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dHYM connections coupled to a variable Kähler metric

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Introduction

Chapter contents

1 Introduction

This thesis introduces a system of coupled equations for a Kähler metric on a compact complex manifold and a connection on an holomorphic line bundle over it. It is obtained intertwining two major problems in complex geometry. The first one, i.e. the existence of a Kähler metric of constant scalar curvature (or *cscK equation*), traces back to the 1950s [Cal54, Cal57] and it was settled more precisely in the 1980s by Calabi [Cal82, Cal85], in order to find a suitable notion of canonical Kähler metric. The setting of this problem is the following: for a compact complex manifold (M, J), consider a non-empty Kähler class $\Omega \in H^2(M, \mathbb{R})$. The cscK equation is

$$s(\omega) = \hat{s}, \tag{1.1}$$

where $s(\omega)$ denotes the scalar curvature of the Riemannian metric associated to ω and J and $\hat{s}$ is a constant, depending only on the topology of M and the choice of Ω. An historical overview on this problem is provided in Section 1.3.1.

The second problem is more recent and comes from string theory. Near 2000, Leu-Yau-Zalsow [LYZ01] and Mariño-Minasian-Moore-Strominger [MMMS00], using different approaches, found the same supersymmetric constraint for D-branes in a superstring theory of type IIB: this condition was called *deformed Hermite-Yang-Mills equation (dHYM equation)*. In the string-theoretical setting, such branes correspond to holomorphic line bundles, supported on analytic subvarieties of a Calabi-Yau manifold; however, since the dHYM equation does not depend on the Calabi-Yau ambient, Jacob and Yau [JY17] initiated the study of the dHYM equation on a compact Kähler manifold, from a purely geometric and analytic point of view. In this case, consider an holomorphic line bundle L over a compact Kähler manifold (M, J, ω); then we look for an hermitian metric h on L, satisfying

$$\operatorname{Im}\left(e^{i\hat{\theta}}\left(\omega - F(h)\right)^n\right) = 0, \tag{1.2}$$

where $F(h)$ denotes the Chern curvature form associated to h and $\hat{\theta}$ is a topological constant depending on $c_1(L)$, Ω and M, which we always assume $\hat{\theta} \neq 0$. For an historical overview on the dHYM equation and its string theoretical interpretation, we refer to Section 1.2.1.

The coupling of these equations is quite natural and exploits the fact that both equations (1.1) and (1.2) admit a moment map interpretation. In complex geometry and mathematical physics, it is common to study a PDE from an (infinite-dimensional) symplectic point of view, whenever it corresponds to the vanishing of a moment map associated to some Hamiltonian action. Many interesting problems fit in this framework; besides the aforementioned equations, other noticeable examples are the Hermite-Yang-Mills equation for a connection on a vector bundle, the Hitchin's equations for Higgs bundles or, due to Donaldson, the condition for exact Lagrangian immersions.

This approach is motivated by the Kempf-Ness theorem, that relates GIT quotients and symplectic reductions, albeit in a finite-dimensional setting. This result implies that the set of zeroes of a moment map corresponds to "stable" orbits of the complexified action, in a suitable sense; moreover, the GIT-stability of an orbit can be checked using different criteria, the most noticeable, the so-called *Hilbert-Mumford criterion*, being a simple numerical condition. For a more in-depth exposition of the finite-dimensional setting, which provides a conceptual guideline for the study of a geometric PDE arising from a moment map, we refer to Section 1.1.

Despite the lack of a canonical way to generalize the Kempf-Ness theorem to an infinite-dimensional setting and to define the relevant stability notion, there is a case in which this approach proved to be extremely fruitul, namely the Hermite-Yang-Mills equation. The setting is the following: let $L \to M$ denote an Hermitian vector bundle L over a compact Kähler manifold M. Then we look for an Hermitian connection A satisfying

$$\begin{cases} F(A)^{2,0} = 0, \\ \Lambda_\omega F(A) = z\mathrm{Id}, \end{cases} \tag{1.3}$$

where $F(A)$ denotes the curvature form associated to the connection A, Λ_ω is the contraction operator on the form-part defined by the metric ω and z is a topological constant. Due to the work of Atiyah-Bott [AB83], the set of Hermitian connections satisfying (1.3) was identified with the symplectic reduction associated to the Hamiltonian action of the gauge group $\mathcal{G}$ on a suitable subspace $\mathcal{A}^{1,1}$ of the space of connections $\mathcal{A}$. More concretely, the condition (1.3) was proved to be equivalent to the vanishing of the moment map associated to the standard action $\mathcal{G} \curvearrowright \mathcal{A}^{1,1}$.

This point of view led to a fundamental result: already in 1963, Mumford had proposed a notion of *slope-stability* for an holomorphic vector bundle, necessary to state an analogue of the Kempf-Ness theorem in this setting, i.e. the *Kobayashi-Hitchin correspondence*. Its content is quite straightforward: the HYM equation (1.3) admits a solution if and only if L is slope-stable. This result had already been proved in 1965 by Narasimhan-Seshadri [NS65] for Riemann surfaces and subsequently generalized by Donalson [Don85] in the case $n = 2$ and by Uhlenbeck-Yau [UY86] for $n > 2$, being the first success in the application of the moment map framework to the study of geometric

PDEs. An overview on the basic example of the HYM problem is provided in 1.1.2 and 1.1.4.

The moment map interpretation of the dHYM equation (1.2) is closely related to the GIT picture for the Hermite-Yang-Mills equation. The relevant Hamiltonian action is again the standard action $\mathcal{G} \curvearrowright \mathcal{A}^{1,1}$, but the linear symplectic structure Ω^{AB} on the space $\mathcal{A}^{1,1}$ proposed by Atiyah-Bott in [AB83] is replaced with a non-linear symplectic form Ω^{dHYM}, which is skew-symmetric and closed, but fails to be non-degenerate [CY18]. Despite this issue, one still hopes to use the GIT framework as a guideline to study this problem.

From this picture, one can deduce a noticeable feature of the dHYM equation: in the large volume limit, i.e. replacing the symplectic form ω by $t\omega$ for $t >> 1$ and expanding Ω^{dHYM} in power series of t, the leading order term is equivalent to the standard form Ω^{AB}. Similarly, the leading order term in t in the dHYM equation is precisely the HYM equation. The GIT picture for the dHYM equation is explained in detail in Section 1.2.2.

The interpretation of the cscK equation (1.1) as a condition equivalent to the vanishing of a moment map traces back to the 1990s, due to the work of Fujiki [Fuj90] and Donaldson [Don99]. In this case, the relevant Hamiltonian action is the action by pull-back of the group $\mathcal{H}$ of Hamiltonian symplectomorphisms on the space $\mathcal{J}$ parametrizing almost-complex structures compatible with a fixed Kähler form ω.

As in the Kobayashi-Hitchin correspondence, this framework aims to define an algebro-geometric notion of stability for the cscK equation which should be equivalent to the existence of a solution of (1.1). In 1997, Tian introduced the notion of *K-stability* for a Fano manifold [Tia97], which was subsequently generalized by Donaldson for any polarized Kähler variety [Don02] and led to the famous Yau–Tian–Donaldson conjecture. While this result remains one of the most important open problems in complex geometry, it proved to hold in many important examples, as for Kähler toric surfaces [Don05a, Don08, Don09] or for Fano manifolds, when the Kähler class Ω is proportional to $c_1(M)$ [CDS14, CDS15a, CDS15b, CDS15c].

In the standard setting for the dHYM problem (1.2), the unique variable is the connection A, while the Kähler structure is fixed. We propose a system of coupled equations, in which the first one is the dHYM equation, but the Kähler metric ω is allowed to vary, being coupled to the connection A via the second equation, which corresponds to the cscK equation plus an additional term. The coupling is obtained quite naturally using the moment map picture as a guideline and, following the ideas of [AGG13], depends on the symmetry groups $\mathcal{H}$ and $\mathcal{G}$ associated respectively to the cscK problem and the dHYM problem.

Our system of PDEs has also a moment map interpretation, associated to the action of the extended gauge group $\widetilde{\mathcal{G}}$, i.e. the group of unitary automorphisms of a line bundle L, covering an Hamiltonian symplectomorphism on the base manifold M. The group $\widetilde{\mathcal{G}}$ is an extension of $\mathcal{H}$ by the gauge group $\mathcal{G}$ and fits into the exact sequence of (infinite-dimensional) Lie groups

$$1 \to \mathcal{G} \xrightarrow{\iota} \widetilde{\mathcal{G}} \xrightarrow{p} \mathcal{H} \to 1.$$

In order to describe a GIT picture for our equations, we consider a suitable subset of the product

$$\mathcal{P} \subset \mathcal{A} \times \mathcal{J},$$

which is an infinite-dimensional manifold with a natural complex structure, such that $\mathcal{P}$ is a complex submanifold. A symplectic structure Ω_α on $\mathcal{P}$ is produced by pulling-back the symplectic forms on the factors $\mathcal{A}$ and $\mathcal{J}$ associated to the GIT pictures of the dHYM problem and the cscK problem, i.e.

$$\Omega_\alpha = n\alpha\Omega^{\mathrm{dHYM}} + \Omega^{\mathrm{DF}}.$$[1]

The parameter $\alpha \in \mathbb{R}$ plays the role of a coupling constant; when $\alpha > 0$, the symplectic form Ω_α on $\mathcal{P}$ is also Kähler.

Finally, we observe that there is a Hamiltonian action of $\widetilde{\mathcal{G}}$ on $\mathcal{A} \times \mathcal{J}$, preserving $\mathcal{P}$ and its symplectic (possibly Kähler) structure, and an associated moment map

$$\mu_\alpha : \mathcal{P} \to \mathrm{Lie}(\widetilde{\mathcal{G}})^*,$$

(see Theorem 2.1.1 and Corollary 2.1.2). The vanishing of μ_α is equivalent of a system of *two* equations because, for each $A \in \mathcal{A}$, there is an operator θ_A inducing an equivariant vector space splitting

$$\mathrm{Lie}(\widetilde{\mathcal{G}})^* = \mathrm{Lie}(\mathcal{G})^* \bigoplus_{\theta_A} \mathrm{Lie}(\mathcal{H})^*. \tag{1.4}$$

With this procedure, we obtain a system of PDEs corresponding to the *dHYM equation coupled to a variable Kähler metric*

$$\begin{cases} \mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right) = 0 \\ s(\omega) - \alpha \dfrac{\mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right)}{\omega^n} = c, \end{cases} \tag{1.5}$$

where c is a topological constant; we point out that, due to the definition of $\mathcal{P}$, there is also the additional constraint

$$F(A)^{2,0} = 0.$$

This construction is exposed in details in Section 2.1.

In order to rephrase the system (1.5) in a more manageable way, we introduce the *radius function* $r_\omega(A)$ and the *Lagrangian phase operator* $\Theta_\omega(A)$, respectively defined as

$$r_\omega(A) = \sqrt{\left(\prod_{i=1}^{n}\left(1 + \lambda_i^2\right)\right)},$$

$$\Theta_\omega(A) = \sum_{i=1}^{n} \arctan(\lambda_i),$$

[1] The form Ω^{dHYM} fails to be non-degenerate everywhere; we will address this issue in the following.

where the λ_i's denote the real eigenvalues of the Hermitian endomorphism of the tangent bundle given by $\omega^{-1}\left(\sqrt{-1}F(A)\right)$. The system (1.5) we propose is equivalent to

$$\begin{cases} \Theta_\omega(h) = \hat{\theta} \mod 2\pi \\ s(\omega) - \alpha r_\omega(h) = \hat{s} - \alpha \hat{r}, \end{cases}$$

where $\hat{s}$, $\hat{r}$ denote respectively the average scalar curvature and the average radius. In a suitable limit, or with a special choice of the parameters, our system (1.5) corresponds to the systems introduced by Álvarez-Cónsul, Garcia-Fernandez and García-Prada [AGG13] (in the large radius limit, see Proposition 2.3.1) and by Hultgren-Witt Nyström [HWN19] (on complex surfaces, see Section 2.2 and in particular Corollary 2.2.1).

In Chapter 2 we establish several general results about the system (1.5). Following [AGG13] Chapter 3, we adapt the usual Futaki invariant for the cscK equation [Fut83], producing an obstruction to solutions of the coupled system (see Section 2.1.3).

In Section 2.2, we describe the cohomological condition which allows to lift the dependence in the equations (1.5) on the metric ω to the Ricci curvature; the resulting system should be interpreted as the coupling of the dHYM problem with the Kähler-Einstein equation. While for a Kähler manifold of higher dimension this condition is not very explicit (see Formula 2.2.1), the case of a complex surface is quite special, due to the fact that the dHYM equation is equivalent to a complex Monge-Ampère equation.

When $n = 2$, the existence of a dHYM connection A on the line bundle L is equivalent to a geometric stability condition, i.e. the positivity of the class

$$\cos\hat{\theta}[\omega] - \sin\hat{\theta}[F] > 0, \tag{1.6}$$

where $[F]$ is a class in $H^2(M, \mathbb{R})$ corresponding to $[F] = -[\sqrt{-1}F(A)]$. Assuming the additional cohomological condition

$$c_1(X) + \frac{\alpha}{\sin\hat{\theta}}[F] = \lambda[\omega], \tag{1.7}$$

in which λ is a constant depending on the topology of the bundle L, $c_1(X)$, Ω and the choice of α, the coupled equations (2.1.4) become the system of complex Monge-Ampère equations

$$\begin{cases} \left(\sqrt{-1}\sin(\hat{\theta})F(A) + \cos(\hat{\theta})\omega\right)^2 = \omega^2 \\ \mathrm{Ric}(\omega) = \lambda\omega + \frac{\alpha}{\sin(\hat{\theta})}\sqrt{-1}F(A). \end{cases} \tag{1.8}$$

The reduced system (1.8) provides also a first example of solutions of (1.5), on a del Pezzo surface with discrete automorphisms (see Corollary 2.2.1).

Notice that the conditions (1.6) and (1.7) are not simply linear in $[\omega]$ and $[F]$; $\cos\hat{\theta}$ and $\sin\hat{\theta}$ depend non-linearly on such classes. However, assuming to have a solution of (1.6) and (1.7) on a surface M, it is possible to induce a new solution on the blow-up $\widetilde{M} = Bl_p(M)$ (see Lemmas 2.2.2 and 2.2.3).

In Section 2.3, we consider the family of Kähler forms

$$\omega_t = t\omega,\ t \in \mathbb{R}_{>0}$$

and analyze the leading behavior of the system (1.5), in the *large volume limit* ($t \to \infty$) and in the *small volume limit* ($t \to 0$). In the first case, our equations degenerate to the Kähler-Yang-Mills coupled equations introduced in [AGG13], in the particular case of a line bundle. In the small volume limit, provided that $F = -\sqrt{-1}F(A)$ is a Kähler form, we obtain a system coupling the famous *J-equation* of Donaldson [Don00] and Chen [Che04] with the cscK equation; this system does not seem to appear in the literature, except in the case of a complex surface [DP20].

In Chapter 3, we study the coupled equations and their limits on abelian varieties. In this special case, we consider a slightly different problem; note that any constant coefficient representative for an integral class $[F] \in H^2(M, \mathbb{Z})$ and for a Kähler class Ω is a solution of (1.5), when M is abelian. We will consider the problem of finding a pair $(A, J) \in \mathcal{P}$ such that, in the splitting (1.4) given by θ_A, the moment map μ_α vanishes on $\mathrm{Lie}(\mathcal{G})$ and acts as some prescribed element $-f \in C^\infty_0(M, \omega) \cong \mathrm{Lie}(\mathcal{H})^*$ on Hamiltonian vector fields. This is equivalent to the system

$$\begin{cases} \mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right) = 0 \\ s(\omega) - \alpha \dfrac{\mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right)}{\omega^n} = f + c. \end{cases} \tag{1.9}$$

The string-theoretical origins of the dHYM equation is the first motivation for considering such class of varieties: homological mirror symmetry for B-models on a complex torus has been studied in detail [Fuk02, PZ98]; in this context, abelian varieties play a special role also as fibres of holomorphic Lagrangian fibrations (see e.g. [GW00]). Moreover, in this setting it is easier to consider the additional problem of realizing effectively solutions of (1.9) in the B-model (see Proposition 3.1.9).

From an analytic point of view, the system (1.9) is closely related with the the problem of finding cscK metrics on complex tori, or more generally periodic solutions of Abreu's equation, as studied by Feng and Székelyhidi [FS11] (see also [LS15]).

In the special case of abelian surfaces, our main existence results are the following:

Theorem 1.1. *Let X be an abelian surface with Kähler classes $[\omega]$, $[F]$. Suppose the phase $e^{\sqrt{-1}\hat{\theta}}$ satisfies*

$$\sin(\hat{\theta}) < 0,\ \cos(\hat{\theta}) > 0.$$

Consider the equations (2.1.4), with coupling constant

$$\alpha = \alpha' \cos(\hat{\theta}),\ \alpha' > 0$$

and datum f given by the image of any function A, as above, under Legendre duality, that is

$$f(\nabla u(x)) = A(x).$$

Then, these are solvable provided the classes $[F]$ *and* $[\omega]$ *are sufficiently close, depending only on* α' *and* $\sup|A|$.

By rescaling F and ω appropriately, it follows:

Corollary 1.2. *Fix negative line bundles* L, N *on the abelian surface* M*. Then for all sufficiently large* k*, depending only on* α', A*, the equations (2.1.4) with coupling constant* $\alpha' \cos(\hat{\theta})$ *and datum* f *as in Theorem 3.1.3 are solvable on the line bundle* $L^{\otimes k} \otimes N^{-1}$*, with respect to the Kähler class* $-kc_1(L)$*.*

The positivity of L^{-1} is a key ingredient in the proof of Theorem 1.1 and Corollary 1.2; using a similiar argument, we obtain analogous existence results in the large radius limit, for arbitrary dimension (see Proposition 3.1.5 and Theorem 3.1.6).

Instead of considering an arbitrary datum f, we may assume that f is invariant under translations with respect to all but one of the symplectic coordinates; in this special case, we are able to obtain much more precise results (see Theorem 3.1.7 and 3.1.8 and Proposition 3.1.9).

In Chapter 4, we study the dHYM equation (1.2) and the coupled equations (1.5) on ruled surfaces, providing many new examples of dHYM connections coupled to a variable Kähler metric (see Theorem 4.1.2). We manage to exhibit very explicit solutions, using the momentum construction, which we set up in Section 4.2, following [KTF12].

It turns out that the unique requirement for solving the system (1.5) is the geometric stability condition for the dHYM equation (1.6), which for a ruled surface can be stated in a quite explicit way. Parametrizing a Kähler class $[\omega]$ and an integral class $[F] \in H^2(M, \mathbb{Z})$ in terms of suitable variables $0 < x < 1$ and $k_i \in \mathbb{Z}$ for $i = 1, 2$, the stability condition (1.6) is equivalent to

$$\left(1 + (k_1 + k_2)^2\right) > x\left(1 + (k_1 - k_2)^2\right). \tag{1.10}$$

The main limitation of Theorem 4.1.2 is that solutions necessarily require a negative coupling constant. In Section 4.5 we allow the background metric to develop conical singularities along certain divisors E_0,E_∞, respectively with cone angles $2\pi\beta_0$, $2\pi\beta_\infty$. For $0 < \beta_0 \leq 1$ and provided that the stability condition (1.10) is satisfied, the coupled system (1.5) admits a solution for a unique value of the angle $\beta_\infty > 1$ and the constant α, which for $\beta_0 \neq 1$ has possibly a positive sign (see Corollary 4.1.4).

In Section 4.6 we find an explicit condition, in term of the parameters involved in the momentum construction, under which the coupled system may be lifted to the Ricci curvature.

In Section 4.7 we consider the degenerations of the coupled equations; in the large radius limit, the stability condition 1.10 becomes trivial and our solutions converge smoothly to the solutions found by Keller and Tønnesen-Friedman in [KTF12]. In the small radius limit, we have smooth convergence provided that the condition

$$(k_1 + k_2)^2 > x(k_1 - k_2)^2$$

holds.

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1 Preliminaries

Chapter contents

1.1 Hamiltonian actions, GIT and stability

In this Section we provide some background material for the rest of the thesis, recalling some basic results in symplectic and algebraic geometry. This introduction has the main purpose of reviewing the moment map construction and the Kempf-Ness theorem, in a finite-dimensional setting; the following exposition should clarify the interest for studying geometric PDEs from a GIT perspective, considering also the noticeable example of the Hermite-Yang-Mills equation.

1.1.1 Hamiltonian actions and symplectic quotients

Let (M, J, ω) be a n-dimensional compact Kähler manifold without boundaries and $g = \omega(J\cdot, \cdot)$ the associated Riemannian metric. In the following, we will focus on certain subgroups of $\mathrm{Diff}(M)$, which we now define.

Definition 1.1.1. The group $\mathrm{Sp}(M, \omega)$ of symplectomorphisms of (M, ω) is the group of diffeomorphisms f preserving the symplectic structure, i.e. $f^*\omega = \omega$.

The identity component $\mathrm{Sp}_0(M, \omega)$ is an infinite dimensional Lie group, whose Lie algebra $\mathfrak{sp}(M, \omega)$ corresponds to the space of real vector fields X satisfying $\mathcal{L}_X\omega = 0$ or equivalently, via Cartan formula, such that the 1-form $\iota_X\omega$ is closed.

The subgroup of Hamiltonian symplectomorphisms Let $\{H_t : M \to \mathbb{R}\}$ be a family of smooth functions for $t \in [0, a]$; we will assume that $H_t \in C_0^\infty(M, \mathbb{R})$., i.e. each function is normalized $\int_X H_t \omega^n = 0$ to have zero average with respect to the reference volume form. The *Hamiltonian construction* associates to $\{H_t\}$ a (time-dependent) Hamiltonian vector field X_{H_t} satisfying

$$dH_t = -\iota_{X_{H_t}} \omega$$

or, in term of the metric g,

$$X_{H_t} = J \mathrm{grad} H_t .$$

Let Φ_H^t denote the flow generated by X_{H_t}; we will refer to the family of diffeomorphisms Φ_H^t as the *Hamiltonian flow* of $\{H_t\}$.

Definition 1.1.2. The group $\mathrm{Ham}(M, \omega)$ of Hamiltonian symplectomorphisms of (M, ω) is the group of diffeomorphisms in the Hamiltonian flow of some time-dependent Hamiltonian function.

Obviously $\mathrm{Ham}(M, \omega) \subset \mathrm{Diff}_0(M)$, but the relevant property is that these transformations also preserve the symplectic structure, i.e. $\mathrm{Ham}(M, \omega) \subset \mathrm{Sp}_0(M, \omega)$. To see this, we observe that, for each (possibly time-dependent) Hamiltonian function H, it holds

$$\mathcal{L}_{X_H} = d\left(\iota_{X_H} \omega\right) + \iota_{X_H} d\omega = -ddH = 0.$$

Proposition 1.1.3. *If* $\mathrm{H}^1(M, \mathbb{R}) = 0$, *then* $\mathrm{Ham}(M, \omega) = \mathrm{Sp}_0(M, \omega)$.

Proof. Let ξ_t be the time-dependent vector field generated by a path of symplectomorphisms. The 1-form $\iota_{\xi_t} \omega$ is closed and exact, so there is a function satisfying $dH_t = -\iota_{\xi_t} \omega$, which is unique if we require the standard normalization. □

In order to characterize the Lie algebra $\mathfrak{ham}(M, \omega)$ of the group $\mathrm{Ham}(M, \omega)$, we point out that a flow of Hamiltonian symplectomorphisms is also the Hamiltonian flow of some function H_t. This important result, which is not a priori obvious, was established in [Ban78].

Proposition 1.1.4. $\mathfrak{ham}(M, \omega)$ *corresponds the Lie algebra of Hamiltonian vector fields; it is isomorphic (as a Lie algebra) to* $C_0^\infty(X, \mathbb{R})$, *endowed with the Poisson bracket.*

Proof. The proof follows immediately from our previous consideration and from the formula

$$\left[J \mathrm{grad} f, J \mathrm{grad} g\right] = -J \mathrm{grad} \left\{f, g\right\},$$

where $[\cdot, \cdot]$ denotes the usual commutator and $\{\cdot, \cdot\}$ the Poisson bracket. □

Remark 1.1.5. Due to its definition, a complete description of $\mathrm{Ham}(X, \omega)$ is not easy in the general case of a symplectic manifold. It is quite straightforward to prove that $\mathrm{Ham}(M, \omega) \subset \mathrm{Sp}_0(M, \omega)$ is a normal (and obviously path-connected) subgroup [MS17, Chapter 10]. Endowing $\mathrm{Sp}(M, \omega)$ with the C^1-topology, it is still an open question whether $\mathrm{Ham}(M, \omega)$ is C^1-closed or, equivalently, a submanifold of $\mathrm{Symp}(M, \omega)$. This so-called *Flux conjecture* is known to hold at least when M is a Kähler manifold [Ban78]; we refer to [Pol01, Chapter 14] for a more in-depth discussion on this problem.

Hamiltonian group actions and moment maps Let K be a compact Lie group, acting on (M, ω) by symplectomorphisms. In this setting, there is not a general procedure to define a meaningful quotient on M, which inherits its symplectic structure. However, when the action $K \curvearrowright M$ is Hamiltonian, one can exploit a noticeable construction, called *Marsden-Weinstein quotient* or *symplectic reduction*; in order to define such quotient, we need some additional concepts, which we define subsequently.

For any element $\xi \in \mathrm{Lie}(K) := \mathfrak{K}$, we define the infinitesimal action as the map $\mathfrak{K} \to \mathrm{Vec}(X)$. given by

$$X_\xi(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi)p.$$

It is immediate to check that

$$X_{[\xi,\chi]} = \left[X_\xi, X_\chi\right]$$

and

$$X_{\mathrm{Ad}(k)^{-1}\xi} = k^* X_\xi,$$

for any $k \in K$; moreover, since X_ξ is a symplectic vector field, $\iota_{X_\xi}\omega$ is closed.

Definition 1.1.6. Let K be acting on M via symplectomorphisms; the action is called Hamiltonian if

1. For every $\xi \in \mathfrak{K}$, the infinitesimal action X_ξ is an Hamiltonian vector field for some potential H_ξ, i.e. $-\iota_{X_\xi}\omega = dH_\xi$. Hence there is a map $\rho : \mathfrak{K} \to C^\infty(M, \mathbb{R})$, associating to each element of the Lie algebra ξ its Hamiltonian potential H_ξ.

2. The map $\rho : \mathfrak{K} \to C^\infty(M, \mathbb{R})$ is a Lie algebra homomorphism, with respect to the Poisson structure on $C^\infty(M, \mathbb{R})$.

We notice that this definition implies *i)* $\rho\,(\mathrm{Lie}(K)) \subset \mathfrak{ham}(M, \omega) \cong C^\infty(M, \mathbb{R})/\mathbb{R}$ and *ii)* that there exists a lift to $C^\infty(M, \mathbb{R})$ that respects the relevant Lie algebra structures. Such lift, which fixes the average of H_ξ possibly adding a real constant, can always be chosen so that ρ is a linear map; the obstruction to having an actual Hamiltonian action is stated in terms of an element in $H^2(M, \mathfrak{K})$; for more details on this, see [MS17].

To interpret $C^\infty(M, \mathbb{R})$ as a Lie algebra which is acting infinitesimally, assume that M is a polarized Kähler manifold with respect to some line bundle L; this implies that for some hermitian metric h on L, the curvature 2-form of the Chern connection of h is $F(h) = 2\pi\sqrt{-1}\omega$. Consider now the group $\widetilde{\mathrm{Ham}}(L, \omega)$ of unitary automorphisms of (L, h) covering an Hamiltonian symplectomorphism on M. Identifying $\mathrm{Lie}(\widetilde{\mathrm{Ham}}) \cong C^\infty(M, \mathbb{R})$, the infinitesimal action of an element H is given by the vector field

$$\widetilde{X}_H + \sqrt{-1}H,$$

where $\widetilde{X}_H$ is the horizontal lift to L of the Hamiltonian vector field X_H on M, while $\sqrt{-1}H(p)$ has to be identified with the Lie algebra $\sqrt{-1}\mathbb{R}$ of the isometry group $U(1)$ of the fiber $L_p \cong \mathbb{C}$ at p.

To give an equivalent definition of Hamiltonian actions, suppose that the action of a group K is Hamiltonian according to Definition 1.1.6; the map ρ defines a K-equivariant linear functional on Lie(K), collecting together nicely all of the Hamiltonians functions associated to its infinitesimal action; it is an easy check that

$$H_{\mathrm{Ad}(k^{-1})\xi} = H_\xi \circ k,$$

for any $k \in K$. We define the moment map μ_K as the function

$$\mu_K : M \longrightarrow \mathfrak{K}^*$$

satisfying the condition

$$\langle \mu(p), \xi \rangle = H_\xi(p) \tag{1.1.1}$$

or, equivalently,

$$d\langle \mu, \xi \rangle = \omega\,(X_\xi, \cdot)\,. \tag{1.1.2}$$

Definition 1.1.6 implies that the moment map is K-equivariant, with respect to the co-adjoint action on $\mathfrak{K}^*$, i.e.

$$\mu_K(k(p)) = k \cdot \mu_K(p) \cdot k^{-1}.$$

Hamiltonian action of extended Lie group Suppose now that $K \subset \widetilde{K}$ is a normal subgroup, such that exists a Lie group extension

$$1 \longrightarrow K \xrightarrow{\iota} \widetilde{K} \xrightarrow{p} H \longrightarrow 1\,. \tag{1.1.3}$$

Assuming that $\widetilde{K}$ admits an Hamiltonian action on M, we want to characterize the associated moment map in term of the normal subgroup K and the quotient group H, following [AGG13]. In order to do so, we need a unique, and in general quite strong, technical assumption, which we now describe.

From 1.1.3, we consider the associated Lie algebra extension

$$0 \longrightarrow \mathrm{Lie}K \xrightarrow{\iota} \mathrm{Lie}\widetilde{K} \xrightarrow{p} \mathrm{Lie}H \longrightarrow 0 \tag{1.1.4}$$

and let $W \subset \mathrm{Hom}\left(\mathrm{Lie}\widetilde{K}, \mathrm{Lie}K\right)$ be the affine space of vector space splitting of the short exact sequence 1.1.4.

Suppose now that exists a smooth, $\widetilde{K}$-equivariant map $\theta : M \longrightarrow W$, with respect to the action on $\theta \in W$ given by

$$k \cdot \theta = \mathrm{Ad}(k) \circ \theta \circ \mathrm{Ad}(k^{-1}) \quad \text{for } k \in \widetilde{K}.$$

Notice that each splitting $\theta \in W$ uniquely defines a lift $\theta^\perp : \mathrm{Lie}H \longrightarrow \mathrm{Lie}\widetilde{K}$ by

$$\mathrm{Id}_{\mathrm{Lie}\widetilde{K}} = \iota \circ \theta + \theta^\perp \circ p,$$

and, considering the $\widetilde{K}$-action on Hom $\left(\mathrm{Lie}H, \mathrm{Lie}\widetilde{K}\right)$ given by

$$k \cdot \theta^{\perp} = \mathrm{Ad}(k) \circ \theta^{\perp} \circ \mathrm{Ad}(p(k^{-1})),$$

the equivariance for the map θ implies the same property for $\theta^{\perp}$.

With this assumption, we can split the moment map associated to the action $\widetilde{K} \curvearrowright M$

$$\langle \mu_{\widetilde{K}}, \xi \rangle = \langle \mu_{\widetilde{K}}, \iota\theta\,(\xi) \rangle + \langle \mu_{\widetilde{K}}, \theta^{\perp} p\,(\xi) \rangle \quad \text{for } \xi \in \mathrm{Lie}\widetilde{K}$$

and define the maps $\mu_K : M \longrightarrow \mathrm{Lie}K^*$ and $\sigma_\theta : M \longrightarrow \mathrm{Lie}H^*$ given by

$$\langle \mu_{\widetilde{K}}, \iota\,(\xi) \rangle = \langle \mu_K, \xi \rangle, \quad \forall \xi \in \mathrm{Lie}K$$

and

$$\langle \mu_{\widetilde{K}}, \theta^{\perp}\eta \rangle = \langle \sigma_\theta, \eta \rangle, \quad \forall \eta \in \mathrm{Lie}H.$$

The maps σ_θ and μ_K are $\widetilde{K}$-equivariant, since K is a normal subgroup, and it is immediate to check that μ_K does not depend on the choice of θ, being the moment map for the Hamiltonian action $K \curvearrowright M$.

Proposition 1.1.7 ([AGG13])**.** *The $\widetilde{K}$-action on M is Hamiltonian if and only if the K-action is Hamiltonian, with a $\widetilde{K}$-equivariant moment map μ_K, and there is a $\widetilde{K}$-equivariant map $\sigma_\theta : M \longrightarrow \mathrm{Lie}H^*$ satisfying*

$$\omega\left(X_{\theta^{\perp}\eta}, \cdot\right) = \langle \mu_K, \langle d\theta, \eta \rangle \rangle + d\langle \sigma, \eta \rangle, \quad \forall\, \xi \in \mathrm{Lie}\widetilde{K}. \tag{1.1.5}$$

Then the equivariant moment map for the $\widetilde{K}$-action is given by

$$\langle \mu_{\widetilde{K}}, \xi \rangle = \langle \mu_K, \theta\,(\xi) \rangle + \langle \sigma_\theta, p\,(\xi) \rangle, \quad \forall \xi \in \mathrm{Lie}\widetilde{K}.$$

Notice that σ_θ is the piece of the moment map $\mu_{\widetilde{K}}$ associated to the H-action, with the condition 1.1.5 generalizing the moment map equation 1.2.1. When $d\theta = 0$, i.e. when the extension 1.1.3 splits, σ_θ coincides with μ_H.

Symplectic reduction For any submanifold N of a symplectic manifold (M, ω), let us denote by TN^{ω} the symplectic complement of the tangent bundle of N; the submanifold N is *isotropic* (resp. *coisotropic*) when $TN \subset TN^{\omega}$ (resp. $TN^{\omega} \subset TN$). The basic idea underlying the Marsden-Weinstein quotient is that any coisotropic submanifold is foliated by isotropic leaves and the quotient, if smooth, inherits a symplectic structure from the ambient space. Since the zero level of a moment map is a coisotropic submanifold, one can prove the following:

Theorem 1.1.8 (Marsden-Weinstein quotient)**.** *Let the action of compact Lie group K on M be Hamiltonian and assume that $M//K := \mu_K^{-1}(0)/K$ is smooth. Then $M//K$ is a symplectic manifold of dimension equal to dim$M$ - 2 dim$K$.*

We mention that a similar result holds for any coisotropic orbit in $\mathfrak{K}^*$ rather than just the set of the zeroes $\mu_K^{-1}(0)$, which is obviously a K-invariant subset of M due to the equivariance of the moment map. This construction is also relevant in Kähler geometry; if the action of a compact group K on a Kähler manifold (M, ω, J) is both Hamiltonian and holomorphic, then the quotient $M//K$ in Theorem 1.1.8 is a Kähler manifold.

1.1.2 Example, part I: an infinite-dimensional Kähler reduction

Inspired by the Narasimhan–Seshadri theorem, Atiyah and Bott [AB83] proved in 1982 that stable bundles over a compact Riemann surface correspond to special unitary connections, satisfying the Hermite-Yang-Mills condition. This was achieved interpreting the HYM equation as arising from an infinte-dimensional moment map construction, as we are going to see. We will complete this example in the end of this Section, referring the reader to [Kob87] and [IN90] for additional details.

Let (L, h) be an Hermitian vector bundle over a n-dimensional compact Kähler manifold (M, J, ω). We consider the space $\mathcal{A}(L, h)$ of connections on L, i.e.

$$\mathcal{A}(L) = A_0 + \sqrt{-1}\Omega_1\left(M, \mathrm{End}L\right),$$

where A_0 denotes any fixed connection. Notice that $\mathcal{A}$ is an infinite-dimensional affine Fréchet space, with $T_A\mathcal{A} = \Omega_1\left(M, \mathrm{End}L\right)$, for any $A \in \mathcal{A}$. In particular, we focus on the submanifold $\mathcal{A}^{1,1} \subset \mathcal{A}$, the space of unitary, J-integrable connections

$$\mathcal{A}^{1,1} = \left\{A \in \mathcal{A} | \text{unitary}, F_A^{0,2} = 0, \right\}$$

which can be endowed with a Kähler structure. The complex structure J on M induces a complex structure on $\mathcal{A}^{1,1}$, which we will also denote by J, given by

$$J\alpha = -\sqrt{-1}\alpha^{1,0} + \sqrt{-1}\alpha^{0,1},$$

for $\alpha \in \Omega_1\left(M, \mathrm{End}L\right)$ and $\alpha^{1,0}$ (resp. $\alpha^{0,1}$) denoting the $(1, 0)$ part (resp. $(0, 1)$) of α. The complex structure J extends to the complexification of the tangent bundle of $\mathcal{A}$, i.e. to $\Omega_1\left(M, \mathrm{End}L\right) \otimes \mathbb{C}$, so that $\mathcal{A}^{1,1}$ is a complex submanifold of the space of connection. A Riemannian metric on $\mathcal{A}$ is induced by the L^2-product

$$g\left(\alpha, \beta\right) = -\int_M tr\left(\alpha \wedge *\beta\right).$$

The metric g and the complex structure J are compatible, meaning that the 2-form $\Omega(\cdot, \cdot) = g\left(J\cdot, \cdot\right)$ is closed, skew-symmetric and non-degenerate, defining a symplectic structure; it is easy to check that these properties hold for

$$\Omega\left(\alpha, \beta\right) = -\frac{1}{(n-1)!}\int_M tr\left(\alpha \wedge \beta\right) \wedge \omega^{n-1}.$$

Endowed with the symplectic structure Ω and the complex structure J, the space of integrable connections $\mathcal{A}^{1,1}$ is an infinite dimensional Kähler manifold, locally modelled on some Frechét space.

Consider now the (infinite dimensional, Frechét) Lie group $\mathcal{G}$ of gauge transformations, i.e. the group of the unitary automorphisms of the vector bundle L, covering the identity on M. The natural action of $\mathcal{G}$ on $\mathcal{A}$ leaves invariant the submanifold $\mathcal{A}^{1,1}$ and preserves the symplectic and the complex structure.

Theorem 1.1.9 (Donaldson, [Don85]). *The natural action of $\mathcal{G}$ on $(\mathcal{A}^{1,1}, J, \Omega)$ is Hamiltonian.*

In the infinite dimensional setting, to prove that an action by symplectomorphisms is Hamiltonian, it is easier to check the existence of a moment map. To prove Theorem 1.1.9, we will exhibit a map

$$\mu : \mathcal{A}^{1,1} \to \mathrm{Lie}\mathcal{G}^* = \Omega^{2n}(M, \mathrm{End}L)$$

which is $\mathcal{G}$-equivariant and satisfies a condition equivalent to equation (1.1.1), i.e.

$$d_{\mathcal{A}} \left(\int_M \mu(A) f \right) (\beta) = \Omega \left(\beta, X_f \right), \tag{1.1.6}$$

where $\beta \in \mathrm{T}_A\mathcal{A}$, $f \in \mathrm{Lie}\mathcal{G}$ and X_f denotes the infinitesimal action of f.

Consider now the map

$$\mu_{\mathcal{G}}(A) = -\frac{1}{(n-1)!} F(A) \wedge \omega^{n-1}, \tag{1.1.7}$$

where $F(A)$ denotes the curvature $(1,1)$-form of the connection A. Under the action of a gauge transformation g, we get

$$F(g \cdot A) = \mathrm{Ad}(g^{-1}) F_A,$$

so $\mu_{\mathcal{G}}$ is equivariant.

