{z}\|_{L^1(0,T;H)}K,\\
|(\dot{u}(t), \dot{z}(t))| &\leq \|\dot{z}\|_{C^0([0,T];H)}K,\\
|(u^1, \dot{z}(0))| &\leq \|\dot{z}\|_{C^0([0,T];H)}K.
\end{aligned}$$

It remains to study the last two terms, which are

$$\begin{aligned}
&\int_0^t (\mathbb{G}(\tau)(eu(\tau) - eu^0), e\dot{z}(\tau))\mathrm{d}\tau + \int_0^t\int_0^\tau (\dot{\mathbb{G}}(\tau - r)(eu(r) - eu(\tau)), e\dot{z}(\tau))\mathrm{d}r\mathrm{d}\tau\\
&= \int_0^t (\mathbb{G}(0)(eu(\tau) - eu^0), e\dot{z}(\tau))\mathrm{d}\tau + \int_0^t\int_0^\tau (\dot{\mathbb{G}}(\tau - r)(eu(r) - eu^0), e\dot{z}(\tau))\mathrm{d}r\mathrm{d}\tau.
\end{aligned}$$

Since $z \in W^{2,1}(0,T;V_0)$, arguing as in Proposition 3.2.3 we can deduce that the function

$$p(t) := \int_0^t (\mathbb{G}(t-\tau)(eu(\tau) - eu^0), e\dot{z}(t))\mathrm{d}\tau$$

is absolutely continuous on $[0,T]$. In particular

$$p(t) - p(0) = \int_0^t \dot{p}(\tau)\mathrm{d}\tau,$$

which gives

$$\int_0^t (\mathbb{G}(\tau)(eu(\tau)-eu^0), e\dot{z}(\tau))\mathrm{d}\tau + \int_0^t\int_0^\tau (\dot{\mathbb{G}}(\tau-r)(eu(r)-eu(\tau)), e\dot{z}(\tau))\mathrm{d}r\mathrm{d}\tau$$
$$= \int_0^t (\mathbb{G}(t-\tau)(eu(\tau)-eu^0), e\dot{z}(t))\mathrm{d}\tau - \int_0^t\int_0^\tau (\mathbb{G}(\tau-r)(eu(r)-eu^0), e\ddot{z}(\tau))\mathrm{d}r\mathrm{d}\tau. \quad (3.67)$$

Hence, we deduce

$$\Big|\int_0^t (\mathbb{G}(\tau)(eu(\tau)-eu^0), e\dot{z}(\tau))\mathrm{d}\tau + \int_0^t\int_0^\tau (\dot{\mathbb{G}}(\tau-r)(eu(r)-eu(\tau)), e\dot{z}(\tau))\mathrm{d}r\mathrm{d}\tau\Big|$$
$$\le 2(\|e\dot{z}\|_{C^0([0,T];H)} + \|e\ddot{z}\|_{L^1(0,T;H)})\|\mathbb{G}\|_{L^1(0,T;B)}E.$$

Therefore, since

$$\mathscr{E}(0) \le \frac{1}{2}\|u^1\|^2 + \frac{C_{\mathbb{A}}}{2}\|eu^0\|^2,$$

by (3.49) we deduce the following estimate

$$\|\dot{u}(t)\|^2 + c_{\mathbb{A}}\|eu(t)\|^2 \le C_0 + C_1K + C_2E \quad \text{for every } t\in[0,T],$$

where

$$C_0 = C_0(z,N,f,u^0,u^1,\mathbb{A}), \quad C_1 = C_1(f,z,N), \quad C_2 = C_2(z,N,\mathbb{A},\|\mathbb{G}\|_{L^1(0,T;B)}).$$

In particular, being the right-hand side independent of $t\in[0,T]$, we conclude

$$K^2 + c_{\mathbb{A}}E^2 \le 2C_0 + 2C_1K + 2C_2E \quad \text{for every } t\in[0,T].$$

This implies the existence of a constant $M = M(C_0, C_1, C_2)$ for which (3.66) is satisfied. □

Remark 3.2.15. By the previous estimate, we can easily derive a uniform bound also for $\dot{u}$ in $H^1(0,T;(V_0^D)')$, which unfortunately depends on $\mathbb{G}$ via $\|\mathbb{G}(0)\|_B$. Indeed, let us assume that z, N, f, u^0, u^1, $\mathbb{A}$, and $\mathbb{G}$ satisfy (3.2)–(3.1) and (3.12)–(3.15) and let u be the generalized solution of Theorem 3.2.13. Thanks to (3.49) and (3.66) there exists a constant $\overline{M} = \overline{M}(z,N,f,u^0,u^1,\mathbb{A},\|\mathbb{G}\|_{L^1(0;T;B)})$ such that for every $t\in[0,T]$

$$\|eu(t)\|^2 + (\mathbb{G}(t)(eu(t)-eu^0), eu(t)-eu^0) + \int_0^t (-\dot{\mathbb{G}}(t-\tau)(eu(t)-eu(\tau)), eu(t)-eu(\tau))\mathrm{d}\tau \le \overline{M}.$$

By equation (3.17) it is easy to see that $\dot{u}\in H^1(0,T;(V_0^D)')$ and that $\ddot{u}$ satisfies for a.e. $t\in(0,T)$ and for every $v\in V_0^D$

$$|\langle \ddot{u}(t), v\rangle_{(V_0^D)'}|$$
$$\le C_{\mathbb{A}}\|eu(t)\|\|ev\| + \sqrt{(\mathbb{G}(t)(eu(t)-eu^0), eu(t)-eu^0)}\sqrt{(\mathbb{G}(t)ev, ev)}$$
$$+ \sqrt{\int_0^t (-\dot{\mathbb{G}}(t-\tau)(eu(t)-eu(\tau)), eu(t)-eu(\tau))\mathrm{d}\tau}\sqrt{\int_0^t (-\dot{\mathbb{G}}(t-\tau)ev, ev)\mathrm{d}\tau}$$
$$+ \|f(t)\|\|v\| + \|N(t)\|_{H_N}\|v\|_{H_N}.$$

Hence, we derive

$$|\langle \ddot{u}(t), v\rangle_{(V_0^D)'}|^2 \le 5C_{\mathbb{A}}^2\overline{M}\|ev\|^2 + 5\overline{M}(\mathbb{G}(t)ev, ev) + 5\overline{M}\int_0^t (-\dot{\mathbb{G}}(t-\tau)ev, ev)\mathrm{d}\tau$$
$$+ 5\|f(t)\|^2\|v\|^2 + 5C_{tr}^2\|N(t)\|_{H_N}^2\|v\|_{V_0}^2$$

$$= 5C_{\mathbb{A}}^2\overline{M}\|ev\|^2 + 5\overline{M}(\mathbb{G}(0)ev, ev) + 5\|f(t)\|^2\|v\|^2 + 5C_{tr}^2\|N(t)\|_{H_N}^2\|v\|_{V_0}^2,$$

which gives

$$\|\ddot{u}\|^2_{L^2(0,T;(V_0^D)')} \leq 5C_{\mathbb{A}}^2\overline{M}T + 5\overline{M}T\|\mathbb{G}(0)\|_B + 5\|f\|^2_{L^2(0,T;H)} + 5C_{tr}^2\|N\|^2_{L^2(0,T;H_N)}.$$

Therefore the bounds on $\ddot{u}$ depends on $\|\mathbb{G}(0)\|_B$ even when $z \in W^{2,1}(0,T;V_0)$.

As explained in the previous remark, we cannot deduce a uniform bound for $\dot{u}$ in the space $H^1(0,T;(V_0^D)')$ depending on $\mathbb{G}$ only via its L^1-norm. On the other hand, the bound on $\dot{u}$ in $H^1(0,T;(V_0^D)')$ is useful if we want to prove the existence of a generalized solution u^* to the fractional Kelvin-Voigt's system (3.9), especially to show that $\dot{u}^* \in C_w^0([0,T];H)$. To overcome this problem, we introduce another function that is related to $\dot{u}$ and for which is possible to derive a uniform bound. Let us consider the auxiliary function $\alpha\colon [0,T] \to (V_0^D)'$ defined for every $v \in V_0^D$ and $t \in [0,T]$ as

$$\langle \alpha(t), v\rangle_{(V_0^D)'} := (\dot{u}(t), v) + \int_0^t (\mathbb{G}(t-\tau)(eu(\tau) - eu^0), ev)\mathrm{d}\tau.$$

Notice that $\alpha \in C_w^0([0,T];(V_0^D)')$. Indeed, given $t^* \in [0,T]$ and

$$\{t_k\}_k \subset [0,T] \quad \text{such that} \quad t_k \xrightarrow[k\to\infty]{} t^*,$$

we have for every $v \in V_0^D$ the following convergence

$$\begin{aligned}\langle \alpha(t_k), v\rangle_{(V_0^D)'} &= (\dot{u}(t_k), v) + \int_0^{t_k} (\mathbb{G}(t_k-\tau)(eu(\tau) - eu^0), ev)\mathrm{d}\tau \\ &\xrightarrow[k\to\infty]{} (\dot{u}(t^*), v) + \int_0^{t^*} (\mathbb{G}(t^*-\tau)(eu(\tau) - eu^0), ev)\mathrm{d}\tau = \langle \alpha(t^*), v\rangle_{(V_0^D)'},\end{aligned}$$

since

$$\dot{u}(t_k) \xrightarrow[k\to\infty]{H} \dot{u}(t^*),$$

$$\int_0^{t_k} (\mathbb{G}(t_k-\tau)(eu(\tau) - eu^0), ev)\mathrm{d}\tau \xrightarrow[k\to\infty]{} \int_0^{t^*} (\mathbb{G}(t^*-\tau)(eu(\tau) - eu^0), ev)\mathrm{d}\tau.$$

The second convergence is true because

$$\begin{aligned}&\int_0^{t_k} (\mathbb{G}(t_k-\tau)(eu(\tau) - eu^0), ev)\mathrm{d}\tau \\ &\quad = \int_0^{t^*} (eu(\tau) - eu^0, \mathbb{G}(t_k-\tau)ev)\mathrm{d}\tau - \int_{t_k}^{t^*} (eu(\tau) - eu^0, \mathbb{G}(t_k-\tau)ev)\mathrm{d}\tau.\end{aligned}$$

Clearly

$$\mathbb{G}(t_k - \cdot)ev \xrightarrow[k\to\infty]{L^1(0,t^*;H)} \mathbb{G}(t^* - \cdot)ev$$

while $eu \in L^\infty(0,t^*;H)$. Therefore

$$\begin{aligned}\int_0^{t^*} (eu(\tau) - eu^0, \mathbb{G}(t_k-\tau)ev)\mathrm{d}\tau &\xrightarrow[k\to\infty]{} \int_0^{t^*} (eu(\tau) - eu^0, \mathbb{G}(t^*-\tau)ev)\mathrm{d}\tau \\ &= \int_0^{t^*} (\mathbb{G}(t^*-\tau)(eu(\tau) - eu^0), ev)\mathrm{d}\tau.\end{aligned}$$

Moreover

$$\Big|\int_{t_k}^{t^*}(eu(\tau)-eu^0,\mathbb{G}(t_k-\tau)ev)\mathrm{d}\tau\Big|\le 2M\|ev\|\Big|\int_0^{t_k-t^*}\|\mathbb{G}(\tau)\|_B\mathrm{d}\tau\Big|\xrightarrow[k\to\infty]{}0.$$

For this function α is possible to find a uniform bound in $H^1(0,T;(V_0^D)')$ which depends on $\|\mathbb{G}\|_{L^1(0,T;B)}$.

Corollary 3.2.16. *Assume* (3.2)–(3.1) *and* (3.12)–(3.15). *Then* $\alpha\in H^1(0,T;(V_0^D)')$ *and there exists a constant* $\tilde{M}=\tilde{M}(z,N,f,u^0,u^1,\mathbb{A},\|\mathbb{G}\|_{L^1(0,T;B)})$ *such that*

$$\|\alpha\|_{H^1(0,T;(V_0^D)')}\le\tilde{M}. \tag{3.68}$$

Proof. First, by Lemma 3.2.14 we have

$$\|\alpha(t)\|_{(V_0^D)'}\le M(1+2\|\mathbb{G}\|_{L^1(0,T;B)})\quad\text{for every } t\in[0,T].$$

Moreover, by the definition of generalized solution, we deduce that for every $\psi\in C_c^1(0,T)$ and $v\in V_0^D$ it holds

$$\begin{aligned}&-\int_0^T\langle\alpha(t),v\rangle_{(V_0^D)'}\dot{\psi}(t)\mathrm{d}t\\&\qquad=-\int_0^T(\mathbb{A}eu(t),ev)\psi(t)\mathrm{d}t+\int_0^T(f(t),v)\psi(t)\mathrm{d}t+\int_0^T(N(t),v)_{H_N}\psi(t)\mathrm{d}t.\end{aligned}$$

This gives that there exists $\dot{\alpha}\in L^2(0,T;(V_0^D)')$ such that for every $v\in V_0^D$ and for a.e. $t\in(0,T)$ we have

$$\langle\dot{\alpha}(t),v\rangle_{(V_0^D)'}=-(\mathbb{A}eu(t),ev)+(f(t),v)+(N(t),v)_{H_N}.$$

In particular, $\alpha\in C^0([0,T];(V_0^D)')$ and

$$\|\dot{\alpha}\|^2_{L^2(0,T;(V_0^D)')}\le 3C_{\mathbb{A}}^2M^2T+3\|f\|^2_{L^2(0,T;H)}+3C_{tr}^2\|N\|^2_{L^2(0,T;H_N)},$$

which gives (3.68). □

3.3 The fractional Kelvin-Voigt's model

In this section we prove the existence of a generalized solution to (3.9) for a tensor $\mathbb{F}$ which is not necessary bounded at $t=0$, as it happens in (5). Here, we assume that our data $z,N,f,u^0,u^1,\mathbb{A}$, and $\mathbb{F}$ satisfy the conditions (3.2)–(3.8). To prove the existence of a generalized solution to (3.9) under these assumptions, we first regularize $\mathbb{F}$ by a parameter $\varepsilon>0$ and we consider system (3.11) associated to this regularization. Then, we take the solution u^ε given by Theorem 3.2.13 and thanks to Lemma 3.2.14 and Corollary 3.2.16 we obtain a generalized solution to (3.9).

Let us regularize $\mathbb{F}$ by defining

$$\mathbb{G}^\varepsilon(t):=\mathbb{F}(t+\varepsilon)\quad\text{ for } t\in[0,T] \text{ and } \varepsilon\in(0,\delta_0).$$

Clearly $\mathbb{G}^\varepsilon$ satisfies (3.12)–(3.15). Moreover, we have $\mathbb{G}^\varepsilon\to\mathbb{F}$ in $L^1(0,T;B)$ since $\mathbb{F}\in L^1(0,T_0;B)$. For every fixed $\varepsilon\in(0,\delta_0)$ we can consider the generalized solution u^ε to system (3.11) with $\mathbb{G}$ replaced by $\mathbb{G}^\varepsilon$ of Theorem 3.2.13. By Lemma 3.2.14 and Corollary 3.2.16 we deduce the following compactness result:

Lemma 3.3.1. *Assume (3.2)–(3.8). For every $\varepsilon \in (0, \delta_0)$ let u^ε be the generalized solution associated to system (3.11) with $\mathbb{G}$ replaced by $\mathbb{G}^\varepsilon$ given by Theorem 3.2.13. Then there exists a function $u^* \in \mathcal{C}_w$ and a subsequence of ε, not relabeled, such that*

$$u^\varepsilon \xrightharpoonup[\varepsilon\to 0^+]{L^2(0,T;V)} u^*, \quad \dot{u}^\varepsilon \xrightharpoonup[\varepsilon\to 0^+]{L^2(0,T;H)} \dot{u}^*, \tag{3.69}$$

and for every $t \in [0, T]$

$$u^\varepsilon(t) \xrightharpoonup[\varepsilon\to 0^+]{V} u^*(t), \quad \dot{u}^\varepsilon(t) \xrightharpoonup[\varepsilon\to 0^+]{H} \dot{u}^*(t). \tag{3.70}$$

Moreover, $u^*(0) = u^0$ in V_0, $\dot{u}^*(0) = u^1$ in H, and $u^*(t) - z(t) \in V_t^D$ for every $t \in [0, T]$.

Proof. Thanks to Lemma 3.2.14 we deduce

$$\|\dot{u}^\varepsilon(t)\| + \|eu^\varepsilon(t)\| \le M \quad \text{for every } t \in [0, T] \text{ and } \varepsilon \in (0, \delta_0),$$

with a constant M independent of ε since $\|\mathbb{G}^\varepsilon\|_{L^1(0,T;B)} \le \|\mathbb{F}\|_{L^1(0,T_0;B)}$. Hence, by Banach-Alaoglu's theorem and Lemma 3.2.6 there exists

$$u^* \in C_w^0([0, T]; V) \cap W^{1,\infty}(0, T; H)$$

and a not relabeled subsequence of ε such that

$$u^\varepsilon \xrightharpoonup[\varepsilon\to 0^+]{L^2(0,T;V)} u^*, \quad \dot{u}^\varepsilon \xrightharpoonup[\varepsilon\to 0^+]{L^2(0,T;H)} \dot{u}^*, \quad u^\varepsilon(t) \xrightharpoonup[\varepsilon\to 0^+]{V} u^*(t) \quad \text{for every } t \in [0, T]. \tag{3.71}$$

In particular, we deduce that $u^*(0) = u^0$ in V_0, $u^*(t) \in V_t$ and $u^*(t) - z(t) \in V_t^D$ for every $t \in [0, T]$.

It remains to show that $\dot{u}^* \in C_w^0([0, T]; H)$, $\dot{u}^*(0) = u^1$ in H, and that for every $t \in [0, T]$

$$\dot{u}^\varepsilon(t) \xrightharpoonup[\varepsilon\to 0^+]{H} \dot{u}^*(t).$$

To this aim we consider the auxiliary function defined at the end of the previous section. More precisely, for every $\varepsilon \in (0, \delta_0)$ let $\alpha^\varepsilon \colon [0, T] \to (V_0^D)'$ be defined for every $v \in V_0^D$ and $t \in [0, T]$ as

$$\langle \alpha^\varepsilon(t), v \rangle_{(V_0^D)'} := (\dot{u}^\varepsilon(t), v) + \int_0^t (\mathbb{G}^\varepsilon(t - \tau)(eu^\varepsilon(\tau) - eu^0), ev) \mathrm{d}\tau.$$

In view of Corollary 3.2.16, we have

$$\|\alpha^\varepsilon\|_{H^1(0,T;(V_0^D)')} \le \tilde{M} \quad \text{for every } \varepsilon \in (0, \delta_0),$$

with $\tilde{M}$ independent of $\varepsilon > 0$ being $\|\mathbb{G}^\varepsilon\|_{L^1(0,T;B)} \le \|\mathbb{F}\|_{L^1(0,T_0;B)}$. Hence, up to extract a further subsequence, there exists $\alpha^* \in H^1(0, T; (V_0^D)')$ such that

$$\alpha^\varepsilon \xrightharpoonup[\varepsilon\to 0^+]{H^1(0,T;(V_0^D)')} \alpha^*, \qquad \alpha^\varepsilon(t) \xrightharpoonup[\varepsilon\to 0^+]{(V_0^D)'} \alpha^*(t) \quad \text{for every } t \in [0, T]. \tag{3.72}$$

In particular, since $\alpha^\varepsilon(0) = u^1$ in $(V_0^D)'$ we conclude that $\alpha^*(0) = u^1$ in $(V_0^D)'$. For every $v \in V_0^D$ and for a.e. $t \in (0, T)$ we claim that

$$\langle \alpha^*(t), v \rangle_{(V_0^D)'} = (\dot{u}^*(t), v) + \int_0^t (\mathbb{F}(t - \tau)(eu^*(\tau) - eu^0), ev) \mathrm{d}\tau.$$

Indeed, for every $\varphi \in C_c^\infty(0,T;V_0^D)$ we have

$$\int_0^T \langle \alpha^\varepsilon(t), \varphi(t)\rangle_{(V_0^D)'}\mathrm{d}t = \int_0^T (\dot{u}^\varepsilon(t), \varphi(t))\mathrm{d}t + \int_0^T \int_0^t (\mathbb{G}^\varepsilon(t-\tau)(eu^\varepsilon(\tau) - eu^0), e\varphi(t))\mathrm{d}\tau\mathrm{d}t$$
$$\xrightarrow[\varepsilon\to 0^+]{} \int_0^T (\dot{u}^*(t), \varphi(t))\mathrm{d}t + \int_0^T \int_0^t (\mathbb{F}(t-\tau)(eu^*(\tau) - eu^0), e\varphi(t))\mathrm{d}\tau\mathrm{d}t.$$

Notice that this convergence is true thanks to (3.71) and

$$\mathbb{G}^\varepsilon(t-\cdot) \xrightarrow[\varepsilon\to 0^+]{L^1(0,t;B)} \mathbb{F}(t-\cdot),$$

which gives

$$\int_0^T (\dot{u}^\varepsilon(t), \varphi(t))\mathrm{d}t \xrightarrow[\varepsilon\to 0^+]{} \int_0^T (\dot{u}^*(t), \varphi(t))\mathrm{d}t,$$
$$\int_0^t (\mathbb{G}^\varepsilon(t-\tau)(eu^\varepsilon(\tau) - eu^0), e\varphi(t))\mathrm{d}\tau \xrightarrow[\varepsilon\to 0^+]{} \int_0^t (\mathbb{F}(t-\tau)(eu^*(\tau) - eu^0), e\varphi(t))\mathrm{d}\tau.$$

Hence by the dominated convergence theorem we have

$$\int_0^T \int_0^t (\mathbb{G}^\varepsilon(t-\tau)(eu^\varepsilon(\tau) - eu^0), e\varphi(t))\mathrm{d}\tau\mathrm{d}t$$
$$\xrightarrow[\varepsilon\to 0^+]{} \int_0^T \int_0^t (\mathbb{F}(t-\tau)(eu^*(\tau) - eu^0), e\varphi(t))\mathrm{d}\tau\mathrm{d}t.$$

Therefore, for a.e. $t \in (0,T)$ we deduce for every $v \in V_0^D$ that

$$\langle \dot{u}^*(t), v\rangle_{(V_0^D)'} = (\dot{u}^*(t), v) = \langle \alpha^*(t), v\rangle_{(V_0^D)'} - \int_0^t (\mathbb{F}(t-\tau)(eu^*(\tau) - eu^0), ev)\mathrm{d}\tau.$$

Notice the function on the right-hand side is well defined in $(V_0^D)'$ for every $t \in [0,T]$. Therefore, we can extend $\dot{u}^*$ to a function defined in the whole interval $[0,T]$ with values in $(V_0^D)'$. In particular, we deduce $\dot{u}^* \in C_w^0([0,T];(V_0^D)')$, arguing in a similar way as we did in the previous section for α, and thanks to the fact that $\dot{u}^*(0) = \alpha^*(0) = u^1$ in $(V_0^D)'$. Therefore, since $\dot{u}^* \in C_w^0([0,T];(V_0^D)') \cap L^\infty(0,T;H)$ we derive that $\dot{u}^* \in C_w^0([0,T];H)$ (thanks to Lemma 3.2.6), and that $\dot{u}^*(0) = u^1$ in H. Finally, we have

$$\dot{u}^\varepsilon(t) \xrightarrow[\varepsilon\to 0^+]{(V_0^D)'} \dot{u}^*(t) \quad \text{for every } t \in [0,T] \tag{3.73}$$

by definition of $\dot{u}^*$ and by (3.71) and (3.72). The convergence (3.73) combined with

$$\|\dot{u}^\varepsilon(t)\| \le M \quad \text{for every } t \in [0,T],$$

give us the last convergence required. □

We can now prove the main existence result of Theorem 3.1.4 for the fractional Kelvin-Voigt's system involving Caputo's derivative.

