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Functional and geometric rigidities of RCD spaces and bi-Lipschitz Reifenberg's theorem in metric spaces

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Preamble

This thesis is divided in two independent parts.
In the first one we will present the results concerning the theory of metric measure spaces satisfying synthetic Ricci curvature lower bounds, obtained in [**131**, **132**, **185**]. The focus will be on the extension of some analytical tools to this setting and on the derivation on both geometric and analytical rigidities and almost-rigidities.
In the second part instead we will present the works in [**130**] and [**209**] about the bi-Lipschitz version of Cheeger-Colding's intrinsic Reifenberg's theorem in abstract metric spaces.

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Symbols and notations

ω_N	generalized unit N-dimensional ball volume	page 3
σ_N	generalized unit N-dimensional sphere volume	page 3
$\mathsf{Eucl}(N,p)$	sharp Euclidean-Sobolev constant	page 3
$\simeq$	isomorphism of metric measure spaces	page 3
lip	local Lipschitz-constant	page 3
d_p	distance function from p	page 3
$L^0(\mu)$	equivalence classes up to μ-a.e. equality of Borel functions	page 4
$\lvert Df\rvert$	minimal weak upper gradient	page 5
$\lambda^{1,2}(\mathrm{X})$	first eigenvalue of the Laplcian	page 5
(q,p)-Poincaré inequality		page 5
PI-space	locally doubling m.m.s. with $(1,2)$-Poincaré inequality	page 5
$W^{1,2}_{\mathsf{loc}}$, $W^{1,2}(\Omega)$, $W^{1,2}_0(\Omega)$	local Sobolev spaces	page 6
Δ, $\mathbf{\Delta}$	Laplacian and measure valued Laplacian	page 7
$\mathsf{D}(\mathbf{\Delta},\Omega)$, $\mathsf{D}_V(\Delta,\Omega)$	domain of the Laplacian in Ω (with Laplacian in the space V)	page 7
$\mathbf{div}$, div	(measure-valued) divergence	page 8
$\mathsf{D}(\mathbf{div},\Omega)$, $\mathsf{D}_V(\mathrm{div},\Omega)$	domain of the divergence in Ω (with divergence in the space V)	page 8
Cap	capacity	page 9
QCR	quasi-continuous representative map	page 9
$\overline{\mathrm{Cap}}$	variational capacity	page 9
$\mathsf{BV}(\mathrm{X})$	functions of bounded variation	page 10
$\lvert Du\rvert_1$	a.c. part of the total variation	page 10
$\mathfrak{m}^+(E)$	Minkowski content	page 10
$L^0(T\mathrm{X})$, $L^2(T\mathrm{X})$, $L^\infty(T\mathrm{X})$	tangent module	page 12
∇f, $\lvert\nabla f\rvert$	gradient and modulus of the gradient	page 13
$\sigma^{(t)}_{K,N}$, $\tau^{(t)}_{K,N}$	distortion coefficients	page 14
I_N	one-dim. model space for the $\mathrm{CD}(N-1,N)$ condition	page 14
$v_{K,N}$	volume of balls in the K,N-model space	page 15
AVR	asymptotic volume ratio	page 15
$\theta_{N,r}$, θ_N	Bishop-Gromov ratios and density	page 15
$p_t(.,.)$	heat kernel	page 17
$\dim_{\mathsf{loc}}$	local dimension of an $\mathrm{RCD}(K,N)$	page 18
$[0,\pi]\times^N_{\sin}\mathrm{Z}$	spherical suspension	page 18
$W^{2,2}(\mathrm{X})$	functions with Hessian	page 20
$W^{1,2}_C(T\mathrm{X})$, $H^{1,2}_C(T\mathrm{X})$	spaces of Sobolev vector fields	page 22
tr_E, π_E	trace and projection w.r.t. perimeter measure	page 24
$L^2_E(T\mathrm{X})$	tangent module over ∂E	page 24
pmGH/mGH	(pointed) measure Gromov Hausdorff convergence	page 26
extrinsic approach		page 26
(p,q)-Sobolev inequality		page 93
α_p	first optimal constant in the Sobolev inequality	page 93
A^{opt}_q	second optimal constant in the Sobolev inequality	page 93
u^*_N	monotone rearrangement into I_N	page 94
$I_{0,N}$	Euclidean model space	page 95
$u^*_{0,N}$	Euclidean-monotone rearrangement into $I_{0,N}$	page 95

Introduction

PART I

Spaces with synthetic Ricci curvature bounds

The theory of singular spaces with Ricci curvature bounds finds its roots and motivations in the *Gromov's precompactness theorem* [**133**], which states that the class of smooth Riemannian manifolds with dimension bounded above by $N \in [1, \infty)$, Ricci curvature bounded below by $K \in \mathbb{R}$ and uniformly bounded diameter, is pre-compact in the Gromov-Hausdorff topology (see also [**112**]). The spaces in the closure of this class are in general not manifolds and are called *Ricci-limits*. The study of these singular-limits has been initiated at the end of the Nineties by Cheeger and Colding in a series of papers [**73**–**75**] in which a deep structure theory was developed. In the following years important results have been added to the theory, among these let us also mention the so-called *constancy of the dimension* [**89**] and the recent rectifiability of the singular strata [**77**] (improving [**78**]). A natural question initially raised in the seminal papers by Cheeger and Colding (see [**73**, Appendix 2]) is whether Ricci-limits have 'Ricci curvature bounded below' in some synthetic sense. Rephrasing, we ask if it is possible to give a notion of metric space with Ricci curvature bounds, without relying neither on a smooth structure nor on an approximating sequence. There are several motivations to look for such a synthetic formulation:

(*i*) being able to work directly on the space with synthetic methods, without passing through the study of smooth Riemannian manifolds,
(*ii*) to get new results for Riemannian manifolds by proving a result first on a limit space and then deduce properties on the smooth approximate sequence,
(*iii*) to have a larger class of spaces which is stable with respect to natural geometric constructions, for example it is not clear if the cross section of a conical Ricci-limit is again a Ricci-limit, while this will be true in an (appropriate) synthetic setting,
(*iv*) if one is interested in the effect of lower Ricci curvature bounds on Riemannian manifolds, it is conceptually useful to work with objects where these Ricci curvature bounds are the only assumption.

This is an instance of the general strategy of studying smooth objects by considering a larger, but more easy to handle class of (possibly) irregular objects, in analogy with smooth functions which are often analysed by looking at Sobolev functions.

Since Ricci curvature is tightly linked to volumes, the right setting for synthetic Ricci curvature bounds has proven to be the one of *metric measure spaces*, that are metric spaces endowed with a non-negative Borel measure. The notion of synthetic Ricci bounds originated in the independent works by Lott-Villani [**173**] and Sturm [**200**, **201**], where it was introduced the concept of metric measure space with synthetic Ricci curvature bounded below by $K \in \mathbb{R}$ and dimension bounded above by $N \in [1, \infty]$, the so-called $\mathrm{CD}(K, N)$ *Curvature-Dimension condition*. This definition is given via optimal transport and is based on the equivalence between Ricci curvature bounds and convexity property of entropies proved in [**173**] and [**90**]. A slightly weaker condition than $\mathrm{CD}(K, N)$ is the reduced curvature-dimension condition, $\mathrm{CD}^*(K, N)$, introduced by Bacher and Sturm [**35**] to deal with the local-to-global problem. After the deep work of Cavalletti and Milman [**63**] the $\mathrm{CD}^*(K, N)$ and the $\mathrm{CD}(K, N)$ assumptions are actually known be equivalent in the essentially non-branching case with finite mass, while they are known to be different without further assumptions, as shown in [**189**].

Already from its introduction the $\mathrm{CD}(K, N)$ condition has shown to be sufficient to prove many geometric and analytic results typical of Ricci curvature bounds and comparison geometry. Some of the most notable are the Bishop-Gromov volume comparison [**201**], the Brunn-Minkowski inequality [**35**, **201**], the Lichnerowicz's spectral gap inequality [**172**] (see also [**65**] for a general p), the Bonnet-Myers' diameter bound [**155**], the Levy-Gromov's isoperimetric inequality [**62**, **64**] and the Laplacian comparison [**117**] (see also [**67**]). Nevertheless the $\mathrm{CD}(K, N)$ class of spaces is for some purposes too large. Indeed it contains Finsler manifolds, which are not present in Ricci-limits (see [**75**]) and fails to have some rigidity-properties like the Cheeger-Gromoll splitting theorem [**76**]. To exclude Finsler structures it was proposed by Ambrosio, Gigli and Savarè [**21**] the *Riemannian Curvature-Condition* $\mathrm{RCD}(K,\infty)$ (see also [**17**]). The definition in the finite dimensional case $\mathrm{RCD}(K,N)$ was instead given by Gigli [**117**], who also proved it to be sufficient for the splitting theorem to hold [**115, 116**]. The main idea of the RCD condition is to add to the CD condition the linearity of the heat flow, or equivalently the assumption that the Sobolev space $W^{1,2}$ is Hilbert. In the same way, adding the linearity of the heat flow to the reduced $\mathrm{CD}^*(K,N)$ condition one defines the $\mathrm{RCD}^*(K,N)$ condition. It is conjectured that the $\mathrm{RCD}(K,N)$ condition and the $\mathrm{RCD}^*(K,N)$ conditions are actually the same and from the results in [**63**] (and since the $\mathrm{RCD}(K,N)$ condition implies essentially-nonbranching, [**190**]), this is known at least in the case of finite reference measure.

An important question settled in [**22**] is the equivalence between the Eulerian-formulation of the $\mathrm{RCD}(K,\infty)$ condition via optimal transport and the Lagrangian approach of the so-called Bakry-Emery condition $\mathrm{BE}(K,\infty)$ coming from the setting of Dirchlet-forms ([**38–40**]). The finite dimensional counterpart of this result was proved independently in [**107**] and [**26**], where it was obtained the equivalence between the $\mathrm{BE}(K,N)$ condition and the $\mathrm{RCD}^*(K,N)$ condition.

Functional and geometric rigidities and almost-rigidities. An advantage of working with spaces satisfying the RCD condition, as opposed to the CD-condition, is that they admit rigidity results analogous the ones of Riemannian manifolds. The most notable are the generalization of the Cheeger-Gromoll splitting theorem [**115**], the maximal diameter theorem [**155**], the rigidity of the spectral gap [**156**] (see also [**65**] for the p-spectral gap), the volume cone-to-metric cone theorem [**95**], the rigidity of the Levy-Gromov isoperimetric inequality [**64**] and the rigidity of the Polya-Szego inequality [**182**].

Rigidities in the RCD setting, besides generalizing results for manifolds, sometimes come also with some new informations even in the smooth case. Examples are the splitting theorem and the volume-cone-to-metric cone theorem, where the *"splitting factor"* is also known to satisfy synthetic Ricci curvature bounds, information that is missing in the classical smooth counterpart of these results.

In addition to this, the most important consequence of rigidities in the RCD setting, is that they often allow to deduce by compactness their *almost-rigid* counterpart. In particular with this scheme it is possible to recover (see [**95**],[**115**] and also [**30**]), in non quantitative form, the almost-volume-cone-to-metric cone theorems by Cheeger and Colding [**72**] and the sphere-theorems by Colding ([**81, 82**]), see ([**143, 151**]). Furthermore many almost-rigidity results that have been obtained in the RCD-class are new even in the smooth case. Among these we mention the Levy-Gromov isoperimetric inequality ([**62, 64**]) and the almost-rigidity of the p-spectral gap [**65**] (see also [**182**] for the same result concerning the Dirichlet p-spectral gap).

Monotonicity formula for the electrostatic potential on RCD spaces

In this section we review the results obtained in [**132**], that will be presented in detail in Chapter 2.

On a smooth n-dimensional Riemannian manifold M with non-negative Ricci curvature, the classical Bishop-Gromov inequality implies that for every point $p\in M$ the quantity $\frac{|\partial B_r(p)|}{r^{n-1}}$ is non-increasing in r. Recalling that $|\nabla \mathsf{d}_p|=1$ (where d_p denotes the distance from p), this is equivalent to the quantity

$$\frac{1}{r^{n-1}}\int_{\{\mathsf{d}_p=r\}}|\nabla \mathsf{d}_p|\,\mathrm{d}\sigma\downarrow \tag{1}$$

being monotone non-increasing. In a series of works Colding and Minicozzi ([**84, 86–88**]) proved that an analogous result holds integrating the Green's function for the Laplacian. In particular they proved that on a *non-parabolic* n-dimensional Riemannian manifold, $n\ge 3$, with non-negative Ricci curvature, the quantities

$$A_\beta(r):=\frac{1}{r^{n-1}}\int_{\{b=r\}}|\nabla b|^{\beta+1}\,\mathrm{d}\sigma,\quad r\in(0,\infty) \tag{2}$$

are non-increasing for every $\beta\ge\frac{n-2}{n-1}$, where $b:=G(p,.)^{\frac{1}{2-n}}$, G being the (minimal positive) Green's function and $p\in M$ is fixed. As a remarkable geometric application of the above monotonicity, the same authors proved the uniqueness of tangent cones for Ricci-flat manifolds [**87**]. Their idea, roughly said, was to use A_β as a 'smooth-substitute' of the Bishop-Gromov volume ratio to have a better asymptotic control on the distance of the manifold from the nearest cone (see also Remark 0.6 below). Let us mention that Colding and Minicozzi derived the existence of other monotone quantities analogous to A_β, which however will not be discussed here.

More recently the monotonicity for the Green function has been revisited and generalized by Agostiniani Fogagnolo and Mazzieri [**2**] to the case of an *electrostatic potential* (building upon the previous work of Agostiniani and Mazzieri on the Euclidean space [**3**]). We recall that given a set E, bounded, open and with smooth boundary in a (non-compact) Riemannian manifold M, its electrostatic potential is defined as the unique solution to:

$$\begin{cases} u = 1 & \text{in } \partial E, \\ \Delta u = 0 & \text{in } M \setminus \bar{E}, \\ u(x) \to 0, & \text{as } \mathsf{d}(x, E) \to +\infty, \end{cases}$$

which is well known to exist if and only if the manifold is non-parabolic (see e.g. [**166**] or [**2**]). It has been proved in [**2**] that, for an electrostatic potential u, the function

$$U_\beta(t) := \frac{1}{t^{\beta \frac{N-1}{N-2}}} \int_{\{u=t\}} |\nabla u|^{\beta+1} \, \mathrm{d}\sigma, \quad t \in (0,1], \tag{3}$$

is differentiable and monotone non-decreasing for every $\beta \geq \frac{n-2}{n-1}$. Note that this result contains the one of Colding-Minicozzi, as can be seen by taking as u the Green function (see also [**2**, Appendix]). As the main geometric application, the monotonicity of U_β has been used by the same authors in [**2**] to prove sharp Willmore-type inequalities on manifolds with Euclidean volume growth, that we will discuss more in Section "Willmore inequalities on RCD spaces" below.

Additionally in [**2**] it is proved a rigidity result for the monotonicity of U_β, saying that $U_\beta(t)' = 0$ for some $t \in (0,1]$, if and only if the sublevel $\{u < t\}$ is a truncated Riemannian cone. The same rigidity is then inherited and present in the equality case of the Willmore-inequalities (see (W)) as well.

In Chapter 2 we will extend the monotonicity-rigidity for the function U_β to the setting of non-smooth $\mathrm{RCD}(0, N)$ spaces. The main problems that we will need to address are the existence of the electrostatic potential, which can be dealt with more or less standard machinery, and the natural lack of regularity of the objects involved due to the non-smoothness of the ambient space, that requires more recent tools.

Existence of electrostatic potential on $\mathrm{RCD}(0, N)$ spaces. In Section 2.2 we prove the existence of the electrostatic potential for an open set E with sufficiently regular boundary in the setting of non-parabolic $\mathrm{RCD}(0, N)$ spaces (Theorem 2.2.13). In this setting the electrostatic potential is defined exactly as in the smooth case (see Definition 2.2.11).

They key points is that on an $\mathrm{RCD}(0, N)$ space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ satisfying the non-parabolicity assumption

$$\int_1^\infty \frac{s}{\mathfrak{m}(B_s(x_0))} \, \mathrm{d}s < +\infty, \tag{4}$$

for some (and thus any) $x_0 \in \mathrm{X}$, we have the existence of a positive Green's function. This has been shown by Bruè and Semola [**58**], who also proved that the Green's function satisfies an analogue of the classical estimate by Li and Yau [**168**] and in particular it vanishes at infinity. Recall also that from a result of Varopolous [**206**], (4) is equivalent in the case of smooth manifolds to the usual non-parabolicity condition.

The presence of the Green's function will be one of the key ingredients to prove the existence of the electrostatic potential. More precisely with a comparison argument with the Green's function we will be able to force the potential to have the required vanishing-behaviour at infinity.

Concerning the regularity assumption on the boundary of the set, it will be enough to satisfy the so-called *interior corkscrew-condition*:

Definition 0.1 (Interior corkscrew-condition)**.** We say that an open set E satisfies the interior corkscrew condition if for every $x \in \partial E$ there exist two numbers $r > 0$ and $\lambda \in (0,1)$ such that for every $s \in (0, r)$ there exists a ball of radius λs contained in $B_r(x) \cap E$.

The crucial fact about this condition is that it is relatively easy to be satisfied, examples are balls or enlargements of arbitrary sets (see Section 2.2.1). Actually it will be enough to have a weaker (and more abstract) regularity condition, called Cap-*fatness* (see Definition 2.2.1).

This regularity on the set E will be needed to get continuity of the solution up to the boundary and to satisfy the boundary data ($u = 1$ in ∂E) characterizing the electrostatic potential. To obtain the desired behaviour at the boundary we will rely on tools coming from a well-developed theory of elliptic equations and potential theory on metric measure spaces (we refer to [**46**] for an overview on these topics). In particular we will exploit and re-prove in the $\mathrm{RCD}(K, N)$ setting part of the Wiener criterion for boundary regularity already obtained in the setting of PI-spaces in [**47**–**49**] (see also [**171**, **211**] for the Euclidean case).

Monotonicity result on $\mathrm{RCD}(0,N)$ **spaces.** Having a notion of electrostatic potential, we can now present our main rigidity result, that will be proved in Section 2.4

We first see how to write the candidate monotone quantity in this setting. Given an electrostatic potential u (see Def. 2.2.11) on an non-parabolic $\mathrm{RCD}(0,N)$ space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ we define (in analogy with (3)) for every $\beta\ge\frac{N-2}{N-1}$ the function

$$U_\beta(t):=\frac{1}{t^{\beta\frac{N-1}{N-2}}}\int|\nabla u|^{\beta+1}\,\mathrm{dPer}(\{u<t\}),\quad t\in(0,1),\tag{5}$$

(see Section 1.2.1 for the definition of $|\nabla u|$), where we have fixed a Borel representative of $|\nabla u|$ (which is defined only $\mathfrak{m}$-a.e.). It turns out that different choices of representative change the value of U_β only a.e. and thus that U_β is a well defined equivalence class (up to a.e.-equality) of functions.

To better explain our result we first recall the general scheme used in [**2**,**84**,**88**] to prove the monotonicity of either U_β or A_β in the smooth case. The main idea is to integrate the divergence of suitable vector fields (see also the introduction of [**84**] where this is discussed). More precisely it is a general fact that, whenever we have a (smooth) vector field with non-negative divergence, we can produce monotone quantities. Indeed by the divergence theorem we have that (neglecting integrability issues)

$$\int_{\{u=t\}}v\cdot\frac{\nabla u}{|\nabla u|}\mathrm{d}\sigma-\int_{\{u=s\}}v\cdot\frac{\nabla u}{|\nabla u|}\,\mathrm{d}\sigma=\int_{\{s<u<t\}}\mathrm{div}(v)\,\mathrm{dVol}\ge 0,\tag{6}$$

where: v is a smooth vector field satisfying $\mathrm{div}(v)\ge 0$, u is a smooth function and s,t are regular values for u. Note that (6) is saying precisely that the function $r\mapsto\int_{\{u=r\}}v\cdot\frac{\nabla u}{|\nabla u|}\mathrm{d}\sigma$ is monotone non-decreasing. Moreover applying the coarea formula on the right hand side of (6) we can even compute its derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\{u=t\}}v\cdot\frac{\nabla u}{|\nabla u|}\mathrm{d}\sigma\right)=\int_{\{u=t\}}\frac{\mathrm{div}(v)}{|\nabla u|}\,\mathrm{d}\sigma.\tag{7}$$

This scheme in our case is applied choosing u as the electrostatic potential and with the vector field $v:=u^2\nabla|\nabla u^{\frac{1}{2-n}}|^\beta$, n-being the dimension of the manifold. This actually does not prove directly the monotonicity of U_β, but it shows instead that the function $U'_\beta(t)t^2$ is non-decreasing. The monotonicity of U_β is then derived integrating its derivative with an argument due to Colding [**84**] (see also the proof of Theorem 2.4.4).

There are two main issues in performing the above scheme in the setting of RCD-spaces and are both linked to the lack of regularity of the objects involved. One is the missing information on the critical set of the electrostatic potential. Indeed gradients are defined only up to measure zero sets, as they formally belong to the space of L^0-sections of the tangent bundle (see Sec. 1.2.7 for the precise definition), and thus it is not clear how to define the critical set in general. Additionally, and more importantly, in the case of an harmonic function it is not known whether the critical set (even if defined up-to measure zero) can have positive measure. Even worst, the (weak) unique-continuation problem is still open in RCD setting, which means that we cannot exclude that harmonic functions are constant on some open set.

The second main problem is that the object $\mathrm{div}(u^2\nabla|\nabla u^{\frac{1}{2-n}}|^\beta)$ is only a measure (see Corollary 2.3.5) and this clearly poses fundamental problems in both applying the divergence theorem as in (6) and in writing (7).

These issues will be resolved in Section 2.4.2 by arguing in more distributional sense, rather than considering single sublevels (as in [**84**] and [**2**]), heavily exploiting the coarea formula, which is available on metric measure spaces in its full form (see [**179**] and also Section 1.2.6). This however will reflect on a weaker form of monotonicity and regularity of the functional U_β. Indeed we will prove:

Theorem 0.2 (Theorem 2.4.4). *For every $\beta\ge\frac{N-2}{N-1}$ the function U_β belongs to $W^{1,1}(0,1)$, $U'_\beta\in\mathsf{BV}_{\mathsf{loc}}(0,1)$ and*

$$U'_\beta(t)\ge\frac{C}{t^2}\int_{\{u<t\}}u^2|\nabla|\nabla v|^{\frac{\beta}{2}}|^2\mathrm{d}\mathfrak{m},\qquad a.e.\ in\ (0,1),\tag{8}$$

where $v:=u^{\frac{1}{2-N}}$ and $C=C_{\beta,N}\ge 0$. In particular U_β has a continuous non-decreasing representative.

The main differences with the result in the smooth setting is that we get lower regularity of U_β and only a lower bound on its derivative, while in [**2**] (and also [**84**,**88**]) a pointwise stronger (than (8)) identity is valid. Nevertheless, the lower bound in (8) is enough to deduce a rigidity result (see Section "Willmore inequalities on RCD space" below). We also point out that, even if we prove the continuity of U_β, we do not have an explicit formula for its continuous representative, since (5) only coincides with it outside a set of points depending on the representative of $|\nabla u|$. It would be natural to take the quasi-continuous representative of $|\nabla u|$ (see Section 1.2.5 for the definition of this object), in which case we know that formula (5) coincides with U_β at least at every 'first-order regular sublevel' (see Section 3.3 for the precise definition) thanks to Theorem 3.3.4.

It is worth to mention that the 'distributional' technique that we developed for the proof of the monotonicity formula has proven to be helpful even in the smooth setting in the recent [**44**], where it has been adapted to improve the previous monotonicity result for the p-capacitary potential (see [**1**]).

New estimates for harmonic functions. As described above, the monotonicity of the function U_β is proved by 'integrating' suitable differential inequalities. More precisely to prove the main monotonicity result we will first prove (in Section 2.3) a lower bound on $\mathrm{div}(u^2\nabla|\nabla u^{\frac{1}{2-n}}|^\beta)$. However to approach this estimate we will first need to prove some new non-trivial regularity and higher-order bounds on harmonic functions on $\mathrm{RCD}(0,N)$ spaces. This regularity can be formally summarised as: for every u harmonic, the functions

$$|\nabla u|^\alpha, \quad \forall\, \alpha \geq \frac{1}{2}\frac{N-2}{N-1},$$

are in $W^{1,2}$. Note that this is not obvious since $\frac{N-2}{N-1} < 1$ and thus we are allowed to take α below 1 (and even below 1/2). Our precise result is the following, which actually holds in an arbitrary $\mathrm{RCD}(K,N)$ space (see Sec. 1.2.4 for the definition of measure-valued Laplacian).

Theorem 0.3 (Theorem 2.3.1)**.** *Let* X *be an* $\mathrm{RCD}(K,N)$ *space with* $N \in [2,\infty)$*,* $\Omega \subset \mathrm{X}$ *be open,* u *be harmonic in* Ω *and* $\beta > \frac{N-2}{N-1}$*. Then* $|\nabla u|^{\beta/2} \in W^{1,2}_{\mathsf{loc}}(\Omega)$ *and*

$$\mathbf{\Delta}(|\nabla u|^\beta) \geq C_{\beta,N}\, |\nabla|\nabla u|^{\frac{\beta}{2}}|^2\, \mathfrak{m} + \beta K|\nabla u|^2\mathfrak{m}, \tag{9}$$

where $C = C_{\beta,N} > 0$*.*

The estimate (9) is similar and strongly inspired by a similar result in [**79**, Sec. 3.1], where a stronger identity is proven in the case of Riemannian manifolds. Notably the argument in [**79**] requires bounds from above on the size of the critical set of u, while for us to derive (9) these are not needed (and currently not available). The main tool to prove this result will be the improved weak Bochner inequality by Han [**135**] (see also (1.4.7)) and a Kato-inequality that will be proved in Lemma 2.3.3.

Thanks to the regularity and estimates provided by the above result, we will be able to get the key inequality which will imply our main monotonicity result. In particular we will prove that for an electrostatic potential u in an $\mathrm{RCD}(0,N)$ space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ it holds that

$$\mathbf{div}(u^2\nabla|\nabla u^{\frac{1}{2-n}}|^\beta) \geq Cu^2|\nabla|\nabla u^{\frac{1}{2-N}}|^{\frac{\beta}{2}}|^2\mathfrak{m}, \quad \forall\beta \geq \frac{N-2}{N-1}, \tag{10}$$

where $C = C_{\beta,N} \geq 0$, see Corollary 2.3.5 for the precise statement. Note also that the right hand side of (10) makes sense precisely due to Theorem 0.3.

As a final remark we mention that the regularity given by Theorem 0.3 will also play a crucial role in Chapter 3, as it will imply that it can be given a notion of mean curvature to the sublevels of harmonic functions.

Functional (almost)-rigidities. The rigidity part of the monotonicity of U_β will be derived from an independent functional rigidity result in the spirit of the Cheeger-Gromoll splitting theorem [**76**] and the volume-cone-to-metric-cone theorem (see e.g. [**72**]). Let us first state this result:

Theorem 0.4 (Theorem 2.5.1)**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(0,N)$ *space with* $N \in [2,\infty)$ *and* $U \subset X$ *be open with* ∂U *bounded. Suppose there exists a positive function* $v \in C(\bar{U})$ *satisfying*

$$\begin{cases} v = 1, & \text{in } \partial U, \\ v > 1, & \text{in } U, \\ \Delta v^2 = 2N,\ |\nabla v| = 1, & \text{in } U. \end{cases} \tag{11}$$

Then U *is (locally isometric to)* $Y \setminus \bar{B}_1(p)$*, where* Y *is a cone over an* $\mathrm{RCD}(N-1,N-2)$ *space, with tip* p*.*

This should be interpreted as a sort of "outer-functional-cone-to-outer-metric-cone", or in other words as the converse of the fact that in an N-dimensional cone the distance function from the tip satisfies $\Delta\mathsf{d}^2 = 2N$ and $|\nabla\mathsf{d}| = 1$. A-posteriori we actually get that the function v in (11) is a distance function (see (2.5.1)).

To explain how Theorem 2.5.1 is linked to the rigidity of the function U_β we look at the lower bound for the derivative of U'_β in (8). We see that whenever $U'_\beta(t_0) = 0$ for some $t_0 \in (0,1)$ it holds that $|\nabla v|$ is (a non-zero) constant in $\{u < t_0\}$, where $v := u^{\frac{1}{2-N}}$. From this, a simple computation using the fact that u is harmonic shows that (up to rescaling)

$$\Delta v^2 = 2N, \quad |\nabla v| = 1, \quad \mathfrak{m}\text{-a.e. in } \{u < t_0\}. \tag{12}$$

Therefore from Theorem 2.5.1 it follows almost immediately the rigidity of the monotonicity formula for U_β:

Theorem 0.5 (Theorem 2.6.1)**.** *If $U_\beta(t_0)' = 0$ for some $t_0 \in (0, 1]$ then $\{u < t_0\}$ is locally isometric to a truncated cone over an* RCD$(N-2, N-1)$ *space.*

One of the main results of Chapter 2 is the almost-version of the above rigidity, which is in the spirit of the almost-volume-cone-to-almost-metric-cone results in by Cheeger and Colding ([**72**]). We provide here only a formal statement, precise results can be found in Theorems 2.5.5 and 2.5.6.

Almost-functional-cone-to-almost-metric-cone: Suppose that v and U are as in Theorem 2.5.1, but v only satisfying $\Delta v^2 = 2N|\nabla v|^2$ in U and assume that $|\nabla v|$ is almost constant in the following "Sobolev sense": for some $\gamma \geq \frac{1}{2}\frac{N-2}{N-1}$

$$\int_U \frac{|\nabla|\nabla v|^\gamma|}{v^{N-2}} \, \mathrm{d}\mathfrak{m} < \delta(\gamma, \varepsilon, N),$$

Then X is ε-close in the pmGH-sense to another RCD$(0, N)$ space Y which is a truncated cone outside a ball (of controlled radius). Moreover the sets $\{s < u < t\}$ are ε-close (with their intrinsic distance) in the GH-sense to annuli of a fixed cone.

As for the rigidity, from the above result follows an analogous almost-rigidity result for the monotonicity formula (see Theorems 2.6.2 and 2.6.3).

Remark 0.6. Le us point out that an almost rigidity result for the monotonicity formula was already proved by Colding and Minicozzi in [**87**] (and announced in [**84**]) and was one of the main tools used to prove uniqueness of tangent cones for Ricci-flat manifolds in [**87**]. Note however that our almost rigidity result holds for the whole class U_β, $\beta \geq \frac{N-2}{N-1}$, while in [**84**] is stated for $\beta = 2$, moreover we also obtain curvature bounds on the section of the cone. ■

Willmore inequalities on RCD spaces

In this section we present the results obtained in [**131**], that are contained in Chapter 3.

The goal will be both to generalize the Willmore-type inequalities obtained in [**2**], to RCD spaces, and to give a meaning of mean curvature and Willmore energy in this setting. To present our results, we need first to review these inequalities and their proof in the smooth case.

As a consequence of the monotonicity formula, it is proved in [**2**] that on an n-dimensional Riemannian manifold M, with Euclidean volume-growth and non-negative Ricci curvature, given an open subset E with smooth boundary it holds

$$\int_{\partial E} \left| \frac{H}{n-1} \right|^{\beta+1} \mathrm{d}\sigma \geq (\mathsf{AVR}(M)|\mathbb{S}^{n-1}|)^{\frac{\beta}{n-2}} \left(\frac{\overline{\mathrm{Cap}}(E)}{n-2} \right)^{\frac{n-2-\beta}{n-2}}, \quad \forall \beta \geq \frac{n-2}{n-1}, \tag{W}$$

where H is the mean curvature of ∂E and $\mathsf{AVR}(M) := \lim_{r\to+\infty} \frac{\mathrm{Vol}(B_r(p))}{\omega_n r^n}$. They also show that equality holds in (W) if and only if $M \setminus E$ is a truncated Riemannian cone. We mention that the inequality in (W) obtained through setting $\beta = n-2$ generalizes to Riemannian manifolds the so-called *Willmore inequality* in $\mathbb{R}^n$, proved in [**212**] and [**80**] (and in quantitative form in [**3**] using the monotonicity formula).

Remark 0.7. Here $\overline{\mathrm{Cap}}(E)$ denotes the *variational-capacity* (see Def. 1.2.22) and must not be confused with what we will call *capacity* (see (1.2.13)). Note also that here $\overline{\mathrm{Cap}}$ is not renormalized as in [**2**]. ■

Now we briefly recall how inequality (W) is deduced in [**2**] from the monotonicity of U_β. The key point is the expression of the derivative of U'_β ([**2**, Thm 1.3] or Proposition 2.4.5):

$$0 \leq U'_\beta(t) = \beta t^{-\beta\frac{n-1}{n-2}} \int_{\{u=t\}} |\nabla u|^\beta \underbrace{\langle \nabla u, \frac{\nabla|\nabla u|}{|\nabla u|^2}\rangle}_{H} - \left(\frac{n-1}{n-2}\right) \frac{|\nabla u|^{\beta+1}}{u} \, \mathrm{d}\sigma, \tag{13}$$

where u is the electrostatic potential. The main observation is that the quantity H is precisely the mean curvature of $\{u = t\}$ (provided t is a regular value), because from the harmonicity of u, $H = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$.

Therefore taking $t=1$ in (13) and applying the Hölder inequality with exponent β we obtain

$$\frac{n-1}{n-2}\int_{\partial E}|\nabla u|^{\beta+1}\,\mathrm{d}\sigma \le \int_{\partial E}|\nabla u|^{\beta}H\,\mathrm{d}\sigma \le \left(\left(\int_{\partial E}|\nabla u|^{\beta+1}\,\mathrm{d}\sigma\right)^{\beta}\cdot\int_{\partial E}|H|^{\beta+1}\,\mathrm{d}\sigma\right)^{\frac{1}{\beta+1}},$$

elevating to the $\beta+1$ and simplifying we reach

$$\left(\frac{n-1}{n-2}\right)^{\beta+1}U_\beta(1)=\left(\frac{n-1}{n-2}\right)^{\beta+1}\int_{\partial E}|\nabla u|^{\beta+1}\,\mathrm{d}\sigma \le \int_{\partial E}|H|^{\beta+1}\,\mathrm{d}\sigma$$

and finally using the monotonicity of U_β

$$\left(\frac{n-1}{n-2}\right)^{\beta+1}\lim_{t\to 0^+}U_\beta(t)\le\int_{\partial E}|H|^{\beta+1}\,\mathrm{d}\sigma.$$

Inequality (W) is then a consequence of the expression of the limit on the left hand side. In the Euclidean setting this can be computed with explicit pointwise asymptotics both for u and $|Du|$ (see [**3**]), while on Riemannian manifolds it requires a quite involved integral expansion, which is performed in [**2**] and is inspired by previous computations present in [**73**, **84**, **85**].

From the above argument, we deduce that there are two steps the are required for the proof of the Willmore-type inequalities in the RCD setting:

(i) computing the limit $\lim_{t\to 0^+}U_\beta(t)$,

(ii) understanding in which sense the object $\langle \nabla u, \frac{\nabla|\nabla u|}{|\nabla u|^2}\rangle$ represents the *mean curvature*.

The answers to the above will constitute the content of Chapter 3.

Note that, strictly speaking, point (ii) is not needed to prove the above inequalities in the non-smooth setting, but it is essential for giving a meaning to the inequality. This is also motivated by the independent problem of the definition of the mean curvature in RCD spaces, which is in large part still missing (see also the beginning of Section "Mean curvature and Willmore energy on RCD spaces" below). Observe also that question (ii) makes sense on an arbitrary $\mathrm{RCD}(K,N)$ space.

Expansion of the electrostatic potential in RCD setting. In Section 3.7.3 we will answer to problem (i) as follows (see Def. 1.2.22 for the variational-capacity $\overline{\mathrm{Cap}}$):

Theorem 0.8 (Theorem 3.7.9)**.** *Let* X *be an* $\mathrm{RCD}(0,N)$ *space with Euclidean volume growth. Let* u *be the electrostatic potential of a bounded open set* $E\subset\mathrm{X}$ *and let* U_β *as above. Then for every* $\beta\ge\frac{N-2}{N-1}$

$$\lim_{t\to 0^+}U_\beta(t)=(N-2)^{\beta\frac{N-1}{N-2}}\,\overline{\mathrm{Cap}}(E)^{1-\frac{\beta}{N-2}}(\sigma_{N-1}\mathsf{AVR}(\mathrm{X}))^{\frac{\beta}{N-2}}, \tag{14}$$

where $\mathsf{AVR}(\mathrm{X}):=\lim_{r\to+\infty}\frac{\mathfrak{m}(B_r(x_0))}{\omega_N R^N}$.

Our proof of (14) will be along the same lines of [**2**], which has been in turn inspired by the previous works in [**73**, **85**]. In particular we will derive an asymptotic expansion at infinity of the electrostatic potential u (see Theorem 3.7.4) and an integral expansion for its gradient $|\nabla u|$ (Proposition 3.7.8).

We mention that an essential ingredient for our analysis, as in [**2**], is the following asymptotic behaviour at infinity of the Green's function on $\mathrm{RCD}(0,N)$ spaces with Euclidean volume growth (Theorem 3.7.3):

$$\lim_{y\to+\infty}\frac{G(x,y)}{\mathsf{d}(x,y)^{2-N}}=\frac{1}{(N-2)\sigma_{N-1}\mathsf{AVR}(\mathrm{X})},\quad \text{for every } x\in\mathrm{X}. \tag{15}$$

In Riemannian setting (15) is due to [**85**] (see also [**167**]), while on the RCD setting it has been proved only very recently in [**54**]. Actually in [**54**] it is only proved a local expansion, however the argument is easily adapted to deal with our case (see also Remark 2.5 therein). It is worth to say that the proof of (15) is very different from the one in smooth-case, as it is based on a blow-down procedure and relies on the stability properties of the RCD condition and of the heat-kernel under pmGH-convergence (see [**124**] and also [**25**]).

Mean curvature and Willmore energy on RCD spaces. In Sections 3.2 and 3.5 we propose an answer to (ii) above. More precisely we address the problem of the definition of the Willmore energy and of the mean curvature on arbitrary $\mathrm{RCD}(K,N)$ spaces.

Let us mention that definitions of mean curvature have recently appeared in CD spaces in [**157**] and [**61**] (the last one inspired by the previous work in the synthetic-Lorentzian setting [**68**]), which contain generalization to non-smooth setting respectively of the Heinze-Karcher inequality and of the Kasue's inscribed-radius bound. However in relation to our problem, both of these notions seem not suitable (or not yet sufficiently developed) to answer question (*ii*) above. Indeed they are given via *localization-techique* and thus very difficult to link to the second-order calculus on RCD spaces. In addition to this they only describe weak bounds on the mean curvature rather than the mean curvature itself.

Going back to the RCD setting, the main obstacle to define the mean curvature and Willmore energy lies in the fact that they are second-order objects that live on a codimension-1 subset of the ambient space. Indeed, even if a well developed second order calculus is available (see [**118**]), objects like Hessian and covariant derivatives cannot be defined better than up to measure zero sets (with respect to the ambient measure). This is instead possible for first order objects, as recently shown in [**99**] and successfully used in [**56**] to prove a Gauss-Green formula in the RCD setting.

We will give two approaches to the definition of Willmore energy. In the first one we will perform a relaxation of an appropriate Willmore functional, while on the second we will introduce a full notion of mean curvature vector. The main differences between the two is that in the first case we can consider arbitrary sets, while the second is available only for level sets of smooth functions.

Relaxed Willmore energy. In Section 3.2 we propose a notion of Willmore energy for sets in RCD spaces. This will be done via relaxation of a *Willmore-functional* defined on sufficiently smooth functions and will be partially inspired by the classical notion of Willmore (or elastic) functional appearing in the Euclidean setting (see e.g. [**165**, **176**, **177**]).

We will assume that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is an arbitrary $\mathrm{RCD}(K, N)$ space.

For every sufficiently smooth function u we define

$$\mathcal{W}(u) := \int_{\mathrm{X}} \left| \frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla |\nabla u|, \nabla u\rangle}{|\nabla u|^2} \right|^2 |\nabla u| \mathrm{d}\mathfrak{m}, \tag{16}$$

(see Definition 3.2.3). Notice that formally $\frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla |\nabla u|, \nabla u\rangle}{|\nabla u|^2} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ and in particular from the coarea formula (again at a formal level)

$$\mathcal{W}(u) = \int_{\mathbb{R}} \|H_t\|^2_{L^2(\{u=t\})}\, \mathrm{d}t,$$

where H_t is the mean curvature of the level set $\{u = t\}$ (this will be a-posteriori made rigorous in Proposition 3.5.17). Our main result for this part will be to show that there are many functions such that $\mathcal{W}(u) < +\infty$. In particular we show that we have enough functions with finite Willmore energy to approximate (characteristic functions of) compact sets (see Theorem 3.2.6):

$$\{\chi_K \ : \ K \subset \mathrm{X} \text{ compact}\} \subset \overline{\{\mathcal{W} < +\infty\} \cap \mathsf{LIP}_c(\mathrm{X})}^{L^1(\mathfrak{m})}. \tag{17}$$

This fact will heavily rely on the estimates for harmonic functions that are developed in Chapter 2. Indeed from the regularity result that we presented above in Theorem 0.3 we see that for an harmonic functions u in an open set Ω it holds that $\sqrt{|\nabla u|} \in W^{1,2}_{\mathsf{loc}}(\Omega)$, which easily implies that

$$\left| \frac{\langle \nabla |\nabla u|, \nabla u\rangle}{|\nabla u|^2} \right|^2 |\nabla u| \in L^1_{\mathsf{loc}}(\Omega).$$

This shows that, at least locally, harmonic functions have finite Willmore energy. This observation will allow us to construct many example of functions u such that $\mathcal{W}(u) < +\infty$.

Thanks to (17) it is meaningful to define the relaxed Willmore energy for a Borel set $E \subset \mathrm{X}$ as

$$\overline{\mathcal{W}}(E) = \inf\{\varliminf_n W(u_n) \ : \ u_n \to \chi_E \text{ in } L^1_{\mathsf{loc}}(\mathfrak{m}), \mathcal{W}(u_n) < +\infty \}.$$

As a final existence result we will show that (see Proposition 3.2.8):

almost every sublevel of a function u such that $\mathcal{W}(u) < +\infty$, has finite Willmore-energy.

Remark 0.9. We will actually not be able to prove Willmore inequalities with this notion of Willmore energy, but only for the one that we will define as L^2-norm of the mean curvature vector (see below). Also, we de not now whether the two coincide (see Corollary 3.5.18 for the "$\leq$" inequality). ■

Mean curvature. Our approach to define the mean curvature on RCD setting will be via duality with the *tangential divergence*. This is partly inspired by theory of curvature varifolds in the Euclidean space, where a weak notion of mean curvature can be defined in a similar manner (see e.g. [**8**, **175**, **196**]).

In our construction we will use and rely on some objects and definitions introduced in the recent [**56**, **99**], like the notion of outer normal to a set of finite perimeter and the trace operator.

The main idea is to define the mean curvature vector H for a set of finite perimeter E as the one satisfying the so-called *tangential integration by parts formula*:

$$\int \mathrm{div}_T(v)\mathrm{dPer}(E) = \int \langle H, \mathrm{tr}_E(v)\rangle \mathrm{dPer}(E), \tag{18}$$

for every 'smooth enough' vector field v, where $\mathrm{div}_T(v)$ is the tangential divergence and tr_E is the trace operator (see Section 1.5 for a precise definition of trace of a vector field). The main problem however is that it is not obvious how to define neither $\mathrm{div}_T(v)$ nor its integral appearing in the left hand side of (18). Indeed we would be tempted to define

$$\mathrm{div}_T(v) = \mathrm{div}(v) - \langle \nabla_{\nu_E} v, \nu_E\rangle, \tag{19}$$

(where $\nu_E$is the outer normal to E) however the covariant derivative ∇v of a vector field (introduced in [**118**], see also Section 1.4.3) is defined only up to (reference)-measure zero sets, hence it is not obvious how to localize this object with respect to the perimeter measure. We stress again that this is so because we are dealing with second order objects, since first order object are a-posteriori defined better than up to measure-zero (see [**99**]). To avoid this problem we will employ an averaging approach in which we assume that the set E is a sublevel $E = \{u < t\}$ of a smooth function u. We then define for suitably smooth vector fields

$$\int \mathrm{div}_T(v)\mathrm{dPer}(E) := \lim_{h\to 0^+} \frac{1}{2h}\int_{\{t-h<u<t+h\}} \mathrm{div}(v) - \nabla v\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \mathrm{d}\mathfrak{m}, \tag{20}$$

(see Definition 3.4.1 for more details). Note that formally, $\frac{\nabla u}{|\nabla u|}$ is the outer normal to the level sets of u, hence the above formula is an integrated version of (19). In Section 3.4 we will then prove that for smooth functions u the limit (20) exists for many vector fields v and for a.e. t (see Proposition 3.4.6 and Corollary 3.4.7 for precise statements). This will allow us to define, not only the integral of the tangential divergence, but an actual notion of tangential divergence $\mathrm{div}_T(v) \in L^2(\mathrm{Per}(E))$ (see Proposition 3.4.2).

Having a tangential divergence, we can hope to define the mean curvature via formula (18). Clearly this cannot be possible for every set of finite perimeter, but some additional regularity is needed. For this reason in Section 3.5 we will give a notion of sublevels with *second order regularity* (see Definition 3.5.1), that are formally sublevels with L^2-mean curvature.

For these type of second-order regular sublevels we will prove that the tangential divergence $\mathrm{div}_T(v)$ of a vector field depends only on its trace $\mathrm{tr}_E(v)$. Additionally this independence will come with explicit bounds on $\int \mathrm{div}_T(v)\mathrm{dPer}(E)$ in term of the L^2-norm of $\mathrm{tr}_E(v)$ (see Theorem 3.5.5 for a precise statement). This will allow us to define for such sets, via duality, a notion of mean curvature vector H which satisfies (18) (see Theorem 3.5.7).

The final crucial point is that we will show that there are many smooth functions, for which almost every sublevel is regular enough to admit a mean curvature vector, in the sense described above (see Proposition 3.5.9). These examples turns out to be precisely the ones with finite Willmore-functional defined in (16) above, which includes harmonic functions and in particular the electrostatic potential.

Remark 0.10. Note that, even if we are able to define a mean curvature vector and not only the mean curvature, to avoid being too technical we are not making precise here in which sense H is a "vector". Roughly said H belongs to same space of the outer unit normal, which is the L^2-tangent bundle over a set of finite perimeter that was recently introduced in [**56**] (and will be reviewed in Section 1.5). ■

We have yet to see how this notion of mean curvature answers to the question (*ii*) posed at the beginning. The crucial point is that for a smooth function u (with finite Willmore energy) H will satisfy

$$H = \frac{\Delta u}{|\nabla u|} - \langle \nabla u, \frac{\nabla|\nabla u|}{|\nabla u|^2}\rangle$$

(see Proposition 3.5.15 for a precise statement). This in particular answers in a sense to question (*ii*) above, since in the case of an electrostatic potential $\Delta u = 0$.

Willmore inequalities. In Section 3.8 we will prove our version of the Willmore type inequalities for the electrostatic potential of a set E in an RCD$(0.N)$ spaces with Euclidean-volume growth. In particular we show that for every $\beta \in [\frac{N-2}{N-1}, 1]$ and a.e. $t \in (0,1)$ the sublevel $\{u < t\}$ has a mean curvature H and

$$\int \left| \frac{H}{N-1} \right|^{\beta+1} \mathrm{dPer}(\{u<t\}) \geq (\sigma_{N-1}\mathsf{AVR}(\mathrm{X}))^{\frac{\beta}{N-2}} \left(\frac{\overline{\mathrm{Cap}}(\{u \leq t\}^c)}{N-2} \right)^{\frac{N-2-\beta}{N-2}}, \quad \text{for a.e. } t \in (0,1). \tag{21}$$

(see Theorem 3.8.1 for more details). Note that if we could plug in $t = 1$ in the above inequality, we would get exactly the inequality in the smooth case (W).

Remark 0.11. Note that by our construction $|H| \in L^2(\mathrm{Per}(\{u < t\}))$, hence the left hand side of (21) is finite, which makes the inequality meaningful. Indeed (21) holds also for $\beta > 2$, however we do not know whether in these cases we get a non-trivial statement. ■

Rigidity and almost rigidity of Sobolev inequalities on compact RCD spaces

In Chapter 4 we will present the work in [**185**], which contains rigidity and almost-rigidity statements for optimal constants in the Sobolev inequality, under positive Ricci curvature lower bound. To state the main results we first need to recall some notions and facts concerning optimal Sobolev constants in the setting of Riemannian manifolds.

Optimal Sobolev constants on Riemannian manifolds. The standard Euclidean-Sobolev inequality in sharp form reads as

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq \mathsf{Eucl}(n,p)\|\nabla u\|_{L^p(\mathbb{R}^n)}, \qquad \forall u \in W^{1,p}(\mathbb{R}^n), \tag{22}$$

where $p \in (1,n)$ and $p^* := \frac{pn}{n-p}$, and $\mathsf{Eucl}(n,p)$ is the smallest constants for which the inequality (22) remains valid. Its precise value (see (1.1.2)) was computed independently by Aubin [**32**] and Talenti [**202**] (see also [**91**]).

In the setting of compact Riemannian manifolds, the presence of constant functions in the Sobolev space immediately shows that an inequality of the kind of (22) must fail. Yet, Sobolev embeddings are certainly valid also in this context and they can be expressed by calling into play the full Sobolev norm:

$$\|u\|^p_{L^{p^*}(M)} \leq A\|\nabla u\|^p_{L^p(M)} + B\|u\|^p_{L^p(M)}, \qquad \forall u \in W^{1,p}(M), \tag{$\star$}$$

where M is a smooth compact n-dimensional Riemannian manifold and $A, B > 0$. From the presence of the two parameters A, B, it is not straight-forward which is the notion of optimal constant in this case. The issue of defining and determine the best constants in $(\star)$ has been the central role of the celebrated AB-program, we refer to [**136**] for a thorough presentation of this topic (see also [**105**]). The starting point of this program is the definition of the following two different notions of "best Sobolev constants":

$$\alpha_p(M) := \inf\{A \ : \ (\star) \text{ holds for some } B\}, \qquad \beta_p(M) := \inf\{B \ : \ (\star) \text{ holds for some } A.\}$$

The first natural problem is to determine the value of $\alpha_p(M)$ and $\beta_p(M)$. It is rather easy to see that

$$\beta_p(M) = \mathsf{Vol}(M)^{p/p^*-1}$$

indeed the constant function gives automatically $\beta_p(M) \geq \mathsf{Vol}(M)^{p/p^*-1}$, while the other inequality follows from the Sobolev-Poincaré inequality (see, e.g. [**136**, Sec 4.1]). It is instead more subtle to determine whether $\beta_p(M)$ is attained, in the sense that the infimum in its definition is actually a minimum. This is true for $p = 2$ and due to Bakry [**37**] (see also Proposition 4.6.1), but actually false for $p > 2$.

Concerning the other optimal constant $\alpha_p(M)$, a celebrated result of Aubin [**32**] says that $\alpha_p(M)$ is precisely the one of the Euclidean space. More precisely he showed that on any n-dimensional compact Riemannian manifold M with $n \geq 2$, we have

$$\alpha_p(M) = \mathsf{Eucl}(n,p)^p \qquad \forall p \in (1,n). \tag{23}$$

The question whether $\alpha_p(M)$ is attained is instead more delicate and was proved to be true in in [**137**] for the case $p = 2$, answering affirmatively to a conjecture of Aubin.

On the other hand, knowing the value of $\beta_p(M)$ and that is attained for $p = 2$, we can define a further notion of optimal-constant A, "relative" to $B = \beta_2(M)$. More precisely we define

$$A^{\mathrm{opt}}_{2^*}(M) := \mathrm{Vol}(M)^{1-2/2^*} \cdot \inf\{A \ : \ (\star) \text{ for } p = 2 \text{ holds with } A \text{ and } B = \mathrm{Vol}(M)^{2/2^*-1}\}.$$

For generality will actually consider A^{opt} also in the so-called *subcritical case*, meaning that we enlarge the class of Sobolev inequalities and consider for every $q \in (2, 2^*]$

$$\|u\|^2_{L^q(M)} \leq A\|\nabla u\|^2_{L^2(M)} + \mathrm{Vol}(M)^{2/q-1}\|u\|^2_{L^2(M)}, \quad \forall u \in W^{1,2}(M), \tag{$\star\star$}$$

for some constant $A \geq 0$. Then we define

$$A_q^{\rm opt}(M) := {\rm Vol}(M)^{1-2/q} \cdot \inf\{A \ : \ (\star\star) \text{ holds } \}.$$

Observe that the infimum above is always a minimum and that ${\rm Vol}(M)^{2/q-1}$ is the 'minimal B' that we can take in $(\star\star)$.

Remark 0.12. We bring to the attention of the reader the renormalization factor ${\rm Vol}(M)^{1-2/q}$ in the definition of $A_q^{\rm opt}(M)$. This is usually not present in the literature concerning the AB-program (see e.g. [**136**]), however this choice will allow us to to have cleaner inequalities. This also makes $A_q^{\rm opt}$ invariant under rescalings of the volume measure of M. ■

One of the main questions that we will investigate in Chapter 4 concerns the value of $A_q^{\rm opt}(M)$. So far $A_q^{\rm opt}(M)$ is known explicitly only in the case of $\mathbb{S}^n$ and was firstly computed by Aubin in [**31**] in the case of $q = 2^*$ and by Beckner in [**43**] for a general q:

$$A_q^{\rm opt}(\mathbb{S}^n) = \frac{q-2}{n}, \qquad \forall n \geq 3. \tag{24}$$

For positive Ricci curvature we have the following celebrated comparison result originally proven in [**145**] (see also [**40, 163**] for the case of a general q):

Theorem 0.13. *Let M be an n-dimensional Riemannian manifold, $n \geq 3$, with ${\rm Ric} \geq n-1$. Then, for every $q \in (2, 2^*]$, it holds*

$$A_q^{\rm opt}(M) \leq A_q^{\rm opt}(\mathbb{S}^n). \tag{25}$$

Main rigidity and almost-rigidity results. Our first main result is the characterization of the equality in (25). In particular we will show:

Theorem 0.14. *Equality in (25) holds for some $q \in (2, 2^*]$ if and only if M is isometric to $\mathbb{S}^n$.*

It is important to point out that the novelty of the above result is that it covers the case $q = 2^*$. Indeed, for $q < 2^*$, Theorem 0.14 was already established (see e.g., [**40**, Remark 6.8.5]) and follows from an improvement (only for $q < 2^*$) of (25) due to [**111**] involving the spectral gap. On the other hand, up to the author's knowledge, this is the first time that it appears in the critical case $q = 2^*$.

We will also characterize the almost-equality case:

Theorem 0.15 (Almost-rigidity of $A_q^{\rm opt}$)**.** *For every $n \in (2, \infty)$, $q \in (2, n)$ and every $\varepsilon > 0$, there exists $\delta := \delta(n, \varepsilon, q) > 0$ such that the following holds. Let M be an n-dimensional Riemannian manifold, $n \geq 3$, with ${\rm Ric} \geq n-1$ and satisfying*

$$A_q^{\rm opt}(M) \geq A_q^{\rm opt}(\mathbb{S}^n) - \delta,$$

Then, there exists a spherical suspension $({\rm Y}, {\sf d}_{\rm Y}, \mathfrak{m}_{\rm Y})$ (i.e. there exists an ${\rm RCD}(N-2, N-1)$ space $({\rm Z}, {\sf d}_{\rm Z}, \mathfrak{m}_{\rm Z})$ so that ${\rm Y}$ is isomorphic as a metric measure space to $[0, \pi] \times_{\sin}^{N-1} {\rm Z}$) such that

$${\sf d}_{mGH}((M, {\sf d}, {\rm Vol}), ({\rm Y}, {\sf d}_{\rm Y}, \mathfrak{m}_{\rm Y})) < \varepsilon.$$

We point out two important facts concerning the two above statements:

i) For $q < 2^*$ the almost rigidity follows 'directly' from the sharper version of (25) cited above and using the almost-rigidity of the 2-spectral gap [**65, 69**]. Nevertheless, we are not aware of any such statement in the literature and in any case our proof will not use any improved version of (25).

ii) The key feature of Theorems 0.14 and 0.15 is that they include the 'critical' exponent. Indeed, the difference between the 'subcritical' case $q < 2^*$ and $q = 2^*$ is not only technical but a major issue linked to the lack of compactness in the Sobolev embedding. For this reason the proof of the critical case will require several additional arguments that constitute the main part of Chapter 4.

We will actually prove both Theorem 0.14 and Theorem 0.15 in the setting ${\rm RCD}(K, N)$ spaces. We first recall the following generalization to this setting of Theorem 0.13, which has been proved in [**187**] (see also [**41**] for the case of Dirichlet-forms and [**65**] in the case of essentially non-branching CD spaces):

Theorem 0.16. *Let $({\rm X}, {\sf d}, \mathfrak{m})$ be an ${\rm RCD}(N-1, N)$ space, $N \in (2, \infty)$. Then, for every $q \in (2, 2^*]$*

$$A_q^{\rm opt}({\rm X}) \leq \frac{q-2}{N}. \tag{26}$$

Note that in the case of an n-dimensional Riemannian manifold, from (24) we have that (26) is exactly (25) with $N = n$. Our main results will be the following rigidity and almost-rigidity of (26):

Theorem 0.17 (Rigidity of $A_q^{\rm opt}$, Theorem 4.8.1). *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(N-1,N)$ *space for some* $N\in(2,\infty)$ *and let* $q\in(2,2^*]$. *Then*

$$A_q^{\rm opt}(\mathrm{X})=\frac{q-2}{N}, \tag{27}$$

if and only if $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is isomorphic to a spherical suspension, i.e. there exists an* $\mathrm{RCD}(N-2,N-1)$ *space* $(\mathrm{Z},\mathsf{d}_{\mathrm{Z}},\mathfrak{m}_{\mathrm{Z}})$ *such that* $(\mathrm{X},\mathsf{d},\mathfrak{m})\simeq[0,\pi]\times_{\sin}^{N-1}\mathrm{Z}$.

Theorem 0.18 (Almost-rigidity of $A_q^{\rm opt}$, Theorem 4.10.1). *For every* $N\in(2,\infty)$, $q\in(2,N)$ *and every* $\varepsilon>0$, *there exists* $\delta:=\delta(N,\varepsilon,q)>0$ *such that the following holds. Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(N-1,N)$ *space with* $\mathfrak{m}(\mathrm{X})=1$ *and suppose that*

$$A_q^{\rm opt}(\mathrm{X})\ge\frac{(q-2)}{N}-\delta,$$

Then there exists a spherical suspension $(\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\mathfrak{m}_{\mathrm{Y}})$ *(i.e. there exists an* $\mathrm{RCD}(N-2,N-1)$ *space* $(\mathrm{Z},\mathsf{d}_{\mathrm{Z}},\mathfrak{m}_{\mathrm{Z}})$ *so that* Y *is isomorphic as a metric measure space to* $[0,\pi]\times_{\sin}^{N-1}\mathrm{Z}$*) such that*

$$\mathsf{d}_{mGH}((\mathrm{X},\mathsf{d},\mathfrak{m}),(\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\mathfrak{m}_{\mathrm{Y}}))<\varepsilon.$$

Best Sobolev constant in compact RCD **spaces.** The proof of the rigidity (and almost rigidity) of $A_q^{\rm opt}$ in the case $q=2^*$, will force us to study also the best Sobolev constant α_p defined above (see (23)) in the setting of $\mathrm{RCD}(K,N)$ spaces. In particular we will extend to this setting Aubin's result (see (23) above). To state it precisely let us first fix some notations.

Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space with $N\in(1,\infty)$ and $K\in\mathbb{R}$. For any $p\in(1,N)$ set $p^*:=\frac{Np}{N-p}$ and, in the same fashion of $(\star)$, consider:

$$\|u\|^p_{L^{p^*}(\mathrm{X})}\le A\||Du|\|^p_{L^p(\mathrm{X})}+B\|u\|^p_{L^p(\mathrm{X})},\qquad\forall u\in W^{1,p}(\mathrm{X}), \tag{$**$}$$

with constant $A,B\ge0$. We then define:

$$\alpha_p(\mathrm{X}):=\inf\{A\ :\ (**)\text{ holds for some }B\}. \tag{28}$$

For every $x\in\mathrm{X}$ we also recall the N-dimensional density defined as $\theta_N(x):=\lim_{r\to0}\frac{\mathfrak{m}(B_r(x))}{\omega_N r^N}$ (where the limit exists thanks to the Bishop-Gromov inequality). Our result reads as follows:

Theorem 0.19 (Theorem 4.5.1). *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a compact* $\mathrm{RCD}(K,N)$ *space with* $N\in(1,\infty)$. *Then for every* $p\in(1,N)$

$$\alpha_p(\mathrm{X})=\left(\frac{\mathsf{Eucl}(N,p)}{\min_{x\in\mathrm{X}}\theta_N(x)^{\frac1N}}\right)^p. \tag{29}$$

We point out that $\min_{x\in\mathrm{X}}\theta_N(x)$ always exists because θ_N is lower semicontinuous (see Section 1.3.2).

Remark 0.20. Note that if X is a n-dimensional Riemannian manifold, $\theta_n(x)=1$ for every $x\in\mathrm{X}$, hence in this case (29) (with $N=n$) is exactly (23). Recall also that here N needs not to be an integer and thus $\mathsf{Eucl}(N,p)$ has to be defined for arbitrary $N\in(1,\infty)$ (see (1.1.2)). ■

Remark 0.21. We are not assuming $(\mathrm{X},\mathsf{d},\mathfrak{m})$ to be renormalized. In particular observe that if we rescale the reference measure $\mathfrak{m}$ as $c\cdot\mathfrak{m}$, then α_p gets multiplied by $c^{-p/N}$, which is in accordance with the scaling in (29). ■

Remark 0.22. Finally let us say that Theorem 0.19 gives non-trivial information even in the 'collapsed' case, i.e. when $\theta_N=+\infty$ in a set of positive (or even full) measure (see [**96**] for the notion of collapsed/non-collapsed RCD spaces, which generalizes same notion introduced in [**73**] for Ricci-limits). Indeed to have $\alpha_p(\mathrm{X})>0$ it is sufficient that $\theta_N(x)<+\infty$ at a single point $x\in\mathrm{X}$. As an example consider the model space $([0,\pi],|.|,\sin^{N-1}(t)\mathrm{d}t)$ which is $\mathrm{RCD}(N-1,N)$ with $\theta_N(x)<+\infty$ only for $x\in\{0,\pi\}$. ■

Application to the Yamabe equation in RCD **spaces.** As an application of Theorem 0.19 and of the technical tools developed in Chapter 4 (see Section "Additional results" below) we obtain some results concerning the existence of solutions to the Yamabe equation in the RCD setting, together with a continuity result of the Yamabe constant under mGH-convergence.

Existence of solutions to the Yamabe equation. Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a compact $\mathrm{RCD}(K,N)$-space for some $K\in\mathbb{R}, N\in(2,\infty)$ and set $2^*:=2N/(N-2)$. Fix also a function $S\in L^p(\mathfrak{m})$ with $p>N/2$. The *Yamabe equation* is then given by

$$(\mathcal{Y})\qquad -\Delta u=\lambda u^{2^*-1}-u\mathrm{S},\qquad \lambda\in\mathbb{R}$$

(see Section 1.2.4 for the definition of Laplacian in this setting). It is easy to see and classical (see Section 4.11.1) that $(\mathcal{Y})$ is the Eulero-Lagrange equation for the following functional:

$$W^{1,2}(\mathrm{X})\setminus\{0\}\ni u\mapsto Q_{\mathrm{S}}(u):=\frac{\int|Du|^2\,\mathrm{d}\mathfrak{m}+\int S|u|^2\mathrm{d}\mathfrak{m}}{\|u\|^2_{L^{2^*}(\mathfrak{m})}},$$

Observe that by the Sobolev embedding $u\in L^{2^*}(\mathfrak{m})$, hence the assumption on S grants that $Su^2\in L^1(\mathfrak{m})$. We now define the *generalized Yamabe constant* (relative to S) as

$$\lambda_{\mathrm{S}}(\mathrm{X}):=\inf\{Q_{\mathrm{S}}(u)\ :\ u\in W^{1,2}(\mathrm{X})\setminus\{0\}\}. \tag{30}$$

A now standard result on Riemannian manifolds is that whenever

$$\lambda_S(M)<\alpha_2(M)^{-2}, \tag{31}$$

it holds that $(\mathcal{Y})$ has a non-negative and non-zero solution with $\lambda=\lambda_S$. We mention that this fact plays a key role in the solution of the Yamabe problem (see e.g. [**164**,**215**] or Section 4.11 for more on this topic). The same result has been proved in the setting of Ricci-limits by Honda [**141**]. Here we will extend it to the RCD setting as follows.

Theorem 0.23 (Theorem 4.11.2)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a compact $\mathrm{RCD}(K,N)$ space for some $K\in\mathbb{R}$, $N\in(2,\infty)$. Suppose that for some $\mathrm{S}\in L^p(\mathfrak{m})$ with $p>N/2$, it holds that*

$$\lambda_{\mathrm{S}}(\mathrm{X})<\alpha_2(\mathrm{X})^{-2}.$$

Then there exists a non-negative and non-zero solution to $(\mathcal{Y})$ with $\lambda=\lambda_{\mathrm{S}}(\mathrm{X})$ which is a minimizer for (4.11.5)*.*

Remark 0.24. As an improvement from the results in [**141**] note that from Theorem 0.19 we know the explicit expression $\alpha_2(\mathrm{X})^{-2}=\frac{\min_x\theta_N(x)^2}{\mathsf{Eucl}(N,2)^2}$. ■

Continuity of the Yamabe constant under mGH-convergence. We will prove the following continuity result for the Yamabe constant defined above (see also Section 1.6.2 for the notion L^p-weak convergence of functions on different spaces)

Theorem 0.25 (Theorem 4.11.5)**.** *Let $(\mathrm{X}_n,\mathsf{d}_n,\mathfrak{m}_n)$ be a sequence of compact $\mathrm{RCD}(K,N)$-spaces $n\in\bar{\mathbb{N}}$, for some $K\in\mathbb{R},N\in(2,\infty)$ satisfying $\mathrm{X}_n\overset{mGH}{\to}\mathrm{X}_\infty$. Let also $\mathrm{S}_n\in L^p(\mathfrak{m}_n)$ be L^p-weak convergent to $\mathrm{S}\in L^p(\mathfrak{m}_\infty)$, for a given for $p>N/2$. Then,*

$$\lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)=\lambda_{\mathrm{S}}(\mathrm{X}_\infty).$$

Remark 0.26. Theorem 0.25 was proved in [**141**, Thm. 1.7, Cor. 1.8] for Ricci-limits assuming an upper bounds on $\overline{\lim}_n\lambda_{\mathrm{S}_n}(\mathrm{X}_n)$. Here, exploiting the additional information given by concentration compactness we sharpen this result getting rid of this additional assumption. ■

Additional results.

Concentration compactness under mGH-convergence. The almost rigidity result in Theorem 0.15 will be proved using the compactness property of the $\mathrm{RCD}(K,N)$ class. However, in the case $q=2^*$ the Sobolev embedding has a natural lack of compactness. Hence for the proof we will need an additional tool, which is a concentration compactness result under mGH-convergence of compact RCD-spaces. In particular, we will prove a concentration-compactness dichotomy principle (see Lemma 4.7.6 and Theorem 4.7.1) in the spirit of Lions [**170**] (see also the monograph [**198**]), under mGH-convergence. To the author's best knowledge this is the first result of this type dealing with varying spaces and we believe it to be interesting on its own.

Euclidean-type Polya-Szego inequality on $\mathrm{CD}(K,N)$ *spaces.* One of the main tools that we will need to develop is a Polya-Szego inequality (see Section 4.2.2), which is an Euclidean-variant of the Polya-Szego inequality for $\mathrm{CD}(K,N)$ spaces, $K>0$, derived in [**182**]. The main feature of this inequality is that it holds on arbitrary $\mathrm{CD}(K,N)$ spaces, $K\in\mathbb{R}$, but assumes the validity of an isoperimetric inequality of the type

$$\mathrm{Per} E \geq \mathsf{C}_{\mathsf{Isop}}\mathfrak{m}(E)^{\frac{N-1}{N}}, \qquad \forall E\subset\Omega \text{ Borel },$$

for some $\Omega\subset \mathrm{X}$ open and where $\mathsf{C}_{\mathsf{Isop}}$ is a positive constant independent of E. For our purposes this Polya-Szego inequality will be used to derive local Sobolev inequalities of Euclidean-type (see Theorem 4.3.2), however it allows us to obtain also sharp Sobolev inequalities under Euclidean-volume growth assumption (see below).

Sharp Sobolev inequalities under Euclidean-volume growth. We recall that a $\mathrm{CD}(0,N)$ space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ has Euclidean-volume growth if

$$\mathsf{AVR}(\mathrm{X}) := \lim_{R\to+\infty}\frac{\mathfrak{m}(B_R(x_0))}{\omega_N R^N}>0,$$

for some (and thus any) $x_0\in\mathrm{X}$. In this setting we will prove the following sharp Sobolev inequality:

Theorem 0.27 (Theorem 4.4.6)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a* $\mathrm{CD}(0,N)$ *space for some* $N\in(1,\infty)$ *and with Euclidean volume-growth. Then, for every* $p\in(1,N)$*, it holds*

$$\|u\|_{L^{p^*}(\mathfrak{m})}\leq \mathsf{Eucl}(N,p)\mathsf{AVR}(\mathrm{X})^{-\frac{1}{N}}\||Du|\|_{L^p(\mathfrak{m})}, \qquad \forall u\in \mathsf{LIP}_c(\mathrm{X}). \tag{32}$$

Moreover (32) *is sharp.*

This extends a result recently proved in [**42**] in the case of Riemannian manifolds and answer positively to a question also posed in [**42**, Sec. 5.2].

PART II

Bi-Lipschitz version of Cheeger and Colding's metric-Reifenberg's theorem

In the second part of the thesis, which consists of Chapter 5, we will present the content of the works in [**130**, **209**], that are about the bi-Lipschitz version of Reifenberg's theorem in metric spaces.

Classical Reifenberg's and metric-Reifenberg's theorems. The celebrated Reifenberg's theorem [**192**] (see also [**183**, **196**]) states that if a set in $\mathbb{R}^d$ is well approximated at every small scale by n-dimensional affine planes, then it is (locally) an n-dimensional bi-Hölder manifold. More precisely, for a set $S\subset\mathbb{R}^d$ and $n\in\mathbb{N}$, with $n<d$, we set

$$\mathsf{e}(x,r) := r^{-1}\inf_{\Gamma}\mathsf{d}_H(S\cap B_r(x),\Gamma\cap B_r(x)), \quad \text{for every } r>0 \text{ and } x\in S,$$

where d_H is the Hausdorff distance and where the infimum is taken among all the n-dimensional affine planes Γ in $\mathbb{R}^d$ containing x. Then Reifenberg's result reads as follows:

Theorem 0.28 (Reifenberg's theorem,[**192**])**.** *For every* $n,d\in\mathbb{N}$ *with* $n<d$ *and* $\alpha\in(0,1)$ *there exists* $\delta=\delta(n,d,\alpha)$ *such that the following holds. Let* $S\subset\mathbb{R}^d$ *be closed, containing the origin and such that* $\mathsf{e}(x,r)<\delta$ *for every* $x\in S\cap B_1(0)$ *and* $r\in(0,1)$.

Then there exists an α*-bi-Hölder homeomorphism* $F:\Omega\to S\cap B_{1/2}^{\mathbb{R}^d}(0)$*, where* Ω *is an open set in* $\mathbb{R}^n$.

The original motivation in [**192**] to prove this result was the regularity of minimal surfaces. However, from its original formulation, Reifenberg's theorem and more in general the idea behind its proof have found successful generalizations and applications in harmonic analysis, geometric measure theory, rectifiability theory and PDE's (see e.g. [**34**,**36**,**92**,**94**,**106**,**150**,**184**,**203**,**204**] and the references therein).

Coming to the topic of Chapter 5, we will be interested in the Reifenberg's theorem for metric spaces. The generalization of Theorem 0.28 in this setting has been obtained in a famous result by Cheeger and Colding [**73**, Appendix A]. To state it we need to define the metric-analogue of the 'flatness'-coefficients $\mathsf{e}(x,r)$ in Reifenberg's theorem. In this case the comparison with planes, which clearly is not available, is replaced with the Gromov-Hausdorff distance to Euclidean balls. In particular for a metric space (Z,d) and a fixed $n\in\mathbb{N}$ we set

$$\varepsilon(z,r) := r^{-1}\mathsf{d}_{GH}(B_r(z),B_r^{\mathbb{R}^n}(0)), \quad \text{for every } r>0 \text{ and } z\in Z,$$

(we refer to Section 5.1 for the definition of d_{GH}).

Theorem 0.29 (Metric-Reifenberg's theorem, [**73**]). *For every $n \in \mathbb{N}$ and $\alpha \in (0,1)$ there exists $\delta = \delta(n,\alpha)$ such that the following holds. Let (Z, d) be a complete and connected metric space and suppose that $\varepsilon(z,r) < \delta$ for every $z \in Z$ and $r \in (0,1)$.*

Then there exists an α-bi-Hölder homeomorphism $F : M \to Z$, where M is a smooth n-dimensional Riemannian manifold.

A local version of this result, more in the spirit of Theorem 0.28 also exists (see [**73**, Theorem A.1.1]), however for simplicity we will focus only on the global version. We mention also [**93**], where metric spaces satisfying the assumptions of Theorem 0.29 have been further studied.

The original motivation of Cheeger and Colding to prove the metric-Reifenberg's theorem was to prove manifold-regularity of the regular set of "non-collapsed" Ricci-limit spaces ([**73**]). However as a consequence of their construction they obtained also several improvements of some previous stability results for Riemannian manifolds with Ricci curvature bounded below [**81**,**83**,**186**]. In addition, this result has found recently many different applications in the theory of metric measure spaces with synthetic Ricci curvature lower bounds, see for instance ([**55**,**151**,**174**]). It is worth mentioning that also the more general Reifenberg-type constructions connected to rectifiability (as in [**184**]) have found successful applications on spaces with Ricci bounds (see [**55**, **77**] using ideas also from [**149**]).

Bi-Lipschitz version of the classical Reifenberg's theorem. Returning to the Euclidean case, it is now classical that if the approximation by planes improves sufficiently fast as the scale decreases, the bi-Hölder regularity in Reifenberg's result can be improved to bi-Lipschitz. This goes back to Toro [**204**] and the same idea has been further refined and developed in the context of rectifiability e.g. in [**92**, **94**, **106**, **184**].

The right decay condition turns out to be the square summability of the numbers $\mathsf{e}(x,r)$ along dyadic scales:

Theorem 0.30 ([**204**]). *For every $\varepsilon > 0$, $n, d \in \mathbb{N}$ with $n < d$, there exists $\delta = \delta(n,d,\varepsilon) > 0$ such that the following holds. Let $S \subset \mathbb{R}^d$ be closed, containing the origin and such that*

$$\sum_{i=1}^{\infty} \sup_{x \in S \cap B_1(0)} \mathsf{e}(x, 2^{-i})^2 < \delta. \tag{33}$$

Then there exists a $(1+\varepsilon)$-bi-Lipschitz homeomorphism $F : \Omega \to S \cap B^{\mathbb{R}^d}_{1/2}(0)$, where Ω is an open set in $\mathbb{R}^n$.

The square summability assumption is sharp as shown by suitable *snowflake-constructions*.

We will be interested in answering to the following question:

Question: Can Theorem 0.30 be extended to the setting of metric spaces?

Note that as we mentioned above, some results on metric spaces with Ricci bounds concerning rectifiability and Reifenberg-type arguments, which also lead to the construction of biLipschitz maps to the Euclidean space, have been obtained ([**55**, **77**]). However they use the analysis that is available on such spaces, while here we are interested on a 'purely' metric generalization of Theorem 0.30 in the spirit of Cheeger and Colding's result (Theorem 0.29).

We will answer this question in two steps. First we will prove a modified version of Cheeger-Colding's theorem in the biLipschitz case. Then we will show that this version is sharp and that it is the correct generalization of Theorem 0.30.

Bi-Lipschitz version of the Cheeger-Colding's metric Reifenberg's theorem. A careful analysis of the Cheeger and Colding argument reveals that a bi-Lipschitz version of their result can be deduced with small modifications to the proof.

Theorem 0.31. *For every $n \in \mathbb{N}$ the following holds. Let (Z, d) be a complete and connected metric space. Suppose that for some $r > 0$*

$$\sum_{i=1}^{\infty} \sup_{z \in Z} \varepsilon(x, 2^{-i} r) < +\infty. \tag{34}$$

Then for every $\varepsilon > 0$ there exists a smooth n-dimensional Riemannian manifold $(W_\varepsilon, d_\varepsilon)$ and an homeomorphism $F_\varepsilon : W_\varepsilon \to Z$ that is (uniformly) locally $(1+\varepsilon)$-bi-Lipschitz.

We refer to Theorem 5.2.2 for a more detailed statement. Note that the condition that appears in this case is the summability of the numbers $\varepsilon(x,r)$ along dyadic scales.

The above Theorem, even if not present in the literature, seems to be a folklore-result known at least to some of the experts in the field of set-parametrizations (see [**93**] for a discussion about Reifenberg's parametrization of metric spaces).

The first goal of Chapter 5 is to give a detailed and self-contained proof of Theorem 0.31. We will do so for two reasons:

- to get the bi-Lipschitz regularity of the map it is necessary that the error-order of every estimate in the proof is quantified and therefore we need to make explicit every step,
- given the recent growing interest in the metric version of Reifenberg's theorem we believe it useful to have available an expanded version, also with a somewhat different presentation, of the arguments in [**73**].

Comparison between "Hausdorff flatness" and "Gromov-Hausdorff flatness". In the second part of Chapter 5 we discuss the sharpness of Theorem 0.31, in particular we answer to the following question:

Question: Is it possible to weaken the assumption (34) from summability to square summability, at least restricting ourselves to some special class of metric spaces?

Indeed, if we interpret the Gromov-Hausdorff numbers $\varepsilon(x,r)$ as a metric-generalization of the Reifenberg's numbers $\mathsf{e}(x,r)$, it appears that the bi-Lipschitz metric-Reifenberg is weaker than Toro's bi-Lipschitz Reifenberg, Theorem 0.30 (as for sequences square summability is strictly weaker than summability).

Perhaps surprisingly, the answer the the above question turns out to be negative even in the classical case of subsets of the Euclidean space. Even more we have that the above bi-Lipschitz version of the metric-Reifenberg theorem is already equivalent to Toro's theorem, when applied to subsets of $\mathbb{R}^d$. The main insight is that the discrepancy between the result of Toro and Theorem 0.31 is not in the regularity of the metric space in consideration, but rather on the way the flatness is measured.

Indeed we will show that for a set $S \subset \mathbb{R}^d$ it holds

$$\sum_{i=1}^{\infty} \sup_{x\in S\cap B_1(0)} \mathsf{e}(x,2^{-i})^2 < +\infty \iff \sum_{i=1}^{\infty} \sup_{x\in S\cap B_1(0)} \varepsilon(x,2^{-i}), \tag{35}$$

where $\varepsilon(x,2^{-i})$ are computed identifying S as a metric space endowed with the restriction of the Euclidean distance (see Theorem A and Proposition 5.5.9 for more precise statements). This can be formally interpreted as the Gromov-Hausdorff numbers $\varepsilon(x,r)$ behaving as the square of the Reifenberg's numbers $\mathsf{e}(x,r)$. This phenomenon was actually already mentioned (but not rigorously explored) by David and Toro [**93**]. We also point out that, even if (35) can be made more quantitative (see Theorem 5.6.7), the inequality $\varepsilon(x,r) \lesssim \mathsf{e}(x,r)^2$ cannot hold in general, as shown by elementary examples (see Section 5.5.2).

We conclude with the following key-example of a very thin triangle, that gives an intuition on why the numbers $\varepsilon(x,r)$ behave as the square of the numbers $\mathsf{e}(x,r)$ (see also Example 5.5.7 for more details).

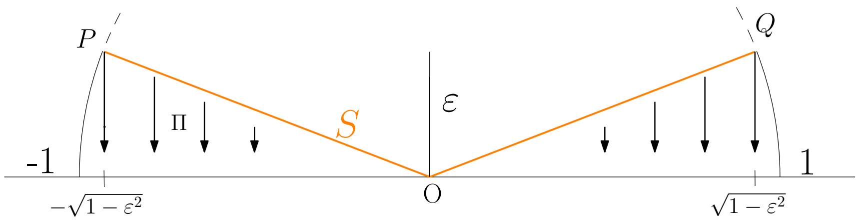

FIGURE 1. An example where $\mathsf{e}(O,1)\sim\varepsilon$, but $\varepsilon(O,1)\sim\varepsilon^2$.

Example 0.32. Fix $0<\varepsilon<1$. Let $P,Q\in\mathbb{R}^2$ be the two points in the upper half plane that are at distance 1 from the origin and at distance $\varepsilon\in(0,1/2)$ from the x-axis. Then define the set $S\subset\mathbb{R}^2$ to be the union of the two segments joining P and Q to the origin. It can be easily checked using the orthogonal projection Π onto the x-axis (see Example 5.5.7 for the details), that $\mathsf{e}(O,1)\sim\varepsilon$, while $\varepsilon(O,1)\sim\varepsilon^2$.

Part 1

Metric measure spaces with synthetic Ricci curvature bounds

CHAPTER 1

Preliminaries

1.1. Basic notations and definitions

1.1.1. Basic constants. We collect here some constants that will often appear along this part of the thesis.

For all $N \in [1,\infty), p \in (1,N)$, we define the generalized[1] *unit ball* and *unit sphere* volumes by

$$\omega_N := \frac{\pi^{N/2}}{\Gamma(N/2+1)}, \quad \sigma_{N-1} := N\omega_N, \tag{1.1.1}$$

where Γ is the Gamma-function, and the *sharp Euclidean-Sobolev constant* by

$$\mathsf{Eucl}(N,p) := \frac{1}{N}\Big(\frac{N(p-1)}{N-p}\Big)^{\frac{p-1}{p}}\Big(\frac{\Gamma(N+1)}{N\omega_N\Gamma(N/p)\Gamma(N+1-N/p)}\Big)^{\frac{1}{N}}. \tag{1.1.2}$$

For $N>2$ and $p=2$, the above reduces to

$$\mathsf{Eucl}(N,2) = \Big(\frac{4}{N(N-2)\sigma_N^{2/N}}\Big)^{\frac{1}{2}}. \tag{1.1.3}$$

We will sometimes need the following identity:

$$\int_0^\pi \sin^{N-1}(t)\,\mathrm{d}t = \frac{\sigma_N}{\sigma_{N-1}}, \quad \forall N>1. \tag{1.1.4}$$

1.1.2. Metric measure spaces. Throughout the exposition a *metric measure space* (abbreviated in *m.m.s.*) will be a triple $(\mathrm{X},\mathsf{d},\mathfrak{m})$ where

> (X,d) is a complete and separable metric space and $\mathfrak{m}$ is a non-negative and non-zero Borel measure on X, finite on bounded sets and such that $\operatorname{supp}\mathfrak{m} = \mathrm{X}$.

We will also use the notion of *pointed metric measure space* (abbreviated in *p.m.m.s.*), which is a quadruple $(\mathrm{X},\mathsf{d},\mathfrak{m},\bar{x})$, where $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is a metric measure space and $\bar{x} \in \mathrm{X}$.

Two metric measure spaces $(\mathrm{X}_i,\mathsf{d}_i,\mathfrak{m}_i)_{i=1,2}$ are said to be *isomorphic*, $\mathrm{X}_1 \simeq \mathrm{X}_2$ in short, if there exists an isometry $\iota : \mathrm{X}_1 \to \mathrm{X}_2$ such that $\iota_*\mathfrak{m}_1 = \mathfrak{m}_2$.

We will denote by $\mathsf{LIP}(\mathrm{X})$, $\mathsf{LIP}_{\mathsf{loc}}(\mathrm{X})$, $\mathsf{LIP}_b(\mathrm{X})$, $\mathsf{LIP}_{bs}(\mathrm{X})$, $\mathsf{LIP}_c(\mathrm{X})$, $C_b(\mathrm{X})$ and $C_{bs}(\mathrm{X})$, respectively the spaces of Lipschitz functions, locally Lipschitz functions, bounded Lipschitz functions, Lipschitz functions with bounded support, Lipschitz functions with compact support, bounded continuous functions and bounded continuous functions with bounded support in (X,d). For an open set $A \subset \mathrm{X}$ we also denote by $\mathsf{LIP}_{\mathsf{loc}}(A)$ and $\mathsf{LIP}_c(A)$ the spaces of locally Lipschitz functions and Lipschitz functions with compact support in A. Moreover for a function $f \in \mathsf{LIP}_{\mathsf{loc}}(A)$ we denote by $\mathrm{lip}f : A \to [0,+\infty)$ its *local-Lipschitz constant* defined by

$$\mathrm{lip}f(x) := \varlimsup_{y\to x} \frac{|f(x)-f(y)|}{\mathsf{d}(x,y)} \tag{1.1.5}$$

taken to be 0 if x is an isolated point.

Given a metric space (X,d) and $B \subset \mathrm{X}$ we will denote by $\mathrm{diam}(B)$ the quantity $\sup_{x,y\in B}\mathsf{d}(x,y)$. Moreover for a point $p \in \mathrm{X}$ we denote by $B_r^{\mathrm{X}}(p) := \{x \in \mathrm{X} \ : \ \mathsf{d}(x,p) < r\}$ the ball of radius $r>0$ centered at p and by $\overline{B}_r(p)$ its topological closure, which in case of a length space coincides with the closed ball $\{x \in \mathrm{X} \ : \ \mathsf{d}(x,p) \le r\}$.

For a metric space (X,d) and a point $p \in X$ we shall indicate by $\mathsf{d}_p : \mathrm{X} \to \mathbb{R}^+$ the distance function from the point p, defined by $\mathsf{d}_p(x) := \mathsf{d}(x,p)$ for every $x \in \mathrm{X}$.

[1]For an integer N, ω_N is the volume of the unit ball in $\mathbb{R}^N$ and σ_N is the volume of the N-sphere $\mathbb{S}^N$.

A subset S of a metric space (X,d) is said to be *δ-dense in* X if $\{x\in \mathrm{X}\ :\ \mathsf{d}(x,S)<\delta\}=\mathrm{X}$. A map between two metric spaces $f:(\mathrm{X},\mathsf{d}_1)\to(Y,\mathsf{d}_2)$ is said to be a *δ-isometry* if $|\mathsf{d}_2(f(x),f(y))-\mathsf{d}_1(x,y)|<\delta$, for every $x,y\in S$.

A m.m.s. $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is said to be *uniformly locally doubling*, if for every $R>0$, there exists a constant $C:=C(R)$ so that

$$\mathfrak{m}(B_{2r}(x))\le C\mathfrak{m}(B_r(x)),\qquad \forall x\in\mathrm{X},r\in(0,R).$$

Whenever $C(R)$ can be taken independent of R we say that $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is doubling. For us a geodesic will be always a constant length speed minimizing curve between its endpoints and defined on $[0,1]$, i.e. a map $\gamma:[0,1]\to\mathrm{X}$ so that $\mathsf{d}(\gamma_t,\gamma_s)=|t-s|\mathsf{d}(\gamma_0,\gamma_1)$, for every $t,s\in[0,1]$. Also, we denote by $\mathsf{Geo}(\mathrm{X})$ the set of all goedesics (endowed with the sup-distance) and, for every $t\in[0,1]$, we call *evaluation map* the assignment $\mathrm{e}_t\colon C([0,1],\mathrm{X})\to\mathrm{X}$ defined via $\mathrm{e}_t(\gamma):=\gamma_t$.

For $1\le p\le\infty$ and a non-negative Borel measure μ on X we denote by $L^p(\mu)$ the space of (μ-a.e. equivalence classes of) p-integrable functions. Moreover we denote by $L^0(\mu)$ the space of (μ-a.e. equivalence classes of) Borel functions.

We will denote by $\mathscr{M}_b(\mathrm{X}),\mathscr{M}_{\mathsf{loc}}(\mathrm{X}),\mathscr{P}(\mathrm{X})$ respectively the spaces of finite Borel measures, boundedly-finite Borel measures and (non-negative) probability measures.

If (X,d) is proper and given $\Omega\subset\mathrm{X}$ an open set, we say that μ is a non-negative Radon measure on Ω is μ is a non-negative Borel measure on Ω which is finite on relatively compact sets in Ω. A signed-Radon measure on Ω will instead denote a set function $\mu:\{\text{Borel sets relatively compact in }\Omega\}\to\mathbb{R}$ which can be written as $\mu(B)=\mu^+(B)-\mu^-(B)$, for some non-negative Radon measures μ^+,μ^-. By Riesz representation theorem we have that every linear functional $T:\mathsf{LIP}_c(\Omega)\to\mathbb{R}$ such that for any compact set $K\subset\Omega$ there exists a constant C_K such that

$$|T(\varphi)|\le C_K\sup|\varphi|,\quad \forall\varphi\in C_c(\Omega)\text{ with }\operatorname{supp}(\varphi)\subset K,$$

can be represented via integration against a (unique) signed Radon measure on Ω.

The following well-known result (see e.g. [**127**]) ensures that it is sufficient to check weak convergence in duality with $C_b(\mathrm{X})$ only against a countable family of functions.

Proposition 1.1.1. *Let (X,d) be complete and separable. Then there exists a countable family $\mathcal{D}\subset C_b(\mathrm{X})$ such that*

$$\mu_n\overset{C_b(\mathrm{X})}{\rightharpoonup}\mu\in\mathscr{M}_b(\mathrm{X})\text{ if and only }\int f\,\mathrm{d}\mu_n\to\int f\mathrm{d}\mu,\quad\forall f\in\mathcal{D}.$$

Proposition 1.1.2. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be proper. Then there exists a countable family $\mathcal{F}\subset\mathsf{LIP}_c(\mathrm{X})$ such that $\mathcal{F}$ is dense in $L^p(\mu)$ for every $\mu\in\mathscr{M}_b(\mathrm{X})$.*

Proof. Fix $x_0\in\mathrm{X}$. Recall that $\mathsf{LIP}_{bs}(\mathrm{X})$ is dense in $L^p(\mu)$ for every $\mu\in\mathscr{M}_b(\mathrm{X})$ and observe that we can write $\mathsf{LIP}_{bs}(\mathrm{X})=\cup_{n\in\mathbb{N}}A_n$, where $A_n:=\{f\in\mathsf{LIP}(\mathrm{X}),\operatorname{supp}(f)\subset\overline{B_n(x_0)}\}$. Moreover, since $\overline{B_n(x_0)}$ is compact and $A_n\subset C(\overline{B_n(x_0)})$, there exists a countable family $\mathcal{F}_n\subset A_n$ dense in A_n with respect to the sup norm. In particular $\mathcal{F}_n$ is dense in A_n also we respect to the $L^p(\mu)$-norm, for every $\mu\in\mathscr{M}_b(\mathrm{X})$. Taking $\mathcal{F}:=\cup_n\mathcal{F}_n$ concludes the proof. □

1.2. Calculus on metric measure spaces

1.2.1. Sobolev spaces via Cheeger energy. The notion of Sobolev space on metric measure spaces is by-now classical and goes back to the seminal works of Cheeger [**71**] and then Shanmugalingam [**195**], in turn inspired by the notion of upper gradients introduced by Heinonen-Koskela [**138, 139**]. Here we adopt the approach via Cheeger energy introduced in [**19**] and proved there to be equivalent to the ones in [**71, 195**]. For an account on the rich theory of Sobolev spaces on metric measure spaces we refer to [**46, 127, 134, 140**].

The *p-Cheeger energy* $\mathsf{Ch}_p\colon L^p(\mathfrak{m})\to[0,\infty]$, $p\in(1,\infty)$, is defined as the convex and lower semicontinuous functional

$$\mathsf{Ch}_p(f):=\inf\Big\{\varliminf_{n\to\infty}\int(\mathrm{lip}f_n)^p\,\mathrm{d}\mathfrak{m}\colon(f_n)\subset L^p(\mathfrak{m})\cap\mathsf{LIP}(\mathrm{X}),\lim_n\|f-f_n\|_{L^p(\mathfrak{m})}=0\Big\}.$$

The p-Sobolev space is then defined as $W^{1,p}(\mathrm{X}):=\{\mathsf{Ch}_p<\infty\}$ and equipped with the norm $\|f\|^p_{W^{1,p}(\mathrm{X})}:=\|f\|^p_{L^p(\mathfrak{m})}+\mathsf{Ch}^p(f)$ which makes it a Banach space. Under the assumption that $\mathfrak{m}$ is doubling, $W^{1,p}(\mathrm{X})$ is reflexive as proven in [**13**] and in particular the class $\mathsf{LIP}_{bs}(\mathrm{X})$ is dense in $W^{1,p}(\mathrm{X})$ (see also the more recent [**108**]). Finally, exploiting the definition by relaxation given for the p-Cheeger energy, it is a standard fact (see [**19**]) that whenever $f \in W^{1,p}(\mathrm{X})$, then there exists a minimal $\mathfrak{m}$-a.e. object $|Df|_p \in L^p(\mathfrak{m})$ called *minimal p-weak upper gradient* so that

$$\mathsf{Ch}_p(f) := \int |Df|_p^p \, \mathrm{d}\mathfrak{m}.$$

Remark 1.2.1. In general, the dependence on p of such object is hidden and not trivial (that is why we introduced the p-subscript in the object $|Df|_p$), see for example [**101**] (or also [**19**, Sec. 5]). Nevertheless we will almost always omit this dependence both because we will mainly work with $p = 2$ and also since we will consider metric measure spaces where this dependence is ruled out (see Theorem 1.2.3) below. ∎

The following elementary calculus rules hold for the weak upper gradient:

- *Locality:* for every $f, g \in W^{1,p}(\mathrm{X})$ it holds that $|Df| = |Dg|$, $\mathfrak{m}$-a.e. in $\{f = g\}$,
- *Leibniz rule:* for every $f, g \in W^{1,p}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$, $fg \in W^{1,p}(\mathrm{X})$ and $|D(fg)| \le |Df|g + f|Dg|$ $\mathfrak{m}$-a.e.,
- *Chain rule:* for every $f \in W^{1,p}(\mathrm{X})$ and $\varphi \in \mathsf{LIP}(\mathbb{R})$ with $\varphi(0) = 0$, $\varphi(f) \in W^{1,2}(\mathrm{X})$ and $|Df| = \varphi'(f)|Df|$ $\mathfrak{m}$-a.e..

We will sometimes need to consider the case when $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is a weighted interval with a weight that is bounded away from zero, i.e. $(\mathrm{X}, \mathsf{d}, \mathfrak{m}) = ([a, b], \mathsf{d}_{eu}, h\mathscr{L}^1)$, $a, b \in \mathbb{R}$ with $a < b$ where $h \in L^1([a, b])$ and for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ so that $h \ge c_\varepsilon$ $\mathscr{L}^1$-a.e. in $[a + \varepsilon, b - \varepsilon]$. In this case, we denote by $W^{1,p}([a, b], \mathsf{d}_{eu}, h\mathscr{L}^1)$ the p-Sobolev space over the weighted interval according to the metric definition relying on the Cheeger energy, while we simply write $W^{1,p}(a, b)$ for the classical definition via integration by parts. It can be easily shown that (using e.g. [**19**, Remark 4.10])

$$f \in W^{1,p}([a, b], \mathsf{d}_{eu}, h\mathscr{L}^1) \quad \Longleftrightarrow \quad f \in W^{1,1}_{loc}(a, b) \text{ with } f, f' \in L^p(h\mathscr{L}^1), \tag{1.2.1}$$

in which case $|Df|_p = |f'|$, $\mathscr{L}^1$-a.e..

Another important quantity that will appear often is the first eigenvalue of the Laplacian, the so-called *spectral gap*:

Definition 1.2.2 (Spectral gap)**.** Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space with finite measure. We define the first non trivial 2-eigenvalue $\lambda^{1,2}(\mathrm{X})$ as the non-negative number given by

$$\lambda^{1,2}(\mathrm{X}) := \inf \left\{ \frac{\int |Df|_2^2 \, \mathrm{d}\mathfrak{m}}{\int |f|^2 \, \mathrm{d}\mathfrak{m}} \ : \ f \in \mathsf{LIP}(\mathrm{X}) \cap L^2(\mathfrak{m}),\ f \ne 0,\ \int f \, \mathrm{d}\mathfrak{m} = 0 \right\}. \tag{1.2.2}$$

Clearly, in light of [**19**], in the above definition one can equivalently take the infimum among all $f \in W^{1,2}(\mathrm{X})$. In the sequel will use this fact without further notice.

A metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is said to support a weak (q, p)-Poincaré inequality with $q, p \in [1, \infty)$ if there exist constants $C > 0$ and $\lambda \ge 1$ such that

$$\left(\fint_{B_r(x)} \left| f - f_{B_r(x)} \right|^q \, \mathrm{d}\mathfrak{m} \right)^{\frac{1}{q}} \le Cr \left(\fint_{B_{\lambda r}(x)} |Df|_p^p \right)^{\frac{1}{p}}, \quad \forall f \in W^{1,p}(\mathrm{X}),\ x \in \mathrm{X},\ r > 0,$$

where $f_{B_r(x)} := \fint_{B_r(x)} f \, \mathrm{d}\mathfrak{m}$. If we can take $\lambda = 1$ we say that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ supports a strong local (q, p)-Poincaré inequality.

A metric measure space which is both locally doubling and supports a weak $(1, 2)$-Poincaré inequality is called *PI-space* . A remarkable property of PI spaces is that the notion of weak-upper gradient does not depend on the exponent p, as follows from a deep result by Cheeger [**71**].

Theorem 1.2.3. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a PI-space. Then for every* $f \in \mathsf{LIP}_{bs}(\mathrm{X})$ *it holds that* $|Df|_p = \mathrm{lip} f$ $\mathfrak{m}$*-a.e..*

A notion that will play a key role in the sequel is the following:

Definition 1.2.4 ([**117**])**.** We say that a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is *Infinitesimally Hilbertian* if the space $W^{1,2}(\mathrm{X})$ is a Hilbert space.

This property reflects that the underlying geometry at small scales looks Riemannian and can equivalently be characterized via the validity of the following parallelogram identity

$$|D(f + g)|_2^2 + |D(f - g)|_2^2 = 2|Df|_2^2 + |Dg|_2^2 \qquad \mathfrak{m}\text{-a.e.}, \quad \forall\, f, g \in W^{1,2}(\mathrm{X}). \tag{1.2.3}$$

This allows to give a notion of scalar product between gradients of Sobolev functions $f, g \in W^{1,2}(\mathrm{X})$:

$$L^1(\mathfrak{m}) \ni \langle \nabla f, \nabla g \rangle := \lim_{\varepsilon \to 0} \frac{|D(f + \varepsilon g)|_2^2 - |Df|_2^2}{\varepsilon}, \tag{1.2.4}$$

where the limit exists and is bilinear on its entries, as it can be easily checked using (1.2.3). Notice that, at this level of discussion, the symbol $\langle \nabla f, \nabla g\rangle$ is purely formal. Nevertheless, after the introduction of the tangent module and the notion of gradient ∇f, this will be made rigorous.

1.2.2. Sobolev spaces via test plans. We briefly recall another definition Sobolev spaces via test plans introduced in [**20**] which is equivalent to the definition via Cheeger energy presented above.

Definition 1.2.5 (Absolutely continuous curve)**.** A curve $\gamma \in C([0,1],\mathrm{X})$ belongs to the space of *absolutely continuous* curves $AC([0,1],\mathrm{X})$ if there exists $f \in L^1(0,1)$ such that $\mathsf{d}(\gamma_t,\gamma_s) \le \int_s^t f(r)\,\mathrm{d}r$, for every $0 \le s < t \le 1$.

For an absolutely continuous curve γ, the limit $|\dot\gamma_t| := \lim_{h\to 0} h^{-1}\mathsf{d}(\gamma_{t+h},\gamma_t)$ exists for a.e. $t \in (0,1)$ and is called *metric speed* at time t. The length $L(\gamma)$ of an absolutely continuous curve γ is defined by

$$L(\gamma) := \int_0^1 |\dot\gamma_t|\,\mathrm{d}t.$$

Definition 1.2.6 (Test plan)**.** A Borel probability measure $\boldsymbol{\pi}$ on $AC([0,1],\mathrm{X})$ is said to be a *test plan* if

$$\exists C > 0 \,:\, \mathrm{e}_{t*}\boldsymbol{\pi} \le C\mathfrak{m}, \quad \forall t \in [0,1],$$
$$\int\int_0^1 |\dot\gamma_t|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi} < +\infty.$$

Definition 1.2.7 (Sobolev class)**.** The Sobolev class $\mathrm{S}^2(\mathrm{X})$ is the space of all functions $f \in L^0(\mathfrak{m})$ such that there exists a non-negative $G \in L^2(\mathfrak{m})$, called weak-upper gradient, for which

$$\int |f(\gamma_1) - f(\gamma_0)|\,\mathrm{d}\boldsymbol{\pi} \le \int\int_0^1 |\dot\gamma_t| G(\gamma_t)\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}, \quad \forall \boldsymbol{\pi} \text{ test plan.} \tag{1.2.5}$$

Thanks to the results in [**19**] it turns out that $S^2(\mathrm{X}) \cap L^2(\mathrm{X}) = W^{1,2}(\mathrm{X})$ (the space defined via Cheeger-energy) and that the minimal 2-weak upper gradient, $|Df|_2$, is an admissible G in (1.2.5) and actually the minimal one in $\mathfrak{m}$-a.e. sense.

1.2.3. Local Sobolev spaces. Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a proper and infinitesimally Hilbertian m.m.s. and let Ω be an open subset of X. We define

$$W^{1,2}_{\mathsf{loc}}(\Omega) := \{u \in L^2_{\mathsf{loc}}(\Omega) \mid u\eta \in W^{1,2}(X), \text{ for every } \eta \in \mathsf{LIP}_{\mathrm{c}}(\Omega)\}.$$

For analogous notions of local Sobolev space and their properties see also ([**24**, **117**, **128**]). Since we will often deal with the above space it is useful to recall the following alternative characterizations:

Proposition 1.2.8 (Characterization of $W^{1,2}_{\mathsf{loc}}(\Omega)$)**.** *Let $\Omega \subset \mathrm{X}$ be open, then $u \in W^{1,2}_{\mathsf{loc}}(\Omega)$ if and only if one of the following holds:*

(i) for every $B \subset\subset \Omega$ Borel there exists $\tilde u \in W^{1,2}(\mathrm{X})$ such that $u = \tilde u$ $\mathfrak{m}$-a.e. in Ω,
(ii) for every $x \in \Omega$ there exists a ball $B_r(x) \subset \Omega$ and a function $\tilde u \in W^{1,2}(\mathrm{X})$ such that $u = \tilde u$ $\mathfrak{m}$-a.e. in Ω.

Proof. Clearly $u \in W^{1,2}_{\mathsf{loc}}(\Omega)$ implies (i). Conversely given $\eta \in \mathsf{LIP}_{\mathrm{c}}(\Omega)$ we take $B := \mathrm{supp}(\eta)$ and $\tilde u \in W^{1,2}(\mathrm{X})$ satisfying $\tilde u = u$ in B and conclude observing that $u\eta = \tilde u\eta \in W^{1,2}(\mathrm{X})$. Hence to conclude it is sufficient to show that (ii) implies (i). We fix $B \subset\subset \Omega$ Borel. From (ii) and compactness there exists balls $(B_i)_{i=1}^m$ and functions $u_i \in W^{1,2}(\mathrm{X})$ such that $2B_i \subset \Omega$ $u = u_i$, $\mathfrak{m}$-a.e. in B_i and $B \subset \cup_i B_i$. Consider then a partition of unity $(\varphi_i)_{i=1}^n$ of functions $\varphi \in \mathsf{LIP}_c(2B_i)$ and such that $\sum_i \varphi = 1$ in $\cup_i B_i$. Then the function $\tilde u := \sum_i \varphi u_i \in W^{1,2}(\mathrm{X})$ clearly satisfies $\tilde u = u$ $\mathfrak{m}$-a.e. in B. □

For any $u \in W^{1,2}_{\mathsf{loc}}(\Omega)$ we define its gradient ∇u as the unique element of $L^0(TX)_{|\Omega}$ such that

$$\nabla u := \nabla(\eta u), \ \mathfrak{m}\text{-a.e. in } \{\eta = 1\}, \quad \forall \eta \in \mathsf{LIP}_{\mathrm{c}}(\Omega),$$

which is well defined thanks to the locality property of the gradient. In particular $|\nabla u| \in L^2_{\mathsf{loc}}(\Omega)$. It is straightforward to check that ∇u satisfies the expected locality property, Leibniz rule and chain rule. We state explicitly a version for the the chain rule and Leibniz rule that we will need:

Proposition 1.2.9 (Local chain rule and Leibniz rule)**.** *Let $u \in W^{1,2}_{\mathsf{loc}}(\Omega)$, then*

- *for every $f \in \mathsf{LIP}_{\mathsf{loc}}(\Omega)$ it holds that $fu \in W^{1,2}_{\mathsf{loc}}(\Omega)$ and $\nabla(fu) = \nabla f u + f\nabla u$,*

- *for every $\varphi \in \mathsf{LIP}(I)$, with I open interval, such that*

$$u(\Omega') \subset\subset I \text{ (up to a } \mathfrak{m}\text{-negligible set), for every } \Omega' \subset\subset \Omega, \tag{1.2.6}$$

it holds that $\varphi(u) \in W^{1,2}_{\mathsf{loc}}(\Omega)$ and $\nabla\varphi(u) = \varphi'(u)\nabla u$.

Observe that, since $\nabla u \in L^0(T\mathrm{X})_{|\Omega}$, it makes sense to compute the scalar product $\langle \nabla u, v\rangle \in L^0(\Omega, \mathfrak{m})$, for every $v \in L^0(T\mathrm{X})_{|\Omega}$, moreover this this scalar product also satisfies $|\langle \nabla u, v\rangle| \le |\nabla u||v|$, $\mathfrak{m}$-a.e. in Ω (recall the discussion in Section 1.2.7).

We also define the spaces

$$W^{1,2}(\Omega) := \{f \in W^{1,2}_{\mathsf{loc}}(\Omega) \mid f, |\nabla f| \in L^2(\Omega)\},$$
$$W^{1,2}_0(\Omega) := \overline{\mathsf{LIP}_c(\Omega)}^{W^{1,2}(X)} \subset W^{1,2}(\mathrm{X}).$$

We conclude with the following technical lemma.

Lemma 1.2.10. *Let $u \in W^{1,2}_{\mathsf{loc}}(\Omega)$ be non-negative and $\alpha \in (0,1)$ be such that*

$$\chi_{\{u>0\}} u^{\alpha-1}|\nabla u| \in L^2_{\mathsf{loc}}(\Omega).$$

Then $u^\alpha \in W^{1,2}_{\mathsf{loc}}(\Omega)$ and $\nabla u = \alpha^{-1} u^{1-\alpha}\, \nabla u^\alpha$, $\mathfrak{m}$-a.e. in Ω.

Proof. For the first part it is enough to show that $f := \eta u^\alpha \in W^{1,2}(\mathrm{X})$ for every $\eta \in \mathsf{LIP}_c(\Omega)$ with $|\eta| \le 1$. Fix $\varepsilon \in (0,1)$ arbitrary and define $f_\varepsilon = \eta(u+\varepsilon)^\alpha$. Then from the non-negativity of u we have $f_\varepsilon \in W^{1,2}(\mathrm{X})$ and recalling that $|\nabla u| = 0$ $\mathfrak{m}$-a.e. in $\{u=0\}$ we have

$$|\nabla f_\varepsilon| \le \mathrm{Lip}\eta\, C_\alpha(u+2) + \alpha\chi_{u>0}u^{\alpha-1}|\nabla u|, \quad \mathfrak{m}\text{-a.e.}.$$

It follows that the family $\{f_\varepsilon\}_{\varepsilon\in(0,1)}$ is bounded in $W^{1,2}(\mathrm{X})$. Moreover $f_\varepsilon \to f$ in L^2, therefore from the lower semicontinuity of the Cheeger energy it follows that $f \in W^{1,2}(\mathrm{X})$. For the second part we observe that $u \wedge n = (u^\alpha \wedge n^\alpha)^{1/\alpha}$ $\mathfrak{m}$-a.e. in Ω and that $(t \wedge n^\alpha)^{1/\alpha} \in \mathsf{LIP}(\mathbb{R})$. Therefore from the locality and the chain rule for the gradient

$$\nabla u = \alpha^{-1}u^{1-\alpha}\, \nabla u^\alpha, \quad \mathfrak{m}\text{-a.e. in } \{u \le n\}$$

and we conclude from the arbitrariness of n. □

1.2.4. Laplacian and divergence operators. In this section we assume $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ to be a proper and infinitesimally Hilbertian m.m.s..

Definition 1.2.11 (Measure-valued Laplacian ([**117**]))**.** Let $\Omega \subset \mathrm{X}$ open. We say that $u \in W^{1,2}_{\mathsf{loc}}(\Omega)$ belongs to the domain of the *(measure-valued) Laplacian* $\mathsf{D}(\mathbf{\Delta}, \Omega)$ if there exists a (unique) signed Radon measure $\mathbf{\Delta}_{|\Omega}u$ in Ω such that

$$-\int_\Omega \langle \nabla f, \nabla u\rangle \mathrm{d}\mathfrak{m} = \int_\Omega f\, \mathrm{d}\mathbf{\Delta}_{|\Omega}u, \tag{1.2.7}$$

for every $f \in \mathsf{LIP}_c(\Omega)$.

Recall Section 1.1.2 for the notion of signed Radon measure and note that it is not a measure on Ω, but is defined only on relatively compact subsets. We mention also [**67**] for a discussion on the notion of Radon measure related to this definition (see also [**12**]).

When no confusion can occur we will drop the subscript Ω and simply write $\mathbf{\Delta}u$. Moreover we will write $\mathsf{D}(\mathbf{\Delta})$ in place of $\mathsf{D}(\mathbf{\Delta}, \mathrm{X})$ and whenever $\mathbf{\Delta} \ll \mathfrak{m}$ we will use the non bold notation Δ. Sometimes, if $\mathbf{\Delta}u \ll \mathfrak{m}$ and Δu belongs to some function space $V \subset L^1_{\mathsf{loc}}(\Omega)$, we will write $u \in \mathsf{D}_V(\Delta, \Omega)$, e.g. $u \in \mathsf{D}_{L^2}(\Delta, \Omega)$ means that $\Delta u \in L^2(\Omega)$.

A function $u \in D(\mathbf{\Delta}, \Omega)$ is said to be *subharmonic* if $\mathbf{\Delta}u \ge 0$, *superharmonic* if $\mathbf{\Delta}u \le 0$ and *harmonic* if $\mathbf{\Delta}u = 0$.

Finally it easily follows from the definition that the Laplacian operator is linear and satisfies the following locality property:

if $u, v \in D(\mathbf{\Delta}, \Omega)$ and $u = v$ $\mathfrak{m}$-a.e. in U, with U open and relatively compact in Ω, then $\mathbf{\Delta}u_{|U} = \mathbf{\Delta}v_{|U}$.

The following existence and comparison result is proven in [**117**, Prop 4.13].

Proposition 1.2.12. *Let $u \in W^{1,2}_{\mathsf{loc}}(\Omega)$ and suppose that there exists $g \in L^1_{\mathsf{loc}}(\Omega)$ such that*

$$-\int_\Omega \langle \nabla f, \nabla u\rangle\, \mathrm{d}\mathfrak{m} \ge \int_\Omega gf\, \mathrm{d}\mathfrak{m}, \quad \forall f \in \mathsf{LIP}_c(\Omega), \text{ with } f \ge 0,$$

Then $u \in D(\mathbf{\Delta}, \Omega)$ and $\mathbf{\Delta}u \ge g\mathfrak{m}_{|\Omega}$.

Remark 1.2.13. Let $u \in D(\mathbf{\Delta}, \Omega)$. Suppose that $\mathbf{\Delta} u \ll \mathfrak{m}$ (resp. $\mathbf{\Delta} u \ge g\mathfrak{m}$) with $\frac{\mathrm{d}\mathbf{\Delta} u}{\mathrm{d}\mathfrak{m}} \in L^2_{\mathsf{loc}}(\Omega)$ (resp. $g \in L^1_{\mathsf{loc}}(\Omega)$). Then, recalling that in an infinitesimally Hilbertian m.m.s. Lipschitz functions are dense in $W^{1,2}(\mathrm{X})$ (see [**19**]), by a truncation and cut off argument it follows that (1.2.7) (resp. $-\int \langle \nabla f, \nabla u\rangle \mathrm{d}\mathfrak{m} \ge \int g f \mathrm{d}\mathfrak{m}$) holds also for every $f \in W^{1,2}(X)$ (resp. $f \in W^{1,2}(X) \cap L^\infty(\mathfrak{m})$, $f \ge 0$) with support compact in Ω. ■

Heat flow. Since Ch_2 is a convex and lower semicontinuous functional in $L^2(\mathfrak{m})$ and its domain of finiteness is dense in $L^2(\mathfrak{m})$ we can define the *heat flow* h_t as the gradient flow in $L^2(\mathfrak{m})$ of Ch_2. If additionally the space is infinitesimally Hilbertian, the heat flow $h_t : L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$, $t \ge 0$ is linear, continuous, self-adjoin and satisfies $h_t(f) \in D(\Delta) \cap W^{1,2}(\mathrm{X})$. Moreover the curve $(0,\infty) \ni t \mapsto h_t f \in L^2(\mathfrak{m})$ is locally absolutely continuous for every $f \in L^2(\mathfrak{m})$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} h_t(f) = \Delta h_t(f) \in L^2(\mathfrak{m}), \quad \text{for a.e. } t > 0,$$

(see [**118**] for further details).

Divergence.

Definition 1.2.14 (Measure-valued divergence, [**122**])**.** Let $v \in L^0(TX)_{|\Omega}$ be such that $|v| \in L^2_{\mathsf{loc}}(\Omega)$, we say that $v \in D(\mathbf{div}, \Omega)$ if there exists a (unique) signed Radon measure $\mathbf{div}_{|\Omega}(v)$ in Ω such that

$$\int_\Omega \langle \nabla f, v\rangle \mathrm{d}\mathfrak{m} = -\int_\Omega f \,\mathrm{d}\,\mathbf{div}_{|\Omega}(v), \tag{1.2.8}$$

for every $f \in \mathsf{LIP}_c(\Omega)$.

It is clear from the definition that given $u \in W^{1,2}_{\mathsf{loc}}(\Omega)$, we have $\nabla f \in D(\mathbf{div}, \Omega)$ if and only if $u \in D(\mathbf{\Delta}, \Omega)$ and in this case $\mathbf{div}_{|\Omega}(\nabla u) = \mathbf{\Delta}_{|\Omega} u$. As for the Laplacian, we will often write simply $\mathbf{div}(v)$ instead of $\mathbf{div}_{|\Omega}(v)$ and whenever $\mathbf{div}(v) \ll \mathfrak{m}$ and $\mathbf{div}(v)$ belongs to some function space $V \subset L^1_{\mathsf{loc}}(\Omega)$, we will write $v \in \mathsf{D}_V(\mathrm{div}, \Omega)$.

Remark 1.2.15. Analogously to the measure valued Laplacian, we have that if $\mathbf{div}(v) \in L^2_{\mathsf{loc}}(\Omega)$, then (1.2.8) holds also for every $f \in W^{1,2}(X)$ with support compact in Ω. ■

Calculus rules for Laplacian and divergence. We collect and prove here some calculus for the Laplacian and divergence that will serve our purposes, that are mainly variants of the ones proved in [**117**].

Proposition 1.2.16 (Leibniz rule for $\mathbf{\Delta}$)**.** *Let $u \in D(\mathbf{\Delta}, \Omega)$ and let $g \in \mathsf{LIP}_{\mathsf{loc}}(\Omega) \cap D(\mathbf{\Delta}, \Omega)$ be such that $\mathbf{\Delta} g \in L^2(\Omega)$. Then $ug \in D(\mathbf{\Delta}, \Omega)$ and*

$$\mathbf{\Delta}(ug) = g\mathbf{\Delta} u + u\mathbf{\Delta} g + 2\langle \nabla u, \nabla g\rangle \mathfrak{m}. \tag{1.2.9}$$

Proof. Let $f \in \mathsf{LIP}_c(\Omega)$. Then using the Leibniz rule for the gradient

$$\int \langle \nabla(ug), \nabla f\rangle \,\mathrm{d}\mathfrak{m} = \int \langle \nabla u, \nabla(fg)\rangle \,\mathrm{d}\mathfrak{m} + \int \langle \nabla g, \nabla(uf)\rangle \,\mathrm{d}\mathfrak{m} - 2\int f \langle \nabla u, \nabla g\rangle \,\mathrm{d}\mathfrak{m}.$$

Since $fg \in \mathsf{LIP}_c(\Omega)$ and $fu \in W^{1,2}(X)$ with support compact in Ω, we conclude from Remark 1.2.13. □

Proposition 1.2.17 (Leibniz rule for $\mathbf{div}$)**.** *Let $v \in D(\mathbf{div}, \Omega)$. Then*

(1) if $g \in \mathsf{LIP}_{\mathsf{loc}}(\Omega)$, then $gv \in D(\mathbf{div}, \Omega)$ and

$$\mathbf{div}(gv) = \langle \nabla g, v\rangle \mathfrak{m} + g\mathbf{div}(v). \tag{1.2.10}$$

(2) if $g \in W^{1,2}_{\mathsf{loc}}(\Omega)$ and $\mathbf{div}(v) \in L^2_{\mathsf{loc}}(\Omega)$, then $gv \in D(\mathbf{div}, \Omega)$ and (1.2.10) *holds.*

Proof. Let $f \in \mathsf{LIP}_c(\Omega)$. Using the Leibniz rule for the gradient we get

$$-\int \langle \nabla f, gv\rangle \mathrm{d}\mathfrak{m} = -\int \langle \nabla(fg), v\rangle \mathrm{d}\mathfrak{m} + \int f \langle \nabla g, v\rangle \mathrm{d}\mathfrak{m}.$$

The conclusion follows in the first case observing that $fg \in \mathsf{LIP}_c(\Omega)$ and in the second case observing that $fg \in W^{1,2}(\mathrm{X})$ with compact support in Ω and recalling Remark 1.2.15. □

Proposition 1.2.18 (Chain rule for $\mathbf{\Delta}$)**.** *Let $u \in D(\mathbf{\Delta}, \Omega)$ and let $\varphi \in C^2(I)$, where I is an open interval such that* (1.2.6) *holds. Then*

(1) if $u \in \mathsf{LIP}_{\mathsf{loc}}(\Omega)$ then $\varphi(u) \in D(\mathbf{\Delta}, \Omega)$ and

$$\mathbf{\Delta}_{|\Omega}(\varphi(u)) = \varphi'(u)\mathbf{\Delta}_{|\Omega} u + \varphi''(u)|\nabla u|^2 \mathfrak{m}_{|\Omega}. \tag{1.2.11}$$

(2) if $\mathbf{\Delta} u \geq g\mathfrak{m}_{|\Omega}$ *for some* $g \in L^1_{\mathsf{loc}}(\Omega)$ *then* $\varphi(u) \in D(\mathbf{\Delta}, \Omega)$ *and*

$$\mathbf{\Delta}_{|\Omega}(\varphi(u)) \geq \left(\varphi'(u)g + \varphi''(u)|\nabla u|^2\right)\mathfrak{m}_{|\Omega}. \tag{1.2.12}$$

Proof. Let $f \in \mathsf{LIP}_c(\Omega)$, then from the chain rule and Leibniz rule for the gradient

$$-\int \langle \nabla(\varphi(u)), \nabla f\rangle \mathrm{d}\mathfrak{m} = -\int \langle \nabla(\varphi'(u)f), \nabla u\rangle \mathrm{d}\mathfrak{m} + \int f \langle \nabla\varphi'(u), \nabla u\rangle \mathrm{d}\mathfrak{m}.$$

In the first case we conclude from the fact that $\varphi'(u)f \in \mathrm{Lip}_c(\Omega)$. In the second case we assume also $f \geq 0$ and observe that $\varphi'(u)f \in W^{1,2}(X) \cap L^\infty(\mathfrak{m})$ is nonnegative with compact support in Ω, hence from Remark 1.2.13 it follows that

$$-\int \langle \nabla(\varphi(u)), \nabla f\rangle \mathrm{d}\mathfrak{m} \geq \int \varphi'(u) f\, g\mathfrak{m} + \int f\varphi''(u)|\nabla u|^2 \mathrm{d}\mathfrak{m}.$$

The conclusion follows applying Proposition 1.2.12. □

1.2.5. Capacity and quasi continuous functions. For a detailed treatment of the capacity in metric measure spaces we refer to [**99**] (see also [**46**]). The *capacity* of a set $E \subset \mathrm{X}$ is defined as

$$\mathrm{Cap}(E) \coloneqq \inf\{\|f\|^2_{W^{1,2}(\mathrm{X})} : f \in W^{1,2}(\mathrm{X}),\ \ f \geq 1\, \mathfrak{m}\text{-a.e. in a nieghborhood of } E\}. \tag{1.2.13}$$

It turns out that Cap is a submodular outer measure on X and satisfies $\mathfrak{m}(E) \leq \mathrm{Cap}(E)$ for every Borel set $E \subset \mathrm{X}$. For a function $f : \mathrm{X} \to [0, \infty]$ the integral of f against Cap is defined as

$$\int f\, \mathrm{dCap} \coloneqq \int_0^\infty \mathrm{Cap}(\{f > t\})\, \mathrm{d}t.$$

Moreover we set $\int_E f\, \mathrm{dCap} \coloneqq \int \chi_E f\, \mathrm{dCap}$.

Definition 1.2.19 (Quasi-continuous functions)**.** A function $f : \mathrm{X} \to \mathbb{R}$ is said to be *quasi-continuous* if for every $\varepsilon > 0$ there exists a set $E \subset \mathrm{X}$ such that $\mathrm{Cap}(E) < \varepsilon$ and $f_{|\mathrm{X}\setminus E}$ is continuous.

Remark 1.2.20. From the very definition of capacity we see that in the above definition it is equivalent to consider only open sets E. In particular for every quasi-continuous function f there exists a sequence of increasing closed sets C_n such that $\mathrm{Cap}(\mathrm{X} \setminus \bigcup_n C_n) = 0$ and f is continuous on C_n. This also shows that any quasi-continuous function is $\mathfrak{m}$-measurable and Cap-a.e. equivalent to a Borel function (see also the discussion in [**99**]). ■

Following [**99**] we denote by $\mathcal{QC}(\mathrm{X})$ the set of all equivalence classes-up to Cap-a.e. equality-of quasi-continuous functions. Always in [**99**] it is proven that Sobolev functions are quasi-continuous in the sense that they admit a quasi-continuous representative. More precisely under the assumption that continuous functions are dense in $W^{1,2}(\mathrm{X})$, there exists a unique map

$$\mathsf{QCR} : W^{1,2}(\mathrm{X}) \to \mathcal{QC}(\mathrm{X})$$

that is linear and such that $\mathsf{QCR}(f)$ is (the Cap-a.e. equivalence class of) a function which is *quasi continuous* and coincides $\mathfrak{m}$-a.e. with f. Moreover QCR satisfies $\mathsf{QCR}(|f|) = |\mathsf{QCR}(f)|$. Recall that if X is infinitesimally Hilbertian, then Lipschitz functions are dense in $W^{1,2}(\mathrm{X})$ ([**19**]), hence the map QCR is available. It is proved in [**99**] that the quasi continuous representative is unique in the sense given by the following proposition. The proof in [**99**] is actually given for $U = \mathrm{X}$, but the same proof works for arbitrary U open.

Proposition 1.2.21. *Let* $f, g : \mathrm{X} \to \mathbb{R}$ *be quasi-continuous functions and let* $U \subset \mathrm{X}$ *be open. Then* $f = g$ $\mathfrak{m}$*-a.e. in* U *implies that* $f = g$ Cap*-a.e. in* U*.*

The above result (or alternatively the uniqueness of the map QCR) implies immediately that:

$$f, g \in W^{1,2} \cap L^\infty(\mathrm{X}) \implies \mathsf{QCR}(fg) = \mathsf{QCR}(f)\mathsf{QCR}(g), \tag{1.2.14}$$

indeed, since $\mathcal{QC}(\mathrm{X})$ is an algebra, $\mathsf{QCR}(f)\mathsf{QCR}(g) \in \mathcal{QC}(\mathrm{X})$ and coincides $\mathfrak{m}$-a.e. with $fg \in W^{1,2}(\mathrm{X})$.

We will also need a convergence result obtained in [**99**]:

$$\text{if } f_n \to f \text{ in } W^{1,2}(\mathrm{X}), \text{ up to passing to a subsequence, } \mathsf{QCR}(f_n) \to \mathsf{QCR}(f) \text{ Cap-a.e.}. \tag{1.2.15}$$

Finally we define also the relative variational capacity as follows:

Definition 1.2.22 (Variational 2-Capacity)**.** Let $\Omega \subset \mathrm{X}$ be open and $E \subset\subset \Omega$. We define

$$\overline{\mathrm{Cap}}(E, \Omega) = \inf\left\{\int_\Omega |\nabla u|^2\, \mathrm{d}\mathfrak{m}\ :\ u \in \mathsf{LIP}_c(\Omega) \text{ and } u \geq 1 \text{ in a neighbourhood of } E\right\}$$

When $\Omega = \mathrm{X}$ we will simply write $\overline{\mathrm{Cap}}(E)$ in place of $\overline{\mathrm{Cap}}(E, \mathrm{X})$. Observe also that the natural setting for the definition of $\overline{\mathrm{Cap}}$ is the one of proper metric measure spaces.

1.2.6. Functions of bounded variation and sets of finite perimeter. We recall the definition of *function of bounded variation* on a metric measure space. For a detailed treatment of this topic see for example [**179**] and [**14**].

Definition 1.2.23 (Functions of bounded variation)**.** We say that function $f \in L^1_{\mathsf{loc}}(\mathfrak{m})$ belongs to the space $\mathsf{BV}(\mathrm{X})$ of functions of bounded variation if there exists a sequence $f_n \in \mathrm{Lip}_{\mathsf{loc}}(\mathrm{X})$ with $f_n \to f$ in $L^1_{\mathsf{loc}}(\mathfrak{m})$ such that

$$\varlimsup_{n\to+\infty} \int \mathrm{lip} f_n \,\mathrm{d}\mathfrak{m} < +\infty,$$

(where $\mathrm{lip} f_n$ was defined in (1.1.5)). By localizing this construction we also define

$$\|Df\|(A) := \inf\left\{ \varliminf_{n} \int_A \mathrm{lip} f_n \,\mathrm{d}\mathfrak{m} \ : \ f_n \in \mathrm{Lip}_{\mathsf{loc}}(A),\ f_n \to f \text{ in } L^1(A) \right\},$$

for any $A \subset \mathrm{X}$ open.

It is proven in [**179**] that this set function is the restriction to open sets of a finite and positive Borel measure on X that we call *total variation* of f and still denoted by $\|Df\|$. Whenever $\|Df\| \ll \mathfrak{m}$, for example when f is also Lipschitz, we denote by $|Du|_1$ the unique function such that

$$\|Du\| = |Du|_1 \mathfrak{m}. \tag{1.2.16}$$

Remark 1.2.24. Note that, following the presentation in [**179**], we are not requiring a function in $\mathsf{BV}(\mathrm{X})$ to be also in $L^1(\mathfrak{m})$. This will be more convenient four our applications and will be possible since we will use functions of bounded variation only to apply the coarea formula, for which integrability is not required. ■

Definition 1.2.25 (Sets of finite perimeter)**.** A Borel set $E \subset \mathrm{X}$ is said to be of finite perimeter if $\chi_E \in \mathsf{BV}(\mathrm{X})$, in which case we denote its total variation $\|D\chi_E\|(.)$ by $\mathrm{Per}(E, .)$

For simplicity and when no confusion can occur, we will often write $\mathrm{Per}(E)$ both for $\mathrm{Per}(E, \mathrm{X})$ and for the perimeter measure $\mathrm{Per}(E, .)$.

It has been proven by Ambrosio [**9**] that on PI-spaces the perimeter measure is absolutely continuous with respect to the codimension-1 Hausdorff measure (which we do not define here). Moreover it can be proved (see e.g. [**56**]) that, again on PI-spaces, the codimension-1 Hausdorff measure is in turn absolutely continuous with respect to the Capacity outer measure. Combining these two results we have:

Proposition 1.2.26. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a PI-space and* $E \subset \mathrm{X}$ *be of finite perimeter. Then* $\mathrm{Per}(E) \ll \mathrm{Cap}$.

For a Borel set $E \subset \mathrm{X}$ of finite measure we also define its Minkowski content as:

$$\mathfrak{m}^+(E) = \varliminf_{\delta\to 0^+} \frac{\mathfrak{m}(E^\delta) - \mathfrak{m}(E)}{\delta},$$

where $E^\delta := \{x \in \mathrm{X} \ : \ \mathsf{d}(x, E) < \delta\}$. In general it only holds that $\mathrm{Per}(E) \le \mathfrak{m}^+(E)$, however in [**15**] it is shown that the relaxation of the Minkowski content gives back the perimeter. We state here a version of this result adapted to our purposes.

Proposition 1.2.27. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and let* $E \subset B_r(x)$ *be Borel with finite perimeter. Then for every* $r' > r$ *there exists a sequence* $E_n \subset B_{r'}(x)$ *of closed sets such that* $\chi_{E_n} \to \chi_E$ *in* $L^1(\mathfrak{m})$ *and*

$$\mathrm{Per}(E) = \lim_n \mathfrak{m}^+(E_n).$$

A very general version of the coarea formula has been proved in [**179**]:

Theorem 1.2.28. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and let* $u \in \mathsf{BV}(\mathrm{X})$*. Then* $\{u < t\}$ *is of finite perimeter for a.e.* $t \in \mathbb{R}$ *and for every Borel function* $f : \mathrm{X} \to [-\infty, \infty]$ *such that* $f \in L^1(\|Du\|)$ *it holds*

$$\int_{\{s<u<t\}} f \,\mathrm{d}\|Du\| = \int_s^t \int f \mathrm{d}\mathrm{Per}(\{u < t\}), \quad \forall s, t \in \mathbb{R},\ s < t. \tag{1.2.17}$$

1.2.7. Tangent and cotangent module.

Theory of normed modules. We start recalling some notions of the theory of normed modules on a m.m.s. $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, developed in [**118**] and partly inspired by [**210**]. We refer to [**118, 127**] for a detailed account on this theory.

Definition 1.2.29 (L^2-normed L^∞-module)**.** An $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module is a triple $(\mathscr{M}, |.|, \|\|_{\mathscr{M}})$, where $\mathscr{M}$ is a module over the commutative ring $L^\infty(\mathfrak{m})$, $(\mathscr{M}, \|\|_{\mathscr{M}})$ is Banach space, $|.| : \mathscr{M} \to L^\infty(\mathfrak{m})$ is a map satisfying (in the $\mathfrak{m}$-a.e. sense)

$$\begin{aligned} |v| &\geq 0 \quad \text{for every } v \in \mathscr{M}, \text{ with equality if and only if } v = 0, \\ |v + w| &\leq |v| + |w| \quad \text{for every } v, w \in \mathscr{M}, \\ |fv| &= |f||v| \quad \text{for every } f \in L^\infty(\mathfrak{m}) \text{ and } v \in \mathscr{M} \end{aligned}$$

and such that $\||v|\|_{L^2(\mathfrak{m})} = \|v\|_{\mathscr{M}}$.

Definition 1.2.30 (Dual of an L^2-normed L^∞-module)**.** Given a an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{M}$, we define its *dual module* $\mathscr{M}^*$ as the space

$$\mathscr{M}^* := \{L : \mathscr{M} \to L^1(\mathfrak{m}) \ : \ L \text{ is continuous and } L^\infty\text{-linear}\},$$

endowed with the following operations and norms:

$$\begin{aligned} &(L' + L)(v) := L'(v) + L(v), \\ &(f \cdot L)(v) := f L(v), \\ &|L| := \operatorname{ess\,sup} \{L(v) \ : \ v \in \mathscr{M}, \ |v| \leq 1 \ \mathfrak{m}\text{-a.e.}\}, \\ &\|L\|_{\mathscr{M}^*} := \||L|\|_{L^2(\mathfrak{m})}, \end{aligned}$$

which make $(\mathscr{M}^*, |.|, \|\|_{\mathscr{M}^*})$ an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module.

An $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{M}$ is a *Hilbert module* if $(\mathscr{M}, \|.\|_{\mathscr{M}})$ is a Hilbert space or equivalently if for every $v, w \in \mathscr{M}$ it holds

$$|v + w|^2 + |v - w|^2 = 2|v|^2 + 2|w|^2, \quad \mathfrak{m}\text{-a.e.}.$$

If this is the case, by polarization we can define a scalar product $\langle , \rangle : \mathscr{M} \times \mathscr{M} \to L^1(\mathfrak{m})$ that is symmetric, $L^\infty(\mathfrak{m})$-bilinear and satisfies $\langle v, v \rangle = |v|^2$, $|\langle v, w \rangle| \leq |v||w|$, $\mathfrak{m}$-a.e. for every $v, w \in \mathscr{M}$. In particular it holds that

$$\langle v, w \rangle_{\mathscr{M}} = \int \langle v, w \rangle \, \mathrm{d}\mathfrak{m}, \quad \forall v, w \in \mathscr{M}, \tag{1.2.18}$$

where $\langle ., . \rangle_{\mathscr{M}}$ is the usual scalar product on $\mathscr{M}$.

Theorem 1.2.31 (Riesz, [**118**])**.** *Let $\mathscr{M}$ be an Hilbert $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module. Then the map $L : \mathscr{M} \to \mathscr{M}^*$ sending v to the element L_v defined by*

$$L_v(w) := \langle v, w \rangle \in L^1(\mathfrak{m}), \quad \forall w \in \mathscr{M},$$

is an isomorphism of modules.

Definition 1.2.32 (L^0-normed L^0-module)**.** An $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module is a triple $(\mathscr{M}, |.|, \tau)$, where $\mathscr{M}$ is a module over the commutative ring $L^0(\mathfrak{m})$, $(\mathscr{M}, \tau)$ is a topological vector space, $|.| : \mathscr{M} \to L^0(\mathfrak{m})$ is a map satisfying (in the $\mathfrak{m}$-a.e. sense)

$$\begin{aligned} |v| &\geq 0 \quad \text{for every } v \in \mathscr{M}, \text{ with equality if and only if } v = 0, \\ |v + w| &\leq |v| + |w| \quad \text{for every } v, w \in \mathscr{M}, \\ |fv| &= |f||v| \quad \text{for every } f \in L^0(\mathfrak{m}) \text{ and } v \in \mathscr{M}, \end{aligned}$$

and such that τ is induced by the distance $\mathsf{d}_0(v, w) := \int |v - w| \wedge 1 \, \mathrm{d}\mathfrak{m}'$ (where $\mathfrak{m}'$ is a probability measure on X such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$), which is also assumed to be complete.

