

SISSA

Scuola
Internazionale
Superiore di
Studi Avanzati

Physics Area - PhD course in
Theoretical Particle Physics

Wandering around
the walls of massive Super-QCD

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Academic Year 2020-21



Declaration of Authorship

I, Paolo SPEZZATI, declare that this thesis titled, “Wandering around the walls of massive Super-QCD” and the work presented in it are my own. Where I have consulted the published work of others, this is always clearly attributed. This thesis is based on two research papers still unpublished

- S. Benvenuti, P. Spezzati “Mildly Flavoring Domain Walls in $Sp(N)$ SQCD”, *arXiv: 2106.08772*
- S. Benvenuti, P. Spezzati, “Mildly Flavoring domain walls in $SU(N)$ SQCD: baryons and monopole superpotential”, *to appear*, 2021

SISSA – SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI

Abstract

Doctor of Philosophy

Wandering around the walls of massive Super-QCD

by Paolo SPEZZATI

In this thesis, we consider supersymmetric domain walls of four-dimensional $\mathcal{N} = 1$ super-QCD for the simply-connected gauge groups $Sp(N)$ and $SU(N)$. The original results of the thesis are the natural continuation of the work [1]. First, we construct the BPS domain walls numerically when the number of flavors $F = N, N + 1$ in the $SU(N)$ case, and $F = N + 1, N + 2$ in the $Sp(N)$ case.

In the second part, we discuss some proposals for the low energy descriptions of the physics on the domain walls. These proposals pass various tests, including dualities and matching of the vacua of the massive $3d$ theory with the $4d$ analysis. However, our analysis is partially successful in some cases. In those cases, we suggest that new purely quantum phenomena are needed to properly describe the low energy physics.

Acknowledgements

I express my gratitude to my advisor Prof. Francesco Benini for his guidance and the numerous discussion that brought this project to life. I want to thank Dr. Sergio Benvenuti deeply for his essential contribution and help to finish this thesis, his support, and our shared passion beside physics, track and field.

I am infinitely indebted to my family, my parents Valentina and Renzo, and my sister Maria. Their support and encouragement were always present throughout my life, whether I faced happy moments or discouraging circumstances.

Special thanks to my colleagues and friends at SISSA, with whom I shared this incredible journey. I also thank all the friends I met in Padua that are now scattered around the globe. We shared our passion for physics and science, and it was awesome.

I also want to thank my friends in Bassano that kept me aware of the real world during my journey into the beauty of mathematical abstraction. I am grateful for all the wonderful moments we shared. Last but not least, I want to thank Rebecca for the part of our lives we shared so far; she was always patient and loving.

Contents

Declaration of Authorship	iii
Abstract	v
Acknowledgements	vii
1 Introduction	1
1.1 Domain Walls	2
1.2 The strategy of domain wall analysis and outline	4
2 Domain walls	7
2.1 Domain walls: generalities	7
2.1.1 BPS domain walls	8
2.1.2 Counting domain walls	10
2.1.3 Brief recap of BPS domain walls in SYM and SQCD	11
3 Four-dimensional construction of BPS domain walls	15
3.1 Numerical analysis of the BPS equations for $Sp(N)$	15
3.1.1 $Sp(N)$ with $F = N + 1$	16
3.1.2 $Sp(N)$ with $F = N + 2$	19
The parity-invariant walls, $k = \frac{N+1}{2}$	23
3.2 Numerical analysis of the BPS equations for $SU(N)$	26
3.2.1 $SU(N)$ with $F = N$	26
3.2.2 $SU(N)$ with $F = N + 1$	32
The parity-invariant walls, $k = \frac{N}{2}$	33
4 Living on the walls	37
4.0.1 Aspects of QFTs in $3d$	37
4.0.2 Particle-vortex duality	39
4.1 Living on the walls	41
4.1.1 $Sp(N)$ with $F = N + 1$	41
4.1.2 $Sp(N)$ with $F = N + 2$	45
4.1.3 $SU(N)$ with $F = N$ flavors	48
4.1.4 $SU(N)$ with $F = N + 1$	52
5 Conclusions and Outlook	57

A	Supersymmetry: Preliminaries	59
A.0.1	$\mathcal{N} = 1$ supersymmetry in four dimensions	59
A.0.2	Supersymmetry in three and two dimensions	64
B	Level-Rank Duality	67
B.0.1	Representations of $SU(N)_K$	67
B.0.2	Representations of $U(K)_N$	68
B.0.3	Representations of $Sp(N)_K$	68
B.0.4	Duality Map	69
C	Alternative effective theory of domain walls of $SU(N)$ $F = N + 1$.	71
D	Vacuum structure analysis for the $3d$ domain wall theory of $SU(N)$ with $N + 1$ flavors	73
E	sQED with a linear monopole superpotential	77

Chapter 1

Introduction

Quantum Field Theory has been the language of theoretical physics for the past seventy years. Although initially developed to understand elementary particles, it has been used in several other branches of physics, such as statistical physics, condensed matter physics, cosmology, and drove the developments of old and new branches of mathematics.

Quantum field theories are notoriously hard to study, physicists being able to achieve success in the understanding of QFTs when dealing with free theories or weakly interacting models using perturbation theory. Anyway, intrinsically quantum phenomena are really important in understanding natural phenomena, with the famous example of Quantum Chromodynamics (QCD) and its low energy behavior. However, not all the strongly interacting QFTs are born equal. In fact, there are some tools and setups that allow studying effectively strongly interacting regimes. One of these setups is when the model studied enjoys supersymmetry (SUSY). What supersymmetry does, is enlarging the Poincarè algebra with fermionic generators (also called supercharges). In this way, relations between matter content and forces of the theory are set. It turns out that with “enough” supersymmetry in three and four dimensions, namely four supercharges or more, the models satisfy some *non-renormalization theorems* which greatly simplify the analysis of the low energy phenomena. However, one can wonder if there is any hope to extend the results obtained in “enough” supersymmetric QFTs to “not-enough”-supersymmetric or even non-supersymmetric QFTs.

Another tool to study the low energy behavior of QFTs is represented by infrared (IR) dualities. In this context, duality means that two (or more) very different QFTs describe the same physics below a certain energy threshold. This means that one can provide a map between operators and states across the two models. This kind of duality, enjoyed by pairs of models, it turns out to be vitally important because usually, one of the two models is strongly coupled in the infrared. Indeed, studying the physics of the weakly coupled side of the duality, answer some question on the strongly coupled side of the duality.

The past decades witnessed many developments in our understanding of the infrared dynamics of strongly coupled quantum field theories, especially in three and four dimensions. This progress has shown exciting connections between the QFTs in these two different dimensions. In particular, we will address some of these connections

that relate “enough”-supersymmetric theories with models that do not have “enough” supersymmetry to enjoy non-renormalization theorems.

1.1 Domain Walls

One of the contexts in which the connection between QFTs with different dimensionality is manifest is when the four-dimensional QFTs admit domain wall solutions. These domain walls are co-dimension one solitonic objects with finite tension that can be present when the vacuum structure of the model consists of multiple isolated gapped vacua.

One concrete example of this setup is offered by Yang-Mills theory and $4d$ massive QCD. It is believed that these theories have two gapped vacua at the special value $\theta = \pi$ of the topological theta term [2]–[5]. Indeed, it has been shown at large N that as we vary θ through π , there is a first-order phase transition associated with the spontaneous breaking of the time-reversal symmetry [6]–[9]. This means that there are two degenerate vacua at $\theta = \pi$. Anomaly considerations allow arguing that this first-order phase transition is there also at finite N [2]. This setup offers the possibility of constructing a domain wall between the two vacua and studying its dynamics at low energies. Since the four-dimensional vacua are gapped, the $3d$ dynamic of the world-volume theory on the wall is decoupled from the bulk $4d$ theory. This is where the interesting connection between $3d$ QFTs and $4d$ QFTs shows up. For example, in the YM case at $\theta = \pi$, the domain wall theory is composed by the center of mass mode and by a $3d$ topological quantum field theory (TQFT) $SU(N)_1$ Chern-Simons theory.

Richer is the story of massive QCD since the dynamic of massless QCD depends on the number of flavors F and colors N . Briefly, the behavior of the massless model follows this pattern: for $F > \frac{11}{2}N$ the model is IR free, for $N^* \leq F < \frac{11}{2}N$ the theory flows to a non-trivial fixed point in the IR, whereas for $F < N^*$ there is chiral symmetry breaking and the theory in the IR is a non-linear sigma model (NLSM) with target space $SU(F)$. The value N^* is yet unknown. Once we consider the massive theory, even if from the $4d$ perspective the model is gapped and smooth till the quark masses $m_{4d} = 0$, the $3d$ theory trapped on the domain wall undergoes a phase transition as we vary m_{4d} . Obviously, the phase transition and the phases before and after such transition depend on the behavior of the massless theory and, therefore, on the range of the parameters F and N we are considering. For example, in one of such range $1 < F < N^*$, changing the four-dimensional mass parameter of the quarks leads to a phase transition on the domain wall, from a Chern-Simon topological theory to a $\mathbb{C}\mathbb{P}^{F-1}$ non-linear sigma model (NLSM).

Other important theories with multiple gapped vacua are $4d$ $\mathcal{N} = 1$ Super Yang-Mills (SYM) and massive SQCD. These theories have “enough” supersymmetry to enjoy non-renormalization theorems and holomorphy. In this context also, supersymmetry allows for a special kind of domain walls: BPS domain walls preserving half of the supercharges, with computable minimal tension. Since there are many vacua and possibly more than one supersymmetric domain wall connecting each pair of vacua,

the zoo of the domain walls in SQCD is considerably richer than in QCD. The study of the zoo of domain walls of these theories is the scope of this Ph.D. thesis.

The study of the domain walls in $4d$ SQCD has been carried out in [1], [10]–[31].

Acharya and Vafa [10] studied the domain walls of pure SYM ($F = 0$ SQCD) for $SU(N)$ gauge group, proposing an appropriate TQFT as the $3d$ effective description of the walls. Due to the presence of a broken discrete symmetry, in this case, the R-symmetry, there are relations between the theories on different domain walls connecting different vacua. In this case, another property of $3d$ $\mathcal{N} = 1$ theories comes into play, namely infrared dualities. Here what is called level-rank duality connects the different domain walls, which are related by the $4d$ discrete symmetry. A review of what level-rank duality is will be given in the appendix B. SYM's with other gauge groups were considered in [1], [30], [31].

The study of domain walls of SQCD has begun in the early 2000 by [25], [26] (and before by [15]–[20]). These first works addressed the BPS domain walls to compute the Cecotti-Fendley-Intriligator-Vafa index for SYM domain walls, and they focused on the case of $F = N$. Interestingly, they correctly found the domain walls giving the CFIV, but they did not study other solutions involving the baryons, which we discover in our analysis. The domain walls of $SU(N)$ SQCD when $F < N$ were at that time neglected and have been addressed only recently by [1]. They [1] studied domain walls in SQCD, with $SU(N)$ and $Sp(N)$ gauge group and the number of flavors less than h , the dual Coxeter number of the gauge algebra ($h = N$ in the case of $SU(N)$ gauge group). In this thesis, we add more flavors to the story of [1], focussing on the case of $Sp(N)$ gauge group with $F = N + 1$ and $F = N + 2$ flavors (F flavors means $2F$ fundamentals) and of $SU(N)$ SQCD with N and $N + 1$ flavors. This thesis is based on the papers [32], [33].

Crucially the study of the physics on the domain walls of SQCD in [1] also relied on other recent developments about $3d$ $\mathcal{N} = 1$ theories. Note that $3d$ $\mathcal{N} = 1$ theories have two supercharges, and therefore their behavior is more similar to a non-supersymmetric theory than to a $\mathcal{N} = 1$ $4d$ model. Part of the strategy to classify the domain walls is to provide a $3d$ worldvolume description of the low energy dynamics on the walls. On the one hand, this allows capturing eventual phase transitions, like the ones described for the non-supersymmetric QCD. These transitions are expected because, as we will see, the low energy effective theories describing massive SQCD are very different depending on whether the quarks' masses are large or small. On the other hand, since we have relations between different domain walls connecting different vacua for the special case of $4d$ $\mathcal{N} = 1$ massive SQCD, we get that $3d$ infrared dualities are involved when describing the Conformal Field Theory (CFT) at the transition point. This kind of infrared dualities, whose concept was introduced in the 70' with the famous particle-vortex duality, have seen rapid growth in the past decade for $3d$ theories with two supercharges and without supersymmetry. In particular a lot of new dualities have been proposed in [30], [34]–[47]. Some of these new dualities have interesting connections with the domain walls, as we will explain in Chapter 4.

1.2 The strategy of domain wall analysis and outline

Our strategy to study the BPS domain walls, as in [1], consists of two separate parts: a $4d$ side and a $3d$ side.

On the $4d$ side, we study the domain wall solutions numerically. Indeed, thanks to a Wess-Zumino-like (WZ) description of the model at low energy in the regime of small quark masses, we solve the BPS equations numerically. Instead, at large quark masses, the SQCD model is described at low energy by the Super Yang-Mills (SYM) theory. The domain walls of this model are known thanks to the work of [1], [10]). We present a possible classification for the solutions of the models $Sp(N)$ with $F = N + 1$, and $F = N + 2$ flavors and for $SU(N)$ with $F = N$ and $F = N + 1$. For each model and number of flavors, we separately study each sector of domain walls, the sectors defined by the vacua at the opposite side of the universe, which are interpolated by the domain walls. Each is defined by an integer k ranging from 1 to the dual Coxeter number of the gauge algebra of the model. We will also describe a non-trivial relation between pairs of these sectors. This analysis will be carried out in Chapter 3.

One interesting special case is the k -wall for $Sp(N)$ with $F = N + 2$ and $k = \frac{N+1}{2}$ ¹. In this case, a naive analysis provides only one $Sp(F)$ invariant solution. This is at odds with expectations from $k \neq \frac{N+1}{2}$. We address this puzzle by making an infinitesimal deformation of the differential equations, which is equivalent to changing the Kähler potential, explicitly breaking the flavor symmetry. These deformed equations allow us to understand better the nature of the seemingly trivial solution found. In fact, the trivial solution can be seen as a superposition of different solutions, whose moduli spaces have the correct Witten Index to match the one expected. However, a complete picture is yet to be understood. We leave the complete analysis of the classification and counting (weighted by the Witten-Index) of these classes of solutions to future work.

Another interesting case is $SU(N)$ with $F = N$ flavors. As we have said, this case was already addressed by [25], [26]. Still, they missed important solutions related to the existence of the baryon and anti-baryon operators as coordinates of the moduli space of massless SQCD. These new walls that we have found do not modify the formula for the CFIV index or the “counting the domain walls”. However, they give rise to particular phenomena in the $3d$ dimensional description of the low energy physics on the walls.

On the $3d$ side, educated guesses, similar to the ones in [1], about the $3d$ effective description of the physics on the domain wall are made, in terms of $3d$ $\mathcal{N} = 1$ Chern-Simons-matter models with gauge group G (see [35], [39], [40], [48]–[50] for recent progress on $\mathcal{N} = 1$ $3d$ gauge theories). The massless theories sit at a phase transition between a set of vacua (corresponding to the domain walls at small $4d$ mass) and a single vacuum (corresponding to the domain walls at large $4d$ mass, that is $4d$ SYM).

¹These kind of phenomenon is present also for $SU(N)$ with $F = N + 1$ flavors. This is because once we integrate out the baryons, the Wess-Zumino model, which describes the effective dynamics, is the same of $Sp(N)$ with $F = N + 2$.

These vacua host a product of a TQFT and an NLSM. Our $3d$ proposals are argued to satisfy a non-trivial infrared duality, incarnating the $4d$ equivalence between different sectors of domain walls. We stress the rationale behind such $3d$ $\mathcal{N} = 1$ dualities, namely their close relation with known and tested $\mathcal{N} = 2$ dualities. All these proposals and considerations are described in Chapter 4.

A check that worked well in [1] is the comparison of the semiclassical vacua of the massive theory across the duality and with the $4d$ analysis. In the cases studied in this paper, such a comparison works perfectly for $Sp(N)$ with $F = N + 1$, while it works only partially for $F = N + 2$ and $SU(N)$ with $F = N$ and $F = N + 1$. We ascribe such a failure to strong coupling effects present in our $3d$ models, basically due to the smallness of the Chern-Simons level. These models probably require a new non-trivial approach to the study of their vacua because the Chern-Simons level of the models is of the same order as the rank of the three-dimensional gauge group. This gives rise to strong coupling effects and possibly to phenomena similar to those described by [37]. We do not have a definite answer, but we will highlight some hints that this is the case.

Before digging into the description of $4d$ domain walls and the $3d$ Chern-Simons matter models that describe the low energy physics on the walls, we briefly give a summary on domain walls, BPS domain walls, and all the tools and concepts we will use throughout this thesis. This is done in Chapter 2. Finally, in the Appendix A we give the notation and basic definition to properly address supersymmetric models.

Chapter 2

Domain walls

2.1 Domain walls: generalities

In this introductory chapter, we will talk about domain walls in supersymmetric theories in four dimensions. These extended objects exist when a field theory possesses more than one degenerate and isolated vacuum state. We note that the degeneracy property in supersymmetric theories is always realized if we consider supersymmetric vacua. Indeed, supersymmetry dictates that the energy of all vacuum states is zero, hence ensuring degeneracy.

Assume now that we are considering a theory that has more than one discrete vacuum state. Assume also that at the two opposite ends (at $x = -\infty$ and $x = \infty$) of the universe the fields are in different vacuum configurations. Then we call domain wall the smooth configuration of the fields which interpolates between the two different vacuum configurations. This configuration is topologically stable also in non-supersymmetric theories. On the transition region, the energy density is higher than the vacuum energy, and the field configuration obviously organizes itself such that the excess energy due to such region is minimized. This, in turn, implies that classically the wall is flat in the directions perpendicular to the x axis. The difference between the vacuum energy and the energy of the domain wall is proportional to the area of the wall and can be written as

$$E_{wall} - E_{vac} = AT, \quad (2.1)$$

where T is the so-called tension of the wall. If we are considering Minkowski spacetime, we see that the difference in energy is infinite because the area A is infinite. Note, however, that the tension is a finite quantity. The infinity of the energy of the domain wall is what prevents it to dynamically relax into a unique vacuum state on the whole universe. This, in turn, implies that once the system has different vacuum configurations at the ends of the universe, the dynamics generated by the equations of motion (EOMs) or by local non-singular sources cannot evolve the system into a different configuration of the fields at $x = \pm\infty$. The class of different maps that send $(x = -\infty, x = \infty)$ to the corresponding vacuum configuration of the fields is a topological property of the various sectors one can define. One can also introduce a topological 2-brane current trivially conserved, i.e., no EOMs are required for its conservation.

Note that a field configuration that interpolates between two different vacua at the two ends of the universe is not a setup constrained to be in four-dimensional spacetime. Indeed, a similar setup can also be studied in two dimensions. In this number of dimensions, there is the particular feature that these field configurations have finite energy. These configurations are called kinks. In reality, this seemingly trivial observation is crucial because to study some of the properties of the domain walls, we can use the results of the studying of kinks in two dimensions. In particular, we are interested in the kinks of $2d \mathcal{N} = (2, 2)$ theories related via dimensional reduction to the $4d \mathcal{N} = 1$ supersymmetric theories [51], [52]. Alternatively, one says that domain walls are obtained elevating kinks from two to four dimensions.

We will study a particular kind of domain walls in the following subsection: BPS domain walls. Since we are diving into the supersymmetric realm, we summarize notation and basic concepts of supersymmetry in the Appendix A.

2.1.1 BPS domain walls

Since we will deal with four-dimensional $\mathcal{N} = 1$ models, we can consider a particular type of domain walls, namely the ones that preserve half of the four real supercharges. In fact, the $4d \mathcal{N} = 1$ supersymmetry algebra admits central charges [53], [54]. These central charges are elements of the supersymmetry algebra which commute with all the other generators of the SUSY algebra. Strictly speaking, the central charges we are referring to are non-Lorentz invariant and commute only with the generators of supersymmetry Q_α , $\bar{Q}_{\dot{\alpha}}$, and the translation P_μ , but not with the Lorentz transformations because they carry Lorentz indices. Nonetheless, we are going to refer to them as central charges.

The possible extension of the $\mathcal{N} = 1$ algebra consist of 10 central charges, which can be decomposed in the $(1, 0) + (0, 1) + (\frac{1}{2}, \frac{1}{2})$ Lorentz representation¹:

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= 2(\gamma^\mu)_{\alpha\dot{\alpha}}(P_\mu + Z_\mu) \\ \{Q_\alpha, Q_\beta\} &= (\Sigma^{\mu\nu})_{\alpha\beta} Z_{[\mu\nu]} \\ \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} &= (\bar{\Sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} \bar{Z}_{[\mu\nu]} \end{aligned} \tag{2.2}$$

where $(\Sigma^{\mu\nu})_{\alpha\beta} = (\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\nu)^{\dot{\alpha}\beta}$ ². The antisymmetric tensors $Z_{[\mu\nu]}$ and $\bar{Z}_{\mu\nu}$ are related with the domain walls and are usually called “brane charges”, whereas Z_μ is perpendicular to P_μ and it is related with flux tubes. Moreover, the antisymmetric tensor $Z_{\mu\nu}$ can be written as a complex number Z and a spatial vector orthogonal to the domain wall. The complex number Z is what we call the central charge related to the domain wall. The relation of this central charge with domain walls can be derived by expressing the supersymmetric charges Q_α as integral of their currents and computing the commutator $\{Q_\alpha, Q_\beta\}$. Considering a Wess-Zumino (WZ) model for simplicity, one can see that this anticommutator is proportional to the difference $\Delta\mathcal{W}$ of the

¹For the supersymmetry notation see Appendix A which follows [52]

²Here $(\sigma^\mu)_{\alpha\dot{\beta}} = (1, \vec{\sigma})$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = (1, -\vec{\sigma})$. The indices $\alpha, \dot{\alpha}$ are raised and lowered using the antisymmetric tensors $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$.

superpotential \mathcal{W} of the model along a direction specified by a given vector. Now, if we are considering a topological sector in which the system is in two different vacuum configurations at the two ends of the universe, the anticommutator is different from zero, being $\Delta\mathcal{W} \neq 0$. This is exactly the setup in which we are considering domain walls.

If we are considering a BPS domain walls, their tension is fixed by the ‘‘central charge’’ Z

$$T = 2|Z|. \quad (2.3)$$

If the model has a Wess-Zumino effective description with superpotential given by the holomorphic function W , the central charge $Z = W(v_i) - W(v_j)$ is equal to the difference of the superpotential evaluated at the two vacua v_i, v_j at $x = \pm\infty$. So we see that the tension of a BPS domain wall does not depend on D-term; hence it is insensitive to changes of the Kähler potential. It is somehow protected and determined only by the F-terms.

Moreover, for WZ model

$$L = \int d^2\theta d^2\bar{\theta} \mathcal{K}(\Phi^a, \bar{\Phi}^a) + \left(\int d^2\theta W(\Phi^a) + h.c. \right) \quad (2.4)$$

we have also an explicit first order differential equation to compute BPS domain wall solutions [55], [56]:

$$\partial_x \Phi^a = e^{i\gamma} \mathcal{K}^{a\bar{b}} \bar{\partial}_b \bar{W}, \quad (2.5)$$

where Φ^a are the chirals of the WZ model, $\mathcal{K}^{a\bar{b}}$ is the inverse Kähler metric and $e^{i\gamma} = \frac{\Delta W}{|\Delta W|}$. These differential equations are easily derived from the correspondent two-dimensional system, looking for solutions with finite energy. This two-dimensional model is the dimensional reduced model on a flat torus. For a WZ model, the correspondent two-dimensional system is obtained considering the chiral fields as $2d$ chiral fields, the same functions of the chiral fields as superpotential and the Kähler potential. So considering the bosonic part of the $2d$ action obtained from the Lagrangian (2.4)

$$S = \int dx dy \mathcal{K}_{a\bar{b}} \partial_\mu \Phi^a \partial_\nu \bar{\Phi}^{\bar{b}} + \mathcal{K}^{\bar{b}a} \partial_a W(\Phi) \bar{\partial}_b \bar{W}(\bar{\Phi}), \quad (2.6)$$

we get that the energy of the kink between two vacua v_i and v_j is

$$E_{ij} = \int dx \left| \partial_x \Phi^a - e^{i\gamma} \mathcal{K}^{\bar{b}a} \bar{\partial}_b \bar{W} \right|^2 + 2\text{Re}(e^{-i\gamma} \Delta W(\Phi)), \quad (2.7)$$

where $\Delta W = W(v_i) - W(v_j)$ and $e^{i\gamma}$ an arbitrary phase. One can easily see that there is a minimum of the energy if one chooses $e^{i\gamma} = \frac{\Delta W}{|\Delta W|}$, with the energy bounded by

$$E_{ij} \geq 2|\Delta W|. \quad (2.8)$$

We see that this bound is saturated when (2.5) is satisfied. This is what is called

the Bogomolnyi bound. Furthermore, we point out that the fields that satisfy (2.5) automatically satisfy the EOMs of the theory. The converse, in general, is not true. Another crucial observation is that the supersymmetry transformations of the fermionic components of the superfields Φ^a (A.14) are proportional to (2.5), when we consider some specific parameters of the supersymmetric transformations. Therefore, along the domain wall where (2.5) is satisfied, some supersymmetry is preserved. Indeed, one can see that these configurations preserve half of the supersymmetry. In systems where we do not have a WZ-like description, one can still find domain wall solutions imposing that half of the supersymmetry transformations of the fermionic part of the superfields considered are equal to zero.

Note also that the trajectory of the domain wall in the W-space, in other words the image of $W(\Phi^a)$ along the domain wall solution, is a straight line

$$\partial_x W = \partial_a W \cdot \partial_x \Phi^a = e^{i\gamma} |\partial W|^2. \quad (2.9)$$

This fact will come in hand in the following because it gives us an integral of motion of (2.5), namely

$$\text{Im}(e^{-i\gamma} W(\Phi^a)) = \text{const}. \quad (2.10)$$

One here should point out that the very existence of the domain walls does not depend on the D-terms [57]. In other words, it is insensitive to the choice of the Kähler metric. This will allow us, in the following, to find domain wall solutions, to choose a Kähler metric as we like, provided that the Kähler metric does not have singularities along the domain wall solution.

2.1.2 Counting domain walls

BPS domain walls live in short multiplets since they preserve half of the supersymmetry. In two dimensions Cecotti Fendley Intriligator and Vafa proposed an “index” [58] counting the number of the kinks multiplets

$$I_{ij} = \text{Tr}_{ij}(F(-1)^F), \quad (2.11)$$

where F is the fermion number, and the trace is taken over the states of a particular sector in which the fields are at the vacua v_i and v_j at the two ends of the universe. The “index” (2.11) is not an actual index since it depends on some relevant and marginal deformations. However, these deformations are encoded in the F-terms, which are chiral and anti-chiral superfields integrated on half of the superspace. It is, therefore, an index with respect to D-terms deformations. Changing the Kähler potential smoothly is a D-terms deformation, therefore to compute the CFIV index, we have some freedom in choosing the Kähler potential. Using CPT, one can see that $I_{ii} = 0$ and that $I_{ij} = -I_{ji}^*$.