Proof of Theorem 3.1.4. It is enough to show that the function u^* given by Lemma 3.3.1 is a generalized solution to (3.9). To this aim, it remains to prove that u^* satisfies (3.10). For every $\varphi \in \mathcal{C}_c^1$ we know that the function $u^\varepsilon \in \mathcal{C}_w$ satisfy for every $\varepsilon \in (0,\delta_0)$ the following equality

$$-\int_0^T (\dot{u}^\varepsilon(t), \dot{\varphi}(t))\mathrm{d}t + \int_0^T (\mathbb{A}eu^\varepsilon(t), e\varphi(t))\mathrm{d}t - \int_0^T \int_0^t (\mathbb{G}^\varepsilon(t-\tau)(eu^\varepsilon(\tau) - eu^0), e\dot{\varphi}(t))\mathrm{d}\tau\mathrm{d}t$$

$$= \int_0^T (f(t), \varphi(t))\mathrm{d}t + \int_0^T (N(t), \varphi(t))_{H_N}\mathrm{d}t.$$

Let us pass to the limit as $\varepsilon \to 0^+$. Clearly, by (3.69) we have

$$\int_0^T (\dot{u}^\varepsilon(t), \dot{\varphi}(t))\mathrm{d}t \xrightarrow[\varepsilon\to 0^+]{} \int_0^T (\dot{u}^*(t), \dot{\varphi}(t))\mathrm{d}t,$$
$$\int_0^T (\mathbb{A}eu^\varepsilon(t), e\varphi(t))\mathrm{d}t \xrightarrow[\varepsilon\to 0^+]{} \int_0^T (\mathbb{A}eu^*(t), e\varphi(t))\mathrm{d}t.$$

It remains to study the behaviour as $\varepsilon \to 0^+$ of

$$\int_0^T \int_0^t (\mathbb{G}^\varepsilon(t-\tau)(eu^\varepsilon(\tau) - eu^0), e\dot{\varphi}(t))\mathrm{d}\tau\mathrm{d}t.$$

We define for every $\varepsilon \in (0, \delta_0)$ the function

$$v^\varepsilon(t) := \int_0^t (\mathbb{G}^\varepsilon(t-\tau) - \mathbb{F}(t-\tau))(eu^\varepsilon(\tau) - eu^0)\mathrm{d}\tau \quad \text{for } t \in [0,T].$$

By (3.66) for every $t \in [0,T]$ it holds

$$\|v^\varepsilon(t)\| \le \|\mathbb{G}^\varepsilon - \mathbb{F}\|_{L^1(0,T;B)}\|eu^\varepsilon - eu^0\|_{L^\infty(0,T;H)} \le 2M\|\mathbb{G}^\varepsilon - \mathbb{F}\|_{L^1(0,T;B)}, \tag{3.74}$$

with M independent of ε being $\|\mathbb{G}^\varepsilon\|_{L^1(0,T;B)} \le \|\mathbb{F}\|_{L^1(0,T_0;B)}$. Notice that

$$\int_0^T \int_0^t (\mathbb{G}^\varepsilon(t-\tau)(eu^\varepsilon(\tau) - eu^0), e\dot{\varphi}(t))\mathrm{d}\tau\mathrm{d}t$$
$$= \int_0^T (v^\varepsilon(t), e\dot{\varphi}(t))\mathrm{d}t + \int_0^T \int_0^t (\mathbb{F}(t-\tau)(eu^\varepsilon(\tau) - eu^0), e\dot{\varphi}(t))\mathrm{d}\tau\mathrm{d}t,$$

and thanks to (3.74) and to the fact that $\mathbb{G}^\varepsilon \to \mathbb{F}$ in $L^1(0,T;B)$ as $\varepsilon \to 0^+$, we get

$$\left|\int_0^T (v^\varepsilon(t), e\dot{\varphi}(t))\mathrm{d}t\right| \le \int_0^T \|v^\varepsilon(t)\|\|e\dot{\varphi}(t)\|\mathrm{d}t \le 2M\|\mathbb{G}^\varepsilon - \mathbb{F}\|_{L^1(0,T;B)}\|e\dot{\varphi}\|_{L^1(0,T;H)} \xrightarrow[\varepsilon\to 0^+]{} 0.$$

On the other hand, since $\tau \mapsto \int_\tau^T \mathbb{F}(t-\tau)e\dot{\varphi}(t)\mathrm{d}t$ belongs to $L^\infty(0,T;H)$, we can write

$$\int_0^T \int_0^t (\mathbb{F}(t-\tau)(eu^\varepsilon(\tau) - eu^0), e\dot{\varphi}(t))\mathrm{d}\tau\mathrm{d}t$$
$$= \int_0^T (eu^\varepsilon(\tau) - eu^0, \int_\tau^T \mathbb{F}(t-\tau)e\dot{\varphi}(t)\mathrm{d}t)\mathrm{d}\tau \xrightarrow[\varepsilon\to 0^+]{} \int_0^T (eu^*(\tau) - eu^0, \int_\tau^T \mathbb{F}(t-\tau)e\dot{\varphi}(t)\mathrm{d}t)\mathrm{d}\tau$$
$$= \int_0^T \int_0^t (\mathbb{F}(t-\tau)(eu^*(\tau) - eu^0), e\dot{\varphi}(t))\mathrm{d}\tau\mathrm{d}t.$$

As a consequence, u^* is a generalized solution to system (3.9). □

We conclude this section by showing that for the fractional Kelvin-Voigt's model, the generalized solution $u^* \in \mathcal{C}_w$ to (3.9) found before satisfies an energy-dissipation inequality. As before, for $t \in (0,T]$ we define the functions $\mathscr{E}^*(t)$ and $\mathscr{D}^*(t)$ as

$$\mathscr{E}^*(t) := \frac{1}{2}\|\dot{u}^*(t)\|^2 + \frac{1}{2}(\mathbb{A}eu^*(t), eu^*(t))\mathrm{d}t + \frac{1}{2}(\mathbb{F}(t)(eu^*(t) - eu^0), eu^*(t) - eu^0)$$
$$- \frac{1}{2}\int_0^t (\dot{\mathbb{F}}(t-\tau)(eu^*(t) - eu^*(\tau)), eu^*(t) - eu^*(\tau))\mathrm{d}\tau,$$

$$\mathscr{D}^*(t) := -\frac{1}{2}\int_0^t (\dot{\mathbb{F}}(\tau)(eu^*(\tau) - eu^0), eu^*(\tau) - eu^0)\mathrm{d}\tau + \frac{1}{2}\int_0^t \int_0^\tau (\ddot{\mathbb{F}}(\tau - r)(eu^*(\tau) - eu^*(r)), eu^*(\tau) - eu^*(r))\mathrm{d}r\mathrm{d}\tau.$$

Notice that the integrals in $\mathscr{E}^*$ and $\mathscr{D}^*$ are well-posed, eventually with values ∞. Furthermore, we define the total work $\mathscr{W}^*_{tot}(t)$ for $t \in [0, T]$ as

$$\begin{aligned}\mathscr{W}^*_{tot}(t) :&= \int_0^t [(f(\tau), \dot{u}^*(\tau) - \dot{z}(\tau)) + (\mathbb{A}eu^*(t), e\dot{z}(t)) + (\mathbb{F}(t-\tau)(eu^*(\tau) - eu^0), e\dot{z}(t))]\mathrm{d}\tau \\ &- \int_0^t \int_0^\tau (\mathbb{F}(\tau - r)(eu^*(r) - eu^0), e\ddot{z}(\tau))\mathrm{d}r\mathrm{d}\tau \\ &- \int_0^t (\dot{N}(\tau), u^*(\tau) - z(\tau))_{H_N}\mathrm{d}\tau + (N(t), u^*(t) - z(t))_{H_N} - (N(0), u^0 - z(0))_{H_N} \\ &- \int_0^t (\dot{u}^*(\tau), \ddot{z}(\tau))\mathrm{d}\tau + (\dot{u}^*(t), \dot{z}(t)) - (u^1, \dot{z}(0)).\end{aligned} \tag{3.75}$$

We point out the total work $\mathscr{W}^*_{tot}$ is continuous in $[0, T]$ and that the definition given in (3.75) is coherent with the one of (3.50) thanks to identity (3.67).

Theorem 3.3.2. *Assume (3.2)–(3.8). Then the generalized solution $u^* \in \mathcal{C}_w$ to system (3.9) of Theorem 3.1.4 satisfies for every $t \in (0, T]$ the following energy-dissipation inequality*

$$\mathscr{E}^*(t) + \mathscr{D}^*(t) \leq \frac{1}{2}\|u^1\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0) + \mathscr{W}^*_{tot}(t). \tag{3.76}$$

In particular, $\mathscr{E}^*(t)$ and $\mathscr{D}^*(t)$ are finite for every $t \in (0, T]$.

Proof. Let us fix $t \in (0, T]$. For every $\varepsilon \in (0, \delta_0)$ let $u^\varepsilon \in \mathcal{C}_w$ be the generalized solution to system (3.11) with $\mathbb{G}$ replaced by $\mathbb{G}^\varepsilon$ given by Lemma 3.3.1. Thanks to Proposition 3.2.10 we know that the function u^ε satisfies the energy-dissipation inequality (3.49) and we can rewrite the total work (3.50) as in (3.75) since $z \in W^{2,1}(0, T; V_0)$ (as suggested by formula (3.67)). The convergences (3.70) of Lemma 3.3.1, and the lower semicontinuous property of the maps $v \mapsto \|v\|^2$, $w \mapsto (\mathbb{A}w, w)$ (by (3.1)), and $w \mapsto (\mathbb{F}(t)w, w)$ (by (3.6)), imply

$$\|\dot{u}^*(t)\|^2 \leq \liminf_{\varepsilon\to 0^+} \|\dot{u}^\varepsilon(t)\|^2, \tag{3.77}$$

$$(\mathbb{A}eu^*(t), eu^*(t)) \leq \liminf_{\varepsilon\to 0^+}(\mathbb{A}eu^\varepsilon(t), eu^\varepsilon(t)), \tag{3.78}$$

$$(\mathbb{F}(t)(eu^*(t) - eu^0), eu^*(t) - eu^0) \leq \liminf_{\varepsilon\to 0^+}(\mathbb{F}(t)(eu^\varepsilon(t) - eu^0), eu^\varepsilon(t) - eu^0). \tag{3.79}$$

Moreover, by (3.5) we have

$$\begin{aligned}|((\mathbb{F}(t) - \mathbb{G}^\varepsilon(t))(eu^\varepsilon(t) - eu^0), eu^\varepsilon(t) - eu^0)| &\leq \|\mathbb{F}(t) - \mathbb{G}^\varepsilon(t)\|_B \|eu^\varepsilon(t) - eu^0\|^2 \\ &\leq 4M^2\|\mathbb{F}(t) - \mathbb{F}(t+\varepsilon)\|_B \xrightarrow[\varepsilon\to 0^+]{} 0,\end{aligned}$$

being M independent of ε. Hence (3.79) reads as

$$(\mathbb{F}(t)(eu^*(t) - eu^0), eu^*(t) - eu^0) \leq \liminf_{\varepsilon\to 0^+}(\mathbb{G}^\varepsilon(t)(eu^\varepsilon(t) - eu^0), eu^\varepsilon(t) - eu^0). \tag{3.80}$$

Similarly, by (3.5), (3.7), and (3.70), for every $\tau \in (0, t)$ we have

$$(-\dot{\mathbb{F}}(t-\tau)(eu^*(t) - eu^*(\tau)), eu^*(t) - eu^*(\tau))$$

$$\leq \liminf_{\varepsilon\to 0^+}(-\dot{\mathbb{G}}^\varepsilon(t-\tau)(eu^\varepsilon(t)-eu^\varepsilon(\tau)),eu^\varepsilon(t)-eu^\varepsilon(\tau)).$$

In particular, we can use Fatou's lemma to obtain

$$\int_0^t(-\dot{\mathbb{F}}(t-\tau)(eu^*(t)-eu^*(\tau)),eu^*(t)-eu^*(\tau))\mathrm{d}\tau$$
$$\leq \liminf_{\varepsilon\to 0^+}\int_0^t(-\dot{\mathbb{F}}(t-\tau)(eu^\varepsilon(t)-eu^\varepsilon(\tau)),eu^\varepsilon(t)-eu^\varepsilon(\tau))\mathrm{d}\tau.$$

By arguing in a similar way, we can derive

$$\int_0^t(-\dot{\mathbb{F}}(\tau)(eu^*(\tau)-eu^0),eu^*(\tau)-eu^0)\mathrm{d}\tau\leq \liminf_{\varepsilon\to 0^+}\int_0^t(-\dot{\mathbb{G}}^\varepsilon(\tau)(eu^\varepsilon(\tau)-eu^0),eu^\varepsilon(\tau)-eu^0)\mathrm{d}\tau.$$

For the term involving $\ddot{\mathbb{F}}$, we argue as we already did for $\dot{\mathbb{F}}$ and by using two times Fatou's lemma we get

$$\int_0^t\int_0^\tau(\ddot{\mathbb{F}}(\tau-r)(eu^*(\tau)-eu^*(r)),eu^*(\tau)-eu^*(r))\mathrm{d}r\mathrm{d}\tau$$
$$\leq \liminf_{\varepsilon\to 0^+}\int_0^t\int_0^\tau(\ddot{\mathbb{G}}^\varepsilon(\tau-r)(eu^\varepsilon(\tau)-eu^\varepsilon(r)),eu^\varepsilon(\tau)-eu^\varepsilon(r))\mathrm{d}r\mathrm{d}\tau.$$

It remains to study the right-hand side of (3.49) with the formulation of the total work as in (3.75). Thanks to Lemma 3.3.1 and the fact that $\mathbb{G}^\varepsilon\to\mathbb{F}$ in $L^1(0,T;B)$ we deduce

$$\int_0^t(f(\tau),\dot{u}^\varepsilon(\tau))\mathrm{d}\tau\xrightarrow[\varepsilon\to 0^+]{}\int_0^t(f(\tau),\dot{u}^*(\tau))\mathrm{d}\tau, \tag{3.81}$$

$$\int_0^t(\mathbb{A}eu^\varepsilon(\tau),e\dot{z}(\tau))\mathrm{d}\tau\xrightarrow[\varepsilon\to 0^+]{}\int_0^t(\mathbb{A}eu^*(\tau),e\dot{z}(\tau))\mathrm{d}\tau, \tag{3.82}$$

$$\int_0^t(\mathbb{G}^\varepsilon(t-\tau)(eu^\varepsilon(\tau)-eu^0),e\dot{z}(\tau))\mathrm{d}\tau\xrightarrow[\varepsilon\to 0^+]{}\int_0^t(\mathbb{F}(t-\tau)(eu^*(\tau)-eu^0),e\dot{z}(\tau))\mathrm{d}\tau, \tag{3.83}$$

$$(\dot{u}^\varepsilon(t),\dot{z}(t))-\int_0^t(\dot{u}^\varepsilon(\tau),\ddot{z}(\tau))\mathrm{d}\tau\xrightarrow[\varepsilon\to 0^+]{}(\dot{u}^*(t),\dot{z}(t))-\int_0^t(\dot{u}^*(\tau),\ddot{z}(\tau))\mathrm{d}\tau, \tag{3.84}$$

$$(N(t),u^\varepsilon(t))_{H_N}-\int_0^t(N(\tau),\dot{u}^\varepsilon(\tau))_{H_N}\mathrm{d}\tau\xrightarrow[\varepsilon\to 0^+]{}(N(t),u^*(t))_{H_N}-\int_0^t(\dot{N}(\tau),u^*(\tau))_{H_N}\mathrm{d}\tau. \tag{3.85}$$

It remains to study the term

$$\int_0^t\int_0^\tau(\mathbb{G}^\varepsilon(\tau-r)(eu^\varepsilon(r)-eu^0),e\ddot{z}(\tau))\mathrm{d}r\mathrm{d}\tau.$$

For a.e. $\tau\in(0,t)$ we have

$$\int_0^\tau(\mathbb{G}^\varepsilon(\tau-r)(eu^\varepsilon(r)-eu^0),e\ddot{z}(\tau))\mathrm{d}r\xrightarrow[\varepsilon\to 0^+]{}\int_0^\tau(\mathbb{F}(\tau-r)(eu^*(r)-eu^0),e\ddot{z}(\tau))\mathrm{d}r$$
$$\left|\int_0^\tau(\mathbb{G}^\varepsilon(\tau-r)(eu^\varepsilon(r)-eu^0),e\ddot{z}(\tau))\mathrm{d}r\right|\leq 2M\|\mathbb{F}\|_{L^1(0,T_0;B)}\|e\ddot{z}(\tau)\|\in L^1(0,t),$$

with M independent of ε. By the dominated convergence theorem we conclude

$$\int_0^t\int_0^\tau(\mathbb{G}^\varepsilon(\tau-r)(eu^\varepsilon(r)-eu^0),e\ddot{z}(\tau))\mathrm{d}r\mathrm{d}\tau\xrightarrow[\varepsilon\to 0^+]{}\int_0^t\int_0^\tau(\mathbb{F}(\tau-r)(eu^*(r)-eu^0),e\ddot{z}(\tau))\mathrm{d}r\mathrm{d}\tau. \tag{3.86}$$

By combining (3.77)–(3.86) we deduce the energy-dissipation inequality (3.76) for every $t\in(0,T]$. □

Remark 3.3.3. Although we do not have any information about L^1-integrability of $\dot{\mathbb{F}}$ and $\ddot{\mathbb{F}}$ in $t = 0$, for the generalized solution u^* of Theorem 3.1.4 we obtain that the energy terms $\mathscr{E}^*$ and $\mathscr{D}^*$ are finite.

Corollary 3.3.4. *Assume* (3.2)–(3.8)*. Then the generalized solution* $u^* \in \mathcal{C}_w$ *to system* (3.9) *of Theorem 3.1.4 satisfies*

$$\lim_{t\to 0^+} \mathscr{E}^*(t) = \frac{1}{2}\|u^1\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0). \tag{3.87}$$

In particular, (3.76) *holds true also in* $t = 0$ *and*

$$\lim_{t\to 0^+} \|u^*(t) - u^0\|_V = 0, \qquad \lim_{t\to 0^+} \|\dot{u}^*(t) - u^1\| = 0.$$

Proof. By (3.76) for every $t \in (0, T]$ we have

$$\frac{1}{2}\|\dot{u}^*(t)\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0) \le \mathscr{E}^*(t) \le \frac{1}{2}\|u^1\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0) + \mathscr{W}^*_{tot}(t).$$

Since $u^* \in C^0_w([0,T];V)$ and $\dot{u}^* \in C^0_w([0,T];H)$ we get

$$\frac{1}{2}\|u^1\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0) \le \liminf_{t\to 0^+} \mathscr{E}^*(t) \le \limsup_{t\to 0^+} \mathscr{E}^*(t) \le \frac{1}{2}\|u^1\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0).$$

Therefore, we get (3.87). As consequence of this, we derive

$$\lim_{t\to 0^+} \|\dot{u}^*(t)\|^2 = \|u^1\|^2, \qquad \lim_{t\to 0^+} (\mathbb{A}eu^*(t), eu^*(t)) = (\mathbb{A}eu^0, eu^0),$$

and this conclude the proof. □

For the fractional Kelvin-Voigt's model (3.9) we expect to have uniqueness of the solution, as it happens in [13, 48] for the classic Kelvin-Voigt's one. Unfortunately, the technique used in the cited papers cannot be applied here, and we are able to prove it only when the crack is not moving (see Section 3.4). We point out that the uniqueness of the solution is still an open problem even for the pure elastic case ($\mathbb{B} = 0$), unless the family of cracks is sufficiently regular (see [6, 16]).

Moreover, according the theory of dynamic fracture, we do not expect to have the equality in (3.76). Indeed, we should add also the energy used to the increasing crack, which is postulated to be proportional to the area increment of the crack itself, in line with Griffith's criterion [27]. More precisely, we would like to have

$$\mathscr{E}^*(t) + \mathscr{D}^*(t) + \mathcal{H}^{d-1}(\Gamma_t \setminus \Gamma_0) = \frac{1}{2}\|u^1\|^2 + \frac{1}{2}(\mathbb{A}eu^0, eu^0) + \mathscr{W}^*_{tot}(t) \quad \text{for } t \in [0,T]. \tag{3.88}$$

However, with our approach we are not able to show the previous identity, which again is unknown even in the pure elastic case. We underline that there are no results regarding the validity of (3.88) for the fractional Kelvin-Voigt's model (3.9) even when the crack is not moving.

3.4 Uniqueness for a not moving crack

Let us consider the case of a domain with a fixed crack, i.e. $\Gamma_T = \Gamma_0$ (possibly $\Gamma_T = \emptyset$). In this case we can show that the generalized solution to (3.9) is unique. As we explained in the introduction, uniqueness results for fractional type systems can be found in the literature, but they are proved only for regular sets Ω (without cracks) and in particular cases (for $\mathbb{F}$ given by (5) or when eu is replaced by ∇u).

The proof of the uniqueness is based on a particular energy estimate which holds for the primitive of a generalized solution. To this aim, we need to estimate

$$\int_0^t \int_0^\tau (\mathbb{F}(\tau - r)eu(r), eu(\tau))\mathrm{d}r\mathrm{d}\tau$$

and we start with the following identity which is true for a regular tensor $\mathbb{K}$ (see also [51, Lemma 2.1]).

Lemma 3.4.1. *Let $\mathbb{K} \in C^1([0,T]; B)$ and $v \in L^2(0,T;V_0)$. Then, for every $t \in [0,T]$*

$$\begin{aligned}\int_0^t (\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \mathbb{K}(\tau - r)ev(r)\mathrm{d}r, ev(\tau))\mathrm{d}\tau \\ = \frac{1}{2}\int_0^t (\mathbb{K}(t-\tau)ev(\tau), ev(\tau))\mathrm{d}\tau + \frac{1}{2}\int_0^t (\mathbb{K}(\tau)ev(\tau), ev(\tau))\mathrm{d}\tau \\ - \frac{1}{2}\int_0^t \int_0^\tau (\dot{\mathbb{K}}(\tau - r)(ev(\tau) - ev(r)), ev(\tau) - ev(r))\mathrm{d}r\mathrm{d}\tau. \end{aligned} \tag{3.89}$$

Proof. Let us fix $t \in [0,T]$ and let us analyze the right hand-side of (3.89). We have

$$\begin{aligned} -\frac{1}{2}\int_0^t \int_0^\tau (\dot{\mathbb{K}}(\tau - r)(ev(\tau) - ev(r)), ev(\tau) - ev(r))\mathrm{d}r\mathrm{d}\tau \\ = \int_0^t \int_0^\tau (\dot{\mathbb{K}}(\tau - r)ev(r), ev(\tau))\mathrm{d}r\mathrm{d}\tau - \frac{1}{2}\int_0^t \int_0^\tau (\dot{\mathbb{K}}(\tau - r)ev(r), ev(r))\mathrm{d}r\mathrm{d}\tau \\ - \frac{1}{2}\int_0^t \int_0^\tau (\dot{\mathbb{K}}(\tau - r)ev(\tau), ev(\tau))\mathrm{d}r\mathrm{d}\tau. \end{aligned} \tag{3.90}$$

Notice that

$$-\frac{1}{2}\int_0^t \int_0^\tau (\dot{\mathbb{K}}(\tau - r)ev(\tau), ev(\tau))\mathrm{d}r\mathrm{d}\tau \tag{3.91}$$

$$= -\frac{1}{2}\int_0^t (\left(\int_0^\tau \dot{\mathbb{K}}(\tau - r)\mathrm{d}r\right) ev(\tau), ev(\tau))\mathrm{d}r\mathrm{d}\tau$$

$$= -\frac{1}{2}\int_0^t (\mathbb{K}(\tau)ev(\tau), ev(\tau))\mathrm{d}\tau + \frac{1}{2}\int_0^t (\mathbb{K}(0)ev(\tau), ev(\tau))\mathrm{d}\tau, \tag{3.92}$$

and that for a.e. $\tau \in (0,t)$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau (\mathbb{K}(\tau - r)ev(r), ev(r))\mathrm{d}r = (\mathbb{K}(0)ev(\tau), ev(\tau)) + \int_0^\tau (\dot{\mathbb{K}}(\tau - r)ev(r), ev(r))\mathrm{d}r,$$

from which we deduce

$$-\frac{1}{2}\int_0^t (\mathbb{K}(t-\tau)ev(\tau), ev(\tau))\mathrm{d}\tau \tag{3.93}$$

$$= -\frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau (\mathbb{K}(\tau - r)ev(r), ev(r))\mathrm{d}r\mathrm{d}\tau$$

$$= -\frac{1}{2}\int_0^t (\mathbb{K}(0)ev(\tau), ev(\tau))\mathrm{d}\tau - \frac{1}{2}\int_0^t \int_0^\tau (\dot{\mathbb{K}}(\tau - r)ev(r), ev(r))\mathrm{d}r\mathrm{d}\tau. \tag{3.94}$$

By (3.90)–(3.93) we can say

$$-\frac{1}{2}\int_0^t \int_0^\tau (\dot{\mathbb{K}}(\tau - r)(ev(\tau) - ev(r)), ev(\tau) - ev(r))\mathrm{d}r\mathrm{d}\tau$$

$$= \int_0^t \int_0^\tau (\dot{\mathbb{K}}(\tau - r)ev(r), ev(\tau))\mathrm{d}r\mathrm{d}\tau + \int_0^t (\mathbb{K}(0)ev(\tau), ev(\tau))\mathrm{d}\tau$$
$$- \frac{1}{2}\int_0^t (\mathbb{K}(\tau)ev(\tau), ev(\tau))\mathrm{d}\tau - \frac{1}{2}\int_0^t (\mathbb{K}(t-\tau)ev(\tau), ev(\tau))\mathrm{d}\tau,$$

and thanks to the following relation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \mathbb{K}(\tau - r)ev(r)\mathrm{d}r = \mathbb{K}(0)ev(\tau) + \int_0^\tau \dot{\mathbb{K}}(\tau - r)ev(r)\mathrm{d}r \quad \text{for a.e. } \tau \in (0,t),$$

we can conclude the proof. □

Lemma 3.4.2. *Let $\mathbb{F}$ be satisfying (3.5)–(3.8) and $u \in C_w^0([0,T];V_0)$. Then for every $t \in [0,T]$ it holds*

$$\int_0^t \int_0^\tau (\mathbb{F}(\tau - r)eu(r), eu(\tau))\mathrm{d}r\mathrm{d}\tau \geq 0. \tag{3.95}$$

Proof. First, we fix $\varepsilon \in (0,\delta_0)$ and we consider for every $t \in [0,T]$ the following regularized kernel

$$\mathbb{G}^\varepsilon(t) := \mathbb{F}(t+\varepsilon).$$

Moreover, we fix $t \in [0,T]$ and we define for every $\tau \in [0,t]$ a primitive of u in the following way

$$v(\tau) := -\int_r^t u(r)\mathrm{d}r.$$

Clearly $\mathbb{G}^\varepsilon \in C^2([0,T];B)$ and after an integration by parts, since $ev(t) = 0$, we obtain

$$\int_0^t \int_0^\tau (\mathbb{G}^\varepsilon(\tau - r)eu(r), eu(\tau))\mathrm{d}r\mathrm{d}\tau$$
$$= \int_0^t \int_0^\tau (\mathbb{G}^\varepsilon(\tau - r)eu(r), e\dot{v}(\tau))\mathrm{d}r\mathrm{d}\tau$$
$$= -\int_0^t (\mathbb{G}^\varepsilon(0)e\dot{v}(\tau), ev(\tau))\mathrm{d}\tau - \int_0^t \int_0^\tau (\dot{\mathbb{G}}^\varepsilon(\tau - r)eu(r), ev(\tau))\mathrm{d}r\mathrm{d}\tau$$
$$= \frac{1}{2}(\mathbb{G}^\varepsilon(0)ev(0), ev(0)) - \int_0^t \int_0^\tau (\dot{\mathbb{G}}^\varepsilon(\tau - r)eu(r), ev(\tau))\mathrm{d}r\mathrm{d}\tau.$$

Moreover, we have

$$\int_0^\tau \dot{\mathbb{G}}^\varepsilon(\tau - r)eu(r)\mathrm{d}r = \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \dot{\mathbb{G}}^\varepsilon(\tau - r)ev(r)\mathrm{d}r - \dot{\mathbb{G}}^\varepsilon(\tau)ev(0).$$

Therefore, by (3.89) we can write

$$\int_0^t \int_0^\tau (\dot{\mathbb{G}}^\varepsilon(\tau - r)eu(r), ev(\tau))\mathrm{d}r\mathrm{d}\tau$$
$$= \int_0^t (\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \dot{\mathbb{G}}^\varepsilon(\tau - r)ev(r)\mathrm{d}r - \dot{\mathbb{G}}^\varepsilon(\tau)ev(0), ev(\tau))\mathrm{d}\tau$$
$$= \frac{1}{2}\int_0^t (\dot{\mathbb{G}}^\varepsilon(t-\tau)ev(\tau), ev(\tau))\mathrm{d}\tau + \frac{1}{2}\int_0^t (\dot{\mathbb{G}}^\varepsilon(\tau)ev(\tau), ev(\tau))\mathrm{d}\tau$$
$$- \frac{1}{2}\int_0^t \int_0^\tau \ddot{\mathbb{G}}^\varepsilon(\tau - r)(ev(\tau) - ev(r)), ev(\tau) - ev(r))\mathrm{d}r\mathrm{d}\tau$$
$$- \int_0^t (\dot{\mathbb{G}}^\varepsilon(\tau)ev(0), ev(\tau))\mathrm{d}\tau,$$

which implies

$$\begin{aligned}
&\int_0^t \int_0^\tau (\mathbb{G}^\varepsilon(\tau - r)eu(r), eu(\tau))\mathrm{d}r\mathrm{d}\tau \\
&\qquad = \frac{1}{2}(\mathbb{G}^\varepsilon(0)ev(0), ev(0)) + \int_0^t (\dot{\mathbb{G}}^\varepsilon(\tau)ev(0), ev(\tau))\mathrm{d}\tau \\
&\qquad\quad - \frac{1}{2}\int_0^t (\dot{\mathbb{G}}^\varepsilon(t-\tau)ev(\tau), ev(\tau))\mathrm{d}\tau - \frac{1}{2}\int_0^t (\dot{\mathbb{G}}^\varepsilon(\tau)ev(\tau), ev(\tau))\mathrm{d}\tau \\
&\qquad\quad + \frac{1}{2}\int_0^t \int_0^\tau (\ddot{\mathbb{G}}^\varepsilon(\tau - r)(ev(\tau) - ev(r)), ev(\tau) - ev(r))\mathrm{d}r\mathrm{d}\tau \\
&\qquad \geq \frac{1}{2}(\mathbb{G}^\varepsilon(0)ev(0), ev(0)) + \frac{1}{2}\int_0^t (\dot{\mathbb{G}}^\varepsilon(\tau)ev(0), ev(0))\mathrm{d}\tau \\
&\qquad\quad - \frac{1}{2}\int_0^t (\dot{\mathbb{G}}^\varepsilon(t-\tau)ev(\tau), ev(\tau))\mathrm{d}\tau \\
&\qquad\quad + \frac{1}{2}\int_0^t \int_0^\tau (\ddot{\mathbb{G}}^\varepsilon(\tau - r)(ev(\tau) - ev(r)), ev(\tau) - ev(r))\mathrm{d}r\mathrm{d}\tau \\
&\qquad = \frac{1}{2}(\mathbb{G}^\varepsilon(t)ev(0), ev(0)) - \frac{1}{2}\int_0^t (\dot{\mathbb{G}}^\varepsilon(t-\tau)ev(\tau), ev(\tau))\mathrm{d}\tau \\
&\qquad\quad + \frac{1}{2}\int_0^t \int_0^\tau \ddot{\mathbb{G}}^\varepsilon(\tau - r)(ev(\tau) - ev(r)), ev(\tau) - ev(r))\mathrm{d}r\mathrm{d}\tau \geq 0.
\end{aligned}$$

By sending $\varepsilon \to 0^+$ we conclude. □

We can now state our uniqueness result.