To check that 1.1.7 satisfy 1.1.6, recall that the infinitesimal action associated to an element $f \in \mathrm{Lie}\mathcal{G} = \Omega^0(M, \mathrm{End}L)$ is the vector field $d_A f \in T_A\mathcal{A} = \Omega_1$; moreover

$$\left(d_{\mathcal{A}} F \right) (\beta) = d_A \beta.$$

Then we have

$$\begin{aligned} d_{\mathcal{A}} \left(\int_M \mu(A) f \right) (\beta) &= \frac{1}{(n-1)!} \int_M tr \left(f d_A \beta \right) \wedge \omega^{n-1} \\ &= -\frac{1}{(n-1)!} \int_M tr \left(\beta \wedge d_A f \right) \wedge \omega^{n-1} \\ &= \Omega \left(\beta, X_f \right). \end{aligned}$$

Notice that we can identify $\mathrm{Lie}\mathcal{G} = \Omega^0(M, \mathrm{End}L)$ with its dual $\mathrm{Lie}\mathcal{G}^* = \Omega^{2n}$, by $f \to f\omega^n$; moreover, we are free to add a constant term z to $\mu_{\mathcal{G}}$. Rewriting the moment map 1.1.7 as

$$\mu_{\mathcal{G}}(A) = -\frac{1}{n!} \left(\Lambda_\omega F(A) - z\mathrm{Id} \right),$$

we observe that the set of J-integrable connections in $\mu_{\mathcal{G}}^{-1}(0)$ coincides exactly with the solutions of the Hermite-Yang-Mills equation (1.3). Hence the moduli space of HYM connections (up to gauge equivalence) can be described as a Kähler quotient, which is essential in studying its geometry.

Finally, we observe that the parameter $z \in \mathbb{R}$ is fixed by the topology of the problem, depending only on $c_1(L)$ and $[\omega]$ according to the identity

$$z = \frac{2\pi n}{\mathrm{Vol}(M)\mathrm{rank}(L)} \langle c_1(L) \smile \left[\omega^{n-1}\right], [M]\rangle.$$

Remark 1.1.10. We point out that Proposition 1.1.7 holds even in the infinite-dimensional setting. In Chapter 2 we will use this result to study the Lie group extension

$$1 \to \mathcal{G} \xrightarrow{\iota} \widetilde{\mathcal{G}} \xrightarrow{p} \mathcal{H} \to 1,$$

where the extended gauge group $\widetilde{\mathcal{G}}$ denotes the group of automorphisms of L covering an Hamiltonian symplectomorphism of M. In the following, we will define a suitable $\widetilde{\mathcal{G}}$-action on $\mathcal{A}$; for now we point out that, since $\mathcal{A}$ is a space of connection, there is a natural choice for θ.

Since $\mathrm{Lie}\widetilde{\mathcal{G}} \subset \mathrm{Lie}(\mathrm{Aut}L)$, we can identify its elements with vector fields on L and similarly for $\mathrm{Lie}\mathcal{H} \subset \mathrm{Lie}(\mathrm{Diff}X)$. Hence we can define θ mapping each connection A to the associated vertical projection θ_A, so that $\theta_A^{\perp}$ will correspond to the horizontal lift defined by A.

1.1.3 Geometric Invariant Theory (GIT)

We consider now $M \subset \mathbb{CP}^n$ a (smooth) complex projective variety, with the Kähler structure induced by the Fubini-Study metric and the standard complex structure J on $\mathbb{CP}^n$. Let G be a complex reductive Lie group acting on M via a linear representation $\rho : G \to GL(n+1, \mathbb{C})$ on the vector space $\mathbb{C}^{n+1}$ underlying $\mathbb{CP}^n$. Geometric invariant theory (GIT) gives a way to construct a *categorical* quotient for the action $G \curvearrowright M$, within the category of projective varities. For a comprehensive exposition on this topic, we refer to [MFK94], [Dol03] and [Sze14] Chapter 5.

Definition 1.1.11. A categorical quotient for the action of an algebraic group G on a algebraic variety M is a morphism $\pi : M \to Y$ such that

1. *(G-invariance)* $\pi \circ \sigma = \pi \circ p_2$, where $\sigma : G \times M \to M$ denotes the group action and p_2 the projection on the second factor.

$$\begin{array}{ccc} G \times M & \xrightarrow{\sigma} & M \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\pi} & Y \end{array}$$

2. *(Universal property)* Every G-invariant morphism $\psi : M \to Z$ satisfying (1) factors uniquely through π, i.e. there is a unique morphism $\chi : Y \to Z$ such that $\psi = \chi \circ \pi$.

Notice that Definition 1.1.11 does not imply that each point in Y corresponds to a *single* G-orbit; it does not even require π to be surjective, even if this will hold in our setting.

In general, since M is compact but G is not, any non-trivial G-action σ is not proper; consequently the topological quotient is not even Hausdorff, due to the existence of non-closed G-orbits, whose closure contains orbits of lower dimension. In order to get a compact categorical quotient we need to remove certain "bad" (*unstable*) G-orbits and restrict only to the open subset $M^{ss} \subset M$ of *semistable* orbits.

We will briefly describe how to define the GIT quotient. The projective variety M, equipped with the ample line bundle $O(1)|_M$, is

$$M = \mathrm{Proj} \oplus_{r \geq 0} H^0(M, O(r)|_M).$$

The basic idea underlying this construction is to use G-invariant global sections of $O(r)|_M$, i.e. regular invariant functions on the affine cone $\mathbb{C}^{n+1}$, to separate G-orbits in the quotient; let $H^0(M, O(r)|_M)^G$ denote the space of G-*invariant* global sections.

Lemma 1.1.12. *Let G be a reductive algebraic group acting linearly on a projective variety $M \subset \mathbb{CP}^n$; then the ring $\oplus_r H^0(M, O(r)|_M)^G$ is finitely generated.*

We simply define the GIT quotient to be

$$M//G = \mathrm{Proj} \oplus_{r \geq 0} H^0(M, O(r)|_M)^G. \tag{1.1.8}$$

Remark 1.1.13. To define G-invariant sections, we observe that there is a natural *linearization* of the action on the line bundle $O(1)_M$. Since G is acting through a linear representation of $GL(n+1, \mathbb{C})$, the action can be lifted to the affine cone $\mathbb{C}^{n+1}$ over $\mathbb{CP}^n$. Identifying the blow-up at the origin of $\mathbb{C}^{n+1}$ with $O(-1)_{\mathbb{CP}^n}$, so that $O(-1) \subset \mathbb{C}^{n+1} \times \mathbb{CP}^n$, the natural action of G on $\mathbb{C}^{n+1} \times \mathbb{CP}^n$ preserves $O(-1)$; this is called *linearization of the G-action* on $O(1)_M$. Dually, there is an induced action on $O(1)_{\mathbb{CP}^n}$ and, provided that $X \subset \mathbb{CP}^n$ is G-invariant, also on $O(1)_M$.

In order to clarify Definition 1.1.8, we need to understand what the points in $M//G$ are representing. We give the following definitions:

- $x \in M$ is **semistable** if there exists $\sigma \in H^0(M, O(r)|_M)^G$ for $r > 0$ such that $\sigma(x) \neq 0$.
- $x \in M$ is **stable** if $\mathrm{Stab}_G(x)$, i.e. the stabilizer of x, is finite and it exists $\sigma \in H^0(M, O(r)|_M)^G$ separating the orbits near x.
- $x \in M$ is **unstable** if it is not semistable.

Remark 1.1.14. The request that σ separates G-orbits near x means the following. Since x is also semistable, there is an invariant section $\sigma(x) \neq 0$. Restricting to the open set $U_x \subset M$ where $\sigma \neq 0$, we use it to trivialize $O(r)_M$, dividing its sections by σ. Then we ask that each orbit in U_x can be distinguished by $G.x$ using $H^0(M, O(r)|_M)^G$, namely there is an invariant section which has different values on the two orbits.

In the following, we will restrict only to the subset $M^{ss} \subset M$ of semistable points, because G-invariant sections cannot be used to separate unstable points in the quotient. To understand this, we consider $M//G$ as defined in 1.1.8, together with a projection map

$$\pi : M \to M//G$$

induced by the inclusion

$$H^0(M, O(r)|_M)^G \hookrightarrow H^0(M, O(r)|_M).$$

From the previous definitions, it is immediate to check that unstable points of M are not mapped by π to closed points in the quotient $M//G$. Equivalently, Proj $\oplus_{r\geq 0} H^0(M, O(r)|_M)^G$ is the image of M under the linear system $H^0(M, O(r)|_M)^G$, for some $r >> 0$ and the Koidara embedding to $\mathbb{P}\left(H^0(M, O(r)|_M)^G\right)$ is not well-defined for unstable points.

Let $M^s \subset M^{ss}$ denotes the subset of stable points; it can be proved that M^s and M^{ss} are both open subsets of M. Notice that restriction of the quotient

$$\pi^s : M^s \to M^s//G$$

is *geometric* in the sense of [MFK94], i.e. each point of the quotient is parametrizing a single G-orbit. On M^{ss} the quotient is only categorical, but from our definitions it can be checked that the projection π is surjective.

Criteria for GIT stability In the framework of GIT, it is essential to determine whether an G-orbit in M is stable, semistable or unstable. This is not easy in the general case, but there are essentially two results for characterizing the type of the G-orbits, which we now state.

Proposition 1.1.15 (Topological criterion for stability)**.** *Let G be a reductive algebraic group acting linearly on a projective variety $M \subset \mathbb{CP}^n$; for any $x \in M$, let $\tilde{x}$ denotes a lift of x to the affine cone $\mathbb{C}^{n+1}$. Then:*

- *x is semistable $\iff 0 \notin \overline{G\tilde{x}}$.*
- *x is polystable $\iff 0 \notin \overline{G\tilde{x}}$, $G\tilde{x}$ is closed.*
- *x is stable $\iff 0 \notin \overline{G\tilde{x}}$, $G\tilde{x}$ is closed and $Stab_G(\tilde{x})$ is finite.*

We have introduced the additional notion of *polystable* orbits for the reason that, even if $M//G$ is not a geometric quotient, the (closed) points of the GIT quotient are in bijective correspondence with the polystable orbits. We will not prove Proposition 1.1.15, but notice that at least one direction should be clear. Any invariant section in $H^0(M, O(r)|_M)^G$, lifted to a map on the affine cone $\mathbb{C}^{n+1}$, is constant on a G-orbit and its closure. Hence the closure of a semistable orbits cannot contain the origin, or every invariant section should vanish. Similarly, if an orbit is stable, it will coincide with the zeroes of a collection of invariant sections, since by definition invariant sections are separating the stable orbit; consequently, it is closed. Notice that proposition 1.1.15 does not depend on the particular choice of the lift $\tilde{x}$ to $\mathbb{C}^{n+1}$, because two different lifts differ only by a scaling factor.

One of Mumford's major result in GIT was a numerical criterion for stability, which in the literature is usually called *Hilbert-Mumford criterion*. The basic idea is that, instead of taking into account the entire group G, it is enough to check (semi-)stability for 1-parameter subgroups of G.

For $\lambda : \mathbb{C}^* \hookrightarrow G$, i.e. a 1-PS of G, and for any $x \in \mathbb{CP}^n$, let x_0 be the limit

$$x_0 = \lim_{t \to 0} \lambda(t)x.$$

The limit point $x_0 \in \mathbb{CP}^n$ is fixed by the action of the 1-PS $\lambda \subset G$, hence the action on any lift $\tilde{x_0}$ is given by

$$\lambda(t)\tilde{x_0} = t^{\rho(x)}\tilde{x_0},$$

with $\rho(x) \in \mathbb{Z}$.

Proposition 1.1.16 (Numerical criterion for stability)**.** *With the previous notation, it holds*

- *x is stable $\Longleftrightarrow \rho(x) < 0$ for every 1-PS of G.*
- *x is polystable $\Longleftrightarrow \rho(x) < 0$ for every 1-PS of G for which $x_0 \notin Gx$.*
- *x is semistable $\Longleftrightarrow \rho(x) \leq 0$ for every 1-PS of G.*
- *x is unstable $\Longleftrightarrow \rho(x) > 0$ for some 1-PS of G.*

The Kempf-Ness Theorem The assumption of G being a reductive group is crucial to relate the GIT quotient to a suitable symplectic reduction that can be defined in the GIT framework. A reductive group G is the complexification of a compact maximal real subrgroup $K < G$, such that $\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{K} + \sqrt{-1}\mathfrak{K}$. Since K is compact, up to a change of basis, we can assume that it is a subgroup of $U(n+1)$, acting on $\mathbb{CP}^n$ by its standard representation on $\mathbb{C}^{n+1}$. To see this, one can average any hermitian form on $\mathbb{C}^{n+1}$ over K, using the Haar measure, to get a new hermitian form which is now preserved by K. It is a well-known fact that the standard action $U(n+1) \curvearrowright \mathbb{CP}^n$ is Hamiltonian, with respect to the Fubini-Study form, and consequently the same holds for the action of K on $M \subset \mathbb{CP}^n$, equipped with the standard Kähler structure. Consequently, in addition to the GIT quotient $M//G$, we can consider also the symplectitc reduction $\mu_K^{-1}(0)/K$.

Theorem 1.1.17 (Kempf-Ness)**.** *Let G be a reductive algebraic group acting linearly on a projective variety M. A point $x \in M$ is polystable if and only if the orbit Gx contains a zero of the moment map μ_K. If x is polystable, the intersection $\mu_K^{-1}(0) \cap Gx$ is a unique K-orbit. Moreover the inclusion $\mu_K^{-1}(0) \subset M^{ss}$ induces the homeomorphism*

$$\mu_K^{-1}(0)/K \xrightarrow{\sim} M//G.$$

The Kempf-Ness theorem, together with the different notions of GIT stability, is the main reason we study geometrical PDEs in terms of an Hamiltonian action. An infinite-dimensional analogue of GIT would imply that a solution, i.e. the zero of a moment map, lies on a polystable orbit of the complexified action and, instead of solving a PDE, we could use the Hilbert-Mumford criterion to check polystability.

In general there is not yet a canonical way to generalize these results to an infinite-dimensional setting, but this approach proved to be fruitful in many cases. We outlined one side of this correspondence in Subsection 1.1.2, where we regarded the space $\mathcal{A}^{1,1}$ of integrable unitary connections of an Hermitian vector bundle (L, h) as an infinite-dimensional Kähler manifold. The gauge group $\mathcal{G}$ acts on $\mathcal{A}^{1,1}$ by Hamiltonian symplectomorphisms and the set of zeroes of the associated moment map is the space of Hermitian-Yang-Mills connections.

1.1.4 Example, part II: an infinite-dimensional Kempf-Ness theorem

Let us consider again the setting of Section 1.1.2; to reconstruct the GIT picture, we have to complexify an infinite-dimensional Lie group. As we will see in Section 1.3, this is not always possible, but it might be sufficient to complexify the action *infinitesimally*, in order to define complex orbits. For the gauge group $\mathcal{G}$, this problem is avoided and its complexification is

$$\mathcal{G}^{\mathbb{C}} = \left\{g \in \Omega^0\left(M, \mathrm{End}L\right) | g \text{ is invertible}\right\}.$$

The group action $\mathcal{G} \curvearrowright \mathcal{A}^{1,1}$ can be extended to a $\mathcal{G}^{\mathbb{C}}$-action given by

$$\begin{aligned} \overline{\partial}_{g(A)} &= g^{-1} \circ \overline{\partial}_A \circ g, \\ \partial_{g(A)} &= g^* \circ \overline{\partial}_A \circ g^{*-1}, \end{aligned}$$

where g^* denotes the adjoint of g with respect to the fixed hermitian metric h; it is straightforward to check that $g(A)$ is unitary and integrable. Notice that, for $g \in \mathcal{G}$, then $g^* = g^{-1}$ and we recover the usual action

$$d_{g(A)} = g^{-1} \circ d_A \circ g.$$

Associating to each connection $A \in \mathcal{A}^{1,1}$ the corresponding holomorphic structure for L, through the Cauchy-Riemann operator $\overline{\partial}_A$, it follows that $\mathcal{G}^{\mathbb{C}}$-orbits correspond to equivalent holomorphic structure on L, i.e.

$$\mathcal{A}^{1,1}/\mathcal{G}^{\mathbb{C}} \simeq \left\{\text{isomorphism class of holomorphic structures on } L\right\},$$

and, to be precise, we need a suitable notion of stability to construct this quotient.

Let $\mathcal{M}$ denote the space of Hermitian metrics on L and consider the $\mathcal{G}^{\mathbb{C}}$-action given by

$$g \cdot h(\cdot, \cdot) = h\left(g^{-1}\cdot, g^{-1}\cdot\right).$$

For any two metric $h, h' \in \mathcal{M}$ there is an endomorphism $w = h^{-1}h'$ or, equivalently,

$$h' = w^{-1/2} \cdot h.$$

This implies that the $\mathcal{G}^{\mathbb{C}}$-action on $\mathcal{M}$ is transitive and that

$$\mathcal{G}^{\mathbb{C}}/\mathcal{G} \simeq \mathcal{M},$$

where $\mathcal{G}$ corresponds to the stabilizer of a fixed metric h. Notice that for $h = w^{-1/2} \cdot h_0$, we have

$$tr_\omega F(h) = tr_\omega F(h_0) + tr_\omega \overline{\partial}_A \left(w^{-1}\partial_A w\right) = w^{-1/2} \circ tr_\omega F(w^{-1/2} \cdot A_0) \circ w^{1/2}, \qquad (1.1.9)$$

where $F(h)$ denotes the Chern curvature associated to h, with respect to the fixed holomorphic structure $\overline{\partial}$ and A_0 is the Chern connection associated to h_0. It follows from 1.1.9 that it is equivalent to study the HYM equation fixing a Cauchy-Riemann operator $\overline{\partial}$ on L and varying the Hermitian metrics, rather than along $\mathcal{G}^{\mathbb{C}}$-orbits in the space of connection $\mathcal{A}^{1,1}$; in fact, the first choice seems to be easier.

The Kempf-Ness Theorem suggests that, with the appropriate notion of stability, in each polystable $\mathcal{G}^{\mathbb{C}}$-orbit there is a unique Hermite-Yang-Mills metric (i.e. such that the associated Chern connection solves the HYM equation); equivalently, keeping the metric h fixed, a polystable holomorphic structure $\overline{\partial}$ on L should be isomorphic to the $(0,1)$-part of an Hermitian-Yang-Mills connection A, unique up to the action of $\mathcal{G}$.

Definition 1.1.18 (Mumford, 1963)**.** Let $L \to M$ be an *holomorphic* vector bundle and define

$$\mu(L) = \frac{\int_M c_1(L) \wedge \omega^{n-1}}{\text{rank} L}$$

to be the *slope* of L. Then

- L is *slope stable* $\Longleftrightarrow \mu(F) < \mu(L)$ for every holomorphic sub-bundle F.
- L is *slope semistable* $\Longleftrightarrow \mu(F) \leq \mu(L)$ for every holomorphic sub-bundle F.
- L is *slope polystable* $\Longleftrightarrow L = \oplus_i L_i$, where L_i is stable and $\mu(L_i) = \mu(L)$.

The notion of slope stability proved to be the correct one and, for Riemann surfaces, the analogue of the Kempf-Ness Theorem was proved by Narasimhan and Seshadri [NS65] in 1965, even if the link with HYM connections was discovered in the 80s by Atiyah and Bott [AB83]. The generalization of this result to complex manifold of larger dimension, called *the Kobayashi–Hitchin correspondence*, is due to Donaldson [Don85] in the case of Kähler surfaces and due to Uhlenbeck and Yau [UY86] for any dimension.

Theorem 1.1.19 (Donaldson-Uhlenbeck-Yau)**.** *Let (M, ω) be a compact Kähler manifold, and let L be a holomorphic vector bundle over M. Then L admits a Hermite-Yang-Mills metric if and only if L is polystable; the Hermite-Yang-Mills connection is unique, up to the action of $\mathcal{G}$.*

Remark 1.1.20. We conclude this Section with a remark on finite-dimensional GIT. These results, and the Kempf-Ness Theorem, are usually stated in an algebro-geometric setting, assuming to have a reductive alegbraic group acting on a non-singular projective variety $M \subset \mathbb{CP}^n$, such that the action can be lifted to an holomorphic line bundle.

In a series of different and independent works [Geo16, Tel06, Ban04, MiR99], the authors developed a framework that allows to drop the projectivity assumption and adapted the basic concepts of GIT to a differential-geometric setting, dealing with a general Hamiltonian Lie group action on a Kähler manifold, without requiring the symplectic form to be rational or the existence of a linerization. GIT stability is replaced by the notion of μ-stability and, for example, the Hilbert-Mumford criteria are recoverd; in this case, the Mumford numerical invariants need not to be integers, as in the usual GIT theory.

1.2 Deformed Hermite-Yang-Mills equation

In this Section we briefly give an historical overview on the dHYM equation and its string-theoretical interpretation; in 1.2.2 we review the moment map interpretation of this problem and some relevant developments.

1.2.1 Overview

In the 1990s, at the beginning of the so-called *second superstring revolution*, the spacetime of a realistic particle physics model was assumed to be 10-dimensional, consisting of a 4-dimensional Lorenztian manifold, usually M^{3+1}, dS_4 or AdS_4, with the other six extra dimensions compactified in the form of a Calabi-Yau 3-manifold. Moreover, theorical physicists were able to describe five different and perturbatively well-defined superstring theories: type I, type IIA, type IIB and heterotic models with gauge group either $E_8 \times E_8$ or $SO(32)$. We refer to [BBS06] for an introduction on this subject.

The existence of so many possibly inequivalent theories, both in terms of geometries and models, was addressed in several ways. Witten and others proposed to interpret them as different limits of a single, 11-dimensional theory, called *M-theory.* While this picture is still conjectural, it was also discovered that many superstring models were equivalent, being related by equivalence transformations, or *string dualities*: S-duality, U-duality, T-duality and mirror symmetry. The other groundbreaking development in the second superstring revolution, due to Polchinski, was the introduction of other higher-dimensional objects than the fundamental strings, called D-branes, see [Asp03] and [Joh03].

Open strings are required to satisfy Dirichlet boundary conditions for their ending points, which have to lie on such D-branes, the latter being an ingredient also of the aforementioned string dualities. On a type IIA model compactified on a Calabi 3-fold $(\check{X}, \check{\omega})$, D-branes, usually called A-branes, are understood as Lagrangian submanifolds of $\check{X}$, endowed with a flat unitary bundle. On a B-model compactified on (X, ω), B-branes correspond to the derived category of coherent sheaves on X, but for our purposes is enough to consider them as holomorphic vector bundles, supported on an analytic subset $Y \subset X$, which should correspond to the support of a coherent sheaf over X. However not every D-brane is physically realistic and one has to restrict only to the minima of an appropriate brane-action functional, these solutions being called *supersymmetric D-branes* or *BPS states*.

The supersymmetric constraint for A-branes is well understood [BBS95]; in a type IIA theory, they correspond to special Lagrangian submanifolds (together with a flat unitary bundle), i.e. a 3-dimensional submanifold $\check{Y} \hookrightarrow \check{X}$ such that

$$\check{\omega}|_{\check{Y}} = 0 \text{ and } \mathrm{Im} e^{-\sqrt{-1}\theta}\Omega|_{\check{Y}} = 0,$$

with Ω denoting a nowhere vanishing holomorphic volume form on $\check{X}$. The discovery of a moment map interpretation for the sLag condition, due to Thomas [Tho01], led also to a conjectural stability notion; more precisely, considering a Hamiltonian deformation class of Lagrangians $[L]$, there should exists a suitable notion of stability for $[L]$ which is equivalent to admitting a special Lagrangian representative [TY02].

A precise characterization for the BPS state of a B-model is much more complicated; at least in the large volume limit, i.e. rescaling the Kähler form as $t\omega$ for $t \to \infty$, supersymmetric B-branes correspond to holomorphic vector bundles with an Hermite-Yang-Mills connection. Mirror symmetry, which has already proved to be a powerful tool in theoretical physics and geometry (see for example [COGP91]) led to a better understanding of supersymmetric B-branes.

This duality relates a pair of Calabi-Yau manifold (X, ω) and $(\check{X}, \check{\omega})$, exchanging the complex structures moduli space of X with the complexified Kähler moduli space of $\check{X}$; physically, this is realized as an equivalence between a type IIA model and a type IIB model, compactified on a mirror pair $(\check{X}, X)$. In this sense, mirror symmetry predicts a correspondence between supersymmetric A-branes on $\check{X}$ and supersymmetric B-branes on X [Kon95].

In 2000, using the SYZ picture for mirror symmetry [SYZ96] and a Fourier-Mukai transform, Leung, Yau and Zaslow described the mirror object of supersymmetric A-branes [LYZ01]. For a mirror pair of dual torus fibrations X and $\check{X}$ over a torus T, the so-called *semi-flat setting* for mirror symmetry, they identified supersymmetric B-branes for abelian gauge theories with holomorphic linde bundles, supported on an holomorphic, possibly not proper, submanifold $Y \overset{\iota}{\hookrightarrow} X$ and equipped with an hermitian metric h solving

$$\begin{aligned} &\mathrm{Im}\left(e^{i\theta}\left(\omega - F(h)\right)^n\right) = 0, \\ &\mathrm{Re}\left(e^{i\theta}\left(\omega - F(h)\right)^n\right) > 0, \end{aligned} \tag{1.2.1}$$

where $F(h)$ denots the Chern curvature associated to h and θ is a topological constant.

The system 1.2.1 is now called *deformed Hermite-Yang-Mills equation* (dHYM); using different techniques, Mariño, Minasian, Moore and Strominger derived independently the dHYM condition as the appropriate supersymmetric constraint for a B-brane

[MMMS00]. Notice also that, in the large volume limit, the equation 1.2.1 corresponds to the usual Hermite-Yang-Mills condition, which is the well-established degeneration of a BPS state in such limit.

We point out that, since the deformed Hermite-Yang-Mills equation does not depend on the ambient Calabi-Yau X, we will consider 1.2.1 simply as an equation for the hermitian metric of a line bundle over a compact Kähler manifold, exiting the domain of mirror symmetry; it is an additional and interesting problem to understand when such solutions can be realized as B-branes.

As a final remark, we mention that one can include in the string theoretical picture also a *B-field* term, i.e. the equivalent of a background electromagnetic field of a classical field theory. This object is described on the ambient Calabi-Yau as a collection of locally defined 2-forms, such that their pull-backs along $\iota : Y \hookrightarrow X$ defines a global, closed 2-form in $\beta \in H^2(Y, \mathbb{R})$. Turning on the B-field action in the equation 1.2.1 amounts to $F(h) \to F(h) + \sqrt{-1}\beta$; in the following we will mainly consider on the case $\beta = 0$.

1.2.2 GIT framework for the dHYM equation

A rigorous analysis of the deformed Hermite-Yang-Mills equation, putting aside its string theoretical interpretation and regarding it as a purely geometric PDE, was started in [JY17] and [JYC20]. In [CY18], the authors proposed a GIT framework for studying the dHYM equation from the variational point of view, which we are now going to describe. Notice that this setting is a generalization of the example of the Hermite-Yang-Mills problem (for a line bundle) that we have sketched in Subsections 1.1.2 and 1.1.4, the two equations coinciding in the large volume limit.

Let $L \to M$ denote an holomorphic line bundle over a n-dimensional compact Kähler manifold; assume also that L has a unitary structure induced by a fixed hermitian metric h. We consider again the Frechét affine space $\mathcal{A}^{1,1} \subset \mathcal{A}$ of unitary, integrable connections, where the integrability constraint for a connection $\nabla = d + A$ is equivalent to $\bar{\partial} A^{0,1} = 0$. For a line bundle, we can identify $T_A\mathcal{A} = \Omega_1(M, \sqrt{-1}\mathbb{R})$, so that

$$T_A\mathcal{A}^{1,1} = \left\{ \alpha \in \Omega_1(M, \sqrt{-1}\mathbb{R}) \,|\, \bar{\partial}\alpha^{0,1} = 0. \right\}$$

As in the HYM example, a complex structure J on M induces a complex structure on $\mathcal{A}^{1,1}$, again denoted by J, defined by

$$J\alpha = -\sqrt{-1}\alpha^{1,0} + \sqrt{-1}\alpha^{0,1}.$$

Assuming that it does not vanish, we define uniquely a phase $e^{\sqrt{-1}\hat{\theta}} \in U(1)$ by requiring

$$\int_M (\omega - F(A))^n \in \mathbb{R}_{>0} e^{\sqrt{-1}\hat{\theta}}. \tag{1.2.2}$$

Notice that the constant angle $\hat{\theta}$ depends only on the topology of the line bundle and the Kähler class $[\omega]$.

Consider now the 2-form defined as

$$\Omega^{\mathrm{dHYM}}(\alpha,\beta) = -\int_M \alpha\wedge\beta\wedge \mathrm{Re}\left(e^{\sqrt{-1}\hat{\theta}}(\omega - F(A))^{n-1}\right),$$

for $\alpha,\beta \in T_A\mathcal{A}^{1,1}$. In general, Ω^{dHYM} is closed and skew-symmetric, but fails to be non-degenerate; howeverm it can be checked that this holds at least in a neighborhood of a solution of the dHYM equation. The associated Hermitian metric, defined by $g(\cdot,\cdot) = \Omega^{\mathrm{dHYM}}(\cdot, J\cdot)$, is also degenerate. Regardless this issue, one still hopes to use the ideas coming from finite-dimensional GIT to study this problem.

We consider again the group of gauge transformations of (L,h), denoted by $\mathcal{G}$. Under its standard action on $\mathcal{A}$, the submanifold $\mathcal{A}^{1,1}$ is invariant; moreover J and Ω^{dHYM} are preserved. We prove now a Theorem that should be compared with Theorem 1.2.1.

Theorem 1.2.1 ([CY18]). *The natural action of $\mathcal{G}$ on $(\mathcal{A}^{1,1}, J, \Omega^{\mathrm{dHYM}})$ is Hamiltonian.*

Proof. We will prove that

$$\mu_{\mathcal{G}}^{\mathrm{dHYM}} = \frac{\sqrt{-1}}{n}\mathrm{Im}\left(e^{\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right) \tag{1.2.3}$$

is a moment map for the action $\mathcal{G} \curvearrowright \mathcal{A}^{1,1}$.

Since $\mathrm{Lie}\mathcal{G} = C^\infty(M, \sqrt{-1}\mathbb{R})$, its dual coincides with the space of purely imaginary $2n$ forms and there is a non-degenerate pairing

$$(\phi,\gamma) = -\int_M \phi\gamma, \quad \text{for } \gamma \in \mathrm{Lie}\mathcal{G}^*, \phi \in \mathrm{Lie}\mathcal{G}.$$

Moreover, for any $\phi \in \mathrm{Lie}\mathcal{G}$, the associated infinitesimal action on $\mathcal{A}^{1,1}$ corresponds to the vector field $d\phi$.

The expression 1.2.3 is clearly $\mathcal{G}$-invariant, so we only need to prove that the condition

$$\Omega^{\mathrm{dHYM}}(d\phi,\alpha) = d(\mu_{\mathcal{G}}^{\mathrm{dHYM}},\phi)(\alpha)$$

holds for any $\alpha \in T_A\mathcal{A}^{1,1}$. It is immediate to check that

$$\begin{aligned}
d(\mu_{\mathcal{G}}^{\mathrm{dHYM}},\phi)(\alpha) &= \frac{\sqrt{-1}}{n}\int_M \phi\, \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Im}\left(e^{\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right)\\
&= \int_M \phi d\alpha \wedge \mathrm{Re}\left(e^{\sqrt{-1}\hat{\theta}}(\omega - F(A))^{n-1}\right)\\
&= -\int_A d\phi\wedge\alpha\wedge \mathrm{Re}\left(e^{\sqrt{-1}\hat{\theta}}(\omega - F(A))^{n-1}\right)\\
&= \Omega^{\mathrm{dHYM}}(d\phi,\alpha).
\end{aligned}$$

□

Let us introduce the Hermitian endomorphism of the tangent bundle given by $\omega^{-1}\left(\sqrt{-1}F(A)\right)$, with real eigenvalues λ_i. Then

$$\begin{aligned}\frac{(\omega - F(A))^n}{\omega^n} &= \prod_i \left(1+\sqrt{-1}\lambda_i\right)\\ &= r_\omega(A)e^{\sqrt{-1}\Theta_\omega(A)},\end{aligned}$$

where we have introduced the *radius function* $r_\omega(A)$ and the *Lagrangian phase operator* $\Theta_\omega(A)$, respectively defined as

$$\begin{aligned} r_\omega(A) &= \sqrt{\left(\prod_{i=1}^n \left(1+\lambda_i^2\right)\right)},\\ \Theta_\omega(A) &= \sum_{i=1}^n \arctan(\lambda_i). \end{aligned} \tag{1.2.4}$$

Notice that we can rephrase the dHYM equation

$$\operatorname{Im}\left(e^{\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right) = 0$$

in term of the Lagrangian phase, as

$$\Theta_\omega(A) = \hat{\theta} \bmod 2\pi.$$

To exploit the GIT framework for studying the dHYM equation, recall that in the finite-dimensional setting, instead of solving the equation $\mu(x) = 0$, we can directly check GIT-stability; moreover, a point x is polystable if any lift $\hat{p}$ to $\mathbb{C}^{n+1}$ has closed orbits under the action of 1-parameter subgroups of G (and finite stabilizer, if we require proper stability).

In order to adapt the Kempf-Ness theorem to this setting, let us sketch briefly its proof:

1. We consider the space G/K endowed with a Riemannian metric, so to be a symmetric space with non-positive sectional curvature, with infinite geodesics given by 1-parameter subgroups $[e^{\sqrt{-1}t\xi}g]$, for $\xi \in \mathrm{Lie}K$ and $g \in G$.
2. We associate to each point x a K-invariant *Kempf-Ness functional* f_x defined on G/K, with the properties that *i*) $[g]$ is a critical point if and only if $\mu(g \cdot x) = 0$ and *ii*) is convex along geodesics, i.e. 1-PS of G.
3. It follows that x is stable if and only if f_x is a convex, proper function, when restricted to infinite geodesic rays; this can be evaluated by the limiting behavior along 1-PS, which is used the define the numerical invariants appearing in Proposition 1.1.16.

4. From the definition of f_x, it follows also that the K-orbit contains a zero of the moment map μ if and only if x is polystable, which is the standard statement of the Kempf-Ness theorem.

In [JYC20], the authors introduce a space $\mathcal{H}_\omega \subset C^\infty(M,\mathbb{R})$, which should be the infinite-dimensional analogue of G/K.

Definition 1.2.2. For a Kähler form $\omega \in [\omega]$, $[\sqrt{-1}F] \in H^2(M,\mathbb{Z})$ and setting $F_\phi = F + \partial\overline{\partial}\phi$, we define

$$\mathcal{H}_\omega = \left\{\phi \in C^\infty(M,\mathbb{R}) \,|\, \mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F_\phi)^n\right) > 0.\right\}$$

Notice that $\mathcal{H}_\omega$ depends on the Kähler form $\omega \in [\omega]$; moreover, Definition 1.2.2 is similar to the definition of the space of ω-plurisubharmonic functions, which plays the same role of $\mathcal{H}_\omega$ in the GIT interpretation of the cscK equation.

The tangent space is $T_\phi\mathcal{H} = C^\infty(M,\mathbb{R})$, so there is a natural choice of Riemannian structure given by

$$\langle f_1, f_2\rangle_\phi = \int_M f_1 f_2 \mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F_\phi)^n\right),$$

which allows us to define geodesic rays in $\mathcal{H}$.

In [CY18] are also introduced certain functionals Z, C and $\mathcal{J}$ on $\mathcal{H}_\omega$, the latter playing the same role of the Kempf-Ness functional, i.e. it is convex along *smooth* geodesics and its critical points correspond to solution of the dHYM equation.

The main problem that arises in the infinite-dimensional setting is that smooth geodesics do not always exists; one usually tries to prove the existence of suitably weak geodesics, so that the functionals Z, C and $\mathcal{J}$ mantain their properties. Without going into further details about these regularity issues, the hope is to use the limiting behavior of the functionals along geodesic rays to produce algebro-geometric obstructions to the existence of dHYM connections.

Analytic and algebraic obstructions for the dHYM equation In this paragraph, we will survey some results about the existence of obstructions or solutions for the dHYM equation.

The first obstruction, being non-trivial in dimensions $n \geq 3$, is the following result.

Lemma 1.2.3 ([CXY17])**.** *If there exists a solution of the dHYM equation, then*

$$\int_M (\omega - F(h))^n \in \mathbb{C}^*.$$

On the space of hermitian metrics h on L, we define

$$V(h) = \int_M r_\omega(h)\omega^n.$$

The functional $V(h)$ has remarkable properties: its critical points are hermitian metrics such that the related Chern connections solve the dHYM equation. Moreover, these critical points are always absolute minima. Using the properties of $V(h)$, it can be proved a remarkable uniqueness result:

Theorem 1.2.4 ([JY17]). *Let L be a holomorphic line bundle over a compact Kähler manifold M. Suppose there exists a metric h on L such that $F(h)$ solves the dHYM equation. Then any other solution is a real constant multiple of h.*

Let us now introduce a crucial notion into the analysis of the dHYM equation, i.e. the *lifted angle* $\hat{\theta}$. From 1.2.2, the topological angle $\hat{\theta}$ is defined only mod 2π. However, we can rewrite the equation in terms of the Lagrangian phase operator, as

$$\Theta_\omega(h) = \sum_{i=1}^{n} \arctan(\lambda_i) = \hat{\theta} \bmod 2\pi,$$

and from Theorem 1.2.4 it follows that, if there is a solution, there is also a unique lift of $\hat{\theta}$ to $\mathbb{R}$. The space $\mathcal{H}_\omega$ previously defined can be concretely defined in terms of $\Theta_\omega(h)$.

Lemma 1.2.5. *An equivalent definition of $\mathcal{H}_\omega$ is*

$$\mathcal{H}_\omega = \left\{\phi \in C^\infty(M,\mathbb{R}) \,|\, \Theta_\omega(F_\phi) \in \left(\hat{\theta} - \frac{\pi}{2}, \hat{\theta} + \frac{\pi}{2}\right) \bmod 2\pi.\right\}$$

Moreover, from the maximum principle, it follows also

Lemma 1.2.6. *If $\mathcal{H}_\omega \neq 0$, then there is a well-defined and unique lift of $\hat{\theta}$ to $\mathbb{R}$.*

Assuming that such lift exists, we will introduce two notions which will play an important role into the existence results that we are going to list.

Definition 1.2.7. Let $\hat{\theta}$ denotes the lifted angle; we will refer to

$$|\hat{\theta}| > (n-2)\frac{\pi}{2}$$

as *supercritical phase condtion* and to

$$|\hat{\theta}| > (n-1)\frac{\pi}{2}$$

as *hypercritical phase condtion*.

In the following, we will assume that $\hat{\theta} > 0$; this can always be obtained, possibly considering the dHYM equation for the line bundle L^{-1} instead of L, which amounts to replace $F \longrightarrow -F$. Consider now the real 2-form defined by

$$\Omega = \cot \hat{\theta}\omega + \sqrt{-1}F(h).$$

We will define the line bundle L *stable* if there is some Hermitian metric h such that $\Omega > 0$, i.e. Ω is a Kähler form.