This story derived in the context of $2d$ theories can be extended to the $4d$ case, provided an infrared regulator is used to make the total wall energy finite.

An alternative way to see the CFIV index is that it counts the number of supersymmetric vacua of the $3d$ theory that lives on the domain wall. The counting of supersymmetric vacua, taking into account their fermion number, is given by the Witten Index

$$I_{ij} = \text{Tr}_{3d}(-1)^F. \quad (2.12)$$

This will be the point of view we are going to have throughout this thesis. So, we will always talk about Witten Indices even in the context of counting the domain walls, even if the CFIV index is more appropriate in those situations.

2.1.3 Brief recap of BPS domain walls in SYM and SQCD

In this next subsection we are going to briefly recap the known facts about domain walls of SYM with gauge group G^3 and in SQCD when the number of flavors $F < h$, $h = c_2(\mathfrak{g})^4$ being the dual Coxeter number associated to the gauge algebra \mathfrak{g} of G .

Consider $\mathcal{N} = 1$ SYM with gauge group G . The classical $U(1)_R$ R-symmetry, which assigns R -charge to the gaugini $R(\lambda) = 1$, is anomalous, and it is broken down to \mathbb{Z}_{2h} .

At low energy, strong coupling dynamics gives rise to an effective superpotential,

$$W_{SYM} = h\Lambda^3, \quad (2.13)$$

where Λ is the dynamically generated scale. This fact tells us that there is a gaugino condensate and that there are h vacua

$$\langle \lambda\lambda \rangle = -\frac{\partial W_{SYM}}{\partial \log \Lambda^{3h}} = \Lambda^3 \omega^k \quad (2.14)$$

where $\omega^h = 1$. Note that since the gauginos condensate, the R-symmetry is broken $\mathbb{Z}_{2h} \rightarrow \mathbb{Z}_2$. One can see this looking at the operator $\langle \lambda\lambda \rangle$, which takes VEV and has R -charge equals to two. The action of the \mathbb{Z}_h broken R-symmetry generators relates the different vacua sending

$$\mathbb{Z}_h = \{\sigma \mid \sigma^h = 1\}, \quad \sigma^j(\omega^k) = \omega^{j+k}. \quad (2.15)$$

Since the vacua are gapped⁵, it is possible to construct BPS domain walls between different vacua. The tension of the domain walls connecting the j^{th} vacuum and the $(j+k)^{\text{th}}$ vacuum is

$$T = 2|\Delta W_{SYM}| = 2h\Lambda^3 \left| \omega^k - 1 \right|. \quad (2.16)$$

The walls connecting the j^{th} vacuum with the $(j+k)^{\text{th}}$ vacuum will be called k -walls. In other words, the inequivalent types of BPS walls are classified by the elements of

³Here we consider only gauge group which are simply-connected with simple gauge algebra \mathfrak{g}

⁴Recall that $c_2(\mathfrak{su}(N)) = N$, $c_2(\mathfrak{sp}(N)) = N + 1$, $c_2(\mathfrak{so}(N)) = N - 2$, $c_2(\mathfrak{e}_6) = 12$, $c_2(\mathfrak{e}_7) = 18$, $c_2(\mathfrak{e}_8) = 30$, $c_2(\mathfrak{f}_4) = 9$, $c_2(\mathfrak{g}_2) = 4$.

⁵The vacua of SYM must be gapped because they are isolated and supersymmetric. If they were not gapped, we would have a moduli space and not isolated vacua.

the broken part \mathbb{Z}_h of the R-symmetry group: there are h different sectors of domain walls, classified by the element $\sigma^k \in \mathbb{Z}_h$ that relates the vacuum configuration at $x = +\infty$ to the one at $x = -\infty$. Since the vacua are related by the broken generators of the R-symmetry, the physics of k -walls is the parity reversed of the physics of $(h - k)$ -walls. Indeed, performing a parity transformation along the axis orthogonal to the worldvolume of the wall, we get that the wall between the 0^{th} vacuum on $x = +\infty$ and the k^{th} vacuum on $x = -\infty$ get transformed to the domain wall connecting the k^{th} vacuum $x = +\infty$ and the 0^{th} vacuum on $x = -\infty$. Then applying the R-symmetry generator σ^{-k} on both sides, we get that this configuration is the same as the configuration which interpolates between the 0^{th} vacuum on the left and the $(h - k)^{\text{th}}$ vacuum on the right.

As we said in the introduction, the studying of the existence and features of the domain walls has been carried out for $SU(N)$ SYM by Acharya and Vafa [10]. From their construction which uses the realization of $4d \mathcal{N} = 1$ $SU(N)$ SYM using a G_2 -holonomy geometry in M-theory, they proposed as low energy description on the k -domain wall, the model⁶

$$3d \quad \mathcal{N} = 1 \quad U(k)_{N-\frac{k}{2}, N}^{\mathcal{N}=1} \quad \text{gauge theory} \quad + \quad \text{singlet scalar multiplet.} \quad (2.17)$$

The singlet multiplet is composed of the goldstone associated with the broken translations and its fermionic partner, the goldstino of the broken supersymmetry generators. It describes the center-of-mass motion of the wall perpendicular to its worldvolume. It has only derivative couplings that are suppressed at low energy, therefore we consider it decoupled from the gauge part of the model. At low energy is a free chiral. This part of the domain wall theory is always there when considering a BPS domain wall and will be omitted for the rest of the thesis. We are going to talk about the Chern-Simons-matter model (CS) in Chapter 4.

The proposal for the low energy effective model of the domain walls of SYM for $Sp(N)$ gauge groups has been done in [1] and it is

$$3d \quad \mathcal{N} = 1 \quad Sp(k)_{N-\frac{k-3}{2}}^{\mathcal{N}=1} \quad \text{gauge theory} \quad + \quad \text{singlet scalar multiplet.} \quad (2.18)$$

These models (2.17) and (2.18) enjoy an infrared duality relating precisely the k -walls to the $h - k$ walls. These dualities are called level-rank duality (see C) and they are

$$\begin{aligned} U(k)_{N-\frac{k}{2}, N}^{\mathcal{N}=1} &\iff U(N - k)_{-\frac{N+k}{2}, -N}, \\ Sp(k)_{N-\frac{k-3}{2}}^{\mathcal{N}=1} &\iff Sp(N + 1 - k)_{-1-\frac{N+k}{2}}^{\mathcal{N}=1}. \end{aligned} \quad (2.19)$$

Consider now $4d \mathcal{N} = 1$ massive SQCD, with flavors $F < h$. These cases have been studied in [1]. Here the story is much more complicated even if there are still h vacua. The masses of the quarks control the low-energy behavior of the model: we are going to use a single mass parameter m_{4d} which breaks the global symmetry group of the

⁶For the definition of the Chern-Simons indices see Section 4.0.1.

Flavors	k -wall					
	$\text{Gr}(k, N-1)$	$\text{Gr}(k-1, N-1)$	$G(k-2, N-2)$	$G(k-2, N-k) \times U(2)_{k-1, k}$	$G(1, N-k) \times U(k-1)_{\frac{k+1}{2}, k}$	$U(k)_{\frac{k}{2}, k}$
$N-1$	$\text{Gr}(k, N-2)$	$\text{Gr}(k-1, N-2) \times U(1)_2$	\vdots	\vdots	\vdots	\vdots
$N-2$	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
$N-k$	$\text{Gr}(k, N-k)$	$\text{Gr}(k-1, N-k) \times U(1)_k$	$G(k-2, k-1, N-k)$	$G(k-2, k) \times U(2)_{N-k-1, N-k}$	$G(1, k) \times U(k-1)_{N-\frac{3k+1}{2}, k}$	$U(k)_{N-\frac{3k}{2}, N-k}$
k	$\text{Gr}(k, k) \sim \text{gap}$	$\text{Gr}(k-1, k) \times U(1)_{N-k}$	$G(k-2, k-1) \times U(2)_{N-k, N-k+1}$	\vdots	$G(1, k-1) \times U(k-1)_{N-\frac{k+1}{2}, N-1}$	$U(k)_{N+1-\frac{3k}{2}, N-k+1}$
$k-1$	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
1	\vdots	\vdots	\vdots	\vdots	\vdots	$U(k)_{N-1-\frac{k}{2}, N-1}$
0	\vdots	\vdots	\vdots	\vdots	\vdots	$U(k)_{N-\frac{k}{2}, N}$

TABLE 2.1: Domain wall solutions found for 4d $\mathcal{N} = 1$ $SU(N)$ SQCD with $F < N$ flavors in the regime when $m_{4d} \ll \Lambda$. Here $k \leq \frac{N}{2}$ and for each number of flavors, i.e. for each row, there are at maximum $k+1$ solutions. In particular, there are $k+1$ solutions only when the flavors $k \leq F \leq N-k$.

massless theory to the bigger global symmetry subgroup provided all the quarks are massive. First of all, when $m_{4d} \gg \Lambda$, one can integrate out the quarks and get back to the SYM case. Instead, if $m_{4d} \ll \Lambda$ there is a Wess-Zumino like description in the IR, which gives many different solutions. We only report here the summary of what has been found in Table 2.1 for the $SU(N)$ case. In [1] the authors also propose a $3d$ low energy description on the walls, and it is

$$3d \mathcal{N} = 1 U(k)_{N-\frac{k+F}{2}, N-\frac{F}{2}}^{\mathcal{N}=1} \quad \text{with } F \text{ fundamentals,} \quad (2.20)$$

for the $SU(N)$ k -walls, whereas they propose

$$3d \mathcal{N} = 1 Sp(k)_{N-\frac{k-3+F}{2}}^{\mathcal{N}=1} \quad \text{with } F \text{ fundamentals,} \quad (2.21)$$

for the $Sp(N)$ k -walls. These models enjoy a duality relating the k -walls with the $(h - k)$ -walls.

Chapter 3

Four-dimensional construction of BPS domain walls

In this chapter we are going to analyze the equation (2.5) in order to find domain wall solutions for $SU(N)$ and $Sp(N)$ with $F = h$ and $F = h + 1$.¹

3.1 Numerical analysis of the BPS equations for $Sp(N)$

Let us consider four-dimensional $\mathcal{N} = 1$ SQCD with gauge group $Sp(N)$ and $2F$ fundamental flavors Q . The symplectic group $Sp(N)$ is the subgroup of $SU(2N)$ which leaves invariant the $2N \times 2N$ symplectic form $\Omega = \mathbb{1}_N \otimes i\sigma_2$. The dimension of this group is $N(2N + 1)$ and it has rank equals N . The IR behavior of the models is well known [59]–[62]. If the quarks are massless, the physics at low energies crucially depends on the rank of the gauge group and on the number of flavors. Instead, if the quarks are massive, the theory always has $N + 1$ distinct and massive vacua, regardless of the number of flavors. The vacua arise from spontaneous breaking of the \mathbb{Z}_{2N+2} R-symmetry of massive SQCD down to \mathbb{Z}_2 ; therefore, they are all related by \mathbb{Z}_{N+1} R-symmetry rotations. The BPS domain wall sectors k and $N + 1 - k$ are related by a parity transformation.

As has been said, the case of $F \leq N$ has been considered in [1]. We now discuss in some detail the cases $F = N + 1$ and $F = N + 2$.

The massless theory has $2F$ flavors of quarks Q_I , where $I = 1, \dots, 2F$, in the fundamental representation (the number of flavor must be even because of a global gauge anomaly [63]) and no superpotential. The non-anomalous continuous global symmetry is $SU(2F) \times U(1)_R$. Regarding $Sp(N)$ as the subgroup of $SU(2N)$ that leaves the $2N \times 2N$ symplectic form $\Omega = \mathbb{1}_N \otimes i\sigma_2$ invariant, we indicate the flavors as Q_I^α with $\alpha = 1, \dots, 2N$. We introduce the antisymmetric meson matrix $M_{IJ} = \Omega_{\alpha\beta} Q_I^\alpha Q_J^\beta$. The low energy behavior of this theory was discussed in detail in [62].

¹We want to mention that we restricted ourselves to these cases because the four-dimensional analysis provided a possible complete set of solutions. We also investigate other cases when $F > h + 1$, for example, $F = h + 2, h + 3$, but we do not yet have a complete picture in those cases. This is partly due to the emergence of gauge symmetry at low energy, discovered thanks to Seiberg's duality.

3.1.1 $Sp(N)$ with $F = N + 1$

For $F = N + 1$, the massless theory has a moduli space of vacua. It is parametrized by a meson matrix M_{IJ} that transforms in the rank-two antisymmetric representation of $Sp(F)$. The meson matrix satisfies the quantum-deformed constraint²

$$\text{Pf } M = \Lambda^{2(N+1)} , \quad (3.1)$$

in terms of the dynamically-generated scale Λ . We turn on a diagonal mass term for the flavors,

$$W_m = \frac{m_{4d}}{2} M_{IJ} \Omega^{IJ} , \quad (3.2)$$

where Ω^{IJ} is the symplectic form of $Sp(F)$ (in the following, we will often indicate all symplectic forms as Ω , irrespective of their dimension, and will not distinguish between upper and lower indices). This explicitly breaks the $SU(2F)$ flavor symmetry to $Sp(F)$, while leaving a discrete $\mathbb{Z}_{2(N+1)}$ R-symmetry unbroken, and it also lifts most of the moduli space. With the introduction of the mass term, the R-symmetry group breaks because the flavors get fixed R-charge $R(Q) = 1^3$, and therefore they do not contribute to the anomaly of the R-current. This implies that the anomaly of the R-current is given by the gauge part of the theory, and therefore the R-symmetry group is the same as SYM. The quantum constraint (3.1) on the would-be moduli space can be implemented with a Lagrange multiplier A . Therefore, the low-energy physics is described by the following effective superpotential on the mesonic space:

$$W = \frac{m_{4d}}{2} M_{IJ} \Omega^{IJ} - A \left(\text{Pf } M - \Lambda^{2(N+1)} \right) . \quad (3.3)$$

The F-term equations

$$\begin{aligned} m_{4d} M_{KI} \Omega^{KJ} - A (\text{Pf } M) \delta_I^J &= 0 \\ \text{Pf } M &= \Lambda^{2(N+1)} \end{aligned} \quad (3.4)$$

lead to $N + 1$ gapped vacua with gaugino condensation and spontaneous R-symmetry breaking $\mathbb{Z}_{2(N+1)} \rightarrow \mathbb{Z}_2$:

$$M = \widetilde{M} \Omega_{2F} , \quad \widetilde{M}^{N+1} = \Lambda^{2(N+1)} , \quad (3.5)$$

while $A = m_{4d} \widetilde{M} / \Lambda^{2(N+1)}$ and $\langle \lambda \lambda \rangle = \partial W / \partial \log \Lambda^{2(N+1)} = m_{4d} \widetilde{M}$.

When the quark mass is small, $|m_{4d}| \ll |\Lambda|$, the effective description as a Wess-Zumino model on the mesonic space is reliable. On the other hand, when the quark mass is large, $|m_{4d}| \gg |\Lambda|$, we can integrate the quarks out first and remain with pure $Sp(N)$ SYM, with the very same $N + 1$ vacua as above.

A small complication, with respect to other values of F , arises because the expectation value of M in (3.5) does not depend on the mass parameter m_{4d} but only on the

²The Pfaffian of a $2F \times 2F$ antisymmetric matrix M is $\text{Pf } M = \frac{1}{2^F F!} \epsilon^{I_1 \dots I_{2F}} M_{I_1 I_2} \dots M_{I_{2F-1} I_{2F}}$ so that $\det M = (\text{Pf } M)^2$. The variation is $\delta \text{Pf } M = \frac{1}{2} \text{Pf } M \cdot \text{Tr}(M^{-1} \delta M)$. Moreover $\text{Pf } \Omega = 1$.

³Recall that the fermions of the chiral superfield have R-charge $R(Q) - 1$ and only the fermions contribute to the anomaly.

dynamically-generated scale Λ . If we were able to make $|M| \gg |\Lambda|$, the theory would go in a Higgsed semiclassical regime: the low energy theory would be well described by the Wess-Zumino model (3.3) with the Kähler potential for M , $\mathcal{K} = \text{Tr} \sqrt{M\Omega M^*\Omega}$, induced by the canonical Kähler potential in terms of quarks Q_I . This was the situation in [1]. On the other hand, if we were able to make $|M| \ll |\Lambda|$, the theory would focus around a smooth point of its moduli space, and for very low energies the Kähler potential would essentially be the canonical one in terms of M (up to rescalings) $\mathcal{K} = \text{Tr}(M\Omega M^*\Omega)$. In our case, instead, $|M| \sim |\Lambda|$ and so we do not have control over the Kähler potential, except for the fact that it is smooth. However it has been shown in [58] that the Cecotti–Fendley–Intriligator–Vafa index, which counts the number of BPS domain walls with signs, is independent of smooth deformations of the Kähler potential. Therefore we assume that a smooth deformation of the Kähler potential does not affect the existence of the domain walls we want to study. In summary, to find the domain solutions, we solve the equation (2.5), making a sensible choice for the Kähler metric, which is the canonical Kähler potential for the fields M .

Furthermore we will assume that there exist a point along the domain wall solution where the expectation value of the meson matrix is diagonalizable with the flavor symmetry, namely $M = \text{diag}(\xi_1, \dots, \xi_{N+1}) \otimes i\sigma_2$. As shown in [1], it follows that M is diagonal everywhere on the domain wall. The problem further simplifies if we also assume that the eigenvalues split in two sets of different values ξ_1, ξ_2 . In this case, we see that it is not even necessary to solve (2.5), since we can find solutions by other means. Let us call⁴ $k_{1,2} = 1, \dots, N$ the number of eigenvalues equal to $\xi_{1,2}$, respectively, with $k_1 + k_2 = N + 1$. The superpotential takes the form

$$W = m_{4d}(k_1\xi_1 + k_2\xi_2) - A\left(\xi_1^{k_1}\xi_2^{k_2} - \Lambda^{2(N+1)}\right). \quad (3.6)$$

To simplify further, we impose the constraint, and moreover we express the meson matrix M in units of $(\Lambda^{3(N+1)-F}/m_{4d}^{N+1-F})^{1/(N+1)} \equiv \Lambda^2$ and set the remaining dimensionful constant $\Lambda^2 m_{4d}$ to one. The superpotential then reduces to

$$W = k_1 \xi_1 + k_2 \xi_1^{-k_1/k_2}. \quad (3.7)$$

As we explained in Chapter 2, each domain wall solution traces in the complex W -plane a straight line connecting the values of the superpotential at the two vacua (the direction of such a line is $e^{i\gamma}$). Therefore, up to reparametrizations, the solutions can be found by simply inverting the equation

$$W\left(M\Big|_{x=+\infty}\right)t + W\left(M\Big|_{x=-\infty}\right)(1-t) = k_1 \xi_1(t) + k_2 \xi_1(t)^{-k_1/k_2} \quad (3.8)$$

in terms of $\xi_1(t)$, where t is some reparametrization of x . Some examples of the solutions we found using this procedure are sketched in Figure 3.1 and Figure 3.2. It

⁴Here we consider the situation where both $k_{1,2}$ are non-zero. If all eigenvalues are equal, say to ξ , then the constraint imposes $\xi^{N+1} = \Lambda^{2(N+1)}$ leading to the $N + 1$ vacua and no domain wall solution exists.

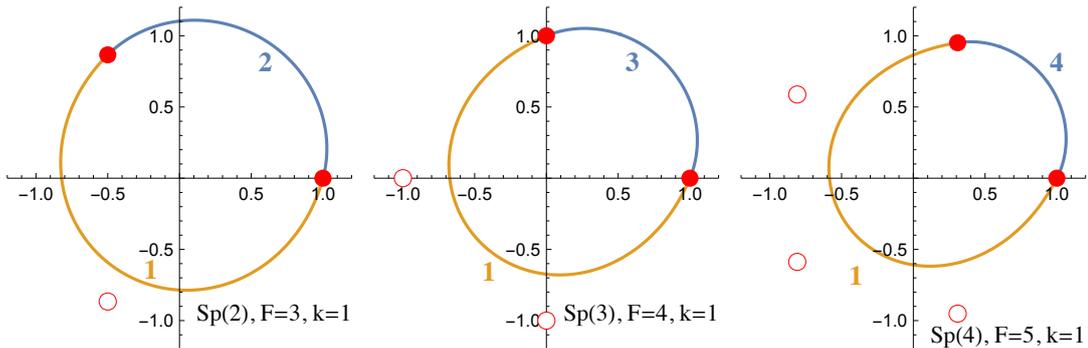


FIGURE 3.1: Examples of 1-walls in $Sp(N)$ SQCD with $F = N + 1$ flavors. We draw the trajectories of the eigenvalues of the meson matrix M in the complex plane along the domain-wall transverse direction x . The filled red circles represents the expectation values of the vacua at $x = \pm\infty$, whereas the unfilled red disks represent the expectation values of the other vacua. The eigenvalues along 1-walls split into a group of $k_1 = 1$ (in yellow) and a group of $k_2 = N$ (in blue) elements. The number of eigenvalues that follow a given trajectory is indicated in the figures with a number colored like the trajectory it refers to.

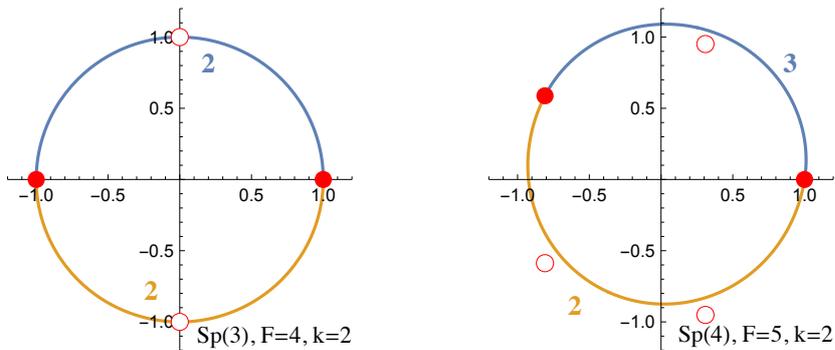


FIGURE 3.2: Examples of 2-walls in $Sp(N)$ SQCD with $F = N + 1$ couples of flavors. The notation is as in Figure 3.1.

turns out that k -wall solutions exist for $k_1 = k$ and $k_2 = N + 1 - k$.

The solutions we have found, in which the $N + 1$ eigenvalues of M split into two groups of k and $N + 1 - k$ elements, break the flavor symmetry of the vacua according to the pattern $Sp(N + 1) \rightarrow Sp(k) \times Sp(N + 1 - k)$. Hence, they represent a symplectic (or quaternionic) Grassmannian

$$\text{HGr}(k_1, N + 1) = \frac{Sp(N + 1)}{Sp(k) \times Sp(N + 1 - k)} \quad (3.9)$$

worth of domain walls. The low-energy theory on the domain walls is given by a $3d$ $\mathcal{N} = 1$ NLSM of Goldstone fields with the quaternionic Grassmannian as target. We summarize the k -wall solutions we have found in Table 3.1.

The solutions we found rely on the assumption that the eigenvalues split into at most two groups. We were not able to find solutions with splitting into more than

Wall	Effective theory	Witten index
k	$\text{HGr}(k, N + 1)$	$\binom{N + 1}{k}$

TABLE 3.1: Domain wall solutions found for 4d $\mathcal{N} = 1$ $Sp(N)$ SQCD with $F = N + 1$ flavors in the regime when $m_{4d} \ll \Lambda$. For each k -wall sector are included also the various contributions to the Witten Index of the low energy theory on the domain wall from each solution.

two groups⁵. However, finding such solutions requires solving ODEs, which is a much more difficult task and we might have missed solutions.

A check of the completeness of our set of solutions comes from Witten indices of the low energy theories living on the domain walls at large mass, which are [1] the TQFT's

$$3d \quad Sp(k)_{N - \frac{k-3}{2}}^{\mathcal{N}=1}, \quad (3.11)$$

and have Witten Index $\binom{N+1}{k}$. The Witten Index of the TQFT (valid at large masses) is equal to the Witten Index of the NLSM we found here (valid at small masses). See Table 3.1. (See also [1], [31] for computation of Witten Indexes in TQFT's and in NLSM's on Grassmannians.)

In Sec. 4.1.1 we discuss a $3d$ $\mathcal{N} = 1$ SCFT describing the phase transition between the TQFT vacuum and the NLSM vacuum.

3.1.2 $Sp(N)$ with $F = N + 2$

Let us now move to $F = N + 2$.⁶ In this case, the low-energy $4d$ physics has a weakly-coupled description [60], [62] in terms of a Wess-Zumino model of chiral multiplets $M_{IJ} = -M_{JI}$, with $I, J = 1, \dots, 2F$ and superpotential

$$W = -\frac{1}{\Lambda^{2N+1}} \text{Pf } M. \quad (3.12)$$

In the UV description M_{IJ} is the meson matrix $\Omega_{\alpha\beta} Q_I^\alpha Q_J^\beta$. The moduli space of the Wess-Zumino model is parametrized by antisymmetric matrices M_{IJ} with rank $M \leq N$, which coincides with the classical constraint in the UV SQCD theory. Adding a diagonal

⁵Assuming that the Kähler potential is the canonical one for the fields M_{IJ} , the equations (2.5) we have to study for the eigenvalues of the meson matrix are

$$\partial_x \xi_i = e^{i\gamma} \left[\left(1 - \frac{\prod_{k \neq i} |\xi_k|^2}{D} \right) \left(\frac{\xi_i^* \prod_j \xi_j^* - 1}{\xi_i^* \prod_j \xi_j^*} \right) + \sum_{j \neq i} \left(\frac{\xi_i \xi_j^* \prod_{h \neq i, j} |\xi_h|^2}{D} \right) \left(\frac{\xi_j^* \prod_k \xi_k^* - 1}{\xi_j^* \prod_k \xi_k^*} \right) \right], \quad (3.10)$$

where $D = \prod_k |\xi_k|^4 + \frac{1}{2} \sum_{k \neq j} |\xi_k|^2 |\xi_j|^2$. These equations have been obtained first evaluating the constraint $\prod_k \xi_k = 1$, expressing the $\xi_{N+1} = \frac{1}{\prod_{i=1}^N \xi_i}$. Then substituting the expression for ξ_{N+1} into the superpotential (3.3) appropriately rescaled — obtaining the expression $W = \sum_{i=1}^N \xi_i + \prod_{j=1}^N \frac{1}{\xi_j}$ — and into the Kähler potential $\mathcal{K} = \sum_{i=1}^N |\xi_i|^2 + \prod_{i=1}^N \frac{1}{|\xi_i|^2}$.