Theorem 3.4.3. *Assume* (3.2)–(3.8) *and* $\Gamma_T = \Gamma_0$. *Then there exists at most one generalized solution to system* (3.9).

Proof. Let $u_1, u_2 \in \mathcal{C}_w$ be two generalized solutions to (3.9). Then $u := u_1 - u_2$ satisfies equality (3.10) with $z = N = f = u^0 = u^1 = 0$. Consider the function $\beta\colon [0,T] \to (V_0^D)'$ defined for every $\tau \in [0,T]$ as

$$\langle \beta(\tau), v\rangle_{(V_0^D)'} := (\dot{u}(\tau), v) + \int_0^\tau (\mathbb{C}eu(r), ev)\mathrm{d}r + \int_0^\tau (\mathbb{F}(\tau - r)eu(r), ev)\mathrm{d}r$$

for every $v \in V_0^D$. Clearly $\beta \in C_w^0([0,T];(V_0^D)')$, $\beta(0) = 0$ since $\dot{u}(0) = 0$ in $(V_0^D)'$, and by (3.10) we derive

$$\int_0^T \langle \beta(\tau), v\rangle_{(V_0^D)'}\dot{\psi}(\tau)\mathrm{d}\tau = 0 \quad \text{for every } v \in V_0^D \text{ and } \psi \in C_c^1(0,T).$$

Therefore β is constant in $[0,T]$, which gives $\beta(\tau) = 0$ in $(V_0^D)'$ for every $\tau \in [0,T]$, namely for every $v \in V_0^D$ and $\tau \in [0,T]$ we have

$$(\dot{u}(\tau), v) + \int_0^\tau (\mathbb{C}eu(r), ev)\mathrm{d}r + \int_0^\tau (\mathbb{F}(\tau - r)eu(r), ev)\mathrm{d}r = 0.$$

In particular, for every $t \in [0,T]$ we deduce

$$\int_0^t (\dot{u}(\tau), u(\tau))\mathrm{d}\tau + \int_0^t \int_0^\tau (\mathbb{C}eu(r), eu(\tau))\mathrm{d}r\mathrm{d}\tau + \int_0^t \int_0^\tau (\mathbb{F}(\tau - r)eu(r), eu(\tau))\mathrm{d}r\mathrm{d}\tau = 0.$$

Hence, by (3.95) we conclude

$$\frac{1}{2}\|u(t)\|^2 + \frac{1}{2}(\mathbb{A}\Big(\int_0^t eu(\tau)\mathrm{d}\tau\Big), \int_0^t eu(\tau)\mathrm{d}\tau) \leq 0 \quad \text{for every } t \in [0,T].$$

Therefore, since both terms are non-negative, we get that $u(t) = 0$ for every $t \in [0,T]$. □

Chapter 4

Quasistatic limit of a dynamic viscoelastic model with memory

The chapter is organized as follows. In Section 4.1 we fix the notation adopted throughout the chapter and we prove some properties of the solutions to (21) such as the energy balance (4.24) of Proposition 4.1.7. In Section 4.2 we state our main results (Theorems 4.2.6 and 4.2.7). In Section 4.3, under the assumption of the compatibility condition (4.38), we prove the uniform convergence of Theorem 4.2.6 of the solutions of dynamic evolution problem (21) to the solution of stationary problem (22) by means of energy estimate (4.44) of Lemma 4.2.8, derived by energy balance (4.24). In Section 4.4 we recall the main properties of the Laplace Transform and of the Inverse Laplace Transform for functions with values in Hilbert spaces. In particular, in Subsection 4.4.1 we develop the Laplace Transform tools, consequently in Subsection 4.4.2 we study the equation satisfied by the Laplace Transforms of the solutions to (21) and (22) (see (4.80) and (4.81)), and finally in Subsection 4.4.3 we prove the convergence in L^2 of the solutions of (4.80) to the solution of (4.81). Thanks to the theory developed in Section 4.4 and to energy inequality (4.44) of Lemma 4.2.8, under general assumptions, we prove in Section 4.5 and 4.6 the convergence in L^2 and the local uniform convergence of the solution of dynamic evolution problem (21) to the solution of stationary problem (22).

The results presented here are obtained in collaboration with Prof. G. Dal Maso and are contained in [18].

4.1 Hypotheses and statement of the problem

Let d be a positive integer and let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. We use standard notation for Lebesgue and Sobolev spaces. For convenience we set for every $m \in \mathbb{N}$ the space $H := L^2(\Omega;\mathbb{R}^m)$ and we always identify the dual of H with H itself. Moreover, we define

$$V := H^1(\Omega;\mathbb{R}^d), \quad V_0 := H^1_0(\Omega;\mathbb{R}^d), \quad V_0' := H^{-1}(\Omega;\mathbb{R}^d).$$

The symbols $(\cdot,\cdot)$ and $\|\cdot\|$ denote the scalar product and the norm in H. The duality product between V_0' and V_0 is denoted by $\langle\cdot,\cdot\rangle$. Given $u \in V$ we denote with eu its strain, which is defined as the symmetric part of the gradient.

Under these assumptions, the Second Korn Inequality (see, e.g., [39, Theorem 2.4]) states that there exists a positive constant $C_K = C_K(\Omega)$ such that

$$\|\nabla u\| \le C_K \left(\|u\|^2 + \|eu\|^2\right)^{1/2} \quad \text{for every } u \in V. \tag{4.1}$$

Moreover, there exists a positive constant $C_P = C_P(\Omega)$ such that the following Korn-Poincaré

Inequality holds (see, e.g., [39, Theorem 2.7]):

$$\|u\| \le C_P \|eu\| \qquad \text{for every } u \in V_0. \tag{4.2}$$

Thanks to (4.1) we can use on the space V the equivalent norm

$$\|u\|_V := (\|u\|^2 + \|eu\|^2)^{1/2} \quad \text{for every } u \in V.$$

Let $\mathscr{L}(\mathbb{R}^{d\times d}_{sym}; \mathbb{R}^{d\times d}_{sym})$ be the space of all linear operators from $\mathbb{R}^{d\times d}_{sym}$ into itself. We assume that the elasticity and viscosity tensors $\mathbb{A}$ and $\mathbb{B}$ satisfy the following assumptions:

$$\mathbb{A}, \mathbb{B} \in L^\infty(\Omega; \mathscr{L}(\mathbb{R}^{d\times d}_{sym}; \mathbb{R}^{d\times d}_{sym})), \tag{4.3}$$

and for a.e. $x \in \Omega$

$$\mathbb{A}(x)\xi_1 \cdot \xi_2 = \xi_1 \cdot \mathbb{A}(x)\xi_2, \qquad \mathbb{B}(x)\xi_1 \cdot \xi_2 = \xi_1 \cdot \mathbb{B}(x)\xi_2 \qquad \text{for every } \xi_1, \xi_2 \in \mathbb{R}^{d\times d}_{sym}, \tag{4.4}$$

$$c_{\mathbb{A}}|\xi|^2 \le \mathbb{A}(x)\xi \cdot \xi \le C_{\mathbb{A}}|\xi|^2, \quad c_{\mathbb{B}}|\xi|^2 \le \mathbb{B}(x)\xi \cdot \xi \le C_{\mathbb{B}}|\xi|^2 \qquad \text{for every } \xi \in \mathbb{R}^{d\times d}_{sym}, \tag{4.5}$$

where $c_{\mathbb{A}}$, $c_{\mathbb{B}}$, $C_{\mathbb{A}}$, and $C_{\mathbb{B}}$ are positive constants independent of x, and the dot denotes the Euclidean scalar product of matrices.

Let us fix $T > 0$ and $\beta > 0$. To give a precise meaning to the notion of solution to problem (12)–(21) we introduce the function spaces

$$\begin{aligned}
\mathcal{V} &:= L^2(0,T;V) \cap H^1(0,T;H) \cap H^2(0,T;V_0') \\
\mathcal{V}_0 &:= L^2(0,T;V_0) \cap H^1(0,T;H) \cap H^2(0,T;V_0'), \\
\mathcal{V}_{loc} &:= L^2_{loc}(-\infty,T;V) \cap H^1_{loc}(-\infty,T;H) \cap H^2_{loc}(-\infty,T;V_0').
\end{aligned}$$

Remark 4.1.1. By the Sobolev Embedding Theorem, if $u \in \mathcal{V}$ (resp. $u \in \mathcal{V}_{loc}$), then

$$u \in C^0([0,T];H) \cap C^1([0,T];V_0') \quad (\text{resp. } u \in C^0((-\infty,T);H) \cap C^1((-\infty,T);V_0')).$$

We study problem (12)–(21) with ℓ, z, and u_{in} depending on ε. Let us consider $\varepsilon > 0$ and

$$f_\varepsilon \in L^2(0,T;H), \quad g_\varepsilon \in H^1(0,T;V_0'), \quad z_\varepsilon \in H^2(0,T;H) \cap H^1(0,T;V), \tag{4.6}$$

$u_{\varepsilon,in} \in C^0((-\infty,T);H) \cap C^1((-\infty,T);V_0')$ such that

$$\begin{gathered}
u_{\varepsilon,in}(0) \in V, \quad u_{\varepsilon,in}(0) - z_\varepsilon(0) \in V_0, \quad \dot u_{\varepsilon,in}(0) \in H, \\
\int_{-\infty}^0 \frac{1}{\beta\varepsilon} \mathrm{e}^{\frac{\tau}{\beta\varepsilon}} \|u_{\varepsilon,in}(\tau)\|_V \mathrm{d}\tau < +\infty.
\end{gathered} \tag{4.7}$$

The notion of solution to (12)–(21) is made precise by the following definition.

Definition 4.1.2. We say that u_ε is a solution to the viscoelastic dynamic system (12)–(21), with forcing term $\ell = f_\varepsilon + g_\varepsilon$, boundary condition z_ε, and initial condition $u_{\varepsilon,in}$, if

$$u_\varepsilon \in \mathcal{V}_{loc} \quad \text{and} \quad u_\varepsilon - z_\varepsilon \in \mathcal{V}_0, \tag{4.8a}$$

$$\varepsilon^2 \ddot u_\varepsilon(t) - \operatorname{div}((\mathbb{A}+\mathbb{B})eu_\varepsilon(t)) + \int_{-\infty}^t \frac{1}{\beta\varepsilon} \mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}} \operatorname{div}(\mathbb{B}eu_\varepsilon(\tau))\mathrm{d}\tau = f_\varepsilon(t) + g_\varepsilon(t)$$
$$\text{for a.e. } t \in [0,T], \tag{4.8b}$$

$$u_\varepsilon(t) = u_{\varepsilon,in}(t) \quad \text{for every } t \in (-\infty, 0]. \tag{4.8c}$$

In the next remark we shall see that (4.8) can be reduced to the following problem starting from 0:

$$u_\varepsilon \in \mathcal{V} \quad \text{and} \quad u_\varepsilon - z_\varepsilon \in \mathcal{V}_0, \tag{4.9a}$$

$$\varepsilon^2 \ddot{u}_\varepsilon(t) - \operatorname{div}((\mathbb{A}+\mathbb{B})eu_\varepsilon(t)) + \int_0^t \frac{1}{\beta\varepsilon} \mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}} \operatorname{div}(\mathbb{B}eu_\varepsilon(\tau))\mathrm{d}\tau = \varphi_\varepsilon(t) + \gamma_\varepsilon(t)$$
$$\text{for a.e. } t \in [0,T], \tag{4.9b}$$

$$u_\varepsilon(0) = u_\varepsilon^0 \text{ in } H \quad \text{and} \quad \dot{u}_\varepsilon(0) = u_\varepsilon^1 \text{ in } V_0', \tag{4.9c}$$

with $\varphi_\varepsilon \in L^2(0,T;H)$, $\gamma_\varepsilon \in H^1(0,T;V_0')$, $u_\varepsilon^0 \in V$, $u_\varepsilon^0 - z_\varepsilon(0) \in V_0$, $u_\varepsilon^1 \in H$.

Remark 4.1.3. It is easy to see that u_ε is a solution according to Definition 4.1.2 if and only if its restriction to $[0,T]$, still denoted by u_ε, solves (4.9) with

$$\varphi_\varepsilon = f_\varepsilon, \quad \gamma_\varepsilon = g_\varepsilon - p_\varepsilon, \quad u_\varepsilon^0 = u_{\varepsilon,in}(0), \quad u_\varepsilon^1 = \dot{u}_{\varepsilon,in}(0), \tag{4.10}$$

where

$$p_\varepsilon(t) := \mathrm{e}^{-\frac{t}{\beta\varepsilon}} g_\varepsilon^0 \quad \text{with} \quad g_\varepsilon^0 := \int_{-\infty}^0 \frac{1}{\beta\varepsilon} \mathrm{e}^{\frac{\tau}{\beta\varepsilon}} \operatorname{div}(\mathbb{B}eu_{\varepsilon,in}(\tau))\mathrm{d}\tau. \tag{4.11}$$

To solve problem (4.9) it is enough to study the corresponding problem with homogeneous boundary condition:

$$v_\varepsilon \in \mathcal{V}_0, \tag{4.12a}$$

$$\varepsilon^2 \ddot{v}_\varepsilon(t) - \operatorname{div}((\mathbb{A}+\mathbb{B})ev_\varepsilon(t)) + \int_0^t \frac{1}{\beta\varepsilon} \mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}} \operatorname{div}(\mathbb{B}ev_\varepsilon(\tau))\mathrm{d}\tau = h_\varepsilon(t) + \ell_\varepsilon(t)$$
$$\text{for a.e. } t \in [0,T], \tag{4.12b}$$

$$v_\varepsilon(0) = v_\varepsilon^0 \text{ in } H \quad \text{and} \quad \dot{v}_\varepsilon(0) = v_\varepsilon^1 \text{ in } V_0', \tag{4.12c}$$

with

$$h_\varepsilon \in L^2(0,T;H), \quad \ell_\varepsilon \in H^1(0,T;V_0'), \quad v_\varepsilon^0 \in V_0, \quad v_\varepsilon^1 \in H. \tag{4.13}$$

Remark 4.1.4. The function u_ε is a solution to (4.9) if and only if $v_\varepsilon = u_\varepsilon - z_\varepsilon$ solves (4.12) with

$$h_\varepsilon(t) = \varphi_\varepsilon(t) - \varepsilon^2 \ddot{z}_\varepsilon(t), \quad \ell_\varepsilon(t) = \gamma_\varepsilon(t) + \operatorname{div}((\mathbb{A}+\mathbb{B})ez_\varepsilon(t)) - \int_0^t \frac{1}{\beta\varepsilon} \mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}} \operatorname{div}(\mathbb{B}ez_\varepsilon(\tau))\mathrm{d}\tau,$$
$$v_\varepsilon^0 = u_\varepsilon^0 - z_\varepsilon(0), \quad v_\varepsilon^1 = u_\varepsilon^1 - \dot{z}_\varepsilon(0), \tag{4.14}$$

Therefore, existence and uniqueness for (4.12) imply existence and uniqueness for (4.9).

Remark 4.1.5. In [11] problem (4.12) has been studied with initial conditions taken in the sense of interpolation spaces. Given two Hilbert spaces X and Y, the symbol $[X,Y]_\theta$ denotes the interpolation space between X and Y of exponent $\theta \in (0,1)$. Thanks to [34, Theorem 3.1] we have the following inclusions:

$$L^2(0,T;V_0) \cap H^1(0,T;H) \subset C^0([0,T];V_0^{\frac{1}{2}}),$$
$$L^2(0,T;H) \cap H^1(0,T;V_0') \subset C^0([0,T];V_0^{-\frac{1}{2}}),$$

where $V_0^{\frac{1}{2}} := [V_0,H]_{\frac{1}{2}}$ and $V_0^{-\frac{1}{2}} := [H,V_0']_{\frac{1}{2}}$. Consequently

$$\mathcal{V}_0 \subset C^0([0,T];V_0^{\frac{1}{2}}) \cap C^1([0,T];V_0^{-\frac{1}{2}}).$$

Therefore, the initial conditions in (4.12) are satisfied also in the stronger sense

$$v_\varepsilon(0) = v_\varepsilon^0 \text{ in } V_0^{\frac{1}{2}} \quad \text{and} \quad \dot{v}_\varepsilon(0) = v_\varepsilon^1 \text{ in } V_0^{-\frac{1}{2}}. \tag{4.15}$$

The following proposition provides the main properties of the solutions. We recall that, if X is a Banach space, $C^0_w([0,T];X)$ denotes the space of all weakly continuous functions from $[0,T]$ to X, namely, the vector space of all functions $u\colon [0,T]\to X$ such that for every $x'\in X'$ the function $t\mapsto \langle x', u(t)\rangle$ is continuous from $[0,T]$ to $\mathbb{R}$.

Proposition 4.1.6. *Given $\varepsilon>0$, assume (4.6) and (4.7). Then there exists a unique solution u_ε to the viscoelastic dynamic system (4.8). Moreover, it satisfies*

$$u_\varepsilon\in C^0([0,T];V)\cap C^1([0,T];H). \tag{4.16}$$

Proof. By Remarks 4.1.3 and 4.1.4 it is enough to prove the theorem for (4.12). Existence and uniqueness are proved in [11], taking into account Remark 4.1.5 about the equivalence between the initial conditions in the sense of (4.12) and (4.15).

After an integration by parts with respect to time, it easy to see that the weak formulation (4.12) is equivalent to the following one:

$$\begin{aligned}-\varepsilon^2\int_0^T(\dot v_\varepsilon(t),\dot\varphi(t))\mathrm{d}t+\int_0^T((\mathbb{A}+\mathbb{B})ev_\varepsilon(t),e\varphi(t))\mathrm{d}t-\int_0^T\int_0^t\frac{1}{\beta\varepsilon}\mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}}(\mathbb{B}ev_\varepsilon(\tau),e\varphi(t))\mathrm{d}\tau\mathrm{d}t\\ =\int_0^T(h_\varepsilon(t),\varphi(t))\mathrm{d}t+\int_0^T\langle\ell_\varepsilon(t),\varphi(t)\rangle\mathrm{d}t\quad\text{for every }\varphi\in C_c^\infty(0,T;V).\end{aligned}\tag{4.17}$$

In [44], in a more general context, it has been proved that if v_ε satisfies (4.17) and the initial conditions in the sense of (4.12), then it satisfies also

$$\begin{aligned}&v_\varepsilon\in C^0_w([0,T];V)\quad\text{and}\quad \dot v_\varepsilon\in C^0_w([0,T];H),\\ &\lim_{t\to0^+}\|v_\varepsilon(t)-v^0_\varepsilon\|_V=0\quad\text{and}\quad\lim_{t\to0^+}\|\dot v_\varepsilon(t)-v^1_\varepsilon\|=0.\end{aligned}\tag{4.18}$$

We fix $s\in[0,T)$. We want to prove

$$\lim_{t\to s^+}\|v_\varepsilon(t)-v_\varepsilon(s)\|_V=0\qquad\text{and}\qquad\lim_{t\to s^+}\|\dot v_\varepsilon(t)-\dot v_\varepsilon(s)\|=0. \tag{4.19}$$

Thanks to the theory developed in [11] there exists a unique $\tilde v_\varepsilon\in L^2(s,T;V_0)\cap H^1(s,T;H)\cap H^2(s,T;V_0')$ such that

$$\begin{aligned}&\varepsilon^2\ddot{\tilde v}_\varepsilon(t)-\operatorname{div}((\mathbb{A}+\mathbb{B})e\tilde v_\varepsilon(t))+\int_s^t\frac{1}{\beta\varepsilon}\mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}}\operatorname{div}(\mathbb{B}e\tilde v_\varepsilon(\tau))\mathrm{d}\tau\\ &\quad=h_\varepsilon(t)+\ell_\varepsilon(t)-\int_0^s\frac{1}{\beta\varepsilon}\mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}}\operatorname{div}(\mathbb{B}ev_\varepsilon(\tau))\mathrm{d}\tau\quad\text{for a.e. }t\in[s,T],\end{aligned}\tag{4.20}$$

$$\lim_{t\to s^+}\|\tilde v_\varepsilon(t)-v_\varepsilon(s)\|=0\quad\text{and}\quad\lim_{t\to s^+}\|\dot{\tilde v}_\varepsilon(t)-\dot v_\varepsilon(s)\|_{V_0'}=0. \tag{4.21}$$

By the results in [44] the function $\tilde v_\varepsilon$ satisfies also

$$\lim_{t\to s^+}\|\tilde v_\varepsilon(t)-v_\varepsilon(s)\|_V=0\quad\text{and}\quad\lim_{t\to s^+}\|\dot{\tilde v}_\varepsilon(t)-\dot v_\varepsilon(s)\|=0. \tag{4.22}$$

Since clearly v_ε satisfies (4.20) and (4.21), by uniqueness we have $\tilde v_\varepsilon(t)=v_\varepsilon(t)$ for every $t\in[s,T]$. In particular, from (4.22) we deduce that (4.19) holds.

To complete the proof we need the following proposition about the energy-dissipation balance, where $\mathscr{W}_\varepsilon(t)$ represents the work done in the interval $[0,t]$.

Proposition 4.1.7. *Given $\varepsilon > 0$, we assume (4.13). Let v_ε be the solution to (4.12) and let $w_\varepsilon\colon [0,T] \to H$ be defined by*

$$w_\varepsilon(t) := \mathrm{e}^{-\frac{t}{\beta\varepsilon}} \int_0^t \frac{1}{\beta\varepsilon} \mathrm{e}^{\frac{\tau}{\beta\varepsilon}} ev_\varepsilon(\tau) \mathrm{d}\tau \quad \text{for every } t \in [0,T]. \tag{4.23}$$

Then $w_\varepsilon \in H^1(0,T;H)$ and the following energy-dissipation balance holds for every $t \in [0,T]$:

$$\begin{aligned}\frac{\varepsilon^2}{2}\|\dot v_\varepsilon(t)\|^2 + \frac{1}{2}(\mathbb{A}ev_\varepsilon(t), ev_\varepsilon(t)) + \frac{1}{2}(\mathbb{B}(ev_\varepsilon(t) - w_\varepsilon(t)), ev_\varepsilon(t) - w_\varepsilon(t))\\ + \beta\varepsilon \int_0^t (\mathbb{B}\dot w_\varepsilon(\tau), \dot w_\varepsilon(\tau))\mathrm{d}\tau = \frac{\varepsilon^2}{2}\|v_\varepsilon^1\|^2 + \frac{1}{2}((\mathbb{A}+\mathbb{B})ev_\varepsilon^0, ev_\varepsilon^0) + \mathscr{W}_\varepsilon(t),\end{aligned} \tag{4.24}$$

where

$$\mathscr{W}_\varepsilon(t) := \int_0^t (h_\varepsilon(\tau), \dot v_\varepsilon(\tau))\mathrm{d}\tau - \int_0^t \langle \dot\ell_\varepsilon(\tau), v_\varepsilon(\tau)\rangle \mathrm{d}\tau + \langle \ell_\varepsilon(t), v_\varepsilon(t)\rangle - \langle \ell_\varepsilon(0), v_\varepsilon^0\rangle.$$

Proof. It is convenient to extend the data of our problem to the interval $[0,2T]$ by setting

$$h_\varepsilon(t) := 0 \quad \text{and} \quad \ell_\varepsilon(t) := \ell_\varepsilon(T) \quad \text{for every } t \in (T,2T].$$

It is clear that $h_\varepsilon \in L^2(0,2T;H)$ and $\ell_\varepsilon \in H^1(0,2T;V_0')$. By uniqueness of the solution to (4.12), the solution on $[0,2T]$ is an extension of v_ε, still denoted by v_ε. We also consider the extension of w_ε on $[0,2T]$ defined by (4.23).