Theorem 1.2.8 ([JY17])**.** *Let $L \longrightarrow M$ be an holomoprhic line bundle over a Kähler surface. Then a dHYM metric exists if and only if L is stable.*

This results and Theorem 1.2.4 completely clarify the questions about existence and uniqueness for Kähler surface; we mention that $n = 2$ is a very special case, because the dHYM equation can be rephrased as a complex Monge-Ampére equation; consequently, these ideas cannot be generalized in higher dimensions. In the supercritical phase, the stability of L can be proved to be at least necessary, but possible not a sufficient condition.

In higher dimensions, one usually study the so-called *line bundle mean curvature flow*, i.e. a flow of Hermitian metrics $h(t) = e^{f(t)}h_0$ defined by the equation

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(t) = \Theta_\omega(F(h(t)) - \hat{\theta}.$$

Working with this evolution equation, in [JY17] is proved the following result.

Theorem 1.2.9. *Let $L \longrightarrow M$ be an ample line bundle over a compact Kähler manifold M with non-negative orthogonal bisectional curvature. Then there exists $k \in \mathbb{N}$ so that $L^{\otimes k}$ admits a solution to the dHYM equation. This solution is constructed via a smoothly converging family of metrics along the line bundle mean curvature flow.*

Let us point out that the request for L to be ample it is crucial, ensuring that the 2-form $\sqrt{-1}F(t)$ remains positive definite along the flow and that the map $h \longrightarrow \Theta_\omega(F(h(t))$ is concave. We conclude this section with a conjecture and a very recent result. For any subvariety $N \subseteq M$, we define

$$\Theta_N = \operatorname{Arg} \int_N (\omega - F(h))^{\dim N}$$

Conjecture 1.2.10. *[JYC20] The existence of solutions to the dHYM equation is equivalent to the condition*

$$\Theta_N > \Theta_M - (n - \dim N)\frac{\pi}{2} \tag{1.2.5}$$

for every proper subvariety N.

The condition appearing in this conjecture has been proved to be at least necessary; in dimension 2, it is also sufficient.

In 2020, G. Chen [Che21] proposed a different stability notion, which turns out to be equivalent to the existence of a dHYM connection in the supercritical phase, regardless the dimension of M; such condition is understood in terms of integrals on subvarieties. We will state his main result:

Theorem 1.2.11 ([Che21])**.** *Fix a compact, n-dimensional Kähler manifold M with a Kähler metric ω and a real closed $(1,1)$-form F. Assume that there exists a constant $\theta \in (0, \pi)$ such that*

$$\int_M \operatorname{Re}(F_\phi - \sqrt{-1}\omega)^n - \operatorname{Im}(F_\phi - \sqrt{-1}\omega)^n \cot\theta = 0.$$

Then F_ϕ is the curvature of a dHYM connection if and only if, for every smooth test family F_t, there exists a constant $\epsilon_1 > 0$ independent of t, V such that for any $t \geq 0$ and any p-dimensional subvariety V

$$\int_V \mathrm{Re}(F_t - \sqrt{-1}\omega)^p - \mathrm{Im}(F_t - \sqrt{-1}\omega)^p \cot\theta \geq (n-p)\,\epsilon_1 \int_V \omega^p.$$

Here, the author calls a smooth family $F_t, t \in [0, \infty)$ of real closed $(1,1)$-forms a *test family* if and only if

1. $F_0 = F$;
2. For $s > t$, $F_s - F_t$ is positive definite;
3. There is $T > 0$ such that $F_t - \cot\frac{\theta}{n}\omega$ is positive definite for every $t > T$.

1.3 Constant scalar curvature equation for a Kähler metric

In this Section we give an historical overview of the cscK problem in 1.3.1 and explain its moment map interpretation in 1.3.2.

1.3.1 Overview

Consider a compact Kähler manifold M and a non-empty Kähler class $[\omega]$. A natural question is whether it exists or not a suitable notion of a canonical metric in $[\omega]$, with nice properties of existence and uniqueness and possibly parameterized by a finite-dimensional moduli space. We will briefly review the history of this problem, which has been central in the development of Kähler geometry.

In complex dimension $n = 1$, the question was completely answered by the famous Uniformization Theorem, proved in 1907 by Poincaré and Koebe. It implies that in each Kähler class there is a unique metric of constant curvature, with a sign depending on the genus of the surface and induced by a metric on its universal covering. On higher dimensional manifolds, this problem is much more involved.

As a first attempt, consider the class of Kähler manfiolds whose first Chern class $c_1(M)$ has a definite sign. Looking at Kähler classes suitably proportional to $c_1(M)$, one could investigate the existence of a (possibly unique) metric solving the Kähler-Einstein condition

$$\mathrm{Ric}(\omega) = \lambda\omega. \tag{1.3.1}$$

In the 1950s, Calabi proposed a closely related conjecture, regarding the possibility of realizing any representative of $2\pi c_1(M)$ as the Ricci form of a unique Kähler metric (the *Calabi conjecture*, see [Cal54, Cal57]). This was proved by Yau in 1976 [Yau78], implying the existence of a unique Ricci-flat metric in each Kähler class when $c_1(M) = 0$.

In 1976, the case $c_1(M) < 0$ was solved independently by Aubin [Aub76] and Yau [Yau77, Yau78], proving that Kähler manifolds of general type admit a K-E metric, unique up to scaling.

The case $c_1(M) > 0$ proved to be much harder. Let us denote the group of biholomorphisms of a complex manifold (M, J) by $\mathrm{Aut}(M, J)$. Already in 1950s, Matsushima showed that the existence of a K-E metric on a Fano manifold it is not guarenteed [Mat57]. He proved that a solution of 1.3.1 with $c_1(M) > 0$ implies that $\mathrm{Aut}(M, J)$ is reductive; this obstruction signals, for example, that the projective plane blown-up at one or two points cannot admit a K-E metric.

The existence of obstructions to solutions of equation 1.3.1 should not be surprising. The Chern connection of a K-E metric solves the Hermite-Yang-Mills equation, with respect to the same metric, and this would require the holomorphic tangent bundle to be polystable.

Returning to the problem of finding a canonical metric, when $c_1(M) \neq 0$ the Kähler-Einstein equation only applies to Kähler classes proportional to $c_1(M)$. In order to find a more general notion, some other conditions can be considered. The most famous one is the constant scalar curvature equation for a Kähler metric ω, i.e.

$$s(\omega) = \hat{s}. \tag{1.3.2}$$

In the attempt to find an more general condition, so to have a unique canonical metric in each Kähler classes, Calabi proposed, in two papers [Cal82, Cal85] published in 1982 and 1985, the notion of *extremal metric*

$$\bar{\partial}\mathrm{grad}^{1,0}_{\omega} s(\omega) = 0. \tag{1.3.3}$$

Extremal metrics are critical points of the Calabi energy functional

$$\mathrm{Cal}(\omega) := \int_M s(\omega)^2 \frac{\omega^n}{n!}, \tag{1.3.4}$$

the associated Euler-Lagrange equation being precisely the condition 1.3.3.

From the 1980s, new obstructions to the cscK equation were found, the most noticeable being the *Futaki invariant*, introduced by Futaki originally as an obstruction to solve the K-E equation in the Fano case [Fut83]. Consider a metric $\omega \in 2\pi c_1(M) > 0$, let $\rho(\omega)$ be the associated Ricci form and $g_\omega \in C^\infty(M, \mathbb{R})$ the function defined by $\rho(\omega) - \omega = \sqrt{-1}\partial\bar{\partial} g_\omega$. The Futaki invariant is a complex-valued function defined on $\eta(M, J)$, the Lie algebra of $\mathrm{Aut}(M, J)$ or equivalently the space of holomorphic vector fields, defined by

$$F_M(X) = \int_M X(g_\omega)\omega^n.$$

Futaki showed that F_M does not depend on the choice of ω; if a K-E metric exists, necessarily $F_M = 0$. In [Fut83], the author gave also an exampe of a 3-dimensional Fano manifold, with $\mathrm{Aut}(M, J)$ reductive but $F_M \neq 0$, hence not admitting a K-E metric. In [Tia97], Tian proved that for a *del Pezzo surface*, i.e. a complex surface with $c_1(M) > 0$, a K-E metric exists if and only if $\mathrm{Aut}(M, J)$ is reductive. In particular, a solution of the K-E equation automatically exists on a del Pezzo surface with $\mathrm{Aut}(M, J) = \{1\}$. The same author, in [Tia90] provided an example of a 3-manifold not admitting a K-E metric, based on obstructions coming from algebro-geometric considerations.

A new approach to this problem was proposed independently by Fujiki [Fuj90], for a Kähler manifold, and by Donaldson [Don99], in the more general almost Kähler setting. Taking a GIT point of view, we consider a suitable Hamiltonian action on the space of almost complex structures, compatible with a fixed symplectic form ω. The associated moment map turns out to be the Hermitian scalar curvature, which coincides with the Riemannian scalar curvature for an integrable complex structure J, so that the pair (J, ω) defines a Kähler structure.

In the GIT framework, solving the cscK equation 1.3.2 is equivalent to finding the zeroes of a moment map. When this is the case, as we have discussed for the dHYM equation, one hopes to find a notion of stability for the manifold in an algebro-geometric sense, obtaining a sufficient and necessary condition for the existence of solutions. In [Mab87], Mabuchi defined the *K-Energy* functional on the space of Kähler potentials, corresponding to the integral of the moment map from the GIT point of view. Its critical points correspond to cscK metric and their existence is equivalent to the properness of the K-Energy [Tia99]. In 1997 Tian proposed for the Fano case the analytic notion of *K-stability* [Tia97], so called after the K-Energy functional. In [Don02] Donaldson propesed an algebraic notion of *K-stability*, for any polarized Kähler variety, which agrees with Tian's stability for a Fano manifold.

Conjecture 1.3.1 (Yau–Tian–Donaldson conjecture)**.** *A polarized manifold* (M, L) *with discrete* $\mathrm{Aut}(M, J)$ *admits a cscK metric in the class of* $c_1(L)$ *if and only if the pair* (X, L) *is K-polystable.*

The Fano version of this conjecture, suggesting that the K-polistability of the pair $(M, -K_M)$ should be equivalent to the existence of a K-E metric, was proved in 2012 by Chen, Donaldson and Sun [CDS14, CDS15a, CDS15b, CDS15c], using a continuity method with respect to the cone angle of Kähler metrics with conical singularities. Several other proofs followed shortly: firstly by Tian using similar ideas [Tia15], followed by Datar and Székelyhidi [Sze16], [DS16] via a more standard continuity method, by Chen, Sun and Wang [CSW18] studying the Kähler-Ricci flow and by Berman, Boucksom and Jonsson via variational techniques [BBJ21].

In the general case, the "if" part of the YTD conjecture was confirmed by the work of Donaldson [Don05b], Mabuchi [Mab08] and Stoppa [Sto09], the latter using a blow-up argument and the glueing theorem of Arezzo and Pacard [AP06]. In 2009, Donaldson [Don05a, Don08, Don09] proved the full conjecture for toric Kähler surfaces.

Remark 1.3.2. There is a version of the YTD conjecture, when M is allowed to admit holomorphic vector fields, extremal metrics replace cscK metrics in the statement and is introduced the notion of *relative K-stability* . Since extremal metrics will not play any role in this thesis, we will only sketch how they fits into the GIT framework, following [Sze14]. The key observation is interpreting the Calabi functional 1.3.4 as the norm squared of the moment map and extremal metrics as its critical points.

In the finite-dimensional setting for an Hamiltonian action $K \curvearrowright M$, suppose to have an inner product on $\mathrm{Lie}(K) = \mathfrak{K}$, which allows to identify $\mathfrak{K}^* \cong \mathfrak{K}$, so the moment map becomes a map $\mu_K : M \longrightarrow \mathfrak{K}$. Then

Lemma 1.3.3. *A point $x \in M$ is a critical point of $||\mu_K||^2$ if and only if $\mu(x)$ is in the Lie algebra of the stabilizer group of x.*

Proof. The lemma follows immediately by a direct computation and the moment map equation:

$$d\langle \mu_K, \mu_K \rangle(x) = 2\langle d\mu_K(x), \mu(x) \rangle = -2\omega_x(X_{\mu(x)}, \cdot)\,.$$

□

Now, suppose to have a subgroup $H \subset K$, with $p_H : \mathfrak{K} \longrightarrow \mathfrak{h}$ denoting the orthogonal projection of Lie algebras; a natural moment map μ_H for the Hamiltonian action of the subgroup H is $\mu_K \circ p_H$.

The idea underlying the definition of relative stability is quite simple. If x is a critical point of $||\mu_K||^2$, we consider a subgroup H so that $\mathfrak{h}$ is orthogonal to the Lie algebra of the stabilizer of x. Then $\mu_H(x) = 0$ and we can apply the standard Kempf-Ness theorem.

1.3.2 GIT framework for the cscK equation

Let us discuss the GIT framework for the cscK equation, proposed by Fujiki [Fuj90] in the Kähler setting and, independently, by Donaldson [Don97] in the almost Kähler setting; an additional reference on this topic is [Tia99].

Consider a $2n$-dimensional, compact, symplectic manifold (M, ω); for simplicity, we will assume M to be simply-connected, so that $H^1(M, \mathbb{R}) = 0$. Let $\mathcal{J}$ denote the set of almost-complex structures, compatible with the form ω, i. e.

$$\mathcal{J} = \left\{ J \in \mathrm{End}(TM) | J^2 = -\mathrm{Id}, \omega(\cdot, J\cdot) > 0, \omega(J\cdot, J\cdot) = \omega(\cdot, \cdot) \right\}$$

and let $\mathcal{J}^{int} \subset \mathcal{J}$ be the subset of integrable almost-complex structures. Here $\mathcal{J}$ plays the role of infinite-dimensional manifold, with tangent space at J given by

$$T_J\mathcal{J} = \{A \in \mathrm{End}(TM) | AJ = -JA, \omega(JA\cdot, \cdot) = \omega(\cdot, JA\cdot).\}$$

Notice that if an endomorphism $A \in T_J\mathcal{J}$, then also $JA \in T_J\mathcal{J}$; this defines an (integrable) complex structure on $\mathcal{J}$. We can also endow $\mathcal{J}$ with an Hermitian metric, defined by

$$\langle A, B \rangle_J = \int_M \langle A, B \rangle_{g_J} \frac{\omega^n}{n!},$$

which is also compatible with the complex structure, defining at the point J the symplectic 2-form

$$\Omega_J(A,B) = \langle JA, B\rangle_J.$$

Again, the spaces $\mathcal{J}$ and $\mathcal{J}^{int}$ turn out to be infinite-dimensional Kähler manifolds.

Definition 1.3.4 (Hermitian scalar curvature)**.** Let (M,ω,J) be an almost-Kähler manifold and let $\overline{\partial}_J : \Omega^{0,1}_J \to \Omega^{1,1}_J$ be the Dolbeault operator induced by J. Consider the unique g_J-unitary connection ∇_J on TM such that $\nabla^{0,1}_J = \overline{\partial}_J$ and let $\sqrt{-1}\rho_J$ be the curvature 2-form of the induced connection on the canonical line bundle $K_M = \Lambda^n_{\mathbb{C}} T^*M$. The function $s(J)$ defined as

$$\omega^{n-1} \wedge \rho_J n! = s(J)\omega^n$$

is called *Hermitian scalar curvature*.

Consider now the group of Hamiltonian symplectomorphisms $\mathrm{Ham}(M,\omega)$ and the action by pullback $\mathrm{Ham}(M,\omega) \curvearrowright \mathcal{J}$ given by

$$f \cdot J = df \circ J \circ df^{-1}$$

for $f \in \mathrm{Ham}(M,\omega)$. This action preserves the Kähler structure on $\mathcal{J}$; moreover, $\mathcal{J}^{int}$ is an invariant subspace. Denoting the Nijenhuis tensor by N_J, it is easy to check that, for every diffeomorphism f, it holds

$$N_{f_*J}[X,Y] = f_*^{-1} N_J[f_*X, f_*Y].$$

As we have already discussed in Subsection 1.1.1, the Lie algebra $\mathfrak{ham}$ can be identified with the space $C^\infty_0(M,\mathbb{R})$ and consequently with its dual $\mathfrak{ham}^* = \mathfrak{ham}$ via the standard L^2-pairing

$$(f,g) = \int_M fg\frac{\omega^n}{n!}.$$

Theorem 1.3.5 (Donaldson, Fujiki)**.** *The action of* $\mathrm{Ham}(M,\omega)$ *on* $\mathcal{J}$*, endowed with the aforementioned Kähler structure, is Hamiltonian. Identifying* $C^\infty_0(M,\mathbb{R})$ *with its dual via the* L^2*-product, an equivariant moment map is*

$$\begin{aligned} \mu : &\mathcal{J} \to C^\infty_0(M,\mathbb{R}) \\ &J \to s(J) - \hat{s}, \end{aligned}$$

where $s(J)$ *denotes the Hermitian scalar curvature and* $\hat{s}$ *its average.*

Remark 1.3.6. Notice that the Hermitian scalar curvature of J only coincides with the Riemannian scalar curvature of the metric g_J, when the almost-complex structure J is integrable. Moreover, the constant $\hat{s}$ appearing in Theorem 1.3.5 is fixed by the topology of the manifold and the choice of Kähler class, being equal to

$$\hat{s} = \langle c_1(M) \cup [\omega^{n-1}], M\rangle.$$

In this framework, even if we restrict to $\mathcal{J}^{int}$, we are looking for solutions of the cscK equation fixing the Kähler form ω and varying among the ω-compatible complex structures. On the contrary, the usual point of view is to fix the complex structure J and to move in the Kähler class $[\omega]$. We will now prove that these two pictures are equivalent.

Consider $J, J' \in \mathcal{J}$ such that $J' = f^*J$ for some diffeomorphism f, so that the associated Hermitian metrics are related by

$$g(J', \omega) = f^* g(J, (f^{-1})^* \omega). \tag{1.3.5}$$

When $f \in \mathrm{Ham}(M, \omega)$, g and g' are isometric, so we need to complexify $\mathrm{Ham}(M, \omega)$ to obtain a non-trivial orbits to study. While a "complexified group" $\mathrm{Ham}^c(M, \omega)$ does not exist, an infinitesimal complexification of the Lie algebra is possible,

$$\mathfrak{ham}^c = C_0^\infty(M, \mathbb{C}).$$

Using the complex structure defined on $\mathcal{J}^{int}$, we can complexify also the infinitesimal action, as

$$J\mathcal{L}_{X_h} J = \mathcal{L}_{JX_h} J = \mathcal{L}_{X_{\sqrt{-1}h}} J,$$

defining $X_{\sqrt{-1}h} := JX_h$. The idea is that the complexified action induces a foliation of $\mathcal{J}^{int}$, whose leaves should correspond to the "complex" orbits of Ham^c.

For the orbit $(\mathrm{Ham} \cdot J)$, the tangent space is

$$T_J\,(\mathrm{Ham} \cdot J) = \left\{ \mathcal{L}_{X_f} J, \ \text{for } f \in \mathfrak{ham} \right\},$$

hence we consider the distribution $\mathcal{D}$ defined by

$$\mathcal{D}_J = \left\{ \mathcal{L}_{X_f} J, \mathcal{L}_{JX_f} J, \ \text{for } f \in \mathfrak{ham}. \right\}$$

Lemma 1.3.7. $\mathcal{D}_J \subset T_J \mathcal{J}$.

Proof. The only non-trivial thing to prove is that $J\mathcal{L}_{JX_f} J$ is skew-symmetric with respect to ω.

Consider

$$\begin{aligned}
\left(\mathcal{L}_{JX_f} J\right) Y &= \mathcal{L}_{JX_f}(JY) - J\mathcal{L}_{JX_f} Y \\
&= [JX_f, JY] - J[JX_f, Y] \\
&= [X_f, Y] + J[X, JY] \\
&= \mathcal{L}_{X_f} Y + J\mathcal{L}_{X_f}(JY) \\
&= -\mathcal{L}_{X_f}(J^2 Y) + J\mathcal{L}_{X_f}(JY) \\
&= -(\mathcal{L}_{X_f} J)(JY).
\end{aligned}$$

Then, using the identity $-(\mathcal{L}_{X_f}J)Y = \mathcal{L}_{JX_f}(JY)$, we get

$$\begin{aligned}\omega(J(\mathcal{L}_{JX_f}J)v,w)+\omega(v,J(\mathcal{L}_{JX_f}J)w) &= -\omega(J(\mathcal{L}_{X_f}J)Jv,w)-\omega(v,J(\mathcal{L}_{X_f}J)Jw)\\ &= \omega((\mathcal{L}_{X_f}J)Jv,Jw)+\omega(Jv,(\mathcal{L}_{X_f}J)Jw)\\ &= 0.\end{aligned}$$

□

Lemma 1.3.8. *$\mathcal{D}$ is integrable.*

Proof. While $\mathcal{D}$ is closed under Lie bracket, in an infinite-dimensional setting this does not ensure that, at every point $J \in \mathcal{J}^{int}$, there are leaves integrating the distribution $\mathcal{D}$.

To see this, let $\mathcal{K}_\omega$ denote the space of Kähler metrics in $[\omega]$, so that

$$\mathcal{K}_\omega = \left\{\omega_f = \omega + \sqrt{-1}\partial\bar{\partial} f > 0\right\},$$

with $f \in C^\infty(M,\mathbb{R})$. Consider now the bundle $S \xrightarrow{\pi} \mathcal{K}$, where the fiber $S_f = \pi^{-1}(\omega_f)$ is the connect component of group of symplectomorphisms of ω_f; due to our assumption $H^1(M,\mathbb{R}) = 0$, the structure group of S is isomorphic to $\mathrm{Ham}(M,\omega)$. By Moser's lemma, for each $\omega_f \in \mathcal{K}$, there is a diffeomorphism ψ_f such that $\psi_f^*\omega_f = \omega$.

Let us define the map

$$\begin{aligned}\tau_J : &\mathcal{K} \times S_0 \to \mathcal{J}^{int}\\ &(\omega_f,\phi) \to \phi^*\left((\psi_f^{-1})^*J\right).\end{aligned}$$

It is easy to check that ω and any complex structure $J' \in \tau_J(\mathcal{K} \times S_0)$ are compatibile. A much more involved computation shows that

$$D\tau_J|_{\omega_f,\phi} \in \mathcal{D}_{\phi^*\left((\psi_f^{-1})^*J\right)}$$

and that $D\tau_J$ is surjective, proving that the distribution is integrable. □

In this picture, the image $\tau_J(\mathcal{K} \times S_0)$ should correspond to the orbit of J under the complexification of $\mathrm{Ham}(M,\omega)$. As observed in 1.3.5, we get something isometric if, instead of exploring $\tau_J(\mathcal{K} \times S_0)$, we fix J and we move along the flow of ω with respect to the vector field $-X_{JX_f}$. A direct compurtation shows that the derivative of ω along this flow is

$$-\mathcal{L}_{JX_f}\omega = 2\sqrt{-1}\partial\bar{\partial} f,$$

hence we are moving inside the Kähler class of ω.

2 dHYM equation with variable Kähler metric

Chapter contents

In this Chapter we introduce and study the dHYM equation with a variable Kähler metric, involving at the same time both the Lagrangian phase and the radius function defined in (1.2.4). The basic idea is that the radius function should be used to couple the dHYM metric h to a variable Kähler metric ω. The equations we propose are

$$\begin{cases} \Theta_{\omega}(h) = \hat{\theta} \mod 2\pi \\ s(\omega) - \alpha r_{\omega}(h) = \hat{s} - \alpha \hat{r}, \end{cases} \tag{2.0.1}$$

where $s(\omega)$, $\hat{s}$, $\hat{r}$ denote the scalar curvature and its average, respectively the average radius, and $\alpha > 0$ is an arbitrary coupling constant. The quantities $\hat{s}$, $\hat{r}$ are fixed by cohomology, and in particular

$$\hat{r} = \frac{1}{n! \operatorname{Vol}(M, \omega)} \left| \int_X (\omega - F(h))^n \right|.$$

Our coupled equations (2.0.1) are natural as they are obtained by lifting the moment map interpretation of dHYM connections, due to Collins-Yau [CY18] and mirror to Thomas' moment map for special Lagrangians [Tho01], to the extended gauge group of bundle automorphisms covering Hamiltonian symplectomorphisms (see Theorem 2.1.1). The well-known Donaldson-Fujiki picture of scalar curvature as a moment map then allows coupling to the underlying Kähler metric, through the scalar (or, in special cases, the Ricci) curvature (see Corollary 2.1.2 and Section 2.2). The resulting moment map partial differential equations (2.0.1) describe special pairs formed by a holomorphic line bundle L, regarded as a B-model object, and a Kähler class, and it is natural to expect that these should satisfy a mixture of Bridgeland-type stability (as in [CY18], see e.g. the inequality (1.2.5)), and K-stability (see e. g. [Don97]).

In special limits, or in special cases, we recover the very interesting systems introduced by Álvarez-Cónsul, Garcia-Fernandez and García-Prada [AGG13] (in the large radius limit, see Proposition 2.3.1) and Hultgren-Witt Nyström [HWN19] (on complex surfaces, see Section 2.2 and in particular Corollary 2.2.1).

Remark 2.0.1. When $\dim_{\mathbb{C}} M = 1$, the coupled system (2.0.1) is equivalent to the cscK equation and the existence and uniqueness of its solutions are a consequence of the Uniformization theorem. Fixing a Kähler structure ω on M, it is straightforward to check that each line bundle L, with first Chern class $c_1(L) = k[\omega] \in H^2(M, \mathbb{Z})$, admits a dHYM connection with curvature form $F = 2\pi k\omega$. Moreover, the second equation in (2.0.1) is equivalent to $s(\omega) = \hat{s}$, without any dependence on the coupling constant α or the line bundle L.

Remark 2.0.2. One can allow a general class in $H^{1,1}(M, \mathbb{R})$ in the dHYM equation (1.2.1), not necessarily the first Chern class of a line bundle. At least in the absence of holomorphic 2-forms, this can be interpreted as allowing a nontrivial B-field, as discussed in [CY18] Section 8. Thus our coupled equations (2.0.1) also admit a different (although closely related) interpretation: we may replace $[\sqrt{-1}F(h)]$ with a minimal lift of a B-field class $[B] \in H^2(M, \mathbb{R})/H^2(M, \mathbb{Z})$ (which exists under suitable assumptions), and regard the equations as trying to prescribe a *canonical representative* $\omega + \sqrt{-1}B$ of a complefixied Kähler class $[\omega] + \sqrt{-1}[B]$, much as the cscK equation $s(\omega) = \hat{s}$ tries to find a canonical representative for $[\omega]$. Note that in the Calabi-Yau case, at zero coupling $\alpha = 0$ and in the large radius limit discussed in Proposition 2.3.1, these equations for $\omega + \sqrt{-1}B$ reduce to the conditions

$$\begin{cases} \Delta_\omega B = 0 \\ \mathrm{Ric}(\omega) = 0, \end{cases}$$

which are standard in the physics literature (see e.g. [ABC+09] Section 1.1).

In the following, we will study the coupled equations (2.0.1) from the moment map point of view, establishing several general results and analysing concrete cases in more detail in the next Chapters.

2.1 Main results

Thomas [Tho01] gave a moment map interpretation of the special Lagrangian equation on submanifolds of $\check{M}$. Mirror to this, there is a moment map description of the dHYM equation (1.2.1), due to Collins and Yau, which in fact is intrinsic to M. As we have explained in details in Section 1.2.2, one considers for a fixed metric h the space $\mathcal{A}^{1,1}$ of h-unitary integrable connections on L, endowed with the natural action of the gauge group $\mathcal{G}$ of unitary bundle automorphisms of L covering the identity on M, and the (nonstandard, nonlinear, possibly degenerate[1]) symplectic form given at $A \in \mathcal{A}^{1,1}$ by

$$\Omega^{\mathrm{dHYM}}_A(a,b) = -\int_X a \wedge b \wedge \mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^{n-1}\right),$$

with $a, b \in T_A\mathcal{A}^{1,1} \subset \mathcal{A}^1(X, \sqrt{-1}\mathbb{R})$. According to [CY18] Section 2 the action of $\mathcal{G}$ on $\mathcal{A}^{1,1}$ is Hamiltonian, with equivariant moment map at $A \in \mathcal{A}^{1,1}$, evaluated on $\varphi \in \mathrm{Lie}(\mathcal{G})$, given by

$$\langle \mu_{\mathcal{G}}(A), \varphi \rangle = \frac{\sqrt{-1}}{n}\int_X \varphi \,\mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^{n}\right). \tag{2.1.1}$$

A standard argument then shows that the dHYM equation (1.2.1) becomes precisely the problem of finding zeroes of the moment map $\mu_{\mathcal{G}}$ inside the orbits of the complexified gauge group $\mathcal{G}^{\mathbb{C}}$.

2.1.1 Lift to the extended gauge group

Fix a Hermitian metric h on L as above. The *extended gauge group* $\widetilde{\mathcal{G}}$ of L consists of unitary bundle automorphisms of (L, h) which cover a Hamiltonian symplectomorphism of M, with respect to the fixed symplectic (in fact Kähler) form ω. It can be shown that $\widetilde{\mathcal{G}}$ fits into an exact sequence of infinite-dimensional Lie groups

$$1 \to \mathcal{G} \xrightarrow{\iota} \widetilde{\mathcal{G}} \xrightarrow{p} \mathcal{H} \to 1, \tag{2.1.2}$$

where $\mathcal{H}$ denotes the group of Hamiltonian symplectomorphisms of (M, ω).

As observed by Álvarez-Cónsul, Garcia-Fernandez and García-Prada [AGG13], there is a natural action of $\widetilde{\mathcal{G}}$ on the space of all unitary connections $\mathcal{A}$, given by thinking of a connection A as a projection operator θ_A on the vertical bundle,

$$\theta_A : TL \to VL, \quad g \cdot \theta_A = g_* \circ \theta_A \circ (g_*)^{-1}. \tag{2.1.3}$$

The resulting action was studied in detail in [AGG13], in the much more general context of arbitrary principal bundles for a compact real Lie group. The main application considered in [AGG13] concerns the case when the space of connections is endowed with the standard, linear Atiyah-Bott symplectic form, which for a line bundle is

$$\Omega^{\mathrm{AB}}_A(a,b) = -\int_X a \wedge b \wedge \omega^{n-1}.$$

However we can use this general setup to obtain a result for the symplectic form Ω^{dHYM}.

[1] According to [CY18] Section 2, it is nondegenerate at least in an open neighbourhood of a solution to the dHYM equation.

Theorem 2.1.1. *The action of the extended gauge group $\widetilde{\mathcal{G}}$ on the space of all unitary connections $\mathcal{A}$, endowed with obvious extension of the symplectic form Ω^{dHYM}, is Hamiltonian, with equivariant moment map at $A \in \mathcal{A}$, evaluated on $\zeta \in \mathrm{Lie}(\widetilde{\mathcal{G}})$, given by*

$$\langle \mu_{\widetilde{\mathcal{G}}}(A), \zeta \rangle = \langle \mu_{\mathcal{G}}(A), \theta_A(\zeta) \rangle + \frac{1}{n} \int_X p_*(\zeta)\, \mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}} (\omega - F(A))^n \right),$$

where $\mu_{\mathcal{G}}$ is defined as in (2.1.1) and p is the projection appearing in (2.1.2).

Notice that we are identifying $\mathfrak{ham}$ with the space $C^\infty_0(X, \omega)$ of Hamiltonian functions with zero average. Theorem 2.1.1 is proved in Section 2.4. Its analogue for the standard symplectic form Ω^{AB} is (a special case of) [AGG13], Proposition 1.6: we will return to this below.

2.1.2 Coupling to a variable metric

It was shown by Jacob and Yau [JY17] that solutions of the dHYM equation are unique. It follows that the naive problem of looking for zeroes of the extended moment map $\mu_{\widetilde{\mathcal{G}}}$ is overdetermined.

The fundamental work of Donaldson [Don97] and Fujiki [Fuj90] on scalar curvature as a moment map suggests that the right thing to do instead is to let the extended gauge group act on a larger space. Let $\mathcal{A}$ be the space of all unitary connections on (L, h), as above, and let $\mathcal{J}$ be the space of ω-compatible almost complex structures on M. We endow $\mathcal{A}$ with the symplectic form given by the obvious extension of Ω^{dHYM} and $\mathcal{J}$ with the Donaldson-Fujiki form Ω^{DF}. We consider the induced action of the extended gauge group $\widetilde{\mathcal{G}}$ on the product $\mathcal{A} \times \mathcal{J}$, which on the second factor is given by

$$g \cdot J = p(g)_* \circ J \circ p(g)_*^{-1}.$$

This preserves the space $\mathcal{P} \subset \mathcal{A} \times \mathcal{J}$ consisting of pairs (A, J) of a unitary connection A and an integrable complex structure J, such that A is integrable with respect to J. We denote by $s(J)$ the scalar curvature of the metric determined by ω and J, and let α be a real *positive* "coupling constant".

Corollary 2.1.2. *The action of $\widetilde{\mathcal{G}}$ on $\mathcal{A} \times \mathcal{J}$, endowed with the symplectic form*

$$\Omega_\alpha = n\alpha \Omega^{\mathrm{dHYM}} + \Omega^{\mathrm{DF}},$$

for $\alpha > 0$, is Hamiltonian, with equivariant moment map at (A, J), acting on $\zeta \in \mathrm{Lie}(\widetilde{\mathcal{G}})$, given by

$$\langle \mu_\alpha(A, J), \zeta \rangle = -\int_X p_*(\zeta) s(J) \frac{\omega^n}{n!} + n\alpha \langle \mu_{\widetilde{\mathcal{G}}}(A), \zeta \rangle.$$

The proof is given in Section 2.4. The analogue of this for the Atiyah-Bott symplectic form is [AGG13], Proposition 2.1.

For each $A \in \mathcal{A}$, the operator θ_A induces an equivariant *vector space* splitting

$$\mathrm{Lie}(\widetilde{\mathcal{G}}) = \mathrm{Lie}(\mathcal{G}) \bigoplus_{\theta_A} \mathrm{Lie}(\mathcal{H}).$$

We may consider the general problem of finding a pair $(A, J) \in \mathcal{P}$ such that, in the above splitting with respect to θ_A, the moment map μ_α vanishes on $\mathrm{Lie}(\mathcal{G})$ and acts as some prescribed element $-f \in C^\infty_0(M, \omega) \cong \mathrm{Lie}(\mathcal{H})^*$ on Hamiltonian vector fields:

$$\langle \mu_\alpha(A, J), \zeta \rangle = \langle \mu_\alpha(A, J), \zeta' \oplus_{\theta_A} \zeta'' \rangle = -\int_X \zeta'' f \frac{\omega^n}{n!}.$$

This is equivalent to the equations

$$\begin{cases} \mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right) = 0 \\ s(J) - \alpha \frac{\mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right)}{\omega^n} = f + c, \end{cases}$$

where the only possible constant c is determined once the coupling α is chosen, depending only on $[\omega]$ and $c_1(L)$. Note that the problem of finding a zero of the moment map μ_α corresponds to the choice $f \equiv 0$.

Formally complexifying the action of $\widetilde{\mathcal{G}}$, following Donaldson [Don97], we keep the complex structures on L and M fixed and vary the Hermitian metric h on L and the Kähler form ω in its Kähler class instead. Thus we arrive at the *dHYM equation coupled to a variable Kähler metric*

$$\begin{cases} \mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(h))^n\right) = 0 \\ s(\omega) - \alpha \frac{\mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(h))^n\right)}{\omega^n} = f, \end{cases} \tag{2.1.4}$$

where we absorbed the constant c in the datum f.

These equations are the main object of study in the present thesis. Important motivation for this study comes from the fact that, when the datum f is constant, so that we are looking for zeroes of the moment map, it is natural to expect that solutions should satisfy a mixture of Bridgeland-type stability and K-stability. Note that it is straightforward to rewrite (2.1.4) in terms of the Lagrangian phase operator and the radius function, as we did in (2.0.1).

Remark 2.1.3. Given the origin of the dHYM equation, it is natural to ask whether a given solution of the coupled equations (2.1.4) can be effectively realised in the B-model, that is if the pair (M, ω) underlying a solution can be embedded holomorphically and isometrically in a Calabi-Yau $\mathcal{M}$. Note that, at the infinitesimal level, this is always possible: for example, by the results of [Kal99], we may embed (M, ω) isometrically as the zero section of the holomorphic cotangent bundle T^*M, endowed with a hyperkähler metric defined in a *formal* neighbourhood of the zero section.

Remark 2.1.4. As we mentioned in the Remark 2.0.2, the dHYM equation makes sense and plays a role in mirror symmetry even when we replace $c_1(L)$ with some arbitrary class $[F]$ of type $(1,1)$, not necessarily rational. The same holds for the coupled equations (2.1.4). When $[F]$ is rational, we can obtain solutions in a class $c_1(L)$ from solutions in $[F]$ by rescaling F and ω appropriately. We will often use this fact, sometimes without further comment.

2.1.3 Futaki invariant

We find a first obstruction to the existence of solutions of the coupled equations (2.1.4), which generalises the classical Futaki character [Fut83], following closely the approach of [AGG13]. Fix a complex line bundle $L \to M$ and the associated principal $GL(1,\mathbb{C})$-bundle $\mathcal{L} \to M$. Let $\mathcal{J}_{\mathcal{L}}$ be the space of holomorphic structures on $\mathcal{L}$, namely the integrable $GL(1,\mathbb{C})$-equivariant almost complex structures on $\mathcal{L}$, acting as multiplication by $\sqrt{-1}$ on the vertical bundle $V\mathcal{L} \cong \mathcal{L}\times\mathfrak{gl}(1,\mathbb{C})$. An element $\check{J} \in \mathcal{J}_{\mathcal{L}}$ determines uniquely a complex structure J on M and an holomorphic structure on L. Let $\mathrm{Aut}(\mathcal{L},\check{J})$ denote the group of automorphisms g of the holomorphic principal bundle $(\mathcal{L},\check{J})$, covering an automorphism g of the complex manifold (M,J). Each $\check{\zeta} \in \mathrm{Lie}\,\mathrm{Aut}(\mathcal{L},\check{J})$ covers a (unique) real holomorphic vector field ζ on (M,J). For any symplectic 2-form ω on M, which is J-compatible, we have the Hodge-type decomposition

$$\zeta = \eta_{\phi_1} + J\eta_{\phi_2} + \beta,$$

where η_{ϕ_i} denotes the Hamiltonian vector field of $\phi_i \in C_0^\infty(M,\omega)$, while β is the Riemannian dual of an harmonic 1-form, with respect to the metric $\omega(\cdot, J\cdot)$ (see [LS94]). Fixing also a Hermitian metric h over the line bundle L, we define a $\mathbb{C}$-linear map

$$\mathcal{F}_I : \mathrm{Lie}\,\mathrm{Aut}(\mathcal{L},J) \longrightarrow \mathbb{C}$$

given by

$$\begin{aligned}\mathcal{F}_I(\check{\zeta}) =& \alpha\sqrt{-1}\int_X \theta_h\check{\zeta}\,\mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(h))^n\right)\\ &- \int_X \phi\left(s(\omega)\omega^n - \alpha\,\mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(h))^n\right)\right),\end{aligned}$$

where $\phi = \phi_1 + \sqrt{-1}\phi_2$. One can show that $\mathcal{F}_I$ is a character of $\mathrm{Lie}\,\mathrm{Aut}(\mathcal{L},I)$ and does not depend on the choice of ω and h. The proof is essentially the same as in [AGG13], Section 3, up to replacing Ω^{AB} with Ω^{dHYM}. Then clearly $\mathcal{F}_I$ must vanish identically if the coupled equations (2.1.4) have a solution.