⁶We consider only $k \neq \frac{N+1}{2}$ at first. The parity-invariant case $k = \frac{N+1}{2}$ requires a special procedure, discussed in Sec. 3.1.2.

mass term, the IR superpotential becomes

$$W = \frac{m_{4d}}{2} M_{IJ} \Omega^{IJ} - \frac{1}{\Lambda^{2N+1}} \text{Pf } M . \quad (3.13)$$

In the massive theory, the moduli space reduces to $N + 1$ gapped vacua

$$M = \widetilde{M} \Omega_{2F} , \quad \widetilde{M}^{N+1} = m_{4d} \Lambda^{2N+1} . \quad (3.14)$$

Notice that in this case $M \rightarrow 0$ as $m_{4d} \rightarrow 0$, so in the small mass limit the IR physics is well described by the Wess-Zumino model (3.13) with canonical Kähler potential in terms of M , $\mathcal{K} = \text{Tr}(M \Omega M^* \Omega)$.

To find the domain wall solutions, we study the differential equations (2.5), with superpotential given by (3.13) and Kähler potential $\mathcal{K} = \text{Tr}(M \Omega M^* \Omega)$. In order to simplify the equations, we express M in units of $(m_{4d}^{F-N-1} \Lambda^{3(N+1)-F})^{\frac{1}{N+1}}$ and we set $(m_{4d}^F \Lambda^{3(N+1)-F})^{\frac{1}{N+1}} = 1$.

We make a diagonal ansatz for the meson matrix:

$$M = \text{diag}(\xi_1, \dots, \xi_F) \otimes i\sigma_2 . \quad (3.15)$$

With this ansatz, the "off-diagonal" differential equations are automatically satisfied, as in [1], and we are left with the "diagonal" equations.

In order to write the F complex equations for the F complex eigenvalues ξ_i , we pass to polar coordinates. Expressing the eigenvalues in polar form, $\xi_j = \rho_j e^{i\phi_j}$ the $2F$ real differential equations read

$$\begin{aligned} \partial_x \rho_i &= - \left(\prod_{j \neq i} \rho_j \right) \cos \left(\sum_{j=1}^{N+2} \phi_j - \gamma \right) + \cos(\phi_i - \gamma), \\ \partial_x \phi_i &= \rho_i^{-1} \left(\prod_{j \neq i} \rho_j \right) \sin \left(\sum_{j=1}^{N+2} \phi_j - \gamma \right) - \rho_i^{-1} \sin(\phi_i - \gamma) \end{aligned} \quad (3.16)$$

These differential equations can be seen as the Hamiltonian system

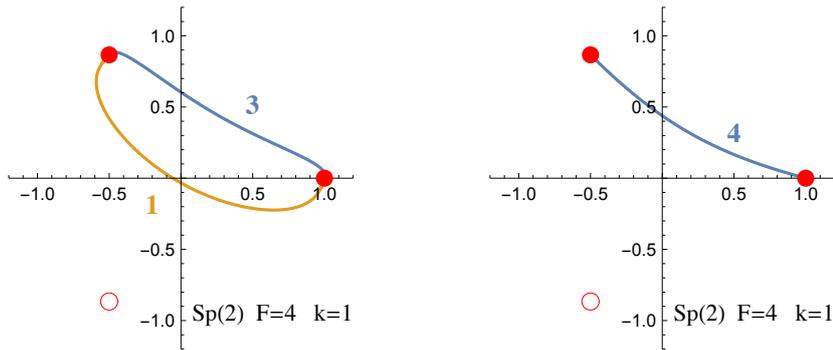
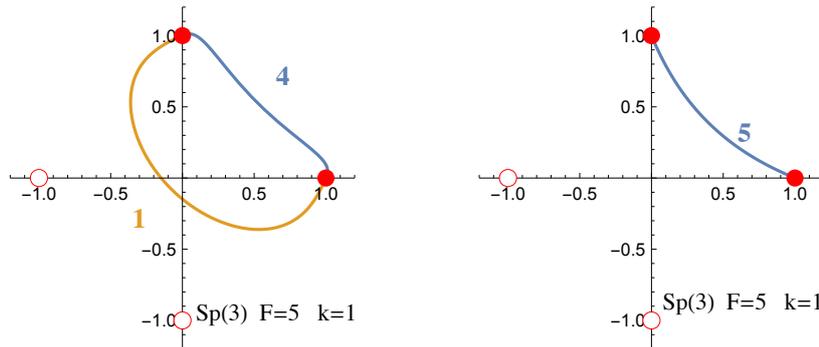
$$\rho_i = \frac{1}{\rho_i} \frac{\partial H}{\partial \phi_i}, \quad \phi_i = - \frac{1}{\rho_i} \frac{\partial H}{\partial \rho_i}, \quad (3.17)$$

whose Hamiltonian is⁷

$$H = - \left(\prod_i \rho_i \right) \sin \left(\sum_i \phi_i - \gamma \right) + \sum_i \rho_i \sin(\phi_i - \gamma). \quad (3.19)$$

⁷The Hamiltonian can also be written as $H = \text{Im}(e^{-i\gamma} W(\xi_i))$. Here the Poisson tensor is not the canonical one, but it is $J = \text{diag}(\rho_1^{-1} \otimes i\sigma_2, \dots, \rho_{N+2}^{-1} \otimes i\sigma_2)$ and the reduced superpotential $W(\xi_i)$

$$W(\xi_i) = \sum_{i=1}^{N+2} \xi_i - \prod_{i=1}^{N+2} \xi_i . \quad (3.18)$$

FIGURE 3.3: Examples of 1-wall in $Sp(2)$, $F = 4$.FIGURE 3.4: Examples of 1-wall solutions of $Sp(3)$, $F = 5$.

The solutions of the differential equations (3.16), that we found numerically, split the eigenvalues into at most two sets: J plus $F - J$.

We plot the solution for $N \leq 4$ in Figure 3.3, Figure 3.4 and Figure 3.5. We only display $k < \frac{N+1}{2}$. The domain wall sector $k = \frac{N+1}{2}$ will be treated in sec. 3.1.2. The cases $k > \frac{N+1}{2}$ are the parity reversed of $N + 1 - k < \frac{N+1}{2}$.

We find a k -wall solution for any $J = 0, \dots, k$, so there are $k + 1$ different solutions. These solutions break the $Sp(F)$ flavor symmetry to $Sp(J) \times Sp(F - J)$. Therefore, these are families of solutions parametrized by the symplectic Grassmannian

$$\text{HGr}(J, F = N + 2). \quad (3.20)$$

The low energy theory on the domain walls is given by a $3d \mathcal{N} = 1$ NLSM of Goldstone fields with target $\text{HGr}(J, N + 2)$. We sum up the various k -walls we have found in Table 3.2.

The solutions found have the property that the eigenvalues of the meson matrix split into two groups, and not more. We were not able to find solutions where the eigenvalues split into three or more groups.

A check that the solutions we found are the full set of solutions comes from the Witten Index. The alternating sum⁸ of Witten indices in the k sector of Table 3.2 coincides with the Witten Index of the k -wall of pure $Sp(N)$ Super-Yang-Mills.

⁸See [48] for the explanation for the alternating sign of the sum. This is due to the number of fermions with negative mass that are integrated out.

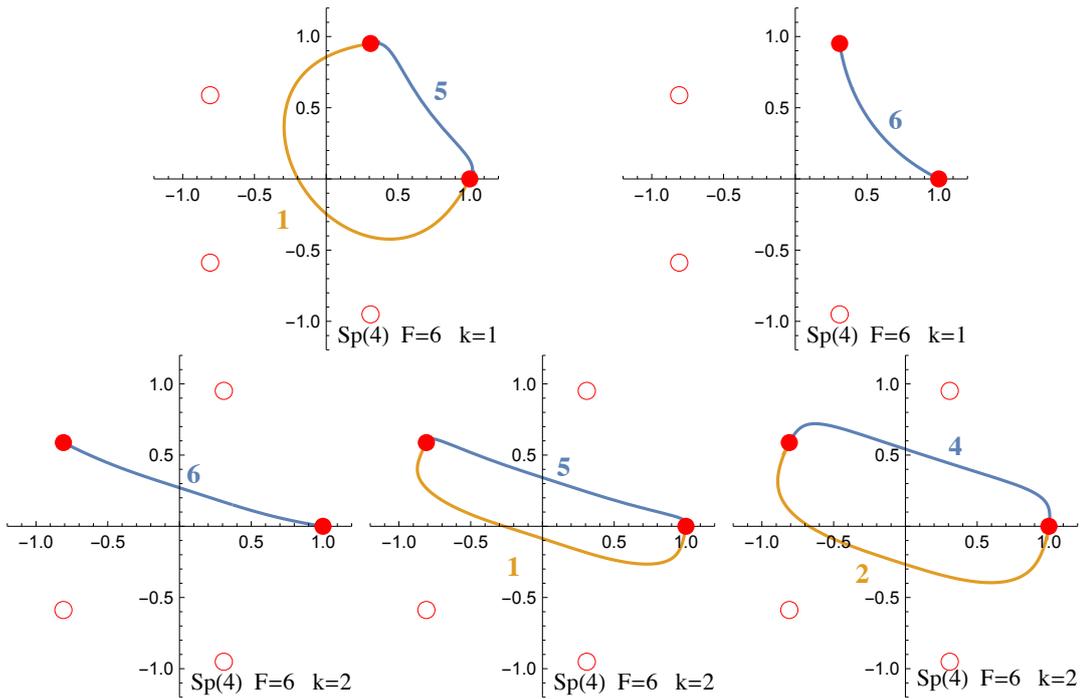


FIGURE 3.5: Examples of domain walls in $Sp(4)$, $F = 6$: on the first row there are the 1-wall solutions, whereas on the second row there are the 2-wall solutions.

Wall	Effective theory	Witten Index
k	$HGr(J, N + 2)$, $J \in \{0, \dots, k\}$	$\binom{N+1}{k} = \sum_{j=0}^k (-1)^{j+k} \binom{N+2}{j}$

TABLE 3.2: The $k + 1$ domain wall solutions of $4d \mathcal{N} = 1$ $Sp(N)$ SQCD with $F = N + 2$ flavors in the regime $m_{4d} \ll \Lambda$. On the right we show how the various contributions to the Witten index from each solution sum up the Witten index of the pure $Sp(N)$ SYM.

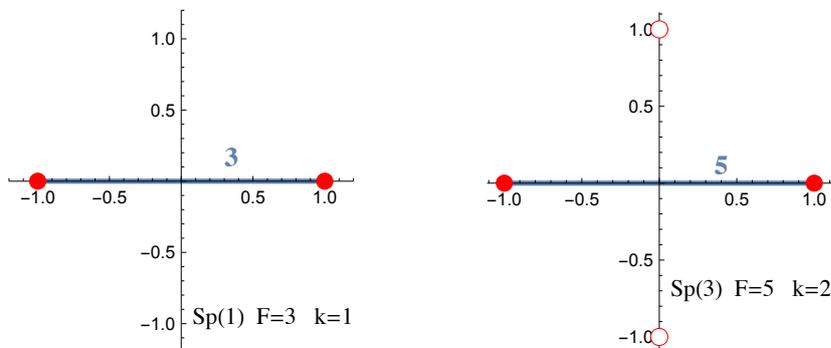


FIGURE 3.6: Examples of $k = \frac{N+1}{2}$ domain walls : on the left a 1-wall solution of $Sp(1)$ with $F = 3$ flavors, on the right a 2-wall solution of $Sp(3)$ with $F = 5$ flavors.

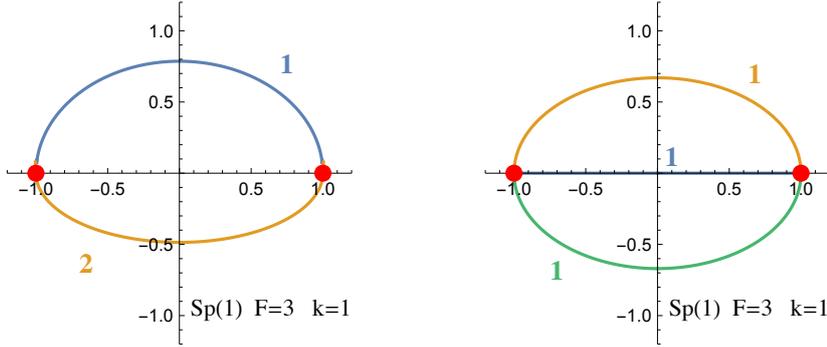


FIGURE 3.7: Examples of deformed domain walls solutions in $Sp(1)$, $F = 3$: on the left the parameters of the deformation (3.21) are $\epsilon_i = (0.005, 0.005, 0)$. On the right the parameters are $\epsilon_i = (0, -1, -1)$. The parameters of the right figure cannot be considered small, but they have been chosen such that the figure could be easy to read. There is also a third solution, corresponding to the charge-conjugated of left-figure, which we do not display.

The parity-invariant walls, $k = \frac{N+1}{2}$

If N is odd, the $k = \frac{N+1}{2}$ wall exists and must be equivalent to its $4d$ parity transformed. For this reason we dub such domain walls *parity-invariant*.

In this case, a naive numerical analysis yields only a single domain wall, the one with all the $N + 2$ eigenvalues following the same trajectory (so the global $Sp(F)$ symmetry is unbroken), which is an horizontal straight line connecting the vacuum $M = +\Omega_{2F}$ to the vacuum $M = -\Omega_{2F}$ along the real line (see Figure 3.6). This fact is in contrast with expectations from the other k -walls, with $k < \frac{N+1}{2}$, where we find $k + 1$ different solutions (parameterized by $J = 0, 1, \dots, k$ splitting the F eigenvalues into J and $F - J$). Analogously, for $k > \frac{N+1}{2}$, the k -wall, being the parity transformed $N + 1 - k < \frac{N+1}{2}$ wall, admits $N + 2 - k$ solutions. So it is natural to expect $\frac{N+1}{2} + 1$ solutions for the parity-invariant walls of $Sp(N)$ with $N + 2$ flavors, not just a single solution.⁹

In this subsection we give our interpretation of this puzzle.

In the case of the parity-invariant wall, we found that upon making a small deformation of the system of ODE's, more solutions appear. All these additional solutions collapse to the straight line solution if we tune the deformation to zero. Notice that this is not true for other k 's: generically, a small deformations does not generate additional solutions on top of the ones discussed previously. One can think of such small deformation as a regularization of the problem of finding and counting the solutions of the system of differential equations.

The deformations we are considering are equivalent to a deformation of the Kähler potential that break the global $Sp(F)$ symmetry to a product of smaller Sp factors. The deformations can be parametrized by the block-diagonal matrix

⁹Other more exotic options are of course possible.

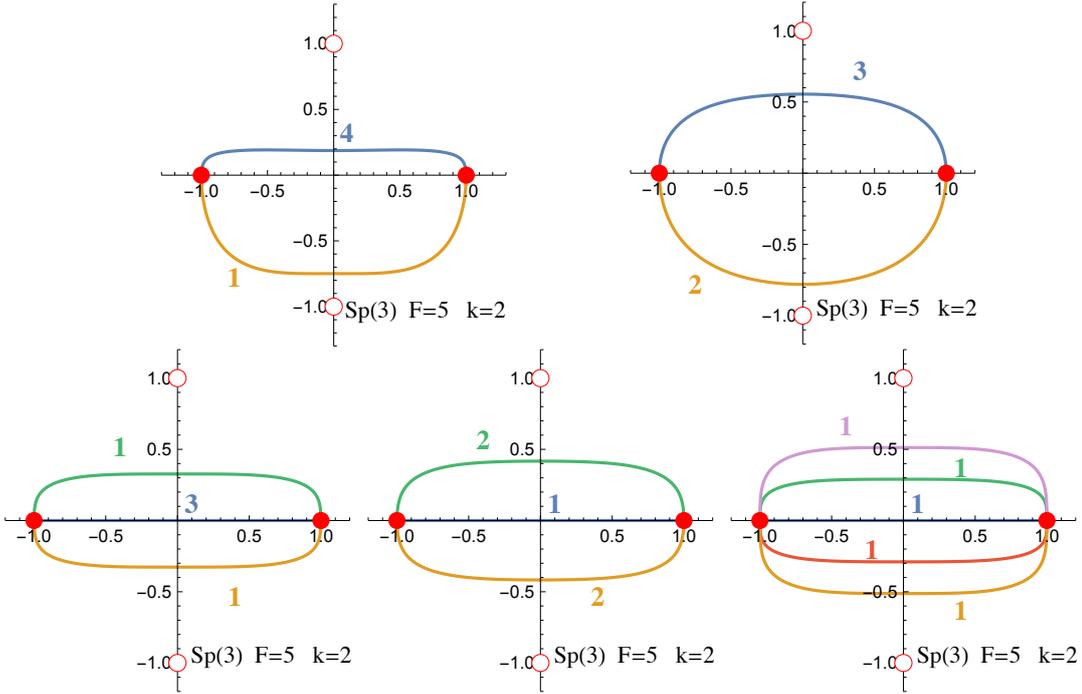


FIGURE 3.8: Examples of domain walls in $Sp(3)$, $F = 5$: the parameters of the deformations are, from left to right, top to bottom, $(0, -1, -1, -1, -1, -1)$, $(-1, -1, -2, -2, -2)$, $(0.01, 0.01, -0.1, -0.1, -0.1)$, $(0.01, 0.01, 0.01, 0.01, -0.1)$, $(0.1, 0.1, 0, 0.2, 0.2)$.

$J = \text{diag}(\epsilon_1, \dots, \epsilon_{N+2}) \otimes i\sigma_2$. The variation of the Kähler potential is

$$\delta\mathcal{K} = \frac{1}{4} \text{Tr}(MJ) \text{Tr}(M^*J). \quad (3.21)$$

Note that when $\epsilon_i \neq \epsilon_j$ for some i and j then the $Sp(F)$ is explicitly broken.

The solutions we found are depicted in Figure 3.7 (1-wall of $Sp(1)$), Figure 3.8 (2-wall of $Sp(3)$) and Figure 3.9 (3-wall of $Sp(5)$), where we also specify the coefficients ϵ_i of the deformations.

With this deformation, we find the expected solutions where the F eigenvalues split into J plus $F - J$. However, we also find additional unexpected solutions, splitting the eigenvalues into more than two different sets. So the global symmetry $Sp(F)$ can be broken to a product of many smaller Sp factors.

In order to find the moduli space of such solutions, in principle we need to quotient the explicitly broken flavor group (which is a product of many Sp factors) with the sub-group preserved by the eigenvalues trajectories. Doing so, we do not automatically recover the Grassmannians. However, let us discuss a possible way of obtaining the Grassmannians.

For instance, in the case of the 1-walls of $Sp(1)$ with 3 flavors, we find three solutions of the deformed equations:

- One solution (left of Figure 3.7) has explicit global symmetry $Sp(2) \times Sp(1)$, which is preserved by the solution, so its moduli space is trivial (Witten Index

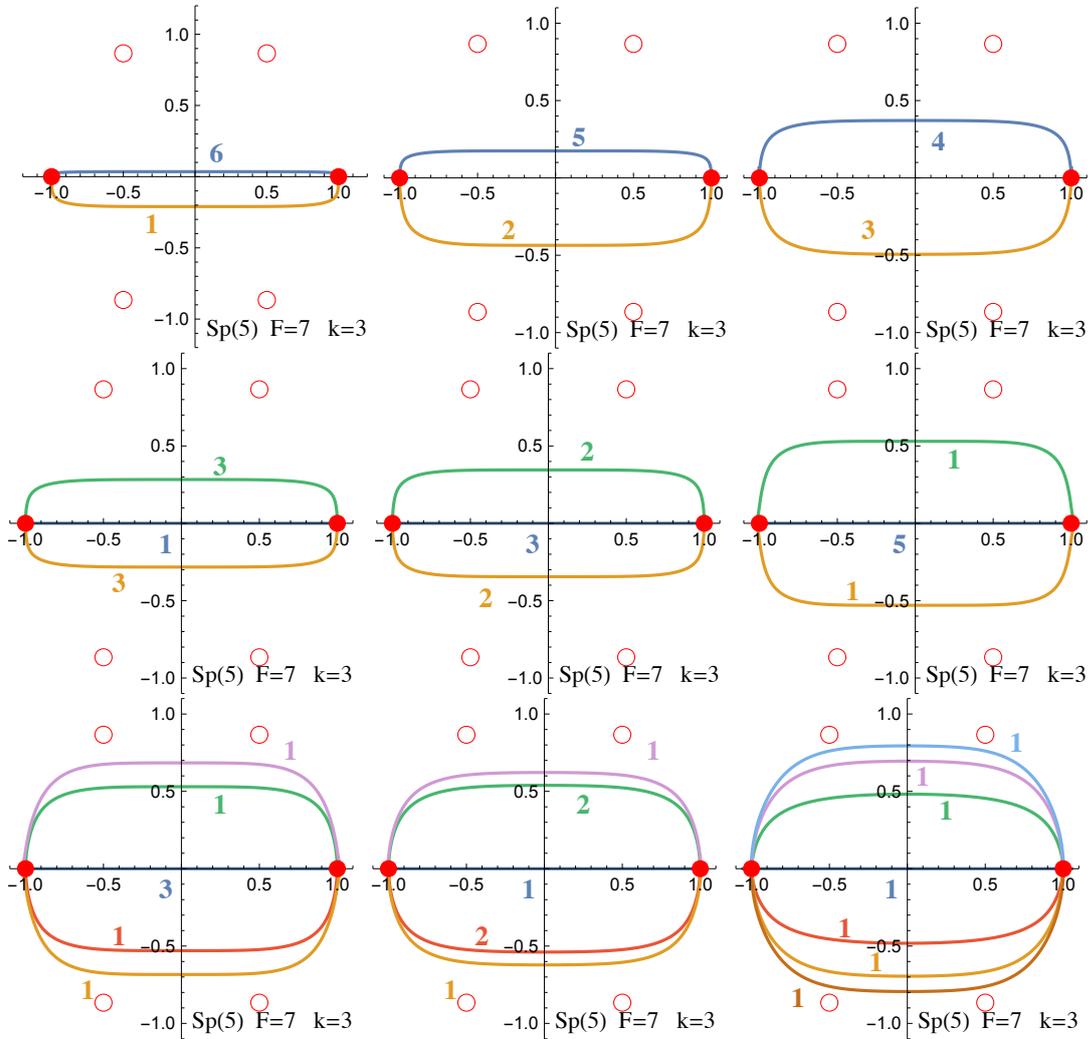


FIGURE 3.9: Examples of domain walls in $Sp(5)$, $F = 7$: here are listed the parameters of the deformations used to compute these solutions. First line from left to right, the ϵ_i are $(0, -0.1, -0.1, -0.1, -0.1, -0.1, -0.1)$, $(0.1, 0.1, -0.1, -0.1, -0.1, -0.1, -0.1)$, $(0.1, 0.1, 0.1, -0.1, -0.1, -0.1, -0.1)$. Second line from left to right, $(0, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1)$, $(0, 0, 0, 0.1, 0.1, 0.1, 0.1)$, $(-0.1, -0.1, -0.1, -0.1, -0.1, 0.1, 0.1)$. Third line from left to right $(0.1, 0.1, 0.2, 0.2, -0.1, -0.1, -0.1)$, $(0.05, 0.05, 0.05, 0.05, -0.1, 0.1, 0.1)$, $(0.05, 0.05, 0.2, 0.2, 0.3, 0.3, -0.1)$.

± 1).

- Another solution is the charge-conjugated of left of Figure 3.7, so its moduli space is trivial (Witten Index ± 1).
- The solution on the right of Figure 3.7 has explicit global symmetry $Sp(2) \times Sp(1)$, broken to $Sp(1) \times Sp(1) \times Sp(1)$ by the eigenvalues trajectories. The moduli space is $\frac{Sp(2)}{Sp(1) \times Sp(1)} = \text{HGr}(1, 2)$, (Witten Index $\pm \binom{2}{1}$).¹⁰

In this case, $Sp(1)$ with 3 flavors, there is a simple way of organizing the three deformed solutions into the two expected solutions (that is a trivial moduli space with $WI = -1$ and a $\text{HGr}(1, 3)$ with $WI = +3$). We combine one trivial solution ($WI = +1$) with a $\text{HGr}(1, 2)$ ($WI = +2$) together, to get the $\text{HGr}(1, 3)$ expected solution, while the other trivial solution ($WI = -1$) provides the expected trivial solution.

Unfortunately, we do not have a complete analysis of this kind for the parity-invariant walls of $Sp(N)$ for generic N . We leave this issue to future work.

3.2 Numerical analysis of the BPS equations for $SU(N)$

Consider $\mathcal{N} = 1$ SQCD with gauge group $SU(N)$ and F flavors of quarks Q_I , \tilde{Q}^I in the fundamental and antifundamental representation. The IR behavior of the massless model is well known [59]–[61]. The UV massless model has no superpotential. The non-anomalous continuous global symmetry group is $SU(F) \times SU(F) \times U(1)_B \times U(1)_R$. The gauge invariant operators are the mesons $M_I^J = Q_I^\alpha \tilde{Q}_\alpha^J$, the baryon $B^{I_1 \dots I_N} = \epsilon^{I_1 \dots I_N I_{N+1} \dots I_F} \epsilon_{\alpha_1 \dots \alpha_N} Q_{I_1}^{\alpha_1} \dots Q_{I_N}^{\alpha_N}$ and anti-baryon $\tilde{B}_{I_{N+1} \dots I_F} = \epsilon_{I_1 \dots I_N I_{N+1} \dots I_F} \epsilon^{\alpha_1 \dots \alpha_N} \tilde{Q}_{\alpha_1}^{I_1} \dots \tilde{Q}_{\alpha_N}^{I_N}$. Baryon and anti-baryon are present only if $F \geq N$. Obviously depending on the number of flavors, some of those operators may be redundant and therefore there are relations (both present at the classical level or arising quantum-mechanically) between mesons, baryons and anti-baryons. For the cases we are going to study we will give this precise relation in the following.

3.2.1 $SU(N)$ with $F = N$

Consider $\mathcal{N} = 1$ SQCD with gauge group $SU(N)$ and N flavors of quarks Q_I , \tilde{Q}^I in the fundamental representation.