Since $ev_\varepsilon \in L^2(0,2T;H)$, it follows from (4.23) that $w_\varepsilon \in H^1(0,2T;H)$, and

$$\beta\varepsilon \dot w_\varepsilon(t) = ev_\varepsilon(t) - w_\varepsilon(t) \quad \text{for a.e. } t \in [0,2T]. \tag{4.25}$$

Thanks to (4.19) in $[0,2T]$ and (4.25) there exists a representative of $\dot w_\varepsilon$ such that

$$\lim_{t\to s^+} \|\dot w_\varepsilon(t) - \dot w_\varepsilon(s)\| = 0 \quad \text{for every } s \in [0,2T). \tag{4.26}$$

Moreover, since v_ε satisfies (4.12) in $[0,2T]$, we have

$$\varepsilon^2 \ddot v_\varepsilon(t) - \operatorname{div}(\mathbb{A}ev_\varepsilon(t)) - \operatorname{div}(\mathbb{B}(ev_\varepsilon(t) - w_\varepsilon(t))) = h_\varepsilon(t) + \ell_\varepsilon(t) \quad \text{for a.e. } t \in [0,2T]. \tag{4.27}$$

Multiplying (4.25) and (4.27) by $\psi \in H$ and $\varphi \in V_0$, respectively, and then integrating over Ω and adding the results, we get

$$\begin{aligned}\varepsilon^2\langle \ddot v_\varepsilon(t), \varphi\rangle + (\mathbb{A}ev_\varepsilon(t), e\varphi) + (\mathbb{B}(ev_\varepsilon(t) - w_\varepsilon(t)), e\varphi - \psi)\\ + \beta\varepsilon(\mathbb{B}\dot w_\varepsilon(t), \psi) = (h_\varepsilon(t), \varphi) + \langle \ell_\varepsilon(t), \varphi\rangle \quad \text{for a.e. } t \in [0,2T].\end{aligned} \tag{4.28}$$

Given a function r from $[0,2T]$ into a Banach space X, for every $\eta > 0$ we define the sum and the difference functions $\sigma^\eta r, \delta^\eta r\colon [0,2T-\eta] \to X$ by $\sigma^\eta r(t) := r(t+\eta) + r(t)$ and $\delta^\eta r(t) := r(t+\eta) - r(t)$. For a.e. $t \in [0,2T-\eta]$ we have $\sigma^\eta v_\varepsilon(t), \delta^\eta v_\varepsilon(t) \in V_0$ and $\sigma^\eta w_\varepsilon(t), \delta^\eta w_\varepsilon(t) \in H$. For a.e. $t \in [0,2T-\eta]$ we use (4.28) first at time t and then at time $t+\eta$, with $\varphi := \delta^\eta v_\varepsilon(t)$ and $\psi := \delta^\eta w_\varepsilon(t)$. By summing the two expressions and then integrating in time on the interval $[0,t]$ we get

$$\int_0^t [\varepsilon^2 K_\eta(\tau) + A_\eta(\tau) + B_\eta(\tau) + \varepsilon D_\eta(\tau)]\mathrm{d}\tau = \int_0^t W_\eta(\tau)\mathrm{d}\tau, \tag{4.29}$$

where for a.e. $\tau \in [0,2T-\eta]$

$$K_\eta(\tau) := \langle \sigma^\eta \ddot v_\varepsilon(\tau), \delta^\eta v_\varepsilon(\tau)\rangle,$$

$$\begin{aligned}
A_\eta(\tau) &:= (\mathbb{A}\,\sigma^\eta ev_\varepsilon(\tau), \delta^\eta ev_\varepsilon(\tau)),\\
B_\eta(\tau) &:= (\mathbb{B}(\sigma^\eta ev_\varepsilon(\tau) - \sigma^\eta w_\varepsilon(\tau)), \delta^\eta ev_\varepsilon(\tau) - \delta^\eta w_\varepsilon(\tau)),\\
D_\eta(\tau) &:= \beta(\mathbb{B}\,\sigma^\eta \dot w_\varepsilon(\tau), \delta^\eta w_\varepsilon(\tau)),\\
W_\eta(\tau) &:= (\sigma^\eta h_\varepsilon(\tau), \delta^\eta v_\varepsilon(\tau)) + \langle \sigma^\eta \ell_\varepsilon(\tau), \delta^\eta v_\varepsilon(\tau)\rangle.
\end{aligned}$$

An integration by parts in time gives

$$\begin{aligned}
\int_0^t K_\eta(\tau)\mathrm{d}\tau &= (\sigma^\eta \dot v_\varepsilon(t), \delta^\eta v_\varepsilon(t)) - (\sigma^\eta \dot v_\varepsilon(0), \delta^\eta v_\varepsilon(0)) - \int_0^t (\sigma^\eta \dot v_\varepsilon(\tau), \delta^\eta \dot v_\varepsilon(\tau))\mathrm{d}\tau\\
&= \int_t^{t+\eta} (\sigma^\eta \dot v_\varepsilon(t), \dot v_\varepsilon(\tau))\mathrm{d}\tau - \int_0^\eta (\sigma^\eta \dot v_\varepsilon(0), \dot v_\varepsilon(\tau))\mathrm{d}\tau - \int_0^t \|\dot v_\varepsilon(\tau+h)\|^2\mathrm{d}\tau + \int_0^t \|\dot v_\varepsilon(\tau)\|^2\mathrm{d}\tau\\
&= \int_t^{t+\eta} \big[(\sigma^\eta \dot v_\varepsilon(t), \dot v_\varepsilon(\tau)) - \|\dot v_\varepsilon(\tau)\|^2\big]\mathrm{d}\tau - \int_0^\eta \big[(\sigma^\eta \dot v_\varepsilon(0), \dot v_\varepsilon(\tau)) - \|\dot v_\varepsilon(\tau)\|^2\big]\mathrm{d}\tau. \qquad (4.30)
\end{aligned}$$

Moreover

$$\int_0^t A_\eta(\tau)\mathrm{d}\tau = \int_t^{t+\eta} (\mathbb{A}ev_\varepsilon(\tau), ev_\varepsilon(\tau))\mathrm{d}\tau - \int_0^\eta (\mathbb{A}ev_\varepsilon(\tau), ev_\varepsilon(\tau))\mathrm{d}\tau, \qquad (4.31)$$

$$\begin{aligned}
\int_0^t B_\eta(\tau)\mathrm{d}\tau &= \int_t^{t+\eta} (\mathbb{B}(ev_\varepsilon(\tau) - w_\varepsilon(\tau)), ev_\varepsilon(\tau) - w_\varepsilon(\tau))\mathrm{d}\tau\\
&\quad - \int_0^\eta (\mathbb{B}(ev_\varepsilon(\tau) - w_\varepsilon(\tau)), ev_\varepsilon(\tau) - w_\varepsilon(\tau))\mathrm{d}\tau, \qquad (4.32)
\end{aligned}$$

$$\int_0^t D_\eta(\tau)\mathrm{d}\tau = \beta \int_0^t \int_\tau^{\tau+\eta} (\mathbb{B}\,\sigma^\eta \dot w_\varepsilon(\tau), \dot w_\varepsilon(s))\mathrm{d}s\mathrm{d}\tau, \qquad (4.33)$$

$$\begin{aligned}
\int_0^t W_\eta(\tau)\mathrm{d}\tau &= \int_0^t \int_\tau^{\tau+\eta} (\sigma^\eta h_\varepsilon(\tau), \dot v_\varepsilon(s))\mathrm{d}s\mathrm{d}\tau - \int_\eta^t \int_{\tau-\eta}^{\tau+\eta} \langle \dot\ell_\varepsilon(s), v_\varepsilon(\tau)\rangle \mathrm{d}s\mathrm{d}\tau\\
&\quad + \int_{t-\eta}^t \langle \sigma^\eta \ell_\varepsilon(\tau), v_\varepsilon(\tau+\eta)\rangle\mathrm{d}\tau - \int_0^\eta \langle \sigma^\eta \ell_\varepsilon(\tau), v_\varepsilon(\tau)\rangle\mathrm{d}\tau. \qquad (4.34)
\end{aligned}$$

We now divide by η all terms of (4.30)–(4.34). Observing that

$$\begin{aligned}
&\sigma^\eta h_\varepsilon \xrightarrow[\eta\to 0^+]{L^2(0,T;H)} 2h_\varepsilon,\\
&\int_0^t \Big\| \fint_\tau^{\tau+\eta} \dot v_\varepsilon(s)\mathrm{d}s - \dot v_\varepsilon(\tau)\Big\|^2 \mathrm{d}\tau \xrightarrow[\eta\to 0^+]{} 0,\\
&\int_\eta^t \Big\| \fint_{\tau-\eta}^{\tau+\eta} \dot\ell_\varepsilon(s)\mathrm{d}s - \dot\ell_\varepsilon(\tau)\Big\|^2_{V_0'} \mathrm{d}\tau \xrightarrow[\eta\to 0^+]{} 0,
\end{aligned}$$

thanks to (4.19) in $[0,2T)$ and (4.26), we can pass to the limit as $\eta\to 0^+$, and from (4.29) we obtain that (4.24) is satisfied for every $t\in[0,T]$. □

Proof of Proposition 4.1.6 (Continuation). Now we want to prove (4.16). By using (4.24), for every $t\in[0,T]$ we can write

$$\begin{aligned}
\frac{\varepsilon^2}{2}\|\dot v_\varepsilon(t)\|^2 + \frac12((\mathbb{A}+\mathbb{B})ev_\varepsilon(t), ev_\varepsilon(t)) &= \frac{\varepsilon^2}{2}\|v^1_\varepsilon\|^2 + \frac12((\mathbb{A}+\mathbb{B})ev^0_\varepsilon, ev^0_\varepsilon) + \mathscr{W}_\varepsilon(t)\\
&\quad - \frac12(\mathbb{B}w_\varepsilon(t), w_\varepsilon(t)) + (\mathbb{B}ev_\varepsilon(t), w_\varepsilon(t)) - \beta\varepsilon\int_0^t (\mathbb{B}\dot w_\varepsilon(\tau), \dot w_\varepsilon(\tau))\mathrm{d}\tau. \qquad (4.35)
\end{aligned}$$

Let $\Psi_\varepsilon\colon [0,T]\to[0,+\infty)$ be defined by

$$\Psi_\varepsilon(t) := \frac{\varepsilon^2}{2}\|\dot v_\varepsilon(t)\|^2 + \frac12((\mathbb{A}+\mathbb{B})ev_\varepsilon(t), ev_\varepsilon(t));$$

since $w_\varepsilon \in C^0([0,T];H)$, thanks to (4.18) and (4.35) we have $\Psi_\varepsilon \in C^0([0,T])$.

Now we fix $t \in [0,T]$. Given a sequence $\{t_k\}_k \subset [0,T]$ such that $t_k \to t$ as $k \to +\infty$, we define

$$\mathscr{E}_k := \frac{\varepsilon^2}{2}\|\dot v_\varepsilon(t_k) - \dot v_\varepsilon(t)\|^2 + \frac{1}{2}((\mathbb{A}+\mathbb{B})(ev_\varepsilon(t_k) - ev_\varepsilon(t)), ev_\varepsilon(t_k) - ev_\varepsilon(t)).$$

By elementary computations we have

$$\mathscr{E}_k = \Psi_\varepsilon(t_k) + \Psi_\varepsilon(t) - \varepsilon^2(\dot v_\varepsilon(t_k), \dot v_\varepsilon(t)) - ((\mathbb{A}+\mathbb{B})ev_\varepsilon(t_k), ev_\varepsilon(t)),$$

therefore, by (4.2) and (4.5) there exists a positive constant $C = C(\mathbb{A},\mathbb{B},\Omega)$ such that

$$\begin{aligned}&\varepsilon^2\|\dot v_\varepsilon(t_k) - \dot v_\varepsilon(t)\|^2 + \|v_\varepsilon(t_k) - v_\varepsilon(t)\|_V^2\\ &\qquad\le C\Big(\Psi_\varepsilon(t_k) + \Psi_\varepsilon(t) - \varepsilon^2(\dot v_\varepsilon(t_k), \dot v_\varepsilon(t)) - ((\mathbb{A}+\mathbb{B})ev_\varepsilon(t_k), ev_\varepsilon(t))\Big).\end{aligned}$$

The right-hand side of the previous inequality tends to 0 as $k \to +\infty$ because of (4.18) and the continuity of Ψ_ε. Since $z_\varepsilon \in C^0([0,T];V)$, by (4.6), and $u_\varepsilon = v_\varepsilon + z_\varepsilon$, we obtain (4.16). □

4.2 Statement of the main results

In this section we present the main results about the convergence, as $\varepsilon \to 0^+$, of the solutions u_ε. We assume the following hypotheses on the dependence on $\varepsilon > 0$ of our data:

(H1) $\{f_\varepsilon\}_\varepsilon \subset L^2(0,T;H)$, $f \in L^2(0,T;H)$, $\{g_\varepsilon\}_\varepsilon \subset H^1(0,T;V_0')$, $g \in W^{1,1}(0,T;V_0')$,

$$f_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^2(0,T;H)} f, \quad \text{and} \quad g_\varepsilon \xrightarrow[\varepsilon\to 0^+]{W^{1,1}(0,T;V_0')} g;$$

(H2) $\{z_\varepsilon\}_\varepsilon \subset H^2(0,T;H) \cap H^1(0,T;V)$, $z \in W^{2,1}(0,T;H) \cap W^{1,1}(0,T;V)$, and

$$z_\varepsilon \xrightarrow[\varepsilon\to 0^+]{W^{2,1}(0,T;H)\cap W^{1,1}(0,T;V)} z;$$

(H3) $\{u_{\varepsilon,in}\}_\varepsilon \subset C^0((-\infty,0];V) \cap C^1((-\infty,0];H)$, $u_{0,in} \in C^0((-\infty,0];V)$, and there exist $a > 0$ such that

$$u_{\varepsilon,in} \xrightarrow[\varepsilon\to 0^+]{C^0([-a,0];V)} u_{0,in}, \qquad \varepsilon\dot u_{\varepsilon,in} \xrightarrow[\varepsilon\to 0^+]{C^0([-a,0];H)} 0,$$

$$\int_{-\infty}^{-a} \frac{1}{\beta\varepsilon} e^{\frac{\tau}{\beta\varepsilon}} \|u_{\varepsilon,in}(\tau)\|_V \,\mathrm{d}\tau \xrightarrow[\varepsilon\to 0^+]{} 0, \qquad \int_{-\infty}^{-a} \frac{1}{\beta\varepsilon} e^{\frac{\tau}{\beta\varepsilon}} \|u_{0,in}(\tau)\|_V \,\mathrm{d}\tau \xrightarrow[\varepsilon\to 0^+]{} 0.$$

Remark 4.2.1. Let $u_\varepsilon^0 = u_{\varepsilon,in}(0)$, $u_\varepsilon^1 = \dot u_{\varepsilon,in}(0)$, and $u^0 = u_{0,in}(0)$. Hypothesis (H3) implies

$$u_\varepsilon^0 \xrightarrow[\varepsilon\to 0^+]{V} u^0 \quad \text{and} \quad \varepsilon u_\varepsilon^1 \xrightarrow[\varepsilon\to 0^+]{H} 0.$$

Our purpose is to show that the solutions u_ε converge, as $\varepsilon \to 0^+$, to the solution u_0 of the stationary problem (22) with boundary condition (12). The notion of solution to this problem is the usual one:

$$\begin{cases} u_0(t) \in V, \quad u_0(t) - z(t) \in V_0, & \text{for a.e. } t \in [0,T],\\ -\operatorname{div}(\mathbb{A}eu_0(t)) = f(t) + g(t) & \text{for a.e. } t \in [0,T]. \end{cases} \tag{4.36}$$

Remark 4.2.2. The existence and uniqueness of a solution u_0 to (4.36) follows easily from the Lax-Milgram Lemma. Since $f + g \in L^2(0,T;V_0')$, the estimate for the solution implies also $u_0 \in L^2(0,T;V)$.

We shall sometimes use the corresponding problem with homogeneous boundary conditions:

$$\begin{cases} v_0(t) \in V_0 & \text{for a.e. } t \in [0,T], \\ -\operatorname{div}(\mathbb{A}ev_0(t)) = h(t) + \ell(t) & \text{for a.e. } t \in [0,T], \end{cases} \tag{4.37}$$

with $h \in L^2(0,T;H)$ and $\ell \in H^1(0,T;V_0')$.

Remark 4.2.3. The function u_0 is a solution to (4.36) if and only if $v_0 = u_0 - z$ is a solution to (4.37) with

$$h(t) = f(t) \quad \text{and} \quad \ell(t) = g(t) + \operatorname{div}(\mathbb{A}ez(t)).$$

The following lemma will be used to prove the regularity with respect to time of the solution to (4.36).

Lemma 4.2.4. *Let $m \in \mathbb{N}$ and $p \in [1,+\infty)$. If $f = 0$, $g \in W^{m,p}(0,T;V_0')$, and $z \in W^{m,p}(0,T;V)$, then the solution u_0 to problem (4.36) satisfies $u_0 \in W^{m,p}(0,T;V)$.*

Proof. By Remark 4.2.3 it is enough to consider the case $z = 0$. Let $R : V_0' \to V_0$ be the resolvent operator defined as follows:

$$R(\psi) = \varphi \iff \begin{cases} \varphi \in V_0, \\ -\operatorname{div}(\mathbb{A}e\varphi) = \psi. \end{cases}$$

Since $u_0(t) = R(g(t))$, the conclusion follows from the continuity of the linear operator R. □

Remark 4.2.5. In the case $f = 0$, since $g \in W^{1,1}(0,T;V_0')$ and $z \in W^{1,1}(0,T;V)$, we can apply Lemma 4.2.4 to obtain that the solution u_0 to (4.36) belongs to $W^{1,1}(0,T;V)$, hence $u_0 \in C^0([0,T];V)$.

In the final statement of the next theorem, besides (H1)–(H3) we assume $f_\varepsilon = 0$ and the following compatibility condition: there exists an extension of g (still denoted by g) such that

$$g \in W^{1,1}(-a,T;V_0') \quad \text{and} \quad -\operatorname{div}(\mathbb{A}eu_{in}(t)) = g(t) \quad \text{for } t \in [-a,0]. \tag{4.38}$$

The meaning of (4.38) is that $u_{in}(t)$ is in equilibrium with external loads for $t \in [-a,0]$. This condition must be required if we want to obtain uniform convergence of u_ε to u_0 also near $t = 0$.

We are now in position to state the main results of this chapter.

Theorem 4.2.6. *Let us assume (H1)–(H3). Let u_ε be the solution to the viscoelastic dynamic system (4.8) and let u_0 be the solution to the stationary problem (4.36). Then*

$$u_\varepsilon \xrightarrow[\varepsilon\to0^+]{L^2(0,T;V)} u_0, \tag{4.39}$$

$$\varepsilon\dot{u}_\varepsilon \xrightarrow[\varepsilon\to0^+]{L^2(0,T;H)} 0. \tag{4.40}$$

If, in addition, $f_\varepsilon = 0$ for every $\varepsilon > 0$, then

$$u_\varepsilon \xrightarrow[\varepsilon\to0^+]{L^\infty(\eta,T;V)} u_0 \quad \text{and} \quad \varepsilon\dot{u}_\varepsilon \xrightarrow[\varepsilon\to0^+]{L^\infty(\eta,T;H)} 0 \qquad \text{for every } \eta \in (0,T). \tag{4.41}$$

If $f_\varepsilon = 0$ for every $\varepsilon > 0$ and the compatibility condition (4.38) holds, then we have also

$$u_\varepsilon \xrightarrow[\varepsilon\to0^+]{L^\infty(0,T;V)} u_0 \quad \text{and} \quad \varepsilon\dot{u}_\varepsilon \xrightarrow[\varepsilon\to0^+]{L^\infty(0,T;H)} 0. \tag{4.42}$$

In the case of solutions to problems (4.9) we have the following results, assuming that

$$u_\varepsilon^0 \xrightarrow[\varepsilon\to 0^+]{V} u^0 \quad \text{and} \quad \varepsilon u_\varepsilon^1 \xrightarrow[\varepsilon\to 0^+]{H} 0. \tag{4.43}$$

Theorem 4.2.7. *Let us assume (H1), (H2), and (4.43). Let u_ε be the solution to the viscoelastic dynamic system (4.9), with $\varphi_\varepsilon = f_\varepsilon$ and $\gamma_\varepsilon = g_\varepsilon$, and let u_0 be the solution to the stationary problem (4.36). Then (4.39) and (4.40) hold. Moreover, if $f_\varepsilon = 0$ for every $\varepsilon > 0$, then (4.41) holds.*

Theorems 4.2.6 and 4.2.7 will be proved in several steps. First, we prove (4.42) when $f_\varepsilon = 0$ and the compatibility condition (4.38) holds (Theorem 4.3.1). For $g \in H^2(0,T;V_0')$ the proof is based on the estimate in Lemma 4.2.8 below, which is derived from the energy-dissipation balance (4.24). The general case is obtained by an approximation argument based on the same estimate.

Next, we prove that (4.39) holds for the solution to (4.9) if $\gamma_\varepsilon = \gamma = 0$, $z_\varepsilon = 0$, $u_\varepsilon^0 = 0$, and $u_\varepsilon^1 = 0$ (Proposition 4.5.1). The proof is obtained by means of a careful estimate of the solutions to the elliptic system (4.80) obtained from (4.12) via Laplace Transform (Section 4.4). Under the general assumptions (H1), (H2), and (4.43) the same result is deduced from the previous one by an approximation argument based again on Lemma 4.2.8 below.

Then, (4.40) is obtained from (4.39) using a suitable test function in (4.9) (Theorem 4.5.3). A further approximation argument gives (4.39) and (4.40) under the assumptions (H1), (H2), and (H3) (Theorem 4.5.4).

Finally, if $f_\varepsilon = 0$, we obtain (4.41) from (4.39) and (4.40) (Lemma 4.6.1), concluding the proof of Theorems 4.2.6 and 4.2.7.

The following lemma, derived from the energy-dissipation balance (4.24), will be frequently used to approximate the solution to (4.12) by means of solutions corresponding to more regular data.

Lemma 4.2.8. *Given $\varepsilon > 0$, $\varphi_\varepsilon \in L^2(0,T;H)$, $\ell_\varepsilon \in H^1(0,T;V_0')$, $v_\varepsilon^0 \in V_0$, and $v_\varepsilon^1 \in H$, let v_ε be the solution to (4.12) with $h_\varepsilon = \varepsilon\varphi_\varepsilon$. Then there exists*

$$\begin{aligned}\varepsilon^2\|\dot v_\varepsilon\|^2_{L^\infty(0,T;H)} &+ \|v_\varepsilon\|^2_{L^\infty(0,T;V)} \\ &\le C_E\Big(\varepsilon^2\|v_\varepsilon^1\|^2 + \|v_\varepsilon^0\|_V^2 + \|\varphi_\varepsilon\|^2_{L^1(0,T;H)} + \|\ell_\varepsilon\|^2_{W^{1,1}(0,T;V_0')}\Big).\end{aligned} \tag{4.44}$$

a positive constant $C_E = C_E(\mathbb{A},\mathbb{B},\Omega,T)$, independent of ε, such that

Proof. By the energy-dissipation balance (4.24) proved in Proposition 4.1.7 and by (4.2) and (4.5) there exists a positive constant $C = C(\mathbb{A},\mathbb{B},\Omega)$ such that

$$\varepsilon^2\|\dot v_\varepsilon(t)\|^2 + \|v_\varepsilon(t)\|_V^2 \le C\Big(\varepsilon^2\|v_\varepsilon^1\|^2 + \|v_\varepsilon^0\|_V^2 + \mathscr{W}_\varepsilon(t)\Big) \quad \text{for every } t\in[0,T], \tag{4.45}$$

where the work is now defined by

$$\mathscr{W}_\varepsilon(t) = \langle \ell_\varepsilon(t), v_\varepsilon(t)\rangle - \langle \ell_\varepsilon(0), v_\varepsilon^0\rangle - \int_0^t \langle \dot\ell_\varepsilon(\tau), v_\varepsilon(\tau)\rangle \mathrm{d}\tau + \int_0^t (\varphi_\varepsilon(\tau), \varepsilon\dot v_\varepsilon(\tau))\mathrm{d}\tau. \tag{4.46}$$

Let $K_\varepsilon := \varepsilon\|\dot v_\varepsilon(t)\|_{L^\infty(0,T;H)}$ and $E_\varepsilon := \|v_\varepsilon(t)\|_{L^\infty(0,T;V)}$, which are finite by (4.16). Thanks to (4.45) and (4.46) for every $t\in[0,T]$ we get

$$\begin{aligned}\varepsilon^2\|\dot v_\varepsilon(t)\|^2 &+ \|v_\varepsilon(t)\|_V^2 \\ &\le C\Big(\varepsilon^2\|v_\varepsilon^1\|^2 + \|v_\varepsilon^0\|_V^2 + \big(3+\tfrac{2}{T}\big)\|\ell_\varepsilon\|_{W^{1,1}(0,T;V_0')}E_\varepsilon + \|\varphi_\varepsilon\|_{L^1(0,T;H)}K_\varepsilon\Big).\end{aligned}$$

By passing to the supremum with respect to t and using the Young Inequality we can find a positive constant $C_E = C_E(\mathbb{A}, \mathbb{B}, \Omega, T)$ such that

$$K_\varepsilon^2 + E_\varepsilon^2 \le C_E\Big(\varepsilon^2\|v_\varepsilon^1\|^2 + \|v_\varepsilon^0\|_V^2 + \|\varphi_\varepsilon\|^2_{L^1(0,T;H)} + \|\ell_\varepsilon\|^2_{W^{1,1}(0,T;V_0')}\Big),$$

which concludes the proof. □

In the proof of Theorem 4.2.6 we shall use the following lemma, which ensure that it is enough to consider the case $z_\varepsilon = 0$ and $z = 0$.

Lemma 4.2.9. *If Theorem 4.2.6 holds when $z_\varepsilon = 0$ for every $\varepsilon > 0$, then it holds for arbitrary $\{z_\varepsilon\}_\varepsilon$ and z satisfying (H2).*

Proof. It is not restrictive to assume $\operatorname{div}(\mathbb{B}ez_\varepsilon(0)) = \operatorname{div}(\mathbb{B}ez(0)) = 0$. Indeed, if this is not the case, we can consider the solutions z_ε^0 and z^0 to the stationary problems

$$\begin{cases} z_\varepsilon^0 \in V_0, \\ -\operatorname{div}(\mathbb{B}ez_\varepsilon^0) = \operatorname{div}(\mathbb{B}ez_\varepsilon(0)), \end{cases} \quad \text{and} \quad \begin{cases} z^0 \in V_0, \\ -\operatorname{div}(\mathbb{B}ez^0) = \operatorname{div}(\mathbb{B}ez(0)), \end{cases}$$

and we can replace $z_\varepsilon(t)$ and $z(t)$ by $\tilde z_\varepsilon(t) := z_\varepsilon(t) + z_\varepsilon^0$ and $\tilde z(t) := z(t) + z^0$. It is clear that $\operatorname{div}(\mathbb{B}e\tilde z_\varepsilon(0)) = \operatorname{div}(\mathbb{B}e\tilde z(0)) = 0$ and that problems (4.8) and (4.36) do not change passing from z_ε and z to $\tilde z_\varepsilon$ and $\tilde z$, respectively.

Let $\psi_\varepsilon, \psi\colon [0,T] \to V_0'$ be the functions defined by

$$\psi_\varepsilon(t) := \begin{cases} 0 & \text{if } t \in (-\infty, 0), \\ \operatorname{div}(\mathbb{B}ez_\varepsilon(t)) & \text{if } t \in [0,T], \\ \operatorname{div}(\mathbb{B}ez_\varepsilon(T)) & \text{if } t \in (T, +\infty), \end{cases} \qquad \psi(t) := \begin{cases} 0 & \text{if } t \in (-\infty, 0), \\ \operatorname{div}(\mathbb{B}ez(t)) & \text{if } t \in [0,T], \\ \operatorname{div}(\mathbb{B}ez(T)) & \text{if } t \in (T, +\infty). \end{cases} \tag{4.47}$$

Since $\operatorname{div}(\mathbb{B}ez_\varepsilon(0)) = \operatorname{div}(\mathbb{B}ez(0)) = 0$, $z_\varepsilon \in H^1(0,T;V)$, and $z \in W^{1,1}(0,T;V)$, we have $\psi_\varepsilon \in H^1_{loc}(\mathbb{R};V_0')$ and $\psi \in W^{1,1}_{loc}(\mathbb{R};V_0')$. Moreover, thanks to (H2) we have

$$\psi_\varepsilon \xrightarrow[\varepsilon\to 0^+]{W^{1,1}_{loc}(\mathbb{R};V_0')} \psi. \tag{4.48}$$

Since u_ε is the solution to (4.8), by Remark 4.1.3 it solves (4.9) with $\gamma_\varepsilon = g_\varepsilon - p_\varepsilon$ and initial conditions defined by (4.10), where p_ε is defined by (4.11). By Remark 4.1.4 the function $v_\varepsilon = u_\varepsilon - z_\varepsilon$ is the solution to (4.12) with

$$\begin{aligned} h_\varepsilon(t) &= f_\varepsilon(t) - \varepsilon^2 \ddot z_\varepsilon(t), \\ \ell_\varepsilon(t) &= g_\varepsilon(t) - p_\varepsilon(t) + \operatorname{div}((\mathbb{A}+\mathbb{B})ez_\varepsilon(t)) - \int_0^t \frac{1}{\beta\varepsilon} \mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}} \operatorname{div}(\mathbb{B}ez_\varepsilon(\tau))\mathrm{d}\tau, \end{aligned} \tag{4.49}$$

and initial conditions v_ε^0 and v_ε^1 defined by (4.14). We define the family of convolution kernels $\{\rho_\varepsilon\}_\varepsilon \subset L^1(\mathbb{R})$ by

$$\rho_\varepsilon(t) := \begin{cases} \frac{1}{\beta\varepsilon}\mathrm{e}^{-\frac{t}{\beta\varepsilon}} & \text{if } t \in [0, +\infty), \\ 0 & \text{if } t \in (-\infty, 0), \end{cases} \tag{4.50}$$

and notice that, by (4.47), the integral in (4.49) coincides with $(\rho_\varepsilon * \psi_\varepsilon)(t)$, hence

$$\ell_\varepsilon(t) = g_\varepsilon(t) - p_\varepsilon(t) + \operatorname{div}(\mathbb{A}ez_\varepsilon(t)) + \psi_\varepsilon(t) - (\rho_\varepsilon * \psi_\varepsilon)(t) \quad \text{for every } t \in [0,T].$$

By Remark 4.2.3 the function $v_0 = u_0 - z$ is the solution to (4.37) with $h = f$ and $\ell = g + \operatorname{div}(\mathbb{A}ez)$. By the definition of v_ε and v_0 it is clear that to prove the theorem it is enough to show that the conclusions of Theorem 4.2.6 holds for v_ε and v_0. To this aim, we introduce the solution $\tilde{v}_\varepsilon$ to (4.12) with $h_\varepsilon = f_\varepsilon$, $\ell_\varepsilon = g_\varepsilon - p_\varepsilon + \operatorname{div}(\mathbb{A}ez_\varepsilon)$, and v^0_ε, v^1_ε defined by (4.14). Then the function $\bar{v}_\varepsilon := v_\varepsilon - \tilde{v}_\varepsilon$ satisfies (4.12) with $h_\varepsilon = -\varepsilon^2\ddot{z}_\varepsilon$, $\ell_\varepsilon = \psi_\varepsilon - \rho_\varepsilon * \psi_\varepsilon$, and homogeneous initial conditions. By Lemma 4.2.8 we can write

$$\varepsilon^2\|\dot{\bar{v}}_\varepsilon\|^2_{L^\infty(0,T;H)} + \|\bar{v}_\varepsilon\|^2_{L^\infty(0,T;V)} \le C_E\Big(\varepsilon^2\|\ddot{z}_\varepsilon\|^2_{L^1(0,T;H)} + \|\psi_\varepsilon - \rho_\varepsilon * \psi_\varepsilon\|^2_{W^{1,1}(0,T;V_0')}\Big). \quad (4.51)$$

By (4.48) and by classical results on convolutions we obtain

$$\psi_\varepsilon - \rho_\varepsilon * \psi_\varepsilon \xrightarrow[\varepsilon\to0^+]{W^{1,1}(0,T;V_0')} 0.$$

Since $\{\ddot{z}_\varepsilon\}_\varepsilon$ is bounded in $L^1(0,T;H)$ by (H2), from (4.51) we deduce

$$v_\varepsilon - \tilde{v}_\varepsilon \xrightarrow[\varepsilon\to0^+]{L^\infty(0,T;V)} 0 \quad \text{and} \quad \varepsilon(\dot{v}_\varepsilon - \dot{\tilde{v}}_\varepsilon) \xrightarrow[\varepsilon\to0^+]{L^\infty(0,T;H)} 0. \quad (4.52)$$

By Remark 4.1.3 the function $\tilde{v}_\varepsilon$ is the solution to (4.8) with g_ε replaced by $g_\varepsilon + \operatorname{div}(\mathbb{A}ez_\varepsilon)$ and $z_\varepsilon = 0$. Thanks to (H1) and (H2) we have

$$g_\varepsilon + \operatorname{div}(\mathbb{A}ez_\varepsilon) \xrightarrow[\varepsilon\to0^+]{W^{1,1}(0,T;V_0')} g + \operatorname{div}(\mathbb{A}ez).$$

Since by hypothesis, Theorem 4.2.6 holds in the case of homogeneous boundary condition, its conclusions are valid for $\tilde{v}_\varepsilon$ and v_0. Thanks to (4.52) the same results hold for v_ε and v_0. This concludes the proof. □

In a similar way we can prove the following result.