An L^0-normed module $\mathscr{M}$ is a Hilbert module if for every $v, w \in \mathscr{M}$ it holds

$$|v + w|^2 + |v - w|^2 = 2|v|^2 + 2|w|^2, \quad \mathfrak{m}\text{-a.e.}.$$

As above, by polarization we can define a scalar product $\langle , \rangle : \mathscr{M} \times \mathscr{M} \to L^0(\mathfrak{m})$ that is symmetric, $L^0(\mathfrak{m})$-bilinear and satisfies $\langle v, v \rangle = |v|^2$, $|\langle v, w \rangle| \leq |v||w|$, $\mathfrak{m}$-a.e. for every $v, w \in \mathscr{M}^0$.

The *dual of an $L^0(\mathfrak{m})$-normed module $\mathscr{M}^0$* is defined exactly as in Definition 1.2.30 except that we replace both $L^1(\mathfrak{m})$ and $L^\infty(\mathfrak{m})$ with $L^0(\mathfrak{m})$ and skip the definition of $\|\|_{\mathscr{M}^*}$. The resulting space $(\mathscr{M}^0)^*$ has a structure of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module, when endowed with the distance d_0 as in Definition 1.2.32.

Given an L^0-normed module $\mathscr{M}$ and E a Borel subset of X we define the localized module

$$\mathscr{M}_{|E} := \{\chi_E v \ : \ v \in \mathscr{M}\}.$$

$\mathscr{M}_{|E}$ inherits naturally a structure of L^0-normed module and if $\mathscr{M}$ is a Hilbert module then $\mathscr{M}_{|E}$ is Hilbert as well with $\langle .,. \rangle_{\mathscr{M}_{|E}} = \chi_E \langle .,. \rangle_{\mathscr{M}}$. $\mathscr{M}_{|E}$ can also be seen as the quotient of $\mathscr{M}$ by the equivalence relation '$v \sim w$ if and only if $|v - w| = 0$ $\mathfrak{m}$-a.e. in E'. This identification will be used in the rest of the work without further notice.

Finally given an $L^0(\mathfrak{m})$-normed module $\mathscr{M}$ we have that

$$\mathscr{M}^2 := \{v \in \mathscr{M} \ : |v| \in L^2(\mathfrak{m})\}$$

inherits naturally a structure of L^2-normed L^∞ module and if $\mathscr{M}$ is a Hilbert module then $\mathscr{M}^2$ is Hilbert as well.

Proposition 1.2.33 (L^0-completion)**.** *Given an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{M}$, there exists a unique couple $(\mathscr{M}^0, \iota)$ where $\mathscr{M}^0$ is an $L^0(\mathfrak{m})$-normed module and $\iota : \mathscr{M} \to \mathscr{M}^0$ is linear, with dense image and preserves the pointwise norm. Moreover*

- *$(\mathscr{M}^0)^*$ coincides with L^0-completion of $\mathscr{M}^*$,*
- *$\iota(\mathscr{M}) = \{v \in \mathscr{M}^0 \ : |v| \in L^2(\mathfrak{m})\}$, or in other words $\mathscr{M}$ can be identified as the set of elements in $\mathscr{M}^0$ with pointwise norm in $L^2(\mathfrak{m})$,*
- *$\mathscr{M}$ is a Hilbert module if and only if $\mathscr{M}^0$ is a Hilbert module and*

$$\langle v, w \rangle = \langle \iota(v), \iota(w) \rangle, \quad \mathfrak{m}\text{-a.e.}.$$

We now recall the notion of local dimension and local basis for an L^0-module.

Definition 1.2.34 (Local basis)**.** Given an $L^0(\mathfrak{m})$-module $\mathscr{M}$, a Borel set $E \subset \mathrm{X}$ with $\mathfrak{m}(E) > 0$ and $\{v_i\}_{i=1}^n \subset \mathscr{M}$

- we say that $\{v_i\}_{i=1}^n$ *generate $\mathscr{M}$ on E* if the set of sums of the form $\sum_{i=1}^n v_i \chi_{E_i}$ with $\{E_i\}_{i=1}^n$ a Borel partition of E is dense in $\mathscr{M}_{|E}$ or equivalently if for every $v \in \mathscr{M}$ there exists $\{f_i\}_{i=1}^n \subset L^0(\mathfrak{m})$ such that $\chi_E v = \sum_{i=1}^n f_i v_i$,
- $\{v_i\}_{i=1}^n$ are said to be *independent on E* if for every $\{f_i\}_{i=1}^n \subset L^0(\mathfrak{m})$ such that $\sum_{i=1}^n f_i v_i = 0$ $\mathfrak{m}$-a.e. in E, we have that $f_i = 0$ $\mathfrak{m}$-a.e. in E for every $i = 1, ..., n$,
- $\{v_i\}_{i=1}^n$ are called a *local basis for $\mathscr{M}$ in E* if they are both independent on E and generate $\mathscr{M}$.

Definition 1.2.35 (Local dimension)**.** Given an $L^0(\mathfrak{m})$-module $\mathscr{M}$ and a Borel set $E \subset \mathrm{X}$ with $\mathfrak{m}(E) > 0$ we say that $\mathscr{M}$ has local dimension in E equal to $n \in \mathbb{N}$ if there exists a local basis in E of cardinality n. If instead $\mathscr{M}$ has no local dimension on E we say that it has *infinite dimension on E*.

It can be checked that the local dimension is well defined, since two local bases must have the same cardinality.

Remark 1.2.36. If $\mathscr{M}$ has *local dimension* in E and is also an Hilbert module a standard Gram-Schmidt procedure shows the existence of an *orthonormal basis* $e_1, ..., e_n \in \mathscr{M}$ which satisfies $\langle e_i, e_j \rangle = \delta_{i,j}$ $\mathfrak{m}$-a.e. in E. In particular for every $v \in \mathscr{M}$ it holds that $\chi_E v = \sum_{i=1}^n \chi_E \langle v, e_i \rangle e_i$. ■

The following result can be found in [**118**]

Theorem 1.2.37 (Existence of the dimensional decomposition)**.** *Let $\mathscr{M}$ be an $L^0(\mathfrak{m})$ normed module. Then there exists a unique (up to $\mathfrak{m}$-a.e. sets) Borel partition $\{E\}_{i \in \mathbb{N} \cup \{\infty\}}$ such that: for every E_i with $i \in \mathbb{N}$ and $\mathfrak{m}(E_i) > 0$ $\mathscr{M}$ has dimension i on E_i; $\mathscr{M}$ has infinite dimension on every $E \subset E_\infty$ with $\mathfrak{m}(E) > 0$.*

Tangent and cotangent module. We are ready to give the notion of cotangent module.

Theorem 1.2.38 (Cotangent module, [**118**])**.** *Suppose that* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *metric measure space. Then there exists a (unique) couple* $(L^2(T^*\mathrm{X}), \mathrm{d})$*, where* $L^2(T^*\mathrm{X})$ *is an* $L^2(\mathfrak{m})$*-normed* $L^\infty(\mathfrak{m})$*-module and* $\mathrm{d} : W^{1,2}(\mathrm{X}) \to L^2(T^*\mathrm{X})$*, called* differential operator*, is a linear and continuous map such that*

$$|\mathrm{d}f| \textit{ coincides with the minimal weak upper gradient of } f,$$

$$\left\{ \sum_{i=1}^n \chi_{E_i} \mathrm{d}f_i \ : \{E_i\}_{i=1}^n \textit{ Borel partition of } \mathrm{X}, \{f_i\}_{i=1}^n \subset W^{1,2}(\mathrm{X}) \right\} \textit{ is dense in } L^2(T^*\mathrm{X}).$$

The differential operator has the following calculus rules, which are parallel to the ones of the weak upper gradient:

Locality: $\mathrm{d}f = \mathrm{d}g$, $\mathfrak{m}$-a.e. in $\{f = g\}$,

Leibniz rule: $\mathrm{d}(fg) = g\mathrm{d}f + f\mathrm{d}g$, for every $f, g \in W^{1,2}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$,

Chain rule: $\mathrm{d}(\varphi(f)) = \varphi'(f)\mathrm{d}f$, for every $f \in W^{1,2}(\mathrm{X})$, $\varphi \in \mathsf{LIP}(\mathbb{R})$ with $\varphi(0) = 0$.

The *tangent module* is then defined as follows.

Definition 1.2.39 (Tangent module)**.** The *tangent module* $L^2(T\mathrm{X})$ is defined as the dual module of $L^2(T^*\mathrm{X})$ and its element are called *vector fields.*

The following result states that when $W^{1,2}(\mathrm{X})$ is a Hilbert space, we can essentially identify the tangent and cotangent modules.

Theorem 1.2.40 ([**118**])**.** *A metric measure space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is infinitesimally Hilbertian if and only if* $L^2(T^*\mathrm{X})$ *(and thus also* $L^2(T\mathrm{X})$*) is a Hilbert module. Moreover in this case for every* $f \in W^{1,2}(\mathrm{X})$ *we define its* gradient $\nabla f \in L^2(T\mathrm{X})$ *as the one corresponding to* $\mathrm{d}f$ *via the Riesz isomorphism.*

Remark 1.2.41. Note that as a consequence of the definitions and constructions, the gradient operator $\nabla : W^{1,2}(\mathrm{X}) \to L^2(T\mathrm{X})$ is linear, its image generates $L^2(T\mathrm{X})$ as a module and

$$|\nabla f| = |Df|, \quad \mathfrak{m}\text{-a.e..}$$

For this reason whenever $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian we will freely interchange $|\nabla f|$ and $|Df|$ without further notice. ■

The gradient operator also satisfies the expected calculus rules:

Locality: $\nabla f = \nabla g$, $\mathfrak{m}$-a.e. in $\{f = g\}$,

Leibniz rule: $\nabla(fg) = g\nabla f + f\nabla g$, for every $f, g \in W^{1,2}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$,

Chain rule: $\nabla(\varphi(f)) = \varphi'(f)\nabla f$, for every $f \in W^{1,2}(\mathrm{X})$, $\varphi \in \mathsf{LIP}(\mathbb{R})$ with $\varphi(0) = 0$.

Remark 1.2.42. Observe also that the two objects $\langle \nabla f, \nabla g\rangle_{L^2(T\mathrm{X})}$ and $\langle \nabla f, \nabla g\rangle$, for $f, g \in W^{1,2}(\mathrm{X})$, where the latter is defined as in (1.2.4), coincide. ■

Definition 1.2.43 (Measurable tangent bundle)**.** For an infinitesimally Hilbertian metric measure X we define $L^0(T\mathrm{X})$ as the L^0-completion of $L^2(T\mathrm{X})$. In particular $L^0(T\mathrm{X})$ is an Hilbert $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module and we can identify $L^2(T\mathrm{X})$ as the set $\{v \in L^0(T\mathrm{X}) \ : \ |v| \in L^2(\mathfrak{m})\}$.

Finally for convenience of notation we set

$$L^\infty(T\mathrm{X}) \coloneqq \{v \in L^0(T\mathrm{X}) \ : \ |v| \in L^\infty(\mathfrak{m})\}.$$

1.3. CD and RCD spaces

1.3.1. $\mathrm{CD}(K, N)$ **spaces.** To give a precise definition of the $\mathrm{CD}(K, N)$ condition we need first to introduce some basic notions and definitions of optimal transport, we refer to [**16, 207**] for a detailed introduction to this topic.

Optimal transport tools. On a complete and separable metric space (X, d) we denote by $\mathscr{P}_2(\mathrm{X})$ the space of probability measures with finite second moment, that is the set of all measures $\mu \in \mathscr{P}(\mathrm{X})$ such that for some (and thus all) $x \in \mathrm{X}$ it holds $\int_{\mathrm{X}} \mathsf{d}(x, y)^2 \, \mathrm{d}\mu(y) < +\infty$. For any couple $\mu_0, \mu_1 \in \mathscr{P}_2(\mathrm{X})$ we define their 2-Kantorovich-Wasserstein distance as

$$W_2(\mu_0, \mu_1)^2 \coloneqq \inf_{\pi \in \mathrm{Adm}(\mu_0, \mu_1)} \int \mathsf{d}(x, y)^2 \, \mathrm{d}\pi, \tag{1.3.1}$$

where $\mathrm{Adm}(\mu_0, \mu_1) \subset \mathscr{P}(\mathrm{X} \times \mathrm{X})$ is the set of admissible transport plans between μ_0 and μ_1, i.e. the set of all probability measures π such that $(P_1)_*\pi = \mu_1$ and $(P_2)_*\pi = \mu_1$. Here $P_i : \mathrm{X} \times \mathrm{X} \to \mathrm{X}$, $i = 1, 2$, denotes the projection respectively onto the first and second coordinate. It turns out that (1.3.1) is actually a minimum and we denote by $\mathrm{Opt}(\mu_0, \mu_1)$ the set of all π realizing it. It can be checked that W_2 is indeed a distance on $\mathscr{P}_2(\mathrm{X})$ that is complete if (X, d) is complete. Moreover the metric space $(\mathscr{P}_2(\mathrm{X}), W_2)$ is geodesic if and only if (X, d) is also geodesic. A key fact is that any W_2-geodesic μ_t between μ_0 and μ_1 can be lifted to probability measure $\nu \in \mathscr{P}(\mathsf{Geo}(\mathrm{X}))$ that satisfies $(\mathrm{e}_t)_*\nu = \mu_t$ for every $t \in [0, 1]$ and $(\mathrm{e}_0, \mathrm{e}_1)_*\nu \in \mathrm{Opt}(\mu_0, \mu_1)$. We define the set of *optimal geodesic plans* $\mathrm{OptGeo}(\mu_0, \mu_1)$ as the set of all $\nu \in \mathscr{P}(\mathsf{Geo}(\mathrm{X}))$ such that $(\mathrm{e}_0, \mathrm{e}_1)_*\nu \in \mathrm{Opt}(\mu_0, \mu_1)$, which in particular is non-empty if (X, d) is geodesic.

Curvature Dimension condition. The $\mathrm{CD}(K,N)$-Curvature-Dimension condition for metric measure spaces has been introduced in the two independent seminal works [**173**] and [**200, 201**].

Following the presentation in [**201**], we give the definition of distortion coefficients: for every $K \in \mathbb{R}, N \in (0,\infty), t \in [0,1]$ set

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} +\infty, & \text{if } K\theta^2 \geq N\pi^2, \\ \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})}, & \text{if } 0 < K\theta^2 < N\pi^2, \\ t, & \text{if } K\theta^2 < 0 \text{ and } N = 0 \text{ or if } K\theta^2 = 0, \\ \frac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})}, & \text{if } K\theta^2 \leq 0 \text{ and } N > 0. \end{cases}$$

Set also, for $N > 1$, $\tau_{K,N}^{(t)}(\theta) := t^{\frac{1}{N}} \sigma_{K,N-1}^{(t)}(\theta)^{1-\frac{1}{N}}$ while $\tau_{K,1}^{(t)}(\theta) = t$ if $K \leq 0$ and $\tau_{K,1}^{(t)}(\theta) = \infty$ if $K > 0$.

The last ingredient that we need is the *Rènyi-entropy functional* $\mathcal{E}_N : \mathscr{P}_2(\mathrm{X}) \to [-\infty, 0]$, given by

$$\mathcal{E}_N(\mu) := -\int \rho^{1-1/N} \, \mathrm{d}\mathfrak{m},$$

where $\mu = \rho\mathfrak{m} + \mu^\perp$ with $\mu^\perp$ singular with respect to $\mathfrak{m}$.

Definition 1.3.1 ($\mathrm{CD}(K,N)$-spaces)**.** Let $K \in \mathbb{R}$ and $N \in [1,\infty)$. A metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ satisfies the *curvature dimension condition* $\mathrm{CD}(K,N)$ if, for every couple of probability measures $\mu_0 = \rho_o\mathfrak{m}, \mu_1 = \rho_1\mathfrak{m} \in \mathscr{P}_2(\mathrm{X})$ absolutely continuous with respect to $\mathfrak{m}$ and with bounded supports, there exists an optimal geodesic plan $\pi \in \mathrm{OptGeo}(\mathrm{X}))$ between μ_0, μ_1 such that $\mu_t := (\mathrm{e}_t)_*\nu \ll \mathfrak{m}$ and for every $N' \geq N$ we have

$$\mathcal{E}_{N'}(\mu_t) \leq -\int \left(\tau_{K,N'}^{(1-t)}(\mathsf{d}(\gamma_1,\gamma_0))\rho_0(\gamma_0)^{-\frac{1}{N}} + \tau_{K,N'}^{(t)}(\mathsf{d}(\gamma_1,\gamma_0))\rho_1(\gamma_1))^{-\frac{1}{N}} \right) \mathrm{d}\pi(\gamma). \tag{1.3.2}$$

Since in the smooth case the curvature is a local object one would would like the $\mathrm{CD}(K,N)$ condition to be local as well. However it turns out that in general this is not true as shown by a counterexample by Rajala ([**189**]). To deal with the *local-to-global* problem Bacher and Sturm [**35**] introduced the so called *reduced curvature dimension condition* $\mathrm{CD}^*(K,N)$.

Definition 1.3.2 ($\mathrm{CD}^*(K,N)$-spaces)**.** Let $K \in \mathbb{R}$ and $N \in [1,\infty)$. A metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ satisfies the *curvature dimension condition* $\mathrm{CD}(K,N)^*$ if, for every couple of probability measures $\mu_0 = \rho_o\mathfrak{m}, \mu_1 = \rho_1\mathfrak{m} \in \mathscr{P}_2(\mathrm{X})$ absolutely continuous with respect to $\mathfrak{m}$ and with bounded supports, there exists a *dynamical optimal plan* $\pi \in \mathrm{OptGeo}(\mathrm{X}))$ between μ_0, μ_1 such that $\mu_t := (\mathrm{e}_t)_*\nu \ll \mathfrak{m}$ and for every $N' \geq N$ we have

$$\mathcal{E}_{N'}(\mu_t) \leq -\int \left(\sigma_{K,N'}^{(1-t)}(\mathsf{d}(\gamma_1,\gamma_0))\rho_0(\gamma_0)^{-\frac{1}{N}} + \sigma_{K,N'}^{(t)}(\mathsf{d}(\gamma_1,\gamma_0))\rho_1(\gamma_1))^{-\frac{1}{N}} \right) \mathrm{d}\pi(\gamma). \tag{1.3.3}$$

The key fact about this definition is that the $\mathrm{CD}^*(K,N)$ condition do enjoy the local-to-global property as shown again by Bacher and Sturm [**35**]. It can be verified that it always hold that $\tau_{K,N}^{(t)}(\theta) \geq \sigma_{K,N}^{(t)}(\theta)$ and in particular the $\mathrm{CD}^*(K,N)$ condition is weaker than the $\mathrm{CD}(K,N)$ condition. However in the work of Cavalletti and Milmann [**63**] they are able to show that the $\mathrm{CD}(K,N)$ and $\mathrm{CD}^*(K,N)$ conditions agree on essentially non-branching metric measures spaces with finite measure and in particular proving that the $\mathrm{CD}(K,N)$ condition satisfies the local-to-global property on such spaces.

To conclude this part we recall the notion of one-dimensional model space for the $\mathrm{CD}(N-1,N)$ condition.

Definition 1.3.3 (One dimensional model space)**.** For every $N > 1$ we define $I_N := ([0,\pi], |.|, \mathfrak{m}_N)$, where $|.|$ is the Euclidean distance restricted on $[0,\pi]$ and

$$\mathfrak{m}_N := \tfrac{1}{c_N} \sin(t)^{N-1} \mathscr{L}^1|_{[0,\pi]},$$

with $c_N := \int_{[0,\pi]} \sin(t)^{N-1} \, \mathrm{d}t$.

1.3.2. Geometric and functional inequalities on $\mathrm{CD}(K,N)$ **spaces.** Already from its introduction the $\mathrm{CD}(K,N)$ condition turned out to be sufficient to prove geometric and functional inequalities. We start recalling the Brunn-Minkowski inequality.

Theorem 1.3.4 ([**201**])**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a* $\mathrm{CD}(K,N)$ *space with* $N \in [1,\infty)$, $K \in \mathbb{R}$. *For any couple of Borel sets* $A_0, A_1 \subset \mathrm{X}$ *it holds that*

$$\mathfrak{m}(A_t)^{\frac{1}{N}} \geq \sigma_{K,N}^{(1-t)}(\theta)\mathfrak{m}(A_0)^{\frac{1}{N}} + \sigma_{K,N}^{(t)}(\theta)\mathfrak{m}(A_1)^{\frac{1}{N}}, \qquad \forall t \in [0,1], \tag{1.3.4}$$

where

$$A_t \coloneqq \{\gamma_t \;:\; \gamma \text{ geodesic such that } \gamma_0 \in A_0,\ \gamma_1 \in A_1\}$$

and

$$\theta \coloneqq \begin{cases} \inf_{(x_0,x_1)\in A_0\times A_1} \mathsf{d}(x_0,x_1), & \text{if } K \geq 0,\\ \sup_{(x_0,x_1)\in A_0\times A_1} \mathsf{d}(x_0,x_1), & \text{if } K < 0.\end{cases}$$

We remark that (1.3.4) is actually weaker than the statement appearing in [**201**] and it holds for the (a priori) larger class of $\mathrm{CD}^*(K,N)$ spaces (see [**35**]). A consequence of the Brunn-Minkowski inequality is the famous *Bishop-Gromov inequality* (see [**201**]):

$$\frac{\mathfrak{m}(B_R(x))}{v_{K,N}(R)} \leq \frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)}, \quad \text{for any } 0<r<R\leq \pi\sqrt{\frac{N-1}{K^+}} \text{ and any } x\in \mathrm{X}, \tag{1.3.5}$$

where the quantities $v_{K,N}(r)$, $N\in[1,\infty)$ $K\in\mathbb{R}$ are defined as

$$v_{K,N}(r) \coloneqq \sigma_{N-1}\int_0^r |s_{K,N}(t)|^{N-1}\,\mathrm{d}t,$$

where $s_{K,N}(t)$ denotes respectively the function $\sin\left(t\sqrt{\frac{K}{N-1}}\right)$, if $K>0$, $\sinh\left(t\sqrt{\frac{|K|}{N-1}}\right)$, if $K<0$ and t if $K=0$. In particular $\mathrm{CD}(K,N)$ spaces are uniformly locally doubling and thus proper. We also note that in the case $K=0$ the Bishop-Gromov inequality implies that the limit

$$\mathsf{AVR}(\mathrm{X}) \coloneqq \lim_{r\to+\infty}\frac{\mathfrak{m}(B_r(x))}{\omega_N r^N}$$

exists finite and does not depend on the point $x\in\mathrm{X}$. We call the quantity $\mathsf{AVR}(\mathrm{X})$ *asymptotic volume ratio* of X and if $\mathsf{AVR}(\mathrm{X})>0$ we say that X has *Euclidean-volume growth.* We also define the Bishop-Gromov ratios and density as follows:

$$\theta_{N,r}(x) \coloneqq \frac{\mathfrak{m}(B_r(x))}{\omega_N r^N}, \quad \theta_N(x) \coloneqq \lim_{r\to0^+}\theta_{N,r}(x), \qquad \forall\, r>0, x\in\mathrm{X}.$$

Observe that the above limit exists thanks to the Bishop-Gromov inequality and the fact that

$$\lim_{r\to0^+}\frac{\omega_N r^N}{v_{K,N}(r)} = 1$$

for every $K\in\mathbb{R}$, $N\in[1,\infty)$. This in particular grants that

$$\theta_N(x) = \lim_{r\to0}\frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)} = \sup_{r>0}\frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)}. \tag{1.3.6}$$

This and the fact that $\mathfrak{m}(\partial B_r(x))=0$ for every $r>0$ and $x\in\ X$ (which follows from the Bishop-Gromov inequality), implies that $\theta_N(x)$ is a lower-semicontinuous function of x.

It has been proven in [**190**] that essentially non-branching $\mathrm{CD}(K,N)$ spaces satisfy a weak local $(1,1)$-Poincaré inequality. Thanks to Theorem 1.2.3 this also shows that there is no dependence on p on the weak upper gradient.

Moreover, it goes back to the seminal work [**134**] that on a doubling metric measure space a $(1,p)$-Poincaré inequality improves to a (q,p)-Poincaré inequality with $q>1$ depending on the doubling constant. In the case of $\mathrm{CD}(K,N)$ spaces, from the Bishop-Gromov inequality we can obtain the following Sobolev-Poincaré inequality.

Theorem 1.3.5 ((p^*,p)-Poincaré inequality)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an essentially non-branching* $\mathrm{CD}(K,N)$ *space for some* $N\in(1,\infty)$*,* $K\in\mathbb{R}$*. Fix also* $p\in(1,N)$ *and* $R>0$*. Then, for every* $B_r(x)\subset\mathrm{X}$ *with* $r\leq R$ *it holds*

$$\Big(\fint_{B_r(x)}|u-u_{B_r(x)}|^{p^*}\,\mathrm{d}\mathfrak{m}\Big)^{\frac{1}{p^*}} \leq C(K,N,p,R)r\Big(\fint_{B_{2r}(x)}|Du|^p\,\mathrm{d}\mathfrak{m}\Big)^{\frac{1}{p}}, \qquad \forall\, u\in W^{1,p}(\mathrm{X}), \tag{1.3.7}$$

where $p^*\coloneqq pN/(N-p)$ *and* $u_{B_r(x)}\coloneqq\fint_{B_r(x)}u\,\mathrm{d}\mathfrak{m}$*.*

We end this part recalling the sharp Sobolev-inequality on the N model space $I_N=([0,\pi],|.|,c_N\sin^{N-1}\mathrm{d}t)$, for $N\in(2,\infty)$ (see Def. 1.3.3):

$$\|u\|^2_{L^q(\mathfrak{m}_N)} \leq \frac{q-2}{N}\,\||Du|\|^2_{L^2(\mathfrak{m}_N)}+\|u\|^2_{L^2(\mathfrak{m}_N)}, \quad \forall u\in W^{1,2}([0,\pi],\mathsf{d}_{eu},\mathfrak{m}_N)), \tag{1.3.8}$$

for every $q\in(2,2^*]$, with $2^*\coloneqq 2N/(N-2)$, (see e.g. [**163**] and also [**65**]).

1.3.3. $\mathrm{RCD}(K,N)$ **spaces.** To exclude Finsler geometry, which is included in the CD class, it was introduced in [**21**] the *Riemannian curvature dimension condition* (see also [**17**] for the case of σ-finite reference measure). This was then extended to the finite dimensional case in [**117**]:

Definition 1.3.6. A metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ satisfies the $\mathrm{RCD}(K,N)$ (resp. $\mathrm{RCD}^*(K,N)$) condition for $K\in\mathbb{R}$ and $N\in[1,\infty)$ if it satisfies the $\mathrm{CD}(K,N)$ (resp. $\mathrm{CD}^*(K,N)$) condition and it is infinitesimally Hilbertian.

Recall from Section 1.2.1 that a space is said to be infinitesimally Hilbertian if the Sobolev space $W^{1,2}(\mathrm{X})$ is Hilbert, condition which is equivalent to the linearity of the heat flow.

Since $\mathrm{RCD}(K,N)$ spaces are essentially non-branching (see [**190**]) we can apply the result in [**63**] to deduce that the $\mathrm{RCD}(K,N)$ and $\mathrm{RCD}^*(K,N)$ conditions are equivalent at least in the case of a finite reference measure. For further references and details on the theory of CD and RCD spaces we refer to the survey [**11**].

The $\mathrm{RCD}(K,N)$ condition can be characterized by the validity of the weak version of the Bochner inequality (see also the previous [**21, 22**] for the analogue in the infinite dimensional case).

Theorem 1.3.7 ([**22, 26, 107**])**.** *A metric measure space* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *satisfies that* $\mathrm{RCD}(K,N)^*$ *condition if and only if the following hold:*

- $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is infinitesimally Hilbertian,*
- *there exists* $a,b>0$ *such that* $\mathfrak{m}(B_r(x))\le ae^{br^2}$*, for every* $r>0$ *and* $x\in\mathrm{X}$*,*
- *the* Sobolev-to-Lipschitz *property holds: every* $f\in W^{1,2}(\mathrm{X})$ *with* $|Df|\le 1$ *has a* 1*-Lipschitz representative,*
- *for every* $f\in W^{1,2}(\mathrm{X})\cap\mathsf{D}_{L^2}(\Delta)$ *with* $\Delta f\in W^{1,2}(\mathrm{X})$ *and every* $g\in W^{1,2}(\mathrm{X})\cap\mathsf{D}(\Delta)$ *with* $\Delta g\in L^\infty(\mathrm{X})$

$$\frac{1}{2}\int \Delta g|\nabla f|^2\,\mathrm{d}\mathfrak{m}\ge\int\frac{(\Delta f)^2}{N}g+\langle\nabla f,\nabla\Delta f\rangle g+K|\nabla f|^2g\mathrm{d}\mathfrak{m}. \tag{1.3.9}$$

The formulation via the Bochner inequality can be seen as a sort of Lagrangian formulation of the Ricci curvature bound, as opposed to the Eulerian characterization given by optimal transport and the convexity properties of the entropy.

We will also need the following local variant of the Sobolev-to-Lipschitz property.

Proposition 1.3.8 (Local Sobolev-to-Lipschitz property)**.** *Let* X *be an* $\mathrm{RCD}(K,N)$ *space,* $K\in\mathbb{R}$ *and* $N\in[1,+\infty)$ *and let* $\Omega\subset\mathrm{X}$ *be open. Suppose* $f\in W^{1,2}_{\mathsf{loc}}(\Omega)$ *is such that* $\||\nabla f|\|_{L^\infty(\Omega)}<+\infty$*, then* f *has a locally-Lipschitz representative. Moreover for such representative it holds*

$$|f(x)-f(y)|\le\||\nabla f|\|_{L^\infty(\Omega)}\,\mathsf{d}(x,y), \tag{1.3.10}$$

for every $x,y\in\Omega$ *such that* $\mathsf{d}(x,y)\le\mathsf{d}(x,\partial\Omega)$*.*

Proof. The fact that f has a locally-Lipschitz representative it follows from the Sobolev-to-Lipschitz property and a cut-off argument.

For the second part we observe that it is sufficient to consider the case $\mathsf{d}(x,y)<\mathsf{d}(x,\partial\Omega)$, since the equality case follows by continuity. Hence for some $r>0$ we have $\mathsf{d}(x,y)<r<\mathsf{d}(x,\partial\Omega)$ and we can consider $\tilde f\in W^{1,2}(\mathrm{X})$ such that $\tilde f=f$ $\mathfrak{m}$-a.e. in $B_r(x)$. Then for $\varepsilon<(r-\mathsf{d}(x,y))/4$ we define $\mu^0_\varepsilon:=\mathfrak{m}|_{B_\varepsilon(x)}\mathfrak{m}(B_\varepsilon(x))^{-1}$, $\mu^1_\varepsilon:=\mathfrak{m}|_{B_\varepsilon(y)}\mathfrak{m}(B_\varepsilon(y))^{-1}$. thanks to the results in [**188**] and [**191**] there exists a unique $\boldsymbol{\pi}^\varepsilon\in\mathsf{OptGeo}(\mu^0_\varepsilon,\mu^1_\varepsilon)$ such that $\mathrm{e}_{t*}\boldsymbol{\pi}^\varepsilon\le C\mathfrak{m}$, $\forall t\in[0,1]$, for some constant C depending on ε. In particular $\boldsymbol{\pi}^\varepsilon$ is a test plan. Moreover from the triangle inequality it follows that $\boldsymbol{\pi}^\varepsilon$ is concentrated on curves γ with support contained in $B_r(x)$. Therefore from (1.2.5)

$$\left|\int f\,\mathrm{d}\mu^1_\varepsilon-\int f\,\mathrm{d}\mu^0_\varepsilon\right|\le\int|\tilde f(\gamma_1)-\tilde f(\gamma_0)|\,\mathrm{d}\boldsymbol{\pi}^\varepsilon\le\||\nabla\tilde f|\|_{L^\infty(B_r(x))}\int\int_0^1|\dot\gamma_t|\mathrm{d}td\boldsymbol{\pi}^\varepsilon\le\||\nabla f|\|_{L^\infty}W_2(\mu^0_\varepsilon,\mu^1_\varepsilon).$$

Letting $r\to0^+$ from the continuity of f we obtain (1.3.10). □

From Proposition 1.3.8 it also follows that

$$\text{if }\Omega\text{ is connected, }u\in W^{1,2}_{\mathsf{loc}}(\Omega)\text{ and }|\nabla u|=0,\ \mathfrak{m}\text{-a.e., then }u\text{ is constant in }\Omega. \tag{1.3.11}$$

Remark 1.3.9. Being PI-spaces, $\mathrm{RCD}(K,N)$ spaces enjoy the independence of the weak upper gradient of the Sobolev exponent (recall Theorem 1.2.3). In addition to this fact on $\mathrm{RCD}(K,N)$ spaces (and more generally on proper $\mathrm{RCD}(K,\infty)$ spaces) we have also a non-trivial identification for Sobolev functions between the total variation and the weak-upper gradient due to Gigli and Han [**120**]. In particular we have that whenever $f \in W^{1,2}_{\mathsf{loc}}(\Omega)$ with $|Df| \in L^1(\mathfrak{m})$ it holds that $f \in BV(\mathrm{X})$ with $\|Df\| = |Df|\mathfrak{m}$ (see [**120**, Remark 3.5]). ■

Thanks to the above remark, on $\mathrm{RCD}(K,N)$ spaces the following local version of the coarea formula holds (recall (1.2.17) for the standard coarea formula).

Corollary 1.3.10 (Local coarea formula)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space and* $\Omega \subset \mathrm{X}$ *be open. Let* $u \in \mathsf{LIP}_{\mathsf{loc}}(\Omega)$ *be such that for some* $a, b \in \mathbb{R}$ *with* $a < b$ *it holds that* $\{a < u < b\} \subset\subset \Omega$ *and* $\mathsf{d}(\{u < b\}, \Omega^c) > 0$*. Then* $\{u < t\}$ *has finite perimeter for a.e.* $t \in (a,b)$ *and for any* $f : \Omega \to [-\infty,+\infty]$ *Borel and such that* $f \in L^1_{\mathsf{loc}}(|\nabla u|\mathfrak{m}, \Omega)$ *it holds that*

$$\int_{\{a<u<b\}} f\,|\nabla u|\,\mathrm{d}\mathfrak{m} = \int_a^b \int f\,\mathrm{dPer}(\{u<t\})\,\mathrm{d}t. \tag{1.3.12}$$

Proof. Define $\tilde u \coloneqq a \vee u \wedge b$ in Ω and $\tilde u \coloneqq b$ in Ω^c. It is easily seen that the assumptions grant that $\tilde u \in \mathsf{LIP}(\mathrm{X})$ with $\mathrm{lip}\tilde u = 0$ in $\{a < u < b\}^c$, hence $\tilde u \in \mathsf{BV}(\mathrm{X})$. Then the result follows applying (1.2.17), recalling from Remark 1.3.9 that $\|D\tilde u\| = |D\tilde u|\mathfrak{m}$, the locality of the minimal weak upper gradient and observing that $\{\tilde u < t\} = \{u < t\}$ for every $t \in (a,b)$. □

We go on by recalling the so-called *Laplacian comparison* for $\mathrm{RCD}(K,N)$ spaces proved in [**117**] (see also [**67**]):

Theorem 1.3.11. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1,\infty)$*. Then for every* $x_0 \in \mathrm{X}$ *we have* $\mathsf{d}(x_0,.)^2 \in D(\mathbf{\Delta})$ *with*

$$\mathbf{\Delta}\mathsf{d}(x_0,.)^2 \le 2N\, l_{K,N}(\mathsf{d}_{x_0})\mathfrak{m}, \tag{1.3.13}$$

where $l_{K,N} : [0,\infty) \to [0,\infty)$ *is a continuous function.*

We will not need the precise expression for $l_{K,N}$, which can be found in [**117**], we only recall that $l_{0,N} \equiv 1$. We stress also that thanks to Theorem 1.2.3 and the fact that $\mathrm{CD}(K,N)$ spaces admits a $(1,1)$-Poincaré inequality we have that

$$|\nabla \mathsf{d}(x_0,.)| = 1, \quad \mathfrak{m}\text{-a.e.}. \tag{1.3.14}$$

For a more direct proof of this fact in the setting of $\mathrm{RCD}(K,N)$ spaces see also [**125**, Prop. 3.1].

We go on reporting some further regularities of the heat flow in the setting of $\mathrm{RCD}(K,N)$ spaces. From the results in ([**21**] and [**20**]) we have the existence of the so-called heat kernel $p_t : \mathrm{X}\times\mathrm{X} \to [0,+\infty]$ that represents the heat flow via integration:

$$h_t f(x) = \int p_t(x,y) f(y)\,\mathrm{d}\mathfrak{m}(y), \ \mathfrak{m}\text{-a.e. } x \in \mathrm{X}, \quad \forall f \in L^2(\mathfrak{m}).$$

As a consequence of the results in [**199**] and the validity of a weak local Poincaré inequality, p_t has a locally Hölder-continuous representative, which satisfies the following pointwise bounds ([**148**]):

$$\begin{aligned} &\frac{1}{C_1\mathfrak{m}(B(x,\sqrt{t}))}\exp\left\{-\frac{\mathsf{d}^2(x,y)}{3t} - ct\right\} \le p_t(x,y) \le \frac{C_1}{\mathfrak{m}(B(x,\sqrt{t}))}\exp\left\{-\frac{\mathsf{d}^2(x,y)}{5t} + ct\right\},\\ &|\nabla p_t(x,\cdot)(y)| \le \frac{C_1}{\sqrt{t}\mathfrak{m}(B(x,\sqrt{t}))}\exp\left\{-\frac{\mathsf{d}^2(x,y)}{5t} + ct\right\} \quad \text{for } \mathfrak{m}\text{-a.e. } y \in X, \end{aligned} \tag{1.3.15}$$

for any $x, y \in X$, for any $t > 0$ and where c, C_1 are positive constants depending only on K, N such that $c = 0$ if $K = 0$. Observe that this combined with the Sobolev-to-Lipschitz property shows that p_t has a Lipschitz representative.

Finally h_t has the so called L^∞*-to Lipschitz regularization property* (see [**21**]), i.e. there exists a constant $C(K) > 0$ such that for every $f \in L^\infty \cap L^2(\mathfrak{m})$ it holds that $|\nabla h_t f| \in L^\infty(\mathfrak{m})$ and

$$\||\nabla h_t f|\|_{L^\infty(\mathfrak{m})} \le \frac{C(K)}{\sqrt{t}}\|f\|_{L^\infty(\mathfrak{m})}, \quad \forall t \in (0,1). \tag{1.3.16}$$

In an $\mathrm{RCD}(K,N)$ space the tangent module $L^0(T\mathrm{X})$ (see Section 1.2.7 for the definition of this object) has local dimension which is at most $\lfloor N \rfloor$, more precisely we have the following result (see e.g. [**135**] or also [**126**] for a proof).

Theorem 1.3.12. *Let* X *be any* $\mathrm{RCD}(K,N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1,\infty)$ *and let* $\{E_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ *be the dimensional decomposition of* $L^0(T\mathrm{X})$ *(recall Theorem 1.2.37). Then* $\mathfrak{m}(E_n) = 0$ *for every* $n > \lfloor N \rfloor$*.*

Following [**135**] we also define the function $\dim_{\mathsf{loc}} : \mathrm{X} \to \mathbb{N} \cup \{\infty\}$ defined by setting $\dim_{\mathsf{loc}}(x) = n$ in E_n, which is well defined as an $\mathfrak{m}$-a.e. function. The above result can be equivalently restated as: $\dim_{\mathsf{loc}} \le \lfloor N \rfloor$ $\mathfrak{m}$-a.e.. Recall also that from Remark 1.2.36 we have for every E_i with $\mathfrak{m}(E_n) > 0$, the existence of an orthonormal base $\{e_1, ..., e_n\} \subset L^0(T\mathrm{X})$ such that $\langle e_i, e_j \rangle = \delta_{i,j}$ $\mathfrak{m}$-a.e. for every $i, j = 1, ..., n$. In particular for every $v \in L^0(T\mathrm{X})$ it holds that $v = \sum_{i=1}^n v_i e_i$, where $v_i := \langle v, e_i \rangle \in L^0(\mathfrak{m})$.

Remark 1.3.13. It is worth recalling that, even if the above dimensional decomposition is abstract, thanks to a well developed structure theory for $\mathrm{RCD}(K, N)$ spaces and after many contributions ([**57, 97, 123, 125, 126, 153, 181**]) we know much more on the sets E_n. In particular the set E_n can actually be characterized as the set where the tangent cone is unique and isomorphic to the n-dimensional Euclidean space and turns out to be (up to measure zero) bi-Lipschitz to a subset of $\mathbb{R}^n$. ■

Remark 1.3.14. A long-standing open problem has been the *constancy of the dimension* which consists in proving that actually $\mathfrak{m}(E_n) = 0$ for every $n \in \mathbb{N}$ except one. This was recently shown to be true in a breakthrough result in [**58**], however we will not take advantage of this fact. We also mention that the same result for Ricci-limits was previously solved by Colding and Naber [**89**], but with different techniques, which have been recently extended also in the RCD case by Deng [**100**]. ■

1.3.4. Stability and rigidity results on RCD spaces. One of the main advantages to work with the RCD class is that it satisfies some useful stability and rigidity properties. We will list here some of them, which will be useful for our purposes (see also Theorem 1.6.3 for the stability of the RCD condition under measure Gromov-Hausdorff convergence).

We recall the definition of spherical suspension and cone over a metric measure space. For any $N \in [1, \infty)$ the Euclidean N-cone over a metric measure space $(\mathrm{Z}, \mathfrak{m}_{\mathrm{Z}}, \mathsf{d}_{\mathrm{Z}})$ is defined to be the space $C_N(\mathrm{Z}) := [0, \infty) \times \mathrm{Z}/(\{0\} \times \mathrm{Z})$ endowed with the following distance and measure

$$\begin{aligned} &\mathsf{d}((t, z), (s, z')) := \sqrt{s^2 + t^2 - 2st\cos(\mathsf{d}_{\mathrm{Z}}(z, z') \wedge \pi)}, \\ &\mathfrak{m} := t^{N-1}\mathrm{d}t \otimes \mathfrak{m}_{\mathrm{Z}}. \end{aligned}$$

Analogously the N-spherical suspension over Z is defined to be the space $([0, \pi] \times_{\sin}^N \mathrm{Z}) := \mathrm{Z} \times [0, \pi] / (\mathrm{Z} \times \{0, \pi\})$ endowed with the following distance and measure

$$\begin{aligned} &\mathsf{d}((t, z), (s, z')) := \cos^{-1}\left(\cos(s)\cos(t) + \sin(s)\sin(t)\cos\left(\mathsf{d}_{\mathrm{Z}}(z, z') \wedge \pi\right)\right), \\ &\mathfrak{m} := \sin^{N-1}(t)\mathrm{d}t \otimes \mathfrak{m}_{\mathrm{Z}}. \end{aligned}$$

It turns out that the RCD condition is stable under the action of taking cones and spherical suspensions, more precisely it has been proven in [**155**] that

$$\begin{aligned} &C_N(\mathrm{Z}) \text{ (resp. } [0, \pi] \times_{\sin}^N \mathrm{Z}),\ N \ge 2, \text{ is an } \mathrm{RCD}(0, N) \text{ (resp. } \mathrm{RCD}(N-1, N)) \text{ space,} \\ &\text{if and only if } \mathrm{diam}(\mathrm{Z}) \le \pi \text{ and Z is an } \mathrm{RCD}(N-2, N-1) \text{ space.} \end{aligned} \tag{1.3.17}$$

We can now recall the two main rigidity statements that we will need in Chapter 4: the maximal diameter theorem and the Obata theorem for $\mathrm{RCD}(K, N)$ spaces:

Theorem 1.3.15 (Maximal diameter theorem, [**155**])**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(N-1, N)$ *space with and* $N \in [2, \infty)$ *and suppose that diam*$(\mathrm{X}) = \pi$. *Then* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is isomorphic to a spherical suspension, i.e. there exists a* $\mathrm{RCD}(N-2, N-1)$ *space* $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$ *with diam*$(\mathrm{Z}) \le \pi$ *satisfying* $\mathrm{X} \simeq [0, \pi] \times_{\sin}^N \mathrm{Z}$.

Theorem 1.3.16 (Obata's theorem, [**156**])**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(N-1, N)$ *space with and* $N \in [2, \infty)$. *Then* $\lambda^{1,2}(\mathrm{X}) = N$ *if and only if* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is isomorphic to a spherical suspension, i.e. there exists a* $\mathrm{RCD}(N-2, N-1)$ *space* $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$ *with diam*$(\mathrm{Z}) \le \pi$ *satisfying* $\mathrm{X} \simeq [0, \pi] \times_{\sin}^N \mathrm{Z}$.

1.3.5. (Sub)harmonic functions in RCD spaces. In this subsection $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ m.m.s. with $N \in [1, \infty)$.

Definition 1.3.17. Let $\Omega \subset \mathrm{X}$ be open. We say that a function $u \in \mathbf{\Delta}(\Delta, \Omega)$ is *subharmonic* (resp. *superharmonic*) in Ω if $\mathbf{\Delta}_{|\Omega} u \ge 0$ (resp. ≤ 0). Moreover if $\mathbf{\Delta}_{|\Omega} u = 0$ we say that u is *harmonic* in Ω.

We will need the following variational characterization of (sub/super)harmonic functions (see [**129**, Theorem 2.5] and also [**122**], [**117**]).

Proposition 1.3.18. *Let* $\Omega \subset \mathrm{X}$ *be open. A function* $u \in W^{1,2}(\Omega)$ *is superharmonic (resp. subharmonic) in* Ω *if and only if*

$$\int_\Omega |\nabla u|^2 \mathrm{d}\mathfrak{m} \le \int_\Omega |\nabla(u + \varphi)|^2 \mathrm{d}\mathfrak{m},$$

for every $\varphi \in \mathsf{LIP}_c(\Omega)$ *with* $\varphi \geq 0$ *(resp.* $\varphi \leq 0$*) or equivalently for every* $\varphi \in W^{1,2}_0(\Omega)$ *with* $\varphi \geq 0$ *(resp.* $\varphi \leq 0$*)* $\mathfrak{m}$*-a.e..*

It is well established (see e.g. [**46**]) that the maximum principle for subharmonic functions holds in the general setting of PI-spaces, for the present formulation and an alternative proof in the special case of $\mathrm{RCD}(K,N)$ spaces we refer to [**129**].

Proposition 1.3.19 (Maximum principle)**.** *Let* $\Omega \subset \mathrm{X}$ *be an open and bounded and let* $u \in D(\mathbf{\Delta}, \Omega)$ *be such that* $\mathbf{\Delta} u \geq 0$*. Then*

- **weak maximum principle***: if* u *is upper semicontinuous in* $\bar{\Omega}$*, then*
$$\operatorname*{ess\,sup}_{\Omega} u \leq \sup_{\partial\Omega} u,$$
- **strong maximum principle***:*
$$\textit{if } u \in C(\Omega),\ \Omega \textit{ is connected and } u(x) = \sup_{\Omega} u \textit{ for some } x \in \Omega, \textit{ then } u \textit{ is constant.}$$

Proof. For the first part suppose by contradiction that $\operatorname{ess\,sup}_{\Omega} u > \sup_{\partial\Omega} u$. Then there exists $c \in \mathbb{R}$ such that $\sup_{\partial\Omega} u < c < \operatorname{ess\,sup}_{\Omega}$, in particular from the upper semicontinuity and compactness of $\partial\Omega$ we have that $f - \min(f, c) \in W^{1,2}(\mathrm{X})$ with compact support in Ω. From this point the proof continue exactly as in [**129**, Thm. 2.3].

For the second part we apply the strong-maximum principle in [**129**, Thm. 2.8] to a ball $B_r(x) \subset \Omega$, obtaining that u is constantly equal to $\sup_{\Omega} u$ in $B_r(x)$. In particular the set $\{u = \sup_{\Omega} u\} \cap \Omega$ is open (and closed in Ω) and thus must coincide with Ω. □

Since on PI-spaces a class of Sobolev embedding is available (see [**134**] or also Theorem 1.3.5), a Moser iteration can be performed to obtain Harnack inequalities for subharmonic functions (see e.g. [**46**, Chap. 4-5] or also [**158**] for an alternative approach). For the case of $\mathrm{RCD}(0,N)$ spaces (which fall into the category of in PI-space) see also [**144**].

Proposition 1.3.20. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *m.m.s. with* $N \in [1,\infty)$*. For every* $R_0 > 0$ *there exists two positive constants* $C_i = C_i(R_0, K^-, N)$*,* $i = 1, 2$*, such that the following hold for any* $R < R_0$*:*

(1) if u *is subharmonic function in a ball* $B_{2R}(x)$*, then*
$$\operatorname*{ess\,sup}_{B_R(x)} u \leq C_2 \fint_{B_{2R}(x)} |u| \mathrm{d}\, \mathfrak{m},$$
(2) if u *is a nonnegative superharmonic function in a ball* $B_{2R}(x)$*, then*
$$\operatorname*{ess\,inf}_{B_R(x)} u \geq C_1 \fint_{B_{2R}(x)} u \, \mathrm{d}\mathfrak{m}.$$

The above Harnack inequalities imply that harmonic functions have a locally Hölder continuous representative and that superharmonic functions have a lower semicontinuous representative (see for example [**46**, Theorem 8.22]). From now on we will always tacitly consider these special representatives.

The Lipschitz continuity of harmonic functions is instead false on arbitrary PI-spaces and a counterexample can be found in [**159**]. However in same work the authors proved that Lipschitz continuity holds under the assumptions of Ahlfors-regularity (see also [**146**]). Subsequently this result has been extended by Jiang [**147**] to the case of lower Ricci curvature bound (see also [**152**]):

Theorem 1.3.21 (Gradient estimate, [**147**])**.** *For every* $R > 0$*,* $N \in [1,\infty)$ *and* $K \in \mathbb{R}$ *there exist a positive constants* $C = C(N,K)$*,* $c = c(N,K,R)$ *such that the following holds. Let* X *be an* $\mathrm{RCD}(K,N)$ *space and let* $u \in D(\mathbf{\Delta}, B_{2r}(x))$ *be positive and harmonic in* $B_{2r}(x)$*, then*
$$\left\| \frac{|Du|}{u} \right\|_{L^\infty(B_r(x))} \leq C\left(\frac{1}{r} + c\right). \tag{1.3.18}$$
Moreover we can take $c = 0$ *if* $K = 0$*.*

Thanks to the local version of the Sobolev-to-Lipschitz property (see Proposition 1.3.8) we have the following 'pointwise' version of the above result:

Lemma 1.3.22. *For every* $N \in [1,\infty)$ *there exist two constants* $C = C(N) > 0$*,* $\lambda = \lambda(N) < 1$ *such that the following holds. Let* X *be an* $\mathrm{RCD}(0,N)$ *space and let* u *be positive and harmonic in* $B_R(x_0)$*, then*
$$\| |\nabla u| \|_{L^\infty(B_{\lambda R})} \leq \frac{C u(x_0)}{R}.$$

Proof. From the gradient estimate (1.3.18) we have

$$|\nabla u|(x) \leq \frac{Cu(x)}{R}, \quad \text{for } \mathfrak{m}\text{-a.e. } x \in B_{R/2}(x_0),$$

for some constant $C = C(N) > 0$. In particular, from the local Sobolev-to-Lipschitz property (Proposition 1.3.8) for any $\lambda < 1/2$ we have

$$\||\nabla u|\|_{L^\infty(B_{\lambda R}(x_0))} \leq \frac{C\|u\|_{L^\infty(B_{\lambda R}(x_0))}}{R} \leq \frac{Cu(x_0)}{R} + C\lambda\||\nabla u|\|_{L^\infty(B_{\lambda R})},$$

from which the conclusion follows taking $\lambda < C/2$. □

Even if not relevant for our purposes we observe that Lemma 1.3.22 immediately implies the following result which generalizes a famous result by Yau [**216**] on Riemannian manifolds.

Theorem 1.3.23. *On an* $\mathrm{RCD}(0, N)$ *space with* $N \in [1, \infty)$ *every positive harmonic function on* X *is constant.*

We conclude with the following variant of Ascoli-Arzelà theorem for harmonic functions.

Lemma 1.3.24. *Let* X *be an* $\mathrm{RCD}(0, N)$ *space and let* $\{u_i\}$ *be a sequence of harmonic and continuous functions in* Ω. *Suppose that*

$$\sup_\Omega |u_i| < C,$$

then there exists a subsequence u_{i_k} *that converges locally uniformly to a function* u *harmonic in* Ω.

Proof. The existence of a (non relabelled) subsequence u_i converging locally uniformly to a continuous function u follows from the gradient estimate (1.3.18) and Ascoli-Arzelà. It remains to prove that u is harmonic in Ω. Fix $\Omega' \subset\subset \Omega$ and $\eta \in \mathrm{Lip}_c(\Omega)$ with $\eta = 1$ in Ω'. Then $\eta u_i \in W^{1,2}(\mathrm{X})$ converges to $u\eta$ in $L^2(\Omega')$. Moreover, again by (1.3.18), we have

$$\sup_i \||\nabla(\eta u_i)|\|_{L^2(\mathfrak{m})} < +\infty.$$

In particular (recall that $W^{1,2}(\mathrm{X})$ is Hilbert) up to a subsequence $u_i\eta \rightharpoonup \eta u$ in $W^{1,2}(\mathrm{X})$ and therefore (from the locality of the gradient) $\int_\Omega \langle \nabla u, \nabla f\rangle \, d\mathfrak{m} = 0$ for every $f \in \mathsf{LIP}_c(\Omega')$ (see also [**117**, Prop 5.19] for a similar limiting argument). From the arbitrariness of Ω' we deduce both that $u \in W^{1,2}_{\mathsf{loc}}(\Omega)$ and that u is harmonic in Ω. □

1.4. Second order calculus on RCD spaces

1.4.1. Hessian. In [**118**] it has been showed that the presence of Ricci curvature lower bounds on a metric measure space permits to push the calculus to the second order and in particular define objects such as Hessians and covariant derivatives. The basic objects to build this calculus is the algebra of test functions $\mathsf{Test}(\mathrm{X})$ firstly introduced by Savarè [**193**] and defined as

$$\mathsf{Test}(\mathrm{X}) := \{f \in L^\infty(\mathfrak{m}) \cap \mathsf{LIP}(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta) \cap W^{1,2}(\mathrm{X}) \mid \Delta f \in W^{1,2}(\mathrm{X})\}.$$

The crucial point is that, thanks to the regularizing properties of the heat flow, the class of test function is rich. In particular $h_t \in \mathsf{Test}(\mathrm{X})$ for $t > 0$ whenever $f \in L^2 \cap L^\infty(\mathfrak{m})$ and $\mathsf{Test}(\mathrm{X})$ is dense in $W^{1,2}(\mathrm{X})$.

A crucial regularity property, which essentially comes from the validity of the Bochner inequality is that $|\nabla f|^2 \in W^{1,2}(\mathrm{X})$ for every $f \in \mathsf{Test}(\mathrm{X})$ and in particular by polarization $\langle \nabla f, \nabla g\rangle \in W^{1,2}(\mathrm{X})$ for every $f, g \in W^{1,2}(\mathrm{X})$ (see [**193**], then [**118**] and also [**127**]). The key observation to define the Hessian is the following identity valid in the smooth setting

$$2\mathrm{Hess}(f)(\nabla g_2, \nabla g_2) = \nabla g_1 \cdot \nabla(\nabla f \cdot \nabla g_2) + \nabla g_2 \cdot \nabla(\nabla f \cdot \nabla g_1) - \nabla f \cdot (\nabla g_1 \cdot \nabla g_2)$$

combined with the observation that the right hand side makes sense at least if $f, g_1, g_2 \in \mathsf{Test}(\mathrm{X})$. In what follows, we refer to [**118**] for the definition of the tensor-module $L^2((T^*)^{\otimes^2}\mathrm{X})$ and we limit ourselves to recall that it has structure of Hilbert $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$ module.

Definition 1.4.1. We define $W^{2,2}(\mathrm{X})$ as the space of all functions $f \in W^{1,2}(\mathrm{X})$ such that there exists $T \in L^2((T^*)^{\otimes^2}\mathrm{X})$ such that for every $h, g_1, g_2 \in \mathsf{Test}(\mathrm{X})$ it holds

$$2\int hT(\nabla g_1, \nabla g_2)\mathrm{d}\mathfrak{m} =$$
$$-\int \langle \nabla f, \nabla g_1\rangle \mathrm{div}(h\nabla g_1) + \nabla f, \nabla g_2 \mathrm{div}(h\nabla g_2) - \mathrm{div}(h\nabla f)\langle \nabla g_1, \nabla g_2\rangle \mathrm{d}\mathfrak{m},$$

in which case such element T is unique, called Hessian of f and denoted by $\mathrm{Hess}(f)$.

We endow the space $W^{2,2}(\mathrm{X})$ with the following norm:

$$\|f\|^2_{W^{2,2,}(\mathrm{X})} := \|f\|^2_{L^2(\mathfrak{m})} + \||Df|\|^2_{L^2(\mathfrak{m})} + \||\mathrm{Hess}(f)|_{HS}\|^2_{L^2(\mathfrak{m})},$$

where $|\mathrm{Hess}(f)|_{HS}$ denotes the Hilbert Schmidt norm of $\mathrm{Hess}(f)$, that is the pointwise norm of $\mathrm{Hess}(f)$ in $L^2((T^*)^{\otimes^2}\mathrm{X})$. It is proved in [**118**] that the space $(W^{2,2}(\mathrm{X}), \|.\|_{W^{2,2,}(\mathrm{X})})$ is a separable Hilbert space. The crucial point is again the fact that we have many $W^{2,2}$ functions as shown by the following result which is [**118**, Corollary 3.3.9]:

Theorem 1.4.2. *It holds that* $\mathsf{D}_{L^2}(\Delta) \cap W^{1,2}(\mathrm{X}) \subset W^{2,2}(\mathrm{X})$ *and the following quantitative estimate holds*

$$\int |\mathrm{Hess}(f)|^2_{HS}\,\mathrm{d}\mathfrak{m} \le \int (\Delta f)^2\,\mathrm{d}\mathfrak{m} - K\int |\nabla f|^2\,\mathrm{d}\mathfrak{m}, \quad \forall f \in \mathsf{D}_{L^2}(\Delta) \cap W^{1,2}(\mathrm{X}). \tag{1.4.1}$$

We proceed to define the space $H^{2,2}(\mathrm{X})$ as the closure in $W^{2,2}(\mathrm{X})$ of $\mathsf{Test}(\mathrm{X})$.

Proposition 1.4.3 ([**118**, Prop. 3.3.18])**.** $H^{2,2}(\mathrm{X})$ *coincides with the closure of* $\mathsf{D}_{L^2}(\Delta) \cap W^{1,2}(\mathrm{X})$ *in* $W^{2,2}(\mathrm{X})$.

The Hessian enjoys also the usual calculus rules as shown in [**118**], we recall the locality:

Proposition 1.4.4. *Let* $f, g \in H^{2,2}(\mathrm{X})$*, then* $\mathrm{Hess}(f) = \mathrm{Hess}(g)$ $\mathfrak{m}$*-a.e. in* $\{f = g\}$.

It is also possible to apply the Hessian to a single vector field to obtain another vector field. More precisely the Hessian induces an L^0-linear and continuous map $\mathrm{Hess}(f) : L^0(T\mathrm{X}) \to L^0(T\mathrm{X})$ characterized by

$$\langle \mathrm{Hess}(f)(v), w\rangle = \mathrm{Hess}(f)(v, w), \quad \mathfrak{m}\text{-a.e.}, \quad \forall w \in L^0(T\mathrm{X})$$

and which satisfies $|\mathrm{Hess}(f)(v)| \le |\mathrm{Hess}(f)|_{HS}|v|$. We recall also the following fact proved in [**118**, Prop. 3.3.22].

Proposition 1.4.5. *For every* $f \in H^{2,2}(\mathrm{X}) \cap \mathsf{LIP}(\mathrm{X})$ *it holds that* $|\nabla f|^2 \in W^{1,2}(\mathrm{X})$ *and*

$$2\mathrm{Hess}(f)(\nabla f) = \nabla|\nabla f|^2, \quad \forall f \in H^{2,2,}(\mathrm{X}). \tag{1.4.2}$$

As a consequence we obtain the following regularity result

Proposition 1.4.6. *For every* $f \in \mathsf{D}_{L^2}(\Delta) \cap W^{1,2}(\mathrm{X})$ *it holds that* $|\nabla f| \in W^{1,2}(\mathrm{X})$.

Proof. Assume first that $f \in \mathsf{Test}(\mathrm{X})$. In particular f is also Lipschitz. Then from Proposition (1.4.5) we have that $|\nabla f|^2 \in W^{1,2}(\mathrm{X})$ with $|\nabla|\nabla f|^2| \le 2|\mathrm{Hess}(f)|_{HS}|\nabla f|$ $\mathfrak{m}$-a.e.. Then the desired result follows directly applying Lemma 1.2.10. For a general $f \in \mathsf{D}_{L^2}(\mathrm{X}) \cap W^{1,2}(\mathrm{X})$ the statement follows via approximation recalling that from Proposition 1.4.3 there exists a sequence $f_n \in \mathsf{Test}(\mathrm{X})$ converging to f strongly in $W^{2,2}(\mathrm{X})$. □

Thanks to the finite dimensionality of $L^0(T\mathrm{X})$ (recall Theorem 1.3.12) and the existence of local orthonormal basis, in $\mathrm{RCD}(K, N)$ spaces, $N < +\infty$, we can write objects in coordinates. More precisely for every E_n with $n \in \{1, ..., \lfloor N \rfloor\}$ with $\mathfrak{m}(E_n) > 0$ there exists an orthonormal basis $e_1, ..., e_n$ (recall Remark 1.2.36) of $L^0(T\mathrm{X})$ in E_i. Moreover for a function $f \in W^{2,2}(\mathrm{X})$ we denote by $(Hf)_{i,j}$ the functions $\mathrm{Hess}(f)(e_i, e_j) \in L^2(\mathfrak{m})$, for $i, j \in \{1, ..., n\}$. Then the following identities can be verified:

$$\chi_{E_n}\mathrm{Hess}(f)(v) = \chi_{E_n} \sum_{1\le i,j\le n} (Hf)_{i,j} v_j\, e_i, \quad \forall v \in L^0(T\mathrm{X}), \tag{1.4.3}$$

$$|\mathrm{Hess}(f)|^2_{HS} = \sum_{1\le i,j\le n} (Hf)^2_{i,j}, \quad \mathfrak{m}\text{-a.e. in } E_n. \tag{1.4.4}$$

Additionally it can be also defined the trace of the Hessian as

$$\mathrm{trHess}(f) := \sum_{1\le i\le n} (Hf)_{i,i}, \quad \mathfrak{m}\text{-a.e. in } E_n, \tag{1.4.5}$$

which uniquely determines a function $\mathrm{trHess}(f) \in L^2(\mathfrak{m})$, well defined in the sense that it does not depend on the choice of the basis, as can be easily verified by a direct computation. In general it does not hold that the trace of the Hessian is the Laplacian, however we have the following result by Han [**135**]:

Theorem 1.4.7. *Let* X *be any* $\mathrm{RCD}(K, N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$*. Then for every* $f \in \mathsf{Test}(\mathrm{X})$ *it holds*

$$\mathrm{trHess}(f) = \Delta f, \quad \mathfrak{m}\text{-a.e. in } \{\dim_{\mathsf{loc}} = N\}. \tag{1.4.6}$$

1.4.2. Improvement and localization of the Bochner inequality. With the notion of Hessian available, in [**135**] it is proved the following powerful improved version of the Bochner inequality (see [**107**],[**26**] or also (1.3.9) for the "basic version" of the Bochner inequality.)

Theorem 1.4.8 (Improved Bochner-inequality)**.** *Let* X *be any* $\mathrm{RCD}(K,N)$ *space with* $K\in\mathbb{R}$ *and* $N\in[1,\infty)$*. Then for any* $f\in\mathsf{Test}(\mathrm{X})$ *it holds that* $|\nabla f|^2\in D(\mathbf{\Delta})$ *and*

$$\mathbf{\Delta}\left(\frac{|\nabla f|^2}{2}\right)\ge\left(|\mathrm{Hess}(f)|_{HS}^2+K|\nabla f|^2+\langle\nabla f,\nabla\Delta f\rangle+\frac{(\Delta f-\mathsf{tr}\mathrm{Hess}(f))^2}{N-\dim_{\mathsf{loc}}(\mathrm{X})}\right)\mathfrak{m},\tag{1.4.7}$$

where $\frac{(\Delta f-\mathsf{tr}\mathrm{Hess}(f))^2}{N-\dim_{\mathsf{loc}}(\mathrm{X})}$ *is taken to be 0 in whenever* $\dim_{\mathsf{loc}}=N$*.*

Being an inequality between functions it is natural to expect that the Bochner inequality holds also in local form. This is indeed true and follows from the existence of the following good cut-off constructed in [**181**] (see also [**26**] for similar constructions).

Proposition 1.4.9 (Good cut-off functions)**.** *Let* X *be an* $\mathrm{RCD}(K,N)$ *space,* $K\in\mathbb{R}$ *and* $N\in(1,\infty)$*. Then for every* $0<r<R<+\infty$*, every compact set* P *and every open set* U *containing* P*, such that diam*$(U)<R$ *and* $\mathsf{d}(P,U^c)>r$*, there exists a function* $\eta\in\mathsf{Test}(\mathrm{X})$ *such that*

(1) $0\le\eta\le1$*,* $\eta=1$ *in* P *and* $\operatorname{supp}\eta\subset U$*,*
(2) $r|\nabla\eta|+r^2|\Delta\eta|\le C(R,N,K)$*,*

moreover C *can be taken independent of* R *in the case* $K=0$*.*

Fix Ω an open subset of X. The relevant class of functions is the following

$$\mathsf{Test}_{\mathsf{loc}}(\Omega):=\{u\in\mathsf{LIP}_{\mathsf{loc}}(\Omega)\cap D(\mathbf{\Delta},\Omega)\mid\mathbf{\Delta}u\in W^{1,2}_{\mathsf{loc}}(\Omega)\}.$$

This definition is motivated by the following observation:

$$\eta\in\mathsf{Test}(X)\text{ with }\operatorname{supp}\eta\subset\subset\Omega,\ u\in\mathsf{Test}_{\mathsf{loc}}(\Omega)\implies\eta u\in\mathsf{Test}(X),\tag{1.4.8}$$

Indeed from Proposition 1.4.6, the fact that $\eta u\in\mathsf{D}_{L^2}(\mathrm{X})\cap W^{1,2}(\mathrm{X})$ and the Leibniz rule for the Laplacian follows that

$$\text{if } u\in\mathsf{D}_{L^2_{\mathsf{loc}}}(\Omega)\text{, then }|\nabla u|\in W^{1,2}_{\mathsf{loc}}(\Omega)\tag{1.4.9}$$

and in particular $|\nabla u|^2\in W^{1,2}_{\mathsf{loc}}(\Omega)$ for every $u\in\mathsf{Test}_{\mathsf{loc}}(\Omega)$. Then (1.4.8) follows again from the Leibniz rule for the Laplacian which ensures that $\Delta(u\eta)\in W^{1,2}(\mathrm{X})$. Thanks to (1.4.8) we can define for every $u\in\mathsf{Test}_{\mathsf{loc}}(\Omega)$ the functions $|\mathrm{Hess}(u)|_{HS},\mathsf{tr}\mathrm{Hess}(u)\in L^2_{\mathsf{loc}}(\Omega)$ as

$$\begin{aligned}|\mathrm{Hess}(u)|_{HS}&\coloneqq|\mathrm{Hess}(\eta u)|_{HS},\ \mathfrak{m}\text{-a.e. in }\Omega',\\ \mathsf{tr}\mathrm{Hess}(u)&\coloneqq\mathsf{tr}\mathrm{Hess}(\eta u),\ \mathfrak{m}\text{-a.e. in }\Omega',\end{aligned}$$

for every $\eta\in\mathsf{Test}(\mathrm{X})$ with compact support in Ω and such that $\eta=1$ in $\Omega'\subset\subset\Omega$. This definition is well posed thanks to both the locality property of the Hessian (Proposition 1.4.4) and the existence of many such functions η as granted by Proposition 1.4.9. The following is a direct consequence of (1.4.7) and the above definitions, together with the locality property of the gradient and Laplacian.

Proposition 1.4.10 (Local improved Bochner Inequality)**.** *Let* $u\in\mathsf{Test}_{\mathsf{loc}}(\Omega)$*, then* $|\nabla u|^2\in D(\mathbf{\Delta},\Omega)$ *and*

$$\mathbf{\Delta}_{|\Omega}(|\nabla u|^2)\ge2\left(|\mathrm{Hess}(u)|_{HS}^2+\frac{(\Delta u-\mathsf{tr}\mathrm{Hess}(u))^2}{N-\dim(\mathrm{X})}+\langle\nabla u,\nabla\Delta u\rangle+K|\nabla u|^2\right)\mathfrak{m}|_{\Omega},\tag{1.4.10}$$

where $\frac{(\Delta u-\mathrm{trHess}(u))^2}{N-\dim(\mathrm{X})}$ *is taken to be 0 in the case* $\dim(\mathrm{X})=N$*.*

Remark 1.4.11. Even if we will not need this fact, let us recall that also the converse of the above holds, that is to say that a local Bochner inequality implies a global one as shown in ([**26**]). Since the Bochner inequality is equivalent to the $\mathrm{RCD}^*(K,N)$ condition (recall Theorem 1.3.7 and also [**26**, **107**]), this is an instance of the local-to-global property of the $\mathrm{RCD}^*(K,N)$ condition. ■

1.4.3. Covariant derivative. Starting from the existence of many second-order regular functions it has been built in [**118**] a notion of vector fields with covariant derivative.

In what follows, we refer to [**118**] for the definition of the tensor-module $L^2(T^{\otimes^2}\mathrm{X})$ and we just recall that it has structure of Hilbert $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$ module and that it can be identified as the dual module of $L^2((T^*)^{\otimes^2}\mathrm{X})$ (recall Def. 1.2.30).

Definition 1.4.12. We define $W_C^{1,2}(T\mathrm{X})$ as the space of all $v \in L^2(T\mathrm{X})$ such that there exists $T \in L^2(T^{\otimes^2}\mathrm{X})$ such that for every $h, g_1, g_2 \in \mathsf{Test}(\mathrm{X})$ it holds that

$$2\int hT(\nabla g_1 \otimes \nabla g_2)\mathrm{d}\mathfrak{m} = -\int \langle v, \nabla g_1\rangle \mathrm{div}(h\nabla g_2) + h\mathrm{Hess}(g_1)(v, \nabla f)\mathfrak{m},$$

in this case such element T is unique, called covariant derivative of v and denoted by ∇v.

We endow the space $W_C^{1,2}(T\mathrm{X})$ with the following norm:

$$\|v\|^2_{W_C^{1,2}(T\mathrm{X})} := \|v\|^2_{L^2(T\mathrm{X})} + \||\nabla v|_{HS}\|^2_{L^2(\mathfrak{m})},$$

where $|\nabla v|_{HS}$ denotes the the pointwise Hilbert-Schmidt norm of ∇v in $L^2(T^{\otimes^2}\mathrm{X})$. As shown in [**118**] (see also) the space $(W_C^{1,2}(T\mathrm{X}), \|.\|^2_{W_C^{1,2}(T\mathrm{X})})$ is a separable Hilbert space. As usual the key fact is that we have many vector fields that admits a covariant derivative:

Proposition 1.4.13 ([**118**])**.** *For every $f \in W^{2,2}(\mathrm{X})$ it holds that $\nabla f \in W_C^{1,2}(T\mathrm{X})$ and*

$$\nabla(\nabla f)(v, w) = \mathrm{Hess}(f)(v, w), \quad \forall v, w \in L^0(T\mathrm{X}). \tag{1.4.11}$$

The identity (1.4.11) can be equivalently stated via the musical isomorphism (see [**118**, Section 3.2]) as $(\mathrm{Hess}(f))^\# = \nabla(\nabla f)$. From (1.4.11), (1.4.2) and Proposition 1.4.6 we deduce that for every $f \in \mathsf{D}_{L^2}(\Delta) \cap W^{1,2}(\mathrm{X})$ and $w \in L^0(T\mathrm{X})$ it holds

$$|\nabla f|\langle \nabla|\nabla f|, w\rangle = \nabla(\nabla f)(\nabla f, w) \quad \mathfrak{m}\text{-a.e.}. \tag{1.4.12}$$

Proposition 1.4.14 (Leibniz rule, [**118**])**.** *For every $f \in W^{1,2} \cap L^\infty(\mathrm{X})$ and every $v \in W_C^{1,2}(T\mathrm{X})$ it holds that $fv \in W_C^{1,2}(T\mathrm{X})$ and*

$$\nabla(fv) = f\nabla v + \nabla f \otimes v. \tag{1.4.13}$$

Proposition 1.4.13 and the Leibniz rule motivate the following notion of test vector fields:

$$\mathrm{TestV}(\mathrm{X}) := \left\{\sum_{i=1}^n g_i \nabla f_i \ : \ n \in \mathbb{N},\ f_i, g_i \in \mathsf{Test}(\mathrm{X})\right\} \subset W_C^{1,2}(T\mathrm{X}).$$

It can be checked that $\mathrm{TestV}(\mathrm{X})$ is dense in $L^2(T\mathrm{X})$. We proceed defining the set $H_C^{1,2}(T\mathrm{X})$ as the closure in $W_C^{1,2}(T\mathrm{X})$ of $\mathrm{TestV}(\mathrm{X})$. From Proposition 1.4.13 we clearly have that

$$\nabla f \in H_C^{1,2}(T\mathrm{X}) \text{ for every } f \in H^{2,2}(\mathrm{X}). \tag{1.4.14}$$

The following elementary stability result will be needed:

Proposition 1.4.15. *Let $v \in H_C^{1,2} \cap L^\infty(T\mathrm{X})$ and $f \in W^{1,2} \cap L^\infty(\mathrm{X})$, then $fv \in H_C^{1,2}(T\mathrm{X})$.*

Proof. We first consider the case in which $f \in \mathrm{TestV}(\mathrm{X})$. By assumption there exists a sequence $(v_n) \subset \mathrm{TestV}(\mathrm{X})$ such that $v_n \to v$ strongly in $W_C^{1,2}(T\mathrm{X})$. By definition $fv_n \in \mathrm{TestV}(\mathrm{X})$ as well and from the Leibniz rule for the covariant derivative $|\nabla(fv_n)|_{HS} \le |f||\nabla v_n|_{HS} + |\nabla f||v_n|$. Since $v_n \to v$ strongly in $W_C^{1,2}(T\mathrm{X})$ we have that $|v_n|, |\nabla v_n||_{HS}$ are uniformly bonded in $L^2(\mathfrak{m})$, which implies that (fv_n) is uniformly bounded in $W_C^{1,2}(T\mathrm{X})$. In particular fv_n weakly converges to fv in $W_C^{1,2}(T\mathrm{X})$. However $\mathrm{TestV}(\mathrm{X})$ is a vector space and its weak closure coincides with its strong closure, hence $f_v \in H_C^{1,2}(T\mathrm{X})$ as desired.

Consider now a general $f \in W^{1,2} \cap L^\infty(\mathrm{X})$. From heat flow-approximation there exists $(f_n) \subset \mathsf{Test}(\mathrm{X})$ such that $f_n \to f$ strongly in $W^{1,2}(\mathrm{X})$ and $\sup_n \|f_n\|_{L^\infty}(\mathfrak{m}) < +\infty$. From the first part we have that $f_n v \in H_C^{1,2}(T\mathrm{X})$, moreover it is immediate to check that $f_n v \to fv$ strongly in $L^2(T\mathrm{X})$. Again from the Leibniz rule for the covariant derivative we have that $f_n v \in W_C^{1,2}(T\mathrm{X})$ with $|\nabla(f_n v)|_{HS} \le |f_n||\nabla v|_{HS} + |\nabla f_n||v|$, which is uniformly bounded in $L^2(\mathfrak{m})$. Therefore fv is in the weak closure of $H_C^{1,2}(T\mathrm{X})$ which however, being a vector space, coincides with its strong closure. □

We conclude with the important fact about the compatibility of the covariant derivative with the scalar product induced by Sobolev spaces:

Proposition 1.4.16 (Compatibility with the metric, [**118**])**.** *Let $v_1, v_2 \in H_C^{1,2} \cap L^\infty(T\mathrm{X})$, then $\langle v_1, v_2\rangle \in W^{1,2}(\mathrm{X})$ and*

$$\langle \nabla\langle v_1, v_2\rangle, w\rangle = \nabla v_1(w, v_2) + \nabla v_2(w, v_1), \quad \forall w \in L^0(T\mathrm{X}). \tag{1.4.15}$$

A consequence of the above is that for every $v \in H^{1,2}_C \cap L^\infty(T\mathrm{X})$ it holds $|v|^2 \in W^{1,2}(\mathrm{X})$ and

$$|D|v|^2| \le 2|\nabla v|_{HS}|v|, \tag{1.4.16}$$

which combined with Lemma 1.2.10 and an approximation argument (see e.g. [**99**] Lemma 2.5) we obtain that

$$\text{for every } v \in H^{1,2}_C \text{ it holds that } |v| \in W^{1,2}(\mathrm{X}). \tag{1.4.17}$$

1.5. Tangent module to a set of finite perimeter

Building on top of the existence of many Sobolev vector fields it was shown in [**99**] that on $\mathrm{RCD}(K,N)$ spaces it can be given a notion of *quasi continuous vector field.* In particular in [**99**] it has been built the tangent $L^0(\mathrm{Cap})$-module, $L^0_{\mathrm{Cap}}(T\mathrm{X})$, of vector fields defined Cap-a.e.. Subsequently, exploiting the fact that the perimeter measure is absolutely continuous with respect to the capacity (recall Proposition 1.2.26), by quotienting this object it has been defined in [**56**] the tangent module to a set of finite perimeter.

To state this results more precisely we start with the notion of $L^0(\mathrm{Cap})$-module (recall that Cap is only an outer measure, hence this object does not fall into the category of 'standard' L^0-modules as in Section 1.2.7).

Definition 1.5.1 ($L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module,[**99**])**.** An $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module is a triple $(\mathscr{M}, |.|, \tau)$, where $\mathscr{M}$ is a module over the commutative ring $L^0(\mathrm{Cap})$, $(\mathscr{M},\tau)$ is a topological vector space, $|.| : \mathscr{M} \to L^0(\mathrm{Cap})$ is a map satisfying (in the Cap-a.e. sense)

$$\begin{aligned} |v| &\ge 0 && \text{for every } v \in \mathscr{M}, \text{ with equality if and only if } v = 0,\\ |v+w| &\le |v| + |w| && \text{for every } v, w \in \mathscr{M},\\ |fv| &= |f||v| && \text{for every } f \in L^0(\mathrm{Cap}) \text{ and } v \in \mathscr{M}, \end{aligned}$$

and such that distance on $\mathscr{M}$ given by

$$\mathsf{d}_{\mathscr{M}}(v,w) := \sum_{k\in\mathbb{N}} \frac{1}{\mathrm{Cap}(A_k)\vee 1}\int_{A_k} |v-w| \wedge 1 \,\mathrm{dCap}, \quad \text{for all } v, w \in \mathscr{M},$$

is complete and induces τ.

One of the main results in [**99**] is the construction of the $L^0(\mathrm{Cap})$ tangent module:

Theorem 1.5.2 (Tangent $L^0(\mathrm{Cap})$-module, [**99**])**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,\infty)$ space. Then there exists a unique couple $\big(L^0_{\mathrm{Cap}}(T\mathrm{X}), \tilde\nabla\big)$, where $L^0_{\mathrm{Cap}}(T\mathrm{X})$ is a $L^0(\mathrm{Cap})$-module over X and the operator $\tilde\nabla : \mathrm{Test}(\mathrm{X}) \to L^0_{\mathrm{Cap}}(T\mathrm{X})$ is linear, such that the following properties hold:*

- *For any $f \in \mathrm{Test}(\mathrm{X})$ we have that the equality $|\tilde\nabla f| = \mathsf{QCR}\big(|Df|\big)$ holds Cap-a.e. on X (recall that $|Df| \in W^{1,2}(\mathrm{X})$ by Proposition 1.4.6)*
- *The space of $\sum_{n\in\mathbb{N}} \chi_{E_n}\tilde\nabla f_n$, with $(f_n)_n \subseteq \mathrm{Test}(\mathrm{X})$ and $(E_n)_n$ Borel partition of X, is dense in $L^0_{\mathrm{Cap}}(T\mathrm{X})$.*

Additionally (see [**99**] Theorem 2.14) there exists a linear map $\bar{\mathsf{QCR}} : H^{1,2}_C(T\mathrm{X}) \to L^0_{\mathrm{Cap}}(T\mathrm{X})$ which sends a vector field to its quasi continuous representative (see [**99**] for a precise definition of quasi continuous vector fields). Moreover recalling that $|v| \in W^{1,2}(\mathrm{X})$ for every $v \in H^{1,2}_C(T\mathrm{X})$ (see [**99**] Lemma 2.5) we also have $|\bar{\mathsf{QCR}}(v)| = \mathsf{QCR}(|v|)$ Cap-a.e. for every $v \in H^{1,2}_C(T\mathrm{X})$ (where QCR is the quasi continuous representative map for Sobolev functions, recall Section 1.2.5).

To define the tangent module for a set of finite perimeter we start recalling from Proposition 1.2.26 that in the $\mathrm{RCD}(K,N)$ setting it holds that $\mathrm{Per}(E) \ll \mathrm{Cap}$ for every set E of finite perimeter. In particular there exists the projection map $\pi_E : L^0(\mathrm{Cap}) \to L^0(\mathrm{Per}(E))$ for the following equivalence relation: $f \sim g$ if $f = g$ $\mathrm{Per}(E)$-a.e. for any $f, g \in L^0(\mathrm{Cap})$. Then following [**56**] we can define the trace operator $\mathrm{tr}_E : W^{1,2}(\mathrm{X}) \to L^0(\mathrm{Per}(E))$ as $\mathrm{tr}_E := \pi_E \circ \mathsf{QCR}$.

Theorem 1.5.3 (Tangent module over ∂E, [**56**])**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space. Let $E \subset \mathrm{X}$ be a set of finite perimeter. Then there exists a unique couple $\big(L^2_E(T\mathrm{X}), \bar\nabla\big)$ – where $L^2_E(T\mathrm{X})$ is an Hilbert $L^2(|D\chi_E|)$-normed $L^\infty(|D\chi_E|)$-module and $\bar\nabla : \mathrm{Test}(X) \to L^2_E(T\mathrm{X})$ is linear – such that:*

i) *The equality $|\bar\nabla f| = \mathrm{tr}_E(|\nabla f|)$ holds $|D\chi_E|$-a.e. for every $f \in \mathrm{Test}(\mathrm{X})$.*
ii) *$\big\{\sum_{i=1}^n \chi_{E_i}\bar\nabla f_i \;\big|\; (E_i)_{i=1}^n$ Borel partition of X, $(f_i)_{i=1}^n \subset \mathrm{Test}(X)\big\}$ is dense in $L^2_E(T\mathrm{X})$.*

Finally $L^2_E(TX)$ can be identified as the quotient $\{v \in L^0_{\mathrm{Cap}}(TX) \;:\; |v| \in L^2(|D\chi_E|)\}/\sim_{\mathrm{Per}(E)}$ defined via the equivalent relation $v \sim_{\mathrm{Per}(E)} w$ if and only if $|v - w| = 0$ $\mathrm{Per}(E)$-a.e., which inherits a natural structure of Hilbert $L^2(\mathrm{Per}(E))$-normed $L^\infty(\mathrm{Per}(E))$-module.

We are now able to define the trace of a vector field, that with a slight abuse of notation we denote by $\mathrm{tr}_E : W^{1,2}_C \cap L^\infty(T\mathrm{X}) \to L^2_E(T\mathrm{X})$ and which is defined by

$$\mathrm{tr}_E := \bar{\pi}_E \circ \mathsf{Q}\bar{\mathsf{C}}\mathsf{R},$$

where $\bar{\pi}_E : L^0_{\mathrm{Cap}}(T\mathrm{X}) \to L^2_E(T\mathrm{X})$ is the quotient map of the equivalent relation $\sim_{\mathrm{Per}(E)}$ as above. Notice that since $\mathsf{Q}\bar{\mathsf{C}}\mathsf{R}$ is linear then tr_E is linear as well.

Since $|\bar{\pi}_E(v)| = \pi_E(|v|)$ and $|\mathsf{Q}\bar{\mathsf{C}}\mathsf{R}(v)| = \mathsf{QCR}(|v|)$, we can verify that $|\mathrm{tr}_E(v)| = \mathrm{tr}_E(|v|)$ for every $v \in W^{1,2}_C \cap L^\infty(T\mathrm{X})$. From [**56**, Lemma 2.7] and the results in [**99**] we can deduce the following density result.

Proposition 1.5.4. *There exists a countable set* $\mathcal{V} \subset H^{1,2}_C \cap L^\infty(T\mathrm{X}) \cap \mathsf{D}_{L^2}(\mathrm{div})$ *of vector fields with bounded support, such that* $\mathrm{tr}_E(\mathcal{V})$ *is dense in* $L^2_E(T\mathrm{X})$ *for every* E *of finite perimeter.*

Proof. It is enough to show the statement without the bounded support-requirement, indeed the desired result would then follow immediately with a cut-off argument with test cut-off functions (recall Proposition 1.4.9).

We start observing that $\mathrm{TestV}(\mathrm{X}) \subset H^{1,2}_C \cap L^\infty(T\mathrm{X}) \cap \mathsf{D}_{L^2}(\mathrm{div})$. In particular

$$\mathrm{TestV}(\mathrm{X}) \subset \cup_{m\in\mathbb{N}} \mathrm{TestV}_m(\mathrm{X}),$$

where $\mathrm{TestV}_m(\mathrm{X}) := \{v \in \mathrm{TestV}(\mathrm{X}) \ : \ |v| \le m \, \mathfrak{m}\text{-a.e.}\}$. Moreover the space $H^{1,2}_C(T\mathrm{X})$ is separable (recall 1.4.3) and by its very definition $\mathrm{TestV}(\mathrm{X}) \subset H^{1,2}_C(T\mathrm{X})$, hence the sets $\mathrm{TestV}_m(\mathrm{X})$ are separable as well with respect to the $\|.\|_{W^{1,2}_C(T\mathrm{X})}$-norm, for every m. We consider $\mathcal{V}_m \subset \mathrm{TestV}_m(\mathrm{X})$ dense in $\mathrm{TestV}_m(\mathrm{X})$ with respect to the $\|.\|_{W^{1,2}_C(T\mathrm{X})}$-norm. We claim that $\mathcal{V} := \cup_{m\in\mathbb{N}}\mathcal{V}_m$ satisfies the desired requirements. From [**56**, Lemma 2.7] we know that $\mathrm{tr}_E(\mathrm{TestV}(\mathrm{X})) = \cup_m \mathrm{tr}_E(\mathcal{V}_m)$ is dense in $L^2_E(TX)$. The conclusion would then follow if we were able to show that tr_E is continuous on $\mathrm{TestV}_m(\mathrm{X})$ for every m. To see this take $v_n \to v$ in $W^{1,2}_C(T\mathrm{X})$ with $v_n, v \in \mathrm{TestV}_m(\mathrm{X})$. Then from (1.4.16) we have that $|v_n - v|^2 \in W^{1,2}(\mathrm{X})$ with $|\nabla|v_n - v|^2| \le 2|v_n - v||\nabla(v_n - v)| \le 4m|\nabla(v_n - v)| \to 0$ in $L^2(\mathfrak{m})$. Therefore $|v_n - v| \to 0$ in $W^{1,2}(\mathrm{X})$ and from (1.2.15) we know that every subsequence n_k has a further non-relabelled subsequence such that $|v_{n_k} - v|^2 \to 0$ Cap-a.e. and in particular also such that $|\mathrm{tr}_E(v_{n_k} - v)| = \mathrm{tr}_E(|v_{n_k} - v|) \to 0$ $\mathrm{Per}(E)$-a.e.. By dominated convergence we deduce that $\mathrm{tr}_E(v_{n_k}) \to \mathrm{tr}_E(v)$ in $L^2_E(T\mathrm{X})$ and from the arbitrariness of the initial subsequence we obtain also that $\mathrm{tr}_E(v_n) \to \mathrm{tr}_E(v)$ in $L^2_E(T\mathrm{X})$, which concludes the proof. □

Recall that, being $L^2_E(T\mathrm{X})$ an Hilbert module, it admits a scalar product $\langle , \rangle : L^2_E(T\mathrm{X}) \times L^2_E(T\mathrm{X}) \to L^1(\mathrm{Per}(E))$ given by

$$\langle v, w \rangle_{L^2_E(T\mathrm{X})} := \frac{1}{2}\left(|v + w|^2 - |v|^2 - |w|^2\right), \quad \forall\, v, w \in L^2_E(T\mathrm{X}).$$

In particular from the above discussion and the linearity of the trace we deduce that

$$\langle \mathrm{tr}_E v, \mathrm{tr}_E w \rangle_{L^2_E(T\mathrm{X})} = \mathrm{tr}_E(\langle v, w \rangle), \quad \forall v, w \in H^{1,2}_C \cap L^\infty(T\mathrm{X}). \tag{1.5.1}$$

Remarkably in [**56**] it has been shown that for a set of finite perimeter it is well defined a notion of exterior normal, which belongs to $L^2_E(T\mathrm{X})$ and is characterized by the validity of a Gauss-Green formula (see also [**59**]):

Theorem 1.5.5. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and let* $E \subset \mathrm{X}$ *be of finite perimeter and* $\mathfrak{m}(E) < +\infty$. *Then there exists a unique vector field* $\nu_E \in L^2_E(T\mathrm{X})$ *such that* $|\nu_E| = 1$ $\mathrm{Per}(E)$*-a.e. and*

$$\int \langle \mathrm{tr}_E(v), \nu_E \rangle \, \mathrm{dPer}(E) = \int_E \mathrm{div}(v) \, \mathrm{d}\mathfrak{m}, \quad \forall v \in H^{1,2}_C \cap L^\infty(T\mathrm{X}) \cap \mathsf{D}_{L^2}(\mathrm{div}). \tag{1.5.2}$$

Moreover the same is true if $\mathfrak{m}(E^c) < +\infty$, *but assuming in* (1.5.2) *that* v *has also bounded support.*

Proof. The first part is exactly [**56**, Theorem 2.2], hence we assume that $\mathfrak{m}(E^c) < +\infty$. In particular since E^c has also finite perimeter, (from the finite measure case) there exists $\nu_{E^c} \in L^2_{E^c}(T\mathrm{X})$. We claim that $\nu_E := -\nu_{E^c}$ has the desired properties. First we observe that ν_E is in the right space, indeed $L^2_{E^c}(T\mathrm{X}) = L^2_E(T\mathrm{X})$, which is immediate from the fact that $\mathrm{Per}(E) = \mathrm{Per}(E^c)$ (as measures) and by how $L^2_E(T\mathrm{X})$ is defined (see the second part of Theorem 1.5.3). To verify (1.5.2) we first observe that for every $v \in L^2(T\mathrm{X}) \cap \mathsf{D}_{L^2}(\mathrm{div})$ with bounded support it holds $\int_\mathrm{X} \mathrm{div}(v)\, \mathrm{d}\mathfrak{m} = 0$. Therefore from (1.5.2) for E^c we have

$$\int_E \mathrm{div}(v)\, \mathrm{d}\mathfrak{m} = -\int_{E^c} \mathrm{div}(v)\, \mathrm{d}\mathfrak{m} = -\int \langle \mathrm{tr}_E(v), \nu_{E^c} \rangle \, \mathrm{dPer}(E) = \int \langle \mathrm{tr}_E(v), \nu_E \rangle \, \mathrm{dPer}(E),$$

for every $v \in H^{1,2}_C \cap L^\infty(T\mathrm{X}) \cap \mathsf{D}_{L^2}(\mathrm{div})$ with bounded support. This shows that (1.5.2) holds also for E. The uniqueness of ν_E follows immediately from the density result in Proposition 1.5.4. □

1.6. pmGH-convergence and stability properties

1.6.1. Notion of pmGH-convergence. We recall the definition of *pointed-measure Gromov Hausdorff convergence* of pointed metric measure spaces. We refer to ([**60**], [**133**] [**208**] and [**124**]) for discussions on different notions of convergence of metric spaces and their relations.

Definition 1.6.1 (Pointed-measure Gromov Hausdorff convergence)**.** We say that the sequence $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$, $n \in \mathbb{N}$, of p.m.m.s. *pointed-measure Gromov Hausdorff*-converges (pmGH-converges in short) to the p.m.m.s. $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$, if there are sequences $R_n \uparrow +\infty$, $\varepsilon_n \downarrow 0$ and Borel maps $f_n : X_n \to X_\infty$ such that

1) $f_n(x_n) = x_\infty$,
2) $\sup_{x,y \in B_{R_n}(x_n)} |\mathsf{d}_n(x, y) - \mathsf{d}_\infty(f_n(x), f_n(y))| \le \varepsilon_n$,
3) the ε_n-neighbourhood of $f_n(B_{R_n}(x_n))$ contains $B_{R_n-\varepsilon_n}(x_\infty)$,
4) for any $\varphi \in C_{bs}(X_\infty)$ it holds $\lim\limits_{n\to\infty} \int \varphi \circ f_n \, \mathrm{d}\mathfrak{m}_n = \int \varphi \, \mathrm{d}\mathfrak{m}_\infty$.

In [**124**] it is shown that in the case of a sequence of uniformly locally doubling metric measure spaces the pmGH-convergence is equivalent to the so-called *pointed-measure Gromov convergence* defined as follows:

Definition 1.6.2 (Pointed-measure Gromov convergence, [**124**])**.** We say that the sequence $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$, $n \in \mathbb{N}$, of p.m.m.s. *pointed-measure Gromov*-converges (pmG-converges in short) to the p.m.m.s. $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$, if there exist isometric embeddings $\iota_n : \mathrm{X}_n \to (Z, \mathsf{d})$, $n \in \mathbb{N}$, $\iota_\infty : \mathrm{X}_\infty \to (Z, \mathsf{d})$ into a common metric space (Z, d) such that

$$\iota_{n*}\mathfrak{m}_n \rightharpoonup \iota_{\infty*}\mathfrak{m}_\infty \text{ in duality with } C_{bs}(Z) \text{ and } \iota_n(x_n) \to \iota_\infty(x_\infty).$$

This equivalence allows in the case of uniformly doubling metric measure spaces to adopt the to so-called *extrinsic approach* (see [**124**]), where the pmGH-convergence is realized by a proper metric space (Y, d) where X_n, $n \in \bar{\mathbb{N}}$, are identified as subsets of Z, with $\mathrm{supp}(\mathfrak{m}_n) = \mathrm{X}_n$, $n \in \bar{n}$, and so that $\mathsf{d}_{\mathrm{Z}}|_{\mathrm{X}_n \times \mathrm{X}_n} = \mathsf{d}_n$, $n \in \bar{\mathbb{N}}$, $x_n \to x_\infty$ in Z and $\mathfrak{m}_n \rightharpoonup \mathfrak{m}_\infty$ in duality with $C_{bs}(\mathrm{Z})$. Moreover (under the uniformly locally doubling assumption) this realization can be taken so that $\mathsf{d}_H^{\mathrm{Z}}(B_R^{\mathrm{X}_n}(x_n), B_R^{\mathrm{X}_\infty}(x_\infty)) \to 0$ for every $R > 0$, where $\mathsf{d}_H^{\mathrm{Z}}$ is the Hausdorff distance in Z.

It is proven in [**124**] that there exists a distance d_{pmGH} that metrizes the pmGH-convergence for the class of $\mathrm{CD}(K, N)$ (or $\mathrm{RCD}(K, N)$) spaces with $K \in \mathbb{R}$ and $N < +\infty$ fixed, and more generally for every family of uniformly locally doubling metric measure spaces.

After the works in [**200**],[**201**], [**173**], [**124**], [**21**], [**114**] and in view of the Gromov compactness theorem [**133**, Sec. 5.A], the following fundamental compactness and stability result holds:

Theorem 1.6.3. *Let* $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ *be a sequence of pointed* $\mathrm{CD}(K_n, N_n)$ *(resp.* $\mathrm{RCD}(K_n, N_n)$*) spaces, with* $\mathfrak{m}(B_1(x_n)) \in [v^{-1}, v]$*, for* $v > 1$ *and* $K_n \to K \in \mathbb{R}, N_n \to N \in [1, \infty)$*. Then, there exists a subsequence* (n_k) *and a pointed* $\mathrm{RCD}(K, N)$ *(resp.* $\mathrm{RCD}(K, N)$*) space* $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$ *satisfying*

$$\lim_{k\to\infty} \mathsf{d}_{pmGH}\big((\mathrm{X}_{n_k}, \mathsf{d}_{n_k}, \mathfrak{m}_{n_k}, x_{n_k}), (\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)\big) = 0.$$

When dealing with a uniformly bounded compact metric spaces the pmGH-convergence can be reduced to measure-Gromov Hausdorff convergence, where we ignore the convergence of the base points.

Definition 1.6.4 (*Measure Gromov Hausdorff convergence*)**.** We say that the sequence $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$ of m.m.s. *measure Gromov Hausdorff*-converges (mGH-converges in short) to a compact m.m.s. $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$, if there are Borel maps $f_n : X_n \to X_\infty$ such that

1) $\sup_{x,y \in B_{R_n}(x_n)} |\mathsf{d}_n(x, y) - \mathsf{d}_\infty(f_n(x), f_n(y))| \le \varepsilon_n$,
2) $f_n(\mathrm{X}_n)$ is ε_n-dense in X_∞,
3) $f_{n*}\mathfrak{m}_n \rightharpoonup \mathfrak{m}_\infty$ in duality with $C(\mathrm{X}_\infty)$.

As in the non-compact case there exists a distance d_{mGH} which metrizes the mGH-convergence (see [**124, 200**]) on every family of uniformly doubling and uniformly bounded metric spaces. Moreover the implication

$$(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n) \overset{mGH}{\to} (\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty) \implies \mathsf{d}_{mGH}((\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n), (\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)) \to 0,$$

holds without any doubling assumptions.

Remark 1.6.5. For the case of uniformly doubling and uniformly bounded metric measure spaces, the mGH-convergence can also be characterized with the extrinsic approach exactly as above, but with the exception that the common ambient space (Z, d) can be also taken to be compact. ■

We end recalling the definitions of Gromov-Hausdorff distance and convergence. For any couple of compact metric spaces (X_1, d_1), (X_2, d_2) we define their *Gromov Hausdorff distance* as

$$\mathsf{d}_{GH}((X_1, \mathsf{d}_2), (\mathrm{X}_2, \mathsf{d}_2)) := \inf\{\varepsilon > 0 : \exists f : X_1 \to X_2 \text{ such that } f(\mathrm{X}_1) \text{ is } \varepsilon\text{-dense in } \mathrm{X}_2 \\ \text{and } |\mathsf{d}_1(x, y) - \mathsf{d}_2(f(x), f(y))| \le \varepsilon,\ \forall x, y \in X_1\}.$$

For a sequence of compact metric spaces (X_n, d_n) converging to (X, d) in the GH-sense we say that a sequence of maps $f_n : \mathrm{X}_n \to \mathrm{X}$ realizes the convergence if there exists a sequence $\varepsilon_n \to 0$ such that $f(\mathrm{X}_n)$ is ε_n-dense in X and $|\mathsf{d}_n(x, y) - \mathsf{d}(f_n(x), f_n(y))| \le \varepsilon_n$, $\forall x, y \in X_n$.

Note also that mGH-convergence always implies GH-convergence.

1.6.2. Convergence of functions under pmGH-convergence. We now recall some stability and convergence results for functions along pmGH-convergence. For additional details and analogous results we refer to [**23**, **124**] and also [**18**, Sec 5.2].

In this subsection $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$, $n \in \mathbb{N}$, is a sequence of $\mathrm{CD}(K, N)$ spaces, $N < +\infty$, pmGH-converging to a $\mathrm{CD}(K, N)$ space $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$ and (Z, d) is a proper metric space which realizes such convergence through the extrinsic approach (see the previous subsection).

We start defining the basic notion of pointwise and L^p convergence.

Definition 1.6.6 (Locally uniform / uniform convergence)**.** Let $f_n : \mathrm{X}_n \to \mathbb{R}$ $n \in \bar{\mathbb{N}}$. We say that f_n converges *locally uniformly* to f_∞ if for every $y \in \mathrm{X}_\infty$ and every sequence $y_n \in \mathrm{X}_n$ such that $\mathsf{d}_Y(y_n, y) \to 0$ it holds that $\lim_n f_n(y_n) = f_\infty(y)$. We say that f_n *converges uniformly* to f_∞ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f_n(y_n) - f_\infty(y)| < \varepsilon$ for every $n \ge \delta^{-1}$ and every y_n such that $\mathsf{d}_Y(y_n, y) < \delta$.

We point out that in the case of a fixed proper metric space the two notions of convergence in the above definition coincide respectively with the usual locally uniform and uniform convergence.

The following is a version of the Ascoli-Arzelà theorem for varying metric spaces (see also [**208**, Prop. 27.20]). The proof can be achieved arguing as in case of a fixed (proper) metric metric space and we will skip it.

Proposition 1.6.7. *Let $f_i : \mathrm{X}_n \to \mathbb{R}$ be equi-Lipschitz, equibounded functions with* $\operatorname{supp} f_n \subset B_R(x_n)$*, for some $R > 0$ fixed, then there exists a subsequence that converges uniformly to a Lipschitz function $f : \mathrm{X}_\infty \to \mathbb{R}$.*

Definition 1.6.8. Let $p \in (1, \infty)$, we say that

(i) $f_n \in L^p(\mathfrak{m}_n)$ *converges L^p-weak* (or weakly in L^p) to $f_\infty \in L^p(\mathfrak{m}_\infty)$, provided $\sup_{n\in\mathbb{N}} \|f_n\|_{L^p(\mathfrak{m}_n)} < \infty$ and $f_n\mathfrak{m}_n \rightharpoonup f_\infty\mathfrak{m}_\infty$ in $C_{bs}(Z)$,

(ii) $f_n \in L^p(\mathfrak{m}_n)$ *converges L^p-strong* (or strongly in L^p) to $f_\infty \in L^p(\mathfrak{m}_\infty)$, provided it converges L^p-weak and
$$\sup_n \|f_n\|_{L^p(\mathfrak{m}_n)} \le \|f_\infty\|_{L^p(\mathfrak{m}_\infty)},$$

(iii) $f_n \in W^{1,2}(\mathrm{X}_n)$ *converges $W^{1,2}$-weak* to $f_\infty \in W^{1,2}(\mathrm{X})$ provided it converges L^2-weak and
$$\sup_{n\in\mathbb{N}} \||Df_n|\|_{L^2(\mathfrak{m}_n)} < \infty,$$

(iv) $f_n \in W^{1,2}(\mathrm{X}_n)$ *converges $W^{1,2}$-strong* to $f_\infty \in W^{1,2}(\mathrm{X})$ provided it converges L^2-strong and $\||Df_n|\|_{L^2(\mathfrak{m}_n)} \to \||Df_\infty|\|_{L^2(\mathfrak{m}_\infty)}$,

Remark 1.6.9. From the results in [**18**, Sec. 5.2] (see also [**23**, Remark 3.2]) we know that the strong convergence in L^p for $p > 1$ can be equivalently characterized by
$$(\mathsf{id} \times f_n)_*\mathfrak{m}_n \rightharpoonup (\mathsf{id} \times f_\infty)_*\mathfrak{m}_\infty$$
in duality with $C_p(Z \times \mathbb{R}) := \{\psi \in C(Z \times \mathbb{R}) \ : \psi(z, t) \le C|t|^p\}$. ■

Moreover, we say that f_n is uniformly bounded in L^p if $\sup_n \|f_n\|_{L^p(\mathfrak{m}_n)} < \infty$. We start with a list of useful properties of L^p-convergence.

Proposition 1.6.10 (Properties of L^p-convergence)**.** *For all $p \in (1, \infty)$, it holds*

(i) If f_n converges L^p-strong to f_∞, then $\varphi(f_n)$ converges L^p-strong to $\varphi(f_\infty)$ for every $\varphi \in \mathsf{LIP}(\mathbb{R})$ with $\varphi(0) = 0$, and if f_n has uniformly bounded support also for every $\varphi \in C(\mathbb{R})$ such that $|\varphi(t)| \le C(1 + |t|)$, for some positive constant $C > 0$,

(ii) If f_n (resp. g_n) converges L^p-strong to f_∞ (resp. g_∞), then $f_n + g_n$ converge L^p-strong to $f_\infty + g_\infty$,

(iii) if f_n converges L^p-weak to f, then $\|f_\infty\|_{L^p(\mathfrak{m}_\infty)} \le \varliminf_n \|f_n\|_{L^p(\mathfrak{m}_n)}$,

(iv) *suppose that* $\sup_n \|f_n\|_{L^p(\mathfrak{m}_n)} < +\infty$, *then up to a subsequence* f_n *converges* L^p*-weak to some* $f_\infty \in L^p(\mathfrak{m}_\infty)$,

(v) *If* f_n *converges* L^p*-strong (resp.* L^p*-weak) to* f_∞, *then* φf_n *converges* L^p*-strong (resp.* L^p*-weak) to* φf_∞, *for all* $\varphi \in C_b(Z)$,

(vi) *If* $f_n, g_n \in L^2(\mathfrak{m}_n)$ *are uniformly bounded in* $L^\infty(\mathfrak{m}_n)$ *and converge strongly in* L^2 *respectively to* $f, g \in L^2(\mathfrak{m}_\infty)$, *then* $f_n g_n$ *converge strongly in* L^2 *to* fg,

(vii) *for every* $f \in L^p(\mathfrak{m}_\infty)$ *there exists a sequence* $f_n \in L^p(\mathfrak{m}_n)$ *converging* L^p*-strong to* f,

(viii) *if* f_n *are non-negative and converges in* L^p *strong to* f, *then for every* $q \in (1,\infty)$, $f_n^{p/q}$ *converge* L^q*-strong to* $f^{p/q}$,

(ix) *Fix* $p, q \in (1,\infty]$ *so that* $p < q$. *If the sequence* (f_n) *is uniformly bounded in* L^q *and converges* L^p*-strong to* f_∞, *then it converges also* L^r*-strong to* f_∞ *for every* $r \in [p,q)$,

(x) *if* $f_n : \mathrm{X}_n \to \mathbb{R}$, *with* $\operatorname{supp} f_n \subset B_R^{\mathrm{X}_n}(x_n)$, *converge uniformly to a bounded function* $f : \mathrm{X}_\infty \to \mathbb{R}$ *then* f_n *converge in* L^2*-strong to* f.

Proof. A proof for (ii) to (vi) and the first part of (i) can be found in [**23**, Prop. 3.3]. For the second part of (i) from Remark 1.6.9 it enough to show that

$$\int \psi(x, \varphi(f_n(x)) \,\mathrm{d}\mathfrak{m}_n(x) \to \int \psi(x, \varphi(f_\infty(x)) \,\mathrm{d}\mathfrak{m}_\infty(x), \quad \forall \psi \in C_p(Z \times \mathbb{R}).$$

Take $\eta \in C_{bs}(Z)$ such that $\eta = 1$ on all the supports of f_n $n \in \bar{\mathbb{N}}$ and notice that $\psi(x, \varphi(u_n(x))) = \eta\psi(z, \varphi_n(z))$. Moreover $\eta(z)\psi(z, \varphi(t)) \in C(Z \times \mathbb{R})$ and satisfies $|\eta(z)\psi(z,\varphi(t))| \le \tilde{C}\eta((1 + C|t|)^p) \le c_p\eta + \tilde{c}_p|t|^p$. The conclusion now follows from [**124**, (6.6)]. (vii) can instead be found in [**124**]. $(viii)$ follows immediately from Remark 1.6.9. For (ix), the case $q = \infty$ follows immediately from item (i) (see also [**23**, e) of Prop. 3.3]), hence we can assume $q < +\infty$. Fix $r \in [p, q)$. Clearly from the Hölder inequality f_n is uniformly bounded in L^r, hence by definition f_n converges L^r-weakly to f_∞. Moreover from item $iii)$ we known that $f_\infty \in L^r(\mathfrak{m}_\infty)$, therefore by truncation and diagonalization we can suppose that $f_\infty \in L^\infty(\mathfrak{m}_\infty)$. From (vii) then there exists a sequence $g_n \in L^r(\mathfrak{m}_n)$ converging to f in L^r-strong and by item $i)$ we can also assume that g_n are uniformly bounded in L^∞. Then, from $(viii)$ in the case $q = \infty$ we have that g_n converge also in L^p-strong to f_∞. Then by (ii) we have that $g_n - f_n$ converges to 0 in L^p-strong and in particular $\|f_n - g_n\|_{L^p(\mathfrak{m}_n)} \to 0$. Finally by the Hölder inequality (since f_n, g_n are both uniformly bounded in L^q) we have that $\|f_n - g_n\|_{L^r(\mathfrak{m}_n)} \to 0$. In particular $\lim_n \|f_n\|_{L^r(\mathfrak{m}_n)} = \lim_n \|g_n\|_{L^r(\mathfrak{m}_n)} = \|f_\infty\|_{L^r(\mathfrak{m}_\infty)}$, which concludes the proof. For (x) we first observe that from the properness of Y it follows that the functions f_n are equibounded, moreover they have uniform bounded support by hypothesis. Therefore we can apply the generalized version of Fatou lemma for varying measure (see e.g. [**96**, Lemma 2.5]), first to $f_n\varphi$ and then to $-f_n\varphi$ to obtain that $\int f_n\varphi \,\mathrm{d}\mathfrak{m}_n \to \int f\varphi \,\mathrm{d}\mathfrak{m}$, where φ is an arbitrary function in $C_{bs}(Y)$. Applying the same lemma also the functions $f_n^2, -f_n^2$ we deduce that $\int f_n^2 \,\mathrm{d}\mathfrak{m}_n \to \int f^2 \,\mathrm{d}\mathfrak{m}$, concluding the proof. □

We now pass to some convergence and stability results related to Sobolev spaces. We start with the following generalized version of the compact embedding of $W^{1,2} \hookrightarrow L^2$:

Proposition 1.6.11 ([**124**]). *Suppose that* $f_n \in W^{1,2}(\mathrm{X}_n)$ *satisfies*

$$\sup_{n\in\mathbb{N}} \Big(\int |f_n|^2 \,\mathrm{d}\mathfrak{m}_n + \int |Df_n|^2 \,\mathrm{d}\mathfrak{m}_n \Big) < \infty,$$

$$\lim_{R\to+\infty} \sup_n \int_{\mathrm{X}_n \setminus B_R(x_n)} |f|^2 \,\mathrm{d}\mathfrak{m}_n = 0.$$

Then (f_n) *has a* L^2*-strongly convergent subsequence.*

Next we recall the Γ-convergences of the 2-Cheeger energies proven in [**124**]:

- (Γ-$\underline{\lim}$): for every $f_n \in L^2(\mathfrak{m}_n)$ L^2-strong converging to $f_\infty \in L^2(\mathfrak{m}_\infty)$, it holds
$$\int |Df_\infty|^2 \,\mathrm{d}\mathfrak{m}_\infty \le \varliminf_{n\to\infty} \int |Df_n|^2 \,\mathrm{d}\mathfrak{m}_n;$$
- (Γ-$\overline{\lim}$): for every $f_\infty \in L^2(\mathfrak{m}_\infty)$, there exists a sequence $f_n \in L^2(\mathfrak{m}_n)$ converging L^2-strong to f_∞ so that
$$\varlimsup_{n\to\infty} \int |Df_n|^2 \,\mathrm{d}\mathfrak{m}_n \le \int |Df_\infty|^2 \,\mathrm{d}\mathfrak{m}_\infty.$$

We will also need the (Γ-$\overline{\lim}$) inequality for the p-Cheeger energies as proved in [**23**, Theorem 8.1]: for every $p \in (1,\infty)$ and every $f_\infty \in L^p(\mathfrak{m}_\infty)$, there exists $f_n \in L^p(\mathfrak{m}_n)$ converging L^p-strong to f_∞ so that

$$\overline{\lim_{n\to\infty}} \int |Df_n|^p \,\mathrm{d}\mathfrak{m}_n \le \int |Df_\infty|^p \,\mathrm{d}\mathfrak{m}_\infty.$$

The above is stated in [**23**] only for $\mathrm{RCD}(K,\infty)$ spaces, but it easily seen that the proof works without modification also in the case of $\mathrm{CD}(K,\infty)$ spaces.

In the reminder of this subsection we make the additional assumption that the spaces X_n $n \in \bar{\mathbb{N}}$ satisfies the $\mathrm{RCD}(K,N)$ condition. First we recall the continuity result of the spectral gap (see [**124**]): if X_n, $n \in \bar{\mathbb{N}}$, are all compact it holds

$$\lambda^{1,2}(\mathrm{X}_\infty) = \lim_{n\to\infty} \lambda^{1,2}(\mathrm{X}_n). \tag{1.6.1}$$

Remark 1.6.12. The continuity of the spectral gap was previously obtained in the setting of Ricci-limit spaces by Cheeger and Colding [**75**]. ■

We conclude with the following results about stability of Laplacian and gradient with respect to strong L^2 convergence. Here Δ_n (resp. Δ_∞) represents the Laplacian operator in X_n (resp. X_∞) and ∇_n (resp. ∇_∞) represents the gradient operator in X_n (resp. X_∞) (recall that we are assuming X_n $n \in \bar{\mathbb{N}}$ to be $\mathrm{RCD}(K,N)$ spaces).

Theorem 1.6.13 ([**24**, Theorem 2.7, Theorem 2.8])**.** *Let $f_n \in D(\Delta_n)$ be such that*

$$\sup_n \|f_n\|_{L^2(\mathfrak{m}_n)} + \|\Delta_n f_n\|_{L^2(\mathfrak{m}_n)} < +\infty$$

and assume that f_n converge strongly in L^2 to f. Then $f \in D(\Delta_\infty)$, $\Delta_n f \to \Delta_\infty f$ weakly in L^2 and $|\nabla_n f_n| \to |\nabla_\infty f|$ strongly in L^2.

Lemma 1.6.14 ([**23**, Lemma 5.8])**.** *Let $f_n \in W^{1,2}(\mathrm{X}_n)$ be such that $\sup_n \||\nabla f_n|\|_{L^2(\mathfrak{m}_n)} < +\infty$ and converging strongly in L^2 to $f \in W^{1,2}(\mathrm{X}_\infty)$, then for any non-negative $\varphi \in C_b(\mathrm{Z})$*

$$\int \varphi |\nabla_\infty f|^2 \,\mathrm{d}\mathfrak{m}_\infty \le \underline{\lim}_n \int \varphi |\nabla_n f|^2 \,\mathrm{d}\mathfrak{m}_n.$$

Additionally for any $A \subset \mathrm{X}_\infty$ open it holds

$$\int_A |\nabla_\infty f|^2 \,\mathrm{d}\mathfrak{m}_\infty \le \underline{\lim}_n \int_A |\nabla_n f|^2 \,\mathrm{d}\mathfrak{m}_n.$$

1.7. Regular Lagrangian flows

Here we recall the basics of the theory of flows for Sobolev vector fields in metric measure spaces. In the Euclidean setting this theory has been firstly developed by Di Perna and Lions [**103**] and subsequently by Ambrosio [**10**]. In the work of Ambrosio and Trevisan ([**28**]) this theory was extended to very general metric measure spaces and in particular for $\mathrm{RCD}(K,\infty)$ spaces. We restate here their main results about existence and uniqueness using the language introduced in [**118**] and reviewed in Section 1.4.

Definition 1.7.1. Let $v_t : [0,T] \to L^2(T\mathrm{X})$ be Borel, we say that a map $F : [0,T] \times X \to X$ is a Regular Lagrangian Flow associated to v_t if the following are satisfied:

(1) There exists $C > 0$ such that

$$F_{t*}\mathfrak{m} \le C\mathfrak{m}, \quad \text{for every } t \in [0,T], \tag{1.7.1}$$

(2) for $\mathfrak{m}$-a.e. $x \in X$ the function $[0,T] \ni t \mapsto F_t(x)$ is continuous and satisfies $F_0(x) = x$.

(3) for every $f \in \mathsf{Test}(X)$ it holds that for $\mathfrak{m}$-a.e. $x \in X$ the function $(0,T) \ni t \mapsto f \circ F_t(x)$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ F_t(x) = \langle \nabla f, v_t \rangle \circ F_t(x), \quad \text{for a.e. } t \in (0,T). \tag{1.7.2}$$

Notice that in (1.7.2) we are implicitly choosing for every $t \in (0,T)$ a Borel representative of $\langle \nabla f, v_t \rangle$, however (1.7.1) ensures that the validity of item 3 in Definition 1.7.1 is independent of this choice.

Observe also that in Definition 1.7.1 we are assuming that the map F is pointwise defined, however the definition is stable under modification in a negligible set of trajectories in the following sense. If $F_t(x)$ is a Regular Lagrangian Flow for v_t (as in Definition 1.7.1) and for $\mathfrak{m}$-a.e. x, $\tilde{F}_t(x) = F_t(x)$ holds for every $t \in [0,T]$, for some map $\tilde{F} : [0,T] \times X \to X$ then $\tilde{F}$ is also a Regular Lagrangian Flow for v_t. In any case to avoid technical issues, in our discussion we prefer to fix a pointwise defined representative for the flow map F.

Remark 1.7.2. If F_t is a regular Lagrangian flow for a vector field v_t, then for $\mathfrak{m}$-a.e. x it holds that the curve $[0,T] \ni t \mapsto F_t(x)$ is absolutely continuous and its metric speed is given by

$$|\dot{F_t(x)}| = |v_t| \circ F_t(x), \quad \text{a.e. } t \in [0,T]. \tag{1.7.3}$$

This follows from [**28**, Lemma 7.4 and 9.2]. Observe that this statement is independent of the chosen representative of $|v_t|$, thanks to (1.7.1). ■

We will see below in Theorem 1.7.4 that the existence and uniqueness of a Regular Lagrangian Flow is linked to the existence and uniqueness of a solution to the continuity equation ([**119**]):

Definition 1.7.3. Let $v_t : [0,T] \to L^2(T\mathrm{X})$ be Borel and $t \mapsto \mu_t \in \mathscr{P}(\mathrm{X})$, $t \in [0,T]$ be also a Borel map. Suppose also that $\||v_t|\|_{L^2(\mathfrak{m})} \in L^1(0,T)$ and $\mu_t \le C\mathfrak{m}$ for every $t \in [0,T]$ and some positive constant C. We say that μ_t with is a weak solution of the continuity equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t + \mathrm{div}(v_t\mu_t) = 0,$$

with initial datum μ_0, if for every $f \in \mathrm{Lip}_{bs}(\mathrm{X})$ the function $[0,T] \ni t \mapsto \int f \,\mathrm{d}\mu_t$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f \,\mathrm{d}\mu_t = \int \langle \nabla f, v_t\rangle \,\mathrm{d}\mu_t \qquad \text{for } a.e.\, t \in (0,T). \tag{1.7.4}$$

Recall Section 1.4.3 for the definition of the space $W^{1,2}_C(T\mathrm{X})$ while we refer to[**118**, Sec. 3.4] for the the symmetric part of the covariant derivative $\nabla_{sym} v \in L^2(T^{2\otimes}\mathrm{X})$. For our purposes it sufficient to know that for any $f \in \mathsf{Test}(\mathrm{X})$ we have $\nabla f \in W^{1,2}_C(T\mathrm{X})$, with $\nabla_{sym}\nabla f = \nabla(\nabla f) = \mathrm{Hess}(f)^\sharp$ (see Proposition 1.4.13 or [**118**, Sec. 3.2]).

Theorem 1.7.4 ([**28**]). *Let $v_t : [0,T] \to L^2(T\mathrm{X})$ be Borel and such that $v_t \in D(\mathrm{div})$ for every $t \in [0,T]$. Assume furthermore that $\||v_t|\|_{L^2(\mathfrak{m})} \in L^1(0,T)$, $\|\mathrm{div}(v_t)^-\|_{L^\infty} \in L^\infty(0,T)$ and $\|\nabla_{sym} v_t\|_{L^2(T^{\otimes 2}X)} \in L^1(0,T)$. Then*

(1) there exists a unique Regular Lagrangian flow F_t associated to v_t,
(2) for every initial datum $\mu_0 \in \mathscr{P}(\mathrm{X})$ with $\mu_0 \le C\mathfrak{m}$ there exists a unique weak solution μ_t to the continuity equation and it is given given by $\mu_t := F_{t*}\mu_0$.

We remark that the uniqueness of the Regular Lagrangian Flow in Theorem 1.7.4 has to be intended up to modification in a negligible set of trajectories, as discussed above.

Remark 1.7.5. Let v_t be as in Theorem 1.7.4 and autonomous, i.e. $v_t \equiv v$ for some $v \in D(\mathrm{div}) \cap W^{1,2}_C(TX)$ with $\mathrm{div}(v)^- \in L^\infty(\mathfrak{m})$. Then, thanks to the uniqueness given by Theorem 1.7.4, the Lagrangian flow F_t relative to v can be extended uniquely (up to a set of negligible trajectories) to a map $F : [0,\infty) \times X \to X$ which satisfies the following group property

$$F_s \circ F_t = F_{s+t}, \quad \mathfrak{m}\text{-a.e.} \tag{1.7.5}$$

for every $s,t \in [0,\infty)$.

Moreover if also $\mathrm{div}(v) \in L^\infty(\mathfrak{m})$, it can be shown (see for example [**128**, Lemma 3.18]) that, denoting by F_t^{-v} the Lagrangian flow relative to $-v$ (which exists unique for all times $t \ge 0$, thanks to Theorem 1.7.4 and the previous observation)

$$F_t^{-v} \circ F_t = \mathsf{id}, \quad \mathfrak{m}\text{-a.e}, \tag{1.7.6}$$

for every $t \ge 0$. Hence setting $F_{-t} := F_t^{-v}$ we can extend F to $F : (-\infty,+\infty) \times X \to X$, for which (1.7.5) is satisfied for every $s,t \in \mathbb{R}$. ■

CHAPTER 2

Monotonicity formulas for the electrostatic potential on RCD spaces

Structure of the chapter

We will start in Section 2.1 with the definition of *non-parabolic* $\mathrm{RCD}(0, N)$ *spaces* and the discussion of some of their main properties, like the existence of the Green's function and their number of ends. Then in Section 2.2 we will prove the existence of the *relative electrostatic potential* on arbitrary $\mathrm{RCD}(K, N)$ spaces and then of the *global electrostatic potential* on non-parabolic $\mathrm{RCD}(0, N)$ spaces.

In Section 2.3 we will prove the main *differential inequality* (Corollary 2.3.5) which will lead to the monotonicity formula and will also prove some higher order regularity for harmonic function in a general $\mathrm{RCD}(K, N)$ space. Finally in Section 2.4 we will introduce the quantity U_β for an electrostatic potential and prove its *monotonicity.*

In the subsequent sections 2.5 and 2.6 we will deal with the rigidity and almost-rigidity of the monotonicity formula. The key point will be that if the derivative of U_β vanishes at some point, we obtain that (a power of) the electrostatic potential solves a precise PDE. In Section 2.5 we will study the geometric implications of the existence of a solution (or an almost-solution) to this PDE on an $\mathrm{RCD}(0, N)$ space. More precisely we obtain a *functional rigidity and almost-rigidity result*, that says that the space has respectively a conical or almost-conical structure. Some key parts of the argument for the rigidity will be presented in Section 2.5.1, while most of the details will be given in Appendix 2.A. The statement and proof of the almost-rigidity result are found instead in Section 2.5.2.

We will conclude with Section 2.6 where we derive from the previous rigidity and almost-rigidity results the corresponding ones for the monotonicity formula for U_β.

All the results that will be presented in this chapter are contained in [**132**].

2.1. Non-parabolic RCD spaces

In this section we introduce the notion of non-parabolic $\mathrm{RCD}(0, N)$ space, which will be the natural setting to study the electrostatic potential. Indeed already for smooth manifolds with non-negative Ricci curvature the existence of the electrostatic potential implies that the manifold is non-parabolic (see for example Theorem 2.3 in [**2**]).

We recall that a (non-compact) Riemannian manifold is said to be non-parabolic if it admits a positive global Green's function. It has been proved by Varopoulos ([**206**]) that in the case $\mathsf{Ric} \geq 0$ the non-parabolicity is equivalent to (2.1.1). This motivates the following definition.

Definition 2.1.1 (Non-parabolic RCD space)**.** Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0, N)$ space with $N < +\infty$. We say that X is *non-parabolic* if

$$\int_1^{+\infty} \frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s < +\infty, \qquad \text{for every } x \in X. \tag{2.1.1}$$

Observe that the above quantity is finite for one $x \in X$ if and only if it is finite for all $x \in X$.

We point out that condition (2.1.1) in the context of RCD spaces was already introduced in [**58**].

Remark 2.1.2. It follows immediately from the Bishop-Gromov inequality that if X is a non-parabolic $\mathrm{RCD}(0, N)$ space then it is non-compact and $N > 2$. ■

In the following two subsections we develop the two main features of a non-parabolic RCD space X, that we will be crucial in the forthcoming discussion: the existence of a Green's function vanishing at infinity and the fact that X has only one end (see Definition 2.1.5).

2.1.1. The Green's function. It turns out that on a non-parabolic RCD space it can be given a notion of positive global Green's function. Following [**58**] we define the Green's function $G : \mathrm{X} \times \mathrm{X} \to [0, +\infty]$ as

$$G(x, y) := \int_0^{\infty} p_t(x, y)\,\mathrm{d}\mathfrak{m}. \tag{2.1.2}$$

We also set $G_x(y) := G(x,y)$. For any $\varepsilon > 0$ we also define the quasi Green's function $G^\varepsilon : \mathrm{X}\times\mathrm{X} \to [0,+\infty]$ as

$$G^\varepsilon(x,y) := \int_\varepsilon^\infty p_t(x,y)\,\mathrm{d}\mathfrak{m} \tag{2.1.3}$$

and as above we set $G^\varepsilon_x(y) := G^\varepsilon(x,y)$. It is prove in [**58**, Lemma 2.5] that $G^\varepsilon_x \in \mathsf{LIP}(\mathrm{X}) \cap D(\mathbf{\Delta})$ with $\mathbf{\Delta} G^\varepsilon_x = -p_\varepsilon(x,y)\mathfrak{m}$, in particular G^ε_x is superharmonic in the whole X.

Proposition 2.1.3 (Estimates for the Green's functions, [**58**, Prop. 2.3])**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a non-parabolic* $\mathrm{RCD}(0,N)$ *m.m.s. Then there exits a constant* $C = C(N) \geq 1$ *such that*

$$\frac{1}{C}\int_{\mathsf{d}(x,y)}^\infty \frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s \leq G(x,y) \leq C\int_{\mathsf{d}(x,y)}^\infty \frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s, \quad \forall x,y \in \mathrm{X}. \tag{2.1.4}$$

Proposition 2.1.4. *Let* X *be a non-parabolic* $\mathrm{RCD}(0,N)$ *space. Then* G_x *is positive, continuous and harmonic in* $\mathrm{X}\setminus\{x\}$*, for every* $x \in \mathrm{X}$.

Proof. Fix $R,\delta > 0$ with $R > \delta$ and define $A_{\delta,R} = B_R(x) \setminus \bar{B}_\delta(x)$. It enough to prove that G_x is harmonic on $A_{\delta,R}$. Recall that $G^\varepsilon_x \in \mathsf{LIP}(\mathrm{X}) \cap D(\mathbf{\Delta})$ with $\mathbf{\Delta} G^\varepsilon_x = -p_\varepsilon(x,y)\mathfrak{m}$ and that $G^\varepsilon_x \to G_x$ in $L^1_{\mathsf{loc}}(\mathrm{X})$. We now observe that from the heat kernel bounds (1.3.15) we have $\sup_{t\in(0,1)} \|p_t(x,.) + |\nabla p_t(x,.)|\|_{L^\infty(A_{\delta,R})} < +\infty$ and that $\sup_{\varepsilon>0}\|G^\varepsilon_x\|_{L^\infty(A_{\delta,R})} < +\infty$. From this, following the arguments in the proof of [**58**, Lemma 2.5], we can prove that the sequence G^ε_x is Cauchy in $W^{1,2}(A_{\delta,R})$. In particular $G^\varepsilon_x \to G_x$ in $W^{1,2}(A_{\delta,R})$ and, since $p_t(x,.) \to 0$ uniformly in $A_{\delta,R}$, we deduce that G_x is harmonic in $A_{\delta,R}$ (cf. with Lemma 2.3.4).

We pass to the continuity of G_x in $\mathrm{X}\setminus\{x\}$. We first observe that from (1.3.15) we have

$$\||\nabla p_t(x,.)|\|_{L^\infty(\mathrm{X}\setminus B_\delta(x))} \leq C(N)t^{-1/2}\mathfrak{m}(B_{\sqrt{t}}(x))^{-1}e^{\frac{-\delta^2}{5t}} =: \beta(t,\delta)$$

and that, thanks to the Bishop-Gromov inequality and the non-parabolicity assumption, $\int_0^\infty \beta(t,\delta)\,\mathrm{d}t < +\infty$, for every $\delta > 0$. Therefore from the continuity of p_t and Proposition 1.3.8 we deduce that

$$\varlimsup_{z\to y}|G_x(z) - G_x(y)| \leq \left(\int_0^\infty \beta(t,\delta)\,\mathrm{d}t\right)\varlimsup_{z\to y}\mathsf{d}(z,y) = 0, \quad \forall y \in \mathrm{X}\setminus\bar{B}_\delta(x_0),$$

from which the claimed continuity follows. □

2.1.2. Number of ends. Let us introduce the notion of *ends* for a metric space. The definition is usually given for manifolds, however, since the definition is purely metric, it carries over verbatim to metric spaces.

Definition 2.1.5 (Number of ends of a metric space)**.** Let (X,d) be a metric space and $k \in \mathbb{N}$. We say that X has k ends if both the following are true:

(1) for any K compact, $\mathrm{X}\setminus K$ has at most k unbounded connected components,
(2) there exists K' compact such that $\mathrm{X}\setminus K'$ has exactly k unbounded connected components.

The following result generalizes to the non smooth setting a well known result for Riemannian manifolds.

Proposition 2.1.6. *Suppose* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is a non compact* $\mathrm{RCD}(0,N)$ *space,* $N \in [1,\infty)$*. Then exactly one of the following holds:*

(i) X *is a cylinder, meaning that* X *is isomorphic to the product* $(\mathbb{R}\times\mathrm{X}', \mathsf{d}_{Eucl}\times\mathsf{d}', \mathcal{L}^1\otimes\mathfrak{m}')$*, where* $(X',\mathsf{d}',\mathfrak{m}')$ *is a compact* $\mathrm{RCD}(0,N-1)$ *m.m.s. if* $N \geq 2$ *and a single point if* $N \in [1,2)$*,*
(ii) for every C *bounded subset of* X*, there exists* $R > 0$ *such that the following holds: for every couple of points* $x,y \in X$ *satisfying* $\mathsf{d}(x,C),\mathsf{d}(y,C) > R$ *there exists* $\gamma \in \mathrm{Lip}([0,1],\mathrm{X})$ *such that* $\gamma(0) = x$*,* $\gamma(1) = y$*,* $\gamma \subset X\setminus C$ *and* $\mathrm{Lip}(\gamma) \leq 5\mathsf{d}(x,y)$*. In particular* X *has one end.*

Proof. We closely follow [**2**, Prop. 2.10]. Suppose that *ii*) does not hold, it follows that there exists a bounded set C, two sequences of points $(x_k),(y_k) \subset X$ and geodesics (γ_k) between x_k and y_k such that $\mathsf{d}(x_k,C),\mathsf{d}(y_k,C) \to +\infty$ and γ_k intersects C for all k. Indeed for a geodesic, $\mathrm{Lip}(\gamma_k) = \mathsf{d}(x_k,y_k)$. Since X is proper and C is compact, with a compactness argument (re parametrizing all the γ_k by arc length) we deduce that X contains a line. In particular, being X an $\mathrm{RCD}(0,N)$ space, by the splitting theorem [**115**] we infer that X is is isomorphic to the product $(\mathbb{R}\times\mathrm{X}', \mathsf{d}_{Eucl}\times\mathsf{d}', \mathcal{L}^1\otimes\mathfrak{m}')$, where $(X',\mathsf{d}',\mathfrak{m}')$ is an $\mathrm{RCD}(0,N-1)$ space if $N \geq 2$ and a single point if $N \in [1,2)$. It remains to prove that X′ is bounded.

Suppose it is not. We claim that this would imply the validity of *ii*) and thus a contradiction. Indeed suppose that $C \subset X$ is bounded, then $C \subset B_R(p)$ for some $R > 0$ and $p \in \mathrm{X}$. It is enough to show that for every couple of points $x_0,x_1 \in B_{2R}(p)^c$ there exists $\gamma \in \mathrm{Lip}([0,1],\mathrm{X})$ joining them and with image contained in $B_R(p)^c$. In the case $\mathsf{d}(x_0,x_1)\le R$ we conclude immediately by taking a geodesic between x_0 and x_1, hence we can suppose that $\mathsf{d}(x_0,x_1)>R$. Identifying X with $\mathbb{R}\times\mathrm{X}'$ we have that $p=(\bar t,x')$, $x_i=(t_i,x_i')$ $i=0,1$, for some $\bar t,t_0,t_1\in\mathbb{R}$ and $x',x_0',x_1'\in\mathrm{X}'$. Hence $I\times B'\subset B_{2R}(p)$, where $I:=[\bar t-R,\bar t+R]$ and $B':=B_R(x')$. In particular $x_i\in(I\times B')^c$, $i=0,1$, i.e. for every $i=0,1$ either $t_i\notin I$ or $x_i\notin B'$. We claim that it is sufficient to deal with the case $t_0\notin I$ and $x_1'\notin B'$, indeed the other cases follow from this one by concatenating two paths of this type as follows: if $t_0,t_1\notin I$ and $x_0',x_1'\in B'$ we choose $y'\notin B'$ (which exists since X$'$ is unbounded) and we join (t_0,x_0') to (t_0,y') and then (t_0,y') to (t_1,x_1'); if $t_0,t_1\in I$ and $x_0',x_1'\notin B'$ we pick $s\notin I$ and we join (t_0,x_0') to (s,x_0') and then (s,x_0') to (t_1,x_1').