The UV massless model has no superpotential and the non-anomalous continuous global symmetry group is $SU(N) \times SU(N) \times U(1)_B \times U(1)_R$. The gauge-invariant operators which describe the moduli space of the massless theory (at the classical

¹⁰More generally the moduli space of these solutions include, as factors, *flag manifolds*:

$$\mathcal{F}_{\{k_1, \dots, k_i\}} = \frac{Sp(M)}{Sp(k_1) \times \dots \times Sp(k_i)}, \quad \sum_{j=1}^i k_j = M, \quad (3.22)$$

The WI of such manifolds is given by the formula

$$WI(\mathcal{F}_{\{k_1, \dots, k_i\}}) = \frac{M!}{k_1! \dots k_i!}. \quad (3.23)$$

level) are the mesons $M_I^J = Q_I^\alpha \tilde{Q}_\alpha^J$, the baryon $B = \epsilon^{I_1 \dots I_N} \epsilon_{\alpha_1 \dots \alpha_N} Q_{I_1}^{\alpha_1} \dots Q_{I_N}^{\alpha_N}$ and anti-baryon $\tilde{B} = \epsilon_{I_1 \dots I_N} \epsilon^{\alpha_1 \dots \alpha_N} \tilde{Q}_{\alpha_1}^{I_1} \dots \tilde{Q}_{\alpha_N}^{I_N}$. The massless theory has classical moduli space which is singular at the origin when $M_I^J = B = \tilde{B} = 0$. However at the quantum mechanical level there is a constraint

$$\det M - B\tilde{B} = \Lambda^{2N}. \quad (3.24)$$

in terms of the dynamically generated scale Λ . This constraint smooths out the singularity at the origin, yielding a smooth moduli space. This smooth manifold will be called $\mathcal{M}_{N,N}$.¹¹

We then turn on a diagonal mass term for the quarks,

$$W_m = m_{4d} \text{Tr } M. \quad (3.26)$$

This breaks the flavor symmetry $SU(N) \times SU(N) \rightarrow SU(N)$ and it leaves only a discrete R-symmetry unbroken, i.e. \mathbb{Z}_{2N} . Here again, the mass term in the superpotential fixes the R -charge of the quarks such that their fermionic partners are chargeless and do not contribute to the anomaly. Only the gaugini contribute to the anomaly, giving us the same situation of SYM. We implement the quantum constraint on the would-be moduli space using a Lagrange multiplier A . Therefore the low-energy physics is described by the effective superpotential

$$W = m_{4d} \text{Tr } M + A(\det M - B\tilde{B} - \Lambda^{2N}), \quad (3.27)$$

The solutions of the F-term equations are N gapped vacua with R-symmetry breaking $\mathbb{Z}_{2N} \rightarrow \mathbb{Z}_2$:

$$M = \tilde{M} \mathbb{I}_N, \quad \tilde{M}^N = \Lambda^{2N}, \quad B = \tilde{B} = 0, \quad A = \frac{m_{4d} \tilde{M}}{\Lambda^{2N}}. \quad (3.28)$$

We can see that the moduli space is lifted when we introduce a mass for the quarks. When the quark mass is large $|m_{4d}| \gg \Lambda$, one can integrate out the quarks and remain with pure SYM, which has, in turn, N vacua. On the other hand, if $|m_{4d}| \ll \Lambda$ the effective description as a WZ model on the moduli space is reliable. As in the $Sp(N)$ $F = N + 1$ case described in Sec. 3.1.1 and [32], we do not need any information about the Kähler potential, except for the fact that it is smooth.

¹¹ In the special case of $SU(2)$, for which the fundamental representation coincides with the anti-fundamental, this is precisely the constraint (3.1) $\text{Pf } M_{Sp(1)} = \Lambda^{2N}$ which appears in the $Sp(1)$ model with 2 flavors. This can be seen reorganizing the baryons and the mesons into the matrix

$$M_{Sp(1)} = \begin{pmatrix} 0 & B & M_{11} & M_{12} \\ -B & 0 & M_{21} & M_{22} \\ -M_{11} & -M_{21} & 0 & \tilde{B} \\ -M_{12} & -M_{22} & -\tilde{B} & 0 \end{pmatrix} \quad (3.25)$$

that maps the $Sp(1)$ mesons to the $SU(2)$ mesons and baryons.

Non-baryonic walls

To ease our way into the study of $SU(N)$ with N flavors domain walls, firstly, we assume that the baryons are spectators, namely $B = \tilde{B} = 0$ along the solution. (This type of solution was already studied in [25], [26]). This really simplifies our problem if we also assume that the matrix M is diagonalizable by means of the flavor symmetry and the eigenvalues ξ_i of such matrix are split into two groups of k_1 (called ξ_1) and k_2 elements (called ξ_2). Once we have done such simplifications, we can evaluate the constraint (3.24) and write one group of the eigenvalues in terms of the other, obtaining a superpotential that depends on only one superfield ξ_1

$$W = k_1 \xi_1 + k_2 \xi_1^{-\frac{k_1}{k_2}}, \quad (3.29)$$

here the fields ξ_i have been rescaled in units of Λ^2 and $\Lambda^2 m_{4d}$ has been set to one for simplicity. These assumptions allow us to use the fact the domain wall trajectory in the W -space is a straight line, so in order to find the solutions we have simply to invert the equation

$$W\left(M\Big|_{x=+\infty}\right)t + W\left(M\Big|_{x=-\infty}\right)(1-t) = k_1 \xi_1(t) + k_2 \xi_1(t)^{-k_1/k_2} \quad (3.30)$$

Fortunately, this algebraic method does not rely on the choice of a Kähler potential and therefore we can omit its description. It turns out that one can find solutions for the k -wall sector only when $k_1 = k$ and $k_2 = N - k$. Therefore, the flavor symmetry is broken along the solutions into $U(N) \rightarrow U(k) \times U(N - k)$. What we have found is a family of solutions, parametrized by the Grassmannian $\text{Gr}(k, N) = \frac{U(N)}{U(k) \times U(N-k)}$. Examples of such solutions are sketched in Figure 3.10, Figure 3.11.

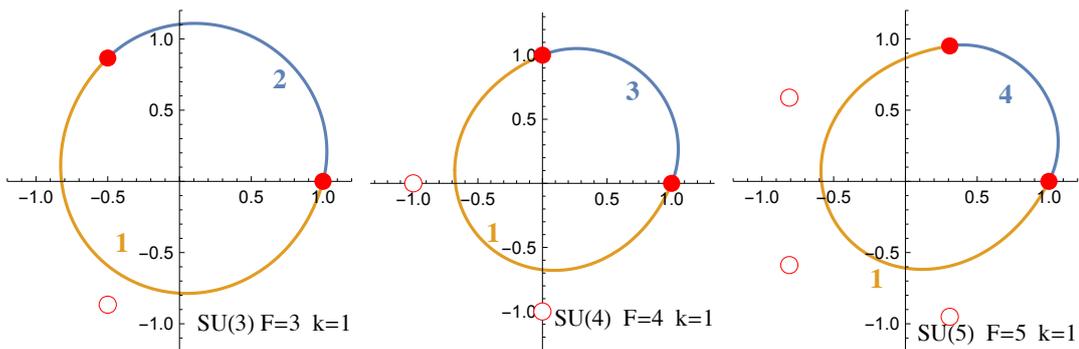


FIGURE 3.10: Examples of 1-wall solutions for $SU(N)$ with $F = N$ flavors for $N = 3, 4, 5$. In each of these cases the eigenvalues of the meson matrix M are split in two groups of $k_1 = 1$ and $k_2 = N - 1$ elements.

Besides finding domain walls solutions by means of the algebraic method we have just described, one can solve the ODE (2.5). Note that if we assume that the matrix M is diagonal at some point along the trajectory of the domain wall, then the form of (2.5) implies the M is always diagonal. The equations (2.5) do not turn on any off-diagonal terms, provided the off-diagonal terms are zero. We were able to reproduce

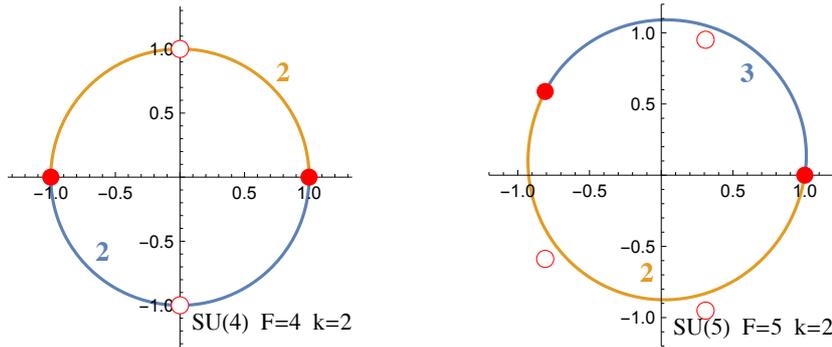


FIGURE 3.11: Examples of 2-wall solution for $SU(N)$ with $F = N$ flavors, where $N = 4, 5$.

all the solutions found with the algebraic method¹² and nothing more. In other words, we looked for solutions where the eigenvalues of M were split into more than two groups, and we did not find any.

Baryonic walls

We implement (2.5) to find solutions in which $B \neq 0$ or $\tilde{B} \neq 0$. Note that the ansatz where all the off-diagonal components of the meson matrix M are set to zero is coherent with the form of the equations (2.5). Indeed, if at some point along the domain wall solution we set the off-diagonal components of M to zero, they remain zero along all the solution. This is due to the fact that the equations (2.5) transform covariantly under the flavor symmetry. Therefore, we assume that the meson matrix can be diagonalized $M = \text{diag}(\xi_1, \dots, \xi_N)$ using the flavor symmetry. Expressing M in units of Λ^2 , B and \tilde{B} in units of Λ^N and setting $A = \frac{m_{4d}\Lambda^2}{\Lambda^{2N}}$ to one, the superpotential becomes

$$W = \sum_i \xi_i + \left(\prod_i \xi_i - B\tilde{B} - 1 \right). \quad (3.31)$$

The differential equations we get with the ansatz that the meson matrix M has diagonal form are the same equations we can derive by considering our system to be given by the eigenvalues ξ_i and the superpotential (3.31). We call this system the associated reduced system and we will study it. We assume that the Kähler potential is the canonical one for the fields ξ_i , B and \tilde{B} , namely $\mathcal{K} = \sum_{i=1}^N \xi_i \xi_i^* + BB^* + \tilde{B}\tilde{B}^*$ ¹³. The system of differential equations we have to solve looks like

$$\partial_x \xi_j = e^{i\gamma} \left(1 + \prod_{i \neq j} \xi_i^* \right) \quad (3.32)$$

$$\partial_x B = -e^{i\gamma} \tilde{B}^* \quad (3.33)$$

$$\partial_x \tilde{B} = -e^{i\gamma} B^* \quad (3.34)$$

¹²We have assumed a canonical Kähler potential $\mathcal{K} = \text{Tr} MM^\dagger$ when studying (2.5)

¹³This choice of the Kähler potential is equivalent to the assumption of canonical Kähler potential for the whole meson matrix in the complete system $\mathcal{K} = \text{Tr}(MM^*) + BB^* + \tilde{B}\tilde{B}^*$

Here we make one observation: the Kähler potential chosen gives us a smooth manifold $\mathcal{M}_{N,N}$ as target space for the chiral fields involved.

It is much easier to solve these differential equations if we make some further assumptions that simplify the problem. First of all, we make the ansatz $\tilde{B} = B$. This ansatz is coherent with the equations (3.33) and (3.34), which is not a surprise because the theory enjoys the $U(1)$ baryonic symmetry. After making this ansatz (3.34) and (3.33) become equal to

$$\partial_x B = -e^{i\gamma} B. \quad (3.35)$$

The next step to take is to evaluate the constraint (3.24) (that looks like $\prod_i \xi_i - B\tilde{B} = 1$ for our reduced system), writing one of the chirals in terms of the other. We write

$$\xi_1 = \frac{1 + B^2}{\prod_{i \neq 1} \xi_i}. \quad (3.36)$$

This particular choice of solving the constraint allows us to have well-defined coordinates on the vacua of the theory, namely when $B = \tilde{B} = 0$. The superpotential in these coordinates reads

$$W = \sum_{i=2}^N \xi_i + \frac{1 + B^2}{\prod_{i \neq 1} \xi_i}. \quad (3.37)$$

Then, projecting the canonical Kähler potential and metric of the ambient space \mathbb{C}^{2N+2} on the manifold $\mathcal{M}_{N,N}$, we get that the Kähler potential looks like

$$\mathcal{K} = \sum_{i=2}^N |\xi_i|^2 + |B|^2 + \frac{|1 + B^2|^2}{\prod_{i \neq 1} |\xi_i|^2}, \quad (3.38)$$

whereas the metric reads

$$\begin{aligned} \mathcal{K}_{a\bar{b}} = & \sum_{i=2}^N |d\xi_i|^2 \left(1 + \frac{|1 + B^2|^2}{\prod_{j \neq 1} |\xi_j|^2 |\xi_i|^2} \right) + \sum_{i \neq j} d\xi_i^* d\xi_j \frac{|1 + B^2|^2}{\prod_{h \neq 1} |\xi_h|^2 \xi_j \xi_i^*} + \\ & + 2 \sum_{i=2}^N \frac{B^*(1 + B^2) dB^* d\xi_i}{\prod_{h \neq 1} |\xi_h|^2 \xi_i} + 2 \sum_{i=2}^N \frac{B(1 + B^{*2}) dB d\xi_i^*}{\prod_{h \neq 1} |\xi_h|^2 \xi_i^*} + |dB|^2 \left(1 + \frac{4|B|^2}{\prod_{h \neq 1} |\xi_h|^2} \right). \end{aligned} \quad (3.39)$$

Then we assume again that the eigenvalues may split in two groups of k_1 and k_2 elements and express each chiral fields as $\xi_i = \rho_i e^{i\phi}$, $B = \beta e^{i\alpha}$. Then to implement the equations (2.5) we need to pay attention: after imposing the constraint (3.36) we can see only $k_1 - 1$ eigenvalues of the first group. So we need to make sure that also the k_1 -th element is equal to the other, imposing the (3.36) along the differential equations. We will not report the full expression taken by the (2.5) because it is lengthy and not illuminating.

However, we will discuss some of the features of the solutions we have found, Figure 3.12, Figure 3.13. First, we point out that the solutions in Figure 3.12 can also be computed using the algebraic method described above. This is a reassuring check

Wall	Effective theory	Witten Index
k	$\text{Gr}(k, N)$ $S^1 \times \text{Gr}(J, N), \quad J \in \{0, \dots, k-1\}$	$\binom{N}{k}$ 0

TABLE 3.3: Domain wall solutions found for 4d $\mathcal{N} = 1$ $SU(N)$ SQCD with $F = N$ flavors in the regime when $m_{4d} \ll \Lambda$. For each $k < \frac{N}{2}$ -wall sector are included also the various contributions to the Witten index from each solution.

for our numerical analysis.

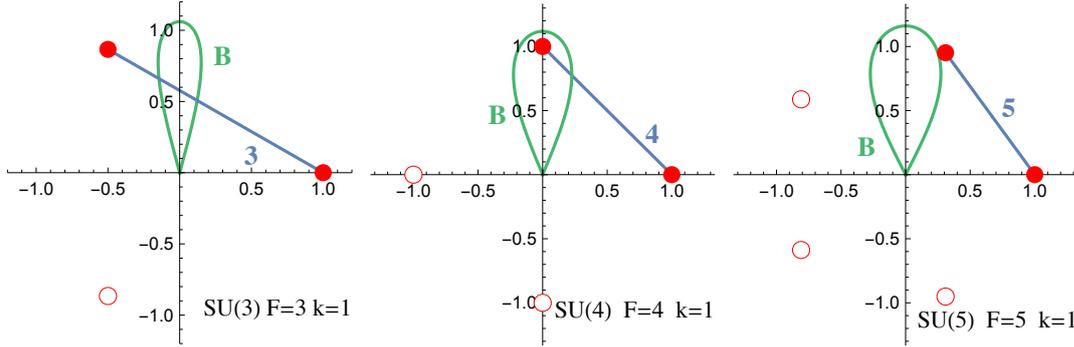


FIGURE 3.12: Examples of domain walls of $SU(N)$ $F = N$: each figure represent a 1-wall solution in which the eigenvalues (in blue) of the meson matrix M do not split. The baryons are depicted in green. The three cases are respectively from left to right $N = 3, 4, 5$.

In these solutions, the baryonic symmetry is broken because the baryons take VEV, therefore we expect a family of solutions that is parametrized by an S^1 . This can be traced back to the fact that $B = \tilde{B}$ fixes the difference of the phase. If the flavor symmetry is also broken (as in Figure 3.13) the family of solutions is described by the Cartesian product of the S^1 and the Grassmannian $\text{Gr}(k_1, N)$.

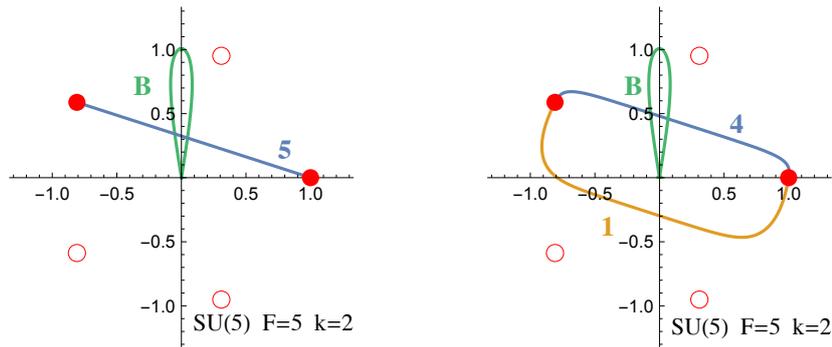


FIGURE 3.13: Examples of 2-wall solutions for $SU(5)$ with $F = 5$ flavors.

We sum up the solutions found in Table 3.3.

The solutions found rely on the assumption that the eigenvalues of the meson matrix split into two groups and not more. We were not able to find solutions where the eigenvalues split into three or more groups.

One can also notice that the case in which $k = \frac{N}{2}$ (parity-invariant wall) when N is even has been left out. This case is particular because the solutions we have found do not exactly follow the pattern of Table 3.3. In fact, we were able to solve only when the eigenvalues of the meson matrix M were evenly split or were not split at all.

Here the weak check involving the comparison of the Witten indices of the 3d $\mathcal{N} = 1$ effective theories on the domain wall and the Witten index of the domain wall theory at large quark mass is even weaker than before because all the ‘‘baryonic’’ domain wall solutions do not contribute to the computation of the Witten index. In fact, the solutions which break the baryonic symmetry have Witten index equal to zero due to the S^1 factor which is present¹⁴. Therefore we cannot say anything conclusive about the exhaustivity of the baryonic BPS k-wall solution we have found, there could be more. However, we can point out that the solutions in which the baryons are spectators ($B = \tilde{B} = 0$) are enough to match the Witten Index computation.

3.2.2 $SU(N)$ with $F = N + 1$

Next, let us consider $\mathcal{N} = 1$ $SU(N)$ gauge theory with $F = N + 1$ flavors, that means $N + 1$ chiral fields in the fundamental Q_a^I and anti-fundamental \tilde{Q}_J^a representations, here $J, I = 1, \dots, N + 1$ and $a = 1, \dots, N$. As is well known, the massless $\mathcal{N} = 1$ $SU(N)$ theory with $N + 1$ flavors is described at low energy by a Wess-Zumino model with fields $M_J^I \leftrightarrow \text{Tr}(Q^I \tilde{Q}_J)$, $B_I \leftrightarrow \epsilon_{II_1 \dots I_N} Q_{I_1} \dots Q_{I_N}$, $\tilde{B}^J \leftrightarrow \epsilon^{JJ_1 \dots J_N} \tilde{Q}_{J_1} \dots \tilde{Q}_{J_N}$. The superpotential of the massless Wess-Zumino model is

$$W = \frac{1}{\Lambda^{2N-1}} (B_I M_J^I \tilde{B}^J - \det(M)). \quad (3.40)$$

This superpotential, which is a purely quantum expression (note that classically the rank of M should be $\text{rk}(M) \leq N$, giving us $\det(M) = 0$), generates the correct moduli space of the massless theory. This moduli space is parametrized by the meson and the baryons of the gauge model, which are related to M_J^I , B_I , \tilde{B}^J . Classically, there are also constraints between baryons and mesons which are precisely the F-term equations derived from the superpotential (3.40).

Once we introduce a mass term $\delta W = m_{4d} \text{Tr} M$, the moduli space of the massless theory is lifted and the N supersymmetric vacua become

$$B_I = \tilde{B}^J = 0, \quad M_J^I = \omega m_{4d}^{\frac{1}{N}} \Lambda^{\frac{2N-1}{N}} \mathbb{I}_{N+1}, \quad \text{where } \omega^N = 1. \quad (3.41)$$

Here m_{4d} is small because (3.40) describes the low energies behavior of SQCD. The WZ is weakly coupled, therefore we use the canonical Kähler potential in terms of the fields B_I , \tilde{B}^J and M_J^I : $\mathcal{K} = M_J^I (M^\dagger)^I_J + B^I B^\dagger_I + \tilde{B}_J \tilde{B}_J^\dagger$.

Let us make the following observation. Away from the origin of the mesons space, the meson matrix takes a VEV, making all the fields B_I , \tilde{B}^J and M_J^I massive. From

¹⁴The Witten index of a NLSM with target manifold S^1 is zero since it is equal to the Euler characteristic of the circle. The Witten index of a NLSM which is a product manifold is equal to the product of the Witten indices of the manifolds

Wall	Effective theory	Witten Index
k	$\text{Gr}(J, N+1), \quad J \in \{0, \dots, k\}$	$\binom{N}{k} = \sum_{j=0}^k (-1)^{j+k} \binom{N+1}{j}$

TABLE 3.4: Domain wall solutions found for 4d $\mathcal{N} = 1$ $SU(N)$ SQCD with $F = N + 1$ flavors in the regime when $m_{4d} \ll \Lambda$. For each $k < \frac{N}{2}$ -wall sector are included also the various contributions to the Witten index from each solution.

(3.40), the mass of the B_I and \tilde{B}^J is proportional to $m_{4d}^{\frac{1}{N}}$, while the mass of M_I^J is proportional to m_{4d} . Since m_{4d} is small, the mass of B_I, \tilde{B}^J is much larger than the mass of the M_I^J . This means that there should not be any solutions of the differential equations where the baryons have a non-zero profile.¹⁵

Since the fields B_I and \tilde{B}^J are much heavier, we can integrate them out, obtaining the reduced superpotential

$$W = -\frac{1}{\Lambda^{2N-1}} \det(M) + m_{4d} \text{Tr } M. \quad (3.42)$$

After the diagonalization of the matrix M using the flavor symmetry, one can easily see that the superpotential (3.42) gives us the same differential equations of the superpotential of $Sp(N)$ with $F = N + 2$ flavors studied in Sec. 3.1.2 and in [32]. So the solutions are the same. The difference is that the flavor group symmetry is $U(F)$ instead of $Sp(F)$, which translates in the change of the moduli of the domain wall from the quaternionic Grassmannian (denoted $\text{HGr}(J, F)$) to the complex Grassmannian (denoted $\text{Gr}(J, F)$). The list of solutions we have found is in Table 3.4 and are in one-to-one correspondence with the solutions found for $Sp(N)$ with $F = N + 2$ of Sec. 3.1.2.

Each k -wall sector consists of $k+1$ walls parameterized by the integer $J = 0, 1, \dots, k$. Each wall hosts a trivial a TQFT and a NLSM with target space

$$\text{Gr}(J, F) = \frac{U(F)}{U(J) \times U(F-J)}, \quad (3.43)$$

the flavor symmetry being broken as $U(F) \rightarrow U(J) \times U(F-J)$.

We display the trajectories of the mesonic eigenvalues in Figures 3.14, 3.15 and 3.16.

The parity-invariant walls, $k = \frac{N}{2}$

Similar to what happens for $Sp(N)$ with $N+2$ flavors in Sec. 3.1.2, the parity-invariant walls of $SU(N)$ with $N+1$ flavors, that is the $k = \frac{N}{2}$ when N is even, require a special treatment.

¹⁵One possible exception to this argument is when the mesonic trajectory of the wall passes through the origin. This will be important in Sec. 3.2.2, where we consider parity invariant walls, for $k = \frac{N}{2}$, that do pass through the origin.

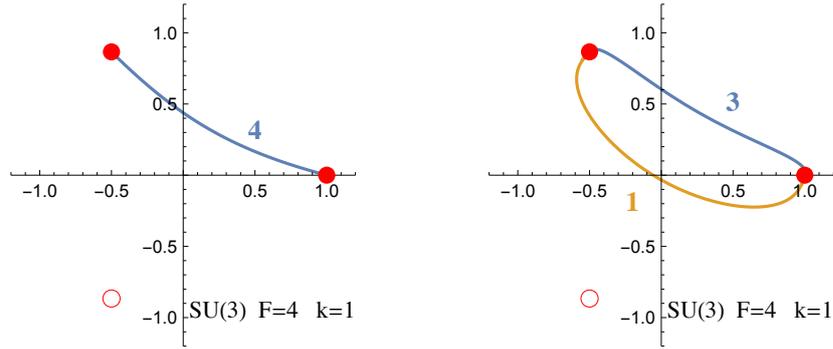


FIGURE 3.14: Examples of 1-wall solutions for $SU(3)$ with $F = 4$ flavors.

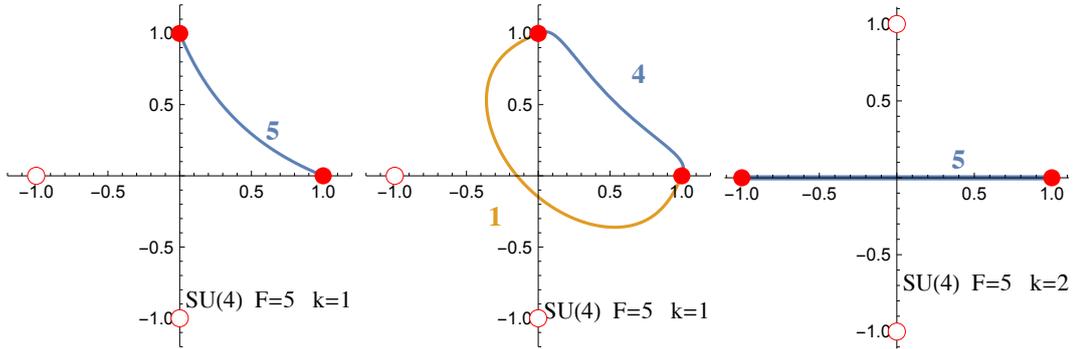


FIGURE 3.15: Examples of 1-wall solutions and 2-wall solutions for $SU(4)$ with $F = 5$ flavors.