Lemma 4.2.10. *If Theorem 4.2.7 holds when $z_\varepsilon = 0$ for every $\varepsilon > 0$, then it holds for arbitrary $\{z_\varepsilon\}_\varepsilon$ and z satisfying (H2).*

4.3 The uniform convergence

In this section we shall prove (4.42) of Theorem 4.2.6 under the compatibility condition (4.38).

Theorem 4.3.1. *Let us assume (H1)–(H3), the compatibility condition (4.38), and $f_\varepsilon = 0$ for every $\varepsilon > 0$. Let u_ε be the solution to the viscoelastic dynamic system (4.8) and let u_0 be the solution to the stationary problem (4.36), with $f = 0$. Then (4.42) holds.*

To prove the theorem we need the following lemma, which gives the result when g is more regular.

Lemma 4.3.2. *Under the assumptions of Theorem 4.3.1, if $g \in H^2(0,T;V_0')$, then (4.42) holds.*

Proof. Thanks to Lemma 4.2.9 we can suppose $z = 0$ and $z_\varepsilon = 0$ for every $\varepsilon > 0$. Let p_ε be defined by (4.10). Since u_ε is the solution to (4.8), thanks to Remark 4.1.3 it solves (4.12) with $h_\varepsilon = 0$, $\ell_\varepsilon = g_\varepsilon - p_\varepsilon$, $v^0_\varepsilon = u_{\varepsilon,in}(0)$, and $v^1_\varepsilon = \dot{u}_{\varepsilon,in}(0)$. We fix $b > a > 0$ and we extend the function g in (4.38) to $(-\infty,T)$ in such a way $g \in W^{1,1}(-\infty,T;V_0')$ and $g(t) = 0$ for every $t \in (-\infty,-b]$. Since $z = 0$ we can extend u_0 by solving the following problem:

$$\begin{cases} u_0(t) \in V_0 & \text{for every } t \in (-\infty,T], \\ -\operatorname{div}(\mathbb{A}eu_0(t)) = g(t) & \text{for every } t \in (-\infty,T]. \end{cases}$$

We observe that $u_0 = 0$ on $(-\infty,-b]$ and $u_0 = u_{0,in}$ on $[-a,0]$ by the compatibility condition (4.38).

Assume $g \in H^2(0,T;V_0')$. By Lemma 4.2.4 (with $z = 0$) we have $u_0 \in H^2(0,T;V)$, hence by (4.36) we get

$$\varepsilon^2\ddot{u}_0(t) - \operatorname{div}((\mathbb{A}+\mathbb{B})eu_0(t)) + \int_0^t \frac{1}{\beta\varepsilon}\mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}}\operatorname{div}(\mathbb{B}eu_0(\tau))\mathrm{d}\tau$$
$$= \varepsilon^2\ddot{u}_0(t) + g(t) - \operatorname{div}(\mathbb{B}eu_0(t)) + (\rho_\varepsilon * \operatorname{div}(\mathbb{B}eu_0))(t) - \tilde{p}_\varepsilon(t) \quad \text{for a.e. } t \in [0,T], \tag{4.53}$$

where ρ_ε is defined by (4.50) and

$$\tilde{p}_\varepsilon(t) := \mathrm{e}^{-\frac{t}{\beta\varepsilon}}\tilde{g}_\varepsilon^0 \quad \text{with} \quad \tilde{g}_\varepsilon^0 := \int_{-\infty}^0 \frac{1}{\beta\varepsilon}\mathrm{e}^{\frac{\tau}{\beta\varepsilon}}\operatorname{div}(\mathbb{B}eu_0(\tau))\mathrm{d}\tau = \int_{-b}^0 \frac{1}{\beta\varepsilon}\mathrm{e}^{\frac{\tau}{\beta\varepsilon}}\operatorname{div}(\mathbb{B}eu_0(\tau))\mathrm{d}\tau.$$

Let $q_\varepsilon := g_\varepsilon - g + \operatorname{div}(\mathbb{B}eu_0) - (\rho_\varepsilon * \operatorname{div}(\mathbb{B}eu_0)) - p_\varepsilon + \tilde{p}_\varepsilon$. By (4.53) the function $\bar{u}_\varepsilon := u_\varepsilon - u_0$ satisfies (4.12) with $h_\varepsilon = -\varepsilon^2\ddot{u}_0$, $\ell_\varepsilon = q_\varepsilon$, $v_\varepsilon^0 = u_{\varepsilon,in}(0) - u_0(0)$, and $v_\varepsilon^1 = \dot{u}_{\varepsilon,in}(0) - \dot{u}_0(0)$.

Since $g \in W^{1,1}(-\infty,T;V_0')$ and $g = 0$ on $(-\infty,-b]$, thanks to Lemma 4.2.4 we obtain $u_0 \in W^{1,1}(-\infty,T;V)$ and therefore $\operatorname{div}(\mathbb{B}eu_0) \in W^{1,1}(-\infty,T;V_0')$. Then the properties of convolutions imply

$$\rho_\varepsilon * \operatorname{div}(\mathbb{B}eu_0) \xrightarrow[\varepsilon\to 0^+]{W^{1,1}(-\infty,T;V_0')} \operatorname{div}(\mathbb{B}eu_0). \tag{4.54}$$

As we have already observed, by the compatibility condition (4.38) we have $u_0 = u_{0,in}$ on $[-a,0]$, hence

$$\|\tilde{g}_\varepsilon^0 - g_\varepsilon^0\|_{V_0'} \le \int_{-\infty}^{-a} \frac{1}{\beta\varepsilon}\mathrm{e}^{\frac{\tau}{\beta\varepsilon}}\|\operatorname{div}(\mathbb{B}(eu_{\varepsilon,in}(\tau))\|_{V_0'}\mathrm{d}\tau + \int_{-b}^{-a} \frac{1}{\beta\varepsilon}\mathrm{e}^{\frac{\tau}{\beta\varepsilon}}\|\operatorname{div}(\mathbb{B}(eu_0(\tau))\|_{V_0'}\mathrm{d}\tau$$
$$+ \|\operatorname{div}(\mathbb{B}(eu_{\varepsilon,in} - eu_{0,in}))\|_{L^\infty(-a,0;V_0')}.$$

Thanks to (H3) we obtain $\tilde{g}_\varepsilon^0 - g_\varepsilon^0 \to 0$ strongly in V_0' as $\varepsilon \to 0^+$. Hence

$$\tilde{p}_\varepsilon - p_\varepsilon \xrightarrow[\varepsilon\to 0^+]{W^{1,1}(0,T;V_0')} 0. \tag{4.55}$$

By (H1), (4.54), and (4.55) we have

$$q_\varepsilon \xrightarrow[\varepsilon\to 0^+]{W^{1,1}(0,T;V_0')} 0. \tag{4.56}$$

Since $u_0(0) = u_{0,in}(0)$, (H3) gives

$$u_{\varepsilon,in}(0) - u_0(0) \xrightarrow[\varepsilon\to 0^+]{V} 0 \quad \text{and} \quad \varepsilon(\dot{u}_{\varepsilon,in}(0) - \dot{u}_0(0)) \xrightarrow[\varepsilon\to 0^+]{H} 0. \tag{4.57}$$

By using Lemma 4.2.8 we get

$$\varepsilon^2\|\dot{\bar{u}}_\varepsilon\|^2_{L^\infty(0,T;H)} + \|\bar{u}_\varepsilon\|^2_{L^\infty(0,T;V)}$$
$$\le C_E\Big(\varepsilon^2\|\dot{u}_{\varepsilon,in}(0) - \dot{u}_0(0)\|^2 + \|u_{\varepsilon,in}(0) - u_0(0)\|_V^2 + \varepsilon^2\|\ddot{u}_0\|^2_{L^1(0,T;H)} + \|q_\varepsilon\|^2_{W^{1,1}(0,T;V_0')}\Big),$$

therefore thanks to (4.56) and (4.57) we obtain the conclusion. □

In the proof of Theorems 4.3.1, 4.5.2, and 4.5.4 we shall use the following density result.

Lemma 4.3.3. *Let X, Y be two Hilbert spaces such that $X \hookrightarrow Y$ continuously, with X dense in Y. Then for every $m, n \in \mathbb{N}$ with $m \le n$, and $p \in [1,2]$ the space $H^n(0,T;X)$ is dense in $W^{m,p}(0,T;Y)$.*

Proof. Since every simple function with values in Y can be approximated by simple functions with values in X, it is easy to see that $L^2(0,T;X)$ is dense in $L^p(0,T;Y)$.

To prove the result for $m=1$ we fix $u \in W^{1,p}(0,T;Y)$. By the density of $L^2(0,T;X)$ in $L^p(0,T;Y)$ we can find a sequence $\{\psi_k\}_k \subset L^2(0,T;X)$ such that $\psi_k \to \dot{u}$ strongly in $L^p(0,T;Y)$ as $k \to +\infty$. By the density of X in Y there exists $\{u_k^0\}_k \subset X$ such that $u_k^0 \to u(0)$ strongly in Y as $k \to +\infty$. Now we define

$$u_k(t) := \int_0^t \psi_k(\tau)\mathrm{d}\tau + u_k^0.$$

It is easy to see that $\{u_k\}_k \subset H^1(0,T;X)$ and $u_k \to u$ strongly in $W^{1,p}(0,T;Y)$ as $k \to +\infty$.

Arguing by induction we can prove that for every integer $m \geq 0$ the space $H^m(0,T;X)$ is dense in $W^{m,p}(0,T;Y)$. Since $H^n(0,T;X)$ is dense in $H^m(0,T;X)$, the conclusion follows. □

We are now in position to deduce Theorem 4.3.1 from Lemma 4.3.2 by means of an approximation argument.

Proof of Theorem 4.3.1. Thanks to Lemma 4.2.9 we can suppose $z=0$ and $z_\varepsilon = 0$ for every $\varepsilon > 0$. We fix $\delta > 0$. By Lemma 4.3.3 there exists a function $\psi \in H^2(0,T;V_0')$ such that

$$\|\psi - g\|_{W^{1,1}(0,T;V_0')} < \delta. \tag{4.58}$$

By (H1) there exists a positive number $\varepsilon_0 = \varepsilon_0(\delta)$ such that

$$\|\psi - g_\varepsilon\|_{W^{1,1}(0,T;V_0')} < \delta \qquad \text{for every } \varepsilon \in (0,\varepsilon_0). \tag{4.59}$$

Let p_ε be defined by (4.11). Since u_ε is the solution to (4.8) with $f_\varepsilon = 0$ and $z_\varepsilon = 0$, thanks to Remark 4.1.3 it solves (4.12) with $h_\varepsilon = 0$, $\ell_\varepsilon = g_\varepsilon - p_\varepsilon$, $v_\varepsilon^0 = u_{\varepsilon,in}(0)$, and $v_\varepsilon^1 = \dot{u}_{\varepsilon,in}(0)$. Moreover, let $\tilde{u}_\varepsilon$ be solution to (4.12) with $h_\varepsilon = 0$, $\ell_\varepsilon = \psi - p_\varepsilon$, $v_\varepsilon^0 = u_{\varepsilon,in}(0)$, and $v_\varepsilon^1 = \dot{u}_{\varepsilon,in}(0)$, and let $\tilde{u}_0$ be the solution to (4.37) with $h = 0$ and $\ell = \psi$. Thanks to Remark 4.1.3 the function $\tilde{u}_\varepsilon$ is the solution to (4.8) with $f_\varepsilon = 0$, $g_\varepsilon = \psi$, and $z_\varepsilon = 0$, hence by Lemma 4.3.2 we have

$$\tilde{u}_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^\infty(0,T;V)} \tilde{u}_0 \quad \text{and} \quad \varepsilon\dot{\tilde{u}}_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^\infty(0,T;H)} 0. \tag{4.60}$$

We now consider the functions $\bar{u}_0 := \tilde{u}_0 - u_0$ and $\bar{u}_\varepsilon := \tilde{u}_\varepsilon - u_\varepsilon$. Since $\bar{u}_0$ is the solution to (4.37), with $h = 0$ and $\ell = \psi - g$, by the Lax-Milgram Lemma we get

$$\|\bar{u}_0\|_{L^\infty(0,T;V)} \leq \tfrac{C_P^2+1}{c_{\mathbb{A}}}\|\psi - g\|_{L^\infty(0,T;V_0')} \leq \tfrac{C_P^2+1}{c_{\mathbb{A}}}(1+\tfrac{1}{T})\|\psi - g\|_{W^{1,1}(0,T;V_0')}. \tag{4.61}$$

Moreover, since $\bar{u}_\varepsilon$ is the solution to (4.12), with $h_\varepsilon = 0$, $\ell_\varepsilon = \psi - g_\varepsilon$, $v_\varepsilon^0 = 0$, and $v_\varepsilon^1 = 0$, thanks to Lemma 4.2.8 we get

$$\varepsilon^2\|\dot{\bar{u}}_\varepsilon\|^2_{L^\infty(0,T;H)} + \|\bar{u}_\varepsilon\|^2_{L^\infty(0,T;V)} \leq C_E\|\psi - g_\varepsilon\|^2_{W^{1,1}(0,T;V_0')}. \tag{4.62}$$

By using (4.58), (4.59), (4.61), and (4.62), we can find a positive constant $C = C(\mathbb{A},\mathbb{B},\Omega,T)$ such that

$$\varepsilon\|\dot{\bar{u}}_\varepsilon\|_{L^\infty(0,T;H)} + \|\bar{u}_\varepsilon\|_{L^\infty(0,T;V)} + \|\bar{u}_0\|_{L^\infty(0,T;V)} \leq C\delta \quad \text{for every } \varepsilon \in (0,\varepsilon_0). \tag{4.63}$$

Since

$$\|u_\varepsilon - u_0\|_{L^\infty(0,T;V)} \leq \|\bar{u}_\varepsilon\|_{L^\infty(0,T;V)} + \|\tilde{u}_\varepsilon - \tilde{u}_0\|_{L^\infty(0,T;V)} + \|\bar{u}_0\|_{L^\infty(0,T;V)},$$
$$\varepsilon\|\dot{u}_\varepsilon\|_{L^\infty(0,T;H)} \leq \varepsilon\|\dot{\bar{u}}_\varepsilon\|_{L^\infty(0,T;H)} + \varepsilon\|\dot{\tilde{u}}_\varepsilon\|_{L^\infty(0,T;H)},$$

by (4.60) and (4.63) we have

$$\limsup_{\varepsilon\to 0^+}\|u_\varepsilon - u_0\|_{L^\infty(0,T;V)} \leq C\delta \quad \text{and} \quad \limsup_{\varepsilon\to 0^+}\|\varepsilon\dot{u}_\varepsilon\|_{L^\infty(0,T;H)} \leq C\delta.$$

The conclusion follows from the arbitrariness of $\delta > 0$. □

4.4 Use of the Laplace Transform

In this section we shall use the Laplace Transform to prepare the proof of the convergence, as $\varepsilon \to 0^+$, of the solution to the problem

$$v_\varepsilon \in \mathcal{V}_0, \tag{4.64a}$$

$$\varepsilon^2 \ddot{v}_\varepsilon(t) - \operatorname{div}((\mathbb{A}+\mathbb{B})ev_\varepsilon(t)) + \int_0^t \frac{1}{\beta\varepsilon} \mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}} \operatorname{div}(\mathbb{B}ev_\varepsilon(\tau))\mathrm{d}\tau = h_\varepsilon(t) \quad \text{for a.e. } t \in [0,T], \tag{4.64b}$$

$$v_\varepsilon(0) = 0 \quad \text{in } H \quad \text{and} \quad \dot{v}_\varepsilon(0) = 0 \quad \text{in } V_0', \tag{4.64c}$$

to the solution v_0 to the problem

$$\begin{cases} v_0(t) \in V_0 & \text{for a.e. } t \in [0,T], \\ -\operatorname{div}(\mathbb{A}ev_0(t)) = h(t) & \text{for a.e. } t \in [0,T], \end{cases} \tag{4.65}$$

when $\{h_\varepsilon\}_\varepsilon \subset L^2(0,T;H)$, $h \in L^2(0,T;H)$, and

$$h_\varepsilon \xrightarrow[\varepsilon \to 0^+]{L^2(0,T;H)} h, \tag{4.66}$$

This partial result will be the starting point for the proof of the convergence in $L^2(0,T;V)$ under the general assumptions of Theorem 4.2.6.

4.4.1 The Laplace Transform for functions with values in Hilbert spaces

Given a complex Hilbert space X, let $r \in L^1_{loc}(0,+\infty;X)$ be a function such that

$$\int_0^{+\infty} \mathrm{e}^{-\alpha t}\|r(t)\|_X \,\mathrm{d}t < +\infty \quad \text{for every } \alpha > 0, \tag{4.67}$$

and let $\mathbb{C}_+ := \{s \in \mathbb{C} : \Re(s) > 0\}$. The Laplace Transform of r is the function $\hat{r} : \mathbb{C}_+ \to X$ defined by

$$\hat{r}(s) := \int_0^{+\infty} \mathrm{e}^{-st} r(t)\mathrm{d}t \quad \text{for every } s \in \mathbb{C}_+. \tag{4.68}$$

Besides $\hat{r}$, we shall also use the notation $\mathcal{L}(r)$, which is sometimes written as $\mathcal{L}_t(r(t))$, with dummy variable t. In the particular case $r \in L^\infty(0,+\infty;X)$ we have

$$\|\hat{r}(s)\|_X \leq \frac{1}{s_1}\|r\|_{L^\infty(0,+\infty;X)} \quad \text{for every } s = s_1 + is_2 \in \mathbb{C}_+, \text{ with } s_1, s_2 \in \mathbb{R}.$$

There is a close connection between the Laplace Transform and the Fourier Transform, defined for every $\rho \in L^1(\mathbb{R};X)$ as the function $\mathcal{F}(\rho) \in L^\infty(\mathbb{R};X)$ given by

$$\mathcal{F}(\rho)(\xi) = \int_{-\infty}^{+\infty} \mathrm{e}^{-i\xi t}\rho(t)\mathrm{d}t \quad \text{for every } \xi \in \mathbb{R}. \tag{4.69}$$

For $\mathcal{F}(\rho)$ we use also the notation $\mathcal{F}_t(\rho(t))$ with dummy variable t. For the main properties of the Fourier and Laplace Transforms of functions with values in Hilbert spaces we refer to [3].

We extend the function r satisfying (4.67) by setting $r(t) = 0$ for every $t < 0$. By (4.68) and (4.69) we have

$$\mathcal{L}_t(r(t))(s) = \mathcal{F}_t(\mathrm{e}^{-s_1 t}r(t))(s_2) \quad \text{for every } s = s_1 + is_2 \in \mathbb{C}_+, \text{ with } s_1, s_2 \in \mathbb{R}.$$

We remark that the Laplace Transform commutes with linear transformations, as shown in the following proposition (see, for instance [3, Proposition 1.6.2]).

Proposition 4.4.1. *Let X and Y be two complex Hilbert spaces, let $r \in L^1_{loc}(0,+\infty;X)$, and let T be a continuous linear operator from X to Y. Then $T \circ r \in L^1_{loc}(0,+\infty;Y)$. If in addition, r satisfies (4.67), then the same property holds also for $T \circ r$, with X replaced by Y, and $\mathcal{L}(T \circ r)(s) = (T \circ \hat{r})(s)$ for every $s \in \mathbb{C}_+$.*

Now we consider the Inverse Laplace Transform. Let $R : \mathbb{C}_+ \to X$ be a function. Suppose that there exists $r \in L^1_{loc}(0,+\infty;X)$ such that (4.67) holds and $\mathcal{L}(r)(s) = R(s)$ for every $s \in \mathbb{C}_+$. In this case we say that r is the Inverse Laplace Transform of R, and we use the notation $r = \mathcal{L}^{-1}(R)$ or $r = \mathcal{L}^{-1}_s(R(s))$ with dummy variable s. It can be proven that r is uniquely determined up to a negligible set (see [3, Theorem 1.7.3]). Moreover, r can be obtained by the Bromwich Integral Formula:

$$r(t) = \mathcal{L}^{-1}(R)(t) = \frac{\mathrm{e}^{s_1 t}}{2\pi} \lim_{k\to+\infty} \int_{-k}^{k} e^{is_2 t} R(s_1 + is_2)\mathrm{d}s_2, \tag{4.70}$$

where s_1 is an arbitrary positive number. Clearly (4.70) can be expressed in terms of the Inverse Fourier Transform, namely

$$r(t) = \mathcal{L}^{-1}_s(R(s))(t) = \mathrm{e}^{s_1 t}\mathcal{F}^{-1}_{s_2}(R(s_1 + is_2))(t), \tag{4.71}$$

where $\mathcal{F}^{-1}_{s_2}(R(s_1 + is_2))$ denotes the Inverse Fourier Transform with respect to the variable s_2.

To use the Laplace Transform, we extend our problems from the interval $[0,T]$ to $[0,+\infty)$. To do this, we extend the functions h_ε and h, introduced in (4.66), by setting them equal to zero in $(T,+\infty)$, and we consider the solution to (4.64) in $[0,+\infty)$, which we still denote v_ε. Moreover, we consider the solution to (4.65) in $[0,+\infty)$, which we still denote v_0. Notice that, thanks to the choice of the extension we have

$$h_\varepsilon \xrightarrow[\varepsilon\to0^+]{L^2(0,+\infty;H)} h.$$

By Proposition 4.2.8 and by using the equality $h_\varepsilon = 0$ on $(T,+\infty)$, we get

$$v_\varepsilon \in L^\infty(0,+\infty;V_0) \quad \text{and} \quad \dot{v}_\varepsilon \in L^\infty(0,+\infty;H). \tag{4.72}$$

Since $h \in L^2(0,T;H)$ and $h = 0$ on $(T,+\infty)$, by means of standard estimates for the solution to (4.65) we obtain

$$v_0 \in L^2(0,+\infty;V_0) \quad \text{and} \quad v_0 = 0 \quad \text{on } (T,+\infty). \tag{4.73}$$

From (4.3), (4.64), and (4.72) we can deduce

$$\ddot{v}_\varepsilon \in L^2(0,T;V_0') \cap L^\infty(T,+\infty;V_0'). \tag{4.74}$$

To study our problem by means of the Laplace Transform we introduce the complexification of the Hilbert spaces H, V_0, and V_0' defined by

$$\hat{H} := L^2(\Omega;\mathbb{C}^d), \quad \hat{V}_0 := H^1(\Omega;\mathbb{C}^d), \quad \hat{V}_0' := H^{-1}(\Omega;\mathbb{C}^d).$$

The symbols $(\cdot,\cdot)$ and $\|\cdot\|$ denote the hermitian product and the norm in $\hat{H}$ or in other complex L^2 spaces. For every $s \in \mathbb{C}_+$ the Laplace Transforms $\hat{h}_\varepsilon(s)$ and $\hat{h}(s)$ of h_ε and h in $\hat{H}$ are well defined. Thanks to (4.72) and (4.73) the Laplace Transforms $\hat{v}_\varepsilon(s)$ and $\hat{v}_0(s)$ in $\hat{V}_0$ are well defined for every $s \in \mathbb{C}_+$. By (4.74) the Laplace Transform $\hat{\ddot{v}}_\varepsilon$ of $\ddot{v}_\varepsilon$ is well defined for every $s \in \mathbb{C}_+$. Using (4.72) we can integrate by parts two times in the integral which defines $\hat{\ddot{v}}_\varepsilon$ and, since $v_\varepsilon(0) = 0$ and $\dot{v}_\varepsilon(0) = 0$, we obtain

$$\hat{\ddot{v}}_\varepsilon(s) = s^2\hat{v}_\varepsilon(s) \quad \text{for every } s \in \mathbb{C}_+. \tag{4.75}$$

By considering the operators $S_{\mathbb{A}}, S_{\mathbb{B}} : \hat{V}_0 \to \hat{V}_0'$ defined by

$$S_{\mathbb{A}}(\psi) := -\operatorname{div}(\mathbb{A}e\psi) \quad \text{and} \quad S_{\mathbb{B}}(\psi) := -\operatorname{div}(\mathbb{B}e\psi),$$

we can rephrase (4.64) and (4.65) as equalities of elements of $\hat{V}_0'$:

$$\varepsilon^2 \ddot{v}_\varepsilon(t) = S_{\mathbb{B}}\Big(\int_0^t \frac{1}{\beta\varepsilon} \mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}} v_\varepsilon(\tau)\mathrm{d}\tau\Big) - (S_{\mathbb{A}} + S_{\mathbb{B}})(v_\varepsilon(t)) + h_\varepsilon(t) \quad \text{for a.e. } t \in [0, +\infty), \tag{4.76}$$

$$S_{\mathbb{A}}(v_0(t)) = h(t) \quad \text{for a.e. } t \in [0, +\infty). \tag{4.77}$$

Now we want to consider the Laplace Transforms, in the sense of $\hat{V}_0'$, of both sides of these equations. By Proposition 4.4.1 we can say

$$\mathcal{L}(S_{\mathbb{A}}(v_\varepsilon)) = S_{\mathbb{A}}(\hat{v}_\varepsilon), \quad \mathcal{L}(S_{\mathbb{B}}(v_\varepsilon)) = S_{\mathbb{B}}(\hat{v}_\varepsilon), \quad \mathcal{L}(S_{\mathbb{A}}(v_0)) = S_{\mathbb{A}}(\hat{v}_0), \tag{4.78}$$

where $\hat{v}_\varepsilon$ and $\hat{v}_0$ are the Laplace Transforms of v_ε and v_0, respectively, in the sense of $\hat{V}_0$. Moreover, since we have

$$\sup_{t\in[0,+\infty)} \Big\| \int_0^t \frac{1}{\beta\varepsilon} \mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}} v_\varepsilon(\tau)\mathrm{d}\tau \Big\|_{V_0} \le \|v_\varepsilon\|_{L^\infty(0,+\infty;V_0)},$$

this integral admits Laplace Transform in the sense of $\hat{V}_0$, which for every $s \in \mathbb{C}_+$ satisfies

$$\mathcal{L}_t\Big(\int_0^t \frac{1}{\beta\varepsilon} \mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}} v_\varepsilon(\tau)\mathrm{d}\tau\Big)(s) = \frac{1}{\beta\varepsilon s + 1}\hat{v}_\varepsilon(s).$$

Hence, by using Proposition 4.4.1 again, we obtain

$$\mathcal{L}_t\Big(S_{\mathbb{B}}\Big(\int_0^t \frac{1}{\beta\varepsilon} \mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}} v_\varepsilon(\tau)\mathrm{d}\tau\Big)\Big)(s) = \frac{1}{\beta\varepsilon s + 1} S_{\mathbb{B}}(\hat{v}_\varepsilon(s)). \tag{4.79}$$

4.4.2 Properties of the Laplace Transform of the solutions

Thanks to (4.75), (4.78), and (4.79) we can apply the Laplace Transform to both sides of (4.76) and (4.77) to deduce the following equalities in $\hat{V}_0'$:

$$\varepsilon^2 s^2 \hat{v}_\varepsilon(s) - \operatorname{div}((\mathbb{A} + \mathbb{B})e\hat{v}_\varepsilon(s)) + \frac{1}{\beta\varepsilon s + 1}\operatorname{div}(\mathbb{B}e\hat{v}_\varepsilon(s)) = \hat{h}_\varepsilon(s) \quad \text{for every } s \in \mathbb{C}_+, \tag{4.80}$$

$$-\operatorname{div}(\mathbb{A}e\hat{v}_0(s)) = \hat{h}(s) \quad \text{for every } s \in \mathbb{C}_+. \tag{4.81}$$

Our purpose is to prove that for every $s_1 > 0$ we have

$$\int_{-\infty}^{+\infty} \|\hat{v}_\varepsilon(s_1 + is_2) - \hat{v}_0(s_1 + is_2)\|^2_{\hat{V}_0} \mathrm{d}s_2 \xrightarrow[\varepsilon\to 0^+]{} 0. \tag{4.82}$$

To prove (4.82) we need two lemmas. In the first one we deduce from (4.80) an estimate for $\hat{v}_\varepsilon(s)$, which is used in the second lemma to prove a convergence result for $\hat{v}_\varepsilon(s)$.