2.2 Reduction to Ricci curvature

We describe a special case in which the dependence on the scalar curvature in the equations (2.1.4) can be reduced to the Ricci curvature. To see this we note that by using the dHYM equation in order to eliminate the top power $(F(h))^n$ we can always write

$$\frac{\mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(h))^n\right)}{\omega^n} = \sum_{i=0}^{n-1} \lambda_i(\hat{\theta})\Lambda^i_\omega(F(h))^i$$

for unique coefficients $\lambda_i(\hat{\theta})$. It follows that we may rewrite the equation in (2.1.4) involving the scalar curvature as

$$\Lambda_\omega\left(\mathrm{Ric}(\omega) - \alpha\lambda_1(\hat{\theta})F(h)\right) - \alpha\sum_{i=2}^{n-1}\lambda_i(\hat{\theta})\Lambda^i_\omega(F(h))^i = c$$

for a unique constant c. Now we can uniquely solve

$$\sqrt{-1}\Lambda_\omega\partial\bar{\partial}h = \Delta_\omega h = \sum_{i=2}^{n-1}\lambda_i(\hat{\theta})\Lambda^i_\omega F^i(h) - \sum_{i=2}^{n-1}\lambda_i(\hat{\theta})\int_X \Lambda^i_\omega(F(h))^i\frac{\omega^n}{n!}$$

with the normalisation $\int_M h\omega^n = 0$. Thus we may rewrite our equation as

$$\mathrm{Ric}(\omega) - \alpha\lambda_1(\hat{\theta})F(h) - \alpha\partial\bar{\partial}h = \lambda\omega$$

for a unique λ, provided the cohomological condition

$$c_1(M) = \lambda[\omega] + \alpha\lambda_1(\hat{\theta})[F(h)] \tag{2.2.1}$$

is satisfied.

2.2.1 The case of complex surfaces

In the special case of complex surfaces the computation above amounts to expressing $(F(h))^2$ in terms of ω^2 and $F(h)\wedge\omega$ by using the dHYM equation. Moreover it is well known that the dHYM equation on surfaces reduces to a complex Monge-Ampère equation (see e.g. the proof of Proposition 3.1.1 below). Thus under the condition (2.2.1)

$$c_1(X) + \frac{\alpha}{\sin\hat{\theta}}[F] = \lambda[\omega], \tag{2.2.2}$$

the coupled equations (2.1.4) on surfaces become the system of complex Monge-Ampère equations

$$\begin{cases}\left(\sqrt{-1}\sin(\hat{\theta})F(h) + \cos(\hat{\theta})\omega\right)^2 = \omega^2\\ \mathrm{Ric}(\omega) = \lambda\omega + \frac{\alpha}{\sin(\hat{\theta})}\sqrt{-1}F(h).\end{cases} \tag{2.2.3}$$

With the assumption

$$\sqrt{-1}\sin(\hat{\theta})F(h) + \cos(\hat{\theta})\omega > 0$$

and provided the equalities

$$\lambda = 1 + \cos(\hat{\theta}),\ \alpha = \sin^2(\hat{\theta})$$

hold, the system (2.2.3) is given precisely by the coupled Kähler-Einstein equations in the sense of Hultgren-Witt Nyström [HWN19]

$$\begin{cases} \mathrm{Ric}\left(\sqrt{-1}\sin(\hat{\theta})F(h) + \cos(\hat{\theta})\omega\right) = \mathrm{Ric}(\omega) \\ \mathrm{Ric}(\omega) = \left(\sqrt{-1}\sin(\hat{\theta})F(h) + \cos(\hat{\theta})\omega\right) + \omega, \end{cases} \tag{2.2.4}$$

of Fano type.

Hultgren-Witt Nyström ([HWN19] Theorem 1.7) showed that the Kähler-Einstein coupled equations are solvable on a Kähler-Einstein Fano manifold with discrete automorphisms, provided the corresponding decomposition of $c_1(M) > 0$ is sufficiently close, in the Kähler cone, to a "parallel" decomposition of the form $c_1(M) = \sum_i (\lambda_i c_1(M))$, $\lambda_i > 0$. By the discussion above, this implies immediately the following existence result for our coupled equations (2.1.4). It is convenient to set $\chi = \sqrt{-1}\sin(\hat{\theta})F(h) + \cos(\hat{\theta})\omega$.

Corollary 2.2.1. *Suppose that M is a del Pezzo surface with discrete $\mathrm{Aut}(M, J)$ and the conditions $[\chi] > 0$, $c_1(M) = [\omega] + [\chi]$ are satisfied. Assume moreover that the classes $[\omega]$, $[\chi]$ are sufficiently close, in the Kähler cone, to (positive) multiples of $c_1(M)$. Then there is a solution to our coupled equations (2.1.4) in the classes $[F(h)]$, $[\omega]$, with coupling constant $\alpha = \sin^2(\hat{\theta})$.*

As the phase $e^{\sqrt{-1}\hat{\theta}}$ depends on $[F(h)]$, $[\omega]$ (through explicit formulae which we give below in (3.1.1)), the conditions appearing in this Corollary are nonlinear constraints in these cohomology classes. To obtain examples in which they are satisfied we consider the choices

$$[\omega] = \frac{1}{2}c_1(M) + t\eta,$$
$$\sqrt{-1}[F(h)] = \frac{1}{2}c_1(M) - t\eta$$

where η is a fixed class satisfying $\int_M c_1(M) \cup \eta = 0$ and the real parameter t is sufficiently small. Then $[\omega]$, $\sqrt{-1}[F(h)]$ are positive and by (3.1.1) we have $\cos(\hat{\theta}) = 0$, $\sin(\hat{\theta}) = 1$. Since clearly $[\omega] + \sqrt{-1}[F(h)] = c_1(M)$, we do obtain solutions to the coupled equations (2.1.4) in these classes, for all sufficiently small t.

Duality We have seen that in special cases our coupled equations on surfaces reduce to the coupled Kähler-Einstein equations (2.2.4), that is, setting

$$\chi = \sqrt{-1}\sin(\hat{\theta})F(h) + \cos(\hat{\theta})\omega,$$

we obtain the equations

$$\begin{cases} \mathrm{Ric}(\chi) = \mathrm{Ric}(\omega) \\ \mathrm{Ric}(\omega) = \chi + \omega. \end{cases}$$

We observe that these are now symmetric in χ, ω, so that the term χ involving the dHYM connection curvature $F(h)$ is interchanged with the variable Kähler form ω. It could be interesting to interpret this duality in terms of the origin of the dHYM equation in the B-model.

Cohomological conditions for surfaces In the following, we consider a compact Kähler surface M and $[F] \in H^2(M, \mathbb{Z}) \cap H^{1,1}(M, \mathbb{R})$ will denote an integral class, to be identified with the Chern class of some holomorphic line bundle over M. We will also assume that the coupling constant α appearing in the coupled equation is positive.

The topological angle $\hat{\theta}$ appearing in the dHYM equation is defined by

$$e^{\sqrt{-1}\hat{\theta}} K = \int_M (\omega - iF)^2$$

with $K > 0$. Consequently,

$$K(\omega, F) = \sqrt{\left(\int_M \omega^2 - F^2\right)^2 + 4\left(\int_M \omega \wedge F\right)^2},$$

$$\cos \hat{\theta} = \frac{\int_M \omega^2 - F^2}{K(\omega, F)},$$

and

$$\sin \hat{\theta} = \frac{-2 \int_M \omega \wedge F}{K(\omega, F)}.$$

Lemma 2.2.2. *For any choice of* $[\omega]$ *and* $[F]$*, there is* $\bar{t} > 0$ *such that the dHYM equation admits a solution in the classes* $[t\omega]$ *and* $[F]$*, for* $t > \bar{t}$*.*

Proof. Under re-scaling of the Kähler class, for $\omega_t = t\omega$ with $t > 0$, it is immediate to derive that $\cos \hat{\theta}_t \to 1$ and $\sin \hat{\theta}_t \to 0$, for $t >> 1$. Consequently, for every choice of $[\omega]$ and $[F]$, up to a suitable re-scaling of the Kähler class $[\omega]$, we can assume that the stability condition for the dHYM equation

$$\cos \hat{\theta}[\omega] - \sin \hat{\theta}[F] > 0 \tag{2.2.5}$$

is satisfied and by Theorem 1.2.8 the dHYM equation admits a solution. □

Consider now the condition

$$c_1(M) + \frac{\alpha}{\sin \hat{\theta}}[F] = \lambda[\omega], \tag{2.2.6}$$

which is the cohomological constraint for to the second equation in 2.2.3. Consider any triple $([\omega],[F],\alpha)$ which is a solution of (2.2.6); setting

$$\alpha_t = t\alpha\frac{K(\omega,F)}{K(t\omega,F)},$$

then the triple $(t[\omega],[F],\alpha_t)$ satisfies (2.2.6) with $\lambda_t = \lambda/t$ and for any $t > 0$. Hence, up to a suitable re-scaling of ω and α, we can assume that any solution of (2.2.6) satisfies also the stability condition (2.2.5).

In order to ease the notation, we assume to normalize (2.2.6) setting $\lambda = \pm 1$, depending on the sign of

$$\int_X \mathrm{Ric}(\omega)\wedge\omega - \frac{\alpha K(\omega,F)}{2}.$$

Lemma 2.2.3. *Let M be a Kähler surface, such that $([\omega],[F],\alpha)$ is a solution of (2.2.6) with $\alpha' = \alpha/\sin\hat{\theta} \in \mathbb{Q}$. Then there exist a solution of (2.2.6) $([\widetilde{\omega}],[\widetilde{F}],\widetilde{\alpha})$ on $\widetilde{M} = Bl_p(M)$.*

Proof. We claim that any solution of (2.2.6) on M will induce a solution on $\widetilde{M} = Bl_p(M)$, provided that the rationality condition $\alpha' = \alpha/\sin\hat{\theta} \in \mathbb{Q}$ is satisfied. Let $p : \widetilde{M} \to M$ denote the projection and consider

$$\begin{aligned} c_1(\widetilde{M}) &= p^*c_1(M) - E,\\ [\widetilde{\omega}] &= p^*[\omega] - \epsilon E,\\ [\widetilde{F}] &= p^*[F] - zE, \end{aligned}$$

where E is the exceptional divisor and ϵ is the size of the blow-up; we will assume that also $\epsilon \in \mathbb{Q}$. Then equation (2.2.6) for $[\widetilde{\omega}]$ and $[\widetilde{F}]$ becomes

$$(1+\alpha' z \pm \epsilon)E - p^*(c_1(M) + \alpha' F \pm \omega) = (1+\alpha' z \pm \epsilon)E = 0,$$

which is satisfied if

$$z = -\frac{1\pm\epsilon}{\alpha'}.$$

Notice that the coupling constant for the equation (2.2.6) on $\widetilde{M}$ is given by

$$\widetilde{\alpha} = \alpha\left(\sin\hat{\hat{\theta}}/\sin\hat{\theta}\right),$$

which we can assume to be positive, provided that ϵ is small enough. In general $z \in \mathbb{Q}$ and consequently $[\widetilde{F}] \in H^2(\widetilde{M},\mathbb{Q})$; in order to produce an actual solution we have consider $[\widetilde{F}'] = [N\widetilde{F}] \in H^2(\widetilde{M},\mathbb{Z})$ for some $N \in \mathbb{N}$, re-scaling also the coupling constant as

$$\alpha_N = \widetilde{\alpha}\frac{K(\omega_Y,F_Y)}{K(\omega_Y,NF_Y)}.$$

□

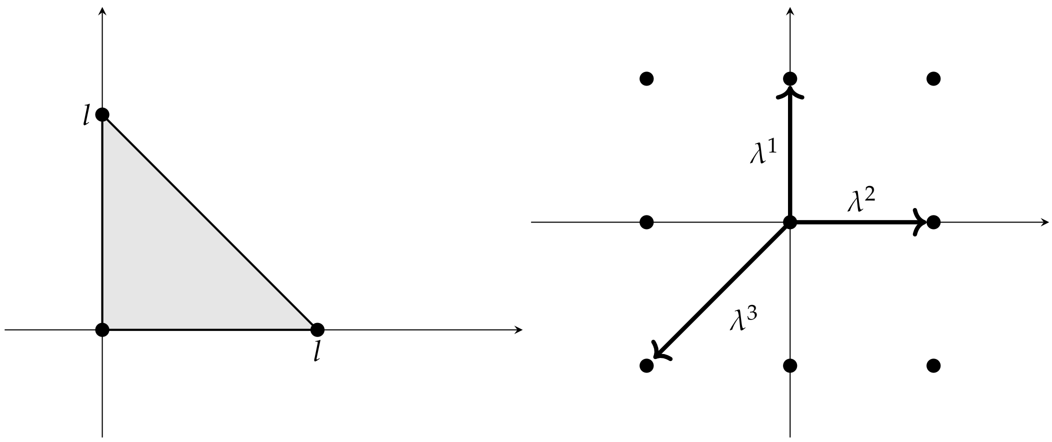

Figure 2.1: The Delzant polytope and the associated fan of $\mathbb{CP}^2$

Example: $\mathbb{CP}^2$ In the following, we will consider the equation (2.2.6) for $\mathbb{CP}^2$, proving that each triple $([\omega], [F], \alpha)$ is a solution. From Lemma (2.2.3), it follows that we can produce infinitely many solutions on the blow-ups of $\mathbb{CP}^2$, satisfying the rationality conditions on α'.

Since $\mathbb{CP}^2$ is a toric surface, we can associate to each Kähler class $[\omega]$ the corresponding Delzant polytope Δ_ω, i.e. the convex hull of the set $\{(0,0),(l,0),(0,l)\}$, for some $l > 0$. To describe $H^2(\mathbb{CP}^2, \mathbb{R})$, we consider the polynomial ring $\mathbb{R}[v_1, v_2, v_3]$, associating each variable to a vertex of the polytope; moreover, let λ^i_k denotes the i-th component of the primitive vector normal to the k-th edge of Δ_ω, with $i = 1, 2$ and $k = 1, 2, 3$. Then

$$H^2(\mathbb{CP}^2, \mathbb{R}) = \frac{\mathbb{R}[v_1, v_2, v_3]}{(\lambda^1_k v_k), (\lambda^2_{k'} v_{k'})} = \frac{\mathbb{R}[v_1, v_2, v_3]}{(v_1 - v_3), (v_2 - v_3)} \simeq \mathbb{R}, \tag{2.2.7}$$

with $H^2(\mathbb{CP}^2, \mathbb{Z})$ corresponding to $\mathbb{Z}$, the Kähler cone to $\mathbb{R}_+$ and the intersection form to the standard scalar product. For an explicit correspondence, we can use the general formula

$$\text{Area}(\Delta_\omega) = \frac{1}{(2\pi)^2}\text{Vol}(X, \omega),$$

so, for example, $\omega_{FS} = 4\pi^3$ and $l_{FS} = 1/2$.

Using (2.2.7), we can rephrase (2.2.6) as the polynomial equation

$$1 + \frac{\alpha}{\sin\hat{\theta}} F_k = \lambda\omega_k + a\lambda^1_k + b\lambda^2_k,$$

for any $a, b \in \mathbb{R}$. Identifying $H^2(\mathbb{CP}^2, \mathbb{R})$ with $\mathbb{R}$ as in (2.2.7), we may assume $F_k = \omega_k = 0$ for $k = 1, 2$, so $a = b = 1$. Setting

$$\widetilde{\alpha} = \frac{\alpha K(\omega, F)}{2},$$

the topological constant λ is given by

$$\lambda = \frac{3\omega_3 - \widetilde{\alpha}}{\omega_3^2}$$

and (2.2.6) becomes the trivial equality

$$1 - \frac{\widetilde{\alpha}}{F_3\omega_3}F_3 = \frac{3\omega_3 - \widetilde{\alpha}}{\omega_3^2}\omega_3 - 2.$$

2.3 Large and small radius limits

Large radius limit Let us consider the family of Kähler forms

$$\omega_t = t\omega,\ t \in \mathbb{R}_{>0}.$$

The *large radius limit*, roughly mirror to a large complex structure limit on $\check{M}$, refers to the leading behaviour of the moment maps $\mu_{\widetilde{\mathcal{G}}}$, μ_α, computed with respect to ω_t, as $t \to \infty$.

Proposition 2.3.1. *Let $F(h) = \sqrt{-1}F$, $c = \frac{n[\omega]^{n-1}\cup[F]}{[\omega]^n}$. As $t \to \infty$, there is an expansion*

$$\begin{aligned}
&\langle \mu_{\widetilde{\mathcal{G}}}(A), \zeta\rangle \\
&= \frac{\sqrt{-1}}{n}\int_X \theta_A(\zeta)\left(\left(-n\omega^{n-1}\wedge F + c\omega^n\right)t^{n-1} + O(t^{n-3})\right) \\
&+ \frac{1}{n}\int_X p_*(\zeta) \\
&\left(\omega^n t^n - \left(\frac{n(n-1)}{2}\omega^{n-2}\wedge F\wedge F - cn\omega^{n-1}\wedge F + \frac{1}{2}c^2\omega^n\right)t^{n-2} + O(t^{n-4})\right).
\end{aligned}$$

As a consequence, the Kähler-Yang-Mills coupled equations introduced in [AGG13], in the particular case of line bundles, arise as the large radius limit of the coupled equations (2.1.4).

Indeed, up to higher order terms as $t \to \infty$, the system (2.1.4), becomes in the large radius limit

$$\begin{cases} n\omega^{n-1}\wedge F = c\omega^n \\ s(\omega_t) - \alpha\dfrac{\omega^n t^n - \left(\frac{n(n-1)}{2}\omega^{n-2}\wedge F\wedge F - cn\omega^{n-1}\wedge F + \frac{1}{2}c^2\omega^n\right)t^{n-2}}{(\omega_t)^n} = f, \end{cases}$$

or equivalently

$$\begin{cases} \Lambda_\omega F = c \\ t^{-1}s(\omega) - \alpha\left(1 - t^{-2}\Lambda_\omega^2(F\wedge F) - \frac{1}{2}t^{-2}c^2\right) = f. \end{cases}$$

Thus choosing the appropriate scaling behaviour for the coupling constant and datum,

$$\alpha = \alpha' t, \quad f = -\alpha' t + \frac{\tilde{f}}{t}$$

we arrive at the equations

$$\begin{cases} \Lambda_\omega F = c \\ s(\omega) + \alpha' \left(\Lambda^2_\omega (F \wedge F) + \frac{1}{2}c^2\right) = \tilde{f}. \end{cases} \tag{2.3.1}$$

When $\tilde{f}$ is a (topologically fixed) constant, these are precisely the *coupled Kähler-Yang-Mills equations* studied in [AGG13], in the particular case of a holomorphic line bundle.

Proof. We prove now Proposition 2.3.1. As usual, it is convenient to write $F(h) = \sqrt{-1}F$ and set $z = \int_X (t\omega - \sqrt{-1}F)^n$. Identifying top classes with their integrals, we may expand as $t \to \infty$

$$z = \left(t^n[\omega]^n - t^{n-2}\frac{n(n-1)}{2}[\omega]^{n-2} \cup [F] \cup [F]\right) - \sqrt{-1}t^{n-1}n[\omega]^{n-1} \cup [F] + O(t^{n-3}).$$

By definition, we have

$$\begin{aligned} e^{-\sqrt{-1}\hat{\theta}} &= \frac{\bar{z}}{|z|} \\ &= \left(1 - \frac{(n[\omega]^{n-1} \cup [F])^2}{2([\omega]^n)^2}\frac{1}{t^2}\right) + \sqrt{-1}\frac{n[\omega]^{n-1} \cup [F]}{[\omega]^n}\frac{1}{t} + O\left(\frac{1}{t^3}\right). \end{aligned}$$

It follows that we have

$$\operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(t\omega - \sqrt{-1}F)^n\right) = n\left(-\omega^{n-1} \wedge F + \frac{[\omega]^{n-1} \cup [F]}{[\omega]^n}\omega^n\right)t^{n-1} + O(t^{n-3}),$$

and similarly

$$\begin{aligned} &\operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(t\omega - \sqrt{-1}F)^n\right) \\ &= \omega^n t^n \\ &- \left(\frac{n(n-1)}{2}\omega^{n-2} \wedge F \wedge F - \frac{n[\omega]^{n-1} \cup [F]}{[\omega]^n}n\omega^{n-1} \wedge F + \frac{(n[\omega]^{n-1} \cup [F])^2}{2([\omega]^n)^2}\omega^n\right)t^{n-2} \\ &+ O(t^{n-4}). \end{aligned}$$

Now Proposition 2.3.1 follows at once from the definition of $\mu_{\tilde{\mathcal{G}}}$. □

Small radius limit The *small radius limit* refers to the leading behaviour of the moment maps $\mu_{\widetilde{\mathcal{G}}}, \mu_\alpha$, computed with respect to ω_t, as $t \to 0$.

Proposition 2.3.2. *Let $F(h) = \sqrt{-1}F$, $c = \frac{n[\omega]\cup[F]^{n-1}}{[F]^n}$. As $t \to 0$, there is an expansion*

$$\begin{aligned}\langle \mu_{\widetilde{\mathcal{G}}}(A), \zeta\rangle = \frac{\sqrt{-1}}{n} \int_X \theta_A(\zeta) \left(\left(n\omega \wedge F^{n-1} - cF^n\right) t + O(t^2)\right) \\ + \frac{1}{n}\int_X p_*(\zeta) \left(F^n + O(t^3)\right).\end{aligned}$$

Thus, up to higher order terms as $t \to 0$, the system (2.1.4), becomes in the small radius limit

$$\begin{cases} n\omega \wedge F^{n-1} = cF^n \\ s(\omega_t) - \alpha\frac{F^n}{(\omega_t)^n} = f, \end{cases}$$

or equivalently, provided F is a Kähler form,

$$\begin{cases} \Lambda_F \omega = c \\ t^{-1}s(\omega) - \alpha\frac{F^n}{\omega^n t^n} = f. \end{cases}$$

With the appropriate scaling behaviour

$$\alpha = t^{n-1}\alpha', \quad f = t^{-1}\tilde{f}$$

we arrive at the system

$$\begin{cases} \Lambda_F \omega = c \\ s(\omega) - \alpha'\frac{F^n}{\omega^n} = \tilde{f}. \end{cases} \tag{2.3.2}$$

This comprises the *J-equation* of Donaldson [Don00] and Chen [Che04], well-known to be a scaling limit of the dHYM equation (see e.g. [CXY17]). However, unlike the large radius limit, it seems that the system (2.3.2) does not appear in the literature, except for the case when M is a complex surface. In that case setting

$$\chi = cF - \omega$$

we may rewrite (2.3.2) as the system

$$\begin{cases} \chi^2 = \omega^2 \\ s(\omega) - \frac{\alpha'}{c^2}\Lambda_\omega \chi = \tilde{f} + \frac{\alpha'}{c^2}. \end{cases}$$

In the particular case when $\tilde{f}$ is a constant this comprises a complex Monge-Ampère equation coupled to a twisted cscK equation, and it is precisely of the type studied by Datar and Pingali [DP20].

Proof. The case of small radius, Proposition 2.3.2, is similar to proof of Proposition 2.3.1. We have, as $t \longrightarrow 0$

$$z = (-\sqrt{-1})^n \left([F]^n + \sqrt{-1}n[\omega] \cup [F]^{n-1} t + O(t^2)\right),$$

so

$$\begin{aligned} e^{-\sqrt{-1}\hat{\theta}} &= \frac{\bar{z}}{|z|} \\ &= (\sqrt{-1})^n \left(1 - \sqrt{-1}\frac{n[\omega] \cup [F]^{n-1}}{[F]^n} t + O(t^2)\right), \end{aligned}$$

and

$$e^{-\sqrt{-1}\hat{\theta}}(t\omega - \sqrt{-1}F)^n = F^n + \sqrt{-1}\left(n\omega \wedge F^{n-1} - \frac{n[\omega] \cup [F]^{n-1}}{[F]^n} F^n\right) t + O(t^2).$$

Proposition 2.3.2 follows immediately. □

2.4 Extended gauge group and scalar curvature

This Section is devoted to the proofs of Theorem 2.1.1 and Corollary 2.1.2. Let M be a compact n-dimensional Kähler manifold, with Kähler form ω, and $L \longrightarrow M$ a complex line bundle with a Hermitian metric h. We consider the space $\mathcal{A}$ of h-unitary connections on L, endowed with the symplectic structure given by Ω^{dHYM}. The $\widetilde{\mathcal{G}}$-equivariant map θ defined as

$$\begin{aligned} \theta \colon \mathcal{A} &\longrightarrow \mathrm{Hom}(\mathrm{Lie}\,\widetilde{\mathcal{G}}, \mathrm{Lie}\,\mathcal{G}) \\ A &\longmapsto \theta_A \end{aligned}$$

associates to each connection A the projection operator θ_A introduced in (2.1.3). We consider also the map $\theta^\perp$ given by

$$\begin{aligned} \theta^\perp \colon \mathcal{A} &\longrightarrow \mathrm{Hom}(\mathrm{Lie}\,\mathcal{H}, \mathrm{Lie}\,\widetilde{\mathcal{G}}) \\ A &\longmapsto \theta_A^\perp \end{aligned}$$

where the *lifting operator* $\theta_A^\perp$ is uniquely defined by $\mathrm{Id} = \iota \circ \theta_A + \theta_A^\perp \circ p$, with ι and p as in (2.1.2). For any $\zeta \in \mathrm{Lie}\,\widetilde{\mathcal{G}}$, Y_ζ denotes the vector field on $\mathcal{A}$ associated to the infinitesimal $\widetilde{\mathcal{G}}$-action on $\mathcal{A}$:

$$Y_{\zeta|A} = \frac{d}{dt}\Big|_{t=0} \exp(t\zeta) \cdot A.$$

In particular we have

$$Y_{\theta^\perp X_\phi|A} = -X_\phi \lrcorner F_A$$

for any $X_\phi \in \operatorname{Lie}\mathcal{H}$ (see e. g. [AGG13], Lemma 1.5). It follows from [AGG13], Proposition 1.3, that the $\widetilde{\mathcal{G}}$-action on $\mathcal{A}$ is Hamiltonian if and only if there is a $\widetilde{\mathcal{G}}$-equivariant map $\sigma\colon \mathcal{A} \to (\operatorname{Lie}\mathcal{H})^*$ satisfying

$$\Omega^{\mathrm{dHYM}}(Y_{\theta^\perp X_\phi|A}, a) = \langle \mu_{\mathcal{G}}, a(X_\phi)\rangle + d\langle \sigma, X_\phi\rangle(a). \tag{2.4.1}$$

We claim that this holds for the equivariant map defined by

$$\langle \sigma(A), X_\phi\rangle = \frac{1}{n}\int_X \phi \operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right),$$

for all $X_\phi \in \operatorname{Lie}\mathcal{H}$.

Since $\sqrt{-1}a \wedge \operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right) = 0$, contracting with X_ϕ we obtain the identity

$$\begin{aligned}
&-\sqrt{-1}na \wedge (X_\phi \lrcorner\, \omega) \wedge \operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^{n-1}\right)\\
&+ na \wedge (X_\phi \lrcorner\, F(A)) \wedge \operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^{n-1}\right)\\
&+ \sqrt{-1}a(X_\phi)\operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right) = 0.
\end{aligned} \tag{2.4.2}$$

On the other hand, by the above identity for the infinitesimal generator, we have

$$\Omega^{\mathrm{dHYM}}(Y_{\theta^\perp X_\phi|A}, a) = -\int_X a \wedge (X_\phi \lrcorner\, F(A)) \wedge \operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^{n-1}\right).$$

Also, by definition,

$$\langle \mu_{\mathcal{G}}, a(X_\phi)\rangle = +\frac{\sqrt{-1}}{n}\int_X a(X_\phi)\operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right),$$

and similarly

$$\begin{aligned}
d\langle \sigma, X_\phi\rangle(a) &= -\frac{1}{n}\int_X \phi \operatorname{Re}\left(n\, da \wedge e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^{n-1}\right)\\
&= -\sqrt{-1}\int_X \phi da \wedge \operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^{n-1}\right)\\
&= -\sqrt{-1}\int_X a \wedge (X_\phi \lrcorner\, \omega) \wedge \operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^{n-1}\right).
\end{aligned}$$

Hence, using the identity (2.4.2), we see that the condition (2.4.1) is satisfied and Theorem 2.1.1 follows.

Now we endow $\mathcal{A} \times \mathcal{J}$ with the infinite dimensional Kähler structure given by the form

$$\Omega_\alpha = \alpha\Omega^{\mathrm{dHYM}} + \Omega^{\mathrm{DF}}$$

with $\alpha > 0$. The $\widetilde{\mathcal{G}}$-action on $\mathcal{A} \times \mathcal{J}$ preserves $\mathcal{P}$, and combining our computation above with the well-known results of Donaldson [Don97] and Fujiki [Fuj90] we see that it is Hamiltonian, with equivariant moment map $\mu_\alpha \colon \mathcal{A} \times \mathcal{J} \to (\mathrm{Lie}\, \widetilde{\mathcal{G}})^*$ given by

$$\begin{aligned}\langle \mu_\alpha(A, J), \zeta \rangle &= -\int_X p_*(\zeta) s(J) \omega^n + \alpha \langle \mu_{\widetilde{\mathcal{G}}}(A), \zeta \rangle \\ &= \sqrt{-1}\alpha \int_X \theta_A \zeta \, \mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right) \\ &\quad - \int_X \phi \left(s(J)\omega^n - \alpha \, \mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right)\right),\end{aligned}$$

for all $(A, J) \in \mathcal{A} \times \mathcal{J}$, $\zeta \in \mathrm{Lie}\, \widetilde{\mathcal{G}}$ and $p_*(\zeta) = X_\phi$.

3 Coupled equations on abelian varieties

Chapter contents

In this Chapter, we focus on the equations and their large/small radius limits on abelian varieties, with a source term, following ideas of Feng and Székelyhidi [FS11]. In particular we prove a priori estimates (see Propositions 3.1.1, 3.1.5) from which we can deduce existence in some cases (see Theorems 3.1.3, 3.1.6, 3.1.7, 3.1.8). Our main results, together with the necessary background, are contained in Section 3.1.

3.1 Background and main results

The equations on abelian varieties After establishing the general results described in Chapter 2, here we focus on studying the coupled equations (2.1.4), and their scaling limits, when M is an abelian variety. Note that in this special case the equations (2.1.4) for constant f are always solvable by taking constant coefficients representatives, so in fact it is necessary here to include a suitable non-constant source term.

Considering abelian varieties is motivated in part by the origin of the dHYM equations in the B-model: for example, homological mirror symmetry for abelian varieties has been studied in detail [Fuk02, PZ98]; moreover abelian varieties also play a special role in this context as fibres of holomorphic Lagrangian fibrations (see e.g. [GW00]).

A more analytic reason is that the coupled equations (2.1.4) interact nicely with the theory of the scalar curvature of Kähler metrics on complex tori, or more generally of periodic solutions of Abreu's equation, as developed by Feng and Székelyhidi [FS11] (see also [LS15]). This is exploited in our results Theorems 3.1.3, 3.1.6 and 3.1.7.

Further motivation comes from the fact that the problem of realising solutions of the coupled equations (2.1.4) effectively in the B-model, as in Remark 2.1.3, is more tractable when M is a complex torus. We explain this, in a special case, in Proposition 3.1.9.

We can now discuss our existence results on abelian varieties. As in the work of Feng and Székelyhidi [FS11] we may assume, without loss of generality, that M is the abelian variety $\mathbb{C}^n/(\mathbb{Z}^n \oplus \sqrt{-1}\mathbb{Z}^n)$ and $[\omega_0]$ is the class of the constant metric $\omega_0 = \sqrt{-1}\sum_i dz_i \wedge \overline{dz}_i$.

The group $U(1)^n$ acts on M, by translations in the direction $\sqrt{-1}\mathbb{R}^n$. We will restrict to $U(1)^n$-invariant tensors and thus work effectively over the real torus $T = \mathbb{R}^n/\mathbb{Z}^n$. Note that an invariant Kähler metric g is given by the real Hessian of a convex function $v(y)$ on $\mathbb{R}^n$ of the form

$$v(y) = \frac{1}{2}|y|^2 + \psi(y)$$

where $\psi\colon \mathbb{R}^n \to \mathbb{R}$ is $\mathbb{Z}^n$-periodic (so we have $\omega_g = \sqrt{-1}\sum_{i,j} v_{ij} dz_i \wedge \overline{dz_j}$). Such a function has a well-defined *Legendre transform* $u(x)\colon \mathbb{R}^n \to \mathbb{R}$, where the "symplectic coordinates" x and "real holomorphic" coordinates y are related by the diffeomorphism $y = \nabla u(x)$.

We begin by studying the special case when M is an *abelian surface*. As a preliminary step we derive a priori estimates for invariant solutions of (2.1.4), under a semipositivity condition, and a condition on the phase $e^{\sqrt{-1}\hat{\theta}}$. These rely heavily on the results of [FS11] and may be of independent interest.

As before, it is convenient to set $F(h) = \sqrt{-1}F$, for a real 2-form F, and to formulate our results in terms of F.

Proposition 3.1.1. *Let M be an abelian surface, and (ω_g, F) be a $U(1)^2$-invariant solution of the coupled equations (2.1.4), for a fixed function f. Suppose we have $F \geq 0$, and the phase $e^{\sqrt{-1}\hat{\theta}}$ satisfies*

$$\sin(\hat{\theta}) < 0,\ \cos(\hat{\theta}) > 0.$$

Then there are a priori C^k estimates on (g, F) of all orders, with respect to the background flat metric ω_0, depending only on f, the phase $e^{\sqrt{-1}\hat{\theta}}$, and the coupling constant α. Moreover the metric g is uniformly positive, depending on $\sup|f|$, α and $e^{\sqrt{-1}\hat{\theta}}$.

Remark 3.1.2. The necessary conditions that the class $[F]$ is (semi)positive and that we have $\sin(\hat{\theta}) < 0$, $\cos(\hat{\theta}) > 0$ are indeed compatible. A straightforward computation shows that on a complex surface we have

$$\cos(\hat{\theta}) = \frac{\int \omega^2 - F^2}{\left(\left(\int \omega^2 - F^2\right)^2 + 4\left(\int F\wedge\omega\right)^2\right)^{1/2}},$$
$$\sin(\hat{\theta}) = -\frac{2\int F\wedge\omega}{\left(\left(\int \omega^2 - F^2\right)^2 + 4\left(\int F\wedge\omega\right)^2\right)^{1/2}}, \tag{3.1.1}$$

so it is enough to choose a positive class $[F]$ with smaller volume than $[\omega]$.

Proposition 3.1.1 is proved in Section 3.2. Similarly to the work of Feng-Székelyhidi [FS11] in the case of the scalar curvature, using the Legendre transform we can apply Proposition 3.1.1 to obtain an existence result for the coupled equations (2.1.4) on an abelian surface M. Let A be any $U(1)^2$-invariant function on M, satisfying the necessary cohomological condition

$$\int_X A\frac{\omega^2}{2} = -\alpha\int_X \mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(h))^2\right).$$

Theorem 3.1.3. *Let M be an abelian surface with Kähler classes $[\omega]$, $[F]$. Suppose the phase $e^{\sqrt{-1}\hat{\theta}}$ satisfies*

$$\sin(\hat{\theta}) < 0,\ \cos(\hat{\theta}) > 0.$$

Consider the equations (2.1.4), with coupling constant

$$\alpha = \alpha' \cos(\hat{\theta}),\ \alpha' > 0$$

and datum f given by the image of any function A, as above, under Legendre duality, that is

$$f(\nabla u(x)) = A(x).$$

Then, these are solvable provided the classes $[F]$ and $[\omega]$ are sufficiently close, depending only on α' and $\sup|A|$.

Theorem 3.1.3 is proved in Section 3.2. The following application follows at once, by rescaling suitably (recall Remark 2.1.4).

Corollary 3.1.4. *Fix negative line bundles L, N on the abelian surface M. Then for all sufficiently large k, depending only on α', A, the equations (2.1.4) with coupling constant $\alpha' \cos(\hat{\theta})$ and datum f as in Theorem 3.1.3 are solvable on the line bundle $L^{\otimes k} \otimes N^{-1}$, with respect to the Kähler class $-kc_1(L)$.*

Theorem 3.1.3 suggests a similar approach, based on the positivity of L^{-1}, in the case of the large radius limit, that is, of the Kähler-Yang-Mills equations studied in [AGG13],

$$\begin{cases} \Lambda_g F = \mu \\ s(g) + \alpha \Lambda_g^2 (F \wedge F) = f, \end{cases} \tag{3.1.2}$$

for a smooth function f. Indeed it turns out that in this case we can obtain analogues of Proposition 3.1.1 and Theorem 3.1.3, for arbitrary dimension. Following [FS11] as above, we may assume, without loss of generality, that M is the abelian variety $\mathbb{C}^n/(\mathbb{Z}^n \oplus \sqrt{-1}\mathbb{Z}^n)$ and $[\omega_0]$ is the class of the metric $\omega_0 = \sqrt{-1}\sum_i dz_i \wedge \overline{dz_i}$.

Proposition 3.1.5. *Let (g, h) be a $U(1)^n$-invariant solution of (3.1.2) on a line bundle $L \to M$, for a fixed function f. Suppose we have $F \geq 0$. Then there are a priori C^k estimates on (g, h) of all orders, with respect to the background flat metric ω_0, depending only on f, the dimension n, the degree μ of L^{-1}, and the coupling constant α. Moreover the metric g is uniformly positive, depending only on $\sup|f|$, α and μ.*

Let A be any $U(1)^n$-invariant function on M, satisfying

$$\int_M A \frac{\omega^n}{n!} = \lambda \frac{[\omega]^n}{n!},$$

where the topological constant λ is given by

$$\lambda = \alpha n(n-1) \frac{(c_1(L))^2 \cup [\omega]^{n-2}}{[\omega]^n}.$$

Suppose $[\omega]$ is the class of the curvature of an ample line bundle N on M. We also fix a negative line bundle L, with degree $-\mu$.

Theorem 3.1.6. *There exists an integer $K > 0$ such that, for all $k \geq K$, there are a Hermitian metric h on the fibres of $L^n \otimes (k-1)\mu N$ and an invariant Kähler metric $g \in c_1(N)$, solving the equations*

$$\begin{cases} \Lambda_g F = nk\mu \\ s(g) + \frac{\alpha}{(nk)^2}\Lambda_g^2(F \wedge F) = f, \end{cases}$$

where f is the image of any function A, as above, under Legendre duality, that is

$$f(\nabla u(x)) = A(x).$$

The integer K depends only on $\sup|f|$, the dimension n, the degree μ, and the coupling constant α.

Note that Theorem 3.1.6 is not obtained by "perturbation" around $\alpha = 0$ and the solution of the corresponding problem $s(g) = f$ found by Feng-Székelyhidi. Indeed for all k as in the statement we have

$$\int_M \Lambda_g^2(F \wedge F)\frac{\omega_g^n}{n!} = (nk)^2\lambda = O(k^2),$$

so the term $\frac{\alpha}{(nk)^2}\Lambda_g^2(F \wedge F)$ coupling the metric to the connection is of order $O(1)$ and the actual coupling constant is α, not $\frac{\alpha}{k^2}$.