Hence we can assume that $t_0\in\mathbb{R}\setminus I$ and $x_1'\in\mathrm{X}'\setminus B'$. To build the required path, consider a geodesic $\eta:[0,1]\to X'$ going from x_0' to x_1' and define the function $s:[0,1]\to\mathbb{R}$ as $s(t)=t_1\,t+(1-t)\,t_0$. Then the curve

$$\gamma(t)=\begin{cases}(t_0,\eta(t)), & t\in[0,1),\\ (s(t),x_1'), & t\in[1,2],\end{cases}$$

is Lipschitz and has image contained in $(I\times B')^c\subset B_R(p)^c$, hence (up to a reparametrization) satisfies all the requirements.

To estimate $\mathrm{Lip}(\gamma)$ we observe that, up to a reparametrization, we can assume that $\mathrm{Lip}(\gamma)=L(\gamma)$, hence it is sufficient to bound the length of γ. In the case $t_0\in\mathbb{R}\setminus I$ and $x_1'\in\mathrm{X}'\setminus B'$ it is sufficient to observe that $L(\gamma)=\mathsf{d}'(x_0',x_1')+|t_0-t_1|\le 2(\mathsf{d}_{Eucl}\times\mathsf{d}((t_0,x_0'),(t_1,x_1')))$, where γ is the curve constructed above. The general case follows concatenating two paths as described above, where we pick y' and s so that $\mathsf{d}'(y',x')<2R$, $|\bar t-s|<2R$. Indeed it can be easily checked these two resulting paths have respectively length not grater than $2R$ and $2R+\mathsf{d}(x_0,x_1)$. Since we are assuming $\mathsf{d}(x_0,x_1)\ge R$, this concludes the proof. □

Corollary 2.1.7. *If* X *is a non-parabolic* RCD$(0,N)$ *space, then it is not a cylinder and in particular item ii) of Proposition 2.1.6 holds and* X *has only one end.*

Proof. Suppose by contradiction X is a cylinder $\mathbb{R}\times\mathrm{X}'$. Then for any $r>0$ and any $(x,t)\in\mathrm{X}'\times\mathbb{R}$ we have

$$\mathfrak{m}(B_r((x,t)))=(\mathcal{L}^1\otimes\mathfrak{m}')(B_r((x,t)))\le(\mathcal{L}^1\otimes\mathfrak{m}')(\mathrm{X}'\times[t-r,t+r])=\mathfrak{m}'(\mathrm{X}')\,2r,$$

which clearly contradicts the fact that X is non-parabolic. □

2.2. The electrostatic potential on RCD spaces

As we mentioned in Section 2.1 the existence of the electrostatic potential on RCD spaces is not granted, in particular already for Riemannian manifolds with non-negative Ricci curvature, its existence is equivalent to the manifold being non-parabolic. In this section we prove that on non-parabolic RCD$(0,N)$ spaces there exists the electrostatic potential for open sets with sufficiently regular boundary (Theorem 2.2.13).

We will first prove the existence of a relative electrostatic potential of a set E, with regular boundary (Theorem 2.2.3), with respect to a larger open set. Then, following a classical argument via comparison and exhaustion (see e.g. [**166**] or [**2**]) we will prove by compactness the existence of a global electrostatic potential.

To deal with the existence of a relative electrostatic potential for sufficiently regular sets, we will make use of the so-called Wiener criterion, developed for PI-spaces in [**47–49**]. To make the work self-contained we will revisit and reprove in details the results that we need in the setting of RCD(K,N) spaces and in doing so we will mainly follow the exposition in [**46**].

As a corollary of the construction of the relative electrostatic potential we will prove the existence of 'harmonic-cut-off' functions that will be used in the next Chapter (see Corollary 2.2.4 and Section 3.2 respectively). For this reason and for generality, even if in this chapter we are working with $K=0$, in this section we will consider arbitrary K.

2.2.1. Relative electrostatic potential.

We first need to introduce the notion of boundary regularity that we will use:

Definition 2.2.1 (Cap-fat boundary points)**.** We say that an open set E is $\overline{\mathrm{Cap}}$-fat at a point $x\in\partial E$ if there exists $r,c>0$ such that

$$\frac{\overline{\mathrm{Cap}}(B_s(x)\cap E,B_{2s}(x))}{\overline{\mathrm{Cap}}(B_s(x),B_{2s}(x))}\ge c\,,\quad\forall\,s\in(0,r).$$

Moreover we say that E has (uniformly) Cap-fat boundary if it is Cap-fat at every point $x \in \partial E$ (with global parameters $c, r > 0$).

A geometric condition that is enough to ensure Cap-fatness of the boundary is the following interior corkscrew condition. This follows essentially from the doubling property of the measure and the Poincaré inequality (see for example [**46**, Prop. 6.16]).

Definition 2.2.2 (Corkscrew-condition)**.** Let $\lambda \in (0,1)$ and $r > 0$. We say that E satisfies the (interior) (λ, r)-corkscrew condition at $x \in \partial E$ if for every $s \in (0, r)$ there exists an ball of radius λs contained in $B_s(x) \cap E$.

It is easily verified that any ball of radius $> \delta$ satisfies the (interior) $(1/4, \delta)$-corkscrew condition. Moreover arbitrary unions of sets satisfying the (interior) (λ, r)-corkscrew condition still satisfies the (interior) $(\lambda/2, r)$-corkscrew condition. In particular union of balls with radius uniformly bounded below satisfies the interior corkscrew condition. It follows that any ε-enlargements of a set, i.e. a set of the form $S^\varepsilon = \{x \ : \ \mathsf{d}(x, S) < \varepsilon\}$, with $\varepsilon > 0$ and S an arbitrary set, satisfies the interior corkscrew condition.

Our main result will be to prove the existence of relative electrostatic potential for sets with Cap-fat boundary.

Theorem 2.2.3 (Existence of relative electrostatic potential)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *metric measure space with* $N < +\infty$. *Let* $\Omega, E \subset \mathrm{X}$ *be bounded open sets with* $E \subset\subset \Omega$. *Suppose also that* E *has* Cap*-fat boundary. Then there exists* $u \in W_0^{1,2}(\Omega) \cap C(\Omega)$, *superharmonic in* Ω *and harmonic in* $\Omega \setminus \bar{E}$ *with* $0 \le u \le 1$, $u = 1$ *in* $\overline{E}$ *and*

$$\overline{\mathrm{Cap}}(E, \Omega) = \int_\Omega |\nabla u|^2 \, \mathrm{d}\mathfrak{m}.$$

Moreover the following continuity estimate holds: for every $x \in \partial E$ *it holds*

$$1 - u(y) \le C_x \mathsf{d}(y, x)^{\alpha_x}, \quad \forall y \in B_{r_x/2}(x) \cap \Omega,$$

for some positive constants $C_x = C_x(r_x, c_x, K, N, \delta) > 0$, $\alpha_x = \alpha(r_x, c_x, K, N, \delta) > 0$, *where* r_x, c_x *are the* Cap*-fatness parameters of* x *and* $\delta > 0$ *is such that* $\mathsf{d}(E, \Omega^c) \ge \delta$.
Finally u *satisfies the following comparison principle: for every* $v \in W^{1,2}(\Omega)$ *superharmonic and such that* $v \ge \chi_E$ $\mathfrak{m}$*-a.e. in* Ω, *it holds that*

$$u \le v, \quad \mathfrak{m}\text{-a.e. in } \Omega.$$

Before passing to the proof of the above result we deduce from it the existence of a class of harmonic cut-off functions, which is of independent interest and will be used in Chapter 3

Corollary 2.2.4 (Existence of harmonic cut-off functions)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *metric measure space with* $N < +\infty$. *Then for every compact set* $P \subset \mathrm{X}$ *and any* U *open set with* $P \subset U$, *there exists a continuous function* $u \in C(\mathrm{X}) \cap W_0^{1,2}(U)$ *such that* $0 \le u \le 1$, $u = 1$ *in* P, $u = 0$ *in* U^c *and* u *is harmonic in* $\{0 < u < 1\}$.

Proof. If $U = \mathrm{X}$ to there is nothing to prove since we can just take $u \equiv 1$. Hence we can assume that $U^c \neq \emptyset$ and in particular $\mathsf{d}(P, U^c) > 0$. We choose numbers $r_1, r_2 \in \mathbb{R}^+$ so that $r_1 < r_2 < \mathsf{d}(P, U^c)$. We consider the sets P^{r_1}, P^{r_2}, where $P^r := \{x \ : \ \mathsf{d}(x, P) < r\}$, which obviously satisfy $P \subset\subset P^{r_1} \subset\subset P^{r_2} \subset U$. In particular $C := \mathrm{X} \setminus P^{r_2} \neq \emptyset$. We then fix $\delta \in (0, (r_2 - r_1)/2)$ and take $V := \{x \ : \ \mathsf{d}(x, C) > \delta\}$. Clearly by the triangle inequality $P \subset\subset P^{r_1} \subset\subset V \subset P^{r_2}$. Moreover, P^{r_1} being the enlargement of a set, it satisfies the interior cork-screw condition. Therefore we can apply Theorem 2.2.3 to have existence of a function $u \in C(V)$, with $0 \le u \le 1$, $u = 1$ in P^{r_1} and harmonic in $V \setminus \overline{P^{r_1}}$. To conclude is then sufficient to show that $u(x) \to 0$ as $x \to \partial V$. To do so we fix $\bar{x} \in \partial V$ arbitrary. We will follow a barrier-type argument. More precisely we aim to exploit the comparison principle in Theorem 2.2.3 to compare u near $\bar{x}$ with a suitable superharmonic function. We claim that for every $r > 0$ small enough there exists a ball $B_r(y) \subset V^c$ with $\mathsf{d}(x, y) = r$. Indeed observe that by definition $\mathsf{d}(\bar{x}, C) = \delta$. Therefore by compactness and by the fact that X is geodesic, there exists a geodesic γ connecting $\bar{x}$ with a point $c \in C$ such that $\mathsf{d}(x, c) = \delta$. From this the claim easily follows recalling that by definition $V = \{x \ : \ \mathsf{d}(x, C) > \delta\}$. For every $r > 0$ and y as in the claim we define the function $f_r : V \to \mathbb{R}$ as $f_r := (r)^{-2\alpha} - \mathsf{d}_y^{-2\alpha}$, with $\alpha \gg 1$ fixed and to be chosen later. By definition $f_r \in \mathsf{LIP}(V) \subset W^{1,2}(V)$. Moreover by construction $f_r \ge 0$ in V and $f_r|_{P^{r_1}} \ge (r)^{-2\alpha} - ((r_2 - r_1)/2)^{-2\alpha}$, in particular if $r > 0$ is small enough we have $f_r \ge 1$ in P^{r_1} (we stress that the choice of α will not depend on r and there is no circular reasoning). Therefore $f_r \ge \chi_{P^{r_1}}$. We claim that f_r is superharmonic in V. Indeed from the chain rule for the Laplacian 1.2.16

and the Laplacian comparison (1.3.13) (recalling also (1.3.14)) we have that $f_r \in \mathsf{D}(\mathbf{\Delta}, V)$ with

$$\begin{aligned}\mathbf{\Delta} f_r &= \alpha \mathsf{d}_y^{2(-\alpha-1)} \mathbf{\Delta} \mathsf{d}_y^2 + 4\alpha(-\alpha-1)|D\mathsf{d}_{x_0}|^2 \mathsf{d}_y^{2(-\alpha-1)} \mathfrak{m} \le \alpha \mathsf{d}_y^{2(-\alpha-1)} 2N\, l_{K,N}(\mathsf{d}_{x_n}) - 4\alpha(\alpha+1)\mathsf{d}_y^{2(-\alpha-1)} \\ &= \alpha \mathsf{d}_y^{2(-\alpha-1)}(2N l_{K,N}(\mathsf{d}_y) - 4(\alpha+1))\mathfrak{m}.\end{aligned}$$

Since $l_{K,N} : [0,\infty) \to [0,\infty)$ is a continuous function it is locally bounded, hence $l_{K,N}(\mathsf{d}_y) \le C$ in V for some constant C depending on the diameter of V and on the numbers N and K. Therefore if we take α large enough we obtain that $\mathbf{\Delta} f_r \le 0$ and thus the desired superharmonicity of f_r. From this, the comparison principle in Theorem 2.2.3 and the continuity of both u and f_r ensure that $u \le f_r$ in V. Then

$$0 \le \overline{\lim_{U \ni z \to x}} u(y) \le \overline{\lim_{U \ni z \to x}} f_r(y) = \overline{\lim_{U \ni z \to x}} f_r(z) = f_r(x) = 0.$$

From the arbitrariness of $x \in \partial V$ the conclusion follows. □

The relative electrostatic potential will be constructed as a solution of the obstacle problem.

The obstacle problem. Given an open and bounded set $\Omega \subset \mathrm{X}$ and a (Borel) set $E \subset\subset \Omega$ we consider the following minimization problem

$$Obs(E,\Omega) := \inf_{u \in \mathcal{F}_{E,\Omega}} \int_\Omega |\nabla u|^2 \, \mathrm{d}\mathfrak{m}, \tag{O}$$

where $\mathcal{F}_{E,\Omega} = \{u \in W_0^{1,2}(\Omega) \ : u \ge \chi_E \ \mathfrak{m}\text{-a.e. in } \Omega\}$. Note that $\mathcal{F}_{E,\Omega}$ is non-empty as soon as $E \subset\subset \Omega$.

It is clear that if $E \subset\subset \Omega$ is open, then

$$Obs(E,\Omega) = \overline{\mathrm{Cap}}(E,\Omega).$$

The proof of the following result is a straightforward application of the direct method of the calculus of variations, recalling that the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^2(\mathrm{X})$ is compact (see Proposition 1.6.11) and from the lower semi continuity and (strict) convexity of the Cheeger energy.

Proposition 2.2.5. *Whenever $\mathcal{F}_{E,\Omega} \neq \emptyset$, there exists a unique minimizer to* (O). *Moreover this minimizer is superharmonic in Ω.*

We now show the two main properties of a minimizer u of (O): the first is that u is harmonic far from the obstacle E and the second says that u is essentially the smallest superharmonic function which stays above χ_E.

We will need the following technical lemma, whose simple proof is omitted.

Lemma 2.2.6. *Let $u \in W_0^{1,2}(\Omega)$, then $u^+ \in W_0^{1,2}(\Omega)$.*
Let $v \in W^{1,2}(\Omega)$, $u \in W_0^{1,2}(\Omega)$ be such that $0 \le v \le u$, then $v \in W_0^{1,2}(\Omega)$

Proposition 2.2.7. *Let u be the minimum of* (O) *for some $E \subset \Omega$. Then $u = 1$ $\mathfrak{m}$-a.e. in E and the following hold:*

(1) u is harmonic in $\Omega \setminus \bar{E}$,
(2) comparison principle: for every $v \in W^{1,2}(\Omega)$ superharmonic and such that $v \ge \chi_E$, $\mathfrak{m}$-a.e., it holds that

$$u \le v, \quad \mathfrak{m}\text{-a.e. in } \Omega.$$

Proof. We start by showing that $u \le 1$ $\mathfrak{m}$-a.e. in Ω. Indeed $u \wedge 1 \in \mathcal{F}_{E,\Omega}$ and $\int_\Omega |\nabla(u \wedge 1)|^2 \le \int_\Omega |\nabla u|^2 \mathrm{d}\mathfrak{m}$, from which the claim follows. Since $u \ge \chi_E$, $\mathfrak{m}$-a.e. it also follows that $u = 1$ $\mathfrak{m}$-a.e. in E.

We pass to the harmonicity. Fix $\varphi \in \mathsf{LIP}_c(\Omega \setminus \bar{E})$. Clearly $(u+\varphi)^+ \in \mathcal{F}_{E,\Omega}$, therefore

$$\begin{aligned}\int_{\Omega\setminus\bar{E}} |\nabla(u+\varphi)|^2 \, \mathrm{d}\mathfrak{m} &\ge \int_{\Omega\setminus\bar{E}} |\nabla(u+\varphi)^+|^2 \, \mathrm{d}\mathfrak{m} \\ &= \int_\Omega |\nabla(u+\varphi)^+|^2 \, \mathrm{d}\mathfrak{m} - \int_{\bar{E}} |\nabla u|^2 \, \mathrm{d}\mathfrak{m} \ge \int_{\Omega\setminus\bar{E}} |\nabla u|^2 \mathrm{d}\mathfrak{m},\end{aligned}$$

where in the equality step we have used that $\varphi = 0$ in $\bar{E}$ and the locality of the gradient. This and Proposition 1.3.18 prove the claimed harmonicity.

It remains to prove the comparison principle. We start claiming that $(u-v)^+ \in W_0^{1,2}(\Omega)$ and $\min(u,v) \in \mathcal{F}_{E,\Omega}$. Indeed we have that $0 \le (u-v)^+ \le u$, $\mathfrak{m}$-a.e. in Ω and $\chi_E \le \min(u,v) \le u$ $\mathfrak{m}$-a.e. in Ω, therefore the claim follows applying Lemma 2.2.6.

Observe that $\max(u,v) = v + (u-v)^+$, hence from the superharmonicity of v, Proposition 2.2.5, and the locality of the gradient we have

$$\int_{\{u>v\}} |\nabla u|^2 \, \mathrm{d}\mathfrak{m} \ge \int_{\{u>v\}} |\nabla v|^2 \, \mathrm{d}\mathfrak{m}.$$

Therefore from the locality of the gradient it follows that

$$\int_\Omega |\nabla \min(u,v)|^2 \,\mathrm{d}\mathfrak{m} \le \int_\Omega |\nabla u|^2 \,\mathrm{d}\mathfrak{m},$$

that combined with $\min(u,v) \in \mathcal{F}_{E,\Omega}$ and the uniqueness of the solution to (O) implies that $\min(u,v) = u$ $\mathfrak{m}$-a.e. in Ω. □

We conclude this part with the following technical result.

Lemma 2.2.8. *Let u be the minimum of* (O) *for some $E \subset\subset \Omega$. Then for every $m \in (0,1]$, $\mathcal{F}_{\{u>m\},\Omega}$ is non-empty and the function $\frac{u}{m} \wedge 1$ is the minimum of* (O) *in Ω with $E = \{u > m\}$.*

Proof. Set $u_m = \frac{u}{m} \wedge 1$. Observe that $u_m \ge \chi_{\{u>m\}}$ and that $u_m \in W^{1,2}_0(\Omega)$ by Lemma 2.2.6, hence $u_m \in \mathcal{F}_{\{u>m\},\Omega}$ and in particular $\mathcal{F}_{\{u>m\},\Omega}$ is non-empty. Fix now $v \in \mathcal{F}_{\{u>m\},\Omega}$. Define the function $\bar u \coloneqq u + m(v - u_m)$ and observe that $\bar u \in W^{1,2}_0(E)$. Moreover $\bar u \ge u \ge 0$ $\mathfrak{m}$-a.e. in Ω and, since from Proposition 2.2.7 $u = 1$ $\mathfrak{m}$-a.e. in E, we also have that $\bar u = 1$ $\mathfrak{m}$-a.e. in E. Therefore $\bar u \in \mathcal{F}_{E,\Omega}$. This and the fact that $\bar u = mv$ $\mathfrak{m}$-a.e. in $\{u \le m\}$ and $\bar u = u$ $\mathfrak{m}$-a.e. in $\{u > m\}$ gives

$$\int_\Omega |\nabla v|^2 \,\mathrm{d}\mathfrak{m} \ge \int_{\{u\le m\}} |\nabla v|^2 \,\mathrm{d}\mathfrak{m} \ge \frac{1}{m^2}\int_{\{u\le m\}} |\nabla u|^2 \,\mathrm{d}\mathfrak{m} = \int_\Omega |\nabla u_m|^2 \,\mathrm{d}\mathfrak{m}.$$

Since $v \in \mathcal{F}_{\{u>m\},\Omega}$ was arbitrary we conclude. □

Wiener-criterion on RCD *spaces.* We now come to the core of this section. The Wiener criterion is used to characterize the regular boundary points in the Perron-method for the solution of the Dirichlet problem for the Laplacian operator (see [**46**, Chp. 10]). Here we will not prove such a general result, however we will use core ideas coming from this theory, which will allow us to control from below the solution of the obstacle problem in term of the capacity of balls at the boundary of the obstacle.

Proposition 2.2.9. *For every $r_0 < 4diam(\mathrm{X})$ there exists $C = C(r_0, K, N) > 0$ such that the following holds. Let $E \subset B_r(x)$ be open, $r < r_0$ let $2B = B_{2r}(x)$ and let u be the solution to* (O) *for E in $2B$. Then*

$$u \ge C\frac{\overline{\mathrm{Cap}}(E, 2B)}{\overline{\mathrm{Cap}}(B, 2B)}, \quad \mathfrak{m}\text{-a.e. in } B_r(x).$$

Proof. Set $B' = B_{\frac{3}{2}r}(x)$ and observe that, since $r_0 < 4\mathrm{diam}(\mathrm{X})$, $\partial B' \ne \emptyset$. Define $m \coloneqq \max_{\partial B'} u$, which exists because u is continuous in $\partial B'$. We claim that $m > 0$. Indeed if $m = 0$, from the maximum principle (Proposition 1.3.19) we would have that $u = 0$ in the ball $2B$ (recall that balls are connected), and thus $u = 0$ in E, which contradicts the fact that $u = 1$ $\mathfrak{m}$-a.e. in E with E open. We now claim that

$$u \le m, \quad \text{in } 2B \setminus B'. \tag{2.2.1}$$

To see this let $m' > m$ and observe that $(u - m')^+ \le u^+$ hence by Lemma 2.2.6 $(u - m')^+ \in W^{1,2}_0(2B)$. Moreover, from the continuity of u and the definition of m we have that $(u-m')^+ = 0$ in a neighbourhood of $\partial B'$. These two observations together imply that $(u-m')^+ \in W^{1,2}_0(2B\setminus\overline{B'})$. Observe that $\min(u, m') = u - (u - m')^+$ hence from harmonicity of u we deduce that

$$\int_{2B\setminus\bar B'} |\nabla u|^2 \,\mathrm{d}\ \mathfrak{m} \le \int_{2B\setminus\bar B'} |\nabla \min(u,m')|^2 \,\mathrm{d}\mathfrak{m},$$

which combined with the locality of the gradient gives that $|\nabla u| = 0$ $\mathfrak{m}$-a.e. in $\{2B \setminus \bar B'\} \cap \{u \ge m'\}$. Therefore again by locality $|\nabla(\max(u, m'))| = 0$ $\mathfrak{m}$-a.e. in $2B\setminus\bar B'$. This implies that $\max(u, m')$ is constant in every connected component of $2B \setminus \bar B'$. However the boundary of any such connected component must intersect $\partial B'$ (because $2B$ is connected), hence we deduce that $u \le m'$ in $2B \setminus \bar B'$. Since $m' > m$ was arbitrary (2.2.1) follows.

Define the functions $u_1 = \frac{u}{m} \wedge 1$, $u_2 = \frac{u - mu_1}{1-m}$ and observe that $u_1, u_2 \in \mathcal{F}_{E,2B}$. In particular for every $t \in (0,1)$ $tu_1 + (1-t)u_2 \in \mathcal{F}_{E,2B}$ and

$$\int_{2B} |\nabla u|^2 \,\mathrm{d}\mathfrak{m} \le t^2 I_1 \,\mathrm{d}\mathfrak{m} + (1-t)^2 I_2,$$

where $I_i = \int_{2B} |\nabla u_i|^2 \,\mathrm{d}\mathfrak{m}$. Optimizing in t we obtain that

$$\frac{1}{\int_{2B} |\nabla u|^2 \,\mathrm{d}\mathfrak{m}} \ge \frac{1}{I_1} + \frac{1}{I_2}. \tag{2.2.2}$$

Observe now that $u_2 = 0$ in $\{u \le m\}$ and $u_2 = (1-m)^{-1}(u-1)$ in $\{u > m\}$, therefore $|\nabla u_2| = \chi_{\{u>m\}}|\nabla u|(1-m)^{-1}$ $\mathfrak{m}$-a.e. in $2B$. In particular $I_2 \le (1-m)^{-2}\int_{2B}|\nabla u|^2\,\mathrm{d}\mathfrak{m}$, that combined with (2.2.2) gives

$$\overline{\mathrm{Cap}}(E, 2B) = \int_{2B}|\nabla u|^2\,\mathrm{d}\mathfrak{m} \le (2m - m^2)I_1 \le 2mI_1.$$

This combined with Lemma 2.2.8 gives

$$\overline{\mathrm{Cap}}(E, 2B) \le 2m\,Obs(\{u > m\}, 2B) \le 2m\,Obs(B', 2B) = 2m\overline{\mathrm{Cap}}(B', 2B), \tag{2.2.3}$$

where in the second inequality we have used (2.2.1).

From the definition of m there exists a ball $B'' = B_{r/2}(y)$ with $y \in \partial B'$ such that $\sup_{B''} u \ge m$. Applying twice the Harnack inequality, recalling that u is harmonic in B'' and superharmonic in B, and using the doubling property we obtain that

$$m \le \sup_{B''} u \le C(r_0, K, N)\operatorname*{ess\,inf}_B u.$$

The conclusion then follows from (2.2.3) and recalling that thanks to the doubling condition and the Poincaré inequality we have $\overline{\mathrm{Cap}}(B', 2B) \le c\overline{\mathrm{Cap}}(B, 2B)$, for some constant c depending only on r_0, K and N (see [**46**, Prop. 6.16]) □

Theorem 2.2.10. *For every $r_0 < 4diam(\mathrm{X})$ there exists $C = C(r_0, K, N) > 0$ such that the following holds. Let $E \subset B_r(x)$ be open, $r < r_0$, and set $B_i := B_{2^{1-i}r}(x)$ for $i \in \mathbb{N}_0$. Let u be the capacitary potential for E in B_0, then for every $i \ge 1$ it holds that*

$$1 - u \le \exp\left(-C\sum_{j=1}^{i}\frac{\overline{\mathrm{Cap}}(E \cap B_j, B_{j-1})}{\overline{\mathrm{Cap}}(B_j, B_{j-1})}\right), \quad \mathfrak{m}\text{-}a.e.\ in\ B_i.$$

Proof. Let u_i be the solution to (O) for $E \cap B_i$ in B_{i-1} (in particular $u = u_1$) and define $a_i := \frac{\overline{\mathrm{Cap}}(E\cap B_i, B_{i-1})}{\overline{\mathrm{Cap}}(B_i, B_{i-1})}$ for $i \in \mathbb{N}$. Proposition 2.2.9 ensures that

$$\operatorname*{ess\,inf}_{B_i} u_i \ge Ca_i \ge 1 - e^{-Ca_i}. \tag{2.2.4}$$

Define the functions $v_i \in W_0^{1,2}(B_0)$ inductively as $v_1 = u_1$ and $v_i = 1 - e^{Ca_{i-1}}(1 - v_{i-1})$, for $i \ge 2$. Observe that, since u_1 is superharmonic in B_0, v_i is superharmonic in B_0 for all $i \ge 1$. We claim that

$$v_i \ge 0, \quad \mathfrak{m}\text{-a.e. in } B_{i-1}. \tag{2.2.5}$$

We will actually show the stronger estimate $v_i \ge u_i$ $\mathfrak{m}$-a.e. in B_{i-1}. We proceed by induction. By definition $v_1 = u_1$, now suppose that $v_i \ge u_i$ in B_{i-1}. It follows from (2.2.4) that $v_{i+1} \ge 1 - e^{Ca_i}(1 - u_i) \ge 0$ $\mathfrak{m}$-a.e. in B_i. Moreover, since $u_1 = 1$ in $E \cap B_1$, evidently $v_{i+1} = 1$ $\mathfrak{m}$-a.e. in $E \cap B_{i+1}$. Combining these two observations we obtain that $v_{i+1} \ge \chi_{E\cap B_{i+1}}$ $\mathfrak{m}$-a.e. in B_i. Recalling that v_i is superharmonic in B (and thus also on B_i) we can apply the comparison principle of Proposition 2.2.7 to deduce that $v_{i+1} \ge u_{i+1}$ $\mathfrak{m}$-a.e. in B_i. This proves the claim. Therefore from (2.2.5)

$$1 - u = 1 - v_1 = e^{-C(a_1+\ldots+a_{i-1})}(1 - v_i) \le e^{-C(a_1+\ldots+a_{i-1})}, \quad \mathfrak{m}\text{-a.e. in } B_{i-1},$$

that concludes the proof. □

With the above powerful result, we can now easily proof the existence of the relative electrostatic potential.

Proof of Theorem 2.2.3. Fix $x \in \partial E$ and let c, r be its Cap-fat parameters. Let $B := B_{r_0}(x)$ with $r_0 := (\delta \wedge r)/4$ and let u to be the solution to (O) for $E \cap B$ in $2B$. Fix $y \in B \setminus \bar{E}$ with $\mathsf{d}(y, x) < r/2$. There exists $i \in \mathbb{N}_0$ such that $2^{-i-1}r_0 < \mathsf{d}(x, y) < 2^{-i}r_0 < r$. Therefore from Theorem 2.2.10 (observe that we must have $r_0 < \mathrm{diam}(\mathrm{X})$) and the continuity of u in $B \setminus \bar{E}$ we have

$$1 - u(y) \le (e^{-i})^{c\cdot C} \le (2^{-i})^{c\cdot C} \le (2r_0^{-1}\mathsf{d}(x, y))^{c\cdot C} = (8\delta \wedge r)^{-c\cdot C}(\mathsf{d}(x, y))^{c\cdot C}. \tag{2.2.6}$$

Consider now any $\Omega \subset \mathrm{X}$ open with $E \subset\subset \Omega$ and $\mathsf{d}(E, \Omega^c) \ge \delta$ and let $\bar{u}$ to be the solution of (O) for E in Ω. Fix $x \in \partial E$ and let u as in the previous part of the proof. Since $B_{2r_0}(x) \subset \Omega$, from the comparison principle of Proposition 2.2.7 we have that $\bar{u} \ge u$ $\mathfrak{m}$-a.e. in $B_{r_0}(x)$ and since both $\bar{u}$ and u are continuous in $B_{r_0}(x) \setminus \bar{E}$ we have that (2.2.6) holds for $\bar{u}$ and every $y \in B \setminus \bar{E}$ with $\mathsf{d}(y, x) < r/2$. This proves that $\lim_{B_{r_0}(x)\setminus\bar{E}\ni y\to x}\bar{u}(y) = 1$ for every $x \in \partial E$ (recall that $\bar{u} \le 1$) and since $\bar{u}$ is also lower semicontinuous we deduce that $\bar{u} = 1$ in $\bar{E}$.

The comparison principle is already contained in Proposition 2.2.7. □

2.2.2. Existence of electrostatic potential on non-parabolic $\mathrm{RCD}(0,N)$ **spaces.**

Definition 2.2.11 (Electrostatic potential)**.** Given $(\mathrm{X},\mathsf{d},\mathfrak{m})$ an (unbounded) infinitesimally Hilbertian m.m.s. and $E\subset\mathrm{X}$ open and bounded, an *electrostatic potential for* E is a function $u\in D(\mathbf{\Delta},\mathrm{X}\setminus\bar E)\cap C(\mathrm{X}\setminus E)$ solution to

$$\begin{cases}\mathbf{\Delta}|_{\mathrm{X}\setminus\bar E}u=0,\\ u=1, & \text{in } \partial E,\\ u(x)\to 0 & \text{as } \mathsf{d}(x,\partial E)\to+\infty.\end{cases}$$

Remark 2.2.12. If X is a non-parabolic $\mathrm{RCD}(0,N)$ space, than the electrostatic potential (if it exists) is unique. This follows immediately from the maximum principle (see Proposition 1.3.19) and the fact that $\mathrm{X}\setminus\bar E$ has only one unbounded connected component (recall Corollary 2.1.7). ■

We pass to our main existence result for the electrostatic potential, which holds for sets with sufficiently regular boundary, namely with Cap-fat regular boundary. See the previous section for the definition of Cap-fat regular boundary and for examples of sets satisfying this condition.

Theorem 2.2.13. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a non-parabolic* $\mathrm{RCD}(0,N)$ *m.m.s. and let* $E\subset\mathrm{X}$ *be open and bounded with* Cap*-fat boundary. Then the electrostatic potential* u *for* E *exists. Moreover the following continuity estimate holds: for every* $x\in\partial E$ *it holds*

$$1-u(y)\le\mathsf{d}(y,x)^{\alpha_x},\quad\forall y\in B_{r_x/2}(x)\cap E^c,\tag{2.2.7}$$

for some positive constant $\alpha_x=\alpha(r_x,c_x,N)>0$*, where* r_x,c_x *are the* Cap*-fatness parameters of* x*.*

Finally, extending u *to be identically 1 in* $\bar E$*, it holds that* $u\in W^{1,2}_{\mathsf{loc}}(\mathrm{X})$ *and*

$$\int_{\mathrm{X}}|\nabla u|^2\,\mathrm{d}\mathfrak{m}\le\overline{\mathrm{Cap}}(E).\tag{2.2.8}$$

Proof. We will actually build a globally defined function $\tilde u$ and then define u to be the restriction of $\tilde u$ to E^c. The argument is by compactness. Fix $x_0\in E$ and set $B_n:=B_n(x_0)$ with $n\in\mathbb{N}$ and $n>\mathrm{diam}(E)+100$. Theorem 2.2.3 grants the existence of a function $u_n\in W^{1,2}_0(B_n)\cap C(B)$ harmonic in $B_n\setminus\overline{E}$, such that $0\le u_n\le 1$, $u_n=1$ on $\bar E$ and $\int_{\mathrm{X}}|\nabla u_n|^2\,\mathrm{d}\mathfrak{m}=\overline{\mathrm{Cap}}(E,B_n)$. Moreover from the comparison principle in Theorem 2.2.3 we must have that $u_n\le u_{n+1}$ in B_n.

It follows from Lemma 1.3.24 that, up to a (non relabelled) subsequence, u_n converges in $\mathrm{X}\setminus\bar E$ uniformly on compact sets to a function $\tilde u\in C(\mathrm{X}\setminus\bar E)$ harmonic in $\mathrm{X}\setminus\bar E$. In particular $u_n\to\tilde u$ $\mathfrak{m}$-a.e.. Moreover, since $\overline{\mathrm{Cap}}(E,B_r(x_0))$ is decreasing in r, we have that $\sup_n\||\nabla u_n|\|_{L^2(\mathrm{X})}<+\infty$. Then with a cut-off argument it can be easily seen that $\tilde u\in W^{1,2}_{\mathsf{loc}}(\mathrm{X})$ and that (see e.g. Lemma 1.6.14)

$$\int_{\mathrm{X}}|\nabla\tilde u|^2\,\mathrm{d}\mathfrak{m}\le\varliminf_n\int_{\mathrm{X}}|\nabla u_n|^2\,\mathrm{d}\mathfrak{m}\le\varliminf_n\overline{\mathrm{Cap}}(E,B_n)=\lim_{r\to+\infty}\overline{\mathrm{Cap}}(E,B_r(x_0))\overline{\mathrm{Cap}}(E).$$

The continuity estimate follows directly from the fact that $u_n\le\tilde u\le 1$ for every n and from the continuity estimate in Theorem 2.2.3.

It remains to show that u goes to 0 at infinity. We prove it by comparison with the quasi Green's function $G^1_{x_0}$ (recall (2.1.3)). We have that $G^1_{x_0}$ is Lipschitz and superharmonic in X. Moreover $G^1_{x_0}$ is positive, hence $\lambda G^1_{x_0}\ge\chi_E$ for a large enough constant $\lambda>0$. In particular the comparison principle in Theorem 2.2.3 implies that $u_n\le\lambda G$ for every n, which in turn gives $u\le\lambda G^1_{x_0}$. Finally from the estimate for the Green's function in (2.1.4) we have

$$G^1_{x_0}\le G_{x_0}(x)\le\int_{\mathsf{d}(x,x_0)}^{\infty}\frac{s}{\mathfrak{m}(B_s(x))}\,ds,$$

in particular $G^1_{x_0}(x)\to 0$ at infinity. This concludes the proof. □

2.3. New estimates for harmonic functions

2.3.1. Laplacian estimates. The following result can be interpreted as a generalization of the well know fact that in a Riemannian manifold with nonnegative Ricci curvature the square norm of the gradient of an harmonic function is subharmonic.

Theorem 2.3.1. *Let* X *be an* $\mathrm{RCD}(K,N)$ *space with* $N\in[2,\infty)$*, let* u *be harmonic in* Ω *and* $\beta>\frac{N-2}{N-1}$*. Then* $|\nabla u|^{\beta/2}\in W^{1,2}_{\mathsf{loc}}(\Omega)$*,* $|\nabla u|^\beta\in D(\mathbf{\Delta},\Omega)$ *and*

$$\mathbf{\Delta}(|\nabla u|^\beta)\ge C_{\beta,N}\,|\nabla|\nabla u|^{\frac{\beta}{2}}|^2\,\mathfrak{m}_{|\Omega}+\beta K|\nabla u|^2\mathfrak{m}_{|\Omega},\tag{2.3.1}$$

where $C_{\beta,N} = \frac{4}{\beta}\left(\beta - \frac{N-2}{N-1}\right)$. *Moreover* $|\nabla u|^\beta \in D(\mathbf{\Delta},\Omega)$ *also for* $\beta = \frac{N-2}{N-1}$ *with* (2.3.1) *holding without the term containing* $C_{\beta,N}$.

Before passing to the proof of Theorem 2.3.1 we isolate an immediate but important corollary, which will crucial in Chapter 3.

Corollary 2.3.2. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathsf{RCD}(K,N)$ *space,* $N \in [2,\infty)$*, and let* u *be harmonic in* $\Omega \subset \mathrm{X}$. *Then* $\sqrt{|\nabla u|} \in W^{1,2}_{\mathsf{loc}}(\Omega)$ *and in particular*

$$\frac{|\nabla|\nabla u||^2}{|\nabla u|} \in L^1_{\mathsf{loc}}(\Omega), \tag{2.3.2}$$

where the whole function is taken to be zero whenever $|\nabla u| = 0$.

The main ingredient for the proof of Theorem 2.3.1 is the following new Kato-type inequality. Observe that on the set $\{\dim_{\mathsf{loc}} = N\}$, where $\mathsf{tr}\mathrm{Hess}(u) = \Delta u$ (recall (1.4.6)), letting $t \to 0$ we recover the standard refined Kato-inequality for harmonic functions.

Lemma 2.3.3 (Generalized refined Kato inequality)**.** *Let* X *be an* $\mathrm{RCD}(K,N)$*space, with* $N \in [1,+\infty)$. *Then for any* $u \in \mathsf{Test}_{\mathsf{loc}}(\Omega)$ *(see Sec. 1.4.2) it holds*

$$\frac{t+\dim_{\mathsf{loc}}}{t+\dim_{\mathsf{loc}}-1}|\nabla|\nabla u||^2 \le |\mathrm{Hess}(u)|^2_{HS} + \frac{(\mathsf{tr}\mathrm{Hess}(u))^2}{t}, \quad \mathfrak{m}\text{-}a.e.\ in\ \Omega,\ \forall t>0. \tag{2.3.3}$$

Proof. We make the following preliminary observation. Let A be any symmetric $n \times n$ real matrix, then the following inequality holds for every $t > 0$

$$\frac{t+n}{t+n-1}|A\cdot v|^2 \le \frac{|v|^2(\mathsf{tr}A)^2}{t} + |v|^2|A|^2, \quad \forall v \in \mathbb{R}^n, \quad \forall N \ge n. \tag{2.3.4}$$

where $|A|$ is the Hilbert-Schmidt norm of A. To prove it we can assume that A is diagonal with diagonal entries $\lambda_1,...,\lambda_n$, where $\lambda_n \ge \lambda_i$, $i \le n$ and also that $|v| \le 1$. Applying twice Cauchy-Schwartz we obtain

$$\begin{aligned}\frac{(\lambda_1+...+\lambda_n)^2}{t} + \lambda_1^2+...+\lambda_{n-1}^2+\lambda_n^2 &\ge \frac{(\lambda_1+...+\lambda_n)^2}{t} + \frac{(\lambda_1+...+\lambda_{n-1})^2}{n-1} + \lambda_n^2\\ &\ge \frac{\lambda_n^2}{t+n-1} + \lambda_n^2 \ge \frac{t+n}{t+n-1}|A\cdot v|^2,\end{aligned}$$

which proves (2.3.4).

Observe that it is enough to prove (2.3.3) for $u \in \mathsf{Test}(\mathrm{X})$. Moreover it is enough to prove (2.3.3) in every E_n with $\mathfrak{m}(E_n) > 0$, where $E_n = \{\dim_{\mathsf{loc}} = n\}$ for every $n \in \{1,...,\lfloor N\rfloor\}$ (recall Theorem 1.3.12). Hence we fix one of such E_n. Let $e_1,...,e_n \in L^0(T\mathrm{X})$ be an orthonormal base for $L^0(T\mathrm{X})$ in E_n. Consider the $n\times n$ real matrix $A : \mathrm{X} \to L^0(\mathfrak{m})^{n^2}$ defined by $\{A_{i,j}(x)\}_{i,j} = \{\mathrm{Hess}(u)(e_i,e_j)(x)\}_{i,j}$. Define also the vector $v : \mathrm{X} \to L^0(\mathfrak{m})^n$ as $v_i(x) \coloneqq \langle\nabla u, e_i\rangle(x)$. From (1.4.2) and the formula (1.4.3) we have that

$$2\chi_{E_n}|\nabla u|\nabla|\nabla u| = \chi_{E_n}\nabla|\nabla u|^2 = 2\chi_{E_n}\sum_{i=1}^n\sum_{j=1}^n A_{i,j}v_je_i, \quad \text{in } L^0(T\mathrm{X}).$$

Taking the square pointwise norm on both sides we obtain

$$|\nabla u|^2|\nabla|\nabla u||^2 = |A\cdot v|^2, \quad \mathfrak{m}\text{-a.e. in } E_n.$$

Since from (1.4.4) and (1.4.5) we have that $|\mathrm{Hess}(u)|^2_{HS} = |A|^2_{HS}$ $\mathfrak{m}$-a.e. in E_n and $\mathsf{tr}\mathrm{Hess}(u) = \mathsf{tr}A$ $\mathfrak{m}$-a.e. in E_n, the conclusion follows combining the above identity with (2.3.4). □

The second ingredient for the proof is the following simple technical lemma (cf. with [**117**, Prop. 4.15])

Lemma 2.3.4. *Let* X *be an* $\mathrm{RCD}(K,N)$*space, with* $N < +\infty$. *Let* $(u_n) \subset D(\mathbf{\Delta},\Omega)$ *and* $u \in W^{1,2}_{\mathsf{loc}}(\Omega)$ *be such that* $|\nabla u_n - \nabla u|^2 \to 0$ *in* $L^1_{\mathsf{loc}}(\Omega)$. *Moreover assume that* $\mathbf{\Delta}u_n \ge g_n\mathfrak{m}$ *for some* $g_n \in L^1_{\mathsf{loc}}(\Omega)$ *such that* $\int g_nf\ \mathrm{d}\mathfrak{m} \to \int gf\,\mathrm{d}\mathfrak{m} + \int hf\,\mathrm{d}\mathfrak{m}$*, for every* $f \in \mathsf{LIP}_c(\Omega)$ *with* $f \ge 0$*, for some fixed functions* $g \in L^0(\Omega,\mathfrak{m})$ *and* $h \in L^1_{\mathsf{loc}}(\Omega)$*, with* $g \ge 0$ $\mathfrak{m}$*-a.e.. Then* $g \in L^1_{\mathsf{loc}}(\Omega)$*,* $u \in D(\mathbf{\Delta},\Omega)$ *and* $\mathbf{\Delta}u \ge (g+h)\mathfrak{m}$.

Proof. The assumptions grant that $\int\langle\nabla u_n,\nabla f\rangle\mathrm{d}\mathfrak{m} \to \int\langle\nabla u,\nabla f\rangle\mathrm{d}\mathfrak{m}$, for every $f \in \mathsf{LIP}_c(\Omega)$, therefore we can pass to the limit on both sides of $-\int\langle\nabla u_n,\nabla f\rangle\mathrm{d}\mathfrak{m} \ge \int g_nf\,\mathrm{d}\mathfrak{m}$ to obtain that

$$-\int\langle\nabla u,\nabla f\rangle\mathrm{d}\mathfrak{m} \ge \int gf\,\mathrm{d}\mathfrak{m} + \int hf\,\mathrm{d}\mathfrak{m}, \quad \forall f \in \mathsf{LIP}_c(\Omega), \text{ with } f \ge 0.$$

From this it follows that $g \in L^1_{\mathsf{loc}}(\Omega)$, indeed we can take for any K compact in Ω a function $f \in \mathsf{LIP}_c(\Omega)$ such that $f \geq 0$ and $f = 1$ in K and then bring $\int hf \, \mathrm{d}\mathfrak{m}$ to the other side of the inequality. The conclusion then follows applying Proposition 1.2.12. $\square$

Proof of Theorem 2.3.1. The proof is based on a inductive bootstrap argument. We make the following claim

$$\text{if } \beta > \frac{N-2}{N-1} \text{ is such that } |\nabla u|^\beta \in W^{1,2}_{\mathsf{loc}}(\Omega), \text{ then } |\nabla u|^{\beta/2} \in W^{1,2}_{\mathsf{loc}}(\Omega) \text{ and (2.3.1) holds.}$$

Observe that, since we already know that $|\nabla u|^\beta \in W^{1,2}_{\mathsf{loc}}(\Omega)$ for every $\beta \geq 1$ (recall (1.4.9)), the first part of the conclusion follows iterating the above statement.

We pass to the proof of the claim, hence we fix $\beta > \frac{N-2}{N-1}$ such that $|\nabla u|^\beta \in W^{1,2}_{\mathsf{loc}}(\Omega)$. Since $u \in \mathsf{Test}_{\mathsf{loc}}(\Omega)$, from the local Bochner inequality (1.4.10) combined with the Kato inequality (2.3.3) (with $t = N - \dim_{\mathsf{loc}}$ whenever $\dim_{\mathsf{loc}} < N$ and letting $t \to 0$ in the set where $N = \dim_{\mathsf{loc}}$ (recalling that in the latter case $0 = \Delta u = \mathsf{tr}\mathrm{Hess}(u)$ from (1.4.6)) we have $|\nabla u|^2 \in D(\mathbf{\Delta}, \Omega)$ and

$$\mathbf{\Delta}(|\nabla u|^2) \geq \left(\frac{2N}{N-1} |\nabla |\nabla u||^2 + 2K |\nabla u|^2 \right) \mathfrak{m}_{|\Omega}, \tag{2.3.5}$$

Hence from the chain rule for the Laplacian (second version in Proposition 1.2.18, applied with $\varphi \in C^2(\mathbb{R})$ as $\varphi(t) = (t+\varepsilon)^{\frac{\beta}{2}}$) we have that $(|\nabla u|^2 + \varepsilon)^{\frac{\beta}{2}} \in D(\mathbf{\Delta}, \Omega)$ and

$$\begin{aligned} \mathbf{\Delta}((|\nabla u|^2+\varepsilon)^{\frac{\beta}{2}}) \geq \Bigg[&\beta(|\nabla u|^2+\varepsilon)^{\frac{\beta}{2}-1} |\nabla|\nabla u||^2 \left(\frac{N}{N-1} + \frac{(\beta-2)|\nabla u|^2}{|\nabla u|^2+\varepsilon} \wedge (\beta-2) \right) \\ &+ K\beta (|\nabla u|^2+\varepsilon)^{\frac{\beta}{2}} \frac{|\nabla u|^2}{|\nabla u|^2+\varepsilon} \Bigg] \mathfrak{m}_{|\Omega}, \end{aligned} \tag{2.3.6}$$

for every $\varepsilon > 0$. Note that the above holds without the term "$\wedge(\beta-2)$", however this will be useful to deal with the case $\beta < 2$. Setting $v_\varepsilon := \nabla(|\nabla u|^2+\varepsilon)^{\frac{\beta}{2}}$ it is easy to see using dominated convergence that $|v_\varepsilon - \nabla(|\nabla u|^\beta)|^2 \to 0$ in $L^1_{\mathsf{loc}}(\Omega)$ as $\varepsilon \to 0^+$. Moreover for every $\beta > \frac{N-2}{N-1}$, denoting by $g_{\beta,\varepsilon} \in L^1_{\mathsf{loc}}(\Omega)$ the function on the right hand side of (2.3.6), we have that $\int_\Omega g_{\beta,\varepsilon} f \mathrm{d}\mathfrak{m} \to \int_\Omega g_\beta f \mathrm{d}\mathfrak{m} + 2K\beta \int_\Omega |\nabla u|^\beta f \mathrm{d}\mathfrak{m}$, for every $f \in \mathrm{Lip}_c(\Omega)$ with $f \geq 0$, where $g_\beta \in L^0(\Omega, \mathfrak{m})$ is given by

$$g_\beta := \beta \left(\beta - \frac{N-2}{N-1} \right) \chi_{|\nabla u|>0} |\nabla u|^{\beta-2} |\nabla|\nabla u||^2.$$

This can be seen applying dominated convergence for the second term in (2.3.6) and using respectively dominated convergence in the case $\beta \geq 2$ and monotone convergence in the case $\frac{N-2}{N-1} < \beta < 2$, for the first term. We are therefore in position to apply Lemma 2.3.4 and deduce both that $g_\beta \in L^1_{\mathsf{loc}}(\Omega)$ and that $|\nabla u|^\beta \in D(\mathbf{\Delta}, \Omega)$ with $\mathbf{\Delta}(|\nabla u|^\beta) \geq (g_\beta + K\beta |\nabla u|^\beta) \mathfrak{m}_{|\Omega}$. Moreover the fact that $g_\beta \in L^1_{\mathsf{loc}}(\Omega)$ together with Lemma 1.2.10 implies that $|\nabla u|^{\beta/2} \in W^{1,2}_{\mathsf{loc}}(\Omega)$. This shows the claim and thus concludes the proof of the first part.

We pass to the case $\beta = \frac{N-2}{N-1}$. From the previous part we know that $|\nabla u|^\beta \in W^{1,2}_{\mathsf{loc}}(\Omega)$, hence we can repeat the above argument and observe that in this case $g_\beta = 0$, from which the conclusions follows. $\square$

2.3.2. Divergence estimates.

Corollary 2.3.5. *Let* X *be an* $\mathrm{RCD}(K,N)$ *space with* $N \in (2, \infty)$. *Suppose* u *is positive and harmonic in* Ω, *set* $v = u^{\frac{-1}{N-2}}$ *and let* $\beta > \frac{N-2}{N-1}$. *Then* $|\nabla v|^{\beta/2} \in W^{1,2}_{\mathsf{loc}}(\Omega)$, $u^2 \nabla |\nabla v|^{\beta/2} \in D(\mathbf{div}, \Omega)$ *and*

$$\mathbf{div}(u^2 \nabla |\nabla v|^\beta) \geq \tilde{C}_{\beta,N} \, u^2 |\nabla |\nabla v|^{\frac{\beta}{2}}|^2 \, \mathfrak{m}_{|\Omega} + \beta K u^{-\beta \frac{N-1}{N-2}} |\nabla u|^\beta \, \mathfrak{m}_{|\Omega}, \tag{2.3.7}$$

where $\tilde{C}_{\beta,N} = \frac{4(N-2)^\beta}{\beta} \left(\beta - \frac{N-2}{N-1} \right)$. *Moreover* $u^2 \nabla |\nabla v|^{\beta/2} \in D(\mathbf{div}, \Omega)$ *also for* $\beta = \frac{N-2}{N-1}$ *with* (2.3.7) *holding without the term containing* $\tilde{C}_{\beta,N}$.

Proof of Corollary 2.3.5. We start observing that from the positivity and local Lipschitzianity of u follows that $u^{-1} \in \mathrm{Lip}_{\mathsf{loc}}(\Omega)$ and thus $u^\alpha \in W^{1,2}_{\mathsf{loc}}(\Omega)$ for every $\alpha \in \mathbb{R}$. Moreover, from the chain rule for the Laplacian (first version in Proposition 1.2.18, applied with u and $\varphi(t) = t^\alpha$, $\alpha \in \mathbb{R}$) and by the harmonicity of u, we deduce that $u^\alpha \in D(\mathbf{\Delta}, \Omega)$ with $\mathbf{\Delta}(u^\alpha) = \alpha(\alpha-1) u^{\alpha-2} |\nabla u|^2 \mathfrak{m}_{|\Omega}$. Hence from the Leibniz rule for the Laplacian (Proposition 1.2.16) and Theorem 2.3.1 we have that $|\nabla u|^\beta u^\alpha \in D(\mathbf{\Delta}, \Omega)$ with

$$\mathbf{\Delta}(|\nabla u|^\beta u^\alpha) \geq \left(u^\alpha g_\beta + \alpha(\alpha-1) u^{\alpha-2} |\nabla u|^{\beta+2} + 2\alpha u^{\alpha-1} \langle \nabla |\nabla u|^\beta, \nabla u \rangle \right) \mathfrak{m}_{|\Omega} + K\beta u^\alpha |\nabla u|^\beta \, \mathfrak{m}_{|\Omega},$$

for every $\beta \ge \frac{N-2}{N-1}$ and every $\alpha \in \mathbb{R}$, where g_β is the same as in the proof of Theorem 2.3.1.

Applying the Leibniz rule for the divergence (Proposition 1.2.17) we deduce that $u^2\nabla(|\nabla u|^\beta u^\alpha) \in D(\mathbf{div}, \Omega)$ with

$$\begin{aligned}\mathbf{div}(u^2\nabla(|\nabla u|^\beta u^\alpha)) \ge &\Big(u^{\alpha+2}g_\beta + \alpha(\alpha+1)u^\alpha|\nabla u|^{\beta+2} + 2(\alpha+1)u^{\alpha+1}\langle\nabla|\nabla u|^\beta, \nabla u\rangle\Big)\mathfrak{m}|_\Omega \\ &+ K\beta u^{\alpha+2}|\nabla u|^\beta\,\mathfrak{m}|_\Omega,\end{aligned} \tag{2.3.8}$$

for every $\beta \ge \frac{N-2}{N-1}$ and every $\alpha \in \mathbb{R}$.

We now assume that $\beta > \frac{N-2}{N-1}$. Since $|\nabla v| = (N-2)^{-1}u^{\frac{1-N}{N-2}}|\nabla u|$ and since from Theorem 2.3.1 we have $|\nabla u|^{\beta/2} \in W^{1,2}_{\mathsf{loc}}(\Omega)$, it follows that $|\nabla v|^{\beta/2} \in W^{1,2}_{\mathsf{loc}}(\Omega)$.

To see (2.3.7) we just take $\alpha = -\beta\frac{N-1}{N-2}$ in (2.3.8). Then a direct computation gives that the right hand side of (2.3.8) equals the right hand side of (2.3.7).

Finally choosing $\beta = \frac{N-2}{N-1}$, $\alpha = -1$ in (2.3.8) and recalling that in this case $g_\beta = 0$, shows also the second part of the statement, thus finishing the proof. $\square$

2.4. The monotonicity formula

2.4.1. Decay estimates for the electrostatic potential. Throughout this subsection $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is a non-parabolic $\mathrm{RCD}(0, N)$ space (recall from Remark 2.1.2 that in this case $N > 2$) and u is the electrostatic potential (recall Def. 2.2.11) of an open and bounded set $E \subset \mathrm{X}$. The main goal will be to derive some explicit decay estimates and controls on u and its gradient.

We start setting $\Omega \coloneqq \mathrm{X} \setminus \bar{E}$ and we fix $x_0 \in E$. From the maximum principle (Proposition 1.3.19)

$$0 < u \le 1, \quad \text{in } \Omega.$$

Proposition 2.4.1. *Set $R_0 \coloneqq 3diam(E) + 1$, then*

$$\frac{|\nabla u|(x)}{u(x)} \le \frac{C}{\mathsf{d}(x, x_0)}, \qquad \textit{for } \mathfrak{m}\textit{-a.e. } x \in B_{R_0}(x_0)^c, \tag{2.4.1}$$

where $C = C(N)$ is a positive constant depending only on N.

Proof. Immediate from the gradient estimate (1.3.18) with $R = \mathsf{d}(x, x_0)/4$. $\square$

Proposition 2.4.2. *For every $D > 0$ and $N \in (2, \infty)$ there exists a positive constant $C_2 = C_2(N, D)$ such that the following holds. Let u, Ω, E and $x_0 \in E$ as above and assume that $diam(E) \le D$. Then setting $\delta \coloneqq \mathsf{d}(x_0, \{u \le 1/2\}) \wedge 1$ we have*

$$\frac{\delta^{N-2}}{2}\mathsf{d}(x_0, x)^{2-N} \le u(x) \le C_2\int_{\mathsf{d}(x,x_0)}^{+\infty} s\frac{\mathfrak{m}(B_1(x_0))}{\mathfrak{m}(B_s(x_0))}\,\mathrm{d}s, \quad \forall x \in B_{R_0}(x_0)^c, \tag{2.4.2}$$

where and $R_0 \coloneqq 3diam(E) + 1$ and the first inequality actually holds in $\Omega \cap B_\delta(x_0)^c$.

Before passing to the proof we notice that, since $u = 1$ in ∂E, we have $\mathsf{d}(x_0, \{u \le 1/2\}) > 0$, in particular the left inequality in (2.4.2) is non-trivial.

Proof of Proposition 2.4.2. We start with the first inequality.

From Laplacian comparison (1.3.13) we know that $d^2_{x_0} \in D(\mathbf{\Delta})$ and $\mathbf{\Delta} d^2_{x_0} \le 2N\mathfrak{m}$. Moreover from (1.3.14) $|\nabla d^2_{x_0}|^2 = 4d^2_{x_0}$. Define now the function $h = d^{2-N}_{x_0}$, then from the chain rule for the Laplacian we have that $h \in D(\mathbf{\Delta}, \mathrm{X} \setminus \{x_0\})$, and

$$\mathbf{\Delta} h = \frac{2-N}{2}d^{-N}_{x_0}\mathbf{\Delta} d^2_{x_0} + \frac{2-N}{2}\left(\frac{2-N}{2} - 1\right)4d^2_{x_0}d^{-N-2}_{x_0} = \frac{N-2}{2}d^{-N}_{x_0}(2N\mathfrak{m} - \mathbf{\Delta} d^2_{x_0}) \ge 0.$$

Hence h is subharmonic in $\mathrm{X} \setminus \{x_0\}$. Moreover we have that $\lambda h \le 1/2$ in $B_\delta(x_0)^c$, where $\lambda \coloneqq \delta^{N-2}/2$. Finally from the assumption $\mathsf{d}(x_0, \{u \le 1/2\}) > \delta$ we have $u \ge \lambda h$ in $\Omega \cap B_\delta(x_0)^c$. Fix now $r > 0$ and define the open set $\Omega^r \coloneqq \{x \in \Omega \ : \ \mathsf{d}(x, \Omega^c) > r\}$. Observe that the function $\lambda h - u$ is subharmonic in Ω^r. Therefore from the weak maximum principle (see Proposition 1.3.19) we deduce that

$$\begin{aligned}\sup_{\Omega^r \cap B_R(x_0) \cap B_\delta(x_0)^c}(\lambda h - u) &\le \max_{\partial\Omega^r \cap B_\delta(x_0)^c}(\lambda h - u) \vee \max_{\partial B_\delta(x_0) \cap \Omega^r}(\lambda h - u) \vee \max_{\partial B_R(x_0)}(\lambda h - u) \\ &\le \max_{\partial\Omega^r}(1/2 - u) \vee 0 \vee \max_{\partial B_R(x_0)}(\lambda h - u)\,,\end{aligned}$$

for every $R > R_0$. Sending R to $+\infty$ and r to 0, recalling that both h and u vanish at infinity (since $N > 2$) and that $\underline{\lim}_{x \to \partial\Omega} u(x) \ge 1$, we conclude that $\lambda h \le u$ in $\Omega \cap B_\delta(x_0)^c$. This proves the first inequality in (2.4.2).

We now pass to the second inequality in (2.4.2). We argue by comparison with the quasi Green's function $G^1(x) \coloneqq G^1(x_0, x)$ (recall its definition in (2.1.3)). Recall that G^1 is superharmonic in X. Moreover, using the upper bound for the Green's function and the estimates of the heat kernel we deduce that

$$c_1^{-1} \int_1^{+\infty} \frac{e^{-\frac{\mathsf{d}(x_0,x)^2}{3s}}}{\mathfrak{m}(B_{\sqrt{s}}(x_0))} \, \mathrm{d}s \overset{(1.3.15)}{\leq} G^1(x) \leq G(x, x_0) \overset{(2.1.4)}{\leq} c_1 \int_{\mathsf{d}(x,x_0)}^{+\infty} \frac{s}{\mathfrak{m}(B_s(x_0))} \, \mathrm{d}s, \tag{2.4.3}$$

for every $x \in X \setminus \{x_0\}$, for some positive constant $c_1 = c_1(N) > 1$. From Bishop-Gromov inequality and using the change of variable $s = t\mathsf{d}(x_0, x)^2$ we obtain

$$G^1(x) \geq c_1^{-1} \frac{\mathsf{d}(x_0, x)^{2-N}}{\mathfrak{m}(B_1(x_0))} \int_{\frac{1}{\mathsf{d}(x_0,x)^2}}^{+\infty} \frac{e^{-\frac{1}{3t}}}{t^{\frac{N}{2}}} \, \mathrm{d}t \geq C_1 \frac{\mathsf{d}(x_0, x)^{2-N}}{\mathfrak{m}(B_1(x_0))}, \quad \forall x \in B_1(x_0)^c,$$

for some constant C_1 depending only on N. Therefore, taking $\lambda \coloneqq \mathfrak{m}(B_1(x_0)) \frac{R_0^{N-2}}{C_1}$, we have $\lambda G^1 \geq 1 \geq u$ in $\partial B_{R_0}(x_0)$. Hence, since $\lambda G^1 - u$ is superharmonic in Ω, from the weak maximum principle it follows that for every $R > R_0$

$$\inf_{B_R(x_0) \cap B_{R_0}(x_0)^c} (\lambda G^1 - u) = \min_{\partial B_R(x_0) \cup \partial B_{R_0}(x_0)} (\lambda G^1 - u) \geq \min(0, \min_{\partial B_R(x_0)} (\lambda G^1 - u)).$$

Sending R to $+\infty$ and recalling that both G^1 and u go to 0 at infinity we conclude that $u \leq \lambda G^1$ in $B_{R_0}(x_0)^c$, which combined with the second bound in (2.4.3) gives the second inequality in (2.4.2). □

2.4.2. Monotonicity. Here we prove our main monotonicity result for the electrostatic potential (see Theorem 2.4.4 below).

As in the previous section $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is assumed to be a non-parabolic $\mathrm{RCD}(0, N)$ space, u is the electrostatic potential of an open and bounded set $E \subset \mathrm{X}$ and $\Omega \coloneqq \mathrm{X} \setminus \bar{E}$.

We start with the following simple remark, which allows to define our candidate monotone quantity U_β and will be needed to justify several applications of the coarea formula along all this section.

Remark 2.4.3. Since u is locally Lipschitz (recall Proposition 2.4.1), satisfies $u = 1$ in $\partial\Omega$ and vanishes at infinity, it follows that u satisfies the hypotheses needed to apply the coarea formula (1.3.12) in Ω. In particular for every $f \in L^1_{\mathsf{loc}}(\Omega)$ with $f\mathfrak{m}_{|_\Omega} \ll |\nabla u|\mathfrak{m}_{|_\Omega}$ and every $\varphi : [0, 1] \to \mathbb{R}$ Borel, with $\mathrm{supp}(\varphi) \subset (0, 1)$, we have

$$\int_0^1 \varphi(t) \int g \, \mathrm{dPer}(\{u < r\}) \, \mathrm{d}r = \int_\Omega \varphi(u) f \, \mathrm{d}\mathfrak{m} < +\infty, \tag{2.4.4}$$

where g is any Borel representative of the function $\frac{f}{|\nabla u|}$ taken to be zero whenever $|\nabla u| = 0$. Therefore, by the arbitrariness of φ, we also deduce that:

$$\begin{gathered} \text{for any } f \in L^1_{\mathsf{loc}}(\Omega) \text{ with } f\mathfrak{m}_{|_\Omega} \ll |\nabla u|\mathfrak{m}_{|_\Omega} \text{ and for any choice of the Borel representative } g, \\ \text{the function } (0, 1) \ni r \mapsto \int g \, \mathrm{dPer}(\{u < r\}) \text{ is in } L^1_{\mathsf{loc}}(0, 1) \text{ and (its a.e. equivalence class)} \\ \text{does not depend on the choice of the representative } g. \end{gathered} \tag{2.4.5}$$

■

Choosing in the above remark $f = \frac{|\nabla u|^{\beta+2}}{u^{\beta \frac{N-1}{N-2}}} \in L^1_{\mathsf{loc}}(\Omega)$, with $\beta > -2$, and observing that $\mathrm{supp}(\mathrm{Per}(\{u < t\})) \subset \{u = t\}$ we deduce that, fixed a Borel representative of $|\nabla u|$, the function

$$U_\beta(t) \coloneqq \frac{1}{t^{\beta \frac{N-1}{N-2}}} \int |\nabla u|^{\beta+1} \, \mathrm{dPer}(\{u < t\}) \in L^1_{\mathsf{loc}}(0, 1), \tag{2.4.6}$$

is well defined and independent of the representative chosen for $|\nabla u|$.

We are ready to state our main monotonicity result.

Theorem 2.4.4. *Let* X *be a non-parabolic* $\mathrm{RCD}(0, N)$ *space and let* u *be an electrostatic potential for a bounded open set* E *(see Def. 2.2.11). Letting* $U_\beta \in L^1_{\mathsf{loc}}(0, 1)$, *with* $\beta \geq \frac{N-2}{N-1}$, *be the function defined in* (2.4.6), *it holds that* $U_\beta \in W^{1,1}_{\mathsf{loc}}(0, 1)$, $U'_\beta \in BV_{\mathsf{loc}}(0, 1)$ *and*

$$U_\beta'^{-}(t) \geq \frac{\tilde{C}_{\beta,N}}{t^2} \int_{\{u<t\}} u^2 |\nabla |\nabla u^{\frac{1}{2-N}}|^{\frac{\beta}{2}}|^2 \mathrm{d}\mathfrak{m}, \quad \forall t \in (0, 1], \tag{2.4.7}$$

(recall that $|\nabla u^{\frac{1}{2-N}}|^{\frac{\beta}{2}} \in W^{1,2}_{loc}(\Omega)$ *for every* $\beta > \frac{N-2}{N-1}$ *by Corollary 2.3.5) where* $\tilde{C}_{\beta,N} = (N-2)^{\beta}\frac{4}{\beta}\left(\beta - \frac{N-2}{N-1}\right)$, $U_{\beta}^{'-}$ *being the left continuous representative of* U'_β *and where the left hand side is taken to be 0 if* $\beta = \frac{N-2}{N-1}$*. In particular* U_β *is non-decreasing.*

To prove Theorem 2.4.4 we start computing the first derivative of U_β (which does not evidently carry a sign).

Proposition 2.4.5. *With the same assumptions as in Theorem 2.4.4, the function* U_β *belongs to* $W^{1,1}_{\mathsf{loc}}(0,1)$ *and its derivative is given by*

$$U'_\beta(t) = \int \langle \frac{\nabla u}{|\nabla u|}, \nabla\left(\frac{|\nabla u|^\beta}{u^{\beta\frac{N-1}{N-2}}}\right)\rangle \mathrm{dPer}(\{u<t\}), \quad a.e.\ t\in(0,1), \tag{2.4.8}$$

where the right hand side has to be intended as in (2.4.5) *with* $f = \langle \nabla u, \nabla\left(\frac{|\nabla u|^\beta}{u^{\beta\frac{N-1}{N-2}}}\right)\rangle$.

Proof. Consider the vector field $v := \frac{\nabla u |\nabla u|^\beta}{u^{\beta\frac{N-1}{N-2}}} \in L^0(TX)_{|\Omega}$ for $\beta \geq \frac{N-2}{N-1}$ and observe that from the Leibniz rule for the divergence (second version in Proposition 1.2.17) $v \in D(\mathrm{div},\Omega)$ with

$$\mathrm{div}(v) = \langle \nabla u, \nabla\left(\frac{|\nabla u|^\beta}{u^{\beta\frac{N-1}{N-2}}}\right)\rangle \in L^1_{\mathsf{loc}}(\Omega),$$

thanks to the harmonicity of u. In particular $\mathrm{div}(v)\mathfrak{m} \ll |\nabla u|\mathfrak{m}$, hence recalling (2.4.4) and integrating by parts we have

$$\int_0^1 \int \frac{\mathrm{div}(v)}{|\nabla u|}\mathrm{dPer}(\{u<t\})\varphi(t)\mathrm{d}t \overset{(2.4.4)}{=} \int \mathrm{div}(v)\varphi(u)\mathrm{d}\mathfrak{m} = -\int \frac{|\nabla u|^{\beta+2}}{u^{\beta\frac{N-1}{N-2}}}\varphi'(u)\mathrm{d}\mathfrak{m} \overset{(1.3.12)}{=} -\int_0^1 U_\beta(t)\varphi'(t)\mathrm{d}t,$$

for every $\varphi \in C^1_c(0,1)$, where in the last step we used that $\mathrm{supp}(\mathrm{Per}(\{u<t\},.)) \subset \{u=t\}$ and having fixed a Borel representative of $\frac{\mathrm{div}(v)}{|\nabla u|}$. The conclusion follows. □

To prove that U'_β is nonnegative we need to push our analysis to the second order and in particular to compute the derivative of $U'_\beta(t)t^2$. The reason for the term t^2 is that the key vector field with nonnegative divergence of Corollary 2.3.5 presents a term u^2.

Proposition 2.4.6. *With the same assumptions as in Theorem 2.4.4, the function* $U_\beta(t)'t^2$ *belongs to* $BV_{\mathsf{loc}}(0,1)$ *and*

$$(U'_\beta(t)t^2)' \geq \tilde{C}_{\beta,N}\left(\int \frac{u^2|\nabla|\nabla u^{\frac{1}{2-N}}|^{\frac{\beta}{2}}|^2}{|\nabla u|}\mathrm{dPer}(\{u<t\})\right)\mathcal{L}^1|_{(0,1)} \geq 0, \tag{2.4.9}$$

where $\tilde{C}_{\beta,N} = (N-2)^\beta\frac{4}{\beta}\left(\beta - \frac{N-2}{N-1}\right)$ *and where the right hand side has to be intended as in* (2.4.5) *with* $f = u^2|\nabla|\nabla u^{\frac{1}{2-N}}|^{\frac{\beta}{2}}|^2$ *when* $\beta > \frac{N-2}{N-1}$ *(recall also that from Corollary 2.3.5* $|\nabla u^{\frac{1}{2-N}}| \in W^{1,2}_{\mathsf{loc}}(\Omega)$*), and identically 0 in the case* $\beta = \frac{N-2}{N-1}$.

Proof. Consider any non-negative $\varphi \in C^1_c(0,1)$. Applying formula (2.4.8) and the coarea formula (2.4.4)

$$\begin{aligned}\int_0^1 (U'_\beta(t)t^2)\varphi'(t)\mathrm{d}t &= \int_0^1 \int \langle \frac{\nabla u}{|\nabla u|}, u^2\nabla\left(\frac{|\nabla u|^\beta}{u^{\beta\frac{N-1}{N-2}}}\right)\rangle\varphi'(u)\mathrm{dPer}(\{u<t\}) \\ &\overset{(2.4.4)}{=} \int \langle \nabla(\varphi(u)), u^2\nabla\left(\frac{|\nabla u|^\beta}{u^{\beta\frac{N-1}{N-2}}}\right)\rangle\,\mathrm{d}\mathfrak{m},\end{aligned}$$

observing that $\varphi(u) \in \mathsf{LIP}_c(\Omega)$ and recalling from Corollary 2.3.5 that $u^2\nabla\left(\frac{|\nabla u|^\beta}{u^{\beta\frac{N-1}{N-2}}}\right) \in D(\mathbf{div},\Omega)$, we obtain

$$-\int_0^1 (U'_\beta(t)t^2)\varphi'(t)\mathrm{d}t = \int \varphi(u)\,\mathrm{d}\mathbf{div}\left(u^2\nabla\left(\frac{|Du|^\beta}{u^{\beta\frac{N-1}{N-2}}}\right)\right).$$

We now plug in (2.3.7) and (when $\beta > \frac{N-2}{N-1}$) apply the coarea formula (2.4.4) (observe that $||\nabla|\nabla u^{\frac{1}{2-N}}|^{\frac{\beta}{2}}|^2|\mathfrak{m} \ll |\nabla u|\mathfrak{m}$) to obtain

$$-\int_0^1 (U'_\beta(t)t^2)\varphi'(t)\mathrm{d}t \geq \tilde{C}_{\beta,N}\int_0^1 \int u^2|\nabla u|^{-1}|\nabla|\nabla u^{\frac{1}{2-N}}|^{\frac{\beta}{2}}|^2\,\mathfrak{m}_{|\Omega}\mathrm{dPer}(\{u<t\})\varphi(t)\,\mathrm{d}t \geq 0, \tag{2.4.10}$$

for any Borel representative of $u^2|\nabla u|^{-1}|\nabla|\nabla u^{\frac{1}{2-N}}|^{\frac{\beta}{2}}|^2$. The proof is concluded observing that (2.4.10) gives at once that the distributional derivative of $U'(t)t^2$ is a locally finite measure (it is positive) and that (2.4.9) holds. □

Justified by Proposition 2.4.5, from this point on we will identify U_β with its continuous representative. Moreover Proposition 2.4.6 grants that $U'_\beta \in BV_{\mathsf{loc}}(0,1)$, thus we will denote by $U_\beta^{'-}$ its representative which is left-continuous in $(0,1]$ (notice that $U_\beta^{'-}$ might take value $+\infty$ at $t=1$). We observe also that (2.4.9) implies that

$$(0,1] \ni t \mapsto U_\beta^{'-}(t)t^2 \text{ is a non decreasing function.} \tag{2.4.11}$$

To prove Theorem 2.4.4 we aim to integrate (2.4.9), however to do so we still need to know that U_β is bounded close to 0. In particular we prove the following:

Proposition 2.4.7. *With the same assumptions as in Theorem 2.4.4,*

$$U_\beta \in L^\infty(0,1/2). \tag{2.4.12}$$

Proof. It is enough to show that

$$\left|\int_0^{\frac{1}{2}} U_\beta \varphi \,\mathrm{d}t\right| \le C\int_0^{\frac{1}{2}} |\varphi|, \quad \forall \varphi \in C^1_c(0,1/2), \tag{2.4.13}$$

for some positive constant C independent of φ.

We start observing that, integrating by parts and applying the coarea formula (2.4.4),

$$0=\int_\Omega \Delta u \varphi(u)\,\mathrm{d}\mathfrak{m} = -\int_\Omega |\nabla u|^2\varphi'(u),\mathrm{d}\mathfrak{m} = -\int_0^1\int |\nabla u|\,\mathrm{dPer}(\{u<r\})\varphi'(r)\mathrm{d}r, \quad \forall \varphi \in C^1_c(0,1),$$

in particular $\int |\nabla u|\,\mathrm{dPer}(\{u<r\}) = D$ for a.e. $r\in(0,1)$, for some constant D. Therefore using again the coarea formula

$$\begin{aligned}\left|\int_0^{\frac{1}{2}} U_\beta \varphi \,\mathrm{d}t\right| &\le \int u^{\beta\frac{1-N}{N-2}}|\nabla u|^{\beta+2}|\varphi(u)|\,\mathrm{d}\mathfrak{m} \le \left\||\nabla u|^\beta u^{\beta\frac{1-N}{N-2}}\right\|_{L^\infty(\{u\le 1/2\})}\int |\nabla u|^2|\varphi(u)|\mathrm{d}\mathfrak{m}\\ &\overset{(2.4.4)}{=} \left\||\nabla u|^\beta u^{\beta\frac{1-N}{N-2}}\right\|_{L^\infty(\{u\le 1/2\})}\int_0^{\frac{1}{2}}\int |\nabla u|\,\mathrm{dPer}(\{u<r\})|\varphi(r)|\,\mathrm{d}r\\ &= D\left\||\nabla u|^\beta u^{\beta\frac{1-N}{N-2}}\right\|_{L^\infty(\{u\le 1/2\})}\int_0^{\frac{1}{2}}|\varphi|, \quad \forall \varphi\in C^1_c(0,1/2).\end{aligned}$$

Therefore to prove (2.4.13) it remains to show that $\left\||\nabla u|^\beta u^{\beta\frac{1-N}{N-2}}\right\|_{L^\infty(\{u\le 1/2\})} < +\infty$. Let R_0 be as in Proposition 2.4.2 and observe that u, being positive and satisfying $u=1$ on $\partial\Omega$, is bounded away from zero in $B_{R_0}(x_0)\cap\Omega$. Moreover again thanks to $u=1$ on $\partial\Omega$ we have $\mathsf{d}(\partial\Omega,\{u\le 1/2\})>0$. Therefore, since $u\in\mathsf{LIP}_{\mathsf{loc}}(\Omega)$, we have

$$\left\||\nabla u|^\beta u^{\beta\frac{1-N}{N-2}}\right\|_{L^\infty(\{u\le 1/2\}\cap B_{R_0}(x_0))} < +\infty.$$

Moreover combining Proposition 2.4.1 and the lower bound in (2.4.2) we obtain

$$\||\nabla u|^\beta u^{\beta\frac{1-N}{N-2}}\|_{L^\infty(\Omega\cap B_{R_0}(x_0)^c)} \le \left\|\left(\frac{C(N)}{\mathsf{d}(.,x_0)^{N-2}u}\right)^\beta\right\|_{L^\infty(\Omega\cap B_{R_0}(x_0)^c)} < +\infty.$$

Combining the two estimates we conclude. □

We are now ready to prove the main monotonicity result.

Proof of Theorem 2.4.4. We start observing that, thanks to (2.4.9), setting $\mu := (U_\beta(t)'t^2)' \ge 0$ we have

$$U_\beta^{'-}(t)t^2 - U_\beta^{'-}(s)s^2 = \mu([s,t)) \ge \int_s^t\int g\mathrm{dPer}(\{u<r\})\,\mathrm{d}r \overset{(2.4.4)}{=} \int_{\{s<u<t\}}\Phi(u)\mathrm{d}\mathfrak{m}, \tag{2.4.14}$$

for every $0<s<t\le 1$, with $\Phi(u) = \tilde{C}_{\beta,N}u^2|\nabla|\nabla u^{\frac{1}{2-N}}|^{\frac{\beta}{2}}|^2$ when $\beta > \frac{N-2}{N-1}$, $\Phi(u)=0$ when $\beta=\frac{N-2}{N-1}$ and where g is a Borel representative of $\frac{\Phi(u)}{|\nabla u|}$. Therefore to conclude it is enough to prove that there exists a sequence $s_n\to 0^+$ such that $U_\beta^{'-}(s_n)s_n^2 \to 0$.

To achieve this we first prove that $U_\beta^{'-}(t) \geq 0$ for every $t \in (0,1)$. We assume by contradiction that there exists $T \in (0,1)$ such that $U_\beta^{'-}(T) < 0$. From (2.4.11)

$$U_\beta^{'-}(s) \leq U_\beta^{'-}(T)\frac{T^2}{s^2}, \quad \forall s < T,$$

from which integrating with respect to s on the interval (t,T)

$$U_\beta(T) - U_\beta(t) \leq U_\beta^{'-}(T)T^2\left(\frac{1}{t} - \frac{1}{T}\right).$$

Sending $t \to 0^+$ and recalling that $U_\beta^{'-}(T) < 0$ we obtain $U_\beta(t) \to +\infty$ as $t \to 0^+$, which however contradicts (2.4.12).

Since $U_\beta^{'-}(t) \geq 0$ we have that U_β is non-decreasing and also non-negative, hence it admits a limit as $t \to 0^+$. In particular $U_\beta^{'} \in L^1(0, \frac{1}{2})$, therefore

$$a_n := \int_{2^{-(n+1)}}^{2^{-n}} U_\beta^{'}(t)\mathrm{d}t \to 0, \text{ as } n \to +\infty.$$

Moreover from Markov inequality we have $|\{U_\beta^{'-} > a_n 2^{n+2}\} \cap (2^{-(n+1)}, 2^{-n})| \leq \frac{1}{2}2^{-(n+1)}$, thus for every n we can find $s_n \in (2^{-(n+1)}, 2^{-n})$ such that $U_\beta^{'-}(s_n) \leq a_n 2^{n+2}$. Therefore

$$U_\beta^{'-}(s_n)s_n \leq a_n 2^{n+2}s_n < 4a_n \to 0$$

and the proof is complete. □

2.5. Functional versions of the rigidity and almost rigidity

2.5.1. From outer functional cone to outer metric cone. The following result is a variant of the "from volume cone to metric cone" theorem for RCD spaces (see [**95**]). The two main differences with the work in [**95**] are that here we start from a function satisfying an equation (instead that from a condition on the measure), from which we deduce a conical structure on the complement of a bounded set (instead that on a ball).

Theorem 2.5.1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* RCD$(0,N)$ *space with $N \in [2,\infty)$ and $U \subset X$ be open with ∂U bounded. Suppose there exists a positive $\mathbf{u} \in D(\boldsymbol{\Delta}, U) \cap C(\bar{U})$ such that $\Delta \mathbf{u} = N$ $\mathfrak{m}$-a.e. in U, $|\nabla\sqrt{2\mathbf{u}}|^2 = 1$ $\mathfrak{m}$-a.e. in U, $\mathbf{u}_0 := \overline{\lim}_{x\to\partial U} u(x) < +\infty$ and $\{\mathbf{u} > \mathbf{u}_0\} \neq \emptyset$.*

Then

- *i) there exists unique an* RCD$(N-2, N-1)$ *space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ with diam$(Z) \leq \pi$ and a bijective measure preserving local isometry $S : \{\mathbf{u} > \mathbf{u}_0\} \to Y \setminus \overline{B_r(O_Y)}$, with $r := \sqrt{2\mathbf{u}_0}$ and where $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ is the Euclidean N-cone built over Z with vertex O_Y,*
- *ii)*
  - *if $D_Z :=$ diam$(Z) < \pi$ then the local isometry of point i) is an isometry between $Y \setminus \overline{B_{r_Z}(O_Y)}$ and $\{\mathbf{u} > r_Z^2/2\}$, where $r_Z := r(1 - \sin D_Z/2)^{-1} > r$,*
  - *if diam$(Z) = \pi$, then $(X, \mathsf{d}, \mathfrak{m})$ isomorphic to $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$,*
- *iii) the function $\mathbf{u}$ has the following explicit form*

$$\mathbf{u}(x) = \frac{1}{2}\mathsf{d}_Y(S(x), O_Y)^2 = \frac{1}{2}(\mathsf{d}(x, \partial\{\mathbf{u} > \mathbf{u}_0\}) + \sqrt{2\mathbf{u}_0})^2, \quad \forall x \in \{\mathbf{u} > \mathbf{u}_0\}, \tag{2.5.1}$$

in particular the level set $\{u = \frac{t^2}{2}\}$, for every $t > \mathbf{u}_0$, is Lipschitz-path connected and isometric (with its induced intrinsic distance) to $(Z, t\mathsf{d}_Z)$.

We observe that the uniqueness part of Theorem 2.5.1 is an immediate consequence of the rest of the statement. Indeed, from the last part of *iii)* we deduce that the metric space (Z, d_Z) (and thus (Y, d_Y)) is uniquely determined up to isometries. Moreover, since S is measure preserving, the measure $\mathfrak{m}_Y$ is uniquely determined as well, hence from the definition of the measure in an N-cone we obtain that also $\mathfrak{m}_Z$ is uniquely determined.

As already said, the proof of the above Theorem is mainly an adaptation of the proof in [**95**]. However some parts will require new arguments. The first main point is that the in [**95**] the starting point is the gradient flow of the distance function d, which is used to deduce analytical informations on d. Here instead we start from an analytical information, i.e. a PDE, and we want to build a flow. This will be done through the tool of Regular Lagrangian Flows. One of the main tools we need to develop in this regard is an a-priori estimate of local type, which seems to be missing in literature and does not follows immediately from the standard global a-priori estimates in [**28**].

The second main difference is that here the analysis takes place in the complementary of a bounded set, while in [**95**] all the work is done inside a fixed ball. Among other things, this difference will mainly affect the way in which we deduce that the cone is itself an $\mathrm{RCD}(0,N)$ space. Indeed in [**95**] this follows from the fact that a whole ball with center x_0 is isometric to a ball centered at the tip of the cone, therefore any blow up of the space at x_0 will converge to the said cone. Then from the closedness of the $\mathrm{RCD}(0,N)$ condition the conclusion follows. However in our case the same argument cannot be applied, indeed our isometry is by nature far from the tip of the cone. This issue will be overcome noticing that our isometry is almost global, meaning that the space is isometric to the cone outside a bounded set. This allow us to deduce that any blow down of the space will converge to the cone, which gives the conclusion again by the closedness of the $\mathrm{RCD}(0,N)$ class.

Since they are interesting on their own and independent of the rest of the proof, we isolate the two ingredients that we just described in the following two subsections. The remaining part of the argument will be outlined in Appendix 2.A.

The blow down argument.

Proposition 2.5.2. *Let $(\mathrm{X},\mathsf{d}_\mathrm{X},\mathfrak{m}_\mathrm{X})$ be a m.m.s. and let $U\subset \mathrm{X}$ be closed and bounded. Suppose that there exists an Euclidean N-cone $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ over a m.m.s. Z, $N\in[1,\infty)$, with tip O_Y and a bijective local isometry $T:U^c\to Y\setminus B_R(O_Y)$, which is measure-preserving, i.e. $T_*\mathfrak{m}_\mathrm{X}|_{U^c}=\mathfrak{m}_Y|_{B_R(O_Y)}$.*

Then for every $x_0\in\mathrm{X}$ and every sequence $r_n\to+\infty$ it holds that

$$(\mathrm{X},r_n^{-1}\mathsf{d}_\mathrm{X},r_n^{-N}\mathfrak{m}_\mathrm{X},x_0)\overset{pmGH}{\longrightarrow}(Y,\mathsf{d}_Y,\mathfrak{m}_Y,O_Y).$$

In particular if X is an $\mathsf{RCD}(0,N)$ space, then Y is an $\mathsf{RCD}(0,N)$ space as well.

Finally if X is $\mathsf{RCD}(0,N)$ and $diam(Z)=\pi$ then X is isomorphic to Y as m.m.s..

Proof. Fix $x_0\in\mathrm{X}$ and observe that, up to increase R and enlarge U, it is not restrictive to assume that $x_0\in U$.

Set $D:=\mathrm{diam}(U)$, $\delta_n:=\frac{1}{4r_n}(D+R)$ and define $\mathrm{X}_n:=(\mathrm{X},r_n^{-1}\mathsf{d}_\mathrm{X},r_n^{-N}\mathfrak{m}_\mathrm{X},x_0)$. Without loss of generality we will assume that $r_n\geq 1$.

We consider the map $i_n:Y_n\to Y$, defined as $i_n(t,z):=(t/r_n,z)$ in polar coordinates, which satisfies $\mathsf{d}_Y(i_n(y_1),i_n(y_2))=r_n^{-1}\mathsf{d}_Y(y_1,y_2)$ for every $y_1,y_2\in Y$ and $i_n(B_r^Y(O_Y))=B^Y_{r_n^{-1}r}(O_Y)$ for every $r>0$. Observe that in particular $i_{n*}\mathfrak{m}_Y=r_n^N\mathfrak{m}_Y$.

We extend T to the whole X by setting $T(x)=O_Y$ for every $x\in U$ and we denote this new map again by T. It is straightforward to check that

$$|\mathsf{d}_X(x_1,x_2)-\mathsf{d}_Y(T(x_1),T(x_2))|\leq 2(R+D),\quad \forall\, x_1,x_2\in X. \tag{2.5.2}$$

Define now the map $T_n:\mathrm{X}_n\to Y$ as $T_n=i_n\circ T$. It follows from (2.5.2) and the properties of i_n that T_n is a δ_n-isometry. Moreover it can be readily checked that the R-neighbourhood of $T(B_r^\mathrm{X}(x_0))$ contains $B^Y_{r-R-D}(O_Y)$, for any $r\gg R+D$. In particular it follows that the δ_n neighbourhood of $T_n(B_r^{\mathrm{X}_n}(x_0))$ contains $B^Y_{r-\delta_n}(O_Y)$, for every $r>R+D$. Finally we let $\varphi\in C_b(Y)$ be of bounded support, since T is measure preserving we have

$$\begin{aligned}r_n^{-N}\int\varphi\circ T_n\mathrm{d}\mathfrak{m}&=r_n^{-N}\int_U\varphi\circ T_n\mathrm{d}\mathfrak{m}+r_n^{-N}\int_{Y\setminus B_R(O_Y)}\varphi\circ i_n\mathrm{d}\mathfrak{m}_Y\\&=r_n^{-N}\int_U\varphi\circ T_n\mathfrak{m}+\int_{Y\setminus B_{r_n^{-1}R}(O_Y)}\varphi\mathrm{d}\mathfrak{m}_Y.\end{aligned}$$

Passing to the limit, observing that the first term on the right hand side vanishes as $r_n\to+\infty$, we obtain $r_n^{-N}\int\varphi\circ T_n\mathrm{d}\mathfrak{m}\to\int\varphi\mathrm{d}\mathfrak{m}_Y$ as $r_n\to+\infty$. This concludes the first part.

The second part follows immediately from the closedness of the $\mathrm{RCD}(0,N)$ condition under pmGH-convergence.

Suppose now that X is an $\mathrm{RCD}(0,N)$ space and $\mathrm{diam}(Z)=\pi$. Then Y must contain a line. Therefore, since from the previous part Y is an $\mathrm{RCD}(0,N)$ space, it follows from the splitting theorem ([**115**], [**116**]) that Y is isomorphic to $(\mathbb{R}\times Y',\mathsf{d}_{Eucl}\times\mathsf{d}',\mathcal{L}^1\otimes\mathfrak{m}'_Y)$ for some m.m.s. $(Y',\mathsf{d}',\mathfrak{m}_{Y'})$. In particular $O_Y=(\bar t,\bar y)$ for some $\bar t\in\mathbb{R}$ and $\bar y\in Y'$ and $\mathfrak{m}_Y(B_r(O_Y))=\mathfrak{m}_Y(B_r(s,\bar y))$, for any $r>0$ and any $s\in\mathbb{R}$.

Therefore taking s big enough we have that $O':=(s,\bar y)\in Y$ satisfies $O'\in\{\mathsf{d}_Y(.,O_Y)>R+1\}$. Therefore, since $T|_{U^c}$ is a measure preserving local isometry, $\mathfrak{m}_Y(B_r(O'))=\mathfrak{m}_X(B_r(T^{-1}(O')))$ holds for every $r\in(0,1)$. Hence

$$\lim_{r\to0^+}\frac{\mathfrak{m}(B_r(T^{-1}(O')))}{r^N}=\lim_{r\to0^+}\frac{\mathfrak{m}_Y(B_r(O'))}{r^N}=\lim_{r\to0^+}\frac{\mathfrak{m}_Y(B_r(O_Y))}{r^N}=\mathfrak{m}_Y(B_1(O_Y))=:\theta,$$

since O_Y is the vertex of Y. On the other hand, since $\mathrm{X}_n := (\mathrm{X}, r_n^{-1}\mathsf{d}_\mathrm{X}, r_n^{-N}\mathfrak{m}_\mathrm{X}, x_0) \overset{\mathrm{pm}GH}{\longrightarrow} (Y, \mathsf{d}_Y, \mathfrak{m}_Y, O_Y)$ we have

$$\lim_{r_n \to +\infty} \frac{\mathfrak{m}(B_{r_n}(T^{-1}(O')))}{r_n{}^N} = \lim_{r_n \to +\infty} \frac{\mathfrak{m}_X(B_{r_n}(x_0))}{r_n^N} = \lim_n \mathfrak{m}_{X_n}(B_1(x_0)) = \mathfrak{m}_Y(B_1(O_Y)) = \theta.$$

From Bishop-Gromov inequality we deduce that $\frac{\mathfrak{m}(B_r(T^{-1}(O')))}{r^N} = \theta$ for every $r > 0$ and from [**95**, Thm. 1.1] we must have that X is a cone, which must evidently coincide with Y. ☐

Local a priori estimate for Regular Lagrangian Flows in RCD spaces. The following local version of the a priori estimates in [**28**, Prop. 4.6] will be crucial for the argument of Appendix 2.A to work (see Proposition 2.A.6).

Proposition 2.5.3. *Let $\{v_t, \mu_t\}_{t\in[0,T]}$ be as in Theorem 1.7.4. Assume additionally that $\{\mu_t\}_{t\in[0,T]}$ are all concentrated in a common bounded Borel set B. Then setting $\rho_t := \frac{\mathrm{d}\rho_t}{\mathrm{d}\mathfrak{m}}$, $t \in [0,T]$, it holds that*

$$\sup_{t\in(0,T)} \|\rho_t\|_{L^\infty} \le \|\rho_0\|_{L^\infty} e^{\int_0^T \|\mathrm{div}(v_t)^-\|_{L^\infty(B_t)}\,\mathrm{d}t}, \tag{2.5.3}$$

for any family $\{B_t\}_{t\in[0,T]}$ of Borel sets such that $\mathrm{supp}(\rho_t) \subset B_t$ *and the map $(x,t) \mapsto \chi_{B_t}(x)$ is Borel.*

For the proof of Proposition 2.5.3 we will need the following :

Lemma 2.5.4 (Commutator estimate [**28**, Lemma 5.8])**.** *Let* X *be an* $\mathrm{RCD}(K,\infty)$ *m.m.s., then there exists a positive constant $C = C(K) > 0$ such that the following holds. Let $v \in W^{1,2}_C(TX)$ with* $\mathrm{div}(v) \in L^\infty(\mathfrak{m})$, *then*

$$\int \langle \nabla h_t(f), v\rangle g \mathrm{d}\mathfrak{m} + \int f \mathrm{div}(h_t(g)v_t)\mathrm{d}\mathfrak{m} \le C\left(\|\nabla v\|_{L^2(T^{\otimes 2}X)} + \|\mathrm{div}(v)\|_{L^\infty}\right)\|f\|_{L^2\cap L^4}\|g\|_{L^2\cap L^4}, \tag{2.5.4}$$

for every $f, g \in L^2(\mathfrak{m}) \cap L^4(\mathfrak{m})$ and every $t > 0$. In particular for fixed g the left hand side of (2.5.4) *defines a functional in $(L^2(\mathfrak{m}) \cap L^4(\mathfrak{m}))^* = L^2(\mathfrak{m}) + L^{4'}(\mathfrak{m})$, denoted by $\mathscr{C}^t(g,v)$, which satisfies*

$$\|\mathscr{C}^t(g,v)\|_{L^2(\mathfrak{m})+L^{4'}(\mathfrak{m})} \le C\left(\|\nabla_{sym} v\|_{L^2(T^{\otimes 2}X)} + \|\mathrm{div}(v)\|_{L^\infty}\right)\|g\|_{L^2\cap L^4}. \tag{2.5.5}$$

Moreover it holds that

$$\|\mathscr{C}^t(g,v)\|_{L^2(\mathfrak{m})+L^{4'}(\mathfrak{m})} \to 0, \qquad \text{as } t \to 0^+. \tag{2.5.6}$$

We can now pass to the proof of the local a priori estimate.

Proof of Proposition 2.5.3. We start with the preliminary observation that, combining the fact that $\mu_t = F_{t*}\mu_0$ with (1.7.1) and recalling that B is bounded, we have

$$\sup_{t\in[0,T]} \|\rho_t\|_{L^q(\mathfrak{m})} < +\infty, \quad \forall q \in [1,\infty]. \tag{2.5.7}$$

To conclude it is sufficient to prove that for every $p > 1$ the function $[0,T] \ni t \mapsto \int (\rho_t)^p \mathrm{d}\mathfrak{m}$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (\rho_t)^p \mathrm{d}\mathfrak{m} \le (p-1)\int (\rho_t)^p \mathrm{div}(v_t)^- \mathrm{d}\mathfrak{m}, \quad \text{for a.e. } t \in (0,T). \tag{2.5.8}$$

Indeed (2.5.3) would follow first applying Gronwall Lemma (noticing that $\rho_t = 0$ $\mathfrak{m}$-a.e. outside B_t) and then letting $p \to +\infty$.

So we fix $p > 1$. Pick a sequence $s_n \downarrow 0$ and define $\rho_t^n := h_{s_n}\rho_t$. From the fact that ρ_t is a solution of the continuity equation and the selfadjointedness of the heat flow, we obtain that for every $f \in \mathsf{LIP}_{bs}(\mathrm{X})$, the function $t \mapsto \int f\rho_t^n \mathrm{d}\mathfrak{m}$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f\rho_t^n \mathrm{d}\mathfrak{m} = \int \langle \nabla h_{s_n} f, v_t\rangle \rho_t\, \mathrm{d}\mathfrak{m} = \int f[\mathscr{C}^{s_n}(\rho_t, v_t) - \mathrm{div}(\rho_t^n v_t)]\mathrm{d}\mathfrak{m}, \quad \text{for a.e. } t \in (0,T), \tag{2.5.9}$$

where $\mathscr{C}^{s_n}(\rho_t, v_t)$ is defined as in Lemma 2.5.4. Set now $\eta_t^n := \mathscr{C}^{s_n}(\rho_t, v_t) - \mathrm{div}(\rho_t^n v_t)$. From the Leibniz rule and the L^∞-to Lipschitz regularization of the heat flow (1.3.16) we have that

$$\|\mathrm{div}(\rho_t^n v_t)\|_{L^2} \le \|\rho_t\|_{L^2}\|\mathrm{div}(v_t)\|_{L^\infty} + c(K)\frac{\|\rho_t\|_{L^\infty}}{\sqrt{s_n}}\||v_t|\|_{L^2}.$$

This bound together with (2.5.5), (2.5.7) and the hypotheses on v_t, grants that $\eta_t^n \in L^1((0,T), L^2(\mathfrak{m}) + L^{4'}(\mathfrak{m}))$. Denote by V the Banach space $L^2(\mathfrak{m}) + L^{4'}(\mathfrak{m})$ and observe that $L^2 \cap L^4 = V^*$. Then (2.5.9) can be restated as: for a weakly*-dense set of $\varphi \in V^*$ the function $[0,T] \ni t \mapsto \varphi(\rho_t^n)$ is absolutely continuous and $\frac{\mathrm{d}}{\mathrm{d}t}\varphi(\rho_t^n) = \varphi(\eta_t^n)$. It follows (see e.g. Remark 4.9 in [**28**]) that ρ_t^n is absolutely continuous in $L^1((0,T), V)$ and strongly differentiable a.e. with $\frac{\mathrm{d}}{\mathrm{d}t}\rho_t^n = \eta_t^n$.

Pick a convex function $\beta : [0+\infty) \to [0+\infty)$ such that $\beta(t) = t^p$ for every $t \le 2\sup_{t\in(0,T)} \|\rho_t\|_{L^\infty} < +\infty$ and such that β' is globally bounded. In particular from the maximum principle for the heat flow we have that $\beta(\rho_t^n) = (\rho_t^n)^p$ for every t and n. Moreover, since ρ_t are uniformly bounded in $L^q(\mathfrak{m})$ for every $1 \le q \le \infty$, from the contractivity of the heat flow we have also that ρ_t^n are bounded in $L^q(\mathfrak{m})$ for every $1 \le q \le \infty$, uniformly in t and n. Finally, observing that $\beta'(t)/t$ is globally bounded, we deduce that $\beta'(\rho_t^n)$ are again bounded in $L^q(\mathfrak{m})$ for every $1 \le q < \infty$, uniformly in t and n.

Observe now that from the convexity of β we have that

$$\int \beta(\rho_t^n) - \beta(\rho_s^n)\mathrm{d}\mathfrak{m} \le \int \beta'(\rho_t^n)(\rho_t^n - \rho_s^n)\mathrm{d}\mathfrak{m}, \quad \forall t,s \in [0,T]. \tag{2.5.10}$$

This in turn gives

$$\int \beta(\rho_t^n) - \beta(\rho_s^n)\mathrm{d}\mathfrak{m} \le \sup_{t\in[0,T]} \|\beta'(\rho_t^n)\|_{L^2\cap L^4(\mathfrak{m})}\|\rho_t^n - \rho_s^n\|_{L^2+L^{4'}} \le \sup_{t\in[0,T]} \|\beta'(\rho_t^n)\|_{L^2\cap L^4} \int_s^t \|\eta_r^n\|_{L^2+L^{4'}}\mathrm{d}r,$$

for every $t, s \in [0,T]$, with $s \le t$. Hence the function $\int \beta(\rho_t^n)\mathrm{d}\mathfrak{m}$ is absolutely continuous in $[0,T]$ and from (2.5.10) we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \beta(\rho_t^n)\mathrm{d}\mathfrak{m} \le \int \beta'(\rho_t^n)\eta_t^n\mathrm{d}\mathfrak{m}, \qquad \text{for a.e. } t \in (0,T).$$

Then from the definition of η_t^n and β and integrating by parts we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \beta(\rho_t^n)\mathrm{d}\mathfrak{m} &\le -\int [\beta'(\rho_t^n)\rho_t^n - \beta(\rho_t^n)]\mathrm{div}(v_t)\mathrm{d}\mathfrak{m} + \int \mathscr{C}^{s_n}(\rho_t, v_t)\beta'(\rho_t^n)\mathrm{d}\mathfrak{m}\\ &\le (p-1)\int (\rho_t^n)^p\mathrm{div}(v_t)^-\mathrm{d}\mathfrak{m} + p\int \mathscr{C}^{s_n}(\rho_t, v_t)(\rho_t^n)^{p-1}\mathrm{d}\mathfrak{m},\end{aligned}$$

for a.e. $t \in (0,T)$. Observe now that combining (2.5.5) with (2.5.6), an application of dominated convergence gives that $\int_0^T \|\mathscr{C}^{s_n}(\rho_t, v_t)\|_{L^2+L^{4'}} \to 0$ as $s_n \to 0$. Then, recalling that $\mathrm{div}(v_t)^- \in L^\infty((0,T), L^\infty(\mathfrak{m}))$ and (2.5.7), we can let $s_n \to 0$ in the above and obtain at once the absolute continuity of $\int (\rho_t)^p\mathrm{d}\mathfrak{m}$ together with (2.5.8). □

2.5.2. From almost outer functional cone to almost outer metric cone. Our aim in this section is to prove the following.

Theorem 2.5.5. *For every $\varepsilon \in (0,1/3)$, $R_0 > 0$, $\gamma > \frac{1}{2}\frac{N-2}{N-1}$, $N \in [2,\infty)$, $L > 0$ there exists $0 < \delta = \delta(\varepsilon, \gamma, N, R_0, L)$ such that the following holds. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m}, x_0)$ be a pointed $\mathrm{RCD}(-\delta, N)$ m.m.s. with $\mathfrak{m}(B_1(x_0)) \in (\varepsilon, \varepsilon^{-1})$. Let $U \subset \mathrm{X}$ be open with $U^c \subset B_{R_0}(x_0)$ and $v \in D(\mathbf{\Delta}, U)\cap C(U)$ be positive and such that $\overline{\lim}_{x\to\partial U} v(x) \le 1$, $\Delta v = N|\nabla\sqrt{2v}|^2$, $v \ge 1+\varepsilon$ in $B_R(x_0)^c \ne \emptyset$, with $R < R_0$ and $\||\nabla\sqrt{v}|\|_{L^\infty(U)} \le L$. Suppose furthermore that*

$$\int_U \frac{1}{v^{N-2}} \left|\nabla|\nabla\sqrt{v}|^\gamma\right|^2 \mathrm{d}\mathfrak{m} < \delta, \tag{2.5.11}$$

where it is intended that $v^{N-2} \equiv 1$ in the case $N = 2$.

Then there exists a pointed $\mathrm{RCD}(0,\mathrm{N})$ space $(\mathrm{X}', \mathsf{d}', \mathfrak{m}', x')$ such that

$$\mathsf{d}_{pmGH}((\mathrm{X}, \mathsf{d}, \mathfrak{m}, x_0), (\mathrm{X}', \mathsf{d}', \mathfrak{m}', x')) < \varepsilon$$

and $(\mathrm{X}', \mathsf{d}', \mathfrak{m}', x')$ is a truncated cone outside a compact set $K \subset B_{(1+\varepsilon)R}(x')$, i.e. there exists an $\mathrm{RCD}(0,\mathrm{N})$ N-cone Y, with vertex O_Y, over an $\mathrm{RCD}(N-2, N-1)$ space Z, and a measure preserving local isometry $T : \mathrm{X}' \setminus K \to Y \setminus \bar{B}_r(O_Y)$, for some $r > L^{-1}$.

Observe that (2.5.11) makes sense thanks to Corollary 2.3.5.

Notice also that the assumption $u \ge 1+\varepsilon$ in $B_{R_0}(x_0)^c$ is necessary as the function $v \equiv 1$ shows. We will also prove another version of the above result, which is Theorem 2.5.6. Before passing to its statement we need to introduce the notion of intrinsic metric.

Finally, for any (X, d) complete metric space and $A \subset \mathrm{X}$ we define the *intrinsic metric on A* to be the distance function $\mathsf{d}^A : A \to [0,+\infty]$ defined by

$$\mathsf{d}^A(x,y) := \inf_\gamma L(\gamma), \quad \forall x,y \in A,$$

where the infimum is taken among all curves $\gamma \in AC([0,1],\mathrm{X})$ with values in A and such that $\gamma(0) = x$, $\gamma(1) = y$.

If we also assume that A is relatively compact, then for every $x, y \in A$ such that $\mathsf{d}(x,y) < +\infty$ there exists an absolutely continuous curve γ with values in $\bar{A}$ such that $\mathsf{d}^A(x,y) = L(\gamma)$.

We are ready to state the second version of Theorem 2.5.5, which is more in the spirit of the 'volume annulus implies metric annulus' theorem of Cheeger and Colding (see [**72**, Theorem 4.85]).

Theorem 2.5.6. *For every $\varepsilon \in (0,1/3)$, $R_0>0$, $\gamma > \frac{1}{2}\frac{N-2}{N-1}$, $N \in [2,\infty)$, $L>0$, $\eta \in (0,\varepsilon^{-1})$ there exists $0<\delta=\delta(\varepsilon,\gamma,N,R_0,L,\eta)$ such that, given X, U and v as in Theorem 2.5.5, there exists an* $\mathrm{RCD}(0,\mathrm{N})$ *N-cone $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ with vertex O_Y, over an* $\mathrm{RCD}(N-2,N-1)$ *space Z, and a constant $\lambda \in (0,L)$ such that the following holds.*

For every $1+\varepsilon+\eta<t_1\le t_2<\varepsilon^{-1}$ satisfying $\{\sqrt{v}\le t_2+2\eta\}\subset B_{R_0}(x_0)$, it holds

$$\mathsf{d}_{GH}\left((\{t_1\le \sqrt{v}\le t_2\},\mathsf{d}^{\eta}_{\mathrm{X}}),(\{t_1\le \lambda \mathsf{d}_{O_Y}\le t_2\},\mathsf{d}^{\eta}_{Y})\right)<\varepsilon, \tag{2.5.12}$$

where $\mathsf{d}_{O_Y}:=\mathsf{d}_Y(.,O_Y)$ and $\mathsf{d}^{\eta}_{\mathrm{X}}$ and d^{η}_{Y} denote the intrinsic metrics on $\{t_1-\eta<\sqrt{v}<t_2+\eta\}$ and on $\{t_1-\eta<\lambda\mathsf{d}_{O_Y}<t_2+\eta\}$ (see definition above). Moreover, provided that $t_1+\varepsilon<t_2$,

$$\mathsf{d}_{mGH}\left(\left(\{t_1\le \sqrt{v}\le t_2\},\mathsf{d}^{\varepsilon}_{\mathrm{X}},\mathfrak{m}|_{\{t_1\le\sqrt{v}\le t_2\}}\right),\left(\{t_1\le \lambda \mathsf{d}_{O_Y}\le t_2\},\mathsf{d}^{\varepsilon}_{Y},\mathfrak{m}_Y|_{\{t_1\le\lambda\mathsf{d}_{O_Y}\le t_2\}}\right)\right)<\varepsilon. \tag{2.5.13}$$

Moreover the cone Y can be taken so that the conclusion of Theorem 2.5.5 holds (with the same ε, R_0 and L) with Y and for some $\mathrm{RCD}(0,N)$ *space* X'.

We point out that in general we cannot say anything better than $\lambda\le L$. This is immediately seen by taking $v=L^2|x|^2$ in $\mathbb{R}^n$ and $U=\mathbb{R}^n\setminus \bar{B}_{1/L}(0)$.

It is important to notice that the information in (2.5.12) is not contained in (2.5.13), indeed it is not clear if, fixed $\varepsilon,\gamma,N,R_0,L$ as in Theorem 2.5.6, the metric measure spaces $(\{\sqrt{t_1}\le\sqrt{v}\le\sqrt{t_2}\},\mathsf{d}^{\eta}_{\mathrm{X}},\mathfrak{m}|_{\{t_1\le\sqrt{v}\le t_2\}})$, for arbitrary v,t_1,t_2 as in the hypotheses, satisfy some uniform doubling condition.

For the proof of Theorem 2.5.6 we will need the following elementary lemma. The proof is a direct consequence of the definition of distance in a cone and will be omitted.

Lemma 2.5.7. *Let (Y,d) be an Euclidean cone of vertex O_Y and for any $0<a<b$ let $\mathsf{d}_{a,b}$ be the intrinsic metric on $\{a<d(.,O_Y)<b\}$. Then for every $0<\varepsilon<a<b$ it holds*

$$\mathsf{d}_{a-\varepsilon,b+\varepsilon}\le \mathsf{d}_{a,b}\le \frac{a}{a-\varepsilon}\mathsf{d}_{a-\varepsilon,b+\varepsilon},\qquad \text{in } \{a<d(.,O_Y)<b\}.$$

Moreover for any two sequences $(a_n),(b_n)$ such that $a_n\to a$, $b_n\to b$ it holds

$$(\{a_n\le d(.,O_Y)\le b_n\},\mathsf{d}_{a_n-\varepsilon,b_n+\varepsilon})\overset{GH}{\to}(\{a\le d(.,O_Y)\le b\},\mathsf{d}_{a-\varepsilon,b+\varepsilon})$$

and the map $f_n(t,z):=\left(\frac{(t-a_n)(b-a)}{(b_n-a_n)}+a,z\right)$ (in polar coordinates) realizes such convergence.

Proof of Theorem 2.5.5 and Theorem 2.5.6. The proof of Theorem 2.5.5 is essentially the same as Theorem 2.5.6, except that it stops earlier. For this reason we will prove both theorems together. The reader interested only in the proof of the first result can ignore the second half of the argument.

Proof of Theorem 2.5.5:

We argue by contradiction. Suppose that there exist numbers $\varepsilon\in(0,1/3), N\in[2,\infty)$, $R_0>0$, $\gamma\ge\frac{1}{2}\frac{N-2}{N-1}$, $L>0$, a sequence $\delta_n\to 0$, a sequence $(X_n,\mathsf{d}_n,\mathfrak{m}_n,x_n)$ of $\mathrm{RCD}(-\delta_n,N)$ m.m.s., a sequence of open sets $U_n\subset X_n$, with $U_n^c\subset B_{R_0}(x_n)$, a sequence R_n with $R_n<R_0$, functions $v_n\in D(\mathbf{\Delta},U_n)$ satisfying $\Delta v_n=2N|\nabla\sqrt{v_n}|^2$ such that:

a) $\overline{\lim}_{x\to\partial U_n}v_n(x)\le 1$, $v_n\ge 1+\varepsilon$ in $B_{R_n}(x_0)^c\ne\emptyset$, $\||\nabla\sqrt{v}_n|\|_{L^\infty(U_n)}\le L$,
b) $\mathfrak{m}_n(B_1(x_n))\in(\varepsilon,\varepsilon^{-1})$,
c) (2.5.11) holds (with v_n, $\mathfrak{m}_n$, γ and $\delta=\delta_n$),
d) for every n the conclusion of Theorem 2.5.5 does not hold.

We first observe that, since $v_n\ge 1+\varepsilon$ in $B_{R_0}(x_0)^c$, removing the set $\{v_n\le 1\}$ from U_n does not effect neither the hypotheses of the theorems nor their conclusions, therefore it is not restrictive to assume that $v_n>1$ in U_n.

Moreover by compactness, up to passing to a non relabelled subsequence, we can assume that the p.m.m. spaces $(X_n,\mathsf{d}_n,\mathfrak{m}_n,x_n)$ pmGH-converge to a to an $\mathrm{RCD}(0,\mathrm{N})$ pointed m.m.s. $(X_\infty,\mathsf{d}_\infty,\mathfrak{m}_\infty,x_\infty)$.

Passing to the extrinsic approach, we consider a proper metric space (Y,d_Y) that realizes such convergence, in particular we identify the metric spaces X_n and X_∞ with the corresponding subsets of Y such that $\mathsf{d}_Y(x_n,x_\infty)\to 0$, $\mathfrak{m}_n\rightharpoonup\mathfrak{m}_\infty$ in duality with $C_{bs}(Y)$ and $\mathsf{d}^Y_H(B_R^{\mathrm{X}_n}(x_n),B_R^{\mathrm{X}_\infty}(x_\infty))\to 0$ for every $R>0$. In particular for every $x\in\mathrm{X}_\infty$ there exists a sequence $y_n\in\mathrm{X}_n$ such that $\mathsf{d}_Y(x,y_n)\to 0$ and conversely for every $R>0$ and every sequence $y_n \in B_R^{\mathrm{X}_n}(x_n)$ there exists a subsequence converging to a point $x \in \mathrm{X}_\infty$. This two facts will be used repeatedly in the proof without further notice.

Define the compact sets $K_n = U_n^c \subset \mathrm{X}_n$ and observe that, since $K_n \subset B_{R_0}(x_n)$, they are all contained in a common ball in Y centered at x_∞. Hence from the metric version of Blaschke's theorem (see [**60**, Theorem 7.3.8]) there exists a compact set $K_\infty \subset \mathrm{X}_\infty$ such that, up to a subsequence, $\mathsf{d}_H^Y(K_\infty, K_n) \to 0$.

Define the open (in the topology of X_∞) set $U_\infty \coloneqq X_\infty \setminus K_\infty$ and for every $r>0$, define the open sets $U_n^{<r} = \{x \in X_n \ : \ \mathsf{d}_n(x, K_n) < r\}$ and $U_\infty^{<r} = \{x \in X_\infty \ : \ \mathsf{d}_\infty(x, K_\infty) < r\}$. Analogously we define the sets $U_n^{>r}, U_n^{\leq r}, U_n^{\geq r}$ and the corresponding ones for $n = \infty$.

From the assumptions $\overline{\lim}_{x \to \partial U_n} v_n(x) \leq 1$ and $\||\nabla \sqrt{v}_n|\|_{L^\infty(U_n)} \leq L$, applying Proposition 1.3.8 it follows that

$$v_n \leq (1+RL)^2, \text{ in } B_R(x_n) \cap U_n, \quad v_n \leq 1 + \varepsilon/4 \text{ in } U_n^{\leq 4\rho} \cap U_n, \tag{2.5.14}$$

for every $R>0$ and for some small constant $0<\rho = \rho(\varepsilon, L) < \varepsilon$, independent of n.

For every n and every $k \in \mathbb{N}$ with $k \geq R_0 + 100$, thanks to Proposition 1.4.9, there exists a cut off function $\eta_k^n \in \mathsf{Test}(X)$, $0 \leq \eta_k^n \leq 1$, such that $\operatorname{supp} \eta_n^k \subset U_n^{>\rho/2} \cap B_{k+2}(x_n)$, $\eta_n^k = 1$ on $U_n^{\geq \rho} \cap B_{k+1}(x_n)$ and $\mathrm{Lip}\eta_n^k + |\Delta \eta_n^k| \leq C$, for some constant C depending only on ε, N, L. Observe also that we can choose η_n^k so that $\eta_n^k = \eta_n^{k+1}$ in $B_{k+1}(x_n)$, for every k.

Define the functions $v_n^k \coloneqq v_n \eta_n^k$, $\tilde{v}_n^k \coloneqq \sqrt{v_n}\eta_n^k$ and observe that from (2.5.14) and the assumption $\||\nabla \sqrt{v}_n|\|_{L^\infty(U_n)} \leq L$ they are equi-Lipschitz, equi-bounded in n and all supported on $B_{k+2}(x_n)$. Hence by Ascoli-Arzelà (see Prop. 1.6.7), up to a subsequence, as $n \to +\infty$ they converge uniformly to functions $v_\infty^k, \tilde{v}_\infty^k \in C(X_\infty)$ with support in $\bar{B}_{k+2}(x_\infty)$.

From Proposition 1.6.10 it follows that $v_n^k, \tilde{v}_n^k$ converge also strongly in L^2 respectively to $v_\infty^k, \tilde{v}_\infty^k$. It is clear from the construction and the uniform convergence that $v_\infty^k = v_\infty^{k+1}$ on $B_k(x_\infty)$. Therefore the assignment $v_\infty \coloneqq v_\infty^k$ in $B_k(x_\infty)$ for every k, well defines a function $v_\infty \in C(\mathrm{X}_\infty)$. Analogously we can define $\tilde{v}_\infty \in C(\mathrm{X}_\infty)$ and we observe that $\tilde{v}_\infty = \sqrt{v_\infty}$ in $U_\infty^{\geq 2\rho}$.

We make the following two claims:

(**A**) $v_\infty \leq 1 + \varepsilon/4$ in $U_\infty^{\leq 3\rho}$,

(**B**) $\{v_\infty > 1+\varepsilon/4\} \neq \emptyset$, $v_\infty \in D(\mathbf{\Delta})$ and (up to multiplying v_∞ by a positive constant C_0) it holds that $\Delta v_\infty = N$, $|\nabla v_\infty|^2 = 2 v_\infty$ $\mathfrak{m}$-a.e. in $\{v_\infty > 1 + \varepsilon/4\}$.

We start with claim (**A**). It is clearly enough prove that

$$v_\infty^k \leq 1 + \varepsilon/4, \quad \text{ in } U_\infty^{\leq 3\rho}$$

for any k. Pick any $y \in U_\infty^{\leq 3\rho}$, then there exists a sequence $y_n \in X_n$ such that $y_n \to y$ in Y and by uniform convergence $v_n^k(y_n) \to v_\infty^k(y)$. Moreover it must hold that $\mathsf{d}_Y(y_n, K_n) < 4\rho$ if n is big enough. If $y_n \notin U_n$, then $v_n^k(y_n) = 0$ by construction. If instead $y_n \in U_n$ from the second in (2.5.14) and the fact that $\eta_k^n \leq 1$ we deduce that $v_n^k(y_n) \leq v_n(y_n) \leq 1 + \varepsilon/4$. Combining these two observations we get claim (**A**).

We pass to the proof of claim (**B**). It is easy to check, since $v_n \geq 1$ in U_n (recall the observation made at the beginning of the proof), that $\sup_n \|\Delta v_n^k\|_{L^\infty(X_n)} < +\infty$ and $\sup_n \|\Delta \tilde{v}_n^k\|_{L^\infty(X_n)} < +\infty$. Moreover by Bishop-Gromov inequality we have $\sup_n \mathfrak{m}_n(B_R(x_n)) < +\infty$, for every $R \geq 1$. Therefore $\sup_n \|\Delta v_n^k\|_{L^2(X_n)}$, $\sup_n \|\Delta \tilde{v}_n^k\|_{L^2(X_n)} < +\infty$ and applying Theorem 1.6.13 we deduce that $v_\infty^k, \tilde{v}_\infty^k \in D(\Delta)$, that Δv_n^k converges to Δv_∞^k weakly in L^2 and that $|\nabla \tilde{v}_n^k| \to |\nabla \tilde{v}_\infty^k|$ strongly in L^2. From the locality of Laplacian follows that $v_\infty \in D(\mathbf{\Delta})$. Additionally, since $L' \coloneqq \sup_n \||\nabla \tilde{v}_n^k|\|_{L^\infty} < +\infty$ and v_n^k have uniformly bounded support, applying (i) of Proposition 1.6.10 (with $\varphi(t) = (t \wedge L')^\alpha$) we also deduce that $|\nabla \tilde{v}_n^k|^\alpha \to |\nabla \tilde{v}_\infty^k|^\alpha$ strongly in L^2, for every $\alpha > 0$. We make the intermediate claim that

$$\Delta v_\infty = N |\nabla \sqrt{2 v_\infty}|^2, \quad \mathfrak{m}\text{-a.e. in } U_\infty^{>2\rho} \cap B_k(x_\infty). \tag{2.5.15}$$

In particular from Corollary 2.3.5 this implies that $|\nabla \sqrt{v_\infty}|^\gamma \in W^{1,2}_{\mathsf{loc}}(U_\infty^{>2\rho} \cap B_k(x_\infty))$. To prove (2.5.15) pick any $\varphi \in \mathsf{LIP}_c(U_\infty^{>2\rho} \cap B_k^{\mathrm{X}_\infty}(x_\infty))$. Consider also a function $\eta \in \mathsf{LIP}(Y)$ such that $\eta \equiv 1$ in $\operatorname{supp} \varphi$, $\mathsf{d}_Y(\operatorname{supp} \eta, K_\infty) > 2\rho$ and $\operatorname{supp} \eta \subset B_k^Y(x_\infty)$. Moreover, since $\mathsf{d}_H^Y(K_n, K_\infty) \to 0$, for n big enough we have $\{y \ : \ \mathsf{d}_Y(y, K_\infty) > 2\rho\} \subset \{y \ : \ \mathsf{d}_Y(y, K_n) > \rho\}$ and analogously, since $x_n \to x_\infty$ in Y, for n big enough $B_k^Y(x_\infty) \subset B_{k+1}^Y(x_n)$. Therefore $\operatorname{supp} \eta \cap X_n \subset U_n^{>\rho} \cap B_{k+1}^{\mathrm{X}_n}(x_n)$ for n big enough. We now extend φ to a function $\varphi' \in \mathsf{LIP}(Y)$ and define $\bar{\varphi} = \eta \varphi' \in \mathsf{LIP}_{bs}(Y)$. Since by the locality of the Laplacian and gradient we have $\Delta v_n^k = \Delta v_n = N |\nabla \sqrt{2 v_n}|^2 = 2N |\nabla \tilde{v}_n^k|^2$ $\mathfrak{m}_n$-a.e. in $\operatorname{supp} \bar{\varphi}$, we can compute

$$\begin{aligned}\int \varphi \Delta v_\infty^k \mathrm{d}\mathfrak{m}_\infty &= \int \bar{\varphi} \Delta v_\infty^k \mathrm{d}\mathfrak{m}_\infty = \lim_n \int \bar{\varphi} \Delta v_n^k \mathrm{d}\mathfrak{m}_n = \lim_n \int \bar{\varphi} 2N |\nabla \tilde{v}_n^k|^2 \mathrm{d}\mathfrak{m}_n \\ &= \int \bar{\varphi} 2N |\nabla \tilde{v}_\infty^k|^2 \mathrm{d}\mathfrak{m}_\infty = \int \bar{\varphi} 2N |\nabla \sqrt{v}_\infty|^2 \mathrm{d}\mathfrak{m}_\infty.\end{aligned}$$

This and the locality of the Laplacian prove (2.5.15).

For every n and every k as above we consider a cut-off function $\xi_n^k \in \mathsf{LIP}(\mathrm{X}_n)$ analogous to η_n^k but with smaller support, more precisely we require that $0 \le \xi_k^n \le 1$, $\operatorname{supp} \xi_n^k \subset U_n^{>\rho} \cap B_{k+2}(x_n)$, $\xi_n^k = 1$ on $U_n^{\ge 2\rho} \cap B_{k+1}(x_n)$ and $\mathrm{Lip}\xi_n^k \le C'$, for some constant C' depending only on ε, N, L. Up to a subsequence, from Ascoli-Arzelà we have that $\xi_n^k \to \xi_\infty^k$ uniformly, for some $\xi_\infty^k \in \mathsf{LIP}(\mathrm{X}_\infty)$ satisfying $\xi_\infty^k = 1$ in $U_\infty^{\ge 2\rho} \cap B_k^{\mathrm{X}_\infty}(x_\infty)$. In particular the same convergence holds also strongly in L^2.

We set $w_{n,k} \coloneqq \xi_n^k |\nabla \sqrt{v_n}|^\gamma \in W^{1,2}(\mathrm{X}_n)$ and $w_{\infty,k} \coloneqq \xi_\infty^k |\nabla \tilde{v}_\infty^k|^\gamma \in W^{1,2}(\mathrm{X}_\infty)$ and observe that by construction and the locality of the gradient $w_{n,k} = \xi_n^k |\nabla \tilde{v}_n^k|^\gamma$ $\mathfrak{m}_n$-a.e.. In particular, since we proved that $|\nabla \tilde{v}_n^k|^\gamma \to |\nabla \tilde{v}_\infty^k|^\gamma$ strongly in L^2 and recalling $\sup_n \| |\nabla \tilde{v}_n^k| \|_{L^\infty} < +\infty$, we have from Proposition 1.6.10 that $w_{n,k} \to w_{\infty,k}$ strongly in L^2.

Combining (2.5.11) with the first in (2.5.14) and $\mathrm{Lip}\xi_{n,k} \le C'$ we deduce that $\sup_n \| |\nabla w_{n,k}| \|_{L^2(\mathfrak{m}_n)} < +\infty$. We now apply Lemma 1.6.14 with the open set $A = \{\mathsf{d}_Y(K_\infty, .) > 3\rho\} \cap B_k^Y(x_\infty)$ that, combined with the observation that $A \cap X_n \subset U_n^{\ge 2\rho} \cap B_{k+1}(x_n)$ for n big enough, gives

$$\begin{aligned} \int_{U_\infty^{\ge 3\rho} \cap B_k(x_\infty)} |\nabla |\nabla \tilde{v}_\infty^k|^\gamma|^2 \, \mathrm{d}\mathfrak{m}_\infty &\le \varliminf_n \int_{U_n^{\ge 2\rho} \cap B_{k+1}(x_n)} |\nabla |\nabla \sqrt{v_n}|^\gamma|^2 \, \mathrm{d}\mathfrak{m}_n \\ &\overset{(2.5.14)}{\le} (1+RL)^{2N-4} \varliminf_n \int_{U_n} v_n^{2-N} |\nabla |\nabla \sqrt{v_n}|^\gamma|^2 \, \mathrm{d}\mathfrak{m}_n \overset{(2.5.11)}{=} 0. \end{aligned}$$

Therefore, from the locality of the gradient and the arbitrariness of k we obtain that $|\nabla |\nabla \sqrt{v_\infty}|^\gamma| = 0$ $\mathfrak{m}_\infty$-a.e. in $U_\infty^{\ge 3\rho}$. Consider now the open set $\{v_\infty > 1+\varepsilon/4\} \subset U_\infty^{\ge 3\rho}$. Observe that from the assumption $v_n \ge 1+\varepsilon$ in $B_{R_0}(x_n)^c \ne \emptyset$ and (2.5.14) we deduce that for every n there exists $y_n \in \mathrm{X}_n \cap B_{2R_0}(x_n) \cap U_n^{>4\rho}$ such that $v_n^k(y_n) = v_n(y_n) \ge 1+\varepsilon$ for every k, therefore by compactness and uniform convergence we deduce that $\{v_\infty > 1+\varepsilon/4\} \ne \emptyset$. From (**A**) it holds $\partial\{v_\infty > 1+\varepsilon/4\} = \{v_\infty = 1+\varepsilon/4\}$, in particular since $v_\infty^{(2-N)/2}$ ($\ln(v_\infty^{-1/2})$ if $N=2$) is harmonic in $U_\infty^{\ge 2\rho}$ (recall (2.5.15)), from the maximum principle it follows that the connected components of $\{v_\infty > 1+\varepsilon/4\}$ are unbounded. Let U' be one of these components. It follows that $|\nabla \sqrt{v_\infty}| \equiv C$ $\mathfrak{m}$-a.e. in U' for some constant C, that must be positive. Indeed if $C = 0$, we would have that v_∞ is constant in U', but since $\partial U' \subset \{v_\infty = 1+\varepsilon/4\}$, v_∞ should be constantly equal to $1+\varepsilon/4$, which contradicts $U' \subset \{v_\infty > 1+\varepsilon/4\}$. Finally the assumption $v_n \ge 1+\varepsilon$ in $B_{R_0}(x_n)^c$ ensures that $\mathrm{X}_\infty \setminus U' \subset B_{2R_0}(x_\infty)$. It follows that the function $(2C^2)^{-1} {v_\infty}_{|U'}$ satisfies the hypotheses of Theorem 2.5.1 with $U = U'$. In particular X_∞ has Euclidean volume growth and from Corollary 2.1.7 it has one end, from which we deduce that $\{v_\infty > 1+\varepsilon/4\}$ is connected. Therefore repeating the above argument for $U' = \{v_\infty > 1+\varepsilon/4\}$ proves claim (**B**) with $C_0 \coloneqq (2C^2)^{-1}$.

Combining (**A**) and (**B**), from Theorem 2.5.1 we deduce the existence of an RCD$(0,N)$ N-cone $(Y', \mathsf{d}_{Y'}, \mathfrak{m}_{Y'})$ with vertex $O_{Y'}$ and a bijective measure preserving local isometry $T : \{1+\varepsilon/4 < v_\infty\} \to \{\, r < \mathsf{d}_{Y'}(O_{Y'}, .)\}$, for some $r > 0$ which also satisfies (recall (2.5.1))

$$\sqrt{v_\infty}(x) = \lambda \mathsf{d}_{Y'}(O_{Y'}, T(x)), \qquad \text{for every } x \in \{v_\infty > 1+\varepsilon/4\}, \tag{2.5.16}$$

where $\lambda \coloneqq (2C_0)^{-1/2}$ (C_0 being the constant in (**B**)). We claim that $\lambda \le \sup_n \| |\nabla \sqrt{v_n}| \|_{L^\infty}$, which in particular gives that $r \ge \lambda^{-1}(1+\varepsilon/4) \ge L^{-1}$. Indeed from (2.5.16), the fact that Y' is geodesic and the fact that T is a local isometry we deduce that for every $x \in \{v_\infty > 1+\varepsilon/4\}$ and every $r' > 0$ small enough, there exists $y \in B_{r'}^{\mathrm{X}_\infty}(x)$ such that $|\sqrt{v_\infty(x)} - \sqrt{v_\infty(y)}| = \lambda \mathsf{d}_\infty(x,y)$,. The claim now follows from uniform convergence and the Sobolev to Lipschitz property. Since clearly for every n big enough it holds $\{1+\varepsilon/4 < v_\infty\}^c \subset B_{(1+\varepsilon)R_n}(x_\infty)$, the conclusion of Theorem 2.5.5 holds for every sufficiently large n, which contradicts item d) above. **This concludes the proof of Theorem 2.5.5**.

Proof of Theorem 2.5.6: We argue by contradiction exactly as above, except that we substitute the assumption d) with the following:

d') there exist a number $\eta > 0$ and two sequences $(t_1^n), (t_2^n) \subset (1+\varepsilon+\eta, \varepsilon^{-1})$ with $t_1^n \le t_2^n$, such that $\{\sqrt{v}_n \le t_2^n + 2\eta\} \subset B_{R_0}(x_n)$ and the conclusion of Theorem 2.5.6 is false with t_1^n, t_2^n, for every n.

Since assumption d) was not used until the very end of the proof of Theorem 2.5.5 above, we can, and will, repeat all the first part of the proof up to this point together with all the constructions and objects introduced along the argument.

Up to passing to a subsequence we can assume that $t_i^n \to t_i^\infty \in [1+\varepsilon+\eta, \varepsilon^{-1}]$, $i = 1, 2$.

It will be useful later to remark that

$$\sqrt{v_n} = \tilde{v}_n^k \quad \text{ in } \{1+\varepsilon/2 \le \sqrt{v_n} \le t_2^n + 2\eta\}, \text{ for every } k, \tag{2.5.17}$$

which follows from the second in (2.5.14) and the assumption $\{\sqrt{v_n} \le t_2^n + 2\eta\} \subset B_{R_0}(x_n)$.

We claim that

$$a_n \coloneqq \sup_{s \in (t_1^\infty - \eta, t_2^\infty + \eta)} \mathsf{d}_H^Y(\{\sqrt{v_n} = s\}, \{\sqrt{v_\infty} = s\}) \to 0, \quad \text{as } n \to +\infty. \tag{2.5.18}$$

(We point out that this does not follows form the uniform convergence, indeed we need first to prove some regularity of the level sets of $\sqrt{v_\infty}$.) The key observation is that for every $\varepsilon' > 0$ there exists $\delta' > 0$ such that for every $t \in [1+\varepsilon, 2\varepsilon^{-1}]$ it holds

$$B_{\varepsilon'}^{X_\infty}(x) \cap \{\sqrt{v_\infty} = t'\} \neq \emptyset, \quad \forall x \in \{\sqrt{v_\infty} = t\}, \forall t' \in [t - \delta', t + \delta']. \tag{2.5.19}$$

This is an immediate consequence of (2.5.16), the fact that T is a local isometry and the fact that, since Y' is a cone, for every $y' \in Y'$ there exists a ray emanating from $O_{Y'}$ and passing through y'.

Suppose now that (2.5.18) does not hold. Then, up to a passing to non relabelled subsequence, there exists a sequence $(s_n) \subset (t_1^\infty - \eta, t_2^\infty - \eta)$ and $\varepsilon' > 0$ such that $s_n \to \bar{s} \in [t_1^\infty - \eta, t_2^\infty + \eta] \subset [1+\varepsilon, 2\varepsilon^{-1}]$ and

$$\mathsf{d}_H^Y(\{\sqrt{v_n} = s_n\}, \{\sqrt{v_\infty} = s_n\}) > \varepsilon', \quad \forall n.$$

Therefore, up to passing to a further subsequence, there exists either a sequence $y_n \in \{\sqrt{v}_n = s_n\}$ such that $\mathsf{d}_Y(y_n, \{\sqrt{v_\infty} = s_n\}) > \varepsilon'$, for all n, or a sequence $y_n \in \{\sqrt{v_\infty} = s_n\}$ such that $\mathsf{d}_Y(y_n, \{v_n = s_n\}) > \varepsilon'$, for all n. In the first case, since by assumption $\{\sqrt{v}_n = s_n\} \subset \{t_1^n - 2\eta \le \sqrt{v_n} \le t_2^n + 2\eta\} \subset B_{R_0}(x_n)$ for n big enough, up to passing to a further subsequence we have that $y_n \to y_\infty \in \mathrm{X}_\infty$ and by uniform convergence (recall (2.5.17)) that $\sqrt{v_\infty}(y_\infty) = \bar{s}$. In particular $\mathsf{d}(y_\infty, \{\{\sqrt{v_\infty} = s_n\}\}) > \varepsilon'/2$ for every n big enough, which contradicts (2.5.19). In the second case, again up to a subsequence and from the continuity of $\sqrt{v_\infty}$, we have that $y_n \to y_\infty \in \{\sqrt{v_\infty} = \bar{s}\}$. Moreover from (2.5.19) it follows the existence of a $\delta' > 0$ and points $y_\infty^+, y_\infty^- \in B_{\varepsilon'/4}^{\mathrm{X}_\infty}(y_\infty)$ such that $\sqrt{v_\infty}(y_\infty^\pm) = \bar{t} \pm \delta'$. Finally, from uniform convergence (recall again (2.5.17)), for every n big enough there exist $y_n^+, y_n^- \in \mathrm{X}^n$ such that $\mathsf{d}_Y(y_n^\pm, y_\infty^\pm) < \varepsilon'/4$ and $|v_n(y_n^\pm) - (\bar{s} \pm \delta')| < \delta'/2$. In particular by the continuity of v_n, for every n big enough, there exists z_n which lies on a geodesic connecting y_n^+ and y_n^- such that $z_n \in \{v_n = s_n\}$. From the triangle inequality it follows that $\mathsf{d}(z_n, y_n) < \varepsilon'$ if n is big enough, which is a contradiction since $y_n \in \{\sqrt{v_\infty} = s_n\}$.