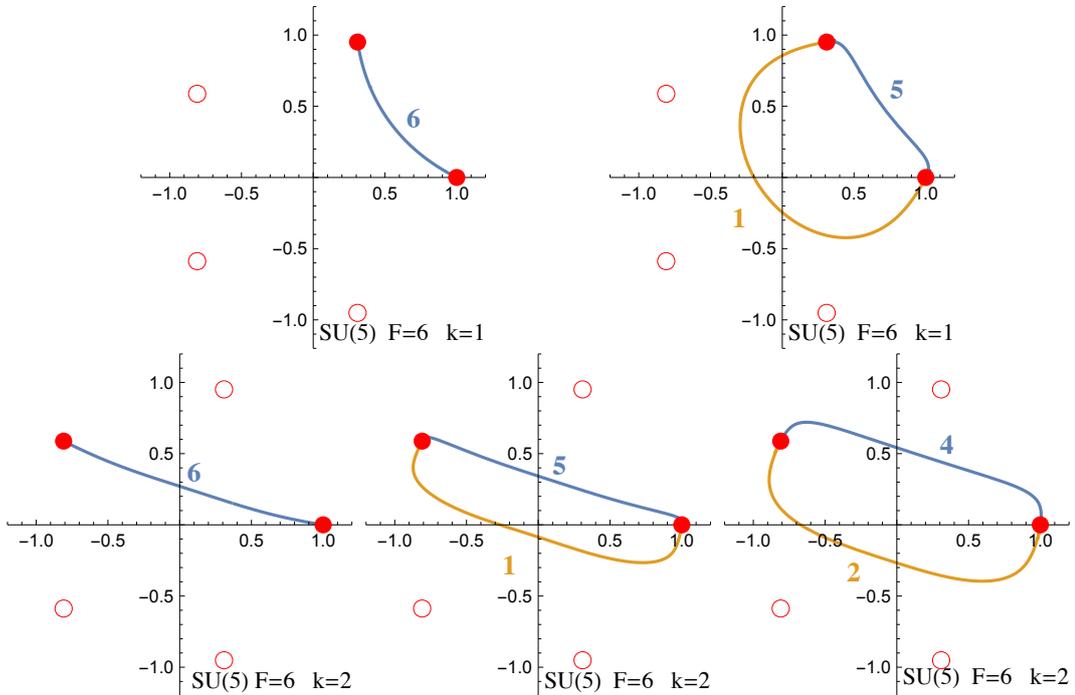


FIGURE 3.16: Examples of 1-wall and 2-wall solutions for $SU(5)$ with $F = 6$ flavors.

In this case, the numerical analysis of (2.5) yields a single domain wall that connects the two vacua $M = \mathbb{I}_{N+1}$ and $M = -\mathbb{I}_{N+1}$ (here we have rescaled M in units

of $m_{4d}^{\frac{1}{N}} \Lambda^{\frac{2N-1}{N}}$) along the real line, hence passing through the origin, see right picture in Figure 3.15. Having a single solution, invariant under the global symmetry, is in contrast with the expectations coming from $k \neq \frac{N}{2}$, where we find $k + 1$ domain walls.

In [32] we showed that the single naive solution of the parity invariant wall must be interpreted as the coalescence of many different solutions. The strategy was to deform the Kähler potential. Upon making an infinitesimal deformation of the Kähler potential, more solutions appear. Such deformations, however, break the flavor symmetry, so there is no automatic recipe to obtain the full moduli space of solutions. The saturation of the Witten index is also problematic.

The solutions found in Sec. 3.1.2 and in [32] for $Sp(N - 1)$ with $N + 1$ flavors carry over to $SU(N)$ with $N + 1$ flavors, so we do not repeat the same analysis here.

One last comment about possible baryonic solutions. Since the solutions pass through the origin of the mesonic space, the argument given before (3.42) does not apply here. As we show below in the special case of $SU(2) = Sp(1)$, upon deforming the Kähler potential, there are baryonic solutions.

For $SU(2)$ with 3 flavors, which has global symmetry $Sp(3)$, one of the deformations studied in Sec. 3.1.2 is

$$\delta\mathcal{K} = \frac{1}{4} \text{Tr}(MJ) \text{Tr}(M^*J), \quad (3.44)$$

where $J = i\sigma_2 \otimes \text{diag}(0.1, 0.1, 0.0)$. This deformation breaks explicitly the flavor symmetry $Sp(3) \rightarrow Sp(2) \times Sp(1)$. The solutions we have found after such deformation are in Figure 3.17. We see that after the deformation a solution involving the baryon

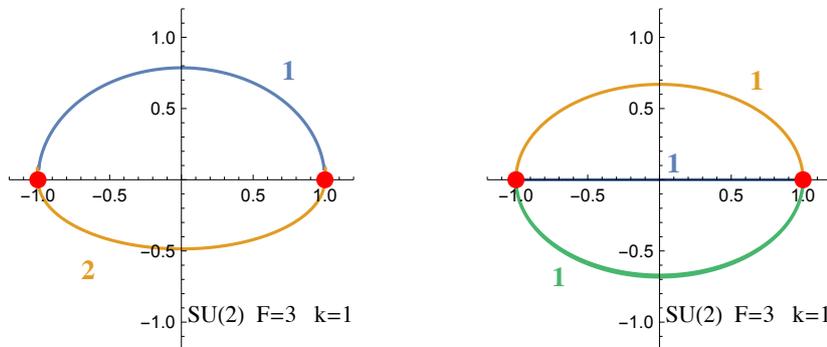


FIGURE 3.17: Example of 1-wall solutions for $SU(2)$ with $F = 3$ flavors.

is possible. Indeed, using the residual flavor symmetry $Sp(2)$ we can rotate the figure on the right into the Figure 3.18 where a baryon is different from zero. One can see that using the map between the operators in the $SU(2)$ language and the $Sp(1)$ language

$$M_{Sp(1)} = \begin{pmatrix} 0 & q_3 & -q_2 & M_{11} & M_{12} & M_{13} \\ -q_3 & 0 & q_1 & M_{21} & M_{22} & M_{23} \\ q_2 & -q_1 & 0 & M_{31} & M_{32} & M_{33} \\ -M_{11} & -M_{21} & -M_{31} & 0 & \tilde{q}_3 & -\tilde{q}_2 \\ -M_{12} & -M_{22} & -M_{32} & -\tilde{q}_3 & 0 & \tilde{q}_1 \\ -M_{13} & -M_{23} & -M_{33} & \tilde{q}_2 & -\tilde{q}_1 & 0 \end{pmatrix} \quad (3.45)$$

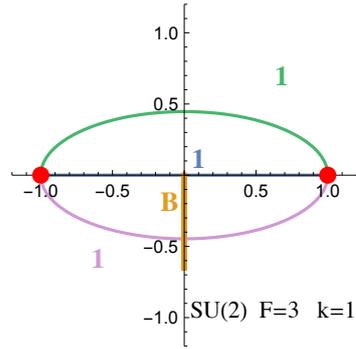


FIGURE 3.18: Example of 1-wall solution for $SU(2)$ with $F = 3$ flavors, where the baryons have a non zero profile. We used different scales to plot these trajectories: the imaginary axis is zoomed in $10\times$.

and the rotation matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in Sp(2) \times Sp(1) \quad (3.46)$$

Chapter 4

Living on the walls

This section discusses the $3d$ effective theories that describe the low energy behavior of the domain walls. However, before dive into that, we set the notation we will be using throughout this chapter and we will present a simple example of $3d$ duality to show how these dualities work.

4.0.1 Aspects of QFTs in $3d$

A very peculiar aspect of $3d$ gauge theories is that they admit the possibility of adding a Chern-Simons term to the Lagrangian. Therefore the gauge part of the Lagrangian looks like

$$L = -\frac{1}{2g^2} \text{Tr}(F \wedge *F) + \frac{k}{4\pi} \text{Tr}(A \wedge dA - \frac{2i}{3} A \wedge A \wedge A). \quad (4.1)$$

k is called the Chern-Simons level. We will denote a $3d$ theory with gauge group G and CS level k G_k . The second term of the Lagrangian (4.1) gives a mass to the gauge fields as we can see from the EOMs:

$$D_\nu F^{\nu\mu} + \frac{kg^2}{2\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho} = 0. \quad (4.2)$$

There are massive modes propagating with mass $m = \frac{kg^2}{2\pi}$, whereas the zero modes correspond to flat connection $F = 0$. We will consider only connected and simply-connected gauge groups G . We note that the second term of (4.1) can be written as

$$S_{CS}[A] = \frac{k}{4\pi} \int_{\mathcal{M}_3} \text{Tr}\left(A \wedge dA - \frac{2i}{3} A \wedge A \wedge A\right) = \frac{k}{4\pi} \int_{X_4} \text{Tr}(F \wedge F), \quad \text{where } \partial X_4 = \mathcal{M}_3 \quad (4.3)$$

The action (4.3) is manifestly gauge invariant, while the action come from the lagrangian (4.1) it is not. However, the action (4.3) depends on the choice of the 4-dimensional manifold X_4 for which $\partial X_4 = \mathcal{M}_3$. If we want the action to be independent also from the choice of 4-dimensional manifold we require $k \in \mathbb{Z}$. In this way, if we choose a different 4-manifold X'_4 we get that the difference between the two actions is

$$S_{X_4}[A] - S_{X'_4}[A] = \frac{k}{4\pi} \int_{X_4 \cup \bar{X}'_4} \text{Tr}(F \wedge F). \quad (4.4)$$

Here \bar{X}_4 it means that we have changed the orientation of X'_4 . Now $X_4 \cup \bar{X}'_4$ is a four dimensional closed manifold and

$$\frac{1}{8\pi^2} \int_{X_4 \cup \bar{X}'_4} \text{Tr}(F \wedge F) \in \mathbb{Z}. \quad (4.5)$$

Therefore, the exponential of the difference of the actions is

$$e^{i(S_{X_4}[A] - S_{X'_4}[A])} = e^{2\pi i k n} = 1, \quad (4.6)$$

and it is invariant over the choice of 4-manifold. When only gauge fields are involved, the time-reversal transformation T sends the CS level

$$k \rightarrow -k \quad (4.7)$$

This means that when $k \neq 0$, the models are not T invariant. However, if fermionic fields are present, the T transformation becomes

$$k \rightarrow -k + \sum_f T(R_f), \quad (4.8)$$

where the sum is performed over all the Majorana fermions and $T(R_f)$ is the index of the gauge representation of the fermions. To simplify the reading, we will then indicate as CS level always

$$K = k - \frac{1}{2} \sum_f T(R_f), \quad (4.9)$$

such that K transforms under T as $T(K) = -K$. We denote the CS theory as G_K where G is the gauge group, and K is the CS level. When fermions are integrated out, the CS level shifts as

$$K_{IR} = K + \frac{1}{2} \sum_f \text{sgn}(m_f) T(R_f). \quad (4.10)$$

This result is one loop exact. Note that K_{IR} must always be an integer, whereas K can be integer or half-integer depending on the number of fermions and their representation.

Given massive matter and CS level different from zero, at low energy, we are left only with topological degrees of freedom due to the EOM $F = 0$. Our theory does not have any local observable. However, one can construct Wilson lines

$$W_R[\gamma] = \text{Tr}_R e^{i \int_\gamma A}, \quad (4.11)$$

R is the selected representation of the gauge group, and γ is the path of the Wilson line. In the seminal paper [64] the relation between CS theories and $2d$ rational conformal field theory RCFT was established. So, counting the line operators in a CS model is equivalent to counting *integrable* representation in the Kac-Moody algebra \mathfrak{g}_K . See the Appendix B for the computation of integrable representation in the case of $SU(N)_K$,

$U(N)_{K,K^1}$ and $Sp(N)_K$. The number of different lines also corresponds to the number of vacuum states of the CS theory on a torus and, therefore, in a supersymmetric theory, to the Witten index [65].

When we consider a $3d \mathcal{N} = 1$ CS theory, there is also a Majorana fermion λ in the adjoint representation of the gauge group in the same multiplet of the gauge fields. The Lagrangian reads

$$L = -\frac{1}{2g^2} \text{Tr}(F \wedge *F) + i \text{Tr} \lambda \not{D} \lambda + \frac{k}{4\pi} \text{Tr} \left(A \wedge dA - \frac{2i}{3} A \wedge A \wedge A \right) - \frac{kg^2}{2\pi} \text{Tr} \lambda \lambda. \quad (4.13)$$

Note that a $\mathcal{N} = 1$ CS theory G_K has Witten index different from zero if $2K \geq c_2(\mathfrak{g})$. In fact, the presence of the massive Majorana fermion in the adjoint is such that $K_{IR} = K - \frac{c_2(\mathfrak{g})}{2}$. Witten showed [65] that when K_{IR} changes sign when integrating out the Majorana fermion, supersymmetry is broken. In [36] they showed that when $2K < c_2(\mathfrak{g})$ at low energy, there is a goldstino, coming from the breaking of the supersymmetry, and a TQFT generated by the strong coupling effects of the CS model.

4.0.2 Particle-vortex duality

One of the first examples of $3d$ dimensional duality is given by two non-supersymmetric theories: the Abelian-Higgs model and the $O(2)$ scalar model (see [66], [67]).

The Abelian-Higgs is a $U(1)_0$ gauge theory with one complex scalar field ϕ of charge 1 and a potential

$$V(\phi) = m^2 |\phi|^2 + |\phi|^4. \quad (4.14)$$

When $m^2 < 0$ the vacuum breaks the gauge symmetry, there is Higgs mechanism, and thus the model is gapped. In this phase, we can construct solitons, which are usually called *vortices*. These solitons winds around the spatial infinity, and their expression in polar coordinates is

$$\phi(r, \varphi) = \phi_0(r) e^{i\varphi}, \quad \frac{1}{2\pi} \int_{\mathbb{R}^2} F = n. \quad (4.15)$$

The energy of this configuration is finite, and therefore the Abelian-Higgs model has finite-mass vortices. The idea is that, at low energies, the effective physics is described by weakly interacting vortices that we treat like particles. These particles are treated like complex fields $\tilde{\phi}$, with positive mass $\tilde{m}^2 > 0$, and potential

$$V(\tilde{\phi}) = \tilde{m}^2 |\tilde{\phi}|^2 + |\tilde{\phi}|^4 \quad (4.16)$$

This is, in fact, the $O(2)$ model when the mass is positive.

¹The CS theory $U(N)_{k_1, k_2}$ has two CS levels because we can write

$$U(N)_{k_1, k_2} = \frac{SU(N)_{k_1} \times U(1)_{Nk_2}}{\mathbb{Z}_N}. \quad (4.12)$$

The CS level appearing in the lagrangian is k_1 for the $SU(N)$ part of the gauge fields and Nk_2 for the $U(1)$ part. Consistency of the model requires $k_1 = k_2 \pmod N$.

When $m^2 > 0$ the complex scalar is massive and can be integrated out, leaving a $U(1)_0$ model which is equivalent to a compact scalar σ living on S^{12} . This can be seen dualizing the gauge field in the usual way

$$d\sigma = *F. \quad (4.17)$$

In this phase, there cannot be a potential for σ because σ can be seen as the goldstone boson of the $U(1)_M$ magnetic symmetry, whose current is $J = *F$. This current is topologically conserved because of the Bianchi identity. This symmetry shifts σ , which is its Goldstone boson.

We have already seen that for $\tilde{m}^2 > 0$ the $O(2)$ has the same IR physics of the Abelian-Higgs model for $m^2 < 0$. If we now consider $\tilde{m}^2 < 0$ in the $O(2)$ model with potential (4.16) we see that there is symmetry breaking, and at low energy, we have a Goldstone boson on a circle. This is the same IR physics of the Abelian-Higgs model.

We have found two phases in both theories: one is gapped, and the other is an S^1 Goldstone boson. There should be a transition between these two phases. In the $O(2)$ model, it is known that this transition is second order, and it is described by the $O(2)$ Wilson-Fisher fixed point. We are led to conjecture that this is also the case for the Abelian-Higgs model. This is not an obvious statement, but it has passed numerous tests, including lattice Monte-Carlo simulation [68]. We are going to use the following notation to write this duality

$$U(1)_0 \text{ with } \phi \Leftrightarrow O(2) \tilde{\phi} \text{ (WF)} \quad (4.18)$$

Moreover, note that on both sides of the duality, the global symmetries match. The global symmetry of the $O(2)$ model is indeed $O(2) \cong U(1) \times \mathbb{Z}_C^2$, where \mathbb{Z}_C^2 is charge conjugation and parity (time-reversal) symmetry. The Abelian Higgs model has charge conjugation, time reversal, and the magnetic symmetry $U(1)_M$. To conclude, what is the operator in the Abelian-Higgs model that creates vortices and it has the dual role of $\tilde{\phi}$? This operator is called *monopole operator* \mathfrak{M} . This is a "defect" or "disorder" operator, not defined as a polynomial of the fundamental fields of the Lagrangian, but as singular boundary conditions at a point x_0 in the path integral. Around x_0 , we remove a small S^2 and impose

$$\frac{1}{2\pi} \int_{S^2} F = 1 \quad \text{basic monopole} \quad (4.19)$$

in the path integral. There is, therefore, the following operator map:

$$\mathfrak{M} \leftrightarrow \tilde{\phi}. \quad (4.20)$$

²Consider the path-integral over $S^2 \times S^1$. There are field configurations where $\frac{1}{2\pi} \int_{S^2} F = n$. They are mapped to configurations where $\frac{1}{2\pi} \int_{S^1} d\sigma = n$. This make sense only if $\sigma \cong \sigma + 2\pi$

4.1 Living on the walls

Let us now turn to the description of our proposal for the $3d$ models that describes the low energy physics trapped on the walls. Such a purely $3d$ description exists because the vacua of massive $4d$ SQCD develop a mass gap due to strong interactions, and so the $4d$ IR dynamics below the strong scale Λ is trivial. This, in turn, tells us that the degrees of freedom of the theory below the energy scale Λ are frozen on the domain wall. As a result, the $3d$ system is decoupled from the $4d$ bulk.

As we have said, the theory living on the domain wall has $3d$ $\mathcal{N} = 1$ supersymmetry because the domain walls considered do not break completely the $4d$ $\mathcal{N} = 1$ supersymmetry, but preserves two supercharges. Remember, there is a universal part of this $3d$ theory, described by a $3d$ free chiral field associated with the breaking of translational invariance and supersymmetry. We will assume that this part of the $3d$ theory is always there, and in the following, we will omit it.

The world-volume theories we propose follow some general requirements.

The flavor symmetry of the world-volume $3d$ theory matches the symmetry of the (massive) $4d$ theory, which is $Sp(F)$.

There is a free parameter in the $3d$ theory, and by tuning such a parameter, we end up in different massive phases. This comes about because the $4d$ model has different IR descriptions depending on the mass parameter of the quarks: if $|m_{4d}| \gg \Lambda$, the low energy description of the $4d$ theory is pure $\mathcal{N} = 1$ SYM, instead if $|m_{4d}| \ll \Lambda$ the low energy model is the Wess-Zumino model we discussed in the previous sections. Therefore, we request that the different phases of $3d$ world-volume theory describe the different domain walls we found in the two different IR four-dimensional descriptions.

Another important feature is that the $3d$ theory for the k -wall is IR dual to the theory for the $(h - k)$ -wall, up to a parity transformation. Indeed the k -wall sector is related to the $(h - k)$ sector because the k vacuum and $(h - k)$ vacuum are related in $4d$ by parity and R-symmetry transformations.

4.1.1 $Sp(N)$ with $F = N + 1$

Let us start from the bulk theory $4d$ $Sp(N)$ with $F = N + 1$ flavors, whose global symmetry is $SU(2N + 2) \times U(1)$ at zero mass and $Sp(N + 1)$ at non-zero mass (which is the situation of interest for us).

We propose that the $3d$ theory living at low energy on the k -domain wall is

$$3d \quad Sp(k)_{\frac{N-k+2}{2}}^{\mathcal{N}=1} \quad \text{with } N + 1 \text{ fundamentals } X. \quad (4.21)$$

The superfields X are in the fundamental representation of the gauge group and are denoted by the matrix X_{aI} , where $a = 1, \dots, 2k$ is the gauge index and $I = 1, \dots, 2N + 2$. We impose the reality condition $X_{aI} = \Omega^{ab} \Omega^{IJ} X_{bJ}^*$. In this representation the $Sp(N + 1)$ flavor symmetry is manifest. Gauge invariants are constructed in terms of $X_{IJ}^2 = X_{aI} X_{bJ} \Omega^{ab}$. X_{IJ}^2 is manifestly skew-symmetric. Notice that the $4d$ $Sp(N)$ theory has $2F$ massive fundamentals, while the $3d$ $Sp(k)$ theory has F fundamentals. The $\mathcal{N} = 1$

superpotential is

$$\mathcal{W} = \frac{1}{4} \text{Tr}(X^2 \Omega X^2 \Omega) + \frac{\alpha}{4} \text{Tr}(X^2 \Omega)^2 + m \text{Tr} X^2 \Omega, \quad (4.22)$$

the SCFT being at zero mass, $m = 0$.

We assume that a fixed point of the RG flow exists in the region where $\alpha > -\frac{1}{k}$. The overall scale of the superpotential has been fixed for convenience and, for the sake of studying the model's vacuum structure, we can set $\alpha = 0$. The 3d parameter m is the effective IR mass and is related to the m_{4d} parameter. The precise relation between these two parameters is not known, but it is not important because we are interested in the two regimes, large and small four-dimensional mass (compared to the strong scale Λ), which are related to positive and negative three-dimensional mass. As we show below, this model has two different phases: a single gapped vacuum for $m > 0$ and multiple vacua for $m < 0$, hence the proposal meets some of the requirements we demanded for our low energy theory. Moreover, the transition between the two phases is smooth, thanks to the unbroken $\mathcal{N} = 1$ supersymmetry.

Furthermore, in order to fulfill the demand that the k -domain wall sector is the parity reversal of the $N + 1$ -wall sector, these models should enjoy the following infrared duality:

$$\begin{array}{ccc} Sp(k)_{\frac{N-k+2}{2}}^{\mathcal{N}=1} & & Sp(N+1-k)_{\frac{k+1}{2}}^{\mathcal{N}=1} \\ \text{with } N+1 \text{ fundamentals } X & \iff & \text{with } N+1 \text{ fundamentals } X \\ \mathcal{W} \sim |X|^4 & & \mathcal{W} \sim -|X|^4 \end{array} \quad (4.23)$$

We call the theory on the left Theory A, whereas the model on the right Theory B.

Duality (4.23) is similar to the family of dualities used in [1]; however, the parameter F it is the limit of the range of validity of the duality in [1]. Therefore it seems that our domain wall studies suggest that the dualities are valid not only for $0 < F < N + 1$ but also for $F = N + 1$.

$\mathcal{N} = 1$ duality (4.23) is expected to be an $\mathcal{N} = 1$ deformation of the $\mathcal{N} = 2$ duality duality for Sp gauge group with non-zero CS, found and tested by Willet and Yaakov [69]:³

$$\begin{array}{ccc} Sp(k)_{2+N-\frac{F}{2}}^{\mathcal{N}=2} & & Sp(N+1-k)_{-N-2+\frac{F}{2}}^{\mathcal{N}=2} \\ \text{with } F \text{ fundamentals} & \iff & \text{with } F \text{ fundamentals} \\ \mathcal{W}_{\mathcal{N}=2} = 0 & & \mathcal{W}_{\mathcal{N}=2} = \mu_{ij} \text{tr}(p_i p_j) \end{array} \quad (4.25)$$

³This duality is obtained from the well known 3d $\mathcal{N} = 2$ Aharony duality [70]

$$\begin{array}{ccc} Sp(N_c) \text{ with } 2N_f \text{ fundamentals} & \iff & Sp(N_f - N_c - 1) \text{ with } 2N_f \text{ fundamentals} \\ \mathcal{W}_{\mathcal{N}=2} = 0 & & \mathcal{W}_{\mathcal{N}=2} = \mu_{ij} \text{tr}(p^i p^j) + \mu \mathfrak{M} \end{array} \quad (4.24)$$

giving real masses to $2N_f - F$ fundamentals. CS levels $\pm(N_f - F/2)$ are generated (with positive/negative sign on the electric/magnetic side) and μ becomes massive. Changing variables as $N_c = k$, $N_f = 2 + N$, we get (4.25).

which holds for any $F > 0$ and $k = 1, \dots, N$. One goes from (4.25) to (4.23) turning on a quartic $\mathcal{N} = 1$ superpotential term, and along the way, on the r.h.s., the gauge singlets μ_{ij} become massive. See [33] for an example of such a deformation discussed with more details, in the case of $U(k) \leftrightarrow U(N - k)$ CSM dualities. Notice that, for generic k and F , the $\mathcal{N} = 2$ theories have $U(F)$ global symmetry, which is enhanced to $Sp(F)$ at the end of the RG flow that lands on the $\mathcal{N} = 1$ SCFTs. This means that the $U(F)$ -invariant $Sp(F)$ -breaking interactions flow to zero. It is not known that the (4.25) duality can be deformed to an $\mathcal{N} = 1$ duality for any value of k, N, F . If $F \leq N + 1$ the semiclassical analysis of the massive vacua match across the $\mathcal{N} = 1$ duality, so it is very likely that such an $\mathcal{N} = 1$ is correct. In Sec. 4.1.2 we extend this story to $F = N + 2$. It would be very interesting to investigate more the regime $F > N + 1$, where the models are strongly coupled.

In the following we provide further checks of the duality (4.23), studying the two different phases. Varying the mass parameter m from positive to negative values, the vacuum structures of Theory A and of Theory B are the same. The mapping between the right and left side mass parameter is $m \rightarrow -m$.

Analysis of the massive vacua of Theory A

Let us discuss the vacuum structure of the theory A. To do so we have to 2×2 -block diagonalize the matrix $X^2\Omega$ using the gauge and flavor symmetry. The entries λ_i of the 2×2 -antisymmetric blocks are real because we have imposed a reality condition on X_{aI} . Note that the maximal rank of the matrix $X^2\Omega$ is $2k$ and therefore the index i runs from 1 to k . Once we have diagonalized $X^2\Omega$, the F-term equations for the ‘‘eigenvalues’’ are

$$\lambda_i(\lambda_i^2 + m) = 0 \quad i \in \{0, \dots, k\}. \quad (4.26)$$

When $m \neq 0$ these equations have $k + 1$ solutions that we will parametrize by $J = 0, \dots, k$. Each solution has only J non-vanishing eigenvalues:

$$\text{solution } J: \quad \lambda_1^2, \dots, \lambda_J^2 = -m, \quad \lambda_{J+1}, \dots, \lambda_k = 0 \quad (4.27)$$

Depending on the sign of m , not all these solutions are acceptable. This gives us a different number of vacua for the two phases of the model.