Theorems 3.1.3 and 3.1.6 apply when the datum f (or rather its Legendre transform A with respect to the unknown g) is arbitrary. We can obtain much more precise results when A has a particular form. We consider here the case when A is invariant under translations with respect to all but one of the symplectic coordinates, say x_1. For simplicity we analyse the case when M is the abelian surface $\mathbb{C}^2/(\mathbb{Z}^2 \oplus \sqrt{-1}\mathbb{Z}^2)$, although similar results hold much more generally.

Theorem 3.1.7. *Suppose M is the abelian surface $\mathbb{C}^2/(\mathbb{Z}^2 \oplus \sqrt{-1}\mathbb{Z}^2)$. Then, the coupled equations (2.1.4) are solvable on any L, with respect to the class $[\omega_0]$, with coupling constant $\alpha > 0$, and datum $f(y_1)$ given by the image of any function $A(x_1)$, as above, under Legendre duality, that is*

$$f(\nabla u(x)) = A(x_1).$$

We also prove an analogue of this result for the large and small radius limits.

Theorem 3.1.8. *In the situation of Theorem 3.1.7, the coupled Kähler-Yang-Mills equations (2.3.1) are always solvable on L, with respect to the class $[\omega_0]$, with coupling constant $\alpha > 0$, and datum $f(y_1)$ given by the image of any function $A(x_1)$ under Legendre duality.*

The same holds for the small radius limit coupled equations (2.3.2), under the condition $\det(F^0) > 0$, *where $F^0(h) = \sum_{i,j} F^0_{ij} dz_i \wedge \overline{dz_j}$ is a* constant *curvature form for a metric on L.*

Let us return to the question of realising the solutions of the coupled equations (2.1.4) effectively in the B-model. As recalled in Remark 2.1.3, it is always possible to embed (M, ω) isometrically as the zero section of the holomorphic cotangent bundle T^*M, endowed with a hyperkähler metric defined in a *formal* neighbourhood of the zero section. It is natural to ask when this metric extends at least to an open neighbourhood of the zero section, in the analytic topology. By the main result of [Fei01], this is the case if and only if ω is real analytic.

Proposition 3.1.9. *Suppose the datum $A(x_1)$ is real analytic. Then, the metric ω underlying a solution of the coupled equations (2.1.4) given by Theorem 3.1.7 is also real analytic. It follows that these solutions can be realised effectively in the B-model of an open neighbourhood of $M \subset T^*M$ endowed with a hyperkähler metric, extending ω.*

The proof is given in Section 3.4. Naturally it would be interesting to understand when these extensions are complete.

The Appendix to this Chapter is devoted to the linearised Kähler-Yang-Mills equations in symplectic coordinates on a torus. In particular we prove that these linearised equations correspond to a scalar linear differential operator which has trivial kernel and is formally self-adjoint, with respect to the Lebesgue measure. Besides its application in our proof of Theorem 3.1.6, we believe this may be a useful result in view of future applications.

3.2 A priori estimates and Theorem 3.1.3

We consider the coupled equations (2.1.4) when M is an abelian surface $\mathbb{C}^2/\Lambda$. In particular we will soon assume that L is semipositive. In order to simplify our exposition, following [FS11], we can further assume that $M = \mathbb{C}^2/(\mathbb{Z}^2 \oplus \sqrt{-1}\mathbb{Z}^2)$, and that the background Kähler form is $\omega_0 = \sqrt{-1}\sum_i dz_i \wedge \overline{dz}_i$. The general case only differs by slightly more complicated notation.

The group $U(1)^2$ acts on M, by translations in the direction $\sqrt{-1}\mathbb{R}^n$, and we will look for invariant solutions, so we are effectively considering equations on the real torus $\mathbb{R}^2/\mathbb{Z}^2$. Following [FS11], Section 5 we may formulate the problem (using invariant *complex coordinates*, with real part y) in terms of a convex function

$$v(y) = \frac{1}{2}|y|^2 + \psi(y)$$

where $\psi\colon \mathbb{R}^n \to \mathbb{R}$ is periodic, with fundamental domain $\Omega = [0,1] \times [0,1]$, normalised by $\psi(0) = 0$. The invariant metric g is given by the real Hessian of $v(y)$, namely $\omega_g = \sqrt{-1}\sum_{i,j} v_{ij} dz_i \wedge \overline{dz}_j$. Then, by a standard formula in Kähler geometry, we have

$$s(g) = -\frac{1}{4}v^{ij}[\log\det(v_{ab})]_{ij}.$$

We set $F(h) = \sqrt{-1}F$, and abusing notation slightly we think of F as a periodic function with values in symmetric matrices (so we have $F(h) = \sum_{i,j} F_{ij} dz_i \wedge \overline{dz_j}$). Then the coupled equations (2.1.4) become

$$\begin{cases} \operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}} \det(v_{ij} - \sqrt{-1}F_{ij})\right) = 0 \\ -\frac{1}{4}v^{ij}[\log\det(v_{ab})]_{ij} - \alpha \frac{\operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}} \det(v_{ij} - \sqrt{-1}F_{ij})\right)}{\det(v_{ab})} = f, \end{cases} \tag{3.2.1}$$

where the phase $e^{\sqrt{-1}\hat{\theta}}$ is determined by the cohomological condition

$$\int_\Omega \det(v_{ij} - \sqrt{-1}F_{ij}) \in \mathbb{R}_{>0} e^{\sqrt{-1}\hat{\theta}} \tag{3.2.2}$$

and the datum f must satisfy the necessary cohomological constraint

$$\int_\Omega f \det(v_{ab}) = -\alpha \int_\Omega \operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}} \det(v_{ij} - \sqrt{-1}F_{ij})\right). \tag{3.2.3}$$

It is especially convenient to formulate the problem in terms of *symplectic coordinates*, that is, in terms of the Legendre transform $u(x)$ of $v(y)$, see [FS11] Section 5. Recall that $u(x)$ is defined by the equation

$$u(x) + v(y) = x \cdot y,$$

where we set $x = \nabla v(y)$. Then $u(x)$ has the form

$$u(x) = \frac{1}{2}|x|^2 + \phi(x). \tag{3.2.4}$$

where $\phi\colon \mathbb{R}^n \to \mathbb{R}$ is periodic, with domain Ω, and we have the inverse relation

$$y = \nabla u(x).$$

Using a well-known result of Abreu for the scalar curvature in symplectic coordinates (see [Abr98]), as well as the fundamental Legendre duality property

$$v^{ij}(y) = u_{ij}(x),$$

the coupled equations (2.1.4) become

$$\begin{cases} \operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}} \det(u^{ij} - \sqrt{-1}F_{ij}(\nabla u))\right) = 0 \\ -\frac{1}{4}[u^{ij}]_{ij} - \alpha \operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}} \det(u^{ij} - \sqrt{-1}F_{ij}(\nabla u))\right) \det(u_{ab}) = A, \end{cases} \tag{3.2.5}$$

where the datum A is given by the relation $A(x) = f(y) = f(\nabla u)$.

A key advantage of formulating the problem in symplectic coordinates is that it is now trivial to take the cohomological constraint (3.2.3) into account: the datum $A(x)$ must simply satisfy

$$\int_{\Omega} A(x)d\mu(x) = -\alpha \int_{\Omega} \mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}} \det(v_{ij} - \sqrt{-1}F_{ij})\right)$$

with respect to the *fixed* Lebesgue measure $d\mu(x)$.

In the following we start from the equations in symplectic coordinates with datum $A(x)$ and *define* $f(y)$ through the relation

$$f(\nabla u(x)) = A(x).$$

Our first task is to establish the necessary a priori estimates for this problem, Proposition 3.1.5. As in [FS11], the first step for this is obtaining a uniform bound for the determinant.

Lemma 3.2.1. *Suppose the phase* $e^{\sqrt{-1}\hat{\theta}}$ *satisfies*

$$\sin(\hat{\theta}) < 0, \cos(\hat{\theta}) > 0.$$

Then, there are uniform constants $c_1, c_2 > 0$*, depending only on the coupling constant* α*, the phase* $e^{\sqrt{-1}\hat{\theta}}$ *and on* $\sup |f|$*, such that a solution of* (3.2.1) *with* $F \geq 0$ *satisfies*

$$0 < c_1 < \det(v_{ab}) < c_2.$$

(Recall $A(x), f(y)$ *and* $u(x), v(y)$ *are related by Legendre transform).*

Proof. Feng-Székelyhidi [FS11] study Abreu's equation

$$[u^{ij}]_{ij} = \tilde{A}$$

in all dimensions and for an arbitrary smooth periodic function $\tilde{A}$ (with zero average). In loc. cit. Section 3 it is shown that solutions satisfy a uniform bound of the form

$$0 < c'_1 < \det(u_{ab}) < c'_2$$

where the constants c'_1, c'_2 depend only on the dimension n and a bound on $\sup |\tilde{A}|$. In our case, we can write the equations in symplectic coordinates (3.2.5) in the form $[u^{ij}]_{ij} = \tilde{A}$ with the choice

$$\tilde{A} = -4\alpha \,\mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}} \det(u^{ij} - \sqrt{-1}F_{ij}(\nabla u))\right) \det(u_{ab}) - 4A.$$

We claim that, under our assumptions, there is a uniform a priori bound for $\sup |\tilde{A}|$, depending only on $\sup |A|$, α, $e^{\sqrt{-1}\hat{\theta}}$. Equivalently, we claim that there is a uniform bound for the quantity

$$\frac{\mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}} \det(v_{ij} - \sqrt{-1}F_{ij})\right)}{\det(v_{ab})} = \frac{\mathrm{Re}(e^{-\sqrt{-1}\hat{\theta}}(\omega - \sqrt{-1}F)^2)}{\omega^2}.$$

In order to see this, note that in the two-dimensional case the coupled equations (2.1.4) may be written as

$$\begin{cases} F^2\sin(\hat{\theta}) - 2F\wedge\omega\cos(\hat{\theta}) - \omega^2\sin(\hat{\theta}) = 0 \\ s(\omega) - \alpha\frac{-F^2\cos(\hat{\theta})-2F\wedge\omega\sin(\hat{\theta})+\omega^2\cos(\hat{\theta})}{\omega^2} = f. \end{cases}$$

In particular the dHYM equation is

$$\sin(\hat{\theta})\frac{F^2}{\omega^2} - 2\frac{F\wedge\omega}{\omega^2}\cos(\hat{\theta}) = \sin(\hat{\theta}).$$

So under the conditions $\sin(\hat{\theta}) < 0$, $\cos(\hat{\theta}) > 0$, together with semipositivity $F \geq 0$, the dHYM equation implies the a priori bounds

$$\frac{F^2}{\omega^2} < 1, \frac{F\wedge\omega}{\omega^2} < \frac{|\tan(\hat{\theta})|}{2}, \tag{3.2.6}$$

which immediately give the required bounds on $\tilde{A}$. □

It is possible to obtain higher order estimates from the bound on the determinant given by Lemma 3.2.1. Following [FS11] Section 4 the key idea is to write the second equation in (3.2.5) in the form

$$U^{ij}w_{ij} = \tilde{A} \tag{3.2.7}$$

where U^{ij} is the cofactor matrix of the Hessian u_{ij}, while

$$\begin{aligned} &w = (\det(u_{ab}))^{-1}, \\ &\tilde{A} = -4\alpha\,\mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}\det(u^{ij} - \sqrt{-1}F_{ij}(\nabla u))\right)\det(u_{ab}) - 4A. \end{aligned}$$

Note that this rewriting is possible because of the identity $[U^{ij}]_i = 0$. Then (3.2.7) can be regarded as a non-homogeneous linearised Monge-Ampère equation satisfied by w.

Lemma 3.2.2. *Suppose the phase $e^{\sqrt{-1}\hat{\theta}}$ satisfies*

$$\sin(\hat{\theta}) < 0, \cos(\hat{\theta}) > 0.$$

Then, there are uniform constants $0 < \Lambda_0 < \Lambda_1$, $0 < \delta < 1$, $\Lambda_{2,\delta} > 0$, depending only on $e^{\sqrt{-1}\hat{\theta}}$, α and A, such that for a solution u, F of (3.2.5) with $F \geq 0$ we have

$$\Lambda_0 I < u_{ij} < \Lambda_1 I, \; ||u||_{C^{2,\delta}} < \Lambda_{2,\delta}.$$

Proof. We first observe that by [FS11] Lemma 4 we have a uniform C^1 bound on u. This is independent of the equation satisfied by u and holds simply by periodicity, positivity $u_{ij} > 0$ and the normalisation $\psi(0) = 0$. In particular we have a uniform C^0 bound on u.

Let us now show that we have an estimate on $||u||_{C^{2,\delta}}$ for some $\delta > 0$, depending only on $\sup|A|$, α, $e^{\sqrt{-1}\hat{\theta}}$. In [TW] Section 3.7, Corollary 3.2, Trudinger-Wang give an interior

Hölder estimate for a C^2 solution of the non-homogeneous linearised Monge-Ampère equation on the ball $B_1(0) \subset \mathbb{R}^n$, for all n. Note that, adapting to our present notation, they would actually write (3.2.7) as

$$U^{ij} w_{ij} = \left(\frac{\tilde{A}}{\det(u_{ab})}\right) \det(u_{ab}).$$

Then their estimate takes the form

$$||w||_{C^{\delta}(B_{1/2}(0))} \leq C \left(||w||_{C^0(B_1(0))} + \int_{B_1(0)} \left|\frac{\tilde{A}}{\det(u_{ab})}\right|^n \det(u_{ab}) d\mu\right),$$

where $d\mu$ is the Lebesgue measure (so $\det(u_{ab})d\mu$ is the Monge-Ampère measure associated with u), where the constants $\delta, C > 0$ depend only on n and a pinching for the quantity $\det(u_{ab})$, that is, on constants $c'_1, c'_2 > 0$ such that

$$0 < c'_1 < \det(u_{ab}) < c'_2.$$

In the proof of Lemma 3.2.1 we have shown that our current assumptions on $e^{\sqrt{-1}\hat{\theta}}$ and F imply a uniform C^0 bound for the function $\tilde{A}$, depending only on $\tan(\hat{\theta})$, α, $\sup|A|$, see (3.2.6). Moreover Lemma 3.2.1 shows that the pinching constants $c'_1, c'_2 > 0$ can be chosen uniformly, depending only on the same quantities. Recalling that $w = (\det(u_{ab}))^{-1}$, we find that there is a uniform C^{δ} bound on w on the ball $B_{1/2}(0)$, in terms of $\tan(\hat{\theta})$, α, $\sup|A|$. With our current conventions, the ball $B_{1/2}(0)$ does not contain a period domain $\Omega = [0,1] \times [0,1]$, but this is only a matter of notation. For example we could have started with the lattice $\Lambda = \frac{1}{4}\mathbb{Z}^2 \oplus \frac{\sqrt{-1}}{4}\mathbb{Z}^2$. So we get a uniform a priori C^{δ} bound on w everywhere.

By the definition of w, and the regularity we just obtained, the function u satisfies the Monge-Ampère equation

$$w(x)\det(u_{ab}(x)) = 1,$$

with C^{δ} coefficients. A well-known Schauder estimate due to Caffarelli [Caf90] shows that then there is a uniform a priori $C^{2,\delta}$ bound on $u(x)$, depending only on $\sup|A|$, α, $e^{\sqrt{-1}\hat{\theta}}$.

We claim that this implies a uniform bound $\Lambda_0 I < u_{ij} < \Lambda_1 I$. Equivalently, we need to show that the eigenvalues of the Hessian u_{ij} are uniformly bounded, and bounded away from 0, in terms of the usual quantities. But this follows immediately from the uniform bound on the determinant $0 < c'_1 < \det(u_{ab}) < c'_2$ and the uniform bound on $||u||_{C^2}$, which we established above. □

Remark 3.2.3. The proof and (3.2.6) actually show that the bounds only depend on an upper bound for the quantities α, $\alpha|\tan(\hat{\theta})|$ and $\sup|A|$.

We can now complete the proof of Proposition 3.1.1. Recall that this claims that there are a priori bounds of all orders on solutions (g, F) of (3.2.1) and on the positivity of the metric g, depending only on $e^{\sqrt{-1}\hat{\theta}}$, α, f, under the conditions

$$\sin(\hat{\theta}) < 0, \cos(\hat{\theta}) > 0, F \geq 0.$$

It is convenient to write

$$F_{ij} = \left[\frac{1}{2}y^T By + \varphi(y)\right]_{ij}$$

where B is a fixed, symmetric positive semidefinite matrix, and $\varphi(x)$ is periodic, with period domain Ω, and satisfies $\varphi(0) = 0$.

We will use the well-known reduction of the dHYM equation to complex Monge-Ampère equation in the case of a complex surface (see e.g. [CXY17]). As in the proof of Lemma 3.2.1, we write the dHYM equation as

$$-F^2 \sin(\hat{\theta}) + 2F \wedge \omega \cos(\hat{\theta}) = -\omega^2 \sin(\hat{\theta}).$$

We consider a general equation of the form

$$(c_1 F + c_2 \omega)^2 = c_3 \omega^2$$

for some choice of constants c_i. This is of course

$$c_1^2 F^2 + 2c_1 c_2 F \wedge \omega = (c_3 - c_2^2)\omega^2.$$

Recall we have $\sin(\hat{\theta}) < 0$, $\cos(\hat{\theta}) > 0$. Then choosing

$$c_1 = (-\sin(\hat{\theta}))^{1/2},\ c_2 = \frac{\cos(\hat{\theta})}{(-\sin(\hat{\theta}))^{1/2}},\ c_3 = -\frac{1}{\sin(\hat{\theta})}$$

shows that the dHYM condition becomes the complex Monge-Ampère equation

$$\left((-\sin(\hat{\theta}))^{1/2} F - \frac{\cos(\hat{\theta})}{(-\sin(\hat{\theta}))^{1/2}}\omega\right)^2 = -\frac{1}{\sin(\hat{\theta})}\omega^2$$

or equivalently

$$\chi^2 = \omega^2$$

where

$$\chi = -\sin(\hat{\theta})F + \cos(\hat{\theta})\omega.$$

We should think of this as an equation for χ, and so F, given ω. Note that χ is automatically a Kähler class. By the Calabi-Yau Theorem, this Monge-Ampère equation is solvable iff

$$\int \chi^2 = \int \omega^2,$$

which of course determines $e^{\sqrt{-1}\hat{\theta}}$ just as before. In our situation, this reduces to the real Monge-Ampère

$$\det(-\sin(\hat{\theta})F_{ij} + \cos(\hat{\theta})v_{ij}) = \det(v_{ij}). \tag{3.2.8}$$

By Lemma 3.2.2 and the Legendre transform we have a uniform estimate on $||v||_{C^{2,\delta}}$, depending only on $\sup|A|$, α, $e^{\sqrt{-1}\hat{\theta}}$. Moreover, just as in the proof of that Lemma, we observe that by [FS11] Lemma 4 we have a uniform C^1 bound on φ. This is independent of the equation satisfied by φ and holds simply by periodicity, (semi)positivity $F_{ij} \geq 0$ and the normalisation $\psi(0) = 0$. In particular we have a uniform C^0 bound on φ. Caffarelli's Hölder estimates for the real Monge-Ampère equation then give a uniform bound on $||\varphi||_{C^{2,\delta}}$, depending only on $\sup|A|$, α, $e^{\sqrt{-1}\hat{\theta}}$. In particular we have a uniform C^δ bound on F_{ij}.

We use the latter estimate on the bundle curvature F in the linearised Monge-Ampère equation (3.2.7), yielding a uniform $C^{2,\delta}$ bound on $w = (\det(u_{ab}))^{-1}$ and so in turn a uniform bound on $||u||_{C^{4,\delta}}$, depending only on $||A||_{C^\delta}$, α, $e^{\sqrt{-1}\hat{\theta}}$.

We can now proceed inductively, using the equations (3.2.7) and (3.2.8), to obtain estimates of all orders on v and φ, depending only on A, α, $e^{\sqrt{-1}\hat{\theta}}$. Proposition 3.1.1 follows.

Given the a priori estimates of Proposition 3.1.1, we are in a position to prove Theorem 3.1.3. Recall that this involves the choice of coupling constant

$$\alpha = \alpha' \cos(\hat{\theta})$$

for some fixed $\alpha' > 0$. The proof relies on the continuity method. We apply this to the family of equations, depending on a parameter $t \in [0,1]$, given by

$$\begin{cases} \operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}}\det(u^{ij} - \sqrt{-1}F_{ij}(\nabla u))\right) = 0 \\ [u^{ij}]_{ij} = \tilde{A}_t - 4(1-t)\int_\Omega A(x)d\mu(x), \end{cases} \tag{3.2.9}$$

where we set

$$\tilde{A}_t = -4\alpha'\cos(\hat{\theta})\operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}}\det(u^{ij} - \sqrt{-1}F_{ij}(\nabla u))\right)\det(u_{ab}) - 4tA.$$

For $t = 0$ the equations are solvable by choosing $v_{ij} = u^{ij}$ and F_{ij} to be *constant* representatives of their cohomology classes.

By Proposition 3.1.1 the set of times $t \in [0,1]$ for which the equations (3.2.9) are solvable is closed as long as the solution satisfies $F \geq 0$. We claim that if the classes $[F]$ and $[\omega]$ are sufficiently close, depending only on $\sup|A|$ and α', then the bundle curvature actually remains negative, i.e. the condition $F > 0$ is *closed* along the continuity path. To see this we use the complex Monge-Ampère equation

$$(-\sin(\hat{\theta})F + \cos(\hat{\theta})\omega)^2 = \omega^2$$

satisfied by F. We have shown in the proof of Lemma 3.2.1 that, assuming only semi-posivity $F \geq 0$, one has a priori bounds on ω,

$$0 < \Lambda_0\omega_0 < \omega < \Lambda_1\omega_0,\ ||\omega||_{C^\delta(\omega_0)} < \Lambda_{2,\delta},$$

depending only on an upper bound for the quantities α, $\alpha|\tan(\hat{\theta})|$, $\sup|A|$ (see Remark 3.2.3). Thus, with our choice of coupling constant $\alpha = \alpha'\cos(\hat{\theta})$ for some fixed α', we see that the constants $\Lambda_0, \Lambda_1, \Lambda_{2,\delta}$ above can be chosen uniformly in terms of α', $\sup|A|$ only, and in particular do not depend on $t \in [0,1]$ and on $[F]$. Then with our assumptions, if $[F] - [\omega] \to 0 \in H^{1,1}(X, \mathbb{R})$ we have $\cos(\hat{\theta}) \to 0^+$, $-\sin(\hat{\theta}) \to 1^-$ and $||F - \omega||_\omega \to 0$. So by choosing $[F]$ and $[\omega]$ to be sufficiently close, depending only on α', $\sup|A|$ we can make sure that F remains strictly positive along the continuity path.

It remains to check openness of the continuity path. The condition $F > 0$ is clearly open, so we only need to show that, with our choice $\alpha = \alpha'\cos(\hat{\theta})$, the linearisation of the equations (3.2.9) at any point of the path are always solvable provided $[F]$, $[\omega]$ are sufficiently close, in a uniform way. Consider the equations obtained in the limiting case $\cos(\hat{\theta}) = 0$:

$$\begin{cases} F^2 = \omega^2 \\ s(\omega) = f(\nabla u). \end{cases}$$

By the results of [FS11] Section 2, the corresponding linearised equations are uniquely solvable, so the same holds for the linearisation of (3.2.9) for $\cos(\hat{\theta})$ sufficiently small, depending only on our a priori estimates on solutions of (3.2.9), and so on α', A. This completes the proof of Theorem 3.1.3.

3.3 Kähler-Yang-Mills equations on abelian varieties and Theorem 3.1.6

We proceed to prove our main results concerning the Kähler-Yang-Mills system on an abelian variety M of arbitrary dimension n, Proposition 3.1.5 and Theorem 3.1.6. Recall the system is given by

$$\begin{cases} \Lambda_g F = \mu \\ s(g) + \alpha\Lambda_g^2(F \wedge F) = f, \end{cases}$$

where $F(h) = \sqrt{-1}F$ is the curvature of a Hermitian metric on the fibres of a holomorphic line bundle $L \to M$, of degree $-\mu$, and $f \in C^\infty(M)$ is a prescribed function. Note that a solution g must satisfy the cohomological constraint

$$\int_M f\frac{\omega_g^n}{n!} = \alpha\int_M F \wedge F \wedge \frac{\omega_g^{n-2}}{(n-2)!}. \tag{3.3.1}$$

Using the identity (involving pointwise norms)

$$\Lambda_g^2(F \wedge F) = 2||\Lambda_g F||_g^2 - 2||F||_g^2$$

the equations can be written in the form

$$\begin{cases} \Lambda_g F = \mu, \\ s(g) = f + 2\alpha \|F\|_g^2 - 2\alpha\mu^2. \end{cases}$$

Following [FS11] we may assume, without loss of generality, that M is the abelian variety $\mathbb{C}^n/(\mathbb{Z}^n \oplus \sqrt{-1}\mathbb{Z}^n)$ and $[\omega_0]$ is the class of the metric $\omega_0 = \sqrt{-1}\sum_i dz_i \wedge \overline{dz_i}$. Then $U(1)^n$ acts on M, by translations in the $\sqrt{-1}\mathbb{R}^n$ direction. We suppose that f is $U(1)^n$-invariant and look for $U(1)^n$-invariant solutions. As in the previous Section, we formulate the problem in terms of the convex function

$$v(y) = \frac{1}{2}|y|^2 + \psi(y)$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$ is periodic, with period domain $\Omega = [0,1]^n$, and of its Legendre transform

$$u(x) = \frac{1}{2}|x|^2 + \phi(x). \tag{3.3.2}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is periodic with the same period domain Ω. As before, the invariant metric g is given by the real Hessian of $v(y)$, namely $\omega_g = \sqrt{-1}\sum_{i,j} v_{ij} dz_i \wedge \overline{dz_j}$. Then we have

$$\begin{aligned} s(g) &= -\frac{1}{4} v^{ij}[\log\det(v_{ab})]_{ij}, \\ \|F\|_g^2 &= v^{ij}v^{kl}F_{il}F_{kj}. \end{aligned}$$

So the equations become

$$\begin{cases} v^{ij}F_{ij} = \mu \\ v^{ij}[\log\det(v_{ab})]_{ij} = -4f + 8\alpha\mu^2 - 8\alpha v^{ij}v^{kl}F_{il}F_{kj}. \end{cases} \tag{3.3.3}$$

Here $F = F_{ij}$ is regarded as a periodic function with values in symmetric matrices (so we have $F(h) = -\sum_{i,j} F_{ij} dz_i \wedge \overline{dz_j}$).

In terms of the Legendre transform, the equations are

$$\begin{cases} u_{ij}F_{ij}(\nabla u) = \mu \\ [u^{ij}]_{ij} = -4A + 8\alpha\mu^2 - 8\alpha u_{ij}u_{kl}F_{il}(\nabla u)F_{kj}(\nabla u), \end{cases} \tag{3.3.4}$$

where we set $A(x) = f(y) = f(\nabla u)$. From this symplectic viewpoint, it is trivial to take the constraint (3.3.1) into account: the datum $A(x)$ must simply satisfy

$$\int_\Omega A(x)d\mu(x) = \alpha \int_M F \wedge F \wedge \frac{\omega_g^{n-2}}{(n-2)!}$$

with respect to the fixed Lebesgue measure $d\mu(x)$. As in the previous Section, we start from the equations in symplectic coordinates with datum $A(x)$ and *define* $f(y)$ through the relation $f(\nabla u(x)) = A(x)$.

We can now establish our a priori estimates, Proposition 3.1.5. A simple computation shows that we have

$$\begin{aligned} v^{ij}F_{ij} &= \operatorname{tr}(\operatorname{Hess}(v)^{-1}F), \\ v^{ij}v^{kl}F_{il}F_{kj} &= \operatorname{tr}\left((\operatorname{Hess}(v)^{-1}F)^2\right). \end{aligned}$$

So our equations, in invariant complex coordinates, are equivalent to

$$\begin{cases} \operatorname{tr}(\operatorname{Hess}(v)^{-1}F) = \mu, \\ v^{ij}[\log\det(v_{ab})]_{ij} = -4f + 8\alpha\mu^2 - 8\alpha\operatorname{tr}\left((\operatorname{Hess}(v)^{-1}F)^2\right). \end{cases}$$

Suppose v, F give a solution with $F \geq 0$ (so in particular the bundle L is seminegative). Note that $\operatorname{Hess}(v)^{-1}F$ is a product of symmetric matrices and so it is similar to a symmetric matrix, hence it has real eigenvalues λ_i. Since both $\operatorname{Hess}(v)^{-1}$ and F are positive definite and semidefinite respectively, by assumption, we have in fact $\lambda_i \geq 0$. The condition

$$\operatorname{tr}(\operatorname{Hess}(v)^{-1}F) = \mu > 0$$

immediately gives the bound $0 \leq \lambda_i \leq \mu$. Therefore

$$0 \leq \operatorname{tr}\left((\operatorname{Hess}(v)^{-1}F)^2\right) = \sum_i \lambda_i^2 < n\mu^2.$$

It follows immediately that under the semipositivity assumption $F \geq 0$ there is a uniform C^0 bound for the image under Legendre duality of the quantity

$$\tilde{A} = -4A + 8\alpha\mu^2 - 8\alpha u_{ij}u_{kl}F_{il}(\nabla u)F_{kj}(\nabla u), \tag{3.3.5}$$

depending only on $\sup|f|$,n,μ. This bound is preserved under pullback by the diffeomorphism induced by Legendre duality. On the other hand the second equation in (3.3.4) is precisely $[u^{ij}]_{ij} = \tilde{A}$. If follows that we have a uniform a priori bound on the quantity $[u^{ij}]_{ij}$, depending only on $\sup|f|$, n, μ. From here, proceeding exactly as in the proof of Lemma 3.2.2, we find that there are uniform constants $0 < \Lambda_0 < \Lambda_1$, $0 < \delta < 1$, $\Lambda_{2,\delta} > 0$, depending only on A, n, μ, such that for a solution u, F of (3.3.4) with $F \geq 0$ we have

$$\begin{aligned} &\Lambda_0 I < u_{ij} < \Lambda_1 I, \\ &||u||_{C^{2,\delta}} < \Lambda_{2,\delta}. \end{aligned} \tag{3.3.6}$$

Let us now consider the bundle curvature, or equivalently the form F. Recall that, on the universal cover, this is given by the Hessian of a function,

$$F_{ij} = \left[\frac{1}{2}y^T By + \varphi(y)\right]_{ij},$$

where B is a fixed, symmetric positive semidefinite matrix, and $\varphi(x)$ is periodic, with period domain Ω. We can normalise φ so that $\varphi(0) = 0$. The HYM equation $\Lambda_g F = \mu$ satisfied by h can be seen as a second order linear elliptic PDE, with periodic coefficients, satisfied by the periodic function φ,

$$v^{ij}\varphi_{ij} = \mu - v^{ij}B_{ij}. \tag{3.3.7}$$

By standard Schauder theory and periodicity there is a bound

$$||\varphi||_{C^{2,\delta}(\Omega)} \leq C_{2,\delta}\left(||\varphi||_{C^0(\Omega)} + ||\mu - v^{ij}B_{ij}||_{C^{2,\delta}(\Omega)}\right),$$

where $C_{2,\delta} > 0$ depends only on $||v^{ij}||_{C^{2,\delta}(\Omega)}$ and the ellipticity constants. By our previous a priori bounds (3.3.6) and the Legendre transform, both quantities are uniformly bounded in terms of $\sup|f|$, n, μ. It follows that in fact we have an a priori bound of the form

$$||\varphi||_{C^{2,\delta}(\Omega)} \leq C_{2,\delta}||\varphi||_{C^0(\Omega)} + C'_{2,\delta}$$

where the constants $C_{2,\delta}, C'_{2,\delta} > 0$ depend only on $\sup|f|$, n, μ. Moreover, just as in the proof of Lemma 3.2.2 and by [FS11] Lemma 4, we have a uniform C^1 bound on φ, so we see that $||\varphi||_{C^{2,\delta}(\Omega)}$ is controlled only by $\sup|f|$, n, μ.

As in the proof of Proposition 3.1.1, we can use this estimate on φ in the linearised Monge-Ampère equation (3.2.7), with $\tilde{A}$ now given by (3.3.5). The resulting C^δ bound on $\tilde{A}$ gives a $C^{2,\delta}$ bound on $w = (\det(u_{ab}))^{-1}$ and so a uniform bound on $||u||_{C^{4,\delta}}$, depending only on $||A||_{C^\delta}$, μ, n.

We can now proceed inductively, using the linearised Monge-Ampère (3.2.7) (with right hand side given by (3.3.5)) and the Poisson equation (3.3.7), to obtain estimates of all orders on v and φ, depending only on A, μ, n. Proposition 3.1.5 follows.

3.3.1 Proof of Theorem 3.1.6

We are in a position to prove Theorem 3.1.6. Recall for this result we have a negative line bundle L, respectively an ample line bundle N on M, where L has degree $-\mu$ and $[\omega] = c_1(N)$. We are concerned with the system

$$\begin{cases} u_{ij}F_{ij}(\nabla u) = k\mu \\ [u^{ij}]_{ij} = -4A + 8\frac{\alpha}{k^2}\mu^2 - 8\frac{\alpha}{k^2}u_{ij}u_{kl}F_{il}(\nabla u)F_{kj}(\nabla u). \end{cases}$$

Here $\sqrt{-1}F$ is the curvature of a metric on the fibres of $L \otimes (k-1)\mu N$ for some $k \geq 1$. We claim that by taking k sufficiently large, depending only on $\sup|A|$, n, μ, we can find u and F solving the equations.

For the proof it is convenient to work instead with the $\mathbb{Q}$-line bundle $(1-\beta)L + \frac{\beta}{n}N$. Here β is a parameter in the construction, to be chosen appropriately. At the end of the argument we will see how to obtain from this a solution on a genuine line bundle. So we have

$$F_{ij} = (1-\beta)\tilde{F}_{ij} + \beta\frac{\mu}{n}v_{ij},\ \beta \in (0,1)\cap\mathbb{Q}$$

where $\sqrt{-1}\tilde{F}$ is the curvature of some metric on the fibres of L.

The proof relies on the continuity method. We apply this to the family of equations, depending on the parameter β, given by

$$\begin{cases} u_{ij}F_{ij}(\nabla u) = \mu \\ [u^{ij}]_{ij} = -4tA - 4(1-t)\int_{\Omega} A(x)d\mu(x) + 8\alpha\mu^2 - 4u_{ij}u_{kl}F_{il}(\nabla u)F_{kj}(\nabla u) \\ F > 0, \end{cases} \tag{3.3.8}$$

for $t \in [0,1]$.

When $t = 0$ a solution of (3.3.8) is given in complex coordinates by taking

$$v(y) = \frac{1}{2}|y|^2,\ F_{ij} = (1-\beta)B_{ij} + \beta\frac{\mu}{n}v_{ij} > 0$$

for all $\beta \in (0,1)\cap\mathbb{Q}$, where $B > 0$ is a constant symmetric matrix.

We will show in the Appendix that the linearised equations corresponding to the system (3.3.4) are uniquely solvable. This implies that the set of times $t \in [0,1]$ for which (3.3.8) has a solution is open.

We claim that this set is also closed. By Proposition 3.1.5 we have a priori C^k estimates of all orders on solutions of (3.3.8), as well as on the positivity of the solution metric g, which only depend on A, n, μ, and in particular are independent of $t \in [0,1]$. Then it follows from the Ascoli-Arzelà Theorem that we can take the limit of a (sub)sequence of solutions, corresponding to times $t_i \to t \in [0,1]$ and obtain a solution at time t.

It remains to be seen that the positivity condition $F > 0$ is also closed. Arguing by contradiction we assume that for the limit solution we have $F \geq 0$ but not $F > 0$. So we have

$$(1-\beta)\tilde{F}_{ij}(\xi) + \beta\frac{\mu}{n}v_{ij}(\xi) = 0$$

for some unit vector ξ. Recall that $\tilde{F}$ takes the form

$$\tilde{F}_{ij} = \left[\frac{1}{2}y^T By + \varphi(y)\right]_{ij}$$

for a periodic function φ, so we find

$$\mathrm{Hess}(\varphi)(\xi) = -B(\xi) - \frac{\beta}{1-\beta}\frac{\mu}{n}\,\mathrm{Hess}\, v(\xi). \tag{3.3.9}$$

The equation satisfied by $F_{ij} = (1-\beta)\tilde{F}_{ij} + \beta\frac{\mu}{n}v_{ij}$ in complex coordinates is

$$v^{ij}\big((1-\beta)\tilde{F}_{ij} + \beta\frac{\mu}{n}v_{ij}\big) = \mu.$$

It follows that $\tilde{F}$ satisfies the equation

$$v^{ij}\tilde{F}_{ij} = \mu,$$

or is terms of φ

$$v^{ij}\varphi_{ij} = \mu - v^{ij}B_{ij}.$$

Recall we are free to normalise φ with an additive constant. In particular we can assume that φ is L^2-orthogonal with respect to the metric g to constant functions, that is, to the kernel of the Laplacian Δ_g. With this assumption we have a standard Schauder estimate

$$||\varphi||_{C^{2,\delta}(\Omega)} \leq K_{2,\delta}||\mu - v^{ij}B_{ij}||_{C^{\delta}(\Omega)},$$

where $K_{k,\delta} > 0$ depends only on $||v^{ij}||_{C^{\delta}(\Omega)}$ and the ellipticity constants. By the proof of Proposition 3.1.5 all these quantities, with $||\mu - v^{ij}B_{ij}||_{C^{\delta}(\Omega)}$, are uniformly bounded in terms of $\sup|A|$, n, μ, assuming only $F \geq 0$ (in particular, independently of $t \in [0,1]$ and $\beta \in (0,1) \cap \mathbb{Q}$). It follows that we have a uniform bound on $||\varphi||_{C^2(\Omega)}$. But (3.3.9), together with the strictly positive uniform lower bound on $\mathrm{Hess}(v)$ given by Proposition 3.1.5, implies that $\mathrm{Hess}(\varphi)(\xi)$ can be made arbitrarily large by taking β sufficiently close to 1. This is a contradiction, arising from our assumption that at some time F is only semipositive.

The upshot is that for all rational β sufficiently close to 1, depending only on $\sup|A|$, n, μ, we have g, F, providing a solution on the $\mathbb{Q}$-line bundle $(1-\beta)L + \beta\frac{\mu}{n}N$. In general, if g, F give a solution to the Kähler-Yang-Mills equations (2.3.1) on a line bundle, then the same g together with the rescaled 2-form ρF solve the system

$$\begin{cases} \Lambda_g(\rho F) = \rho\mu I \\ s(g) + \frac{\alpha}{\rho^2}\Lambda_g^2((\rho F)\wedge(\rho F)) = \lambda. \end{cases}$$

We apply this simple consideration to our solution g, F above, choosing $\beta = 1 - \frac{1}{k}$ for sufficiently large k, and with scaling factor $\rho = nk$. This yields a solution defined on the fibres of $L^n \otimes (k-1)\mu N$, and with parameter $\frac{\alpha}{(nk)^2}$, as claimed by Theorem 3.1.6.