From (2.5.18) it follows that $\mathsf{d}_H^Y(\{t_1^n \le \sqrt{v_n} \le t_2^n\}, \{t_1^n \le \sqrt{v_\infty} \le t_2^n\}) \to 0$ as $n \to +\infty$. Moreover it is clear that $\mathsf{d}_H^Y(\{t_1^n \le \sqrt{v_\infty} \le t_2^n\}, \{t_1^\infty \le \sqrt{v_\infty} \le t_2^\infty\}) \to 0$ as $n \to +\infty$ (recall (2.5.16)), therefore

$$b_n \coloneqq \mathsf{d}_H^Y(\{t_1^n \le \sqrt{v_n} \le t_2^n\}, \{t_1^\infty \le \sqrt{v_\infty} \le t_2^\infty\}) \to 0, \quad \text{as } n \to +\infty. \tag{2.5.20}$$

In particular, since both sets are compact, we can build a Borel map $f_n : \{t_1^n \le \sqrt{v_n} \le t_2^n\} \to \{t_1^\infty \le \sqrt{v_\infty} \le t_2^\infty\}$ that has b_n-dense image and such that $\mathsf{d}_Y(x, f_n(x)) \le 2b_n$ for all $x \in \{t_1^n \le \sqrt{v_n} \le t_2^n\}$.

We claim that

$$f_{n*}\left(\mathfrak{m}_n|_{\{t_1^n \le \sqrt{v}_n \le t_2^n\}}\right) \rightharpoonup \mathfrak{m}_\infty|_{\{t_1^\infty \le \sqrt{v}_\infty \le t_2^\infty\}}, \quad \text{in duality with } C(\{t_1^\infty \le \sqrt{v}_\infty \le t_2^\infty\}). \tag{2.5.21}$$

From the fact that $\mathsf{d}_Y(., f_n(.)) \le 2b_n$ and using dominated convergence it is enough to show that

$$\mathfrak{m}_n|_{\{t_1^n \le \sqrt{v}_n \le t_2^n\}} \rightharpoonup \mathfrak{m}_\infty|_{\{t_1^\infty \le \sqrt{v}_\infty \le t_2^\infty\}}, \quad \text{in duality with } C_{bs}(Y).$$

To prove the above we first define for every $\delta > 0$ the closed set $C_\delta \coloneqq \{y \in Y \ : \ \mathsf{d}_Y(y, \{\sqrt{v_\infty} = t_1^\infty\} \cup \{\sqrt{v_\infty} = t_2^\infty\}) \le \delta\}$ and observe that for every $\varepsilon' > 0$ there exists δ' such that

$$\mathfrak{m}_\infty(C_{\delta'}) < \varepsilon'. \tag{2.5.22}$$

This can be seen using the fact that T is a measure preserving local isometry, the Bishop-Gromov inequality and formula (2.5.16).

We also define for any $\delta > 0$ the sets $A_\delta \coloneqq \{y \in Y \ : \ \mathsf{d}_Y(y, X_\infty \setminus \{t_1^\infty \le \sqrt{v_\infty} \le t_2^\infty\}) \ge \delta\}$ and $B_\delta \coloneqq \{y \in Y \ : \ \mathsf{d}_Y(y, \{t_1^\infty \le \sqrt{v_\infty} \le t_2^\infty\} \le \delta)\}$. We claim that $B_{\delta_1} \setminus A_{\delta_2} \subset C_{2\delta_1 + 2\delta_2}$, for every $\delta_1, \delta_2 > 0$. To see this let $y \in B_{\delta_1} \setminus A_{\delta_2}$, which implies $\mathsf{d}(y, \{t_1^\infty \le \sqrt{v_\infty} \le t_2^\infty\}) \le \delta_1, \mathsf{d}(y, X_\infty \setminus \{t_1^\infty \le \sqrt{v_\infty} \le t_2^\infty\}) < \delta_2$. Taking two points $y_1, y_2 \in \mathrm{X}_\infty$ which realize these two distances we must have that $\mathsf{d}_\infty(y_1, y_2) = \mathsf{d}_Y(y_1, y_2) \le \delta_1 + \delta_2$. Moreover any geodesics in X_∞ from y_1 to y_2, by the continuity of v_∞, must intersects $\{\sqrt{v_\infty} = t_1^\infty\} \cup \{\sqrt{v_\infty} = t_2^\infty\}$, from which the claim follows.

We finally fix $\varphi \in C_{bs}(Y)$ and $\varepsilon' > 0$ arbitrary. Let $\delta' = \delta'(\varepsilon')$ be the one given by (2.5.22) and pick $\eta \in C_b(Y)$ such that $0 \le \eta \le 1$, $\eta = 1$ in $A_{\delta'/4}$ and $\mathrm{supp}(\eta) \subset A_{\delta'/8}$. Observe that by uniform convergence (recall also (2.5.17)), for n big enough we have that $A_{\delta'/8} \cap \mathrm{supp}(\varphi) \cap X_n \subset \{t_1^n \le \sqrt{v_n} \le t_2^n\} \subset B_{\delta'/4}$, therefore

$$\begin{aligned}
\overline{\lim_n}\left|\int \varphi \, d\mathfrak{m}_n|_{\{t_1^n\le\sqrt{v_n}\le t_2^n\}} - \int \varphi \, d\mathfrak{m}_\infty|_{\{t_1^\infty\le\sqrt{v_\infty}\le t_2^\infty\}}\right| &\le \overline{\lim_n}\left|\int \varphi\eta \, d\mathfrak{m}_n - \int\varphi\eta\, d\mathfrak{m}_\infty\right| + \\
&+ \overline{\lim_n}\|\varphi\|_\infty \mathfrak{m}_n(B_{\delta'/4}\setminus A_{\delta'/4}) + \|\varphi\|_\infty\mathfrak{m}_\infty(\{t_1^\infty\le\sqrt{v_\infty}\le t_2^\infty\}\setminus A_{\delta'/4})\\
&\le \overline{\lim_n}\,\mathfrak{m}_n(C_{\delta'}) + \mathfrak{m}_\infty(C_{\delta'}) \overset{(2.5.22)}{\le} 2\varepsilon'.
\end{aligned}$$

From the arbitrariness of ε' and $\varphi \in C_{bs}(Y)$ the convergence in (2.5.21) follows.

We now pass to the study of the behaviour of f_n with respect to the intrinsic metrics. More precisely for every $\tau > 0$ we set $A_n^\tau := \{t_1^n - \tau < \sqrt{v_n} < t_2^n + \tau\}$, $A_\infty^\tau := \{t_1^\infty - \tau < \sqrt{v_\infty} < t_2^\infty + \tau\}$, $A_{\infty,n}^\tau := \{t_1^n - \tau < \sqrt{v_\infty} < t_2^n + \tau\}$ and denote by $\mathsf{d}_n^\tau, \mathsf{d}_\infty^\tau, \mathsf{d}_{\infty,n}^\tau$ the intrinsic metrics on A_n^τ, A_∞^τ, $A_{\infty,n}^\tau$ respectively (see the beginning of this section). It is clear that the metrics $\mathsf{d}_n^\tau, \mathsf{d}_\infty^\tau, \mathsf{d}_{\infty,n}^\tau$ induce on the sets $\{t_1^n \le \sqrt{v_n} \le t_2^n\}, \{t_1^\infty \le \sqrt{v_\infty}\le t_2^\infty\}, \{t_1^n \le \sqrt{v_\infty} \le t_2^n\}$ the same topology induced by the metrics $\mathsf{d}_n, \mathsf{d}_\infty$.

Notice also that, from (2.5.16) and since T is a local isometry on $\{\sqrt{v}_\infty > 1+\varepsilon/4\}$,

(2.5.23) $(\{s \le \sqrt{v_\infty} \le t\}, \mathsf{d}_\infty^{s,t,\tau})$ is isometric to $(\{s \le \lambda \mathsf{d}_{O_{Y'}} \le t\}, \mathsf{d}_{Y'}^{s,t,\tau})$, $\forall \tau \in (0,\eta)$, $\forall\, t \ge s > 1+\varepsilon+\eta$,

where $\mathsf{d}_\infty^{s,t,\tau}$ and $\mathsf{d}_{Y'}^{s,t,\tau}$ are the intrinsic metrics respectively on $\{s-\tau < \sqrt{v_\infty} < t+\tau\}$ and on $\{s - \tau < \lambda\mathsf{d}_{O_{Y'}} < t+\tau\}$, the isometry being T itself, which also measure preserving. In particular there exists a constant $D > 0$ such that for every $\tau \in (0,\eta)$ it holds $\mathrm{diam}(\{t_1^\infty \le \sqrt{v_\infty}\le t_2^\infty\}, \mathsf{d}_\infty^\tau) \le D$.

Observe that from (2.5.17) we deduce that the functions $\sqrt{v_n}, \sqrt{v_\infty}$ are equi-Lipschitz on $\{t_1^n - \eta \le \sqrt{v_n} \le t_2^n + \eta\}$, $\{t_1^\infty - \eta \le \sqrt{v_\infty} \le t_1^\infty + \eta\}$ and we fix $M \ge 2$ a bound on their Lipschitz constant.

Putting $\varepsilon_n := 2\max(b_n, a_n)$ (where a_n, b_n are the ones in (2.5.18) and (2.5.20)) it is not restrictive to assume both that $\sqrt{\varepsilon_n} < \eta/(2M)$ and that $|t_1^n - t_1^\infty|, |t_2^n - t_2^\infty| < \varepsilon_n$, for every n.

Pick any $x_0, x_1 \in \{t_1^n \le \sqrt{v_n} \le t_2^n\}$ and set $y_i = f_n(x_i) \in \{t_1^\infty \le \sqrt{v_\infty}\le t_2^\infty\}$, $i = 0,1$, where f_n was defined above and recall that $\mathsf{d}_Y(x_i, f_n(x_i)) \le \varepsilon_n$. Consider an absolutely continuous curve $\gamma : [0,1] \to \overline{A_\infty^{\varepsilon - 2M\sqrt{\varepsilon_n}}}$ such that $\gamma(i) = y_i$, $i = 0,1$ and $L(\gamma) = \mathsf{d}_\infty^{\eta-2M\sqrt{\varepsilon_n}}(y_0,y_1) \le D$. Letting $N_n := \lfloor 2D/\sqrt{\varepsilon_n}\rfloor$, there exist $0 = t_0 < t_1 < ... < t_{N_n} = 1$ such that $\mathsf{d}_\infty(\gamma(t_i), \gamma(t_{i+1}) \le L(\gamma|_{[t_i,t_{i+1}]})) \le L(\gamma)/N_n$, for every $i = 0, ..., N_n - 1$. Thanks to (2.5.18) and the M-Lipschitzianity of $\sqrt{v_n}$ there exist points $x_i \in A_n^{\eta - M\sqrt{\varepsilon_n}}$ $i = 1, ..., N_n - 1$, such that $\mathsf{d}_Y(x_i, \gamma(t_i)) < \varepsilon_n$, $i = 1, ..., N_n - 1$, and in particular $|\mathsf{d}_n(x_i, x_{i+1}) - \mathsf{d}_\infty(\gamma(t_i), \gamma(t_{i+1})| \le 2\varepsilon_n$, for every $i = 0, ..., N_n$. Therefore $\mathsf{d}_n(x_i, x_{i+1}) < L(\gamma)/N_n + 2\varepsilon_n \le \sqrt{\varepsilon_n}$, and thus any geodesic (in X_n) γ_i from x_i to x_{i+1} has image contained in A_n^η. We define $\bar\gamma : [0,1] \to A_n^\eta$ as the concatenation of all the geodesics γ_i (appropriately reparametrized), in particular

$$\mathsf{d}_n^\eta(x_0,x_1) \le L(\bar\gamma) \le N_n\left(\frac{L(\gamma)}{N_n} + 2\varepsilon_n\right) \le \mathsf{d}_\infty^{\eta-2M\sqrt{\varepsilon_n}}(f_n(x_0), f_n(x_1)) + 4D\sqrt{\varepsilon_n}. \tag{2.5.24}$$

Conversely pick an absolutely continuous curve $\bar\gamma : [0,1] \to \overline{A_n^\eta}$ such that $\bar\gamma(i) = x_i$, $i = 0,1$ and $L(\bar\gamma) = d_n^\eta(x,y) \le 2D$, which exists thanks to (2.5.24). Arguing exactly as above we can construct an absolutely continuous curve $\gamma : [0,1] \to A_\infty^{\eta+2M\sqrt{\varepsilon_n}}$ such that $\gamma(i) = y_i$, $i = 0,1$ and

$$\mathsf{d}_\infty^{\eta+2M\sqrt{\varepsilon_n}}(f_n(x_0), f_n(x_1)) \le L(\gamma) \le \mathsf{d}_n^\eta(x_0,x_1) + 4D\sqrt{\varepsilon_n}.$$

Recalling (2.5.23) we are in position to apply Lemma 2.5.7 and deduce that $\mathsf{d}_\infty^{\eta\pm2M\sqrt{\varepsilon_n}}(f_n(x_0), f_n(x_1)) \to \mathsf{d}_\infty^\eta(f_n(x_0), f_n(x_1))$ as $n \to +\infty$, uniformly in $x_0, x_1 \in \{t_1^n \le \sqrt{v_n} \le t_2^n\}$. Moreover again from Lemma 2.5.7 we have that the image of f_n is cb_n-dense in $\{t_1^\infty \le \sqrt{v}_\infty \le t_2^\infty\}$ w.r.t. the metric d_∞^η, for some constant c independent of n.

Combing this with the above inequalities and (2.5.21) we obtain

$$(\{t_1^n \le \sqrt{v_n} \le t_2^n\}, \mathsf{d}_n^\eta, \mu_n) \overset{mGH}{\to} (\{t_1^\infty \le \sqrt{v_\infty}\le t_2^\infty\}, \mathsf{d}_\infty^\eta, \mu_\infty), \tag{2.5.25}$$

with $\mu_n := \mathfrak{m}_n|_{\{t_1^n\le\sqrt{v}_n\le t_2^n\}}$, $\mu_\infty := \mathfrak{m}_\infty|_{\{t_1^\infty\le\sqrt{v}_\infty\le t_2^\infty\}}$ and where, if $t_1^\infty = t_2^\infty$, the convergence is intended only in the GH-sense. Finally from Lemma 2.5.7 and recalling (2.5.23) we have that $\left(\{t_1^n \le \sqrt{v_\infty}\le t_2^n\}, \mathsf{d}_{\infty,n}^\eta\right) \overset{GH}{\to} \left(\{t_1^\infty \le \sqrt{v_\infty} \le t_2^\infty\}, \mathsf{d}_\infty^\eta\right)$, and that such convergence can be realized by a map $g_n : \{t_1^n \le \sqrt{v_\infty} \le t_2^n\} \to \{t_1^\infty \le \sqrt{v_\infty}\le t_2^\infty\}$ that (if $t_1^n \ne t_2^n$) also satisfies $g_{n*}\left(\mathfrak{m}_\infty|_{\{t_1^n\le\sqrt{v_\infty}\le t_2^n\}}\right) = \frac{(t_2^\infty)^N - (t_1^\infty)^N)}{((t_2^n)^N - (t_1^n)^N)}\mathfrak{m}_\infty|_{\{t_1^\infty\le\sqrt{v_\infty}\le t_2^\infty\}}$. In particular

$$\left(\{t_1^n \le \sqrt{v_\infty} \le t_2^n\}, \mathsf{d}_{\infty,n}^\eta, \mu_{\infty,n}\right) \overset{mGH}{\to} (\{t_1^\infty \le \sqrt{v_\infty} \le t_2^\infty\}, \mathsf{d}_\infty^\eta, \mu_\infty), \tag{2.5.26}$$

with $\mu_{\infty,n} := \mathfrak{m}_\infty|_{\{t_1^n\le\sqrt{v_\infty}\le t_2^n\}}$ and where in the case $t_1^\infty = t_2^\infty$ the convergence is intended only in the GH-sense.

If $t_1^\infty = t_2^\infty$, combining (2.5.25) with (2.5.26), and recalling (2.5.23), we obtain that (2.5.12) holds for n big enough. Therefore, since we are assuming d') (see above), up to a subsequence, we must have that either (2.5.13) is false every n big enough or the last claim about Y in Theorem 2.5.6 is false. The latter cannot happen, indeed in the first half of the proof we proved precisely that Theorem 2.5.5 holds with the same Y and with the same ε, L, R_0. Hence we must be n the first case and in particular $t_1^\infty+\varepsilon<t_2^\infty$ and from (2.5.25) and (2.5.26) and obtain that (2.5.13) holds for n big enough, which is a contradiction. **This concludes the proof of Theorem 2.5.6**. □

2.6. Rigidity and almost rigidity from the monotonicity formula

2.6.1. Rigidity. The following rigidity result follows almost immediately combining the explicit lower bound on the derivative of U_β' in (2.4.7) and Theorem 2.5.1 about "from outer functional cone to outer metric cone".

Theorem 2.6.1. *Let* X, Ω, u, U_β, *with* $\beta > \frac{N-2}{N-1}$, *be as in Theorem 2.4.4 and suppose that* $U_\beta^{'-}(t_0)=0$ *for some* $t_0\in(0,1]$. *Then the hypotheses of Theorem 2.5.1, and in particular also its conclusions, are satisfied choosing* $\mathbf{u} = Cu^{\frac{2}{2-N}}$, $\mathbf{u}_0 = Ct_0^{\frac{2}{2-N}}$ *and* $U=\{u<t_0\}$, *for some constant* $C>0$.

Proof. Suppose $U_\beta^{'-}(t_0)=0$ for some $t_0\in(0,1]$ and observe that, thanks to (2.4.7), since $\tilde{C}_{\beta,N}>0$, we must have that $|\nabla|\nabla u^{\frac{1}{2-N}}|^{\beta/2}|=0$ $\mathfrak{m}$-a.e. in $\{u<t_0\}$. We claim that $\{u<t_0\}$ is connected. Indeed, if $t_0<1$, from the continuity of u follows that $\partial\{u<t_0\}\subset\{u=t_0\}$, hence from the strong maximum principle we deduce that all the connected components of $\{u<t_0\}$ are unbounded. Moreover $\partial\{u<t_0\}$ is bounded and thus from Corollary 2.1.7 it follows that that $\{u<t_0\}$ is connected. If $t_0=1$, we conclude observing that $\{u<1\}$ is the union of the sets $\{u<t\}$ with $t<1$. Therefore we have that $|\nabla u^{\frac{1}{2-N}}|^2\equiv C$ $\mathfrak{m}$-a.e. in $\{u<t_0\}$, for some constant C. We now claim that $C>0$. Indeed if $C=0$ we would have that $\nabla u = -(2-N)u^{\frac{N-1}{N-2}}\nabla u^{\frac{1}{2-N}}=0$ $\mathfrak{m}$-a.e. in $\{u<t_0\}$ and therefore u would be constant in $\{u<t_0\}$ (recall (1.3.11)). However u goes to 0 at $+\infty$ and $\{u<t_0\}$ is unbounded, therefore $u\equiv 0$ in $\{u<t_0\}$, but this violates the positivity of u. Setting $v=u^{\frac{1}{2-N}}$, by the chain rule for the Laplacian, the harmonicity of u and by locality we have

$$\Delta\frac{v^2}{2} = \frac{1}{2}\Delta(u^{\frac{2}{2-N}}) = \frac{N}{(2-N)^2}\frac{|\nabla u|^2}{u^{2\frac{N-1}{N-2}}} = CN, \quad \mathfrak{m}\text{-a.e. in } \{u<t_0\}.$$

Moreover $|\nabla v^2/2|^2 = v^2|\nabla v|^2 = 2C\frac{v^2}{2}$. Therefore the function $\mathbf{u} = C^{-1}v^2/2$ satisfies the hypotheses of Theorem 2.5.1 with $U=\{u<t_0\}$ and $\mathbf{u}_0 = C^{-1}t_0^{2/(2-N)}/2$. This concludes the proof. □

2.6.2. Almost rigidity. The goal of this subsection is to prove the following.

Theorem 2.6.2. *For all numbers* $\varepsilon\in(0,1/3)$, $R_0>0$ $\beta>\frac{N-2}{N-1}$, $N\in(2,\infty)$ *and for every function* $f:(1,+\infty)\to\mathbb{R}^+$ *in* $L^1(1,+\infty)$ *there exists* $0<\delta=\delta(\varepsilon,\beta,N,f)$ *such that the following holds. Let* $(\mathrm{X},\mathsf{d},\mathfrak{m},x_0)$ *be a pointed* RCD$(0,N)$ *m.m.s. such that* $\mathfrak{m}(B_1(x_0))\le\varepsilon^{-1}$ *and* $\frac{s}{\mathfrak{m}(B_s(x_0))}\le f(s)$ *for* $s\ge1$. *Let u be an electrostatic potential for some* $E\subset B_{R_0}(x_0)$ *and suppose that there exists* $t\in(\varepsilon,1]$ *satisfying* $\mathsf{d}(x_0,\{u\le t\})>\varepsilon$, $\||\nabla u|\|_{L^\infty(\{u<t\})}\le\varepsilon^{-1}$ *and*

$$U_\beta^{-'}(t)<\delta. \tag{2.6.1}$$

Then there exists a pointed RCD(0, N) *space* $(\mathrm{X}',\mathsf{d}',\mathfrak{m}',x')$ *such that*

$$\mathsf{d}_{pmGH}((\mathrm{X},\mathsf{d},\mathfrak{m},x_0),(\mathrm{X}',\mathsf{d}',\mathfrak{m}',x'))<\varepsilon$$

and $(\mathrm{X}',\mathsf{d}',\mathfrak{m}',x')$ *is a truncated cone outside a compact set* $K\subset B_{(1+\varepsilon)R}(x')$, *where* $\{u<t-t\varepsilon\}^c\subset B_R(x_0)$, *i.e. there exists an* RCD(0, N) *Euclidean N-cone Y, with vertex O_Y, over an* RCD$(N-2,N-1)$ *space Z and a measure preserving local isometry* $T:\mathrm{X}'\setminus K\to Y\setminus\bar{B}_r(O_Y)$, *for some* $r>0$.

We will also prove the following alternative version of the above statement (see Section 2.5.2 for the definition of intrinsic metric).

Theorem 2.6.3. *For all numbers* $\varepsilon\in(0,1/3)$, $R_0>0$ $\beta>\frac{N-2}{N-1}$, $N\in(2,\infty)$, $\eta>0$ *and for every function* $f:(1,+\infty)\to\mathbb{R}^+$ *in* $L^1(1,+\infty)$ *there exists* $0<\delta=\delta(\varepsilon,\beta,N,f,\eta)$ *such that, given* X, v *and t as in Theorem 2.5.5, there exists an* RCD(0, N) *Euclidean N-cone* $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$, *with vertex* O_Y, *over an* $\mathrm{RCD}(N-2, N-1)$ *space* Z *and a constant* $\lambda > 0$ *such that the following holds. For every* $(1+\varepsilon+\eta)t^{\frac{1}{2-N}} < t_1 < t_2 < \varepsilon^{-1}t^{\frac{1}{2-N}}$ *it holds*

$$\mathsf{d}_{GH}\left((\{t_1 \le u^{\frac{1}{2-N}} \le t_2\}, \mathsf{d}^{\eta}_{\mathrm{X}}), (\{t_1 \le \lambda \mathsf{d}_{O_Y} \le t_2, \mathsf{d}^{\eta}_{Y})\right) < \varepsilon,$$

where $\mathsf{d}_{O_Y} := \mathsf{d}_Y(.,O_Y)$ *and* $\mathsf{d}^{\eta}_{\mathrm{X}}$ *and* d^{η}_{Y} *denote the intrinsic metrics on* $\{t_1 - \eta t^{\frac{1}{2-N}} < u^{\frac{1}{2-N}} < t_2 + \eta t^{\frac{1}{2-N}}\}$ *and on* $\{t_1 - \eta < \lambda \mathsf{d}_{O_Y} < t_2 + \eta\}$*. Moreover, provided that* $t_1 + \varepsilon t < t_2$,

$$\mathsf{d}_{mGH}\left((\{t_1 \le u^{\frac{1}{2-N}} \le t_2\}, \mathsf{d}^{\eta}_{\mathrm{X}}, \mathfrak{m}|_{\{t_1 \le u^{\frac{1}{2-N}} \le t_2\}}), (\{t_1 \le \lambda \mathsf{d}_{O_Y} \le t_2\}, \mathsf{d}^{\eta}_{Y}, \mathfrak{m}_Y|_{\{t_1 \le \lambda \mathsf{d}_{O_Y} \le t_2\}})\right) < \varepsilon.$$

Before passing to the proof we explain why the bound on $\||\nabla u|\|_{L^\infty(\{u<t\})}$ is natural and often satisfied. The immediate observation is that from the gradient bound for harmonic functions (1.3.18) we deduce

$$|\nabla u| \le \frac{C(N)}{\varepsilon}, \quad \mathfrak{m}\text{-a.e. in } \{u<1\} \cap \{x \,:\, \mathsf{d}(x,\partial\Omega) > \varepsilon\}, \quad \forall\, \varepsilon > 0. \tag{2.6.2}$$

In particular for fixed $\varepsilon > 0$, thanks to the assumption $\underline{\lim}_{x\to\partial\Omega} u(x) \ge 1$, for t sufficiently small (but depending on u) the gradient bound $\||\nabla u|\|_{L^\infty(\{u<t\})} \le C(N)\varepsilon^{-1}$ is always satisfied. An estimate on the value of t can be given in the case $B_\varepsilon(x_0) \subset \Omega$. Indeed applying the lower bound for u given by (2.4.2), it is immediately seen that $\{u<t\} \subset \{x \,:\, \mathsf{d}(x,\partial\Omega) > \varepsilon\}$ for any $0 < t < \frac{1}{2}\left(\frac{\varepsilon}{\mathrm{diam}(\Omega^c)+\varepsilon}\right)^{N-2}$.

Something more explicit can be said if we consider u to be an electrostatic potential. Indeed combining (2.6.2) with the continuity estimate (2.2.7) one can easily prove the following:

Proposition 2.6.4. *For all numbers* $\varepsilon \in (0,1/3)$, $N \in (2,\infty)$ *there exists* $0 < C = C(\varepsilon, N)$ *such that the following holds. Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a non compact RCD*$(0,N)$ *m.m.s. and let* $E \subset \mathrm{X}$ *be open and bounded with uniformly* Cap*-fat boundary with parameters* $(\varepsilon,\varepsilon)$ *(see Def. 2.2.1). Let* u *be the capacitary potential relative to* E *(see Theorem 2.2.13). Then*

$$|\nabla u| \le C(\varepsilon, N), \quad \mathfrak{m}\text{-}a.e. \text{ in } \{u<t\}, \text{ for every } t \in (0, 1-\varepsilon).$$

We pass to the proof of Theorem 2.6.2 and Theorem 2.6.3, which are almost corollaries of Theorem 2.5.5 and Theorem 2.5.6.

Proof of Theorem 2.6.2 and Theorem 2.6.3. Observe first that by Bishop Gromov inequality

$$\mathfrak{m}(B_1(x_0)) \ge \mathfrak{m}(B_{\varepsilon^{-1}})\varepsilon^N \ge \varepsilon^{N-1}/f(\varepsilon^{-1}).$$

From the second inequality in (2.4.2) we have that there exist a positive constant $C_1 = C_1(\varepsilon, R_0, N)$, such that

$$u(x) \le C_1 \int_{\mathsf{d}(x,x_0)}^{\infty} f(s)\,\mathrm{d}s, \quad \forall\, x \in B_{4R_0+1}(x_0)^c. \tag{2.6.3}$$

In particular, since $t > \varepsilon$, $\mathrm{diam}(\{u<t\}^c) \le \mathrm{diam}(\{u<\varepsilon\}^c) \le C_2$, for some constant C_2 depending only on ε, R_0, N, f. Therefore, again since $t > \varepsilon$, up to rescaling u as ut^{-1} we can assume that $t=1$ (observe also that, called $\tilde{U}_\beta$ the function relative to $t^{-1}u$, it holds that $\tilde{U}_\beta(s) = U_\beta(ts)t^{\beta\frac{N-1}{N-2}-\beta-1}$, $s \in (0,1)$).

Define $v := u^{\frac{2}{2-N}}$ and set $\Omega' = \{u<1\}$. From the harmonicity of u we have that $\Delta v = N|\nabla\sqrt{2v}|^2$, $\mathfrak{m}$-a.e. in Ω' and that $\overline{\lim}_{x\to\partial\Omega'} v(x) \le 1$. Moreover (2.6.3) ensures that $v \ge 1+\varepsilon$ in $B_{\bar{R}}(x_0)^c$ for some $\bar{R} = \bar{R}(\varepsilon, N, f, R_0)$.

To apply Theorems 2.5.5 and 2.5.6 it still remains to check the bounds on $|\nabla\sqrt{v}|$.

Observe that from the assumption $\mathsf{d}(x_0, \{u \le 1\}) > \varepsilon$ and the first in (2.4.2) we deduce that

$$u_n(x) \ge c\,\mathsf{d}(x_n, x)^{2-N}, \text{ for every } x \in \Omega', \tag{2.6.4}$$

for some positive constant $c = c(\varepsilon)$. Combining (2.6.4), the assumption $\||\nabla u|\|_{L^\infty(\{u<1\})} \le \varepsilon^{-1}$ and the gradient estimate (1.3.18), it easily follows that

$$|\nabla\sqrt{v}| = (N-2)^{-1}|\nabla u|u^{\frac{1-N}{N-2}} \le C_3,$$

for some positive constant $C_3 = C_3(\varepsilon)$. Finally from (2.4.7) and (2.6.1) we have

$$\int_{\{u<1\}} \frac{1}{v^{N-2}}\left|\nabla|\nabla\sqrt{v}|^{\beta/2}\right|^2 \mathrm{d}\mathfrak{m} \le \tilde{C}^{-1}_{\beta,N}U^{-\prime}_{\beta}(1) < C^{-1}_{\beta,N}\delta.$$

We are therefore in position to apply Theorem 2.5.5 and conclude the proof of Theorem 2.6.2.

Theorem 2.6.3 follows from Theorem 2.5.6 and observing that $\{u = s^{\frac{1}{2-N}}\} = \{\sqrt{v} = s\}$ and that, thanks to (2.6.3), for every $t > 0$

$$\{\sqrt{v} \le t\} \subset B_{R_1}(x_0),$$

for some $R_1 = R_1(t, \varepsilon, N, f, R_0)$. □

2.A. Appendix: From outer functional cone to outer metric cone - additional details

This appendix is devoted to the proof of Theorem 2.5.1 given the two results presented in Section 2.5.1. Since the proof is in large part the same as [**95**] some steps will be only outlined, however there will be differences and new arguments that will be explained in detail and emphasized along the exposition.

2.A.1. The gradient flow of u and its effect on the measure. From the chain rule for the Laplacian (1.2.11), the positivity of $\mathbf{u}$ and recalling that $\Delta\mathbf{u} = N$ and $|\nabla\mathbf{u}|^2 = 2\mathbf{u}$ $\mathfrak{m}$-a.e. in U, it follows that

$$v := \begin{cases} \mathbf{u}^{\frac{2-N}{2}}, & \text{if } N > 2, \\ \ln(\frac{1}{\sqrt{\mathbf{u}}}), & \text{if } N = 2, \end{cases} \quad \text{is harmonic in } U.$$

In particular the maximum principle and the fact that $\mathbf{u}_0 = \overline{\lim}_{x\to\partial U} \mathbf{u}(x)$ ensure that

$$\text{every connected component of } \{\mathbf{u} > \mathbf{u}_0\} \text{ is unbounded.} \tag{2.A.1}$$

Since by assumption $\{\mathbf{u} > \mathbf{u}_0\}$ is nonempty we must have that X is unbounded.

For technical reasons we will work locally, in particular we fix a set V open and relatively compact in U and consider $\eta \in \mathsf{Test}(\mathrm{X})$ such that $\eta = 1$ in $\overline{V}$, $0 \le \eta \le 1$ and $\operatorname{supp}\eta \subset U$, which exists thanks to Proposition 1.4.9. We then define

$$u := \eta\mathbf{u}.$$

Since $\mathbf{u} \in \mathsf{Test}_{\mathsf{loc}}(U)$ from (1.4.8) we deduce that $u \in \mathsf{Test}(X)$.

We point out that we would like to take right away V to be of the form $\{t_0 < \mathbf{u} < T_0\}$, however to ensure that this set is relatively compact in U we need first to know that $\mathbf{u}$ blows up at infinity. This will be proved in Lemma 2.A.2.

We now consider the regular Lagrangian flow $F : [0,T] \times X \to X$ associated to the autonomous vector field $v = -\nabla u$. Observe that since $\Delta u \in L^\infty(\mathfrak{m})$ and $u \in \mathsf{Test}(\mathrm{X})$ the assumptions of Theorem 1.7.4 are satisfied. In particular the flow F exists unique. Moreover, again thanks to $\Delta u \in L^\infty(\mathfrak{m})$ and Remark 1.7.5 we can extend the map F to $(-\infty, +\infty) \times X$.

Proposition 2.A.1. *(1) For $\mathfrak{m}$-a.e. $x \in X$ it holds that $F_t(F_s(x)) = F_{s+t}(x)$ for every $s, t \in \mathbb{R}$.*

(2) For $\mathfrak{m}$-a.e. $x \in U$ it holds that $(-\infty, \infty) \ni t \mapsto F_t(x)$ is continuous. Moreover denoted by (a_x, b_x) the maximal interval such that $F_t(x) \in U$ for all $t \in (a_x, b_x)$ (which in particular satisfies $a_x < 0, b_x > 0$ and possibly $a_x = -\infty$ or $b_x = +\infty$), it holds

$$\mathbf{u}(F_t(x)) = e^{-2t}\mathbf{u}(x), \quad \forall t \in (a_x, b_x) \tag{2.A.2}$$

and

$$\mathsf{d}(F_s(x), F_t(x)) \le |e^{-t} - e^{-s}|\sqrt{2\mathbf{u}(x)}, \quad \forall t, s \in (a_x, b_x). \tag{2.A.3}$$

Proof. The first is just (1.7.5).

(2.A.2) follows observing that from (1.7.2), since $|\nabla u|^2 = 2u$ $\mathfrak{m}$-a.e. in U, for $\mathfrak{m}$-a.e. $x \in U$ it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t} u(F_t(x)) = -2u(F_t(x)), \quad \text{for a.e. } t \in (a_x, b_x).$$

(2.A.3) instead can be derived from the fact that, recalling Remark 1.7.2, for $\mathfrak{m}$-a.e. $x \in U$ it holds $|\dot{F}_t(x)| = \sqrt{2u(F_t(x))}$ for a.e. $t \in (a_x, b_x)$. □

Recall that, as remarked at the beginning of the section, X is unbounded, hence the following result makes sense.

Lemma 2.A.2.

$$\mathbf{u}(x) \to +\infty \quad \text{as } \mathsf{d}(x, U^c) \to +\infty. \tag{2.A.4}$$

Proof. Suppose (2.A.4) is false. Then we can find a ball $B_{2R}(\bar{x}) \subset U$ such that $R > 100\sqrt{\mathbf{u}(\bar{x})} + 1$. We choose $\eta \in \mathsf{Test}(\mathrm{X})$ such that $\eta = 1$ in $\overline{B}_R(\bar{x})$, $0 \le \eta \le 1$ and $\operatorname{supp}\eta \subset U$, which exists from Proposition 1.4.9. We define $u := \mathbf{u}\eta \in \mathsf{Test}(\mathrm{X})$ and consider the regular Lagrangian flow F_t relative to $-\nabla u$. Then from (2.A.3) (with the choice $V = B_R(\bar{x})$), the continuity of $\mathbf{u}$ and the choice of R we can find $x' \in B_1(\bar{x})$ such that the curve $F_t(x')$ is contained in $B_R(\bar{x})$ for all $t > 0$. This together with (2.A.2) contradicts the positivity of $\mathbf{u}$. □

From now until the very last part of the proof we fix $t_0, T_0 \in \mathbb{R}^+$ such that $\mathbf{u}_0 < t_0 + 1 < T < T_0 - 1$, where T is to be chosen later.

Thanks to both (2.A.4) and $\mathbf{u}_0 = \overline{\lim}_{x \to \partial U} \mathbf{u}(x)$ we have that $\{t_0 < \mathbf{u} < T_0\}$ is compactly contained in U. Hence we can pick a cut off function $\eta \in \mathsf{Test}(\mathrm{X})$ such that $\eta = 1$ in $\{t_0 \le \mathbf{u} \le T_0\}$, $0 \le \eta \le 1$, $\operatorname{supp} \eta \subset U$ and define $u \coloneqq \eta \mathbf{u} \in \mathsf{Test}(\mathrm{X})$. As above we consider F_t the flow relative to $-\nabla u$, which is defined for all positive and negative times.

Define for every $a, b \in [\mathbf{u}_0, \infty)$ the open set

$$A_{a,b} \coloneqq \{a < \mathbf{u} < b\}.$$

From the definition of u, the hypotheses on $\mathbf{u}$ and the locality of the gradient and the Laplacian we have

$$\begin{aligned} &\Delta u = N, \quad \mathfrak{m}\text{-a.e. in } A_{t_0,T_0}. \\ &|\nabla u|^2 = 2u, \quad \mathfrak{m}\text{-a.e. in } A_{t_0,T_0}. \end{aligned} \tag{2.A.5}$$

The following can be proven arguing as in [**95**, sec. 3.6.1], however we give a shorter proof, which use the improved Bochner inequality (1.4.7).

Proposition 2.A.3.

$$\mathrm{Hess}(u) = \mathsf{id} \quad \mathfrak{m}\text{-}a.e. \text{ } in \text{ } A_{t_0,T_0}. \tag{2.A.6}$$

Proof. Localizing (1.4.7) to A_{t_0,T_0} and recalling (2.A.5) we obtain

$$N \ge |\mathrm{Hess}(u)|_{HS}^2 + \frac{(N - \mathsf{tr}\mathrm{Hess}(u))^2}{N - \dim_{\mathsf{loc}}} \quad \mathfrak{m}\text{-a.e. in } A_{t_0,T_0}.$$

By Cauchy-Swartz and recalling (1.4.4), (1.4.5) we observe that

$$\begin{aligned} |\mathrm{Hess}(u)|_{HS}^2 &= \sum_{1 \le i,j \le \dim(\mathrm{X})} \mathrm{Hess}(u)(e_i, e_j)^2 \\ &\ge \sum_{i=1}^{\dim_{\mathsf{loc}}} \mathrm{Hess}(u)(e_i, e_i)^2 \ge \frac{\mathsf{tr}\mathrm{Hess}(u)^2}{\dim_{\mathsf{loc}}}. \end{aligned}$$

Plugging this in the above inequality and applying again Cauchy-Swartz we obtain

$$N \ge \frac{\mathsf{tr}\mathrm{Hess}(u)^2}{\dim_{\mathsf{loc}}} + \frac{(N - \mathsf{tr}\mathrm{Hess}(u))^2}{N - \dim_{\mathsf{loc}}} \ge N \quad \mathfrak{m}\text{-a.e. in } A_{t_0,T_0}.$$

Hence all the inequality we used were actually equalities, in particular $\mathrm{Hess}(u)(e_i, e_j) = 0$ $\mathfrak{m}$-a.e. in A_{t_0,T_0}, for every $i \ne j$ and $\mathrm{Hess}(u)(e_i, e_i) = 1$ $\mathfrak{m}$-a.e. in A_{t_0,T_0} for every $i, j = 1, ..., \dim_{\mathsf{loc}}$, which concludes the proof. ☐

Proposition 2.A.4. *(1) For $\mathfrak{m}$-a.e. $x \in X$ it holds that $F_t(F_s(x)) = F_{s+t}(x)$ for every $s, t \in \mathbb{R}$.*

(2) For $\mathfrak{m}$-a.e. $x \in A_{t_0,T_0}$ it holds that $F_t(x) \in A_{t_0,T_0}$ and

$$u(F_t(x)) = e^{-2t} u(x), \tag{2.A.7}$$

for every $t \in (\frac{1}{2} \log \frac{u(x)}{T_0}, \frac{1}{2} \log \frac{u(x)}{t_0})$, moreover

$$\mathsf{d}(F_s(x), F_t(x)) = |e^{-t} - e^{-s}| \sqrt{2u(x)}, \tag{2.A.8}$$

for every $s, t \in (\frac{1}{2} \log \frac{u(x)}{T_0}, \frac{1}{2} \log \frac{u(x)}{t_0})$, in particular the curve $(\frac{1}{2} \log \frac{u(x)}{T_0}, \frac{1}{2} \log \frac{u(x)}{t_0}) \ni t \mapsto F_t(x)$ is supported on a geodesic.

Proof. Everything except for the equality in (2.A.8) follow Proposition 2.A.1 together with the observation that in this case $a_x = \frac{1}{2} \log \frac{u(x)}{T_0}$ and $b_x = \frac{1}{2} \log \frac{u(x)}{t_0}$.

To show equality in (2.A.8) it is enough to show it for $t = 0$ and $s > 0$. Hence we fix $s \in (0, \frac{1}{2} \log \frac{u(x)}{t_0})$. Thanks to (2.A.3) we only need to show that

$$\mathsf{d}(F_s(x), F_t(x)) \ge (1 - e^{-s}) \sqrt{2u(x)}. \tag{2.A.9}$$

We make the intermediate claim that

$$\mathsf{d}(x, U^c) \ge \sqrt{2\mathbf{u}(x)} - \sqrt{2\mathbf{u}_0}, \quad \forall x \in U. \tag{2.A.10}$$

To prove it we first observe that, since $\mathbf{u}$ is positive, we have that $\sqrt{\mathbf{u}} \in W^{1,2}_{\mathsf{loc}}(U)$ and $|\nabla \sqrt{2\mathbf{u}}| = 1$ $\mathfrak{m}$-a.e. in U.

The properness of the space X ensures that there exists $\bar{x} \in \partial U$ such that $\mathsf{d}(x, \bar{x}) = \mathsf{d}(x, \partial U)$. Moreover, since X is geodesic, there exists a sequence $(x_n) \subset U$ such that $x_n \to x$ and $\mathsf{d}(x, x_n) \leq \mathsf{d}(x, \partial U)$. Hence recalling that $|\nabla\sqrt{2\mathbf{u}}| = 1$ $\mathfrak{m}$-a.e., we are in position to apply (1.3.10) and deduce that

$$\mathsf{d}(x, \bar{x}) = \lim_n \mathsf{d}(x, x_n) \geq \varliminf_n \sqrt{2\mathbf{u}(x)} - \sqrt{2\mathbf{u}(x_n)} \geq \sqrt{2\mathbf{u}(x)} - \sqrt{2\mathbf{u}_0},$$

which proves (2.A.10).

From (2.A.3) and by how we chose s we have that $\mathsf{d}(F_s(x), x) \leq (1 - e^{-s})\sqrt{2\mathbf{u}(x)} \leq \sqrt{2\mathbf{u}(x)} - \sqrt{2t_0} \leq \sqrt{2\mathbf{u}(x)} - \sqrt{2\mathbf{u}_0}$. Hence from (2.A.10) we deduce $\mathsf{d}(F_s(x), x) \leq \mathsf{d}(x, \partial U)$ and applying again (1.3.10) combined with (2.A.7) we obtain (2.A.9). □

Lemma 2.A.5. *For every $t \in (0, \frac{1}{2}\log\frac{T_0}{t_0})$ it holds that*

$$\mathfrak{m}(A_{e^{2t}t_0, T_0}) = e^{Nt}\mathfrak{m}(A_{t_0, e^{-2t}T_0}). \tag{2.A.11}$$

Proof. The argument is essentially the same as in [**95**, Prop. 3.7] but reversed, indeed here we start from $\Delta u = N$ and $|\nabla u|^2 = 2u$ and deduce information on the measure. □

The following result has not a direct counterpart in [**95**], however it morally substitutes the bound (3.1) in [**95**, Prop. 3.2] (cf. with (2.A.13) below). We remark that the proof of the following Proposition relies on the local estimate of Proposition 2.5.3.

Proposition 2.A.6. *For every $t \in (0, \frac{1}{2}\log\frac{T_0}{t_0})$ it holds that*

$$(F_{t*}\mathfrak{m})|_{A_{t_0, e^{-2t}T_0}} = F_{t*}(\mathfrak{m}|_{A_{e^{2t}t_0, T_0}}) = e^{Nt}\mathfrak{m}|_{A_{t_0, e^{-2t}T_0}}. \tag{2.A.12}$$

Proof. Consider the probability measure $\mu_0 = \frac{\mathfrak{m}|_{A_{e^{2t}t_0, T_0}}}{\mathfrak{m}(A_{e^{2t}t_0, T_0})}$. By Theorem 1.7.4 $\{F_{s*}\mu_0\}_{s\in[0,t]}$ are all Borel probability measures, absolutely continuous with respect to $\mathfrak{m}$ and solve the continuity equation with initial datum μ_0. Moreover (2.A.7) implies that $F_{s*}\mu_0$ is concentrated on A_{t_0, T_0} for every $s \in [0, t]$. Therefore setting $F_{t*}\mu_0 = \rho_t\mathfrak{m}$ we are in position to apply (2.5.3), that combined with (2.A.5) gives

$$\|\rho_t\|_\infty \leq \mathfrak{m}(A_{e^{2t}t_0, T_0})^{-1}e^{Nt}. \tag{2.A.13}$$

However applying now (2.A.11) and observing that $F_{t*}\mu_0$ is concentrated in $A_{t_0, e^{-2t}T_0}$, again thanks to (2.A.7), we can compute

$$0 \leq \int_{A_{t_0, e^{-2t}T_0}} \mathfrak{m}(A_{e^{2t}t_0, T_0})^{-1}e^{Nt} - \rho_t \mathrm{d}\mathfrak{m} = \mathfrak{m}(A_{e^{2t}t_0, T_0})^{-1}e^{Nt}\mathfrak{m}(A_{t_0, e^{-2t}T_0}) - 1 = 0,$$

that gives the second in (2.A.12).

The first in (2.A.12) follows directly from (2.A.7). □

Having at our disposal (2.A.12) and (2.A.8), we can argue exactly as in [**95**, Cor. 3.8] to obtain the following

Proposition 2.A.7 (Continuous disintegration)**.** *We have*

$$u_*\mathfrak{m}|_{A_{t_0, T_0}} = cr^{\frac{N}{2}-1}\mathcal{L}^1|_{(t_0, T_0)}, \tag{2.A.14}$$

where $c := \frac{N}{2}\frac{\mathfrak{m}(A_{t_1, t_2})}{t_1^{N/2} - t_2^{N/2}}$, for any $t_1, t_2 \in \mathbb{R}^+$ with $t_0 \leq t_2 < t_2 \leq T_0$. Moreover there exists a weakly continuous family of Borel measures $(t_0, T_0) \ni r \mapsto \mathfrak{m}_r \in \mathcal{P}(X)$ such that

$$\int \varphi \mathrm{d}\mathfrak{m} = c\int_{t_0}^{T_0}\int \varphi \mathrm{d}\mathfrak{m}_r r^{N/2-1}\mathrm{d}r, \quad \forall \varphi \in C_c(A_{t_0, T_0}). \tag{2.A.15}$$

Finally, for every $t \in (0, \log\frac{T_0}{t_0})$ the measures $\mathfrak{m}_r$ satisfies

$$F_{t*}\mathfrak{m}_r = \mathfrak{m}_{e^{-2t}r}, \quad \text{for a.e. } r \in (e^{2t}t_0, T_0). \tag{2.A.16}$$

The following result has not a counterpart in [**95**], since it deals with large scales, while the analysis in [**95**] is local.

Corollary 2.A.8. X *has Euclidean volume growth, in particular $\{\mathbf{u} > \mathbf{u}_0\}$ is unbounded, connected with $\{\mathbf{u} > \mathbf{u}_0\}^c$ bounded.*

Proof. Combining (2.A.7) and (2.A.8) it can be shown that $A_{t_0,T_0} \subset B_{4\sqrt{T_0}+C}(x_0)$ for every $T_0 > 2t_0$ for some fixed constant $C > 0$ (recall that what we proved so far holds for an arbitrary $T_0 > T$). Therefore $A_{t_0,\frac{(R-C)^2}{16}} \subset B_R(x_0)$ for every R big enough and the conclusion follows using (2.A.15).

Since X has Euclidean volume growth it is not compact and not a cylinder in the sense of $i)$ of Proposition 2.1.6. Now observe that $\partial\{\mathbf{u} > \mathbf{u}_0\}$ is bounded (as a consequence of (2.A.4)), hence compact, and that each connected component of $\{\mathbf{u} > \mathbf{u}_0\}$ is unbounded (by (2.A.1)). Hence the conclusion follows from Proposition 2.1.6 and the fact that $\{\mathbf{u} > \mathbf{u}_0\}$ is not empty. □

Lemma 2.A.9 ([**95**, Lemma 3.11])**.** *Let $f \in L^p(\mathfrak{m})$ with $p < +\infty$, then the map $t \mapsto f \circ F_t$ is continuous in $L^p(\mathfrak{m})$. Moreover if $f \in W^{1,2}(X)$ the map $t \mapsto f \circ F_t$ is C^1 in L^2 and its derivative is given by*

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ F_t = -\langle \nabla f, \nabla u\rangle \circ F_t.$$

2.A.2. Effect on the Dirichlet Energy. Thanks to (2.A.5) and (2.A.12) we can repeat almost verbatim the analysis done in [**95**, Sec. 3.2]. Indeed all the proofs contained there rely only on the analogous properties of the function b and Fl_t, i.e. "$|D\mathsf{b}|^2 = 2\mathsf{b}, \Delta\mathsf{b} = N$ and $\mathsf{Fl}_{t*}\mathfrak{m} = e^{Nt}\mathfrak{m}$".

This said, we will only state, adapted to our case and without proof, the final result in [**95**, Sec. 3.2] (i.e. Corollary 3.17), since it is the only statement that is needed for the rest of the argument.

Theorem 2.A.10. *Let $t \in (0, \log\frac{T_0}{t_0})$, and $f \in L^2(\mathfrak{m})$ with support in $A_{t_0,e^{-2t}T_0}$. Then $f \in W^{1,2}(X)$ if and only if $f \circ F_t \in W^{1,2}(X)$ and in this case*

$$|\nabla(f \circ F_t)| = e^{-t}|\nabla f| \circ F_t, \mathfrak{m} - a.e. \tag{2.A.17}$$

2.A.3. Precise representative of the flow. The following proposition is the analogous of [**95**, Thm. 3.18]. We point out that the proof in [**95**] contains an oversight in the proof that Fl_t has a locally-Lipschitz representative. Indeed it is claimed that this follows from the fact that Fl_t is Lipschitz in $\mathsf{Fl}_t^{-1}(B_r(x_0))$ for every small enough ball $B_r(x_0)$. However, since Fl_t is not yet proven to be continuous, we do not know enough information on the sets $\mathsf{Fl}_t^{-1}(B_r(x_0))$ to 'patch' them and obtain the claimed local Lipschitzianity.

For this reason we will give a complete proof which also fixes the original argument.

Proposition 2.A.11. *Let $\mathcal{U}_{t_0,T_0} \subset \mathbb{R} \times \mathrm{X}$ be the open set given by*

$$\mathcal{U}_{t_0,T_0} := \left\{(x,t) \ : x \in A_{t_0,T_0} \text{ and } t \in \left(\frac{1}{2}\log\frac{u(x)}{T_0}, \frac{1}{2}\log\frac{u(x)}{t_0}\right)\right\}.$$

Then the map

$$F : \mathcal{U}_{t_0,T_0} \to A_{t_0,T_0},$$

has a continuous representative w.r.t the measure $\mathcal{L}^1 \otimes \mathfrak{m}$. Moreover for such representative (which we denote again by F) the map $F_t : A_{e^{2t}t_0,T_0} \to A_{t_0,e^{-2t}T_0}$ is locally e^{-t}-Lipschitz having F_{-t} as its inverse, which is locally e^t-Lipschitz. Also for every $x \in A_{t_0,T_0}$ and every $s,t \in (\frac{1}{2}\log\frac{u(x)}{T_0}, \frac{1}{2}\log\frac{u(x)}{t_0})$

$$\mathsf{d}(F_t(x), F_s(x)) = |e^{-s} - e^{-t}|\sqrt{2u(x)} \tag{2.A.18}$$

and

$$u(F_t(x)) = e^{-2t}u(x). \tag{2.A.19}$$

Finally for every $t \in (0, \log\frac{T_0}{t_0})$ and every curve γ with values in $A_{e^{2t}t_0,T_0}$, putting $\tilde{\gamma} := F_t \circ \gamma$ we have

$$|\dot{\tilde{\gamma}}_s| = e^{-t}|\dot{\gamma}_s|, \quad \text{for a.e. } s \in [0,1], \tag{2.A.20}$$

meaning that one is absolutely continuous if and only if the other is absolutely continuous, in which case (2.A.20) *holds.*

Proof. Fix $t \in [0, \log\frac{T_0}{t_0})$. We start claiming that

$$\begin{aligned} &F_t|_{A_{e^{2t}t_0,T_0}} \text{ has a continuous representative that we denote by } \bar{F}_t \text{ and} \\ &\bar{F}_t(A_{e^{2t}t_0,T_0}) = A_{t_0,e^{-2t}T_0}. \end{aligned} \tag{2.A.21}$$

For the first part it is sufficient to show that for every $a,b \in \mathbb{R}$ such that $e^{2t}t_0 < a < b < T_0$ the map $F_t|_{A_{a,b}}$ has a continuous representative. Hence we fix such $a,b \in \mathbb{R}$ and define the open sets $A' := A_{e^{-2t}a,e^{-2t}b}$ and $A := A_{t_0,e^{-2t}T_0}$. Observe that the continuity of u implies $\mathsf{d}(A', A^c) =: \delta > 0$. Consider now the countable family of 1-Lipschitz functions $\mathcal{D} \subset \mathsf{LIP}(X)$ defined as

$$\mathcal{D} := \{f_{n,k} \mid n,k \in \mathbb{N}\} = \{\max(\min(\mathsf{d}(.,x_n), k - \mathsf{d}(.,x_n)), 0) \mid n,k \in \mathbb{N}\},$$

where $\{x_n\}_{n\in\mathbb{N}}$ is dense subset of A. We pick a cut off function $\eta \in \mathsf{LIP}_c(A)$ such that $0 \le \eta \le 1$ and $\eta \equiv 1$ in A' and define the set $\eta\mathcal{D} \subset \mathsf{LIP}_c(A)$ as $\eta\mathcal{D} := \{\eta f \mid f \in \mathcal{D}\}$. For any $f \in \mathcal{D}$ it holds

$$\mathrm{Lip}(f\eta) \le \mathrm{Lip}\eta \sup_A |f| + \mathrm{Lip}f \le \mathrm{Lip}\eta\,\mathrm{diam}(A) + 1 =: L,$$

hence the functions in $\eta\mathcal{D}$ are L-Lipschitz. We now make the key observation that

$$\mathsf{d}(x,y) = \sup_{f\in\mathcal{D}} |f(x) - f(y)| = \sup_{f\in\mathcal{D}} |\eta f(x) - \eta f(y)| = \sup_{f\in\eta\mathcal{D}} |f(x) - f(y)|, \tag{2.A.22}$$

for every $x, y \in A'$. Thanks to Corollary (2.A.10) we know that $f \circ F_t \in W^{1,2}(X)$ for every $f \in \eta\mathcal{D}$ and $|D(f \circ F_t)| = e^{-t}|Df| \circ F_t \le e^{-t}L$ $\mathfrak{m}$-a.e. Then from the Sobolev-to-Lipschitz property of X, we deduce that $f \circ F_t$ has an L-Lipschitz representative. Thus there exists an $\mathfrak{m}$-negligible set $N \subset X$ such that for every $f \in \eta\mathcal{D}$ the restriction of $f \circ F_t$ to $X \setminus N$ is L-Lipschitz. Moreover from (2.A.7) it follows the existence of an $\mathfrak{m}$-negligible set N' such that $F_t(A_{a,b} \setminus N') \subset A'$. Therefore from (2.A.22) it follows that for every $x, y \in A_{a,b} \setminus (N \cup N')$

$$\mathsf{d}(F_t(x), F_t(y)) = \sup_{f\in\eta\mathcal{D}} |f(F_t(x)) - f(F_t(y))| \le e^{-t}L\mathsf{d}(x,y).$$

This proves the first part of (2.A.21). We now show $\subset$ of the second part. From (2.A.7) it follows the existence of a negligible set N such that for every set U relatively compact in $A_{e^{2t}t_0,T_0}$ we have that $\bar{F}_t(U\setminus N)$ is relatively compact in $A_{t_0,e^{-2t}T_0}$. Moreover, since negligible sets have empty interior we deduce that $\overline{U \setminus N}$ contains U. Therefore $\bar{F}_t(U) \subset \bar{F}_t(\overline{U \setminus N}) \subset \overline{\bar{F}_t(U \setminus N)}$ which is contained in $A_{e^{2t}t_0,T_0}$ thanks to the first observation. We now show $\supset$. Again thanks to (2.A.7) the set $N := A_{t_0,e^{-2t}T_0} \setminus \bar{F}_t(A_{e^{2t}t_0,T_0})$ is negligible. Pick any set U relatively compact in $A_{t_0,e^{-2t}T_0}$ and define $V := \bar{F}_t^{-1}(U \setminus N)$ which is relatively compact in $A_{e^{2t}t_0,T_0}$. Therefore, since $\bar{F}_t$ is continuous, the set $F_t(\bar{V})$ is compact in $A_{t_0,e^{-2t}T_0}$ and, since negligible sets have empty interior, contains U. This concludes the proof of (2.A.21).

For any $t \in (\log \frac{t_0}{T_0}, 0)$ we can now argue exactly as above, to deduce that

$$\begin{aligned} &F_t|_{A_{t_0,e^{2t}T_0}} \quad \text{has a continuous representative that we denote by } \bar{F}_t \text{ and} \\ &\bar{F}_t(A_{t_0,e^{2t}T_0}) = A_{e^{-2t}t_0,T_0}. \end{aligned} \tag{2.A.23}$$

In particular, since $F_t(F_{-t}) = \mathsf{id}$ $\mathfrak{m}$-a.e., we deduce that $\bar{F}_{-t}$ is the continuous inverse of $\bar{F}_t$.

Having proved (2.A.21) and (2.A.23) we can complete the proof arguing as in [**95**] with the obvious modifications. □

From now on we denote by F a representative of $F : \mathbb{R} \times \mathrm{X} \to \mathrm{X}$ which is continuous (in space and time) on $\mathcal{U}_{t_0,T_0}$.

2.A.4. Properties of level set $\{\mathbf{u} = T\}$. In this short subsection we prove that the level set $\{\mathbf{u} = T\}$ is Lipschitz-path connected when T is big enough. We remark that the argument will rely on Proposition 2.1.6 and is different from the one used in [**95**] to prove the same property for the "sphere".

For the following result recall from Section 2.A.1 that in our construction we first choose T and then we choose t_0 and T_0 accordingly.

Lemma 2.A.12. *There exists $T > \mathbf{u}_0$ and a constant $c > 0$ (depending only on T, t_0 and $\mathbf{u}$) such that, if T_0 is big enough, for every couple of points $x, y \in \{\mathbf{u} = T\}$ there exists $\gamma \in \mathsf{LIP}([0,1], \mathrm{X})$ joining x and y and such that $\gamma \subset A_{t_0+1,T_0-1}$ and $\mathrm{Lip}(\gamma) \le 5\mathsf{d}(x,y)$.*

Proof. From (2.A.4) we know that $\mathsf{d}(\{\mathbf{u} = T\}, \{\mathbf{u} > t_0 + 1\}^c) \to +\infty$ as $T \to +\infty$. Moreover $\{\mathbf{u} > t_0 + 1\}^c$ is bounded and, as proven in Corollary 2.A.8, X has Euclidean volume growth and in particular is not compact and not a cylinder in the sense of $i)$ of Proposition 2.1.6. Therefore from Proposition 2.1.6, provided T is big enough, we have that for every couple of points $x, y \in \{\mathbf{u} = T\}$ there exists a Lipschitz path γ from x to y and such that $\gamma \subset \{\mathbf{u} > t_0 + 1\}$ and $\mathrm{Lip}(\gamma) \le 5\mathsf{d}(x,y)$. Therefore, again from (2.A.4) and since $\{\mathbf{u} = T\}$ is bounded, if T_0 is big enough then it holds that $\gamma \subset A_{t_0+1,T_0-1}$. □

Consider now the closed set $S_T := \{\mathbf{u} = T\}$ and the projection map $\mathsf{Pr} : A_{t_0,T_0} \to S_T$ defined as

$$\mathsf{Pr}(x) := F_{\frac{1}{2}\log\frac{u(x)}{T}}(x).$$

Note that from Proposition 2.A.11 and (2.A.18), we have that Pr is well defined and locally Lipschitz.

Proposition 2.A.13. *The set S_T is Lipschitz path connected, meaning that for every couple of points $x, y \in S_T$ there exists a Lipschitz curve γ taking values in S_T, joining x and y. Moreover we can choose γ so that $\mathrm{Lip}(\gamma) \le c\,\mathsf{d}(x,y)$, for some uniform constant $c > 0$.*

PROOF. From Proposition 2.A.12 we now that there exists a Lipschitz path connecting x and y taking values in A_{t_0+1,T_0-1}, which is relatively compact in A_{t_0,T_0}, then we simply consider the curve

$$\tilde{\gamma} := \mathsf{Pr} \circ \gamma,$$

which remains a Lipschitz curve, since Pr is locally Lipschitz. The claimed bound on $\mathrm{Lip}(\tilde{\gamma})$ follows from the bound on $\mathrm{Lip}(\gamma)$ given in Proposition 2.A.12 and again by the local Lipschitzianity of Pr. □

2.A.5. The level set equipped with the induced distance and measure.

Definition 2.A.14. We put $\mathrm{X}' := S_T$. For $x', y' \in S_T$ we define $\mathsf{d}'(x', y')$ as

$$\mathsf{d}'(x', y')^2 := \inf \int_0^1 |\dot{\gamma}_t|^2 \mathrm{d}t, \tag{2.A.24}$$

where the infimum is taken among all Lipschitz path $\gamma : [0,1] \to X' \subset X$ joining x' and y' and the metric speed is computed w.r.t. the distance d.

Lemma 2.A.15.

$$\mathsf{d}(x,y) \le \mathsf{d}'(x,y) \le c\mathsf{d}(x,y), \qquad \textit{for every } x, y \in \mathrm{X}', \tag{2.A.25}$$

where $c > 0$ is the constant given in Proposition 2.A.13

PROOF. The first in (2.A.25) is immediate from the definition of d', while the second follows directly from Proposition 2.A.13. □

Corollary 2.A.16. *The topology induced by d' on X' is the same as the one induced by the inclusion $X' \subset X$.*

We now define the measure $\mathfrak{m}'$ on X' as

$$\mathfrak{m}' := \mathfrak{m}_T, \tag{2.A.26}$$

where $\mathfrak{m}_T$ is given in Proposition 2.A.7. Observe that, thanks to Corollary 2.A.16, $\mathfrak{m}'$ is a Borel probability measure on X'.

A straightforward computation, exploiting (2.A.15), gives also that

$$\mathrm{Pr}_* \mathfrak{m}|_{A_{a,b}} = c_{a,b}\, \mathfrak{m}', \tag{2.A.27}$$

for every $a, b \in \mathbb{R}$ such that $t_0 \le a < b \le T_0$, where $c_{a,b} = c \int_a^b r^{N/2-1} \mathrm{d}r$, with c as in Proposition 2.A.7.

The proof of the following Proposition, which is the analogous of [**95**, Prop. 3.23], is postponed to the following section.

Proposition 2.A.17. *Let $[a,b] \subset (t_0, T_0)$ and let π be a test plan on X such that $u(\gamma_t) \in [a,b]$ for every $t \in [0,1]$ and $\boldsymbol{\pi}$-a.e. γ. Then for $\boldsymbol{\pi}$-a.e. γ the curve $\tilde{\gamma} := \mathsf{Pr} \circ \gamma$ is absolutely continuous and satisfies*

$$|\dot{\tilde{\gamma}}_t| \le \sqrt{\frac{T}{u(\gamma_t)}} |\dot{\gamma}_t|, \qquad \textit{a.e. } t.$$

The following result relies on Proposition 2.A.17 and it is the counterpart of [**95**, Thm. 3.24]. Since its proof use the exact same arguments as in [**95**], we will omit it.

Theorem 2.A.18. *Let $[a,b] \subset (t_0, T_0)$, $h \in \mathsf{LIP}(\mathbb{R})$ with support in (t_0, T_0) and identically 1 on $[a,b]$ and $f \in L^2(\mathrm{X})$ of the form $f(x) = g(\mathsf{Pr}(x))h(u(x))$ for some $g \in L^2(\mathfrak{m}')$.*

Then $f \in W^{1,2}(\mathrm{X})$ if and only if $g \in W^{1,2}(\mathrm{X}')$ and in this case we have

$$|\mathsf{D}f|_{\mathrm{X}}(x) = \sqrt{\frac{T}{u(x)}} |\mathsf{D}g|_{\mathrm{X}'}(\mathsf{Pr}(x)), \qquad \textit{for } \mathfrak{m}\textit{-a.e. } x \textit{ such that } u(x) \in [a,b]. \tag{2.A.28}$$

The following Proposition can be proved following verbatim the argument of [**95**, Prop. 3.26]. We refer to [**121**] for the definition of measured-length space.

Proposition 2.A.19. $(\mathrm{X}', \mathsf{d}', \mathfrak{m}')$ *is infinitesimally Hilbertian, doubling and a measured-length space.*

2.A.6. Speed of projection. This section is devoted to the proof of Proposition 2.A.17. The arguments and the results are the same as in [**95**, Sec. 3.6.2], however some differences are present in the set up, since we will use the language of Regular Lagrangian Flows.

To avoid confusion we point out that we will continue to indicate by F_t the continuous representative of the flow relative to $-\nabla u$, as in Proposition 2.A.11.

Fix $\bar{t_0}, \bar{T_0} \in \mathbb{R}^+$ such that $T \in (\bar{t_0}, \bar{T_0}) \subset [\bar{t_0}, \bar{T_0}] \subset (t_0, T_0)$ and pick a function $\psi \in C^2(\mathbb{R})$ with compact support contained in (t_0, T_0) and such that $\psi(z) = (\sqrt{z} - \sqrt{T})^2$ for every $z \in [\bar{t_0}, \bar{T_0}]$. Define also the function $\hat{u} := \psi \circ u|_{A_{t_0,T_0}}$ and notice that, thanks to expression of ψ, the chain rule for the gradient, Laplacian and Hessian and recalling (2.A.5) we have that $\hat{u} \in \mathsf{TestX}$ and

$$\begin{aligned}&\nabla \hat{u} = \psi'(u)\nabla u, \quad \mathrm{Hess}(\hat{u}) = \psi'(u)\mathsf{id} + \psi''(u)\nabla u \otimes \nabla u, \qquad \mathfrak{m}\text{-a.e.},\\ &|\nabla \hat{u}|^2 = 2\hat{u}, \quad \mathrm{Hess}(\hat{u}) = \psi'(u)\mathsf{id} + \frac{\sqrt{T}}{2u\sqrt{u}}\nabla u \otimes \nabla u \qquad \mathfrak{m}\text{-a.e. in } A_{\bar{t}_0,\bar{T}_0}.\end{aligned} \tag{2.A.29}$$

It follows from Theorem 1.7.4 and Remark 1.7.5 that there exists a unique Regular Lagrangian Flow relative to the autonomous vector field $v_t \equiv -\nabla u$, which is defined for all times $t \in (-\infty, +\infty)$.

Our goal is now to build an explicit representation for the flow map and prove some of its useful properties. To this aim we need the following definition. For every $r \in \mathbb{R}^+$ define the function $\mathsf{rep}_t(r) \in C^1(\mathbb{R})$ as the one such that $\mathsf{rep}_0(r) = 0$ and

$$\partial_t \mathsf{rep}_t(r) = \psi'(e^{-2\mathsf{rep}_t(r)}r), \quad t \in \mathbb{R}. \tag{2.A.30}$$

It is clear that

$$|\mathsf{rep}_t(r)| \le t\|\psi'\|_\infty, \quad t \in \mathbb{R},\ r \in \mathbb{R}^+. \tag{2.A.31}$$

Let $a, b \in \mathbb{R}^+$ be such that $\operatorname{supp}\psi \subset [a, b] \subset (t_0, T_0)$, then

$$\begin{aligned}&\mathsf{rep}_t(r) \equiv 0, \quad t \in \mathbb{R},\ r \in \mathbb{R}^+ \setminus (a, b),\\ &\mathsf{rep}_t(r) \in \left(\frac{1}{2}\log\frac{r}{T_0}, \frac{1}{2}\log\frac{r}{t_0}\right), \quad t \in \mathbb{R},\ r \in (t_0, T_0).\end{aligned} \tag{2.A.32}$$

We also define the function $\mathbb{R} \times \mathrm{X} \mapsto \mathsf{rep}_t^x \in \mathbb{R}$ as $\mathsf{rep}_t^x = \mathsf{rep}_t(u(x))$ and the function $\hat{F}_t : \mathbb{R} \times \mathrm{X} \to \mathrm{X}$ as

$$\hat{F}_t(x) = F_{\mathsf{rep}_t^x}(x).$$

Observe that, from (2.A.32), the continuity of u, the continuity of $\mathsf{rep}_t(r)$ w.r.t. r and the continuity of $F_t(x)$, jointly in space and time, as given by Proposition 2.A.21, it is easy to check that the map $\mathrm{X} \ni x \mapsto F_{\mathsf{rep}_t^x}(x)$ is well defined and continuous for every $t \in \mathbb{R}$.

If follows from the continuity of $x \mapsto \hat{F}_t(x)$, (2.A.19) and the second in (2.A.32) that

$$\hat{F}_t(A_{t_0,T_0}) \subset A_{t_0,T_0}, \quad t \in \mathbb{R}$$

and from the first in (2.A.32) that

$$\hat{F}_t = \mathsf{id}, \qquad \text{on } \mathrm{X} \setminus \overline{A_{a,b}},\ \ t \in \mathbb{R},$$

where a, b are as in (2.A.32).

In the following result we refer to [**95**] for the definition of a map of bounded deformation and its differential. Its proof can be carried out as in [**95**, Sec. 3.6.2] using the above properties of rep_t and it will be omitted. We limit ourselves to observe that in the smooth setting the estimate given in (2.A.33) are an immediate consequence of the fact that the flow map $\hat{F}_t$ (provided $\hat{u}$ is smooth) is differentiable both in time and space and $\hat{F}_0 = \mathsf{id}$

Theorem 2.A.20. *The map $\hat{F} : \mathbb{R} \times X \to \mathrm{X}$ is the unique regular Lagrangian flow for the vector field $-\nabla \hat{u}$.*

Moreover For every $t \in \mathbb{R}$ the map $\hat{F}_t : \mathrm{X} \to \mathrm{X}$ is of bounded deformation with inverse of bounded deformation. Moreover there exists a positive constant C such that

$$|\mathrm{d}(f \circ \hat{F}_s)| \le (1 + |s|C)|\mathrm{d}f| \circ \hat{F}_s, \qquad \textit{and} \qquad |\mathrm{d}\hat{F}_s(v)| \circ \hat{F}_s \le (1 + |s|C)|v| \tag{2.A.33}$$

$\mathfrak{m}$-a.e. for every $f \in W^{1,2}(\mathrm{X})$, $v \in L^2(T\mathrm{X})$ and $s \in (-1, 1)$.

Notice now that (1.7.2) and $\hat{u} \in \mathsf{Test}(\mathrm{X})$ give that $[0, \infty) \ni t \mapsto \hat{u}(\hat{F}_t(x))$ is non increasing for $\mathfrak{m}$-a.e. x. Therefore, since $\hat{u} = (\sqrt{u} - \sqrt{T})^2$ in $A_{\bar{t}_0,\bar{T}_0}$ and $T \in (\bar{t}_0, \bar{T}_0)$, we deduce that for every $x \in A_{\bar{t}_0,\bar{T}_0}$ the map $[0, \infty) \ni t \mapsto |u(\hat{F}_t(x)) - T|$ is non increasing. In particular this entails that

$$\hat{F}_t(A_{\bar{t}_0,\bar{T}_0}) \subset A_{\bar{t}_0,\bar{T}_0}, \quad t \in [0, \infty). \tag{2.A.34}$$

Lemma 2.A.21. *The map $\hat{F}_t$ converges to* Pr *as $t \to +\infty$ uniformly in $A_{\bar{t}_0,\bar{T}_0}$.*

Proof. Similarly to the proof of (2.A.2), recalling also that both $\hat{F}_t$ and $\hat{u}$ are continuous, we can prove that $\hat{u}(\hat{F}_t(x)) = e^{-2t}\hat{u}(x)$, for every $t \geq 0$ and every $x \in A_{\bar{t}_0,\bar{T}_0}$. Combining this with (2.A.34) and the definition of $\hat{u}$ we obtain

$$\left(\sqrt{u(\hat{F}_t(x))} - \sqrt{T}\right)^2 = e^{-2t}\hat{u}(x), \quad t \geq 0,\ x \in A_{\bar{t}_0,\bar{T}_0}. \tag{2.A.35}$$

From this using (2.A.18) and (2.A.19) we conclude. □

We now pass to the proof the following result, which is the analogue of Proposition 3.30 in [**95**]. The argument there however is not completely correct, since it makes use of the notion of essential limsup (and liminf), which however does not seem to be the right one to perform the proof that is presented. We here substitute this notion with the one below in Definition 2.A.23 and correct the original proof. Let us stress that in doing so we still cannot prove the $\mathfrak{m}$-a.e. convergence claimed in [**95**], which fortunately is never actually used.

Proposition 2.A.22. *Let $v \in L^2(T\mathrm{X})$ and put $v_s := \mathrm{d}\hat{F}_s(v)$. Then the map $s \mapsto \frac{1}{2}|v_s|^2 \circ \hat{F}_s \in L^1(\mathfrak{m})$ is C^1 and its derivative in $L^1(\mathfrak{m})$ exists and it is given by the formula*

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}|v_s|^2 \circ \hat{F}_s = -\mathrm{Hess}(\hat{u})(v_s, v_s) \circ \hat{F}_s. \tag{2.A.36}$$

If v is also bounded, then the curve $s \mapsto \frac{1}{2}|v_s|^2 \circ \hat{F}_s$ is C^1 also when seen with values in $L^2(\mathrm{X})$ and in this case the difference quotients in (2.A.36) also converge in $L^2(\mathrm{X})$ to the right hand side.

To prove such proposition it is useful to recall the following definition.

Definition 2.A.23. Let $\{f_s\}_{s\geq 0} \subset L^0(\mathfrak{m})$, $g \in L^0(\mathfrak{m})$, we say that $\mathfrak{m}\text{-}\overline{\lim}_{s\to 0} f_s \leq g$ if $\mathfrak{m}(\{f_s > g + \varepsilon\}) \to 0$ as $s \to 0$, for every $\varepsilon > 0$. Analogously we say that $\mathfrak{m}\text{-}\underline{\lim}_{s\to 0} f_s \geq g$ if $\mathfrak{m}(\{f_s < g - \varepsilon\}) \to 0$ as $s \to 0$, for every $\varepsilon > 0$.

Remark 2.A.24. It can be easily proved that, in the case $\mathfrak{m}(\mathrm{X}) < +\infty$, $\mathfrak{m}\text{-}\overline{\lim}_{s\to 0} f_s \leq g$ is equivalent to the the statement "for every sequence $s_n \to 0$, there exists a subsequence s_{n_k} such that $\overline{\lim}_{k\to\infty} f_{s_{n_k}}(x) \leq g(x)$ $\mathfrak{m}$-a.e. x." The analogous is true for $\mathfrak{m}\text{-}\underline{\lim}_{s\to 0} f_s \geq g$. ■

Proposition 2.A.25. *(i)* $\mathfrak{m}\text{-}\underline{\lim}_{s\to 0} f^i_s \geq g^i$ $i = 1,2 \Longrightarrow \mathfrak{m}\text{-}\underline{\lim}_{s\to 0}(f^1_s + f^2_s) \geq g^1 + g^2$,

(ii) $\mathfrak{m}\text{-}\underline{\lim}_{s\to 0} f_s \geq g \Longrightarrow \mathfrak{m}\text{-}\underline{\lim}_{s\to 0} f_s\chi_B \geq g\chi_B$, *for every $B \subset X$ Borel,*

(iii) $f_s \geq g_s$ $\mathfrak{m}$*-a.e. and* $g_s \xrightarrow{L^p} g \Longrightarrow \mathfrak{m}\text{-}\underline{\lim}_{s\to 0} f_s \geq g$,

(iv) $f_s \xrightarrow{L^p} f$ *and* $\mathfrak{m}\text{-}\underline{\lim}_{s\to 0} f_s \geq g \Longrightarrow f \geq g$ $\mathfrak{m}$*-a.e.,*

(v) $g^n \xrightarrow{L^p} g$, $f^n_s \xrightarrow{L^p} f_s$ *uniformly in s and* $\mathfrak{m}\text{-}\underline{\lim}_{s\to 0} f^n_s \geq g^n$, $n \in \mathbb{N} \Longrightarrow \mathfrak{m}\text{-}\underline{\lim}_{s\to 0} f_s \geq g$,

(vi) $F : X \to X$ *of bounded compression,* $\mathfrak{m}\text{-}\underline{\lim}_{s\to 0} f_s \geq g \Longrightarrow \mathfrak{m}\text{-}\underline{\lim}_{s\to 0} f_s \circ F \geq g \circ F$.

The analogous statements exchanging $\geq$ with $\leq$, hold for the $\mathfrak{m}\text{-}\overline{\lim}$.

Proof. We only prove only (v), since the others are straightforward. Observe that for every $\varepsilon > 0$ and $n \in \mathbb{N}$ the following inclusion holds

$$\{f_s - g < -\varepsilon\} \subset \{f^n_s < g^n - \frac{\varepsilon}{2}\} \cup \{|f^n_s - g^n - (f_s - g)| > \frac{\varepsilon}{2}\},$$

applying Markov inequality

$$\mathfrak{m}(\{f_s - g < -\varepsilon\}) \leq \mathfrak{m}(\{f^n_s < g^n - \frac{\varepsilon}{2}\}) + \|f^n_s - g^n - (f_s - g)\|_{L^p}\frac{2^p}{\varepsilon^p}.$$

Passing to the $\overline{\lim}$ in s and then letting $n \to +\infty$ (recalling that $f^n_s \xrightarrow{L^p} f_s$ uniformly in s) we get the conclusion. □

Proof of Proposition 2.A.22. **Step 1: v is the gradient of a test function and $s = 0$.** Assume that $v = \nabla\varphi$ for some $\varphi \in \mathsf{Test}(\mathrm{X})$. For any $s \in \mathbb{R}$ we have

$$\frac{1}{2}|v_s|^2 \circ \hat{F}_s \geq \mathrm{d}\varphi(v_s) \circ \hat{F}_s - \frac{1}{2}|\mathrm{d}\varphi|^2 \circ \hat{F}_s \qquad \mathfrak{m} - a.e., \tag{2.A.37}$$

with equality $\mathfrak{m}$-a.e. for $s=0$. The right hand side of this last inequality is C^1 when seen as a curve depending on s with values in $L^2(\mathrm{X})$ and its derivative at $s=0$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathrm{d}\varphi(v_s)\circ\hat{F}_s-\frac{1}{2}|\mathrm{d}\varphi|^2\circ\hat{F}_s\Big)_{|s=0}=-\mathrm{Hess}(\hat{u})(v,v),$$

(see [**95**] for details). From (2.A.37) and (iii) of Proposition 2.A.25 we obtain

$$\mathfrak{m}\text{-}\overline{\lim_{s\uparrow 0}}\frac{|v_s|^2\circ\hat{F}_s-|v|^2}{2s}\leq-\mathrm{Hess}(\hat{u})(v,v)\leq\mathfrak{m}\text{-}\underline{\lim_{s\downarrow 0}}\frac{|v_s|^2\circ\hat{F}_s-|v|^2}{2s},\tag{2.A.38}$$

Step 2: v is locally the gradient of a test function and $s=0$. From the locality property of the differential and items $(i),(ii)$ of Proposition 2.A.25 we deduce that (2.A.38) holds for v of the form $\sum_i\chi_{A_i}\nabla\varphi_i\in L^2(T\mathrm{X})$ for a given Borel partition $(A_i)_{i\in\mathbb{N}}$ of X and functions $\varphi_i\in\mathsf{Test}(\mathrm{X})$.
Step 3: generic $v\in L^2(T\mathrm{X})$ and $s=0$. Let $Q_s:L^2(T\mathrm{X})\to L^1(\mathrm{X})$ be the quadratic form defined by

$$Q_s(v):=\frac{|v_s|^2\circ\hat{F}_s-|v|^2}{s}.$$

Arguing exactly as in [**95**], using (2.A.33), we can prove that the Q_s's are uniformly continuous in $s\in(-1,1)$.

We pick a sequence $v_n\subset L^2(T\mathrm{X})$, made of elements as in Step 2, that converges to v strongly in $L^2(T\mathrm{X})$. Then the boundedness of $\mathrm{Hess}(\hat{u})$, given by (2.A.29), and the local uniform continuity of the Q_s's, grant that $Q_s(v_n)\to Q_s(v)$ strongly in $L^1(\mathrm{X})$, uniformly on s, and that $\mathrm{Hess}(\hat{u})(v_n,v_n)\to\mathrm{Hess}(\hat{u})(v,v)$ in $L^1(\mathrm{X})$. Therefore from (v) of Proposition 2.A.25 we deduce that (2.A.38) holds also for a generic $v\in L^2(T\mathrm{X})$. **Step 4: conclusion.** Since the $\hat{F}_s$'s form a group, by what we just proved and (vi) of Proposition 2.A.25 we know that for any $v\in L^2(T\mathrm{X})$ and any $s_0\in\mathbb{R}$ we have

$$\mathfrak{m}\text{-}\overline{\lim_{s\uparrow s_0}}\frac{|v_s|^2\circ\hat{F}_s-|v_{s_0}|^2\circ\hat{F}_{s_0}}{2(s-s_0)}\leq-\mathrm{Hess}(\hat{u})(v_{s_0},v_{s_0})\circ\hat{F}_{s_0}\leq\mathfrak{m}\text{-}\underline{\lim_{s\downarrow s_0}}\frac{|v_s|^2\circ\hat{F}_s-|v_{s_0}|^2\circ\hat{F}_{s_0}}{2(s-s_0)}.\tag{2.A.39}$$

Now assume for a moment that $v\in L^2\cap L^\infty(T\mathrm{X})$, so that using again the group property and keeping in mind the bound (2.A.33) we have that $\mathbb{R}\ni s\mapsto\frac{1}{2}|v_s|^2\circ\hat{F}_s\in L^2(\mathrm{X})$ is Lipschitz and hence - since Hilbert spaces have the Radon-Nikodym property - it is a.e. differentiable. Letting s_0 be a point of differentiability, from (2.A.39) and using twice (iv) of Proposition 2.A.25 we deduce that

$$\lim_{s\to s_0}\frac{|v_s|^2\circ\hat{F}_s-|v_{s_0}|^2\circ\hat{F}_{s_0}}{2(s-s_0)}=-\mathrm{Hess}(\hat{u})(v_{s_0},v_{s_0})\circ\hat{F}_{s_0},\tag{2.A.40}$$

the limit being in $L^2(\mathfrak{m})$.

Finally, let $v\in L^2(T\mathrm{X})$ be arbitrary and notice that the bound (2.A.33) grants that for any $s_0\in\mathbb{R}$ the different quotients $\frac{|v_s|^2\circ\hat{F}_s-|v_{s_0}|^2\circ\hat{F}_{s_0}}{2(s-s_0)}$ are, for $s\in[s_0-1,s_0+1]$, dominated in $L^1(\mathrm{X})$. To conclude, put $v_n:=\chi_{\{|v|\leq n\}}v\in L^2\cap L^\infty(TX)$, $n\in\mathbb{N}$, and notice that for a.e. $s_0\in\mathbb{R}$ the L^2-limit in in (2.A.40) holds with v_n in place of v, for every $n\in\mathbb{N}$, in particular the same limit holds also in measure. Therefore from the identity $|\mathrm{d}\hat{F}_s(v_n)|\circ\hat{F}_s=|v_s|\circ\hat{F}_s\chi_{\{|v|\leq n\}}$ $\mathfrak{m}$-a.e., for all $s\in\mathbb{R}$, we deduce that (2.A.40) holds also for v, the limit being intended in measure. Recalling that the different quotients are dominated in $L^1(\mathfrak{m})$, we conclude that (2.A.40) holds for v in strong $L^1(\mathfrak{m})$ topology.

The stated C^1 regularity is then a consequence of the continuity in $L^1(\mathrm{X})$ of $s\mapsto\mathrm{Hess}(\hat{u})(v_s,v_s)\circ\hat{F}_s$ which in turn is a consequence of the $L^2(T\mathrm{X})$ continuity of $s\mapsto v_s$ and the boundedness of $\mathrm{Hess}(\hat{u})$. □

From this point on the proof of Proposition 2.A.17 follows exactly as in [**95**], combining Proposition 2.A.22, Lemma 2.A.21 and (2.A.29), and we shall therefore skip it.

2.A.7. Cone over the rescaled level set. Let us define the $(Z,\mathsf{d}_Z,\mathfrak{m}_Z)$ as $\Big(X',\frac{\mathsf{d}'}{\sqrt{2T}},\mathfrak{m}'\Big)$ and note that $f:Z\to\mathbb{R}$ belongs to $W^{1,2}(Z)$ if and only if it belongs to $W^{1,2}(X')$ and in this case

$$|\mathsf{D}f|_Z=\sqrt{2T}|\mathsf{D}f|_{X'},\quad\mathfrak{m}_Z\text{-a.e. (equivalently }\mathfrak{m}'\text{-a.e).}\tag{2.A.41}$$

We then define the metric measure space $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ as the N-Euclidean cone over the metric measure space $(Z,\mathsf{d}_Z,\mathfrak{m}_Z)$.

2.A.8. Isometry of Sobolev spaces. For every $t_0 \le a < b \le T_0$ we introduce the set $A^Y_{a,b} = \{y \in Y : \mathsf{d}_Y(y, O_Y) \in (\sqrt{2a}, \sqrt{2b})\} = \{(r, z) \in Y : r \in (\sqrt{2a}, \sqrt{2b})\} \subset Y$.

We then introduce the map $T : A^Y_{t_0,T_0} \to A_{t_0,T_0}$ as

$$T((r, z)) \coloneqq F_{\frac{1}{2}\log\frac{2T}{r^2}}(z), \tag{2.A.42}$$

which is well defined thanks to (2.A.19), and the map $S : A_{t_0,T_0} \to A^Y_{t_0,T_0}$ defined as

$$S(x) \coloneqq (\sqrt{2u(x)}, \mathrm{Pr}(x)). \tag{2.A.43}$$

It is immediate from the definition that $S(A_{a,b}) = A^Y_{a,b}$ and $T(A^Y_{a,b}) = A_{a,b}$ for every $t_0 \le a < b \le T_0$.

Moreover it is clear from the definition of Pr, (2.A.19) and the fact that $F_{-t} = F_t^{-1}$, that S and T are one the inverse of the other.

The following result follows from the definitions, Proposition 2.A.11, Proposition 2.A.7 and the local Lipschitzianity of Pr (see [**95**] for details).

Proposition 2.A.26. *The maps $S : A_{t_0+\varepsilon,T_0-\varepsilon} \to A^Y_{t_0+\varepsilon,T_0-\varepsilon}$ and $T : A^Y_{t_0+\varepsilon,T_0-\varepsilon} \to A_{t_0+\varepsilon,T_0-\varepsilon}$ are Lipschitz for every $\varepsilon \in (0, \frac{T_0-t_0}{4})$. Moreover*

$$S_*\mathfrak{m}|_{A_{t_0,T_0}} = \mathfrak{m}_Y|_{A^Y_{t_0,T_0}} \quad \text{and} \quad T_*\mathfrak{m}_Y|_{A^Y_{t_0,T_0}} = \mathfrak{m}|_{A_{t_0,T_0}}. \tag{2.A.44}$$

The following result follows combining Theorem 2.A.18, Proposition 2.A.26, Lemma 2.A.9, (2.A.41) and Proposition 2.A.19. The proof is identical to the one of [**95**, Theorem 3.37] and we shall skip it.

Theorem 2.A.27. *For every $\varepsilon \in (0, \frac{T_0-t_0}{4})$ we have that $f \circ S \in W^{1,2}_0(A_{t_0+\varepsilon,T_0-\varepsilon})$ if and only if $f \in W^{1,2}_0(A^Y_{t_0+\varepsilon,T_0-\varepsilon})$ and in this case*

$$|\mathsf{D}(f \circ S)|_X = |\mathsf{D}f|_Y \circ S, \quad \mathfrak{m}\text{-a.e. in } A^Y_{t_0+\varepsilon,T_0-\varepsilon}. \tag{2.A.45}$$

2.A.9. Back to the metric and conclusion.

Proof of Theorem 2.5.1. Arguing exactly as in [**95**, Sec. 3.9] we can show, exploiting Theorem 2.A.27, that the maps S and T are locally isometries from A_{t_0,T_0} to $A^Y_{t_0,T_0}$ and viceversa.

Pick now any $t_0', T_0' \in [\bar{u}_0, \infty)$ such that $t_0' < t_0 < T < T_0 < T_0'$, then we can repeat all the arguments in the previous sections with t_0', T_0' in place of t_0, T_0 (but using the same T to define X' as in subsection 2.A.4) to obtain a map $S' : A_{t_0',T_0'} \to A^Y_{t_0',T_0'}$ that is a local isometry, with an inverse T', which is also a local isometry. The key observation is that S' agrees with S in A_{t_0,T_0}. Indeed from (2.A.43) we deduce that S' on A_{t_0,T_0} depends only on the value of the function u' and the map Pr' on A_{t_0,T_0}. From the construction is clear that u' agrees with u on A_{t_0,T_0}, since both agree with $\mathbf{u}$ on this set. Therefore we need to show that the two projection maps $\mathrm{Pr}, \mathrm{Pr}' : A_{t_0,T_0} \to S_T$ agree. Suppose they do not, i.e. there exists $x \in A_{t_0,T_0}$ such that $\mathrm{Pr}(x) \ne \mathrm{Pr}'(x)$. Recall that $\mathrm{Pr}(x) = F_{\frac{1}{2}\log\frac{u(x)}{T}}(x)$ and that the curve $\gamma^1_t = F_t(x)$ for $t \in [0, \frac{1}{2}\log\frac{u(x)}{T}]$ is (up to a reparametrization) a minimizing geodesic joining x to $\mathrm{Pr}(x)$ and with values in A_{t_0,T_0}, as shown in Proposition 2.A.4. With the same argument we deduce the existence of a geodesic γ^2 joining x and $\mathrm{Pr}'(x)$ with values in A_{t_0,T_0}. Moreover from (2.A.18) we have that $\mathsf{d}(x, \mathrm{Pr}(x)) = \mathsf{d}(x, \mathrm{Pr}'(x)) = \sqrt{2}|\sqrt{T} - \sqrt{u(x)}|$, in particular γ^1, γ^2 are geodesics with same length. Since S is a local isometry we have that the curves $S(\gamma^i_t)$ are both geodesics in Y with the same length. In particular $\mathsf{d}_Y(S(x), S(\mathrm{Pr}(x))) = \mathsf{d}_Y(S(x), S(\mathrm{Pr}'(x)))$ which using the expression for S gives

$$\sqrt{2}|\sqrt{T} - \sqrt{u(x)}| = \mathsf{d}_Y((\sqrt{2u(x)}, \mathrm{Pr}(x)), (\sqrt{2T}, \mathrm{Pr}(x))) = \mathsf{d}_Y((\sqrt{2u(x)}, \mathrm{Pr}(x)), (\sqrt{2T}, \mathrm{Pr}'(x))).$$

However, recalling that $\mathrm{Pr}(x) \ne \mathrm{Pr}'(x)$ and from the definition of d_Y we easily deduce that the rightmost term in the above identity is strictly bigger than $\sqrt{2}|\sqrt{T} - \sqrt{u(x)}|$, which is a contradiction.

We can now send $t_0 \to \mathbf{u}_0$ and $T_0 \to +\infty$ and obtain a map $\mathbf{S} : \{\mathbf{u} > \mathbf{u}_0\} \to Y \setminus \overline{B_{\sqrt{2\mathbf{u}_0}}}(O_Y)$ which is a surjective and measure preserving local isometry. Moreover extending analogously the maps $T : A^Y_{t_0,T_0} \to A_{t_0,T_0}$, which are the inverses of the maps S, we obtain a map $\mathbf{T} : Y \setminus \overline{B_{\sqrt{2\mathbf{u}_0}}}(O_Y) \to U$, which is the inverse of $\mathbf{S}$ and a local isometry as well.

Observe now that, since $\mathbf{S}$ and $\mathbf{T}$ are a local isometries, they send geodesics to geodesics. This easily implies that

$$\mathsf{d}(x, \partial\{\mathbf{u} > \mathbf{u}_0\}) = \mathsf{d}_Y(\mathbf{S}(x), B_{\sqrt{2\mathbf{u}_0}}(O_Y)) = \sqrt{2u(x)} - \sqrt{2\mathbf{u}_0}, \tag{2.A.46}$$

from which (2.5.1) follows.

We are now in position to apply Proposition 2.5.2 to obtain that Y is an RCD$(0, N)$ space, which is the unique tangent cone at infinity to X. Moreover from the fact that Y is an RCD$(0, N)$ and from (1.3.17) it follows that $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ is an RCD$(N-2, N-1)$ space satisfying $\mathrm{diam}(Z) \le \pi$.

Suppose now that $\mathrm{diam}(Z) = \pi$, then again from Proposition 2.5.2 we obtain that X is isomorphic to Y.

The fact that X has Euclidean volume growth was already proved in Corollary 2.A.8.

It remains to prove the first part of ii). Let $r = \sqrt{2\mathbf{u}_0}$ and r_Z as in the statement. It is enough to show that for every couple of points $y_1, y_2 \in Y$ such that $\mathsf{d}_Y(y_i, O_Y) > r_Z$, $i = 1, 2$, all the geodesics connecting them are contained in $\{\mathsf{d}(., O_Y) > r\}$. Moreover we can clearly restrict ourselves to consider points y_1, y_2 of the form $y_i = (t, z_i)$, with $z_i \in Z$, $i = 1, 2$ and $t > r_Z$. For such points we have that

$$\mathsf{d}_Y(y_1, y_2) = t\sqrt{2 - 2\cos(\mathsf{d}(z_1, z_2))} \leq t\sqrt{2 - 2\cos(\mathrm{diam}(Z))}$$

Let now γ be a geodesic between y_1 and y_2, then by the triangle inequality

$$\mathsf{d}(\gamma_t, O_Y) \geq t - \frac{\mathsf{d}_Y(y_1, y_2)}{2} > r_Z\left(1 - \sqrt{\frac{1 - \cos(\mathrm{diam}(Z))}{2}}\right) = r, \quad \forall t \in [0, 1],$$

where the last identity follows from the definition of r_Z. $\square$

CHAPTER 3

Willmore inequalities on $\mathrm{RCD}(K, N)$ spaces

Structure of the chapter

First, we will recall in Section 3.1 some objects and notations that will be used frequently. Then in the first main Section 3.2 we will define a notion of *relaxed Willmore energy* for sets in an $\mathrm{RCD}(K, N)$ space and prove the existence of sets with finite Willmore energy. Then in Section 3.3 we introduce a notion of regular sublevels for a smooth function on an $\mathrm{RCD}(K, N)$ space, by requiring a sort of infinitesimal version of the coarea formula and a non-vanishing of the gradient. We will also prove in this part that almost every sublevel of a sufficiently smooth function is regular, which will be used in the subsequent sections to define the tangential divergence and the mean curvature with an averaging procedure.

The definition of *tangential divergence* will be given in Section 3.4, where we will study and prove some of its main properties. Then in Section 3.5 we will define the notion of *mean curvature* for regular sublevels on $\mathrm{RCD}(K, N)$ spaces, in duality with the tangential divergence.

In the last three sections we will restrict ourselves to the setting of $\mathrm{RCD}(0, N)$ spaces with Euclidean volume growth. First in Section 3.6 we will relate the variational capacity $\overline{\mathrm{Cap}}$ with the 'electrostatic-capacity' obtained using the electrostatic potential. Section 3.7 will contain the asymptotic analysis of the electrostatic potential, which will ultimately lead to the computation of $\lim_{t\to 0^+} U_\beta$, where U_β is the monotone quantity for the electrostatic potential introduced in the previous chapter. We will finally combine this result with the notion of mean curvature to derive in Section 3.8 a class of Willmore-type inequalities for $\mathrm{RCD}(0, N)$ spaces with Euclidean volume growth.

All the results that will be presented in this chapter are contained in [**131**].

3.1. Notations and facts used frequently

In this section we recall some objects and notations used frequently in the chapter. Here we assume $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ to be an $\mathrm{RCD}(K, N)$ space.

A *quasi-continuous function* is a function $f : \mathrm{X} \to \mathbb{R}$ that is continuous outside a (open) set of arbitrary small capacity (see Def. 1.2.19). We denote by $\mathcal{QC}(\mathrm{X})$ the algebra of equivalence classes of up to Cap-a.e. equality of quasi-continuous functions.

For a set of finite perimeter $E \subset \mathrm{X}$ it holds that $\mathrm{Per}(E) \ll \mathrm{Cap}$ (see Prop. 1.2.26). Hence there is a natural *projection map*

$$\pi_E : L^0(\mathrm{Cap}) \to L^0(\mathrm{Per}(E)).$$

We clearly have that $\pi_E(fg) = \pi_E(f)\pi_E(g)$ and $\pi_E(f^{-1}) = \pi_E(f)^{-1}$ for every $f, g \in L^0(\mathrm{Cap})$.

Moreover there exists a unique linear *quasi-continuous representative map*

$$\mathsf{QCR} : W^{1,2}(\mathrm{X}) \to \mathcal{QC}(\mathrm{X})$$

such that $\mathsf{QCR}(f) = f$ $\mathfrak{m}$-a.e., for every $f \in W^{1,2}(\mathrm{X})$ (see Sec. 1.2.5). Note that QCR does not behave as well as π_E with respect to products and ratios, but it holds that

$$\mathsf{QCR}(fg) = \mathsf{QCR}(f)\mathsf{QCR}(g) \quad \forall f, g \in W^{1,2} \cap L^\infty(\mathrm{X})$$

(see (1.2.14)). The *trace map* $\mathrm{tr}_E : W^{1,2}(\mathrm{X}) \to L^0(\mathrm{Per}(E))$ is defined by

$$\mathrm{tr}_E \coloneqq \pi_E \circ \mathsf{QCR}.$$

For a set E of finite perimeter $L^2_E(T\mathrm{X})$ denotes the "tangent bundle over the boundary" of E (see Theorem 1.5.3) and is an $L^2(\mathrm{Per}(E))$-normed $L^\infty(\mathrm{Per}(E))$-module (see Sec. 1.2.7). The *outer unit normal to E* is the vector $\nu_E \in L^2_E(T\mathrm{X})$ characterized by the *Gauss-Green formula*:

$$\int \langle \mathrm{tr}_E(v), \nu_E\rangle \,\mathrm{dPer}(E) = \int_E \mathrm{div}(v)\,\mathrm{d}\mathfrak{m}, \quad \forall v \in H^{1,2}_C \cap L^\infty(T\mathrm{X}) \cap \mathsf{D}_{L^2}(\mathrm{div})$$

(see Theorem 1.5.5). There exists a *vector-trace map* $\mathrm{tr}_E : H^{1,2}_C \cap L^\infty(T\mathrm{X}) \to L^2_E(T\mathrm{X})$ that satisfies:

$$|\mathrm{tr}_E(v)| = \mathrm{tr}_E(|v|), \quad \mathrm{tr}_E(\langle v, w\rangle) = \langle \mathrm{tr}_E(v), \mathrm{tr}_E(w)\rangle_{L^2_E(T\mathrm{X})},$$

for every $v, w \in H^{1,2}_C \cap L^\infty(T\mathrm{X})$.

3.2. Relaxed Willmore energy

We start recalling two results from Chapter 2 concerning harmonic functions on RCD spaces, which are fundamental for our arguments in this section. These are respectively Corollaries 2.3.2 and 2.2.4, that we restate here for the convenience of the reader.

Corollary 3.2.1 (Second order estimates for harmonic functions)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K,N)$ *space,* $N \in [2,\infty)$*, and let* u *be harmonic in* $\Omega \subset \mathrm{X}$*. Then* $\sqrt{|\nabla u|} \in W^{1,2}_{\mathsf{loc}}(\Omega)$ *and in particular*

$$\frac{|\nabla |\nabla u||^2}{|\nabla u|} \in L^1_{\mathsf{loc}}(\Omega), \tag{3.2.1}$$

where the whole function is taken to be zero whenever $|\nabla u| = 0$.

Corollary 3.2.2 (Existence of harmonic cut-off functions)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K,N)$ *metric measure space with* $N < +\infty$*. For every compact set* $P \subset \mathrm{X}$ *and any* U *open set with* $P \subset U$*, there exists a continuous function* $u \in C(\mathrm{X}) \cap W^{1,2}_0(U)$ *such that* $0 \le u \le 1$*,* $u = 1$ *in* P*,* $u = 0$ *in* U^c *and* u *is harmonic in* $\{0 < u < 1\}$.

We now define the *Willmore energy functional* for smooth functions:

Definition 3.2.3 (Willmore functional)**.** We define the functional $\mathcal{W} : W^{1,2}(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta) \to [0, +\infty]$ as

$$\mathcal{W}(u) := \int_{\mathrm{X}} \left| \frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla |\nabla u|, \nabla u\rangle}{|\nabla u|^2} \right|^2 |\nabla u| \mathrm{d}\mathfrak{m}, \tag{3.2.2}$$

with the convention that whenever $|\nabla u| = 0$, the whole integrand is taken to be 0.

Observe that the expression in (3.2.2) makes sense since $|\nabla u| \in W^{1,2}(\mathrm{X})$ (recall Proposition 1.4.6).

The intuition behind the functional $\mathcal{W}$ is that formally $\frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla|\nabla u|, \nabla u\rangle}{|\nabla u|^2} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ and in particular from the coarea formula

$$\mathcal{W}(u) = \int_{\mathbb{R}} \|H_t\|^2_{L^2(\{u=t\})} \,\mathrm{d}t,$$

where H_t is formally the mean curvature of the level set $\{u = t\}$ (this will be a-posteriori made rigorous in Proposition 3.5.17, when we will develop a notion of mean curvature).

We denote by $\mathsf{D}(\mathcal{W})$ the domain of finiteness of $\mathcal{W}$, that is

$$\mathsf{D}(\mathcal{W}) := \{u \in W^{1,2}(\mathrm{X}) \ : \mathcal{W}(u) < +\infty\}.$$

For an arbitrary Borel set we define the *Willmore energy* as the L^1-lower semicontinuous-relaxation of the above functional.

Definition 3.2.4 (Relaxed Willmore energy)**.** For every $E \subset \mathrm{X}$ Borel we define

$$\overline{\mathcal{W}}(E) = \inf\{\varliminf_n W(u_n) \ : u_n \in \mathsf{D}(\mathcal{W}) \ u_n \to \chi_E \text{ in } L^1_{\mathsf{loc}}(\mathfrak{m})\}. \tag{3.2.3}$$

Moreover, whenever $\overline{\mathcal{W}}(E) < +\infty$, we say that E has finite Willmore energy.

Remark 3.2.5. An alternative reasonable definition of energy would be to take instead the relaxation of the functional $\mathcal{W}(u) + \int |\nabla u| \,\mathrm{d}\mathfrak{m}$, so that a set with finite energy would have also finite perimeter. Another possibility would be to ask also the convergence of the term $\int |\nabla u| \,\mathrm{d}\mathfrak{m}$ (to the perimeter) along the approximating sequence, in order to have a sort order-one (instead of an order-zero) convergence, indeed the functional that we are considering is of second order.

We will not however investigate these directions in this work. ■

The above definition of Willmore energy is motivated by the following approximation result.

Theorem 3.2.6. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space with* $N \in [2,\infty)$*. Then*

$$\{\chi_P \ : \ P \subset \mathrm{X} \text{ compact}\} \subset \overline{\mathsf{D}(\mathcal{W}) \cap \mathsf{LIP}_c(\mathrm{X})}^{L^1(\mathfrak{m})}. \tag{3.2.4}$$

The proof of the above Theorem relies on the following technical lemma.

Lemma 3.2.7. *Let* u *be harmonic in* Ω*. Then for every* $\varphi \in C^2(\mathbb{R})$ *with* φ', φ'' *bounded it holds that* $\varphi(u) \in \mathrm{Lip}_{\mathsf{loc}}(\Omega) \cap \mathsf{D}_{L^2_{\mathsf{loc}}}(\Delta, \Omega)$*,* $|\nabla \varphi(u)| \in W^{1,2}_{\mathsf{loc}}(\Omega)$ *and*

$$\int_{\Omega'} \left| \frac{\Delta\varphi(u)}{|\nabla\varphi(u)|} - \frac{\langle \nabla|\nabla\varphi(u)|, \nabla\varphi(u)\rangle}{|\nabla\varphi(u)|^2} \right|^2 |\nabla\varphi(u)| \mathrm{d}\mathfrak{m} < +\infty,$$

for every $\Omega' \subset\subset \Omega$*, where the whole integrand is taken to be 0 whenever* $|\nabla\varphi(u)| = 0$.

PROOF. We first prove the regularity claims on $\varphi(u)$. Recall from Theorem 1.3.21 that $u \in \mathsf{LIP_{loc}}(\Omega)$, hence from the chain rule for the Laplacian (see Proposition 1.2.11) we have that $\varphi(u) \in \mathsf{D}(\Delta, \Omega)$ with $\Delta(\varphi(u)) = \varphi''(u)|\nabla u|^2 \in L^2_{\mathsf{loc}}(\Omega)$. From the chain rule (Proposition 1.2.9) for the gradient we have $|\nabla \varphi(u)| = |\varphi'(u)||\nabla u|$ $\mathfrak{m}$-a.e. in Ω. Moreover, since $\varphi'(u) \in \mathsf{LIP_{loc}}(\Omega)$, again from the chain rule for the gradient we have $|\varphi'(u)| \in \mathsf{LIP_{loc}}(\Omega)$ with $\nabla|\varphi'(u)| = \mathsf{sgn}(\varphi'(u))|\varphi''(u)|\nabla u|$. Recall also that from (1.4.9) we have that $|\nabla u| \in W^{1,2}_{\mathsf{loc}}(\Omega)$. Combining these observations with the Leibniz rule for the gradient (Proposition 1.2.9) we obtain that $|\nabla \varphi(u)| \in W^{1,2}_{\mathsf{loc}}(\Omega)$ with

$$\nabla|\nabla\varphi(u)| = \mathsf{sgn}(\varphi'(u))(\varphi''(u)\nabla u|\nabla u| + \varphi'(u)\nabla|\nabla u|), \quad \mathfrak{m}\text{-a.e. in } \Omega.$$

Combining the above formulas we obtain

$$\frac{\Delta\varphi(u)}{|\nabla\varphi(u)|} - \frac{\langle \nabla|\nabla\varphi(u)|, \nabla\varphi(u)\rangle}{|\nabla\varphi(u)|^2} = -\mathsf{sgn}(\varphi'(u))\frac{\langle\nabla|\nabla u|, \nabla u\rangle}{|\nabla u|^2}, \quad \mathfrak{m}\text{a.e. in } \Omega$$

where both sides are taken to be 0 whenever $|\nabla u| = 0$ or $|\nabla\varphi(u)| = 0$. Combining this with (3.2.1) gives the desired conclusion. □

We can prove the main approximation result of sets with functions with finite Willmore energy.

PROOF OF THEOREM 3.2.6. Fix $K \subset \mathrm{X}$ and let $\varepsilon > 0$ be arbitrary. Define the open set $K^\varepsilon := \{x : \mathsf{d}(x, K) < \varepsilon\}$. From Corollary 3.2.2 we know that there exists a function $u \in C(\mathrm{X}) \cap W^{1,2}_0(K^\varepsilon)$ with $0 \le u \le 1$, $u = 1$ in K, $\mathrm{supp}(u) \subset K^\varepsilon$ and which is harmonic in $\Omega := \{0 < u < 1\}$. Choose a function $\varphi \in C^2(\mathbb{R})$ so that $0 \le \varphi \le 1$, $\varphi \equiv 0$ in $(-\infty, 1/3)$ and $\varphi \equiv 1$ in $(2/3, +\infty)$. Obviously $0 \le \varphi(u) \le 1$, $\varphi(u) = 1$ in 1 and $\mathrm{supp}(\varphi(u)) \subset K^\varepsilon$, hence $\varphi(u) \to \chi_K$ in $L^1(\mathfrak{m})$. It remains to show that $\varphi(u) \in \mathsf{D}(\mathcal{W})$. Since $u \in \mathsf{LIP_{loc}}(\Omega)$ we have that $\varphi(u) \in \mathsf{LIP}(\mathrm{X}) \cap W^{1,2}(\mathrm{X})$. Moreover from Lemma 3.2.7 we have that $\varphi(u) \in \mathsf{D}_{L^2_{\mathsf{loc}}}(\Delta, \Omega)$ and $|\nabla\varphi(u)| \in W^{1,2}_{\mathsf{loc}}(\Omega)$. Additionally from the locality property of the gradient and Laplacian we have that

$$\Delta_{|\Omega}\varphi(u) = |\nabla|\nabla\varphi(u)|| = 0, \quad \mathfrak{m}\text{-a.e. in } \Omega \setminus \{1/3 \le u \le 2/3\}.$$

In particular, since by continuity $\{1/3 \le u \le 2/3\} \subset\subset \Omega$, from Lemma 3.2.7 we obtain that

$$\int_\Omega \left| \frac{\Delta_{|\Omega}\varphi(u)}{|\nabla\varphi(u)|} - \frac{\langle\nabla|\nabla\varphi(u)|, \nabla\varphi(u)\rangle}{|\nabla\varphi(u)|^2} \right|^2 |\nabla\varphi(u)| \mathrm{d}\mathfrak{m} < +\infty.$$

Therefore we would have concluded if we were able to show that $\varphi(u) \in \mathsf{D}_{L^2}(\Delta, \mathrm{X})$ and $|\nabla\varphi(u)| \in W^{1,2}_{\mathsf{loc}}(\mathrm{X})$. The latter is clear from the fact that $|\nabla\varphi(u)| = 0$ $\mathfrak{m}$-a.e. in a neighbourhood of $\partial\Omega$ (and outside Ω) (recall Proposition 1.2.8). To show that $\varphi(u) \in \mathsf{D}_{L^2}(\Delta)$ we argue as follows. let again $\eta \in \mathsf{LIP_c}(\mathrm{X})$ be arbitrary and let $\eta' \in \mathsf{LIP}_c(\Omega)$ be such that $\eta' \equiv 1$ in $\mathrm{supp}\,\eta \cap \{1/3 \le u \le 2/3\}$. From the linearity of the gradient operator

$$\begin{aligned}\int \langle\nabla\varphi(u), \nabla\eta\rangle \, \mathrm{d}\mathfrak{m} &= \int \langle\nabla\varphi(u), \nabla(\eta\eta')\rangle \, \mathrm{d}\mathfrak{m} + \int_{\mathrm{X}\setminus\{1/3\le u\le 2/3\}} \langle\nabla\varphi(u), \nabla(\eta(1-\eta'))\rangle \, \mathrm{d}\mathfrak{m} \\ &= \int \langle\nabla\varphi(u), \nabla(\eta\eta')\rangle \, \mathrm{d}\mathfrak{m},\end{aligned}$$

where in the third equality we used that $|\nabla\varphi(u)| = 0$ $\mathfrak{m}$-a.e. in $\mathrm{X} \setminus \{1/3 \le u \le 2/3\}$. Moreover since now $\eta\eta' \in \mathsf{LIP}_c(\Omega)$ we can use that $\varphi(u) \in \mathsf{D}(\boldsymbol{\Delta}, \Omega)$ with to deduce that

$$\int \langle\nabla\varphi(u), \nabla\eta\rangle \, \mathrm{d}\mathfrak{m} = -\int \eta'\eta\Delta\varphi(u)\mathfrak{m} = -\int_\Omega \eta'\eta\Delta_{|\Omega}\varphi(u) \, \mathrm{d}\mathfrak{m},$$

which implies that $\varphi(u) \in \mathsf{D}(\boldsymbol{\Delta}, \mathrm{X})$ with $\Delta\varphi(u) = \chi_\Omega\Delta_{|\Omega}\varphi(u)$. However, since $\Delta_{|\Omega}\varphi(u) = 0$ in $\Omega \setminus \{1/3 \le u \le 2/3\}$ we also deduce that $\Delta\varphi(u) \in L^2(\mathrm{X})$, which is what we wanted. □

We have not shown so far whether there exists non-trivial sets with finite Willmore energy. We answer positively in the following result (recall that $\mathsf{D}(\mathcal{W}) \cap \mathsf{LIP}_c(\mathrm{X})$ is a rich set thanks to Theorem 3.2.6).

Proposition 3.2.8 (Existence of sets with finite Willmore energy)**.** *For every $u \in \mathsf{D}_{L^2}(\Delta) \cap \mathsf{LIP}_c(\mathrm{X})$ and every $t \in \mathbb{R}$ it holds*

$$\overline{W}(\{u < t\}) \le \varliminf_{h\to 0^+} \frac{1}{2h}\int_{\{t-h<u<t+h\}} \left|\frac{\Delta u}{|\nabla u|} - \frac{\langle\nabla|\nabla u|, \nabla u\rangle}{|\nabla u|^2}\right|^2 |\nabla u| \mathrm{d}\mathfrak{m}. \tag{3.2.5}$$

Moreover if also $u \in \mathsf{D}(\mathcal{W})$ then (3.2.5) *is finite for a.e. $t \in \mathbb{R}$ and in particular*

$$\overline{W}(\{u < t\}) < +\infty, \quad \text{for a.e. } t \in \mathbb{R}. \tag{3.2.6}$$

Proof. The argument is similar to Lemma 3.2.7.

For every $\varepsilon>0$ we consider a function $\varphi\in C^2(\mathbb{R})$ such that $\varphi=0$ in $[1,+\infty)$, $\varphi=1$ in $(-\infty,0]$ and $|\varphi'|\le 1+\varepsilon$. Such function can easily be built via mollification and rescaling. Then for every $n\in\mathbb{N}$ we define the function $\varphi_n\in C^2(\mathbb{R})$ as $\varphi_n(x)=\varphi(nx-nt_0+1)$. In particular $\varphi_n=0$ in $[t_0,+\infty)$, $\varphi_n=1$ in $(-\infty,t_0-\frac{1}{n}]$ and $|\varphi_n'|\le(1+\varepsilon)n$.

Using the Dominated Convergence Theorem it is easily seen that $\varphi_n(u)\to\chi_{\{u<t_0\}}$ in $L^1_{\mathsf{loc}}(\mathfrak{m})$.

From the chain rule for the gradient (Proposition 1.2.9) we have $\varphi_n(u)\in W^{1,2}_{\mathsf{loc}}(\mathrm{X})$ with $|\nabla\varphi_n(u)|=|\varphi_n'|(u)|\nabla u|$ $\mathfrak{m}$-a.e.. Moreover from $u\in\mathsf{D}_{L^2_{\mathsf{loc}}}(\Delta)$ and recalling (1.4.9) we have $|\nabla u|\in W^{1,2}_{\mathsf{loc}}(\mathrm{X})$. Moreover, since $u\in\mathsf{LIP}_{\mathsf{loc}}(\mathrm{X})$, we also have that $|\varphi_n'|(u)\in\mathsf{LIP}_{\mathsf{loc}}(\mathrm{X})$. Therefore again by the chain rule we deduce $|\nabla\varphi_n(u)|\in W^{1,2}_{\mathsf{loc}}(\mathrm{X})$ with

$$\nabla|\nabla\varphi_n(u)|=\mathsf{sgn}(\varphi_n'(u))(\varphi_n''(u)\nabla u|\nabla u|+\varphi_n'(u)\nabla|\nabla u|).$$

Finally the chain rule for the Laplacian (Proposition 1.2.11) we have $\varphi_n(u)\in\mathsf{D}_{L^2_{\mathsf{loc}}}(\Delta)$ with

$$\Delta(\varphi_n(u))=\varphi_n'(u)\Delta u+\varphi_n''(u)|\nabla u|^2$$

Combining the above identities, as in the proof of Lemma 3.2.7, we get

$$\frac{\Delta\varphi_n(u)}{|\nabla\varphi_n(u)|}-\frac{\langle\nabla|\nabla\varphi_n(u)|,\nabla\varphi_n(u)\rangle}{|\nabla\varphi_n(u)|^2}=\begin{cases}\mathsf{sgn}(\varphi_n')\left(\frac{\Delta u}{|\nabla u|}-\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|^2}\right), & \mathfrak{m}\text{-a.e. in } \{t_0<u<t_0+\frac{1}{n}\},\\ 0 & \mathfrak{m}\text{-a.e. in } \mathrm{X}\setminus\{t_0<u<t_0+\frac{1}{n}\},\end{cases}$$

where both sides are taken to be 0 whenever $|\nabla u|=0$ or $|\nabla\varphi_n(u)|=0$. Therefore from (3.2.7)

$$\begin{aligned}W_2(\varphi_n(u))&\le\int_{\{t_0<u<t_0+\frac{1}{n}\}}\left|\frac{\Delta u}{|\nabla u|}-\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|^2}\right|^2|\nabla u||\varphi_n'|\mathrm{d}\mathfrak{m}\\&\le(1+\varepsilon)n\int_{\{t_0<u<t_0+\frac{1}{n}\}}\left|\frac{\Delta u}{|\nabla u|}-\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|^2}\right|^2|\nabla u|\mathrm{d}\mathfrak{m},\end{aligned}$$

where we are always taking the integrand to be 0, whenever $|\nabla u|$ vanishes. This, combined with the arbitrariness of $\varepsilon>0$, concludes the proof of (3.2.5). It remains to prove that

$$\varliminf_{h\to0^+}\frac{1}{2h}\int_{\{t-h<u<t+h\}}\left|\frac{\Delta u}{|\nabla u|}-\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|^2}\right|^2|\nabla u|\mathrm{d}\mathfrak{m}<+\infty,\quad\text{for } a.e.\ t\in\mathbb{R}. \tag{3.2.7}$$

To show the above fix f a Borel representative of the function $\chi_{|\nabla u|>0}\left|\frac{\Delta u}{|\nabla u|}-\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|^2}\right|^2$. Then by the coarea formula (1.3.12) (recalling also Remark 1.3.9) and the fact that $u\in\mathsf{D}(\mathcal{W})$ we get both that the function $g(t):=\int f\mathrm{dPer}(\{u>t\},.)$ is in $L^1(\mathbb{R})$ and that

$$⨍_{t-h}^{t+h}g(r)\,\mathrm{d}r=\frac{1}{2h}\int_{\{t-h<u<t+h\}}f|\nabla u|\mathrm{d}\mathfrak{m},\forall t\in\mathbb{R},\ h\in(0,1).$$

Then (3.2.7) holds for every t which is Lebesgue point for g.

□

We summarize what we obtained. We have proven that there is a rich family of functions with finite Willmore energy, which approximates in L^1 every compact set. Moreover we have proved that there exist many sets with finite Willmore energy (i.e. sublevels of functions with finite Willmore energy). However we can not exclude at the moment, that every set has either zero or infinite Willmore energy. In particular the following question arises naturally:

Question: Is there a set with finite and non-zero Willmore energy? Or more generally can we give a of lower bound for the Willmore energy?

Unfortunately we are not able to answer to this question at this time, however in the case of RCD$(0,N)$ spaces with Euclidean volume growth we will obtain a sort of weak-lower bound (given by the Willmore-type inequality in Theorem 3.8.1) which indicates that the answer to the above question should be positive.

As a final remark we anticipate that in Section 3.5 we will give a notion of level sets with L^2-mean curvature vector, which will be related to the notion of Willmore energy defined here. In particular sets with L^2-mean curvature vector will have finite relaxed Willmore energy and we will be able to give a representation of the functional $\mathcal{W}(u)$ in term of the mean curvature of its sublevels

3.3. Sublevels of first-order regularity

In this section we will develop some technical tools that will be used in the next two sections to build notions of tangential divergence and mean curvature. Concretely we will provide sufficient conditions to be able to express the integral of a quasi-continuous function, with respect to a perimeter measure, as a limit of suitable averages.

The following notion will be central:

Definition 3.3.1 (Sublevels of first order regularity)**.** Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$ be continuous with bounded sublevels. We say that a sublevel $\{u < t\}$ is first-order regular if the following hold

(i) $\{u < t\}$ is of finite perimeter,
(ii) $\mathrm{tr}_{\{u<t\}}(|\nabla u|) \neq 0$ $\mathrm{Per}(\{u < t\})$-a.e.,
(iii)

$$\mathfrak{m}(\{u < t\})' := \varlimsup_{h\to 0^+} \frac{\mathfrak{m}(\{t-h < u < t+h\})}{2h} < +\infty, \tag{3.3.1}$$

(iv)

$$\mu_t^h := \frac{|\nabla u|}{2h}\mathfrak{m}|_{\{t-h<u<t+h\}} \rightharpoonup \mathrm{Per}(\{u < t\}) \text{ as } h \to 0^+, \text{ in duality with } C_b(\mathrm{X}). \tag{3.3.2}$$

Remark 3.3.2. Note that (ii) makes sense, since the assumptions on u grant that $|\nabla u| \in W^{1,2}(\mathrm{X})$ as stated in Proposition 1.4.6. ■

They key feature of this definition is that it allows the existence of many non-trivial first order regular sublevels. This is the content of the following Proposition, whose short proof will be given at the end of this subsection.

Proposition 3.3.3. *Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$, then for a.e. $t \in \mathbb{R}$ the set $\{u < t\}$ is first order regular.*

The goal of this subsection is to prove the following convergence result.

Theorem 3.3.4. *Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$. Let $t \in \mathbb{R}$ be such that $\{u < t\}$ is first order regular. Then*

$$\lim_{h\to 0^+} \int f \,\mathrm{d}\mu_t^h = \int \pi_E(f) \,\mathrm{dPer}(\{u < t\}), \tag{3.3.3}$$

for every $f \in \mathcal{QC}(\mathrm{X}) \cap L^\infty(\mathrm{Cap})$. Moreover (3.3.3) *holds also for every function $f \in L^\infty(\mathrm{Cap})$ and such that $f\mathsf{QCR}(|\nabla u|)^2 \in \mathcal{QC}(\mathrm{X})$ (recall that $|\nabla u| \in W^{1,2}(\mathrm{X})$ by Proposition 1.4.6).*

The proof of Theorem 3.3.4 is done via approximation with continuous functions and relies on the following Capacity estimate.

Proposition 3.3.5. *Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$. Let $t \in \mathbb{R}$ be such that $\{u < t\}$ has finite perimeter and* (3.3.2) *hold. Then*

$$\varlimsup_{h\to 0^+} \mu_t^h(A)^2 \le 2\mathfrak{m}(\{u < t\})' \sqrt{\mathrm{Cap}(A)} \left(\|\Delta u\|_{L^2(\mathfrak{m})} + \||\nabla u|\|_{L^2(\mathfrak{m})}\right), \tag{3.3.4}$$

for every $A \subset \mathrm{X}$ Borel.

Proof of Theorem 3.3.4 given Proposition 3.3.5. Let $f \in \mathcal{QC}(\mathrm{X})$ be bounded. With a slight abuse of notation we denote by f its quasi-continuous representative. From Remark 1.2.20 and Tietze extension theorem we have the existence of functions $g_n \in C_b(\mathrm{X})$ and open sets $\Omega_n \subset \mathrm{X}$ such that $\mathrm{Cap}(\Omega_n) \to 0^+$, $g_n = f$ in $X \setminus \Omega_n$ and $\|g_n\|_\infty \le \|f\|_\infty$. In particular $g_n \to f$ Cap-a.e.. We proceed with the following estimate

$$\begin{aligned} &\left| \int f \mathrm{d}\mu_t^h - \int f \mathrm{dPer}(\{u < t\}) \right| \\ &\qquad \le \int |f - g_n| \mathrm{d}\mu_t^h + \left| \int g_n \mathrm{d}\mu_t^h - \int g_n \mathrm{dPer}(\{u < t\}, .) \right| + \int |g_n - f| \mathrm{dPer}(\{u < t\}, .). \end{aligned}$$

Observing that the middle term on the right hand side vanishes as $h \to 0^+$ thanks to (3.3.2) and estimating the first term using Proposition 3.3.5 we reach

$$\varlimsup_{h\to 0^+} \left| \int f \mathrm{d}\mu_t^h - \int f \mathrm{dPer}(\{u < t\}) \right| \le C_u \|f\|_\infty \sqrt{\mathfrak{m}(\{u < t\})'} \, (\mathrm{Cap}(\Omega_n))^{\frac{1}{4}} + \int |g_n - f| \mathrm{dPer}(\{u < t\}, .),$$

where C_u is a positive constant depending only on the function u. Sending $n \to +\infty$, recalling that by hypothesis $\mathfrak{m}(\{u < t\})' < +\infty$ and applying dominated convergence to the last term, we obtain (3.3.3).

We pass now to the second part of the statement.

Let $f\in L^0(\mathrm{Cap})$ be bounded and such that $f\mathsf{QCR}(|\nabla u|)^2\in\mathcal{QC}(\mathrm{X})$. For every $\varepsilon>0$ we have that $\frac{\mathsf{QCR}(|\nabla u|)^2}{(\mathsf{QCR}(|\nabla u|)+\varepsilon)^2}f\in\mathcal{QC}\cap L^\infty(\mathrm{X})$. Therefore the first part of the theorem gives that

$$\lim_{h\to0^+}\int\frac{|\nabla u|^2}{(|\nabla u|+\varepsilon)^2}f\mathrm{d}\mu_t^h=\int\frac{\mathrm{tr}_{\{u<t\}}(|\nabla u|)^2}{(\mathrm{tr}_{\{u<t\}}(|\nabla u|)+\varepsilon)^2}\pi_{\{u<t\}}f\,\mathrm{dPer}(\{u<t\}),\tag{3.3.5}$$

where we have used that by definition $\mathrm{tr}_{\{u<t\}}=\pi_{\{u<t\}}\circ\mathsf{QCR}$. Moreover, recalling that $\mathrm{tr}_{\{u<t\}}(|\nabla u|)\neq0$ $\mathrm{Per}(\{u<t\},.)$-a.e., we can apply dominated convergence to infer

$$\lim_{\varepsilon\to0^+}\int\frac{\mathrm{tr}_{\{u<t\}}(|\nabla u|)^2}{(\mathrm{tr}_{\{u<t\}}(|\nabla u|)+\varepsilon)^2}\pi_{\{u<t\}}f\,\mathrm{dPer}(\{u<t\})=\int\pi_{\{u<t\}}f\,\mathrm{dPer}(\{u<t\}).\tag{3.3.6}$$

Finally a straight-forward computation gives the following estimate

$$\left|1-\frac{|\nabla u|^2}{(|\nabla u|+\varepsilon)^2}\right|\le\frac{2\varepsilon}{|\nabla u|+\varepsilon},\quad\mathfrak{m}\text{-a.e..}$$

Therefore we obtain

$$\varlimsup_{h\to0^+}\int\left|\frac{|\nabla u|^2}{(|\nabla u|+\varepsilon)^2}f-f\right|\mathrm{d}\mu_t^h\le2\|f\|_\infty\mathfrak{m}(\{u<t\})'\varepsilon.\tag{3.3.7}$$

Combining (3.3.7), (3.3.6) and (3.3.5) and we conclude. □

It remains to prove Proposition 3.3.5. Its proof passes through the following maximal-type estimate.

Proposition 3.3.6. *Let $u\in\mathsf{LIP}_c(\mathrm{X})\cap\mathsf{D}_{L^2}(\Delta)$. Let $t\in\mathbb{R}$ be such that $\{u<t\}$ has finite perimeter and (3.3.2) hold. Then for every $f\in W^{1,2}\cap L^\infty(\mathrm{X})$*

$$\varlimsup_{h\to0^+}\left|\int f|\nabla u|\mathrm{d}\mu_t^h\right|\le2\|f\|_{W^{1,2}(\mathrm{X})}\left(\|\Delta u\|_{L^2(\mathfrak{m})}+\||\nabla u|\|_{L^2(\mathfrak{m})}\right).\tag{3.3.8}$$

Proof. From the coarea formula (1.3.12) (recall also Remark 1.3.9) we have that the function $g(s):=\int f|\nabla u|\mathrm{dPer}(\{u<s\},.)$ is in $L^1(\mathbb{R})$ and does not depend on the choice of the Borel representative of $f|\nabla u|$. We claim that $g\in\mathsf{BV}(\mathbb{R})$ and

$$|Dg|(\mathbb{R})\le2\|f\|_{W^{1,2}(\mathrm{X})}\left(\|\Delta u\|_{L^2(B_\delta)}+\||\nabla u|\|_{L^2(B_\delta)}\right).\tag{3.3.9}$$

To see this fix $\varphi\in C^1_c(\mathbb{R})$ and observe that $\varphi(u)\in W^{1,2}(\mathrm{X})$. Moreover by the Leibniz rule for the divergence (Proposition 1.2.17) $f\nabla u\in D_{L^1}(\mathrm{div})$ with $\mathrm{div}(f\nabla u)=f\Delta u+\langle\nabla f,\nabla u\rangle$. Hence from the coarea formula (1.3.12) and integrating by parts

$$\int\varphi'(s)g(s)\mathrm{d}t=\int\varphi'(u)f|\nabla u|^2\mathrm{d}\mathfrak{m}=\int\langle\nabla\varphi(u),\nabla uf\rangle\mathrm{d}\mathfrak{m}=-\int\varphi(u)\mathrm{div}(f\nabla u)\mathrm{d}\mathfrak{m},$$

which yields

$$\left|\int\varphi'(t)g(t)\mathrm{d}t\right|\le\|\varphi\|_\infty\|\mathrm{div}(\nabla uf)\|_{L^1(\mathfrak{m})}\le2\|\varphi\|_\infty\|f\|_{W^{1,2}(\mathrm{X})}\left(\|\Delta u\|_{L^2(\mathfrak{m})}+\||\nabla u|\|_{L^2(\mathfrak{m})}\right),$$

which proves the claim.

Recall that, since $g\in\mathsf{BV}(\mathbb{R})$ the functions $Dg(-\infty,s))$, $Dg((-\infty,s])$ are respectively left and right continuous and coincide with g a.e.. Hence

$$\left|\lim_{h\to0^+}\fint_{t-h}^{t+h}g(s)\mathrm{d}s\right|=\left|\frac{Dg((t-\delta,t))+Dg((t-\delta,t)]}{2}\right|\le|Dg|(\mathbb{R}).$$

Applying again coarea to the left hand side and combining it with (3.3.9) we get the conclusion. □

We can finally prove the capacity estimate in Proposition 3.3.5, which was the main tool to prove Theorem 3.3.4.

Proof of Proposition 3.3.5. Fix $f\in W^{1,2}\cap L^\infty(\mathrm{X})$. From of Holder inequality we have

$$\left|\frac{1}{2h}\int_{\{t-h<u<t+h\}}f|\nabla u|\mathrm{d}\mathfrak{m}\right|^2\le\frac{\mathfrak{m}(\{t-h<u<t+h\})}{2h}\int f^2|\nabla u|\mathrm{d}\mu_t^h.$$

Passing to the $\varlimsup_{h\to0^+}$, observing that $f^2\in W^{1,2}(\mathrm{X})$ and applying (3.3.8) we obtain

$$\varlimsup_{h\to0^+}\left|\int f|\nabla u|\mathrm{d}\mu_t^h\right|^2\le2\mathfrak{m}(\{u<t\})'\|f^2\|_{W^{1,2}(\mathrm{X})}\left(\|\Delta u\|_{L^2(\mathfrak{m})}+\||\nabla u|\|_{L^2(\mathfrak{m})}\right).\tag{3.3.10}$$

We now pick any Borel set A and consider $f \in W^{1,2}(\mathrm{X})$ with $0 \le f \le 1$ and such that $f \ge 1$ in a neighbourhood of A. From (3.3.10) and $|\nabla f^2| \le 2|\nabla f|$ $\mathfrak{m}$-a.e., we deduce

$$\overline{\lim_{h\to 0^+}}\, \mu_t^h(A)^2 \le 2\mathfrak{m}(\{u<t\})' \left(\|\Delta u\|_{L^2(\mathfrak{m})} + \||\nabla u|\|_{L^2(\mathfrak{m})}\right) \|f\|_{W^{1,2}(\mathrm{X})},$$

from which passing to the infimum among all functions f of this kind we obtain the desired conclusion. □

We conclude this subsection with the proof of Proposition 3.3.3, which was stated at the beginning of this section and ensures that there are many sublevels of first order regularity. This in turn shows that the hypotheses of Theorem 3.3.4 are not too much restrictive.