Lemma 4.4.2. *For every $s \in \mathbb{C}_+$ there exists a positive constant $M(s)$ such that*

$$\|\hat{v}_\varepsilon(s)\|_{\hat{V}_0} \le M(s)\|\hat{h}_\varepsilon(s)\| \quad \textit{for every } \varepsilon \in (0, 1). \tag{4.83}$$

Proof. We fix $\varepsilon \in (0,1)$ and for every $s \in \mathbb{C}_+$ we define the operator $S_\varepsilon(s) \colon \hat{V}_0 \to \hat{V}_0'$ in the following way:

$$S_\varepsilon(s)(\psi) := \varepsilon^2 s^2 \psi - \operatorname{div}((\mathbb{A}+\mathbb{B})e\psi) + \frac{1}{\beta\varepsilon s+1}\operatorname{div}(\mathbb{B}e\psi) \qquad \text{for every } \psi \in \hat{V}_0.$$

Since $S_\varepsilon(s)(\hat{v}_\varepsilon(s)) = \hat{h}_\varepsilon(s)$ by (4.80), the Lax-Milgram Lemma, together with the Korn-Poincaré Inequality (4.2), implies (4.83) if we can show that for every $s \in \mathbb{C}_+$ there exists a positive constant $K(s)$, independent of ε, such that

$$|\langle S_\varepsilon(s)(\psi), \psi\rangle| \geq c_{\mathbb{A}} K(s) \|e\psi\|^2 \quad \text{for every } \psi \in \hat{V}_0, \tag{4.84}$$

where

$$|\langle S_\varepsilon(s)(\psi), \psi\rangle| = \frac{|(\beta\varepsilon^3 s^3 + \varepsilon^2 s^2)\|\psi\|^2 + \beta\varepsilon s((\mathbb{A}+\mathbb{B})e\psi, e\psi) + (\mathbb{A}e\psi, e\psi)|}{|\beta\varepsilon s+1|}.$$

We can suppose $\psi \in \hat{V}_0 \setminus \{0\}$, otherwise the inequality is trivially satisfied, and we set

$$a := \frac{(\mathbb{A}e\psi, e\psi)}{\|\psi\|^2} \quad \text{and} \quad b := \frac{((\mathbb{A}+\mathbb{B})e\psi, e\psi)}{\|\psi\|^2},$$

which satisfy, thanks to the Korn-Poincaré Inequality (4.2) and to (4.3)–(4.5), the following relations

$$a \geq \frac{c_{\mathbb{A}}\|e\psi\|^2}{\|\psi\|^2} \geq \frac{c_{\mathbb{A}}}{C_P^2} =: a_0, \qquad b \geq \frac{(c_{\mathbb{A}}+c_{\mathbb{B}})\|e\psi\|^2}{\|\psi\|^2} \geq \frac{c_{\mathbb{A}}+c_{\mathbb{B}}}{C_P^2} =: b_0, \qquad a \leq c_0 a \leq b \leq c_1 a, \tag{4.85}$$

where $c_0 := 1 + \frac{c_{\mathbb{B}}}{C_{\mathbb{A}}}$ and $c_1 := 1 + \frac{C_{\mathbb{B}}}{c_{\mathbb{A}}}$. Therefore, to prove (4.84) it is enough to obtain

$$\left|\frac{\beta\varepsilon^3 s^3 + \varepsilon^2 s^2 + \beta b\varepsilon s + a}{\beta\varepsilon s+1}\right| \geq K(s)\, a \quad \text{for every } s \in \mathbb{C}_+. \tag{4.86}$$

For simplicity of notation we set $z = \varepsilon s$ and we consider two cases.
Case $b > \frac{2}{3\beta^2}$. In this situation, thanks to (4.150) we know that the polynomial $\beta z^3 + z^2 + \beta b z + a$ has one real root z_0 and two complex and conjugate ones w and $\bar{w}$. Therefore, thanks to Lemmas 4.7.1 and 4.7.2, we can write

$$\begin{aligned}\left|\frac{\beta z^3 + z^2 + \beta b z + a}{\beta z+1}\right| &= \left|\frac{\beta(z-z_0)(z-w)(z-\bar{w})}{\beta z+1}\right| \geq \left|\frac{\beta(z-z_0)}{\beta z+1}\right| |\Re(w)||\Im(w)| \\ &= \left|\frac{\beta(z-z_0)}{\beta z+1}\right| |\Re(w)| \sqrt{3|\Re(w)|^2 + \frac{2}{\beta}\Re(w) + b} \geq \left|\frac{\beta(z-z_0)}{\beta z+1}\right| \alpha\sqrt{b - \frac{1}{3\beta^2}} \\ &\geq \left|\frac{\beta(z-z_0)}{\beta z+1}\right| \alpha\sqrt{\frac{b}{2}} \geq \frac{\alpha}{\sqrt{3}}\left|\frac{z}{\beta z+1}\right|,\end{aligned} \tag{4.87}$$

where in the last inequality we used $z_0 < 0$.

If $a \leq 2|z|^2$, then $|z| \geq \frac{a}{2|z|}$ and, thanks to (4.87), we deduce

$$\left|\frac{\beta z^3 + z^2 + \beta b z + a}{\beta z+1}\right| \geq \frac{\alpha}{2\sqrt{3}} \frac{a}{|z(\beta z+1)|}. \tag{4.88}$$

For $a > 2|z|^2$ we have

$$\frac{1}{a}\left|\frac{\beta z^3 + z^2 + \beta b z + a}{\beta z+1}\right| = \left|\frac{z^2}{a} + \frac{\beta b z + a}{a(\beta z+1)}\right| \geq \left|\frac{\beta b z + a}{a(\beta z+1)}\right| - \frac{1}{2},$$

and, by writing $z = x + iy$, we obtain

$$\left|\frac{\beta bz + a}{a(\beta z + 1)}\right| = \left|\frac{\beta bx + a + i\beta by}{\beta ax + a + i\beta ay}\right| = \sqrt{\frac{(\beta bx + a)^2 + \beta^2 b^2 y^2}{(\beta ax + a)^2 + \beta^2 a^2 y^2}} \geq 1,$$

which implies

$$\left|\frac{\beta z^3 + z^2 + \beta bz + a}{\beta z + 1}\right| \geq \frac{a}{2}. \tag{4.89}$$

By (4.88) and (4.89) in the case $b > \frac{2}{3\beta^2}$ we conclude

$$\left|\frac{\beta z^3 + z^2 + \beta bz + a}{\beta z + 1}\right| \geq \min\left\{\frac{1}{2}, \frac{\alpha}{2\sqrt{3}} \frac{1}{|z(\beta z + 1)|}\right\} a. \tag{4.90}$$

Case $b_0 \leq b \leq \frac{2}{3\beta^2}$. In this case, thanks to (4.85), we have $a_0 \leq a \leq \frac{2}{3\beta^2}$. We define

$$R := \sqrt{\frac{2(2 + c_1)}{3\beta^2}}.$$

Then for $z \in \mathbb{C}_+$, with $|z| > R$, we get

$$\left|\frac{z^2}{a} + \frac{\beta bz + a}{a(\beta z + 1)}\right| \geq \frac{3\beta^2 |z|^2}{2} - \frac{b}{a}\left|\frac{\beta z}{\beta z + 1}\right| - \frac{1}{|\beta z + 1|} \geq 2 + c_1 - c_1 - 1 = 1, \tag{4.91}$$

where we used the inequalities $|\beta z| \leq |\beta z + 1|$ and $1 \leq |\beta z + 1|$.

To deal with the case $z \in \mathbb{C}_+$, with $|z| \leq R$, we define

$$\gamma := \min\left\{\left|\frac{\beta z^3 + z^2 + \beta bz + a}{a(\beta z + 1)}\right| : \Re(z) \geq 0, \quad |z| \leq R, \quad b_0 \leq b \leq \frac{2}{3\beta^2}, \quad a_0 \leq a \leq \frac{2}{3\beta^2}\right\},$$

and we claim $\gamma > 0$. Indeed the function under examination is continuous with respect to (z, a, b), and by Lemma 4.7.1 it does not vanish in the compact set considered in the minimum problem. By using also (4.91) we conclude that for $b_0 \leq b \leq \frac{2}{3\beta^2}$ we have

$$\left|\frac{\beta z^3 + z^2 + \beta bz + a}{\beta z + 1}\right| \geq \min\{\gamma, 1\} a. \tag{4.92}$$

for every $z \in \mathbb{C}_+$ and every a satisfying (4.85). Since $\varepsilon \in (0, 1)$ we have

$$\frac{1}{|\varepsilon s(\beta \varepsilon s + 1)|} \geq \frac{1}{|s(\beta s + 1)|},$$

therefore, by setting

$$K(s) := \min\left\{\frac{1}{2}, \frac{\alpha}{2\sqrt{3}} \frac{1}{|s(\beta s + 1)|}, \gamma\right\},$$

from (4.90) and (4.92) we obtain (4.86), which concludes the proof. □

4.4.3 Convergence of the Laplace Transform of the solutions

We begin by proving the pointwise convergence.

Lemma 4.4.3. *For every $s \in \mathbb{C}_+$ we have*

$$\hat{v}_\varepsilon(s) \xrightarrow[\varepsilon \to 0^+]{\hat{V}_0} \hat{v}_0(s).$$

Proof. Thanks to (4.66) and to the Hölder Inequality for every $s \in \mathbb{C}_+$ we get

$$\|\hat{h}_\varepsilon(s) - \hat{h}(s)\| \le \int_0^{+\infty} \mathrm{e}^{-\Re(s)t}\|h_\varepsilon(t) - h(t)\|\mathrm{d}t \le \frac{1}{\sqrt{2\Re(s)}}\|h_\varepsilon - h\|_{L^2(0,T;H)} \xrightarrow[\varepsilon\to 0^+]{} 0. \tag{4.93}$$

Consequently, thanks to Lemma 4.4.2, for every $s \in \mathbb{C}_+$ there exist two constants $\bar{M}(s) > 0$ and $\varepsilon(s) \in (0,1)$ such that

$$\|\hat{v}_\varepsilon(s)\|_{\hat{V}_0} \le \bar{M}(s) \qquad \text{for every } \varepsilon \in (0, \varepsilon(s)). \tag{4.94}$$

By (4.94) we can say that for every $s \in \mathbb{C}_+$ there exist a sequence $\varepsilon_j \to 0^+$ and $v^*(s) \in \hat{V}_0$ such that

$$\hat{v}_{\varepsilon_j}(s) \xrightarrow[j\to+\infty]{\hat{V}_0} v^*(s). \tag{4.95}$$

Thanks to (4.4) and (4.95) for every $\psi \in \hat{V}_0$ we deduce

$$((\mathbb{A}+\mathbb{B})e\hat{v}_{\varepsilon_j}(s), e\psi) \xrightarrow[j\to+\infty]{} ((\mathbb{A}+\mathbb{B})ev^*(s), e\psi), \;\; |\varepsilon_j^2 s^2(\hat{v}_{\varepsilon_j}(s),\psi)| \le \varepsilon_j^2|s|^2\bar{M}(s)\|\psi\| \xrightarrow[j\to+\infty]{} 0,$$

$$\begin{aligned}&\left|\frac{1}{\beta\varepsilon_j s+1}(\mathbb{B}e\hat{v}_{\varepsilon_j}(s), e\psi) - (\mathbb{B}ev^*(s), e\psi)\right|\\ &\qquad\le \left|(\mathbb{B}(e\hat{v}_{\varepsilon_j}(s) - ev^*(s)), e\psi)\right| + \frac{\beta\varepsilon_j|s|}{|\beta\varepsilon_j s+1|}|(\mathbb{B}e\hat{v}_{\varepsilon_j}(s), e\psi)|\\ &\qquad\le \left|(e\hat{v}_{\varepsilon_j}(s) - ev^*(s), \mathbb{B}e\psi)\right| + \beta\varepsilon_j|s|C_\mathbb{B}\bar{M}(s)\|e\psi\| \xrightarrow[j\to+\infty]{} 0.\end{aligned}$$

Therefore by (4.93) we have

$$\begin{cases} v^*(s) \in \hat{V}_0,\\ -\operatorname{div}(\mathbb{A}ev^*(s)) = \hat{h}(s). \end{cases} \tag{4.96}$$

Since, by (4.81), $\hat{v}_0(s)$ is a solution to (4.96), by uniqueness we have $v^*(s) = \hat{v}_0(s)$. Moreover, since the limit does not depend on the subsequence, the whole sequence satisfies

$$\hat{v}_\varepsilon(s) \xrightarrow[\varepsilon\to 0^+]{\hat{V}_0} \hat{v}_0(s) \qquad \text{for every } s \in \mathbb{C}_+. \tag{4.97}$$

To prove the strong convergence we use $\hat{v}_\varepsilon(s)$ and $\hat{v}_0(s)$ as test function in (4.80) and (4.81), respectively. By subtracting the two equalities, we obtain

$$\begin{aligned}&(\mathbb{A}e\hat{v}_\varepsilon(s), e\hat{v}_\varepsilon(s)) - (\mathbb{A}e\hat{v}_0(s), e\hat{v}_0(s))\\ &\qquad = (\hat{h}_\varepsilon(s), \hat{v}_\varepsilon(s)) - (\hat{h}(s), \hat{v}_0(s)) - \varepsilon^2 s^2\|\hat{v}_\varepsilon(s)\|^2 - \frac{\beta\varepsilon s}{\beta\varepsilon s+1}(\mathbb{B}e\hat{v}_\varepsilon(s), e\hat{v}_\varepsilon(s)),\end{aligned}$$

from which we deduce

$$\begin{aligned}&|(\mathbb{A}e\hat{v}_\varepsilon(s), e\hat{v}_\varepsilon(s)) - (\mathbb{A}e\hat{v}_0(s), e\hat{v}_0(s))|\\ &\qquad \le |(\hat{h}_\varepsilon(s), \hat{v}_\varepsilon(s)) - (\hat{h}(s), \hat{v}_0(s))| + \varepsilon^2|s|^2\|\hat{v}_\varepsilon(s)\|^2 + \beta\varepsilon|s|C_\mathbb{B}\|e\hat{v}_\varepsilon(s)\|^2.\end{aligned}$$

By using again (4.93), (4.94), and (4.97), we can deduce

$$\lim_{\varepsilon\to 0^+}(\mathbb{A}e\hat{v}_\varepsilon(s), e\hat{v}_\varepsilon(s)) = (\mathbb{A}e\hat{v}_0(s), e\hat{v}_0(s)) \quad \text{for every } s \in \mathbb{C}_+. \tag{4.98}$$

Thanks to the coerciveness assumption (4.5), the conclusion follows from the weak convergence (4.97) together with (4.98). □

Now we are in position to prove the following result about the convergence in the space L^2 on the lines $\{s_1 + is_2 : s_2 \in \mathbb{R}\}$.

Proposition 4.4.4. *The functions $\hat{v}_\varepsilon$ and $\hat{v}_0$ satisfy (4.82).*

Proof. As before, we set

$$a := \frac{(\mathbb{A}e\hat{v}_\varepsilon(s), e\hat{v}_\varepsilon(s))}{\|\hat{v}_\varepsilon(s)\|^2} \quad \text{and} \quad b := \frac{((\mathbb{A}+\mathbb{B})e\hat{v}_\varepsilon(s), e\hat{v}_\varepsilon(s))}{\|\hat{v}_\varepsilon(s)\|^2}, \tag{4.99}$$

and we observe that (4.85) holds. For every $s \in \mathbb{C}_+$, by using $\hat{v}_\varepsilon(s)$ as test function in (4.80) we obtain

$$\frac{1}{\beta\varepsilon s + 1}\Big(\beta\varepsilon^3 s^3 + \varepsilon^2 s^2 + \beta b\varepsilon s + a\Big)\|\hat{v}_\varepsilon(s)\|^2 = (\hat{h}_\varepsilon(s), \hat{v}_\varepsilon(s)). \tag{4.100}$$

Therefore, thanks to (4.87), Lemma 4.7.1, and (4.92) we can deduce

$$\begin{aligned}
&\left|\frac{\beta\varepsilon^3 s^3 + \varepsilon^2 s^2 + \beta b\varepsilon s + a}{\beta\varepsilon s + 1}\right| \ge \left|\frac{\beta(\varepsilon s - z_0)}{\beta\varepsilon s + 1}\right| \alpha\sqrt{\frac{b}{2}} \ge \beta\,|z_0|\,\alpha\sqrt{\frac{a}{2}} \ge \frac{\beta\alpha^2}{\sqrt{2}}\sqrt{a} \qquad \text{for } b > \frac{2}{3\beta^2},\\
&\left|\frac{\beta\varepsilon^3 s^3 + \varepsilon^2 s^2 + \beta b\varepsilon s + a}{\beta\varepsilon s + 1}\right| \ge \min\{\gamma, 1\}a \ge \min\{\gamma, 1\}\sqrt{a_0}\sqrt{a} \qquad \text{for } b \le \frac{2}{3\beta^2},
\end{aligned}$$

where in the first line we used the inequality $|z_0(\beta\varepsilon s + 1)| \le |\varepsilon s - z_0|$ for every $s \in \mathbb{C}_+$, which follows from the condition $z_0 < 0$.

As a consequence of these inequalities and of (4.100) there exists a positive constant $C = C(\alpha, \beta, \gamma, a_0)$ such that

$$\|\hat{v}_\varepsilon(s)\|^2 = \left|\frac{\beta\varepsilon s + 1}{\beta\varepsilon^3 s^3 + \varepsilon^2 s^2 + \beta b\varepsilon s + a}\right| |(\hat{h}_\varepsilon(s), \hat{v}_\varepsilon(s))| \le \frac{C}{\sqrt{a}}\|\hat{h}_\varepsilon(s)\|\|\hat{v}_\varepsilon(s)\| \quad \text{for every } s \in \mathbb{C}_+.$$

Therefore, by using (4.99) and the coerciveness assumption (4.5), we can write

$$\sqrt{c_{\mathbb{A}}}\|e\hat{v}_\varepsilon(s)\| \le \sqrt{(\mathbb{A}e\hat{v}_\varepsilon(s), e\hat{v}_\varepsilon(s))} \le C\|\hat{h}_\varepsilon(s)\|,$$

from which, recalling (4.2), we deduce

$$\|\hat{v}_\varepsilon(s)\|_{\hat{V}_0} \le (C_P + 1)\frac{C}{\sqrt{c_{\mathbb{A}}}}\|\hat{h}_\varepsilon(s)\| \qquad \text{for every } s \in \mathbb{C}_+. \tag{4.101}$$

By extending the function h_ε to $(-\infty, 0)$ with value 0, we can write

$$\hat{h}_\varepsilon(s) = \int_0^{+\infty} \mathrm{e}^{-st} h_\varepsilon(t)\mathrm{d}t = \int_{-\infty}^{+\infty} \mathrm{e}^{-st} h_\varepsilon(t)\mathrm{d}t = \mathcal{F}_t(\mathrm{e}^{-s_1 t} h_\varepsilon(t))(s_2).$$

Since for every $s = s_1 + is_2 \in \mathbb{C}_+$ the function $t \mapsto \mathrm{e}^{-s_1 t} h_\varepsilon(t)$ belongs to $L^2(\mathbb{R}; H)$, by the properties of the Fourier Transform we deduce that $s_2 \mapsto \hat{h}_\varepsilon(s_1 + is_2)$ belongs to $L^2(\mathbb{R}; \hat{H})$ for every $\varepsilon > 0$. Moreover, by using (4.66) and the Plancherel Theorem, we can write

$$\begin{aligned}
\int_{-\infty}^{+\infty} \|\hat{h}_\varepsilon(s_1 + is_2) - \hat{h}(s_1 + is_2)\|^2 \mathrm{d}s_2 &= \int_{-\infty}^{+\infty} \|\mathcal{F}_t(\mathrm{e}^{-s_1 t}(h_\varepsilon(t) - h(t)))(s_2)\|^2 \mathrm{d}s_2\\
&= \int_{-\infty}^{+\infty} \|\mathrm{e}^{-s_1 t}(h_\varepsilon(t) - h(t))\|^2\,\mathrm{d}t \le \int_0^T \|h_\varepsilon(t) - h(t)\|^2\,\mathrm{d}t \xrightarrow[\varepsilon \to 0^+]{} 0.
\end{aligned} \tag{4.102}$$

Since $\hat{v}_\varepsilon(s) \to \hat{v}_0(s)$ strongly in $\hat{V}_0$ by Lemma 4.4.3 and $\hat{h}_\varepsilon(s) \to \hat{h}(s)$ strongly in $\hat{H}$ by (4.93), thanks to (4.101) and (4.102) we can apply the Generalized Dominated Convergence Theorem to get the conclusion. □

4.5 L^2 convergence

In this section we shall prove (4.39) and (4.40) under the assumptions of Theorems 4.2.6 and 4.2.7. We begin by proving the following partial result.

Proposition 4.5.1. *Let $\{h_\varepsilon\}_\varepsilon \subset L^2(0,T;H)$ and $h \in L^2(0,T;H)$ be such that (4.66) holds. Let v_ε and v_0 be the solutions to problems (4.64) and (4.65). Then*

$$v_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^2(0,T;V)} v_0.$$

Proof. By the Plancherel Theorem we deduce from (4.71) and Proposition 4.4.4 that for every $s_1 > 0$ there exists a positive constant $C = C(s_1,T)$ such that

$$\begin{aligned}\|v_\varepsilon - v_0\|^2_{L^2(0,T;V)} &= \int_0^T \|v_\varepsilon(t) - v_0(t)\|_V^2\,\mathrm{d}t = \int_0^T \|\mathcal{L}^{-1}(\hat v_\varepsilon - \hat v_0)(t)\|_V^2\,\mathrm{d}t\\ &\le C(s_1,T)\int_{-\infty}^{+\infty} \|\mathcal{F}^{-1}_{s_2}(\hat v_\varepsilon(s_1+is_2) - \hat v_0(s_1+is_2))(t)\|_V^2\mathrm{d}t\\ &= C(s_1,T)\int_{-\infty}^{+\infty}\|\hat v_\varepsilon(s_1+is_2) - \hat v_0(s_1+is_2)\|^2_{\hat V_0}\mathrm{d}s_2 \xrightarrow[\varepsilon\to 0^+]{} 0,\end{aligned}$$

which concludes the proof. □

Theorem 4.5.2. *Let us assume (H1), (H2), and (4.43). Let u_ε be the solution to the viscoelastic dynamic system (4.9), with $\varphi_\varepsilon = f_\varepsilon$ and $\gamma_\varepsilon = g_\varepsilon$, and let u_0 be the solution to the stationary problem (4.36). Then (4.39) holds.*

Proof. Thanks to Lemma 4.2.10 it is enough to prove the theorem in the case $z = 0$ and $z_\varepsilon = 0$ for every $\varepsilon > 0$. We divide the proof into two steps.

Step 1. The case $u^1_\varepsilon = 0$. We reduce the problem to the case of homogeneous initial conditions by considering the functions

$$v_\varepsilon(t) := u_\varepsilon(t) - u^0_\varepsilon \qquad \text{and} \qquad v_0(t) := u_0(t) - u^0 \qquad \text{for a.e. } t \in [0,T]. \tag{4.103}$$

Let us define

$$q_\varepsilon(t) := g_\varepsilon(t) + \operatorname{div}(\mathbb{A}eu^0_\varepsilon) + \mathrm{e}^{-\frac{t}{\beta\varepsilon}}\operatorname{div}(\mathbb{B}eu^0_\varepsilon) \qquad \text{for every } t\in[0,T], \tag{4.104}$$

$$q(t) := g(t) + \operatorname{div}(\mathbb{A}eu^0) \qquad \text{for every } t\in[0,T]. \tag{4.105}$$

Since $u^1_\varepsilon = 0$, it is easy to see that v_ε satisfies (4.12) with $h_\varepsilon = f_\varepsilon$, $\ell_\varepsilon = q_\varepsilon$, $v^0_\varepsilon = 0$, and $v^1_\varepsilon = 0$, while v_0 satisfies (4.37) with $h = f$ and $\ell = q$. By (4.43) and (4.103), to prove (4.39) it is enough to show that

$$v_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^2(0,T;V)} v_0 \tag{4.106}$$

In order to apply Proposition 4.5.1, we approximate the forcing terms of the problems for v_ε and v_0 by means of functions in $H^1(0,T;H)$ and we consider the corresponding solutions $\tilde v_\varepsilon$ and $\tilde v_0$, for which Proposition 4.5.1 yields $\tilde v_\varepsilon \to \tilde v_0$ strongly in $L^2(0,T;V)$ as $\varepsilon \to 0^+$. Finally we show that $\|\tilde v_\varepsilon - v_\varepsilon\|_{L^2(0,T;V)}$ and $\|\tilde v_0 - v_0\|_{L^2(0,T;V)}$ are small uniformly with respect to ε, and this leads to the proof of (4.106).