3.4 Proof of Theorems 3.1.7 and 3.1.8

In the present context M is the abelian surface $\mathbb{C}^2/(\mathbb{Z}^2 \oplus \sqrt{-1}\mathbb{Z}^2)$ and the datum f only depends on a single coordinate, say y_1. Equivalently, its Legendre transform A only depends on x_1. The coupled equations (2.1.4) are equivalent to the system

$$\begin{cases} \mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}\det(v_{ij} - \sqrt{-1}F_{ij})\right) = 0 \\ -\frac{1}{4}v^{ij}[\log\det(v_{ab})]_{ij} - \alpha\frac{\mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}\det(v_{ij}-\sqrt{-1}F_{ij})\right)}{\det(v_{ab})} = f(y_1), \end{cases}$$

to be solved for matrices v, F of the form

$$v_{ij} = \begin{pmatrix} 1+\psi''(y_1) & 0 \\ 0 & 1 \end{pmatrix},\; F_{ij} = F^0_{ij} + \varphi''(y_1)\delta_{1i}\delta_{1j} = \begin{pmatrix} a+\varphi''(y_1) & b \\ b & c \end{pmatrix}.$$

Here we have, a priori, $1+\psi''(y_1) > 0$, while we are not imposing positivity conditions on F_{ij}. The phase $e^{\sqrt{-1}\hat{\theta}}$ is determined by the constraint

$$\int_{\Omega} \det(v_{ij} - \sqrt{-1}F_{ij}) \in \mathbb{R}_{>0} e^{\sqrt{-1}\hat{\theta}}.$$

We have

$$\det(v_{ij} - \sqrt{-1}F_{ij}) = \left(1 - \det(F^0) + \psi'' - c\varphi''\right) - \sqrt{-1}\left(\operatorname{tr}(F^0) + \varphi'' + c\psi''\right),$$

so integrating using the periodic boundary conditions shows

$$e^{\sqrt{-1}\hat{\theta}} = \frac{1-\det(F^0) - \sqrt{-1}\operatorname{tr}(F^0)}{\left(\left(1-\det(F^0)\right)^2 + \left(\operatorname{tr}(F^0)\right)^2\right)^{1/2}}.$$

Similarly we have

$$\begin{aligned} &\operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}}\det(v_{ij} - \sqrt{-1}F_{ij})\right) \\ &= -\sin(\hat{\theta})\left(1-\det(F^0) - c\varphi'' + \psi''\right) - \cos(\hat{\theta})\left(\operatorname{tr}(F^0) + c\psi'' + \varphi''\right), \end{aligned}$$

so that the dHYM equation becomes the algebraic identity

$$\varphi'' = -\frac{\sin(\hat{\theta})\left(1-\det(F^0)\right) + \cos(\hat{\theta})\operatorname{tr}(F^0) + \psi''(x)(c\cos(\hat{\theta}) + \sin(\hat{\theta}))}{\cos(\hat{\theta}) - c\sin(\hat{\theta})}.$$

Using this identity for φ'' gives

$$\begin{aligned} &\frac{\operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}}\det(v_{ij} - \sqrt{-1}F_{ij})\right)}{\det(v_{ab})} \\ &= \frac{b^2}{\left(\psi''+1\right)(\cos(\hat{\theta}) - c\sin(\hat{\theta}))} - \frac{c^2+1}{c\sin(\hat{\theta}) - \cos(\hat{\theta})}. \end{aligned}$$

On the other hand the scalar curvature is given by

$$-\frac{1}{4}v^{ij}[\log\det(v_{ab})]_{ij} = -\frac{1}{4}\frac{(\log(1+\psi''))''}{1+\psi''},$$

so that the coupled equations (2.1.4) become the single nonlinear ODE

$$-\frac{1}{4}\frac{(\log(1+\psi''))''}{1+\psi''} - \frac{\alpha b^2}{\left(\psi''+1\right)(\cos(\hat{\theta}) - c\sin(\hat{\theta}))} + \frac{\alpha(c^2+1)}{c\sin(\hat{\theta}) - \cos(\hat{\theta})} = f(y_1).$$

Now consider the Legendre transform of the convex function *of a single variable*

$$\frac{1}{2}y_1^2 + \psi(y_1).$$

This takes the form

$$\frac{1}{2}x_1^2 + \phi(x_1)$$

for some periodic $\phi(x_1)$, and by the standard Legendre property

$$1 + \phi''(x_1) = \frac{1}{1 + \psi''(y_1)}$$

together with the one-dimensional case of Abreu's formula for the scalar curvature

$$-\frac{1}{4}\frac{(\log(1 + \psi(y_1)''))''}{1 + \psi(y_1)''} = -\frac{1}{4}\left(\frac{1}{1 + \phi''(x_1)}\right)''$$

we find that the above nonlinear ODE, to which we reduced the coupled equations (2.1.4), can be written in terms of $\phi(x_1)$ as

$$-\frac{1}{4}\left(\frac{1}{1 + \phi''}\right)'' - \frac{\alpha b^2(1 + \phi'')}{\cos(\hat{\theta}) - c\sin(\hat{\theta})} + \frac{\alpha(c^2 + 1)}{c\sin(\hat{\theta}) - \cos(\hat{\theta})} - A(x_1) = 0, \tag{3.4.1}$$

where $A(x_1)$ denotes the image of $f(y_1)$ under the Legendre transform diffeomorphism, as usual. We are assuming of course the cohomological compatibility condition

$$\int_0^1 A(x_1)dx_1 = -\frac{\alpha b^2}{\cos(\hat{\theta}) - c\sin(\hat{\theta})} - \frac{\alpha(c^2 + 1)}{c\sin(\hat{\theta}) - \cos(\hat{\theta})}.$$

In order to prove the existence of a periodic solution ϕ, satisfying $1 + \phi'' > 0$, we argue precisely as in the proofs of Theorems 3.1.3 and 3.1.6, relying on the same continuity method and the results of Feng-Székelyhidi. Thus in order to obtain closedness it is enough to prove that a periodic solution ϕ of (3.4.1), with $1 + \phi'' > 0$, would satisfy a priori a uniform C^0 bound on the scalar curvature. Equivalently it is enough to prove a uniform a priori C^0 bound for the quantity

$$\left|\frac{\alpha b^2(1 + \phi'')}{\cos(\hat{\theta}) - c\sin(\hat{\theta})}\right|$$

for a solution of (3.4.1). But since we have $1 + \phi'' > 0$ by assumption, we only need to show that there is a uniform a priori bound from above, $\phi'' < C$. Thus suppose $\bar{x}_1$ is a point at which $1 + \phi''(x_1)$ attains its maximum. Then at $\bar{x}_1$ the quantity $(1 + \phi''(x_1))^{-1}$ attains its minimum, so we have

$$\left(\frac{1}{1 + \phi''(\bar{x}_1)}\right)'' \geq 0.$$

Using the equation (3.4.1) this shows

$$\frac{\alpha b^2(1+\phi''(\bar{x}_1))}{\cos(\hat{\theta})-c\sin(\hat{\theta})}-\frac{\alpha(c^2+1)}{c\sin(\hat{\theta})-\cos(\hat{\theta})}\leq -A(\bar{x}_1)\leq \sup|A|.$$

Since we already know $1+\phi''(\bar{x}_1)>0$, and we have $\alpha>0$, if we further assume the condition

$$\frac{b^2}{\cos(\hat{\theta})-c\sin(\hat{\theta})}>0$$

the above inequality immediately gives the required uniform bound $\phi''<C$. But a little computation shows that we have in fact

$$\frac{\alpha b^2}{\cos(\hat{\theta})-c\sin(\hat{\theta})}=\frac{b^2}{1+b^2+c^2}\left((1-\det(F^0))^2+(\mathrm{tr}(F^0))^2\right)^{1/2}$$

so this quantity is nonnegative, and only vanishes for $b=0$, in which case (3.4.1) reduces to the (solvable) Abreu equation.

To obtain openness for the continuity method, we need to show that the operator given by the left hand side of (3.4.1), mapping $C^{k,\alpha}_0(S^1,d\mu)$ to $C^{k-4,\alpha}_0(S^1,d\mu)$, has surjective differential at a solution. The differential maps $\dot{\phi}$ to $L\left(\frac{\dot{\phi}''}{(1+\phi'')^2}\right)$, where we set, for any $\beta\in C^{k,\alpha}(S^1,d\mu)$,

$$L(\beta)=\frac{1}{4}\beta''-\frac{\alpha b^2(1+\phi'')^2\beta}{\cos(\hat{\theta})-c\sin(\hat{\theta})}.$$

The operator L acting on $C^{k,\alpha}(S^1,d\mu)$ is formally self-adjoint and has trivial kernel by the condition $\frac{\alpha b^2}{\cos(\hat{\theta})-c\sin(\hat{\theta})}>0$. Thus the equation $L(\beta)=\gamma$ is uniquely solvable for all periodic γ, and if $\gamma\in C^{k-2,\alpha}_0(S^1,d\mu)$ the unique solution $\beta\in C^{k,\alpha}(S^1,d\mu)$ satisfies $\int(1+\phi'')^2\beta=0$, so in turn the equation $\frac{\dot{\phi}''}{(1+\phi'')^2}=\beta$ is solvable.

Theorem 3.1.8 follows from similar arguments. The large and small radius limit equations are given by

$$\begin{cases} v^{ij}F_{ij}=\mu\\ -\frac{1}{4}v^{ij}[\log\det(v_{ab})]_{ij}-2\alpha v^{ij}v^{kl}F_{il}F_{kj}+2\alpha\mu^2=f(y_1),\end{cases}$$

respectively

$$\begin{cases} F^{ij}v_{ij}=\kappa\\ -\frac{1}{4}v^{ij}[\log\det(v_{ab})]_{ij}-\alpha\frac{\det(F_{ab})}{\det(v_{ab})}=f(y_1),\end{cases}$$

where

$$\mu=\mathrm{tr}(F^0),\ \kappa=\frac{\mathrm{tr}(F^0)}{\det(F^0)}.$$

In both cases, the HYM (i.e. Poisson) equation and the J-equation can be solved explicitly, giving the identities

$$\varphi'' = a\psi'',$$

respectively

$$\varphi'' = \frac{c\det(F^0)}{b^2+c^2}\psi''.$$

Using these identities, we find that the large radius limit becomes the nonlinear ODE

$$-\frac{1}{4}\frac{(\log(1+\psi''))''}{1+\psi''} - 2\alpha(a^2+c^2) - \frac{4\alpha b^2}{1+\psi''(x)} + 2\alpha(\mathrm{tr}(F^0))^2 = f(y_1),$$

and similarly the small radius limit becomes

$$-\frac{1}{4}\frac{(\log(1+\psi''))''}{1+\psi''} - \frac{\alpha b^2\det(F^0)}{(b^2+c^2)\left(1+\psi''(x)\right)} - \frac{\alpha c^2\det(F^0)}{b^2+c^2} = f(y_1).$$

Taking the Legendre transform $\frac{1}{2}x_1^2+\phi$ of the convex function $\frac{1}{2}y_1^2+\psi$, we obtain the ODE

$$-\frac{1}{4}\left(\frac{1}{1+\phi''(x_1)}\right)'' - 2\alpha(a^2+c^2) - 4\alpha b^2(1+\phi''(x)) + 2\alpha(\mathrm{tr}(F^0))^2 = A(y_1),$$

respectively

$$-\frac{1}{4}\left(\frac{1}{1+\phi''(x_1)}\right)'' - \frac{\alpha b^2\det(F^0)\left(1+\phi''(x)\right)}{(b^2+c^2)} - \frac{\alpha c^2\det(F^0)}{b^2+c^2} = A(y_1).$$

We can now apply the same maximum principle argument used in the proof of Theorem 3.1.7, to conclude that the large radius limit equations are solvable provided the condition

$$\alpha b^2 > 0$$

is satisfied, and the same holds for the small radius limit equations under the condition

$$\frac{\alpha b^2\det(F^0)}{(b^2+c^2)} > 0.$$

Of course the large radius limit condition holds unless $b=0$, in which case we reduce to the (solvable) Abreu equation. On the other hand the small radius limit condition $\det(F^0)>0$ does give a nontrivial constraint.

Finally, let us prove Proposition 3.1.9. Recall that in the proof of Theorem 3.1.7 we reduced the coupled equations (2.1.4) to the single nonlinear ODE (3.4.1) for the

Legendre transform $1+\phi''(x_1)$. We only need to show that if $A(x_1)$ is real analytic, then so is $1+\phi''(x_1)$. Setting $U(x_1) = (1+\phi''(x_1))^{-1}$, (3.4.1) is equivalent to the system

$$\begin{cases} U' = V, \\ V' = -\left(\frac{4\alpha b^2}{\cos(\hat{\theta})-c\sin(\hat{\theta})}\right)\frac{1}{U} - \frac{4\alpha(c^2+1)}{c\sin(\hat{\theta})-\cos(\hat{\theta})} - 4A(x_1). \end{cases}$$

If (U,V) is a solution with $U>0$ and $A(x_1)$ is real analytic, it follows from the Cauchy-Kovalevskaya Theorem that U,V are also real analytic. The solution constructed in Theorem 3.1.7 satisfies $U>0$, so Proposition 3.1.9 follows.

3.A Appendix - Linearised equations

This Appendix studies the linearisation of the Kähler-Yang-Mills equations formulated in symplectic coordinates on a torus. In particular we prove that these linearised equations correspond to a scalar linear differential operator which has trivial kernel and is formally self-adjoint, with respect to the Lebesgue measure.

Thus we consider the system (3.3.4), replacing the datum A with $A_t = tA + (1-t)\int_\Omega A(x)d\mu(x)$ for $t\in[0,1]$. In complex coordinates

$$F_{ij}(y) = B_{ij} + \varphi_{ij}(y), \tag{3.A.1}$$

where B_{ij} denotes a constant symmetric matrix; using this notation, the system (3.3.4) in symplectic coordinates has the form:

$$\begin{cases} u_{ij}B_{ij} + \partial_i u^{ij}\partial_j\varphi = \mu \\ [u^{ij}]_{ij} + (\partial_i(u^{lm}\varphi_m))(\partial_l(u^{in}\varphi_n)) \\ +B_{ik}B_{jl}u_{ij}u_{kl} + 2(\partial_j(u^{kn}\varphi_n))u_{kl}B_{jl} + A_t = 0, \end{cases} \tag{3.A.2}$$

where, with a small abuse of notation, φ denotes also the Legendre transform of the function in (3.A.1). In order to study the linearization of (3.A.2), we consider the linear operator $L(\dot{u})$ associated to the second equation, which has the form

$$\begin{aligned} L(\dot{u}) = &-\partial^2_{ij}(u^{ia}\dot{u}_{ab}u^{bj}) + 2(\partial_i(u^{lm}\dot{\varphi}(\dot{u})_m))(\partial_l(u^{in}\varphi_n)) \\ &-2(\partial_i(u^{la}\dot{u}_{ab}u^{bm}\varphi_m))(\partial_l(u^{in}\varphi_n)) + 2B_{ik}B_{jl}\dot{u}_{ij}u_{kl} \\ &+2(\partial_j(u^{kn}\dot{\varphi}(\dot{u})_n))u_{kl}B_{jl} + 2(\partial_j(u^{kn}\varphi_n))\dot{u}_{kl}B_{jl} \\ &-2(\partial_j(u^{ka}\dot{u}_{ab}u^{bn}\varphi_n))u_{kl}B_{jl}, \end{aligned} \tag{3.A.3}$$

where $\dot{\varphi}(\dot{u})$ is the unique solution of the linearization of the first equation in (3.A.2):

$$\dot{u}_{ij}B_{ij} + \Delta\dot{\varphi} - \partial_i(u^{ia}\dot{u}_{ab}u^{bj}\varphi_j) = 0 \tag{3.A.4}$$

with the normalization $\int_M \dot{\varphi}d\mu = 0$ and with $\Delta = \partial_i u^{ij}\partial_j$. In order to prove that the operator $L\colon C^{N,\alpha}_0(T^n)\to C^{N-4,\alpha}_0(T^n)$, at a solution of (3.A.2), is invertible for a sufficently large $N \in \mathbb{N}$ and $0 < \alpha < 1$, is enough to show that L is injective and formally self-adjoint with respect to the L^2-product defined by the volume form $d\mu$. Then, by the implicit function theorem, if we have a smooth solution of (3.A.2) for $\bar{t} \in [0,1]$, there exist $C^{N,\alpha}$ solutions for $\bar{t} + \varepsilon$, with $\varepsilon << 1$. By standard bootstraping technique, these solutions are actually smooth.

We prove that, for any $\gamma, \xi \in C_0^{N,\alpha}(T^n)$ with $N >> 1$, $\int_X \xi L(\gamma) = \int_X \gamma L(\xi)$, omitting in the following the volume form, in order to ease the notation. We split $L(\gamma) = L_0(\gamma) + L_1(\dot{\varphi}(\gamma))$, with

$$\begin{aligned} L_0(\gamma) = &- \partial^2_{ij}(u^{ia}\gamma_{ab}u^{bj}) + 2(\partial_j(u^{kn}\varphi_n))\gamma_{kl}B_{jl} \\ &- 2(\partial_i(u^{la}\gamma_{ab}u^{bm}\varphi_m))(\partial_l(u^{in}\varphi_n)) + 2B_{ik}B_{jl}\gamma_{ij}u_{kl} \\ &- 2(\partial_j(u^{ka}\gamma_{ab}u^{bn}\varphi_n))u_{kl}B_{jl}, \end{aligned} \tag{3.A.5}$$

$$L_1(\gamma) = + 2(\partial_j(u^{kn}\dot{\varphi}(\gamma)_n))u_{kl}B_{jl} + 2(\partial_i(u^{lm}\dot{\varphi}(\gamma)_m))(\partial_l(u^{in}\varphi_n)). \tag{3.A.6}$$

We will show that

$$\int_M (\xi L_0(\gamma) - \gamma L_0(\xi)) = - \int_M (\xi L_1(\gamma) - \gamma L_1(\xi)).$$

At a solution of (3.A.2), integrating by parts, we get the following identities for the different terms of $\int_M \xi L_0(\gamma)$:

1.
$$\int \xi \partial^2_{ij}(u^{ia}\gamma_{ab}u^{bj}) = \int \gamma \partial^2_{ij}(u^{ia}\xi_{ab}u^{bj});$$

2.
$$\begin{aligned} &\int \xi \gamma_{kl} B_{jl} \partial_j(u^{kn}\varphi_n) = \\ &\int \xi_{kj} u^{kn}\varphi_n \gamma_l B_{jl} + \int \xi_k u^{kn}\varphi_n \gamma_{lj} B_{lj} - \int \xi \gamma_l B_{jl} \partial_j \Delta\varphi; \end{aligned}$$

3.
$$\begin{aligned} &\int \xi(\partial_i(u^{la}\gamma_{ab}u^{bm}\varphi_m))(\partial_l(u^{in}\varphi_n)) = \\ &\int \xi_i u^{in}\varphi_n \partial_j(u^{ja}\gamma_{ab}u^{bl}\varphi_l) - \int \gamma_b u^{bm}\varphi_m \partial_j(u^{ja}\xi_{ab}u^{bl}\varphi_l) \\ &+ \int \gamma(u^{la}\xi_{ab}u^{bm}\varphi_m)\partial_l(\Delta\varphi) - \int \xi(u^{la}\gamma_{ab}u^{bm}\varphi_m)\partial_l(\Delta\varphi) \\ &+ \int \gamma \partial_a(u^{bm}\varphi_m)\partial_b(u^{al}\xi_{li}u^{in}\varphi_n); \end{aligned}$$

4.

$$\int \xi B_{ik} B_{jl} \gamma_{ij} u_{kl} =$$
$$\int \gamma B_{ik} B_{jl} u_{kl} \xi_{ij} + \int \gamma B_{ik} B_{jl} u_{klj} \xi_i - \int \xi B_{ik} B_{jl} u_{kli} \gamma_j;$$

5.

$$\int \xi u_{kl} B_{jl} \partial_j (u^{ka} \gamma_{ab} u^{bn} \varphi_n) =$$
$$\int \gamma_b u^{bn} \varphi_n \xi_{ja} B_{ja} - \int \gamma \xi_{jb} B_{ja} \partial_a (u^{bn} \varphi_n) - \int \gamma \xi_j B_{ja} \partial_a \Delta\varphi$$
$$+ \int \xi \gamma_b u^{bn} \varphi_n u^{ka}_a u_{jkl} B_{jl} + \int \xi u^{ka} u_{jkl} B_{lj} \gamma_b \partial_a (u^{bn} \varphi_n)$$
$$+ \int \xi \gamma_b u^{ka} u^{bn} \varphi_n u_{jkla} B_{lj} - \int \gamma \xi_a u^{ka}_b u^{bn} \varphi_n u_{jkl} B_{lj}$$
$$- \int \gamma \xi_a u^{ka} u_{jkl} B_{lj} \Delta\varphi - \int \gamma u^{bn} \varphi_n \xi_a u^{ak} u_{jklb} B_{lj}$$
$$+ \int (\partial_j (u^{ka} \xi_{ab} u^{bn} \varphi_n)) u_{kl} B_{lj} \gamma + \int \xi_{ab} u^{ka} u^{bn} \varphi_n u_{kl} B_{lj} \gamma_j.$$

Hence, after several cancellations, we get

$$- \int_M \xi L_0(\gamma) + \int_M \gamma L_0(\xi) =$$
$$= \underbrace{\int \xi_i u^{in} \varphi_n \partial_j (u^{ja} \gamma_{ab} u^{bl} \varphi_l)}_{(i)} - \underbrace{\int (\gamma_b u^{bm} \varphi_m) \partial_j (u^{ja} \xi_{ab} u^{bl} \varphi_l)}_{(ii)}$$
$$+ \underbrace{\int \xi_{ja} B_{ja} \gamma_b u^{bn} \varphi_n}_{(iii)} - \underbrace{\int \gamma_{ja} B_{ja} \xi_b u^{bn} \varphi_n}_{(iv)} .$$

Consider now (i) and (iv); using (3.A.4), we get:

$$
\begin{aligned}
&\int \xi_i u^{in} \varphi_n (\partial_j (u^{ja} \gamma_{ab} u^{bl} \varphi_l) - \gamma_{ja} B_{ja}) \\
&= \int \xi_i u^{in} \varphi_n \Delta \dot{\varphi}(\gamma) \\
&= -\int u^{kn} \dot{\varphi}(\gamma)_n \xi_i \partial_k (u^{in} \varphi_n) - \int \dot{\varphi}(\gamma)_n u^{kn} \xi_{ik} u^{im} \varphi_m \\
&= \int \xi (\partial_i (u^{kn} \dot{\varphi}(\gamma)_n))(\partial_k (u^{in} \varphi_n)) + \int \xi u^{kn} \dot{\varphi}(\gamma)_n \partial_k \Delta \varphi \\
&- \int \dot{\varphi}(\gamma)_n u^{kn} \xi_{ik} u^{im} \varphi_m \\
&= \int \xi (\partial_i (u^{kn} \dot{\varphi}(\gamma)_n))(\partial_k (u^{in} \varphi_n)) + \int \dot{\varphi}(\gamma) \partial_n (u^{nk} \xi_{ki} u^{im} \varphi_m) \\
&- \int \dot{\varphi}(\gamma)_n u^{kn} \xi u_{klm} B_{lm} \\
&= \int \xi (\partial_i (u^{kn} \dot{\varphi}(\gamma)_n))(\partial_k (u^{in} \varphi_n)) + \int \dot{\varphi}(\gamma) \partial_n (u^{nk} \xi_{ki} u^{im} \varphi_m) \\
&+ \int \xi (\partial_m (u^{kn} \dot{\varphi}(\gamma)_n)) u_{kl} B_{lm} + \int \xi_m \dot{\varphi}(\gamma)_n B_{nm} \\
&= \int \xi (\partial_i (u^{kn} \dot{\varphi}(\gamma)_n))(\partial_k (u^{in} \varphi_n)) + \int \xi u_{kl} B_{lm} \partial_m (u^{kn} \dot{\varphi}(\gamma)_n) \\
&+ \int \dot{\varphi}(\gamma) \Delta \dot{\varphi}(\xi)\,.
\end{aligned}
$$

Notice that the first two terms coincide with $\int_M \xi L_1(\gamma)$, while the third one is symmetric in ξ and γ. An identical computation for (ii) and (iii) shows that L is formally self-adjoint.

A similar argument using repeated integration by parts proves that $\int_M \gamma L(\gamma) \leq 0$, with $\int_M \gamma L(\gamma) = 0$ if and only if $\gamma = 0$, so that L has trivial kernel.

4 Coupled equations on rules surfaces

Chapter contents

In this Chapter, we study deformed Hermitian Yang-Mills (dHYM) connections on ruled surfaces explicitly, using the momentum construction. As a main application we provide many new examples of dHYM connections coupled to a variable background Kähler metric. These are solutions of the moment map partial differential equations given by the Hamiltonian action of the extended gauge group, coupling the dHYM equation to the scalar curvature of the background. The large radius limit of these coupled equations is the Kähler-Yang-Mills system of Álvarez-Cónsul, Garcia-Fernandez and García-Prada, and in this limit our solutions converge smoothly to those constructed by Keller and Tønnesen-Friedman [KTF12]. We also discuss other aspects of our examples including conical singularities, realisation as B-branes, the small radius limit and canonical representatives of complexified Kähler classes. While this is a classical test bed for equations in complex differential geometry, here we allow a rather general setup, as we now discuss. Our main results, together with the necessary background, are contained in Section 4.1. In Section 4.2 we set up the momentum construction on our ruled surfaces. Section 4.3 solves the dHYM equation on our ruled surfaces explicitly using the momentum construction, under the necessary "stability condition" (4.1.2). This result is applied in Section 4.4 in order to solve the coupled equations (2.0.1). All of this is extended to allow conical singularities in Section 4.5; the main advantage is that in this case there exist solutions with positive coupling constants. Finally Section 4.7 contains our results on the large and small radius limits.

4.1 Background and main results

Let Σ be a compact Riemann surface of genus h, with Kähler metric g_Σ of constant scalar curvature $2s_\Sigma$, and let $\mathcal{L} \xrightarrow{p} \Sigma$ denote a holomorphic line bundle of degree $k \in \mathbb{Z}_{>0}$, with $2\pi c_1(\mathcal{L}) = [\omega_\Sigma]$. Since $\mathrm{Vol}(\Sigma) = 2\pi k$, by the Gauss-Bonnet formula we have

$$s_\Sigma = \frac{1}{\mathrm{Vol}(\Sigma)} \int_\Sigma s_\Sigma \omega_\Sigma = \frac{1}{\mathrm{Vol}(\Sigma)} \int_\Sigma \rho_\Sigma = \frac{2(1-h)}{k},$$

where ρ_Σ denotes the Ricci 2-form of g_Σ.

We will construct solutions of the coupled equations (2.0.1) on ruled surfaces of Hirzebruch type, obtained by the projectivization

$$M = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \to \Sigma,$$

where $\mathcal{O}$ denotes the trivial holomorphic line bundle. (It is well known that such M does not admit cscK metrics). Our solutions are obtained by extending the classical *momentum construction* (also known as the Calabi ansatz, see [HS02]) to the equations (2.0.1): see (4.2.2), (4.3.1) for our ansatz.

Let $E_0 = \mathbb{P}(0 \oplus \mathcal{O})$ and $E_\infty = \mathbb{P}(\mathcal{L} \oplus 0)$ denote respectively the zero section and the infinity section of the $\mathbb{CP}^1$-bundle M over Σ, with general fibre C. We introduce the real parameters k_1, k_2 and $k' > 0$, and consider the cohomology classes

$$\begin{aligned} [\omega] &= 2\pi[2E_0 + k'C], \\ [F] &= 2\pi[2(k_1 - k_2)E_0 + (2kk_2 + k'(k_1 + k_2)C], \end{aligned} \tag{4.1.1}$$

where we slightly abuse the notation and denote the Poincaré duals of E_0 and C respectively by $[E_0]$ and $[C]$. Then $[\omega]$ is a Kähler class and $[F/(2\pi)]$ is integral, provided k_1, k_2, k' are integers and $k' > 0$, so it is possible to find a holomorphic line bundle $L \to X$ such that $-2\pi c_1(L) = [F]$.

Remark 4.1.1. The first equation (2.0.1) is equivalent to

$$\mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}\left(\omega - \sqrt{-1}F\right)^n\right) = 0$$

and the latter condition is preserved when $F \to -F$ and $\hat{\theta} \to -\hat{\theta}$, which should be interpreted geometrically as considering the dHYM equation on L^{-1} instead of L. With our choice of parametrization, this implies that the set of parameters corresponding to solutions of the system (2.0.1) is invariant under $k_i \to -k_i$, for $i = 1, 2$. When $k_2 = 0$, it follows from (4.1.1) that, for any choice of Kähler class, the unique solution of the dHYM equation is given by $F = k_1\omega$. In this case, the Lagrangian radius is also constant $r_\omega(k_1\omega) = (1 + k_1^2)^2$ and, since M does not admit cscK metrics, the system (2.0.1) has no solution. In Section 2 it will be clear that also for $k_1 = 0$ the dHYM equation has a trivial solution; in this case, $\hat{\theta} = 0$ and we can solve also (2.0.1). In the following, we will focus on the nontrivial choices of parameters, i.e.

$$k_1 < 0, \qquad k_2 \neq 0.$$

It is also convenient to introduce the quantity

$$x = \frac{k}{k+k'} \in (0,1).$$

Theorem 4.1.2. *Suppose the "stability condition"*

$$(1+(k_1+k_2)^2) > x(1+(k_1-k_2)^2) \tag{4.1.2}$$

holds. Then, there exist a unique Kähler form ω and curvature form F, with cohomology classes given by (4.1.1)*, such that they are obtained by the momentum construction (see* (4.2.2), (4.3.1)*) and solve the coupled equations* (2.0.1) *on the ruled surface M, for the unique value of the coupling constant*

$$\alpha = \frac{\sqrt{4k_1^2+(1-k_1^2+k_2^2)^2}}{2(1+(k_1-k_2)^2)k_2^2}(-2+s_\Sigma x)\,.$$

If equality holds instead in (4.1.2)*, then there is a smooth solution on $M \setminus E_\infty$, with underlying metric $\omega \in C^{1,1/2}(M) \cap C^\infty(M \setminus E_\infty)$.*

Theorem 4.1.2 is proved in Section 4.4.

4.1.1 Solutions with conical singularities

The main limitation of Theorem 4.1.2 concerns the sign of the coupling constant: it is straightforward to check that in the situation of that result we always have $\alpha < 0$, since $s_\Sigma \leq 2$ and $x < 1$. In order to gain more flexibility we allow the background metric ω to develop conical singularities along the divisors E_0, E_∞. Fix $0 < \beta_0 \leq 1$ and let

$$\beta_\infty = \frac{-2+\beta_0(1+x)}{-1+x} \geq 1.$$

Theorem 4.1.3. *Suppose the "stability condition"* (4.1.2) *holds. Then, there exist a unique Kähler form ω and curvature form F, such that they are obtained by the momentum construction (see* (4.2.2), (4.3.1)*), ω has conical singularities with cone angles $2\pi\beta_0$ along E_0 and $2\pi\beta_\infty$ along E_∞, the corresponding cohomology classes (in the sense of currents) are given by* (4.1.1)*, and they solve the coupled equations* (2.0.1)*, for the unique value of the coupling constant*

$$\alpha = \frac{\sqrt{4k_1^2+(1-k_1^2+k_2^2)^2}}{2(1+(k_1-k_2)^2)k_2^2}\,\frac{3+x+s_\Sigma x^2-3(1+x)\beta_0}{x}.$$

Theorem 4.1.3 is proved in Section 4.5. Note that this gives a generalisation of Theorem 4.1.2: when $\beta_0 = 1$ we recover precisely the smooth solutions provided by that result.

Corollary 4.1.4. *For sufficiently small cone angle $2\pi\beta_0$ and sufficiently large $k' > 0$, the coupling constant α is positive.*

4.1.2 Relation to twisted KE metrics

As usual, under a suitable cohomological condition, the equation in (2.0.1) involving the scalar curvature may be reduced to a condition involving the Ricci curvature. In our case, this condition is given by

$$[\mathrm{Ric}(\omega)] + \frac{\alpha}{2\sin\hat{\theta}}[F] = \frac{\hat{s}-\alpha\hat{r}}{4}[\omega].$$

Then, the equation

$$s(\omega) - \alpha r_{\omega}(F) = \hat{s} - \alpha\hat{r}$$

reduces to the twisted Kähler-Einstein equation

$$\mathrm{Ric}(\omega) + \frac{\alpha}{2\sin\hat{\theta}}F = \frac{\hat{s}-\alpha\hat{r}}{4}\omega. \tag{4.1.3}$$

We provide an explicit criterion for when this reduction occurs for the class of examples provided by Theorem 4.1.3 (in which case $\mathrm{Ric}(\omega)$, F and ω extend to closed currents on M).

Proposition 4.1.5. *The condition $s(\omega)-\alpha r_{\omega}(F) = \hat{s}-\alpha\hat{r}$ reduces to the twisted Kähler-Einstein equation (4.1.3) iff we have*

$$\begin{aligned}&(1+k_1^2+k_2^2)(x-1)\left(s_{\Sigma}x^2 - 3\beta_0(x+1) + x + 3\right)\\ &= 2k_1k_2\left(-3\beta_0 + s_{\Sigma}x^3 - x^2(\beta_0 + s_{\Sigma} - 1) + 3\right).\end{aligned} \tag{4.1.4}$$

Morevoer, there are infinitely many admissible values of k_1, k_2, k' which satisfy this equality for some β_0 and for which the "stability condition" (4.1.2) holds (so that the corresponding coupled equations are solvable).

This result is proved in Section 6. Writing the dHYM equation on the surface M in Monge-Ampère form (as in [JYC20]) we see that in the twisted Kähler-Einstein case the coupled equations (2.0.1) become

$$\begin{cases}\left(-\sin(\hat{\theta})F + \cos(\hat{\theta})\omega\right)^2 = \omega^2\\ \mathrm{Ric}(\omega) + \frac{\alpha}{2\sin\hat{\theta}}F = \frac{\hat{s}-\alpha\hat{r}}{4}\omega,\end{cases}$$

and so they are closely related to the systems of coupled Monge-Ampère equations studied by Hultgren and Wytt-Nyström [HWN19].

4.1.3 Realisation as B-branes

Given the origin of the dHYM equation in mirror symmetry, it seems interesting to ask whether the special dHYM connections appearing in Theorem 4.1.2, i.e. solutions of the coupled equations (2.0.1), can in fact be realised as B-branes (i.e. for our purposes, holomorphic submanifolds endowed with a dHYM connection) in some ambient Calabi-Yau manifold (this is how the dHYM equation appears in mathematical physics, see e.g. [CXY17]). Thus we are asking for a Calabi-Yau manifold $\check{N}$ with a Ricci flat Kähler metric $\omega_{\check{M}}$, and a holomorphic embedding $\iota: M \hookrightarrow \check{N}$, such that the Kähler form ω constructed in Theorem 2.0.1 is given by the restriction $\omega = \iota^* \omega_{\check{N}}$. We show that this can be achieved at least locally around M, relying on the classical results on Feix [Fei01] on the hyperkähler extension of real analytic Kähler metrics.

Proposition 4.1.6. *The Kähler form ω and curvature form F provided by Theorem 4.1.2 are real analytic. Thus, ω extends to a hyperkähler metric defined on an open neighbourhood of the zero section in the holomorphic cotangent bundle T^*M, and F extends to the curvature form of a hyperholomorphic line bundle defined on the same open neighbourhood.*

This result is proved in Section 4.4.

4.1.4 Large and small radius limits

In the mathematical physics literature (see e.g. [ABC+09], Chapter 1), the dHYM equation involves a "slope" parameter $\alpha' > 0$ (related to the "string length" by $\alpha' = l_s^2$), which appears simply as a scale parameter for the curvature form, $F \mapsto \alpha' F$. The corresponding coupled equations (2.0.1) are given by

$$\begin{cases} \Theta_\omega(\alpha' F) = \hat{\theta} \mod 2\pi \\ s(\omega) - \alpha r_\omega(\alpha' F) = \hat{s} - \alpha \hat{r}. \end{cases} \tag{4.1.5}$$

The expressions "large radius limit" (or "zero slope limit") refer to the behaviour of the dHYM equations and their solutions as $\alpha' \to 0$. As explained in Chapter 2 Section 2.3, the large radius limit of our coupled equations is the (rank 1 case of) the Kähler-Yang-Mills system introduced by Álvarez-Cónsul, Garcia-Fernandez and García-Prada [AGG13]. We can prove a much stronger result, at the level of solutions, on the ruled surface M.

Theorem 4.1.7. *For all sufficiently small α', depending only on the fixed parameters k_1, k_2, k', (i.e. on the fixed cohomology classes $[\omega], [F]$), the coupled equations (4.1.5) are uniquely solvable on M with the momentum construction. Moreover, as $\alpha' \to 0$, the corresponding solutions $\omega_{\alpha'}$, $F_{\alpha'}$ converge smoothly to a solution of the Kähler-Yang-Mills system*

$$\begin{cases} \Lambda_\omega F = \mu \\ s(\omega) + \tilde{\alpha}\, \Lambda^2_\omega (F \wedge F) = c \end{cases} \tag{4.1.6}$$

for some (explicit) coupling constant $\tilde{\alpha} < 0$.

The particular solutions of the Kähler-Yang-Mills system obtained in this limit are due to Keller and Tønnesen-Friedman [KTF12].

Similarly, the "small radius limit" (or "infinite slope limit") concerns the behaviour of the coupled equations (4.1.5) as $\alpha' \to \infty$.

Theorem 4.1.8. *Fix parameters k_1, k_2, k' (i.e. cohomology classes $[\omega]$, $[F]$) such that the "stability condition"*

$$(k_1 + k_2)^2 > x(k_1 - k_2)^2$$

holds. Then the coupled equations (4.1.5) are uniquely solvable on M with the momentum construction, for all $\alpha' > 0$. Moreover, as $\alpha' \to \infty$, the corresponding solutions $\omega_{\alpha'}$, $F_{\alpha'}$ converge smoothly to a solution of the system

$$\begin{cases} F \wedge \omega = c_1 F^2 \\ s(\omega) - \hat{\alpha}\, \Lambda_\omega F = c_2. \end{cases}$$

for some (explicit) coupling constant $\hat{\alpha}$.

At least in the case when F is Kähler, this system couples the J-equation $\Lambda_F \omega = c_1'$ for F to a twisted cscK equation for ω. In general, these limiting equations belong to a class of coupled PDEs studied by Datar and Pingali [DP20].

Theorems 4.1.7 and 4.1.8 are proved in Section 4.7.

4.1.5 Complexified Kähler classes

Complexified Kähler classes are expressions of the form $[\omega + \sqrt{-1}B]$, where ω is a Kähler form and $[B] \in H^2(M, \mathbb{R})/H^2(M, \mathbb{Z})$ is known as the B-field. They play an important role in mirror symmetry (see e.g. [Tho] Section 2). Let M be a compact Kähler manifold with no holomorphic 2-forms. Collins and Yau (see [CY18] Section 8) consider a dHYM equation on M of the form

$$\Theta_\omega(F + B) = \hat{\theta} \mod 2\pi,$$

where $\sqrt{-1}F$ is the unknown curvature form of a Hermitian holomorphic line bundle $L \to M$ and B is a fixed representative of a (lift of a) B-field. Arguing from mirror symmetry, they propose that the existence of a solution F should be related, conjecturally, to the a suitable notion of stability of the object L with respect to the complexified Kähler class $[\omega + \sqrt{-1}B]$.

In the special case then L is the trivial bundle $\mathcal{O}_M$, the equation becomes

$$\Theta_\omega(B + \sqrt{-1}\partial\bar{\partial}u) = \hat{\theta} \mod 2\pi,$$

so we are effectively trying to find a canonical representative of the B-field $[B]$ with respect to a background Kähler form ω; the existence of such a representative should be related to the stability of the object $\mathcal{O}_M$ with respect to $[\omega + \sqrt{-1}B]$.