PROOF OF PROPOSITION 3.3.3. Clearly by the coarea formula $\{u<t\}$ is of finite perimeter for a.e. $t \in \mathbb{R}$. For (ii) we fix g a quasi-continuous Borel representative of $|\nabla u|$ (recall Remark 1.2.20) and define the Borel set $B := \{g = 0\}$. From the coarea formula we have

$$0 = \int_B g \mathrm{d}\mathfrak{m} = \int_B |\nabla u| \mathfrak{m} = \int_{-\infty}^{\infty} \mathrm{Per}(\{u<t\}, B)\, \mathrm{d}t,$$

which implies that $\mathrm{Per}(\{u<t\}, B) = 0$ for a.e. $t \in \mathbb{R}$. In particular for a.e. $t \in \mathbb{R}$ it holds that $g \neq 0$ $\mathrm{Per}(\{u<t\})$-a.e.. However by definition of the trace we have that $\mathrm{tr}_{\{u<t\}}|\nabla u| = g$ $\mathrm{Per}(\{u<t\})$-a.e. for every t, from which the conclusion follows. We now prove (iii), i.e. that (3.3.1) holds for a.e. $t \in \mathbb{R}$. Denote now by u^+, u^- the positive and negative part of u respectively. Cavalieri's formula gives that $\int_X u^+ \mathfrak{m} = \int_0^{+\infty} \mathfrak{m}(u^+ > t)\, \mathrm{d}t$. This implies that the function $[0,\infty) \ni t \mapsto \mathfrak{m}(u^+ > t)$ belongs to $L^1(\mathbb{R}^+)$ and an application of Lebesgue differentiation Theorem gives that $\overline{\lim}_{h\to 0^+} \frac{\mathfrak{m}(\{t-h<u^+<t+h\})}{2h} < +\infty$ for a.e. $t \in [0,\infty)$. The same holds for u^-. It is also clear that $\{t-h<u<t+h\} = \{t-h<u^+<t+h\}$ whenever $t>0$ and h is small enough and that $\{t-h<u<t+h\} = \{-t-h<u^-<-t+h\}$ whenever $t<0$ and h is small enough. Combining these remarks we deduce (3.3.1). It remains to prove that (3.3.2) holds for a.e. $t \in \mathbb{R}$. Fix $\varphi \in C_b(\mathrm{X})$ with $\varphi \ge 0$. From the coarea formula we get both that the function $g(t) := \int \varphi \mathrm{d}\mathrm{Per}(\{u>t\}, .)$ is in $L^1(\mathbb{R})$ and that

$$⨍_{t-h}^{t+h} g(r)\, \mathrm{d}r = \int \varphi\, \mathrm{d}\mu_t^h, \quad \forall t \in \mathbb{R}, h \in (0,\infty).$$

Hence by Lebesgue differentiation theorem we obtain that $\int \varphi\, \mathrm{d}\mu_t^h \to \int \varphi \mathrm{d}\mathrm{Per}(\{u>t\}, .)$, for a.e. $t \in \mathbb{R}$. Therefore, writing any $\varphi \in C_b(\mathrm{X})$ as the difference between its positive and negative part, we obtain that for every $\varphi \in C_b(\mathrm{X})$, $\int \varphi\, \mathrm{d}\mu_t^h \to \int \varphi \mathrm{d}\mathrm{Per}(\{u>t\}, .)$ holds for a.e. $t \in \mathbb{R}$. The conclusion then follows recalling that it is enough to test weak convergence against a countable family $\mathcal{D} \subset C_b(\mathrm{X})$ (Proposition 1.1.1). □

3.4. Tangential divergence

In this section we give a notion of tangential divergence of a smooth vector field over the level set of a smooth function. We will always assume that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ space with $N < +\infty$.

We start with the following definition.

Definition 3.4.1 (Vector fields with tangential divergence)**.** Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$. Suppose that $\{u<t\}$ is first-order regular (see Def. 3.3.1). We define the domain of the tangential divergence (at $\{u<t\}$) to be the set $\mathsf{D}(\mathrm{div}_{T,\{u<t\}})$ of elements $v \in H^{1,2}_C \cap L^\infty(\mathrm{X}) \cap \mathsf{D}_{L^2}(\mathrm{div})$ satisfying:

(1)

$$C(v) := \overline{\lim_{h\to 0^+}} \int \left(\mathrm{div}(v)^2 + |\nabla v|^2\right) \mathrm{d}\mu_t^h < +\infty, \tag{3.4.1}$$

where as usual $\mu_t^h := (2h)^{-1} \mathfrak{m}|_{\{t-h<u<t+h\}}$,

(2) there exists a set $\mathcal{D} \subset C_b(\mathrm{X})$ dense in $L^2(\mathrm{X}, \mathrm{Per}(\{u<t\}, .))$ such that the limit

$$\mathrm{TanDiv}_v(\varphi) := \lim_{h\to 0^+} \int \varphi \left(\mathrm{div}(v) - \nabla v\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right)\right) \mathrm{d}\mu_t^h \tag{3.4.2}$$

exists for all $\varphi \in \mathcal{D}$, where the integrand is taken to be 0 whenever $|\nabla u| = 0$.

Notice that the integral in (3.4.2) is well defined and finite, since $\mathrm{div}(v) - \nabla v\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \in L^2(\mathfrak{m})$. Moreover when no confusion can occur we will often drop the subscript $\{u<t\}$ and simply write $\mathsf{D}(\mathrm{div}_T)$.

Observe also that in the above we have only formally defined $\mathsf{D}(\mathrm{div}_T)$ as the set of vector fields with suitable properties of the covariant derivative, but we still have to define a notion of tangential divergence. This will be done by showing that the map $\mathcal{D} \ni \varphi \mapsto \mathrm{TanDiv}_v(\varphi)$ extends to a functional on $L^2(\mathrm{Per}(\{u<t\}))$.

Proposition 3.4.2. *Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$. Let $\{u<t\}$ be a first-order regular sublevel (see Def. 3.3.1) and $v \in \mathsf{D}(\mathrm{div}_T)$. Then TanDiv_v (as in Definition 3.4.1) extends uniquely to an element in the dual of $L^2(\mathrm{Per}(\{u<t\}))$ still denoted by TanDiv_v. Moreover the limit in (3.4.2) exists and equals $\mathrm{TanDiv}_v(\varphi)$ also for every $\varphi \in C_b(\mathrm{X})$.*

Proof. We start observing that if limit in (3.4.2) exists for some $\varphi, \varphi' \in C_b(\mathrm{X})$ then it also exists for $a\varphi + b\varphi'$ $a, b \in \mathbb{R}$, with $\mathrm{TanDiv}_v(a\varphi' + b\varphi) = a\mathrm{TanDiv}_v(\varphi') + b\mathrm{TanDiv}_v(\varphi)$. In particular we can assume that $\mathcal{D}$ is a linear subspace of $C_b(\mathrm{X})$, on which TanDiv_v is linear. Moreover applying the Hölder inequality to (3.4.2) and using the first-order regularity of $\{u<t\}$ we obtain

$$|\mathrm{TanDiv}_v(\varphi)| \le 2\sqrt{C(v)} \left(\varlimsup_{h\to 0^+} \int \varphi^2 \mathrm{d}\mu_t^h \right)^{\frac{1}{2}} \overset{(3.3.2)}{=} 2\sqrt{C(v)} \|\varphi\|_{L^2(\mathrm{Per}(\{u<t\},.))},$$

for every $\varphi \in \mathcal{D}$, where $C(v)$ is as in (3.4.1). Therefore, since by assumption $C(v) < +\infty$, we deduce that TanDiv_v is bounded in $L^2(\mathrm{Per}(\{u<t\},.))$ and thus extends uniquely to a linear functional on $L^2(\mathrm{Per}(\{u<t\},.))$.

Fix now a generic $\varphi \in C_b(\mathrm{X})$ and consider a sequence $(\varphi_n) \subset \mathcal{D}$ converging to φ in $L^2(\mathrm{Per}(\{u<t\},.))$. Arguing as in the above estimate we obtain

$$\begin{aligned}
&\varlimsup_{h\to 0^+} \left| \mathrm{TanDiv}_v(\varphi) - \int \varphi \left(\mathrm{div}(v) - \nabla v \left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \right) \right) \mathrm{d}\mu_t^h \right| \le |\mathrm{TanDiv}_v(\varphi - \varphi_n)| \\
&+ \varlimsup_{h\to 0^+} \left| \mathrm{TanDiv}_v(\varphi_n) - \int \varphi_n \left(\mathrm{div}(v) - \nabla v \left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \right) \right) \mathrm{d}\mu_t^h \right| \\
&+ 2\sqrt{C(v)} \|\varphi - \varphi_n\|_{L^2(\mathrm{Per}(\{u<t\},.))},
\end{aligned}$$

which implies that (3.4.2) holds also for φ. □

With the previous result at hand, we can now give a notion of tangential divergence.

Definition 3.4.3 (Tangential divergence)**.** Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$. Suppose that $\{u<t\}$ is of finite perimeter and let $v \in \mathsf{D}(\mathrm{div}_{T,\{u<t\}})$. We define the tangential divergence of v as the unique $\mathrm{div}_{T,\{u<t\}}(v) \in L^2(\mathrm{Per}(\{u<t\}))$ representing TanDiv_v as a functional in $L^2(\mathrm{Per}(\{u<t\}))$.

Remark 3.4.4. We observe that from the second part of Proposition 3.4.2 we have the following representation formula for the tangential divergence: for every $v \in \mathsf{D}(\mathrm{div}_{T,\{u<t\}})$ it holds that

$$\int \varphi \mathrm{div}_T(v)\, \mathrm{dPer}(\{u<t\}) = \lim_{h\to 0^+} \int \varphi \left(\mathrm{div}(v) - \nabla v \left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \right) \right) \mathrm{d}\mu_t^h. \tag{3.4.3}$$

■

As for $\mathsf{D}(\mathrm{div}_T)$, when no confusion can occur we will often drop the subscript $\{u<t\}$ and simply write $\mathrm{div}_T(v)$. Notice also that to define the set $\mathsf{D}(\mathrm{div}_T)$ we only used that $\{u<t\}$ is of finite perimeter, while to define $\mathrm{div}_T(v)$ the first-order regularity of $\{u<t\}$ entered into play.

Proposition 3.4.5 (Linearity of div_T)**.** *The domain of the tangential divergence $D(\mathrm{div}_T)$ is a vector subspace of $H^{1,2}_C \cap L^\infty(\mathrm{X}) \cap \mathsf{D}_{L^2}(\mathrm{div})$ and the tangential divergence operator $\mathrm{div}_T : D(\mathrm{div}_T) \to L^2(\mathrm{Per}(\{u<t\}))$ is linear.*

Proof. Clearly any linear combination of elements in $D(\mathrm{div}_T)$ still satisfies condition 1 in Definition 3.4.1, hence we only need to show that condition 2 is stable for linear combinations. To see this we fix any $v_1, v_2 \in D(\mathrm{div})$. From the last part of Proposition 3.4.2 we know that the limit in (3.4.2) exists both for $v = v_1$ and $v = v_2$ and for every $\varphi \in C_b(\mathrm{X})$. Moreover the expression in (3.4.2) is clearly linear in v, which shows that the same limit exists also for $v = av_1 + bv_2$, $a, b \in \mathbb{R}$. This shows that $\mathsf{D}(\mathrm{div}_T)$ is a vector space. For the linearity of div_T, we notice that the previous observations imply that

$$\mathrm{TanDiv}_{av_1+bv_2}(\varphi) = a\mathrm{TanDiv}_{v_1}(\varphi) + b\mathrm{TanDiv}_{v_2}(\varphi), \quad \forall \varphi \in C_b(\mathrm{X}),$$

from which the claimed linearity follows by density. □

We have yet to show that the above definitions are meaningful in the sense we have many examples of vector field admitting a tangential divergence. This is the content of the following result.

Proposition 3.4.6. *Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$. Then for every $v \in H^{1,2}_C \cap L^\infty(\mathrm{X}) \cap \mathsf{D}_{L^2}(\mathrm{div})$, $v \in \mathsf{D}(\mathrm{div}_{T,\{u<t\}})$ for a.e. $t \in \mathbb{R}$.*
Moreover, fixing a Borel representative of the function $\mathrm{div}(v) - \nabla v\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right)$ (taken to be 0 when $|\nabla u| = 0$), the identity

$$\mathrm{div}_{T,\{u<t\}}(v) = \mathrm{div}(v) - \nabla v\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \quad \mathrm{Per}(\{u<t\})\text{-a.e.,} \tag{3.4.4}$$

holds for a.e. $t \in \mathbb{R}$ (the negligible set of t's depending on the representative chosen).

Proof. Fix $v \in H^{1,2}_C \cap L^\infty(\mathrm{X}) \cap D(\mathrm{div})$. We already know from Proposition 3.3.3 that $\{u < t\}$ is first-order regular for a.e. t. Moreover from the coarea formula and Lebesgue differentiation theorem follows that $\overline{\lim}_{h\to 0^+} \int \left(\mathrm{div}(v)^2 + |\nabla v|^2\right) \mathrm{d}\mu_t^h < +\infty$, for a.e. t. Analogously coarea ensures that the limit in (3.4.2) exists for a.e. t, once we fix $\varphi \in C_b(\mathrm{X})$. The first part of the statement then follows recalling that from Proposition 1.1.2 there exists a countable family $\mathcal{F} \subset C_b(\mathrm{X})$ dense $L^2(\mathrm{Per}(\{u<t\}))$, for every t such that $\{u<t\}$ is of finite perimeter.

For the second part we fix g a Borel representative of $\mathrm{div}(v) - \nabla v\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right)$. We also fix a function $\varphi \in \mathcal{F}$, where $\mathcal{F}$ is as above. By coarea and Lebesgue differentiation we have that $g \in L^2(\mathrm{Per}(\{u<t\}))$ for a.e. $t \in \mathbb{R}$ and

$$\lim_{h\to 0} \int \varphi g \, \mathrm{d}\mu_t^h = \int \varphi g \, \mathrm{d}\mathrm{Per}(\{u<t\}), \quad \text{for a.e. } t \in \mathbb{R}.$$

However from the first part $v \in D(\mathrm{div}_{T,\{u<t\}})$ for a.e. $t \in \mathbb{R}$, hence by Proposition 3.4.2

$$\lim_{h\to 0} \int \varphi g \, \mathrm{d}\mu_t^h = \int \varphi \mathrm{div}_{T,\{u<t\}}(v) \, \mathrm{d}\mathrm{Per}(\{u<t\}), \quad \text{for a.e. } t \in \mathbb{R}.$$

Combining the two identities above and using that $\mathcal{F}$ is countable we reach: for a.e. $t \in \mathbb{R}$

$$\int \varphi \mathrm{div}_{T,\{u<t\}}(v) \, \mathrm{d}\mathrm{Per}(\{u<t\}) = \int \varphi g \, \mathrm{d}\mathrm{Per}(\{u<t\}), \quad \forall \varphi \in \mathcal{F}.$$

The conclusion then follows from the density of $\mathcal{F}$ in $L^2(\mathrm{Per}(\{u<t\}))$, for all t. □

We end this section with a final but fundamental density result, which says that there are many vector fields with tangential divergence also at the level of 'trace'.

Corollary 3.4.7. *Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$. Then $\mathrm{tr}_{\{u<t\}}(\mathsf{D}(\mathrm{div}_{T,\{u<t\}}))$ is dense in $L^2_{\{u<t\}}(T\mathrm{X})$ for a.e. $t \in \mathbb{R}$.*

Proof. From Proposition 1.5.4 there exists a countable set $\mathcal{V} \subset H^{1,2}_C \cap L^\infty(T\mathrm{X}) \cap \mathsf{D}_{L^2}(\mathrm{div})$ such that $\mathrm{tr}_{\{u<t\}}(\mathcal{V})$ is dense in $L^2_{\{u<t\}}(T\mathrm{X})$ for every t such that $\{u<t\}$ has finite perimeter. Moreover from Proposition 3.4.6 we infer that for every $v \in \mathcal{V}$ $\mathrm{tr}_{\{u<t\}}(v) \in \mathsf{D}(\mathrm{TanDiv}_{\{u<t\}})$ for a.e. $t \in \mathbb{R}$, from which the conclusion follows. □

3.5. Mean curvature

The goal of this section is to define a notion of mean curvature starting from the notion of tangential divergence developed in the previous section. Also here it always assumed that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ space with $N < +\infty$.

We start giving a definition of sublevel with L^2-mean curvature:

Definition 3.5.1 (Sublevels of second order regularity)**.** Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$. We say that the sublevel set $\{u<t\}$ is second-order regular if it is first order regular and the following hold:

(1)

$$\nu_{\{u<t\}} = \frac{\mathrm{tr}_{\{u<t\}}(\nabla u)}{\mathrm{tr}_{\{u<t\}}(|\nabla u|)}, \quad \mathrm{Per}(\{u<t\}, .)\text{-a.e.,} \tag{3.5.1}$$

(2)

$$\mathscr{H}(\{u<t\}) := \left(\overline{\lim_{h\to 0^+}} \frac{1}{2h} \int_{\{t-h<u<t+h\}} \left| \frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla |\nabla u|, \nabla u\rangle}{|\nabla u|^2}\right|^2 |\nabla u| \mathrm{d}\mathfrak{m}\right)^{\frac{1}{2}} < +\infty. \tag{3.5.2}$$

.

Notice that we can write (3.5.2) since by Proposition 1.4.6 $|\nabla u| \in W^{1,2}(\mathrm{X})$.

Remark 3.5.2. Note that any sublevel with second-order regularity has also finite relaxed Willmore energy by Proposition 3.2.8. ■

As we did for the first-order regular sublevels, it is important to check that this definition is meaningful in the sense that there exist many non trivial second order regular sublevels. We start showing that (3.5.1) is often satisfied.

Proposition 3.5.3. *Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$. Then for every a.e. $t \in \mathbb{R}$ it holds that* $\mathrm{tr}_{\{u<t\}}(|\nabla u|) \neq 0$ $\mathrm{Per}(\{u<t\})$*-a.e. and*

$$\nu_{\{u<t\}} = \frac{\mathrm{tr}_{\{u<t\}}(\nabla u)}{\mathrm{tr}_{\{u<t\}}(|\nabla u|)}.$$

Proof. Fix $v \in H^{1,2}_C \cap L^\infty(T\mathrm{X}) \cap \mathsf{D}_{L^2}(\mathrm{div})$ with bounded support. Let us define the following two functions

$$f_1^v(t) := \int \mathrm{tr}_{\{u<t\}}(|\nabla u|)\langle \mathrm{tr}_{\{u<t\}} v, \nu_{\{u<t\}}\rangle \mathrm{dPer}(\{u<t\},.),$$
$$f_2^v(t) := \int \langle \mathrm{tr}_{\{u<t\}} v, \mathrm{tr}_{\{u<t\}} \nabla u\rangle \mathrm{dPer}(\{u<t\},.),$$

which are well define and finite for a.e. $t \in \mathbb{R}$. From hypotheses and Proposition 1.4.6 we have that $|\nabla u| \in W^{1,2}(\mathrm{X})$, hence from Proposition 1.4.15 and the Leibniz rule for the divergence (Prop. 1.2.10) we have that $|\nabla u| v \in H^{1,2}_C \cap L^\infty(T\mathrm{X}) \cap \mathsf{D}_{L^2}(\mathrm{div})$. Fix $t \neq 0$ such that $\{u<t\}$ has finite perimeter and observe that either $\{u<t\}$ or $\{u>t\}$ has finite measure. Therefore we are in position to apply the Gauss-Green formula (1.5.2) and coarea to obtain

$$f_1^v(t) = \int_{\{u<t\}} \mathrm{div}(v|\nabla u|)\mathrm{d}\mathfrak{m} = \int_{-\infty}^t \int \frac{\mathrm{div}(v|\nabla u|)}{|\nabla u|}\mathrm{dPer}(\{u<t\}),$$

having chosen a representative of $\frac{\mathrm{div}(v|\nabla u|)}{|\nabla u|}$. Fix now $\varphi \in C^1_c(-\infty,t)$. Then by coarea

$$\int_{\mathbb{R}} \varphi' f_2^v = \int \varphi'(u)|\nabla u|\langle v, \nabla u\rangle \mathrm{d}\mathfrak{m} = -\int \mathrm{div}(v|\nabla u|)\varphi(u) = -\int_{\mathbb{R}} \varphi \int \frac{\mathrm{div}(v|\nabla u|)}{|\nabla u|}\mathrm{dPer}(\{u<t\})\mathrm{d}t.$$

This proves that $f_1^v, f_2^v \in W^{1,1}(\mathbb{R})$ and have the same distributional derivative. Hence for every $v \in H^{1,2}_C \cap L^\infty(T\mathrm{X}) \cap \mathsf{D}_{L^2}(\mathrm{div})$ with bounded support it holds $f_1^v(t) = f_2^v(t)$ for a.e. t. Recall that Proposition 1.5.4 that there exists a countable set $\mathcal{V} \subset H^{1,2}_C \cap L^\infty(T\mathrm{X}) \cap \mathsf{D}_{L^2}(\mathrm{div})$ such that $\mathrm{tr}_{\{u<t\}}(\mathcal{V})$ is dense in $L^2_{\{u<t\}}(T\mathrm{X})$ for every $t \in \mathbb{R}$. Hence by density we obtain that $\mathrm{tr}_{\{u<t\}}(\nabla u) = \nu_{\{u<t\}}\mathrm{tr}_{\{u<t\}}(|\nabla u|)$ as elements of $L^2_{\{u<t\}}(T\mathrm{X})$, for a.e. t. Recalling that by Proposition 3.3.3 $\mathrm{tr}_{\{u<t\}}(|\nabla u|) \neq 0$ $\mathrm{Per}(\{u<t\})$-a.e. for a.e. $t \in \mathbb{R}$ concludes the proof. □

The result below shows that there area many functions with many second-order regular sublevels, indeed recall that that $\mathsf{D}(\mathcal{W}) \cap \mathsf{LIP}_c(\mathrm{X})$ is a rich family thanks to Theorem 3.2.6.

Proposition 3.5.4. *Let $u \in \mathsf{D}(\mathcal{W}) \cap \mathsf{LIP}(\mathrm{X})$, then for a.e. $t \in \mathbb{R}$ the sublevel $\{u<t\}$ is second order regular.*

Proof. In light of propositions 3.3.3 and 3.5.3 we only need to show (3.5.2), which immediately follows by the coarea formula and Lebesgue differentiation theorem. □

The following is the fundamental result of this section. It shows that for sublevels which are second-order regular, the integral of tangential divergence of a vector field is bounded above by the L^2-norm of its trace. Its proof is based on the approximation results developed in Section 3.3.

Theorem 3.5.5. *Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$ and let $\{u<t\}$ be second-order regular. Then for every $v \in \mathsf{D}(\mathrm{div}_T)$ and $\varphi \in \mathsf{LIP}_b(\mathrm{X})$, it holds*

(3.5.3)
$$\left|\int \varphi \mathrm{div}_T(v)\,\mathrm{dPer}(\{u<t\})\right| \le \left(\mathscr{H}(\{u<t\})\|\varphi\|_\infty + 2\mathrm{Lip}\varphi\sqrt{\mathrm{Per}(\{u<t\})}\right)\|\mathrm{tr}_{\{u<t\}}(v)\|_{L^2_{\{u<t\}}(T\mathrm{X})},$$

where $\mathscr{H}(\{u<t\})$ is the constant defined in (3.5.2). *In particular $\mathrm{div}_T(v)$ depends only on $\mathrm{tr}_{\{u<t\}}(v)$ and*

$$\left|\int \mathrm{div}_T(v)\,\mathrm{dPer}(\{u<t\})\right| \le \mathscr{H}(\{u<t\})\|\mathrm{tr}_{\{u<t\}}(v)\|_{L^2_{\{u<t\}}(T\mathrm{X})}, \quad \forall v \in \mathsf{D}(\mathrm{div}_T). \tag{3.5.4}$$

The proof of the above theorem relies on the following technical result, which can be formally interpreted as an averaged version of the integration by-parts formula.

Lemma 3.5.6. *Let $v \in H^{1,2}_C \cap L^\infty(T\mathrm{X}) \cap \mathsf{D}_{L^2}(\mathrm{div})$ and let $u \in W^{1,2}(\mathrm{X}) \cap \mathsf{LIP}(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$ be continuous. Then for every $a, b \in \mathbb{R}$ with $a < b$*

$$\begin{aligned}\int_{\{a<u<b\}} |\nabla u| \left(\mathrm{div}(v) - \nabla v\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right)\right) \mathrm{d}\mathfrak{m} \\ = -\int_{\{a<u<b\}} \langle v, \nabla u\rangle \left(\frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla|\nabla u|, \nabla u\rangle}{|\nabla u|^2}\right) \mathrm{d}\mathfrak{m},\end{aligned} \tag{3.5.5}$$

where both integrand are taken to be 0 whenever $|\nabla u| = 0$.

Proof. We start observing that, since $u \in H^{2,2}(\mathrm{X})$, then $\nabla u \in H^{1,2}_C(T\mathrm{X})$ (recall (1.4.14)). Therefore combining (1.4.15) and (1.4.12) we obtain that $\langle v, \nabla u\rangle \in W^{1,2}(\mathrm{X})$ and

$$\nabla v(\nabla u, \nabla u) = \langle \nabla\langle v, \nabla u\rangle, \nabla u\rangle - |\nabla u|\langle \nabla|\nabla u|, v\rangle, \quad \mathfrak{m}\text{-a.e.}.$$

Fix $\varphi \in C^1(\mathbb{R})$ bounded and fix also $\varepsilon > 0$. Using the above identity, integrating by parts and rearranging the terms we obtain

$$\int \varphi(u)|\nabla u|\mathrm{div}(v) - \frac{\varphi(u)}{|\nabla u| + \varepsilon}\nabla v(\nabla u, \nabla u)\, \mathrm{d}\mathfrak{m} =$$
$$\int \left(\frac{|\nabla u|}{|\nabla u| + \varepsilon} - 1\right)(\langle \nabla u, v\rangle\varphi'(u)|\nabla u| + \langle v, \nabla|\nabla u|\varphi(u)\rangle) + \langle v, \nabla u\rangle \left(\frac{\Delta u}{|\nabla u| + \varepsilon} - \frac{\langle \nabla|\nabla u|, \nabla u\rangle}{(|\nabla u| + \varepsilon)^2}\right)\varphi(u)\, \mathrm{d}\mathfrak{m}.$$

Using the following $\mathfrak{m}$-a.e. bounds:

$$\left|\frac{\nabla v(\nabla u, \nabla u)}{|\nabla u| + \varepsilon}\right| \le |\nabla v|_{HS}|\nabla u|, \qquad \left|\left(\frac{|\nabla u|}{|\nabla u| + \varepsilon} - 1\right)\langle \nabla u, v\rangle|\nabla u|\right| \le \varepsilon|\nabla u||v|,$$
$$\left|\left(\frac{|\nabla u|}{|\nabla u| + \varepsilon} - 1\right)\langle v, \nabla|\nabla u|\rangle\right| \le |\nabla|\nabla u|||v|, \qquad \left|\langle v, \nabla u\rangle\frac{\Delta u}{|\nabla u| + \varepsilon}\right| \le |v||\Delta u|,$$
$$\left|\langle v, \nabla u\rangle\frac{\langle \nabla|\nabla u|, \nabla u\rangle}{(|\nabla u| + \varepsilon)^2}\right| \le |\nabla|\nabla u||\,|v|,$$

where all the functions on the right-hand sides are in $L^1(\mathfrak{m})$, we can apply the dominated converge theorem sending $\varepsilon \to 0^+$ to obtain

$$\int \varphi(u)|\nabla u| \left(\mathrm{div}(v) - \nabla v\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right)\right) \mathrm{d}\mathfrak{m} = \int \varphi(u)\langle v, \nabla u\rangle \left(\frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla|\nabla u|, \nabla u\rangle}{|\nabla u|^2}\right) \mathrm{d}\mathfrak{m},$$

where both integrand are taken to be 0 whenever $|\nabla u| = 0$. Finally, taking a sequence $\varphi_n \in C^1(\mathbb{R})$, uniformly bounded in L^∞ and such that $\varphi_n \to \chi_{(a,b)}$ pointwise, plugging φ_n in the above identity, sending $n \to +\infty$ and applying dominated convergence we reach (3.5.5). □

Proof of Theorem 3.5.5. Fix $\varphi \in \mathsf{LIP}(\mathrm{X})$ bounded and $v \in \mathsf{D}(\mathrm{div}_T)$. From the Leibniz rule for the divergence (Proposition 1.2.17)and the covariant derivative 1.4.13 we obtain the following identity

$$\begin{aligned}\varphi\left(\mathrm{div}(v) - \nabla v\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right)\right) &= \mathrm{div}(\varphi v) - \nabla(\varphi v)\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \\ &\quad - \langle v, \nabla\varphi\rangle + |\nabla u|^{-2}\langle v, \nabla u\rangle\langle \nabla\varphi, \nabla u\rangle, \quad \mathfrak{m}\text{-a.e.},\end{aligned}$$

where as usual both sides are intended to be 0 whenever $|\nabla u| = 0$. Integrating in $\mathrm{d}\mu_t^h$ and passing to the $\overline{\lim}_{h\to 0^+}$, recalling that (3.4.2) holds for any $\varphi \in C_b(\mathrm{X})$ and applying the Hölder inequality to the last two terms we obtain

$$\begin{aligned}|\mathrm{TanDiv}_v(\varphi)| \le \overline{\lim_{h\to 0^+}} \left|\int \mathrm{div}(\varphi v) - \nabla(\varphi v)\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \mathrm{d}\mu_t^h\right| \\ + \overline{\lim_{h\to 0^+}}\ 2\mathrm{Lip}\varphi\sqrt{\mu_t^h(\mathrm{X})}\left(\int |v|^2 \mathrm{d}\mu_t^h\right)^{\frac{1}{2}}.\end{aligned} \tag{3.5.6}$$

The second term can be computed recalling that $\{u < t\}$ is first order regular. More precisely recalling from (1.4.16) that $|v| \in W^{1,2} \cap L^\infty(\mathrm{X})$, we have that $\mathsf{QCR}(|v|)$ is well defined, coincide with $|v|$ $\mathfrak{m}$-a.e. and belongs to $\mathcal{QC}(\mathrm{X}) \cap L^\infty(\mathrm{Cap})$. In particular also $\mathsf{QCR}(|v|)^2 \in \mathcal{QC}(\mathrm{X}) \cap L^\infty(\mathrm{Cap})$ and coincide with $|v|^2$ $\mathfrak{m}$-a.e.. Therefore combining (3.3.2) and (3.3.3) we obtain

$$\lim_{h\to 0^+} 2\mathrm{Lip}\varphi\sqrt{\mu_t^h(\mathrm{X})}\left(\int |v|^2 \mathrm{d}\mu_t^h\right)^{\frac{1}{2}} = 2\mathrm{Lip}\varphi\sqrt{\mathrm{Per}(\{u < t\})}\, \|\mathrm{tr}_{\{u<t\}} v\|_{L^2_{\{u<t\}}(T\mathrm{X})}, \tag{3.5.7}$$

where we have used that $|\pi_{\{u<t\}}(\mathsf{QCR}(|v|)^2)| = |\pi_{\{u<t\}}(\mathsf{QCR}(|v|))|^2 = \mathrm{tr}_{\{u<t\}}(|v|)^2 = |\mathrm{tr}_{\{u<t\}}(v)|^2$. For the first term observe that by Lemma 1.4.15 we have $\varphi v \in H^{1,2}_C(T\mathrm{X})$, hence applying (3.5.5) and recalling that $\{u<t\}$ is second order regular we have

$$\begin{aligned}
&\overline{\lim_{h\to 0^+}} \left| \int \mathrm{div}(\varphi v) - \nabla(\varphi v)\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \mathrm{d}\mu_t^h \right| \\
&\overset{(3.5.5)}{=} \overline{\lim_{h\to 0^+}} \left| \frac{1}{2h}\int_{\{t-h<u<t+h\}} \langle \varphi v, \nabla u\rangle \left(\frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla|\nabla u|, \nabla u\rangle}{|\nabla u|^2}\right) \mathrm{d}\mathfrak{m} \right| \\
&\le \overline{\lim_{h\to 0^+}} \left(\frac{1}{2h}\int_{\{t-h<u<t+h\}} \left|\frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla|\nabla u|, \nabla u\rangle}{|\nabla u|^2}\right|^2 |\nabla u|\mathrm{d}\mathfrak{m}\right)^{\frac12} \overline{\lim_{h\to 0^+}} \left(\frac{1}{2h}\int_{\{t-h<u<t+h\}} \varphi^2|v|^2|\nabla u|\mathrm{d}\mathfrak{m}\right)^{\frac12} \\
&\overset{(3.5.2),(3.3.3)}{\le} \mathscr{H}(\{u<t\})\|\varphi\|_\infty \||\mathrm{tr}_{\{u<t\}}(v)|\|_{L^2(\mathrm{Per}(\{u<t\}))} = \mathscr{H}(\{u<t\})\|\varphi\|_\infty \|v\|_{L^2_E(T\mathrm{X})}.
\end{aligned}$$

Plugging this estimate together with (3.5.7) in (3.5.6), we obtain (3.5.3). For (3.5.4) we simply take 1-Lipschitz functions with bounded support $\varphi_r \in \mathsf{LIP}_{bs}(\mathrm{X})$, $r\to+\infty$, such that $0\le\varphi_r\le 1$ with $\varphi_r = 1$ in $B_r(x_0)$ for some fixed $x_0 \in X$ and apply dominated convergence. □

Thanks to the fundamental estimates proved in Theorem 3.5.5 we can now prove the existence of a mean curvature vector.

Theorem 3.5.7 (Existence of mean curvature vector)**.** *Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$ and suppose that $\{u<t\}$ is second-order regular and that $\mathrm{tr}_{\{u<t\}}(\mathsf{D}(\mathrm{div}_T))$ is dense in $L^2_{\{u<t\}}(T\mathrm{X})$. Then there exists a unique $H \in L^2_{\{u<t\}}(T\mathrm{X})$, called mean curvature vector, such that*

$$\int \langle H, \mathrm{tr}_{\{u<t\}}(v)\rangle \,\mathrm{dPer}(\{u<t\}) = \int \mathrm{div}_T(v)\,\mathrm{dPer}(\{u<t\}), \quad \forall v \in \mathsf{D}(\mathrm{div}_T). \tag{3.5.8}$$

Proof. We define the functional $\mathrm{IntDiv} : \mathsf{D}(\mathrm{div}_T) \to \mathbb{R}$ as

$$\mathrm{IntDiv}(v) \coloneqq \int \mathrm{div}_T(v)\,\mathrm{dPer}(\{u<t\}), \quad \forall v \in \mathsf{D}(\mathrm{div}_T).$$

From the linearity of the tangential divergence (Proposition 3.4.5) it follows that IntDiv is a linear. The key observation is now that, thanks to (3.5.4), the induced map $\overline{\mathrm{IntDiv}} : \mathrm{tr}_{\{u<t\}}(\mathsf{D}(\mathrm{div}_T)) \to \mathbb{R}$ given by

$$\overline{\mathrm{IntDiv}}(\mathrm{tr}_{\{u<t\}}(v)) \coloneqq \mathrm{IntDiv}(v), \quad \forall v \in \mathsf{D}(\mathrm{div}_T),$$

is well defined, linear and satisfies

$$\overline{\mathrm{IntDiv}}(\bar v) \le \mathscr{H}(\{u<t\})\|\bar v\|_{L^2_E(T\mathrm{X})}, \quad \forall\, \bar v \in \mathrm{tr}_{\{u<t\}}(\mathsf{D}(\mathrm{div}_T)). \tag{3.5.9}$$

From the density of $\mathsf{D}(\mathrm{div}_T)$ in $L^2_{\{u<t\}}(T\mathrm{X})$ we have that $\overline{\mathrm{IntDiv}}$ extends uniquely to an element $\bar H$ in the dual $(L^2_{\{u<t\}}(T\mathrm{X}))'$ (in the sense of Hilbert spaces). Then the Riesz theorem (for Hilbert spaces) ensures that $\bar H(.) = \langle H, .\rangle_{L^2_{\{u<t\}}(T\mathrm{X})}$ for some $H \in L^2_{\{u<t\}}(T\mathrm{X})$,. Then recalling (1.2.18) we have:

$$\bar H(v) = \int \langle H, w\rangle \,\mathrm{Per}(\{u<t\}), \quad \forall w \in L^2_{\{u<t\}}(T\mathrm{X}).$$

This concludes the proof. □

Theorem 3.5.7 motivates the following definition:

Definition 3.5.8 (Mean curvature of a sublevel set)**.** Let $u \in \mathsf{LIP}_c(\mathrm{X})\cap\mathsf{D}_{L^2}(\Delta)$. We say that the sublevel set $\{u<t\}$ admits a mean curvature vector if it is second-order regular and $\mathrm{tr}_{\{u<t\}}(\mathsf{D}(\mathrm{div}_T))$ is dense in $L^2_{\{u<t\}}(T\mathrm{X})$. Moreover we define its mean curvature vector to be $H_{\{u<t\}} = H \in L^2_{\{u<t\}}(T\mathrm{X})$, the one given by Theorem 3.5.7.

The following crucial result says that there are many functions u and many sublevels $\{u<t\}$ that admits a mean curvature vector.

Proposition 3.5.9. *Let $u \in \mathsf{D}(\mathcal{W}) \cap \mathsf{LIP}_c(\mathrm{X})$. Then for a.e. $t \in \mathbb{R}$ the set $\{u<t\}$ admits a mean curvature vector.*

Proof. Simply combine Proposition 3.5.4 and Corollary 3.4.7. □

Remark 3.5.10. Notice that the fact that $\{u < t\}$ admits a mean curvature vector (defined as above) is a property both of the set $\{u < t\}$ and of the function u combined. This is due to a limitation of our analysis, since we do not know if the resulting mean curvature vector is intrinsic of the set or truly depends on the function u. However it is easily seen that it depends only on the value of u in a neighbourhood $\{u = t\}$. This is made precise by the following proposition. ■

Proposition 3.5.11 (Locality of the mean curvature)**.** *Let $u, v \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$ and $t \in \mathbb{R}$ such that $\{u < t\} = \{v < t\}$. Suppose furthermore that for some $\delta > 0$ it holds $\{t-\delta < u < t+\delta\} = \{t-\delta < v < t+\delta\}$ and $u = v$ in this set. Then $\{u < t\}$ admits a mean curvature vector if and only it does $\{v < t\}$, in which case the two mean curvature vectors coincide.*

Proof. We first check that $\{u < t\}$ is first order regular if and only if $\{v < t\}$ is first order regular (Def. 3.3.1). Clearly $\{u < t\}$ is of finite perimeter if and only if $\{v < t\}$ is of finite perimeter. Moreover (iii) and (iv) of Definition 3.3.1 are clearly satisfied for $\{u < t\}$ if and only if they are for $\{v < t\}$ by the assumptions and the locality of the weak upper gradient. Finally for (ii) we observe that by Proposition 1.2.21 and the locality of the weak-upper gradient we have that $\mathsf{QCR}(|\nabla u|) = \mathsf{QCR}(|\nabla v|)$ Cap-a.e. in $\{t - \delta < u < t + \delta\} = \{t - \delta < v < t + \delta\}$, from which (ii) follows by the definition of trace. Moreover, by the same argument we deduce that (i) of Definition 3.5.1 is satisfied for $\{u < t\}$ if and only if it is satisfied for $\{v < t\}$. Finally by locality of the gradient and Laplacian the same is true for (ii) of Definition 3.5.1. This proves that $\{u < t\}$ is second order regular if and only if it is $\{v < t\}$. Moreover from Definition 3.4.1 and the locality of the gradient and Laplacian we deduce also that $\mathsf{D}(\mathrm{TanDiv}_{\{u<t\}}) = \mathsf{D}(\mathrm{TanDiv}_{\{v<t\}})$. It remains to check that the mean curvature vector for $\{u < t\} = \{v < t\}$ given by Theorem 3.5.7, when it exists (observe that it exists for u if and if it exists for v), is the same for u and v. However this is immediate from the fact that the mean curvature vector, as defined in Theorem 3.5.7, depends only on the map TanDiv_v (see Definition 3.4.1) which is the same both for v and u. □

Remark 3.5.12. It can also be proved that the mean curvature vector is invariant under composition with smooth functions. That is to say that if $\{u < t\}$ has a mean curvature vector H, also $\{\varphi(u) < \varphi(t)\}$ has the same mean curvature vector, provided $\{u < t\} = \{\varphi(u) < t\}$, $\varphi \in C^2(\mathbb{R})$ and $\varphi'(t) \neq 0$. We will however not prove (or use) this fact here. ■

The mean curvature vector was defined in a rather indirect way via duality, however in the next proposition we derive a more direct representation formula.

Proposition 3.5.13 (Formula for the mean curvature)**.** *Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$ and suppose that $\{u < t\}$ admits a mean curvature vector $H \in L^2_{\{u<t\}}(T\mathrm{X})$. Then for every $w \in H^{1,2}_C \cap L^\infty(T\mathrm{X})$ it holds*

$$\int \langle H, \mathrm{tr}_{\{u<t\}} w\rangle \, \mathrm{dPer}(\{u < t\}) = \lim_{h\to 0+} \int \langle w, \frac{\nabla u}{|\nabla u|}\rangle \left(\frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla|\nabla u|, \nabla u\rangle}{|\nabla u|^2}\right) \mathrm{d}\mu_t^h, \tag{3.5.10}$$

where the integrand on the right hand side is taken to be 0 whenever $|\nabla u| = 0$ and its integral makes sense thanks to assumption 2 in Definition 3.5.1.

Proof. From (3.5.8) we have

$$\int \langle H, \mathrm{tr}_{\{u<t\}} v\rangle \, \mathrm{dPer}(\{u < t\}, .) = \int \mathrm{div}_T(v) \mathrm{dPer}(\{u < t\}, .), \quad \forall v \in \mathsf{D}(\mathrm{div}_T).$$

Moreover from (3.4.3) we know that the right hand side of the above equals $\lim_{h\to 0^+} \int \mathrm{div}(v) - \nabla v \left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \mathrm{d}\mu_t^h$. This combined with (3.5.5) gives

$$\int \langle H, \mathrm{tr}_{\{u<t\}} v\rangle \, \mathrm{dPer}(\{u < t\}, .) = \lim_{h\to 0+} \int \langle v, \frac{\nabla u}{|\nabla u|}\rangle \left(\frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla|\nabla u|, \nabla u\rangle}{|\nabla u|^2}\right) \mathrm{d}\mu_t^h, \quad \forall v \in \mathsf{D}(\mathrm{div}_T).$$

Fix now $w \in H^{1,2}_C \cap L^\infty(T\mathrm{X})$ and consider a sequence $(v_n)_n \subset \mathsf{D}(\mathrm{div}_T)$ such that $\mathrm{tr}_{\{u<t\}} v_n \to \mathrm{tr}_{\{u<t\}} v$ in $L^2_{\{u<t\}}(T\mathrm{X})$. Applying the Hölder inequality we obtain

$$\begin{aligned} &\overline{\lim_{h\to 0^+}} \left| \int \langle v - v_n, \frac{\nabla u}{|\nabla u|}\rangle \left(\frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla|\nabla u|, \nabla u\rangle}{|\nabla u|^2}\right) \mathrm{d}\mu_t^h \right| \\ &\leq \mathscr{H}(\{u < t\}) \overline{\lim_{h\to 0^+}} \left(\int |v - v_n|^2 \, \mathrm{d}\mu_t^h \right) \overset{(3.3.3)}{=} \mathscr{H}(\{u < t\}) \|\mathrm{tr}_{\{u<t\}} v - \mathrm{tr}_{\{u<t\}} v_n\|_{L^2_{\{u<t\}}(T\mathrm{X})}, \end{aligned}$$

where in the last step we argued exactly as in (3.5.7). Since, by hypothesis $\mathscr{H}(\{u < t\}) < +\infty$, this concludes the proof. □

The assumption that the renormalized gradient is the normal to the sublevel, i.e. (3.5.1), on the definition of second-order regular sublevel was not used to prove the existence of the mean curvature. However it is needed to prove that the mean curvature is parallel to the normal:

Proposition 3.5.14 (The mean curvature vector is parallel to the normal)**.** *Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$. Then for a.e. $t \in \mathbb{R}$ it holds that $H = \bar{H}\nu_{\{u<t\}}$ for some function $\bar{H} \in L^2(\mathrm{Per}(\{u<t\}))$.*

Proof. Fix $t \in \mathbb{R}$ so that $\{u<t\}$ admits a mean curvature vector (recall Definition 3.5.8). By how we constructed H we have that $\mathrm{tr}_{\{u<t\}}(\mathsf{D}(\mathrm{div}_T))$ is dense in $L^2_{\{u<t\}}(T\mathrm{X})$ and

$$\int \langle H, \mathrm{tr}_{\{u<t\}}v\rangle \,\mathrm{dPer}(\{u<t\},.) = \int \mathrm{div}_T(v)\mathrm{dPer}(\{u<t\},.), \quad \forall v \in \mathsf{D}(\mathrm{div}_T).$$

This combined with (3.4.2) implies that

$$\left|\int \langle H, \mathrm{tr}_{\{u<t\}}v\rangle \,\mathrm{dPer}(\{u<t\},.)\right|^2 = \lim_{h\to0^+}\left|\int \mathrm{div}(v) - \nabla(v)\left(\frac{\nabla u}{|\nabla u|},\frac{\nabla u}{|\nabla u|}\right)\mathrm{d}\mu_t^h\right|^2.$$

From the assumptions and Proposition 1.4.16 we have that $\langle v,\nabla u\rangle \in W^{1,2}(\mathrm{X})$, hence it makes sense to consider its continuous representative $\mathsf{QCR}(\langle v,\nabla u\rangle)$. Making use of (3.5.5) and subsequently applying the Hölder inequality gives

$$\begin{aligned}
\left|\int \langle H, \mathrm{tr}_{\{u<t\}}v\rangle \,\mathrm{dPer}(\{u<t\},.)\right|^2 &= \left|\lim_{h\to0^+}\frac{1}{2h}\int_{\{t-h<u<t+h\}}\langle v,\nabla u\rangle\left(\frac{\Delta u}{|\nabla u|}-\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|^2}\right)\mathrm{d}\mathfrak{m}\right|^2\\
&\le \mathscr{H}(\{u<t\})\lim_{h\to0^+}\int\frac{\mathsf{QCR}(\langle v,\nabla u\rangle)^2}{\mathsf{QCR}(|\nabla u|)^2}\mathrm{d}\mu_t^h\\
&\overset{(3.3.3)}{=} \mathscr{H}(\{u<t\})\int\frac{\langle \mathrm{tr}_{\{u<t\}}v, \mathrm{tr}_{\{u<t\}}\nabla u\rangle^2}{\mathrm{tr}_{\{u<t\}}(|\nabla u|)^2}\mathrm{dPer}(\{u<t\},.)\\
&\overset{(3.5.1)}{=} \mathscr{H}(\{u<t\})\int\langle \mathrm{tr}_{\{u<t\}}(v),\nu_{\{u<t\}}\rangle^2\mathrm{dPer}(\{u<t\},.),
\end{aligned}$$

where in the penultimate equality we have used that

$$\pi_{\{u<t\}}(\mathsf{QCR}(\langle v,\nabla u\rangle)^2) = (\pi_{\{u<t\}}(\mathsf{QCR}(\langle v,\nabla u\rangle)))^2 = (\mathrm{tr}_{\{u<t\}}\langle v,\nabla u\rangle)^2 \overset{(1.5.1)}{=} \langle \mathrm{tr}_{\{u<t\}}v, \mathrm{tr}_{\{u<t\}}\rangle^2,$$

that $\pi(\mathsf{QCR}(|\nabla u|)) = \mathrm{tr}_{\{u<t\}}(|\nabla u|)$, and the fact $\pi_{\{u<t\}}(f)^{-1} = \pi_{\{u<t\}}(f^{-1})$ for every $f \in L^0(\mathrm{Cap})$. This estimate coupled with the density of $\mathsf{D}(\mathrm{div}_T)$ and the fact that the map $v \mapsto \int\langle v,\nu_{\{u<t\}}\rangle\,\mathrm{dPer}(\{u<t\})$ is continuous in $L^2_{\{u<t\}}(T\mathrm{X})$ (indeed $|\nu_{\{u<t\}}| \le 1$), implies that

$$\int\langle H,w\rangle\,\mathrm{dPer}(\{u<t\},.) = 0, \qquad \forall w \in L^2_{\{u<t\}}(T\mathrm{X}) \;:\; \langle w,\nu_{\{u<t\}}\rangle = 0\,\mathrm{Per}(\{u<t\})\text{-a.e..} \tag{3.5.11}$$

Consider now the vector $H_= \in L^2_{\{u<t\}}(T\mathrm{X})$ defined by $H_= := H - \langle H,\nu\rangle\nu$ and observe that to conclude it is sufficient to show that $H_= = 0$ $\mathrm{Per}(\{u<t\})$-a.e..

We have that $\langle H_=,\nu\rangle = 0$ $\mathrm{Per}(\{u<t\})$-a.e. and that $|H_=|^2 = \langle H,H_=\rangle$ $\mathrm{Per}(\{u<t\})$-a.e.. Therefore from (3.5.11) we deduce that

$$\int |H_=|^2\mathrm{dPer}(\{u<t\}) = \int\langle H,H_=\rangle\mathrm{dPer}(\{u<t\}) = 0,$$

that is what we wanted. □

We now show a representation formula similar to the one for the tangential divergence obtain in Proposition 3.4.6.

Proposition 3.5.15 (Expression of the mean curvature vector)**.** *Suppose that $u \in \mathsf{D}(\mathcal{W}) \cap \mathsf{LIP}(\mathrm{X})$, then for a.e. $t \in \mathbb{R}$ it holds that*

$$H_{\{u<t\}} = \nu_{\{u<t\}}\left(\frac{\Delta u}{|\nabla u|}-\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|^2}\right), \tag{3.5.12}$$

where we have chosen a Borel representative of the function $\frac{\Delta u}{|\nabla u|}-\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|^2}$ (taken to be 0 when $|\nabla u| = 0$) and the negligible set of t's depends on the choice of such representative.

Remark 3.5.16. A posteriori an alternative but equivalent definition of mean curvature and tangential divergence would be to take instead *as definitions* respectively formula (3.5.12) and formula (3.4.4) and then working 'backwards' to prove the tangential integration by parts formula in (3.5.8). ■

Proof of Proposition 3.5.15. Let $\eta \in \mathsf{LIP}_c(\mathrm{X})$ and consider the vector field $v := \frac{\eta\nabla u}{|\nabla u|+\varepsilon}$. Since by assumption $\nabla u \in H^{1,2}_C(T\mathrm{X})$, from Proposition 1.4.15 we have that $v \in H^{1,2}_C \cap L^\infty(T\mathrm{X})$. Therefore we can apply Proposition 3.5.10 (recall also Proposition 3.5.9) to deduce that for a.e. $t \in \mathbb{R}$

$$\int \langle H, \mathrm{tr}_{\{u<t\}} v\rangle \,\mathrm{Per}(\{u<t\}) = \lim_{h\to 0+} \int \frac{\eta|\nabla u|}{|\nabla u|+\varepsilon}\left(\frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla|\nabla u|, \nabla u\rangle}{|\nabla u|^2}\right) \mathrm{d}\mu_t^h. \tag{3.5.13}$$

From Proposition 3.5.14 we have that $H_{\{u<t\}} = \bar{H}_{\{u<t\}}\nu_{\{u<t\}}$ for some $\bar{H}_{\{u<t\}} \in L^2(\mathrm{Per}(\{u<t\}))$. Moreover from (3.5.1) $\nu_{\{u<t\}} = \frac{\mathrm{tr}_{\{u<t\}}\nabla u}{\mathrm{tr}_{\{u<t\}}(|\nabla u|)}$. Therefore (omitting the subscript $\{u<t\}$)

$$\begin{aligned}\int \langle H, \mathrm{tr}(v)\rangle \,\mathrm{Per}(\{u<t\}) &= \int \frac{\bar{H}}{\mathrm{tr}(|\nabla u|)} \langle \mathrm{tr}\nabla u, \mathrm{tr}(v)\rangle \,\mathrm{Per}(\{u<t\}) \overset{(1.5.1)}{=} \int \frac{\bar{H}\mathrm{tr}(\langle \nabla u, v\rangle)}{\mathrm{tr}(|\nabla u|)}\,\mathrm{Per}(\{u<t\})\\ &= \int \frac{\bar{H}\mathrm{tr}(|\nabla u|^2\eta(|\nabla u|+\varepsilon)^{-1})}{\mathrm{tr}(|\nabla u|)}\,\mathrm{Per}(\{u<t\})\\ &\overset{(1.2.14)}{=} \int \eta\frac{\bar{H}\mathrm{tr}(|\nabla u|)}{\mathrm{tr}(|\nabla u|+\varepsilon)}\,\mathrm{Per}(\{u<t\}),\end{aligned}$$

where we have used that η is continuous. Fix a Borel representative g of the function $\frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla|\nabla u|, \nabla u\rangle}{|\nabla u|^2}$ taken to be 0 when $|\nabla u| = 0$. Observe that, since $u \in \mathsf{D}(\mathcal{W})$, $g \in L^2(|\nabla u|\mathfrak{m})$. Therefore, fixing also a representative for ∇u, from the coarea formula we have that for a.e. $t \in \mathbb{R}$

$$\lim_{h\to 0+} \int \frac{\eta|\nabla u|}{|\nabla u|+\varepsilon}\left(\frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla|\nabla u|, \nabla u\rangle}{|\nabla u|^2}\right) \mathrm{d}\mu_t^h = \int \frac{\eta|\nabla u|g}{|\nabla u|+\varepsilon}\,\mathrm{dPer}(\{u<t\}).$$

Combining the two identities above we obtain that for a.e. $t \in \mathbb{R}$

$$\int \eta\frac{\bar{H}\mathrm{tr}(|\nabla u|)}{\mathrm{tr}(|\nabla u|+\varepsilon)}\,\mathrm{Per}(\{u<t\}), = \int \frac{\eta|\nabla u|g}{|\nabla u|+\varepsilon}\,\mathrm{dPer}(\{u<t\}).$$

Recalling that for every representative of $|\nabla u|$ it holds that $|\nabla u| \neq 0$ $\mathrm{Per}(\{u<t\})$-a.e. for a.e. $t \in \mathbb{R}$, we can now send $\varepsilon \to^+$ and use dominated convergence (recall that $g \in L^2(\mathrm{Per}(\{u<t\}))$ for a.e. $t \in \mathbb{R}$) to get

$$\int \eta\bar{H}\,\mathrm{Per}(\{u<t\}), = \int \eta g\,\mathrm{dPer}(\{u<t\}), \qquad \text{for a.e. } t \in \mathbb{R}.$$

The conclusion now follows recalling that there exists a countable family of functions in $\mathsf{LIP}_c(\mathrm{X})$ which is dense in $L^2(\mathrm{Per}(\{u<t\}))$ for every $t \in \mathbb{R}$ (see Proposition 1.1.2). □

As an immediate corollary of the above proposition we obtain the following formula, which connects the mean curvature defined in this section with the Willmore functional defined in the Section 3.2.

Corollary 3.5.17. *Suppose that $u \in \mathsf{D}(\mathcal{W}) \cap \mathsf{LIP}_c(\mathrm{X})$, then for a.e. $t \in \mathbb{R}$ it holds that*

$$\|H\|_{L^2_{\{u<t\}}(T\mathrm{X})} = \lim_{h\to 0^+}\left(\frac{1}{2h}\int_{\{t-h<u<t+h\}}\left|\frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla|\nabla u|, \nabla u\rangle}{|\nabla u|^2}\right|^2 |\nabla u|\mathrm{d}\mathfrak{m}\right)^{\frac{1}{2}} \tag{3.5.14}$$

and in particular

$$\mathcal{W}(u) = \int \left|\frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla|\nabla u|, \nabla u\rangle}{|\nabla u|^2}\right|^2 |\nabla u|\mathrm{d}\mathfrak{m} = \int_{\mathbb{R}}\int |H|^2\,\mathrm{dPer}(\{u<t\})\mathrm{d}t. \tag{3.5.15}$$

We already observed that a sublevel that admits a mean curvature vector has also finite relaxed Willmore energy (see Remark 3.5.2). A natural question is whether the relaxed Willmore energy coincides with the L^2-norm of the mean curvature. Combining (3.5.14) and Proposition 3.2.8 we can prove one of the two inequalities.

Corollary 3.5.18. *Suppose that $u \in \mathsf{D}(\mathcal{W})$, then*

$$\overline{\mathcal{W}}(\{u<t\}) \le \|H\|_{L^2_{\{u<t\}}(T\mathrm{X})}, \quad \textit{for a.e. } t \in \mathbb{R}. \tag{3.5.16}$$

We now prove that the tangential divergence satisfies the expected Leibniz rule. To it state we need first to give a notion of tangential gradient:

Definition 3.5.19 (Tangential gradient)**.** Let $E \subset \mathrm{X}$ be a set of finite perimeter and let $f \in \mathsf{D}_{L^2}(\Delta) \cap W^{1,2}(\mathrm{X}) \cap \mathsf{LIP}_b(\mathrm{X})$. We define its tangential gradient $\nabla_T f \in L^2_E(T\mathrm{X})$ as

$$\nabla_T f := \mathrm{tr}\nabla f - \langle \mathrm{tr}\nabla f, \nu_E\rangle.$$

Proposition 3.5.20 (Leibniz rule). *Let $u \in \mathsf{LIP}_c(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$ and suppose that $v \in \mathsf{D}(\mathrm{div}_{T,\{u<t\}})$ for some $t \in \mathbb{R}$. Then for every $f \in \mathsf{D}_{L^2}(\Delta) \cap W^{1,2}(\mathrm{X}) \cap \mathsf{LIP}_b(\mathrm{X})$ it holds that $fv \in \mathsf{D}(\mathrm{div}_{T,\{u<t\}})$ and*

$$\mathrm{div}_T(fv) = f\mathrm{div}_T(v) + \langle \nabla_T f, \mathrm{tr}_{\{u<t\}} v\rangle, \quad \mathrm{Per}(\{u<t\},.)\text{-a.e.}. \tag{3.5.17}$$

PROOF. Recall that by Proposition 1.4.15 and by (1.4.14) we have that $\nabla f,\ fv \in H^{1,2}_C \cap L^\infty(T\mathrm{X})$. We begin by showing that $fv \in \mathsf{D}(\mathrm{div}_T)$, which amounts to verify conditions 1 and 2 in Definition 3.4.1. We start with 1. From the chain rule for the divergence operator $vf \in \mathsf{D}_{L^2}(\mathrm{div})$

$$\varlimsup_{h\to 0^+} \int \mathrm{div}(fv)^2 \mathrm{d}\mu_t^h \le 2\|f\|^2_{L^\infty} \varlimsup_{h\to 0^+} \int \mathrm{div}(v)^2 \mathrm{d}\mu_t^h + 2(\mathrm{Lip} f)^2 \|v\|^2_{L^\infty} \varlimsup_{h\to 0^+} \int \mathrm{d}\mu_t^h < +\infty,$$

where we used (3.3.2). Analogously, from the Leibniz rule for the covariant derivative (1.4.13) we obtain

$$\sup_{0<h<1} \int |\nabla(fv)|^2 \mathrm{d}\mu_t^h \le 2\|f\|^2_\infty \sup_{0<h<1} \int |\nabla v|^2 \mathrm{d}\mu_t^h + 2(\mathrm{Lip} f)^2 \|v\|^2_\infty \sup_{0<h<1} \int |\mathsf{id}|^2_{HS} \mathrm{d}\mu_t^h < +\infty,$$

this proves point 1. We pass to point 2. From the Leibniz rule for the divergence and the covariant derivative ((1.4.13)) we obtain the following identity

$$\begin{aligned} \mathrm{div}(fv) - \nabla(fv)\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) &= f\left(\mathrm{div}(v) - \nabla v\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right)\right) \\ &+ \langle v, \nabla f\rangle + |\nabla u|^{-2}\langle v, \nabla u\rangle\langle \nabla f, \nabla u\rangle, \quad \mathfrak{m}\text{-a.e.}, \end{aligned} \tag{3.5.18}$$

where both sides are assumed to be zero whenever $|\nabla u| = 0$. From Proposition 1.4.16 follows that all three of the functions $\langle v, \nabla f\rangle, \langle v, \nabla u\rangle, \langle \nabla f, \nabla u\rangle$ belong to $W^{1,2} \cap L^\infty(\mathrm{X})$ and in particular they all admits a quasi-continuous representative that we denote with the same symbol. Therefore setting $g \coloneqq |\nabla u|^{-2}\langle v, \nabla u\rangle\langle \nabla f, \nabla u\rangle \in L^\infty(\mathrm{Cap})$ (taken to be 0 whenever $|\nabla u| = 0$) we have that $\varphi\langle v, \nabla f\rangle, \varphi g|\nabla u|^2 \in \mathcal{QC} \cap L^\infty(\mathrm{X})$, for every $\varphi \in C_b(\mathrm{X})$. Therefore arguing as in the proof of Proposition 3.5.14 using Theorem 3.3.4 we infer

$$\begin{aligned} &\lim_{h\to 0^+} \int \varphi\Big(\langle \nabla f, v\rangle - |\nabla u|^{-2}\langle \nabla f, \nabla u\rangle\langle v, \nabla u\rangle\Big)\mathrm{d}\mu_t^h = \\ &\qquad \int \varphi\left(\langle \mathrm{tr}\nabla f, \mathrm{tr}(v)\rangle - \langle \mathrm{tr}\nabla f, \nu_{\{u<t\}}\rangle\langle \mathrm{tr}(v), \nu_{\{u<t\}}\rangle\right)\mathrm{d}\mathrm{Per}(\{u<t\}), \quad \forall \varphi \in C_b(\mathrm{X}), \end{aligned}$$

where tr stands for $\mathrm{tr}_{\{u<t\}}$ and where we have used also (3.5.1). Moreover, since for v the limit in (3.4.2) exists for all $\varphi \in C_b(\mathrm{X})$, we also have that

$$\lim_{h\to 0+} \int \varphi f\left(\mathrm{div}(v) - \nabla v\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right)\right)\mathrm{d}\mu_t^h = \mathrm{TanDiv}_v(\varphi f) = \int \varphi f \mathrm{div}_T(v)\, \mathrm{d}\mathrm{Per}(\{u<t\}).$$

The two above limits combined with (3.5.18) prove point 2 and thus that fv belongs to the domain of the tangential divergence.

Finally to see that (3.5.17) holds it is enough to observe that $\langle \overline{\nabla} f, \mathrm{tr} v\rangle - \langle \nabla f, \nu\rangle\langle \mathrm{tr} v, \nu\rangle = \langle \nabla_T f, \mathrm{tr}_{\{u<t\}} v\rangle$ $\mathrm{Per}(\{u<t\})$-a.e. and use again the above limits together with (3.4.2) for the vector field fv. □

We end stating an integration by parts formula which is an immediate consequence of (3.5.8) and the Leibniz rule for the divergence proved above.

Proposition 3.5.21 (Tangential-integration by parts formula). *Let $u \in W^{1,2}(\mathrm{X}) \cap \mathsf{LIP}(\mathrm{X}) \cap \mathsf{D}_{L^2}(\Delta)$ and let $t \in \mathbb{R}$ be such that $\{u<t\}$ admits a mean curvature H. Then*

$$\begin{aligned} \int \langle \nabla_T f, \mathrm{tr}_{\{u<t\}} v\rangle\, \mathrm{d}\mathrm{Per}(\{u<t\}) = -\int f \mathrm{div}_T(v)\, \mathrm{d}\mathrm{Per}(\{u<t\}) + \int f\langle \mathrm{tr}_{\{u<t\}} v, H\rangle\, \mathrm{d}\mathrm{Per}(\{u<t\}) \\ \forall f \in \mathsf{Test}(\mathrm{X}),\ \forall v \in \mathsf{D}(\mathrm{div}_T). \end{aligned} \tag{3.5.19}$$

As an immediate corollary of (3.5.19) we get that, if $f, g \in \mathsf{Test}(\mathrm{X})$ are such that $\nabla f, \nabla g \in \mathsf{D}(\mathrm{div}_{T,\{u<t\}})$ and $f = g$ $\mathrm{Per}(\{u<t\})$-a.e., then $\nabla_T f = \nabla_T g$.

3.5.1. The local version. In all the above discussion we defined the mean curvature vector for sublevels for functions defined globally. However when we will prove Willmore inequalities in Section 3.8, we will need to consider level sets of the electrostatic potential, which is not globally defined. This is a minor issue, since all the analysis developed in the previous sections was local in nature. More precisely we can give the following definition:

Definition 3.5.22 (Mean curvature vector - local case)**.** Let $\Omega \subset \mathrm{X}$ be open and let $u \in \mathsf{LIP}_{\mathsf{loc}}(\Omega) \cap \mathsf{D}_{L^2_{\mathsf{loc}}}(\Delta, \Omega)$. We say that the sublevel $\{u < t\}$ admits a mean curvature vector if for some (and thus any) $\tilde{u} \in \mathsf{D}_{L^2}(\Delta) \cap \mathsf{LIP}_c(\mathrm{X})$ satisfying $\{t - \delta < \tilde{u} < t + \delta\} = \{t - \delta < u < t + \delta\}$ for some $\delta > 0$ and with $\tilde{u} = u$ in this set, the sublevel $\{\tilde{u} < t\}$ admits a mean curvature vector (in the sense of Def. 3.5.8 with $\tilde{u}$). In this case we define the mean curvature vector $H \in L^2_{\{u<t\}}(T\mathrm{X})$ as the one given by $\tilde{u}$.

Remark 3.5.23. The fact that the above definition is well-posed follows from Proposition 3.5.11, which in particular shows that the mean curvature vector H is independent of the function $\tilde{u}$. ■

Proposition 3.5.24 (Existence mean curvature - local case)**.** *Let $\Omega \subset \mathrm{X}$ be open and let $u \in \mathsf{LIP}_{\mathsf{loc}}(\Omega) \cap \mathsf{D}_{L^2_{\mathsf{loc}}}(\Delta, \Omega)$. Let $t, s > 0$ be such that $\{s < u < t\}$ is bounded, $\mathsf{d}(\{u < t\}, \Omega^c) > 0$ and $\mathrm{X} \setminus \{u < t\}$ is bounded. Suppose furthermore that*

$$\int_{\{s<u<t\}} \left| \frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla |\nabla u|, \nabla u \rangle}{|\nabla u|^2} \right|^2 |\nabla u| \mathrm{d}\mathfrak{m} < +\infty.$$

Then for a.e. $r \in (s, t)$ the sublevel $\{u < r\}$ admits a mean curvature vector H satisfying

$$H_{\{u<r\}} = \nu_{\{u<r\}} \left(\frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla |\nabla u|, \nabla u \rangle}{|\nabla u|^2} \right). \tag{3.5.20}$$

Proof. Fix $\varepsilon > 0$ small. Consider an increasing function $\varphi \in C^2(\mathbb{R})$ satisfying $\varphi \equiv t$ in $[t, \infty)$, $\varphi \equiv 0$ in $(-\infty, s)$, such that $\varphi(t) = t$ in $(s + \varepsilon, t - \varepsilon)$ a neighbourhood of t and set

$$\tilde{u} := \begin{cases} \varphi(u), & \text{in } \Omega, \\ t, & \text{in } \Omega^c. \end{cases}$$

The assumptions grant that $\tilde{u} \in \mathsf{LIP}_c(\mathrm{X})$. Moreover it holds that $\tilde{u} \in \mathsf{D}_{L^2}(\Delta)$. Indeed from chain rule for the Laplacian (Proposition 1.2.11) and the locality of the Laplacian we have that $\varphi(u) \in \mathsf{D}_{L^2_{\mathsf{loc}}}(\Delta, \Omega)$ with $\Delta\varphi(u) = 0$ $\mathfrak{m}$-a.e. in $\Omega \setminus \{s + \varepsilon < u < t - \varepsilon\}$. Then, since by hypothesis $\mathsf{d}(\{u < t\}, \Omega^c) > 0$ we can argue exactly as in the end of the proof of Theorem 3.2.6 to show that $\tilde{u} \in \mathsf{D}_{L^2}(\Delta)$ with $\Delta\tilde{u} = \chi_\Omega \Delta\varphi(u)$. Finally from Lemma 3.2.7 (observe that in there the harmonicity is used only to show that the integral is finite) we see that

$$\mathcal{W}(\tilde{u})^2 = \int_{\{s+\varepsilon<u<t-\varepsilon\}} \left| \frac{\Delta u}{|\nabla u|} - \frac{\langle \nabla |\nabla u|, \nabla u \rangle}{|\nabla u|^2} \right|^2 |\nabla u| \mathrm{d}\mathfrak{m} < +\infty,$$

(see also the proof of Theorem 3.2.6 for a similar computation). Therefore $\tilde{u} \in \mathsf{D}(\mathcal{W})$ and from Proposition 3.5.9 we deduce that $\{\tilde{u} < r\}$ admits a mean curvature vector for a.e. $r \in \mathbb{R}$. Moreover $\tilde{u}$ is admissible in Definition 3.5.22 for every $\{u < r\}$ with $r \in (s + \varepsilon, t - \varepsilon)$. This proves the first part of the statement. Formula (3.5.10) is then immediate from (3.5.12) and the locality of the gradient and Laplacian. □

Remark 3.5.25. Proposition 3.5.24 is tailored to fit the case of an electrostatic potential, which we will need to consider in the sequel. However it is clear that analogous suitably modified statements holds in different conditions, e.g. in the case $s, t < 0$ or if we replace sublevels with superlevels. ■

3.6. Electrostatic capacity

In this short section we introduce the notion of electrostatic capacity and deduce some of its properties. We assume that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is a non-parabolic $\mathrm{RCD}(0, N)$ space.

Definition 3.6.1 (Electrostatic capacity)**.** Suppose that $E \subset \mathrm{X}$ admits an electrostatic potential u (see Def. 2.2.11), we define its electrostatic capacity as

$$\mathsf{eCap}(E) := \int_{\mathrm{X} \setminus \bar{E}} |\nabla u|^2 \, \mathrm{d}\mathfrak{m}.$$

We start with a basic scaling property:

Lemma 3.6.2. *Suppose that $E \subset \mathrm{X}$ admits an electrostatic potential u, then*

$$\mathsf{eCap}(\{u < t\}) = \frac{\mathsf{eCap}(E)}{t}, \quad \forall t \in (0, 1].$$

Proof. By definition of electrostatic potential we have that u/t is the electrostatic potential for $\{u < t\}$, hence by coarea

$$\mathsf{eCap}(\{u < t\}) = t^{-2} \int_{\{u<t\}} |\nabla u|^2 \, \mathrm{d}\mathfrak{m} = t^{-2} \int_0^t \int |\nabla u| \, \mathrm{d}\mathrm{Per}(\{u < s\} \, \mathrm{d}s.$$

However in Proposition 2.4.7 we showed that there exists a constant $D>0$ such that $\int |\nabla u|\,\mathrm{dPer}(\{u<t\})=D$ for a.e. $t\in(0,1)$, from which the conclusion immediately follows. $\square$

The following technical lemma provides a useful formula for the electrostatic capacity. Observe that morally we could take u instead of $\tilde u$ in the statement, however from a technical point of view we have defined the trace only for everywhere defined functions.

Lemma 3.6.3. *Whenever* $\mathsf{eCap}(E)$ *is defined it holds that*

$$\mathsf{eCap}(E)=\int \langle \mathrm{tr}_{B_R(x)}\nabla\tilde u,\nu\rangle\,\mathrm{dPer}(B_R(x),.),$$

for every ball $B_R(x)$ with finite perimeter and such that $E\subset\subset B_R(x)$, where $\tilde u\in\mathsf{Test}(\mathrm{X})$ is any function such that $\tilde u=u$ in a neighbourhood of $\partial B_R(x)$.

Proof. As proven in Proposition 2.4.7, there exists a constant $D>0$ such that $\int|\nabla u|\,\mathrm{dPer}(\{u<t\})=D$ for a.e. $t\in(0,1)$, where we are fixing a representative of $|\nabla u|$. From the coarea formula follows immediately that $D=\int_{\mathrm{X}\setminus\bar E}|\nabla u|^2=\mathsf{eCap}(E)$. Let now $B_R(x)$ be as in the statement. From the definition of electrostatic potential, there exists $\delta>0$ small enough so that $\partial B_R(x)\subset\{u<1-2\delta\}$ and in particular $B_R(x)^c\subset\{u<1-2\delta\}$. Consider a cut-off function (recall Proposition 1.4.9) $\eta\in\mathsf{Test}(\mathrm{X})$ such that $\eta=1$ in $\{1-2\delta\le u\le 1-\delta\}\cap B_{R+1}(x)$ and $\operatorname{supp}\eta\subset\subset\Omega$. Define the function $\tilde u:=\eta u\in\mathsf{Test}(\mathrm{X})$ (recall (1.4.8)) that clearly satisfies $u=\tilde u$ in a neighbourhood of $\partial B_R(x)$. Moreover by the locality of the Laplacian we have that $\tilde u$ is harmonic in $\{1-2\delta<u<1-\delta\}\cap B_{R+1}(x)$. We observe that it is enough to prove the statement with this particular $\tilde u$, indeed then we would conclude recalling Proposition 1.2.21.

We claim that:

$$\exists\,\bar t\in(1-2\delta,1-\delta)\ :\ \mathsf{eCap}(E)=\int_{\{u<\bar t\}}\Delta\tilde u\,\mathrm{d}\mathfrak{m}. \tag{3.6.1}$$

To show this we define the function $f(t):=\int\langle|\nabla u|^{-1}\nabla\tilde u,\nabla u\rangle\,\mathrm{dPer}(\{u<t\})$ (where we have fixed a Borel representative for $\langle|\nabla u|^{-1}\nabla\tilde u,\nabla u\rangle$ taken to be 0 whenever $|\nabla u|=0$). From the coarea formula (2.4.4) we can deduce that $f\in W^{1,1}_{\mathsf{loc}}(0,1)$ and, denoting again by f its continuous representative, also that $f(t)-f(s)=\int_{\{s<u<t\}}\Delta\tilde u\,\mathrm{d}\mathfrak{m}$, for every $s,t\in(0,1)$ with $s<t$. Moreover $f(s)=0$ for every s close enough to zero. Therefore we $f(t)=\int_{\{u<t\}}\Delta\tilde u\,\mathrm{d}\mathfrak{m}$ for every $t\in(0,1)$. Finally from the locality of the gradient we have that $f(t)=\int|\nabla u|\,\mathrm{dPer}(\{u<t\})=\mathsf{eCap}(E)$ for a.e. $t\in(1-2\delta,1-\delta)$, from which (3.6.1) follows. Recalling that $\tilde u$ is harmonic in $\{1-2\delta<u<1-\delta\}$ and that $B_R(x)^c\subset\{u<1-2\delta\}$ we can write

$$\begin{aligned}0=\int_{\{u<\bar t\}\cap B_R(x)}\Delta\tilde u\,\mathrm{d}\mathfrak{m}&=\int_{\{u<\bar t\}}\Delta\tilde u\,\mathrm{d}\mathfrak{m}-\int_{B_R(x)^c}\Delta\tilde u\,\mathrm{d}\mathfrak{m}\\&\overset{(3.6.1)}{=}\mathsf{eCap}(E)-\int_{B_R(x)}\Delta\tilde u\,\mathrm{d}\mathfrak{m},\end{aligned}$$

where we have used that $\int\Delta\tilde u\,\mathrm{d}\mathfrak{m}=0$. Applying the Gauss-Green formula (1.5.2) we reach the sought conclusion. $\square$

In Theorem 2.2.13 it is shown that the electrostatic capacity is less or equal than the variational capacity: $\mathsf{eCap}(E)\le\overline{\mathrm{Cap}}(E)$. Here prove that under the Euclidean-volume growth assumption they actually coincide.

Proposition 3.6.4. *Suppose that $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(0,N)$ space with $N>2$ and Euclidean volume growth. Then for every set $E\subset\mathrm{X}$ that admits an electrostatic potential it holds*

$$\mathsf{eCap}(E)=\overline{\mathrm{Cap}}(E)$$

(see Def. 1.2.22 for the definition of $\overline{\mathrm{Cap}}$).

Proof. From Theorem 2.2.13 we already know that $\mathsf{eCap}(E)\le\overline{\mathrm{Cap}}(E)$ is always verified, hence we need only to show the converse inequality. Fix $x_0\in\mathrm{X}$. We consider cut-off functions $\eta_k\in\mathsf{LIP}_c(\mathrm{X})$, $k\in\mathbb{N}$, such that $0\le\eta_k\le1$, $\eta_k=1$ in $B_{2^k}(x_0)$, $\mathrm{supp}(\eta_k)\subset B_{2^{k+1}}(x_0)$ and $\mathrm{Lip}(\eta_k)\le 2^{k-1}$. Then we observe that $|D\eta_k|\le 2\mathsf{d}_{x_0}^{-1}$ $\mathfrak{m}$-a.e.. From the Euclidean volume growth assumption and (2.4.2) we have that $u\le C\mathsf{d}_{x_0}^{2-N}$ outside a fixed large ball B. This implies, again from the Euclidean volume growth assumption, that $u^2\mathsf{d}_{x_0}^{-2}\in L^1(B^c)$. Define now the functions $u_k:=(u\wedge(1-1/k)\eta_k\in\mathsf{LIP}_c(\mathrm{X})$ (recall that

u is locally Lipschitz far from ∂E). Therefore from the definition of $\overline{\mathrm{Cap}}(E)$ and locality of the weak upper gradient

$$\begin{aligned}\overline{\mathrm{Cap}}(E) \le \int |Du_k|^2 \, \mathrm{d}\mathfrak{m} &= \int_{\{u<1-1/k\}\cap B_{2^k}(x_0)} |Du|^2 \, \mathrm{d}\mathfrak{m} + \int_{B_{2^k}(x_0)^c} |D(u\eta_k)|^2 \, \mathrm{d}\mathfrak{m} \\ &\le \int_{\{u<1-1/k\}\cap B_{2^k}(x_0)} |Du|^2 \, \mathrm{d}\mathfrak{m} + 2\int_{B_{2^k}(x_0)^c} |Du|^2 + C\mathsf{d}_{x_0}^{-2}u^2 \, \mathrm{d}\mathfrak{m}.\end{aligned}$$

Sending $k \to +\infty$ and recalling that $u^2\mathsf{d}_{x_0}^{-2}$ is integrable outside a large ball, we get the desired conclusion. □

3.7. Asymptotics for the electrostatic potential

The ultimate goal of this section is to prove an asymptotic expansion (at infinity) for the electrostatic potential defined on an $\mathrm{RCD}(0, N)$ space with Euclidean-volume growth.

3.7.1. Asymptotics for the Green's function. In this short section we prove an expansion at infinity for the Green's function, which extends the analogous result on smooth Riemannian manifolds proved in [**85**] (se also [**167**]). The argument is the same of the local asymptotic recently derived in [**54**] (see Remark 2.5 therein). For completeness we report here most of the argument, which is base on a blow-down procedure.

We will assume that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(0, N)$ space with $N > 2$ and Euclidean-volume growth and we will denote by $G(x, y)$ its Green's function (recall Section 2.1.1)

Proposition 3.7.1. *Let* X *be an* $\mathrm{RCD}(0, N)$ *with* $N \in (2, \infty)$ *and Euclidean volume growth and let* $R_i \to +\infty$ *be a sequence of radii such that*

$$\mathrm{X}_i := (\mathrm{X}, R_i^{-1}\mathsf{d}, R_i^{-N}\mathfrak{m}, x_0) \overset{pmGH}{\to} (Y, \rho, \mu, p).$$

Then, denoted by G^Y *the Green's function on* Y*, it holds*

$$\lim_{i\to+\infty} R_i^{N-2}G(x_0, x_i) = G^Y(p, y), \tag{3.7.1}$$

for $\mathrm{X}_i \ni x_i \to y \in Y$ *and* $y \neq p$.

Proof. Called p_t^i the heat kernel of the space X_i, it holds that $p_t^i(x, y) = R_i^N p_{R_i^2 t}(x, y)$ for every $x, y \in \mathrm{X}$. In particular, denoted by G^i, the Green's function for X_i we have that $G^i(x, y) = R_i^{N-2}G(x, y)$ for every $x, y \in \mathrm{X}$. Therefore (3.7.1) would follow after we prove that $G^i(x_0, x_i) \to G^Y(x_0, y)$ for every $\mathrm{X}_i \ni x_i \to y \in Y$ and $y \neq p$. To show this we first observe that

$$p_t^i(x_0, x_i) = R_i^N p_{R_i^2 t}(x_0, x_i) \le C(N)\frac{R_i^N}{\mathfrak{m}(B_{R_i\sqrt{t}}(x_i))}e^{\left(\frac{-\mathsf{d}(x_0,x_i)}{R^i}\right)^2\frac{1}{5t}} \le C(N)\frac{e^{\frac{-\mathsf{d}_Y(x_0,y)^2}{5t}}}{\mathsf{AVR}(\mathrm{X})t^{N/2}},$$

where we have used the heat kernel bounds (1.3.15) and the fact that $r^{-N}\mathfrak{m}(B_r(x)) \ge \mathsf{AVR}(\mathrm{X})$ for every $x \in \mathrm{X}$, $r > 0$, thanks to the Bishop-Gromov inequality. Arguing as in [**54**] we combine the pointwise-convergence of the heat-kernel under pmGH-convergence (see [**25**] and also [**54**]) with the dominated converge theorem (justified by the above estimate) to finally deduce that $G^i(x_0, x_i) = \int_0^\infty p_t^i(x_0, x_i)\, \mathrm{d}t \to \int_0^\infty p_t^Y(x_0, y) = G^Y(x_0, y)$, where p_t^Y denotes the heat kernel for Y. □

The following simple lemma is a well-known consequence of separation of variables. We refer to [**102**] for a proof in the case of Ricci-limit spaces, which easily adapts to this setting (see also [**54**, Lemma 2.7] Lemma 2.7 and [**98**] for analogous computations).

Lemma 3.7.2. *Suppose that* $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ *is an* $\mathrm{RCD}(0, \mathrm{N})$ *space that is also an* N*-cone (see Section 1.3.4) with* $N > 2$. *Then, denoted by* p *the tip of* Y *and by* G^Y *the Green's function on* Y*, we have*

$$G^Y(p, y) = \frac{\mathsf{d}_Y(p, y)^{2-N}}{N(N-2)\mathfrak{m}_Y(B_1(p))}, \quad \textit{for every } y \in Y. \tag{3.7.2}$$

Combining Proposition 3.7.1 and Lemma 3.7.2 we obtain the following asymptotic expansion at infinity for the Green's function.

Theorem 3.7.3. *Let* X *be an* $\mathrm{RCD}(0, N)$ *space with Euclidean volume growth. Then*

$$\lim_{y\to+\infty} \frac{G(x, y)}{\mathsf{d}(x, y)^{2-N}} = \frac{1}{(N-2)\sigma_{N-1}\mathsf{AVR}(\mathrm{X})}, \quad \textit{for every } x \in \mathrm{X}. \tag{3.7.3}$$

Proof. Fix $x \in \mathrm{X}$. and consider a sequence $y_i \in \mathrm{X}$ such that $\mathsf{d}(x, y_i) \to +\infty$. Setting $R_i := \mathsf{d}(x, y_i)$, there exists a subsequence i_k such that $X_k := (\mathrm{X}, R_{i_k}^{-1}\mathsf{d}, R_{i_k}^{-N}\mathfrak{m}, x)$ converges in the pmGH-topology to an $\mathrm{RCD}(0,N)$ space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y, p)$ and such that $y_{i_k} \to y \in Y$. From the Euclidean volume growth assumption we have that for every $r > 0$

$$\mathfrak{m}_Y(B_r(p)) = \lim_i R_{i_k}^{-N}\mathfrak{m}(B_{R_{i_k}r}(x_0)) = r^N\omega_N \mathsf{AVR}(\mathrm{X}).$$

In particular from the volume cone to metric cone theorem [**95**] we deduce that Y is an Euclidean N-cone. Therefore combining (3.7.1) with (3.7.2) we obtain

$$\lim_{i\to+\infty} \mathsf{d}(x, y_{i_k})^{N-2}G(x, y_{i_k}) = \frac{\mathsf{d}_Y(p, y)^{2-N}}{N(N-2)\mathfrak{m}_Y(B_1(p))} = \frac{1}{N(N-2)\omega_N \mathsf{AVR}(\mathrm{X})}.$$

Since the last term of the above identity is independent both of the initial subsequence and of the sequence, it directly implies (3.7.3). □

3.7.2. Expansion of the electrostatic potential. The main goal is to prove the following expansion result of the electrostatic potential, proved in [**2**] in the setting of Riemannian manifolds.

Theorem 3.7.4. *Let* X *be an* $\mathrm{RCD}(0,N)$ *space with Euclidean volume growth. Let* u *be the electrostatic potential of a bounded open set* $E \subset \mathrm{X}$. *Then*

$$\lim_{y\to+\infty} \frac{u(x)}{\mathsf{d}(x, x_0)^{2-N}} = \frac{\overline{\mathrm{Cap}}(E)}{(N-2)\sigma_{N-1}\mathsf{AVR}(\mathrm{X})}, \quad \textit{for every } x_0 \in \mathrm{X}. \tag{3.7.4}$$

We will follow the idea used in [**2**], which are based on representation results for harmonic functions derived in [**167**] (improving the previous [**85**]).

The following lemma make rigorous the fact that G is the fundamental solution for the Laplacian (see also [**58**]).

Lemma 3.7.5. *Let* $u \in L^2(\mathfrak{m}) \cap \mathsf{D}_{L^\infty\cap L^1}(\Delta)$. *Then*

$$u(x) = -\int G(x, y)\Delta u(y)\, d\mathfrak{m}(y), \quad \textit{for } \mathfrak{m}\textit{-a.e. } x \in X. \tag{3.7.5}$$

Proof. From the definition of $G(x, y)$ we have

$$\int G(x, y)\Delta u(y)\, d\mathfrak{m}(y) = \int \int_0^{+\infty} p_t(x, y)\mathrm{d}t\, \Delta u(y)\, d\mathfrak{m}(y) = \int_0^{+\infty} \int p_t(x, y)\Delta u(y)\, d\mathfrak{m}(y)\, \mathrm{d}t.$$

Observe that since $\Delta u \in L^1 \cap L^\infty(\mathfrak{m})$, $G(x, y) \in L^1_{\mathsf{loc}}(\mathfrak{m}(y)) \cap L^\infty_y(\mathrm{X} \setminus B_1(x))$, we have $\int |G(x, y)\Delta u(y)|\, d\mathfrak{m}(y) < +\infty$, hence all the integrals above are well defined and the application of Fubini's theorem is justified (recall also that $p_t \geq 0$). In particular we obtain

$$\int G(x, y)\Delta u(y)\, d\mathfrak{m}(y) = \int_0^{+\infty} h_t(\Delta u)(x)\, \mathrm{d}t, \quad \text{for every } x \in \mathrm{X}. \tag{3.7.6}$$

We now make the intermediate claim that

$$h_t u(x) \to 0 \quad \text{as } t \to +\infty, \quad \text{for every } x \in \mathrm{X}, \tag{3.7.7}$$

where we are taking as representative of $h_t u(x)$ precisely $\int u p_t(x, y)\mathrm{d}\mathfrak{m}(y)$. To see this we fix $x \in \mathrm{X}$, $\varepsilon > 0$ arbitrary and choose $R > 0$ such that $\int_{\mathrm{X}\setminus B_R(x)} u^2 \mathrm{d}\mathfrak{m} < \varepsilon$. From the heat kernel estimates (1.3.15) and the fact that $\int p_t(x, y)\mathrm{d}\mathfrak{m}(y) = 1$, we have for every $t \geq 1$

$$\begin{aligned}
h_t(u)(x) &= \int_{B_R(x)} p_t(x, y)u(y)\, \mathrm{d}\mathfrak{m} + \int_{\mathrm{X}\setminus B_R(x)} p_t(x, y)u(y)\, \mathrm{d}\mathfrak{m} \\
&\leq \|u\|_{L^2(\mathfrak{m})} \int_{B_R(x)} p_t(x, y)^2\, \mathrm{d}\mathfrak{m} + \|p_t(x, .)\|_{L^2(\mathfrak{m})} \int_{\mathrm{X}\setminus B_R(x)} u^2 \mathrm{d}\mathfrak{m} \\
&\leq C(N)\frac{\mathfrak{m}(B_R(x))\|u\|_{L^2(\mathfrak{m})}}{\mathfrak{m}(B_{\sqrt{t}}(x))} + \frac{C(N)}{\mathfrak{m}(B_1(x))}\varepsilon.
\end{aligned}$$

Letting $t \to +\infty$ and from the arbitrariness of ε we deduce the claim.

We have that $\partial_t h_t(u) = h_t(\Delta u)$ for every $t \in [0, \infty)$, the derivative being in the L^2-sense. In particular for every $T > 0$

$$h_T(x) - u(x) = \int_0^T h_t(\Delta u)(x)\, \mathrm{d}t \quad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}.$$

Hence fixed a sequence $T_n \to +\infty$ we have that

$$h_{T_n}(x) - u(x) = \int_0^{T_n} h_t(\Delta u)(x)\,\mathrm{d}t \quad \text{for every } n, \quad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}.$$

Letting $T_n \to +\infty$ and using (3.7.7) and (3.7.6) we obtain the desired conclusion. $\square$

Restricting to the case of an electrostatic potential allows to get a pointwise version of the above integral formula.

Corollary 3.7.6. *Suppose that* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is an* $\mathrm{RCD}(0, N)$ *space with* $N > 2$ *and Euclidean-volume growth. Let* u *be an electrostatic potential of some open and bounded set* $E \subset\subset B_R(x_0)$*. Suppose that* $\tilde{u} \in \mathsf{LIP}_b(\mathrm{X}) \cap \mathsf{D}_{L^\infty \cap L^1}(\Delta)$ *satisfies* $u = \tilde{u}$ *in* $\mathrm{X} \setminus B_R(x_0)$*, then*

$$u(x) = -\int G(x,y)\Delta\tilde{u}(y)\,d\mathfrak{m}(y), \quad \textit{for every } x \in \mathrm{X} \setminus B_R(x_0). \tag{3.7.8}$$

Proof. We start claiming that

$$\tilde{u}(x) = -\int G(x,y)\Delta\tilde{u}(y)\,d\mathfrak{m}(y), \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X. \tag{3.7.9}$$

We would like to we use Lemma 3.7.5, however we do not have the assumption that $\tilde{u} \in L^2(\mathfrak{m})$, hence we need to perform a cut-off argument. From Proposition 1.4.9 there exist cut-off functions $(\eta_k) \in \mathsf{Test}(\mathrm{X})$, $k \in \mathbb{N}$, such that $0 \le \eta_k \le 1$, $\eta_k = 1$ in $B_{2^kR}(x_0)$, $\mathrm{supp}(\eta_k) \subset B_{2^{k+1}R}$ and $2^kR|D\eta| + (2^kR)^2|\Delta\eta| \le C(N)$. In particular we have that

$$\mathsf{d}_{x_0}|\mathrm{D}\eta_k| + \mathsf{d}_{x_0}^2|\Delta\eta_k| \le 4C(N), \quad \mathfrak{m}\text{-a.e., for every } k \in \mathbb{N}. \tag{3.7.10}$$

Define $u_k \coloneqq \eta_k\tilde{u} \in L^2(\mathfrak{m})$. From the Leibniz rule for the Laplacian (Proposition 1.2.16) we have that $u_k \in \mathsf{D}_{L^\infty \cap L^1}(\Delta)$ with

$$\Delta u_k = \eta_k\Delta\tilde{u} + 2\langle\nabla\tilde{u}, \nabla\eta_k\rangle + \tilde{u}\Delta\eta_k.$$

From Lemma 3.7.5 we have

$$u_k(x) = -\int \Delta u_k(y)G(x,y)\,\mathrm{d}\mathfrak{m}(y), \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X. \tag{3.7.11}$$

We want to pass to the limit in the above identity. To do so we first recall that from the estimates in Section 2.4 (in particular from propositions 2.4.1 and 2.4.2) and the Euclidean volume growth assumption we have that $u(x) \le C\mathsf{d}_{x_0}^{2-N}$ in $B_R(x_0)^c$ and $|\nabla u| \le C\mathsf{d}_{x_0}^{1-N}$ $\mathfrak{m}$-a.e. in $B_R(x_0)^c$ for some constant C. Moreover from the estimates for the Green's functions (2.1.4) we have that $G(x,y) \le C_x\mathsf{d}(x,y)^{2-N}$. Combining these estimates with (3.7.10) we obtain that for every $x \in \mathrm{X}$ there exists a radius R_x and a constant $\tilde{C}_x$ such that

$$|D\eta_k||D\tilde{u}||G(x,.)|,\ |\Delta\eta_k||\tilde{u}||G(x,.)| \le \tilde{C}_x\chi_{B_{R_x}(x_0)^c}\mathsf{d}_{x_0}^{4-2N}, \quad \mathfrak{m}\text{-a.e.}. \tag{3.7.12}$$

It is easy to see that, thanks to the Euclidean volume growth assumption, the function on the right hand side of (3.7.12) is in $L^1(\mathfrak{m})$. This allows us to use the dominated convergence theorem in (3.7.11) and deduce (3.7.9).

To prove (3.7.8) it sufficient to prove that the right hand side of (3.7.8) is continuous in $\mathrm{X} \setminus B_R(x_0)$. We first need to observe that for every $\delta > 0$

$$\sup_{\{y \in \bar{B}_R(x_0),\, x \in \mathrm{X}\ :\ \mathsf{d}(x,y) > \delta\}} G(x,y) \le C(B_R(x_0), \delta), \tag{3.7.13}$$

where $C(K)$ is a constant depending only on $B_R(x_0)$ and the parameter δ. Indeed from the estimates for the Green's function in (2.1.4) and the Bishop-Gromov inequality we have

$$G(x,y) \le C\int_\delta^\infty \frac{s}{\mathfrak{m}(B_s(y))}\,\mathrm{d}s \le C(N)\int_{\delta\wedge 4R}^{4R} \frac{s(4R)^N}{s^N\mathfrak{m}(B_R(x_0))}\,\mathrm{d}s + C(N)\int_{4R}^\infty \frac{s}{\mathfrak{m}(B_{s/2}(x_0))}\,\mathrm{d}s,$$

for every $\delta > 0$, every $y \in B_R(x_0)$ and every $x \in \mathrm{X}$ satisfying $\mathsf{d}(x,y) > \delta$. This proves (3.7.13). We can now prove the claimed continuity. Recall from Proposition 2.1.4 that $G(.,y)$ is continuous in in $X \setminus \{y\}$. Thanks to (3.7.13) and the assumption $\Delta u \in L^1(\mathfrak{m})$ we can apply Fatou lemma and obtain

$$\varlimsup_{z \to x} \int |G(x,y) - G(z,y)|\Delta u(y)\,d\mathfrak{m}(y) \le \int_K \varlimsup_{z \to x} |G(x,y) - G(z,y)|\Delta u(y)\,d\mathfrak{m}(y) = 0,$$

for every $x \in \mathrm{X} \setminus K$. $\square$

This result is a weaker version of a stronger result concerning the representation of harmonic functions on Riemannian manifolds proved first in[**85**] (see also [**167**]). In our arguments we will follow [**167**].

Theorem 3.7.7. *Suppose that* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is an* $\mathrm{RCD}(0,N)$ *space with* $N>2$ *and Euclidean-volume growth. Let* u *be an electrostatic potential of some open and bounded set* E. *Then there exists a ball* $B_R(x_0)$ *and a positive constant* $C>0$ *such that* $E\subset B_R(x_0)$ *and*

$$|u(x)-\overline{\mathrm{Cap}}(E)G(x_0,x)|\le \frac{CG(x_0,x)}{\mathsf{d}(x,x_0)},\quad \textit{for every } x\in \mathrm{X}\setminus B_R(x_0). \tag{3.7.14}$$

PROOF. Fix $R>0$ big enough so that $E\subset\subset B_R(x_0)$. Choose a cut off function $\eta\in\mathsf{Test}(\mathrm{X})$ such that $0\le\eta\le 1$, $\eta=1$ in $\bar B_{R+2}\setminus B_{R+1}(x_0)$ and $\operatorname{supp}\eta\subset \mathrm{X}\setminus B_R(x_0)$ (recall Proposition 1.4.9). Define the function $\tilde u$ as

$$\tilde u=\begin{cases}\eta u, & \text{in } B_{R+1}(x_0),\\ u, & \text{in } \mathrm{X}\setminus B_{R+1}(x_0).\end{cases}$$

It immediate to check that $\tilde u\in \mathsf{LIP}_b(\mathrm{X})\cap \mathsf{D}_{L^1\cap L^\infty}(\Delta)$ with $\tilde u$ harmonic in $\mathrm{X}\setminus \bar B_{R+1}(x_0)$. Therefore from Corollary 3.7.6 we have that $u(x)=\int G(x,y)\Delta\tilde u(y)\,\mathrm{d}\mathfrak{m}(y)$ for every $\mathrm{X}\setminus\bar B_{R+1}(x_0)$. We can now compute

$$\begin{aligned}\tilde u(x)&=-\int (G(x,y)\pm G(x,x_0))\Delta\tilde u(y)\,\mathrm{d}\mathfrak{m}(y)=\\ &=-G(x,x_0)\underbrace{\int_{B_{\bar R}(x_0)}\Delta\tilde u(y)\,\mathrm{d}\mathfrak{m}(y)}_{b:=}-\underbrace{\int (G(x,y)-G(x,x_0))\Delta\tilde u(y)\,\mathrm{d}\mathfrak{m}(y)}_{v(x):=},\end{aligned}$$

for every $x\in\mathrm{X}\setminus\bar B_{R+1}(x_0)$ and every $\bar R\in(R+1,R+2)$. Since $\tilde u$ is harmonic in $\mathrm{X}\setminus\bar B_{R+1}(x_0)$ we can estimate the decay of v as follows

$$|v(x)|\le \|\Delta\tilde u\|_{L^1(B_{R+1}(x_0))}\sup_{y\in B_{R+1}(x_0)}|G(x,y)-G(x,x_0)|,$$

for every $x\in\mathrm{X}\setminus\bar B_{R+1}(x_0)$. By the local Sobolev-to Lip property (Proposition 1.3.8) and Lemma 1.3.22 we have

$$\begin{aligned}\sup_{y\in B_{R+1}(x_0)}|G(x,y)-G(x,x_0)|&\le C(R+1)\||\nabla G(x,.)|\|_{L^\infty(B_{R+2}(x_0))}\\ &\le \frac{C(R+1)G(x_0,x)}{\mathsf{d}(x,x_0)},\end{aligned}$$

provided $\mathsf{d}(x,x_0)>\lambda^{-1}(R+2)$, where $\lambda=\lambda(N)<1$ is given by Lemma 1.3.22. Finally if $B_{\bar R}(x_0)$ has finite perimeter the Gauss-Green formula (1.5.2) gives that

$$\int_{B_{\bar R}(x_0)}\Delta\tilde u(y)\,\mathrm{d}\mathfrak{m}(y)=\int_{B_{\bar R}(x_0)}\Delta(\eta u)(y)\,\mathrm{d}\mathfrak{m}(y)=\int\langle \mathrm{tr}_{B_{\bar R}(x_0)}\nabla(\eta u),\nu_{B_{\bar R}(x_0)}\rangle\,\mathrm{dPer}(B_{\bar R}(x_0),.).$$

Then the conclusion follows from Lemma 3.6.3. □

Combining (3.7.3) with Theorem 3.7.7 and Lemma 3.6.3 we obtain the following asymptotic expansion for the electrostatic potential.

3.7.3. Integral asymptotics for $|\nabla u|$. From the 'order-zero' asymptotics derive in (3.7.4) we will now pass to 'first order' integral asymptotics for the gradient of the electrostatic potential. Even if we will closely follow the argument in [**2**], these type of computations and the idea to get higher order bounds from zero-order bounds were already present in [**72**, **85**].

From (3.7.4) (actually the Euclidean volume growth assumption and the decay estimates for u proven in Section 2.4.1 are enough) and the gradient estimate in Proposition 2.4.1 follows immediately that:

$$|\nabla u|\le C\mathsf{d}_{x_0}^{1-N},\quad \mathfrak{m}\text{-a.e. in } \mathrm{X}\setminus B_R(x_0), \tag{3.7.15}$$

for R big enough and with $C=C(N,E)>0$.

Proposition 3.7.8 (Asymptotics for $|\nabla u|$)**.** *Let* X *be an* $\mathrm{RCD}(0,N)$ *space with Euclidean volume growth. Let* u *be the electrostatic potential of a bounded open set* $E\subset\mathrm{X}$. *Then for every* $x_0\in E$

$$\lim_{R\to+\infty}R^{-1}\int_{A_{R,2R}}\left||\nabla u|-l\mathsf{d}_{x_0}^{1-N}\right|\,\mathrm{d}\mathfrak{m}=0, \tag{3.7.16}$$

where $A_{R,2R}:=B_{2R}(x_0)\setminus\bar B_R(x_0)$ *and* $l:=\frac{\mathsf{eCap}(E)}{\sigma_{N-1}\mathsf{AVR}(\mathrm{X})}$.

Proof. Set $L := \frac{\mathsf{eCap}(E)}{(N-2)\sigma_{N-1}\mathsf{AVR}(\mathrm{X})}$ (i.e. the right hand side of (3.7.4)). Fix $x_0 \in E$ and set $A_{r,r'} := B_{r'}(x_0) \setminus \bar{B}_r(x_0)$ for every $0 < r \le r'$. Fix also $\varepsilon > 0$. From the asymptotic estimate for u in (3.7.4) there exists $R_0 > 0$ such that $|u\mathsf{d}_{x_0}^{N-2} - L| \le \varepsilon$ in $\mathrm{X} \setminus B_R(x_0)$. We also fix $R > R_0$. For every $\delta \in (0, R/2)$ there exists a cut-off function $\varphi \in \mathsf{LIP}_c(A_{R,2R})$ such that $0 \le \varphi \le 1$ and satisfying $\varphi = 1$ in $A_{R+\delta,2R-\delta}$, $\mathrm{Lip}\varphi \le 1/\delta$. Integrating by parts (recalling that $\mathsf{d}_{x_0}^{2-N} \in \mathsf{D}(\boldsymbol{\Delta}, \mathrm{X} \setminus \{x_0\})$) and using the harmonicity of u we have

$$\begin{aligned}
&\int_{A_{R,2R}} |\nabla u - L\nabla \mathsf{d}_{x_0}^{2-N}|^2 \varphi \ d\mathfrak{m} \\
&\le L \underbrace{\left| \int_{A_{R,2R}} \varphi(u - L\mathsf{d}_{x_0}^{2-N}) \boldsymbol{\Delta} \mathsf{d}_{x_0}^{2-N} \right|}_{I_1:=} + \underbrace{\int_{A_{R,2R}} |\langle \nabla u - L\nabla \mathsf{d}_{x_0}^{2-N}, \nabla\varphi \rangle| |u - L\mathsf{d}_{x_0}^{2-N}| \, d\mathfrak{m}}_{I_2:=}.
\end{aligned}$$

We now proceed estimating I_1. Recalling that $\boldsymbol{\Delta}\mathsf{d}_{x_0}^{2-N} \ge 0$ (see the proof of Proposition 2.4.2) and integrating again by parts we have

$$\begin{aligned}
I_1 &\le \varepsilon R^{2-N} \int_{A_{R,2R}} \varphi \boldsymbol{\Delta} \mathsf{d}_{x_0}^{2-N} \le \varepsilon R^{2-N} \int_{A_{R,2R}} |\langle \nabla\varphi, \nabla \mathsf{d}_{x_0}^{2-N} \rangle| d\mathfrak{m} \\
&\le \varepsilon R^{2-N} (N-2) \delta^{-1} \int_{A_{R,R+\delta} \cup A_{2R-\delta,2R}} \mathsf{d}_{x_0}^{1-N} d\mathfrak{m} \\
&\le \varepsilon R^{2-N}(N-2) \left(\frac{\mathfrak{m}(A_{R,R+\delta})}{\delta R^{N-1}} + \frac{\mathfrak{m}(A_{2R,2R+\delta})}{\delta(2R-\delta)^{N-1}} \right) \\
&\le C\varepsilon R^{2-N}(N-2) \left(\frac{(R+\delta)^N - R^N}{\delta R^{N-1}} + \frac{(2R)^N - (2R-\delta)^N}{\delta(2R-\delta)^{N-1}} \right) \\
&= C\varepsilon R^{2-N}(N-2) \left(\frac{(1+\delta/R)^N - 1}{\delta/R} + \frac{1-(1-\delta/(2R))^N}{\delta/(2R)(1-\delta/(2R))^{N-1}} \right) \\
&\le C\varepsilon R^{2-N},
\end{aligned}$$

with C a constant depending only on x_0 and N and where in the third-ultimate step we used the Bishop-Gromov inequality. The second term I_2 can be estimated in a similar way using the gradient estimate (3.7.15):

$$\begin{aligned}
I_2 &\le \varepsilon R^{2-N} \int_{A_{R,2R}} \left(|\nabla u| + L|\nabla \mathsf{d}_{x_0}^{2-N}| \right) |\nabla\varphi| d\mathfrak{m} \\
&\le C\varepsilon R^{2-N} \left(\frac{\mathfrak{m}(A_{R,R+\delta})}{\delta R^{N-1}} + \frac{\mathfrak{m}(A_{2R-\delta,2R})}{\delta(2R-\delta)^{N-1}} \right) \le C\varepsilon R^{2-N},
\end{aligned}$$

where $C = C(E, N)$. Plugging in the estimates for I_1 and I_2 and letting $\delta \to 0$ we obtain

$$\int_{A_{R,2R}} |\nabla u - L\nabla \mathsf{d}_{x_0}^{2-N}|^2 \ d\mathfrak{m} \le C\varepsilon R^{2-N}, \quad \text{for every } R > R_0.$$

The conclusion now follows sending $\varepsilon \to 0$, observing that $\nabla \mathsf{d}_{x_0}^{2-N} = (2-N)\mathsf{d}_{x_0}^{1-N}$, applying the Hölder inequality and from the observation that $\overline{\lim}_{R \to +\infty} \mathfrak{m}(A_{2R,R}) R^{-N} < +\infty$, as a consequence of the Euclidean volume growth assumption. □

With the asymptotics for $|\nabla u|$ proved above we can finally compute the limit of the monotone quantity U_β associated to an electrostatic potential. We recall here the definition of U_β for convenience of the reader:

$$U_\beta(t) := t^{-\frac{N-1}{N-2}} \int |\nabla u|^{1+\beta} \, \mathrm{dPer}(\{u < t\}) \in L^1_{\mathsf{loc}}(0,1), \quad \beta \ge \frac{N-2}{N-1},$$

where a representative of $|\nabla u|$ has been fixed. Recall from Section 2.4.2 that the function U_β is well defined a.e. and does not depend on the chosen representative. Moreover from Theorem 2.4.4 we know that U_β has continuous representative which is non-decreasing. From now on we will denote by U_β such representative.

Theorem 3.7.9. *Let* X *be an* $\mathrm{RCD}(0,N)$ *space with Euclidean volume growth. Let* u *be the electrostatic potential of a bounded open set* $E \subset \mathrm{X}$ *and let* U_β *as above. Then*

$$\lim_{t \to 0^+} U_\beta(t) = (N-2)^{\beta \frac{N-1}{N-2}} \overline{\mathrm{Cap}}(E)^{1-\frac{\beta}{N-2}} (\sigma_{N-1} \mathsf{AVR}(\mathrm{X}))^{\frac{\beta}{N-2}}. \tag{3.7.17}$$

Proof. Set $b := (u/L)^{\frac{1}{2-N}}$. From (3.7.4) there exists $R_0 > 0$ such that

$$\frac{3}{4}\mathsf{d}_{x_0} \le b \le \frac{5}{4}\mathsf{d}_{x_0}, \quad \text{in } \mathrm{X} \setminus B_{R_0}(x_0). \tag{3.7.18}$$

In particular $\{\frac{5}{4}R \le b \le \frac{6}{4}R\} \subset \{R \le \mathsf{d}_{x_0} \le 2R\}$, for every $R > R_0$. Combining this with (3.7.16) and the bound on $|\nabla u|$ in (3.7.15) gives

$$\lim_{R\to+\infty} R^{N-2} \int_{\{\frac{5}{4}R\le b\le \frac{6}{4}R\}} ||\nabla u| - l\mathsf{d}_{x_0}^{1-N}|\, |\nabla u| \mathrm{d}\mathfrak{m} = 0, \tag{3.7.19}$$

Moreover

$$\begin{aligned}\varlimsup_{R\to+\infty} R^{-1}\int_{\{\frac{5}{4}R\le b\le \frac{6}{4}R\}} |\mathsf{d}_{x_0}^{1-N} - b^{1-N}|\,\mathrm{d}\mathfrak{m} &\le \varlimsup_{R\to+\infty} R^{-1}\int_{\{\frac{5}{4}R\le b\le \frac{6}{4}R\}} |\mathsf{d}_{x_0}^{1-N} b^{N-1} - 1| b^{1-N}\,\mathrm{d}\mathfrak{m}\\ &\le \varlimsup_{R\to+\infty} (6/4R)^{-N}\mathfrak{m}(A_{2R,R}) \sup_{A_{R,2R}} |\mathsf{d}_{x_0}^{1-N} b^{N-1} - 1| = 0.\end{aligned}$$

This combined with (3.7.19) gives

$$\lim_{R\to+\infty} R^{N-2} \int_{\{\frac{5}{4}R\le b\le \frac{6}{4}R\}} ||\nabla u| - lb^{1-N}|\, |\nabla u| \mathrm{d}\mathfrak{m} = 0. \tag{3.7.20}$$

We now apply the coarea formula together with the change of variable $t = R^{2-N}$, recalling from Section 3.6 that $\overline{\mathrm{Cap}}(E) = \mathsf{eCap}(E) = \int |\nabla u| \mathrm{dPer}(\{u < t\})$ for a.e. $t \in (0, 1)$, and obtain

$$\lim_{t\to 0} t^{-1} \int_{c_1 t}^{c_2 t} \left|\overline{\mathrm{Cap}}(E) - l(sL^{-1})^{\frac{N-1}{N-2}} \mathrm{Per}(\{u < s\})\right| \mathrm{d}s = 0, \tag{3.7.21}$$

with $c_1 := \left(\frac{6}{4}\right)^{2-N} L < \left(\frac{5}{4}\right)^{2-N} L =: c_2$. From (3.7.20), since thanks to (3.7.15) and (3.7.18) the function $||\nabla u| b^{1-N}|$ is bounded in $\mathrm{X} \setminus B_{R_0}(x_0)$ for R_0 big enough, we also obtain

$$\lim_{R\to+\infty} R^{N-2} \int_{\{\frac{5}{4}R\le b\le \frac{6}{4}R\}} |(|\nabla u| b^{N-1})^{1+\beta} - l^{1+\beta}| b^{1-N}\, |\nabla u| \mathrm{d}\mathfrak{m} = 0.$$

Apply again the coarea formula and setting $t = R^{2-N}$ as above we deduce

$$\lim_{t\to 0} t^{-1} \int_{c_1 t}^{c_2 t} \left|L^{\beta\frac{N-1}{N-2}} U_\beta(s) - l^{1+\beta}(sL^{-1})^{\frac{N-1}{N-2}} \mathrm{Per}(\{u < s\})\right| \mathrm{d}s = 0, \tag{3.7.22}$$

with c_1, c_2 as above. Combining (3.7.21) and (3.7.22) we finally reach

$$\lim_{t\to 0} t^{-1} \int_{c_1 t}^{c_2 t} \left|L^{\beta\frac{N-1}{N-2}} l^{-\beta} U_\beta(s) - \overline{\mathrm{Cap}}(E)\right| \mathrm{d}s = 0.$$

This and the Markov-Chebyshev's inequality grant the existence of a sequence $t_n \to 0$ such that

$$L^{\beta\frac{N-1}{N-2}} l^{-\beta} U_\beta(t_n) \to \overline{\mathrm{Cap}}(E).$$

Since U_β is also non decreasing this concludes the proof. □

3.8. Willmore inequality

We can finally put together the expression for the limit of U_β and the notion of mean curvature vector developed in Section 3.5 to produce Willmore-type inequalities:

Theorem 3.8.1 (Willmore-type inequalities). *Let* X *be an* RCD$(0, N)$ *space with Euclidean volume growth. Let u be the electrostatic potential of a bounded open set $E \subset$ X. Then for a.e. $t \in (0, 1)$ the sublevel $\{u < t\}$ admits a mean curvature H (in the sense of Definition 3.5.22) satisfying for every $\beta \ge \frac{N-2}{N-1}$:*

$$\int \left|\frac{H}{N-1}\right|^{\beta+1} \mathrm{dPer}(\{u < t\}) \ge (|\sigma_{N-1}|\mathsf{AVR}(\mathrm{X}))^{\frac{\beta}{N-2}} \left(\frac{\overline{\mathrm{Cap}}(\{u \le t\}^c)}{N-2}\right)^{\frac{N-2-\beta}{N-2}}, \quad \textit{for a.e. } t \in (0, 1).$$

Proof. The fact that $\{u < t\}$ admits a mean curvature vector for a.e. $t \in (0, 1)$(in the sense of Definition 3.5.22) follows immediately from Proposition 3.5.24, which also says that for a.e. $t \in (0, 1)$

$$|H_{\{u<t\}}| = \left|\langle \frac{\nabla u}{|\nabla u|^2}, \nabla |\nabla u|\rangle\right|, \quad \mathrm{Per}(\{u < t\})\text{-a.e.}.$$

We now pass to the proof of the Willmore inequality. From the fact that $U'_\beta \geq 0$ a.e. in $(0,1)$ and the formula for U'_β (see Proposition 2.4.5) we obtain

$$\frac{N-1}{N-2} t^{-\beta\frac{N-1}{N-2}-1} \int |\nabla u|^{\beta+1} \,\mathrm{dPer}(\{u<t\},.) \leq t^{-\beta\frac{N-1}{N-2}} \int |\nabla u|^{\beta} \,\langle \frac{\nabla u}{|\nabla u|^2}, \nabla|\nabla u|\rangle \,\mathrm{dPer}(\{u<t\},.),$$

for a.e. $t \in (0,1)$, applying the Hölder inequality on the right hand side with exponent $p = (\beta+1)/\beta$ and then simplifying, we reach

$$\left(\frac{N-1}{N-2}\right)^{\beta+1} U_\beta(t) \leq t^{1-\frac{\beta}{N-2}} \int \left|\langle \frac{\nabla u}{|\nabla u|^2}, \nabla|\nabla u|\rangle\right|^{\beta+1} \mathrm{dPer}(\{u<t\},.), \quad \text{for a.e. } t \in (0,1).$$

Taking into account the monotonicity of U_β and the expression for its limit in (3.7.17) the conclusion follows (recalling also Lemma 3.6.2). □

Remark 3.8.2. We make a few comments on the above inequality:

- (i) For every $\beta \leq 1$ we know that the inequality is non-trivial in the sense that the left-hand side is finite for a.e. $t \in (0,1)$. This because the mean curvature is in $L^2(\mathrm{Per}(\{u<t\}))$. On the other hand for $\beta > 1$ we do not now whether the same left hand side is identically $+\infty$ for every $t \in (0,1)$,
- (ii) If we could plug in $t = 1$ we would get exactly the Willmore-type inequalities obtained in [**2**] (see also [**109**]) with the same sharp constant. ■

CHAPTER 4

Rigidity and almost-rigidity of Sobolev inequalities in compact $\mathrm{RCD}(K,N)$ spaces

Structure of the chapter

After briefly recalling in Section 4.1 some of the notations that are important for this chapter, we will pass to the first main Section 4.2, which is about rearrangement inequalities on $\mathrm{CD}(K,N)$ spaces. In the first part we will review the *Polya-Szego inequality* for positive K, due to [**182**], while on the second part we will prove an *Euclidean-variant* of this inequality which is valid for arbitrary $\mathrm{CD}(K,N)$ spaces. On Section 4.3 will be derived a class of *local Sobolev inequalities of Euclidean-type* in $\mathrm{CD}(K,N)$ spaces. The proof will rely on *local isoperimetric inequalities of Euclidean-type*, contained in Section 4.3.1, and on the Euclidean-Polya Szego inequality previously obtained.

Section 4.4 deals with the stability of Sobolev embeddings with respect to mGH-convergence. We will employ both blow-up and blow-down arguments to derive some sharp lower bounds on the constant appearing in the Sobolev inequality. These results will be combined in Section 4.5, with the Local Sobolev inequalities mentioned above, to compute the *optimal Sobolev constant α_p on compact $\mathrm{CD}(K,N)$ spaces* (see (4.5.2) or Section 4.1 for the definition of α_p).

The analysis concerning the optimal constant A_q^{opt} (see (4.6.1) or Section 4.1) will start in Section 4.6, where we will extend some of its classical properties and estimates to the setting of CD spaces. Then in Section 4.7 we will develop our main tools in *concentration-compactness under mGH-convergence*. We will apply these results in Section 4.8, together with a quantitative linearization of the Sobolev inequality (in Section 4.8.1), to prove the *rigidity* of A_q^{opt} in RCD setting. In Section 4.10 we will finally prove the continuity of A_q^{opt} under mGH-convergence and use it to prove the *almost-rigidity* of A_q^{opt}.

We will conclude in Section 4.11 applying some of the results and the tools developed in the chapter to analyse the Yamabe equation on $\mathrm{RCD}(K,N)$ spaces. We will prove a classical *existence result* in Section 4.11.1 while in Section 4.11.2 we will show a *continuity result for the generalized Yamabe constant* under mGH-convergence, which is new also for Ricci-limits.

All the results that will be presented in this chapter are contained in [**185**].

4.1. Notations and constants used frequently

For every $N\in[1,\infty)$ and $p\in(1,N)$ the constants

$$\omega_N, \quad \sigma_{N-1}, \quad \mathsf{Eucl}(N,p),$$

will denote respectively the (generalized) volume of the *N-dimensional unit ball*, the (generalized) volume of the *$(N-1)$-dimensional sphere* and the (generalized) *optimal constant in the p-Euclidean-Sobolev inequality* (see (1.1.1) and (1.1.2)). Here generalized means that N is not necessary an integer.

Given a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$, numbers $q,p\in(1,\infty)$ and constants $A,B\geq 0$, we say that $(\mathrm{X},\mathsf{d},\mathfrak{m})$ supports a *(q,p)-Sobolev inequality with constants A,B* if

$$\|u\|_{L^q}^p \leq A\||Du|\|_{L^p}^p + B\|u\|_{L^p}^p, \quad \forall u\in W^{1,p}(\mathrm{X})$$

(note that the norms are raised to the p-th power). We now recall the two main notions of optimal constants for the Sobolev inequality that will be investigated in this chapter.

Given a compact $\mathrm{CD}(K,N)$ metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ and $p\in(1,N)$ and consider the following (p^*,p)-Sobolev inequality:

$$(\star) \qquad \|u\|_{L^{p^*}}^p \leq A\||Du|\|_{L^p}^p + B\|u\|_{L^p}^p, \quad \forall u\in W^{1,p}(\mathrm{X}),$$

where $p^*:=\frac{pN}{N-p}$. We define

$$\alpha_p(\mathrm{X}) := \inf\{A \ : (\star) \text{ holds with } A \text{ for some } B<+\infty\}.$$

Moreover for every $q\in(2,2^*]$ we consider the $(q,2)$-Sobolev inequality:

$$(\star\star) \qquad \|u\|_{L^q}^p \leq A\||Du|\|_{L^2}^2 + \mathfrak{m}(\mathrm{X})^{2/q-1}\|u\|_{L^2}^2, \quad \forall u\in W^{1,p}(\mathrm{X}),$$

and set

$$A_q^{\mathrm{opt}}(\mathrm{X}) := \inf\{A \ : (\star\star) \text{ holds with } A\} \cdot \mathfrak{m}(\mathrm{X})^{1-2/q}.$$

4.2. Rearrangement inequalities on RCD(K,N) spaces

4.2.1. Polya-Szego inequality for positive Ricci curvature. In this part we recall the Polya-Szego inequality for CD(K,N) spaces, with $K>0$, derived in [**182**]. We will also introduce some definitions and collect some technical results from [**182**] that will be used in Section 4.2.2 to prove a variant of this inequality.

Definition 4.2.1 (Distribution function)**.** Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a compact metric measure space, $\Omega \subseteq X$ an open set with $\mathfrak{m}(\Omega) < +\infty$ and $u : \Omega \to [0,+\infty)$ a non-negative Borel function. We define $\mu : [0,+\infty) \to [0,\mathfrak{m}(\Omega)]$, the distribution function of u, as

$$\mu(t) := \mathfrak{m}(\{u > t\}). \tag{4.2.1}$$

For u and μ as above, we let $u^{\#}$ be the generalized inverse of μ, defined by

$$u^{\#}(s) := \begin{cases} \operatorname{ess\,sup} u & \text{if } s = 0, \\ \inf\{t : \mu(t) < s\} & \text{if } s > 0. \end{cases}$$

It can be checked that $u^{\#}$ is non-increasing and left-continuous.

We recall the definition of the one-dimensional CD$(N-1,N)$-model space $I_N = ([0,\pi],\mathsf{d}_{eu},\mathfrak{m}_N)$, where $I_N := ([0,\pi],\mathsf{d}_{eu},\mathfrak{m}_N)$, d_{eu} is the Euclidean distance restricted on $[0,\pi]$ and

$$\mathfrak{m}_N := \tfrac{1}{c_N}\sin(t)^{N-1}\mathscr{L}^1|_{[0,\pi]},$$

with $c_N := \int_{[0,\pi]} \sin(t)^{N-1}\,\mathrm{d}t$.

Then, given $\Omega \subseteq X$ an open set and $u : \Omega \to [0,+\infty)$ a non-negative Borel function, we define the *monotone rearrangement* into I_N as follows: first we consider $r>0$ so that $\mathfrak{m}(\Omega) = \mathfrak{m}_N([0,r])$ and define $\Omega^* := [0,r]$, then we define the monotone rearrangement function $u_N^* : \Omega^* \to \mathbb{R}^+$ as

$$u_N^*(x) := u^{\#}(\mathfrak{m}_N([0,x])), \qquad \forall x \in [0,r].$$

In the sequel, whenever u and Ω are fixed, Ω^* and u_N^* will be implicitly defined as above.

Theorem 4.2.2 (Polya-Szego inequality, [**182**])**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an essentially non branching* CD$(N-1,N)$ *space for some* $N \in (1,\infty)$ *and* $\Omega \subseteq \mathrm{X}$ *be open. Then, for every* $p \in (1,\infty)$*, the monotone rearrangement in* I_N *maps* $L^p(\Omega)$ *(resp.* $W_0^{1,p}(\Omega)$*) into* $L^p(\Omega^*)$ *(resp.* $W^{1,p}(\Omega^*)$*) and satisfies:*

$$\|u\|_{L^p(\Omega)} = \|u_N^*\|_{L^p(\Omega^*)}, \qquad \forall u \in L^p(\Omega) \tag{4.2.2}$$

$$\int_\Omega |Du|^p \,\mathrm{d}\mathfrak{m} \geq \int_{\Omega^*} |Du_N^*|^p \,\mathrm{d}\mathfrak{m}_N, \qquad \forall u \in W_0^{1,p}(\Omega). \tag{4.2.3}$$

We will also need the following rigidity of the Polya-Szego inequality proven in [**182**, Theorem 5.4].

Theorem 4.2.3. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* RCD$(N-1,N)$ *space for some* $N \in [2,\infty)$ *with* $\mathfrak{m}(\mathrm{X}) = 1$ *and* $p \in (1,\infty)$*. Let* $\Omega \subset \mathrm{X}$ *be an open set and assume that there exists a non-negative and non-constant function* $u \in W_0^{1,p}(\Omega)$ *achieving equality in* (4.2.3)*.*

Then $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is isomorphic to a spherical suspension, i.e. there exists an* RCD$(N-2,N-1)$ *space* $(\mathrm{Z},\mathsf{d}_\mathrm{Z},\mathfrak{m}_\mathrm{Z})$ *with* $\mathfrak{m}_\mathrm{Z}(\mathrm{Z}) = 1$ *so that* $\mathrm{X} \simeq [0,\pi] \times_{\sin}^N \mathrm{Z}$.

Remark 4.2.4. Observe that in Theorem 4.2.3 we did not assume that $\mathfrak{m}(\Omega) < 1$, assumption that is actually present in Theorem 5.4 of [**182**]. This is intentional, since we will need to apply Theorem 4.2.3 precisely in the case $\Omega = \mathrm{X}$. This is possible since the arguments in [**182**] work also in the case $\Omega = \mathrm{X}$ without modification. The only part where the argument does not cover the case $\Omega = \mathrm{X}$ is the proof of the approximation Lemma 3.6 in [**182**], which however can be easily adapted (see Lemma 4.2.5 below). ■

The following technical result will be needed in Section 4.2.2. We include a sketch of the argument in the case $\Omega = \mathrm{X}$, to further justify the validity of Theorem 4.2.3 also in this case (see the above Remark). Recall also that $|Du|_1$ denotes the absolutely continuous part of the total variation (see Section 1.2.6).

Lemma 4.2.5 (Approximation with non-vanishing gradients)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a* CD(K,N) *metric measure space with* $N < +\infty$*, and let* $\Omega \subset \mathrm{X}$ *be open with* $\mathfrak{m}(\Omega) < +\infty$*. Then for any non-negative* $u \in \mathsf{LIP}_c(\Omega)$ *there exists a sequence of non-negative* $u_n \in \mathsf{LIP}_c(\Omega)$ *satisfying* $|Du_n|_1 \neq 0$ $\mathfrak{m}$*-a.e. in* $\{u_n > 0\}$ *and such that* $u_n \to u$ *in* $W^{1,p}(\mathrm{X})$*.*

PROOF. The case $\Omega \neq \mathrm{X}$ has been proven in [**182**, Lemma 3.6 and Corollary 3.7]. The proof presented there, as it is written, does not cover the case $\Omega = \mathrm{X}$ with X compact. However, the argument can be easily adapted by considering a sequence $\varepsilon_n \to 0$ such that $\mathfrak{m}(\{\mathrm{lip}(u_n) = \varepsilon_n\}) = 0$ and taking

$$u_n := u + \varepsilon_n v,$$

with $v(x) := \mathsf{d}(x_0, x)$, for an arbitrary fixed point $x_0 \in \mathrm{X}$. Since $v \in \mathsf{LIP}(\mathrm{X})$ and $\mathrm{lip}(v) = 1$ $\mathfrak{m}$-a.e. in X, arguing exactly as in [**182**, Lemma 3.6] we get that $u_n \to u$ in $W^{1,p}(\mathrm{X})$ and $\mathrm{lip}(u_n) \neq 0$ $\mathfrak{m}$-a.e. in $\{u_n > 0\}$. To get the claimed non-vanishing of $|Du_n|_1$, as in [**182**, Corollary 3.7] we use the existence of a constant $c > 0$ such that

$$|Du|_1 \geq c\,\mathrm{lip}(u), \quad \mathfrak{m}\text{-a.e.},$$

for every $u \in \mathsf{LIP}_{loc}(\mathrm{X})$, which holds from the results in [**27**] and the fact that $\mathrm{CD}(K, N)$ spaces are PI-spaces. □

Lemma 4.2.6 (Derivative of the distribution function)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and let* $\Omega \subseteq X$ *be an open subset with* $\mathfrak{m}(\Omega) < +\infty$*. Assume that* $u \in \mathsf{LIP}_c(\Omega)$ *is non-negative and* $|Du|_1(x) \neq 0$ *for* $\mathfrak{m}$*-a.e.* $x \in \{u > 0\}$*. Then its distribution function* $\mu : [0, +\infty) \to [0, \mathfrak{m}(\Omega)]$*, defined in* (4.2.1)*, is absolutely continuous. Moreover it holds*

$$\mu'(t) = -\int \frac{1}{|Du|_1}\,\mathrm{d}\mathrm{Per}(\{u > t\}) \quad \textit{for a.e. } t, \tag{4.2.4}$$

where the quantity $1/|Du|_1$ *is defined to be* 0 *whenever* $|Du|_1 = 0$*.*

4.2.2. Polya-Szego inequality of Euclidean-type. The goal of this section is to produce an Euclidean-version of the Polya-Szego inequality for $\mathrm{CD}(K, N)$ spaces derived in [**182**] and recalled in the previous section. The main difference is that our inequality holds for arbitrary $K \in \mathbb{R}$ and assumes the a-priori validity of an Euclidean-type isoperimetric inequality, while the one in [**182**] is only for $K > 0$ and is based on the Lévy-Gromov isoperimetric inequality for the $\mathrm{CD}(K, N)$ condition. As opposed to Section 4.2.1 where the symmetrization has as target the model space for the $\mathrm{CD}(K, N)$ condition with $K > 0$, we will use a notion of symmetrization that lives in the weighted half line $([0, \infty), \mathsf{d}_{eu}, t^{N-1}\mathscr{L}^1)$. It should be remarked that, in general, there is not a natural curvature model space to symmetrize functions defined on an arbitrary $\mathrm{CD}(K, N)$-space with $K \leq 0$. This is because there is not a unique model-space for the Lévy-Gromov isoperimetric inequality in the case $K \leq 0$ (see [**178**]). Therefore, it is unclear in this high-generality where the rearrangements should live. For this reason we will equip the metric measure spaces under consideration with a (possibly local) isoperimetric inequality of Euclidean-type:

$$\mathrm{Per}(E) \geq C\mathfrak{m}(E)^{\frac{N-1}{N}},$$

for $N > 1$ and C a non-negative constant.

We start with the definition of *Euclidean model-space* $(I_{0,N}, \mathsf{d}_{eu}, \mathfrak{m}_{0,N})$, $N \in (1, \infty)$:

$$I_{0,N} := [0, \infty), \qquad \mathfrak{m}_{0,N} := \sigma_{N-1} t^{N-1}\mathscr{L}^1,$$

where d_{eu} is the Euclidean distance. Next, we define the *Euclidean monotone rearrangement.*

Definition 4.2.7 (Euclidean monotone rearrangement)**.** Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space and $\Omega \subset \mathrm{X}$ be open with $\mathfrak{m}(\Omega) < +\infty$. For any Borel function $u\colon \Omega \to \mathbb{R}^+$, we define $\Omega^* := [0, r]$ with $\mathfrak{m}_{0,N}([0, r]) = \mathfrak{m}(\Omega)$ (i.e. $r^N = \frac{N}{\sigma_{N-1}}\mathfrak{m}(\Omega)$) and the monotone rearrangement $u^*_{0,N} : \Omega^* \to \mathbb{R}^+$ by

$$u^*_{0,N}(x) := u^{\#}(\mathfrak{m}_{0,N}([0, x])) = u^{\#}\Big(\frac{\sigma_{N-1}x^N}{N}\Big), \qquad \forall x \in \Omega^*,$$

where $u^{\#}$ is the generalized inverse of the distribution function of u, as defined in Section 4.2.1.

In the sequel, whenever we fix Ω and $u\colon \Omega \to [0, \infty)$, the set Ω^* and the rearrangement $u^*_{0,N}$ are automatically defined as above.

Proposition 4.2.8. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $\Omega \subset \mathrm{X}$ *be open and bounded with* $\mathfrak{m}(\Omega) < +\infty$*. Let* $u : \Omega \to [0, +\infty)$ *be Borel and let* $u^*_{0,N} : \Omega^* \to [0, +\infty)$ *be its Euclidean monotone rearrangement. Then,* u *and* $u^*_{0,N}$ *have the same distribution function. Moreover*

$$\|u\|_{L^p(\Omega)} = \|u^*_{0,N}\|_{L^p(\Omega^*)}, \qquad \forall 1 \leq p < +\infty, \tag{4.2.5}$$

and the decreasing rearrangement operator $L^p(\Omega) \ni u \mapsto u^*_{0,N} \in L^p(\Omega^*)$ *is continuous.*

The proof of the above proposition is classical, following e.g. [**154**], with straightforward modification for the metric measure setting which plays no role (see also [**182**]). Observe also that, given $u \in L^p(\mathfrak{m})$, its monotone rearrangement must be defined by fixing a Borel representative of u. However, this choice does not affect the outcome object $u^*_{0,N}$, as clearly the distribution function $\mu(t)$ of u is independent of the representative.

We now introduce the additional assumption that will make this section meaningful. For some open set $\Omega \subset \mathrm{X}$ and a number $N \in (1,\infty)$ we require the validity of the following local Euclidean-isoperimetric inequality

$$\mathrm{Per}(E) \ge \mathsf{C}_{\mathsf{Isop}}\mathfrak{m}(E)^{\frac{N-1}{N}}, \qquad \forall\, E \subset \Omega \text{ Borel.} \tag{4.2.6}$$

where $\mathsf{C}_{\mathsf{Isop}}$ is a positive constant independent of E.

Remark 4.2.9. There is a rich literature about Euclidean-type isoperimetric inequalities in metric measure spaces. In particular inequalities as in (4.2.6) have been proven to hold, at least on balls, in the general setting of locally doubling metric measure spaces satisfying a weak $(1,1)$-Poincaré inequality(see, e.g., [**9**, **179**]). In the context of CD(K,N) spaces, local almost-Euclidean isoperimetric inequalities have ben derived in [**66**], while in the recent [**42**], a global version of (4.2.6) is proven to hold in CD$(0,N)$ spaces with Euclidean-volume growth. In our specific case the validity of (4.2.6) will come from Theorem 4.3.1. ■

Proposition 4.2.10 (Lipschitz-to-Lipschitz property of the rearrangement)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space and let* $\Omega \subset \mathrm{X}$ *be open with* $\mathfrak{m}(\Omega) < +\infty$. *Assume furthermore that for some* $N \in (1,\infty)$ *and* $\mathsf{C}_{\mathsf{Isop}} > 0$ *the isoperimetric inequality in* (4.2.6) *holds in* Ω. *Finally, let* $u \in \mathsf{LIP}_c(\Omega)$ *be non-negative with Lipschitz constant* $L \ge 0$ *and such that* $|Du|_1(x) \neq 0$ *for* $\mathfrak{m}$*-a.e.* $x \in \{u > 0\}$.

Then $u^*_{0,N} \in \mathsf{LIP}(\Omega^*)$ *with* $\mathrm{Lip}(u^*_{0,N}) \le N\omega_N^{\frac{1}{N}} L/\mathsf{C}_{\mathsf{Isop}}$.