- $m > 0$. Only the vacuum with $J = 0$ is acceptable. In such a vacuum we can integrate out massive quarks with positive mass, leaving

$$3d \quad Sp(k)_{N - \frac{k-3}{2}}^{\mathcal{N}=1}. \quad (4.28)$$

This theory is indeed the TQFT describing the domain walls of pure $Sp(N)$ SYM [1]. This is exactly what we expected to be the behavior of SQCD at large mass, $m_{4d} \gg \Lambda$. This theory has a single supersymmetric gapped vacuum, it is a Topological Quantum Field Theory and its Witten index is

$$\text{WI} = \binom{N+1}{k}. \quad (4.29)$$

• $m < 0$. In this case all the $k + 1$ vacua are acceptable. Therefore the quarks get a VEV and the matrix $X^2\Omega$ can be put in the a 2×2 -block diagonal form, with the first J blocks different from zero. This implies that on the vacua the flavor symmetry is broken to

$$Sp(N + 1) \rightarrow Sp(J) \times Sp(N + 1 - J), \quad (4.30)$$

leading at low energy to a NLSM with target space

$$\text{HGr}(J, N + 1) = \frac{Sp(N + 1)}{Sp(J) \times Sp(N + 1 - J)}. \quad (4.31)$$

Also the VEVs of the quarks break the gauge group as $Sp(k) \rightarrow Sp(k - J)$. Therefore for every $J \neq k$ there is a gauge group left in the infrared. All the fermions charged under the unbroken gauge group shift the CS level with a positive contribution or a negative one depending on the sign of the effective mass. These masses come either from the potential or from the Higgs mechanism. As a result the CS level of the unbroken gauge group is $Sp(k - J)_{-\frac{\mathcal{N}=1}{k-1-J}}$. The NLSM and the CS theory are decoupled in the IR, thus the low energy theory on a vacuum labelled by J is

$$3d \quad Sp(k - J)_{-\frac{\mathcal{N}=1}{k-1-J}} \times \text{HGr}(J, N + 1) \quad (4.32)$$

But not all these vacua are supersymmetric, in fact due to non-perturbative effects a $\mathcal{N} = 1$ CS theory has a supersymmetric vacuum only if the CS level K and the dual Coxeter number h of the gauge group satisfy $\frac{h}{2} < K$ [65]. In our case this translates into

$$k - J + 1 < k - 1 - J, \quad J < k. \quad (4.33)$$

This relation is never satisfied, so the only acceptable vacuum is the one with $J = k$ on which the gauge group is completely broken. Only in this case the non-perturbative effects due to the strong dynamics of the gauge group are not present. One should also point out that this supersymmetric NLSM has a Wess-Zumino term, which is conveniently specified by describing the NLSM as an $\mathcal{N} = 1$ $Sp(J)_{\frac{N+1-J}{2}}$ gauge theory coupled to $N + 1$ fundamental scalar multiplets getting VEV. The Witten index of the remaining solution is equal to

$$\text{WI} = \binom{N+1}{k}. \quad (4.34)$$

Analysis of the massive vacua of Theory B

Carrying out a similar procedure we can study the vacuum structure of Theory B.

• $m < 0$. We get that there is only one vacuum on which lives a CS theory

$$Sp(N + 1 - k)_{-1 - \frac{N+k}{2}}. \quad (4.35)$$

This model is the level-rank dual of the model (4.30) and its Witten index is $WI = \binom{N+1}{N-k+1}$.

Wall	Effective theory	Witten Index
k	$\text{HGr}(k, N + 1)$	$\binom{N+1}{k}$

TABLE 4.1: IR models on the vacuum structure of (4.21) for $m < 0$. The various contributions to the Witten index are displayed.

• $m < 0$. In this case there are $N + 1 - k$ vacua. On these vacua the flavor symmetry is broken to $Sp(N + 1) \rightarrow Sp(N + 1 - J) \times Sp(J)$, whereas the gauge symmetry is broken to $Sp(N + 1 - k) \rightarrow Sp(N + 1 - k - J)$. As above, integrating out the fermion charged under the unbroken gauge group we shift the CS level. We obtain the low energy theory

$$Sp(N + 1 - k - J)_{-\frac{\mathcal{N}=1}{k+1} + \frac{N+1-J}{2}} \times \text{HGr}(J, N + 1) \quad (4.36)$$

It seems that the number of vacua does not match. But, taking into account the non-perturbative effects, we need to impose $N + 2 - k - J < N - k - J$. This relation is never satisfied, so the only acceptable vacuum is the one with $J = N + 1 - k$, on which the gauge group is completely broken. The only surviving vacuum has $\text{WI} = \binom{N+1}{N+1-k}$ and it matches the vacuum with $m < 0$ in theory A because $\text{HGr}(k, N + 1) = \text{HGr}(N + 1 - k, N + 1)$.

We stress again that in both models at $m = 0$ there is a second order phase transition. The nature of this transition is necessarily second order and therefore it is described by a $\mathcal{N} = 1$ SCFT. This is because the two vacua have the same energy level, and they fuse at $m = 0$, giving us a continuous transition. These two SCFTs are the one dual to each other.

There is a particular case worth noticing, namely the case $N = 1$, where there is an emerging T reversal symmetry at low energy because

$$Sp(1)_1^{\mathcal{N}=1} \quad F = 2 \quad \iff \quad Sp(1)_{-1}^{\mathcal{N}=1} \quad F = 2$$

A summary of the vacua is displayed in Table 4.1.

The vacua we have found in Table 4.1 match precisely the domain wall solutions found in the previous section in Table 3.1 for the NLSM which describes the low energy regime when $m_{4d} \ll \Lambda$. These models are good candidates for the effective theories on the domain walls of SQCD with gauge group $Sp(N)$ and $N + 1$ flavors.

4.1.2 $Sp(N)$ with $F = N + 2$

Let us now add one flavor, considering the 4d theory $Sp(N)$ with $F = N + 2$. We propose that the low energy theory describing the low energy behavior of the k -wall is

$$Sp(k)_{\frac{\mathcal{N}=1}{N-k+1}} \quad \text{with } N + 2 \quad \text{fundamentals } X. \quad (4.37)$$

In this case the amount of evidence that we can provide is weaker than for $Sp(N)$ with $F \leq N + 1$ flavors, or for $SU(N)$ with $F < N$ flavors. We still have the $\mathcal{N} = 2$ duality (4.25)

$$Sp(k)_{1+\frac{N}{2}>0}^{\mathcal{N}=2} \text{ w/ } N+2 \text{ fundamentals} \quad \Longleftrightarrow \quad Sp(N+1-k)_{-1-\frac{N}{2}}^{\mathcal{N}=2} \text{ w/ } N+2 \text{ fundamentals} \\ \mathcal{W}_{\mathcal{N}=2} = 0 \quad \quad \quad \mathcal{W}_{\mathcal{N}=2} = \mu_{ij} \text{tr}(p_i p_j) \quad (4.38)$$

from which we expect to get an $\mathcal{N} = 1$ duality incarnating the equivalence of a k wall with the time-reversed $N + 1 - k$ wall. This duality is

$$Sp(k)_{\frac{N-k+1}{2}>0}^{\mathcal{N}=1} \text{ with } N+2 \text{ fundamentals} \quad \Longleftrightarrow \quad Sp(N+1-k)_{-\frac{k}{2}}^{\mathcal{N}=2} \text{ with } N+2 \text{ fundamentals}, \quad (4.39)$$

we will call the theory on the left Theory A, whereas we will call Theory B the model on the right.

Analysis of the massive vacua of Theory A

Let us now study the vacuum structure of (4.37). The superpotential we are considering is the same we have considered in the case with $F = N + 1$, namely

$$\mathcal{W} = \frac{1}{4} \text{Tr}(X^2 \Omega X^2 \Omega) + \frac{\alpha}{4} \text{Tr}(X^2 \Omega)^2 + m \text{Tr}(X^2 \Omega). \quad (4.40)$$

Here again we assume that an RG flow fixed point exists in the region with $\alpha > -\frac{1}{k}$, and henceforth we set $\alpha = 0$. The vacua analysis is similar to the one we have done for the $F = N + 1$ case and so we recall schematically what we found there. After the block diagonalization, the F-term equations read

$$\lambda_i (\lambda_i^2 + m) = 0 \quad i \in \{0, \dots, k\} \quad (4.41)$$

These equations have $k + 1$ solutions, which will be parametrized by $J = 0, \dots, k$, but depending on the sign of m , not all are acceptable.

- $m > 0$. There is only one solution $J = 0$ and the low energy theory is the TQFT

$$3d \quad Sp(k)_{N-\frac{k-3}{2}}^{\mathcal{N}=1}, \quad (4.42)$$

once we have integrated out the positive mass fermions.

- $m < 0$. In this case all J solutions are acceptable. Therefore the quarks $X^2 \Omega$ take VEVs and break both the flavor symmetry, $Sp(N + 2) \rightarrow Sp(N + 2 - J) \times Sp(J)$, and the gauge symmetry $Sp(k) \rightarrow Sp(k - J)$. The low energy models living on each of the J vacua are

$$Sp(k - J)_{\frac{k-J+1}{2}}^{\mathcal{N}=1} \times \text{HGr}(J, N + 2). \quad (4.43)$$

Contrary to the $F = N + 1$ case, here the strong dynamic does not break supersymmetry because the CS factor of the effective theory is such that there is a unique

supersymmetric vacuum, with trivial TQFT. Therefore all the vacua are supersymmetric vacua. Computing the Witten index is now a subtle task because as pointed out by [48] the sign between the J vacua depend on the number of charged fermions with negative mass. After a careful analysis of the masses of the charged fermions under the gauge group, generated either by the superpotential or via Higgs mechanism, we get that the Witten index is given by

$$\text{WI} = \sum_{J=0}^k (-1)^{J+k} \binom{N+2}{J}. \quad (4.44)$$

Analysis of the massive vacua of Theory B

In this model we consider the superpotential

$$\mathcal{W} = -\frac{1}{4} \text{Tr}(X^2 \Omega X^2 \Omega) - \frac{\alpha}{4} \text{Tr}(X^2 \Omega)^2 + m \text{Tr}(X^2 \Omega), \quad (4.45)$$

and again we consider $\alpha > -\frac{1}{k}$, hence setting $\alpha = 0$ for simplicity. The study of the vacua goes on as we have done in the previous subsection, and we are going only to list the different phases.

- $m < 0$. There is only one solution and the low energy theory is the TQFT

$$3d \quad Sp(N+1-k)_{-\frac{k+N+2}{2}}^{\mathcal{N}=1}, \quad (4.46)$$

once we have integrated out the negative mass fermions.

- $m > 0$. In this case there are $H = 0, \dots, N+1-k$ solutions. The quarks $X^2 \Omega$ take VEVs and break both the flavor symmetry, $Sp(N+2) \rightarrow Sp(N+2-H) \times Sp(H)$, and the gauge symmetry $Sp(N+1-k) \rightarrow Sp(N+1-k-H)$. The low energy models living on each of the H vacua are

$$Sp(N+1-k-H)_{\frac{N+2-k-H}{2}}^{\mathcal{N}=1} \times \text{HGr}(H, N+2). \quad (4.47)$$

This analysis of the massive vacua in some cases provides more vacua than the $4d$ analysis. The vacua expected are always there, but in some cases there are more vacua. Indeed, only when the rank of the gauge group $k \leq \frac{N+1}{2}$ the vacua match the $4d$ domain wall solutions for the Theory A, while when the gauge group $k > \frac{N+1}{2}$ the $3d$ theory presents more vacua at large negative masses. The behaviour of Theory B is the opposite: when $k \leq \frac{N+1}{2}$ there are too many vacua, while when $k > \frac{N+1}{2}$ the number of vacua matches the $4d$ analysis. Moreover, the vacua do not match perfectly across the $\mathcal{N} = 1$ duality.

We interpret this mismatch as follows: our interpretation of the semiclassical analysis of the $3d$ vacua is naive when the $3d$ theories under consideration are strongly coupled. They might present *quantum phases* similar to what happens for non-supersymmetric theories [37]: for fixed rank and number of flavors, if the Chern-Simons

level is too small there are *quantum phases* and a naive analysis of the vacua is incorrect. Indeed the CS level controls the masses of the gauge multiplet; at CS level equal to zero, the gauge fields are massless, and strong-coupling dynamics must be considered. An underlying assumption of our 3d semiclassical analysis is that such effects are not present. In the case of $Sp(N)$ $F = N + 1$, this assumption is supported by the matching of the vacua across the duality and by the matching with the four-dimensional analysis.

To sum up the vacua for $m < 0$ are reported in Table 4.2.

Wall	Effective theory	Witten Index
k	$\text{HGr}(J, N + 2), \quad J \in \{0, \dots, k\}$	$\binom{N+1}{k} = \sum_{j=0}^k (-1)^{j+k} \binom{N+2}{j}$

TABLE 4.2: IR models on the vacuum structure of (4.37) for $m < 0$ and $k \leq \frac{N+1}{2}$. The various contributions to the Witten index are displayed.

So long as $k \leq \frac{N+1}{2}$, this Table 4.2 exactly reproduces the Table 3.2 of solutions we have found for the WZ model which describes SQCD when $m_{4d} \ll \Lambda$.

In the case of the parity-reversed wall $k = (N + 1)/2$, we expect the infrared duality

$$Sp(k)_{\frac{k}{2}}^{\mathcal{N}=1} \quad \text{with } 2k + 1 \quad \leftrightarrow \quad Sp(k)_{-\frac{k}{2}}^{\mathcal{N}=1} \quad \text{with } 2k + 1, \quad (4.48)$$

which tells us that the infrared SCFT is 3d parity invariant. The parameters in these cases are such that strong-coupling effects do not play an important role (this is because the absolute value of the CS level of the $Sp(N_c)$ theory is equal to, not smaller than, $\frac{N_c+1}{2}$).

4.1.3 $SU(N)$ with $F = N$ flavors

We now move to illustrate the 3d theories that can describe the low energy physics trapped on the domain walls of $SU(N)$ SQCD with $F = N$ flavors. We propose as low energy model living on the k -wall the 3d $\mathcal{N} = 1$ theory

$$U(k)_{\frac{N-k}{2}, \frac{N}{2}}^{\mathcal{N}=1} \quad \text{with } N \text{ fundamentals } X, \quad \mathcal{W} \sim +|X|^4 \quad (4.49)$$

The evidence for this proposal, which is somewhat weaker with respect to the cases of $F < N$ flavors, comes from dualities and from the study of massive vacua.

The following duality is known for $\mathcal{N} = 2$ theories [71]:⁴

$$\begin{array}{ccc} U(k)_{\frac{N}{2}}^{\mathcal{N}=2} \quad \text{w/ } (N, 0) \text{ flavors} & \iff & U(N-k)_{-\frac{N}{2}}^{\mathcal{N}=2} \quad \text{w/ } (0, N) \text{ flavors} \\ \mathcal{W}_{\mathcal{N}=2} = 0 & & \mathcal{W}_{\mathcal{N}=2} = \mu \mathfrak{M} \end{array} \quad (4.50)$$

⁴This $\mathcal{N} = 2$ duality can be derived directly from Aharony duality for $U(k)$ with (N, N) flavors, giving real masses to $(0, N)$ flavors.

(μ is a gauge singlet chiral field, \mathfrak{M} is a supersymmetric monopole⁵). Here the notation for the matter field is (n, m) , where n is the number of flavors in the fundamental representation of the gauge group, and m is the number of antifundamentals. Sometimes besides the number of flavors we also specify how they will be denoted, like (n_q, m_p) . This duality is well tested, for instance the supersymmetric partition functions are known to match.

In $\mathcal{N} = 1$ language the above duality reads

$$\begin{aligned}
U(k)_{\frac{N}{2}, \frac{N}{2}}^{\mathcal{N}=1} \text{ with adjoint } \Phi \text{ and } N \text{ } X's & \iff U(N-k)_{-\frac{N}{2}, -\frac{N}{2}}^{\mathcal{N}=1} \text{ with adjoint } \Psi \text{ and } N \text{ } \tilde{X}'s \\
\mathcal{W} = \text{Tr}\left(X^i \Phi X_i^\dagger\right) - \frac{N}{4} \text{Tr}(\Phi^2) & \iff \mathcal{W} = \text{Tr}\left(\tilde{X}^i \Psi \tilde{X}_i^*\right) + \frac{N}{4} \text{Tr}(\Psi^2) + \\
& + (\mu \mathfrak{M} + c.c.)
\end{aligned} \tag{4.51}$$

Integrating out the massive adjoints Φ and Ψ we can write the $\mathcal{N} = 2$ duality as

$$\begin{aligned}
U(k)_{\frac{N-k}{2}, \frac{N}{2}}^{\mathcal{N}=1} \text{ with } N \text{ } X's & \iff U(N-k)_{-\frac{k}{2}, -\frac{N}{2}}^{\mathcal{N}=1} \text{ with } N \text{ } \tilde{X}'s \\
\mathcal{W} = \frac{1}{2N} \text{Tr}\left(X^i X_j^\dagger X^j X_i^\dagger\right) & \iff \mathcal{W} = -\frac{1}{2N} \text{Tr}\left(\tilde{X}^i \tilde{X}_j^\dagger \tilde{X}^j \tilde{X}_i^\dagger\right) + (\mu \mathfrak{M} + c.c.)
\end{aligned} \tag{4.52}$$

Let us emphasize that 4.52 is just 4.50 written in a different notation. We can now deform the $\mathcal{N} = 2$ duality to an $\mathcal{N} = 1$ duality, turning on

$$\delta\mathcal{W} \sim \text{Tr}\left(X^i X_i^\dagger\right)^2 \iff \delta\mathcal{W} \sim \text{Tr}\left(\tilde{X}^i \tilde{X}_i^\dagger\right)^2 + \mu\mu^* \tag{4.53}$$

Along with the $\mathcal{N} = 1$ superpotential deformation quartic in the fundamental, also the deformation $\delta\mathcal{W} = \mu\mu^*$ is generated (since it does not violate any global symmetry and with only $\mathcal{N} = 1$ supersymmetry there are no non-renormalization theorems), so the gauge singlet field μ in 4.50 becomes massive.

We get, in analogy to the case $F < N$, an $\mathcal{N} = 1$ duality enjoyed by 4.49:

$$\begin{aligned}
U(k)_{\frac{N-k}{2}, \frac{N}{2}}^{\mathcal{N}=1} \text{ with } N \text{ fundamentals } X, & \iff U(N-k)_{-\frac{k}{2}, -\frac{N}{2}}^{\mathcal{N}=1} \text{ with } N \text{ fundamentals } \tilde{X}, \\
\mathcal{W} = \text{Tr}\left(X^i X_j^\dagger X^j X_i^\dagger\right) + \text{Tr}\left(X^i X_i^\dagger\right)^2 & \iff \mathcal{W} = -\text{Tr}\left(\tilde{X}^i \tilde{X}_i^\dagger\right)^2 - \text{Tr}\left(\tilde{X}^i \tilde{X}_j^\dagger \tilde{X}^j \tilde{X}_i^\dagger\right)
\end{aligned} \tag{4.54}$$

The duality 4.54 is precisely what we expect from the $4d$ equivalence between a k -wall and a time reversed $N - k$ wall.

⁵In a $U(N)$ we can construct monopole operators as disorder operators. More precisely, the correlation functions are calculated considering the gauge fields having a Dirac monopole type singularity $A \sim \frac{q}{2r}(1 - \cos(\theta))d\phi$ (where we expressed the vector in polar coordinates (r, θ, ϕ) and we considered a monopole with flux $\int_{S^2} F = 2\pi q$). If there is $3d$ $N = 2$ supersymmetry also the scalar field in the vector multiplet has a singularity for $r = 0$. In general, they are characterized by N numbers that are the gauge fluxes around them. These numbers, which are called GNO charges can be chosen to be in the Cartan subalgebra of $U(N)$, $U(1)^N$. In pure YM models these operators are gauge invariant, but when there is a CS coupling they become electrically charged. These operators are also charged under the topological $U(1)_{top}$ symmetry which is present in $U(N)$ gauge theories. Since the CS level is different from zero, they must be “dressed” by modes coming from the matter fields. The monopole under consideration \mathfrak{M} has flux -1 on the last of the $U(1)$'s, giving gauge charge of \mathfrak{M} equal to $\frac{N}{2}$. Then, it is “dressed” with N antifundamentals that contribute to $-\frac{N}{2}$ to the gauge charge yielding a gauge-invariant operator. It is the only gauge-invariant monopole that can be constructed. For more details about monopole operators see [72]–[77]

Another piece of evidence comes from the $\mathcal{N} = 2$ $SU \leftrightarrow U$ duality [78]

$$\begin{array}{ccc} SU(N)_{-1-N/2}^{\mathcal{N}=2} \text{ w/ } (N, 0) \text{ flavors} & \iff & U(1)_{N/2}^{\mathcal{N}=2} \text{ w/ } (0, N) \text{ flavors} \\ \mathcal{W}_{\mathcal{N}=2} = 0 & & \mathcal{W}_{\mathcal{N}=2} = 0 \end{array} \quad (4.55)$$

The $\mathcal{N} = 1$ deformation of the above duality is expected to be

$$\begin{array}{ccc} SU(N)_{-1}^{\mathcal{N}=1} \text{ w/ } N \text{ fundamentals } Y & \iff & U(1)_{N/2}^{\mathcal{N}=1} \text{ w/ } N \text{ fundamentals } X \\ \mathcal{W} \sim -Y^4 & & \mathcal{W} \sim +X^4 \end{array} \quad (4.56)$$

This duality describes the equivalence between the interface theory $SU(N)_{-1}^{\mathcal{N}=1}$ and the 1-wall theory $U(1)_{N/2}$.⁶

So far the story seems equivalent to the cases of $4d$ $SU(N)$ with $F < N$ flavors of [1] and $Sp(N)$ with $F \leq N$ flavors of [1], [32]. The difference is that for $4d$ $SU(N)$ with N flavors, when we study the massive vacua of the $3d$ $\mathcal{N} = 1$ models, we do not find perfect matchings between dual $3d$ theories in (4.54) and (4.56), and with the $4d$ analysis of the previous section. More precisely, the vacuum structure of (4.49) does match the $4d$ analysis if $k < \frac{N}{2}$, but it does not if $k \geq \frac{N}{2}$ (the $3d$ theory (4.49) has additional vacua not seen in $4d$ or in the $3d$ dual).

Analysis of the massive vacua

Let us now study semi-classically the vacuum structure of (4.49). The full superpotential generated by the RG flow has the form

$$\mathcal{W} = \frac{1}{4} \text{Tr} \left(X X^\dagger X X^\dagger \right) + \frac{\alpha}{4} \text{Tr} \left(X X^\dagger \right)^2 + m \text{Tr} X X^\dagger. \quad (4.58)$$

We assume that our SCFT lies in a region of the parameter space with $\alpha > -\frac{1}{k}$. Since the precise value of α does not change the results, hence-forth we set $\alpha = 0$ for simplicity.

The analysis of the vacua is carried out more or less in the same fashion as the $Sp(N)$ cases in [32]. We diagonalize the matrix $X X^\dagger = \text{diag}(\lambda_1^2, \dots, \lambda_N^2)$ using the flavor and gauge symmetry. Since $X X^\dagger$ is semi-positive definite, $\lambda_i^2 \geq 0$. Supersymmetric vacua satisfy the F-term equations

$$\lambda_i(\lambda_i^2 + m) = 0 \quad i \in \{0, \dots, k\}. \quad (4.59)$$

These equations have the following solutions:

- $m > 0$. There is only one solution, the low energy theory is the TQFT of Acharya and Vafa

$$3d \quad U(k)_{N-\frac{k}{2}, N}^{\mathcal{N}=1} \quad (4.60)$$

⁶ This extends the 1-wall \leftrightarrow interface duality of [1]

$$SU(N)_{-1+F/2-N/2}^{\mathcal{N}=1} \text{ w/ } F \text{ fundamentals} \iff U(1)_{N-F/2}^{\mathcal{N}=1} \text{ w/ } F \text{ chirals} \quad (4.57)$$

to the case of $F = N$.

Wall	Effective theory	Witten Index
k	$U(1)_0^{\mathcal{N}=1} \times Gr(J, N) \quad J = 0, \dots, N - k - 1$ $Gr(k, N)$	0 $\binom{N}{k}$

TABLE 4.3: IR models on the vacuum structure of (4.49) for $m < 0$ and $k < \frac{N}{2}$. The various contributions to the Witten index are displayed.

(The Chern-Simons level is obtained integrating out the positive mass fermions).

• $m < 0$. There are $k + 1$ solutions. The quarks XX^\dagger take VEV and break both the flavor symmetry $U(N) \rightarrow U(J) \times U(N - J)$ and the gauge symmetry $U(k) \rightarrow U(k - J)$. The low energy models living on each of the j vacua are

$$U(k - J)_{-\frac{\mathcal{N}=1}{2}, 0} \times Gr(J, N). \quad (4.61)$$

The Chern-Simon levels are computed looking at the mass of the charged fermions under the unbroken gauge group. All these vacua preserve supersymmetry. At low energies, the $SU(k - J)$ part of the $U(k - J)$ group confines, leaving an $U(1)_0$. This is the signal we were looking for if we were searching for domain walls that break baryonic symmetry. In fact, $U(1)_0 \sim S^1$ which is the NLSM expected when baryons take VEV. Moreover, all the low energy models that have an $U(1)_0$ factor have automatically WI = 0. The only domain wall contributing to the WI is the one with $J = k$, in which the gauge group is completely broken. This domain wall was already discovered in [26]. The authors of [26] did not discuss the other domain walls with vanishing Witten-Index.

Notice that all the NLSM in (4.61) have a corresponding Wess-Zumino term that can be specified describing the NLSM as a $\mathcal{N} = 1$ $U(J)_{\frac{N-J}{2}, N-\frac{J}{2}}$ and N fundamental scalar multiplets taking VEV.

In summary, the various vacua are listed in Table 4.3. As one can see the two tables, Table 4.3 and Table 3.3, match.

We now want to stress that the analysis of the vacua we just perform seems legitimate when $k > \frac{N}{2}$. Therefore it seems that for domain walls with $k > \frac{N}{2}$, the domain walls solutions should be $k + 1$. However from the $4d$ analysis, we have shown that this is not the case since the solutions are $N - k + 1$. The vacua of these models, those with $k > \frac{N}{2}$, are

$$U(1)_0 \times Gr(J, N) \quad J = 0, \dots, k. \quad (4.62)$$

As we can see, the first $N - k + 1$ vacua coincide with the vacua found above in the dual model. However, it seems that here there are more vacua, those from $J = N - k + 1$ to $J = k$. This is because our interpretation analysis is naive and does not consider that strong coupling effects may arise due to the smallness of the CS level compared to the rank of the gauge algebra. Indeed, when $k > \frac{N}{2}$ the CS level $\frac{N-k}{2}$ is smaller than half of the dual Coxeter number k of the non-abelian part of the gauge group. This kind of bound can be found in pure CS models with non-abelian gauge group G_k , where

when we have that $c_2(\mathfrak{g}) > 2k$ (\mathfrak{g} being the algebra of G), non-perturbative effects lift the supersymmetric vacua. In the presence of matter fields, these phenomena have not been demonstrated to be present; nonetheless, it is plausible they are there as well. A possible scenario is given by taking into account that the model could have more than one phase transitions, say for $m = 0$ and for $m = m^*$. The phase of the model when $m < 0$ yields $k + 1$ vacua, the wrong vacuum structure according to the $4d$ analysis. Instead, when $0 < m < m^*$ the model has $N - k + 1$ vacua, the correct vacua matching our $4d$ computation. This means that the transition on the walls is captured by the phase transition around the $m = m^*$ and not around $m = 0$. The phase transition around $m = m^*$ is not seen by our semiclassical analysis.

The special case of $SU(2) \simeq Sp(1)$

If $N = 2$, the $3d$ model, in the regime corresponding to the $4d$ description with the constrained Wess- Zumino model $m_{3d} < 0$, has two vacua: $U(1)_0$ and a NLSM with target space $\mathbb{C}\mathbb{P}^1$. From the $3d$ $U(1)$ -gauge theory perspective these seem two disconnected sets of vacua, however, because from the $4d$ perspective the $SU(2)$ with $F = 2$ model is exactly the same as $Sp(1)$ with $F = 2$, we know there should be only one solution, with a NLSM with target space $Sp(2)/Sp(1) \times Sp(1)$.