Our coupled equations

$$\begin{cases} \Theta_\omega(B) = \hat{\theta} \mod 2\pi \\ s(\omega) - \alpha r_\omega(B) = \hat{s} - \alpha \hat{r}, \end{cases} \tag{4.1.7}$$

with $[B]$ a (lift of a) class in $H^2(M,\mathbb{R})/H^2(M,\mathbb{Z})$, can then be thought of as trying to prescribe a canonical representative of the complexified Kähler class $[\omega + \sqrt{-1}B]$. Note that in the Calabi-Yau case, at zero coupling $\alpha = 0$ and in the large radius limit, these equations for the complex form $\omega + \sqrt{-1}B$ reduce to the conditions

$$\begin{cases} \Delta_\omega B = 0 \\ \mathrm{Ric}(\omega) = 0, \end{cases}$$

which are standard in the physics literature (see e.g. [ABC+09] Section 1.1).

As an example we shall discuss the existence of such a canonical representative for the complexified Kähler class

$$[\omega + \sqrt{-1}B] = 2\pi(2E_0 + (k' + \sqrt{-1}k'')C)$$

on our ruled surfaces M, where the Kähler condition is equivalent to $k' > 0$. The key observation is that this can be expressed in the form

$$\begin{aligned} [\omega] &= 2\pi[2E_0 + k'C], \\ [B] &= 2\pi[2(k_1 - k_2)E_0 + (2kk_2 + k'(k_1 + k_2)C] \end{aligned}$$

with the special choices

$$k_1 = k_2 = \frac{k''}{2(k + k')},$$

provided we have $k'' < 0$. Thus we may apply Theorem 4.1.2 (and, more generally, Theorem 4.1.3 in the case of conical singularities) to show that the coupled equations (4.1.7) are solvable, uniquely under the momentum construction, iff the "stability condition"

$$(1 + (k_1 + k_2)^2) = 1 + \left(\frac{k''}{k + k'}\right)^2 > x(1 + (k_1 - k_2)^2) = \frac{k}{k + k'}$$

holds. But, clearly, this is automatically satisfied. By Remark 4.1.1, the same argument works for the case $k'' > 0$.

Corollary 4.1.9. *The complexified Kähler class*

$$[\omega + \sqrt{-1}B] = 2\pi(2E_0 + (k' + \sqrt{-1}k'')C),$$

where $k' > 0$, $k'' \neq 0$, admits a canonical representative. This also holds allowing conical singularities; the corresponding coupling constant is given by

$$\alpha = \frac{2\sqrt{(k + k')^2 + (k'')^2}\left(k^2(-6\beta_0 + s_\Sigma + 4) + (7 - 9\beta_0)kk' - 3(\beta_0 - 1)(k')^2\right)}{k(k'')^2}.$$

Note that a canonical representative with vanishing B-field $B = 0$ would correspond to a cscK metric, which does not exist. The coupling constant α diverges as $k'' \to 0$. It seems interesting that a nontrivial B-field can stabilise the unstable ruled surface X.

4.2 Momentum construction

Let $M = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \to \Sigma$ be a ruled surface as in the Section 4.1. Let

$$E_0 = \mathbb{P}(0 \oplus \mathcal{O}),\ E_\infty = \mathbb{P}(\mathcal{L} \oplus 0)$$

denote respectively the zero section and the infinity section of the $\mathbb{CP}^1$-bundle M over Σ, with general fibre C. We have the straightforward intersection formulae:

$$E_0 \cdot E_0 = -E_\infty \cdot E_\infty = k, \quad C \cdot C = 0, \quad C \cdot E_0 = C \cdot E_\infty = 1. \tag{4.2.1}$$

We will follow the standard *momentum construction* (sometimes called the Calabi ansatz, see e.g. [HS02]) for metrics on the complement of the zero section $M_0 = \mathcal{L} \setminus E_0$, which extend across the zero and infinity sections of M under suitable conditions.

Thus we consider metrics of the form

$$\omega = \frac{p^*\omega_\Sigma}{x} + \sqrt{-1}\partial\bar{\partial} f(s), \tag{4.2.2}$$

where x is a real parameter satisfying $0 < x < 1$, while f is a strictly convex function, such that $f' : M_0 \to (-1, 1)$. The real coordinate s is the log-norm of the Hermitian metric $h(z)$ on $\mathcal{L}$ for which $-\partial_z \bar{\partial}_z \log(h) = F(h) = -\sqrt{-1}\omega_\Sigma$. Considering a trivialization $U \subset \mathcal{L}$ with adapted bundle coordinates (z, w), s is given by

$$s = \log|\,(z, w)\,|_h^2 = \log|w|^2 + \log h(z),$$

and it follows that

$$\sqrt{-1}\partial_w \bar{\partial}_w f(s) = \sqrt{-1} f''(s) \frac{dw \wedge d\bar{w}}{|w|^2}$$

and

$$\sqrt{-1}\partial_z \bar{\partial}_z f(s) = -f'(s)\omega_\Sigma + \sqrt{-1} f''(s) \frac{\partial_z h \bar{\partial}_z h}{h^2}$$

If we choose U such that $d\log h(z_0) = 0$ in (z_0, w_0), at this point all the mixed derivatives vanish and so we find

$$\omega = \frac{1 - x f'(s)}{x}\omega_\Sigma + \sqrt{-1} f''(s) \frac{dw \wedge d\bar{w}}{|w|^2};$$

moreover we also have, globally,

$$\omega^2 = \frac{2}{|w|^2} \frac{1 - x f'(s)}{x} f''(s) \omega_\Sigma \wedge \sqrt{-1} dw \wedge d\bar{w}.$$

Since $f(s)$ is strictly convex, we may consider its Legendre transform $u(\tau)$, a function of the variable $\tau = f'(s)$, and define the *momentum profile*

$$\phi(\tau) = \frac{1}{u''(\tau)} = f''(s),$$

which must satisfy the condition

$$\phi(\tau) > 0, \quad \text{for } -1 < \tau < 1, \tag{4.2.3}$$

required for ω to be positive. Moreover the momentum construction shows that in order to extend ω across $w = 0$ and $w = \infty$, $\phi(\tau)$ must satisfy the boundary conditions

$$\lim_{\tau \to \pm 1} \phi(\tau) = 0, \quad \lim_{\tau \to \pm 1} \phi'(\tau) = \mp 1. \tag{4.2.4}$$

The space $H^2(M, \mathbb{R})$ is generated by the Poincaré duals of E_0 and C. Following [KTF12], we define the 2-form

$$\beta = \frac{x^2}{(1 - xf'(s))^2} \left(\frac{1 - xf'(s)}{x} \omega_\Sigma - \sqrt{-1} f''(s) \frac{dw \wedge d\bar{w}}{|w|^2} \right).$$

A direct computation shows that β is a closed $(1,1)$-form, traceless with respect to ω, and $\{\omega, \beta\}$ is a basis for the space $H^2(M, \mathbb{R})$. We consider now a real $(1,1)$ cohomology class and its representative

$$F_0 = c_1 \omega + c_2 \beta. \tag{4.2.5}$$

In order to identify $\sqrt{-1} F_0$ with the curvature form of a connection on some line bundle over M, $[F_0/(2\pi)]$ must be an integral class. For $[F_0] = a\,[E_0] + b\,[C]$, using the identities (4.2.1), we have

$$a = \int_C F, \qquad b = \int_{E_0} F_0 - k \int_C F_0. \tag{4.2.6}$$

Since $E_0 = (f')^{-1}(-1)$, we get

$$\int_{E_0} \omega = \frac{(1+x)}{x} \int_\Sigma \omega_\Sigma = 2\pi k \frac{(1+x)}{x}$$

and

$$\int_{E_0} \beta = \frac{x}{(1+x)} \int_\Sigma \omega_\Sigma = 2\pi k \frac{x}{(1+x)}.$$

For the general fibre C, let w denote the bundle adapted coordinate along the fibre and define $r = |w|$, such that $s = 2 \log r$ and $d/ds = \frac{r}{2} d/dr$. Using the boundary conditions (4.2.4), we have

$$\begin{aligned}
\int_C \omega &= \int_{\mathbb{C} \setminus \{0\}} \sqrt{-1} f''(s) \frac{dw \wedge d\bar{w}}{|w|^2} \\
&= \int_{-\infty}^{+\infty} \int_0^{2\pi} \frac{d}{dr} f'(s) dr \wedge d\theta \\
&= 2\pi \left(\lim_{s \to \infty} f'(s) - \lim_{s \to -\infty} f'(s) \right) \\
&= 4\pi
\end{aligned}$$

and similarly

$$\int_C \beta = -4\pi \frac{x^2}{1-x^2}.$$

Using (4.2.6), we obtain

$$\left[\frac{F_0}{2\pi}\right] = \left(2c_1 - 2\frac{x^2}{1-x^2}c_2\right)E_0 + \left(\frac{1-x}{x}kc_1 + \frac{x}{1-x}kc_2\right)C.$$

If we introduce the new parameterization

$$x = \frac{k}{k+k'},\ c_1 = k_1,\ c_2 = \frac{1-x^2}{x^2}k_2, \tag{4.2.7}$$

for real k_1, k_2 and $k' > 0$, then a direct calculation shows that the cohomology classes of $[\omega]$ and $[F_0]$ are given by our previous formulae

$$\begin{aligned}[\omega] &= 2\pi[2E_0 + k'C],\\ [F_0] &= 2\pi[2(k_1-k_2)E_0 + (2kk_2 + k'(k_1+k_2)C].\end{aligned}$$

In particular we see that the choices $k' \in \mathbb{Z}_{>0}$ and $k_i \in \mathbb{Z}$ for $i = 1, 2$ correspond to integral classes.

4.3 dHYM on ruled surfaces

In this Section we will solve the dHYM equation on M explicitly, with respect to a fixed Kähler metric ω obtained by the momentum construction (4.2.2). Given a class $[F]$ satisfying the integrality conditions (4.2.7), we may fix a holomorphic line bundle $L \to M$ with first Chern class $-2\pi\,[c_1(L)] = [F]$.

Recall that the parameter $\hat{\theta}$ is a topological constant determined by the condition

$$\int_M (\omega - \sqrt{-1}F)^2 \in \mathbb{R}_{>0}e^{\sqrt{-1}\hat{\theta}}.$$

Lemma 4.3.1. *We have*

$$e^{\sqrt{-1}\hat{\theta}} = \frac{\left(1 - k_1^2 + k_2^2 - 2\sqrt{-1}k_1\right)}{\sqrt{(1-k_1^2+k_2^2)^2 + (2k_1)^2}}.$$

Proof. Since β is traceless with respect to ω, we only need to compute the quantities $\int_M \omega^2, \int_M \beta^2$. We have

$$\begin{aligned}\int_M \omega^2 &= 2\int_M f''(s)\frac{(1-xf'(s))}{x}\omega_\Sigma \wedge \frac{dw\wedge d\bar{w}}{|w|^2}\\ &= 4\pi\int_\Sigma \omega_\Sigma \int_0^\infty \frac{d}{dr}\left(\frac{f'(s)}{x} + \frac{(f'(s))^2}{2}\right)dr\\ &= \frac{16\pi^2 k}{x}\end{aligned}$$

and similarly

$$\int_M \beta^2 = -\frac{16\pi^2 k}{x}\frac{x^4}{(1-x^2)^2}$$

Using (4.2.7), we find

$$\int_M \left(\omega - \sqrt{-1}F\right)^2 = \frac{16\pi^2 k}{x}\left(1 - k_1^2 + k_2^2 - 2\sqrt{-1}k_1\right),$$

from which the claim follows immediately. □

In order to solve the dHYM equation in the class $[F_0]$ we extend the momentum construction by making the ansatz

$$F = F_g = F_0 + \sqrt{-1}\partial\bar{\partial}g(s). \tag{4.3.1}$$

It will be convenient to introduce the function $\nu(\tau)$ given by the image of $g'(s)$ under the Legendre transform diffeomorphism relative to $f(s)$.

Lemma 4.3.2. *The form $\sqrt{-1}\partial\bar{\partial}g(s)$ extends smoothly to an exact form on M iff $\nu(\tau)$ extends smoothly to the interval $[-1,1]$ and vanishes at the boundary points.*

Proof. The component of $\sqrt{-1}\partial\bar{\partial}g(s)$ in the fibre direction is

$$\sqrt{-1}g''(s)\frac{dw\wedge d\bar{w}}{|w|^2} = \sqrt{-1}\nu'(\tau)\phi(\tau)\frac{dw\wedge d\bar{w}}{|w|^2}.$$

So $\sqrt{-1}\partial\bar{\partial}g(s)$ extends smoothly to M iff $\nu(\tau)$ extends smoothly to $[-1,1]$. In order to derive the appropriate boundary behaviour so that this extension is still exact, we compute

$$\int_{E_0}\partial\bar{\partial}g = -2\pi k\left(\lim_{s\to-\infty}g'(s)\right)$$

and

$$\int_C \partial\bar{\partial}g = 2\pi\left(\lim_{s\to\infty}g'(s) - \lim_{s\to-\infty}g'(s)\right).$$

Using (4.2.6), the only conditions we need to impose are

$$\lim_{\tau\to\pm1}\nu(\tau) = 0. \tag{4.3.2}$$

□

Our next result shows how to reduce the dHYM equation to an ODE. It is convenient to introduce the new variable

$$t = 1/x - \tau$$

as well as the auxiliary function

$$H(t) = k_1 t + \frac{k_2}{t}\frac{1-x^2}{x^2} - \nu(t). \tag{4.3.3}$$

Proposition 4.3.3. *Under the momentum construction* (4.2.2), (4.3.1)*, the dHYM equation is equivalent to the ODE*

$$H'(t) = \frac{t\sin\hat{\theta} + H(t)\cos\hat{\theta}}{H(t)\sin\hat{\theta} - t\cos\hat{\theta}}, \tag{4.3.4}$$

together with the boundary conditions

$$\begin{aligned} H\left(\frac{1+x}{x}\right) &= k_1\left(\frac{1+x}{x}\right) + k_2\left(\frac{1-x}{x}\right), \\ H\left(\frac{1-x}{x}\right) &= k_1\left(\frac{1-x}{x}\right) + k_2\left(\frac{1+x}{x}\right). \end{aligned} \tag{4.3.5}$$

Proof. At a point (z_0, w_0) such that $d\log h(z_0) = 0$, we have

$$\begin{aligned} &\omega - iF_g = \\ &\left(\left(1 - \sqrt{-1}k_1\right)\frac{1-xf'}{x} - \sqrt{-1}\frac{k_2}{x}\frac{1-x^2}{1-xf'} + \sqrt{-1}g'\right)\omega_\Sigma + \\ &\left(f''\left(1 - \sqrt{-1}k_1 + \sqrt{-1}k_2\frac{1-x^2}{1-xf'}\right) - \sqrt{-1}g''\right)\sqrt{-1}\frac{dw\wedge d\bar{w}}{|w|^2}. \end{aligned}$$

and we obtain the global identity

$$\begin{aligned} &\frac{1}{2}\operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}}\left(\omega - \sqrt{-1}F_g\right)^2\right)/\sqrt{-1}\frac{dw\wedge d\bar{w}}{|w|^2}\wedge\omega_\Sigma = \\ &-\sin\hat{\theta}\left(f''\frac{1-xf'}{x} + \left(g' - \frac{k_2}{x}\frac{1-x^2}{1-xf'} - \frac{k_1}{x} + k_1f'\right)\left(g'' + k_1f'' - k_2f''\frac{1-x^2}{(1-xf')^2}\right)\right) \\ &+\cos\hat{\theta}\left(\frac{1-xf'}{x}\left(k_2f''\frac{1-x^2}{(1-xf')^2} - g'' - k_1f''\right) + f''\left(g' - \frac{k_2}{x}\frac{1-x^2}{1-xf'} - \frac{k_1}{x} + k_1f'\right)\right). \end{aligned} \tag{4.3.6}$$

This expression becomes much simpler under the Legendre transform diffeomorphism in terms of the variable $\tau = f'(s)$, for which $d\tau/ds = \phi(\tau)$, and the additional affine change of variable $t = 1/x - \tau$. Setting

$$H(t) = k_1 t + \frac{k_2}{t}\frac{1-x^2}{x^2} - v(t),$$

the dHYM equation is equivalent to

$$2\phi\left(\cos\hat{\theta}\,(H + tH') + \sin\hat{\theta}\,(t - HH')\right) = 0$$

and, since $\phi > 0$, also to

$$H' = \frac{t\sin\hat{\theta} + H\cos\hat{\theta}}{H\sin\hat{\theta} - t\cos\hat{\theta}}.$$

A direct computation shows that the boundary conditions (4.3.2) for $g(s)$, rephrased in term of $H(s)$, become the constraints (4.3.5). ☐

Corollary 4.3.4. *The ODE (4.3.4) is solvable with the boundary conditions (4.3.5) iff the "stability condition"*

$$\left(1+(k_1+k_2)^2\right) > x\left(1+(k_1-k_2)^2\right)$$

holds.

Proof. Setting $tv = H$, equation (4.3.4) becomes

$$tv' = -2\frac{\xi(v)}{\xi'(v)}, \tag{4.3.7}$$

with $\xi(v) = v^2 \sin\hat{\theta} - 2v\cos\hat{\theta} - \sin\hat{\theta}$. Solving (4.3.7) by separation of variables, we get

$$\xi(v) = \frac{C}{t^2},$$

which has two solutions given by

$$H_\pm(t) = t\cot\hat{\theta} \pm \sqrt{\left(\cot^2\hat{\theta}+1\right)(t^2+C')}, \tag{4.3.8}$$

with $C' = C\sin\hat{\theta}$. We need to impose the appropriate boundary conditions (4.3.5). The first condition at $1/x+1$ holds iff we choose the solution H_- in (4.3.8) and set

$$C = \frac{-2k_2\left(1+(k_1+k_2)^2 - x^2 - (k_1-k_2)^2 x^2\right)}{x^2\sqrt{\left(1-k_1^2+k_2^2\right)^2+(2k_1)^2}}.$$

In this case, at $1/x-1$ we have

$$\begin{aligned}
H_-\left(\frac{1-x}{x}\right) &= \frac{1}{2xk_1}\left(-k_1^2(-1+x) + \left(1+k_2^2\right)(-1+x)\right) \\
&+ \left|\frac{-1-(k_1+k_2)^2+x+(k_1-k_2)^2 x}{2k_1 x}\right| \\
&= \begin{cases} k_1\left(\frac{1-x}{x}\right)+k_2\left(\frac{1+x}{x}\right) & \text{if } \left(1+(k_1+k_2)^2\right) > x\left(1+(k_1-k_2)^2\right) \\ \frac{\left(1+k_2^2\right)(x-1)-k_1k_2(1+x)}{k_1 x} & \text{if } \left(1+(k_1+k_2)^2\right) < x\left(1+(k_1-k_2)^2\right), \end{cases}
\end{aligned}$$

so the second condition in (4.3.5) holds iff we have

$$\left(1+(k_1+k_2)^2\right) > x\left(1+(k_1-k_2)^2\right).$$

□

Remark 4.3.5. Jacob and Yau [JY17] showed that the solvability of the dHYM equation on compact Kähler surfaces is equivalent to a certain geometric "stability condition". Considering the closed, real $(1,1)$-form

$$\Omega = \cot\hat{\theta}\,\omega - F,$$

the relevant condition is $[\Omega] > 0$. In our setting, when we regard $H^2(M,\mathbb{R})$ as $\mathbb{R}^2$ with the basis provided by the Poincaré duals of E_0 and C and coordinates (a_1, a_2), the Kähler cone is identified with the subset $\{a_1 > 0, a_2 > 0\}$. A computation shows that the $[\Omega]$ is positive precisely when the condition (4.1.2) is satisfied.

Remark 4.3.6. Suppose equality holds instead in our "stability condition" (4.1.2),

$$\left(1 + (k_1 + k_2)^2\right) = x\left(1 + (k_1 - k_2)^2\right).$$

A direct computation then shows that the quantity $t^2 + C'$ vanishes at the endpoint $t = 1/x - 1$. By our explicit formula (4.3.8) we see that the function $H_-(t)$ is smooth on the interval $(1/x - 1, 1/x + 1]$ and extends to a $C^{1/2}$ function on its closure. Thus, for fixed background ω, we obtain a corresponding solution to the dHYM equation which is smooth on $M \setminus E_\infty$ and extends to a form with $C^{1/2}$ coefficients on M. This should be compared with a result of Takahashi [Tak21] which holds for a general compact Kähler surface M, and states that under suitable assumptions, when the class $[\Omega]$ above is only semipositive, then there exists a solution to the dHYM equation which is smooth on the complement of finitely many holomorphic curves of negative self-intersection and which extends to a closed current on M.

4.4 Coupled equations

In the previous Section we solved the dHYM equation in suitable integral classes, determining explicitly the Legendre transform of the curvature form F in terms of the Kähler metric ω. More precisely, let us assume that the “stability condition"

$$\left(1 + (k_1 + k_2)^2\right) > x\left(1 + (k_1 - k_2)^2\right)$$

holds, and let us denote by $F = F(\omega)$ the unique curvature form constructed in the previous Section.

In this Section we will complete the proof of Theorem 4.1.2 by solving the second equation in (2.0.1). We also establish the real analyticity of our solutions, Proposition 4.1.6.

Recall we are concerned with the equation

$$s(\omega) - \alpha\,\mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}\,\frac{\left(\omega - \sqrt{-1}F\right)^2}{\omega^2}\right) = \hat{s} - \alpha\hat{r}, \tag{4.4.1}$$

where the constants $\hat{s}$ and $\hat{r}$ can be computed as

$$\hat{s} = 2xs_{\Sigma} + 2,\ \hat{r} = \sqrt{(1 - k_1^2 + k_2^2)^2 + 4k_1^2}.$$

Lemma 4.4.1. *In terms of the variable $t = 1/x - \tau$ and the function $H(t)$ appearing in (4.3.4), we have*

$$\begin{aligned}&\operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}}\frac{\left(\omega - \sqrt{-1}F\right)^2}{\omega^2}\right)\\ &= \cos\hat{\theta}\left(1 - \frac{H(t)H'(t)}{t}\right) - \sin\hat{\theta}\left(H'(t) + \frac{H(t)}{t}\right).\end{aligned}$$

Proof. As in the proof of Proposition 4.3.3, at a point (z_0, w_0) such that $d\log h(z_0) = 0$, we have the global identities

$$\begin{aligned}&\frac{1}{2}\operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}}\left(\omega - \sqrt{-1}F_g\right)^2\right)/\sqrt{-1}\frac{dw\wedge d\bar{w}}{|w|^2}\wedge\omega_{\Sigma} =\\ &\cos\hat{\theta}\left(f''\frac{1-xf'}{x} + \left(g' - \frac{k_2}{x}\frac{1-x^2}{1-xf'} - \frac{k_1}{x} + k_1 f'\right)\left(g'' + k_1 f'' - k_2 f''\frac{1-x^2}{(1-xf')^2}\right)\right)\\ &+\sin\hat{\theta}\left(\frac{1-xf'}{x}\left(k_2 f''\frac{1-x^2}{(1-xf')^2} - g'' - k_1 f''\right) + f''\left(g' - \frac{k_2}{x}\frac{1-x^2}{1-xf'} - \frac{k_1}{x} + k_1 f'\right)\right).\end{aligned}$$

and

$$\omega^2 = 2f''\frac{1-xf'}{x}\sqrt{-1}\frac{dw\wedge d\bar{w}}{|w|^2}\wedge\omega_{\Sigma}.$$

In terms of the variable t and the auxiliary function $H(t)$ we have

$$\begin{aligned}&\left(f''\frac{1-xf'}{x} + \left(g' - \frac{k_2}{x}\frac{1-x^2}{1-xf'} - \frac{k_1}{x} + k_1 f'\right)\left(g'' + k_1 f'' - k_2 f''\frac{1-x^2}{(1-xf')^2}\right)\right)\\ &= 1 - \frac{H(t)H'(t)}{t},\end{aligned}$$

respectively

$$\begin{aligned}&\left(\frac{1-xf'}{x}\left(k_2 f''\frac{1-x^2}{(1-xf')^2} - g'' - k_1 f''\right) + f''\left(g' - \frac{k_2}{x}\frac{1-x^2}{1-xf'} - \frac{k_1}{x} + k_1 f'\right)\right)\\ &= -H'(t) - \frac{H(t)}{t},\end{aligned}$$

from which our claim follows immediately. □

Lemma 4.4.2. *Equation* (4.4.1) *becomes the ODE for the momentum profile* $\phi(t)$ *given by*

$$\left(\frac{2s_\Sigma}{t} - \frac{1}{t}\left(2t\phi(t)\right)''\right) + 2\alpha\frac{\cos\hat{\theta}}{\sin^2\hat{\theta}} - \frac{\alpha}{\sin^3\hat{\theta}}\frac{t}{\sqrt{\left(\cot^2\hat{\theta}+1\right)(t^2+C')}} - \frac{\alpha}{t\sin\hat{\theta}}\sqrt{\left(\cot^2\hat{\theta}+1\right)(t^2+C')} = \hat{s} - \alpha\hat{r},$$

with the boundary conditions

$$\lim_{t\to\frac{1}{x}\pm 1}\phi(t) = 0, \qquad \lim_{t\to\frac{1}{x}\pm 1}\phi'(t) = \mp 1.$$

Proof. By a standard computation, the scalar curvature of ω can be expressed in terms of the variable τ as

$$s(\omega) = \frac{2s_\Sigma x}{1-x\tau} - \frac{x}{1-x\tau}\left(2\phi(\tau)\frac{1-x\tau}{x}\right)'',$$

with $\phi(\tau)$ satisfying (4.2.4). After the affine change of variable $t = 1/x - \tau$, our claim follows directly from Lemma 4.4.1 and the explicit formula (4.3.8) for $H(t)$. □

Setting $\psi(t) = 2t\phi(t)$, we obtain the ODE

$$\psi''(t) = \left(2\alpha\frac{\cos\hat{\theta}}{\sin^2\hat{\theta}} - \hat{s} + \alpha\hat{r}\right)t - \frac{\alpha}{\sin\hat{\theta}}\sqrt{\left(\cot^2\hat{\theta}+1\right)(t^2+C')} - \frac{\alpha}{\sin^3\hat{\theta}}\frac{t^2}{\sqrt{\left(\cot^2\hat{\theta}+1\right)(t^2+C')}} + 2s_\Sigma \tag{4.4.2}$$

with the boundary conditions

$$\lim_{t\to\frac{1}{x}\pm 1}\psi(t) = 0, \qquad \lim_{t\to\frac{1}{x}\pm 1}\psi'(t) = \mp 2\left(\frac{1}{x}\pm 1\right), \tag{4.4.3}$$

and the positivity condition

$$\psi(t) > 0, \qquad \frac{1}{x} - 1 < t < \frac{1}{x} + 1. \tag{4.4.4}$$

By integrating twice, we get the general solution of (4.4.2) with integration constants d_0, d_1

$$\psi(t) = s_\Sigma t^2 + \left(\frac{\alpha}{3}\frac{\cos\hat{\theta}}{\sin^2\hat{\theta}} - \frac{\hat{s}-\alpha\hat{r}}{6}\right)t^3 - \frac{\alpha}{3}\sin\hat{\theta}\left(\left(\cot^2\hat{\theta}+1\right)(t^2+C')\right)^{\frac{3}{2}} + d_0 + d_1 t, \tag{4.4.5}$$

which satisfies (4.4.3) if and only if we set

$$d_0 = -\frac{(-2+s_\Sigma x)\left(-3-3k_1^2-2k_1k_2-3k_2^2+3\left(1+(k_1-k_2)^2\right)x^2\right)}{3\left(1+(k_1-k_2)^2\right)x^3},$$

$$d_1 = -\frac{\left(-2\left(1+k_1^2+k_2^2\right)+\left(1+(k_1-k_2)^2\right)s_\Sigma x\right)\left(-1+x^2\right)}{4k_1k_2x^2},$$

$$\alpha = \frac{\sqrt{4k_1^2+\left(1-k_1^2+k_2^2\right)^2}}{2\left(1+(k_1-k_2)^2\right)k_2^2}\left(-2+s_\Sigma x\right).$$

In order to check the positivity condition (4.4.4), we observe that

$$\frac{d^4\psi}{dt^4} = -\frac{3}{4}\alpha\left(\frac{C'}{k_1}\right)^2\left(t^2+C'\right)^{-\frac{5}{2}} > 0. \tag{4.4.6}$$

Moreover, setting $t_- = 1/x - 1$ and $t_+ = 1/x + 1$, we get

$$\begin{aligned}&\psi''(t_-) - \psi''(t_+)\\ &= 4\frac{-3\left(1+(k_1+k_2)^2\right)^2+\left(1+(k_1-k_2)^2\right)^2x^2\left(x+s_\Sigma\right)}{-\left(1+(k_1+k_2)^2\right)^2+\left(1+(k_1-k_2)^2\right)^2x^2} > 0,\end{aligned} \tag{4.4.7}$$

since $s_\Sigma + x < 3$. Thus ψ'' is a convex function defined on the interval $[t_-, t_+]$, such that $\psi''(t_-) > \psi''(t_+)$, and this, toghether with (4.4.3), implies the positivity condition (4.4.4).

Finally let us note that if equality holds in our "stability condition",

$$\left(1+(k_1+k_2)^2\right) = x\left(1+(k_1-k_2)^2\right)$$

then the quantity t^2+C' vanishes at the endpoint $t = 1/x-1$ and by our explicit formulae (4.3.8), (4.4.5) we obtain a solution ω, F which is smooth on $M \setminus E_\infty$, and such that F extends to a form with $C^{1/2}$ coefficients on M, while ω extends with $C^{1,1/2}$ coefficients. This completes the proof of Theorem 4.1.2.

Remark 4.4.3. As we will be interested in the small and large limits of the coupled equations, we point out that (4.4.6) and (4.4.7) hold uniformly as the scaling parameter $\alpha' \to 0$ and, provided the "stability condition"

$$(k_1+k_2)^2 > x(k_1-k_2)^2$$

is satisfied, also for $\alpha' \to \infty$.

We can now prove Proposition 4.1.6. We first claim that the Kähler form ω constructed above is real analytic. Recall ω is obtained by the momentum construction (4.2.2),

$$\omega = \frac{p^*\omega_\Sigma}{x} + \sqrt{-1}\partial\bar{\partial}f(s),$$

for a suitable convex function $f : \mathbb{R} \to \mathbb{R}$, where we have $s = \log|w|^2 + \log h(z)$ with respect to bundle adapted holomorphic coordinates (z, w). The hyperbolic metric ω_Σ is real analytic, so we can choose a local holomorphic coordinate z such that its coefficients are real analytic. On the other hand the real function $h(z)$ satisfies $-\sqrt{-1}\partial_z\bar{\partial}_z \log h(z) = \omega_\Sigma$, with the same choice of local coordinate, and so it is also real analytic. So our claim follows if we can show that the function $f : \mathbb{R} \to \mathbb{R}$ is real analytic. But f is related to the momentum profile ϕ by the ODE

$$f''(s) = \phi(\tau) = \phi(f'(s)),$$

and the momentum profile $\phi(\tau)$ of our solution is clearly a real analytic function of the variable $\tau \in (-1, 1)$ by (4.4.5). Thus $f(s)$ is real analytic and our claim on ω follows. In order to see that the curvature form F is also real analytic, recall that it is given by our ansatz (4.3.1), $F = F_0 + \sqrt{-1}\partial\bar{\partial}g(s)$, and that the dHYM equation satisfied by F can be expressed in terms of $g(s)$ as the vanishing of the right hand side of the expression (4.3.6). Thus, the real analitycity of $g(s)$ follows from that of $f(s)$.

4.5 Conical singularities

In the present Section we prove Theorem 4.1.3. This extends our existence result Theorem 4.1.2 to allow a Kähler form ω with conical singularities. Our main motivation for this extension is describing examples of solutions to the coupled equations (2.0.1) with positive coupling constant $\alpha > 0$.

We consider again Kähler forms ω given by the momentum construction (4.2.2),

$$\omega = \frac{p^*\omega_\Sigma}{x} + \sqrt{-1}\partial\bar{\partial}f(s),$$

with momentum profile $\phi(\tau) > 0$ defined on the interval $(-1, 1)$.

Lemma 4.5.1. *The Kähler form ω extends to a form with conical singularities on M, with cone angle $2\pi\beta_0$ along E_0, respectively $2\pi\beta_\infty$ along E_∞, iff the momentum profile satisfies the boundary conditions*

$$\lim_{\tau\to\pm 1}\phi(\tau) = 0,\ \lim_{\tau\to -1}\phi'(\tau) = \beta_0,\ \lim_{\tau\to 1}\phi'(\tau) = -\beta_\infty.$$

Proof. For any open neighborhood $U \subset M$, in term of the bundle adapted coordinates (z, w), $E_0 \cap U = \{w = 0\}$. We assume that, near $r = |w| = 0$, f'' has the form

$$f''(s) = c_0 r^{2\beta_0} + A(r)$$

with $c_0 \neq 0$ and $A(r) = o(r^{2\beta_0})$. Then we have

$$\begin{aligned}
\omega_{z\bar{z}} &= \left(\frac{1 - xf'}{x}\right)\omega_{\Sigma,z\bar{z}} + \sqrt{-1}f''\frac{\partial h\bar{\partial}h}{h^2} = O(1),\\
\omega_{w\bar{z}} &= -\sqrt{-1}\frac{1}{w}f''(s)\bar{\partial}h = O(r^{2\beta_0-1}),\\
\omega_{z\bar{w}} &= \sqrt{-1}\frac{1}{\bar{w}}f''(s)\partial h = O(r^{2\beta_0-1}),\\
\omega_{w\bar{w}} &= \sqrt{-1}r^{2\beta_0-2}\left(1 + A(r)/r^{2\beta_0}\right),
\end{aligned}$$

hence the metric ω given by the momentum construction has a conical singularity along E_0 of angle $2\pi\beta_0$. Since $d/ds = \frac{r}{2}d/dr$, $f''(s) = \phi(\tau)$ and $f'''(s) = \phi(\tau)\phi'(\tau)$, this implies

$$\lim_{\tau\to -1}\phi(\tau) = 0$$

and

$$\lim_{\tau\to -1}\phi'(\tau) = \beta_0.$$

The proof for E_∞ is the same, up to a change of variable. □

As in the previous Section, it is convenient to consider the parameterization

$$x = \frac{k}{k + k'}$$

for $k' > 0$. Similarly, we introduce the $(1,1)$-forms

$$\begin{aligned}
\beta &= \frac{x^2}{\left(1 - xf'(s)\right)^2}\left(\frac{1 - xf'(s)}{x}\omega_\Sigma - \sqrt{-1}f''(s)\frac{dw \wedge d\bar{w}}{|w|^2}\right),\\
F &= k_1\omega + \frac{1 - x^2}{x^2}k_2\beta,
\end{aligned}$$

as well as the ansatz, extending the momentum construction

$$F_g = F + \sqrt{-1}\partial\bar{\partial}g(s).$$

We also denote by $\nu(\tau)$ the image of $g'(s)$ under the Legendre transform diffeomorphism relative to $f(s)$. The proof of the following result is almost identical to the smooth case and we leave it to the reader. A Kähler form ω with conical singularities as above is a closed $(1,1)$-current on M, with cohomology class

$$[\omega] = 2\pi[2E_0 + k'C].$$

Similarly, F is a closed $(1,1)$-current on M with cohomology class

$$[F] = 2\pi[2(k_1 - k_2)E_0 + (2kk_2 + k'(k_1 + k_2)C].$$

Moreover, $\sqrt{-1}\partial\bar{\partial}g(s)$ extends to a closed $(1,1)$-current on M, which has vanishing cohomology class iff $\nu(\tau)$ satisfies the boundary conditions

$$\lim_{\tau\to\pm1}\nu(\tau)=0.$$

We are now in a position to complete the proof of Theorem 4.1.3. Let us first note that, precisely as in the proof of Theorem 4.1.2, under the momentum construction the dHYM equation for ω and F becomes the ODE (4.3.4), together with the boundary conditions (4.3.5). By Lemma 4.5, the cone angles do not play a role in this reduction. It follows that the second of our coupled equations (2.0.1) also reduces to the same ODE (4.4.2) for a single function $\psi(t)>0$ of the variable

$$t=1/x-\tau$$

appearing in the proof of Theorem 4.1.2. By Lemma 4.5.1, the boundary conditions corresponding to general cone angles β_0, β_∞ are

$$\lim_{t\to\pm1}\psi(t)=0,\ \lim_{t\to\frac{1}{x}+1}\psi'(t)=-2\beta_0\left(\frac{1}{x}+1\right),\ \lim_{t\to\frac{1}{x}-1}\psi'(t)=2\beta_\infty\left(\frac{1}{x}-1\right).$$

However, as (4.4.2) is second order ODE, this problem is overdetermined. If we consider the general solution (4.4.5) and impose the boundary condition

$$\lim_{t\to\frac{1}{x}+1}\psi'(t)=-2\beta_0\left(\frac{1}{x}+1\right)$$

corresponding to a cone angle $2\pi\beta_0$ along E_0, we find that the integration constant d_1 can be expressed in terms of β_0 and the coupling constant α as

$$\begin{aligned} d_1 &= \frac{(x+1)(2k_1(x(-2\beta_0+s_\Sigma(x-1)+1)+1)}{2k_1x^2}\\ &\quad-\alpha\frac{k_2(x-1)\sqrt{k_1^4-2k_1^2(k_2^2-1)+(k_2^2+1)^2)}}{2k_1x^2}. \end{aligned} \tag{4.5.1}$$

Similarly, imposing the condition

$$\lim_{t\to\frac{1}{x}+1}\psi(t)=0$$

and using our expression for d_1 gives the relation

$$\begin{aligned} d_0 &= 4\alpha\frac{k_2^2\left(x^3(k_1-k_2)^2+(k_1+k_2)^2+x^3+1\right)}{3x^3\sqrt{\left(-k_1^2+k_2^2+1\right)^2+4k_1^2}}\\ &\quad-\frac{(x+1)^2\sqrt{k_1^4-2k_1^2\left(k_2^2-1\right)+\left(k_2^2+1\right)^2}(x(-6\beta_0+s_\Sigma(2x-1)+2)+2)}{3x^3\sqrt{\left(-k_1^2+k_2^2+1\right)^2+4k_1^2}}. \end{aligned}$$

Further, imposing the condition

$$\lim_{t\to\frac{1}{x}-1}\psi(t)=0$$

and using our expressions for d_0, d_1 determines the coupling constant uniquely as

$$\alpha=\frac{\sqrt{\left(-k_1^2+k_2^2+1\right)^2+4k_1^2}\left(s_\Sigma x^2-3\beta_0(x+1)+x+3\right)}{2k_2^2x\left((k_1-k_2)^2+1\right)}. \tag{4.5.2}$$

We can now compute directly that a solution $\psi(t)$ corresponding to a cone angle $2\pi\beta_0$ along E_0 satisfies

$$\lim_{t\to\frac{1}{x}-1}\psi'(t)=-\frac{2(\beta_0+\beta_0x-2)}{x}=2\frac{-2+\beta_0(1+x)}{-1+x}\left(\frac{1}{x}-1\right),$$

which yields a cone angle $2\pi\beta_\infty$ along E_∞, with

$$\beta_\infty=\frac{-2+\beta_0(1+x)}{-1+x}.$$

In order to prove the positivity of $\psi(t)$, we consider again (4.4.6), with the coupling constant α given by (4.5.2). When

$$3\,(1+x)\,\beta_0-3>x\,(1+s_\Sigma x)\,,$$

we construct solutions for $\alpha<0$ and hence $\frac{d^4\psi}{dt^4}>0$. Moreover

$$\begin{aligned}&\psi''(t_-)-\psi''(t_+)\\&=4\frac{\left(3\,(1+x)\,\beta_0-3\right)\left(1+(k_1+k_2)^2\right)^2-x^2\left(1+(k_1-k_2)^2\right)^2\left(s_\Sigma x^2+x\right)}{\left(1+(k_1+k_2)^2\right)^2x-\left(1+(k_1-k_2)^2\right)^2x^3}>0\end{aligned}$$

and we can use essentially the same argument given in the proof of Theorem 4.1.2. When $\alpha>0$, an explicit analysis of the momentum profile is more complicated and the positivity of $\psi(t)$ is best checked with the assistance of a numerical software package (see Figure 4.1). This completes the proof of Theorem 4.1.3.