Proof. We closely follow [**182**]. Let μ be the distribution function associated to u and denote by $M := \sup u < +\infty$. The assumptions grant that μ is continuous and strictly decreasing. Therefore for any $s,k \ge 0$ such that $s + k \le \mathfrak{m}(\Omega)$ we can find $0 \le t-h \le t \le M$ in such a way that $\mu(t-h) = s+k$ and $\mu(t) = s$. Then from the coarea formula (1.3.12) and the L-Lipschitzianity of u we get

$$\int_{t-h}^{t} \mathrm{Per}(\{u>r\})\,\mathrm{d}r = \int_{\{t-h\le u<t\}} |Du|_1\,\mathrm{d}\mathfrak{m} \le L(\mu(t-h)-\mu(t)) = kL. \tag{4.2.7}$$

Observe that $\{u > r\} \subset \Omega$ for every $r > 0$, therefore we can apply the isoperimetric inequality (4.2.6) and obtain that

$$\mathrm{Per}(\{u>r\}) \ge \mathsf{C}_{\mathsf{Isop}}\mu(r)^{\frac{N-1}{N}}, \qquad \forall r > 0.$$

Therefore from (4.2.7) and the monotonicity of μ we obtain

$$kL \ge \mathsf{C}_{\mathsf{Isop}}\int_{t-h}^{t}\mu(r)^{\frac{N-1}{N}}\,\mathrm{d}r \ge \mathsf{C}_{\mathsf{Isop}}h\mu(t)^{\frac{N-1}{N}},$$

from which, observing that in this case $u^\#$ is the inverse of μ, we reach

$$u^\#(s) - u^\#(s+k) \le s^{-1+1/N}\mathsf{C}_{\mathsf{Isop}}^{-1}kL.$$

In particular $u^\#$ is locally-Lipschitz and at every one of its differentiability points $s \in (0,\mathfrak{m}(\Omega))$ it holds that

$$-\frac{\mathrm{d}}{\mathrm{d}s}u^\#(s) \le s^{1-1/N}\mathsf{C}_{\mathsf{Isop}}^{-1}L.$$

Fix now two arbitrary and distinct points $x,y \in \Omega^*$ and assume without loss of generality that $y > x$. Recalling the definition of $u^*_{0,N}$ we have that $u^*_{0,N}(x) \ge u^*_{0,N}(y)$ and

$$\begin{aligned} u^*_{0,N}(x) - u^*_{0,N}(y) &= u^\#\Big(\frac{\sigma_{N-1}x^N}{N}\Big) - u^\#\Big(\frac{\sigma_{N-1}y^N}{N}\Big) = \int_{\frac{\sigma_{N-1}x^N}{N}}^{\frac{\sigma_{N-1}y^N}{N}} -\frac{\mathrm{d}}{\mathrm{d}s}u^\#(s)\,\mathrm{d}s \\ &\le \int_{\frac{\sigma_{N-1}x^N}{N}}^{\frac{\sigma_{N-1}y^N}{N}} \frac{s^{-1+1/N}}{\mathsf{C}_{\mathsf{Isop}}}L\,\mathrm{d}s = \Big(\frac{\sigma_{N-1}}{N}\Big)^{\frac{1}{N}}\frac{NL}{\mathsf{C}_{\mathsf{Isop}}}|x-y|, \end{aligned}$$

which proves that $u^*_{0,N} : \Omega^* \to [0,\infty)$ is $N\omega_N^{\frac{1}{N}} L/\mathsf{C}_{\mathsf{Isop}}$-Lipschitz. □

The proof of the following result is exactly the same as in Lemma 3.11 of [**182**], since the only relevant fact for the proof is that $\mathfrak{m}_{0,N} = h_N\mathscr{L}^1$ with a weight h_N which is bounded away from zero out of the origin (recall also (1.2.1)).

Lemma 4.2.11. *Let $p \in (1,\infty)$. Let $u \in W^{1,p}([0,r],\mathsf{d}_{eu},\mathfrak{m}_{0,N})$, with $r \in (0,\infty)$, be monotone. Then $u \in W^{1,1}_{loc}(0,r)$ and it holds that*

$$|Du|_1(t) = |u'|(t) = |Du|(t), \quad \text{for a.e. } t \in [0,r].$$

We can finally state and prove our version of the Polya-Szego rearrangement inequality.

Theorem 4.2.12 (Euclidean Polya-Szego inequality)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathrm{CD}(K,N')$ space, $K \in \mathbb{R}$ $N' \in (1,\infty)$ and let $\Omega \subset \mathrm{X}$ be open with $\mathfrak{m}(\Omega) < +\infty$. Assume furthermore that for some $N \in (1,\infty)$ and $\mathsf{C}_{\mathsf{Isop}} > 0$ the isoperimetric inequality in (4.2.6) holds in Ω.*

Then the Euclidean rearrangement maps $W^{1,p}_0(\Omega)$ to $W^{1,p}(\Omega^*,\mathsf{d}_{eu},\mathfrak{m}_{0,N})$ for any $1 < p < +\infty$. Moreover for any $u \in W^{1,p}_0(\Omega)$ it holds

$$\int_\Omega |Du|^p \mathrm{d}\mathfrak{m} \ge \Big(\frac{\mathsf{C}_{\mathsf{Isop}}}{N\omega_N^{1/N}}\Big)^p \int_{\Omega^*} |Du^*_{0,N}|^p \mathrm{d}\mathfrak{m}_{0,N}. \tag{4.2.8}$$

Proof. The proof is a minor modification of the arguments in [**182**], we will however include most of the details. We first prove the result assuming that $u \in \mathsf{LIP}_c(\Omega)$ and $|Du|_1(x) \neq 0$ for $\mathfrak{m}$-a.e. $x \in \{u > 0\}$, then the general case will follow by approximation. Set $M := \sup u$ and define the functions $\varphi,\psi : [0,M] \to \mathbb{R}^+$ as follows

$$\varphi(t) := \int_{\{u>t\}} |Du|_1^p \,\mathrm{d}\mathfrak{m}, \quad \psi(t) := \int_{\{u>t\}} |Du|_1 \,\mathrm{d}\mathfrak{m}.$$

An application of the coarea formula (1.2.17) gives at once that both φ and ψ are absolutely continuous with

$$\varphi'(t) = -\int |Du|_1^{p-1} \,\mathrm{dPer}(\{u > t\}), \quad \psi'(t) = -\mathrm{Per}(\{u < t\}), \quad \text{for a.e. } t \in [0,M],$$

for any fixed Borel representative of $|Du|_1$. For this reason we fix from now until the end of the proof one of such representatives. From the Hölder inequality we have

$$\psi(t-h) - \psi(t) \le (\varphi(t-h) - \varphi(t))^{1/p}(\mu(t-h) - \mu(t))^{(p-1)/p}, \quad 0 \le t-h \le t < M,$$

where μ denotes the distribution function of u. From Lemma 4.2.6, we know that also μ is absolutely continuous, in particular we have that a.e. $t \in [0,M]$ is at the same time a differentiability point for φ,ψ and μ. Choosing one of such t's in the above inequality, dividing by h and passing to the limit as $h \to 0^+$ we obtain

$$-\psi'(t) \le (-\varphi'(t))^{1/p}(-\mu'(t))^{(p-1)/p}, \quad \text{for a.e. } t \in [0,M].$$

Moreover, by the validity of (4.2.6), we have that $\mathrm{Per}(\{u > t\}) \ge \mathsf{C}_{\mathsf{Isop}}\mu(t)^{(N-1)/N}$. Therefore

$$-\varphi'(t) \ge \frac{\mathsf{C}^p_{\mathsf{Isop}}\mu(t)^{\frac{(N-1)p}{N}}}{(-\mu'(t))^{p-1}}, \quad \text{for a.e. } t \in [0,M].$$

and integrating we reach

$$\int_\Omega |Du|_1^p \,\mathrm{d}\mathfrak{m} = \int_0^M -\varphi'(t)\,\mathrm{d}t \ge \int_0^M \frac{\mathsf{C}^p_{\mathsf{Isop}}\mu(t)^{\frac{(N-1)p}{N}}}{(-\mu'(t))^{p-1}}\,\mathrm{d}t. \tag{4.2.9}$$

Recall now from Proposition 4.2.8 that $\mu(t) = \mathfrak{m}(\{u^*_{0,N} > t\})$, where $u^*_{0,N} : \Omega^* \to \mathbb{R}^+$ is the Euclidean monotone rearrangement. Moreover, thanks to the non-vanishing assumptions on $|Du|_1$, we have from Proposition 4.2.10 that $u^*_{0,N} \in \mathsf{LIP}(\Omega^*)$. Additionally $u^*_{0,N}$ is strictly decreasing in $(0,M)$ and in particular $\{u^*_{0,N} > t\} = [0,r_t)$ for some $r_t \in [0,\mathfrak{m}(\Omega)]$, for every $t \in (0,M)$. Combining these observations with Lemma 4.2.6 and recalling also Lemma 4.2.11 we have following expression for the derivative of μ:

$$-\mu'(t) = \int_{\{u^*_{0,N}=t\}} |Du^*_{0,N}|_1^{-1}\mathrm{dPer}(\{u^*_{0,N} > t\}) = \frac{\mathrm{Per}(\{u^*_{0,N} > t\})}{|(u^*_{0,N})'|(r_t)} \quad \text{for a.e. } t \in (0,M),$$

where r_t is as above. It is clear that $\mathrm{Per}([0,r)) = \sigma_{N-1}r^{N-1}$ for every $r \in (0,\infty)$ (where the perimeter is computed in the space $(I_{0,N},\mathsf{d}_{eu},\mathfrak{m}_{0,N})$), therefore

$$\mu(t)^{\frac{N-1}{N}} = \big(\frac{\sigma_{N-1}}{N}\big)^{\frac{N-1}{N}} r_t^{N-1} = \frac{\mathrm{Per}(\{u^*_{0,N} > t\})}{N\omega_N^{\frac{1}{N}}}, \tag{4.2.10}$$

and thus we can finally obtain that

$$-\mu'(t) = N\omega_N^{\frac{1}{N}} \frac{\mu(t)^{\frac{N-1}{N}}}{|(u^*_{0,N})'|(r_t)} \quad \text{for a.e. } t \in [0,M].$$

Plugging this identity in (4.2.9) and using again (4.2.10) (recalling also Lemma 4.2.11)

$$\begin{aligned}\int_\Omega |Du|_1^p\,\mathrm{d}\mathfrak{m} &\geq \mathsf{C}_{\mathsf{Isop}}^p (N\omega_N^{1/N})^{1-p}\int_0^M |(u_{0,N}^*)'|(r_t)^{p-1}\mu(t)^{\frac{(N-1)}{N}}\,\mathrm{d}t\\ &= \big(\tfrac{\mathsf{C}_{\mathsf{Isop}}}{N\omega_N^{1/N}}\big)^p\int_0^M |(u_{0,N}^*)'|(r_t)^{p-1}\mathrm{Per}(\{u_{0,N}^*>t\})\,\mathrm{d}t\\ &= \big(\tfrac{\mathsf{C}_{\mathsf{Isop}}}{N\omega_N^{1/N}}\big)^p\int_0^M\int |(u_{0,N}^*)'|(r_t)^{p-1}\mathrm{d}\mathrm{Per}(\{u_{0,N}^*>t\})\,\mathrm{d}t = \big(\tfrac{\mathsf{C}_{\mathsf{Isop}}}{N\omega_N^{1/N}}\big)^p\int_{\Omega^*}|Du_{0,N}^*|^p\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Recalling that $|Du|_1\leq \mathrm{lip}\ u$ $\mathfrak{m}$-a.e. for every $u\in \mathsf{LIP}_{bs}(\mathrm{X})$, we obtain (4.2.8). In the case of a general $u\in W_0^{1,p}(\Omega)$ the result follows via approximation exploiting Lemma 4.2.5 exactly as in the proof of Theorem 1.4 in [**182**]. □

Remark 4.2.13. It follows from its proof that Theorem 4.2.12 holds with the weaker assumption that $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is uniformly locally doubling and supports a weak $(1,1)$-Poincaré inequality. Recall also from Remark 4.2.9 that under these assumptions an isoperimetric inequality as in (4.2.6) is available. ■

4.3. Local Sobolev inequalities

The main goal of this section is to prove a class of local almost-Euclidean Sobolev inequalities. The strategy of the proof will be first to derive some almost-Euclidean isoperimetric inequalities and then to apply the rearrangement inequality that we developed in Section 4.2.2.

4.3.1. Local almost-Euclidean isoperimetric inequalities. In this short section we are going to prove a class local almost-Euclidean isoperimetric inequalities in the non-smooth setting. Our proof relies on the Brunn-Minkowski inequality and it is mainly inspired by [**42**], where sharp global isoperimetric inequalities for $\mathrm{CD}(0,N)$ spaces have been proved (see also [**29**] for a refinement and the previous [**52**] and [**110**] for the same result in the smooth case). It is worth to mention that a class of 'almost-Euclidean' isoperimetric inequalities in essentially nonbranching CD-spaces, similar to the following ones, were proved in [**66**] via localization-technique. However, the results in [**66**] present a set of assumptions that are not suitable for our purposes. Moreover our argument is different and does not assume the space to be essentially non-branching.

Theorem 4.3.1 (Almost-Euclidean isoperimetric inequality)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathrm{CD}(K,N)$ space for some $N\in(1,\infty), K\in\mathbb{R}$. Then for every $0<r<R<\frac12\sqrt{N/K^-}$ (where $\sqrt{N/K^-}=+\infty$ for $K\geq 0$) and $x\in\mathrm{X}$ we have*

$$\mathrm{Per}E\geq \mathfrak{m}(E)^{\frac{N-1}{N}}N\omega_N^{\frac1N}\theta_{N,R}^{\frac1N}(x)(1-(2C^{1/N}+1)\delta-2\eta),\quad \forall\, E\subset B_r(x),\tag{4.3.1}$$

where $\delta:=\frac{r}{R}$, $\eta:=R\sqrt{K^-/N}$ and $C:=\theta_{N,r}(x)/\theta_{N,R}(x)$.

Proof. It is sufficient to prove (4.3.1) with the Minkowski content $\mathfrak{m}(E)^+$ instead of the perimeter. Indeed we could then apply the approximation result in Proposition 1.2.27 to deduce that for every $r'\in(r,R)$, (4.3.1) holds with $r=r'$ (this time with $\mathrm{Per}(E)$). Noticing that $\theta_{N,r'}(x)\to\theta_{N,r}(x)$ as $r'\downarrow r$, sending $r'\to r$ would give the conclusion.

Let $r,R\in\mathbb{R}^+$ with $r<R$ and fix $E\subset B_r(x_0)$ with $\mathfrak{m}(E)>0$. We aim to apply the Brunn-Minkowski inequality to the sets $A_0:=E$, $A_1:=B_R(x_0)$. The triangle inequality easily yields that $A_t\subset E^{t(r+R)}$ for every $t\in(0,1)$ (recall that E^ε is the ε-enlargement of the set E, while A_t is the set of t-midpoint between A_0,A_1). We consider first the case $K\geq 0$. From the Brunn-Minkowski applied with $K=0$ we obtain

$$\begin{aligned}\mathfrak{m}^+(E)=\varliminf_{\varepsilon\to0^+}\frac{\mathfrak{m}(E^\varepsilon)-\mathfrak{m}(E)}{\varepsilon}&=\varliminf_{t\to0^+}\frac{\mathfrak{m}(E^{t(r+R)})-\mathfrak{m}(E)}{t(r+R)}\\ &\overset{(1.3.4)}{\geq}\varliminf_{t\to0^+}\frac{\big(t\mathfrak{m}(B_R(x_0))^{1/N}+(1-t)\mathfrak{m}(E)^{1/N}\big)^N-\mathfrak{m}(E)}{t(r+R)}\\ &=N\mathfrak{m}(E)^{\frac{N-1}{N}}\frac{\mathfrak{m}(B_R(x_0))^{1/N}-\mathfrak{m}(E)^{1/N}}{r+R}\\ &\geq N\mathfrak{m}(E)^{\frac{N-1}{N}}\frac{\mathfrak{m}(B_R(x_0))^{1/N}-\mathfrak{m}(B_r(x_0))^{1/N}}{r+R},\end{aligned}$$

where we have used that $E\subset B_r(x_0)$. If instead $K<0$, arguing analogously we obtain

$$\mathfrak{m}^+(E)\geq\frac{N\mathfrak{m}(E)^{\frac{N-1}{N}}}{r+R}\Big(\frac{\theta\sqrt{-K/N}}{\sinh(\theta\sqrt{-K/N})}\mathfrak{m}(B_R(x_0))^{\frac1N}-\frac{\theta\sqrt{-K/N}\cosh(\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})}\mathfrak{m}(B_r(x_0))^{\frac1N}\Big),$$

where as above θ denotes the maximal length of geodesics from A_0 to A_1. It is clear that $\theta \le r+R$. Moreover for $t\le 1$ we both have $1-t\le t/\sinh(t)\le 1$ and $\cosh(t)\le 1+t$. In particular if $R\le \frac12\sqrt{-N/K}$ we obtain that

$$\mathfrak{m}^+(E)\ge N\mathfrak{m}(E)^{\frac{N-1}{N}}\frac{(1-\sqrt{-K/N}(r+R))\mathfrak{m}(B_R(x_0))^{1/N}-(1+\sqrt{-K/N}(r+R))\mathfrak{m}(B_r(x_0))^{1/N}}{r+R}.$$

Going back to the case of a general $K\in\mathbb{R}$, combining the above estimates and rearranging the terms we reach

$$\mathfrak{m}^+(E)\ge \frac{\mathfrak{m}(E)^{\frac{N-1}{N}}N\omega_N^{\frac1N}\theta_{N,R}(x)^{\frac1N}}{1+r/R}\Big((1-\sqrt{\tfrac{K^-}{N}}(r+R))-(1+\sqrt{\tfrac{K^-}{N}}(r+R))\frac{r}{R}\Big(\frac{\theta_{N,r}(x)}{\theta_{N,R}(x)}\Big)^{\frac1N}\Big),$$

provided $R\le\frac12\sqrt{N/K^-}$. Setting $\delta:=\frac{r}{R}$, $\eta:=R\sqrt{K^-/N}$ and $C:=\theta_{N,r}(x)/\theta_{N,R}(x)$, the above gives

$$\mathfrak{m}^+(E)\ge\mathfrak{m}(E)^{\frac{N-1}{N}}N\omega_N^{\frac1N}\theta_{N,R}(x)^{\frac1N}\frac{1}{1+\delta}\big((1-2\eta)-(1+2\eta)\delta C^{\frac1N}\big)$$

that easily implies the conclusion. □

4.3.2. Proof of the local Sobolev inequalities. In this section we will prove the following result.

Theorem 4.3.2 (Local Euclidean-Sobolev inequality). *For every $\varepsilon>0$, $N\in(1,\infty)$ and $D>0$ there exists $\delta=\delta(\varepsilon,D,N)>0$ such that the following holds. Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathrm{CD}(K,N)$ space, $N<+\infty$. Let $r,R\in(0,\frac12\sqrt{N/K^-})$ and $x\in\mathrm{X}$ be such that $r<\delta R$, $R<\delta\sqrt{N/K^-}$ (with $\sqrt{N/K^-}:=+\infty$ if $K\ge0$) and $\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_R(x))}\le D\frac{r^N}{R^N}$. Then*

$$\|u\|_{L^{p^*}(\mathfrak{m})}\le(1+\varepsilon)\mathsf{Eucl}(N,p)\left(\frac{\mathfrak{m}(B_R(x))}{R^N\omega_N}\right)^{-\frac1N}\||Du|\|_{L^p(\mathfrak{m})},\quad\forall u\in\mathsf{LIP}_c(B_r(x)). \tag{4.3.2}$$

The strategy of the proof will be to argue via symmetrization exploiting the local isoperimetric inequalities obtained in the previous and the following classical one-dimensional inequality by Bliss [**50**] (see also [**33, 202**]).

Lemma 4.3.3 (Bliss inequality). *Let $u\in\mathrm{AC}_{loc}(\mathbb{R})$. Then for any $1<p<N$ it holds*

$$\Big(\sigma_{N-1}\int_0^\infty|u|^{p^*}t^{N-1}\,\mathrm{d}t\Big)^{\frac{1}{p^*}}\le\mathsf{Eucl}(N,p)\Big(\sigma_{N-1}\int_0^\infty|u'|^p\,t^{N-1}\,\mathrm{d}t\Big)^{\frac1p}, \tag{4.3.3}$$

whenever one side is finite and where $p^*=pN/(N-p)$. Moreover the functions $v_b(r):=(1+br^{\frac{p}{p-1}})^{\frac{p-N}{p}}$, $b>0$, satisfy (4.3.3) with equality.

Proof of Theorem 4.3.2. Fix $u\in\mathsf{LIP}_c(B_r(x))$ and consider $u^*_{0,N}:B_r(x)^*\to[0,\infty)$ be the Euclidean-rearrangement of u as in Definition 4.2.7, where $B_r(x)^*=[0,t]$ for some $t>0$. The local Euclidean-isoperimetric inequality given by Theorem 4.3.1 implies that the hypotheses of Theorem 4.2.12 are fulfilled with $\Omega=B_r(x)$ and $\mathsf{C}_{\mathsf{Isop}}=(1-(2D^{1/N}+1)\delta'-2\eta)N\omega_N^{\frac1N}\theta_{N,R}(x)^{\frac1N}$, with $\delta':=\frac{r}{R}$, $\eta:=R\sqrt{K^-/N}$ and $D:=\theta_{N,r}(x)/\theta_{N,R}(x)$. In particular it holds that $u^*_{0,N}\in W^{1,p}([0,t],|.|,\mathfrak{m}_{0,N})$, which implies (recall (1.2.1)) that $u^*_{0,N}\in W^{1,1}_{\mathsf{loc}}(0,t)$ with $(u^*_{0,N})'\in L^p((0,t),\mathfrak{m}_{0,N})$ and $|Du^*_{0,N}|=|(u^*_{0,N})'|$ a.e. and thus that $u^*_{0,N}$ satisfies the assumptions for the Bliss inequality. Recall also from Proposition 4.2.8 that $\|u^*_{0,N}\|_{L^p(\mathfrak{m}_{0,N})}=\|u\|_{L^p(\mathfrak{m})}$ for every $p\in[1,\infty)$. Therefore we are in position to apply the Euclidean Polya-Szego inequality given by (4.2.8), that combined with the Bliss-inequality (4.3.3) gives

$$\begin{aligned}\|u\|_{L^{p^*}(\mathfrak{m})}=\|u^*_{0,N}\|_{L^{p^*}(\mathfrak{m}_{0,N})}&\overset{(4.3.3)}{\le}\mathrm{Eucl}(N,p)\||Du^*_{0,N}|\|_{L^p(\mathfrak{m}_{0,N})}\\&\overset{(4.2.8)}{\le}\frac{\mathsf{Eucl}(N,p)\theta_{N,R}(x)^{-\frac1N}}{(1-(2D^{1/N}+1)\delta'-2\eta)}\||Du|\|_{L^p(\mathfrak{m})}.\end{aligned}$$

Finally, we manipulate in the above to notice that there is an admissible $\delta:=\delta(\varepsilon,D,N)$ so that (4.3.2) holds. □

We end this section with another simpler variant of local Sobolev inequality. It will be needed to deal with cases in which $\theta_N(x)=+\infty$, when Theorem 4.3.2 degenerates to trivial informations at small scales.

Proposition 4.3.4 (Local Sobolev embedding)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a* $\mathrm{CD}(K,N)$ *space for some* $N\in[1,\infty)$, $K\in\mathbb{R}$. *Then, for every* $p\in(1,N)$ *and every* $B_r(x)\subset \mathrm{X}$ *with* $r\le 1$, *it holds*

$$\Big(\int_{B_r(x)}|u|^{p^*}\,\mathrm{d}\mathfrak{m}\Big)^{\frac{p}{p^*}}\le\Big(\frac{C\mathfrak{m}(B_r(x))}{r^N}\Big)^{-\frac{p}{N}}\int_{B_{2r}(x)}|Du|^p\,\mathrm{d}\mathfrak{m}+2^p\mathfrak{m}(B_r(x))^{-\frac{p}{N}}\int_{B_r(x)}|u|^p\,\mathrm{d}\mathfrak{m},\tag{4.3.4}$$

for every $u\in\mathsf{LIP}(\mathrm{X})$, *where* $p^*=pN/(N-p)$ *and* $C=C(K,N,p)$.

PROOF. Applying (1.3.7) and the Bishop-Gromov inequality

$$\begin{aligned}\Big(\int_{B_r(x)}|u|^{p^*}\,\mathrm{d}\mathfrak{m}\Big)^{\frac{1}{p^*}}&\le C_1r\frac{\mathfrak{m}(B_r(x))^{1/p^*}}{\mathfrak{m}(B_{2r}(x))^{1/p}}\Big(\int_{B_{2r}(x)}|Du|^p\Big)^{\frac1p}+\mathfrak{m}(B_r(x))^{1/p^*}|u_{B_r(x)}|\\&\le C_2r\frac{\mathfrak{m}(B_r(x))^{1/p^*}}{\mathfrak{m}(B_r(x))^{1/p}}\Big(\int_{B_{2r}(x)}|Du|^p\Big)^{\frac1p}+\mathfrak{m}(B_r(x))^{\frac{1}{p^*}-\frac1p}\Big(\int_{B_r(x)}|u|^p\,\mathrm{d}\mathfrak{m}\Big)^{\frac1p},\end{aligned}$$

for suitable positive constants C_1,C_2 depending only on K,N,p. The desired conclusion follows elevating to the p the above inequality. □

4.4. Stability of Sobolev embeddings under mGH-convergence

In this section we show that if a pmGH-converging sequence of spaces satisfy a Sobolev embedding with some constants, then also the limit space satisfy the same embedding with the same constants (see Lemma 4.4.1). We will then combine this fact with both blow-up and blow-down arguments to derive lower bounds for Sobolev constants.

4.4.1. Blow-up analysis of Sobolev constants. We start proving the following stability result for Sobolev embeddings (see also [**141**, Thm. 3.1] for a similar result for Ricci-limits).

Lemma 4.4.1 (pmGH-Stability of Sobolev constants)**.** *Let* $(\mathrm{X}_n,\mathsf{d}_n,\mathfrak{m}_n,x_n)$, $n\in\bar{\mathbb{N}}$, *be a sequence of* $\mathrm{CD}(K,N)$ *spaces for some* $K\in\mathbb{R}$, $N\in(1,\infty)$ *with* $\mathrm{X}_n\overset{pmGH}{\to}X_\infty$. *Suppose* X_n *support a* (q,p)*-Sobolev inequality for* $1<p<q$ *with constants* A,B. *Then also* X_∞ *supports a* (q,p)*-Sobolev inequality with the same constants* A,B.

PROOF. Fix $u\in\mathsf{LIP}_c(\mathrm{X}_\infty)$. From the Γ-$\overline{\lim}$ inequality of the Ch_p energy, there exists a sequence $u_n\in W^{1,p}(\mathrm{X}_\infty)$ such that u_n converges in L^p-strong to u and $\overline{\lim}_n\int|Du|^p\,\mathrm{d}\mathfrak{m}_n\le\int|Du|^p\,\mathrm{d}\mathfrak{m}_\infty$. In particular

$$\begin{aligned}\overline{\lim_n}\,\|u_n\|^p_{L^q(\mathfrak{m}_n)}&\le\overline{\lim_{n\to\infty}}\,A\||Du_n|\|^p_{L^p(\mathfrak{m}_n)}+B\|u_n\|^p_{L^p(\mathfrak{m}_n)}\\&\le A\||Du|\|^p_{L^p(\mathfrak{m}_\infty)}+B\|u\|^p_{L^p(\mathfrak{m}_\infty)}<+\infty.\end{aligned}$$

Therefore u_n converge also L^q-weak to u. From the lower semicontinuity of the L^q-norm with respect to L^q-weak convergence and the arbitrariness of $u\in\mathsf{LIP}_c(\mathrm{X}_\infty)$ the conclusion follows. □

Our goal will be to employ the above result to get lower bounds for the constants in critical Sobolev inequalities. To key fact is the following scaling properties of the Sobolev constants: if a space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ supports a (p^*,p)-Sobolev inequality with constants A,B, where $p\in(1,N)$ and $p^*:=\frac{pN}{N-p}$, then for every $r>0$ we have that

$$(\mathrm{X},\mathsf{d}/r,\mathfrak{m}/r^N)\text{ supports a }(p^*,p)\text{-Sobolev with constants }A,Br^p.\tag{4.4.1}$$

We now show that if a space satisfies $\frac{\mathfrak{m}(B_r(x_0))}{\omega_Nr^N}=1$ for every $r\in(0,\infty)$, then its Sobolev constant is at least the one of the Euclidean space.

We need the following result, that is a consequence of the existence of the disintegration and can be found for example in [**95**, Corollary 3.8].

Lemma 4.4.2. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a* $\mathrm{CD}(0,N)$ *space with* $N\in[1,\infty)$. *Suppose that for some* $x_0\in\mathrm{X}$ *it holds that* $\frac{\mathfrak{m}(B_r(x_0))}{\omega_Nr^N}=1$ *for every* $r\in(0,\infty)$, *then*

$$\int\varphi(\mathsf{d}(x_0,x))\,\mathrm{d}\mathfrak{m}=\sigma_{N-1}\int_0^\infty\varphi(r)r^{N-1}\,\mathrm{d}r,\qquad\forall\varphi\in C_c([0,\infty]).$$

Lemma 4.4.3. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a* $\mathrm{CD}(0,N)$ *space,* $N\in[1,\infty)$ *and set* $p^*:=\frac{pN}{N-p}$. *Suppose that for some* $x_0\in\mathrm{X}$ *it holds that* $\frac{\mathfrak{m}(B_r(x_0))}{\omega_Nr^N}=1$ *for every* $r\in(0,\infty)$. *Then there exists a sequence of non-constant functions* $u_n\in\mathsf{LIP}_c(\mathrm{X})$ *satisfying*

$$\lim_n\frac{\|u_n\|_{L^{p^*}(\mathfrak{m})}}{\||Du_n|\|_{L^p(\mathfrak{m})}}\ge\mathsf{Eucl}(N,p).$$

Proof. Let $v : [0,\infty) \to [0,\infty)$, $v \in C^\infty(0,\infty)$, be an extremal function for the Bliss inequality (4.3.3) as given by Lemma 4.3.3. It can be easily shown that we can approximate v with functions $v_n \in \mathsf{LIP}_c([0,\infty))$ so that $\|v_n\|_{L^{p^*}(h_N\mathscr{L}^1)} \to \|v\|_{L^{p^*}(h_N n\mathscr{L}^1)}$ and $\|v_n'\|_{L^{p^*}(h_N\mathscr{L}^1)} \to \|v'\|_{L^{p^*}(h_N\mathscr{L}^1)}$, where $h_N\,\mathscr{L}^1 = \sigma_{N-1}t^{N-1}\,\mathscr{L}^1$. For example we can take $v_n \coloneqq \varphi_n(u_b)$ with $\varphi_n \in \mathsf{LIP}[0,\infty)$, $\varphi_n \ge 0$, $\varphi_n(t) \le |t|$, $\mathrm{Lip}(\varphi_n) \le 2$, $\varphi_n(t) = t$ in $[2/n,\infty)$ and $\mathrm{supp}(\varphi_n) \subset [1/n,\infty)$. The claimed approximation of the norms then follows immediately from the fact that v is decreasing and vanishing at infinity. Therefore we have

$$\lim_n \frac{\|v_n\|_{L^{p^*}(h_n\mathscr{L}^1)}}{\|v_n'\|_{L^p(h_n\mathscr{L}^1)}} = \mathsf{Eucl}(N,p). \tag{4.4.2}$$

We can now define $u_n \coloneqq v_n \circ \mathsf{d}_{x_0}$, where $\mathsf{d}_{x_0}(.) \coloneqq \mathsf{d}(x_0,.)$. We clearly have that $u_n \in \mathsf{LIP}_c(\mathrm{X})$ and from the chain rule also that $|Du_n| = |v_n'| \circ \mathsf{d}_{x_0}|D\mathsf{d}_{x_0}| \le |v_n'| \circ \mathsf{d}_{x_0}$ $\mathfrak{m}$-a.e., since d_{x_0} is 1-Lipschitz. Hence applying Lemma 4.4.2 we obtain $\|u_n\|_{L^{p^*}(\mathfrak{m})} = \|v_n\|_{L^{p^*}(h_N\mathscr{L}^1)}$ and $\||Du_n|\|_{L^p(\mathfrak{m})} \le \|v_n'\|_{L^p(h_N\mathscr{L}^1)}$. This combined with (4.4.2) (up to passing to a subsequence) gives the conclusion. □

We can now prove our main lower bound on the Sobolev constant in terms of the local density.

Theorem 4.4.4 (Lower bound on Sobolev constants). *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathrm{CD}(K,N)$ space, $K \in \mathbb{R}$, $N \in (1,\infty)$ that supports a (p^*,p)-Sobolev inequality with constants A,B for some $p \in (1,N)$. Then*

$$A \ge \frac{\mathrm{Eucl}(N,p)^p}{\theta_N(x)^{\frac{p}{N}}}, \qquad \forall\, x \in \mathrm{X}. \tag{4.4.3}$$

Proof. If $\theta_N(x) = \infty$, there is nothing to prove. Hence we can assume that $\theta_N(x) < +\infty$. From the compactness and stability of the $\mathrm{CD}(K,N)$ condition, there exists a sequence $r_i \to 0$ such that $\mathrm{X}_i \coloneqq (\mathrm{X},\mathsf{d}/r_i,\mathfrak{m}/{r_i}^N,x)$ pmGH-converge to a $\mathrm{CD}(0,N)$ space $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y},p)$. Moreover, from (4.4.1) we have that X_i supports a (p^*,p)-Sobolev inequality with constants A, r_i^pB. This combined with Lemma 4.4.1 shows that $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y})$ supports a (p^*,p)-Sobolev inequality with constants $A,0$. However we clearly have that $\mathfrak{m}_\mathrm{Y}$ satisfies $\frac{\mathfrak{m}_\mathrm{Y}(B_r(p))}{\omega_N r^N} = \theta_N(x)$ for every $r>0$. Therefore Lemma 4.4.3 ensures that $A \ge \frac{\mathrm{Eucl}(N,p)^p}{\theta_N(x)^{\frac{p}{N}}}$, which is what we wanted. □

4.4.2. Sharp Sobolev inequality under Euclidean volume growth. With the same technique of the previous section, arguing instead via blow-down we recover a result (see Theorem 4.4.5 below) proved in [**160**] (see also [**161**] for the case $p=2$), which is the non-smooth analogue of a result due to Ledoux [**162**] and improved by Xia [**213**]. We mention also [**104**] and [**214**] for analogous statements for different classes of embeddings. In all the cited works the arguments are based on intricate ODE-comparison (originated in Ledoux [**162**] and inspired by the previous[**41**]) and heavily rely on the explicit knowledge of the extremal functions. However we are able to give a short proof of Theorem 4.4.5, which uses a more direct blow-down procedure, that we believe being interesting on its own. The main advantage of this approach is that we will never need, as opposed to the ODE-comparison approach, the explicit expression of the extremals functions in the Euclidean Sobolev inequality.

Theorem 4.4.5. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{CD}(0,N)$, $N \in (1,\infty)$ such that for some $p \in (1,N)$ and $A>0$*

$$\|u\|_{L^{p^*}(\mathfrak{m})} \le A\||Du|\|_{L^p(\mathfrak{m})}, \quad \forall u \in \mathsf{LIP}_c(\mathrm{X}),$$

where $p^* \coloneqq \frac{pN}{N-p}$. Then X has Euclidean volume-growth and

$$\mathsf{AVR}(\mathrm{X}) \ge \Big(\frac{\mathsf{Eucl}(N,p)}{A}\Big)^N. \tag{4.4.4}$$

Proof. The fact that $\mathfrak{m}(\mathrm{X}) = +\infty$ can be immediately seen by plugging in the Sobolev inequality functions $u_R \in \mathsf{LIP}_c(\mathrm{X})$ so that $u_R = 1$ in $B_R(x_0)$ $\mathrm{supp}(u_R) \subset B_{2R}(x_0)$ and $\mathrm{Lip}(u_R) \le 1/R$ and sending $R \to +\infty$. The fact that X has Euclidean volume growth follows by considering instead functions $u_R(.) \coloneqq (R - \mathsf{d}_{x_0}(.))^+$ as $R \to +\infty$ with fixed $x_0 \in \mathrm{X}$ and using the Bishop-Gromov inequality.

It remains to prove (4.4.4). We argue via blow-down. Let $R_i \to +\infty$. From the Euclidean volume-growth property, up to passing to a non relabelled subsequence, the rescaled spaces $(\mathrm{X},\mathsf{d}/R_i,\mathfrak{m}/R_i^N,x_0)$, $x_0 \in \mathrm{X}$, pmGH-converge to an $\mathrm{CD}(0,N)$ space $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_Y,\boldsymbol{o}_\mathrm{Y})$ satisfying $\frac{\mathfrak{m}(B_R(\boldsymbol{o}_\mathrm{Y}))}{\omega_N r^N} = \mathrm{AVR}(\mathrm{X})$. Then (4.4.4) follows from Lemma 4.4.3. □

We conclude combining the above result with the local-Sobolev inequalities in Theorem 4.3.2 to derive sharp Sobolev inequalities on $\mathrm{CD}(0,N)$ spaces with Euclidean volume growth. We mention that the a proof of the following result via symmetrization was suggested in [**42**, Sec. 5.2].

Theorem 4.4.6. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a* $\mathrm{CD}(0,N)$ *space for some* $N\in(1,\infty)$ *and with Euclidean volume growth. Then, for every* $p\in(1,N)$*, it holds*

$$\|u\|_{L^{p^*}(\mathfrak{m})}\le \mathsf{Eucl}(N,p)\mathsf{AVR}(\mathrm{X})^{-\frac{1}{N}}\||Du|\|_{L^p(\mathfrak{m})},\qquad \forall u\in\mathsf{LIP}_c(\mathrm{X}). \tag{4.4.5}$$

Moreover (4.4.5) *is sharp.*

Proof. Fix $x\in\mathrm{X}$. From the definition of $\mathsf{AVR}(\mathrm{X})$, for every r big enough $\theta_{N,r}(x)\le 2\mathsf{AVR}(\mathrm{X})$. Fix one of such $r>0$. From the Bishop-Gromov inequality we also have that $\theta_{N,R}(x)\ge\mathsf{AVR}(\mathrm{X})$ for every $R>0$. In particular $\theta_{N,r}(x)/\theta_{N,R}(x)\le 2$ for every $R>0$. Hence by Theorem 4.3.2 (for $K=0$) we have that for every $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ so that for every $R>r/\delta$ the following local Euclidean Sobolev inequality holds:

$$\|u\|_{L^{p^*}(\mathfrak{m})}\le(1+\varepsilon)\mathsf{Eucl}(N,p)\theta_{N,R}(x)^{-\frac{1}{N}}\||Du|\|_{L^p(\mathfrak{m})},\qquad\forall u\in\mathsf{LIP}_c(B_r(x)).$$

Taking $R\to\infty$ we achieve

$$\|u\|_{L^{p^*}(\mathfrak{m})}\le(1+\varepsilon)\mathsf{Eucl}(N,p)\mathsf{AVR}(\mathrm{X})^{-\frac{1}{N}}\||Du|\|_{L^p(\mathfrak{m})},\qquad\forall u\in\mathsf{LIP}_c(B_r(x)).$$

Finally, since ε was chosen independently of $r>0$, we can first take $\varepsilon\downarrow 0$ and then send $r\to+\infty$ to achieve the first part of the statement. The sharpness of (4.4.5) follows immediately from Theorem 4.4.5. $\square$

4.5. Best Sobolev constant in compact CD spaces

We recall the following notion of best constant in the Sobolev inequality. Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a $\mathrm{CD}(K,N)$ space with $N\in(1,\infty)$, $p\in(1,N)$ and consider the following inequality

$$\|u\|^p_{L^{p^*}(\mathrm{X})}\le A\||Du|\|^p_{L^p(\mathrm{X})}+B\|u\|^p_{L^p(\mathrm{X})},\qquad\forall u\in W^{1,p}(\mathrm{X}). \tag{4.5.1}$$

We define

$$\alpha_p(\mathrm{X})\coloneqq\inf\{A\ :\ (4.5.1)\text{ holds for some }B\}. \tag{4.5.2}$$

Extending a famous result by Aubin for Riemannian manifolds [**32**] we compute the constant $\alpha_p(\mathrm{X})$ in every compact $\mathrm{CD}(K,N)$ space:

Theorem 4.5.1. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a compact* $\mathrm{CD}(K,N)$ *space with* $N\in(1,\infty)$*. The for every* $p\in(1,N)$

$$\alpha_p(\mathrm{X})=\left(\frac{\mathsf{Eucl}(N,p)}{\inf_{x\in\mathrm{X}}\theta_N(x)^{\frac{1}{N}}}\right)^p, \tag{4.5.3}$$

where $\theta_N(x)\coloneqq\lim_{r\to0}\frac{\mathfrak{m}(B_r(x))}{\omega_N r^N}$, $x\in\mathrm{X}$.

In view of Theorem 4.4.4 to prove Theorem 4.5.1 it is sufficient to prove the following result. The strategy of the proof is by-now classical and combines local-Sobolev inequalities and partitions of unity (see [**32**], [**33**, Chp. 2], [**136**, Theorem 4.5] and also [**5**, Prop. 3.3]).

Theorem 4.5.2 (Upper bound on α_p)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a compact* $\mathrm{CD}(K,N)$ *space, for some* $N\in[1,\infty)$*,* $K\in\mathbb{R}$*. Then, for every* $\varepsilon>0$ *and every* $p\in(1,N)$*, there exists a constant* $B=B(\varepsilon,p,\mathrm{X})>0$ *such that*

$$\|u\|^p_{L^{p^*}(\mathfrak{m})}\le\Big(\frac{\mathsf{Eucl}(N,p)^p}{\inf_{\mathrm{X}}\theta_N(x)^{p/N}}+\varepsilon\Big)\||Du|\|^p_{L^p(\mathfrak{m})}+B\|u\|^p_{L^p(\mathfrak{m})},\qquad\forall u\in\mathsf{LIP}(\mathrm{X}). \tag{4.5.4}$$

Proof. We start claiming that the following local version of (4.5.4) holds: for any $x\in\mathrm{X}$ and every $\varepsilon>0$ there exists $r=r(\varepsilon,x)>0$ and $C=C(\varepsilon,p,x)<+\infty$ such that

$$\|u\|^p_{L^{p^*}(\mathfrak{m})}\le\Big(\frac{\mathsf{Eucl}(N,p)^p}{\inf_{\mathrm{X}}\theta_N(x)^{p/N}}+\varepsilon\Big)\||Du|\|^p_{L^p(\mathfrak{m})}+C\|u\|^p_{L^p(\mathfrak{m})},\qquad\forall u\in\mathsf{LIP}_c(B_r(x)). \tag{4.5.5}$$

To show the above we observe first that in the case that $\theta_N(x)=+\infty$, (4.5.5) follows immediately from (4.3.4) for r small enough. We are left with the case $0<\theta_N(x)<+\infty$. We start by fixing $\varepsilon\in(0,1/2)$. From the definition of $\theta_N(x)$, there exists $r'=r'(x,\varepsilon)$ so that for every $r\in(0,r')$ it holds that $\theta_{N,r}(x)\in((1-\varepsilon)\theta_N(x),(1+\varepsilon)\theta_N(x))$. In particular we have that $\frac{\theta_{N,r}(x)}{\theta_{N,R}(x)}\le 4$ for every $r,R\in(0,r')$. We are therefore in position to apply Theorem 4.3.2 and deduce that there exists $\delta=\delta(\varepsilon,N)$ so that for every $r,R\in(0,r'\wedge\delta\sqrt{N/K^-})$, with $r<\delta R$, the following inequality holds for every $u\in\mathsf{LIP}_c(B_r(x))$

$$\|u\|^p_{L^{p^*}(\mathfrak{m})}\overset{(4.3.2)}{\le}(1+\varepsilon)^p\frac{\mathsf{Eucl}(N,p)^p}{\theta_{N,R}(x)^{p/N}}\||Du|\|^p_{L^p(\mathfrak{m})}\le\frac{(1+\varepsilon)^p}{(1-\varepsilon)^{p/N}}\frac{\mathsf{Eucl}(N,p)^p}{\inf_{\mathrm{X}}\theta_N(x)^{p/N}}\||Du|\|^p_{L^p(\mathfrak{m})},$$

where in the second inequality we have used $\theta_{N,R}(x)\ge(1-\varepsilon)\theta_N(x)$. Therefore (4.5.5) (with $C=0$) follows from the above provided we choose ε small enough.

Since X is compact we can extract a finite covering of balls $\{B_i\}_{i=1}^M$ from the covering $\cup_{x\in X}B_{r(\varepsilon,x)/2}(x)$. We also set $C := \max_i C_i$ and

$$A := \frac{\mathsf{Eucl}(N,p)^p}{\inf_{\rm X}\theta_N(x)^{p/N}} + \varepsilon.$$

We claim that there exists a partition of unity made of functions $\{\varphi_i\}_{i=1}^M$ such that $\varphi_i \in \mathsf{LIP}_c(2B_i)$, $0 \le \varphi_i \le 1$ and $\varphi_i^{1/p} \in \mathsf{LIP}_c(2B_i)$ for all i, having denoted $2B_i$, the ball of twice the radius. To build such partition of unity we can argue as follows: start considering functions $\psi_i \in \mathsf{LIP}_c(2B_i)$, such that $0 \le \psi_i \le 1$ and $\psi_i \ge 1$ in B_i. Then we fix $\beta > p$ and take

$$\varphi_i := \frac{\psi_i^\beta}{\sum_{j=1}^M \psi_j^\beta}.$$

Since by construction $\sum_{j=1}^M \psi_j^\beta \ge 1$ everywhere on X, we have that $\varphi_i^{1/p} \in \mathsf{LIP}_c(2B_i)$. Finally it is clear that $\sum_{i=1}^M \varphi_i = 1$.

We are now ready to prove (4.5.4). Fix $u \in \mathsf{LIP}(\mathrm{X})$ and observe that

$$\|u\|_{L^{p^*}(\mathfrak{m})}^{p/p*} = \Big\| \sum_i \varphi_i |u|^p \Big\|_{L^{p^*/p}(\mathfrak{m})} \le \sum_i \|\varphi_i |u|^p\|_{L^{p^*/p}(\mathfrak{m})} = \sum_i \big\|\varphi_i^{1/p}|u|\big\|_{L^{p^*}(\mathfrak{m})}^p.$$

Since $\varphi_i^{1/p}|u| \in \mathsf{LIP}_c(2B_i)$ we can apply (4.5.5) to obtain

$$\begin{aligned}\|u\|_{L^{p^*}(\mathfrak{m})}^{p/p*} &\le \sum_{i=1}^M A\int \Big(|D\varphi_i^{1/p}||u| + |Du|\varphi_i^{1/p}\Big)^p \,\mathrm{d}\mathfrak{m} + C\int \varphi_i|u|^p\,\mathrm{d}\mathfrak{m}\\ &\le \sum_{i=1}^M A\int \varphi_i|Du|^p + c_1|Du|^{p-1}\varphi_i^{\frac{p-1}{p}}|D\varphi_i^{1/p}||u| + c_2|D\varphi_i^{1/p}|^p|u|^p\,\mathrm{d}\mathfrak{m} + C\int \varphi_i|u|^p\,\mathrm{d}\mathfrak{m},\end{aligned}$$

where $c_1, c_2 \ge 0$ are such that $(1+t)^p \le 1+c_1t+c_2t^p$ for all $t \ge 0$. Recalling that the functions $0 \le \varphi_i^{1/p} \le 1$ are Lipschitz we obtain

$$\|u\|_{L^{p^*}(\mathfrak{m})}^{p/p*} \le A\int |Du|^p\,\mathrm{d}\mathfrak{m} + \tilde C\int |Du|^{p-1}|u|\,\mathrm{d}\mathfrak{m} + \tilde C\int |u|^p\,\mathrm{d}\mathfrak{m},$$

where $\tilde C = \tilde C(p, M, L)$, L begin the maximum of the Lipschitz constants of $\varphi^{1/p}$. Finally from the Young inequality we have

$$\int |Du|^{p-1}|u|\,\mathrm{d}\mathfrak{m} \le \frac{p\delta^{\frac{p}{p-1}}}{p-1}\int |Du|^p\,\mathrm{d}\mathfrak{m} + \frac{1}{p\delta^p}\int |u|^p\,\mathrm{d}\mathfrak{m}, \quad \forall \delta > 0$$

and plugging this estimate above, choosing δ small enough (but independent of u), we obtain that

$$\|u\|_{L^{p^*}(\mathfrak{m})}^{p/p*} \le (A+\varepsilon)\int |Du|^p\,\mathrm{d}\mathfrak{m} + C'\int |u|^p\,\mathrm{d}\mathfrak{m},$$

for some $C' = C'(\varepsilon, L, M, p)$. Since $\varepsilon > 0$ and $u \in \mathsf{LIP}(\mathrm{X})$ were arbitrary, this concludes the proof. □

4.6. The constant $A_q^{\rm opt}$ in metric measure spaces

In this section we will prove some upper and lower bounds on $A_q^{\rm opt}(\mathrm{X})$ in the case of metric measure spaces. The results in Section 4.6.3 are actually not used anywhere-else in the chapter, however we include them here for completeness and to give a more clear picture around the value of $A_q^{\rm opt}(\mathrm{X})$. Let us also remark that the results of this part are valid for a general lower bound $K \in \mathbb{R}$.

We start giving a precise definition of $A_q^{\rm opt}(\mathrm{X})$. For a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, with $\mathfrak{m}(\mathrm{X}) = 1$, and for every $q \in (2, +\infty)$ we define $A_q^{\rm opt}(\mathrm{X}) \in [0, +\infty]$ as the minimal constant satisfying

$$\|u\|_{L^q(\mathfrak{m})}^2 \le A_q^{\rm opt}(\mathrm{X})\,\||Du|_2\|_{L^2(\mathfrak{m})}^2 + \|u\|_{L^2(\mathfrak{m})}^2, \quad \forall u \in W^{1,2}(\mathrm{X}), \tag{4.6.1}$$

with the convention that $A := +\infty$ if no such A exists. In the following sections we will prove three type of bounds on $A_q^{\rm opt}(\mathrm{X})$: an upper bound in the case of synthetic Ricci curvature and dimension bounds; a lower bound in terms of the first non-trivial eigenvalue; a lower bound related to the diameter.

4.6.1. Upper bound on $A_q^{\rm opt}$ in terms of curvature and diameter bounds. Here extend to the non-smooth setting a classical estimate on $A_q^{\rm opt}(\mathrm{X})$ depending only on the curvature, dimension and diameter of a manifold (see [**136**]). The two key ingredients for the proof are the Sobolev-Poincaré inequality and an inequality due to Bakry:

Proposition 4.6.1. *For every $K \in \mathbb{R}$ $N \in (2,\infty)$ and $D>0$ there exists a constant $A=A(K,N,D)>0$ such that the following holds. Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a compact* $\mathrm{CD}(K,N)$ *space with $N \in (1,\infty)$, $K\in\mathbb{R}$, $\mathfrak{m}(\mathrm{X})=1$ and diam$(\mathrm{X})\le D$. Then for every $q\in(2,2^*]$ we have*

$$\|u\|^2_{L^q(\mathfrak{m})} \le A\||Du|\|^2_{L^2(\mathfrak{m})} + \|u\|^2_{L^2(\mathfrak{m})}, \qquad \forall u \in W^{1,2}(\mathrm{X}) \tag{4.6.2}$$

and in particular $A_q^{\rm opt}(\mathrm{X}) \le A(K,N,D)$.

Proof. The proof is based on the following inequality: for every $q\in(2,\infty)$

$$\Big(\int |u|^q \,\mathrm{d}\mathfrak{m}\Big)^{2/q} \le (u_{\mathrm{X}})^2 + (q-1)\Big(\int |u-u_{\mathrm{X}}|^q\,\mathrm{d}\mathfrak{m}\Big)^{2/q} \qquad \forall u \in L^q(\mathfrak{m}), \tag{4.6.3}$$

where $u_{\mathrm{X}} = \int u\,\mathrm{d}\mathfrak{m}$. See ([**37**] or [**40**, Prop. 6.2.2]) for a proof of this fact. Then (4.6.2) follows combining (4.6.3) with (1.3.7) and the Jensen inequality. □

Recall that in the case $K>0$ an explicit and sharp upper bound on $A_q^{\rm opt}(\mathrm{X})$ exists and has been proven in [**65**] (see Theorem 0.16). The argument in [**65**] relies on the powerful *localization technique.* However, it is worth to point out that Theorem 0.16 can also be deduced from the Polya-Szego inequality proved in [**182**] (see Theorem 4.2.2) and the Sobolev inequality on the model space (1.3.8).

4.6.2. Lower bound on $A_q^{\rm opt}$ in terms of the first eigenvalue. It is well known that a tight-Sobolev inequality as in (4.6.1) (i.e. with a constant 1 in front of $\|u\|_{L^2}$ when X is normalized with unit volume) implies a Poincaré-inequality (see e.g. [**40**, Prop. 6.2.2]). This can be rephrased as a lower bound on $A_q^{\rm opt}$ in terms of the first non-trivial eigenvalue:

Proposition 4.6.2. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space with $\mathfrak{m}(\mathrm{X})=1$. Then for every $q\in(2,+\infty)$ it holds*

$$A_q^{\rm opt}(\mathrm{X}) \ge \frac{q-2}{\lambda^{1,2}(\mathrm{X})}, \tag{4.6.4}$$

(meaning that if $\lambda^{1,2}(\mathrm{X})=0$, then $A_q^{\rm opt}(\mathrm{X})=+\infty$).

We will give a detailed proof this result, which amounts to a linearization procedure. Indeed a refinement of the same argument will also play a key role on the rigidity and almost-rigidity results in the sequel (see Section 4.8.1).

We start with a preliminary elementary linearization-Lemma.

Lemma 4.6.3. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space with $\mathfrak{m}(\mathrm{X})=1$ and fix $q\in(2,\infty)$. Let $f\in L^2\cap L^q(\mathfrak{m})$ with $\int f\,\mathrm{d}\mathfrak{m}=0$. Then*

$$\begin{aligned}\Big|\Big(\int |1+f|^q\,\mathrm{d}\mathfrak{m}\Big)^{2/q} &- \int (1+f)^2\,\mathrm{d}\mathfrak{m} - (q-2)\int |f|^2\,\mathrm{d}\mathfrak{m}\Big| \\ &\le C_q\Big(\int |f|^{3\wedge q} + |f|^q\,\mathrm{d}\mathfrak{m} + \Big(\int |f|^q\,\mathrm{d}\mathfrak{m}\Big)^2 + \Big(\int |f|^2\,\mathrm{d}\mathfrak{m}\Big)^2\Big),\end{aligned} \tag{4.6.5}$$

where C_q is a constant depending only on q.

Proof. We start defining $I \coloneqq \int |1+f|^q - 1$ and observe that

$$\Big|\Big(\int |1+f|^q\,\mathrm{d}\mathfrak{m}\Big)^{2/q} - 1 - \frac{2}{q}I\Big| \le c_q|I|^2, \tag{4.6.6}$$

which follows from the inequality $||1+t|^{2/q}-1-2t/q| \le c_q t^2$, $t\ge 0$. It remains to investigate the behaviour of I. Exploiting the inequality $||1+t|^q-1-qt| \le \tilde c_q(|t|^2+|t|^q)$, $t\ge0$, and the fact that f has zero mean we have the following simple bound

$$|I| \le \tilde c_q \int |f|^2 + |f|^q\,\mathrm{d}\mathfrak{m}. \tag{4.6.7}$$

We will also need a more precise estimate of I, which will follow from the following inequality

$$\Big||1+t|^q - 1 - qt - \frac{q(q-1)}{2}t^2\Big| \le C_q(|t|^{3\wedge q} + |t|^q), \quad \forall t \in \mathbb{R}, \tag{4.6.8}$$

that can be seen using Taylor expansion when $|t| \le 1/2$ and elementary estimates in the case $|t| \ge 1/2$. Using (4.6.8) we obtain that

$$\Big|I - \int qf + \frac{q(q-1)}{2}|f|^2\,{\rm d}\mathfrak{m}\Big| \le C_q \int |f|^{3\wedge q} + |f|^q\,{\rm d}\mathfrak{m}$$

and since we are assuming that f has zero mean, we deduce

$$\Big|I - \frac{q(q-1)}{2}\int |f|^2\,{\rm d}\mathfrak{m}\Big| \le C_q \int |f|^{3\wedge q} + |f|^q\,{\rm d}\mathfrak{m}. \tag{4.6.9}$$

Combining (4.6.6), (4.6.7) and (4.6.9) we deduce that (4.6.5). □

We can now prove the above lower bound on $A_q^{\rm opt}({\rm X})$.

Proof of Proposition 4.6.2. If $A_q^{\rm opt}({\rm X}) = +\infty$ there is nothing to prove, hence we assume that $A_q^{\rm opt}({\rm X}) < +\infty$. Let $f \in \mathsf{LIP}({\rm X}) \cap L^2(\mathfrak{m})$ with $\int f\,{\rm d}\mathfrak{m} = 0$ and $\|f\|_{L^2(\mathfrak{m})} = 1$. Observe also that, since $A_q^{\rm opt}({\rm X}) < +\infty$, $f \in L^q({\rm X})$. Therefore applying (4.6.5) we obtain

$$\Big(\int |1+\varepsilon f|^q\,{\rm d}\mathfrak{m}\Big)^{2/q} - \int (1+\varepsilon f)^2\,{\rm d}\mathfrak{m} - (q-2)\int |\varepsilon f|^2\,{\rm d}\mathfrak{m} = o(\varepsilon^2),$$

which combined with (4.6.1) gives

$$A_q^{\rm opt}({\rm X})\varepsilon^2 \int |Df|_2^2\,{\rm d}\mathfrak{m} - (q-2)\int |\varepsilon f|^2\,{\rm d}\mathfrak{m} \ge o(\varepsilon^2).$$

Dividing by ε^2 and sending $\varepsilon \to 0$ gives that $\lambda^{1,2}({\rm X}) \ge \frac{q-2}{A_q^{\rm opt}({\rm X})}$, which concludes the proof. □

4.6.3. Lower bound on $A_q^{\rm opt}$ in terms of the diameter. We start recalling the following result, which was proved in [**41**] in the context of Dirichlet-forms and which proof (that we shall omit) works with straightforward modifications also in the setting of metric measure spaces (see also [**136**] for an exposition of the argument on Riemannian manifolds). We stress that, since this result is used only on this section, the exposition of the rest of the chapter remains self-contained.

Theorem 4.6.4. *Let $q \in (2,\infty)$ and define $N(q) := \frac{2q}{q-2}$. Let $({\rm X},\mathsf{d},\mathfrak{m})$ be a compact metric measure with diam$({\rm X}) = \pi$, $\mathfrak{m}({\rm X}) = 1$ and suppose that*

$$\|u\|_{L^q(\mathfrak{m})} \le \frac{q-2}{N(q)}\||Du|\|_{L^2}^2 + \|u\|^2_{L^2(\mathfrak{m})}, \quad \forall u \in W^{1,2}({\rm X}). \tag{4.6.10}$$

Then there exists a non-constant function $f \in \mathsf{LIP}({\rm X})$ realizing equality in (4.6.10).

Note that $q = 2N(q)/(N(q)-2)$, so that in a sense "$q = 2^*(N(q))$".

With Theorem 4.6.4 we can now prove the following lower bound on $A_q^{\rm opt}({\rm X})$. The proof uses a scaling argument due to Hebey [**136**, Proposition 5.11].

Proposition 4.6.5. *Let $({\rm X},\mathsf{d},\mathfrak{m})$ be a compact metric measure space with $\mathfrak{m}({\rm X}) = 1$ and* diam$({\rm X}) \le \pi$. *Then for every $q \in (2,\infty)$ it holds*

$$A_q^{\rm opt}({\rm X}) \ge \Big(\frac{{\rm diam}(X)}{\pi}\Big)^2 \frac{q-2}{N(q)}., \tag{4.6.11}$$

where $N(q) = \frac{2q}{q-2}$.

Proof. Set $D := {\rm diam}(X)$ and, by contradiction, suppose that $A_q^{\rm opt}({\rm X}) < (\frac{D}{\pi})^2\frac{q-2}{N(q)}$. Define the scaled the metric measure space

$$({\rm X}',\mathsf{d}',\mathfrak{m}') := ({\rm X}, \tfrac{1}{D/\pi}\mathsf{d}, \mathfrak{m}).$$

It can be directly checked that ${\rm X}'$ satisfies the hypotheses of Theorem 4.6.4. Hence there exists a non-constant function $u \in \mathsf{LIP}({\rm X})$ satisfying (4.6.10) with equality (in the space ${\rm X}'$), which rewritten on the the original space X reads as

$$\|u\|_{L^q(\mathfrak{m})} = \Big(\frac{D}{\pi}\Big)^2 \frac{q-2}{N(q)}\||Du|\|^2_{L^2(\mathfrak{m})} + \|u\|^2_{L^2(\mathfrak{m})},$$

which however contradicts the assumption $A_{2^*}^{\rm opt}({\rm X}) < (\frac{D}{\pi})^2\frac{q-2}{N(q)}$. □

4.7. Concentration Compactness and mGH-convergence

In this section we assume that $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$ is a sequence of compact RCD(K,N) spaces, for some fixed $K \in \mathbb{R}$, $N \in (2,\infty)$, which converges in mGH-topology to a compact RCD(K,N) space $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$. We will also adopt the extrinsic approach identifying $\mathrm{X}_n, \mathrm{X}_\infty$ as subsets of a common compact metric space $(\mathrm{Z}, \mathsf{d}_\mathrm{Z})$, with $\mathrm{supp}(\mathfrak{m}_n) = \mathrm{X}_n$, $\mathrm{supp}(\mathfrak{m}_\infty) = \mathrm{X}_\infty$, $\mathfrak{m}_n \rightharpoonup \mathfrak{m}_\infty$ in duality with $C_b(\mathrm{Z})$ and $X_n \to \mathrm{X}_\infty$ in the Hausdorff topology of Z. To lighten the discussion, we shall not recall in the following statements these facts and assume $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$, $n \in \bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and (Z, d) to be fixed as just explained. Also, we will set $2^* \coloneqq 2N/(N-2)$ without recalling its expression in the statements.

Our main goal is to prove the following dichotomy for the behaviour of extremizing sequence in the Sobolev inequality on varying metric measure spaces.

Theorem 4.7.1 (Concentration-compactness for Sobolev-extremals)**.** *Suppose that $\mathfrak{m}_n(\mathrm{X}_n)$, $\mathfrak{m}_\infty(\mathrm{X}_\infty) = 1$ and that X_n supports a $(2^*,2)$-Sobolev inequality*

$$\|u\|^2_{L^{2^*}(\mathfrak{m}_n)} \le A\||Du|\|^2_{L^2(\mathfrak{m}_n)} + B\|u\|^2_{L^2(\mathfrak{m}_n)}, \quad \forall u \in W^{1,2}(\mathrm{X}_n),$$

for some constants $A, B > 0$. Suppose that $u_n \in W^{1,2}(\mathrm{X}_n)$ is a sequence of non-zero functions satisfying

$$\|u_n\|^2_{L^{2^*}(\mathfrak{m}_n)} \ge A_n\||Du_n|\|^2_{L^2(\mathfrak{m}_n)} + B_n\|u_n\|^2_{L^2(\mathfrak{m}_n)},$$

for some sequences $A_n \to A$, $B_n \to B$.

Then, setting $\tilde{u}_n \coloneqq u_n\|u_n\|^{-1}_{L^q(\mathfrak{m}_n)}$, there exists a non relabelled subsequence such that only one of the following holds:

I) *$\tilde{u}_n$ converges L^{2^*}-strongly to a function $u_\infty \in W^{1,2}(\mathrm{X}_\infty)$;*
II) *$\|\tilde{u}_n\|_{L^2(\mathfrak{m}_n)} \to 0$ and there exists $x_0 \in \mathrm{X}_\infty$ so that $|u_n|^{2^*}\mathfrak{m}_n \rightharpoonup \delta_{x_0}$ in duality with $C_b(\mathrm{Z})$.*

The principle behind the concentration compactness technique is very general and was originated in [**169**, **170**]. In our case, since we will work in a compact setting, the lack of compactness is formally due to dilations or rescalings (and not to translations) and the fact that we deal with the critical exponent in the Sobolev embedding. The main idea behind the principle is first to prove that in general the failure of compactness can only be realized throughout concentration on a countable number of points. The second step is then exploit a strict sub-additivity property of the minimization problem to show that concentration can happen only at a single point.

4.7.1. Preliminary technical results. We start by proving necessary results towards the proof of Theorem 4.7.1.

A variant of the following appears also in [**142**, Prop. 3.27]. For the sake of completeness, we provide here an complete proof.

Proposition 4.7.2. *Let $p, q \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that u_n converges L^q-strong to u_∞ and that v_n converges L^p-weak to v_∞, then*

$$\int u_n v_n \,\mathrm{d}\mathfrak{m}_n \to \int u_\infty v_\infty \,\mathrm{d}\mathfrak{m}_\infty.$$

Proof. It is sufficient to consider the case $u_n \ge 0, u_\infty \ge 0$, then the conclusion will follow recalling that $u_n^+ \to u_\infty^+$, $u_n^- \to u_\infty^-$ strongly in L^q.

The argument is similar to the one for the case $p = 2$ (see, e.g., in [**23**]), except that we need to consider the functions $u_n^{q/p} + tv_n$, $t \in \mathbb{R}$. Observe first that $u_n^{p/q} \to u_\infty^{q/p}$ strongly in L^p (by $(viii)$ of Prop. 1.6.10). In particular $u_n^{q/p} + tv_n$ converges to $u_\infty^{q/p} + tv_\infty$ weakly in L^p and in particular from (iii) of Prop. 1.6.10 we have

$$\|u_\infty^{q/p} + tv_\infty\|_{L^p(\mathfrak{m}_\infty)} \le \varliminf_n \|u_n^{q/p} + tv_n\|_{L^p(\mathfrak{m}_n)}. \tag{4.7.1}$$

The second ingredient is the following inequality

$$\big||a+|b||^p - |b|^p - pa|b|^{p-1}\big| \le C_p(|a|^{p\wedge 2}|b|^{p-p\wedge 2} + |a|^p), \quad \forall a,b \in \mathbb{R}, \tag{4.7.2}$$

which is easily derived from $\big||1+t|^p - 1 - pt\big| \le C_p(|t|^{p\wedge 2} + |t|^p)$, $\forall t \in \mathbb{R}$. Combining (4.7.2) and (4.7.1) we have

$$\begin{aligned}
&\int |u_\infty|^q \,\mathrm{d}\mathfrak{m}_\infty + pt\int u_\infty v_\infty \,\mathrm{d}\mathfrak{m}_\infty - C_p t^{p\wedge 2}\int |v_\infty|^{p\wedge 2}|u_\infty^{q/p}|^{p-p\wedge 2}\,\mathrm{d}\mathfrak{m}_\infty - C_p t^p \int |v_\infty|^p \,\mathrm{d}\mathfrak{m}_\infty \\
&\le \|u_\infty^{q/p} + tv_\infty\|^p_{L^p(\mathfrak{m}_\infty)} \le \varliminf_n \|u_n^{q/p} + tv_n\|_{L^p(\mathfrak{m}_n)} \\
&\le \varliminf_n \int |u_n|^q \,\mathrm{d}\mathfrak{m}_n + pt\int u_n v_n \,\mathrm{d}\mathfrak{m}_n + C_p t^{p\wedge 2}\int |v_n|^{p\wedge 2}|u_n^{q/p}|^{p-p\wedge 2}\,\mathrm{d}\mathfrak{m}_n + C_p t^p \int |v_n|^p \,\mathrm{d}\mathfrak{m}_n
\end{aligned}$$

Observe that in the case $p<2$ we have

$$\overline{\lim_n}\int |v_n|^{p\wedge 2}|u_n^{q/p}|^{p-p\wedge 2}=\overline{\lim_n}\int |v_n|^p\,\mathrm{d}\mathfrak{m}_n<+\infty,$$

while for $p\geq 2$ using the Hölder inequality

$$\overline{\lim_n}\int |v_n|^{p\wedge 2}|u_n^{q/p}|^{p-p\wedge 2}\leq \overline{\lim_n}\,\|v_n\|^2_{L^p(\mathfrak{m}_n)}\|u_n\|^{q(p-2)/p}_{L^q(\mathfrak{m}_n)}<+\infty.$$

In particular, recalling that $\int |u_n|^q\mathrm{d}\mathfrak{m}_n\to\int |u_\infty|^q\mathrm{d}\mathfrak{m}_\infty$ and choosing first $t\downarrow 0$ and then $t\uparrow 0$ above we obtain the desired conclusion. $\square$

The following is a variant for varying-measure of the famous Brezis-Lieb Lemma [**53**]. The key difference with the classical version of this result, is that in our setting it does not makes sense to write '$|u_\infty-u_n|$', since u_∞ and u_n will be integrated with respect to different measures. Hence we need to replace this term in (4.7.3) with $|v_n-u_n|$, where v_n is sequence approximating u_∞ in a strong sense.

Lemma 4.7.3 (Brezis-Lieb type Lemma). *Suppose that $\mathfrak{m}_n(\mathrm{X}_n),\mathfrak{m}_\infty(\mathrm{X}_\infty)=1$, let $q\in[2,\infty)$ and $q'\in(1,q)$. Suppose that $u_n\in L^q(\mathfrak{m}_n)$ satisfy $\sup_n\|u_n\|_{L^q(\mathfrak{m}_n)}<+\infty$ and that u_n converges to u_∞ strongly in $L^{q'}$ to some $u_\infty\in L^{q'}(\mathfrak{m}_\infty)\cap L^q(\mathfrak{m}_\infty)$. Then for any sequence $v_n\in L^q(\mathfrak{m}_n)$ such that $v_n\to u_\infty$ strongly both in $L^{q'}$ and L^q, it holds*

$$\lim_n\int |u_n|^q\,\mathrm{d}\mathfrak{m}_n-\int |u_n-v_n|^q\,\mathrm{d}\mathfrak{m}_n=\int |u_\infty|^q\,\mathrm{d}\mathfrak{m}_\infty. \tag{4.7.3}$$

Proof. The proof is based on the following inequality:

$$\big||a+b|^q-|b|^q-|a|^q\big|\leq C_p(|a|^q|b|^{q-1}+|a|^{q-1}|b|^q),\quad \forall a,b\in\mathbb{R}. \tag{4.7.4}$$

Indeed, if $a=v_n-u_n$ and $b=v_n$, we get from the above

$$\int \big||u_n|^q-|v_n-u_n|^q-|v_n|^q\big|\,\mathrm{d}\mathfrak{m}_n\leq C_q\int |v_n-u_n||v_n|^{q-1}+|v_n-u_n|^{q-1}|v_n|\,\mathrm{d}\mathfrak{m}_n. \tag{4.7.5}$$

Since $\int|v_n|^q\,\mathrm{d}\mathfrak{m}_n\to\int|u_\infty|^q\,\mathrm{d}\mathfrak{m}_\infty$, to conclude it is sufficient to show that the right hand side of (4.7.5) vanishes as $n\to+\infty$. We wish to apply Proposition 4.7.2. It follows from our assumptions that $|v_n|\to|v_\infty|$ strongly in L^q and $|v_n|^{q-1}\to|v_\infty|^{q-1}$ strongly in L^p, with $p:=q/(q-1)$. Hence it remains only to show that $|v_n-u_n|,|v_n-u_n|^{q-1}$ converges to 0 weakly in L^q and weakly in L^p respectively. We have that $\sup_n\|u_n-v_n\|_{L^q(\mathfrak{m}_n)}<+\infty$, hence by (iv) in Prop. 1.6.10 up to a subsequence $|u_n-v_n|$ converge weakly in L^q to a function $w\in L^q(\mathfrak{m})$. However by assumption the sequences $(v_n),(u_n)$ both converge strongly in $L^{q'}$ to u, hence $v_n-u_n\to 0$ strongly in $L^{q'}$ (recall (ii) in Prop. 1.6.10) and in particular by from (i) of Prop. 1.6.10 we have that $|v_n-u_n|\to 0$ strongly in $L^{q'}$, which implies that $w=0$. Analogously we also get that up to a subsequence $|u_n-v_n|^{q-1}$ converge weakly in L^p to a non-negative function $w'\in L^p(\mathfrak{m})$. Suppose that $q'\leq q-1$. taking $t\in[0,1]$ such that $q-1=tq'+(1-t)q$ we have

$$\int w'\,\mathrm{d}\mathfrak{m}_\infty=\lim_n\int |u_n-v_n|^{q-1}\,\mathrm{d}\mathfrak{m}_n\leq\|v_n-u_n\|^{tq'}_{L^{q'}(\mathfrak{m}_n)}\|v_n-u_n\|^{(1-t)q}_{L^q(\mathfrak{m}_n)}\to 0,$$

where we have used again that $u_n-v_n\to 0$ strongly in $L^{q'}$ and that u_n-v_n is uniformly bounded in L^q. If instead $q'\geq q-1$ by Hölder inequality we have

$$\int w'\,\mathrm{d}\mathfrak{m}_\infty=\lim_n\int |u_n-v_n|^{q-1}\,\mathrm{d}\mathfrak{m}_n\leq\left(\int |u_n-v_n|^{q'}\,\mathrm{d}\mathfrak{m}_n\right)^{(q-1)/q'}\to 0.$$

In both cases we deduce that $w'=0$, which concludes the proof. $\square$

Lemma 4.7.4 (Mixed recovery sequence). *Let $q\in[2,\infty)$ and let $u_\infty\in W^{1,2}(\mathrm{X}_\infty)\cap L^q(\mathfrak{m}_\infty)$. Then, there exists a sequence $u_n\in W^{1,2}(\mathrm{X}_n)\cap L^q(\mathrm{X}_n)$ that converges both L^q-strong and $W^{1,2}$-strong to u_∞.*

Proof. By truncation and a diagonal argument we can assume that $u_\infty\in L^\infty(\mathrm{X}_\infty)$. By the Γ-$\overline{\lim}$ inequality of the Ch_2 energy, consider a sequence $v_n\in W^{1,2}(\mathrm{X}_n)$ converging strongly in $W^{1,2}$ to u_∞. Defining $u_n:=(v_n\wedge C)\vee -C$, with $C\geq\|u_\infty\|_{L^\infty(\mathrm{X}_\infty)}$, we have by (i) of Proposition 1.6.10 that u_n converges in L^2-strong to u_∞. Moreover $|Du_n|\leq|Dv_n|$ $\mathfrak{m}_n$-a.e., therefore $\overline{\lim}_n\int|Du_n|^2\,\mathrm{d}\mathfrak{m}_n\leq\overline{\lim}_n\int|Dv_n|^2=\int|Du_\infty|^2\,\mathrm{d}\mathfrak{m}$, which grants that u_n converges also $W^{1,2}$-strongly to u_∞. Finally, the sequence u_n is uniformly bounded in L^∞ and converges to u_∞ in L^2-strong, hence by (ix) of Proposition 1.6.10. we have that that u_n is also L^q-strongly convergent to u_∞. $\square$

The following statement is the analogous in metric measure spaces of [**170**, Lemma I.1]. We shall omit its proof, since the arguments, presented there in $\mathbb{R}^n$, extend to this setting with obvious modifications (see also Remark I.5 in [**170**]).

Lemma 4.7.5. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space and* $\mu,\nu\in\mathscr{M}_b^+(\mathrm{X})$. *Suppose that*

$$\Big(\int|\varphi|^q\,\mathrm{d}\nu\Big)^{1/q}\le C\Big(\int|\varphi|^p\,\mathrm{d}\mu\Big)^{1/p},\qquad\forall\varphi\in\mathsf{LIP}_b(\mathrm{X}),$$

for some $1\le p<q<+\infty$ *and* $C\ge0$. *Then there exists a countable set of indices* J, *points* $(x_j)_{j\in J}\subset\mathrm{X}$ *and positive weights* $(\nu_j)_{j\in J}\subset\mathbb{R}^+$ *so that*

$$\nu=\sum_{j\in J}\nu_j\delta_{x_j},\qquad\mu\ge C^{-p}\sum_{j\in J}\nu_j^{p/q}\delta_{x_j}.\tag{4.7.6}$$

4.7.2. Lion's concentration-compactness lemma. Next, we present a generalized Concentration-Compactness principle, with underlying varying ambient space. The rough idea is that weakly convergent absolutely continuous measure are converging strongly except at possibly countable points, where Dirac masses eventually appears in the limit. For the sake of generality and for an application to Yamabe constants in Section 4.11, we will be working with a slightly more general Sobolev inequality containing an arbitrary L^q-norm.

Lemma 4.7.6 (Concentration-Compactness Lemma)**.** *Suppose that* $\mathfrak{m}_n(\mathrm{X}_n),\mathfrak{m}_\infty(\mathrm{X}_\infty)=1$ *and that for some fixed* $q\in(1,\infty)$ *the spaces* X_n *satisfy the following Sobolev-type inequalities*

$$\|u\|^2_{L^{2^*}(\mathfrak{m})}\le A_n\||Du|\|^2_{L^2(\mathfrak{m})}+B_n\|u\|^2_{L^q(\mathfrak{m})},\qquad\forall u\in W^{1,2}(\mathrm{X}),\tag{4.7.7}$$

with uniformly bounded constants A_n,B_n. *Let also* $u_n\in W^{1,2}(\mathrm{X}_n)$ *be* $W^{1,2}$*-weak and both* L^2*-strong and* L^q*-strong converging to* $u_\infty\in W^{1,2}(\mathrm{X}_\infty)$ *and suppose that* $|Du_n|^2\mathfrak{m}_n\rightharpoonup\mu$, $|u_n|^{2^*}\mathfrak{m}_n\rightharpoonup\nu$ *in duality with* $C_b(\mathrm{Z})$ *for two given measures* $\mu,\nu\in\mathscr{M}_b^+(\mathrm{Z})$.

Then,

- (i) *there exists a countable set of indices* J, *points* $(x_j)_{j\in J}\subset\mathrm{X}_\infty$ *and positive weights* $(\nu_j)_{j\in J}\subset\mathbb{R}^+$ *so that*
$$\nu=|u_\infty|^{2^*}\mathfrak{m}_\infty+\sum_{j\in J}\nu_j\delta_{x_j};$$
- (ii) *there exist* $(\mu_j)_{j\in J}\subset\mathbb{R}^+$ *satisfying* $\nu_j^{2/2^*}\le\varlimsup_n A_n\mu_j$ *so that*
$$\mu\ge|Du_\infty|^2\mathfrak{m}_\infty+\sum_{j\in J}\mu_j\delta_{x_j}.$$

In particular, we have $\sum_j\nu_j^{2/2^*}<\infty$.

Proof. We subdivide the proof in two steps.

Step 1. We assume that $u=0$. Let $\varphi\in\mathsf{LIP}_b(\mathrm{Z})$ and consider the sequence $(\varphi u_n)\in W^{1,2}(\mathrm{X}_n)$ which plugged in (4.7.7) for each X_n gives

$$\Big(\int|\varphi|^{2^*}|u_n|^{2^*}\,\mathrm{d}\mathfrak{m}_n\Big)^{1/2^*}\le\Big(A_n\int|D(\varphi u_n)|^2\,\mathrm{d}\mathfrak{m}_n+B_n\Big(\int|\varphi|^qu_n^q\,\mathrm{d}\mathfrak{m}_n\Big)^{2/q}\Big)^{1/2},\qquad\forall n\in\mathbb{N}.$$

It is clear that, by weak convergence, the left hand side of the inequality tends to $(\int|\varphi|^{2^*}\,\mathrm{d}\nu)^{1/2^*}$. While for the right hand side we discuss the two terms separately. First, by L^q-strong convergence, we have $\int\varphi^qu_n^q\,\mathrm{d}\mathfrak{m}_n\to0$, while an application of the Leibniz rule gives $\int|D(\varphi u_n)|\,\mathrm{d}\mathfrak{m}_n\le\int|D\varphi||u_n|+|\varphi||Du_n|\,\mathrm{d}\mathfrak{m}_n$. Combining these observations we reach

$$\Big(\int|\varphi|^{2^*}\mathrm{d}\nu\Big)^{1/2^*}\le\big(\varlimsup_n A_n\big)^{1/2}\Big(\int|\varphi|^2\,\mathrm{d}\mu\Big)^{1/2},\qquad\forall\varphi\in\mathsf{LIP}_b(\mathrm{Z}).$$

Thus, Lemma 4.7.5 (applied in the space $(\mathrm{Z},\mathsf{d}_\mathrm{Z})$) gives (i)-(ii), for the case $u_\infty=0$, except for the fact that we currently do no know whether the points $(x_j)_{j\in J}$ are in X_∞. This last simple fact can be seen as follows. Fix $j\in J$. From the weak convergence $|u_n|^{2^*}\mathfrak{m}_n\rightharpoonup\nu$, there must be a sequence $y_n\in\mathrm{supp}(\mathfrak{m}_n)=\mathrm{X}_n$ such that $\mathsf{d}_\mathrm{Z}(y_n,x_j)\to0$. Then the GH-convergence of X_n to X_∞ ensures that $x_j\in\mathrm{X}_\infty$, which is what we wanted.

Step 2. We now consider the case of a general u_∞. Observe that from Lemma 4.4.1 X_∞ supports a $(2^*,2)$-Sobolev inequality hence, $u\in L^{2^*}(\mathfrak{m}_\infty)$. From Lemma 4.7.4 there exists a sequence $\tilde u_n\in W^{1,2}(\mathrm{X}_n)$ such that $\tilde u_n$ converges to u both strongly in $W^{1,2}$ and strongly in L^{2^*}. Consider now the sequence $v_n:=u_n-\tilde u_n$. Clearly v_n converges to zero both in L^2-strong and in $W^{1,2}$-weak. Moreover the measures

$|v_n|^{2^*}\mathfrak{m}_n$ and $|Dv_n|^2\mathfrak{m}_n$ have uniformly bounded mass. Since (Z,d) is compact, passing to a non-relabelled subsequence we have $|v_n|^{2^*}\mathfrak{m}_n \rightharpoonup \bar\nu$ and $|Dv_n|^2\mathfrak{m}_n \rightharpoonup \bar\mu$ in duality with $C_b(\mathrm{Z})$ for some $\bar\nu,\bar\mu \in \mathscr{M}_b^+(\mathrm{Z})$. Therefore we can apply Step 1 to the sequence v_n to get $\bar\nu = \sum_{j\in J}\nu_j\delta_{x_j}$, $\bar\mu \geq \sum_{j\in J}\mu_j\delta_{x_j}$ for a suitable countable family J, $(x_j)\subset \mathrm{X}_\infty$ and weights $(\nu_j),(\mu_j)$ satisfying $\nu_j^{2/2^*} \leq (\overline{\lim}_n A_n)\mu_j$. To carry the properties of v_n to the sequence u_n we invoke Lemma 4.7.3 (with $q'=2$) to deduce that

$$\lim_{n\to\infty}\int |\varphi|^{2^*}|u_n|^{2^*}\,\mathrm{d}\mathfrak{m}_n - \int |\varphi|^{2^*}|v_n|^{2^*}\,\mathrm{d}\mathfrak{m}_n = \int |\varphi|^{2^*}|u_\infty|^{2^*}\,\mathrm{d}\mathfrak{m}_\infty, \tag{4.7.8}$$

and, taking into account the weak convergence, this implies that

$$\int \varphi^{2^*}\,\mathrm{d}\nu - \int \varphi^{2^*}\,\mathrm{d}\bar\nu = \int |u_\infty|^{2^*}\varphi^{2^*}\,\mathrm{d}\mathfrak{m}_\infty,$$

for every non-negative $\varphi \in C_b(\mathrm{Z})$. In particular, this is equivalent to say that $\nu = |u_\infty|^{2^*}\mathfrak{m}_\infty + \bar\nu = |u_\infty|^{2^*}\mathfrak{m}_\infty + \sum_{j\in J}\nu_j\delta_{x_j}$, which proves i). Next, we claim that $\mu \geq \sum_{j\in J}\mu_j\delta_{x_j}$ and, to do so, we consider for each $j\in J$ and $\varepsilon>0$, $\chi_\varepsilon \in \mathsf{LIP}_b(\mathrm{Z})$, $0\leq \chi_\varepsilon \leq 1, \chi_\varepsilon(x_j)=1$ and supported in $B_\varepsilon(x_j)$. The key ingredient is the following estimate

$$\begin{aligned}\Big|\int \chi_\varepsilon|Du_n|^2\,\mathrm{d}\mathfrak{m}_n - \int \chi_\varepsilon|Dv_n|^2\,\mathrm{d}\mathfrak{m}_n\Big| &\leq \int \chi_\varepsilon\big||Du_n|-|Dv_n|\big|\big(|Du_n|+|Dv_n|\big)\,\mathrm{d}\mathfrak{m}_n\\ &\leq \int \chi_\varepsilon|D\tilde u_n|\big(|Du_n|+|Dv_n|\big)\,\mathrm{d}\mathfrak{m}_n\\ &\leq \Big(\int \chi_\varepsilon^2|D\tilde u_n|^2\,\mathrm{d}\mathfrak{m}_n\Big)^{1/2}\Big(\||Du_n|\|_{L^2(\mathfrak{m}_n)} + \||Dv_n|\|_{L^2(\mathfrak{m}_n)}\Big).\end{aligned}$$

Observe now that Theorem 1.6.13 $|D\tilde u_n| \to |Du_\infty|$ strongly in L^2 and in particular $\int \chi_\varepsilon^2|D\tilde u_n|^2\,\mathrm{d}\mathfrak{m}_n \to \int \chi_\varepsilon^2|D\tilde u|^2\,\mathrm{d}\mathfrak{m}_\infty$. Moreover u_n, v_n are uniformly bounded in $W^{1,2}(\mathrm{X}_n)$. Therefore taking in the above inequality first $n\to+\infty$ and afterwards $\varepsilon\to 0^+$ we ultimately deduce that

$$\mu(\{x_i\}) = \bar\mu(\{x_i\}) \geq \mu_j, \quad \forall\, j\in J.$$

In particular $\mu \geq \sum_{j\in J}\mu_j\delta_{x_j}$, as claimed. Finally, by weak lower semicontinuity result in Lemma 1.6.14, we have

$$\int \varphi|Du_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty \leq \varliminf_n \int \varphi|Du_n|^2\,\mathrm{d}\mathfrak{m}_n = \int \varphi\,\mathrm{d}\mu$$

for every non-negative $\varphi\in C_b(\mathrm{Z})$. Therefore, we get $\mu \geq |Du_\infty|^2\mathfrak{m}_\infty$ and, by mutual singularity of the two lower bounds, we have (ii) and the proof is now concluded. □

4.7.3. Proof of Theorem 4.7.1. We are finally ready to prove the main result of this section.

Proof of Theorem 4.7.1. By assumption

$$1 \geq A_n\||D\tilde u_n|\|^2_{L^2(\mathfrak{m}_n)} + B_n\|\tilde u_n\|^2_{L^2(\mathfrak{m}_n)}, \qquad \forall n\in\mathbb{N}. \tag{4.7.9}$$

In particular $\sup_n \|\tilde u_n\|_{W^{1,2}(\mathrm{X}_n)} < \infty$, hence up to a not relabelled subsequence, Proposition 1.6.11 grants that $\tilde u_n$ converges L^2-strongly to a function $u_\infty \in W^{1,2}(\mathrm{X}_\infty)$. Moreover, the measures $|D\tilde u_n|^2\mathfrak{m}_n$, $|\tilde u_n|^{2^*}\mathfrak{m}_n$ have uniformly bounded mass. In particular up to a further not relabelled subsequence, there exists $\mu,\nu \in \mathscr{M}_b^+(\mathrm{Z})$ so that $|D\tilde u_n|^2\mathfrak{m}_n \rightharpoonup \mu$ and $|\tilde u_n|^{2^*}\mathfrak{m}_n \rightharpoonup \nu$ in duality with $C_b(\mathrm{Z})$. We are in position to apply Lemma 4.7.5 to get the existence of at most countably many points $(x_j)_{j\in J}$ and weights $(\nu_j)_{j\in J}$, so that $\nu = |u_\infty|^{2^*}\mathfrak{m}_\infty + \sum_{j\in J}\nu_j\delta_{x_j}$ and $\mu \geq |Du_\infty|^2\mathfrak{m}_\infty + \sum_{j\in J}\mu_j\delta_{x_j}$, with $A\mu_j \geq \nu_j^{2/2^*}$ and in particular $\sum_j \nu_j^{2/2^*} < \infty$. Finally from Lemma 4.4.1 we have that X_∞ supports a $(2^*,2)$-Sobolev inequality with constants A,B. Therefore we can perform the following estimates

$$\begin{aligned}1 = \lim_n \|\tilde u_n\|^2_{L^{2^*}(\mathfrak{m}_n)} &\geq \lim_n A_n\||D\tilde u_n|\|^2_{L^2(\mathfrak{m}_n)} + B\|\tilde u_n\|^2_{L^2(\mathfrak{m}_n)}\\ &= A\mu(\mathrm{X}_\infty) + B\int |u_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty\\ &\geq A\int |Du_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty + B\int |u_\infty|^2\,\mathrm{d}\mathfrak{m}_\infty + \sum_{j\in J}\nu_j^{2/2^*}\\ &\geq \Big(\int |u_\infty|^{2^*}\,\mathrm{d}\mathfrak{m}_\infty\Big)^{2/2^*} + \sum_{j\in J}\nu_j^{2/2^*}\\ &\geq \Big(\int |u_\infty|^{2^*}\,\mathrm{d}\mathfrak{m}_\infty + \sum_{j\in J}\nu_j\Big)^{2/2^*} = \nu(\mathrm{X}_\infty)^{2/2^*} = 1,\end{aligned}$$

where in the last inequality we have used the concavity of the function $t^{2/2^*}$. In particular all the inequalities must be qualities and, since $t^{2/2^*}$ is strictly concave, we infer that every term in the sum $\int |u_\infty|^{2^*}\,\mathrm{d}\mathfrak{m}_\infty + \sum_{j\in J} \nu_j^{2/2^*}$ must vanish except for one that must be equal to 1. If $\int |u_\infty|^{2^*}\,\mathrm{d}\mathfrak{m}_\infty = 1$ then I) in the statement of Theorem 4.7.1 must hold. If instead $\nu_j = 1$ for some $j \in J$, then $u_\infty = 0$ and by definition of ν, $|\tilde u_n|^{2^*}\mathfrak{m}_n \rightharpoonup \delta_{x_j}$, which is exactly II) in the statement of Theorem 4.7.1. □

4.8. Rigidity of $A_q^{\rm opt}$

In this section we prove our main rigidity result that we restate for convenience of the reader:

Theorem 4.8.1 (Rigidity of $A_q^{\rm opt}$). *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* RCD$(N-1,N)$ *space for some* $N\in(2,\infty)$ *and let* $q\in(2,2^*]$. *Then*

$$A_q^{\rm opt}(\mathrm{X}) = \frac{q-2}{N}, \tag{4.8.1}$$

if and only if $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is isomorphic to a spherical suspension, i.e. there exists an* RCD$(N-2,N-1)$ *space* $(\mathrm{Z},\mathsf{d}_\mathrm{Z},\mathfrak{m}_\mathrm{Z})$ *such that* $(\mathrm{X},\mathsf{d},\mathfrak{m}) \simeq [0,\pi]\times_{\sin}^{N-1}\mathrm{Z}$.

The idea of the proof will be to seek the existence of non-constant extremal functions for the Sobolev inequality and then exploit the rigidity of the Polya-Szego inequality proved in [**66**] (see Theorem 4.2.3. The difficult part is that it is not clear if in general such extremal functions exist. However we will show that non-existence will imply some extra information on the space that will allow us to the deduce the rigidity also in that case.

4.8.1. Quantitative linearization. A key point in our argument for the rigidity, and especially for the almost-rigidity, of $A_q^{\rm opt}$ will be a more 'quantitative' version of the linearization of a tight Sobolev inequality (see Section 4.6.2). To state our result, given $q\in(2,\infty)$ and $u\in W^{1,2}(\mathrm{X})$ with $\int |Du|_2\mathrm{d}\mathfrak{m} > 0$, it is convenient to define the Sobolev ratio associated to u as the quantity

$$\mathcal{Q}_q^{\mathrm{X}}(u) := \frac{\|u\|^2_{L^q(\mathfrak{m})} - \|u\|^2_{L^2(\mathfrak{m})}}{\||Du|_2\|^2_{L^2(\mathfrak{m})}}. \tag{4.8.2}$$

Observe that, if $\lambda^{1,2}(\mathrm{X})>0$, $\int |Du|_2\,\mathrm{d}\mathfrak{m}>0$ as soon as uf is not ($\mathfrak{m}$-a.e. equal to a) constant.

Lemma 4.8.2 (Quantitative linearization). *For all numbers* $A,B\ge 0$, $q>2$ *and* $\lambda>0$ *there exists a constant* $C=C(q,A,B,\lambda)$ *such that the following holds. Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space with* $\mathfrak{m}(\mathrm{X})=1$, $\lambda^{1,2}(\mathrm{X})\ge\lambda$ *and supporting a* $(q,2)$*-Sobolev inequality with constants* A,B. *Then, for every non-constant* $f\in W^{1,2}(\mathrm{X})$ *satisfying* $\|f\|_{L^2(\mathrm{X})}\le 1/2$, *it holds*

$$\left|\mathcal{Q}_q^{\mathrm{X}}(1+f) - \frac{(q-2)\int\big(f-\int f\mathrm{d}\mathfrak{m}\big)^2\,\mathrm{d}\mathfrak{m}}{\int |Df|_2^2\,\mathrm{d}\mathfrak{m}}\right| \le C\big(\|f\|^{3\wedge q-2}_{W^{1,2}(\mathrm{X})} + \|f\|^{q-2}_{W^{1,2}(\mathrm{X})} + \|f\|^{2q-2}_{W^{1,2}(\mathrm{X})}\big). \tag{4.8.3}$$

Proof. We claim that it is enough to prove the statement for functions $f\in W^{1,2}(\mathrm{X})$ with zero mean (and arbitrary L^2-norm). Indeed for a generic $f\in W^{1,2}(\mathrm{X})$ satisfying $\|f\|_{L^2(\mathrm{X})}\le 1/2$, we can take $\tilde f := \frac{f-\int f\,\mathrm{d}\mathfrak{m}}{1+\int f\,\mathrm{d}\mathfrak{m}}$, which clearly has zero mean. Then the conclusion would follow observing that the left hand side of (4.8.3) computed at $\tilde f$ coincides with the left hand side of (4.8.3) computed at f and from the fact that

$$\|\tilde f\|_{W^{1,2}(\mathrm{X})} \le \|f\|_{W^{1,2}(\mathrm{X})}\Big(1+\int f\,\mathrm{d}\mathfrak{m}\Big)^{-1} \le \|f\|_{W^{1,2}(\mathrm{X})}(1-\|f\|_{L^2(\mathrm{X})})^{-1} \le 2\|f\|_{W^{1,2}(\mathrm{X})}.$$

Therefore we can now fix $f\in W^{1,2}(\mathrm{X})$ with $\int f\,\mathrm{d}\mathfrak{m}=0$. We start with a basic estimate of the L^r norm of f for $r\in[1,q]$. Combining the Hölder and the $(q,2)$-Sobolev inequality we have

$$\int |f|^r\,\mathrm{d}\mathfrak{m} \le \Big(\int |f|^q\,\mathrm{d}\mathfrak{m}\Big)^{\frac{r}{q}} \le (A^{r/2}+B^{r/2})\|f\|^r_{W^{1,2}(\mathrm{X})} \tag{4.8.4}$$

In the case $r\in(2,q]$ the following refined estimate holds:

$$\begin{aligned}\frac{\int |f|^r\,\mathrm{d}\mathfrak{m}}{\int |Df|_2^2\,\mathrm{d}\mathfrak{m}} &\le C_q A^{r/2}\Big(\int |Df|_2^2\,\mathrm{d}\mathfrak{m}\Big)^{\frac{r}{2}-1} + C_q B^{r/2}\Big(\int |f|^2\,\mathrm{d}\mathfrak{m}\Big)^{\frac{r}{2}-1}\frac{\int_\mathrm{X}|f|^2\,\mathrm{d}\mathfrak{m}}{\int_\mathrm{X}|Df|_2^2\,\mathrm{d}\mathfrak{m}}\\ &\le C_q(A^{r/2}+B^{r/2}\lambda^{-1})\|f\|^{r-2}_{W^{1,2}(\mathrm{X})}.\end{aligned} \tag{4.8.5}$$

We now apply (4.6.3) to f, which we rewrite here for the convenience of the reader:

$$\Big|\Big(\int |1+f|^q \,\mathrm{d}\mathfrak{m}\Big)^{2/q} - \int (1+f)^2 \,\mathrm{d}\mathfrak{m} - (q-2)\int |f|^2 \,\mathrm{d}\mathfrak{m}\Big| \\ \le \tilde{C}_q\Big(\int |f|^{3\wedge q} + |f|^q \,\mathrm{d}\mathfrak{m} + \Big(\int |f|^q \,\mathrm{d}\mathfrak{m}\Big)^2 + \Big(\int |f|^2 \,\mathrm{d}\mathfrak{m}\Big)^2\Big),$$

where $\tilde{C}_q$ is a constant depending only on q. Dividing by $\int |Df|_2^2 \,\mathrm{d}\mathfrak{m}$ the above inequality and rearranging terms, using the definition of $\lambda^{1,2}(\mathrm{X})$ and the estimates (4.8.4), (4.8.5) we obtain (4.8.3). □

4.8.2. Existence of extremal function for the Sobolev inequality. Here we prove a result about existence of extremal functions for the tight Sobolev inequality. This can be summarized as: either there exist non-constant extremals, or we have informations on the first eigenvalue $\lambda^{1,2}(\mathrm{X})$, or we have informations on the density θ_N.

Theorem 4.8.3 (The Sobolev-alternative)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a compact* $\mathrm{RCD}(K,N)$ *space for some $K\in\mathbb{R}$, $N\in(2,\infty)$ and with $\mathfrak{m}(\mathrm{X})=1$. Let $q\in(2,2^*]$, with $2^*:=2N/(N-2)$. Then at least one of the following holds:*

(i) there exists a non-constant function $u\in W^{1,2}(\mathrm{X})$ satisfying

$$\|u\|^2_{L^{2^*}(\mathfrak{m})} = A_q^{\rm opt}(\mathrm{X})\||Du|\|^2_{L^2(\mathfrak{m})} + \|u\|^2_{L^2(\mathfrak{m})}, \tag{4.8.6}$$

(ii) $A_q^{\rm opt}(\mathrm{X}) = \frac{q-2}{\lambda^{1,2}(\mathrm{X})}$,

(iii) $q=2^*$ and $A_{2^*}^{\rm opt}(\mathrm{X}) = \frac{\mathsf{Eucl}(N,2)^2}{\inf\theta_N(x)^{2/N}}$ (see the introduction and (1.1.2) *for the definition of* $\mathsf{Eucl}(N,2)$*).*

Proof. By definition of $A_q^{\rm opt}$ there exists a sequence of non-constant functions $u_n\in\mathsf{LIP}(\mathrm{X})$ such that $\mathcal{Q}_q^{\mathrm{X}}(u_n)\to A_q^{\rm opt}(\mathrm{X})$ (recall (4.8.2)). By scaling we can suppose that $\|u_n\|_{L^{2^*}(\mathfrak{m})}\equiv 1$. In particular (u_n) is bounded in $W^{1,2}(\mathrm{X})$. We distinguish two cases.
Subcritical: $q<2^*$. By compactness (see Proposition 1.6.11), up to passing to a subsequence, $u_n\to u$ strongly in L^q to some function $u\in W^{1,2}(\mathrm{X})$ such that $\mathcal{Q}_q^{\mathrm{X}}(u)=A_q^{\rm opt}(\mathrm{X})$. If u is non-constant (i) holds and we are done, so suppose that u is constant. Then from the renormalization we must have $u\equiv 1$. Moreover, since $\|u_n\|_{L^q(\mathfrak{m})},\|u_n\|_{L^2(\mathfrak{m})}\to 1$ and $\mathcal{Q}_q^{\mathrm{X}}(u_n)\to A_q^{\rm opt}$, we deduce that $\||Du|\|^2_{L^2(\mathfrak{m})}\to 0$. Consider now the functions $f_n:=u_n-1\in\mathsf{LIP}(X)$, which are non-constant and such that $f_n\to 0$ in $W^{1,2}(\mathrm{X})$. We are therefore in position to apply Lemma 4.8.2 and deduce that

$$A_q^{\rm opt}(\mathrm{X}) = \lim_{n\to\infty}\mathcal{Q}_q^{\mathrm{X}}(u_n) = \lim_n \frac{(q-2)\int\big(f_n-\int f_n\mathrm{d}\mathfrak{m}\big)^2\,\mathrm{d}\mathfrak{m}}{\int|Df_n|^2\,\mathrm{d}\mathfrak{m}} \le \frac{q-2}{\lambda^{1,2}(\mathrm{X})}.$$

Combining this with (4.6.4), we get that $A_q^{\rm opt}(\mathrm{X})=\frac{q-2}{\lambda^{1,2}(\mathrm{X})}$ and conclude the proof in this case.
Critical $q=2^*$. We apply the concentration-compactness result in Theorem 4.7.1 and deduce that up to a subsequence: either $u_n\to u$ in L^{2^*} to some $u\in W^{1,2}(\mathrm{X})$ or $\|u_n\|_{L^2(\mathfrak{m})}\to 0$. In the first case we argue exactly as above using Lemma 4.8.2 and deduce that either (i) or (ii) holds. Hence we are left to deal with the case $\|u_n\|_{L^2(\mathfrak{m})}\to 0$. From the definition of $\alpha_2(\mathrm{X})$, for every ε there exits B_ε so that a $(2^*,2)$-Sobolev inequality with constants $\alpha_2(\mathrm{X})+\varepsilon$ and B_ε is valid. Hence we have

$$\mathcal{Q}_{2^*}^{\mathrm{X}}(u_n)\||Du_n|\|^2_{L^2(\mathfrak{m})} + \|u_n\|^2_{L^2(\mathfrak{m})} = \|u_n\|_{L^{2^*}(\mathfrak{m})} \le (\alpha_2(\mathrm{X})+\varepsilon)\||Du_n|\|^2_{L^2(\mathfrak{m})} + B_\varepsilon\|u_n\|^2_{L^2(\mathfrak{m})},$$

which gives

$$\mathcal{Q}_{2^*}^{\mathrm{X}}(u_n) \le (\alpha_2(\mathrm{X})+\varepsilon) + B_\varepsilon\|u_n\|^2_{L^2(\mathfrak{m})}(\||Du_n|\|^2_{L^2(\mathfrak{m})})^{-1}.$$

Observing that $\underline{\lim}_n \||Du_n|\|^2_{L^2(\mathfrak{m})}>0$ (which follows from the Sobolev inequality, $\|u_n\|^2_{L^2(\mathfrak{m})}\to 0$ and $\|u_n\|_{L^{2^*}(\mathfrak{m})}=1$) and letting $n\to+\infty$ we arrive at $A_{2^*}^{\rm opt}(\mathrm{X})\le(\alpha_2(\mathrm{X})+\varepsilon)$. From the arbitrariness ε we deduce that $A_{2^*}^{\rm opt}(\mathrm{X})\le\alpha_2(\mathrm{X})$ and the proof is concluded, indeed $\alpha_2(\mathrm{X})\ge A_{2^*}^{\rm opt}(\mathrm{X})$ is always true. □

4.9. Proof of the rigidity of $A_q^{\rm opt}$

Proof of Theorem 4.8.1. The result will follow from three different rigidity results, one for each of the alternatives in Theorem 4.8.3. Up to scaling the reference measures, we can suppose $\mathfrak{m}(\mathrm{X})=1$.
Case 1: $i)$ in Theorem 4.8.3 holds. Let u be the non-constant function satisfying (4.8.6). Observe that we can assume that u is non-negative. We aim to apply the Polya-Szego inequality with the model space

I_N as in Section 4.2.1. Let $u_N^* : I_N \to [0,\infty]$ be the monotone-rearrangement of u. From the Polya-Szego inequality in Theorem 4.2.2 we have both that $u_N^* \in W^{1,2}(I_N, \mathsf{d}_{eu}, \mathfrak{m}_N)$ and that $\||Du_N^*|\|_{L^2(\mathfrak{m}_N)} \le \||Du|\|_{L^2(\mathfrak{m})}$. Combining this with (1.3.8) we have

$$\begin{aligned}\|u\|^2_{L^q(\mathfrak{m})} = \|u_N^*\|^2_{L^q(\mathfrak{m}_N)} &\le \tfrac{q-2}{N}\||Du_N^*|\|^2_{L^2(\mathfrak{m}_N)} + \|u_N^*\|^2_{L^2(\mathfrak{m}_N)}\\ &\le \tfrac{q-2}{N}\||Du|\|^2_{L^2(\mathfrak{m})} + \|u\|^2_{L^2(\mathfrak{m})} = \|u\|^2_{L^q(\mathfrak{m})}.\end{aligned}$$

Therefore $\||Du_N^*|\|_{L^2(\mathfrak{m}_N)} = \||Du|\|_{L^2(\mathfrak{m})}$ and, since u is non-constant, we are in position to apply the rigidity of the Polya-Szego inequality of Theorem 4.2.3 and conclude the proof in this case.
CASE 2: *ii)* in Theorem 4.8.3 holds. We immediately deduce that $\lambda^{1,2}(\mathrm{X}) = N$ and the thesis follows from the Obata's rigidity (Theorem 1.3.15).
CASE 3: *iii)* in Theorem 4.8.3 holds. from Theorem 4.5.2 we have that

$$\frac{2^*-2}{N} = A^{\rm opt}_{2^*}(\mathrm{X}) = \alpha_2(\mathrm{X}) = \frac{\mathsf{Eucl}(N,2)^2}{\inf_{\mathrm{X}}\theta_N(x)^{2/N}} = \frac{2^*-2}{N\sigma_N^{2/N}\inf_{\mathrm{X}}\theta_N(x)^{2/N}},$$

therefore $\inf_{\mathrm{X}}\theta_N(x) = \sigma_N^{-1}$. On the other hand by the Bishop-Gromov inequality and identity (1.3.6)

$$\tfrac{1}{\sigma_N} = \inf_{\mathrm{X}}\theta_N(x) \ge \frac{\mathfrak{m}(\mathrm{X})}{v_{N-1,N}(\mathrm{diam}(\mathrm{X}))} = \frac{1}{v_{N-1,N}(\mathrm{diam}(\mathrm{X}))},$$

which, from the definition of $v_{N-1,N}$ and (1.1.4) forces $\mathrm{diam}(\mathrm{X}) = \pi$. The conclusion then follows by the rigidity of the maximal diameter (Theorem 1.3.16).

It remains to show that for any spherical suspension it holds that $A^{\rm opt}_q = \frac{q-2}{N}$, however this is immediate from the lower bound in Proposition 4.6.2 (recall also Theorem 1.3.15) and the upper bound given in (4.6.4). □

4.10. Almost rigidity of $A^{\rm opt}_q$

In this section we will prove our main almost-rigidity result.

Theorem 4.10.1 (Almost-rigidity of $A^{\rm opt}_q$)**.** *For every $N \in (2,\infty)$, $q \in (2,N)$ and every $\varepsilon > 0$, there exists $\delta := \delta(N,\varepsilon,q) > 0$ such that the following holds. Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(N-1,N)$ space with $\mathfrak{m}(\mathrm{X}) = 1$ and suppose that*

$$A^{\rm opt}_q(\mathrm{X}) \ge \frac{(q-2)}{N} - \delta,$$

Then, there exists a spherical suspension $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y})$ (i.e. there exists an $\mathrm{RCD}(N-2,N-1)$ space $(\mathrm{Z},\mathsf{d}_\mathrm{Z},\mathfrak{m}_\mathrm{Z})$ so that Y is isomorphic as a metric measure space to $[0,\pi]\times^{N-1}_{\sin}\mathrm{Z}$) such that

$$\mathsf{d}_{mGH}((\mathrm{X},\mathsf{d},\mathfrak{m}),(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y})) < \varepsilon.$$

Theorem 4.10.1 will follow from a continuity result of the constant $A^{\rm opt}_q$ with respect to mGH-convergence (see Theorem 4.10.3) and a compactness argument.

4.10.1. Behaviour at concentration points. The following technical result has the role of replacing in the varying-space case, the Sobolev inequality with constants $\alpha_2(\mathrm{X})+\varepsilon, B_\varepsilon$ which we used in the fixed-space case (see the proof of Theorem 4.8.3). Indeed it is not clear how to control the constant B_ε in a sequence of mGH-converging spaces. Therefore we need a more precise local analysis which fully exploits the local Sobolev inequalities in Theorem 4.3.2 and Proposition 4.3.4).

Lemma 4.10.2 (Behaviour at concentration points)**.** *Let $(\mathrm{X}_n,\mathsf{d}_n,\mathfrak{m}_n,x_n)$, $n \in \bar{\mathbb{N}}$, be a sequence of $\mathrm{RCD}(K,N)$ spaces $K \in \mathbb{R}$, $N \in [1,\infty)$, so that $\mathrm{X}_n \overset{pmGH}{\to} \mathrm{X}_\infty$. Fix $p \in (1,N)$, set $p^* := pN/(N-p)$ and assume that $u_n \in \mathsf{LIP}_c(\mathrm{X}_n)$ is a sequence satisfying*

$$\|u_n\|^p_{L^{p^*}(\mathfrak{m}_n)} \ge A_n\||Du_n|\|^p_{L^p(\mathfrak{m}_n)} - B_n\|u_n\|^p_{L^s(\mathfrak{m}_n)}, \tag{4.10.1}$$

for some constants $A_n, B_n \ge 0$ uniformly bounded and for some fixed $s \in [p,p^*)$. Assume furthermore that $u_n \to 0$ strongly in L^p, $\|u_n\|_{L^{p^*}(\mathfrak{m}_n)} = 1$ and that $|u_n|^{p^*}\mathfrak{m}_n \rightharpoonup \delta_{y_0}$ for some $y_0 \in \mathrm{X}_\infty$ in duality with $C_{bs}(\mathrm{Z})$ (where $(\mathrm{Z},\mathsf{d}_\mathrm{Z})$ is a proper space realizing the convergence in the extrinsic approach).

Then

$$\theta_N(y_0) \le \mathsf{Eucl}(N,p)^N(\varlimsup_n A_n)^{-N/p}, \tag{4.10.2}$$

meaning that if $\theta_N(y_0) = +\infty$, then $\varlimsup_n A_n^{1/p} = 0$.

PROOF. We subdivide the proof in two cases.

CASE 1: $\theta_N(y_0) < +\infty$.

Fix $\varepsilon < \theta_N(y_0)/4$ arbitrary. Since $\theta_{N,r}(y_0) \to \theta_N(y_0)$ as $r \to 0^+$ there exists $\bar r = \bar r(\varepsilon)$ such that

$$|\theta_{N,r}(y_0) - \theta_N(y_0)| \le \varepsilon, \quad \forall r < \bar r. \tag{4.10.3}$$

Let $\delta := \delta(2\varepsilon, D, N)$, with $D = 4$, be the constant given by Theorem 4.3.2 and fix two radii $r, R \in (0, \bar r)$ such that $R < \delta\sqrt{N/K^-}$ and $r < \delta R$. Consider now a sequence $y_n \in \mathrm{X}_n$ such that $y_n \to y_0$. From the convergence of the measures $\mathfrak{m}_n$ to $\mathfrak{m}_\infty$ we have that $\theta_{N,r}(y_n) \to \theta_{N,r}(y_0)$ and $\theta_{N,R}(y_n) \to \theta_{N,R}(y_0)$. In particular by (4.10.3) there exists $\bar n$ such that

$$|\theta_{N,R}(y_n) - \theta_N(y_0)|, |\theta_{N,r}(y_n) - \theta_N(y_0)| \le 2\varepsilon, \qquad \forall n \ge \bar n. \tag{4.10.4}$$

From the choice of ε this also implies that $\theta_{N,r}(y_n)/\theta_{N,R}(y_n) \le 4$ for every $n \ge \bar n$. We are in position to apply Theorem 4.3.2 and get that

$$\|f\|_{L^{p^*}(\mathfrak{m}_n)} \le \frac{(1+2\varepsilon)\mathsf{Eucl}(N,p)}{(\theta_N(y_0) - 2\varepsilon)^{\frac{1}{N}}} \||Df|\|_{L^p(\mathfrak{m}_n)}, \qquad \forall f \in \mathsf{LIP}_c(B_r(y_n)). \tag{4.10.5}$$

Choose $\varphi \in \mathsf{LIP}(\mathrm{Z})$ such that $\varphi = 1$ in $B^{\mathrm{Z}}_{r/8}(y_0)$, $\mathrm{supp}(\varphi) \subset B^{\mathrm{Z}}_{r/4}(y_0)$ and $0 \le \varphi \le 1$. From the assumptions, we have that $\int \varphi |u_n|^{p^*} \mathrm{d}\mathfrak{m}_n \to 1$, in particular up to increasing $\bar n$ it holds that $\int \varphi |u_n|^{p^*} \mathrm{d}\mathfrak{m}_n \ge 1 - \varepsilon$ for all $n \ge \bar n$. Moreover, again up to increasing $\bar n$, we have that $\mathsf{d}_{\mathrm{Z}}(y_n, y_0) \le r/4$ for all $n \ge \bar n$, therefore

$$1 - \varepsilon \le \int_{B_{r/2}(y_n)} |u_n|^{p^*} \mathrm{d}\mathfrak{m}_n, \qquad \forall n \ge \bar n. \tag{4.10.6}$$

For every n we choose a cut-off function $\varphi_n \in \mathsf{LIP}(\mathrm{X}_n)$ such that $\varphi_n = 1$ in $B_{r/2}(y_n)$, $0 \le \varphi_n \le 1$, $\mathrm{supp}(\varphi_n) \subset \mathsf{LIP}_c(B_r(y_n))$ and $\mathrm{Lip}(\varphi_n) \le 2/r$. Plugging the function $u_n\varphi_n \in \mathsf{LIP}_c(B_r(y_n))$ in (4.10.5) and using (4.10.6) we obtain

$$(1-\varepsilon)^{\frac{1}{p^*}} \le \|u_n\varphi_n\|_{L^{p^*}(\mathfrak{m}_n)} \le \frac{(1+2\varepsilon)\mathsf{Eucl}(N,p)}{(\theta_N(y_0) - \varepsilon)^{\frac{1}{N}}} \big(\||Du_n|\|_{L^p(\mathfrak{m}_n)} + \tfrac{2}{r}\|u_n\|_{L^p(\mathfrak{m}_n)}\big). \tag{4.10.7}$$

Moreover recalling that $\|u_n\|_{L^{p^*}(\mathfrak{m}_n)} = 1$ and the assumption (4.10.1), from (4.10.7) we reach

$$(1-\varepsilon)^{\frac{1}{p^*}} \big(A_n^{1/p}\||Du_n|\|_{L^p(\mathfrak{m}_n)} - B_n\|u_n^p\|_{L^s(\mathfrak{m}_n)}\big) \le \frac{(1+2\varepsilon)\mathsf{Eucl}(N,p)}{(\theta_N(y_0) - \varepsilon)^{\frac{1}{N}}} \big(\||Du_n|\|_{L^p(\mathfrak{m}_n)} + \tfrac{2}{r}\|u_n\|_{L^p(\mathfrak{m}_n)}\big).$$

We also observe that from the assumption $\|u_n\|_{L^p(\mathfrak{m}_n)} \to 0$ and the fact that $\|u_n\|_{L^{p^*}(\mathfrak{m}_n)} = 1$, we have by $(viii)$ in Proposition 1.6.10 that $\|u_n\|_{L^s(\mathfrak{m}_n)} \to 0$. Finally by (4.10.7) and the assumption $\|u_n\|_{L^p(\mathfrak{m}_n)} \to 0$ it holds that $\underline{\lim}_n \||Du_n|\|_{L^p(\mathfrak{m}_n)} > 0$. In particular for n big enough we can divide by $\||Du_n|\|_{L^p(\mathfrak{m}_n)}$ the above inequality and letting $n \to +\infty$ we eventually get

$$\overline{\lim_n}\, A_n^{1/p} \le \frac{(1+2\varepsilon)\mathsf{Eucl}(N,p)}{(1-\varepsilon)^{1/p^*}(\theta_N(y_0) - 2\varepsilon)^{\frac{1}{N}}}.$$

From the arbitrariness of ε, the conclusions follows.

CASE 2: $\theta_N(y_0) = \infty$.

The argument is similar to Case 1, but we will use Proposition 4.3.4 instead of Theorem 4.3.2. Let $M > 0$ be arbitrary. There exists $r \le 1$ such that $\theta_{N,r}(y_0) \ge 2M$. As above we choose a sequence $y_n \to y_0$. For n big enough we have that

$$\theta_{N,r}(y_n) \ge M. \tag{4.10.8}$$

Applying Proposition 4.3.4, from (4.10.8) we get that

$$\|f\|_{L^{p^*}(B_r(y_n))} \le \frac{C_{K,N,p}}{M^{\frac{p}{N}}} \||Df|\|_{L^p(B_r(y_n))} + \frac{C_{p,N}\|f\|_{L^p(B_r(y_n))}}{r^{p/N} M^{\frac{p}{N}}}, \quad \forall f \in \mathsf{LIP}(\mathrm{X}_n). \tag{4.10.9}$$

Observing that (4.10.6) is still satisfied with $\varepsilon = 1/M$ and n big enough, we can repeat the above argument, using (4.10.1), to obtain

$$\overline{\lim_n}\, A_n^{1/p} \le \frac{C_{K,N,p}}{(1-1/M)^{1/p^*} M^{\frac{1}{N}}},$$

which from the arbitrariness M implies the conclusion. □

4.10.2. Continuity of $A^{\rm opt}$ under mGH-convergence. In Lemma 4.4.1, we proved that Sobolev embedding are stable with respect to pmGH-convergence. A much more involved task it to prove that *optimal* constants are also continuous: indeed, if $\mathrm{X}_n \overset{mGH}{\to} \mathrm{X}_\infty$, in general Lemma 4.4.1 ensures only that $A^{\rm opt}_q(\mathrm{X}_\infty) \le \underline{\lim}_n A^{\rm opt}_q(\mathrm{X}_n)$. With the concentration compactness tools developed in Section 4.7, the 'quantitative-linearization' result in Lemma 4.8.2 and the technical tool developed in the previous section we can now prove the continuity of $A^{\rm opt}_q(\mathrm{X}_n)$:

Theorem 4.10.3 (Continuity of $A^{\rm opt}_q$ under mGH-convergence)**.** *Let $(\mathrm{X}_n,\mathsf{d}_n,\mathfrak{m}_n)$ be a sequence, $n \in \mathbb{N}\cup\{\infty\}$ of compact* RCD(K,N)*-spaces with $\mathfrak{m}_n(\mathrm{X}_n)=1$ and for some $K\in\mathbb{R}$, $N\in(2,\infty)$ so that $\mathrm{X}_n \overset{mGH}{\to} \mathrm{X}_\infty$. Then, $A^{\rm opt}_q(\mathrm{X}_\infty)=\lim_n A^{\rm opt}_q(\mathrm{X}_n)$, for every $q\in(2,2^*]$.*

Proof. By definition of optimal constants on every X_n, there exists a non-negative sequence $u_n \in \mathsf{LIP}(\mathrm{X}_n)^+$ satisfying

$$\|u_n\|^2_{L^q(\mathfrak{m}_n)} \ge A_n \||Du_n|\|^2_{L^2(\mathfrak{m}_n)} + \|u_n\|^2_{L^2(\mathfrak{m}_n)}, \tag{4.10.10}$$

having set $A_n := A^{\rm opt}_q(\mathrm{X}_n)-\frac{1}{n}$. By scaling invariance, it is not restrictive to suppose $\|u_n\|_{L^q(\mathfrak{m}_n)}=1$ for every $n\in\mathbb{N}$. Observe that thanks to Lemma 4.4.1 we already have that $0<A^{\rm opt}_q(\mathrm{X}_\infty)\le \underline{\lim}_n A^{\rm opt}_q(\mathrm{X}_n)$, hence we only need to show that $A^{\rm opt}_q(\mathrm{X}) \ge \overline{\lim}_n A^{\rm opt}_q(\mathrm{X}_n)$. To this aim, we distinguish two cases.
Subcritical: $q<2^*$. It is clear that A_n is uniformly bounded from below whence the sequence u_n has uniformly bounded $W^{1,2}$ norms. Then, by Proposition 1.6.11 and the Γ-$\underline{\lim}$ inequality of the Ch_2 energy, there exists a (not relabelled) subsequence L^2-strongly converging to some $u_\infty \in W^{1,2}(\mathrm{X}_\infty)$. Suppose first that the function u_∞ is not constant. Then, we can set up the chain of estimates

$$\begin{aligned} 1=\|u_\infty\|^2_{L^q(\mathfrak{m}_\infty)} &\ge \overline{\lim_{n\to\infty}}\, A_n\||Du_n|\|^2_{L^2(\mathfrak{m}_n)}+\|u_n\|^2_{L^2(\mathfrak{m}_n)}\\ &\ge \overline{\lim_{n\to\infty}}\, A^{\rm opt}_q(\mathrm{X}_n)\||Du_\infty|\|^2_{L^2(\mathfrak{m}_\infty)}+\|u_\infty\|^2_{L^2(\mathfrak{m}_\infty)}. \end{aligned}$$

Since u_∞ is not constant this in turn yields $\overline{\lim}_n A^{\rm opt}_q(\mathrm{X}_n)\le A^{\rm opt}_q(\mathrm{X}_\infty)$ which is what we wanted.

Suppose now that u_∞ is constant. Then, necessarily $u_\infty=1$ since $\|u_n\|_{L^q(\mathfrak{m}_n)}=1$ for every $n\in\mathbb{N}$. Define now $f_n := 1-u_n$ and observe that $\|f_n\|_{w^{1,2}(\mathrm{X}_n)}\to 0$, which follows from (4.10.10) and the fact that $\|u_n\|_{L^2(\mathfrak{m}_n)}\to 1$. Moreover from (1.6.1) we have that $\lambda^{1,2}(\mathrm{X}_n)$ are uniformly bounded below away from zero. Therefore we can apply Lemma 4.8.2 to deduce

$$\overline{\lim_{n\to\infty}}\, A_n = \overline{\lim_{n\to\infty}}\, \mathcal{Q}^{\mathrm{X}_n}_q(u_n) = \overline{\lim_{n\to\infty}}\, \frac{(q-2)\int\left|f_n-\int f_n\,\mathrm{d}\mathfrak{m}_n\right|^2\mathrm{d}\mathfrak{m}_n}{\int |Df_n|^2\,\mathrm{d}\mathfrak{m}_n} \le \overline{\lim_{n\to\infty}}\,\frac{(q-2)}{\lambda^{1,2}(\mathrm{X}_n)} \le \frac{(q-2)}{\lambda^{1,2}(\mathrm{X}_\infty)}, \tag{4.10.11}$$

having used, in the last inequality, again the continuity of the spectral gap (1.6.1). This combined with (4.6.4) gives that $\overline{\lim}_n A^{\rm opt}_q(\mathrm{X}_n)\le A^{\rm opt}_q(\mathrm{X}_\infty)$.
Critical exponent: $q=2^*$. Observe that we are now in position to invoke Theorem 4.7.1 and, up to a further not relabelled subsequence, we just need to handle one of the two different situations I),II) occurring in Theorem 4.7.1. If the case I) occurs, we argue exactly as in the case $q<2^*$, to conclude that $\overline{\lim}_n A^{\rm opt}_q(\mathrm{X}_n)\le A^{\rm opt}_q(\mathrm{X}_\infty)$. Hence suppose we are in the situation II), with a concentration point $x_0\in\mathrm{X}_\infty$ of the sequence u_n. Recalling Lemma 4.10.2, either $\theta_N(x_0)=\infty$ and $\overline{\lim}_n A^{\rm opt}_{2^*}(\mathrm{X}_n)=0$ or $\theta_N(x_0)<\infty$. The first situation cannot happen, since $A^{\rm opt}_{2^*}(\mathrm{X}_\infty)>0$. In the second one rearranging in (4.10.2) we have

$$\overline{\lim_{n\to\infty}}\, A^{\rm opt}_{2^*}(\mathrm{X}_n) \overset{(4.10.2)}{\le} \frac{\mathsf{Eucl}(N,2)^2}{\theta_N(x_0)^{2/N}} \overset{(4.5.2)}{\le} \alpha_2(\mathrm{X}_\infty)\le A^{\rm opt}_{2^*}(\mathrm{X}_\infty).$$

□

4.10.3. Proof of the almost rigidity. Combining the rigidity result for $A^{\rm opt}_q$ with the continuity result proved in the previous part we can now prove the almost-rigidity result for $A^{\rm opt}_q$.

Proof of Theorem 4.10.1. We argue by contradiction, and suppose that there exists $\varepsilon>0$, $q\in(2,2^*]$ and a sequence $(\mathrm{X}_n,\mathsf{d}_n,\mathfrak{m}_n)$ of RCD$(N-1,N)$-spaces with $\mathfrak{m}_n(\mathrm{X}_n)=1$ so that so that

$$\mathsf{d}_{mGH}((\mathrm{X}_n,\mathsf{d}_n,\mathfrak{m}_n),(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y}))>\varepsilon, \tag{4.10.12}$$

for every spherical suspension $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y})$ and $\lim_n A^{\rm opt}_q(\mathrm{X}_n)=\frac{q-2}{N}$. From Theorem 1.6.3 (and the assumption $\mathfrak{m}_n(\mathrm{X}_n)=1$) ensures that up to passing to a non-relabelled subsequence we have $\mathrm{X}_n \overset{mGH}{\to}\mathrm{X}_\infty$, for some RCD$(N-1,N)$-space $(\mathrm{X}_\infty,\mathsf{d}_\infty,\mathfrak{m}_\infty)$ with $\mathfrak{m}_\infty(\mathrm{X}_\infty)=1$. Moreover from (4.10.12)

$$\mathsf{d}_{mGH}((\mathrm{X}_\infty,\mathsf{d}_\infty,\mathfrak{m}_\infty),(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y}))\ge\varepsilon, \tag{4.10.13}$$

for every spherical suspension $(\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\mathfrak{m}_{\mathrm{Y}})$. Finally by Theorem 4.10.3 we deduce that

$$A_q^{\mathrm{opt}}(\mathrm{X}_\infty)=\lim_n A_q^{\mathrm{opt}}(\mathrm{X}_n)=\frac{q-2}{N}$$

. Therefore, by invoking the rigidity Theorem 4.8.1, we get that $(\mathrm{X}_\infty,\mathsf{d}_\infty,\mathfrak{m}_\infty)$ is isomorphic to a spherical suspension. This contradicts (4.10.13) and concludes the proof. □

4.11. Application: The Yamabe equation on RCD(K,N) spaces

In this section we apply Theorem 4.5.1 and the concentration compactness results of Section 4.7 to study the Yamabe equation to the RCD(K,N) setting. In particular we prove an existence result for the Yamabe equation and continuity of the generalized Yamabe constants under mGH-convergence, extending and improving some of the results proved in [**141**] in the case of Ricci limits. For results concerning the Yamabe problem and the Yamabe constant in non-smooth spaces see also also [**4**–**6**, **180**].

4.11.1. Existence of solutions to the Yamabe equation on compact RCD spaces. We recall that the Yamabe problem [**215**] asks if a compact Riemannian manifolds admits a conformal metric with constant scalar curvature. This has been completely solved and shown to be true after the works of Aubin, Schoen and Trudinger [**31**, **194**, **205**]. We also refer to [**164**] for an introduction to this problem and a complete and self-contained proof of this result.

The Yamabe problem turns out to be linked to the so-called Yamabe equation:

$$-\Delta u+S\,u=\lambda u^{2^*-1},\qquad \lambda\in\mathbb{R},\ S\in L^\infty(M), \tag{4.11.1}$$

where $2^*=\frac{2n}{n-2}$. Indeed solving the Yamabe problem is equivalent to find a non-negative and non-zero solution to (4.11.1) for some $\lambda\in\mathbb{R}$ and with $S=\mathrm{Scal}$, the scalar curvature of M. In this direction, it is relevant to see that the Yamabe equation is the Euler-Lagrange equation of the following functional:

$$Q(u):=\frac{\int|Du|^2+S|u|^2\,\mathrm{dVol}}{\|u\|^2_{L^{2^*}(M)}},\quad u\in W^{1,2}(M)\setminus\{0\},$$

where Vol is the volume measure of M. One then defines the Yamabe constant as the infimum of the above functional:

$$\lambda_S(M):=\inf_{u\in W^{1,2}(M)\setminus\{0\}}Q(u).$$

A crucial step in the solution of the Yamabe problem is the following by-now classical result (recall that $\mathsf{Eucl}(n,2)$ denotes the optimal constant in the sharp Euclidean Sobolev inequality (22)):

Theorem 4.11.1 ([**31**, **205**, **215**]). *Let M be a compact n-dimensional Riemannian manifold satisfying $\lambda_S(M)<\mathsf{Eucl}(n,2)^{-2}$. Then there is a non-zero solution to* (4.11.1) *with $\lambda=\lambda_S(M)$.*

Recall also that as showed by Aubin [**32**] (see also [**164**]) it always holds that

$$\lambda_S(M)\le\mathsf{Eucl}(n,2)^{-2}. \tag{4.11.2}$$

We will extended this inequality in the setting of RCD spaces at the end of this subsection.

The relevant point for our discussion is that Theorem 4.11.1 turns out to be linked to the notion of optimal Sobolev constant $\alpha_2(M)$ (see e.g. [**136**]). Since we characterized this constant on compact RCD(K,N) spaces (see Theorem 4.5.1), it is natural to ask if an analogue of Theorem 4.11.1 holds also in this setting.

We start by clarifying in which sense (4.11.1) will be intended in the non-smoooth setting. To this aim, we fix $(\mathrm{X},\mathsf{d},\mathfrak{m})$ a compact RCD(K,N) space for some $K\in\mathbb{R},N\in(2,\infty)$ with $\mathfrak{m}(\mathrm{X})=1$. We will also denote by 2^* the Sobolev-exponent defined as $2^*:=2N/(N-2)$. We fix $\mathrm{S}\in\mathscr{M}_b(\mathrm{X})$ satisfying for some $p>\frac{N}{2}$

$$\mathrm{S}\ge g\mathfrak{m},\ g\in L^p(\mathfrak{m})\qquad\text{and}\qquad \mathrm{S}\ll\mathrm{Cap}, \tag{4.11.3}$$

where Cap denotes the capacity of X (see (1.2.22)). We also denote by $|\mathrm{S}|$ the total variation of S which for instance can be characterized by the formula $\mathrm{S}=\mathrm{S}^++\mathrm{S}^-$, being $\mathrm{S}^\pm$ the Hahn's decomposition of a general signed measure. The reason for this more general choice of S is the fact that on RCD(K,N) spaces a 'scalar curvature' that is bounded is not natural (recall that to solve the Yamabe problem one would like to take $\mathrm{S}=\mathrm{Scal}$). Indeed requiring only a synthetic lower bound on the Ricci curvature, it is more desirable to require only lower bounds on S.

Recall from Section 1.2 that every function $u\in W^{1,2}(\mathrm{X})$ has a well defined and unique quasi continuous representative $\mathsf{QCR}(u)$ defined Cap-a.e.. In particular, thanks to (4.11.3), the object $\mathsf{QR}(u)$ is also defined $|\mathrm{S}|$-a.e.. To avoid heavy notation, for any $u\in W^{1,2}(\mathrm{X})$, we shall denote in the sequel by u its quasi-continuous representative without further notice.

The goal is then to discuss positive solutions $u \in D(\mathbf{\Delta}) \cap L^2(|\mathrm{S}|)$ of

$$-\mathbf{\Delta} u = \lambda u^{2^*-1}\mathfrak{m} - u\mathrm{S}, \qquad \lambda \in \mathbb{R}. \tag{4.11.4}$$

Observe that if $u \in D(\mathbf{\Delta}) \subset W^{1,2}(\mathrm{X})$, by the Sobolev embedding we have that $u \in L^{2^*}(\mathfrak{m})$ and thus, the right hand side of (4.11.4) is a well defined Radon measure on X. A solution for this equation will be deduced with a variational approach as described above. More precisely we define the functional $Q_{\mathrm{S}}\colon W^{1,2}(\mathrm{X}) \setminus \{0\} \to \mathbb{R}$ defined as

$$u \mapsto Q_{\mathrm{S}}(u) := \frac{\int |Du|^2 \,\mathrm{d}\mathfrak{m} + \int |u|^2 \,\mathrm{dS}}{\|u\|^2_{L^{2^*}(\mathfrak{m})}}.$$

Observe that from the first in (4.11.3) the integral $\int |u|^2 \,\mathrm{dS}$ makes sense, i.e. its value (possibly $+\infty$) is well defined. We then define

$$\begin{aligned}\lambda_{\mathrm{S}}(\mathrm{X}) &:= \inf\{Q_{\mathrm{S}}(u) \ : \ u \in W^{1,2}(\mathrm{X}) \setminus \{0\}\} \\ &= \inf\{Q_{\mathrm{S}}(u) \ : \ u \in W^{1,2}(\mathrm{X}),\, \|u\|_{L^{2^*}(\mathfrak{m})} = 1\},\end{aligned} \tag{4.11.5}$$

and immediately claim that

$$\lambda_{\mathrm{S}}(\mathrm{X}) \in (-\infty, +\infty). \tag{4.11.6}$$

Indeed, $\lambda_{\mathrm{S}}(\mathrm{X}) < +\infty$ as can be seen considering constant functions. On the other hand for every $u \in W^{1,2}(\mathrm{X})$ with $\|u\|_{L^{2^*}(\mathfrak{m})} = 1$, Holder inequality (recalling that $\mathfrak{m}(\mathrm{X}) = 1$) yields

$$Q_{\mathrm{S}}(u) \geq -\|g\|_{L^p(\mathfrak{m})}\|u\|_{L^{2^*}(\mathfrak{m})} = -\|g\|_{L^p(\mathfrak{m})}.$$

The ultimate goal of this subsection is to prove the following:

Theorem 4.11.2. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a compact* RCD(K,N) *space for some* $K \in \mathbb{R}$, $N \in (2,\infty)$ *with* $\mathfrak{m}(\mathrm{X}) = 1$ *and let* $\mathrm{S} \in \mathscr{M}_b(\mathrm{X})$. *Suppose that* S *satisfies* (4.11.3) *and*

$$\lambda_{\mathrm{S}}(\mathrm{X}) < \frac{\min \theta_N(x)^{2/N}}{\mathsf{Eucl}(N,2)^2}. \tag{4.11.7}$$

Then there exists a non-negative and non-zero $u \in D(\mathbf{\Delta}) \cap L^2(|\mathrm{S}|)$ *which is a minimum for* (4.11.5) *and satisfies* (4.11.4).

We start by showing that (4.11.4) are the Euler-Lagrange equations for the minimization problem (4.11.5).