The interpretation is as follows: the $3d$ $U(1)$ -gauge theory has UV global symmetry $U(2)$, but IR global symmetry $Sp(2)$. Hence when we compute naively (using the UV global symmetry) the NLSM's living on the vacua in the $3d$ $U(1)$ -gauge theory, we get a wrong result. The two sets of disconnected vacua we naively see from the UV $U(1)$ perspective are in fact submanifolds of a bigger connected set of vacua, which is instead seen semiclassically in the $3d$ $Sp(1)$ -gauge theory discussed in Sec. 4.1.1, for which the UV and IR symmetry are the same, namely $Sp(2)$.

4.1.4 $SU(N)$ with $F = N + 1$

We turn now to the analysis of our proposal for the $3d$ effective theory on the domain walls of $4d$ $SU(N)$ with $N + 1$ flavors. Our proposal is inspired by the following $3d$ $\mathcal{N} = 2$ duality [79], [80]:

$$SU(N)_{-(N+1)/2}^{\mathcal{N}=2} \text{ w/ } (0, N + 1) \text{ flavors } Q_i \quad \Longleftrightarrow \quad U(1)_{N/2}^{\mathcal{N}=2} \text{ w/ } ((N + 1)_{q_i}, 1_{\tilde{p}}) \text{ flavors} \\ \mathcal{W}_{\mathcal{N}=2} = 0 \quad \quad \quad \mathcal{W}_{\mathcal{N}=2} = \mathfrak{M}_{BPS}^+ \quad (4.63)$$

Here \mathfrak{M}_{BPS}^+ is the monopole operator which has positive and equal to one $U(1)$ flux around the point of insertion, and it is moreover invariant under the flavor symmetry because it has been made gauge invariant by dressing it with $(N + 1)_q$ flavors of gauge charge $+1$ and the \tilde{p} of charge -1 . The $SU(N)$ theory on the l.h.s. is expected to be related to the interface theory of $4d$ $SU(N)$ with $N + 1$ flavors, so we can also expect that the $U(1)$ theory on the r.h.s. is the $\mathcal{N} = 2$ ancestor of the $3d$ theory living on the 1-wall. In other words we are extending the 1-wall \leftrightarrow interface duality (4.57) to the case of $F = N + 1$. Notice that in the $U(1)$ theory there is a superpotential term

linear in the supersymmetric monopole, denoted \mathfrak{M}_{BPS}^+ . This is a qualitatively new feature, compared to previously discussed cases.

The CS matter theory admits a supersymmetric monopole because the difference between the number of anti-fundamentals minus the number of fundamentals is precisely twice the CS level. The chiral ring⁷ generators in the $SU(N)$ theory are the $N + 1$ baryons $\varepsilon_N Q^N$. These baryons are mapped to the $N + 1$ mesons $\tilde{p}q_i$ in the dual $U(1)$ theory.

Theories with unitary gauge groups and monopole superpotentials were studied in [81]. Duality 8.18-8.19 of [81] (setting $N_c = k$, $N_f = N + 1$, $\kappa = N$) reads

$$\begin{aligned} U(k)_{N/2}^{\mathcal{N}=2} \text{ w/ } (N + 1, 1) \text{ flavors} &\iff U(N - k)_{-N/2}^{\mathcal{N}=2} \text{ w/ } (1_P, (N + 1)_{\tilde{Q}_i}) \text{ flavors} \\ \mathcal{W}_{\mathcal{N}=2} = \mathfrak{M}_{BPS}^+ &\iff \mathcal{W}_{\mathcal{N}=2} = \mu_i \text{ tr}(P\tilde{Q}_i) + \mathfrak{M}_{BPS}^- \end{aligned} \quad (4.64)$$

The chiral ring is generated by the $N + 1$ mesons on the l.h.s., which are mapped to the gauge singlets μ_i . The continuous flavor symmetry is $SU(N + 1) \times U(1)$ (without the monopole superpotential there would be an additional $U(1)$ factor, the monopole superpotential breaks a combination of the axial and the topological, or magnetic, symmetry).

Because of the two dualities just presented, we expect that the $\mathcal{N} = 2$ theories in (4.64) should be related to the k -wall of $4d$ $SU(N)$ with $N + 1$ flavors. The domain wall is described by an $\mathcal{N} = 1$ SCFT obtained with an $\mathcal{N} = 1$ perturbation of the above $\mathcal{N} = 2$ fixed points, that, among other things, gives mass to the gauge singlet fields μ_i . It is important that the monopole term in the superpotential survives the deformation. More precisely in $\mathcal{N} = 1$ language, such term is a dressed monopole of the form

$$\mathfrak{M}_{dressed}^+ = \mathfrak{M}_{bare}^+ p q_1 q_2 \dots q_{N+1} \varepsilon_{N+1} \quad (4.65)$$

which is gauge-invariant (the flavors q_i/p have $+1/-1$ charge under the gauge $U(1) \subset U(k)$) and also invariant under the $SU(N + 1) \times U(1)$ global symmetry. \mathfrak{M}_{bare}^+ is defined as a disorder operator, creating a flux $(1, 0, \dots, 0)$ in the Cartan of the gauge group $U(k)$.

Summing up, our proposal for the k -wall is

$$\begin{aligned} &U(k)_{(N-k)/2, N/2}^{\mathcal{N}=1} \text{ with } N + 1 \text{ fundamentals } q_i \text{ and 1 fundamental } p, \\ \mathcal{W} = \frac{1}{4} \text{ Tr}(q_i^\dagger q^j q_j^\dagger q^i) &+ \frac{A}{4} \text{ Tr}(q_i^\dagger q^i)^2 + \frac{B}{4} \text{ Tr}(p^\dagger p)^2 + \frac{\eta}{2} \text{ Tr}(q_i^\dagger q^i) \text{ Tr}(p^\dagger p) + \\ &+ \frac{\alpha}{2} \text{ Tr}(p^\dagger q_j^\dagger) \text{ Tr}(q^j p) + \mathfrak{M}_{dressed}^+ \end{aligned} \quad (4.66)$$

where A, B, α, η are real parameters.⁸ It will turn out that the vacua of the mass-deformed (4.66) are related to our $4d$ analysis when $A > 0$, $B < 0$ and $\eta < -1$. We

⁷The chiral ring comprises the gauge-invariant chiral operator, which can be constructed in the model considered. It is a ring since the multiplication of two chiral operators is still chiral.

⁸A minor consistency check of our proposal is that we can mass deform the theory 4.66 with the mass term

$$\delta\mathcal{W} = \text{tr}(p q_{N+1}), \quad (4.67)$$

conjecture that an $\mathcal{N} = 1$ SCFT in this region of parameters exists. It would be nice to study the existence of such fixed point further, testing the validity of our assumptions.

Theory (4.66) has two independent $SU(F)$ -invariant mass terms, namely $\text{Tr}(p^\dagger p)$ and $\text{Tr}(q_i^\dagger q^i)$. In order to interpolate between massive phases which are related to the domain walls we turn on the following combination:

$$\delta\mathcal{W} = m \left(\text{Tr}(q_i^\dagger q^i) - \text{Tr}(p^\dagger p) \right) \quad (4.69)$$

In this way, for positive m , $N + 1$ flavors have positive mass and 1 flavor has negative mass, hence the vacuum is described by $U(k)_{N-k/2, N}^{\mathcal{N}=1}$, reproducing the AV TQFT as required.

Analysis of the massive vacua

In order to analyze the vacua of (4.66), we can set $B = -1$ for simplicity and $\alpha = 0$ for the purpose of this section⁹.

We diagonalize the matrix $qq^\dagger = \text{diag}(\lambda_1^2, \dots, \lambda_{N+1}^2)$ using the flavor and gauge symmetry (with $\lambda_i^2 \geq 0$ and with at most k of them $\lambda_i > 0$).

Supersymmetric vacua satisfy the F-term equations

$$\begin{aligned} \lambda_i(\lambda_i^2 + \eta|p|^2 + \alpha p_i^2 + m) &= 0 \\ p_a(B|p|^2 + \eta \sum_j \lambda_j^2 + \alpha \lambda_a^2 - m) &= 0. \end{aligned} \quad (4.70)$$

Notice that if both q_i and p take a vev, the $U(1)$ factor in the global symmetry is broken, hence such vacua should map to would-be-domain walls where the $4d$ baryonic symmetry is broken, and we did not find any such solution in the previous section.

- $m > 0$. There is only one solution, $p = q_i = 0$, the low energy theory is the TQFT

$$3d \quad U(k)_{N-\frac{k}{2}, N}^{\mathcal{N}=1}. \quad (4.71)$$

The monopole superpotential of the UV theory has no effect on these IR vacua.

- $m < 0$. The solutions with $p \neq 0$ typically can appear and disappear changing the real parameters of the quartic superpotential, we analyze them in Appendix D. Here we discuss only the solutions with $p = 0$, which should be the ones related to breaking the global symmetry as

$$SU(N + 1) \times U(1) \rightarrow SU(N) \times U(1). \quad (4.68)$$

The CS level is unchanged (because the real masses of p and q_{N+1} have opposite signs). The monopole superpotential is lifted (otherwise the global symmetry in the IR would only be $SU(N)$). Hence, in the infrared, we end up on the $U(k)_{\frac{N-k}{2}, \frac{N}{2}}^{\mathcal{N}=1}$ theory with N flavors which is our proposal for the domain walls of $SU(N)$ with N flavors (4.49).

⁹The general analysis of the vacuum structure of the model is done in the Appendix D. For different choices of the parameters there are vacuum structures which seem not to be related to domain walls of $4d$ $SU(N)$ with $N + 1$ flavors.

domain walls. There are $k + 1$ different vacua, parametrized by $J = 0, \dots, k$:

$$p = 0, \quad \lambda_i^2 = -\frac{m}{1 + JA} \quad \text{for } i = 0, \dots, J, \quad \lambda_i = 0 \text{ for } i = J + 1, \dots, k. \quad (4.72)$$

The low energy theory includes a TQFT factor $U(k - J)_{-\frac{k-J}{2}, 0}^{\mathcal{N}=1}$, which without the monopole superpotential would be equivalent to $U(1)_0 \sim S^1$. The monopole superpotential lifts the S^1 to two points. This phenomenon occurs because, once we have integrated out all the flavors and the CS level of the $U(1)$ happens to be zero, we can define the gauge invariant monopole as the \mathfrak{M}^+ as the exponential of the dual photon θ , $e^{i\theta}$. Indeed, dualizing a vector A_μ with field strength $F_{\mu\nu}$ in three dimensions yields a $*F_{\mu\nu} \sim B_\mu$ vector which is the “field-strength” of a scalar field θ . This scalar field has value in S^1 and therefore the superpotential $\mathcal{W} = \text{Re}(\mathfrak{M}^+)$ can be written as $\mathcal{W} = \cos(\theta)$. The F-term equations of such superpotential give us two vacua. One can also see this phenomenon studying a QFT for which a convenient duality is known. Such example is explained in appendix E. Hence the low energy theory is

$$\mathbb{Z}_2 \times Gr(J, N + 1). \quad (4.73)$$

Modulo the double degeneracy, these are the $k + 1$ vacua expected from the $4d$ analysis.

Putting this result together with the $\mathcal{N} = 2$ dualities discussed above, we gathered quite a bit of evidence that the theories 4.66, inside an appropriate region of the parameter space, describe the domain walls of $4d$ $SU(N)$ with $N + 1$ flavors. It would be nice to test this proposal further.

Chapter 5

Conclusions and Outlook

In summary, in this thesis, we have studied domain walls of SQCD for the gauge groups $SU(N)$ and $Sp(N)$ when the number of flavors is $F = h$ or $F = h + 1$, where h is the dual Coxeter number of the gauge algebra.

From the $4d$ point of view, we were able to study both cases, using $4d$ low energy descriptions, which are Wess Zumino models. These descriptions allowed us to compute the domain walls numerically using a first-order differential equation. This simplified a lot our $4d$ analysis. Furthermore, the solutions found are enough to saturate the Witten Index, and therefore we can say that our solutions are an exhaustive set of solutions.

However, we should point out that our $4d$ analysis is incomplete when dealing with the models with $F = h + 1$ and the parity invariant walls. Here, we are confident we have found the domain solution that connects the two vacua that sit on the real axis when h is even, the vacua where the gaugino condensate $\langle \lambda\lambda \rangle = \pm \Lambda^3$. However, we failed in understanding the moduli space of such solution. Indeed, it seems like there is no moduli space yielding an incorrect account of the Witten index. However, under a small deformation of the Kähler potential that breaks the flavor symmetry, we recognized that the seemingly alone solution was, in fact, a superposition of different solutions. However, even after this observation, the structure of the moduli space is not completely clear. This is a puzzle yet to be understood, even if we hope to have suggested a way to figure it out.

From the $3d$ point of view, we got some new interesting phenomena. In particular, for the case of $SU(N)$ with $F = N$ flavors and $Sp(N)$ with $F = N + 2$ flavors, our proposed models showed a particular behavior involving strong coupling effects at low energies. Remember that the naive vacua analysis failed when the CS level was smaller than half of the gauge algebra dual Coxeter number. On the one hand, one could say that this is a weakness of our analysis and that we got the wrong duality relating the k and $(h - k)$ -wall. On the contrary, one could interpret, and this is our interpretation, that we were able to study the vacuum structure of models which are strongly coupled at low energy, thanks to a known duality and thanks to the relation of these strongly coupled models to BPS domain walls in $4d$. Therefore what can be regarded as a weakness, could be in reality, a new way of understanding the vacuum structure of strongly coupled models. Indeed, if one is able to recognize, by other means, that $3d$ $\mathcal{N} = 1$ model describes the low energy behavior of the domain walls of

a $4d$ $\mathcal{N} = 1$ theory, one can use easy numerical methods to understand the vacuum structure of the $3d$ model even if the $3d$ model is strongly coupled in the IR. Another evidence that this interpretation might be the correct one comes from the fact that the dualities proposed are the deformation of known and well-tested $\mathcal{N} = 2$ $3d$ dualities. Moreover, usually on one hand of the duality, the deformation from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ is performed in a weakly coupled regime, where the CS levels are much bigger than the dual Coxeter number of the gauge algebra. Therefore, the deformation can be seen as reliable, especially for low values of k . In our opinion, it seems very unlikely that deformations in weakly coupled regimes spoil the duality, and therefore the $\mathcal{N} = 2$ duality gives rise to a $\mathcal{N} = 1$ one.

We have to underline that the case of $SU(N)$ with $F = N + 1$ is a little more complicated, and it is not on such strong grounds as the previous cases. Therefore, further proofs and checks that the proposed model is, in fact, the correct one are much needed. However, we presented our proposal anyway, highlighting every strength and weakness of our statements.

The next step to understanding domain walls in SQCD is to study the regime in which $F > h + 1$. Some of these models enjoy a low energy description which is WZ-like. However, as discovered by Seiberg [59]–[61], the $4d$ model dual to SQCD when $F > h + 1$ has an emergent gauge dynamics. This could be the chance to shed new light on Seiberg duality and its relations with $3d$ $\mathcal{N} = 1$ dualities. When Seiberg duality is there, one could not expect to find a WZ model with single-valued superpotential, but it has to deal with multivalued functions. However, this should not be a huge problem since this was the situation when $F < h$ and a multivalued superpotential was generated by strong coupling effects. Therefore, from the $4d$ perspective, we do not see any obstacle preventing us from classifying domain walls. However, from the $3d$ perspective, this could be challenging because, if we consider the naive extension of our proposed models, the CS level shrink while the dual Coxeter number of the $3d$ gauge group remain the same. This could, in principle, generate new strong coupling phenomena due to the smallness of the mass of the gauge fields and/or prevent us from properly test the proposed $3d$ models against the $4d$ domain wall solutions found.

Another interesting development of this work is to study SQCD with different matter content, maybe involving other $4d$ dualities besides Seiberg’s duality. Note that as the number and representation of the matter field varies, the $4d$ analysis of the equations (2.5) becomes more and more complicated.

Finally, another interesting case is when we vary the gauge group, considering non-simply connected groups, such as $SO(N)$.

Appendix A

Supersymmetry: Preliminaries

In this brief appendix, we sketch the notation for supersymmetry in four three and two dimensions. We also recall some basic concepts and results about supersymmetric theories.

A.0.1 $\mathcal{N} = 1$ supersymmetry in four dimensions

Given that the Lorentz algebra is $SO(1, 3) \cong SU(2) \times SU(2)$, we have two different types of spinor, left-handed and right-handed, respectively

$$\begin{aligned} \text{left-handed: } & \chi_\alpha, \quad \alpha = 1, 2 \\ \text{right-handed: } & \bar{\psi}^{\dot{\alpha}}, \dot{\alpha} = 1, 2. \end{aligned} \tag{A.1}$$

One can distinguish between left-handed and right-handed spinor from the undotted and dotted indices. The left-handed spinor lives in the $(\frac{1}{2}, 0)$ representation of the Lorentz group, whereas the right-handed spinor live in the $(0, \frac{1}{2})$ representation. One can lower or raise the dotted and undotted indices using the tensors

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^{\alpha\dot{\beta}} = -\epsilon_{\alpha\dot{\beta}}. \tag{A.2}$$

The product of two spinor is defined as $\psi^\alpha \chi_\alpha$, $\bar{\psi}_{\dot{\alpha}} \xi^{\dot{\alpha}}$. These spinor are usually referred as Weyl spinor.

One can obtain vectorial quantities, in the $(\frac{1}{2}, \frac{1}{2})$ Lorentz representation, in spinorial notation using the matrix

$$(\sigma^\mu)_{\alpha\dot{\beta}} = \{1_2, \vec{\sigma}\}, \quad (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = \{1_2, -\vec{\sigma}\}, \tag{A.3}$$

where $\vec{\sigma}$ are the three Pauli matrices. Using this matrix we can write a vector $A_\mu = A_\mu (\sigma^\mu)_{\alpha\dot{\beta}} = A_{\alpha\dot{\beta}}$. Other commonly introduced objects, which are needed to parametrize Lorentz transformations, are

$$\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \tag{A.4}$$

To obtain a Dirac spinor we have to combine one left-handed spinor and one right-handed spinor

$$\Psi_D = \begin{pmatrix} \chi_\alpha \\ \psi^{\dot{\alpha}} \end{pmatrix}. \quad (\text{A.5})$$

Using (A.3) one can also get the Dirac matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \quad (\text{A.6})$$

These matrices satisfy the usual Clifford algebra relation $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$. In addition to that $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{i\dagger} = -\gamma^i$ for $i = 1, 2, 3$. The $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, is used to construct the two chiral projectors that project into left-handed spinors $P_L = \frac{1-\gamma_5}{2}$, and right-handed spinors $P_R = \frac{1+\gamma_5}{2}$.

When we consider $\mathcal{N} = 1$ supersymmetry, the Poincaré algebra (whose generators are denoted by P_μ for translations and $M_{\mu\nu}$ for Lorentz generators) is extended by four other generators Q_α and $\bar{Q}^{\dot{\alpha}}$ that have the following commutation relations with the standard Poincaré group generators

$$\begin{aligned} [P_\mu, Q_\alpha] &= [P_\mu, \bar{Q}^{\dot{\alpha}}] = 0 \\ [M^{\mu\nu}, Q_\alpha] &= i(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta, \quad [M^{\mu\nu}, \bar{Q}^{\dot{\alpha}}] = i(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{\dot{\beta}} \\ \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2P_{\alpha\dot{\beta}} \\ \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} &= 0, \quad \{Q_\alpha, Q_\beta\} = 0. \end{aligned} \quad (\text{A.7})$$

Coleman-Mandula's theorem states that all the global symmetry must commute with the Poincaré group. However, when considering the super-Poincaré group, there are two types of global symmetries: those that commute with all the super-Poincaré group, and those that commute only with the Poincaré group. The latter are called R -symmetries, and in the case of $\mathcal{N} = 1$ four-dimensional theories can be as big as $U(1)_R$. The supersymmetry generators are charged under this symmetry

$$[R, Q_\alpha] = -Q_\alpha, \quad [R, \bar{Q}_{\dot{\alpha}}] = \bar{Q}_{\dot{\alpha}}, \quad (\text{A.8})$$

where R is the generator of R -symmetry.

To provide a natural setting to construct actions that are invariant under supersymmetry, we define Superspace. Superspace is labelled by the four coordinates x^μ and four real Grassmann variables θ^β and $\bar{\theta}^{\dot{\alpha}} = (\theta^\beta)^*$. The standard rules for differentiation and integration are

$$\begin{aligned}
\frac{\partial}{\partial\theta^\alpha}\theta^\beta &= \delta_\alpha^\beta, & \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\theta}^{\dot{\beta}} &= \delta_{\dot{\alpha}}^{\dot{\beta}} \\
\int d\theta^\alpha &= 0, & \int d\theta^\alpha\theta_\beta &= \delta_\beta^\alpha \\
\int d\bar{\theta}^{\dot{\alpha}} &= 0, & \int d\bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} &= \delta_{\dot{\beta}}^{\dot{\alpha}} \\
\{\theta^\beta, \theta^\alpha\} &= \{\theta^\beta, \bar{\theta}^{\dot{\alpha}}\} = 0 \\
\left\{ \frac{\partial}{\partial\theta^\beta}, \frac{\partial}{\partial\theta^\alpha} \right\} &= \left\{ \frac{\partial}{\partial\theta^\beta}, \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \right\} = 0.
\end{aligned} \tag{A.9}$$

(In the following when referring to derivative operators in superspace we will denote $\frac{\partial}{\partial\theta^\beta} = \partial_\beta$ and $\partial_{\alpha\dot{\alpha}} = (\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu$). On superspace naturally acts the $\mathcal{N} = 1$ superalgebra. Conventionally to the Grassmann variables are assigned the R -charges $R(\theta) = 1$ and $R(\bar{\theta}) = -1$. As we will see, this implies that the fields in the same superfields have different R -charges.

We now construct superfields $\Phi_{\dots}(x, \theta)$ (where the dots stand for eventual Lorentz indices) that are fields in superspace. The supersymmetry generators act on them as

$$Q_\alpha = -i\partial_\alpha + \bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \quad \bar{Q}_{\dot{\alpha}} = i\partial_{\dot{\alpha}} - \theta^\alpha\partial_{\alpha\dot{\alpha}} \tag{A.10}$$

Other commonly define objects are supercovariant derivatives that anticommute with the supersymmetry generators

$$D_\alpha = \partial_\alpha - i\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \quad \bar{D}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} + i\theta^\alpha\partial_{\alpha\dot{\alpha}}. \tag{A.11}$$

Every integral on the superspace of an arbitrary function of superfields is automatically supersymmetric invariant. Note also that every expansion of superfields in the Grassmann variables has a finite number of terms, giving that $\theta_\alpha\theta_\beta\theta_\gamma = 0$. In other words, a superfield is a finite collection of ordinary fields in superspace.

We now introduce two types of superfields.

Chiral Superfields

Chiral (anti-chiral) superfields Φ 's are superfields that satisfy the constraint

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{A.12}$$

(anti-chiral superfields involve D_α). If use as coordinate $y^\mu = x^\mu + i\bar{\theta}\sigma^\mu\theta$ ($\bar{D}_{\dot{\alpha}}y^\mu = 0$) we can easily expand the chiral superfields Φ as

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y). \tag{A.13}$$

Thereby the chiral fields describe a minimal supermultiplet: the complex scalar field ϕ gives us two bosonic states, and the Weyl spinor ψ^α gives us two fermionic states. F is an auxiliary field. The R charges of a chiral superfield differ by one unit: $R(\phi) = r$, $R(\psi) = r - 1$.

To derive how the supersymmetry transformations act on the various components of the chiral superfield, one can take the differential expression of the supersymmetric charges (A.10) and acts on the superfield Φ . For reference, we report here the expression for a generic chiral field, given the parameters of the supersymmetric transformations ε^α , $\bar{\varepsilon}_{\dot{\alpha}}$

$$\begin{aligned}\delta\phi &= \sqrt{2}\varepsilon^\alpha\psi_\alpha \\ \psi_\alpha &= -\sqrt{2}i\partial_\alpha\dot{\alpha}\phi^\dagger\bar{\varepsilon}^{\dot{\alpha}} + \sqrt{2}\varepsilon_\alpha F \\ \delta F &= i\sqrt{2}(\partial_{\alpha\dot{\alpha}}\psi^\alpha)\bar{\varepsilon}^{\dot{\alpha}}.\end{aligned}\tag{A.14}$$

The matter content is usually described using chiral superfields. The kinetic term is written in superspace like

$$S_{kin} = \int d^x d^2\theta d^2\bar{\theta}\bar{\Phi}\Phi = \int d^4x(\partial_\mu\bar{\phi}\partial^\mu\phi + i\bar{\psi}_{\dot{\alpha}}\partial^{\dot{\alpha}\alpha}\psi_\alpha + \bar{F}F).\tag{A.15}$$

Note that the field F does not have any derivative term and therefore is auxiliary. A crucial aspect of the kinetic terms is given by the fact that the superspace integral is performed over all the Grassmann variables. As for terminology, integrals over *all* the superspace are called D-terms. In general, we can have a real function of different chiral fields when dealing with sigma models. This function is usually called *Kähler potential* $\mathcal{K}(\Phi, \bar{\Phi})$. This comes because, in that case, the chiral fields are the coordinates of a Kähler manifold. The operation of integrating in the θ and $\bar{\theta}$ variables extract from the Kähler potential the Kähler metric $\mathcal{K}_{\phi\bar{\phi}} = \frac{\partial\mathcal{K}}{\partial\phi\partial\bar{\phi}}$, and its connection. To allow interaction power-like interaction for the scalars, we need to consider different integrals over half of the superspace of functions of only the chiral field $\mathcal{W}(\Phi)$. So the interaction part of the action looks like

$$S_{int} = \int d^4x d^2\theta\mathcal{W}(\Phi) + c.c.\tag{A.16}$$

The function $\mathcal{W}(\Phi)$ is called *superpotential* and it is and holomorphic function of the superfield Φ . In components of the superfield, Φ , the action takes the form

$$S_{int} = \int d^4x \frac{\partial^2\mathcal{W}}{\partial\phi^2}(\phi)\psi\psi + \frac{\partial\mathcal{W}}{\partial\phi}(\phi)F.\tag{A.17}$$

Putting together S_{int} and S_{kin} one can see that eliminating the auxiliary field F we get that the potential of the model is $V(\phi) = \left|\frac{\partial\mathcal{W}}{\partial\phi}(\phi)\right|^2$. This means that in order to find supersymmetric vacua, which have by definition zero energy, we have to find where $\frac{\partial\mathcal{W}}{\partial\phi}(\phi) = 0$. These equations are called *F-term* equations for obvious reasons. The superpotential is exact at tree level, which means it does not renormalize at the perturbative level. It could have non-perturbative correction.