Let us fix $\delta > 0$. Thanks to the density of H in V_0' and to Lemma 4.3.2 we can find $\psi \in H^1(0,T;H)$ and $h^0_{\mathbb{A}}, h^0_{\mathbb{B}} \in H$ such that

$$\begin{gathered}\|h^0_{\mathbb{A}} - \operatorname{div}(\mathbb{A}eu^0)\|_{V_0'} < \delta, \quad \|h^0_{\mathbb{B}} - \operatorname{div}(\mathbb{B}eu^0)\|_{V_0'} < \delta,\\ \|\psi - g\|_{W^{1,1}(0,T;V_0')} < \delta.\end{gathered} \tag{4.107}$$

Thanks to (H1) and (4.43) there exist $\varepsilon_0 = \varepsilon_0(\delta) \in (0, \frac{1}{\beta})$ such that for every $\varepsilon \in (0, \varepsilon_0)$

$$\begin{gathered} \|h^0_{\mathbb{A}} - \operatorname{div}(\mathbb{A}eu^0_\varepsilon)\|_{V_0'} < \delta, \quad \|h^0_{\mathbb{B}} - \operatorname{div}(\mathbb{B}eu^0_\varepsilon)\|_{V_0'} < \delta, \\ \|\psi - g_\varepsilon\|_{W^{1,1}(0,T;V_0')} < \delta. \end{gathered} \tag{4.108}$$

Let $\varphi_\varepsilon, \varphi \colon [0,T] \to H$ be defined by

$$\varphi_\varepsilon(t) := \psi(t) + h^0_{\mathbb{A}} + \mathrm{e}^{-\frac{t}{\beta\varepsilon}} h^0_{\mathbb{B}} \quad \text{and} \quad \varphi(t) := \psi(t) + h^0_{\mathbb{A}} \quad \text{for every } t \in [0,T]. \tag{4.109}$$

By (4.104), (4.105), (4.107), (4.108), and (4.109) for every $\varepsilon \in (0, \varepsilon_0)$ we obtain

$$\begin{aligned} \|\varphi_\varepsilon - q_\varepsilon\|_{W^{1,1}(0,T;V_0')} &\le \|\psi - g_\varepsilon\|_{W^{1,1}(0,T;V_0')} + T\|h^0_{\mathbb{A}} - \operatorname{div}(\mathbb{A}eu^0_\varepsilon)\|_{V_0'} \\ &\quad + (\beta\varepsilon + 1)\|h^0_{\mathbb{B}} - \operatorname{div}(\mathbb{B}eu^0_\varepsilon)\|_{V_0'} \le (3+T)\delta, \end{aligned} \tag{4.110}$$

$$\|\varphi - q\|_{L^\infty(0,T;V_0')} \le (1 + \tfrac{1}{T})\|\psi - g\|_{W^{1,1}(0,T;V_0')} \tag{4.111}$$

$$+ \|h^0_{\mathbb{A}} - \operatorname{div}(\mathbb{A}eu^0_\varepsilon)\|_{V_0'} \le (2 + \tfrac{1}{T})\delta. \tag{4.112}$$

Since $t \mapsto \mathrm{e}^{-\frac{t}{\beta\varepsilon}} \psi^0_{\mathbb{B}}$ converges to 0 strongly in $L^2(0,T;H)$ as $\varepsilon \to 0^+$, by (4.109) we have

$$\varphi_\varepsilon \xrightarrow[\varepsilon \to 0^+]{L^2(0,T;H)} \varphi. \tag{4.113}$$

Let $\tilde v_\varepsilon$ be the solution to (4.64) with $h_\varepsilon = f_\varepsilon + \varphi_\varepsilon$ and let $\tilde v_0$ be the solution to (4.65) with $h = f + \varphi$. By (H1) and (4.113) we have

$$f_\varepsilon + \varphi_\varepsilon \xrightarrow[\varepsilon \to 0^+]{L^2(0,T;H)} f + \varphi,$$

hence Proposition 4.5.1 yields

$$\tilde v_\varepsilon \xrightarrow[\varepsilon \to 0^+]{L^2(0,T;V)} \tilde v_0. \tag{4.114}$$

To estimate the difference $\tilde v_\varepsilon - v_\varepsilon$ we observe that it solves (4.12) with $h_\varepsilon = 0$, $\ell_\varepsilon = \varphi_\varepsilon - q_\varepsilon$, $v^0_\varepsilon = 0$, and $v^1_\varepsilon = 0$. Therefore, by Lemma 4.2.8 we have

$$\|\tilde v_\varepsilon - v_\varepsilon\|_{L^2(0,T;V)} \le \sqrt{C_E T}\|\varphi_\varepsilon - q_\varepsilon\|_{W^{1,1}(0,T;V_0')}. \tag{4.115}$$

To estimate the difference $\tilde v_0 - v_0$ we observe that it solves (4.37) with $h = 0$ and $\ell = \varphi - q$. Therefore by the Lax-Milgram Lemma we obtain

$$\|\tilde v_0 - v_0\|_{L^2(0,T;V)} \le \tfrac{\sqrt{T}(C_P^2+1)}{c_{\mathbb{A}}}\|\varphi - q\|_{L^\infty(0,T;V_0')}. \tag{4.116}$$

By (4.110), (4.112), (4.115), and (4.116) there exists a positive constant $C = C(\mathbb{A}, \mathbb{B}, \Omega, T)$ such that

$$\|\tilde v_\varepsilon - v_\varepsilon\|_{L^2(0,T;V)} + \|\tilde v_0 - v_0\|_{L^2(0,T;V)} \le C\delta,$$

hence

$$\begin{aligned} \|v_\varepsilon - v_0\|_{L^2(0,T;V)} &\le \|v_\varepsilon - \tilde v_\varepsilon\|_{L^2(0,T;V)} + \|\tilde v_\varepsilon - \tilde v_0\|_{L^2(0,T;V)} + \|\tilde v_0 - v_0\|_{L^2(0,T;V)} \\ &\le \|\tilde v_\varepsilon - \tilde v_0\|_{L^2(0,T;V)} + C\delta. \end{aligned}$$

This inequality, together with (4.114), gives

$$\limsup_{\varepsilon \to 0^+} \|v_\varepsilon - v_0\|_{L^2(0,T;V)} \le C\delta.$$

By the arbitrariness of $\delta > 0$ we obtain (4.106), which concludes the proof of Step 1.

Step 2. The general case. Let $\tilde{u}_\varepsilon$ be the solution to (4.12) with $h_\varepsilon = f_\varepsilon$, $\ell_\varepsilon = g_\varepsilon$, $v^0_\varepsilon = u^0_\varepsilon$, and $v^1_\varepsilon = 0$. By Step 1

$$\tilde{u}_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^2(0,T;V)} u_0. \tag{4.117}$$

The function $u_\varepsilon - \tilde{u}_\varepsilon$ is the solution to (4.12) with all data equal to 0 except v^1_ε, which is now equal to u^1_ε. Therefore, Lemma 4.2.8 and (4.43) yield

$$\|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^\infty(0,T;V)} \leq \sqrt{C_E}\varepsilon\|u^1_\varepsilon\| \xrightarrow[\varepsilon\to 0^+]{} 0,$$

which, together with (4.117), gives (4.39). □

In the following theorem, under the assumptions of Theorem 4.2.7 we deduce (4.40) from (4.39).

Theorem 4.5.3. *Let us assume (H1), (H2), and* (4.43)*. Let u_ε be the solution to the viscoelastic dynamic system* (4.9)*, with $\varphi_\varepsilon = f_\varepsilon$ and $\gamma_\varepsilon = g_\varepsilon$, and let u_0 be the solution to the stationary problem* (4.36)*. Then* (4.40) *holds.*

Proof. Thanks to Lemma 4.2.10 we can suppose $z = 0$ and $z_\varepsilon = 0$ for every $\varepsilon > 0$. It is convenient to extend the data of our problem to the interval $[0, 2T]$ by setting

$$f_\varepsilon(t) := 0, \quad f(t) := 0, \quad g_\varepsilon(t) := g_\varepsilon(T), \quad g(t) := g(T) \qquad \text{for every } t \in (T, 2T].$$

Since (H1) holds, it is clear that $\{f_\varepsilon\}_\varepsilon \subset L^2(0, 2T; H)$, $\{g_\varepsilon\}_\varepsilon \in H^1(0, 2T; V_0')$,

$$f_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^2(0,2T;H)} f \quad \text{and} \quad g_\varepsilon \xrightarrow[\varepsilon\to 0^+]{W^{1,1}(0,2T;V_0')} g. \tag{4.118}$$

Moreover, the solution to (4.9) on $[0, 2T]$ with the extended data is an extension of u_ε, which is still denoted by u_ε. Similarly, the solution to (4.36) on $[0, 2T]$ is still denoted by u_0. Since (4.118) holds, Theorem 4.5.2 gives

$$u_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^2(0,2T;V)} u_0. \tag{4.119}$$

We further extend u_ε to $\mathbb{R}$ by setting $u_\varepsilon(t) = 0$ for every $t \in \mathbb{R} \setminus [0, 2T]$, and we define

$$w_\varepsilon(t) := \int_0^t \frac{1}{\beta\varepsilon} \mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}} eu_\varepsilon(\tau)\mathrm{d}\tau = (\rho_\varepsilon * eu_\varepsilon)(t) \quad \text{for every } t \in \mathbb{R},$$

where ρ_ε is as in (4.50). By the properties of convolutions and (4.119) we get

$$eu_\varepsilon - w_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^2(\mathbb{R};H)} 0. \tag{4.120}$$

Thanks to (4.119) and (4.120), by using (4.9) and (4.36) we obtain

$$\varepsilon^2 \ddot{u}_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^2(0,2T;V_0')} 0. \tag{4.121}$$

Since

$$\varepsilon^2 \dot{u}_\varepsilon(t) = \varepsilon^2 u^1_\varepsilon + \varepsilon^2 \int_0^t \ddot{u}_\varepsilon(\tau)\mathrm{d}\tau \quad \text{for every } t \in [0, 2T],$$

(4.43) and (4.121) imply

$$\varepsilon^2 \dot{u}_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^2(0,2T;V_0')} 0. \tag{4.122}$$

By (4.119) and (4.122) there exists a sequence $\varepsilon_j \to 0^+$ such that for a.e. $t \in [0, 2T]$ we have

$$u_{\varepsilon_j}(t) \xrightarrow[j\to+\infty]{V} u_0(t) \quad \text{and} \quad \varepsilon_j^2 \dot{u}_{\varepsilon_j}(t) \xrightarrow[j\to+\infty]{V_0'} 0. \tag{4.123}$$

We choose $T_0 \in (T, 2T)$ such that (4.123) holds at $t = T_0$. This implies

$$\varepsilon_j^2(\dot{u}_{\varepsilon_j}(T_0), u_{\varepsilon_j}(T_0)) = \langle \varepsilon_j^2 \dot{u}_{\varepsilon_j}(T_0), u_{\varepsilon_j}(T_0)\rangle \xrightarrow[j\to+\infty]{} 0. \tag{4.124}$$

Since $z_\varepsilon = 0$ for a.e. $t \in [0, T_0]$ we can use $u_\varepsilon(t) \in V_0$ as test function in (4.9). Then we integrate by parts in time on the interval $(0, T_0)$ to obtain

$$\begin{aligned}
-\varepsilon_j^2 \int_0^{T_0} \|\dot{u}_{\varepsilon_j}(t)\|^2 \mathrm{d}t + \int_0^{T_0} (\mathbb{A} e u_{\varepsilon_j}(t), e u_{\varepsilon_j}(t)) \mathrm{d}t + \int_0^{T_0} (\mathbb{B}(e u_{\varepsilon_j}(t) - w_{\varepsilon_j}(t)), e u_{\varepsilon_j}(t)) \mathrm{d}t \\
= \int_0^{T_0} (f_{\varepsilon_j}(t), u_{\varepsilon_j}(t)) \mathrm{d}t + \int_0^{T_0} \langle g_{\varepsilon_j}(t), u_{\varepsilon_j}(t)\rangle \mathrm{d}t - \varepsilon_j^2(\dot{u}_{\varepsilon_j}(T_0), u_{\varepsilon_j}(T_0)) + \varepsilon_j^2(u^0_{\varepsilon_j}, u^1_{\varepsilon_j}).
\end{aligned}$$

Thanks to (4.36), (4.43), (4.118), (4.119), (4.120), and (4.124) the first term on the left-hand side of the previous equation tends to 0 as $j \to +\infty$. Since $T_0 > T$ we have

$$\varepsilon_j^2 \int_0^T \|\dot{u}_{\varepsilon_j}(t)\|^2 \mathrm{d}t \xrightarrow[j\to+\infty]{} 0.$$

By the arbitrariness of the sequence $\{\varepsilon_j\}_j$ we have

$$\varepsilon^2 \int_0^T \|\dot{u}_\varepsilon(t)\|^2 \mathrm{d}t \xrightarrow[\varepsilon\to0^+]{} 0,$$

which concludes the proof. □

We now use Theorems 4.5.2 and 4.5.3 to obtain (4.39) and (4.40) under the assumptions of Theorem 4.2.6.

Theorem 4.5.4. *Let us assume (H1)–(H3). Let u_ε be the solution to the viscoelastic dynamic system* (4.8) *and let u_0 be the solution to the stationary problem* (4.36)*. Then* (4.39) *and* (4.40) *hold.*

Proof. Thanks to Lemma 4.2.9 we can suppose $z = 0$ and $z_\varepsilon = 0$ for every $\varepsilon > 0$. Let p_ε be defined by (4.11). Since $z_\varepsilon = 0$, by Remark 4.1.3 the function u_ε solves (4.12) with $h_\varepsilon = f_\varepsilon$, $\ell_\varepsilon = g_\varepsilon - p_\varepsilon$, $v^0_\varepsilon = u_{\varepsilon,in}(0)$, and $v^1_\varepsilon = \dot{u}_{\varepsilon,in}(0)$. To obtain (4.39) and (4.40) we cannot apply Theorems 4.5.2 and 4.5.3 directly, because $\{p_\varepsilon\}_\varepsilon$ does not converge to 0 in $W^{1,1}(0,T;V_0')$ as $\varepsilon \to 0^+$ and, in general, $p_\varepsilon \notin L^2(0,T;H)$.

To overcome this difficulty we construct a family $\{q_\varepsilon\}_\varepsilon \subset H^1(0,T;H)$ such that the norm $\|q_\varepsilon - p_\varepsilon\|_{W^{1,1}(0,T;V_0')}$ is uniformly small and $q_\varepsilon \to 0$ strongly in $L^2(0,T;H)$ as $\varepsilon \to 0^+$. Then we can apply Theorems 4.5.2 and 4.5.3 to the solutions v_ε to (4.12) with p_ε replaced by q_ε, obtaining that $v_\varepsilon \to u_0$ strongly in $L^2(0,T;V)$ and $\varepsilon \dot{v}_\varepsilon \to 0$ strongly in $L^2(0,T;H)$. Finally, we show that $\|v_\varepsilon - u_\varepsilon\|_{L^2(0,T;V)}$ and $\varepsilon\|\dot{v}_\varepsilon - \dot{u}_\varepsilon\|_{L^2(0,T;H)}$ are small uniformly with respect to ε, and this leads to the proof of (4.39) and (4.40).

To construct q_ε we consider g^0_ε introduced in (4.11) and we define

$$\tilde{g}^0_\varepsilon := \int_{-\infty}^0 \frac{1}{\beta\varepsilon} \mathrm{e}^{\frac{\tau}{\beta\varepsilon}} \operatorname{div}(\mathbb{B} e u_{0,in}(\tau)) \mathrm{d}\tau = (\rho_\varepsilon * \operatorname{div}(\mathbb{B} e u_{0,in}))(0),$$

where ρ_ε is defined by (4.50). By (H3) we get $\operatorname{div}(\mathbb{B} e u_{0,in}) \in C^0((-\infty,0];V_0')$, hence the properties of convolutions imply

$$\tilde{g}^0_\varepsilon \xrightarrow[\varepsilon\to0^+]{V_0'} g^0 := \operatorname{div}(\mathbb{B} e u_{0,in}(0)). \tag{4.125}$$

Since

$$\begin{aligned}\|g^0_\varepsilon-\tilde g^0_\varepsilon\|_{V_0'}\le&\int_{-\infty}^{-a}\frac{1}{\beta\varepsilon}\mathrm{e}^{\frac{\tau}{\beta\varepsilon}}\|\operatorname{div}(\mathbb{B}(eu_{\varepsilon,in}(\tau))\|_{V_0'}\mathrm{d}\tau\\&+\int_{-\infty}^{-a}\frac{1}{\beta\varepsilon}\mathrm{e}^{\frac{\tau}{\beta\varepsilon}}\|\operatorname{div}(\mathbb{B}(eu_{0,in}(\tau))\|_{V_0'}\mathrm{d}\tau\\&+\|\operatorname{div}(\mathbb{B}(eu_{\varepsilon,in}-eu_{0,in}))\|_{L^\infty(-a,0;V_0')},\end{aligned}$$

thanks to (H3) we have $g^0_\varepsilon-\tilde g^0_\varepsilon\to0$ strongly in V_0' as $\varepsilon\to0^+$, hence (4.125) implies

$$g^0_\varepsilon\xrightarrow[\varepsilon\to0^+]{V_0'}g^0. \tag{4.126}$$

Let us fix $\delta>0$. By the density of H in V_0' we can find $h^0\in H$ such that $\|h^0-g^0\|_{V_0'}<\delta$. By (4.126) there exists $\varepsilon_0=\varepsilon_0(\delta)\in(0,\frac{1}{\beta})$ such that

$$\|h^0-g^0_\varepsilon\|_{V_0'}<\delta\quad\text{for every }\varepsilon\in(0,\varepsilon_0). \tag{4.127}$$

Let $q_\varepsilon\in H^1(0,T;H)$ be defined by $q_\varepsilon(t):=\mathrm{e}^{-\frac{t}{\beta\varepsilon}}h^0$ for every $t\in[0,T]$. Then

$$q_\varepsilon\xrightarrow[\varepsilon\to0^+]{L^2(0,T;H)}0. \tag{4.128}$$

Since $p_\varepsilon(t)=\mathrm{e}^{-\frac{t}{\beta\varepsilon}}g^0_\varepsilon$, by (4.127) we have also

$$\|q_\varepsilon-p_\varepsilon\|_{W^{1,1}(0,T;V_0')}\le(\beta\varepsilon+1)\|h^0-g^0_\varepsilon\|_{V_0'}\le2\delta\quad\text{for every }\varepsilon\in(0,\varepsilon_0). \tag{4.129}$$

Let v_ε be the solution to (4.12) with $h_\varepsilon=f_\varepsilon-q_\varepsilon$, $\ell_\varepsilon=g_\varepsilon$, $v^0_\varepsilon=u_{\varepsilon,in}(0)$, and $v^1_\varepsilon=\dot u_{\varepsilon,in}(0)$. By (H1) and (4.128) we have

$$f_\varepsilon-q_\varepsilon\xrightarrow[\varepsilon\to0^+]{L^2(0,T;H)}f\quad\text{and}\quad g_\varepsilon\xrightarrow[\varepsilon\to0^+]{W^{1,1}(0,T;V_0')}g.$$

By (H3) we have

$$u_{\varepsilon,in}(0)\xrightarrow[\varepsilon\to0^+]{V}u_{0,in}(0)\quad\text{and}\quad\varepsilon\dot u_{\varepsilon,in}(0)\xrightarrow[\varepsilon\to0^+]{H}0.$$

Therefore we can apply Theorems 4.5.2 and 4.5.3 to obtain

$$v_\varepsilon\xrightarrow[\varepsilon\to0^+]{L^2(0,T;V)}u_0\quad\text{and}\quad\varepsilon\dot v_\varepsilon\xrightarrow[\varepsilon\to0^+]{L^2(0,T;H)}0. \tag{4.130}$$

To estimate the difference $v_\varepsilon-u_\varepsilon$ we observe that it solves (4.12) with $h_\varepsilon=0$, $\ell_\varepsilon=p_\varepsilon-q_\varepsilon$, $v^0_\varepsilon=0$, and $v^1_\varepsilon=0$. Therefore, by Lemma 4.2.8 and (4.129) we have

$$\varepsilon^2\|\dot v_\varepsilon-\dot u_\varepsilon\|^2_{L^2(0,T;H)}+\|v_\varepsilon-u_\varepsilon\|^2_{L^2(0,T;V)}\le C_E\|q_\varepsilon-p_\varepsilon\|^2_{W^{1,1}(0,T;V_0')}\le4C_E\delta^2. \tag{4.131}$$

Since by (4.131)

$$\begin{aligned}\|u_\varepsilon-u_0\|_{L^2(0,T;V)}&\le\|u_\varepsilon-v_\varepsilon\|_{L^2(0,T;V)}+\|v_\varepsilon-u_0\|_{L^2(0,T;V)}\le\|v_\varepsilon-u_0\|_{L^2(0,T;V)}+2\sqrt{C_E}\delta,\\\varepsilon\|\dot u_\varepsilon\|_{L^2(0,T;H)}&\le\varepsilon\|\dot u_\varepsilon-\dot v_\varepsilon\|_{L^2(0,T;H)}+\varepsilon\|\dot v_\varepsilon\|_{L^2(0,T;H)}\le\varepsilon\|\dot v_\varepsilon\|_{L^2(0,T;H)}+2\sqrt{C_E}\delta,\end{aligned}$$

thanks to (4.130) we have

$$\limsup_{\varepsilon\to0^+}\|u_\varepsilon-u_0\|_{L^2(0,T;V)}\le2\sqrt{C_E}\delta\quad\text{and}\quad\limsup_{\varepsilon\to0^+}\varepsilon\|\dot u_\varepsilon\|_{L^2(0,T;H)}\le2\sqrt{C_E}\delta.$$

By the arbitrariness of $\delta>0$ we obtain (4.39) and (4.40), which concludes the proof. □

4.6 The local uniform convergence

In this section we shall prove (4.41) under the assumptions of Theorems 4.2.6 and 4.2.7. The proof is based on the following lemma.

Lemma 4.6.1. *Let $\{\ell_\varepsilon\}_\varepsilon \subset H^1(0,T;V_0')$ and $\ell \in W^{1,1}(0,T;V_0')$ be such that*

$$\ell_\varepsilon \xrightarrow[\varepsilon\to 0^+]{W^{1,1}(\eta,T;V_0')} \ell \quad \text{for every } \eta \in (0,T). \tag{4.132}$$

Let v_ε be a solution to the viscoelastic dynamic system (4.12) with $h_\varepsilon = 0$ and arbitrary initial data. Moreover, let v_0 be the solution to the stationary problem (4.37) with $h = 0$. We assume that

$$v_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^2(0,T;V)} v_0, \quad \text{and} \quad \varepsilon\dot{v}_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^2(0,T;H)} 0. \tag{4.133}$$

Then

$$v_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^\infty(\eta,T;V)} v_0 \quad \text{and} \quad \varepsilon\dot{v}_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^\infty(\eta,T;H)} 0 \qquad \text{for every } \eta \in (0,T). \tag{4.134}$$

Proof. We divide the proof into two steps.

Step 1. Let us assume $\ell_\varepsilon = \ell \in H^2(0,T;V_0')$ for every $\varepsilon > 0$. By Lemma 4.2.4 (with $z = 0$) we have $v_0 \in H^2(0,T;V)$, hence recalling (4.37) we get

$$\begin{aligned}\varepsilon^2\ddot{v}_0(t) - \operatorname{div}((\mathbb{A}+\mathbb{B})ev_0(t)) + \int_0^t \frac{1}{\beta\varepsilon}\mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}}\operatorname{div}(\mathbb{B}ev_0(\tau))\mathrm{d}\tau \\ = \varepsilon^2\ddot{v}_0(t) + \ell(t) - \operatorname{div}(\mathbb{B}ev_0(t)) + \int_0^t \frac{1}{\beta\varepsilon}\mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}}\operatorname{div}(\mathbb{B}ev_0(\tau)) \quad \text{for a.e. } t\in[0,T].\end{aligned} \tag{4.135}$$

Now we define $\bar{v}_\varepsilon := v_\varepsilon - v_0$ and observe that by (4.133) we have

$$\bar{v}_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^2(0,T;V)} 0 \quad \text{and} \quad \varepsilon\dot{\bar{v}}_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^2(0,T;H)} 0. \tag{4.136}$$

Let us consider

$$q_\varepsilon(t) := \operatorname{div}(\mathbb{B}ev_0(t)) - \int_0^t \frac{1}{\beta\varepsilon}\mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}}\operatorname{div}(\mathbb{B}ev_0(\tau))\mathrm{d}\tau.$$

Since v_ε satisfies (4.12) with $h_\varepsilon = 0$, by (4.135) the function $\bar{v}_\varepsilon$ satisfies (4.12) with $h_\varepsilon = -\varepsilon^2\ddot{v}_0$ and $\ell_\varepsilon = q_\varepsilon$. After two integrations by parts in time we deduce

$$\begin{aligned}\int_0^t \frac{1}{\beta\varepsilon}\mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}}\operatorname{div}(\mathbb{B}ev_0(\tau))\mathrm{d}\tau = \operatorname{div}(\mathbb{B}ev_0(t)) - \mathrm{e}^{-\frac{t}{\beta\varepsilon}}\operatorname{div}(\mathbb{B}ev_0(0)) - \beta\varepsilon\operatorname{div}(\mathbb{B}e\dot{v}_0(t)) \\ + \beta\varepsilon\mathrm{e}^{-\frac{t}{\beta\varepsilon}}\operatorname{div}(\mathbb{B}e\dot{v}_0(0)) + \beta\varepsilon\int_0^t \mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}}\operatorname{div}(\mathbb{B}e\ddot{v}_0(\tau))\mathrm{d}\tau,\end{aligned}$$

hence

$$q_\varepsilon \xrightarrow[\varepsilon\to 0^+]{W^{1,1}(\eta,T;V_0')} 0 \quad \text{for every } \eta \in (0,T). \tag{4.137}$$

Now we fix $\delta \in (0,T)$, and we consider $\eta \in (0,\delta)$ and $\zeta \in (\eta,\delta)$. We define the family of functions $\{\bar{w}_\varepsilon\}_\varepsilon \subset H^1(0,T;H)$ by

$$\bar{w}_\varepsilon(t) := \int_0^t \frac{1}{\beta\varepsilon}\mathrm{e}^{-\frac{t-\tau}{\beta\varepsilon}}e\bar{v}_\varepsilon(\tau)\mathrm{d}\tau = (\rho_\varepsilon * e\bar{v}_\varepsilon)(t) \quad \text{for every } t\in[0,T],$$

where ρ_ε is defined by (4.50) and $\bar{v}_\varepsilon$ is extended to $\mathbb{R}$ by setting $\bar{v}_\varepsilon(t) = 0$ on $\mathbb{R}\setminus[0,T]$. By properties of convolutions we have

$$e\bar{v}_\varepsilon - \bar{w}_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^2(0,T;H)} 0. \tag{4.138}$$

By the energy-dissipation balance (4.24) of Proposition 4.1.7, for every $t \in [\eta, T]$ and $s \in (\eta, \zeta)$ we can write

$$\begin{aligned}\frac{\varepsilon^2}{2}\|\dot{\bar v}_\varepsilon(t)\|^2 + \frac{1}{2}(\mathbb{A}e\bar v_\varepsilon(t), e\bar v_\varepsilon(t)) + \frac{1}{2}(\mathbb{B}(e\bar v_\varepsilon(t) - \bar w_\varepsilon(t), e\bar v_\varepsilon(t) - \bar w_\varepsilon(t))\\ + \beta\varepsilon\int_s^t(\mathbb{B}\dot{\bar w}_\varepsilon(\tau), \dot{\bar w}_\varepsilon(\tau))\mathrm{d}\tau = \frac{\varepsilon^2}{2}\|\dot{\bar v}_\varepsilon(s)\|^2 + \frac{1}{2}(\mathbb{A}e\bar v_\varepsilon(s), e\bar v_\varepsilon(s))\\ + \frac{1}{2}(\mathbb{B}(e\bar v_\varepsilon(s) - \bar w_\varepsilon(s), e\bar v_\varepsilon(s) - \bar w_\varepsilon(s)) + \mathscr{W}_\varepsilon(t,s),\end{aligned} \tag{4.139}$$

where the work is defined by

$$\mathscr{W}_\varepsilon(t,s) = \langle q_\varepsilon(t), \bar v_\varepsilon(t)\rangle - \langle q_\varepsilon(s), \bar v_\varepsilon(s)\rangle - \int_s^t \langle \dot q_\varepsilon(\tau), \bar v_\varepsilon(\tau)\rangle \mathrm{d}\tau - \varepsilon\int_s^t(\ddot v_0(\tau), \varepsilon\dot{\bar v}_\varepsilon(\tau))\mathrm{d}\tau.$$

Now we take the mean value with respect to s of all terms of (4.139) on (η, ζ), and we pass to the supremum with respect to t on $[\eta, T]$. Thanks to (4.2) and (4.5) we deduce

$$\begin{aligned}\frac{\varepsilon^2}{2}\|\dot{\bar v}_\varepsilon\|^2_{L^\infty(\eta,T;H)} + \frac{c_\mathbb{A}}{2(C_P^2+1)}\|\bar v_\varepsilon\|^2_{L^\infty(\eta,T;V)} \le \frac{\varepsilon^2}{2}⨍_\eta^\zeta \|\dot{\bar v}_\varepsilon(s)\|^2\mathrm{d}s + \frac{C_\mathbb{A}}{2}⨍_\eta^\zeta \|\bar v_\varepsilon(s)\|^2_V\mathrm{d}s\\ + \frac{C_\mathbb{B}}{2}⨍_\eta^\zeta \|e\bar v_\varepsilon(s) - \bar w_\varepsilon(s)\|^2\mathrm{d}s + ⨍_\eta^\zeta \sup_{t\in[\eta,T]}|\mathscr{W}_\varepsilon(t,s)|\mathrm{d}s.\end{aligned} \tag{4.140}$$

Notice that for every $s \in (\eta, \zeta)$ we have

$$\sup_{t\in[\eta,T]}|\mathscr{W}_\varepsilon(t,s)| \le \big(3 + \tfrac{2}{T}\big)\|q_\varepsilon\|_{W^{1,1}(\eta,T;V_0')}\|\bar v_\varepsilon\|_{L^\infty(\eta,T;V)} + \varepsilon\|\ddot v_0\|_{L^1(\eta,T;H)}\|\varepsilon\dot{\bar v}_\varepsilon\|_{L^\infty(\eta,T;H)},$$

hence thanks to the Young Inequality and (4.140) there exists a positive constant $C = C(\mathbb{A}, \mathbb{B}, \Omega, T)$ such that

$$\begin{aligned}\varepsilon^2\|\dot{\bar v}_\varepsilon\|^2_{L^\infty(\eta,T;H)} + \|\bar v_\varepsilon\|^2_{L^\infty(\eta,T;V)} \le C\Big(\varepsilon^2⨍_\eta^\zeta \|\dot{\bar v}_\varepsilon(s)\|^2\mathrm{d}s + ⨍_\eta^\zeta \|\bar v_\varepsilon(s)\|^2_V\mathrm{d}s\\ + ⨍_\eta^\zeta \|e\bar v_\varepsilon(s) - \bar w_\varepsilon(s)\|^2\mathrm{d}s + \|q_\varepsilon\|^2_{W^{1,1}(\eta,T;V_0')} + \varepsilon^2\|\ddot v_0\|^2_{L^1(\eta,T;H)}\Big).\end{aligned} \tag{4.141}$$

By passing to the limit in (4.141) as $\varepsilon \to 0^+$, thanks to (4.136), (4.137), and (4.138) we obtain

$$\varepsilon\|\dot v_\varepsilon - \dot v_0\|_{L^\infty(\eta,T;H)} + \|v_\varepsilon - v_0\|_{L^\infty(\eta,T;V)} = \varepsilon\|\dot{\bar v}_\varepsilon\|_{L^\infty(\eta,T;H)} + \|\bar v_\varepsilon\|_{L^\infty(\eta,T;V)} \xrightarrow[\varepsilon\to 0^+]{} 0,$$

which concludes the proof of (4.134) in the case $\ell \in H^2(0,T;V_0')$.