Proposition 4.11.3. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a compact* RCD(K,N)*-space for some* $K \in \mathbb{R}, N \in (2,\infty)$ *with* $\mathfrak{m}(\mathrm{X}) = 1$. *Suppose* $u \in W^{1,2}(\mathrm{X}) \cap L^2(|\mathrm{S}|)$ *is a minimizer for* (4.11.5) *satisfying* $\|u\|_{L^{2^*}(\mathfrak{m})} = 1$. *Then*

$$\int \langle \nabla u, \nabla v \rangle \,\mathrm{d}\mathfrak{m} = -\int uv \,\mathrm{dS} + \lambda_{\mathrm{S}}(\mathrm{X}) \int u^{2^*-1} v \,\mathrm{d}\mathfrak{m}, \qquad \forall v \in \mathsf{LIP}(\mathrm{X}). \tag{4.11.8}$$

Proof. We consider for every $\varepsilon \in (-1,1)$ and $v \in \mathsf{LIP}(\mathrm{X})$, the function $u^\varepsilon := \|u+\varepsilon v\|^{-1}_{L^{2^*}(\mathfrak{m})}(u+\varepsilon v)$, whenever $\|u+\varepsilon v\|_{L^{2^*}(\mathfrak{m})}$ is not zero. It can be seen that for a fixed v then u^ε is well defined at least for ε close to zero. Indeed, the fact that $\int |u|^{2^*}, \mathrm{d}\mathfrak{m} = 1$ grants that $\|u+\varepsilon v\|_{L^{2^*}(\mathfrak{m})} \to 1$ as $\varepsilon \to 0$ (see below) and in particular $\|u+\varepsilon v\|_{L^{2^*}(\mathfrak{m})}$ does not vanish for $|\varepsilon|$ small enough. By minimality we have (recall also (1.2.4))

$$0 \leq \lim_{\varepsilon \downarrow 0} \frac{Q_{\mathrm{S}}(u^\varepsilon) - Q_{\mathrm{S}}(u)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left(\frac{1}{I_\varepsilon^2} - 1\right)\lambda_{\mathrm{S}}(\mathrm{X}) + \frac{2}{I_\varepsilon^2}\int \langle \nabla u, \nabla v\rangle \,\mathrm{d}\mathfrak{m} + \int uv \,\mathrm{dS},$$

where $I_\varepsilon := \|u+\varepsilon v\|_{L^{2^*}(\mathfrak{m})}$. Furthermore, from the elementary estimate $\big||a+\varepsilon b|^q - |a|^q\big| \leq q|\varepsilon b|\big||a+\varepsilon b|^{q-1} + |a|^{q-1}\big|$, with $q = 2^*$, and the fact that $u, v \in L^{2^*}(\mathfrak{m})$, we have that $\int |u+\varepsilon v|^q \,\mathfrak{m} \to 1$ as $\varepsilon \to 0$. Thanks to the same estimates, the dominated convergence theorem grants that

$$\lim_{\varepsilon \downarrow 0} \frac{1 - I_\varepsilon^2}{\varepsilon} = \frac{2}{2^*}\lim_{\varepsilon \downarrow 0} \int \frac{|u|^{2^*} - |u+\varepsilon v|^{2^*}}{\varepsilon}\,\mathrm{d}\mathfrak{m} = -2\int u^{2^*-1} v \,\mathrm{d}\mathfrak{m}.$$

Arguing analogously considering $\varepsilon \uparrow 0$ gives (4.11.8). □

We can now prove Theorem 4.11.2, which thanks to the previous proposition, amounts to show the existence of a minimizer for (4.11.5). We will do so using the concentration-compactness tools developed in Section 4.7, here employed with a fixed space X. We mention that the application of concentration-compactness arguments to the study of the Yamabe equation goes back to the seminal work [**170**].

PROOF OF THEOREM 4.11.2. Let $u_n \in W^{1,2}(\mathrm{X}) \cap L^2(|\mathrm{S}|)$ be such that $Q_{\mathrm{S}}(u_n) \to \lambda_{\mathrm{S}}(\mathrm{X})$ and $\|u_n\|_{L^{2^*}(\mathfrak{m})} = 1$. We claim that u_n are uniformly bounded in $W^{1,2}(\mathrm{X})$. Indeed, this can be seen from the estimate

$$\int |Du_n|^2 + |u_n|^2 \, \mathrm{d}\mathfrak{m} \le \int |Du_n|^2 \, \mathrm{d}\mathfrak{m} + \int |u_n|^2 \, \mathrm{dS} + (1 + \|g\|_{L^p(\mathfrak{m})})\|u_n\|_{L^{2^*}(\mathfrak{m})} = Q_{\mathrm{S}}(u_n) + (1 + \|g\|_{L^p(\mathfrak{m})}),$$

obtained combining the Hölder inequality with (4.11.3). Hence, by compactness (see Proposition 1.6.11), up to a not relabelled subsequence, we have $u_n \to u$ in $L^2(\mathfrak{m})$ for some $u \in W^{1,2}(\mathrm{X})$. Observe that, since $u \in W^{1,2}(\mathrm{X})$, u admits a quasi-continuous representative (still denoted by u) and thus thanks to (4.11.3) it makes sense to integrate u^2 against $|\mathrm{S}|$. We claim that $u \in L^2(|\mathrm{S}|)$ and

$$\int u^2 \, \mathrm{dS} \le \varliminf_n \int u_n^2 \, \mathrm{dS}. \tag{4.11.9}$$

Observe first that, by (4.11.3), we have $S^- \le |g|\mathfrak{m}$. Hence, since $u_n \to u$ in $L^2(\mathfrak{m})$, we get that $u \in L^2(S^-)$ and $u_n \to u$ also in $L^2(S^-)$. To prove (4.11.9) it remains to prove that $\int u^2 \, \mathrm{dS}^+ \le \varliminf_n \int u_n^2 \, \mathrm{dS}^+$. Observe first that up to passing to a further non-relabeled subsequence we can assume that the right hand side is actually a limit. From Mazur's lemma there exists a sequence $(N_n) \subset \mathbb{N}$ and numbers $(\alpha_{n,i})_{i=n}^{N_n} \subset [0,1]$ such that $\sum_{i=n}^{N_n} \alpha_{n_i} = 1$ for every $n \in N$ and $v_n := \sum_{i=n}^{N_n} \alpha_{n_i} u_i$ converges to u strongly in $W^{1,2}(\mathrm{X})$. In particular from (1.2.15) up to a subsequence $v_n \to u$ also Cap-a.e. and thus, since $\mathrm{S}^+ \ll \mathrm{Cap}$ (recall (4.11.3)), also S^+-a.e.. Therefore, from Fatou's Lemma and the convexity of the L^2-norm we have

$$\|u\|_{L^2(\mathrm{S}^+)} \le \varliminf_n \|v_n\|_{L^2(\mathrm{S}^+)} \le \sum_{i=n}^{N_n} \alpha_{n_i} \|u_i\|_{L^2(\mathrm{S})} \le \lim_n \|u_n\|_{L^2(\mathrm{S}^+)},$$

since we are assuming that the last limit exists. This proves the claim.

We now distinguish two cases:

CASE 1. $\lambda_{\mathrm{S}}(\mathrm{X}) < 0$. By lower semicontinuity of the Cheeger-energy and (4.11.9) we have

$$0 > \lambda_{\mathrm{S}}(\mathrm{X}) = \lim_n Q_{\mathrm{S}}(u_n) \ge \int |Du|^2 \, \mathrm{d}\mathfrak{m} + \int u^2 \mathrm{dS}.$$

In particular u is not identically zero and by the lower semicontinuity of the $L^{2^*}(\mathfrak{m})$-norm we have $0 < \|u\|_{L^{2^*}(\mathfrak{m})} \le 1$. Moreover, from the above we have that $\int |Du|^2 \, \mathrm{d}\mathfrak{m} + \int u^2 \mathrm{dS}$ is negative, hence

$$\lambda_{\mathrm{S}}(\mathrm{X}) \ge \|u\|_{L^{2^*}(\mathfrak{m})}^{-2} \left(\int |Du|^2 \, \mathrm{d}\mathfrak{m} + \int u^2 \mathrm{dS} \right) = Q_{\mathrm{S}}(\|u\|_{L^{2^*}(\mathfrak{m})}^{-1} u).$$

Therefore $\|u\|_{L^{2^*}(\mathfrak{m})}^{-1} u$ is a minimizer for $Q_{\mathrm{S}}(u)$.

CASE 2. $\lambda_{\mathrm{S}}(\mathrm{X}) \ge 0$. Recall that the sequence (u_n) is uniformly bounded both in $L^{2^*}(\mathfrak{m})$ and in $W^{1,2}(\mathrm{X})$. Therefore since X is compact, again up to a subsequence, $|Du_n|^2\mathfrak{m} \rightharpoonup \mu$ and $|u_n|^{2^*} \rightharpoonup \nu$ for some $\mu \in \mathscr{M}_b^+(\mathrm{X})$ and $\nu \in \mathscr{P}(\mathrm{X})$ in duality with $C(\mathrm{X})$. By assumption there exists $\varepsilon > 0$ such that $\lambda_{\mathrm{S}}(\mathrm{X}) < \frac{\min \theta_N(x)^{2/N}}{\mathsf{Eucl}(N,2)^2 + \varepsilon} =: \lambda_\epsilon$. We fix one of these $\varepsilon > 0$ and define $A_\varepsilon = \lambda_\varepsilon^{-1}$. From Theorem 4.5.1 there exists a constant $B_\varepsilon > 0$ so that

$$\|u\|_{L^{2^*}(\mathfrak{m})}^2 \le A_\varepsilon \||Du|\|_{L^2(\mathfrak{m})}^2 + B_\varepsilon \|u\|_{L^2(\mathfrak{m})}^2, \quad \forall u \in W^{1,2}(\mathrm{X}).$$

Hence we are in position to apply Lemma 4.7.6 (with fixed space X) to deduce that there exists a countable set of indices J, points $(x_j)_{j \in J} \subset \mathrm{X}$ and weights $(\mu_j) \subset \mathbb{R}^+$, $(\nu_j) \subset \mathbb{R}^+$ such that $\mu_j \ge \lambda_\varepsilon \nu_j^{2/2^*}$ for every $j \in J$ and

$$\nu = |u|^{2^*}\mathfrak{m} + \sum_{j \in J} \nu_j \delta_{x_j}, \quad \mu \ge |Du|^2\mathfrak{m} + \sum_{j \in J} \mu_j \delta_{x_j}.$$

We now observe that

$$\int |Du|^2 \, \mathrm{d}\mathfrak{m} + \int u^2 \, \mathrm{dS} \ge \|u\|_{L^{2^*}(\mathfrak{m})}^2 \lambda_{\mathrm{S}}(\mathrm{X}). \tag{4.11.10}$$

Indeed, this is obvious if $u = 0$ $\mathfrak{m}$-a.e., hence we assume that $u \ne 0$ $\mathfrak{m}$-a.e.. In this case, (4.11.10) follows noticing that $\lambda_{\mathrm{S}}(\mathrm{X}) \le Q_{\mathrm{S}}(u\|u\|_{L^{2^*}(\mathfrak{m})}^{-1}) = \|u\|_{L^{2^*}(\mathfrak{m})}^{-2} \left(\int |Du|^2 \mathrm{d}\mathfrak{m} + \int u^2 \, \mathrm{dS} \right)$. Therefore using again (4.11.9) we have

$$\begin{aligned}\lambda_{\rm S}({\rm X}) &= \lim_n Q_{\rm S}(u_n) \geq \mu({\rm X}) + \int u^2\,{\rm dS} \geq \int |Du|^2 {\rm d}\mathfrak{m} + \lambda_\varepsilon \sum_{j\in J}\nu_j^{2/2^*} + \int u^2\,{\rm dS}\\ &\overset{(4.11.10)}{\geq} \|u\|^2_{L^{2^*}(\mathfrak{m})}\lambda_{\rm S}({\rm X}) + \lambda_\varepsilon\sum_{j\in J}\nu_j^{2/2^*} \geq \lambda_{\rm S}({\rm X})(\|u\|^2_{L^{2^*}(\mathfrak{m})} + \sum_{j\in J}\nu_j^{2/2^*})\\ &\geq \lambda_{\rm S}({\rm X})\Big(\int |u|^{2^*}\,{\rm d}\mathfrak{m} + \sum_{j\in J}\nu_j\Big)^{2/2^*} = \lambda_{\rm S}({\rm X})\nu({\rm X}) = \lambda_{\rm S}({\rm X}),\end{aligned}$$

where in the last line, we used the concavity of the function $t^{2/2^*}$, the fact that $\nu \in \mathscr{P}({\rm X})$ and finally that $\lambda_{\rm S}({\rm X}) \geq 0$. Hence all the inequalities are equalities and in particular from the strict concavity of $t^{2/2^*}$ we deduce that either $\int |u|^{2^*}{\rm d}\mathfrak{m} = 1$ or $u = 0$ (and the numbers ν_j are all zero except one that is equal to one). In the second case, plugging $u = 0$ in the above chain of inequalities, we infer that $\lambda_\varepsilon = \lambda_{\rm S}({\rm X})$ which is a contradiction. Hence, we must have $\|u\|_{L^{2^*}(\mathfrak{m})} = 1$ and $u_n \to u$ strongly in $L^{2^*}(\mathfrak{m})$ and in particular u is a minimizer for (4.11.5). This together with Proposition 4.11.3 concludes the proof. □

We conclude by extending the upper bound of Aubin (4.11.2) to the setting of RCD(K,N) spaces. This in particular shows that (4.11.7) is a reasonable assumption. Unfortunately, at present, we are able to prove this comparison only adding integrability conditions on S.

Proposition 4.11.4. *Let* $({\rm X},\mathsf{d},\mathfrak{m})$ *be a compact* RCD(K,N) *space for some* $K\in\mathbb{R}$, $N\in(2,\infty)$ *and let* ${\rm S}\in L^p(\mathfrak{m})$, *with* $p > \frac{N}{2}$. *Then*

$$\lambda_{\rm S}({\rm X}) \leq \frac{\min\theta_N(x)^{2/N}}{\mathsf{Eucl}(N,2)^2}.$$

Proof. The argument is almost the same as for Theorem 4.4.4. We start noticing that in the case $\min\theta_N(x) = +\infty$, evidently there is nothing to prove. We are left then to deal with the case $0 < \min\theta_N(x) < +\infty$. Let $p\in{\rm X}$ such that $\theta_N(p) = \min\theta_N(x)$. Then there exists a sequence $r_i \to 0$ such that the sequence of metric measure spaces $({\rm X}_i,\mathsf{d}_i,\mathfrak{m}_i,x_i) := ({\rm X},\mathsf{d}/r_i,\mathfrak{m}/r_i^N,p)$ converges in the pmGH-topology to an RCD$(0,N)$ space $({\rm Y},\mathsf{d}_{\rm Y},\mathfrak{m}_{\rm Y},\boldsymbol{o}_{\rm Y})$ satisfying $\mathfrak{m}_{\rm Y}(B_r(\boldsymbol{o}_{\rm Y})) = \omega_N\theta_N(p)r^N$ for every $r>0$ (this space is actually a cone by [**95**]). In particular from Lemma 4.4.3 for every $\varepsilon>0$ there exists a non-zero $u\in\mathsf{LIP}_c({\rm Y})$ such that $\frac{\|u\|^2_{L^{2^*}(\mathfrak{m}_{\rm Y})}}{\||Du|\|^2_{L^2(\mathfrak{m}_{\rm Y})}} \geq \frac{\mathsf{Eucl}(N,2)^2-\varepsilon}{\theta_N(p)^{2/N}}$. Then by the Γ-convergences of the 2-Cheeger energies there exists a sequence $u_i\in W^{1,2}({\rm X}_i)$ such that $u_i\to u$ strongly in $W^{1,2}$. Moreover, since u_i are uniformly bounded in $W^{1,2}$ (meaning in $W^{1,2}({\rm X}_i)$), by the Sobolev embedding (recall also the scaling property in (4.4.1)) we have $\sup_i\|u_i\|_{L^{2^*}(\mathfrak{m}_i)} < +\infty$. In particular from the lower semicontinuity of the L^{2^*}-norm we get

$$\underline{\lim_i}\frac{\|u_i\|^2_{L^{2^*}(\mathfrak{m})}}{\||Du_i|\|^2_{L^2(\mathfrak{m})}} = \underline{\lim_i}\frac{\|u_i\|^2_{L^{2^*}(\mathfrak{m}_i)}}{\||Du_i|_i\|^2_{L^2(\mathfrak{m}_i)}} \geq \frac{\|u\|^2_{L^{2^*}(\mathfrak{m}_{\rm Y})}}{\||Du|\|^2_{L^2(\mathfrak{m}_{\rm Y})}} \geq \frac{\mathsf{Eucl}(N,2)^2-\varepsilon}{\min\theta_N(x)^{2/N}}, \tag{4.11.11}$$

where $|Du_i|_i$ denotes the weak upper gradient computed in the space ${\rm X}_i$.

Denote by $p' := p/(p-1)$ the conjugate exponent of p and observe that by hypothesis $2p' < 2^*$. This and the fact that u_i are bounded in L^{2^*} from Proposition 1.6.10 imply that u_i converges in $L^{2p'}$-strong to u. Finally using the Hölder inequality we can write

$$\begin{aligned}\overline{\lim_i}\, Q_{\rm S}(u_i) &\leq \overline{\lim_i}\frac{\int |Du_i|^2\,{\rm d}\mathfrak{m}}{\|u_i\|^2_{L^{2^*}(\mathfrak{m})}} + \overline{\lim_i}\frac{\int S|u_i|^2\,{\rm d}\mathfrak{m}}{\|u_i\|^2_{L^{2^*}(\mathfrak{m})}}\\ &\overset{(4.11.11)}{\leq} \frac{\min\theta_N(x)^{2/N}}{\mathsf{Eucl}(N,2)^2-\varepsilon} + \overline{\lim_i}\,\|S\|_{L^p(\mathfrak{m})}\frac{\big(\int|u_i|^{2p'}\,{\rm d}\mathfrak{m}\big)^{1/p'}}{\|u_i\|^2_{L^{2^*}(\mathfrak{m})}},\\ &= \frac{\min\theta_N(x)^{2/N}}{\mathsf{Eucl}(N,2)^2-\varepsilon} + \overline{\lim_i}\,\|S\|_{L^p(\mathfrak{m})}r_i^{N\left(\frac{1}{p'}-\frac{2}{2^*}\right)}\frac{\|u_i\|^2_{L^{2p'}(\mathfrak{m}_i)}}{\|u_i\|_{L^{2^*}(\mathfrak{m}_i)^2}} = \frac{\min\theta_N(x)^{2/N}}{\mathsf{Eucl}(N,2)^2-\varepsilon}.\end{aligned}$$

where we have used that $1/p' < 2/2^*$, that $\underline{\lim}_i\,\|u_i\|_{L^{2^*}(\mathfrak{m}_i)} \geq \|u\|_{L^{2^*}(\mathfrak{m}_{\rm Y})} > 0$ and as observed above $\||u_i\|_{L^{2p'}(\mathfrak{m}_i)} \to \|u\|_{L^{2p'}(\mathfrak{m}_{\rm Y})}$. From the arbitrariness of $\varepsilon>0$ the proof is now concluded. □

4.11.2. Continuity of the generalized Yamabe constants under mGH-convergence. In [**141**] it has been proven in the case of Ricci-limits a result about continuity of the generalized Yamabe constant, under some additional boundedness assumption on the sequence. In the following result we extend this fact in the setting of RCD-spaces and we also remove such extra assumption.

Theorem 4.11.5 (Continuity of $\lambda_{\rm S}$ under mGH-convergence)**.** *Let* $({\rm X}_n,\mathsf{d}_n,\mathfrak{m}_n)$ *be a sequence of compact* RCD(K,N) *spaces with* $\mathfrak{m}({\rm X}_n)=1$, $n\in\bar{\mathbb{N}}$, *for some* $K\in\mathbb{R},N\in(2,\infty)$ *satisfying* ${\rm X}_n\overset{mGH}{\to}{\rm X}_\infty$. *Let also* ${\rm S}_n\in L^p(\mathfrak{m}_n)$ *be* L^p*-weak converging to* ${\rm S}\in L^p(\mathfrak{m}_\infty)$, *for some* $p>N/2$. *Then,*

$$\lim_{n\to\infty}\lambda_{{\rm S}_n}({\rm X}_n)=\lambda_{\rm S}({\rm X}_\infty).$$

We start proving that $\lambda_{\rm S}$ is upper semicontinuous under mGH-convergence.

Lemma 4.11.6. *Let* $({\rm X}_n,\mathsf{d}_n,\mathfrak{m}_n)$ *be a sequence of compact* RCD(K,N)*-spaces with* $\mathfrak{m}({\rm X}_n)=1$, $n\in\bar{\mathbb{N}}$, *for some* $K\in\mathbb{R},N\in(2,\infty)$ *and satisfying* ${\rm X}_n\overset{mGH}{\to}{\rm X}_\infty$. *Let also* ${\rm S}_n\in L^p(\mathfrak{m}_n)$ *be* L^p*-weak convergent to* ${\rm S}\in L^p(\mathfrak{m}_\infty)$, *for a given for* $p>N/2$. *Then,*

$$\overline{\lim_{n\to\infty}}\,\lambda_{{\rm S}_n}({\rm X}_n)\le\lambda_{\rm S}({\rm X}_\infty). \tag{4.11.12}$$

Proof. Fix a non-zero $u\in W^{1,2}({\rm X}_\infty)$. By the Sobolev embedding on ${\rm X}_\infty$ we know that $u\in L^{2^*}(\mathfrak{m}_\infty)$, therefore by Lemma 4.7.4 there exists a sequence $u_n\in W^{1,2}({\rm X}_n)$ that converge $W^{1,2}$-strong and L^{2^*}-strong to u. By definition of $\lambda_{{\rm S}_n}({\rm X}_n)$, we have

$$\|u_n\|^2_{L^{2^*}(\mathfrak{m}_n)}\lambda_{{\rm S}_n}({\rm X}_n)\le\int|Du_n|^2\,{\rm d}\mathfrak{m}_n+\int{\rm S}_n|u_n|^2\,{\rm d}\mathfrak{m}_n,\qquad\forall n\in\mathbb{N}.$$

From the assumption that $p>N/2$, we have that its conjugate exponent p' satisfies $2p'<2^*$, therefore from $(viii)$, (ix) in Proposition 1.6.10 we have that $|u_n|^2$ $L^{p'}$-strongly converges to u^2. Recalling Proposition 4.7.2, we get that all the above quantities pass to the limit and thus we reach

$$\|u\|^2_{L^{2^*}(\mathfrak{m}_\infty)}\overline{\lim_{n\to\infty}}\,\lambda_{{\rm S}_n}({\rm X}_n)\le\int|Du|^2\,{\rm d}\mathfrak{m}_\infty+\int{\rm S}|u|^2\,{\rm d}\mathfrak{m}_\infty.$$

By arbitrariness of u, we conclude. □

We shall now come to the proof of the main continuity result for the generalized Yamabe constant.

Proof of Theorem 4.11.5. Observe that by Lemma 4.11.6 we only have to prove that:

$$\underline{\lim_{n\to\infty}}\,\lambda_{{\rm S}_n}({\rm X}_n)\ge\lambda_{\rm S}({\rm X}_\infty).$$

It is also clearly not restrictive to assume that the $\underline{\lim}$ is actually a limit.

For every $n\in\mathbb{N}$, we take $u_n\in W^{1,2}({\rm X}_n)$ non-zero so that $Q_{{\rm S}_n}(u_n)-\lambda_{{\rm S}_n}({\rm X}_n)\le n^{-1}$. In other words

$$\|u_n\|^2_{L^{2^*}(\mathfrak{m}_n)}\big(\lambda_{{\rm S}_n}({\rm X}_n)+\tfrac{1}{n}\big)\ge\int|Du_n|^2\,{\rm d}\mathfrak{m}_n+\int{\rm S}_n|u_n|^2\,{\rm d}\mathfrak{m}_n. \tag{4.11.13}$$

It is also clearly not restrictive to suppose that $u_n\in\mathsf{LIP}_c({\rm X}_n)$ are non-negative and such that $\|u_n\|_{L^{2^*}(\mathfrak{m}_n)}\equiv1$. Hence, arguing as in the proof of Theorem 4.11.2 (using also (4.11.12)), we get that u_n is uniformly bounded in $W^{1,2}$. Then, by compactness (see Proposition 1.6.11), up to a not relabeled subsequence, we have that u_n converge L^2-strong and $W^{1,2}$-weak to some $u_\infty\in W^{1,2}({\rm X}_\infty)$. From $\|u_n\|_{L^{2^*}(\mathfrak{m}_n)}\equiv1$ and the assumption $p>N/2$, Proposition 1.6.10 implies that u_n^2 converges $L^{p/(p-1)}$-strongly to u_∞^2 and that u_n converges $L^{2p/(p-1)}$-strongly to u_∞. From this point we subdivide the proof in three cases.

Case 1: $\lim_n\lambda_{{\rm S}_n}({\rm X}_n)<0$. In this case, by (4.11.13) we know by lower semicontinuity of the 2-Cheeger energy and Proposition 4.7.2, we have that

$$0>\lim_n\lambda_{{\rm S}_n}({\rm X}_n)\ge\int|Du_\infty|^2\,{\rm d}\mathfrak{m}_\infty+\int{\rm S}u_\infty^2\,{\rm d}\mathfrak{m}_\infty.$$

In particular, u_∞ is not $\mathfrak{m}_\infty$-a.e. equal to zero and by weak-lower semicontinuity, we have that $0<\|u_\infty\|_{L^{2^*}(\mathfrak{m}_\infty)}\le1$. Therefore

$$\|u_\infty\|_{L^{2^*}(\mathfrak{m}_\infty)}\lim_n\lambda_{{\rm S}_n}({\rm X}_n)\ge\lim_n\lambda_{{\rm S}_n}({\rm X}_n)\ge\int|Du_\infty|^2\,{\rm d}\mathfrak{m}_\infty+\int{\rm S}u_\infty^2\,{\rm d}\mathfrak{m}_\infty\ge\lambda_S({\rm X}_\infty)\|u_\infty\|_{L^{2^*}(\mathfrak{m}_\infty)},$$

which concludes the proof in this case.

Case 2: $\lim_n\lambda_{{\rm S}_n}({\rm X}_n)>0$. Before starting, notice that by using the Hölder inequality, for any $n\in\mathbb{N}$ and any $u\in W^{1,2}({\rm X}_n)$ we have by the definition of $\lambda_{{\rm S}_n}({\rm X}_n)$ that

$$\|u\|^2_{L^{2^*}(\mathfrak{m}_n)}\le\lambda_{{\rm S}_n}({\rm X}_n)^{-1}\int|Du|^2\,{\rm d}\mathfrak{m}_n+\lambda_{{\rm S}_n}({\rm X}_n)^{-1}\|{\rm S}_n\|_{L^p(\mathfrak{m}_n)}\|u\|^2_{L^{2p/p-1}(\mathfrak{m}_n)}. \tag{4.11.14}$$

Moreover, since all ${\rm X}_n$ are compact and renormalized, there are $\mu\in\mathscr{M}_b^+({\rm Z}),\nu\in\mathscr{P}({\rm Z})$ so that, up to a not relabelled subsequence, $|Du_n|^2\mathfrak{m}_n\rightharpoonup\mu$ and $|u_n|^{2^*}\mathfrak{m}_n\rightharpoonup\nu$ in duality with $C({\rm Z})$ as n goes to infinity, where

$(\mathrm{Z}, \mathsf{d}_\mathrm{Z})$ is a (compact) space realizing the convergences via extrinsic approach. Since we are assuming that $\lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n) > 0$, the constant in (4.11.14) are uniformly bounded (for n big enough) and we are in position to apply Lemma 4.7.6. In particular we get the existence of an at most countable set J, points $(x_j)_{j\in J} \subset \mathrm{X}_\infty$ and weights $(\mu_j), (\nu_j) \subset \mathbb{R}^+$, so that $\mu_j \geq \lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n)\nu_j^{2/2^*}$ with $j \in J$ and

$$\nu = |u_\infty|^{2^*}\mathfrak{m} + \sum_{j\in J}\nu_j\delta_{x_j}, \quad \mu \geq |Du_\infty|^2\mathfrak{m} + \sum_{j\in J}\mu_j\delta_{x_j}.$$

Moreover, recalling Proposition 4.7.2 we have

$$\mu(\mathrm{X}) + \int \mathrm{S}u_\infty^2 \, \mathrm{d}\mathfrak{m}_\infty = \lim_{n\to\infty} Q_{\mathrm{S}_n}(u_n) \overset{(4.11.13)}{\leq} \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n), \tag{4.11.15}$$

and, arguing as in the proof of (4.11.10), u_∞ is so that $\|u_\infty\|_{L^{2^*}(\mathfrak{m}_\infty)}\lambda_\mathrm{S}(\mathrm{X}_\infty) \leq \int |Du_\infty|^2 \, \mathrm{d}\mathfrak{m}_\infty + \int \mathrm{S}|u_\infty|^2 \, \mathrm{d}\mathfrak{m}_\infty$. Finally, we can perform the chain of estimates

$$\begin{aligned}
\lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n) &\overset{(4.11.15)}{\geq} \mu(\mathrm{X}) + \int \mathrm{S}u_\infty^2 \, \mathrm{d}\mathfrak{m}_\infty \geq \int |Du_\infty|^2\mathrm{d}\mathfrak{m}_\infty + \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\sum_{j\in J}\nu_j^{2/2^*} + \int \mathrm{S}u_\infty^2 \, \mathrm{d}\mathfrak{m}_\infty \\
&\geq \lambda_\mathrm{S}(\mathrm{X}_\infty)\|u_\infty\|^2_{L^{2^*}(\mathfrak{m}_\infty)} + \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\sum_{j\in J}\nu_j^{2/2^*} \\
&\overset{(4.11.12)}{\geq} \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\Big(\|u_\infty\|^2_{L^{2^*}(\mathfrak{m}_\infty)} + \sum_{j\in J}\nu_j^{2/2^*}\Big) \\
&\geq \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\Big(\int |u_\infty|^{2^*} \, \mathrm{d}\mathfrak{m}_\infty + \sum_{j\in J}\nu_j\Big)^{2/2^*} \geq \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n),
\end{aligned}$$

where in the last line, we used the concavity of $t^{2/2^*}$, the fact that $\nu \in \mathscr{P}(\mathrm{X})$. In particular, all inequalities must be equalities and by the strict concavity of $t^{2/2^*}$ either $\|u_\infty\|_{L^{2^*}(\mathfrak{m}_\infty)} = 1$ and all $\nu_j = 0$, or $u_\infty = 0$ $\mathfrak{m}_\infty$-a.e. and all the weights are zero except one $\nu_j = 1$. The first situation is the easier one, as in this case the above inequalities which are actually equalities imply that $\lambda_\mathrm{S}(\mathrm{X}_\infty) = \lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n)$, which is what we wanted. Therefore we suppose that we are in the second case, i.e. that there exists a point $y_0 \in \mathrm{X}_\infty$ so that $|u_n|^{2^*}\mathfrak{m}_n \rightharpoonup \delta_{y_0}$ in duality with $C(\mathrm{Z})$ and that u_n converges in L^2-strong to zero. Moreover, from (4.11.13) and Hölder inequality we get

$$\|u_n\|^2_{L^{2^*}(\mathfrak{m}_n)} \geq \big(\lambda_{\mathrm{S}_n}(\mathrm{X}_n) + \tfrac{1}{n}\big)^{-1}\Big(\int |Du_n|^2 \, \mathrm{d}\mathfrak{m}_n - \|\mathrm{S}_n\|_{L^p(\mathfrak{m}_n)}\|u_n\|^2_{L^{2p/(p-1)}(\mathfrak{m}_n)}\Big), \qquad \forall n \in \mathbb{N}.$$

We can therefore apply Lemma 4.10.2 to get that $\theta_N(y_0) \leq \mathsf{Eucl}(N,2)^N \lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n)^{N/2}$. Finally, we can rearrange and invoke Proposition 4.11.4 to get

$$\lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n) \geq \frac{\theta_N(y_0)^{2/N}}{\mathsf{Eucl}(N,2)^2} \geq \lambda_\mathrm{S}(\mathrm{X}_\infty).$$

Case 3: $\lim_n \lambda_{\mathrm{S}_n}(\mathrm{X}_n) = 0$. The argument is the same as in the previous case, only that we replace (4.11.14) with the Sobolev inequality given in Proposition 4.6.1:

$$\|u\|^2_{L^q(\mathfrak{m})} \leq A(K,N,D)\||Du|\|^2_{L^2(\mathfrak{m})} + \|u\|^2_{L^2(\mathfrak{m}_n)}, \qquad \forall u \in W^{1,2}(\mathrm{X}_n), \tag{4.11.16}$$

where $D > 0$ is constant such that $\mathrm{diam}(X_n) \leq D$. Then we can apply exactly as in the previous case Lemma 4.7.6, except that in this case we obtain $\mu_j \geq A(K,N,D)^{-1}\nu_j^{2/2^*}$ for every $j \in J$. Then the above chain of estimates becomes

$$\begin{aligned}
0 = \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n) &\overset{(4.11.15)}{\geq} \mu(\mathrm{X}) + \int \mathrm{S}u_\infty^2 \, \mathrm{d}\mathfrak{m}_\infty \geq \int |Du_\infty|^2\mathrm{d}\mathfrak{m}_\infty + A(K,N,D)^{-1}\sum_{j\in J}\nu_j^{2/2^*} + \int \mathrm{S}u_\infty^2 \, \mathrm{d}\mathfrak{m}_\infty \\
&\geq \lambda_\mathrm{S}(\mathrm{X}_\infty)\|u_\infty\|^2_{L^{2^*}(\mathfrak{m}_\infty)} + A(K,N,D)^{-1}\sum_{j\in J}\nu_j^{2/2^*} \\
&\overset{(4.11.12)}{\geq} \lim_{n\to\infty}\lambda_{\mathrm{S}_n}(\mathrm{X}_n)\|u_\infty\|^2_{L^{2^*}(\mathfrak{m}_\infty)} + A(K,N,D)^{-1}\sum_{j\in J}\nu_j^{2/2^*} \geq 0.
\end{aligned}$$

Therefore we must have that $\nu_j = 0$ for every $j \in J$ and in particular that $\|u_\infty\|^2_{L^{2^*}(\mathfrak{m}_\infty)} = 1$ and thus that $\lambda_\mathrm{S}(\mathrm{X}_\infty) = 0$. This concludes the proof. □

Part 2

Reifenberg's theorem in metric spaces

CHAPTER 5

Bi-Lipschitz metric Reifenberg's theorem

Structure of the chapter

We will start in Section 5.1 with some elementary preliminaries on metric spaces and Gromov-Hausdorff distance. Then we will move to the proof of the bi-Lipschitz version of the Cheeger and Colding's metric Reifenberg's theorem, which is contained in Sections 5.2, 5.3 and 5.4. In Section 5.2 we will first give the main statement and some remarks on it (Section 5.2.1) and then give a proof assuming the existence of a suitable family of approximating manifolds and mappings (Section 5.2.2). Sections 5.3 and 5.4 will be devoted to the constructions of these objects and will constitute the main portion of the proof. In section 5.3 we will develop the two main tools which are at the core of the argument. One is the mappings modification theorem (Section 5.3.2) that will allow us to build the approximating manifolds. Roughly said this result will produce a family of compatible transition functions starting from a set of almost-compatible transition functions. The second tool which we will develop in Section 5.3.3 allows to patch together a family of embeddings from a manifold to another, which almost agree on the intersection of their domains, to obtain a globally defined embedding. This will be used to build the diffeomorphisms between the approximating manifolds. Section 5.4 constitutes the main body of the proof where, starting from the irregular charts for the metric space given by the Gromov-Hausdorff approximations, we will build smooth atlases to construct smooth manifolds that approximate the metric space at smaller and smaller scales.

The second part of the chapter, Sections 5.5, 5.6 and 5.7, will be concerned with the comparison between the "Hausdorff flatness" and "Gromov-Hausdorff flatness" for subsets of the Euclidean space. In Section 5.5.1 we will introduce the parameters e, b that measure the flatness (of a subset of $\mathbb{R}^d$) via comparison with planes and of their metric counterparts ε, β defined in terms of Gromov-Hausdorff distance and almost-isometries. In Section 5.5.2 we will first prove the elementary inequalities $\varepsilon \lesssim \mathsf{e}$, $\beta \lesssim \mathsf{b}$ and then show, with examples, that they cannot be improved in general. Finally we will give a key-example for which $\varepsilon \lesssim \mathsf{e}^2$ and $\beta \lesssim \mathsf{b}^2$. In Section 5.5.3 we will state our main result which roughly says that the numbers ε, β behave as the square of the numbers e, b. The difficult part will be to show that, in a weak sense, $\varepsilon \lesssim \mathsf{e}^2$ and $\beta \lesssim \mathsf{b}^2$, which will be proved in Section 5.6. In the last Section 5.7 we will instead show the more direct inequalities $\mathsf{e}^2 \lesssim \varepsilon$ and $\mathsf{b}^2 \lesssim \mathsf{b}$.

All the results that will be presented in this chapter are contained in [**130**, **209**].

5.1. Preliminaries and notations

In this short preliminary section we list the very few notations and definitions that we will need along the chapter, which are mainly linked to the Gromov-Hausdorff distance between metric spaces and related concepts (some of them were already present in the first part of this thesis, but we will include them for completeness).

The only non-standard result contained here will be Lemma 5.1.11, which gives a quantified approximation of an almost-isometry with an isometry and will be the starting point for the construction in the proof of the bi-Lipschitz metric Reifenberg's theorem.

Notations. For an $n \times n$ real-matrix $A \in \mathbb{R}^{n\times n}$ we will denote by $\|A\|$ its operator norm and by $\{a_{i,j}\}_{i,j\in 1,\dots,n}$ its entries. We recall also the following elementary inequality, which will be used in several proofs:

$$\frac{1}{n^2}\|A\| \le \max_{i,j} |a_{i,j}| \le \sqrt{n}\|A\|, \quad \forall A \in \mathbb{R}^{n\times n}. \tag{5.1.1}$$

Given a C^1 function $f : \mathbb{R}^n \supset A \to \mathbb{R}^n$ we will denote with f_k, $k = 1, .., n$ its components and with $Df : A \to \mathbb{R}^{n\times n}$ its differential, which will be always treated as an $n \times n$ matrix-valued map.

In the first part of this chapter a central role will be played by the following C^k-norm:

Definition 5.1.1 (C^k-scaling invariant norm)**.** Let $f \in C^1(A;\mathbb{R}^n)$ where A is any open set of $\mathbb{R}^n$, for any $t>0$ we write

$$\|f\|_{C^1(A),t} \le C$$

if $|f| \le Ct$ and $\|Df\| \le C$ on A, where $\|.\|$ denotes the operator norm of the matrix Df. If moreover $f \in C^2(A;\mathbb{R}^n)$ and $|\partial_{i,j} f_k| \le C/t$, for every $i,j,k \in \{1,...,n\}$ (f_k being the k-th component of f) we write $\|f\|_{C^2(A),t} \le C$.

Remark 5.1.2. The reason for us to work with this norm is the following observation: $\|f\|_{C^1(A),t} \le C$ (resp. $\|f\|_{C^2(A),t} \le C$) if and only if $\|f_t\|_{C^1(A/t),1} \le C$ (resp. $\|f_t\|_{C^2(A/t),1} \le C$), where $A/t := \{x/t \ : \ x \in A\}$ and $f_t(x) := \frac{f(tx)}{t}$. ■

Given a metric space (X,d) and a point $p \in \mathrm{X}$ we denote by $B_r^{\mathrm{X}}(p) := \{x \in \mathrm{X} \ : \ \mathsf{d}(x,p) < r\}$ the ball of radius $r>0$ centered at p and by $\overline{B}_r(p)$ its topological closure, which in case of a length space coincides with the closed ball $\{x \in \mathrm{X} \ : \ \mathsf{d}(x,p) \le r\}$.

Hausdorff and Gromov-Hausdorff distance. Here we introduce some basics on the Gromov-Hausdorff distance and GH-approximation maps, for a more detailed presentation we refer to [**60**].

Definition 5.1.3 (δ-dense set)**.** Let (X,d) be a metric spaces and fix a number $\delta>0$. We say that a set $S \subset \mathrm{X}$ is δ-dense in X if for every $x \in \mathrm{X}$ there exists $s \in S$ such that $\mathsf{d}(x,s) < \delta$.

Definition 5.1.4 (δ-isometry)**.** Let $(\mathrm{X}_1,\mathsf{d}_1),(\mathrm{X}_2,\mathsf{d}_2)$ be two metric spaces and fix a number $\delta>0$. We say that a function $f : S \to \mathrm{X}_2$ for some $S \subset \mathrm{X}_1$ is a δ-isometry if

$$|\mathsf{d}_2(f(x),f(y)) - \mathsf{d}_1(x,y)| < \delta, \quad \forall x,y \in S.$$

Definition 5.1.5 (Hausdorff- distance)**.** Let (X,d) be a metric space and let $A,B \subset \mathrm{X}$. We define the Hausdorff distance between A and B as the number

$$d_H^{\mathrm{X}}(A,B) := \inf\{r \mid B \subset (A)_r \text{ and } A \subset (B)_r\},$$

where $(S)_r$ denotes the open r-tubular neighbourhood of a set $S \subset \mathrm{X}$.

Definition 5.1.6 (Gromov-Hausdorff distance)**.** Let $(\mathrm{X}_1,\mathsf{d}_1),(\mathrm{X}_2,\mathsf{d}_2)$ be two metric spaces, we define the Gromov-Hausdorff distance between X_1 and X_2 as the number

$$d_{GH}(\mathrm{X}_1,\mathrm{X}_2) := \inf_{\substack{(Z,\mathsf{d}) \\ i_1:\mathrm{X}_1 \to (Z,\mathsf{d}) \\ i_2:\mathrm{X}_2 \to (Z,\mathsf{d})}} d_H^{\mathrm{Z}}(i_1(\mathrm{X}_1), i_2(\mathrm{X}_2)),$$

where the infimum is taken among all the triples $(\mathrm{Z},\mathsf{d}), i_1, i_2$ such that i_1, i_2 are isometric embeddings of $\mathrm{X}_1, \mathrm{X}_2$ into Z.

It will be sometimes more convenient to work with an equivalent characterization of the Gromov-Hausdorff distance through Gromov-Hausdorff approximation maps defined as follows.

Definition 5.1.7 (GH-approximations)**.** Let $(\mathrm{X}_1,\mathsf{d}_1),(\mathrm{X}_2,\mathsf{d}_2)$ be two metric spaces and fix a number $\delta>0$. We say that a function $f : \mathrm{X}_1 \to \mathrm{X}_2$ is a *δ-Gromov-Hausdorff approximation map* (δ-GH-app. in short) if

- f is a δ-isometry,
- $f(X)$ is δ-dense in X_2.

(Observe that we do not require f to be continuous.)

We can now characterize the Gromov-Hausdorff distance as follows.

Theorem 5.1.8 ([**60**, Corollary 7.3.28])**.** *Let $(\mathrm{X}_1,\mathsf{d}_1),(\mathrm{X}_2,\mathsf{d}_2)$ be two metric spaces. Then*

(1) if $d_{GH}(\mathrm{X}_1,\mathrm{X}_2) \le \delta$, then there exists a 2δ-GH-app. $f : \mathrm{X}_1 \to \mathrm{X}_2$,
(2) if there exists a δ-GH-app. $f : \mathrm{X}_1 \to \mathrm{X}_2$, then $d_{GH}(\mathrm{X}_1,\mathrm{X}_2) \le 2\delta$.

We point out that in [**60**] δ-isometry maps are what we here call δ-GH-app. maps.

Properties of the Gromov-Hausdorff distance. The following result shows that GH-approximations between two metric spaces can be chosen to be almost the inverse of each other.

Proposition 5.1.9. *Let* $(\mathrm{X}_1, \mathsf{d}_1), (\mathrm{X}_2, \mathsf{d}_2)$ *be two metric spaces and suppose* $f : \mathrm{X}_1 \to \mathrm{X}_2$ *is a* δ*-GH-app. then there exists a* 3δ*-GH-app.* $g : \mathrm{X}_2 \to \mathrm{X}_1$ *such that*

$$\mathsf{d}_2(f(g(y)), y) < 2\delta, \tag{5.1.2}$$

$$\mathsf{d}_1(g(f(x)), x) < 2\delta, \tag{5.1.3}$$

for every $x \in \mathrm{X}_1$ *and* $y \in \mathrm{X}_2$.

Proof. We can construct g explicitly as follows. For every $y \in \mathrm{X}_2$ by definition there exists at least one $x \in \mathrm{X}_1$ satisfying $\mathsf{d}_2(f(x), y) < \delta$, we define $g(y)$ to be one of such points (chosen arbitrarily). We prove (5.1.2). Pick any $y \in \mathrm{X}_2$, then by definition $g(y)$ is a point in X_1 such that $\mathsf{d}_2(f(g(y)), y) < \delta$ and we conclude. We now prove (5.1.3). Pick any $x \in \mathrm{X}_1$, then by definition of g we have that $\mathsf{d}_2(f(g(f(x))), f(x)) < \delta$ and since f is a δ-isometry we deduce that $\mathsf{d}_1(g(f(x)), x) < 2\delta$. Notice now that (5.1.3) already proves that $g(\mathrm{X}_2)$ is 2δ-dense in X_1. It remains to prove that g is a 3δ-isometry. Pick $y_1, y_2 \in \mathrm{X}_2$. From (5.1.2) we have $\mathsf{d}_2(f(g(y_i)), y_i) < \delta$, therefore by the triangle inequality we have

$$|\mathsf{d}_2(f(g(y_1)), f(g(y_2))) - \mathsf{d}_2(y_1, y_2)| < 2\delta.$$

Recalling that f is a δ-isometry we obtain that

$$|\mathsf{d}_1(g(y_1), g(y_2)) - \mathsf{d}_2(y_1, y_2)| < 3\delta.$$

□

In general a GH-approximation between two balls in different metric spaces needs not to send the center near the center, however this holds if one the spaces is the Euclidean space:

Proposition 5.1.10. *Let* (X, d) *be a metric space and let* $B_1(x_0)$ *the ball of radius* 1 *centered at* $x_0 \in \mathrm{X}$. *Suppose* $f : B_1(x) \to B_1^{\mathbb{R}^k}(0)$ *is a* δ*-GH-app., then*

$$|f(x_0)| \leq 7\delta.$$

Proof. Suppose by contradiction that $|f(x_0)| > 7\delta$, then there exists a point $y \in B_1(0)$ such that $|y - f(x_0)| > 1 + 6\delta$. Let g be the almost-inverse map given by Proposition 5.1.9. Then from (5.1.3) we have $\mathsf{d}(g(f(x_0)), x_0) < 2\delta$. Moreover, since g is a 3δ-isometry, we get

$$\mathsf{d}(g(y), g(f(x_0))) > 1 + 3\delta.$$

Hence by the triangle inequality

$$\mathsf{d}(g(y), x_0) \geq \mathsf{d}(g(y)), g(f(x_0)) - \mathsf{d}(g(f(x_0)), x_0) \geq 1 + \delta,$$

but $g(y) \in B_1(x_0)$ and thus we have a contradiction. □

Isometry-approximation of a δ-isometry. The following technical lemma shows that a δ-isometry between Euclidean spaces is close to an isometry. The crucial point is that the closeness is quantified: proportional to δ if the source and target have the same dimension and proportional to $\sqrt{\delta}$ otherwise. We point out that a non-quantified version of this statement could be more directly derived by compactness, however for our goal of proving the bi-Lipschitz Reifenberg's theorem, this explicit control will be essential. A version of this result in the case $m = n$ can also be found in [**93**, Lemma 7.11].

Lemma 5.1.11. *For every* $n \in \mathbb{N}$ *exists a constant* $C(n)$ *such that the following holds. For every map* $f : \mathbb{R}^n \supset B_t(0) \to \mathbb{R}^m$ *with* $m \geq n$ *such that*

$$||x - y| - |f(x) - f(y)|| \leq \delta t, \tag{5.1.4}$$

for some $\delta < 1$, *there exists an isometry* $I : \mathbb{R}^n \to \mathbb{R}^m$ *with* $I(0) = f(0)$, *such that*

$$|I - f| \leq C(n)\delta t, \quad \text{on } B_t(0), \quad \text{if } m = n, \tag{5.1.5}$$

and

$$|I - f| \leq C(n)\sqrt{\delta} t \quad \text{on } B_t(0), \quad \text{if } m > n. \tag{5.1.6}$$

PROOF. Without loss of generality we can suppose $f(0) = 0$. Moreover, since the statement is scaling invariant, it's also enough to prove the case $t = 2$. We start by noticing that from (5.1.4), the fact that $f(0) = 0$ and from the assumption $\delta < 1$ we have that $f(B_2(0)) \subset B_4(0)$. This fact will be used without further comments on the computations contained in this proof. Moreover we remark that constant $C(n)$ that will appear may change from line to line, but will depend only on n. Denote now by $\{e_1, ..., e_n\} \subset B_2(0)$ the canonical orthonormal frame in $\mathbb{R}^n$. We claim that there exists an orthonormal frame $\{f_1, ..., f_m\}$ in $\mathbb{R}^m$ such that $|f(e_i) - f_i| \leq C(n)\delta$ for every $i \leq n$. To see this it's sufficient to apply Gram-Schmidt orthogonalization procedure to the vectors $\{f(e_1), ..., f(e_n)\}$: define

$$f_1 := \frac{f(e_1)}{|f(e_1)|}$$

and inductively

$$\tilde{f}_k := f(e_k) - \sum_{i=1}^{k-1} \langle f(e_k), f_i \rangle f_i$$

$$f_k := \frac{\tilde{f}_k}{|\tilde{f}_k|}.$$

Then we complete $\{f_1, ..., f_n\}$ to an orthonormal frame $\{f_1, ..., f_m\}$. We need to verify that $|f(e_i) - f_i| \leq C(n)\delta$ for every $i \leq n$. We argue by induction. First observe that by (5.1.4) and $f(0) = 0$ it follows that

$$||f(e_i)| - 1| \leq \delta, \tag{5.1.7}$$

for every $i = 1, .., n$. In particular $||f(e_1)| - 1| \leq \delta$, therefore $|f_1 - f(e_1)| = |f(e_1)\frac{1-|f(e_1)|}{|f(e_1)|}| \leq \delta$, hence the case $i = 1$ is verified. Suppose now the claim is true up to $i = k - 1$. Again by (5.1.4) and induction hypothesis we also have $||f(e_k) - f_i| - \sqrt{2}| \leq ||f(e_k) - f(e_i)| - \sqrt{2}| + |f(e_i) - f_i| \leq C(n)\delta$ for every $i < k$. This and (5.1.7) imply

$$\begin{aligned} |\langle f(e_k), f_i \rangle| &\leq ||f(e_k)|^2 - 1 - 2\langle f(e_k), f_i \rangle| + ||f(e_k)|^2 - 1| = \\ &= ||f(e_k) - f_i|^2 - 2| + ||f(e_k)|^2 - 1| \\ &\leq 8||f(e_k) - f_i| - \sqrt{2}| + 4||f(e_k)| - 1| \leq C(n)\delta. \end{aligned}$$

Therefore

$$|\sum_{i=1}^{k-1} \langle f(e_k), f_i \rangle f_i| \leq C(n)\delta,$$

that implies $|f(e_k) - \tilde{f}_k| \leq C(n)\delta$. From this and (5.1.7) we also deduce that $||\tilde{f}_k| - 1| \leq C(n)\delta$ and combining this two facts, we easily deduce $|f(e_k) - f_k| \leq C(n)\delta$. We define now the isometry I as the unique isometry such that $I(0) = 0$ and $I(e_i) = f_i$, i.e. $I(x) = Ax$ where A is the $n \times n$ matrix having $f_1, ..., f_n$ as columns. Consider now $x \in B_2(0)$, then $x = \sum_{i=1}^n a_i e_i$ for some a_i and $T(x) = \sum_{i=1}^n a_i f_i$. Moreover $f(x) = \sum_{i=1}^n b_i f_i + \sum_{i=n+1}^m c_i f_i$ for some real numbers b_i, c_i. Then

$$a_i = \langle T(x), f_i \rangle = \frac{|T(x)|^2 + 1 - |T(x) - f_i|^2}{2} = \frac{|x|^2 + 1 - |x - e_i|^2}{2}$$

and for every $i \leq n$

$$b_i = \langle f(x), f_i \rangle = \frac{|f(x)|^2 + 1 - |f(x) - f_i|^2}{2}.$$

Define now $f_= = \sum_{i=1}^n b_i f_i$ and $f_\perp = \sum_{i=n+1}^m c_i f_i$ so that $f_= \perp f_\perp$ and $f(x) = f_= + f_\perp$. Then

$$\begin{aligned} |f_= - T(x)| \leq \sum_i |a_i - b_i| &\leq \frac{||f(x)|^2 - |x|^2| + ||f(x) - f_i|^2 - |x - e_i|^2|}{2} \\ &\leq \frac{8||f(x)| - |x|| + 8||f(x) - f_i| - |x - e_i||}{2} \\ &\leq \frac{C(n)\delta + 4|f_i - f(e_i)|}{2} \leq C(n)\delta. \end{aligned}$$

If $n = m$, then $f_= = f(x)$ and (5.1.5) is proved. Suppose now $m > n$. Observe that

$$||f(x)|^2 - |T(x)|^2| = ||f(x)| - |x||(|f(x)| + |x|) \leq 6\delta t,$$

therefore $|f(x)|^2 \leq |T(x)|^2 + 6\delta$. Moreover $|f(x)|^2 = |f_=|^2 + |f_\perp|^2$, thus

$$\begin{aligned}|f_\perp|^2 = |f(x)|^2 - |f_=|^2 &\leq |T(x)|^2 - |f_=|^2 + 6\delta \\ &\leq ||T(x)|^2 - |f_=|^2| + 6\delta \\ &\leq ||T(x)| - |f_=||(|T(x)| + |f(x)|) + 6\delta \leq C(n)\delta.\end{aligned}$$

This implies that $|f(x) - T(x)|^2 = |f_\perp|^2 + |f_= - T(x)|^2 \leq C(n)(\delta^2 + \delta) \leq C(n)\delta$, that is (5.1.6). □

5.2. Cheeger-Colding's Bi-Lipschitz metric Reifenberg's theorem

5.2.1. Statement and remarks. To state the main result we need first to introduce the key quantities of this section, which are the the following Gromov-Hausdorff 'flatness-coefficients'.

Definition 5.2.1. Let (Z, d) be a metric space. For every $r > 0$, $n \in \mathbb{N}$ and $i \in \mathbb{N}_0$ define

$$\varepsilon_i(r, n) := \frac{2^i}{r} \sup_{z \in Z} d_{GH}(B^Z_{2^{-i}r}(z), B^{\mathbb{R}^n}_{2^{-i}r}(0)). \tag{5.2.1}$$

Theorem 5.2.2 (Bi-Lipschitz metric-Reifenberg's theorem)**.** *For every $n \in \mathbb{N}$ the following holds. Let (Z, d) be a connected and complete metric space. Suppose that for some $r > 0$ we have that*

$$\sum_{i \geq 0} \varepsilon_i(r, n) < +\infty. \tag{5.2.2}$$

Then for every $\varepsilon > 0$ there exists a smooth n-dimensional Riemannian manifold $(W_\varepsilon, d_\varepsilon)$ and a surjective homeomorphism $F_\varepsilon : W_\varepsilon \to Z$ that is (uniformly) locally $(1+\varepsilon)$-bi-Lipschitz. That is to say that there exists $\rho > 0$ such that

$$\text{Lip } F_\varepsilon|_{B_\rho(w)},\ \text{Lip } {F_\varepsilon}^{-1}|_{B_\rho(z)} < 1 + \varepsilon, \tag{5.2.3}$$

for every $w \in W_\varepsilon$ and every $z \in Z$. Moreover for every $\varepsilon_1, \varepsilon_2 > 0$ the manifolds $W_{\varepsilon_1}, W_{\varepsilon_2}$ can be chosen to be diffeomorphic to each other.

Remark 5.2.3. Differently from [**73**] we are not assuming that (Z, d) is separable. Indeed separability is a consequence of the assumptions of Theorem 5.2.2, as we will prove at the beginning of Section 5.4.1. ■

Remark 5.2.4. Observe that, since W_ε is complete and connected, the map F_ε is always globally $1+\varepsilon$ Lipschitz. Analogously, if (Z, d) is a length space, we also obtain that F_ε^{-1} is globally $1+\varepsilon$ bi-Lipschitz. On the contrary, without further assumptions of Z we can only say that ${F_\varepsilon}^{-1}$ is Lipschitz on any bounded set, with a Lipschitz constant that depends only on the diameter of the set (see also Example 5.2.5 below). In particular if Z is bounded, then F_ε is always bi-Lipschitz. ■

The following example shows that F_ε in Theorem 5.2.2 cannot be taken to be globally Lipschitz in general.

Example 5.2.5. Take the real line $\mathbb{R}$ with the following distance

$$\mathsf{d}(t, s) := \begin{cases} |s - t| & \text{if } |t - s| \leq 4, \\ 2\sqrt{|t - s|} & \text{otherwise .} \end{cases}$$

Clearly $(\mathbb{R}, \mathsf{d})$ satisfy the hypothesis of Theorem 5.2.2 with $r = 1$ and $n = 1$. Then for any $\varepsilon > 0$ we can take $W_\varepsilon = (\mathbb{R}, |.|)$ and F_ε to be identity. Since F_ε is locally an isometry, it satisfies (5.2.3). However F_ε is not globally bi-Lipschitz. We point out the these W_ε and F_ε are really the ones that come up in the construction in the proof of Theorem 5.2.2, provided that at the beginning of the proof we pick the maps $\alpha_{i,j}, \beta_{i,j}$ to be the identity. This can be checked simply following the steps of the proof and observing that most of the construction trivializes. ■

Local version. A local version of Theorem 5.2.2, more in the spirit of Theorem 0.30 also holds. To state it we need the following variants of the numbers ε_i defined above.

Definition 5.2.6. Let (Z, d) be a metric space and $z_0 \in Z$. For every $n \in \mathbb{N}$ and every $i \in \mathbb{N}_0$ define

$$\varepsilon_i(n) := 2^i \sup_{z \in B_{1-2^{-i}}(z_0)} d_{GH}(B^Z_{2^{-i}}(z), B^{\mathbb{R}^n}_{2^{-i}}(0)). \tag{5.2.4}$$

Theorem 5.2.7 (Bi-Lipschitz metric-Reifenberg's theorem - local version)**.** *For every $n \in \mathbb{N}$ there exist constants $\varepsilon(n) > 0, M = M(n) > 1$ such that the following is true. Let $B_1(z_0)$ be a a ball inside a complete metric space (Z, d). Suppose that $\varepsilon_i(n) \le \varepsilon(n)$ for every $i \ge 0$ and*

$$\sum_{i \ge 1} \varepsilon_i(n) < +\infty. \tag{5.2.5}$$

Then there exists a bi-Lipschitz map $F : B_1^{\mathbb{R}^n}(0) \to Z$ such that $B_{1-M\varepsilon_0}(z_0) \subset F(B_1(0))$.

We observe that, contrary to Theorem 5.2.2, in Theorem 5.2.7 we do not have an arbitrary small bi-Lipschitz constant. This is due to the fact that in Theorem 5.2.2 we have the freedom to choose the manifold to which we compare the metric space, while in Theorem 5.2.7 the manifold is fixed to be the unit Euclidean ball. We will not include the proof of Theorem 5.2.7, which in any case is a minor modification of the one that we will present for Theorem 5.2.2.

It is also worth to point out that we cannot improve Theorem 5.2.7 to have $B_1(z_0) = F(B_1(0))$, indeed it might even happen that $B_1(z_0)$ is not connected, as shown in the following example.

Example 5.2.8. Fix $\varepsilon > 0$ small and $n \in \mathbb{N}$. Consider the metric space (Z, d) obtained as union of two closed Euclidean balls $B_1 := \overline{B_1^{\mathbb{R}^n}(0_1)}, B_2 := \overline{B_1^{\mathbb{R}^n}(0_2)}$ by setting $d(0_2, 0_1) = 1 - \varepsilon$. It is straight-forward to verify that $B_1^Z(0_1)$ satisfies the hypotheses of Theorem 5.2.7 with $\varepsilon_0 = 3\varepsilon$ and $\varepsilon_i = 0$ for every $i \ge 1$. However $B_s^Z(0_1)$ is disconnected for every $1 - \varepsilon < s \le 1$. ■

5.2.2. Setup and main argument.

Notations and constants. We start by fixing once and for all a positive constant $C = C(n)$, that will appear both on the present and on the following sections. Its value will be determined along the proof and may change from line to line, but will in any case remain dependent only on n. Moreover we also pick $\varepsilon(n) = \varepsilon(n, C)$ another positive constant that may change along the proof, but will depend only on C and n. In particular $\varepsilon(n)$ at the end will depend only on n. Since r and n are fixed along all the proof we will also write ε_i in place of $\varepsilon_i(r, n)$, for every $i \in \mathbb{N}_0$.

We observe that Theorem 5.2.2 holds for a metric space (Z, d) if and only if it holds for all the rescaled spaces $(Z, d\lambda)$, with $\lambda > 0$. Moreover, denoted by $\varepsilon_i^\lambda(r, n)$ the numbers in Definition 5.2.1 relative to the rescaled space $(Z, \lambda\mathsf{d})$, it easy to verify that

$$\varepsilon_i^\lambda(\lambda r 2^{-j}, n) = \varepsilon_{i+j}(r, n),$$

for every $i, j \in \mathbb{N}$ and every $r > 0$. Therefore, since by (5.2.2) $\varepsilon_i \to 0^+$ as $i \to +\infty$, up to rescaling the metric d we can assume both that

$$r = 200 \tag{5.2.6}$$

and that

$$\varepsilon_i \le \varepsilon(n) \text{ for every } i \ge 0. \tag{5.2.7}$$

In particular whenever in the present and following sections we will say that ε_i is small enough, we will implicitly assume that the constant $\varepsilon(n)$ is chosen sufficiently small.

Main argument. The rough idea of the proof is to construct for every scale 2^{-i} a manifold (W_i, d_i) that approximates the metric space Z in the sense that there exists a map $f_i : W_i \to Z$ that is roughly an $\varepsilon_i 2^{-i}$-isometry. Moreover we build maps $h_i : W_i \to W_{i+1}$ that are bi Lipschitz with constant $\sim (1 + \varepsilon_i)$. Then the required map F will be obtained as limit of the maps $f_i \circ h_i \circ h_{i-1} \circ \cdots \circ h_1$.

CLAIM: For the proof it is sufficient to construct a sequence of (complete and connected) n-dimensional Riemannian manifolds (W_i^n, d_i), symmetric maps $\rho_i : W_i^n \times W_i^n \to [0, \infty)$, surjective diffeomorphisms $h_i : W_i^n \to W_{i+1}^n$ and maps $f_i : W_i^n \to Z$, for $i \in \mathbb{N}_0$, satisfying the following statements.

(1) For every $w_1, w_2 \in W_i^n$ it holds that

$$\rho_i(w_1, w_2) = \mathsf{d}_i(w_1, w_2),$$

whenever $\rho_i(w_1, w_2) \le 2^{-i}$ or $\mathsf{d}_i(w_1, w_2) \le 2^{-i}$.

(2)

$$\frac{1}{1 + C(\varepsilon_i + \varepsilon_{i+1})} \le \frac{\rho_{i+1}(h_i(w_1), h_i(w_2))}{\rho_i(w_1, w_2)} \le 1 + C(\varepsilon_i + \varepsilon_{i+1}),$$

for every $w_1, w_2 \in W_i^n$,

(3) $|\mathsf{d}(f_i(w_1), f_i(w_2)) - \rho_i(w_1, w_2)| \le C 2^{-i} \varepsilon_i$, for every $w_1, w_2 \in W_i^n$,

(4) $\mathsf{d}(f_{i+1}(h_i(w)), f_i(w)) \le C 2^{-i}(\varepsilon_i + \varepsilon_{i+1})$, for every $w \in W_i^n$,

(5) $f_i(W_i^n)$ is $20 \cdot 2^{-i}$-dense in Z.

We now briefly comment the statement of the claim. The function ρ_i substitutes the Riemannian distance, coincides with d_i at small scales and it is built to approximate the distance d (see item 3). It is needed because we do not have a good control on the relation between d_i and d at large scales. The map f_i is a kind of GH-approximation between W_i and Z. Indeed it is almost surjective (see item 5) and it is a quasi isometry (see item 3), however not with respect to the Riemannian distance, but with respect to ρ_i. Item 2 is the bi-Lipschitz property of the maps h_i. Finally, item 4 ensures that h_i does not move points too much, from the prospective of the metric space.

Proof of the CLAIM: Pick any $\varepsilon > 0$. Start by fixing an integer $m \geq 0$ large enough, that will be chosen later. Set $\theta_m := \sup_{i \geq m} \varepsilon_i$. For every $i > m$ define the function $F_i : W_m^n \to Z$ as $F_i := f_i \circ h_{i-1} \circ ... \circ h_{m+1} \circ h_m$ and set $F_m := f_m$. From 4 we get that for every $w \in W_m^n$

$$\mathsf{d}(F_{i+1}(w), F_i(w)) \leq (\varepsilon_i + \varepsilon_{i+1}) C 2^{-i} \leq 2C\theta_m 2^{-i}, \tag{5.2.8}$$

for every $i \geq m$. Hence the sequence $\{F_i(w)\}_{i \geq m}$ is Cauchy in Z and we call $F(w)$ its limit. We will prove that the manifold W_m^n and the map $F : W_m^n \to Z$ satisfy the conclusions of Theorem 5.2.2 with ε. Notice first that from (5.2.8) we obtain

$$\mathsf{d}(F(w), F_i(w)) \leq 2C2^{-i}\theta_m, \tag{5.2.9}$$

for every $i \geq m$. Consider now any couple of distinct points $w_1, w_2 \in W_m$ and set

$$s_m := \rho_m(w_1, w_2) > 0.$$

Define also inductively for every $i \geq m$ the points $w_1^i, w_2^i \in W_i^n$ and the numbers s_i by setting $w_1^m := w_1, w_2^m := w_2$,

$$w_1^{i+1} := h_i(w_1^i), \ w_2^{i+1} := h_i(w_2^i)$$

and

$$s_i := \rho_i(w_1^i, w_2^i).$$

Observe that from 2

$$(1 + C(\varepsilon_i + \varepsilon_{i+1}))^{-1} s_i \leq s_{i+1} \leq s_i (1 + C(\varepsilon_i + \varepsilon_{i+1})),$$

for every $i \geq m$. Iterating the above inequality we get

$$s_m \prod_{j=m}^{i} (1 + C(\varepsilon_j + \varepsilon_{j+1}))^{-1} \leq s_i \leq s_m \prod_{j=m}^{i} (1 + C(\varepsilon_j + \varepsilon_{j+1})), \tag{5.2.10}$$

for every $i > m$. Moreover from 3 we deduce that

$$|\mathsf{d}(F_i(w_1), F_i(w_2)) - s_i| \leq C\varepsilon_i 2^{-i}, \tag{5.2.11}$$

for every $i \geq m$. This implies that $s_i \to \mathsf{d}(F(w_1), F(w_2))$ as $i \to +\infty$. Then passing to the limit in (5.2.10) we obtain

$$\prod_{j=m}^{+\infty} \frac{1}{1 + C(\varepsilon_j + \varepsilon_{j+1})} \leq \frac{\mathsf{d}(F(w_1), F(w_2))}{s_m} \leq \prod_{j=m}^{+\infty} (1 + C(\varepsilon_j + \varepsilon_{j+1})).$$

Observe now that, since by hypothesis $\sum_{j \geq 0} \varepsilon_j < +\infty$, we have that $\prod_{j=0}^{+\infty} (1 + C(\varepsilon_j + \varepsilon_{j+1})) < +\infty$ and $\prod_{i=0}^{+\infty} (1 + C(\varepsilon_j + \varepsilon_{j+1}))^{-1} > 0$. In particular $\prod_{j=k}^{+\infty} (1 + C(\varepsilon_j + \varepsilon_{j+1})) \downarrow 1$ and $\prod_{j=k}^{+\infty} (1 + C(\varepsilon_j + \varepsilon_{j+1}))^{-1} \uparrow 1$ as k goes to $+\infty$. Therefore, up to choosing m big enough,

$$\frac{1}{1+\varepsilon} \leq \frac{\mathsf{d}(F(w_1), F(w_2))}{s_m} \leq 1 + \varepsilon.$$

Recall now that by 1 we have that $\mathsf{d}_m(w_1, w_2) \leq 2^{-m}$ implies $s_m = \mathsf{d}_m(w_1, w_2)$, hence

$$\frac{1}{1+\varepsilon} \leq \frac{\mathsf{d}(F(w_1), F(w_2))}{d_m(w_1, w_2)} \leq 1 + \varepsilon, \tag{5.2.12}$$

whenever $\mathsf{d}_m(w_1, w_2) \leq 2^{-m}$. This already proves the first part of (5.2.3), provided we take $\rho < 2^{-m-1}$. Combining now (5.2.11) with (5.2.9) for $i = m$ and observing that $\theta_m \to 0^+$ as m goes to $+\infty$, we obtain that

$$|\mathsf{d}(F(w_1), F(w_2)) - s_m| \leq 6C\theta_m 2^{-m} < \frac{1}{4} 2^{-m},$$

provided m is big enough. Therefore, whenever $\mathsf{d}(F(w_1), F(w_2)) \leq 2^{-m-1}$, it holds that $s_m \leq 2^{-m}$ and thus from 1 also that $d_m(w_1, w_2) \leq 2^{-m}$. Combining this observation with (5.2.12) we obtain that

$$\frac{1}{1+\varepsilon} \leq \frac{\mathsf{d}(F(w_1), F(w_2))}{d_m(w_1, w_2)} \leq 1 + \varepsilon, \tag{5.2.13}$$

whenever $\mathsf{d}(F(w_1), F(w_2)) \leq 2^{-m-1}$. This proves both that F is injective. We now claim that F is surjective. We start proving that $F(W_m^n)$ is dense. To see this consider $z \in Z$ and $\delta > 0$. Take now $i \geq m$ such that $20 \cdot 2^{-i} \leq \delta/8$, then by 5 and the fact that the maps h_k are surjective, there exists $w \in W_m^n$ such that $\mathsf{d}(F_i(w), z) < \delta/2$. Moreover from (5.2.9) $\mathsf{d}(F(w), F_i(w)) < 4C2^{-i}\theta_m \leq 4C\delta\theta_m \leq \delta/2$, provided m is big enough. Hence $\mathsf{d}(F(w), z) < \delta$ and thus $F(W_m^n)$ is dense from the arbitrariness of $\delta > 0$. Pick now any $z \in Z$, by density, there exists a sequence $w_k \in W_m$ such that $F(w_k) \to z$. In particular $(F(w_k))$ forms a Cauchy sequence in (Z, d). From (5.2.13) we deduce that also w_k is Cauchy and by the completeness of W_m it converges to a limit $w \in W_m$. From the continuity of F we have that $F(w) = z$. This proves that F is surjective. The second part of (5.2.3) now follows from (5.2.13) provided we choose $\rho < 2^{-m-2}$. Finally the last part of the Theorem follows directly form the fact that the maps $h_k : W_k^n \to W_{k+1}^n$ are diffeomorphisms. This concludes the proof of the **CLAIM**.

5.3. Core tools needed in the proof

5.3.1. Basic modification result. Here we state and prove Theorem 5.3.1, which is the main building block for the results that we will prove in Sections 5.3.2 and 5.3.3. The key feature of this theorem is that the constants appearing in the statement do not depend on the scale t. We also point out that Theorem 5.3.1 is in a sense the C^1 counterpart of Lemma 3.5 in [**70**].

Main statement.

Theorem 5.3.1. *For every $n \in \mathbb{N}$ and $\eta > 0$, there exists constants $C_2 = C_2(n, \eta) > 0, \delta_2 = \delta_2(n, \eta) \in (0, 1)$ with the following property. Let $m \in \{1, 2\}$. Let U_1, U_2 open bounded subsets of $\mathbb{R}^n$ such that $\bar{U}_1 \subset U_2$ and $\mathsf{d}(\bar{U}_1, U_2^c) \geq \eta t$ with $t > 0$. Suppose $H \in C^k(U_2; \mathbb{R}^n)$, with $k \geq 2$, satisfies*

$$\|H - \mathsf{id}\|_{C^m(U_2),t} \leq \varepsilon$$

for some $\varepsilon < \delta_2$.

Then there exists a smooth global diffeomorphism $\hat{H} : \mathbb{R}^n \to \mathbb{R}^n$ such that $H|_{U_1} = \hat{H}|_{U_1}$, $\hat{H}|_{U_2^c} = id$, $\hat{H}(x) = H(x)$ whenever $H(x) = x$ and

$$\|\hat{H} - \mathsf{id}\|_{C^m(\mathbb{R}^n),t} \leq C_2(n, \eta)\varepsilon. \tag{5.3.1}$$

Moreover $\hat{H}^{-1}$ is C^k and

$$\|\hat{H}^{-1} - \mathsf{id}\|_{C^m(\mathbb{R}^n),t} \leq C_2(n, \eta)\varepsilon. \tag{5.3.2}$$

Technical lemmas. For the proof of Theorem 5.3.1 we will need two elementary technical results.

Lemma 5.3.2. *Let $f \in C^k(\mathbb{R}^n; \mathbb{R}^n)$, $k \in \mathbb{N}$, be such that*

$$\|Df(x) - I\| \leq 1/2,$$

$$\{|f(x) - x| \mid x \in \mathbb{R}^n\} \text{ is bounded .}$$

Then f is a diffeomorphism (from $\mathbb{R}^n$ to $\mathbb{R}^n$) with inverse of class C^k.

Proof. Fix $x \in \mathbb{R}^n$. Then for every $v \in \mathbb{R}^n$

$$|v| \leq |v - Df(x)v| + |Df(x)v| \leq 1/2|v| + |Df(x)v|$$

hence $1/2|v| \leq |Df(x)v|$, thus $Df(x)$ is invertible. Then by the Inverse Function Theorem f is a local diffeomorphism and an open map. Now observe that $f - id$ is $1/2-$Lipschitz, thus for every $x, y \in \mathbb{R}^n$

$$|x - y| - |f(x) - f(y)| \leq |(id - f)(x) - (id - f)(y)| \leq |x - y|/2,$$

hence $|x - y| \leq 2|f(x) - f(y)|$ and in particular f is injective. Moreover f^{-1} is C^k again thanks to the Inverse Function Theorem. We claim now that f is proper, i.e. that $f^{-1}(K)$ is compact for every K compact. Indeed K is closed and bounded, hence $f^{-1}(K)$ is closed and it is also bounded by second hypothesis on f, in particular it is compact. Since f is proper, it a closed map, but we already observed that it is also open, hence it is surjective. This concludes the proof. □

Lemma 5.3.3. *For every $n \in \mathbb{N}$ there exist $\delta = \delta(n) > 0$ and $C = C(n) > 0$ with the following property. Let $m \in \{1, 2\}$. Let $H \in C^k(\mathbb{R}^n; \mathbb{R}^n)$, $k \geq m$, be such that $\|H - \mathsf{id}\|_{C^k(\mathbb{R}^n),t} \leq \varepsilon$, for some $\varepsilon \in (0, 1/4)$. Then H is a diffeomorphism (from $\mathbb{R}^n$ to $\mathbb{R}^n$) with C^k inverse and satisfying*

$$\|H^{-1} - \mathsf{id}\|_{C^m(\mathbb{R}^n),t} \leq C\varepsilon.$$

PROOF. The fact that H is a global diffeomorphism of $\mathbb{R}^n$ with C^k inverse follows from Lemma 5.3.2. To prove the require bound we first observe that

$$|H^{-1} - \mathsf{id}| = |H^{-1} - H(H^{-1})| \leq \varepsilon t.$$

Moreover

$$\|D(H^{-1}) - I_n\| = \|[DH(H^{-1})]^{-1} - I_n\| \leq \|[DH(H^{-1})]^{-1}\| \|DH(H^{-1}) - I_n\| \leq 2\varepsilon,$$

where we have used the fact that $\|D(H^{-1})(x)\| \leq 2$ for every $x \in \mathbb{R}^n$, as a can be easily derived from $\|DH(x) - I_n\| \leq 1/2$ for every $x \in \mathbb{R}^n$. Finally, exploiting the formula for the derivative of the inverse matrix, we have

$$\begin{aligned}\partial_j D(H^{-1}) = \partial_j [DH(H^{-1})]^{-1} &= [DH(H^{-1})]^{-1} \partial_j (DH(H^{-1}))[DH(H^{-1})]^{-1} \\ &= D(H^{-1}) \partial_j (DH(H^{-1})) D(H^{-1})\end{aligned}$$

where ∂_j is component-wise derivative. Therefore, recalling (5.1.1) and that $\|DH\|, \|D(H^{-1})\| \leq 2$,

$$\begin{aligned}|\partial_{i,j}(H^{-1})_k| &\leq \sqrt{n}\|\partial_j D(H^{-1})\| \leq 4\sqrt{n}\|\partial_j (DH(H^{-1}))\| \\ &\leq \sqrt{n} n^2 \max_{l,m} |\partial_j (\partial_l H_m(H^{-1}))| \\ &\leq \sqrt{n} n^2 \max_{l,m} \sum_h |\partial_{h,l} H_m(H^{-1})| |\partial_j H_h^{-1}| \\ &\leq n \cdot n^3 \frac{\varepsilon}{t} \|D(H^{-1})\| \leq 2n^4 \frac{\varepsilon}{t}.\end{aligned}$$

$\square$

PROOF OF THEOREM 5.3.1. Is is enough to consider $t = 1$, since the case of a general t follows by scaling observing that the norms $\|.\|_{C^m,t}$ are scaling invariant (recall Remark 5.1.2). Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $\varphi = 1$ on U_1 and $\varphi = 0$ on U_2^c and such that $|\partial_{i,j}\varphi|, |\partial_i \varphi| \leq c = c(n, \eta)$ and $|\varphi| \leq 1$ (for example $\varphi := \chi_{U_1} * \rho$ with an appropriate convolution kernel.) Define

$$\hat{H} := (H - \mathsf{id})\varphi + \mathsf{id}.$$

Then using (5.1.1)

$$|\hat{H} - \mathsf{id}| \leq \varepsilon,$$

$$|\partial_i(\hat{H} - \mathsf{id})_k| \leq |\partial_i(H - \mathsf{id})_k||\varphi| + |(H - \mathsf{id})_k||\partial_i \varphi| \leq \sqrt{n}(1 + c)\varepsilon,$$

$$\begin{aligned}|\partial_{i,j}(\hat{H} - \mathsf{id})_k| &\leq |\partial_{i,j} H_k||\varphi| + |\partial_i(H - \mathsf{id})_k||\partial_j \varphi| + |\partial_j(H - \mathsf{id})_k||\partial_j \varphi| + |(H - \mathsf{id})_k||\partial_{i,j}\varphi| \\ &\leq \tilde{c}(\delta, n)\varepsilon\end{aligned}$$

on $\mathbb{R}^n$, where $\tilde{c}$ is a constant depending only on δ and n. This proves (5.3.1). It's also clear that $H|_{U_1} = \hat{H}|_{U_1}$, $\hat{H}|_{U_2^c} = id$ and $\hat{H}(x) = H(x)$ whenever $H(x) = x$. Moreover from the first and second bound above, if ε is small enough with respect to c and n, $\hat{H}$ is a diffeomorphism and $\hat{H}^{-1}$ is C^k by Lemma 5.3.2. Then (5.3.2) is a direct consequence of Lemma 5.3.3. $\square$

5.3.2. Mappings modification theorem. This section is devoted to the proof of the most important technical result which will be needed in Section 5.4.1 (in particular in the proof of Lemma 5.4.9) when building the manifolds that will approximate the metric space. Roughly said these manifolds will be built starting from a family of transition maps, but without the knowledge of charts, which instead need to be constructed by hand. However these transitions maps (which are actually isometries of $\mathbb{R}^n$) will not in general be compatible with each other and thus need to be suitably modified in order to produce an actual manifold. This modification procedure is precisely the content of this section.

Let us also say that the result that we will prove concerns only mappings of $\mathbb{R}^n$ into itself and thus it is independent of the rest of the chapter.

Before stating the result we introduce a key notion for this section:

Cocyclical maps.

Definition 5.3.4. Let $f, g, h : \mathbb{R}^n \to \mathbb{R}^n$ be bijective maps and fix a radius $r > 0$. Define the maps $\{I_{ab}\}_{a,b \in \{1,2,3\}}$ as follows: $I_{12} := f, I_{23} := g, I_{13} := h$ and then set $I_{ba} = I_{ab}^{-1}$ for every distinct $a, b \in \{1, 2, 3\}$. We say that the maps f, g, h are r-cocyclical if for any distinct $a, b, c \in \{1, 2, 3\}$ we have that

$$\begin{aligned}&\text{for any point } x \in B_r(0) \text{ such that } I_{ba}(x) \in B_r(0), I_{cb}(I_{ba}(x)) \in B_r(0), \\ &\text{it holds } I_{ca}(x) = I_{cb}(I_{ba}(x)).\end{aligned} \tag{5.3.3}$$

We point out that the above definition is independent of the order of the three functions, i.e. f, g, h are r-cocyclical if and only if g, f, h are r-cocyclical and so on. Moreover it is immediate from the definition that f, g, h are r-cocyclical if and only if f^{-1}, g, h are r-cocyclical. Finally observe that if f, g, h are r-cocyclical, then they are also s-cocyclical for any $s < r$.

Remark 5.3.5. It is worth to observe that asking (5.3.3) is equivalent to ask that the following binary relation, defined on the the disjoint union $B_1 \sqcup B_2 \sqcup B_3$, of three copies of the Euclidean ball $B_r^{\mathbb{R}^n}(0)$, is transitive and symmetric:

$$x \sim y \text{ with } x \in B_a,\ y \in B_b \Longleftrightarrow I_{ba}(x) = y,$$

where $\{I_{ab}\}_{a,b\in\{1,2,3\}}$ are defined as above. ■

Remark 5.3.6. We observe that if (5.3.3) is satisfied for a particular choice a, b, c, then it is automatically satisfied also for the choice c, b, a. Therefore it is for example enough to check it with $(a, b, c) = (1, 2, 3)$, $(3, 1, 2)$, $(1, 3, 2)$. ■

The following simple result will be useful to quickly check the cocyclical condition. Roughly said it tells us that if three maps are almost cocyclical at some radius and (5.3.3) is verified for one choice of a, b, c and for that same radius, then they are fully cocyclical on a slightly smaller radius.

Proposition 5.3.7. *Let f, g, h and $\{I_{ab}\}_{a,b\in\{1,2,3\}}$ be as in Definition 5.3.4. Suppose that for every $a, b, c \in \{1, 2, 3\}$ distinct it holds*

$$|I_{cb} \circ I_{ba} - I_{ca}| < \varepsilon, \qquad \text{in } B_r(0)$$

and that

$$\begin{aligned} &\text{for any point } x \in B_r(0) \text{ such that } I_{21}(x) \in B_r(0), I_{32}(I_{21}(x)) \in B_r(0), \\ &\text{it holds } I_{31}(x) = I_{32}(I_{21}(x)). \end{aligned} \tag{5.3.4}$$

Then te maps f, g, h are $(r - \varepsilon)$-cocyclical.

Proof. Thanks to Remark 5.3.6 we need to verify (5.3.3) only for (a, b, c) in the cases $(1, 2, 3)$, $(3, 1, 2)$ and $(1, 3, 2)$. The case $(a, b, c) = (1, 2, 3)$ is true by hypothesis.

Suppose now that x, $I_{13}(x)$, $I_{21}(I_{13}(x)) \in B_{r-\varepsilon}(0)$. From (5.3.4) follows that $|I_{32}(I_{21}(I_{13}(x))) - x| < \varepsilon$. In particular $I_{32}(I_{21}(y)) \in B_r(x)$ with $y := I_{13}(x) \in B_r(x)$. Moreover we are assuming that $I_{21}(y) = I_{21}(I_{13}(x)) \in B_r(x)$, therefore from (5.3.4) we have $x = I_{31}(y) = I_{32}(I_{21}(y)) = I_{32}(I_{21}(I_{13}(x)))$, from which applying the map I_{23} we obtain $I_{23}(x) = I_{21}(I_{13}(x))$.

Suppose now that x, $I_{31}(x)$, $I_{23}(I_{31}(x)) \in B_{r-\varepsilon}(0)$. From (5.3.4) follows that $|I_{23}(I_{31}(x)) - I_{21}(x)| < \varepsilon$. In particular $I_{21}(x) \in B_r(x)$. Moreover again from (5.3.4) $|I_{32}(I_{21}(x)) - I_{31}(x)| < \varepsilon$, hence also $I_{32}(I_{21}(x)) \in B_r(x)$. Therefore from (5.3.4) we have $I_{32}(I_{21}(x)) = I_{31}(x)$, from which applying I_{23} we conclude. □

Statement of the main result. To state the main result of this section we need some notations:

- J is a countable set of indices,
- $\{J_i\}_{i=1}^N$, $N \geq 3$, is a partition of J and for every $j \in J$ we denote by $n(j)$ the unique integer such that $j \in J_{n(j)}$,
- $\mathcal{A} \subset J \times J$ is a set with the following two properties:

$$(j_1, j_2) \in \mathcal{A} \Longrightarrow (j_2, j_1) \in \mathcal{A}, \tag{5.3.5}$$

$$(j_1, j_2), (j_1, j_3) \in \mathcal{A} \Longrightarrow n(j_1) \neq n(j_2) \neq n(j_3) \neq n(j_1). \tag{5.3.6}$$

Theorem 5.3.8 (Modification theorem)**.** *For every $n, M \in \mathbb{N}$, there exist $C = C(n, M) > 0$ and $\bar{\beta} = \bar{\beta}(n, M) > 0$ such that the following holds. Fix $t > 0$ and let J, $\{J_i\}_{1\leq i\leq N}$ and $\mathcal{A}$ be as above and such that $N \leq M$. Suppose $\{I_{j_1,j_2}\}_{(j_1,j_2)\in\mathcal{A}}$ is a family of global isometries of $\mathbb{R}^n$ with the following properties:*

A) $I_{j_2,j_1} = I_{j_1,j_2}^{-1}$,
B) $I_{j_3,j_2}(I_{j_2,j_1}(B_{8t}(0))) \cap B_{9t}(0) \neq \emptyset \Longrightarrow (j_3, j_1) \in \mathcal{A}$,
C) if $(j_1, j_2), (j_2, j_3), (j_3, j_1) \in \mathcal{A}$, then

$$|I_{j_3,j_2} \circ I_{j_2,j_1} - I_{j_3,j_1}| < \beta t, \qquad \text{in } B_{10t}(0), \tag{5.3.7}$$

for some $\beta < \bar{\beta}$.

Then there exists another family $\{\widetilde{I}_{j_1,j_2}\}_{(j_1,j_2)\in\mathcal{A}}$ *of* C^∞*-global diffeomorphisms of* $\mathbb{R}^n$ *such that* $\widetilde{I}_{j_2,j_1} = \widetilde{I}^{-1}_{j_1,j_2}$ *and satisfying the following compatibility condition. For every* $(j_1, j_2), (j_3, j_2) \in \mathcal{A}$ *for which the set* $\{x \in B_{8t}(0) \ : \ \tilde{I}_{j_2,j_1}(x), \tilde{I}_{j_3,j_2}(\tilde{I}_{j_2,j_1}(x)) \in B_{8t}(0)\}$ *is not empty, we have* $(j_3, j_1) \in \mathcal{A}$ *and the maps* $\tilde{I}_{j_2,j_1}, \tilde{I}_{j_3,j_2}, \tilde{I}_{j_3,j_1}$ *are* $8t$*-cocyclical. Moreover for every* $(j_1, j_2) \in \mathcal{A}$ *it holds that*

$$\|I_{i,j_1,j_2} - \widetilde{I}_{i,j_1,j_2}\|_{C^2(\mathbb{R}^n),t} \le C\beta. \tag{5.3.8}$$

We briefly explain the role of the subclasses J_i in which of the family of indices J is partitioned. The key point is that the triples of maps $I_{j_2,j_1}, I_{j_3,j_2}, I_{j_3,j_1}$ for which we need to obtain the compatibility conditions do not contain maps with the same couple of indices (this comes from (5.3.6)). In particular every map takes part only in a finite (and uniform) number of the triples of maps that we need to consider. This will allow to modify every map only a finite (and controlled) number of times, even if we have no control on the total number of maps (which might be even infinite).

Remark 5.3.9. Similarly to Remark 5.3.5, we observe that the compatibility condition required in Theorem 5.3.8 is equivalent to ask that the following binary relation is transitive and symmetric. Let $\bigsqcup_{j\in J} B_j$, the disjoint union of copies of the Euclidean ball $B_{8t}(0)$, indexed by J, set

$$x \sim y \text{ with } x \in B_{j_1},\, y \in B_{j_2} \iff \tilde{I}_{j_2 j_1}(x) = y.$$

■

One step-modification Lemma. The modification procedure to prove Theorem 5.3.8 is based on the iteration of the following result. Formally, it shows that given three maps which are both close to isometries and almost cocyclical at a given scale, we can slightly modify one of them to make them cocyclical at a smaller scale. The crucial part of this result is that this said map is modified only where strictly needed and left unchanged everywhere else (see (5.3.11) and (5.3.12)). This will allow us in the modification algorithm to modify the same map more than once, without disrupting the work done in the previous steps.

Some elementary technical lemmas will be used in the proof, which are postponed to the end of it.

Proposition 5.3.10 (One Step modification)**.** *Fix* $N \in \mathbb{N}$ *and* $n \in \mathbb{N}$*. Then there exists* $C_1 = C_1(n, N)$ *and* $\delta_1(n, N)$ *such that the following holds for every* $t > 0$ *and* $r \ge 2$*. Suppose we have (for* $k \ge 2$*)* C^k*-global diffeomorphisms of* $\mathbb{R}^n$ I_{ab} *for* $a, b = 1, 2, 3$ *and* $a \ne b$ *and for which* $I_{ab} = I^{-1}_{ba}$*. Suppose we have also some corresponding global isometries* I'_{ab} *(again* $I'_{ab} = (I'_{ba})^{-1}$*) for which*

$$|I'_{ab} - I'_{ac} \circ I'_{cb}| \le \varepsilon t \tag{5.3.9}$$

in $B_{rt}(0)$ *for every distinct* a, b, c*. Suppose finally that*

$$\|I_{ab} - I'_{ab}\|_{C^2(\mathbb{R}^n),t} \le \varepsilon \tag{5.3.10}$$

for every a, b*, where* $\varepsilon < \delta_1(n, N)$*.*

Then there exists a C^k*-global diffeomorphism* $\hat{I}_{32}$ *of* $\mathbb{R}^n$ *such that*

$$\hat{I}_{32} = I_{32} \text{ outside } I_{21}(B_{rt}(0)), \tag{5.3.11}$$

$$\hat{I}_{32}(x) = I_{32}(x) \text{ for any } x \in B_{rt}(0) \text{ such that } I_{32}(I_{21}(x)) = I_{31}(x), \tag{5.3.12}$$

$$I_{21}, I_{31}, \hat{I}_{32} \text{ are } \big(1 - \frac{1}{N}\big)rt\text{-cocyclical}, \tag{5.3.13}$$

and finally

$$\|I_{32} - \hat{I}_{32}\|_{C^2(\mathbb{R}^n),t}, \|I_{23} - \hat{I}_{23}\|_{C^2(\mathbb{R}^n),t} \le C_1\varepsilon, \tag{5.3.14}$$

where $\hat{I}_{23} = \hat{I}^{-1}_{32}$*.*

Proof. It's clear from (5.3.9) that

$$|I'_{32}(I'_{21}(I'_{13})) - \mathsf{id}| \le \varepsilon t$$

in $B_{rt}(0)$. Moreover from (5.3.10)

$$|I_{32}(I_{21}(I_{13})) - \mathsf{id}| \le 4\varepsilon t \tag{5.3.15}$$

in $B_{rt}(0)$. Define now the set $A := I_{32}(I_{21}(B_{(1-\frac{1}{2N})rt}(0))) \cap B_{(1-\frac{1}{2N})rt}(0)$. We distinguish two cases:

Case 1: $A = \emptyset$. We simply take $\hat{I}_{32} = I_{32}$. Indeed in this case I_{21}, I_{32}, I_{31} are vacuously $(1 - \frac{1}{N})rt$-cocyclical, i.e. the set of points where we need to check (5.3.3) is empty. To see this denote $B_{(1-\frac{1}{N})rt}(0)$ by B. Suppose that there exists $x \in B$ such that $I_{21}(x), I_{32}(I_{21}(x)) \in B$, then we would have $A \ne \emptyset$.

Suppose instead that there exists $x \in B$ such that $I_{31}(x), I_{23}(I_{31}(x)) \in B$, then applying to $I_{31}(x)$ the map in (5.3.15), if $\varepsilon < 1/(100N)$, we deduce that $A \neq \emptyset$. Finally suppose there exists $x \in B$ such that $I_{13}(x), I_{21}(I_{13}(x)) \in B$ and set $y \coloneqq I_{13}(x) \in B$. Then from (5.3.15) we deduce $I_{32}(I_{21}(y)) \in B_{(1-\frac{1}{2N})rt}(0)$, therefore again $A \neq \emptyset$. From Remark 5.3.6, this is enough to prove that I_{21}, I_{32}, I_{31} are $(1-\frac{1}{N})rt$-cocyclical.
Case 2: $A \neq \emptyset$. Since $S \coloneqq I'_{31}(I'_{12}(I'_{23}))$ is an isometry, from Lemma 5.3.13 (recall $r \geq 2$) we get $\|DS - \mathsf{id}\| \leq 4\varepsilon$ and thus $\|S - \mathsf{id}\|_{C^2(\mathbb{R}^n),t} \leq 4\varepsilon$. Therefore from (5.3.10), assuming $\varepsilon < 1$ and applying Lemma 5.3.12 we obtain

$$\|I_{31}(I_{12}(I_{23})) - \mathsf{id}\|_{C^2(\mathbb{R}^n),t} \leq C\varepsilon, \tag{5.3.16}$$

where C is a constant depending only on n. Define now the following open sets

$$U_2 \coloneqq I_{32}(I_{21}(B_{rt}(0))) \cap B_{rt}(0)$$

$$U_1 \coloneqq A = I_{32}(I_{21}(B_{(1-\frac{1}{2N})rt}(0))) \cap B_{(1-\frac{1}{2N})rt}(0)$$

that are non empty, since we assumed $A \neq \emptyset$. Clearly $U_1 \subset U_2$ and notice that, by (5.3.10), if say $\varepsilon \leq \frac{1}{100N}$, we have $\mathsf{d}(\bar{U}_1, U_2^c) \geq t/(10N)$. Define now $H \coloneqq I_{31} \circ I_{12} \circ I_{23} : U_1 \to \mathbb{R}^n$. Thanks to (5.3.16) we can now apply Theorem 5.3.1 (provided $\varepsilon \leq \delta_2(n, 1/(10N))C^{-1}$, where δ_2 is the one given by Theorem 5.3.1) with H, with U_1,U_2,t and ε to deduce the existence of $\hat{H}$ C^k-global diffeomorphism such that

$$\hat{H}|_{U_1} = I_{31} \circ I_{12} \circ I_{23}, \tag{5.3.17}$$

$$\hat{H}(x) = H(x) \text{ whenever } H(x) = x, \tag{5.3.18}$$

$$\hat{H}|_{U_2^c} = id, \tag{5.3.19}$$

$$\|\hat{H}^{-1} - \mathsf{id}\|_{C^2(\mathbb{R}^n),t}, \|\hat{H} - \mathsf{id}\|_{C^2(\mathbb{R}^n),t} \leq \tilde{C}\varepsilon, \tag{5.3.20}$$

where $\tilde{C}$ is constant depending only on n and N. We define $\hat{I}_{32} = \hat{H} \circ I_{32}$ and call $\hat{I}_{23}$ its inverse. Clearly (5.3.12) is satisfied thanks to (5.3.18). Moreover (5.3.11) holds from (5.3.19), since $I_{32}(I_{21}(B_{rt}(0)^c)) \subset U_2^c$. We now verify (5.3.14). The first bound follows directly from (5.3.20) and (5.3.10), applying Lemma 5.3.11. The bound for the inverse, $\hat{I}_{23}$, follows from Lemma 5.3.3, provided ε is small enough. Finally we need to prove the cocyclical condition in (5.3.13). Suppose first that there exists $x \in B_{(1-\frac{1}{3N})rt}(0)$ such that $I_{21}(x), \hat{I}_{32}(I_{21}(x)) \in B_{(1-\frac{1}{3N})rt}(0)$. Then, if say $\varepsilon < 1/(100N)$, from (5.3.14) we deduce that $I_{32}(I_{21}(x)) \in B_{(1-\frac{1}{2N})rt}(0)$, that implies $I_{32}(I_{21}(x)) \in U_1$, therefore from (5.3.17)

$$\hat{I}_{32}(I_{21}(x)) = \hat{H}(I_{32}(I_{21}(x))) = I_{31} \circ I_{12} \circ I_{23} \circ I_{32}(I_{21}(x)) = I_{31}(x).$$

(5.3.13) then follows from Proposition 5.3.7, indeed notice that its hypotheses are satisfied thanks to (5.3.9) and (5.3.14), provided ε is small enough. □

In the previous proof we used the following elementary technical lemmas.

Lemma 5.3.11. *For every $n \in \mathbb{N}$ and every $\delta > 0$ there exists positive constants $C(n, \delta)$ such that the following holds. Fix $m \in \{1, 2\}$. Let $f_i \in C^m(U_i; \mathbb{R}^n)$ with $i = 1, 2$, with $U_i \subset \mathbb{R}^n$ open sets, and let $I_1, I_2 : \mathbb{R}^n \to \mathbb{R}^n$ be two isometries. Suppose that $\|f_i - I_i\|_{C^m(U_i),t} \leq \delta_i \leq \delta$, then letting W any open set where $f_1 \circ f_2$ is defined,*

$$\|f_1 \circ f_2 - I_1 \circ I_2\|_{C^m(W),t} \leq C(n, \delta)(\delta_1 + \delta_2). \tag{5.3.21}$$

Proof. The proof is just a straightforward computation.

$$\begin{aligned}|f_1 \circ f_2 - I_1 \circ I_2| &\leq |f_1 \circ f_2 - I_1 \circ f_2| + |I_1 \circ f_2 - I_1 \circ I_2| \\ &\leq \delta_1 t + |f_2 - I_2| \leq \delta_1 t + \delta_2 t,\end{aligned}$$

$$\begin{aligned}\|D(f_1 \circ f_2) - D(I_1 \circ I_2)\| &= \|Df_1(f_2)Df_2 - DI_1 DI_2\| \\ &\leq \|Df_1(f_2) - DI_1\|\|Df_2\| + \|DI_1\|\|Df_2 - DI_2\| \\ &\leq \delta_1(1 + \delta_2) + \delta_2 \leq C(\delta)(\delta_1 + \delta_2).\end{aligned}$$

Moreover, recalling (5.1.1) and that $\|Df_i - I_i\| \le \delta_i$,

$$\begin{aligned}|\partial_{ij}(f_1(f_2))_k| &= |\sum_{m,h=1}^{n} \partial_{hm}(f_1)_k(f_2)\partial_j(f_2)_m\partial_i(f_2)_k + \sum_{h=1}^{n}\partial_{ij}(f_2)_k\partial_h(f_1)_k(f_2)|\\ &\le \sqrt{n}(1+\delta_2)^2\sum_{m,h=1}^{n}|\partial_{hm}(f_1)_k(f_2)| + \sqrt{n}(1+\delta_1)\sum_{h=1}^{n}|\partial_{ij}(f_2)_k|\\ &\le C(n,\delta)\frac{\delta_1+\delta_2}{t}.\end{aligned}$$

□

Observe that iterating Lemma 5.3.11 we also get the analogous statement also for more than two maps, for example we can prove the following.

Lemma 5.3.12. *For every $n \in \mathbb{N}$ and every $\delta > 0$ there exists positive constants $C(n,\delta)$ such that the following holds. Fix $m \in \{1,2\}$. Let $f_1, f_2, f_3, f_4 : \mathbb{R}^n \to \mathbb{R}^n$ be C^m functions and let $I_1, I_2, I_3, I_4 : \mathbb{R}^n \to \mathbb{R}^n$ be isometries. Suppose that $\|f_i - I_i\|_{C^m,t}(\mathbb{R}^n) \le \delta_i < \delta$, then*

$$\|f_1 \circ f_2 \circ f_3 \circ f_4 - I_1 \circ I_2 \circ I_3 \circ I_4\|_{C^m(\mathbb{R}^n),t} \le C(n,\delta)(\delta_1+\delta_2+\delta_2+\delta_3+\delta_4). \tag{5.3.22}$$

Lemma 5.3.13. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry such that $|F - \mathsf{id}| \le r\varepsilon$ on some ball $B_r(x)$, then $\|DF - I\| \le 4\varepsilon$.*

Proof. We may suppose $x = 0$. Recall that $F = DF \cdot x + v$ for some vector v and notice that $|v| = |F(0) - 0| \le r\varepsilon$. Hence $|DF \cdot y - y| \le 2\varepsilon r$, thus for every y such that $|y| = r/2$

$$\frac{|DF\cdot y - y|}{|y|} \le 4\varepsilon,$$

the conclusion then follows from the definition of operator norm. □

Proof of Theorem 5.3.8. The maps $\widetilde{I}_{j_1,j_2}$ are constructed by modifying slightly the original maps. This will be achieved by iterating Proposition 5.3.10. The proof is divided in the following steps: first we describe the iterative algorithm that we use to modify the maps; after this we see the effect of this modification; then we prove that the algorithm is applicable, by showing that the hypotheses of Proposition 5.3.10 are satisfied at every step; finally we prove that the maps that we obtain satisfy the compatibility conditions required by Theorem 5.4.9.

Preparation :
We choose $\beta \le \frac{\delta_1}{100M^3(C_1+1)^{M^3}}$ where $\delta_1 = \delta_1(n, M^3), C_1 = C_1(n, M^3)$ are given in Proposition 5.3.10. Define also the numbers $\beta_k := (C_1+1)^k\beta$ for $k = 0, 1, ..., N^3$.

In the modification of the maps will need to proceed in very precise ordered fashion. To formalize such procedure we need to introduce the following notation.

Set $N_3 := \binom{N}{3}(= \frac{N(N-2)(N-1)}{6})$. Define the set

$$\mathscr{T} := \{(a,b,c) \mid 1 \le c < b < a \le N\}$$

and consider the enumeration of the elements of $\mathscr{T} = \{T_1, ..., T_{N_3}\}$ defined as follows. Set $T_1 := (3,2,1)$. If $T_k = (a,b,c)$ then set:

- $T_{k+1} = (a,b,c+1)$ if $c < b-1$,
- $T_{k+1} = (a,b+1,1)$ if $c = b-1$ and $b < a-1$,
- $T_{k+1} = (a+1,2,1)$ if $T_k = (a,a-1,a-2)$.

In other words: $T_1 = (3,2,1), T_2 = (4,2,1), T_3 = (4,3,1), T_4 = (4,3,2), T_5 = (5,2,1)$... and so on.

Modification procedure :
We divide the modification in a finite number of steps $k = 1, ..., N_3$. At every step we produce for every map $I_{j_1,j_2}, (j_1,j_2) \in \mathcal{A}$, a modified map that will be called $I^k_{j_1,j_2}$.
We start by setting $I^0_{j_1,j_2} := I_{j_1,j_2}$ for every $(j_1,j_2) \in \mathcal{A}$.

- For every $k = 1, ..., N_3$ we do the following:

Step k: Consider $T_k = (a_3, a_2, a_1)$ and for every triple of maps $I^{k-1}_{j_2j_1}, I^{k-1}_{j_3j_1}, I^{k-1}_{j_3j_2}$ with $n(j_1) = a_1, n(j_2) = a_2, n(j_3) = a_3$, apply Proposition 5.3.10 (see the next part for the verification of the hypotheses) with

$$\begin{gathered} I_{21} = I^{k-1}_{j_2j_1}, I_{31} = I^{k-1}_{j_3j_1}, I_{32} = I^{k-1}_{j_3j_2}, \\ I'_{21} = I^{0}_{j_2j_1}, I'_{31} = I^{0}_{j_3j_1}, I'_{32} = I^{0}_{j_3j_2}, \\ t = 2^{-i} \\ N = M^2 \\ r = 10 - \frac{k-1}{N^3} \\ \varepsilon = \beta_k \end{gathered}$$

to produce a modified map $I^k_{j_3j_2}$ (that is the map $\hat{I}_{32}$ given by the Lemma) and then set $I^k_{j_2j_3} := (I^k_{j_3j_2})^{-1}$. Moreover set $I^k_{j_1,j_2} := I^{k-1}_{j_1,j_2}, I^k_{j_1,j_3} := I^{k-1}_{j_1,j_3}$ and the same for their inverses.
Finally for every map $I^{k-1}_{j,\bar{j}}$ that does not belong to any triple considered above and neither does its inverse, we simply set $I^k_{j,\bar{j}} := I^{k-1}_{j,\bar{j}}$, $I^k_{\bar{j},j} := I^{k-1}_{\bar{j},j}$.

Important remark: Note that assumption (5.3.6) ensures that every map $I^{k-1}_{j,\bar{j}}$ belong to at most one of triple considered at Step k, hence the above procedure makes sense and we are not modifying the same map more than once in the same step.

At the end of the iteration define $\widetilde{I}_{j_1,j_2} := I^{N_3}_{j_1,j_2}$.

Effect of the modification :

We gather here the properties of the new maps produced by the modification. From Proposition 5.3.10 (given that we can apply it) we have that

$$\|I^k_{j_3j_2} - I^{k-1}_{j_3j_2}\|_{C^2(\mathbb{R}^n),t}, \|I^k_{j_2j_3} - I^{k-1}_{j_2j_3}\|_{C^2(\mathbb{R}^n),t} \leq \beta_k C_1, \tag{5.3.23}$$

$$I^k_{j_3j_2} = I^{k-1}_{j_3j_2} \text{ in } I^{k-1}_{j_2,j_1}(B_{(10-\frac{k-1}{N^3})t}(0))^c. \tag{5.3.24}$$

$$I^k_{j_3j_2}(x) = I^{k-1}_{j_3j_2}(x) \text{ for any } x \in B_{(10-\frac{k-1}{N^3})t}(0) \text{ such that } I^{k-1}_{j_3j_2}(I^{k-1}_{j_2j_1}(x)) = I^{k-1}_{j_3j_1}(x), \tag{5.3.25}$$

and

$$I^k_{j_2j_1}, I^k_{j_3j_1}, I^k_{j_3j_2} \text{ are } \left(10 - \frac{k}{N^3}\right) t\text{-cocyclical}. \tag{5.3.26}$$

Observe that (5.3.26) holds because $N \leq M$ and being r-cocyclical implies being s-cocyclical for any $s < r$.

Verification of the hypotheses needed to apply Proposition 5.3.10:
The fact that $I_{ab} = I^{-1}_{ba}$ and $I'_{ab} = I'^{-1}_{ba}$ is granted by (5.3.5) and the fact that at the end every modification step, we set $I^k_{j_2j_3} := (I^k_{j_3j_2})^{-1}$. Our initial assumption on β implies that $\beta_k \leq \delta_1(n, N)$ for every k, therefore we only need to prove that (5.3.9) and (5.3.10) are satisfied for some $\varepsilon \leq \beta_k$. (5.3.9) is always satisfied by assumption C) with $\varepsilon = \beta \leq \beta_k$. Therefore we only need to prove that

$$\|I^{k-1}_{j_1j_2} - I^0_{j_1j_2}\|_{C^2(\mathbb{R}^n),t} \leq \beta_k. \tag{5.3.27}$$

We can prove this by induction. It is trivial if $k = 1$. So suppose it is true for k, in particular we can perform the above modification at least up to Step k. Then from (5.3.23) (that we are assuming to hold at Step k) we have

$$\|I^k_{j_1j_2} - I^0_{j_1j_2}\|_{C^2(\mathbb{R}^n),t} \leq \|I^{k-1}_{j_1j_2} - I^0_{j_1j_2}\|_{C^2(\mathbb{R}^n),t} + \|I^{k-1}_{j_1j_2} - I^k_{j_1j_2}\|_{C^2(\mathbb{R}^n),t} \leq C\beta_k + \beta_k = \beta_{k+1},$$

that proves (5.3.27). Notice also that from (5.3.27), (5.3.8) already follows, indeed

$$\|\tilde{I}_{j_1j_2} - I_{j_1j_2}\|_{C^2(\mathbb{R}^n),t} = \|I^{N_3}_{j_1j_2} - I^0_{j_1j_2}\|_{C^2(\mathbb{R}^n),t} \leq \beta_{N_3} = \beta(C_1+1)^{N_3} \leq \beta(C_1+1)^{M^3},$$

hence it is sufficient to take $C \geq (C_1+1)^{M^3}$, which depends only on n, M, since C_1 depends only on n.

Proof of compatibility conditions :

To prove Theorem 5.3.8 it remains only to prove the compatibility conditions. We report them here for the convenience of the reader:
Compatiblity conditions: for every $(j_1, j_2), (j_3, j_2) \in \mathcal{A}$ for which the set

$$\{x \in B_{8t}(0) \ : \ \tilde{I}_{j_2,j_1}(x), \tilde{I}_{j_3,j_2}(\tilde{I}_{j_2,j_1}(x)) \in B_{8t}(0)\}$$

is not empty, we have $(j_3, j_1) \in \mathcal{A}$ and that the maps $\tilde{I}_{j_2,j_1}, \tilde{I}_{j_3,j_2}, \tilde{I}_{j_3,j_1}$ are $8t$-cocyclical.

We first describe the idea of the argument.

Idea: After Step k of the procedure we clearly have, thanks to (5.3.26), that the maps relative to the triple T_k are cocyclical at scale $(10-k/N^3)t$. Therefore what we need to do is check that the modification at Step k does not destroy the compatibility conditions created at the previous steps. For this turns out to be crucial the fact that we are decreasing the scale at every step and that we are modifying the maps only where is strictly needed (see in particular (5.3.24) and (5.3.25)).

We pass now to the rigorous part. We claim that to prove the above compatibility conditions is enough to show that the following statement, denoted by $\mathbf{S}(k)$, is true for every $k = 1, ..., N_3$.

S(k)**:** For every $m \leq k$ consider $T_m = (a_3, a_2, a_1)$. Let $I^k_{j_1j_2}, I^k_{j_1j_3}, I^k_{j_2j_3}$ be such that $n(j_1) = a_1, n(j_2) = a_2, n(j_3) = a_3$, then they are $(10 - \frac{k}{N^3})t$-cocyclical.

To see that this would be sufficient to to conclude, suppose there exists two maps $\tilde{I}_{j_2,j_1}, \tilde{I}_{j_3,j_2}$ and a point $x \in B_{8t}(0)$ such that $\tilde{I}_{j_2,j_1}(x)$, $\tilde{I}_{j_3,j_2}(\tilde{I}_{j_2,j_1}(x)) \in B_{8t}(0)$. Then from (5.3.8) and by how we chose β at the beginning, we have $I_{j_3,j_2}(I_{j_2,j_1}(x)) \in B_{9t}(0)$. Then, thanks to assumption B) we have that $(j_3, j_1) \in \mathcal{A}$. Moreover from (5.3.6) we must have that $n(j_1) \neq n(j_2) \neq n(j_3) \neq n(j_1)$, therefore $(n(j_1), n(j_2), n(j_3)) = T_m$, for some $m \leq N_3$. Therefore $\mathbf{S}(N_3)$ implies that $\tilde{I}_{j_2,j_1}, \tilde{I}_{j_3,j_2}, \tilde{I}_{i,j_3,j_1}$ are $8t$-cocyclical (indeed $N_3 \leq 6N^3$.)

Observe that we actually used only statement $\mathbf{S}(N_3)$, however to prove it we will need to argue by induction and prove every $\mathbf{S}(k)$.

Proof of S(k)**:**

We prove it by induction on k. First we observe that after the step k is completed in the modification procedure, any triple of maps $I^k_{j_1j_2}, I^k_{j_1j_3}, I^k_{j_2j_3}$ such that $T_k = (n(j_1), n(j_2), n(j_3))$, is $(10 - \frac{k}{N^3})t$-cocyclical by (5.3.26). Hence $\mathbf{S}(1)$ is clearly true.

Suppose now that $\mathbf{S}(k)$ is true for k. Consider $T_{k+1} = (b_3, b_2, b_1)$. Since $I^{k+1}_{j\bar{j}} \neq I^k_{j\bar{j}}$ only if $j \in J_{b_3}$, $\bar{j} \in J_{b_2}$ (or the opposite), we only need to check $\mathbf{S}(k+1)$ for $T_m = (b_3, b_2, a_1)$ with $a_1 \leq b_1$. Indeed the other cases are true by induction hypothesis. The case $a_1 = b_1$ is immediately verified from the initial observation. Let now $T_m = (b_3, b_2, a_1)$ with $a_1 < b_1$, then $m \leq k$. We need to show that $I^{k+1}_{j_1j_2}, I^{k+1}_{j_1j_3}, I^{k+1}_{j_2j_3}$ are $(10-\frac{k+1}{N^3})t$-cocyclical. To this aim set $B_k := B_{(10-\frac{k}{N^3})t}(0)$, $B_{k+1/2} := B_{(10-\frac{k+1/2}{N^3})t}(0)$ and $B_{k+1} := B_{(10-\frac{k+1}{N^3})t}(0)$ so that $B_{k+1} \subset B_{k+1/2} \subset B_k$. We claim that is sufficient to show that:

$$(\spadesuit) \qquad \begin{aligned} &\text{for any point } x \in B_{k+1/2} \text{ such that } I^{k+1}_{j_3,j_2}(I^{k+1}_{j_2,j_1}(x)), I^{k+1}_{j_2,j_1}(x) \in B_{k+1/2}, \\ &\text{it holds } I^{k+1}_{j_3,j_2}(I^{k+1}_{j_2,j_1}(x)) = I^{k+1}_{j_3,j_1}(x). \end{aligned}$$

Indeed the full cocyclical condition on the smaller ball B_{k+1} would then follow from Proposition 5.3.7, which hypotheses are satisfied thanks to assumption C), (5.3.8) and our initial choice of β.

Proof of $(\spadesuit)$*:* Let $x \in B_{k+1/2}$ be such that $I^{k+1}_{j_3,j_2}(I^{k+1}_{j_2,j_1}(x)), I^{k+1}_{j_2,j_1}(x) \in B_{k+1/2}$. Notice first that from (5.3.23) and how we chose β, we have that $I^k_{j_3,j_2}(I^k_{j_2,j_1}(x)), I^k_{j_2,j_1}(x) \in B_k$, therefore by *S(k)* and induction hypothesis $I^k_{j_3,j_2}(I^k_{j_2,j_1}(x)) = I^k_{j_3,j_1}(x)$. Hence we need to show

$$I^k_{j_3,j_2}(y) = I^{k+1}_{j_3,j_2}(y) \tag{5.3.28}$$

where $y = I^k_{j_2,j_1}(x)$. If the map $I^k_{j_3,j_2}$ was not modified at the step $k+1$, i.e. $I^k_{j_3,j_2} = I^{k+1}_{j_3,j_2}$, there is nothing to prove. Hence we can assume that $I^{k+1}_{j_3,j_2}$ has been modified at step $k+1$ of the modification procedure by applying Proposition 5.3.10 to the maps $I^k_{j_3,j_2}, I^k_{j_2,j_0}, I^k_{j_3,j_0}$ for some $j_0 \in J_{b_1}$. We divide two cases.

Case 1: There is not any $z \in B_k$ such that $I^k_{j_2,j_0}(z) = y$. In this case (5.3.28) follows immediately from (5.3.24).

Case 2: There exists $z \in B_k$ such that $I^k_{j_2,j_0}(z) = y$. The idea is that in this case the map $I^k_{j_3,j_2}$ was already correct and needed not to be modified. Observe first that $x = I^k_{j_1,j_2}(I^k_{j_2,j_0}(z))$, $I^k_{j_2,j_0}(z) \in B_k$. Moreover by (5.3.23) and by how we chose β at the beginning, we have $I_{j_1,j_2}(I_{j_2,j_0}(z)) \in B_{10t}(0)$, hence thanks to assumption B) we have that $(j_1, j_0) \in \mathcal{A}$. Therefore by induction hypothesis, since $T_l = (b_2, b_1, a_1)$ with $l \leq k$, we have $I^k_{j_1,j_0}(z) = x$. From this we infer that $I^k_{j_3,j_1}(I^k_{j_1,j_0}(z)) = I^k_{j_3,j_1}(x) \in B_k$. Therefore again by induction hypothesis since $T_h = (b_3, b_1, a_1)$ with $h \leq k$, we have

$$I^k_{j_3,j_0}(z) = I^k_{j_3,j_1}(I^k_{j_1,j_0}(z)) = I^k_{j_3,j_1}(x) = I^k_{j_3,j_2}(y) = I^k_{j_3,j_2}(I^k_{j_2,j_0}(z)).$$

Therefore Hence from (5.3.25) we deduce (5.3.28). This concludes the proof of Theorem 5.3.8.

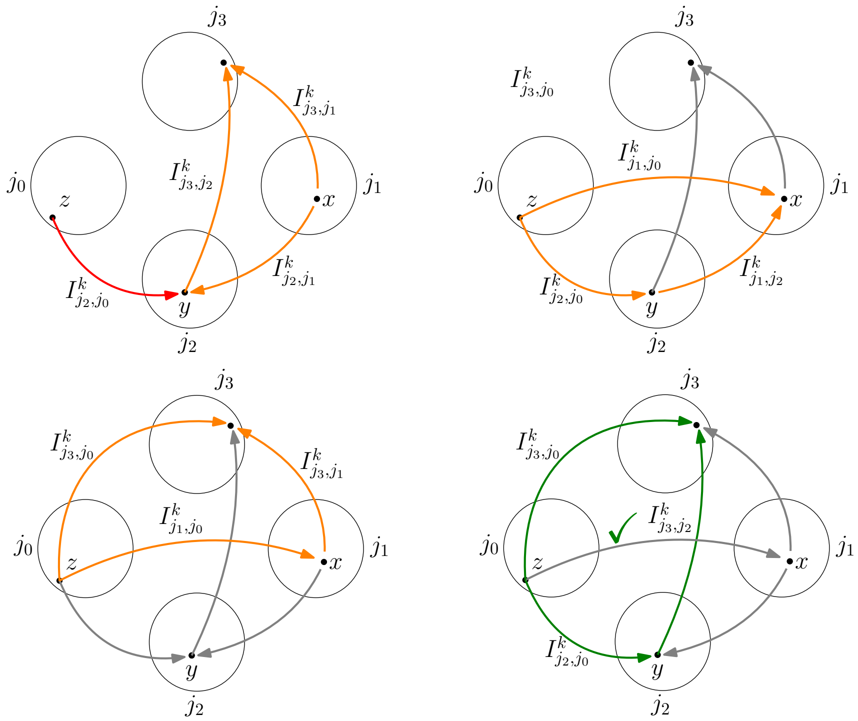

FIGURE 1. Scheme for the proof of (♠).

5.3.3. A tool to show that two manifolds are diffeomorphic. In this section we prove Theorem 5.3.14 stated below, which is essentially a result due to Cheeger [**70**, Lemma 3.5 and Lemma 4.1]. Roughly said this result provides a criterion for two two manifold are diffeomorphic. More in details it says that given a manifold M, a family of coordinate-charts and for each chart a smooth embedding from that chart to another manifold, given that they don't differ too much on the intersection of the charts (condition (5.3.31)), we can modify them by a small amount and glue them together to obtain a smooth immersion map defined in the whole M.

This result will be used to build the maps $h_i : W_i \to W_{i+1}$ in the proof of Theorem 5.2.2 (see Section 5.4.4).

Theorem 5.3.14. *For every $N, n \in \mathbb{N}$ and $L \geq 1$ there exist $C_3 = C_3(n, N, L) > 0$ and $\varepsilon_3(n, N, L)$ with the following property. Let $q \in \mathbb{N}$ with $q \geq 2$. Let M, X be smooth n-dimensional C^q manifolds and let $\varphi_j : B_{2t}(0) \subset \mathbb{R}^n \to M$, $j = 1, ..., m$ (m possibly $+\infty$), be C^q embeddings and set for every $j = 1, \ldots, m$ and $i = 0, \ldots, N$ $B^j_i := \varphi_j(B_{(2-i/N)t}(0))$. Suppose that that $M \subset \cup_j \varphi_j(B_t(0))$ and that for some $t > 0$ for and every $j_1, j_2 = 1, ..., m$*

$$\|D(\varphi_{j_1}^{-1} \circ \varphi_{j_2})\|, |\partial_{ij}(\varphi_{j_1}^{-1} \circ \varphi_{j_2})_k| t \leq L, \tag{5.3.29}$$

on the domain of definition of $\varphi_{j_1}^{-1} \circ \varphi_{j_2}$ (if it is defined). Moreover suppose that we can partition the set of indices $\{1, ..., m\}$ into partitions $I_1, I_2 ..., I_N$ such that for every $j, j_0 \in I_k$ we have $B^j_0 \cap B^{\bar{j}}_0 = \emptyset$. Finally suppose that there exists a family of C^q embeddings $h_j : B^j_0 \to X$ for $j = 1, \ldots, m$ such that for every couple of indexes $j_1 \in I_h, j_2 \in I_l$ with $1 \leq h < l$ for which $B^{j_1}_{l-1} \cap B^{j_2}_{l-1} \neq \emptyset$

$$h_{j_1}(B^{j_1}_{l-2} \cap B^{j_2}_{l-2}) \subset h_{j_2}(B^{j_2}_0), \tag{5.3.30}$$

$$\|H_{j_1,j_2} - \mathsf{id}\|_{C^1\left(\varphi_{j_2}^{-1}(B^{j_1}_{l-2} \cap B^{j_2}_{l-2})\right), t} \leq \varepsilon. \tag{5.3.31}$$

for some $\varepsilon \leq \varepsilon_3$, where $H_{j_1,j_2} = \varphi_{j_2}^{-1} \circ h_{j_2}^{-1} \circ h_{j_1} \circ \varphi_{j_2}$ (that is well defined by (5.3.30)*).*

Then there exists a C^q-immersion $h : M \to X$ such that h is obtained by modifying slightly the maps h_j in the following sense

$$h = h_j \circ \varphi_j \circ H_j \circ \varphi_j^{-1} \tag{5.3.32}$$

in $\varphi_j(B_t(0))$ for every $j \in I_k$ where $H_j : B_{2t}(0) \to B_{2t}(0)$ is a diffeomorphism of class C^q such that $\|H_j - \mathsf{id}\|_{C^1(B_{2t}(0)), t} \leq C_3 \varepsilon$.

Remark 5.3.15. The above theorem does not appear in [**70**, Lemma 4.1] as it is stated here. The main difference is that here we have an infinite number of maps and that we have also an explicit C^1 control on the error of the modification, which will be crucial in our application in Section 5.4.4. This forces us also to assume a C^2 control on the charts (see (5.3.29)), which is not present in [**70**]. For these reasons, even if the proof is analogous to the one in [**70**], we will include it here for completeness. ■

We will need the following elementary technical result.

Lemma 5.3.16. *Let $f : U \to V$ be a bijective C^2 function with C^2 inverse, where $U, V \subset \mathbb{R}^n$ are open sets. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function satisfying $\|g - \mathsf{id}\|_{C^1(\mathbb{R}^n),t} \leq \delta < 1$ for some $t > 0$. Moreover suppose that*

$$\|Df\|, \|Df^{-1}\|, t|\partial_{ij}(f)_k|, t|\partial_{ij}(f^{-1})_k| \leq M,$$

everywhere on U (for f) and everywhere on V (for f^{-1}).

Then letting $W \subset \mathbb{R}^n$ be any open set where the function $f \circ g \circ f^{-1}$ is defined, we have

$$\|f \circ g \circ f^{-1} - \mathsf{id}\|_{C^1(W),t} \leq \delta c,$$

where $c = c(n, M) > 1$ is positive constant that depends only on M and n.

Proof. Observe first that by the mean value theorem

$$\begin{aligned}|f \circ g \circ f^{-1} - \mathsf{id}| &= |f \circ g \circ f^{-1} - f \circ f^{-1}| \\ &\leq \|Df\||g(f^{-1}) - f^{-1}| \leq \|Df\|\delta t.\end{aligned}$$

Moreover

$$\begin{aligned}\|D(f \circ g \circ f^{-1}) - I_n\| &\leq \|Df(g(f^{-1}))Dg(f)Df^{-1} - I_n\| \\ &\leq \|Df(g(f^{-1})) - Df(f^{-1})\|\|Dg\|\|Df^{-1}\| + \|Df(f^{-1})DgDf^{-1} - I_n\|.\end{aligned}$$

To bound the first term we need to exploit the bound on the second derivatives of f as follows. From the mean value theorem and recalling (5.1.1),

$$\begin{aligned}\|Df(g(f^{-1}) - Df(f^{-1})\| &\leq n^2 \sup_{i,k} |\partial_i f_k(g(f^{-1}(x))) - \partial_i f_k(f^{-1})| \\ &\leq n^2 |g(f^{-1}) - f^{-1}| \sup_{i,j,k} |\partial_{ij} f_k| \\ &\leq n^2 \delta t \sup_{i,j,k} |\partial_{ij} f_k|.\end{aligned}$$

To bound the second term, recall that $Df(f^{-1})Df^{-1} = I_n$ and argue in the following way

$$\|Df(f^{-1})DgDf^{-1} - I_n\| = \|Df(f^{-1})(Dg - I_n)Df^{-1}\| \leq \|Df\|\|Df^{-1}\|\delta.$$

□

We can now move to the proof of the main result, which build upon the basic modification result in Section 5.3.1.

Proof of Theorem 5.3.14. Before starting, we need to define constants η_i for $i = 1, ..., N$ in the following way. Let $C_2 = C_2(n, \frac{1}{LN})$, $\delta_2 = \delta_2(n, \frac{1}{LN}) < 1$ the constants given in Theorem 5.3.1. Moreover let $c = C(n, L) > 1$ the constant given in Lemma 5.3.16. Define $\eta_1 := \varepsilon$ and inductively $\eta_{k+1} := \eta_k D$ for some constant $D = D(L, C_2, N, n) > 1$, big enough, to be determined later. Moreover we take $\varepsilon_3 = \varepsilon_3(n, N, L)$ small enough to satisfy

$$\varepsilon_3 \leq \frac{\delta_2}{cLNC_2D^N}. \tag{5.3.33}$$

Before moving to the main body of the proof, we state a preliminary technical claim, which elementary proof will be given at the end.

Claim: Suppose that $B^j_r \cap B^{\bar{j}}_r \neq \emptyset$ for some $r \geq 2$ and some indices $j, \bar{j}$. Define the set $\Omega_r := \varphi_j^{-1}(B^j_r \cap B^{\bar{j}}_r)$. Then the set $\{x \in B_{2t}(0) \mid \mathsf{d}(x, \Omega_r) < \frac{t}{LN}\}$ is contained in the domain of $\varphi_{\bar{j}}^{-1} \circ \varphi_j$.

We can now pass to the core of the argument. We will construct by induction C^q maps

$$\hat{h}_k : \bigcup_{\substack{j \in I_i \\ i=1,..,k}} B^j_k \to X$$

with the following properties. For every $h < k$

$$\hat{h}_k = \hat{h}_h, \quad \text{in} \bigcup_{\substack{j\in I_i \\ i=1,..,h}} B_k^j. \tag{5.3.34}$$

Moreover for every $j_1 \in I_k, j_2 \in I_l$ with $l > k$ such that $B_{l-1}^{j_1} \cap B_{l-1}^{j_2} \neq \emptyset$ then

$$\hat{h}_k \left(B_{l-1}^{j_1} \cap B_{l-1}^{j_2}\right) \subset h_{j_2}(B_0^{j_2}). \tag{5.3.35}$$

Notice that (5.3.35) implies that the map $\hat{H}_{j_1,j_2} := \varphi_{j_2}^{-1} \circ h_{j_2}^{-1} \circ \hat{h}_k \circ \varphi_{j_2}$ is a well defined map $\hat{H}_{j_1,j_2} : \varphi_{j_2}^{-1}\left(B_{l-1}^{j_1} \cap B_{l-1}^{j_2}\right) \to B_{2t}(0)$, then we also require

$$\|\hat{H}_{j_1,j_2} - \mathsf{id}\|_{C^1,t} \le \eta_k \tag{5.3.36}$$

on $\varphi_{j_2}^{-1}\left(B_{l-1}^{j_1} \cap B_{l-1}^{j_2}\right)$.

To start the induction for $k = 1$ we set $\hat{h}_1 := h_{j_1}$ in $B_1^{j_1}$ for every $j_1 \in I_1$. Recall that $B_1^{j_1}$, with $j_1 \in I_1$, are all disjoint and hence $\hat{h}_1$ is well defined. Then we need only to check (5.3.35), (5.3.36), but these are clearly satisfied thanks to (5.3.30) and (5.3.31), since $\eta_1 = \varepsilon$.

Suppose now we have constructed maps $\hat{h}_1, ..., \hat{h}_k$ and consider any $j_{k+1} \in I_{k+1}$. By induction hypothesis for every $j_h \in I_h$ with $h \le k$ by the map $\hat{H}_{j_h,j_{k+1}} := \varphi_{j_{k+1}}^{-1} \circ h_{j_{k+1}}^{-1} \circ \hat{h}_h \circ \varphi_{j_{k+1}}$ is well defined and satisfies $\|\hat{H}_{j_h,j_{k+1}} - \mathsf{id}\|_{C^1,t} \le \eta_h \le \eta_k$ on on $\varphi_{j_{k+1}}^{-1}\left(B_k^{j_h} \cap B_k^{j_{k+1}}\right)$. Moreover from (5.3.34) $\hat{h}_h = \hat{h}_k$ on $B_k^{j_h}$ for every j_h as above. Thus we can patch the maps $\hat{H}_{j_h,j_{k+1}}$ together to get a map $\hat{H}_{j_{k+1}} := \varphi_{j_{k+1}}^{-1} \circ h_{j_{k+1}}^{-1} \circ \hat{h}_k \circ \varphi_{j_{k+1}}$ defined on the whole set $U_2 := \varphi_{j_{k+1}}^{-1}\left((\bigcup_{i=1}^k \bigcup_{j\in I_i} B_k^j) \cap B_k^{j_{k+1}}\right)$ and satisfying $\|\hat{H}_{j_{k+1}} - \mathsf{id}\|_{C^1(U_2),t} \le \eta_k \le \delta_2$ (by (5.3.33)). Set now $U_1 := \varphi_{j_{k+1}}^{-1}\left((\bigcup_{i=1}^k \bigcup_{j\in I_i} B_{k+1}^j) \cap B_{k+1}^{j_{k+1}}\right)$. Clearly $U_1 \subset U_2$ and we also claim that

$$\mathsf{d}_{\mathbb{R}^n}(\bar{U}_1, U_2^c) \ge t\frac{1}{LN} \tag{5.3.37}$$

To prove this is sufficient to show that

$$\mathsf{d}_{\mathbb{R}^n}\left(\varphi_{j_{k+1}}^{-1}(B_{k+1}^j \cap B_k^{j_{k+1}}), (\varphi_{j_{k+1}}^{-1}((B_k^j)^c \cap B_k^{j_{k+1}})\right) \ge t\frac{1}{LN},$$

for every $j \in J_i$, and $i = 1, ..., k$. This can be seen using (5.3.29). Indeed suppose the above is false, then there exist $x \in B_{(2-(k+1)/N)t}(0)$ and $y \in \varphi_{j_{k+1}}^{-1}((B_k^j)^c \cap B_k^{j_{k+1}})$ such that $\mathsf{d}_{\mathbb{R}^n}(\varphi_{j_{k+1}}^{-1}(\varphi_j(x)), y) < t\frac{1}{LN}$. Then the **Claim** above implies that the whole segment joining $\varphi_{j_{k+1}}^{-1}(\varphi_j(x))$ and y is in the domain of $\varphi_j^{-1} \circ \varphi_{j_{k+1}}$. Hence by (5.3.29) we must have that $\mathsf{d}_{\mathbb{R}^n}(x, \varphi_j^{-1}(\varphi_{j_{k+1}}(y))) < t/N$, however by construction $\varphi_j^{-1}(\varphi_{j_{k+1}}(y)) \in B_{(2-k/N)t}(0)^c$ which is a contradiction. This prove (5.3.37). Therefore we can apply Theorem 5.3.1 with $U_1, U_2, H = \hat{H}_{j_{k+1}}, \varepsilon = \eta_k, \eta = \frac{1}{LN}, t = t$. Thus we obtain, after an obvious restriction, a C^q-diffeomorphism $\tilde{H}_{j_{k+1}} : B_{2t}(0) \to B_{2t}(0)$ such that $\tilde{H}_{j_{k+1}}|_{U_1} = \hat{H}_{j_{k+1}}|_{U_1}$ and

$$\|\tilde{H}_{j_{k+1}} - \mathsf{id}\|_{C^1(B_{2t}(0)),t} \le C_2\eta_k < \frac{1}{NLc}, \tag{5.3.38}$$

where the last inequality follows by (5.3.33). We now define the function

$$\tilde{h}_{j_{k+1}} := h_{j_{k+1}} \circ \varphi_{j_{k+1}} \circ \tilde{H}_{j_{k+1}} \circ \varphi_{j_{k+1}}^{-1}, \tag{5.3.39}$$

on $B_0^{j_{k+1}}$. Clearly $\tilde{h}_{j_{k+1}} = \hat{h}_k$ on $\varphi_{j_{k+1}}(U_1) = (\bigcup_{i=1}^k \bigcup_{j\in I_i} B_{k+1}^j) \cap B_{k+1}^{j_{k+1}}$. Repeat now the above construction and define maps $\tilde{h}_{j_{k+1}}$ for every $j_{k+1} \in I_{k+1}$. From the previous observation and the fact that $B_0^{j_{k+1}} \cap B_0^{\overline{j_{k+1}}} = \emptyset$ for every $j_{k+1}, \overline{j_{k+1}} \in I_{k+1}$ the map

$$\hat{h}_{k+1} := \begin{cases} \hat{h}_k & \text{on } \bigcup_{i=1}^k \bigcup_{j\in I_i} B_{k+1}^j, \\ \tilde{h}_{j_{k+1}} & \text{on } B_{k+1}^{j_{k+1}} \text{ for } j_{k+1} \in I_{k+1}, \end{cases} \tag{5.3.40}$$

is a well defined C^q map $\hat{h}_{k+1} : \bigcup_{i=1}^{k+1} \bigcup_{j\in I_i} B_{k+1}^j \to X$. Moreover it's clear from the definition and from the induction hypothesis that (5.3.34) is verified for $\hat{h}_{k+1}$. We need now to verify (5.3.35) and (5.3.36). Observe that it is enough to check these two conditions for $j_1 \in I_{k+1}, j_2 \in I_l$ with $l > k + 1$, since in the other cases they are true by (5.3.40) and induction hypothesis.