4.6 Twisted Kähler-Einstein equation

This Section is devoted to the proof of Proposition 4.1.5, which states explicitly when the equation in (2.0.1) involving the scalar curvature of ω reduces to a twisted Kähler-Einstein equation. For a general complex surface, we should require that

$$[\mathrm{Ric}(\omega)]+\frac{\alpha}{2\sin\hat{\theta}}[F]=\frac{\hat{s}-\alpha\hat{r}}{4}[\omega], \tag{4.6.1}$$

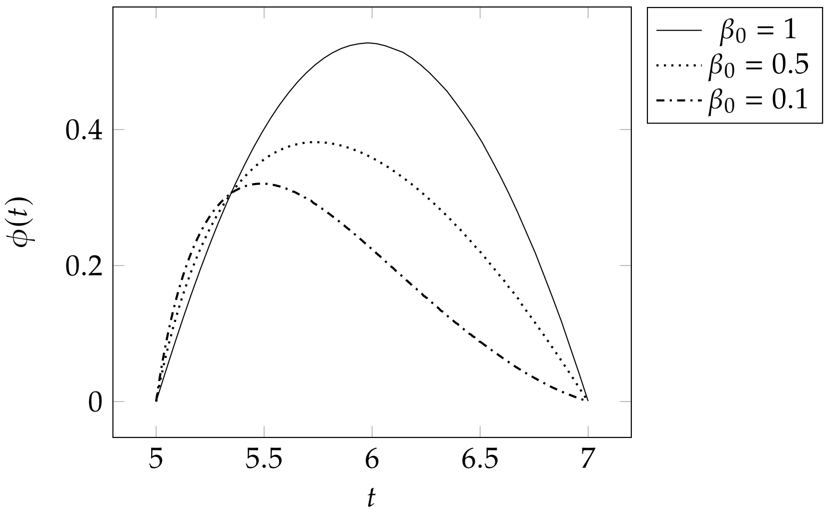

Figure 4.1: The momentum profile $\phi(t)$ of the solution when $k_2 = -k_1 = 1$, $h = 0$ and $x = 1/6$.

and we will make this condition explicit in our current setting.

For any Kähler form ω on M given by the momentum construction, with cone angle $2\pi\beta_0$ along E_0, respectively $2\pi\beta_\infty$ along E_∞, the cohomology class of $\mathrm{Ric}(\omega)$ is given by

$$\left[\frac{\mathrm{Ric}(\omega)}{2\pi}\right] = \left(\beta_0 + \beta_\infty\right)[E_0] + \left(2\,(1-h) - k\beta_\infty\right)[C]\,.$$

Proof. We recall that

$$\begin{aligned}\mathrm{Ric}(\omega) &= -\sqrt{-1}\partial\overline{\partial}\log\det\omega \\ &= -\sqrt{-1}\partial\overline{\partial}\log\det\left(\frac{2}{|w|^2}\left(\frac{1}{x} - f'\right)f''\omega_\Sigma\right) \\ &= -\sqrt{-1}\partial\overline{\partial}\log\det\left(\left(\frac{1}{x} - f'\right)f''\omega_\Sigma\right),\end{aligned}$$

hence, by a straightforward calculation, we get

$$-\sqrt{-1}\partial_z\partial_{\overline{z}}\log\det\left(\left(\frac{1}{x} - f'\right)f''\omega_\Sigma\right) = \left(\frac{f'''}{f''} + \rho_\Sigma - \frac{x}{(1-xf')}f''\right)\omega_\Sigma$$

and

$$-\sqrt{-1}\partial_w\partial_{\overline{w}}\log\det\left(\left(\frac{1}{x} - f'\right)f''\omega_\Sigma\right) = \sqrt{-1}\frac{1}{|w|^2}\frac{d}{ds}\left(\frac{x}{(1-xf')}f'' - \frac{f'''}{f''}\right)dw\wedge d\overline{w}.$$

Using the identities $f'''(s)/f''(s) = \phi'(\tau)$, $f''(s) = \phi(\tau)$ and the boundary conditions required for the momentum profile and its derivative, we compute

$$\begin{aligned}\int_{E_0} \mathrm{Ric}(\omega) &= \left(\phi'(-1) + \frac{2(1-h)}{k} - \frac{\phi(-1)}{x^{-1}-1}\right)\int_{\Sigma} \omega_{\Sigma} dz \wedge d\overline{z} \\ &= 2\pi\left(2(1-h) + k\beta_0\right)\end{aligned}$$

and

$$\begin{aligned}\int_{C} \mathrm{Ric}(\omega) &= \int_{\mathbb{C}\setminus 0} \sqrt{-1}\frac{dw \wedge d\overline{w}}{|w|^2}\frac{d}{ds}\left(\frac{x}{(1-xf')}f'' - \frac{f'''}{f''}\right) \\ &= \int_{-\infty}^{\infty}\int_0^{2\pi} \frac{d}{dr}\left(\frac{x}{(1-xf')}f'' - \frac{f'''}{f''}\right) dr \wedge d\theta \\ &= 2\pi\left(\beta_0 + \beta_\infty\right),\end{aligned}$$

so our claim follows directly from (4.2.6). □

For the following computations, it is convenient to introduce the quantity

$$\Gamma = \frac{3 + x + s_{\Sigma}x^2 - 3(1+x)\beta_0}{x} = 4 - 6\beta_0 + 3\frac{k'}{k}(1-\beta_0) + 2\frac{1-h}{k+k'}.$$

Using Lemma 4.5 and Lemma 4.6, we can then rephrase the general condition (4.6.1) as the system of equations

$$\begin{cases} \frac{1+(k_1+k_2)^2}{2k_1k_2}\Gamma = 2 + 4\frac{1-h}{k+k'} - 2\left(\beta_0 + \beta_\infty\right), \\ \frac{1+(k_1+k_2)^2}{2k_1k_2}\Gamma\left(\frac{k'}{2} + k\right) = 2(1-h)\frac{2k+k'}{k+k'} - 2k\beta_\infty - k'. \end{cases} \tag{4.6.2}$$

Notice that the two equations in (4.6.2) actually coincide when

$$-2(1-h)\frac{2k+k'}{k+k'} + 2k\beta_\infty + k' = (2k+k')\left(\beta_0 + \beta_\infty - 1 - 2\frac{1-h}{k+k'}\right),$$

or, equivalently,

$$2(k'+k) = (2k+k')\beta_0 + k'\beta_\infty. \tag{4.6.3}$$

Recall however that in order to have solutions to our equations in the momentum construction the cone angle $2\pi\beta_0$ is not arbitrary but must satisfy

$$\beta_\infty = \frac{-2 + \beta_0(1+x)}{(-1+x)},$$

in which case (4.6.3) holds automatically. Then the general condition (4.6.1) corresponds to

$$\begin{aligned}\frac{1+(k_1+k_2)^2}{2k_1k_2}\Gamma &= 2 + 4\frac{1-h}{k+k'} - 2\left(\beta_0 + \beta_\infty\right) \\ &= 2\left(2\frac{1-h}{k+k'} + 2\frac{k}{k'}\left(\beta_0 - 1\right) - 1\right).\end{aligned} \tag{4.6.4}$$

In order to show that this coincides with the condition (4.1.4) spelled out in Proposition 4.1.5, we rewrite the latter as

$$(-1+x)\left(1+(k_1+k_2)^2\right)\Gamma - 4k_1k_2\left(1+s_\Sigma - x\left(-1+s_\Sigma+2\beta_0\right)\right) = 0,$$

which implies

$$\begin{aligned}
\frac{1+(k_1+k_2)^2}{2k_1k_2}\Gamma &= 2\frac{1}{-1+x}\left(1+s_\Sigma - x\left(-1+s_\Sigma+2\beta_0\right)\right)\\
&= 2\left(s_\Sigma x + 2\beta_0\frac{x}{1-x} + \frac{1+x}{-1+x}\right)\\
&= 2\left(s_\Sigma x + 2\frac{x}{1-x}\left(\beta_0-1\right)-1\right)\\
&= 2\left(2\frac{1-h}{k+k'} + 2\frac{k}{k'}\left(\beta_0-1\right)-1\right).
\end{aligned}$$

Reading these identities backwards shows that the two conditions (4.1.4), (4.6.4) are indeed equivalent.

It remains to establish the second claim of Proposition 4.1.5, namely that the condition (4.6.4) actually holds for infinitely many solutions of the system (2.0.1). It is convenient to rewrite (4.6.4) in the form

$$F(k_1,k_2) = H(k,k',h,\beta_0), \tag{4.6.5}$$

with

$$F(k_1,k_2) = \frac{1+(k_1+k_2)^2}{2k_1k_2}$$

and

$$\begin{aligned}
H(k,k',h,\beta_0) &= \frac{2}{\Gamma}\left(2\frac{1-h}{k+k'} + 1 - \beta_0 - \beta_\infty\right)\\
&= 2\frac{2\frac{1-h}{k+k'} + 2(\beta_0-1)\frac{k}{k'} - 1}{2\frac{1-h}{k+k'} + 3\frac{k'}{k}(1-\beta_0) + 4 - 6\beta_0}.
\end{aligned}$$

We assume that $k_2 < 0$, so the stability condition (4.1.2) is automatically satisfied, and the system (2.0.1) is solvable. We observe that, under this assumption, the l.h.s. of (4.6.5) satisfies

$$F(k_1,k_2) > 2.$$

On the other hand $H(k,k',h,\beta)$, as a function of the single variable β, has a vertical asymptote at

$$\overline{\beta} = \frac{\frac{4}{3}k + k' + \frac{2(1-h)k}{3(k+k')}}{k'+2k}$$

and it is easy to check that $0 < \overline{\beta} < 1$, for $k' > M(k,h) > 0$. Moreover, at $\beta = 1$ we have

$$0 < H(k,k',h,1) = \frac{k + k' + 2\,(h-1)}{k + k' + h - 1} < 2,$$

and

$$\frac{d}{d\beta}H(k,k',h,1) = \frac{4k}{k'\left(-2 + \frac{2(1-h)}{k+k'}\right)} + \frac{2\left(6 + \frac{3k}{k'}\right)\left(-1 + \frac{2(1-h)}{k+k'}\right)}{\left(-2 + \frac{2(1-h)}{k+k'}\right)^2} < 0,$$

(Figure 4.2 shows the graph of $H\left(\beta\right)$ for $k = k' = 1$ and $h = 6$). This implies that

$$F(k_1,k_2) \in (2,\infty) \subset \operatorname{im} H|_{\beta\in(0,1)},$$

which completes the proof of Proposition 4.1.5.

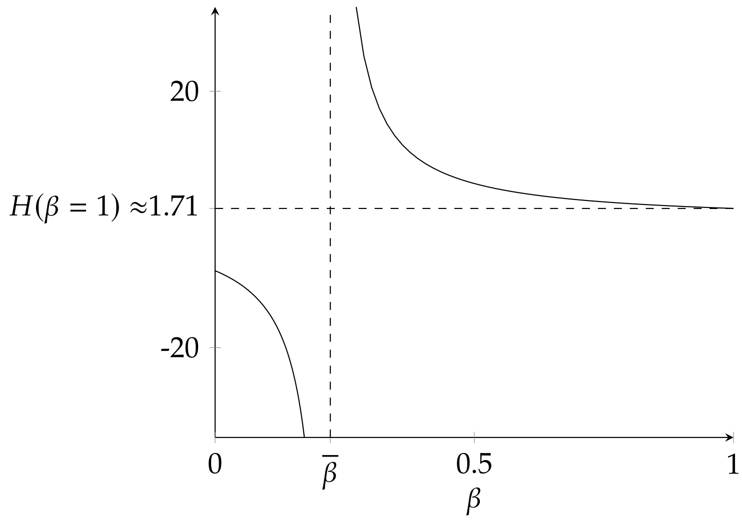

Figure 4.2: $H(k,k',h,\beta)$ for $k = k' = 1$ and $h = 6$.

Remark 4.6.1. A direct computation using (4.5.1), shows that the condition (4.1.4) holds precisely when the coefficient of the linear term d_1 in $\psi(t)$ vanishes, i.e. a solution ω is twisted Kähler-Einstein precisely when the linear term is missing from the momentum profile.

4.7 Large and small radius limits

Let us first prove Theorem 4.1.7. As we have already observed, the "slope parameter" α' appears in the coupled equations (4.1.5) simply as a scale factor for the curvature form F. In other words, a pair ω, F solves (4.1.5) iff the pair $\omega, \alpha' F$ solves (2.0.1): the cohomology parameters are simply rescaled $(k_1,k_2) \mapsto (\alpha' k_1, \alpha' k_2)$. Thus, according to Theorem 4.1.2, there exists a (unique) solution of (4.1.5) given by the momentum construction iff the "stability condition"

$$1 + (\alpha')^2 (k_1 + k_2)^2 > x \left(1 + (\alpha')^2 (k_1 - k_2)^2\right)$$

holds. Since $x < 1$ by construction, this inequality holds for all sufficiently small α', depending only on k_1, k_2 and k'. Let us write $\omega_{\alpha'}, F_{\alpha'}$ for the corresponding family of solutions.

The attached function $H_{\alpha'}(t) = H_-(t)$ appearing in (4.3.4) is also obtained from (4.3.8) simply by rescaling $(k_1, k_2) \mapsto (\alpha' k_1, \alpha' k_2)$, and so it can be computed explicitly as

$$H_{\alpha'}(t) = t \cot \hat{\theta}_{\alpha'} - \sqrt{\left(\cot^2 \hat{\theta}_{\alpha'} + 1\right)(t^2 + C'_{\alpha'})},$$

$$e^{\sqrt{-1}\hat{\theta}_{\alpha'}} = \frac{\left(1 - (\alpha')^2 k_1^2 + (\alpha')^2 k_2^2 - 2\sqrt{-1}\alpha' k_1\right)}{\sqrt{\left(1 - (\alpha')^2 k_1^2 + (\alpha')^2 k_2^2\right)^2 + \left(2(\alpha')^2 k_1\right)^2}},$$

$$\cot \hat{\theta}_{\alpha'} = -\frac{-(\alpha')^2 k_1^2 + (\alpha')^2 k_2^2 + 1}{2\alpha' k_1},$$

$$C'_\alpha = 4(\alpha')^2 k_1 k_2 \left(\frac{1}{x^2\left((\alpha' k_1 - \alpha' k_2)^2 + 1\right)} - \frac{1}{(\alpha' k_1 + \alpha' k_2)^2 + 1}\right). \tag{4.7.1}$$

By elementary computations using these explicit formulae, recalling that we also have $k_1 < 0$, we find

$$H_{\alpha'}(t) = \left(k_1 t + \frac{k_2}{t}\left(-1 + \frac{1}{x^2}\right)\right)\alpha' + (\alpha')^2 R(\alpha', t) \tag{4.7.2}$$

for some function $R(\alpha', t)$, smooth up to $\alpha' = 0$.

As a first consequence, we can show that the sequence of Kähler forms $\omega_{\alpha'}$ converges smoothly to a Kähler form ω as $\alpha' \to 0$. It will be enough to show the smooth convergence of the momentum profiles $\phi_{\alpha'}(t)$. According to Lemma 4.4.2 and the subsequent explicit formulae for the coupling constant α and average radius $\hat{r}$, the profile $\phi_{\alpha'}(t)$ is obtained by integrating twice the identity

$$\frac{2s_\Sigma}{t} - \frac{1}{t}\left(2t\phi(t)\right)'' = \alpha\left(\cos\hat{\theta}\left(1 - \frac{H(t)H'(t)}{t}\right) - \sin\hat{\theta}\left(H'(t) + \frac{H(t)}{t}\right)\right) + \hat{s} - \alpha\hat{r}, \tag{4.7.3}$$

where all quantities are understood as evaluated at $(\alpha' k_1, \alpha' k_2)$, and in particular

$$\alpha = \frac{\sqrt{4(\alpha')^2 k_1^2 + \left(1 - (\alpha')^2 k_1^2 + (\alpha')^2 k_2^2\right)^2}}{2\left(1 + (\alpha' k_1 - \alpha' k_2)^2\right)(\alpha')^2 k_2^2}(-2 + s_\Sigma x)$$

$$\hat{r} = \sqrt{\left(1 - (\alpha')^2 k_1^2 + (\alpha')^2 k_2^2\right)^2 + 4(\alpha')^2 k_1^2}. \tag{4.7.4}$$

By the latter explicit formulae and (4.7.2), the quantity

$$\alpha\left(\cos\hat{\theta}\left(1-\frac{H(t)H'(t)}{t}\right)-\sin\hat{\theta}\left(H'(t)+\frac{H(t)}{t}\right)-\hat{r}\right)$$

has a smooth limit as $\alpha'\to 0$, so the same holds for the right hand side of (4.7.3) and for the momentum profile $\phi(t)=\phi_{\alpha'}(t)$. The positivity of $\phi_{\alpha'}(t)$ and its limit for $\alpha'\to 0$ follows from Remark 4.4.3. We can now show that the curvature forms $F_{\alpha'}$ also converge smoothly as $\alpha'\to 0$. By construction we have $F_{\alpha'}=F_{0,\alpha'}+\sqrt{-1}\partial\bar{\partial}(\alpha')^{-1}g_{\alpha'}(s)$, where $F_0=c_1\omega_{\alpha'}+c_2\beta_{\alpha'}$ and the potential $g_{\alpha'}(s)$ corresponds to the solution for the parameters $(\alpha' k_1,\alpha' k_2)$ (i.e. for the cohomology class $\alpha'[F_0]$). By the smooth convergence of the Kähler forms $\omega_{\alpha'}$, which we just established, it will be enough to show that the potentials $(\alpha')^{-1}g_{\alpha'}(s)$ converge smoothly. In fact they converge smoothly to the zero potential. Indeed by (4.3.3) and (4.7.2) we have

$$\begin{aligned}(\alpha')^{-1}g_{\alpha'}(s)&=(\alpha')^{-1}v_{\alpha'}(\tau)\\&=(\alpha')^{-1}\left(\alpha' k_1 t+\alpha'\frac{k_2}{t}\frac{1-x^2}{x^2}-H_{\alpha'}(t)\right)\\&=\alpha' R(\alpha',t)\end{aligned}$$

where $R(\alpha',t)$ is smooth in a neighbourhood of $\alpha'=0$. It follows that we have, smoothly as $\alpha'\to 0$,

$$F_{\alpha'}\to F_0=c_1\omega+c_2\beta,$$

which is indeed a solution of the HYM equation $\Lambda_\omega F=\mu$.

Finally, this allows to write down the equation satisfied by the limit Kähler form ω. Recall $\omega_{\alpha'}$ solves the equation

$$s(\omega_{\alpha'})-\alpha_{\alpha'}\operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}_{\alpha'}}\frac{\left(\omega_{\alpha'}-\sqrt{-1}\alpha' F_{\alpha'}\right)^2}{\omega_{\alpha'}^2}\right)=\hat{s}-\alpha_{\alpha'}\hat{r}_{\alpha'}.$$

Expanding around $\alpha'=0$ we find

$$\begin{aligned}&\operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}_{\alpha'}}(\omega_{\alpha'}-\sqrt{-1}\alpha' F_{\alpha'})^2\right)\\&=\omega_{\alpha'}^2-\left(F_{\alpha'}\wedge F_{\alpha'}-z_1\omega_{\alpha'}\wedge F_{\alpha'}+z_2\omega_{\alpha'}^2\right)(\alpha')^2+O(\alpha'^4)\end{aligned}$$

for certain cohomological constants z_1,z_2. Similarly,

$$\begin{aligned}\alpha_{\alpha'}&=\frac{1}{(\alpha')^2}\left(\frac{-2+s_\Sigma x}{2k_2^2}+O(\alpha')\right),\\\hat{r}_{\alpha'}&=1+O(\alpha').\end{aligned}$$

Thus, taking the smooth limit as $\alpha' \longrightarrow 0$, and using our result that the limit curvature form satisfies $\Lambda_\omega F = \mu$, we see that ω satisfies

$$s(\omega) + \tilde{\alpha}\Lambda^2_\omega(F \wedge F) = c$$

where

$$\tilde{\alpha} = \frac{-2 + s_\Sigma x}{2k_2^2}$$

and c is a cohomological constant. This completes the proof of Theorem 4.1.7.

The proof of Theorem 4.1.8 is quite similar. Our assumption

$$(k_1 + k_2)^2 > x(k_1 - k_2)^2$$

implies that, for any $\alpha' > 0$, the "stability condition"

$$1 + (\alpha' k_1 + \alpha' k_2)^2 > x(1 + (\alpha' k_1 - \alpha' k_2))^2$$

holds. Thus, by Theorem 4.1.2, the coupled equations (4.1.5) are uniquely solvable with the momentum construction. We denote the corresponding solutions by $\omega_{\alpha'}$, $F_{\alpha'}$ as before. By (4.7.1), as $\alpha' \longrightarrow \infty$, we have an expansion

$$\begin{aligned} H_{\alpha'}(t) = {} & \frac{\sqrt{\left(k_1^2 - k_2^2\right)^2 \left(k_1 k_2 \left(\frac{4}{x^2(k_1-k_2)^2} - \frac{4}{(k_1+k_2)^2}\right) + t^2\right)} + k_1^2 t - k_2^2 t}{2k_1}\alpha' \\ & + S((\alpha')^{-1}, t) \end{aligned} \tag{4.7.5}$$

where $S(y, t)$ is a smooth function near $y = 0$. By this expansion and (4.7.4), the quantity

$$\alpha\left(\cos\hat{\theta}\left(1 - \frac{H(t)H'(t)}{t}\right) - \sin\hat{\theta}\left(H'(t) + \frac{H(t)}{t}\right) - \hat{r}\right)$$

has a smooth limit as $\alpha' \longrightarrow \infty$, so the same holds for the right hand side of (4.7.3) and for the momentum profile $\phi(t) = \phi_{\alpha'}(t)$. Since we are assuming $(k_1 + k_2)^2 > x(k_1 - k_2)^2$, $\phi_{\alpha'}(t)$ and its limit satisfy the positivity condition, by Remark 4.4.3. Thus the sequence of Kähler forms $\omega_{\alpha'}$ converges smoothly to a Kähler form ω as $\alpha' \longrightarrow \infty$.

Considering now the curvature forms $F_{\alpha'} = F_{0,\alpha'} + \sqrt{-1}\partial\bar{\partial}(\alpha')^{-1}g_{\alpha'}(s)$ as before, we find

$$\begin{aligned} (\alpha')^{-1}g_{\alpha'}(s) &= (\alpha')^{-1}\nu_{\alpha'}(\tau) \\ &= (\alpha')^{-1}\left(\alpha' k_1 t + \alpha' \frac{k_2}{t}\frac{1 - x^2}{x^2} - H_{\alpha'}(t)\right) \\ &= k_1 t + \frac{k_2}{t}\frac{1 - x^2}{x^2} - (\alpha')^{-1}H_{\alpha'}(t), \end{aligned}$$

where by (4.7.5) we have the smooth convergence, as $\alpha' \to \infty$,

$$(\alpha')^{-1} H_{\alpha'}(t) \to \frac{k_1^2 - k_2^2}{2k_1} K_{\pm}(t),$$

where

$$K_{\pm}(t) = t \pm \sqrt{t^2 + \hat{C}},$$
$$\hat{C} = 4k_1 k_2 \left(\frac{1}{x^2 (k_1 - k_2)^2} - \frac{1}{(k_1 + k_2)^2} \right)$$

and the sign $\pm$ agrees with the sign of the quantity $k_1^2 - k_2^2$. Thus, by the convergence of the Kähler forms $\omega_{\alpha'}$, the curvature forms $F_{\alpha'}$ also have a smooth limit F as $\alpha' \to \infty$.

Finally we may write down the equations satisfied by the limit Kähler form ω and curvature form F. By our previous results we have expansions, as $\alpha' \to \infty$,

$$\operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}_{\alpha'}} (\omega_{\alpha'} - \sqrt{-1}\alpha' F_{\alpha'})^2 \right) = \left(Z_1 \omega_{\alpha'} \wedge F_{\alpha'} - Z_2 F_{\alpha'}^2 \right) \alpha' + O(1),$$
$$\operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}_{\alpha'}} (\omega_{\alpha'} - \sqrt{-1}\alpha' F_{\alpha'})^2 \right) = F_{\alpha'}^2 (\alpha')^2 + O(1),$$

for some cohomological constants Z_1, Z_2. Similarly,

$$\alpha_{\alpha'} = \frac{|k_1^2 - k_2^2|}{2(k_1 - k_2)^2 k_2^2} \frac{(-2 + s_\Sigma x)}{(\alpha')^2} + O\left(\frac{1}{(\alpha')^3} \right),$$
$$\hat{r}_{\alpha'} = |k_1^2 - k_2^2| (\alpha')^2 + O(\alpha').$$

Thus, passing to the limit as $\alpha' \to \infty$ in the equations (4.1.5), we find that ω, F satisfy the equations

$$\begin{cases} F \wedge \omega = c_1 F^2 \\ s(\omega) - \alpha_\infty \frac{F^2}{\omega^2} = c_2 \end{cases}$$

for a unique α_∞ and cohomological constants c_1, c_2. Using the first equation, the second can also be written in the twisted cscK form as

$$s(\omega) - \hat{\alpha} \Lambda_\omega F = c_2$$

for some unique $\hat{\alpha}$. This completes the proof of Theorem 4.1.8.

Bibliography

[AB83] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615.

[ABC⁺09] P. S. Aspinwall, T. Bridgeland, A. Craw, M. R. Douglas, M. Gross, A. Kapustin, G. W. Moore, G. Segal, B. SzendrHoi, and P. M. H. Wilson, *Dirichlet branes and mirror symmetry*, Clay Mathematics Monographs, vol. 4, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2009.

[Abr98] M. Abreu, *Kähler geometry of toric varieties and extremal metrics*, Internat. J. Math. **9** (1998), no. 6, 641–651.

[AGG13] L. Alvarez-Cònsul, M. Garcìa-Fernàndez, and O. Garcìa-Prada, *Coupled equations for Kähler metrics and Yang-Mills connections*, Geom. Topol. **17** (2013), no. 5, 2731–2812.

[AP06] C. Arezzo and F. Pacard, *Blowing up and desingularizing constant scalar curvature Kähler manifolds*, Acta Math. **196** (2006), no. 2, 179–228.

[Asp03] P. S. Aspinwall, *D-branes on calabi–yau manifolds*, Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2003): Recent Trends in String Theory **3** (2003), 1–152.

[Aub76] T. Aubin, *Équations du type monge-ampère sur les variétés kähleriennes compactes*, C. R. Acad. Sci. Paris Sér. A-B **283** (1976), 339–411.

[Ban78] A. Banyaga, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique*, Comment. Math. Helv. **53** (1978), no. 2, 174–227.

[Ban04] D. Banfield, *The geometry of coupled equations in gauge theory*, International Journal of Mathematics **15** (2004), no. 2, 183–209, 27.

[BBJ21] R. Berman, S. Boucksom, and M. Jonsson, *A variational approach to the yau–tian–donaldson conjecture*, J. Amer. Math. So (2021).

[BBS95] K. Becker, M. Becker, and A. Strominger, *Fivebranes, membranes and nonperturbative string theory*, Nuclear Phys. B **456** (1995), 130–152.

[BBS06] K. Becker, M. Becker, and J. H. Schwarz, *String theory and m-theory: A modern introduction*, Cambridge University Press, 2006.

[Caf90] L. A. Caffarelli, *Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation*, Ann. of Math. (2) **131** (1990), no. 1, 135–150.

[Cal54] E. Calabi, *The space of kähler metrics*, Proc. Internat. Congress Math. Amsterdam **2** (1954), 206–207.

[Cal57] ______, *On kähler manifolds with vanishing canonical class*, Algebraic geometry and topology. A symposium in honor of S. Lefschetz (1957), 78–89.

[Cal82] ______, *Extremal Kähler metrics*, 259–290.

[Cal85] ______, *Extremal Kähler metrics. II*, 95–114.

[CDS14] X. X. Chen, S. K. Donaldson, and S. Sun, *Kähler-Einstein metrics and stability*, Int. Math. Res. Not. IMRN (2014), no. 8, 2119–2125.

[CDS15a] ______, *Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities*, J. Amer. Math. Soc. **28** (2015), no. 1, 183–197.

[CDS15b] ______, *Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than* 2π, J. Amer. Math. Soc. **28** (2015), no. 1, 199–234.

[CDS15c] ______, *Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches* 2π *and completion of the main proof*, J. Amer. Math. Soc. **28** (2015), no. 1, 235–278.

[Che04] X. X. Chen, *A new parabolic flow in Kähler manifolds*, Comm. Anal. Geom. **12** (2004), no. 4, 837–852.

[Che21] Gao Chen, *The J-equation and the supercritical deformed Hermitian-Yang-Mills equation*, Invent. Math. **225** (2021), no. 2, 529–602. MR 4285141

[COGP91] P. Candelas, X. De La Ossa, P. S. Green, and L. Parkes, *A pair of calabi-yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. B **359** (1991), 21–74.

[CSW18] X. X. Chen, S. Sun, and B. Wang, *Kähler-Ricci flow, Kähler-Einstein metric, and K-stability*, Geom. Topol. **22** (2018), no. 6, 3145–3173.

[CXY17] T. Collins, D. Xie, and S.-T. Yau, *The deformed Hermitian-Yang-Mills equation in geometry and physics*.

[CY18] T. Collins and S.-T. Yau, *Moment maps, nonlinear pde and stability in mirror symmetry, i: Geodesics*, Annals of PDE **7** (2018), 1–73.

[Dol03] I. Dolgachev, *Lectures on invariant theory*, Cambridge University Press, 2003.

[Don85] S. K. Donaldson, *Anti-self-dual yang–mills connections on a complex algebraic surface and stable vector bundles*, Proc. London Math. Soc. **50** (1985), no. 3, 1–26.

[Don97] ______, *Remarks on gauge theory, complex geometry and 4-manifold topology*, Fields Medallists' lectures, World Sci. Ser. 20th Century Math., vol. 5, World Sci. Publ., River Edge, NJ, 1997, pp. 384–403. MR 1622931

[Don99] ______, *Symmetric space, kähler geometry and hamiltonian dynamics*, 13–33.

[Don00] ______, *Moment maps and diffeomorphisms*, Surveys in differential geometry, Surv. Differ. Geom., vol. 7, Int. Press, Somerville, MA, 2000, pp. 107–127.

[Don02] ______, *Scalar curvature and stability of toric varieties*, J. Differential Geom. **62** (2002), no. 2, 289–349.

[Don05a] ______, *Interior estimates for solutions of Abreu's equation*, Collect. Math. **56** (2005), no. 2, 103–142.

[Don05b] ______, *Lower bounds on the Calabi functional*, J. Differential Geom. **70** (2005), no. 3, 453–472.

[Don08] ______, *Extremal metrics on toric surfaces: a continuity method*, J. Differential Geom. **79** (2008), no. 3, 389–432.

[Don09] ______, *Constant scalar curvature metrics on toric surfaces*, Geom. Funct. Anal. **19** (2009), no. 1, 83–136.

[DP20] V. Datar and V. P. Pingali, *On coupled constant scalar curvature Kähler metrics*, J. Symplectic Geom. **18** (2020), no. 4, 961–994.

[DS16] V. Datar and G. Székelyhidi, *Kähler-Einstein metrics along the smooth continuity method*, Geom. Funct. Anal. **26** (2016), no. 4, 975–1010.

[Fei01] B. Feix, *Hyperkähler metrics on cotangent bundles*, J. Reine Angew. Math. **532** (2001), 33–46.

[FS11] R. Feng and G. Szekelyhidi, *Periodic solutions of Abreu's equation*, Math. Res. Lett. **18** (2011), no. 6, 1271–1279.

[Fuj90] A. Fujiki, *The moduli spaces and Kähler metrics of polarized algebraic varieties*, Sūgaku **42** (1990), no. 3, 231–243.

[Fuk02] K. Fukaya, *Mirror symmetry of abelian varieties and multi-theta functions*, J. Algebraic Geom. **11** (2002), no. 3, 393–512.

[Fut83] A. Futaki, *An obstruction to the existence of Einstein Kähler metrics*, Invent. Math. **73** (1983), no. 3, 437–443.

[Geo16] V. Georgoulas, *A differential geometric approach to git and stability*, ETH Zürich (2016).

[GW00] M. Gross and P. M. H. Wilson, *Large complex structure limits of K3 surfaces*, J. Differential Geom. **55** (2000), no. 3, 475–546.

[HS02] A.D. Hwang and M. A. Singer, *A momentum construction for circle-invariant Kähler metrics*, Trans. Amer. Math. Soc. **354** (2002), no. 6, 2285–2325.

[HWN19] J. Hultgren and D. Witt Nyström, *Coupled Kähler-Einstein metrics*, Int. Math. Res. Not. IMRN (2019), no. 21, 6765–6796.

[IN90] M. Itoh and H. Nakajima, *Yang-mills connections and einstein-hermitian metrics*, Advanced Studies in Pure Mathematics **18** (1990), no. II, 395–457.

[Joh03] C. V. Johnson, *D-branes*, Cambridge University Press, 2003.

[JY17] A. Jacob and S.-T. Yau, *A special Lagrangian type equation for holomorphic line bundles*, Math. Ann. **369** (2017), no. 1-2, 869–898.

[JYC20] A. Jacob, S.-T. Yau, and T. Collins, *(1, 1) forms with specified lagrangian phase: A priori estimates and algebraic obstructions*, Cambridge Journal of Mathematics **8** (2020), 407–452.

[Kal99] D. Kaledin, *A canonical hyperkähler metric on the total space of a cotangent bundle*, Quaternionic structures in mathematics and physics (Rome, 1999), Univ. Studi Roma "La Sapienza", Rome, 1999, pp. 195–230.

[Kob87] S. Kobayashi, *Differential geometry of complex vector bundles*, Iwanami Shoten, Publishers and Princeton University Press, 1987.

[Kon95] M. Kontsevich, *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians **1** (1995), 120–139.

[KTF12] J. Keller and C. Tönnesen-Friedman, *A non trivial example of coupled equations for kähler metrics and yang-mills connections*, Central European Journal of Math. **10(5)** (2012), 1673–1687.

[LS94] C. LeBrun and S. R. Simanca, *Extremal Kähler metrics and complex deformation theory*, Geom. Funct. Anal. **4** (1994), no. 3, 298–336.

[LS15] M. Lejmi and G. Székelyhidi, *The J-flow and stability*, Adv. Math. **274** (2015), 404–431.

[LYZ01] C. Leung, S.-T. Yau, and E. Zaslow, *From special lagrangian to hermitian-yang-mills via fourier-mukai transform*, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999) **23** (2001), 209–225.

[Mab87] T. Mabuchi, *Some symplectic geometry on compact Kähler manifolds. I*, Osaka J. Math. **24** (1987), no. 2, 227–252.

[Mab08] ———, *K-stability of constant scalar curvature polarization*, https://arxiv.org/abs/0812.4093 (2008).

[Mat57] Y. Matsushima, *Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne*, Nagoya Math. J. **11** (1957), 145–150.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, Springer-Verlag, Berlin, 1994.

[MiR99] I. Mundet i Riera, *Symplectic stability, analytic stability in non-algebraic complex geometry*, D. Phil. thesis (1999).

[MMMS00] M. Mariño, R. Minasian, G. Moore, and A. Strominger, *Nonlinear instantons from supersymmetric p-branes*, J. High Energy Phys. (2000), no. 1, Paper 5, 32.

[MS17] D. MacDuff and D. Salamon, *Introduction to symplectic topology*, Oxford University Press, 2017.

[NS65] M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on compact riemann surfaces*, Annals of Mathematics **38** (1965), 540–567.

[Pol01] L. Polterovich, *The geometry of the group of symplectic diffeomorphism*, Birkhäuser Basel, 2001.

[PZ98] A. Polishchuk and E. Zaslow, *Categorical mirror symmetry: the elliptic curve*, Adv. Theor. Math. Phys. **2** (1998), no. 2, 443–470.

[Sto09] J. Stoppa, *K-stability of constant scalar curvature Kähler manifolds*, Adv. Math. **221** (2009), no. 4, 1397–1408.

[SYZ96] A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror symmetry is t-duality*, Nuclear Phys. B **479** (1996), no. 1-2, 243–259.

[Sze14] G. Szekelyhidi, *Introduction to extremal metrics*, American Mathematical Society, 2014.

[Sze16] ______, *The partial C^0-estimate along the continuity method*, J. Amer. Math. Soc. **29** (2016), no. 2, 537–560.

[Tak21] R. Takahashi, *Collapsing of the line bundle mean curvature flow on Kähler surfaces*, Calc. Var. Partial Differential Equations **60** (2021), no. 1, Paper No. 27, 18.

[Tel06] A. Teleman, *Symplectic stability, analytic stability in non-algebraic complex geometry*, D. Phil. thesis (2006).

[Tho] R. P. Thomas, *The geometry of mirror symmetry*, in Encyclopaedia of Mathematical Physics, pp. 467–524.

[Tho01] ______, *Moment maps, monodromy, and mirror manifolds*, Symplectic geometry and mirror symmetry (Seoul, 2000) **World Sci. Publ.** (2001), 467–498.

[Tia90] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. **101** (1990), no. 1, 101–172.

[Tia97] ———, *Kähler-einstein metrics with positive scalar curvature*, Invent. Math. **130** (1997), no. 1, 1–37.

[Tia99] ———, *Canonical metrics on kähler manifolds*, Birkhäuser, 1999.

[Tia15] ———, *K-stability and Kähler-Einstein metrics*, Comm. Pure Appl. Math. **68** (2015), no. 7, 1085–1156. MR 3352459

[TW] N. S. Trudinger and X.-J. Wang, *The Monge-Ampère equation and its geometric applications*, Handbook of geometric analysis. No. 1, Adv. Lect. Math. (ALM), vol. 7, pp. 467–524.

[TY02] R. P. Thomas and S.-T. Yau, *Special lagrangians, stable bundles, and mean curvature flow*, Comm. Anal. Geom. **10** (2002), no. 5, 1075–1113.

[UY86] K. Uhlenbeck and S.-T. Yau, *On the existence of hermitian–yang–mills connections in stable vector bundles*, Communications on Pure and Applied Mathematics **39** (1986), 257–293.

[Yau77] S.-T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), 1798–1799.

[Yau78] ———, *On the ricci curvature of a compact kähler manifold and the complex monge-ampére equation, i*, Commun. Pure Appl. Math. **31** (1978), 339–411.