Real Superfields

Another type of superfields are real superfields V , which are defined by imposing the constraint

$$V^\dagger = V.\tag{A.18}$$

This kind of superfield contains a vector field as one of its components. Therefore can be used to describe the vector multiplet. Note also that, given a chiral field Φ , $\Phi + \bar{\Phi}$ is a real superfield, and the transformation $V \rightarrow V + \Phi + \bar{\Phi}$ is precisely the gauge transformation for the abelian vector inside V . In a particular gauge, the expansion of the real superfields looks like¹

$$V_{WZ} = 2\theta\sigma^\mu\bar{\theta}v_\mu + i\theta^2(\bar{\theta}\bar{\lambda}) - 2i\bar{\theta}^2(\theta\lambda) + \theta^2\bar{\theta}^2 D, \quad (\text{A.19})$$

where v_μ is the vector field, λ_α and $\bar{\lambda}^{\dot{\alpha}}$ are called gaugini and D is an auxiliary field. To construct the kinetic term for the vector field, one has to consider the superfields

$$W_\alpha = \frac{1}{8}\bar{D}^2 D_\alpha V. \quad (\text{A.20})$$

This superfield is invariant under the gauge transformation $V \rightarrow V + \Phi + \bar{\Phi}$ and it contains the field-strength $F_{\mu\nu}$. One can write the kinetic term for the gauge fields as

$$S_{kin} = \frac{1}{4e^2} \int d^4x d^2\theta d^2\bar{\theta} W_\alpha \bar{W}^\alpha = \frac{1}{4e^2} \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\lambda}_{\dot{\alpha}} \partial^{\dot{\alpha}\alpha} \lambda_\alpha + 2D^2 \right), \quad (\text{A.21})$$

here e^2 is the gauge coupling.

To describe a minimal coupling interaction between matter fields and gauge fields, we write the kinetic term as

$$S_{m.c.} = \int d^4x d^2\theta d^2\bar{\theta} \bar{\Phi} e^{qV} \Phi, \quad (\text{A.22})$$

where q is the charge of the matter fields under the gauge symmetry.

If we have a real superfield in our model, to find out supersymmetric vacua, it is no longer sufficient to study F-term equations, but one has to look at the contribution to the potential of the theory coming from integrating out the auxiliary field D . This contribution is

$$V = \frac{1}{2e^2} D^2, \quad (\text{A.23})$$

therefore the full expression for the potential is

$$V(\phi, \bar{\phi}) = \frac{1}{2e^2} D(\phi, \bar{\phi})^2 + \left| \frac{\partial \mathcal{W}}{\partial \phi}(\phi) \right|^2. \quad (\text{A.24})$$

In order to find supersymmetric vacua that have $V = 0$, we must solve the equations

$$D(\phi, \bar{\phi}) = 0, \quad F = \frac{\partial \mathcal{W}}{\partial \phi}(\phi) = 0. \quad (\text{A.25})$$

The first equation determines the D-flat directions, which are also given by the single-trace gauge invariant operator modulo classical relations, while the second equation gives us the F-flat directions. The intersection of D-flat and F-flat directions is called

¹In case of non-abelian gauge groups the expressions $v_\mu = v_\mu^a T^a$, $\lambda = \lambda^a T^a$, and $D = D^a T^a$.

moduli space, and it is the space of supersymmetric vacua of the model. The moduli space is not lifted by quantum correction, as was the case without supersymmetry.

A.0.2 Supersymmetry in three and two dimensions

The second part of this thesis deals with $\mathcal{N} = 1$ supersymmetry in three dimensions. We briefly recap the notations and some important notions.

In three dimensions, the Dirac spinor has two complex components. The Clifford algebra is constructed with the matrices

$$\gamma^0 = \sigma_2, \quad \gamma^1 = -i\sigma_1, \quad \gamma^2 = -i\sigma_3. \quad (\text{A.26})$$

There is no notion of chirality. Since all the gamma matrices are purely imaginary, one can define Majorana fermion, with two real components $\psi^\alpha = (\psi^+, \psi^-)$. To raise and lower spinor indices one uses $\varepsilon = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Here in three dimensions, the superspace is composed by the three coordinates x^μ and two real Grassmann variables θ^α .

The matter content can be organized in a scala multiplet which reads

$$\Phi(x, \theta) = \phi(x) + \theta\psi(x) - \theta^2 F, \quad (\text{A.27})$$

where $\phi(x)$ and $F(x)$ are real fields, whereas $\psi^\alpha(x)$ is a real Majorana fermion. The gauge multiplet is instead embedded in a superfield connection Γ_α which contains a vector fields v_μ and the gaugini λ_α , as well as other fields that can be gauged away.

The kinetic terms for the matter and gauge superfields looks like

$$S_{kin} = \int d^3x d^2\theta \left[-\frac{1}{2} (D_\alpha \Phi)^2 + W_\alpha W^\alpha \right], \quad (\text{A.28})$$

where $W_\alpha = \frac{1}{2} D^\beta D_\alpha \Gamma_\beta$. Instead power like interaction are constructed using a real function of the scalar fields called superpotential $W(\Phi)$ and they read

$$S_{int} = \int d^3x d^2\theta W(\phi). \quad (\text{A.29})$$

Note that both the superpotential and the kinetic term are real functions and receive quantum correction at the perturbative level. In this regard, the supersymmetry is not too constraining, and the models behave more like non-supersymmetric models.

In two dimensions, the Lorentz group is composed only by the boost. The Dirac spinor is a two component complex spinor $\Psi_D = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. One can define the Dirac matrices as

$$\gamma^0 = \sigma^2, \quad \gamma^1 = -i\sigma^1, \quad (\text{A.30})$$

and the $\gamma^5 = \gamma^0\gamma^1$. Here there is chirality which reduces a chiral spinor to be a one component complex spinor.

In this thesis, we will briefly use the fact that the dimensional reduction of $\mathcal{N} = 1$ four-dimensional supersymmetry on a torus T^2 yields a two-dimensional $\mathcal{N} = (2, 2)$ model. In this case, the superspace can be easily obtained by discarding two coordinates from the four-dimensional superspace and rearranging the Grassmann variables according to light-cone coordinates [51]. Therefore a four-dimensional chiral field in two dimensions becomes schematically

$$\Phi \sim \phi + \theta^+ \psi_+ + \theta^- \psi_- + \theta^+ \theta^- F. \quad (\text{A.31})$$

One usually calls ψ_+ , θ^+ right-moving, while ψ_- , θ^- left-moving. The same terminology goes on as in four dimensions as long we talk about chiral fields, Kähler potential, and superpotential. There is one big difference when it comes to describing gauge fields, which in two dimensions are described by a twisted chiral field. We are not interested in these details because, in this thesis, we deal with only chiral fields.

Appendix B

Level-Rank Duality

In the seminal paper [64], Witten showed how the Wilson lines of TQFTs at low energy of CS theories are related to rational conformal field theories (RCFT) in $2d$. RCFT are 2d conformal field theories with a finite number of primary fields \mathcal{O}_i of conformal dimension h_i . In these models the conformal dimensions of the primary fields and the central charges are rational numbers. These operators satisfy a closed OPE algebra $\mathcal{O}_i(z)\mathcal{O}_j(w) = \sum_k (z-w)^{-h_i-h_j+h_k} c^{ijk} \mathcal{O}_k(w)$. Wilson lines of CS models are related to the representations of these algebras.

We will describe the map between integral representations of the affine algebras that compose the RCFT that are level-rank dual to each other. These maps were first studied in the context of level-rank duality of RCFT [82]–[86]

The first step is to classify integrable representations of the Kac-Moody algebra of $SU(N)_K$, $U(K)_N$, and $Sp(N)_K$. Then we will describe the map between representations. After that, we check that the conformal weights of the representations related by duality coincide up to one-half. The one-half discrepancy is because we are considering spin-TQFT¹. Therefore, the Wilson lines can be tensored with a fermionic line transparent with respect to the other lines. Doing so, the weights match, and we have a duality between spin-TQFT. Moreover, other quantities match across the duality, such as the fusion coefficients, which are the integer numbers N_{ijk} of the non-vanishing channels of $\langle \mathcal{O}^i | \mathcal{O}^j(z) | \mathcal{O}^k \rangle$. Fusion is the decomposition of the tensor product of representations. A transparent line is defined such that its fusion with all the other lines is trivial; an example of a transparent line is the line of the trivial representation.

B.0.1 Representations of $SU(N)_K$

Integrable representations of $SU(N)_K$ are in one to one correspondence with Young diagram with rows $\leq N-1$ and columns $\leq K$. Given N and K , the number of representations is $\frac{(N+K-1)!}{K!(N-1)!}$. The conformal weight of the representation R represented by the Young diagram $(l_1, l_2, \dots, l_{N-1})$, where l_i are the number of boxes in the i -th

¹Spin-TQFT are topological quantum field theories defined over manifolds that admit a spin structure, whose correlation functions possibly depend on the choice of spin structure, For spin-TQFT see [87], [88]

row and $l_i \geq l_{i+1}$, $l_j \leq K \quad \forall j$, is given by:

$$h_R = \frac{c_2(R)}{2(K+N)}, \quad c_2(R) = Nr - \frac{r^2}{N} + T(R) = c_2(\mathfrak{su}(N)), \quad (\text{B.1})$$

where $r = \sum_{i=1}^{N-1} l_i$ is the number of boxes of the (reduced) Young diagram and $T(R) = \sum_{i=1}^{N-1} l_i(l_i - 2i + 1)$.

The action of the simple current (that generates \mathbb{Z}_n) on the representation, $r \rightarrow \sigma(R)$, is given by adding on the top of the young diagram a row of K boxes and deleting the columns with N boxes. This corresponds to the action of the outer automorphism on the Dynkin labels of the representation.

B.0.2 Representations of $U(K)_N$

Representations of $U(K)_N = \frac{SU(K)_N \times U(1)_{NK}}{\mathbb{Z}_K}$ are given by (R, Q) where R is a representation of $SU(K)_N$, and Q is the $U(1)$ charge. The charge is related to the $SU(K)_N$ representation

$$Q = r \pmod{K}. \quad (\text{B.2})$$

Since $U(K)_N$ is a quotient, not all the pairs (R, Q) are different representations of $U(K)_N$, but we have to identify the representations

$$(R, Q) \cong (\sigma(R), Q + N), \quad (\text{B.3})$$

where $\sigma(R)$ is the representation obtained by applying the simple current that generates \mathbb{Z}_K to R . In general, a representation $\mathcal{R} = (R, Q)$ of $U(K)$ can be related to a generalized Young diagram $Y(\mathcal{R})$ that is constructed from the Young diagram $Y(R)$ of the representation of $SU(K)$ to which one adds s columns of length K (note that $s \in \mathbb{Z}$ so one can draw the boxes with different colors whether s is positive or negative). After the identification (B.3) the list of representations of $U(K)_N$ is given by the generalized Young diagrams with rows $\leq K$ and columns $\leq N - 1$. The action \mathbb{Z}_K generator can be represented by adding on the top of the generalized Young diagram a row with N boxes and deleting the columns with more than K boxes. This operation gives equivalent representations in the quotient $U(K)_N$.

The conformal weight of a representation $\mathcal{R} = (R, Q)$ is

$$h_{\mathcal{R}} = h_R + h_Q = \frac{c_2(R)}{2(K+N)} + \frac{Q^2}{2NK}. \quad (\text{B.4})$$

B.0.3 Representations of $Sp(N)_K$

Integrable representations R of $Sp(N)_K$ are given by the Young diagram with N rows and at most K columns. The Dynkin label of the representations $\vec{\lambda} = (\lambda_1, \dots, \lambda_N)$ and the number of boxes in each row are $l_i = \sum_{j=i}^N \lambda_j$, $i = 1, \dots, N$. The conformal

weight of the representation R is calculated

$$h_R = \frac{(\vec{\lambda}, \vec{\lambda} + 2\vec{\rho})}{2(K + N + 1)}, \quad (\text{B.5})$$

where $\vec{\rho} = (1, \dots, 1)$ and the scalar product (\cdot, \cdot) is defined by the matrix

$$F = \frac{1}{2} \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & \cdots & 2 & 2 \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ 1 & 2 & \vdots & N-1 & N-1 \\ 1 & 2 & \vdots & N-1 & N \end{pmatrix}. \quad (\text{B.6})$$

The action of the simple current σ that generates \mathbb{Z}_2 sends the Young diagram in its complement w.r.t. the rectangle $N \times K$.

B.0.4 Duality Map

The duality map is

$$\mathbf{Rep} Y(R) \text{ of } S(N)_K \quad \Leftrightarrow \quad \mathbf{Rep} Y(\mathcal{R}) = Y(R)^T \text{ of } U(k)_{-N},$$

where $Y(R)^T$ is the generalized Young diagram obtained transposing the Young diagram $Y(R)$. Here $\mathbf{Rep} Y(R)$ is a short notation for integrable representation related to the Young diagram $Y(R)$. First of all, note that the weight is meaningful only mod one. Second, one can obtain the lines of $U(K)_N$ simply inverting the orientation of the base manifold. This operation sends $h_{\mathcal{R}} \rightarrow -h_{\mathcal{R}}$. After the identification above, the conformal weights do not match perfectly. In fact, when the charge of the representation \mathcal{R} is odd, one has to tensor this line with the chargeless line ψ of spin $\frac{1}{2}$. Then the lines match.

In summary the duality map f is

$$f : \text{Rep}(SU(N)_K) \mapsto \text{Rep}(U(K)_{-N}) \quad (\text{B.7})$$

$$(Y(R), r) \mapsto \begin{cases} (Y(\mathcal{R}) = Y(R)^T) & \text{if } r \text{ even} \\ (Y(\mathcal{R}) = Y(R)^T) \times \psi & \text{if } r \text{ odd} \end{cases} \quad (\text{B.8})$$

The duality map for the $Sp(N)_K$ is similar to the $SU(N)_K$ case. The representations of $Sp(N)_K$ and are related to representations of $Sp(K)_N$ by the transposition of the Young diagram

$$\mathbf{Rep} Y(R) \text{ of } Sp(N)_K \quad \Leftrightarrow \quad \mathbf{Rep} Y(R') = Y(R)^T \text{ of } Sp(K)_{-N}.$$

If the number of boxes of the Young diagram is even, the relation between the representations is exact, mod 1. In contrast, if the number of boxes of $Y(R)$ is odd, we need to tensor one of the lines on one side of the duality with the chargeless line

of spin $\frac{1}{2} \psi$. In fact, the relation between the conformal weights corresponding to a young diagram and its transposed is

$$h_R + h_{R'} = \frac{1}{2}r, \quad (\text{B.9})$$

where r number of boxes of the $Y(R)$.

In summary the duality map f is

$$f : \text{Rep}(Sp(N)_K) \mapsto \text{Rep}(Sp(K)_{-N}) \quad (\text{B.10})$$

$$(Y(R), r) \mapsto \begin{cases} (Y(R') = Y(R)^T) & \text{if } r \text{ even} \\ (Y(R') = Y(R)^T) \times \psi & \text{if } r \text{ odd} \end{cases} \quad (\text{B.11})$$

Appendix C

Alternative effective theory of domain walls of $SU(N)$ $F = N + 1$.

In this appendix, we analyze what would happen if we considered as effective low energy theory trapped on the k -wall of $SU(N)$ with $F = N + 1$ instead of (4.66):

$$U(k)_{\frac{N-k-1}{2}, \frac{N-1}{2}}^{\mathcal{N}=1} \quad \text{with } N + 1 \text{ fundamentals } X, \quad \mathcal{W} \sim +|X|^4. \quad (\text{C.1})$$

This model is the natural extension of the proposal (4.49), adding one fundamental and changing the CS levels accordingly. First, we see that the global symmetries meet the requirements. Second, if we restrict ourselves to $k < \frac{N}{2}$ we can see that the vacua of this model almost reproduce the domain wall solutions we have found in $4d$ analysis. The vacua analysis follows the same path of 4.1.3. We are considering the following superpotential

$$\mathcal{W} = \frac{1}{4} \text{Tr} \left(X X^\dagger X X^\dagger \right) + \frac{\alpha}{4} \text{Tr} \left(X X^\dagger \right)^2 + m \text{Tr} X X^\dagger \quad (\text{C.2})$$

The analysis of the vacua is carried out diagonalizing the matrix $X X^\dagger = \text{diag}(\lambda_1^2, \dots, \lambda_N^2)$ using the flavor and gauge symmetry. Since $X X^\dagger$ is semi-positive definite, $\lambda_i^2 \geq 0$. Supersymmetric vacua satisfy the F-term equations

$$\lambda_i(\lambda_i^2 + m) = 0 \quad i \in \{0, \dots, k\}. \quad (\text{C.3})$$

These equations have the following solutions:

- $m > 0$. There is only one solution, the low energy theory is the TQFT of Acharya and Vafa

$$3d \quad U(k)_{N-\frac{k}{2}, N}^{\mathcal{N}=1}. \quad (\text{C.4})$$

(The Chern-Simons level is obtained integrating out the positive mass fermions).

- $m < 0$. There are $k + 1$ solutions. The quarks $X X^\dagger$ take VEV and break both the flavor symmetry $U(N) \rightarrow U(J) \times U(N - J)$ and the gauge symmetry $U(k) \rightarrow U(k - J)$. The low energy models living on each of the J vacua are

$$U(k - J)_{-\frac{k-J+2}{2}, -1}^{\mathcal{N}=1} \times Gr(J, N). \quad (\text{C.5})$$

Appendix D

Vacuum structure analysis for the 3d domain wall theory of $SU(N)$ with $N + 1$ flavors

In this appendix we are going to study the vacuum structure of the model (4.66) in full generality. To ease the reading we report here the superpotential of (4.66)

$$\mathcal{W} = \frac{1}{4} \text{Tr}(q_i^\dagger q^j q_j^\dagger q^i) + \frac{A}{4} \text{Tr}(q_i^\dagger q^i)^2 + \frac{B}{4} \text{Tr}(p^\dagger p)^2 + \frac{\eta}{2} \text{Tr}(q_i^\dagger q^i) \text{Tr}(p^\dagger p) + \frac{\alpha}{2} \text{Tr}(p^\dagger q_j^\dagger) \text{Tr}(q^j p) + \mathfrak{M}_{dressed}^+ \quad (\text{D.1})$$

Using the flavor and the gauge symmetry we can put in a semi-diagonal form the matrix $q_a^i = \begin{pmatrix} L_{k \times k} & Z_{k \times F-k} \end{pmatrix}$, where $L = \text{diag}(\lambda_1, \dots, \lambda_k)$ and Z has zeros in all the entries. Since $q_i^\dagger q^i$ is positive definite, $\lambda_i \geq 0$. Having performed these simplifications, the F-term equations are:

$$\begin{aligned} \lambda_i(\lambda_i^2 + A \sum_j \lambda_j^2 + \eta|p|^2 + \alpha|p_i|^2 + m) &= 0 \\ p_a(B|p|^2 + \eta \sum_j \lambda_j^2 + \alpha\lambda_a^2 - m) &= 0. \end{aligned} \quad (\text{D.2})$$

Computing the second derivative of the superpotential we get the fermion mass matrix. This matrix is important because in order to know which is the shift of the CS terms due to integrating out massive fermions, we need to know the sign of the fermion masses. The mass matrix reads

$$\begin{aligned} \frac{\partial \mathcal{W}}{\partial \lambda_i^2} &= \lambda_i^2 + A \sum_j \lambda_j^2 + \eta|p|^2 + \alpha|p_i|^2 + m + 2\lambda_i^2(1 + A) \\ \frac{\partial \mathcal{W}}{\partial \lambda_i \partial \lambda_j} &= 2A\lambda_i \lambda_j \\ \frac{\partial \mathcal{W}}{\partial \lambda_i \partial p_a} &= \lambda_i(\eta p_a^* + 2\alpha \delta_i^a p_a^*) \\ \frac{\partial \mathcal{W}}{\partial p_a^* \partial p_b} &= \delta_{ab}(B|p|^2 + \eta \sum_j \lambda_j^2 + \alpha\lambda_a^2 - m) + B p_b^* p_a \end{aligned} \quad (\text{D.3})$$

The list of solutions for generic A, B, η, α and m are the following.

1. The trivial solution in which $\lambda_i = p_i = 0$ is always present regardless the various coefficients of the superpotential. For positive masses it gives us the AV phase, while for negative masses gives us two vacua due to the presence of the monopole in the superpotential which lifts the remaining S^1 .
2. Another solution is given by

$$\lambda_i^2 = 0 \quad \text{and} \quad |p|^2 = \frac{m}{B}. \quad (\text{D.4})$$

If we want that the only vacuum for positive masses is the trivial one we need to assume that $B < 0$. Under this assumption, we get at low energy a TQFT $U(k-1)_{-k/2+1/2,0}$ which is equivalent to $U(1)_0 \sim S^1$, lifted to two points by the monopole superpotential. The existence of this vacuum it is trouble for us. It is not seen in the 4d analysis, yet it is found semiclassically. If $\frac{\eta+\alpha}{B} < -1$, which changes the masses of the charged fermions, we have a TQFT $U(k-1)_{N-(k-1)/2,0}$, which is not related to the 4d analysis.

3. The next set of k vacua are given by the solution

$$\lambda_i^2 = -\frac{m}{(1+JA)} \quad \text{and} \quad p_i = 0, \quad J = 1, \dots, k, \quad (\text{D.5})$$

Let us call these vacua mesonic J -vacua. If $\frac{\eta+\alpha}{1+JA} < 1$ the low energy theory includes a TQFT factor $U(k-J)_{-\frac{k-J}{2},0}^{\mathcal{N}=1}$, which is equivalent to $U(1)_0 \sim S^1$. The monopole superpotential lifts the S^1 to two points. Hence the low energy theory is

$$\mathbb{Z}_2 \times Gr(J, N+1). \quad (\text{D.6})$$

Modulo the double degeneracy, these are the $k+1$ vacua expected from the 4d analysis. If $\frac{\eta+\alpha}{1+JA} > 1$, at low energy we get a non-trivial TQFT $U(k-J)_{-(k-J+2)/2,0}$ which seems not to be related to the 4d analysis.

4. The next set of k vacua are given by

$$\lambda_i^2 = -\frac{m(B+\eta)}{(B+JAB-J\eta^2)} \quad \text{and} \quad |p|^2 = \frac{m(1+JA+J\eta)}{(B+JAB-J\eta^2)}. \quad (\text{D.7})$$

Let us call these vacua baryonic J -vacua. Such vacua are not seen in the 4d analysis. So since we have assumed that $B < 0$, they exist only if small η , that is $B+\eta$ and $1+AJ+J\eta$ have opposite sign. Therefore in order to discard such solutions we assume that at our fixed point $|\eta|$ is big enough.

5. Last, we have other $k(k - 1)$ vacua, which are parametrized by $J, H = 1, \dots, k$ and $J \geq H$, given by

$$\begin{aligned}\lambda_i^2 &= -m \frac{J(B + \eta) + \alpha(1 + (J - H)(A + \eta))}{D}, \\ |p|^2 &= mH \frac{1 + \alpha + J(A + \eta)}{D}, \\ D &= (H(1 + JA)B - (\alpha + H\eta)^2 - (J - H)(A\alpha^2 + H\eta^2)).\end{aligned}\tag{D.8}$$

These vacua exist on if $J(B + \eta) + \alpha(1 + (J - H)(A + \eta))$ and $1 + \alpha + J(A + \eta)$ have opposite sign. Needless to say that for our purposes it is easy to tune the parameter α to assure the discarding of such solutions. For example, if $\alpha = 0$ we see that, provided the other parameters have been chosen so that the baryonic vacua (D.7) are not present, are automatically not there.

Appendix E

sQED with a linear monopole superpotential

In this short appendix we analyze the vacuum structure of the 3d model $\mathcal{N} = 1$ $U(1)_{\frac{1}{2}}$ with one charge-1 chiral Q . This model is well suited to understand the effects of a deformation with a linear monopole superpotential term $\delta\mathcal{W} = \text{Re}(\mathfrak{M})$. This theory has a dual model which has been described in [40]. The duality we are considering is

$$\begin{aligned}
 U(1)_{\frac{1}{2}}^{\mathcal{N}=1} \text{ with one chiral } Q &\iff \text{WZ model with a real } H \text{ and a complex } P \text{ superfields} \\
 \mathcal{W} = -\frac{1}{4}|Q|^4 &\iff \mathcal{W} = H|P|^2 - \frac{H^3}{3}
 \end{aligned}
 \tag{E.1}$$

The basic operator map is

$$\left\{ \begin{array}{c} \mathfrak{M} \\ |Q|^2 \end{array} \right\} \iff \left\{ \begin{array}{c} P \\ H \end{array} \right\},
 \tag{E.2}$$

where \mathfrak{M} is the dressed monopole operator of the sQED. We can check that the duality (E.1) is a sound proposal, studying the massive vacua of both models. If we deform sQED with a mass term $\delta\mathcal{W} = m|Q|^2$, we get that the F-term equation is $Q(-|Q|^2 + m) = 0$. This in turn gives us the vacuum structure:

- $m > 0$ there are two vacua $Q = 0$ and $|Q| = m$. Both vacua support a gapped theory, one because of the triviality of the TQFT $U(1)_1$ and the other after the Higgs mechanism.
- $m < 0$ there is only one vacuum $Q = 0$. The low energy model is a $U(1)_0$ which can be dualized in a NLSM with target space S^1 .

In the dual model the mass deformation maps into $\delta\mathcal{W} = mH$ and the F-term equations become $|P|^2 - H^2 + m = 0$ and $PH = 0$. The vacuum structure of the model is:

- $m > 0$ $H = \pm\sqrt{m}$ and $P = 0$. These two solutions support a gapped vacua which match the $m > 0$ phase of sQED.
- $m < 0$ $H = 0$ and $|P|^2 = -m$. The global $U(1)$ symmetry is broken by the VEV of the complex superfield P giving us a Goldstone boson which lives in S^1 , matching the $m < 0$ phase of sQED.

Here we are interested in deformed the above sQED model with $\delta\mathcal{W} = \text{Re}(\mathfrak{M})$, which maps to $\delta\mathcal{W} = \text{Re}(P)$ in the dual Wess-Zumino model. So we want to study the phases varying the parameter m of the Wess-Zumino model with superpotential

$$\mathcal{W} = H|P|^2 - \frac{H^3}{3} + \frac{P + P^\dagger}{2} + mH. \quad (\text{E.3})$$

The F-term equations are

$$\begin{aligned} |P|^2 - H^2 + m &= 0, \\ HP + \frac{1}{2} &= 0, \quad \text{and} \quad HP^\dagger + \frac{1}{2} = 0. \end{aligned} \quad (\text{E.4})$$

The second and third equations tell us $P = P^\dagger$ and that $P = -\frac{1}{2H}$. So substituting into the first equation we get

$$H^4 - mH^2 - \frac{1}{4} = 0 \quad \Rightarrow \quad H^2 = \frac{m + \sqrt{m^2 + 1}}{2}. \quad (\text{E.5})$$

We see that, while for $m > 0$ we still get two gapped vacua, we get two gapped vacua also when $m < 0$, $H = \pm\sqrt{\frac{m + \sqrt{m^2 + 1}}{2}}$ and $P = \mp\sqrt{\frac{2}{m + \sqrt{m^2 + 1}}}$. Therefore we can see that the linear monopole deformation of the superpotential lifts the S^1 vacuum to two points.

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