Step 2. In the general case $\ell \in W^{1,1}(0,T;V_0')$ we use an approximation argument. Given $\delta > 0$, by Lemma 4.3.3 there exists a function $\psi \in H^2(0,T;H)$ such that

$$\|\psi - \ell\|_{W^{1,1}(0,T;V_0')} < \delta. \tag{4.142}$$

Thanks to (4.132) for every $\sigma \in (0,T)$ there exists a positive number $\varepsilon_0 = \varepsilon_0(\delta,\sigma)$ such that

$$\|\psi - \ell_\varepsilon\|_{W^{1,1}(\sigma,T;V_0')} < \delta \qquad \text{for every } \varepsilon \in (0, \varepsilon_0). \tag{4.143}$$

Let $\tilde v_\varepsilon$ be the solution to (4.12) in the interval $[\sigma, T]$ with $h_\varepsilon = 0$, $\ell_\varepsilon = \psi$, $\tilde v_\varepsilon(\sigma) = v_\varepsilon(\sigma)$, and $\dot{\tilde v}_\varepsilon(\sigma) = \dot v_\varepsilon(\sigma)$, and let $\tilde v_0$ be the solution to (4.37) in the interval $[0,T]$ with $h = 0$ and $\ell = \psi$. By applying Step 1 in the interval $[\sigma, T]$ we obtain

$$\tilde v_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^\infty(\eta,T;V)} \tilde v_0 \quad \text{and} \quad \varepsilon\dot{\tilde v}_\varepsilon \xrightarrow[\varepsilon\to 0^+]{L^\infty(\eta,T;H)} 0 \qquad \text{for every } \eta \in (\sigma, T). \tag{4.144}$$

We set $\bar v_0 := \tilde v_0 - v_0$ and $\bar v_\varepsilon := \tilde v_\varepsilon - v_\varepsilon$. We observe that $\bar v_0$ is the solution to (4.37) with $h = 0$ and ℓ replaced by $\psi - \ell$, hence by the Lax-Milgram Lemma we get

$$\|\bar v_0\|_{L^\infty(0,T;V)} \le \tfrac{C_P^2+1}{c_{\mathbb{A}}}\|\psi - \ell\|_{L^\infty(0,T;V_0')} \le \tfrac{C_P^2+1}{c_{\mathbb{A}}}(1+\tfrac{1}{T})\|\psi-\ell\|_{W^{1,1}(0,T;V_0')}. \tag{4.145}$$

Moreover, $\bar v_\varepsilon$ is the solution to (4.12) in the interval $[\sigma, T]$ with $h_\varepsilon = 0$, ℓ_ε replaced by $\psi - \ell_\varepsilon$, and homogeneous initial conditions. Thanks to Lemma 4.2.8 we obtain

$$\varepsilon\|\dot{\bar v}_\varepsilon\|^2_{L^\infty(\sigma,T;H)} + \|\bar v_\varepsilon\|^2_{L^\infty(\sigma,T;V)} \le C_E\|\psi - \ell_\varepsilon\|^2_{W^{1,1}(\sigma,T;V_0')}. \tag{4.146}$$

By combining (4.142), (4.143), (4.145), and (4.146), we can find a positive constant $C = C(\mathbb{A}, \mathbb{B}, \Omega, T)$ such that

$$\varepsilon\|\dot{\bar v}_\varepsilon\|_{L^\infty(\sigma,T;H)} + \|\bar v_\varepsilon\|_{L^\infty(\sigma,T;V)} + \|\bar v_0\|_{L^\infty(\sigma,T;V)} \le C\delta. \tag{4.147}$$

Since for every $\eta \in (\sigma, T)$ we have

$$\begin{aligned}
\|v_\varepsilon - v_0\|_{L^\infty(\eta,T;V)} &\le \|\bar v_\varepsilon\|_{L^\infty(\eta,T;V)} + \|\tilde v_\varepsilon - \tilde v_0\|_{L^\infty(\eta,T;V)} + \|\bar v_0\|_{L^\infty(\eta,T;V)},\\
\varepsilon\|\dot v_\varepsilon\|_{L^\infty(\eta,T;H)} &\le \varepsilon\|\dot{\bar v}_\varepsilon\|_{L^\infty(\eta,T;H)} + \varepsilon\|\dot{\tilde v}_\varepsilon\|_{L^\infty(\eta,T;H)},
\end{aligned}$$

thanks to (4.144) and (4.147) we obtain

$$\limsup_{\varepsilon\to 0^+}\|v_\varepsilon - v_0\|_{L^\infty(\eta,T;V)} \le C\delta \quad \text{and} \quad \limsup_{\varepsilon\to 0^+}\|\varepsilon\dot v_\varepsilon\|_{L^\infty(\eta,T;H)} \le C\delta,$$

for every $\eta \in (\sigma, T)$. By the arbitrariness of $\delta > 0$ and $\sigma > 0$ we conclude. □

Now we are in position to prove (4.41).

Theorem 4.6.2. *Let us assume (H1), (H2), (4.43), and $f_\varepsilon = 0$ for every $\varepsilon > 0$. Let u_ε be the solution to the viscoelastic dynamic system (4.9), with $\varphi_\varepsilon = 0$ and $\gamma_\varepsilon = g_\varepsilon$, and let u_0 be the solution to the stationary problem (4.36), with $f = 0$. Then (4.41) holds.*

Proof. By Theorems 4.5.2 and 4.5.3 we obtain (4.39) and (4.40). Since $g_\varepsilon \to g$ strongly in $W^{1,1}(0,T;V_0')$ as $\varepsilon \to 0^+$ by (H1) and $f_\varepsilon = 0$, we can apply Lemma 4.6.1 to conclude. □

Theorem 4.6.3. *Let us assume (H1)–(H3) and $f_\varepsilon = 0$ for every $\varepsilon > 0$. Let u_ε be the solution to the viscoelastic dynamic system (4.8) and let u_0 be the solution to the stationary problem (4.36), with $f = 0$. Then (4.41) holds.*

Proof. Thanks to Lemma 4.2.9 we can suppose $z = 0$ and $z_\varepsilon = 0$ for every $\varepsilon > 0$. By Theorem 4.5.4 we obtain (4.39) and (4.40). Since u_ε is a solution to (4.8) with $f_\varepsilon = 0$, by Remark 4.1.3 it solves (4.12) with $h_\varepsilon = 0$ and $\ell_\varepsilon = g_\varepsilon - p_\varepsilon$, where p_ε is defined by (4.11). Since

$$g_\varepsilon - p_\varepsilon \xrightarrow[\varepsilon\to 0^+]{W^{1,1}(\eta,T;V_0')} g \quad \text{for every } \eta \in (0,T),$$

we can apply Lemma 4.6.1 to conclude. □

Finally we can prove Theorems 4.2.6 and 4.2.7.

Proof of Theorem 4.2.6. It is enough to combine Theorems 4.3.1, 4.5.4, and 4.6.3. □

Proof of Theorem 4.2.7. It is enough to combine Theorems 4.5.2, 4.5.3, and 4.6.2. □

4.7 Appendix

Throughout this section we fix $a_0 > 0$, $b_0 > 0$, and $c_1 \geq c_0 > 1$. For every a, b with

$$c_0 a \leq b \leq c_1 a, \qquad b \geq b_0, \qquad a \geq a_0, \tag{4.148}$$

we consider the polynomial $p(z) := \beta z^3 + z^2 + \beta b z + a$ depending on the complex variable z. The following result about the roots of this polynomial is used in the proof of Lemma 4.4.2 and Proposition 4.4.4.

Lemma 4.7.1. *There exists a positive constant $\alpha = \alpha(\beta, a_0, b_0, c_0, c_1)$ such that, for every $a, b \in \mathbb{R}$ satisfying (4.148), the roots of the polynomial p have real parts in the interval $(-\frac{1}{\beta}, -\alpha)$.*

Proof. Let us set $z := x + iy$ with $x, y \in \mathbb{R}$. Then $p(z) = 0$ if and only if

$$\begin{cases} \beta x^3 + x^2 + \beta b x - (3\beta x + 1)y^2 + a = 0, \\ y(-\beta y^2 + 3\beta x^2 + 2x + \beta b) = 0, \end{cases}$$

from which we derive

$$\begin{cases} q(x) := \beta x^3 + x^2 + \beta b x + a = 0, \\ y = 0, \end{cases} \tag{4.149}$$

$$\begin{cases} r(x) := 8\beta x^3 + 8x^2 + 2\Big(\frac{1}{\beta} + \beta b\Big)x + b - a = 0, \\ y^2 = 3x^2 + \frac{2}{\beta}x + b. \end{cases} \tag{4.150}$$

By recalling $a > 0$ and $b - a \geq (c_0 - 1)a > 0$, for every $x \geq 0$ we have $q(x) > 0$ and $r(x) > 0$, and so the real part of the roots cannot be positive or zero. Moreover, since for every $x \leq -\frac{1}{\beta}$ we have $\beta x^3 + x^2 \leq 0$, we obtain

$$q(x) \leq -b + a \leq (1 - c_0)a < 0 \qquad \text{and} \qquad r(x) \leq b - a - 2\left(\frac{1}{\beta^2} + b\right) = -b - a - \frac{2}{\beta^2} < 0,$$

which imply that the real part of the roots does not belong to $(-\infty, -\frac{1}{\beta}]$. Therefore, by calling $z_1, z_2, z_3 \in \mathbb{C}$ the three roots of the polynomial p, we can say

$$\Re(z_i) \in (-\tfrac{1}{\beta}, 0) \quad \text{for } i = 1, 2, 3. \tag{4.151}$$

Case 1: there is only one real root. In this case by (4.150) there exists a unique $x_1 \in (-\frac{1}{\beta}, 0)$ which satisfies $r(x_1) = 0$ and $3x_1^2 + \frac{2}{\beta}x_1 + b > 0$. Indeed by setting $y_1 := \sqrt{3x_1^2 + \frac{2}{\beta}x_1 + b}$ we obtain that $x_1 + iy_1$ and $x_1 - iy_1$ are two distinct non-real roots of p. Since

$$\begin{aligned} r(-\tfrac{1}{2\beta}) &= -\tfrac{1}{\beta^2} + \tfrac{2}{\beta^2} - \tfrac{1}{\beta^2} - b + b - a = -a < 0, \\ r(-\tfrac{\beta(b-a)}{2(b\beta^2+1)}) &= \tfrac{\beta^2(b-a)^2((a+b)\beta^2+2)}{(b\beta^2+1)^3} > 0, \end{aligned}$$

then $x_1 \in (-\frac{1}{2\beta}, -\frac{\beta(b-a)}{2(b\beta^2+1)})$. Moreover

$$\begin{aligned} q(-\tfrac{1}{\beta}) &= -\tfrac{1}{\beta^2} + \tfrac{1}{\beta^2} - b + a = -b + a < 0, \\ q(-\tfrac{a}{\beta b}) &= -\tfrac{a^3}{b^3\beta^2} + \tfrac{a^2}{b^2\beta^2} - a + a = \tfrac{a^2(b-a)}{b^3\beta^2} > 0, \end{aligned}$$

hence there exists $x_0 \in (-\frac{1}{\beta}, -\frac{a}{\beta b})$ such that $q(x_0) = 0$. As a consequence of this, $(x_0, 0)$ satisfies (4.149), which implies that x_0 is the real root of p, hence we have

$$\Re(z_i) \in (-\frac{1}{\beta}, \max\{-\frac{a}{\beta b}, -\frac{\beta(b-a)}{2\,(b\beta^2+1)}\}).$$

Thanks to (4.148) we can say $-\frac{a}{\beta b} \leq -\frac{1}{c_1\beta}$ and $-\frac{\beta(b-a)}{2(b\beta^2+1)} \leq \frac{\beta(1-c_0)a}{2(c_1 a\beta^2+1)} \leq \frac{\beta(1-c_0)a_0}{2(c_1 a_0\beta^2+1)}$, where in the last inequality we use the decreasing property of the function $a \mapsto \frac{\beta(1-c_0)a}{2(c_1 a\beta^2+1)}$. This implies

$$\Re(z_i) \in (-\tfrac{1}{\beta}, \max\{-\tfrac{1}{c_1\beta}, \tfrac{\beta(1-c_0)a_0}{2(c_1 a_0\beta^2+1)}\}) \quad \text{for } i = 1, 2, 3. \tag{4.152}$$

Case 2: there are only real roots. In this case we have $b \leq \frac{1}{3\beta^2}$, otherwise $q'(x) > 0$ for every $x \in \mathbb{R}$, which forces p to have also non-real roots. Thanks to (4.148) we have also $a < b \leq \frac{1}{3\beta^2}$. By setting $\tilde{b}_0 := 1 - \sqrt{1 - 3b_0\beta^2}$, we can write

$$-\tilde{b}_0 a_0\beta \geq -\tilde{b}_0 a\beta \geq -(1 - \sqrt{1 - 3b\beta^2})a\beta > \tfrac{-1+\sqrt{1-3b\beta^2}}{3\beta} > -\tfrac{1}{\beta},$$

which implies

$$q'(x) > 0 \quad \text{ for every } x \in [-\tilde{b}_0 a_0\beta, +\infty). \tag{4.153}$$

Since

$$q(-\tilde{b}_0 a_0\beta) \geq \beta^2\tilde{b}_0^2 a_0^2(1 - \beta^2\tilde{b}_0 a_0) + a_0(1 - \beta^2\tilde{b}_0 b) > a_0(1 + \beta^2\tilde{b}_0^2 a_0)(1 - \beta^2\tilde{b}_0 b) > 0,$$

thanks to (4.148), (4.151), and (4.153) we get

$$\Re(z_i) \in (-\tfrac{1}{\beta}, -\tilde{b}_0 a_0\beta), \quad \text{for } i = 1, 2, 3. \tag{4.154}$$

By combining (4.152) and (4.154), we obtain the conclusion with

$$\alpha := \min\{\tilde{b}_0 a_0\beta, \frac{1}{c_1\beta}, \frac{\beta(c_0-1)a_0}{2\,(c_1 a_0\beta^2+1)}\}.$$

□

The following easy estimate is used in the proof of Lemma 4.4.2.

Lemma 4.7.2. *For every $z, w \in \mathbb{C}$ with $\Re(z) > 0$ and $\Re(w) < 0$ the following inequality holds:*

$$|(z - w)(z - \bar{w})| \geq |\Re(w)||\Im(w)|.$$

Proof. Without loss of generality we can suppose $\Im(w) > 0$, otherwise we exchange the role of w with $\bar{w}$. If $\Im(z) > 0$, then

$$|z - w| \geq |\Re(z - w)| = |\Re(z) + \Re(-w)| = \Re(z) + \Re(-w) \geq |\Re(w)|,$$
$$|z - \bar{w}| \geq |\Im(z - \bar{w})| = |\Im(z) + \Im(w)| = \Im(z) + \Im(w) \geq |\Im(w)|,$$

which give the conclusion in this case. If $\Im(z) < 0$, then

$$|z - w| \geq |\Im(z - w)| = |-\Im(-z) - \Im(w)| = \Im(-z) + \Im(w) \geq |\Im(w)|,$$
$$|z - \bar{w}| \geq |\Re(z - \bar{w})| = |\Re(z) + \Re(-w)| = \Re(z) + \Re(-w) \geq |\Re(w)|,$$

which conclude the proof. □

Bibliography

[1] V. Agostiniani: Second order approximations of quasistatic evolution problems in finite dimension, *Discrete Contin. Dyn. Syst.* **32** (2010), 1125–1167.

[2] L. Ambrosio, N. Gigli, and G. Savaré: Gradient flows in metric spaces and in the space of probability measures. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.

[3] W. Arendt, C. Batty, M. Hieber, F. Neubrander: Vector-valued Laplace Transforms and Cauchy Problems, Monographs in Mathematics **96**. Birkhäuser, Basel, 2001.

[4] T.M. Atanackovic, M. Janev, L. Oparnica, S. Pilipovic, and D. Zorica: Space-time fractional Zener wave equation. *Proc. A.* **471** (2015), 25 pp.

[5] L. Boltzmann: Zur Theorie der elastischen Nachwirkung, *Sitzber. Kaiserl. Akad. Wiss. Wien, Math.-Naturw. Kl.* **70** (1874), 275–300.

[6] M. Caponi: Linear hyperbolic systems in domains with growing cracks, *Milan J. Math.* **85** (2017), 149–185.

[7] M. Caponi, F. Sapio: A dynamic model for viscoelastic materials with prescribed growing cracks, *Ann. Mat. Pura Appl.* **198** (2019).

[8] M. Caponi, F. Sapio: An existence result for the fractional Kelvin-Voigt's model on time-dependent cracked domains, *J. Evol. Equ.* (2021).

[9] A. Carbotti and G. Comi: A note on Riemann-Liouville fractional Sobolev spaces. *Commun. Pure Appl. Anal.* **20** (2021), 17–54.

[10] S. Carillo, V. Valente, G. Vergara Caffarelli: Heat conduction with memory: A singular kernel problem. *Evol. Equ. Control Theory* **3** (2014), 399–410.

[11] C. Dafermos: An abstract Volterra equation with applications to linear viscoelasticity, *J. Differential Equations* **7** (1970), 554–569.

[12] C. Dafermos: Asymptotic stability in viscoelasticity, *Arch. Rational Mech. Anal.* **37** (1970), 297–308.

[13] G. Dal Maso and C.J. Larsen: Existence for wave equations on domains with arbitrary growing cracks, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **22** (2011), 387–408.

[14] G. Dal Maso, C.J. Larsen, and R. Toader: Existence for constrained dynamic Griffith fracture with a weak maximal dissipation condition. *J. Mech. Phys. Solids* **95** (2016), 697–707.

[15] G. Dal Maso, C.J. Larsen, and R. Toader: Existence for elastodynamic Griffith fracture with a weak maximal dissipation condition. *J. Math. Pures Appl.* **127** (2019), 160–191.

[16] G. Dal Maso and I. Lucardesi: The wave equation on domains with cracks growing on a prescribed path: existence, uniqueness, and continuous dependence on the data, *Appl. Math. Res. Express* **2017** (2017), 184–241.

[17] G. Dal Maso and R. Toader: On the Cauchy problem for the wave equation on time–dependent domains, *J. Differential Equations* **266** (2019), 3209-3246.

[18] G. Dal Maso and F. Sapio: Quasistatic limit of a dynamic viscoelastic model with memory, *Accepted Paper: Milan Journal of Mathematics* (2021).

[19] R. Dautray and J.L. Lions: Mathematical Analysis and Numerical Methods for Science and Technology, Volume 1 Physical Origins and Classical Methods, Original French edition published by Masson, S.A., Paris, 1984

[20] R. Dautray and J.L. Lions: Analyse mathématique et calcul numérique pour les sciences et les techniques. Vol. 8. Évolution: semi-groupe, variationnel, Masson, Paris, 1988.

[21] S. Dipierro, E. Valdinoci, and V. Vespri: Decay estimates for evolutionary equations with fractional time-diffusion. *J. Evol. Equ.* **19**, (2019) 435–462.

[22] C.S. Drapaca and S. Sivaloganathan: A fractional model of continuum mechanics. *J. Elasticity* **107** (2012), 105–123.

[23] M. Fabrizio: Fractional rheological models for thermomechanical systems. Dissipation and free energies. *Fract. Calc. Appl. Anal.* **17** (2014), 206–223.

[24] M. Fabrizio, C. Giorgi, V. Pata: A New Approach to Equations with Memory, *Arch. Rational Mech. Anal.* **198** (2010), 189-232.

[25] M. Fabrizio and A. Morro: Mathematical Problems in Linear Viscoelasticity. SIAM Studies in Applied Mathematics **12** 2001.

[26] P. Gidoni and F. Riva: A vanishing inertia analysis for finite dimensional rate-independent systems with nonautonomous dissipation and an application to soft crawlers, *Calc. Var. Partial Differential Equations* **60** (2021) Paper 191.

[27] A.A. Griffith: The phenomena of rupture and flow in solids, *Philos. Trans. Roy. Soc. London* **221-A** (1920), 163–198.

[28] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo: Theory and applications of fractional differential equations. North-Holland Mathematics Studies, Amsterdam, 2006.

[29] S. Konjik, L. Oparnica, and D. Zorica: Waves in fractional Zener type viscoelastic media. *J. Math. Anal. Appl.* **365**, (2010) 259–268.

[30] O.A. Ladyzenskaya: On integral estimates, convergence, approximate methods, and solution in functionals for elliptic operators, *Vestnik Leningrad. Univ.* **13** (1958), 60–69.

[31] G. Lazzaroni and L. Nardini: On the quasistatic limit of dynamic evolutions for a peeling test in dimension one, *J. Nonlinear Sci.* **28** (2018), 269–304.

[32] L. Li and J.G. Liu: A generalized definition of Caputo derivatives and its application to fractional ODEs. *SIAM J. Math. Anal.* **50**, (2018) 2867–2900.

[33] J.L. Lions: Équations Différentielles Opérationnelles et Problèmes aux Limites, Springer Berlin Heidelberg, 1961.

[34] J.L. Lions and E. Magenes: Non-Homogeneous Boundary-Value Problems and Applications. Vol. 1 **181**, Springer-Verlag, 1972.

[35] F. Mainardi: Fractional calculus and waves in linear viscoelasticity. Imperial College Press, London, 2010.

[36] N.F. Mott: Brittle fracture in mild steel plates, *Engineering* **165** (1948), 16–18.

[37] L. Nardini: A note on the convergence of singularly perturbed second order potential-type equations, *J. Dynam. Differential Equations* **29** (2017), 783–707.

[38] S. Nicaise and A.M. Sändig: Dynamic crack propagation in a 2D elastic body: the out–of–plane case, *J. Math. Anal. Appl.* **329** (2007), 1–30.

[39] O.A. Oleinik, A.S. Shamaev, and G.A. Yosifian: Mathematical problems in elasticity and homogenization. Studies in Mathematics and its Applications, 26. North-Holland Publishing Co., Amsterdam, 1992.

[40] L. Oparnica and E. Süli: Well-posedness of the fractional Zener wave equation for heterogeneous viscoelastic materials. *Fract. Calc. Appl. Anal.* **23**, (2020) 126–166.

[41] I. Podlubny Fractional Differential Equations. Academic Press, San Diego, 1999.

[42] F. Riva: On the approximation of quasistatic evolutions for the debonding of a thin film via vanishing inertia and viscosity, *J. Nonlinear Sci.* **30** (2020), 903–951.

[43] S.G. Samko, A.A. Kilbas, and O.I. Marichev: Fractional integrals and derivatives. Theory and applications. Edited and with a foreword by S. M. Nikol'skii. Translated from the 1987 Russian original. Revised by the authors. Gordon and Breach Science Publishers, Yverdon, 1993.

[44] F. Sapio: A dynamic model for viscoelasticity in domains with time-dependent cracks, *Accepted Paper: Nonlinear Differential Equations and Applications NoDEA* (2021).

[45] G. Scilla and F. Solombrino: A variational approach to the quasistatic limit of viscous dynamic evolutions in finite dimension, *J. Differential Equations* **267** (2019), 6216–6264.

[46] H. Schiessel, R. Metzler, A. Blumen, and T.F. Nonnenmacher: Generalized viscoelastic models: their fractional equations with solutions. *J. Phys. A: Math. Gen.* **28** (1995), 6567–6584.

[47] L.I. Slepyan: Models and phenomena in fracture mechanics, Foundations of Engineering Mechanics. Springer-Verlag, Berlin, 2002.

[48] E. Tasso: Weak formulation of elastodynamics in domains with growing cracks. *Ann. Mat. Pura Appl.*, **199** (2020), 1571–1595.

[49] V. Volterra: Sur la théorie mathématique des phénomènes héréditaires, *J. Math. Pures Appl.* **7** (1928), 249–298.

[50] H.H. Zhu, L.C. Liu, H.F. Pei, and B. Shi: Settlement analysis of viscoelastic foundation under vertical line load using a fractional Kelvin-Voigt model. *Geomech. Eng.*, **4** (2012), 67–68.

[51] R. Zacher: Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces. *Funkcial. Ekvac.* **52** (2009), 1–18.