For (5.3.35) take $j_1 \in I_{k+1}, j_2 \in I_l$ with $l > k+1$, such that $B^{j_1}_{l-1} \cap B^{j_2}_{l-1} \neq \emptyset$ and pick x in such set. In particular $x = \varphi_{j_1}(y) = \varphi_{j_2}(z)$ for some $y, z \in B_{(2-\frac{l-1}{N})t}(0)$. Since $B^{j_1}_{l-1} \subset B^{j_1}_{k+1}$, $\hat{h}_{k+1}(x) = \tilde{h}_{j_1}(x)$. Thus from (5.3.38) we have that

$$\tilde{h}_{j_1}(x) = h_{j_1} \circ \varphi_{j_1} \circ \tilde{H}_{j_1}(y) \in h_{j_1}(\varphi_{j_1}(B_{(2-\frac{l-2}{N})t}(0))) = h_{j_1}(B^{j_1}_{l-2}). \tag{5.3.41}$$

On the other hand we have that

$$\tilde{h}_{j_1}(x) = h_{j_1} \circ \varphi_{j_1} \circ \tilde{H}_{j_1} \circ \varphi^{-1}_{j_1}(\varphi_{j_2}(z)),$$

moreover from the **Claim** and (5.3.38), we deduce that $\tilde{H}_{j_1}(\varphi^{-1}_{j_1}(x))$ is in the domain of $\varphi_{j_1} \circ \varphi^{-1}_{j_2}$. Therefore we can write

$$\tilde{h}_{j_1}(x) = h_{j_1} \circ \varphi_{j_2} \circ (\varphi^{-1}_{j_2} \circ \varphi_{j_1}) \circ \tilde{H}_{j_1} \circ (\varphi^{-1}_{j_1} \circ \varphi_{j_2})(z).$$

From the above, using (5.3.38) and Lemma 5.3.16 we deduce that

$$\tilde{h}_{j_1}(x) \in h_{j_1}(\varphi_{j_2}(B_{(2-\frac{l-2}{N})t}(0))) = h_{j_1}(B^{j_2}_{l-2}). \tag{5.3.42}$$

Then (5.3.35) follows combining (5.3.41) and (5.3.42), using (5.3.30). It remains to show (5.3.36) for $j_1 \in I_{k+1}, j_2 \in I_l$ for $l > k+1$. Observe that by (5.3.35) that we just proved we can define the map $\hat{H}_{j_1,j_2}$ on $\varphi^{-1}_{j_2}(B^{j_1}_{l-1} \cap B^{j_2}_{l-1})$ as above. Moreover since $\varphi^{-1}_{j_2}(B^{j_1}_{l-1} \cap B^{j_2}_{l-1}) \subset \varphi^{-1}_{j_2}(B^{j_1}_{k+1})$ we deduce from (5.3.40)

$$\begin{aligned}
\hat{H}_{j_1,j_2} &= \varphi^{-1}_{j_2} \circ h^{-1}_{j_2} \circ \hat{h}_k \circ \varphi_{j_2} = \varphi^{-1}_{j_2} \circ h^{-1}_{j_2} \circ \tilde{h}_{j_1} \circ \varphi_{j_2} = \\
&= \varphi^{-1}_{j_2} \circ h^{-1}_{j_2} \circ (h_{j_1} \circ \varphi_{j_1} \circ \tilde{H}_{j_1} \circ \varphi^{-1}_{j_1}) \circ \varphi_{j_2} = \\
&= (\varphi^{-1}_{j_2} \circ h^{-1}_{j_2} \circ h_{j_1} \circ \varphi_{j_2}) \circ \varphi^{-1}_{j_2} \circ \varphi_{j_1} \circ \tilde{H}_{j_1} \circ \varphi^{-1}_{j_1} \circ \varphi_{j_2} = \\
&= H_{j_1,j_2} \circ (\varphi^{-1}_{j_2} \circ \varphi_{j_1}) \circ \tilde{H}_{j_1} \circ (\varphi^{-1}_{j_1} \circ \varphi_{j_2}),
\end{aligned}$$

on $\varphi^{-1}_{j_2}(B^{j_1}_{l-1} \cap B^{j_2}_{l-1})$, where H_{j_1,j_2} is the map in the hypothesis of the theorem. (Observe that in order to plug in the term $\varphi_{j_2} \circ \varphi^{-1}_{j_2}$ in the third line, we used the **Claim** as above). Set now $f \coloneqq \varphi^{-1}_{j_2} \circ \varphi_{j_1}$ and $g \coloneqq f \circ \tilde{H}_{j_1} \circ f^{-1}$. With this notation $\hat{H}_{j_1,j_2} = H_{j_1,j_2} \circ g$. From (5.3.38), (5.3.29) and Lemma 5.3.16 we have that

$$\|g - \mathsf{id}\|_{C^1,t} \leq cC_2\eta_k < 1, \tag{5.3.43}$$

on $\varphi^{-1}_{j_2}(B^{j_1}_{l-1} \cap B^{j_2}_{l-1})$. Therefore combining (5.3.43), (5.3.31) and Lemma 5.3.11 we obtain

$$\|\hat{H}_{j_1,j_2} - \mathsf{id}\|_{C^1,t} \leq \tilde{C}(\eta_k + \varepsilon) \leq \eta_k(\tilde{C} + 1),$$

on $\varphi^{-1}_{j_2}(B^{j_1}_{l-1} \cap B^{j_2}_{l-1})$, where $\tilde{C}$ is constant depending only on C_2 and L. Hence (5.3.36) follows provided we choose $D \geq \tilde{C} + 1$.

Since the induction procedure is now proved, we can define the required map as $h \coloneqq \hat{h}_N$. The assumption $M \subset \cup_j \varphi_j(B_t(0))$ and (5.3.40) grant that h is defined on the whole M and it of class C^q. Moreover (5.3.38),(5.3.39) and (5.3.40) imply (5.3.32) together with the bound

$$\|\tilde{H}_j - \mathsf{id}\|_{C^1(B_{2t}(0)),t} \leq C_2\eta_N \leq \varepsilon D^N C_2,$$

hence it is sufficient to take $C_3 \geq D^N C_2$. Recall also that $\hat{h}_1 \coloneqq h_{j_1}$ in $B^{j_1}_1$ for every $j_1 \in I_1$. Thus, up to proving the **Claim**, the proof is concluded.

Proof of the Claim: Define the sets $\Omega_{r-1} \coloneqq \varphi^{-1}_j(B^j_{r-1} \cap B^{\bar{j}}_{r-1})$, $\Omega_0 \coloneqq \varphi^{-1}_j(B^j_0 \cap B^{\bar{j}}_0)$, which are both contained in the domain of $\varphi^{-1}_j \circ \varphi_{\bar{j}}$. Thus it is enough to prove that

$$\{x \in B_{2t}(0) \mid \mathsf{d}(x, \Omega_r) < \frac{t}{LN}\} \subset \Omega_{r-1}.$$

Pick $x \in B_{2t}(0) \setminus \Omega_{r-1}$ and and set $\rho \coloneqq \mathsf{d}(x, \Omega_r) > 0$. Fix $\varepsilon > 0$ arbitrary. There exists a point $y \in \Omega_r$ such that $|y - x| \leq \rho + \varepsilon$. Moreover, since $y \in \Omega_r$, $x \in \Omega^c_{r-1}$ and $\Omega_r \subset \Omega_{r-1}$, then there must be at least one point p, lying in the segment joining x and y, with $p \in \partial\Omega_{r-1}$. However, since the maps $\varphi_j, \varphi_{\bar{j}}$ are homeomorphisms, we must have that $\Omega_{r-1} \subset\subset \Omega_0$ and

$$\partial\Omega_{r-1} \subset \partial B_{(2-\frac{r-1}{N})t}(0) \cup \varphi^{-1}_j \circ \varphi_{\bar{j}}(\partial B_{(2-\frac{r-1}{N})t}(0)).$$

Moreover $\Omega_r \subset B_{(2-\frac{r}{N})t}(0) \cap \varphi^{-1}_j \circ \varphi_{\bar{j}}(B_{(2-\frac{r}{N})t}(0))$. Hence if $p \in \partial B_{(2-\frac{r-1}{N})t}(0)$ it holds $|p - x| \geq t/N \geq 1/(LN)$. If instead $p \in \varphi^{-1}_j \circ \varphi_{\bar{j}}(\partial B_{(2-\frac{r-1}{N})t}(0))$, since the map $\varphi^{-1}_{\bar{j}} \circ \varphi_j$ is L-Lipschitz by (5.3.29), we must have that

$$|x - y| \geq |x - p| \geq \frac{t}{LN}.$$

Hence $\rho + \varepsilon \geq \frac{t}{LN}$ and from the arbitrariness of ε we conclude. □

5.4. Proof of the metric bi-Lipschitz Reifenberg theorem

In this section we build all the objects that were introduced in Section 5.2.2 where we also proved that they imply the validity of the main Theorem 5.4.20. We will first build all the required objects and at the end verify that they satisfy all the properties presented in Section 5.2.2. The construction of the manifolds in Section 5.4.1 and the construction of the manifold-to-manifold mappings in Section 5.4.4 will heavily rely on the tools developed in Section 5.3.

Along all this section we assume that (Z, d) satisfies the hypotheses of Theorem 5.4.20 with $n \in \mathbb{N}$. Moreover $C = C(n)$, $\varepsilon(n)$, $(\varepsilon_i)_{i=1}^{\infty}$, are the same chosen in the initial setup done in Section 5.2.2. Recall also that (as shown in Section 5.2.2) we can assume both that $r = 200$ and $\varepsilon_i \leq \varepsilon(n)$, in particular whenever we will say that ε_i is small enough we mean that $\varepsilon(n)$ has been chosen small enough (always only with respect to n).

5.4.1. Construction of the approximating manifolds: W_i^n.

Construction of the coverings. We start by showing that under our assumptions (Z, d) is separable and locally compact.

To prove this we exploit the following result due to Alexandroff.

Theorem 5.4.1 ([**7**])**.** *Let (Z, d) be a connected metric space that is locally separable, i.e. for every $z \in Z$ there exists a separable ball that contains z. Then Z is separable.*

Therefore to prove that (Z, d) is separable it's enough to prove the following Lemma.

Lemma 5.4.2. *$B_r(z_0)$ (with r the one in the hypotheses of Theorem 5.2.2) is totally bounded for every $z_0 \in Z$.*

Proof. We start with the following claim. For every $z \in Z$ and every $t = r2^{-k}$ with $k \in \mathbb{N}$, there exists a finite set $S_{z,t} \subset B_t(z)$ that is $t/2$-dense in $B_t(z)$. To prove this first observe there exists a finite subset E_t of $B_t^{\mathbb{R}^n}(0)$ that is $t/8$ dense in $B_r^{\mathbb{R}^n}(0)$. Moreover by hypothesis $d_{GH}(B_t^Z(z), B_t^{\mathbb{R}^n}(0)) < \varepsilon(n)t \leq t/16$, therefore there exists a $t/8$-GH approximation $f : B_t^{\mathbb{R}^n}(0) \to B_t(z)$. Define $S_{t,z} := f(E_t)$. By definition of GH-approximation, for every $p \in B_t(z)$, there exists $x \in B_t^{\mathbb{R}^n}(0)$ such that $\mathsf{d}(f(x), p) \leq t/8$. Moreover there exists $e \in E_t$ with $|e - x| \leq t/8$, hence $\mathsf{d}(f(e), p) \leq 3/8t < t/2$ and the claim is proved. We start defining sets S_k with $k \in \mathbb{N}$ inductively as follows. Set $S_1 := S_{z_0, r/2}$ and

$$S_{k+1} := \bigcup_{z \in S_k} S_{z, \frac{r}{2^{k+1}}}.$$

Then define

$$A_k := \bigcup_{z \in S_k} B_{\frac{r}{2^k}}(z).$$

Observe that $A_k \subset A_{k+1}$, indeed

$$A_k = \bigcup_{z \in S_k} B_{\frac{r}{2^k}}(z) \subset \bigcup_{z \in S_k} \bigcup_{\bar{z} \in S_{z, \frac{r}{2^{k+1}}}} B_{\frac{r}{2^{k+1}}}(\bar{z}) = \bigcup_{z \in S_{k+1}} B_{\frac{r}{2^{k+1}}}(z) = A_{k+1}.$$

Notice also that that $B_r(z_0) \subset \bigcup_{z \in S_1} B_{\frac{r}{2}}(z) = A_1 \subset A_k$ for every k. Moreover A_k is union of a finite number of balls of radius $r/2^k$, therefore $B_r(z_0)$ is totally bounded. □

Remark 5.4.3. Instead of Theorem 5.4.1 we could have used the fact that a metric space is paracompact and then the fact that every connected, locally compact and paracompact tolopogical space is separable (see e.g. [**197**, Appendix A]). ■

Consider for every $i \in \mathbb{N}$ a set $X_i \subset Z$, such that for every $x_1, x_2 \in X_i$ it holds that $\mathsf{d}(x_1, x_2) \geq 2^{-i}$ and X_i is maximal with this property with respect to inclusion. In particular X_i is also 2^{-i}-dense. Moreover, since the balls of radius 2^{-i-1} centered in X_i are pairwise disjoint and Z is separable, X_i is countable.

For every i label the elements of X_i as $X_i = \{x_{i,1}, x_{i,2}, ...\} = \{x_{i,j}\}_{j \in J_i}$ where $J_i = \{1, 2, ..., \#|X_i|\}$. For every i we partition every set X_i into disjoint subsets in the following way. Let $Q_1^i \subset X_i$, maximal (with respect to inclusion), with the property that $\mathsf{d}(x, y) \geq 100 \cdot 2^{-i}$ for every $x, y \in Q_1^i$. Consider also $Q_2^i \subset X_i \setminus Q_1^i$ maximal such that $\mathsf{d}(x, y) \geq 100 \cdot 2^{-i}$ for every $x, y \in Q_2^i$. Keep defining inductively (non-empty) sets $Q_1^i, Q_2^i, ..., Q_{N_i}^i$. Partition also the set of indices J_i as $J_i = \bigcup_{k=1}^{N_i} J_i^k$ where $J_i^k = \{j \in J_i \,|x_{i,j} \in Q_k^i\}$. The next result shows that cardinality of these partitions is uniformly bounded in i.

Lemma 5.4.4. *Under the initial assumptions there exists an integer $N = N(n)$ such that $N_i \le N(n)$ for every i.*

PROOF. Suppose that $Q^i_{\bar{k}}$ is non-empty for some $\bar{k} > 1$ and fix any $x \in Q^i_{\bar{k}}$. In particular $x \in X_i \setminus (Q^i_1 \cup ... \cup Q^i_{\bar{k}-1})$. We claim that $B_{100\cdot 2^{-i}}(x) \cap Q^i_k \neq \emptyset$ for every $1 \le k \le \bar{k}$. Suppose not and let k be such that $B_{100\cdot 2^{-i}}(x) \cap Q^i_k = \emptyset$. Observe that $\mathsf{d}(\bar{x}, x) \ge 100 \cdot 2^{-i}$ for every $\bar{x} \in Q^i_k$. Moreover, as we observed earlier, $x \in X_i \setminus (Q^i_1 \cup ... \cup Q^i_{k-1})$. This two facts, together with the maximality of Q^i_k, imply that we could have added x to Q^i_k during its construction, but this is a contradiction. Set now $S \coloneqq X_i \cap B_{100\cdot 2^{-i}}(x)$. The above claim implies that $S \cap Q^i_k \neq \emptyset$ for every $1 \le k \le \bar{k}$ and in particular $\bar{k} \le \#|S|$.

We now aim to give an upper bound on $\#|S|$. Recall that by hypothesis there exists a $C\varepsilon_i 2^{-i}$-isometry $f : B_{200\cdot 2^{-i}}(x) \to B_{200\cdot 2^{-i}}(0)$. Recall also that by construction the points in S are at distance at least 2^{-i} from each other. Therefore, assuming ε_i small enough, we deduce both $f|_S$ is injective and that the points in $f(S) \subset B_{200\cdot 2^{-i}}(0)$ are at distance at least $2^{-i}/4$ from each other. Thus $\#|S| = \#|f(S)| \le N(n)$, for some integer $N(n)$ depending only on n, that in turn implies $\bar{k} \le N(n)$. □

Construction of the irregular transition maps. From hypothesis, combining Theorem 5.1.8, Proposition 5.1.9 and Proposition 5.1.10 (recalling that we are assuming $r = 200$) we have that for every $j \in J_i$ there exist maps

$$\alpha_{i,j} : B^{\mathbb{R}^n}_{200\cdot 2^{-i}}(0) \to B^{Z}_{200\cdot 2^{-i}}(x_{i,j})$$

and

$$\beta_{i,j} : B^{Z}_{200\cdot 2^{-i}}(x_{i,j}) \to B^{\mathbb{R}^n}_{200\cdot 2^{-i}}(0)$$

that are $C\varepsilon_i 2^{-i}$-Gromov-Hausdorff approximations, such that

$$d_{\mathbb{R}^n}(\beta_{i,j} \circ \alpha_{i,j}, id),\, d_Z(\alpha_{i,j} \circ \beta_{i,j}, id) \le C\varepsilon_i 2^{-i} \tag{5.4.1}$$

and

$$\alpha_{i,j}(0) \in B_{C\varepsilon_i 2^{-i}}(x_{i,j}), \;\; \beta_{i,j}(x_{i,j}) \in B_{C\varepsilon_i 2^{-i}}(0). \tag{5.4.2}$$

For simplicity and to avoid very heavy notations we will often drop the index i on the symbols $\alpha_{i,j}, \beta_{i,j}$ and $x_{i,j}$. Since for most part of the proof and constructions the index i is fixed, this convention should cause no confusion. Nevertheless when some ambiguity might occur we will use the complete notation with the double subscript.

An immediate consequence of (5.4.2) is that, provided $\varepsilon(n) < \frac{1}{200C}$,

$$\alpha_j(B_s(0)) \subset B_{s+\frac{2^{-i}}{100}}(x_j), \;\; \beta_j(B_s(x_j)) \subset B_{s+\frac{2^{-i}}{100}}(x_j), \tag{5.4.3}$$

for every $s < 200 \cdot 2^{-i}$.

The maps that we just defined allow us to think of Z as a very irregular manifold. In particular β_j have the role of charts for the metric space Z and $\beta_{j_2} \circ \alpha_{j_1}$ (when defined) can be viewed as transition maps. However all these functions are very rough and could be not even continuous. Therefore the main idea to build the manifold W_i is to approximate the maps $\beta_{j_2} \circ \alpha_{j_1}$ with diffeomorphisms $\tilde{I}_{j_2,j_1}$ and then build a manifold having $\tilde{I}_{j_2,j_1}$ as transition maps. The main issue in this procedure is that to be able to actually build a manifold we need these diffeomorphisms to be compatible with respect to each other. This will require the modification procedure that we developed in Section 5.3.2.

Approximation of the irregular transition maps with isometries. We start approximating $\beta_{j_2} \circ \alpha_{j_1}$ with isometries as follows.

Lemma 5.4.5. *Suppose that $j_1, j_2 \in J_i$ are such that $\mathsf{d}(x_{j_1}, x_{j_2}) < 30\cdot 2^{-i}$. Then there exist two isometries $I_{i,j_2,j_1}, I_{i,j_1,j_2} : \mathbb{R}^n \to \mathbb{R}^n$ such that $I_{i,j_2,j_1} = I^{-1}_{i,j_1,j_2}$ and*

$$|I_{i,j_2,j_1}(x) - \beta_{i,j_2} \circ \alpha_{i,j_1}(x)| \le C\varepsilon_i 2^{-i} \tag{5.4.4}$$

$$|I_{i,j_1,j_2}(x) - \beta_{i,j_1} \circ \alpha_{i,j_2}(x)| \le C\varepsilon_i 2^{-i} \tag{5.4.5}$$

for every $x \in B_{45\cdot 2^{-i}}(0)$.

PROOF. Observe that by hypothesis $B_{90\cdot 2^{-i}}(x_{j_2}) \subset B_{200\cdot 2^{-i}}(x_{j_1})$. Thus (recalling (5.4.3)) the map $\beta_{j_1} \circ \alpha_{j_2} : B_{80\cdot 2^{-i}}(0) \to B_{200\cdot 2^{-i}}(0)$ is well defined and is a $C\varepsilon_i 2^{-i}$ isometry. Therefore, if ε_i is small enough, we are in position to apply Lemma 5.1.11 and deduce that there exists a global isometry $I_{i,j_2,j_1} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$|I_{i,j_1,j_2}(x) - \beta_{j_1} \circ \alpha_{j_2}(x)| \le C\varepsilon_i 2^{-i} \tag{5.4.6}$$

for every $x \in B_{80\cdot 2^{-i}}(0)$. This already proves (5.4.4). We define $I_{i,j_2,j_1} := I^{-1}_{i,j_1,j_2}$. Using (5.4.6) and (5.4.1)

$$\begin{aligned}|I_{i,j_2,j_1}(x) - \beta_{j_2}\circ\alpha_{j_1}(x)| &= |x - I_{i,j_1,j_2}(\beta_{j_2}\circ\alpha_{j_1}(x))|\\ &\overset{(5.4.6)}{\leq} |x - \beta_{j_1}(\alpha_{j_2}(\beta_{j_2}(\alpha_{j_1}(x))))| + C\varepsilon_i 2^{-i}\\ &\leq |x - \beta_{j_1}(\alpha_{j_1}(x))| + C\varepsilon_i 2^{-i}\\ &\leq C\varepsilon_i 2^{-i}\end{aligned}$$

for every $x \in B_{45\cdot 2^{-i}}(0)$. This proves (5.4.5). Observe that to justify the use of (5.4.6) above we need to check that

$$\beta_{j_2}(\alpha_{j_1}(B_{45\cdot 2^{-i}}(0))) \subset B_{80\cdot 2^{-i}}(0). \tag{5.4.7}$$

To prove this observe that from (5.4.3) and the fact that $\mathsf{d}(x_{j_1}, x_{j_2}) < 30\cdot 2^{-i}$, it follows that $\alpha_{j_1}(B_{45\cdot 2^{-i}}(0)) \subset B_{46\cdot 2^{-i}}(x_{j_1}) \subset B_{76\cdot 2^{-i}}(x_{j_2})$. From this, using again (5.4.3), we obtain (5.4.7). □

Definition 5.4.6. For any couple of indices $j_1, j_2 \in J_i$ such that $\mathsf{d}(x_{j_1}, x_{j_2}) < 30\cdot 2^{-i}$ we choose once and for all a couple of maps $I_{i,j_1,j_2}, I_{i,j_2,j_1}$ that satisfies the thesis of Lemma 5.4.5.

We then denote by $\mathcal{I}_i$ the union of all these maps, i.e. we set

$$\mathcal{I}_i = \bigcup_{j_1,j_2\in J_i \text{ such that } B_{15\cdot 2^{-i}}(x_{j_1})\cap B_{15\cdot 2^{-i}}(x_{j_2})\neq\emptyset} \{I_{i,j_1,j_2}, I_{i,j_2,j_1}\}.$$

Again for simplicity we will often write I_{j_2,j_1} in place of I_{i,j_2,j_1}.

Remark 5.4.7. For any two maps $I_{j_1,j_2}, I_{j_1,j_3} \in \mathcal{I}_i$ with $j_2 \neq j_3$, it holds that $j_1 \in J_{i,a_1}, j_2 \in J_{i,a_2}, j_3 \in J_{i,a_3}$ with $a_1 \neq a_2 \neq a_3 \neq a_1$. This follows from the definition of the sets $J_{i,k}$ and the fact that $\mathsf{d}(x_{j_1}, x_{j_2}) < 30\cdot 2^{-i}$, $\mathsf{d}(x_{j_1}, x_{j_3}) < 30\cdot 2^{-i}$ and thus $\mathsf{d}(x_{j_2}, x_{j_3}) < 60\cdot 2^{-i}$. ■

Lemma 5.4.8. *Suppose that for some triple of indices $j_1, j_2, j_3 \in J_i$ the maps $I_{j_3,j_2}, I_{j_2,j_1}, I_{j_3,j_1}$ are defined. Then*

$$|I_{j_3,j_2}(I_{j_2,j_1}(x)) - I_{j_3,j_1}(x)| \leq C\varepsilon_i 2^{-i}$$

for every $x \in B_{12\cdot 2^{-i}}(0)$.

Proof. Observe that the existence of lala the maps $I_{j_3,j_2}, I_{j_2,j_1}, I_{j_3,j_1}$ is equivalent to the fact that $\mathsf{d}(x_{j_1}, x_{j_2}) < 30\cdot 2^{-i}$, $\mathsf{d}(x_{j_1}, x_{j_3}) < 30\cdot 2^{-i}$ and $\mathsf{d}(x_{j_3}, x_{j_2}) < 30\cdot 2^{-i}$. Applying (5.4.4) multiple times and the triangle inequality we get

$$\begin{aligned}|I_{j_3,j_2}(I_{j_2,j_1}(x)) - I_{j_3,j_1}(x)| &\leq |I_{j_3,j_2}(I_{j_2,j_1}(x)) - \beta_{j_3}(\alpha_{j_1}(x))| + C\varepsilon_i 2^{-i}\\ &\leq |I_{j_3,j_2}(\beta_{j_2}(\alpha_{j_1}(x))) - \beta_{j_3}(\alpha_{j_1}(x))| + C\varepsilon_i 2^{-i}\\ &\overset{(5.4.4)}{\leq} |\beta_{j_3}(\alpha_{j_2}(\beta_{j_2}(\alpha_{j_1}(x)))) - \beta_{j_3}(\alpha_{j_1}(x))| + C\varepsilon_i 2^{-i}\\ &\leq |\beta_{j_3}(\alpha_{j_1}(x)) - \beta_{j_3}(\alpha_{j_1}(x))| + C\varepsilon_i 2^{-i} = C\varepsilon_i 2^{-i},\end{aligned}$$

for any $x \in B_{12\cdot 2^{-i}}(0)$, where in the last inequality we used (5.4.1). Observe that, to justify the use of (5.4.4) in the third inequality above we need to check that

$$\beta_{j_2}\circ\alpha_{j_1}(B_{12\cdot 2^{-i}}(0)) \subset B_{45\cdot 2^{-i}}(0), .$$

To see this observe that by (5.4.3) $\alpha_{j_1}(B_{12\cdot 2^{-i}}(0)) \subset B_{13\cdot 2^{-i}}(x_{j_1})$. Then, since $\mathsf{d}(x_{j_1}, x_{j_2}) < 30\cdot 2^{-i}$, we have $\alpha_{j_1}(B_{12\cdot 2^{-i}}(0)) \subset B_{43\cdot 2^{-i}}(x_{j_2})$. Now applying again (5.4.3) we conclude. □

Modification of the isometries to get compatibility. Our plan is now to construct the manifold W_i^n by gluing together a number of copies of Euclidean balls in the following way:

$$W_i^n := \bigsqcup_{j\in J_i} B^j_{10\cdot 2^{-i}}(0)\,/\!\sim,$$

where $x \sim y$ for $x \in B^{j1}_{10\cdot 2^{-i}}(0), y \in B^{j2}_{10\cdot 2^{-i}}(0)$ if and only if $Ii, j_2, j_1(x) = y$ (we set also $x \sim x$ for every x). Notice that this is an equivalent relation if and only if the following compatibility relation is true. For every couple of maps $I_{i,j_1j_2}, I_{i,j_3j_2}$ for which there exists $x \in B_{10\cdot 2^{-i}}(0)$ such that $I_{i,j_2,j_1}(x), I_{i,j_3j_2}(I_{i,j_2,j_1}(x)) \in B_{10\cdot 2^{-i}}(0)$ we have that the map I_{i,j_3,j_1} is defined and we have

$$I_{i,j_3,j_2}\circ I_{i,j_2,j_1}(x) = I_{i,j_3,j_1}(x).$$

However there is no reason for this to be true in general for the maps that we have defined. However it is almost true in a quantitative sense given by Lemma (5.4.8). Thanks to this we can perform a small modification of the maps I_{i,j_1,j_2} to obtain new maps $\widetilde{I}_{i,j_1,j_2}$ such that the above compatibility relations are satisfied. This modification is quite involved and is actually independent of the metric space (Z, d), instead takes place fully in the Euclidean space. For this reason this procedure has been developed separately in Section 5.3.2 and is contained in Theorem 5.3.8, which immediately implies the following:

Lemma 5.4.9. *For every I_{i,j_1,j_2} defined as above there exists a C^∞ diffeomorphism $\widetilde{I}_{i,j_1,j_2} : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\|I_{i,j_1,j_2} - \widetilde{I}_{i,j_1,j_2}\|_{C^2(\mathbb{R}^n),2^{-i}} \le C\varepsilon_i, \tag{5.4.8}$$

$\widetilde{I}_{i,j_2,j_1} = \widetilde{I}^{-1}_{i,j_1,j_2}$ and the following compatibility condition holds: for every couple of maps $\tilde{I}_{i,j_1j_2}, \tilde{I}_{i,j_3j_2}$ for which the set $\{x \in B_{8\cdot 2^{-i}}(0) \ : \ \tilde{I}_{i,j_2,j_1}(x), \tilde{I}_{i,j_3j_2}(\tilde{I}_{i,j_2,j_1}(x)) \in B_{8\cdot 2^{-i}}(0)\}$ is non empty, we have that the map $\tilde{I}_{i,j_3,j_1}$ is defined and

$$\tilde{I}_{i,j_3,j_2} \circ \tilde{I}_{i,j_2,j_1}(x) = \tilde{I}_{i,j_3,j_1}(x),$$

for every x in the above set.

Proof. It is enough to apply Theorem 5.3.8 with $t = 2^{-i}$ and $\mathcal{A} = \mathcal{I}_i$, the family of maps in Definition 5.4.6. Indeed observe that thanks Lemma 5.4.4 the upper bound M on the cardinality of the partition depends only on n, therefore the constants appearing in Theorem 5.3.8 will depend in this case only on n. Thus we just need to check the hypotheses of Theorem 5.3.8:

(5.3.5) follows by construction and (5.3.6) follows by Remark 5.4.7. Property A) is true again by construction. Property C) is the content of Lemma 5.4.8. It remains to check property B). Suppose $I_{j_3,j_2}(I_{j_2,j_1}(x)) \in B_{9\cdot 2^{-i}}(0)$ for some $x \in B_{8\cdot 2^{-i}}(0)$. Recalling now (5.4.4), provided ε_i is small enough, we deduce that $y := \beta_{j_3} \circ \alpha_{j_2} \circ \beta_{j_2} \circ \alpha_{j_1}(x) \in B_{10\cdot 2^{-i}}(0)$. Observe that we can apply (5.4.4) since $\beta_{j_2} \circ \alpha_{j_1}(x) \in B_{45\cdot 2^{-i}}(x_{j_2})$, thanks to (5.4.3) and the fact that $\mathsf{d}(x_{j_1}, x_{j_2}) < 30 \cdot 2^{-i}$ which holds since the map I_{j_1,j_2} is defined. Therefore from (5.4.3) it holds that $\alpha_{j_1}(x) \in B_{10\cdot 2^{-i}}(x_{j_1})$ and $\alpha_{j_3}(y) \in B_{11\cdot 2^{-i}}(x_{j_3})$. Moreover from (5.4.1) we have that $\mathsf{d}(\alpha_{j_3}(y), \alpha_{j_1}(x)) \le 2^{-i}$, if ε_i is small enough. This implies that $B_{15\cdot 2^{-i}}(x_{i,j_3}) \cap B_{15\cdot 2^{-i}}(x_{i,j_1}) \neq \emptyset$, hence the map I_{i,j_3,j_1} exists. This proves property B. □

Construction of the topological manifolds. We can now construct W_i^n as we mentioned above. Specifically we will construct it starting from its transition maps and their domains of definition. This kind of construction, even if very natural, needs to be done carefully to ensure that the resulting space is a well defined Hausdorff manifold. In doing this we will refer to the formalization through *Sets of Gluing Data* described in [**113**, Thereom 3.1]. We start defining maps $\hat{I}_{i,j_1,j_2}$ that are restriction of the maps $\widetilde{I}_{i,j_1,j_2}$. In particular we set

$$\Omega^i_{j_1,j_2} := \widetilde{I}^{-1}_{i,j_1,j_2}(B_{8\cdot 2^{-i}}(0)) \cap B_{8\cdot 2^{-i}}(0) \tag{5.4.9}$$

and $\hat{I}_{i,j_1,j_2} := \widetilde{I}_{i,j_1,j_2}|_{\Omega^i_{j_1,j_2}}$. Observe that the sets $\Omega^i_{j_1,j_2}$ are open subsets of $\mathbb{R}^n$. Then we define

$$W_i^n := \textstyle\bigsqcup_{j\in J_i} B^j_{8\cdot 2^{-i}}(0) /_\sim,$$

where $x \sim y$ (with $x \neq y$) if and only if $x \in B^{j1}_{8\cdot 2^{-i}}(0), y \in B^{j2}_{8\cdot 2^{-i}}(0)$ and $\hat{I}_{i,j_2,j_1}(x) = y$. We impose also $x \sim x$ for every x. This is now a valid equivalence relation by Lemma 5.4.9 (referring to the construction in [**113**], the compatibility condition ensured by Lemma 5.4.9 implies condition (3.c) in Definition 3.1 of [**113**]). The set W_i^n comes with a natural quotient map

$$\Pi : \bigsqcup_{j\in J_i} B^j_{8\cdot 2^{-i}}(0) \to W_i^n,$$

defined as $\Pi(x) := [x]$, where $[x]$ is the equivalent class containing x. Define now for every $j \in J_i$ the map

$$p_j := \Pi \circ \mathsf{in}_j : B_{8\cdot 2^{-i}}(0) \to W_i,$$

where $B_{8\cdot 2^{-i}}(0) \overset{\mathsf{in_j}}{\hookrightarrow} \bigsqcup_{j\in J_i} B^j_{8\cdot 2^{-i}}(0)$ is the natural inclusion map on the j-th component. Observe that, by construction, the map Π is injective on every $B^j_{8\cdot 2^{-i}}(0)$ and thus also p_j is injective. We endow W_i^n with the topology τ defined by

$$\tau = \{U \subset W_i \ : \ p_j^{-1}(U) \text{ is an open subset of } \mathbb{R}^n, \ \forall j \in J_i\}.$$

It is easy to check that τ is indeed a topology. We claim that with this topology, the sets $p_j(B_{8\cdot 2^{-i}}(0))$ are open for every $j \in J_i$ and that the maps $p_j : B_{8\cdot 2^{-i}}(0) \to p_j(B_{8\cdot 2^{-i}}(0))$ are homomorphisms for every $j \in J_i$. To prove the claim it sufficient to show that $p_j(V) \in \tau$, for every $j \in J_i$ and every $V \subset B_{8\cdot 2^{-i}}(0)$ open (possibly taking $V = B_{8\cdot 2^{-i}}(0)$). This in turn follows observing that by construction

$$p_{j_2}^{-1}(p_{j_2}(V)) = \Omega^i_{j_1,j_2} \cap V, \quad \forall j_1, j_2 \in J_i$$

and recalling that the sets $\Omega^i_{j_1,j_2}$ are open.

For every $j \in J_i$ we set $B^i_j := p_j(B_{8\cdot 2^{-i}}(0))$ and define the maps $\psi_{i,j} : B^i_j \to B_{8\cdot 2^{-i}}(0)$ as $\psi_{i,j} := p_j^{-1}$, these maps will be the charts of the manifold. (As usual we will write ψ_j instead of $\psi_{i,j}$ when there will be no ambiguity in doing so.) We need to show that (W^n_i, τ) is Hausdorff. To do this it's enough to observe that

$$\widetilde{I}_{i,j_1,j_2}(\partial(\Omega^i_{j_1,j_2}) \cap B_{8\cdot 2^{-i}}(0)) \subset \partial B_{8\cdot 2^{-i}}(0)$$

that comes from the fact that $\widetilde{I}_{i,j_1,j_2}$ is a global diffeomorphism. This fact implies condition $(3.d)$ in Definition 3.1 of [**113**], which is there shown to be enough to ensure that W^n_i is a Hausdorff space. It remains to endow W^n_i with a smooth structure. Observe that by construction, $\psi_{i,j_2} \circ (\psi_{i,j_1})^{-1} = \hat{I}_{i,j_2,j_1}$ which are diffeomorphisms. Moreover we proved above that the sets B^i_j are open in (W^n_i, τ). This proves that $(B^i_j, \psi_{i,j})_{j\in J_i}$ is a smooth structure for W^n_i.

Lemma 5.4.10. *The manifolds $(W^n_i)_{i=0}^{\infty}$ are all connected.*

Proof. We argue by contradiction. Suppose W^n_i is disconnected. Pick U a non empty connected component of W^n_i. Since $W^n_i \subset U_{j\in J^i} B^i_j$ and each B^i_j is connected (is homeomorphic to an Euclidean ball), we must have that

$$U = \bigcup_{j\in S} B^i_j \tag{5.4.10}$$

for some proper subset S of J^i. Set now $\tilde{S} := J^i \setminus S$. From (5.4.10) we must have $B^i_{j_1} \cap B^i_{j_2} = \emptyset$ for every $j_1 \in S$ and every $j_2 \in \tilde{S}$. We claim that this implies that $\mathsf{d}(x_{j_2}, x_{j_1}) > 5 \cdot 2^{-i}$ for every $j_1 \in S$ and every $j_2 \in \tilde{S}$. Indeed the transition map $\hat{I}_{j_1,j_2}$ must not be defined and this by construction can happen only in two cases. The first case is that the map $\tilde{I}_{j_1,j_2}$ is not defined, the second is that $\tilde{I}_{j_1,j_2}$ is defined but $\tilde{I}^{-1}_{j_1,j_2}(B_{8\cdot 2^{-i}}(0)) \cap B_{8\cdot 2^{-i}}(0) = \emptyset$. In the first case by construction we must have that $\mathsf{d}(x_{j_1}, x_{j_2}) > 15 \cdot 2^{-i}$. In the second case, by (5.4.8) and (5.4.4) we have that $|\beta_{j_1} \circ \alpha_{j_2} - \tilde{I}^{-1}_{j_2,j_1}| \le C\varepsilon_i 2^{-i}$ in $B_{8\cdot 2^{-i}}(0)$. Hence, if ε_i is small enough, $\beta_{j_1}(\alpha_{j_2}(0)) \notin B_{7\cdot 2^{-i}}(0)$, that combined with (5.4.2) and provided ε_i small enough, gives $\mathsf{d}(\alpha_{j_2}(0), x_{j_1}) > 6 \cdot 2^{-i}$. Then again using (5.4.2) we deduce that $\mathsf{d}(x_{j_2}, x_{j_1}) > 5 \cdot 2^{-i}$. Therefore in both cases $\mathsf{d}(x_{j_2}, x_{j_1}) > 5 \cdot 2^{-i}$ and the claim is proved. This implies that

$$\left(\bigcup_{j\in S} B^Z_{2^{-i}}(x_j)\right) \cap \left(\bigcup_{j\in \tilde{S}} B^Z_{2^{-i}}(x_j)\right) = \emptyset,$$

however, since $X_i = \{x_j\}_{j\in J^i}$ is 2^{-i}-dense, this two sets cover Z, which contradicts the connectedness of Z. $\square$

Construction of the Riemannian-metrics. It remains endow the topological manifolds W^i_n with suitable Riemannian metrics g_i.

To construct the metric g_i consider a partition of the unity $\rho_{i,j}$ subordinate to the cover $\{B^i_j\}_{j\in J_i}$ and define

$$g_i := \sum_{j\in J_i} \rho_{i,j} \psi^*_{i,j} g$$

where g is the standard Euclidean metric on $\mathbb{R}^n$. Observe that the covering $\{B^i_j\}_{j\in J_i}$ is locally finite with multiplicity less than $N = N(n)$. This is a consequence of Remark 5.4.7. We denote with d_i the Riemannian distance function induced by the metric g_i (recall that from Lemma 5.4.10 W_i is connected, hence d_i is finite). The first key observation is that, with this metric, the charts $\psi_{i,j}$ turns out to be bi-Lipschitz maps, in particular we have the following.

Lemma 5.4.11. *For every $j \in J_i$ consider ψ_j as a smooth function $\psi_j : B^i_j \to \mathbb{R}^n$, then*

$$\frac{1}{1 + C\varepsilon_i} \le \|D\psi_j\| \le 1 + C\varepsilon_i, \tag{5.4.11}$$

where $\|.\|$ denotes the operator norm.

Proof. Fix $p \in B^i_j$ and pick any $v \in T_p W^n_i$. We can write v in local coordinates with respect to the chart $\psi_j = (x^j_1, ..., x^j_n)$ as $v = a^k \frac{\partial}{\partial x^j_k}$. Set $w := (D\psi_j)_p \cdot v \in \mathbb{R}^n$ and observe that $w = (a_1, ..., a_n)$. We now need to compute $|v|_{T_p W^n_i}$. Recall that $g_p = \sum_{k\in J_i} \rho_{i,k}(p) \psi^*_k g$, thus

$$|v|^2_{T_p W^n_i} = \sum_{k\in J_i} \rho_{i,k}(p) |(D\psi_k)_p v|^2 = \sum_{k\in J_i} \rho_{i,k}(p) |A_{k,j} \bar{a}|^2.$$

where $A_{k,j}$ is the Jacobian matrix of the transition function $\psi_k \circ (\psi_j)^{-1}$, that by construction coincides with $\hat{I}_{k,j}$. Therefore from (5.4.8) we have that $1 - C\varepsilon_i \leq \|A_{k,j}\| \leq 1 + C\varepsilon_i$ (indeed I_{i,j_1,j_2} are isometries). Thus

$$(1 - C\varepsilon_i)|w| \leq |v|_{T_p W_i^n} \leq (1 + C\varepsilon_i)|w|,$$

which gives

$$\frac{1}{1 + C\varepsilon_i}|v| \leq |w| \leq (1 + 2C\varepsilon_i)|v|,$$

that is what we wanted. $\square$

Remark 5.4.12. It holds that

$$B^{W_i}_{\frac{s}{1+2C\varepsilon_i}}(\psi_j^{-1}(x)) \subset \psi_j^{-1}(B_s(x)) \subset B^{W_i}_{s(1+C\varepsilon_i)}(\psi_j^{-1}(x))$$

for every ball $B_s(x) \subset B_{8\cdot 2^{-i}}(0)$. This follows immediately from the fact that ψ_j are $1 + C\varepsilon_i$ bi-Lipschitz homeomorphisms. Indeed suppose the first inclusion does not hold. This means that the (non empty) open set $U := B^{W_i}_{\frac{s}{1+2C\varepsilon_i}}(\psi_j^{-1}(x)) \cap \psi_j^{-1}(B_s(x))$ is a proper subset of $B^{W_i}_{\frac{s}{1+2C\varepsilon_i}}(\psi_j^{-1}(x))$. This means that there is a point $w \in \partial\psi_j^{-1}(B_s(x))$ such that $w \in B^{W_i}_{\frac{s}{1+2C\varepsilon_i}}(\psi_j^{-1}(x))$. However, since ψ_j^{-1} is an homeomorphism, it preserves the boundary, that is $\partial\psi_j^{-1}(B_r(x)) = \psi_j^{-1}(\partial B_s(x))$. Therefore since ψ_j^{-1} is $1 + C\varepsilon_i$ bi-Lipschitz we obtain that $\mathsf{d}_i(w, \psi_j^{-1}(x)) \geq \frac{s}{1+C\varepsilon_i}$, that is a contradiction. The second inclusion follows directly from the Lipschitzianity of ψ_j. ■

Remark 5.4.13. It turns out that the manifolds W_i endowed with the distance d_i are also complete, however we will delay the proof of this fact to the end of the next subsection (see Lemma 5.4.18). ■

5.4.2. The manifold-to-metric space mappings: f_i.

For every $w \in W_i^n$ we define

$$f_i(w) := \alpha_j(\psi_j(w)),$$

where $j \in J_i$ is the smallest such that $w \in B_j^i$. We remark that with this definition $f_i(w)$ will depend on the choice of the index j, indeed the map f_i will not be even continuous in general. However in the following statement we prove that f_i is almost unique, in the sense that the a different choice of j in its definition change its value by at most $\varepsilon_i 2^{-i}$.

Lemma 5.4.14. *Suppose that $x \in W_i^n$, then for any $j \in J_i$ for which and $x \in B_j^i$ we have that*

$$\mathsf{d}(f_i(x), \alpha_j(\psi_j(x))) \leq C\varepsilon_i 2^{-i}. \tag{5.4.12}$$

Proof. By definition $f_i(x) = \alpha_k(\psi_k((x)))$ for some $k \in J_i$ for which $x \in B_k^i$. If $k = j$ there is nothing to prove, so suppose $k \neq j$. Then

$$\begin{aligned}
\mathsf{d}(f_i(x), \alpha_j(\psi_j(x))) &= \mathsf{d}(\alpha_k(\psi_k(x)), \alpha_j(\psi_j(x))) = \\
&= \mathsf{d}(\alpha_k(\psi_k(x)), \alpha_j(\psi_j(\psi_k^{-1}(\psi_k(x)))) = \\
&= \mathsf{d}(\alpha_k(\psi_k(x)), \alpha_j(\hat{I}_{j,k}(\psi_k(x))) \\
&\leq \mathsf{d}(\alpha_k(\psi_k(x)), \alpha_j(\beta_j \circ \alpha_k(\psi_k(x))) + C\varepsilon_i 2^{-i},
\end{aligned}$$

where the last inequality follows combining (5.4.4) and (5.4.8). Recalling now (5.4.1) we obtain (5.4.12). $\square$

By construction f_i sends points in B_i^j inside $B_{10\cdot 2^{-i}}(x_j)$. The following statement is the converse of this, meaning that points mapped by f_i near x_j must belong to the coordinate patch B_i^j.

Lemma 5.4.15. *Suppose that for some $w \in W_i$ it holds that $f_i(w) \in B_{s\cdot 2^{-i}}(x_j)$ with $s < 6$, then*

$$w \in \psi_j^{-1}(B_{(s+C\varepsilon_i)2^{-i}}(0)), \tag{5.4.13}$$

in particular $w \in B_j^i$.

Proof. By definition $f_i(w) = \alpha_k(\psi_k(w))$ for some $k \in J_i$. Observe now that from (5.4.2) we deduce that $\beta_j(\alpha_k(\psi_k(w))) = \beta_j(f_i(w)) \in B_{(s+C\varepsilon_i)2^{-i}}(0)$. Therefore, since by (5.4.8) and (5.4.4) $|\tilde{I}_{j,k} - \beta_j \circ \alpha_k| \leq C\varepsilon_i 2^{-i}$, we also have $\tilde{I}_{j,k}(\psi_k(w)) \in B_{(s+C\varepsilon_i)2^{-i}}(0)$, up to increasing the constant C. Since $s < 6$, if ε_i is small enough we have that $\tilde{I}_{j,k}(\psi_k(w)) \in B_{8\cdot 2^{-i}}(0)$, which implies $\psi_k(w) \in \Omega_{j,k}^i$ (where $\Omega_{j,k}^i$ is as in (5.4.9)). Since $\Omega_{j,k}^i$ is the domain of $\hat{I}_{j,k}$ we can write

$$w = \psi_j^{-1}(\psi_j \circ \psi_k^{-1})(\psi_k(w)) = \psi_j^{-1}(\hat{I}_{j,k}(\psi_k(w)))$$

and since we showed that $\tilde{I}_{j,k}(\psi_k(w)) \in B_{(s+C\varepsilon_i)2^{-i}}(0)$, this proves (5.4.13). $\square$

The following result tells us that, locally, f_i is a $2^{-i}C\varepsilon_i$-isometry.

Lemma 5.4.16. *For every $j \in J_i$ it holds*

$$|\mathsf{d}(f_i(w_1), f_i(w_2)) - \mathsf{d}_i(w_1, w_2)| \le 2^{-i}C\varepsilon_i, \quad \textit{for every } w_1, w_2 \in B^i_j. \tag{5.4.14}$$

Proof. By (5.4.12) we can suppose that $f_i(w_1) = \alpha_j(\psi_j(w_1)), f_i(w_2) = \alpha_j(\psi_j(w_2))$. Thus we need to bound $|\mathsf{d}(\alpha_j(\psi_j(w_1)), \alpha_j(\psi_j(w_2))) - \mathsf{d}_i(w_1, w_2)|$, however since α_j is a $C\varepsilon_i 2^{-i}$ isometry we have

$$|\mathsf{d}(\alpha_j(\psi_j(w_1)), \alpha_j(\psi_j(w_2))) - |\psi_j(w_1) - \psi_j(w_2)|| \le C\varepsilon_i 2^{-i}.$$

Hence we reduced ourselves to estimate $||\psi_j(w_1) - \psi_j(w_2)| - \mathsf{d}_i(w_1, w_2)|$. To do this first observe that by Remark 5.4.12, if ε_i is small enough, we have

$$B^i_j = \psi_j^{-1}(B_{8\cdot 2^{-i}}(0)) \subset B_{10\cdot 2^{-i}}(\psi_j^{-1}(0)),$$

hence $\mathsf{d}_i(w_1, w_2) \le 20 \cdot 2^{-i}$. Then recalling that ψ_j is $1 + C\varepsilon_i$ bi-Lipschitz we have

$$||\psi_j(w_1) - \psi_j(w_2)| - \mathsf{d}_i(w_1, w_2)| = \mathsf{d}_i(w_1, w_2)\left|\frac{|\psi_j(w_1) - \psi_j(w_2)|}{\mathsf{d}_i(w_1, w_2)} - 1\right| \le 20 \cdot 2^{-i}C\varepsilon_i,$$

that is what we wanted.

□

Since the balls centered in X_i of radius 2^{-i} covers Z, it is natural to expect that we can cover W_i with sets that are significantly smaller than $B^i_j = \psi_j^{-1}(B_{10\cdot 2^{-i}}(0))$. This essentially is the content of the following result.

Lemma 5.4.17. *For every $i \in \mathbb{N}_0$ it holds that*

$$W_i^n \subset \bigcup_j \psi_j^{-1}\left(\overline{B_{2\cdot 2^{-i}}(0)}\right) \subset \bigcup_j B^{W_i}_{3\cdot 2^{-i}}(\psi_j^{-1}(0))). \tag{5.4.15}$$

Proof. Let $w \in W_i$ then $f_i(w) \in B_{2^{-i}}(x_j)$ for some $j \in J_i$, since X_i is 2^{-i}-dense in Z by construction. Therefore from Lemma 5.4.15 we deduce that $w \in \psi_j^{-1}(B_{(1+C\varepsilon_i)2^{-i}}(0)) \subset \psi_j^{-1}\left(\overline{B_{(2\cdot 2^{-i}}(0)}\right)$, provided ε_i is small enough. This proves the first inclusion in (5.4.15). The second inclusion follows from Remark 5.4.12. □

Lemma 5.4.18. *The manifold (W_i, d_i) is complete.*

Proof. Let $w \in W_i^n$, then from (5.4.15), there exists $j \in J^i$ such that $w \in B^{W_i}_{3\cdot 2^{-i}}(\psi_j^{-1}(0))$. Hence $B^{W_i}_{2^{-i}}(w) \subset B^{W_i}_{4\cdot 2^{-i}}(\psi_j^{-1}(0))$. From Remark 5.4.12 we deduce, provided ε_i small enough, that

$$B^{W_i}_{2^{-i}}(w) \subset B^{W_i}_{4\cdot 2^{-i}}(\psi_j^{-1}(0)) \subset \psi_j(B_{5\cdot 2^{-i}}(0)) \subset \psi_j\left(\overline{B_{5\cdot 2^{-i}}(0)}\right).$$

Since the last set is compact, we deduce that $\overline{B^{W_i}_{2^{-i}}(w)}$ is also compact and from the arbitrariness of w we conclude. □

5.4.3. Construction of the pseudo-distances: ρ_i. For every $w_1, w_2 \in W_i$ we set

$$\rho_i(w_1, w_2) := \begin{cases} \mathsf{d}_i(w_1, w_2) & \text{if } \mathsf{d}_i(w_1, w_2) \le 2 \cdot 2^{-i}, \\ \mathsf{d}(f_i(w_1), f_i(w_2)) & \text{if } \mathsf{d}_i(w_1, w_2) > 2 \cdot 2^{-i}. \end{cases} \tag{5.4.16}$$

Lemma 5.4.19. *It holds that*

$$\rho_i(w_1, w_2) = \mathsf{d}_i(w_1, w_2),$$

whenever $\rho_i(w_1, w_2) \le 2^{-i}$ or $\mathsf{d}_i(w_1, w_2) \le 2^{-i}$.

Proof. If $\mathsf{d}_i(w_1, w_2) \le 2^{-i}$ the statement is trivial from the definition of ρ_i. So suppose $\rho_i(w_1, w_2) \le 2^{-i}$. If $\rho_i(w_1, w_2) = \mathsf{d}_i(w_1, w_2)$ we are done, so suppose $\rho_i(w_1, w_2) = \mathsf{d}(f_i(w_1), f_i(w_2))$. Then the 2^{-i}-density of X_i implies that $f_i(w_1), f_i(w_2) \in B_{5\cdot 2^{-i}}(x_j)$ for some $j \in J_i$. Hence thanks to Lemma 5.4.15 we have that $w_1, w_2 \in B^i_j$. Applying now (5.4.14) we obtain $\mathsf{d}_i(w_1, w_2) \le 2^{-i} + C\varepsilon_i 2^{-i} \le 2 \cdot 2^{-i}$, but then from (5.4.16) we deduce $\rho_i(w_1, w_2) = \mathsf{d}_i(w_1, w_2)$.

□

5.4.4. Construction of the manifold-to-manifold mappings: h_i. This section is devoted to to proof the following Lemma.

Lemma 5.4.20. *Under the initial assumptions there exists a $1+C(\varepsilon_i+\varepsilon_{i+1})$-bi-Lipschitz diffeomorphism $h_i : W_i \to W^n_{i+1}$ such that*

$$\mathsf{d}(f_{i+1}(h_i(w)), f_i(w)) \le C2^{-i}(\varepsilon_i + \varepsilon_{i+1}), \tag{5.4.17}$$

for every $w \in W_i^n$.

Along this section $(\psi_{i+1,j})_{j\in J_{i+1}}$ denote the charts for the manifold W_{i+1} as defined in Section 5.4.1.

Construction of the new charts for W_i. The idea of the construction of the maps h_i is to build a new atlas for W_i, but instead of using points in X_i, as in its construction, we use the points in X_{i+1}. In particular we will construct charts $(\varphi_{i,j})_{j\in J_{i+1}}$ for W_i in direct relation with the charts $(\psi_{i+1,j})_{j\in J_{i+1}}$ of W_{i+1}. In this way we are able to build maps

$$h_{i,j} := \psi^{-1}_{i+1,j} \circ \varphi^{-1}_{i,j},$$

defined locally, from W_i to W_{i+1}. Then we will patch all these maps using Theorem 5.3.14.

For any $j \in J_{i+1}$ define $k(j) \in J_i$ as the minimal index so that $x_{i+1,j} \in B_{2^{-i}}(x_{i,k(j)})$. $k(j)$ is well defined thanks to the fact that X_i is 2^{-i}-dense. Consider the maps

$$\alpha_{i+1,j} : B_{100\cdot 2^{-i}}(0) \to B_{100\cdot 2^{-i}}(x_{i+1,j})$$

$$\beta_{i,k(j)} : B_{200\cdot 2^{-i}}(x_{i,k(j)}) \to B_{200\cdot 2^{-i}}(0),$$

which are respectively a $C\varepsilon_{i+1}2^{-i}$-GH approximation and a $C\varepsilon_i 2^{-i}$-GH approximation. Analogously we consider the maps $\alpha_{i,k(j)}, \beta_{i+1,j}$. It is crucial here that all these maps are the same considered in Section 5.4.1. Since $B_{100\cdot 2^{-i}}(x_{i+1,j}) \subset B_{200\cdot 2^{-i}}(x_{i,k(j)})$ we have that $\beta_{i,k(j)} \circ \alpha_{i+1,j}$ is well defined on the whole $B_{100\cdot 2^{-i}}(0)$ ad it is a $2^{-i}C(\varepsilon_i + \varepsilon_{i+1})$ isometry.

Lemma 5.4.21. *For every $j \in J_{i+1}$ there exists an isometry $K_{i,j} : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$|K_{i,j} - \beta_{i,k(j)}(\alpha_{i+1,j})| \le C(\varepsilon_i + \varepsilon_{i+1})2^{-i}, \tag{5.4.18}$$

in $B_{15\cdot 2^{-i}}(0)$ and

$$|K^{-1}_{i,j} - \beta_{i+1,j}(\alpha_{i,k(j)})| \le C(\varepsilon_i + \varepsilon_{i+1})2^{-i}, \tag{5.4.19}$$

in $K_{i,j}(B_{15\cdot 2^{-i}}(0))$.

Proof. The existence of an isometry $K_{i,j} : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$|K_{i,j} - \beta_{i,k(j)}(\alpha_{i+1,j})| \le C(\varepsilon_i + \varepsilon_{i+1})2^{-i} \tag{5.4.20}$$

in $B_{20\cdot 2^{-i}}(0)$ is an immediate consequence of Lemma 5.1.11, provided $\varepsilon_i, \varepsilon_{i+1}$ are small enough. To prove (5.4.19) take any $x \in B_{15\cdot 2^{-i}}(0)$, then using (5.4.20)

$$\begin{aligned}|K^{-1}_{i,j}(K_{i,j}(x)) - \beta_{i+1,j}(\alpha_{i,k(j)}(K_{i,j}(x))| &= |x - \beta_{i+1,j}(\alpha_{i,k(j)}(K_{i,j}(x))| \\ &\le |x - \beta_{i+1,j}(\alpha_{i,k(j)}(\beta_{i,k(j)}(\alpha_{i+1,j}(x)))| + C(\varepsilon_i + \varepsilon_{i+1})2^{-i} \\ &\le C(\varepsilon_i + \varepsilon_{i+1})2^{-i},\end{aligned}$$

where in the last inequality we used (5.4.1). This would prove (5.4.19), however these computations needs justification. More precisely we need to check that $\beta_{i+1,j} \circ \alpha_{i,k(j)}$ is defined on $\beta_{i,k(j)}(\alpha_{i+1,j}(B_{15\cdot 2^{-i}}(0)))$ and on $K_{i,j}(B_{15\cdot 2^{-i}}(0))$. Since, as remarked previously $\beta_{i,k(j)} \circ \alpha_{i+1,j}$ is defined on $B_{20\cdot 2^{-i}}(0)$, from (5.4.1) and (5.4.2) we have that

$$\alpha_{i,k(j)}(\beta_{i,k(j)}(\alpha_{i+1,j}(B_{15\cdot 2^{-i}}(0)))) \subset B_{20\cdot 2^{-i}}(x_{i+1,j}), \tag{5.4.21}$$

which is the domain of $\beta_{i+1,j}$. Thus the first condition is verified. For the second we start proving that $\alpha_{i,k(j)}$ is defined on $K_{i,j}(B_{15\cdot 2^{-i}}(0))$. Indeed from (5.4.21) and (5.4.1) we deduce that $\beta_{i,k(j)}(\alpha_{i+1,j}(B_{15\cdot 2^{-i}}(0)) \subset B_{25\cdot 2^{-i}}(0)$. Hence from (5.4.20), provided $\varepsilon_i, \varepsilon_{i+1}$ are small enough, follows that $K_{i,j}(B_{15\cdot 2^{-i}}(0)) \subset B_{30\cdot 2^{-i}}(0)$, that is in the domain of $\alpha_{i,k(j)}$. Now using again (5.4.20), (5.4.21) and assuming $\varepsilon_i, \varepsilon_{i+1}$ small enough we deduce that

$$\alpha_{i,k(j)}(K_{i,j}(B_{15\cdot 2^{-i}}(0))) \subset B_{35\cdot 2^{-i}}(x_{i+1,j}),$$

which is the domain of $\beta_{i+1,j}$. This proves also the second condition and concludes the proof. $\square$

Define for every $j \in J_{i+1}$ the map

$$\varphi_{i,j} := \psi^{-1}_{i,k(j)} \circ K_{i,j}|_{B_{4\cdot 2^{-i}}(0)}.$$

Notice that this is well defined since $K_{i,j}(B_{4\cdot 2^{-i}}(0)) \subset B_{8\cdot 2^{-i}}(0)$, by (5.4.18), assuming $\varepsilon_i, \varepsilon_{i+1}$ small enough. For every $j \in J_{i+1}$ define also the map

$$h_{i,j} := \psi^{-1}_{i+1,j} \circ \varphi^{-1}_{i,j}|_{\varphi_{i,j}(B_{4\cdot 2^{-i}}(0))}. \tag{5.4.22}$$

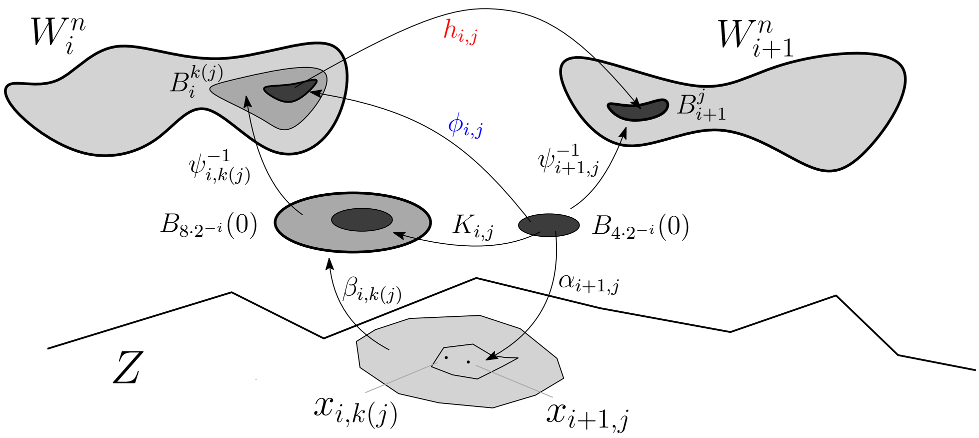

FIGURE 2. Diagram of definition of the maps $\varphi_{i,j}$ and the maps $h_{i,j}$.

The following is the analogous to Lemma 5.4.15 with respect to the new charts, which also imply that the new charts cover the manifold.

Lemma 5.4.22. *Suppose $w \in W_i$ and $x_{i+1,j} \in X_{i+1}$ are such that $\mathsf{d}(f_i(w), x_{i+1,j}) < 2^{-i-1}$, then*

$$w \in \varphi_{i,j}(B_{\frac{5}{4}\cdot 2^{-i}}(0)). \tag{5.4.23}$$

In particular we have

$$W_i^n \subset \bigcup_{j \in J^{i+1}} \varphi_{i,j}(B_{\frac{5}{4}\cdot 2^{-i}}(0)). \tag{5.4.24}$$

PROOF. Observe that $\mathsf{d}(f_i(x), x_{i,k(j)}) < 3/2 \cdot 2^{-i}$. Then by Lemma 5.4.15 we deduce that $x \in B^i_{k(j)}$. Applying now (5.4.12) we have $\mathsf{d}(f_i(w), \alpha_{i,k(j)}(\psi_{i,k(j)}(x))) \le C2^{-i}\varepsilon_i$, that gives $\mathsf{d}(\alpha_{i,k(j)}(\psi_{i,k(j)}(w)), x_{i+1,j}) < 2^{-i}/2 + C\varepsilon_i 2^{-i}$. Then from (5.4.3) $\beta_{i+1,j}(\alpha_{i,k(j)}(\psi_{i,k(j)}(x))) \in B_{2^{-i}}(0)$, provided ε_i is small enough. We can now apply (5.4.19) (we will check the needed hypothesis at the end of the proof) to infer that

$$K^{-1}_{i,j}(\psi_{i,k(j)}(x)) \in B_{\frac{5}{4}\cdot 2^{-i}}(0),$$

provided $\varepsilon_i, \varepsilon_{i+1}$ are small enough. Recalling now the definition of $\varphi_{i,j}$, (5.4.23) follows. Finally (5.4.24) is a consequence of (5.4.23) and the 2^{-i-1}-density of X_{i+1}.
To justify the use of (5.4.19) above, we need to check that $\psi_{i,k(j)}(w) \in K_{i,j}(B_{15\cdot 2^{-i}}(0))$. Call $y := \beta_{i+1,j}(\alpha_{i,k(j)}(\psi_{i,k(j)}(w)))$, $y \in B_{2^{-i}}(0)$, then from (5.4.18)

$$\begin{aligned}
|K_{i,j}(y) - \psi_{i,k(j)}(w)| &\le |\beta_{i,k(j)}(\alpha_{i+1,j}(y)) - \psi_{i,k(j)}(w)| + C(\varepsilon_i + \varepsilon_{i+1})2^{-i} = \\
&= |\beta_{i,k(j)}(\alpha_{i+1,j}(\beta_{i+1,j}(\alpha_{i,k(j)}(\psi_{i,k(j)}(w))))) - \psi_{i,k(j)}(x)| + C(\varepsilon_i + \varepsilon_{i+1})2^{-i} \\
&\overset{(5.4.1)}{\le} C(\varepsilon_i + \varepsilon_{i+1})2^{-i}.
\end{aligned}$$

Now, since $K_{i,j}$ is an isometry and $y \in B_{2^{-i}}(0)$, if $\varepsilon_i, \varepsilon_{i+1}$ are small enough, we conclude that $\psi_{i,k(j)}(w) \in K_{i,j}(B_{2\cdot 2^{-i}}(0))$. $\square$

Construction of the diffeomorphism h_i. Before passing the the actual construction of the map h_i we need a technical lemma. Roughly saying it shows the transition functions relative to the charts $\varphi_{i,j}$ are close to the transition functions $\hat{I}_{i+1,j}$ of the manifold W_{i+1}.

Lemma 5.4.23. *Let $x \in B_{4\cdot 2^{-i}}(0)$ be in the domain of $\varphi_{i,j_1}^{-1} \circ \varphi_{i,j_2}$ for some $j_1, j_2 \in J_{i+1}$, then*

$$|\varphi_{i,j_1}^{-1} \circ \varphi_{i,j_2}(x) - \hat{I}_{i+1,j_1,j_2}(x)| \le C(\varepsilon_i + \varepsilon_{i+1})2^{-i}. \tag{5.4.25}$$

Proof. Consider a point $x \in B_{4\cdot 2^{-i}}(0)$ and two indices $j_1, j_2 \in J_{i+1}$ as in the hypotheses. We start by claiming that

$$\mathsf{d}(x_{i+1,j_1}, x_{i+1,j_2}) \le 9 \cdot 2^{-i}. \tag{5.4.26}$$

Indeed $\varphi_{i,j_2}(x) = \varphi_{i,j_1}(y)$ for some $y \in B_{4\cdot 2^{-i}}(0)$. Then from (5.4.12) we have

$$\mathsf{d}(\alpha_{i,k(j_2)}(\psi_{i,k(j_2)}(\varphi_{i,j_2}(x))), \alpha_{i,k(j_1)}(\psi_{i,k(j_1)}(\varphi_{i,j_1}(y))) \le 2^{-i}C\varepsilon_i,$$

that by definition can be written as

$$\mathsf{d}(\alpha_{i,k(j_2)}(K_{i,j_2}(x))), \alpha_{i,k(j_1)}(K_{i,j_1}(y))) \le 2^{-i}C\varepsilon_i,$$

Applying now (5.4.18) (recalling also (5.4.1) as usual) we deduce

$$\mathsf{d}(\alpha_{i+1,j_2}(x), \alpha_{i+1,j_1}(y)) \le 2^{-i}C(\varepsilon_i + \varepsilon_{i+1}). \tag{5.4.27}$$

Therefore from (5.4.3), if $\varepsilon_i, \varepsilon_{i+1}$ are small enough, we deduce (5.4.26). We can now proceed with the proof of (5.4.25). From the definitions we have $\varphi_{i,j_1}^{-1} \circ \varphi_{i,j_2}(x) = K_{i,j_1}^{-1} \circ \hat{I}_{i,k(j_1),k(j_2)} \circ K_{i,j_2}(x)$. From (5.4.8)

$$|\varphi_{i,j_1}^{-1} \circ \varphi_{i,j_2}(x) - K_{i,j_1}^{-1} \circ I_{i,k(j_1),k(j_2)} \circ K_{i,j_2}(x)| \le C\varepsilon_i 2^{-i}.$$

Moreover, since $x \in B_{4\cdot 2^{-i}}(0)$, from (5.4.18)

$$|\varphi_{i,j_1}^{-1} \circ \varphi_{i,j_2}(x) - K_{i,j_1}^{-1} \circ I_{i,k(j_1),k(j_2)} \circ \beta_{i,k(j_2)} \circ \alpha_{i+1,j_2}(x)| \le C(\varepsilon_i + \varepsilon_{i+1})2^{-i}.$$

Observe now that, from (5.4.3) and the fact that $\mathsf{d}(x_{i+1,j_2}, x_{i,k(j_2)}) \le 2^{-i}$, $\beta_{i,k(j_2)}(\alpha_{i+1,j_2}(x))$ belongs to $B_{10\cdot 2^{-i}}(0)$ (if $\varepsilon_i, \varepsilon_{i+1}$ are small enough), thus we can apply (5.4.4) to obtain

$$|\varphi_{i,j_1}^{-1} \circ \varphi_{i,j_2}(x) - K_{i,j_1}^{-1} \circ \beta_{i,k(j_1)} \circ \alpha_{i,k(j_2)} \circ \beta_{i,k(j_2)} \circ \alpha_{i+1,j_2}(x)| \le C(\varepsilon_i + \varepsilon_{i+1})2^{-i}.$$

Recall now from (5.4.1) that $\alpha_{i,k(j_2)} \circ \beta_{i,k(j_2)}$ is almost the identity, therefore

$$|\varphi_{i,j_1}^{-1} \circ \varphi_{i,j_2}(x) - K_{i,j_1}^{-1} \circ \beta_{i,k(j_1)} \circ \alpha_{i+1,j_2}(x)| \le C(\varepsilon_i + \varepsilon_{i+1})2^{-i}.$$

To write the above we should check that $\alpha_{i+1,j_2}(x)$ is in the domain of $\beta_{i,k(j_1)}$, this follows from the fact that $\alpha_{i+1,j_2}(x) \in B_{5\cdot 2^{-i}}(x_{i+1,j_2}) \subset B_{15\cdot 2^{-i}}(x_{i,k(j_1)})$, where the inclusion is a consequence of (5.4.26) and the fact that $\mathsf{d}(x_{i+1,j_1}, x_{i,k(j_1)}) \le 2^{-i}$. We now apply (5.4.19) to get

$$|\varphi_{i,j_1}^{-1} \circ \varphi_{i,j_2}(x) - \beta_{i+1,j_1} \circ \alpha_{i,k(j_1)} \circ \beta_{i,k(j_1)} \circ \alpha_{i+1,j_2}(x)| \le C(\varepsilon_i + \varepsilon_{i+1})2^{-i}. \tag{5.4.28}$$

Observe that to use (5.4.19) we need to check that $\beta_{i,k(j_1)} \circ \alpha_{i+1,j_2}(x) \in K_{i,j_1}(B_{15\cdot 2^{-i}}(0))$. To see this, we can use (5.4.27) and then (5.4.18) to deduce

$$\begin{aligned}|\beta_{i,k(j_1)} \circ \alpha_{i+1,j_2}(x) - K_{i,j_1}(y)| &\le |\beta_{i,k(j_1)} \circ \alpha_{i+1,j_1}(y) - K_{i,j_1}(y)| + 2^{-i}C(\varepsilon_i + \varepsilon_{i+1}) \\ &\le 2^{-i}C(\varepsilon_i + \varepsilon_{i+1}).\end{aligned}$$

From this, since K_{i,j_1} is a global isometry and $y \in B_{4\cdot 2^{-i}}(0)$, we have $\beta_{i,k(j_1)} \circ \alpha_{i+1,j_2}(x) \in K_{i,j_1}(B_{8\cdot 2^{-i}}(0))$, provided $\varepsilon_i, \varepsilon_{i+1}$ small enough, that is what we wanted. Using once again (5.4.1) in (5.4.28) we obtain that

$$|\varphi_{i,j_1}^{-1} \circ \varphi_{i,j_2}(x) - \beta_{i+1,j_1} \circ \alpha_{i+1,j_2}(x)| \le C(\varepsilon_i + \varepsilon_{i+1})2^{-i},$$

that makes sense since $\alpha_{i+1,j_2}(x) \in B_{14\cdot 2^{-i}}(x_{i+1,j_1})$, which follows from (5.4.26). Finally combining (5.4.4) and (5.4.8) we reach (5.4.25).

□

Lemma 5.4.24. *There exists a map $h_i : W_i^n \to W_{i+1}^n$ that is surjective and*

$$\frac{1}{1 + C(\varepsilon_i + \varepsilon_{i+1})} \le \|Dh_i\| \le 1 + C(\varepsilon_i + \varepsilon_{i+1}). \tag{5.4.29}$$

Moreover for every $j \in J_{i+1}$ it holds that

$$h_i|_{\varphi_{i,j}(B_{2\cdot 2^{-i}}(0))} = \psi_{i+1,j}^{-1} \circ H \circ \varphi_{i,j}^{-1}, \tag{5.4.30}$$

for some diffeomorphism $H : B_{4\cdot 2^{-i}}(0) \to B_{4\cdot 2^{-i}}(0)$ (depending on j) such that $\|H - \mathsf{id}\|_{C^1, 2^{-i}} \le C(\varepsilon_i + \varepsilon_{i+1})$.

PROOF. We plan to apply Theorem 5.3.14 for every i, with $M = W_i^n, X = W_{i+1}^n$, $\varphi_j = \varphi_{i,j}, h_j = h_{i,j}$ (recall their definition in (5.4.22)),$t = 2\cdot 2^{-i}$ and $N = N(n)$ (given in Lemma 5.4.4). We need to check the hypotheses of the Theorem. First notice that $\varphi_{i,j_1}^{-1} \circ \varphi_{i,j_2} = K_{i,j_1}^{-1} \circ \hat{I}_{i,k(j_1),k(j_2)} \circ K_{i,j_2}$, hence from (5.4.8) and the fact that K_{i,j_1} are isometries we deduce that $\|D(\varphi_{i,j_1}^{-1} \circ \varphi_{i,j_2})\| \leq 1 + C\varepsilon_i$ and $|\partial_{ij}(\varphi_{i,j_1}^{-1} \circ \varphi_{i,j_2})_k| \leq C\varepsilon_i/2^{-i}$. Hence (5.3.29) is satisfied. Moreover the required partition of the indices $j \in J_{i+1}$ is naturally induced from the partition $J_{i+1,1}, ..., J_{i+1,N_{i+1}}$. Indeed if $\varphi_{i,j_1}(B_{4\cdot 2^{-i}}(0)) \cap \varphi_{i,j_2}(B_{4\cdot 2^{-i}}(0)) \neq \emptyset$, thanks to Lemma 5.4.23, we have that $B_{j_1}^{i+1} \cap B_{j_2}^{i+1} \neq \emptyset$, therefore the transition function $\hat{I}_{i+1,j_1,j_2}$ exists, and therefore by Remark 5.4.7 j_1, j_2 belong to different sets of the partition. Moreover the required condition that

$$W_i^n \subset \bigcup_{j\in J^{i+1}} \varphi_{i,j}(B_{2\cdot 2^{-i}}(0)). \tag{5.4.31}$$

follows from Lemma 5.4.22.

Finally we need to prove (5.3.30) and (5.3.31). It suffices to prove the following. Define $B_2 = B_{2(2-2/N)2^{-i}}(0)$ and $B_1 = B_{2(2-1/N)2^{-i}}(0)$, then for every j_1, j_2 such that $\varphi_{i,j_1}(B_2) \cap \varphi_{i,j_2}(B_2) \neq \emptyset$, it holds that

$$h_{i,j_1}(\varphi_{i,j_1}(B_1) \cap \varphi_{i,j_2}(B_1) \subset h_{i,j_2}(\varphi_{i,j_2}(B_{4\cdot 2^{-i}}(0)) \tag{5.4.32}$$

and moreover for such j_1, j_2

$$\|\varphi_{i,j_2}^{-1} \circ h_{i,j_2}^{-1} \circ h_{i,j_1} \circ \varphi_{i,j_2} - \mathsf{id}\|_{C^1,2^{-i}} \leq C\varepsilon_i \tag{5.4.33}$$

on $\varphi_{i,j_2}^{-1}(\varphi_{i,j_1}(B_1) \cap \varphi_{i,j_2}(B_1))$. We start with (5.4.32). Suppose

$$\varphi_{i,j_1}(B_2) \cap \varphi_{i,j_2}(B_2) \neq \emptyset$$

and pick $x \in \varphi_{i,j_1}(B_1) \cap \varphi_{i,j_2}(B_1)$. Then $x = \varphi_{i,j_2}(y)$ for some $y \in B_1$. Thus from (5.4.25)

$$|\varphi_{i,j_1}^{-1} \circ \varphi_{i,j_2}(y) - \hat{I}_{i+1,j_1,j_2}(y)| \leq C(\varepsilon_i + \varepsilon_{i+1})2^{-i}.$$

Then applying to the above inequality the map ψ_{i+1,j_1}^{-1}, that is 2-Lipschitz (if ε_{i+1} is small enough), we have that

$$\mathsf{d}_{i+1}(h_{j_1}(x), \psi_{i+1,j_2}^{-1}(y)) = \mathsf{d}_{i+1}(\psi_{i+1,j_1}^{-1} \circ \varphi_{i,j_1}^{-1} \circ \varphi_{i,j_2}(y), \psi_{i+1,j_2}^{-1}(y)) \leq C(\varepsilon_i + \varepsilon_{i+1})2^{-i}.$$

Therefore, since $y \in B_{2(2-1/N)\cdot 2^{-i}}(0)$, from Remark 5.4.12, assuming $\varepsilon_i, \varepsilon_{i+1}$ are small enough, we have

$$h_{j_1}(x) \in \psi_{i+1,j_2}^{-1}(B_{4\cdot 2^{-i}}(0)) = h_{i,j_2}(\varphi_{i,j_2}(B_{4\cdot 2^{-i}}(0)),$$

that proves (5.4.32). It remains only to prove (5.4.33). Notice that from the definitions

$$\varphi_{i,j_2}^{-1} \circ h_{i,j_2}^{-1} \circ h_{i,j_1} \circ \varphi_{i,j_2} = \hat{I}_{i+1,j_2,j_1} \circ \varphi_{i,j_1}^{-1} \circ \varphi_{i,j_2}.$$

Therefore from (5.4.25)

$$|\varphi_{i,j_2}^{-1} \circ h_{i,j_2}^{-1} \circ h_{i,j_1} \circ \varphi_{i,j_2} - \mathsf{id}| \leq C(\varepsilon_i + \varepsilon_{i+1})2^{-i},$$

on $\varphi_{i,j_2}^{-1}(\varphi_{i,j_1}(B_1) \cap \varphi_{i,j_2}(B_1))$. We need now the bound on the first derivatives. To this aim notice that we can also write

$$\varphi_{i,j_2}^{-1} \circ h_{i,j_2}^{-1} \circ h_{i,j_1} \circ \varphi_{i,j_2} = \hat{I}_{i+1,j_1,j_2} \circ K_{j_1}^{-1} \circ \hat{I}_{i,k(j_1),k(j_2)} \circ K_{j_2}$$

and from what we just proved combined with (5.4.8) we deduce

$$|I_{i+1,j_1,j_2} \circ K_{j_1}^{-1} \circ I_{i,k(j_1),k(j_2)} \circ K_{j_2} - \mathsf{id}| \leq C(\varepsilon_i + \varepsilon_{i+1})2^{-i}$$

on $\varphi_{i,j_2}^{-1}(\varphi_{i,j_1}(B_1) \cap \varphi_{i,j_2}(B_1))$. Exploiting Remark 5.4.12 (and the fact that every map $\varphi_{i,j}$ is the composition of $\psi_{i,k(j)}^{-1}$ with an isometry) we can observe that, since $\varphi_{i,j_1}(B_2) \cap \varphi_{i,j_2}(B_2) \neq \emptyset$, the (bigger) set $\varphi_{i,j_1}(B_1) \cap \varphi_{i,j_2}(B_1)$ contains a metric ball in W_i of radius $\frac{2^{-i}}{100N}$, provided ε_i is small enough. Then, again by Remark 5.4.12 we deduce that $\varphi_{i,j_2}^{-1}(\varphi_{i,j_1}(B_1) \cap \varphi_{i,j_2}(B_1))$ contains an Euclidean ball of radius $\frac{2^{-i}}{200N}$, again if ε_{i+1} is small enough. Therefore, since the above map is an isometry, from Lemma 5.3.13 we deduce

$$\|D(I_{i+1,j_1,j_2} \circ K_{j_1}^{-1} \circ I_{i,k(j_1),k(j_2)} \circ K_{j_2}) - I_n\| \leq C(\varepsilon_i + \varepsilon_{i+1})$$

on $\varphi_{i,j_2}^{-1}(\varphi_{i,j_1}(B_1) \cap \varphi_{i,j_2}(B_1))$, that implies using (5.4.8) and Lemma 5.3.12

$$\|D(\hat{I}_{i+1,j_1,j_2} \circ K_{j_1}^{-1} \circ \hat{I}_{i,k(j_1),k(j_2)} \circ K_{j_2}) - I_n\| \leq C(\varepsilon_i + \varepsilon_{i+1})$$

on $\varphi_{i,j_2}^{-1}(\varphi_{i,j_1}(B_1)\cap\varphi_{i,j_2}(B_1))$. This completes the proof of (5.4.33). Therefore if $\varepsilon_i, \varepsilon_{i+1}$ are small enough we can apply Theorem 5.3.14 and get a map $h_i : W_i^n \to W_{i+1}^n$. Thanks to (5.3.32) this map has the property that for every $j \in J_{i+1}$ it holds

$$h_i|_{\varphi_{i,j}(B_{2\cdot 2^{-i}}(0))} = \psi_{i+1,j}^{-1} \circ \varphi_{i,j}^{-1} \circ \varphi_{i,j} \circ H \circ \varphi_{i,j}^{-1} = \psi_{i+1,j}^{-1} \circ H \circ \varphi_{i,j}^{-1}, \tag{5.4.34}$$

for some diffeomorphism $H : B_{2\cdot 2^{-i}}(0) \to B_{2\cdot 2^{-i}}(0)$ (depending on j) with $\|H - \mathsf{id}\|_{C^1,2^{-i}} \le C(\varepsilon_i + \varepsilon_{i+1})$. Thus (5.4.30) is proved. Moreover from this and (5.4.11) we also get $(1 + C(\varepsilon_i + \varepsilon_{i+1}))^{-1} \le \|Dh_i\| \le 1 + C(\varepsilon_i + \varepsilon_{i+1})$. Finally we observe that $h_i(W_i^n) = W_{i+1}^n$ for every i. Indeed $h_i|_{\varphi_{i,j}(B_{2\cdot 2^{-i}}(0))} = \psi_{i+1,j}^{-1} \circ H \circ \varphi_{i,j}^{-1}$, moreover, since $\|H - \mathsf{id}\|_{C^1,t} \le C(\varepsilon_i + \varepsilon_{i+1})$, if $\varepsilon_i, \varepsilon_{i+1}$ are small enough we have $B_{\frac{3}{2}2^{-i}}(0) \subset H(B_{2\cdot 2^{-i}}(0))$. Thus $h_i(\varphi_{i,j}(B_{2\cdot 2^{-i}}(0))) = \psi_{i+1,j}^{-1} \circ H(B_{2\cdot 2^{-i}}(0)) \supset \psi_{i+1,j}^{-1}(B_{\frac{3}{2}2^{-i}}(0))$. Then we conclude by Lemma 5.4.17. □

Lemma 5.4.25. *It holds that*

$$\mathsf{d}(f_{i+1}(h_i(w)), f_i(w)) \le C2^{-i}(\varepsilon_i + \varepsilon_{i+1}), \quad \forall\, w \in W_i^n.$$

Proof. We need to estimate $\mathsf{d}(f_{i+1}(h_i(w)), f_i(w))$. By (5.4.31), there exists an index $j \in J_{i+1}$ such that $w = \varphi_{i,j}(x)$ for some $x \in B_{2\cdot 2^{-i}}(0)$. Since $\varphi_{i,j} = \psi_{i,k(j)} \circ K_{i,j}^{-1}$ we deduce that $w \in B_{k(j)}^i$ and applying (5.4.12) we obtain

$$\mathsf{d}(f_i(w), \alpha_{i,k(j)} \circ \psi_{i,k(j)}(w)) \le C\varepsilon_i 2^{-i}. \tag{5.4.35}$$

We claim now that $h_i(w) \in B_j^{i+1}$. Indeed, since $w \in \varphi_{i,j}(B_{2\cdot 2^{-i}}(0))$ from (5.4.30) and recalling the Lispschitzianity of $\psi_{i+1,j}^{-1}$, we obtain

$$\begin{aligned}\mathsf{d}_{i+1}(h_i(w), h_{i,j}(w)) &= \mathsf{d}_{i+1}(h_i(w), \psi_{i+1,j}^{-1} \circ \varphi_{i,j}^{-1}(w))\\ &\le (1 + C\varepsilon_{i+1})|H(\varphi_{i,j}^{-1}(w)) - \varphi_{i,j}^{-1}(w)| \le C(\varepsilon_i + \varepsilon_{i+1})2^{-i},\end{aligned} \tag{5.4.36}$$

provided ε_{i+1} is small enough. Now by definition $h_{i,j}(w) = \psi_{i+1,j}^{-1}(x) \in \psi_{i+1,j}^{-1}(B_{2\cdot 2^{-i}}(0))$, therefore from (5.4.36) and Remark 5.4.12, if $\varepsilon_i, \varepsilon_{i+1}$ are small enough, we easily deduce that $h_i(w) \in \psi_{i+1,j}^{-1}(B_{4\cdot 2^{-i}}(0)) = B_j^{i+1}$. The claim is proved. In particular we have $h_i(w), h_{i,j}(w) \in B_j^{i+1}$ and combining (5.4.14) with (5.4.36) we find that

$$\mathsf{d}(f_{i+1}(h_i(w)), f_{i+1}(h_{i,j}(w))) \le C2^{-i}(\varepsilon_i + \varepsilon_{i+1}). \tag{5.4.37}$$

Moreover we shall also apply (5.4.12) and deduce

$$\mathsf{d}(f_{i+1}(h_{i,j}(w)), \alpha_{i+1,j}(x)) = \mathsf{d}(f_{i+1}(h_{i,j}(w)), \alpha_{i+1,j} \circ \psi_{i+1,j}(h_{i,j}(w))) \le C\varepsilon_{i+1}2^{-i-1}. \tag{5.4.38}$$

Therefore putting (5.4.38), (5.4.37) and (5.4.35) together we obtain

$$\mathsf{d}(f_{i+1}(h_i(w)), f_i(w)) \le C2^{-i}(\varepsilon_{i+1} + \varepsilon_i) + \mathsf{d}(\alpha_{i,k(j)} \circ \psi_{i,k(j)}(w), \alpha_{i+1,j}(x)).$$

However $w = \varphi_{i,j}(x) = \psi_{i,k(j)}^{-1}(K_{i,j}(x))$, therefore recalling (5.4.18) and (5.4.4)

$$\begin{aligned}\mathsf{d}(f_{i+1}(h_i(w)), f_i(w)) &\le C2^{-i}(\varepsilon_{i+1} + \varepsilon_i) + \mathsf{d}(\alpha_{i,k(j)}(K_{i,j}(x)), \alpha_{i+1,j}(x))\\ &\le C2^{-i}(\varepsilon_{i+1} + \varepsilon_i) + \mathsf{d}(\alpha_{i,k(j)}(\beta_{i,k(j)}(\alpha_{i+1,j}(x))), \alpha_{i+1,j}(x))\\ &\le C2^{-i}(\varepsilon_{i+1} + \varepsilon_i).\end{aligned}$$

□

Proof of Lemma 5.4.20. We already know that h_i is surjective and the estimate in (5.4.29), hence we only need to prove that h_i is injective. Suppose by contradiction that exist two distinct points $w_1, w_2 \in W_i^n$ such that $h_i(w_1) = h_i(w_2)$. First observe that from (5.4.30) we have that $h_i|_{\varphi_{i,j}(B_{2\cdot 2^{-i}}(0))}$ is injective for every $j \in J_{i+1}$. Therefore we cannot have $w_1, w_2 \in \varphi_{i,j}(B_{2\cdot 2^{-i}}(0))$ for any $j \in J_{i+1}$. We claim that this implies

$$\mathsf{d}_i(w_1, w_2) > \frac{1}{4}2^{-i}. \tag{5.4.39}$$

Indeed suppose the contrary. From (5.4.24) we have that $w_1 \in \varphi_{i,j}(B_{\frac{5}{4}\cdot 2^{-i}}(0))$ for some $j \in J_{i+1}$. From definition we have $\varphi_{i,j} = \psi_{i,k(j)}^{-1} \circ K_{i,j}$, where $K_{i,j}$ is an isometry, therefore thanks to Remark 5.4.12 we deduce that

$$\varphi_{i,j}(B_{\frac{5}{4}\cdot 2^{-i}}(0)) \subset B_{\frac{3}{2}\cdot 2^{-i}}^{W_i}(\varphi_{i,j}(0)).$$

In particular $w_2 \in B^{W_i}_{\frac{7}{4}\cdot 2^{-i}}(\varphi_{i,j}(0))$, however again from Remark 5.4.12 we have

$$B^{W_i}_{\frac{7}{4}\cdot 2^{-i}}(\varphi_{i,j}(0)) \subset \varphi_{i,j}(B_{2\cdot 2^{-i}}(0)),$$

but this is a contradiction and (5.4.39) is proved. Observe now that, since $h_i(w_1) = h_i(w_2)$, from Lemma 5.4.25, provided $\varepsilon_i, \varepsilon_{i+1}$ small enough, we have

$$\mathsf{d}(f_i(w_1), f_i(w_2)) \le \frac{1}{8} 2^{-i}. \tag{5.4.40}$$

This, together with the 2^{-i}-density of X_i implies that $f_i(w_1), f_i(w_2) \in B_{2\cdot 2^{-i}}(x_j)$ for some $j \in J_i$. Therefore Lemma 5.4.15 gives that $w_1, w_2 \in B^i_{j_1}$, hence we are in position to apply (5.4.14), that coupled with (5.4.40) provides

$$\mathsf{d}_i(w_1, w_2) \le \frac{1}{8} 2^{-i} + C\varepsilon_i 2^{-i},$$

which, if ε_i is sufficiently small, contradicts (5.4.39). □

5.4.5. Final verifications and conclusion. To prove Theorem 5.2.2 it remains to show that all the objects that we built on the previous sections satisfy the requirements 1, 2, 3, 4 and 5 stated in Section 5.2.2. Almost all the verifications are straightforward:

1- It is the content of Lemma 5.4.19.

3- If $\mathsf{d}_i(w_1, w_2) > 2 \cdot 2^{-i}$ we are done by (5.4.16). Suppose $\mathsf{d}_i(w_1, w_2) \le 2 \cdot 2^{-i}$. Then observe that from (5.4.15) $w_1 \in B_{3\cdot 2^{-i}}(\psi^{-i}_{i,j}(0))$ for some j and thus $w_1, w_2 \in B_{6\cdot 2^{-i}}(\psi^{-i}_{i,j}(0)) \subset B^i_j$ from Remark 5.4.12. Then we can apply (5.4.14) and conclude.

4- This is 5.4.17.

2- Suppose first that $\rho_{i+1}(h_i(w_1), h_i(w_2)) \le 2 \cdot 2^{-(i+1)}$ and $\rho_i(w_1, w_2) \le 2 \cdot 2^{-i}$, and observe that by Lemma 5.4.19 $\rho_{i+1}(h_i(w_1), h_i(w_2)) = \mathsf{d}_{i+1}(h_i(w_1), h_i(w_2))$ and $\rho_i(w_1, w_2) = d_i(w_1, w_2)$. Thus we conclude by the fact that h_i is $1 + C(\varepsilon_i + \varepsilon_{i+1})$ bi-Lipschitz, which comes from Lemma 5.4.20. Suppose now that $\rho_{i+1}(h_i(w_1), h_i(w_2)) > 2 \cdot 2^{-(i+1)}$. By 3 and 4 using triangle inequality we get

$$|\rho_{i+1}(h_i(w_1), h_i(w_2)) - \rho_i(w_1, w_2)| \le C 2^{-i}(\varepsilon_i + \varepsilon_{i+1}).$$

Then dividing by $\rho_{i+1}(h_i(w_1), h_i(w_2))$ follows that

$$\left| 1 - \frac{\rho_i(w_1, w_2)}{\rho_{i+1}(h_i(w_1), h_i(w_2))} \right| \le C(\varepsilon_i + \varepsilon_{i+1})$$

that is what we wanted. The case $\rho_i(w_1, w_2) > 2^{-i}$ is analogous.

5- Take any $x_{i,j} \in X_i$ and take any $w \in B^i_j$, then from (5.4.3) $\alpha_{i,j}(\psi_{i,j}(w)) \in B_{9\cdot 2^{-i}}(x_{i,j})$, if ε_i is sufficiently small. Moreover from (5.4.12) we have $\mathsf{d}(f_i(w), \alpha_{i,j}(\psi_{i,j}(w))) \le C 2^{-i}\varepsilon_i$, therefore, if ε_i is small enough we conclude from the 2^{-i}-density of X_i.

5.5. Comparison between Reifenberg's numbers "e" and Gromov-Huasdorff numbers "ε"

5.5.1. Definitions and notations. Here we define several quantities that measure the flatness of a subsets of the Euclidean space. Basic relations between these quantities will be listed in the following section.

We will assume $S \subset \mathbb{R}^d$ and $n \in \mathbb{N}$ with $n < d$ to be fixed. Moreover we will denote by d_H the Hausdorff distance between sets in $\mathbb{R}^d$ and we will denote by $\mathsf{d}(x, \Gamma)$ the distance between a point $x \in \mathbb{R}^d$ and an affine plane Γ in $\mathbb{R}^d$.

We start with the classical quantities appearing in the Reifenberg's theorem:

Definition 5.5.1 (Reifenberg's e-numbers)**.**

$$\mathsf{e}(x, r) := r^{-1} \inf_{\Gamma} \mathsf{d}_H(S \cap B_r(x), \Gamma \cap B_r(x)), \quad \text{for every } r > 0 \text{ and } x \in S, \tag{5.5.1}$$

where the infimum is taken among all the n-dimensional affine planes Γ containing x.

Then we define the metric-counterpart of the above numbers, where the Husdorff distance from planes is replaced with the more abstract Gromov-Hausdorff distance.

Definition 5.5.2 (Gromov-Hausdorff ε-numbers)**.**

$$\varepsilon(x, r) := r^{-1} \mathsf{d}_{GH}(B^{(S, \mathsf{d}_{Eucl})}_r(x), B^{\mathbb{R}^n}_r(0)), \quad \text{for every } r > 0 \text{ and } x \in S,$$

where (S, d_{Eucl}) is the metric space obtained by endowing S with the Euclidean distance (see Section 5.1 for the definition of d_{GH}).

Next we pass to the one-sided versions of the numbers e's, that is to say that we measure how well the set is approximated by planes, but we ignore how well the plane is approximated by the set.

Definition 5.5.3 (Jones' b-numbers)**.**

$$\mathsf{b}(r,x) := r^{-1} \inf_{\Gamma} \sup_{y \in S \cap B_r(x)} \mathsf{d}(y,\Gamma), \quad \text{for every } r>0 \text{ and } x \in S, \tag{5.5.2}$$

where the infimum is taken among all the n-dimensional affine planes Γ containing x .

The numbers $\mathsf{b}(x,r)$ are usually refer to as L^∞*-Jones' numbers* (see for example [**94**], [**45**] and [**150**]). Finally we define a metric version of the Jones's b-numbers.

Definition 5.5.4 (Metric Jones' β-numbers)**.**

$$\beta(r,x) := r^{-1} \inf \Big\{ \delta \ : \ \text{there exists a } \delta\text{-isometry } f : (S \cap B_r^{\mathbb{R}^d}(x), \mathsf{d}_{Eucl}) \to (B_r^{\mathbb{R}^n}(0), \mathsf{d}_{Eucl}) \Big\}$$

(see Section 5.1 for the definition of δ-isometry).

Remark 5.5.5. It is clear that in the case of a set $S \subset \mathbb{R}^d$ the metric-numbers $\varepsilon(x,r)$ and $\beta(x,r)$ defined above appear overcomplicated, since the approximation by planes is more explicit and natural. However the main features of the metric version of the above quantities is that they can be immediately generalized to an arbitrary metric space. Our goal is here precisely to investigate the behaviour of these abstract quantities in relation to the more classical approach via comparison by planes, in order to understand better how to translate statements from the Euclidean setting to the metric setting. ■

5.5.2. Preliminary observations. In this section we will first show some elementary comparisons between the quantities introduced above. Then we will present an important example to give an intuition on why the metric quantities sometimes behave as the square of the one computed via plane-comparison.

We start observing that the Gromov-Hausdorff numbers are always smaller than the Reifenberg's numbers:

$$\varepsilon(x,r) \le \mathsf{e}(x,r), \quad \text{for every } r>0 \text{ and } x \in S. \tag{5.5.3}$$

Indeed the inclusions $S \cap B_r(x) \hookrightarrow \mathbb{R}^d$, $\Gamma \cap B_r(x) \hookrightarrow \mathbb{R}^d$, with Γ affine n-dimensional plane, are particular cases of the isometric embedding of the metric spaces $(S \cap B_r(x), \mathsf{d}_{Eucl})$, $(B_r^{\mathbb{R}^n}(0), \mathsf{d}_{Eucl})$ into a third metric space (Z,d). Therefore (5.5.3) follows directly from the definitions. Analogously we can show that

$$\beta(x,r) \le \mathsf{b}(x,r), \quad \text{for every } r>0 \text{ and } x \in S \ . \tag{5.5.4}$$

To see this fix $\delta>0$, let Γ be a plane such that $\sup_{y \in S \cap B_r(x)} \mathsf{d}(y,\Gamma) < r(\mathsf{b}(x,r)+\delta)$ and consider $f : S \cap B_r(x) \to \Gamma$ the orthogonal projection onto Γ. Then by the Pythagorean theorem we have

$$\begin{aligned} 0 &\le (|y-z| - |f(y)-f(z)|)(|y-z| + |f(y)-f(z)|) = |y-z|^2 - |f(y)-f(z)|^2 \\ &\le r^2(\mathsf{b}(x,r)+\delta)^2, \quad \forall\, z,y \in S \cap B_r(x), \end{aligned}$$

hence if $|y-z| \ge r\mathsf{b}(x,r)$ we obtain $||y-z|-|f(y)-f(z)|| \le r(\mathsf{b}(x,r)+\delta)^2/\mathsf{b}(x,r)$. On the other hand if $|y-z| \le r\mathsf{b}(x,r)$ we obviously have $||y-z|-|f(y)-f(z)|| \le r\mathsf{b}(x,r)$. From the arbitrariness of $\delta>0$ this shows (5.5.4).

It is also easy to see that (5.5.3) and (5.5.4) are sharp (up to constants). Indeed the following example shows that we can have

$$\mathsf{e}(x,r) \le 4\varepsilon(x,r), \quad \mathsf{b}(x,r) \le 4\beta(x,r)$$

with $\mathsf{b}(x,r), \mathsf{e}(x,r) \neq 0$ and arbitrary small.

Example 5.5.6. Fix $0<\varepsilon<1$. Let $S \subset B_1(0) \subset \mathbb{R}^2$ defined as $S := \big(\{(0,t) \mid t \le 0\} \cup \{(t,\varepsilon) \mid t \ge 0\}\big) \cap B_1(0)$ as in Figure 3. Then it is immediate to see that

$$\mathsf{b}(O,1), \mathsf{e}(O,1) \le \varepsilon,$$

by taking for example as 1-dimensional plane Γ the x-axis. Let now $f : S \to (-1,1)$ be a δ-isometry for some $\delta>0$. Define $P := (-1,0)$, $Q := (0,\varepsilon)$, $R := \{(t,\varepsilon) \mid t \ge 0\} \cap \partial B_1(0)$ and $O := (0,0)$ (see Figure 3). Define the points $x_1 := f(P)$, $x_2 := f(O)$, $x_3 := f(Q)$, $x_4 := f(R)$ (P and R are not really in the domain of S, but there exist points in its domain arbitrary close to them). Then $|x_1-x_2| \ge 1-\delta$, $|x_2-x_3| \ge \varepsilon-\delta$, $|x_3-x_4| \ge 1-\delta$ and $|x_1-x_4| \ge 2-4\varepsilon^2-\delta$ if ε is small enough (indeed $|P-R|^2 \ge 2-8\varepsilon^2$ if ε small enough). Moreover $x_i \in (-1,1)$ for every $i=1,2,3,4$. These conditions together can be easily seen to imply that $\delta \ge \varepsilon/4$. Therefore $\beta(O,1)$, $\varepsilon(O,1) \ge \varepsilon/4$. ■

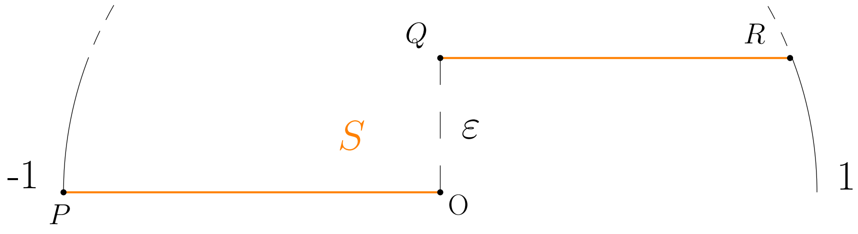

FIGURE 3.

Even if (5.5.3) and (5.5.4) cannot be improved in general, there are important non-trivial examples satisfying

$$\varepsilon(x,r) \leq 4\mathsf{e}(x,r)^2, \quad \beta(x,r) \leq 2\mathsf{b}(x,r)^2,$$

always with $\varepsilon(x,r), \beta(x,r) \neq 0$ and arbitrary small. The key example is the following:

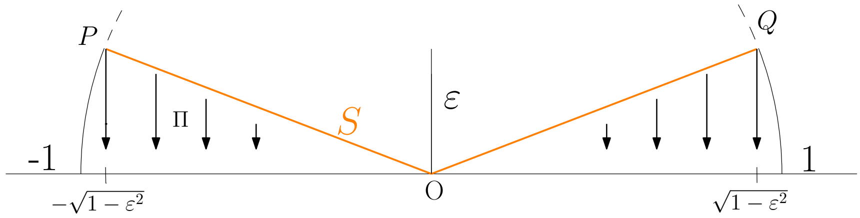

FIGURE 4. An example where $\mathsf{e}(O,1) \sim \varepsilon$, but $\varepsilon(O,1) \sim \varepsilon^2$.

Example 5.5.7. Fix $0 < \varepsilon < 1$. Let $P, Q \in \mathbb{R}^2$ be the two points in the upper half plane that are at distance 1 from the origin and at distance $\varepsilon \in (0, 1/2)$ from the x axis. Then define the set $S \subset \mathbb{R}^2$ to be the union of the two segments joining P and Q to the origin. Observe that for every straight line through the origin $\Gamma \subset \mathbb{R}^2$ we have that

$$d_H(\Gamma \cap B_1(0), S \cap B_1(0)) > \varepsilon, \qquad \sup_{y \in S \cap B_1(O)} \mathsf{d}(y, \Gamma) > \varepsilon,$$

therefore $\mathsf{e}(O,1), \mathsf{b}(O,1) \geq \varepsilon$. However if we consider the vertical projection map Π from S to the x-axis, it follows by the Pythagorean theorem that $1-\varepsilon^2 \leq \sqrt{1-\varepsilon^2} \leq \frac{|\Pi(x)-\Pi(y)|}{|x-y|} \leq 1$ for every $x, y \in S$. Therefore

$$||\Pi(x) - \Pi(y)| - |x-y|| \leq 2\varepsilon^2$$

for every $x, y \in S$ and in particular $\Pi : S \to [-1,1]$ is a $2\varepsilon^2$-isometry. In particular $\beta(O,1) \leq 2\varepsilon^2$ Moreover, again by the Pythagorean theorem $\Pi(P) = -\sqrt{1-\varepsilon^2}$ and $\Pi(Q) = \sqrt{1-\varepsilon^2}$, therefore $\Pi(S) = [-\sqrt{1-\varepsilon^2}, \sqrt{1-\varepsilon^2}]$ and in particular $\Pi : S \to [-1,1]$ is ε^2-dense. This implies that

$$\varepsilon(O,1) = d_{GH}((S, |.|), ([-1,1], |.|) \leq 2\varepsilon^2.$$

■

Our main result will be to make rigorous the above phenomenon, that is to clarify to which extent and in which cases the quantities $\varepsilon(x,r)$ behave like the square of the quantities $\mathsf{e}(x,r)$.

5.5.3. Main results: $\varepsilon \sim \mathsf{e}^2$ and $\mathsf{b} \sim \mathsf{b}^2$. We gather here the results that makes rigorous the fact that the metric quantities $\varepsilon(x,r)$ and $\beta(x,r)$ behave as the square of the classical Reifenberg's and Jones' numbers $\mathsf{e}(x,r)$ and $\mathsf{b}(x,r)$. It turns out that the inequalities of the type $\varepsilon(x,r) \lesssim \mathsf{e}(x,r)^2$, $\beta(x,r) \lesssim \mathsf{b}(x,r)^2$ do hold, while the converse inequalities are more subtle and hold in a more weaker form.

To state our main results we need the following notation: fixed $\delta \in (0, 1/2)$, $S \subset \mathbb{R}^d$ and $n \in \mathbb{N}$ with $n < d$, for every $i \in \mathbb{Z}$ we set

$$\mathsf{e}_i := \sup_{x \in S \cap B_1(0)} \mathsf{e}(x, 2^{-i}), \quad \varepsilon_i := \sup_{x \in S \cap B_{1-\delta}(0)} \varepsilon(x, 2^{-i}), \tag{5.5.5}$$

where we neglected the dependence on δ, n and S. Our main result reads as follows:

Theorem A. *For every $n \in \mathbb{N}$ there exists $\eta(n) > 0$ such that the following holds. Let $S \subset \mathbb{R}^d$, with $d > n$, $\delta \in (0, 1/2)$ and define the numbers $\mathsf{e}_i, \varepsilon_i$ as in* (5.5.5)*. Suppose that $\mathsf{e}_i \le \eta$ for every $i \ge \bar{i} - 2$, for some $\bar{i} \in \mathbb{N}$ with $2^{-\bar{i}} < \delta$, then*

$$\sum_{i \ge \bar{i}} \varepsilon_i^\lambda < C_\lambda \sum_{i \ge \bar{i}-2} \mathsf{e}_i^{2\lambda}, \quad \forall \lambda > 0, \tag{5.5.6}$$

where C_λ is a positive constant depending only on λ and n. In particular for every $\lambda > 0$ it holds that

$$\sum_{i \ge 0} \mathsf{e}_i^{2\lambda} < +\infty \Longrightarrow \sum_{i \ge 0} \varepsilon_i^\lambda < +\infty. \tag{5.5.7}$$

Remark 5.5.8. Notice that there is a small discrepancy between the definition of e_i and ε_i, indeed for the second quantity we had to restrict to the ball $B_{1-\delta}(0)$. However this is only a technical point and it is not relevant from a conceptual and practical point of view. Indeed in applications one always consider the numbers $\mathsf{e}(x, r)$ (or $\varepsilon(x, r)$) for x in some ball to then deduce properties of the set S in a *definite smaller ball* (see e.g. Theorem 0.28). ■

It has to be said that the fact that the numbers $\varepsilon(x, r)$'s "behave" as the square of numbers $\mathsf{e}(x, r)$'s was already noted, at least at an intuitive level, by David and Toro (see [**93**]). However, to the author's best knowledge, both the statement and the proof of Theorem A are new.

Similarly to the numbers $\mathsf{e}_i, \varepsilon_i$ we define for a given $S \subset \mathbb{R}^d$, $n \in \mathbb{N}$ with $n < d$, a fixed $\delta \in (0, 1/2)$ and every $i \in \mathbb{Z}$

$$\mathsf{b}_i := \sup_{x \in S \cap B_1(0)} \mathsf{b}(x, 2^{-i}), \quad \beta_i := \sup_{x \in S \cap B_{1-\delta}(0)} \beta(x, 2^{-i}), \tag{5.5.8}$$

where we neglected the dependence on δ, n and S. Then we can prove the following:

Theorem B. *For every $n \in \mathbb{N}$ there exists $\eta(n) > 0$ such that the following holds. Let $S \subset \mathbb{R}^d$, with $d > n$, $\delta \in (0, 1/2)$ and define the numbers β_i, b_i as in* (5.5.8)*. Suppose that $\mathsf{e}_i \le \eta$ for every $i \ge \bar{i} - 2$, for some $\bar{i} \in \mathbb{N}$ with $2^{-\bar{i}} < \varepsilon$ (where e_i are as in* (5.5.5)*), then*

$$\sum_{i \ge \bar{i}} \beta_i^\lambda < C_\lambda \sum_{i \ge \bar{i}-2} \mathsf{b}_i^{2\lambda}, \quad \forall \lambda > 0, \tag{5.5.9}$$

where C_λ is a positive constant depending only on λ and n.

In particular Theorem B implies that, whenever $\overline{\lim}_{i \to +\infty} \mathsf{e}_i < \eta(n)$ (with $\eta(n)$ as in the statement of the above theorem), we have that

$$\sum_{i \ge 0} \mathsf{b}_i^{2\lambda} < +\infty \Longrightarrow \sum_{i \ge 0} \mathsf{b}_i^\lambda < +\infty, \tag{5.5.10}$$

for every $\lambda > 0$.

Both Theorem A and Theorem B will follow from a 'weak' version of the inequalities $\varepsilon(x, r) \lesssim \mathsf{e}(x, r)^2$, $\beta(x, r) \lesssim \mathsf{b}(x, r)^2$ (see Theorem 5.6.7), which as showed in the previous section cannot hold in its 'strong' form.

As anticipated above however that the inequalities $\mathsf{e}(x, r)^2 \le C\varepsilon(x, r)$ and $\mathsf{b}(x, r)^2 \le C\beta(x, r)$ do hold in general:

Proposition 5.5.9. *Let $S \subset \mathbb{R}^d$ and $n \in \mathbb{N}$ with $n < d$. Then*

$$\mathsf{e}(x, r)^2 \le C\varepsilon(x, r), \quad \textit{for every } x \in S, r > 0,$$

where $\mathsf{e}(x, r)$ is as in (5.5.1) *and $C = C(n) > 0$.*

The following results states instead that $\mathsf{b}(x, r) \le C\sqrt{\beta(x, r)}$, provided $\mathsf{e}(x, r)$ is not big.

Proposition 5.5.10. *Let $S \subset \mathbb{R}^d$ and $n \in \mathbb{N}$ with $n < d$. The for every $x \in S$ and $r > 0$ such that $\mathsf{e}(x, r) < 1/8$, it holds*

$$\mathsf{b}(x, r) \le C(n)\sqrt{\beta(x, r)}.$$

We will actually show a stronger statement of the above, which roughly says that $\mathsf{b}(x, r) \le C\sqrt{\beta(x, r)}$ provided there exist n points which are 'sufficiently independent' in the sense that they span a simplex with large volume.

5.6. Proof of the main comparison result

In this section we will prove Theorem A and Theorem B.

5.6.1. Preliminary technical results. We gather in this section some elementary and well known results that will be needed in the sequel.

We first define a notion of distance between n-dimensional affine planes in $\mathbb{R}^d$.

Definition 5.6.1. Let Γ_1, Γ_2 be two affine n-dimensional planes in $\mathbb{R}^d$, we put

$$\mathsf{d}(\Gamma_1, \Gamma_2) := \mathsf{d}_H(\tilde{\Gamma}_1 \cap B_1(0), \tilde{\Gamma}_2 \cap B_1(0)),$$

where $\tilde{\Gamma}_i$ is the n dimensional plane parallel to Γ_i and passing through the origin.

Notice that the function d just defined clearly satisfies $\mathsf{d}(\Gamma_1, \Gamma_3) \le \mathsf{d}(\Gamma_1, \Gamma_2) + \mathsf{d}(\Gamma_2, \Gamma_3)$.

The following elementary Lemma is well known in literature (see for example Lemma 12.62 in [**94**]). Roughly said, it shows that if n "almost-orthonormal" points in a n-dimensional plane are close to another n-dimensional plane, then the two planes are close in the distance d.

Lemma 5.6.2 (Improvement of plane-distance)**.** *For every $n \in \mathbb{N}$ with n there exists a constant $C = C(n) > 0$ such that the following holds. Suppose that Γ_1, Γ_2 are two affine n-dimensional planes in $\mathbb{R}^d$, with $d > n$, and that there exist points $\{x_i\}_{i=0}^n \subset \Gamma_2 \cap B_1^{\mathbb{R}^d}(0)$ satisfying*

$$|x_i - x_0 - e_i| \le \frac{1}{10}, \quad \text{for every } i = 1, ..., n,$$
$$\mathsf{d}(x_i, \Gamma_1) \le \varepsilon, \quad \text{for every } i = 0, ..., n,$$

where $e_1, ..., e_d$ are orthonormal vectors in $\mathbb{R}^d$ and $\varepsilon \in (0, 1/100)$. Then

$$\mathsf{d}(\Gamma_1, \Gamma_2) \le C\varepsilon.$$

The following elementary lemma says that if two affine planes are sufficiently close with respect to the distance d, then they are not orthogonal to each other.

Lemma 5.6.3. *Let Γ_1, Γ_2 two n-dimensional affine planes in $\mathbb{R}^d$ such that $\mathsf{d}(\Gamma_1, \Gamma_2) < 1$. Write Γ_i as $p_i + V_i$ where $p_i \in \mathbb{R}^d$ and V_i is a n-dimensional subspace of $\mathbb{R}^d$. Then*

$$V_1^\perp \oplus V_2 = \mathbb{R}^d. \tag{5.6.1}$$

In particular for every $p \in \Gamma_1$ there exists $q \in \Gamma_2$ such that $\Pi(q) = p$, where Π is the orthogonal projection onto Γ_1.

Proof. It's enough to prove that $V_1^\perp \cap V_2 = \{0\}$. Suppose $v \in V_1^\perp \cap V_2$, then we can regard V_1, V_2 as affine planes through the origin and parallel to Γ_1, Γ_2. Therefore by hypothesis

$$|v| = \mathsf{d}(v, V_1) \le \mathsf{d}(\Gamma_1, \Gamma_2)|v|$$

and thus $v = 0$. □

The following simple technical result will be the main tool for the proof of Theorem 5.6.7.

Lemma 5.6.4. *Let Γ_1, Γ_2 two n-dimensional affine planes in $\mathbb{R}^d$. Then for any $x \in \Gamma_1$ and any $y \in \mathbb{R}^d$ (different from x)*

$$|x - y|^2 \le |\Pi(x) - \Pi(y)|^2 + |x - y|^2 \left(\mathsf{d}(\Gamma_1, \Gamma_2) + \frac{\mathsf{d}(y, \Gamma_1)}{|x - y|} \right)^2,$$

where Π denotes the orthogonal projection onto Γ_2.

Proof. Let $\mathsf{e} := \mathsf{d}(\Gamma_1, \Gamma_2)$. Up to translating both the plane Γ_1 and the points x, y by the vector $\Pi(x) - x$, we can suppose $x \in \Gamma_2$ and $x = 0$. Let now p be the orthogonal projection of y onto Γ_1. Since both Γ_1 and Γ_2 contain the origin, we have that

$$\mathsf{d}(p, \Gamma_2) \le \mathsf{d}_H(\Gamma_2 \cap B_{|p|}(0), \Gamma_1 \cap B_{|p|}(0)) \le |p|\mathsf{e} \le |y|\mathsf{e}.$$

Therefore $d(y, \Gamma_2) \le \mathsf{d}(y, p) + \mathsf{d}(p, \Gamma_2) = d(y, p) + \mathsf{d}(p, \Gamma_2) \le \mathsf{d}(y, \Gamma_1) + |y|\mathsf{e}$. Then by Pythagoras' theorem

$$|y|^2 = |\Pi(y) - y|^2 + |\Pi(y)|^2 = \mathsf{d}(y, \Gamma_2)^2 + |\Pi(y)|^2 \le (\mathsf{d}(y, \Gamma_1) + |y|\mathsf{e})^2 + |\Pi(y) - \Pi(x)|^2,$$

since $\Pi(x) = 0$. This concludes the proof. □

We conclude with two results about the numbers $\mathsf{e}(x, r)$ and $\mathsf{b}(x, r)$ (recall their definition in (5.5.1), (5.5.2)). The first one shows that there exists a plane which realizes $\mathsf{b}(x, r)$ (i.e. that minimizes (5.5.2)) and at the same time almost realizes $\mathsf{e}(x, r)$. The second is a classical tilting estimates, which says that the orientation of such realizing plane do not vary too much from scale to scale and between points close to each other.

Proposition 5.6.5 (Realizing plane). *For every $n \in \mathbb{N}$ there exists a constant $C = C(n) \geq 1$ such that the following holds. Let $S \subset \mathbb{R}^d$ with $d > n$, let $x \in S$ and $r > 0$ be such that $\mathsf{e}(x,r) \leq 1/100$, then there exists an n-dimensional affine plane Γ_x^r that realizes $\mathsf{b}(x,r)$ (i.e. that minimizes (5.5.2)) and such that*

$$r^{-1} d_H(S \cap B_r(x), \Gamma_x^r \cap B_r(x)) \leq C\mathsf{e}(x,r).$$

Proof. The existence of two planes Γ and Γ' that realize respectively $\mathsf{b}(x,r)$ and $\mathsf{e}(x,r)$, follows by compactness. Without loss of generality we can assume that $x = 0$ and $r = 1$. Since $0 \in \Gamma'$ there exist orthonormal vectors $e_1, ..., e_n \in \Gamma'$ and points $x_1, ..., x_n \in S \cap B_1(0)$ such that $|x_i - e_i| \leq \mathsf{e}(0,1)$, $i = 1, ..., n$. Moreover there exist points $y_0, y_1, ..., y_n \in \Gamma$ such that $|y_i - x_i|, |y_0| \leq \mathsf{b}(0,1) \leq \mathsf{e}(0,1)$, $i = 1, ..., n$. In particular $|y_i - y_0 - e_i| \leq 4\mathsf{e}(0,1)$, $i = 1, ..., n$ and we can apply Lemma 5.6.2 to deduce that $\mathsf{d}_H(\Gamma \cap B_1(0), \Gamma' \cap B_1(0)) \leq C\mathsf{e}(0,1)$, which concludes the proof. □

Proposition 5.6.6 (Tilting estimate). *For any $n \in \mathbb{N}$ there exist $\eta = \eta(n)$, $C = C(n) > 0$ such that the following holds. Let $S \subset \mathbb{R}^d$ with $n > d$ and let $r > 0$ and $x, y \in S$ be such that $\mathsf{e}(x,r), \mathsf{e}(y,r) \leq \eta(n)$ and $|x - y| < \frac{1}{2}r$. Then it holds*

$$\begin{aligned} \mathsf{d}(\Gamma_x^r, \Gamma_x^{2r}) &\leq C(\mathsf{b}(x,r) + \mathsf{b}(x,2r)), \\ \mathsf{d}(\Gamma_x^r, \Gamma_y^r) &\leq C(\mathsf{b}(x,r) + \mathsf{b}(y,r)), \end{aligned}$$

for any choice of realizing planes $\Gamma_x^r, \Gamma_y^r, \Gamma_x^{2r}$ (as given by Prop. 5.6.5).

Proof. We prove only the second, since the first is analogous.

As usual, the scaling and translation invariant nature of the statement allows us to assume that $r = 1$ and x to be the origin. Then (if $\mathsf{e}(x,1)$ is small enough) there exist orthonormal vectors $e_1, ..., e_n \in \Gamma_x^1$ and points $x = x_0, x_1, ..., x_n \in S \cap B_{1/2}(0)$ such that $|x_i - 1/4e_i| \leq C(n)\mathsf{e}(x,1)$, $i = 1, ..., n$. Moreover $x_i \in B_1(y)$, hence there exist points $y_0, ..., y_n \in \Gamma_y^1$ such that $|x_i - y_i| \leq \mathsf{b}(y,1)$, $i = 0, ..., n$. Finally there exist points $z_1, ..., z_n \in \Gamma_x^1$ such that $|z_i - x_i| \leq \mathsf{b}(x,1)$, $i = 1, ..., n$. Putting all together we have $|y_i - y_0 - 1/4e_i| \leq C\mathsf{e}(x,1) + 2\mathsf{b}(y,1) + \mathsf{b}(x,1)$, $i = 1, ..., n$ and $\mathsf{d}(y_i, \Gamma_x^1) \leq \mathsf{b}(0,1) + \mathsf{b}(y,1)$, $i = 0, ..., n$, hence (if $\mathsf{e}(x,1), \mathsf{e}(y,1)$ are small enough) we can apply Lemma 5.6.2 to deduce that $\mathsf{d}(\Gamma_x^1, \Gamma_y^1) \leq C(\mathsf{b}(x,1) + \mathsf{b}(y,1))$. □

5.6.2. Proof of theorems A and B. Both Theorem A and Theorem B will be deduced as corollaries of the following more precise result.

Theorem 5.6.7. *For every $n \in \mathbb{N}$ there exist $C = C(n) > 0, \eta = \eta(n) > 0$ such that the following holds. Let $i \in \mathbb{N}_0$, $S \subset \mathbb{R}^d$ with $d > n$ and assume that $\mathsf{e}_j \leq \eta$ for every $j \geq i - 2$ (where e_j are as in (5.5.5)). Then*

$$\varepsilon(x, 2^{-i}) \leq C \left(\sup_{j \in \mathbb{N}_0} \frac{\big(\mathsf{b}_{i-2} + ... + \mathsf{b}_{i+j}\big)^2}{2^j} \right) \vee C\mathsf{e}_i^2, \quad \forall x \in S, \, |x| \leq 1 - 2^{-i}, \tag{A}$$

$$\beta(x, 2^{-i}) \leq C \sup_{j \in \mathbb{N}_0} \frac{\big(\mathsf{b}_{i-2} + ... + \mathsf{b}_{i+j}\big)^2}{2^j}, \quad \forall x \in S, \, |x| \leq 1 - 2^{-i}, \tag{B}$$

where b_j are as in (5.5.8).

Inequalities (A) and (B) should be thought as weak versions of the formal inequalities "$\varepsilon(x,r) \leq C\mathsf{e}(x,r)^2$" and "$\beta(x,r) \leq C\mathsf{b}(x,r)^2$" that are not true in general since, as we saw in Section 5.5.2, (5.5.3) and (5.5.4) cannot be improved.

Proof of Theorem A and Thoerem B, given Theorem 5.6.7. Let $\delta \in (0,1/2)$ and $\bar{i} \in \mathbb{N}$ be as in the hypotheses of Theorem A and Theorem B. Since $\delta > 2^{-\bar{i}}$, from Theorem 5.6.7 and the definition of the numbers ε_i we have

$$\begin{aligned} (\varepsilon_i)^\lambda \leq \sup_{\substack{x \in S, \\ |x| \leq 1 - 2^{-i}}} \varepsilon(x, 2^{-i})^\lambda &\leq C \left(\sup_{j \in \mathbb{N}_0} \frac{\big(\mathsf{b}_{i-2} + ... + \mathsf{b}_{i+j}\big)^{2\lambda}}{2^{\lambda j}} \right) \vee C\mathsf{e}_i^{2\lambda} \\ &\leq C\mathsf{e}_i^{2\lambda} + C \sum_{j \geq 0} \frac{(j+3)^{2\lambda - 1} \vee 1}{2^{\lambda j}} (\mathsf{b}_{i-2}^{2\lambda} + \cdots + \mathsf{b}_{i+j}^{2\lambda}), \quad \forall i \geq \bar{i}. \end{aligned} \tag{5.6.2}$$

An analogous estimate holds for b_i, $\forall i \geq \bar{i}$. Recalling that $\mathsf{b}_i \leq \mathsf{e}_i$ we obtain

$$\begin{aligned}\sum_{i\geq \bar{i}} \varepsilon_i^\lambda &\leq C\sum_{i\geq\bar{i}} \mathsf{e}_i^{2\lambda} + C\sum_{i\geq\bar{i}}\sum_{j\geq 0}\frac{(j+3)^{2\lambda-1}\vee 1}{2^{\lambda j}}(\mathsf{e}_{i-2}^{2\lambda}+\cdots+\mathsf{e}_{i+j}^{2\lambda})\\ &\leq C\sum_{i\geq\bar{i}} \mathsf{e}_i^{2\lambda} + C\sum_{j\geq 0}\frac{(j+3)^{2\lambda-1}\vee 1}{2^{\lambda j}}\sum_{i\geq\bar{i}}(\mathsf{e}_{i-2}^{2\lambda}+\cdots+\mathsf{e}_{i+j}^{2\lambda})\\ &\leq C\sum_{i\geq\bar{i}} \mathsf{e}_i^{2\lambda} + C\left(\sum_{j\geq 0}\frac{(j+3)^{2\lambda}\vee (j+3)}{2^{\lambda j}}\right)\left(\sum_{i\geq\bar{i}-2}\mathsf{e}_i^{2\lambda}\right),\end{aligned}$$

which proves (5.5.6). The exact same computations yields also (5.5.9). $\square$

Proof of Theorem 5.6.7. Observe that it is sufficient to consider the case $x = 0$ and $i = 0$ for both (A) and (B), since the conclusion then follows by translating and scaling.

We define

$$\theta := C^2\sup_{j\in\mathbb{N}_0}\frac{\left(\mathsf{b}_{-2}+...+\mathsf{b}_j\right)^2}{2^j},$$

$$\theta' := \max(\theta, C\mathsf{e}_0^2),$$

where C is a big enough constant depending only on n, to be determined later. Before proceeding we make the following observation

$$C^2(\mathsf{b}_{-2}+...+\mathsf{b}_j)^2 > \lambda > 0 \implies \theta > \frac{2\lambda}{2^j}. \tag{5.6.3}$$

Along the proof, for a given $x \in S$ and $r > 0$ we will denote by Γ_x^r one of the realizing planes given by Proposition 5.6.5 (the choice of the particular plane is not relevant).

Proof of (B)**:** Let Π be the orthogonal projection onto Γ_0^1. It is sufficient to show that

$$\Pi : S\cap B_1(0) \to \Gamma_0^1 \cap B_1(0), \quad \text{is a } \theta\text{-isometry}, \tag{5.6.4}$$

with respect to the Euclidean distance.

Choose $x, y \in S\cap B_1(0)$ distinct and observe that there exists a unique integer $j \geq 0$ such that

$$\frac{1}{2^j} \leq |x-y| < \frac{1}{2^{j-1}}. \tag{5.6.5}$$

Applying Proposition 5.6.6 multiple times (assuming $\mathsf{e}_j \leq \eta(n)$ for every $j \geq i-2$, with $\eta(n)$ as in the statement of Prop. 5.6.6) and observing that $|x-y| < 2$ we have

$$\begin{aligned}\mathsf{d}(\Gamma_0^1, \Gamma_x^{2^{-j+1}}) &\leq \mathsf{d}(\Gamma_0^1,\Gamma_0^2)+\mathsf{d}(\Gamma_0^2,\Gamma_0^{2^2})+\mathsf{d}(\Gamma_0^{2^2},\Gamma_x^{2^2})+\mathsf{d}(\Gamma_x^{2^2},\Gamma_x^2)+...+\mathsf{d}(\Gamma_x^{2^{-j+2}},\Gamma_x^{2^{-j+1}})\\ &\leq D(\mathsf{b}(0,0)+\mathsf{b}(0,2)+\mathsf{b}(0,2^2)+\mathsf{b}(x,2^2)+...+\mathsf{b}(x,2^{-j+1}))\\ &\leq D(\mathsf{b}_{-2}+...+\mathsf{b}_{j-1}), \qquad (5.6.6)\end{aligned}$$

for some constant D depending only on n. We consider now two cases, when $(D+4)(\mathsf{b}_{-2}+...+\mathsf{b}_{j-1}) > 1$ or the opposite. In the first case, assuming that $C \geq D+4$, from (5.6.3) we have $\theta \geq \frac{2}{2^j}$ and therefore

$$||\Pi(x)-\Pi(y)|-|x-y|| \leq |x-y| \leq \frac{2}{2^j} < \theta,$$

that is what we wanted. Hence we can suppose that $(D+4)(\mathsf{b}_{-2}+...+\mathsf{b}_{j-1}) \leq 1$. Since from (5.6.5) it holds that $|x-y| \geq 2^{-j}$, we have that

$$\mathsf{d}(y, \Gamma_x^{2^{-j+1}}) \leq 4\mathsf{b}(x, 2^{-j+1})|x-y|. \tag{5.6.7}$$

We can now apply Lemma 5.6.4 to the planes $\Gamma_0^1, \Gamma_x^{2^{-j+1}}$, that coupled with (5.6.6) and (5.6.7) gives

$$|\Pi(x)-\Pi(y)| \geq |x-y|\sqrt{1-(D(\mathsf{b}_{-2}+...+\mathsf{b}_{j-1})+4\mathsf{b}_{j-1})^2}.$$

Hence

$$|x-y|-|\Pi(x)-\Pi(y)| \leq |x-y|\left(1-\sqrt{1-((D+4)(\mathsf{b}_{-2}+...+\mathsf{b}_{j-1}))^2}\right).$$

Thanks to the assumption $(D+4)(\mathsf{b}_{-2}+...+\mathsf{b}_{j-1}) \le 1$, we can use the elementary the inequality $1-\sqrt{1-x} \le x$, valid for $0 \le x \le 1$, to finally obtain

$$
\begin{aligned}
||x-y|-|\Pi(x)-\Pi(y)|| &\le |x-y|((D+4)(\mathsf{b}_{-2}+...+\mathsf{b}_{j-1}))^2 \\
&\overset{(5.6.5)}{\le} \frac{((D+4)(\mathsf{b}_{-2}+...+\mathsf{b}_{j-1}))^2}{2^{j-1}} \le \theta,
\end{aligned}
$$

where we have used the definition of θ and assuming $C \ge 2(D+4)$. This concludes the proof of (5.6.4) and thus the proof of (B).

Proof of (A)**:** In view of (5.6.4), we only need to show that Π is also θ'-surjective.

Claim: Let $C' = C'(n) \ge 1$ be the constant given by Proposition 5.6.5. For every $p \in \Gamma_0^1 \cap B_1(0)$ and $x \in S \cap B_1(0)$ such that $C'\mathsf{e}_0 \ge |p-\Pi(x)| \ge \theta'$ it holds

$$
B_{\frac{3}{4}|p-\Pi(x)|}(p) \cap \Pi(S \cap B_1(0)) \ne \emptyset.
$$

Before proving the claim, we show that it implies that Π is θ'-surjective. Indeed suppose it is not, i.e. there exists $p \in \Gamma_0^1 \cap B_1(0)$ such that

$$
R \coloneqq \sup\{r \mid B_r(p) \cap \Pi(S \cap B_1(0)) = \emptyset\} \ge \theta'.
$$

Since $d_H(\Gamma_0^1 \cap B_1(0), S \cap B_1(0)) \le C'\mathsf{e}_0$ (recall that Γ_0^1 was chosen as a realizing plane as given by Prop. 5.6.5), there exists $x \in S \cap B_1(0)$ such that $|x-p| \le C'\mathsf{e}_0$ and in particular $|\Pi(x)-p| \le C'\mathsf{e}_0$. Therefore $R \le C'\mathsf{e}_0$. This implies, from the definition of R, that there exists a point $x' \in S \cap B_1(0)$ such that $\theta' \le R \le |\Pi(x')-p| \le \min(\frac{5}{4}R, C'\mathsf{e}_0)$. However the Claim gives that

$$
\emptyset \ne B_{\frac{3}{4}|\Pi(x')-p|}(p) \cap \Pi(S \cap B_1(0)) \subset B_{\frac{15}{16}R}(p) \cap \Pi(S \cap B_1(0)),
$$

that contradicts the minimality of R.

Proof of the Claim: Set $R \coloneqq |p-\Pi(x)|$. To make the proof more easy to follow we first explain the intuition behind it. The key idea is that near x the set S is distributed in a horizontal manner, near a plane passing through x. We can then move along this plane towards p and thus find a point y in $S \cap B_1(0)$ such that $|\Pi(y)-p| \sim \frac{R}{2}$. However, since p can be near the boundary of $B_1(0)$, in this movement we might go outside the ball $B_1(0)$. To avoid this issue we consider a point q such that $|p-q| \sim \frac{R}{2}$ but placed radially towards the origin and then find a point y (using the idea described above of moving horizontally near x) that projects near q.

Start by noticing that (if e_0 is small enough w.r.t. n) $R \le C'\mathsf{e}_0 < 1/4$. Therefore there exists a unique integer $j \ge 2$ such that

$$
\frac{1}{2^{j+1}} \le R < \frac{1}{2^j}. \tag{5.6.8}
$$

Since by assumption $\theta \le \theta' \le R \le 1/2^j$, from (5.6.3) we have $(C(\mathsf{b}_{-2}+...+\mathsf{b}_j))^2 \le 1/2$. Define now the point $q \in \Gamma_0^1 \cap B_1(0)$ as

$$
q = p - \frac{p}{|p|}\frac{R}{2}.
$$

Then

$$
|q| = \left||p| - \frac{R}{2}\right| \le 1 - \frac{R}{2}, \tag{5.6.9}
$$

indeed $|p| < 1$ and $R < 1$. Moreover $|p-q| = R/2$ and $|q-\Pi(x)| \le |p-\Pi(x)| + |p-q| = 3/2R$. Consider the plane $\Gamma_x^{2^{-j}}$, arguing as in (5.6.6) we can show that

$$
\mathsf{d}(\Gamma_0^1, \Gamma_x^{2^{-j}}) \le C(\mathsf{b}_{-2}+...+\mathsf{b}_j) < 1,
$$

provided C is big enough. Then by Proposition 5.6.3 there exists a point $e \in \Gamma_x^{2^{-j}}$ such that $\Pi(e) = q$. Applying Lemma 5.6.4 we obtain

$$
|e-x|^2 \le |q-\Pi(x)|^2 + |e-x|^2/2
$$

that implies $|e-x| \le \sqrt{2}|q-\Pi(x)| \le 3R \le 1/2^{j-2}$. Therefore there exists $y \in S \cap B_{2^{-j+2}}(x)$ such that $|y-e| \le \mathsf{e}_{j-2}2^{-j+2} < R/4$ (provided e_{j-2} is small enough). Thus

$$
|\Pi(y)-p| \le |\Pi(y)-\Pi(e)| + |p-q| \le |y-e| + R/2 < 3/4R,
$$

that means $\Pi(y) \in B_{\frac{3}{4}R}(p)$. It remains to prove that $y \in B_1(0)$. First we observe that from (5.6.8) and the assumption $R \le C'\mathsf{e}_0$ we have

$$
|y-x| \le \frac{4}{2^j} \le 8R \le 8C'\mathsf{e}_0.
$$

Hence, since $x \in B_1(0)$,

$$\mathsf{d}(y, \Gamma_0^1) \leq |x - y| + \mathsf{d}(x, \Gamma_0^1) \leq 9C'\mathsf{e}_0.$$

From previous computations we know that that $|\Pi(y) - q| = |\Pi(y) - \Pi(e)| \leq |y - e| \leq R/4$, therefore from (5.6.9) $|\Pi(y)| \leq |q| + R/4 \leq 1 - R/4$. From Pythagoras Theorem we obtain

$$\begin{aligned} |y|^2 = |\Pi(y)|^2 + \mathsf{d}(y, \Gamma_0^1)^2 \leq \left(1 - \frac{R}{4}\right)^2 + (9C')^2\mathsf{e}_0^2 = \\ = 1 + R\left(\frac{R}{16} + \frac{(9C')^2\mathsf{e}_0^2}{R} - \frac{1}{2}\right). \end{aligned}$$

Thus to conclude it is enough to show that

$$\frac{R}{16} + \frac{(9C')^2\mathsf{e}_0^2}{R} < \frac{1}{2}.$$

Since by assumption $R \geq \theta'$ and by definition $\theta' \geq C\mathsf{e}_0^2$, we deduce that $\frac{\mathsf{e}_0^2}{R} \leq \frac{1}{C}$. Therefore recalling that $R < 1$, the above inequality is satisfied as soon as $C < 4(9C')^2$. This concludes the proof. □

5.7. Proof of the converse inequalities

We conclude by proving Proposition 5.5.9 and Proposition 5.5.10. We start with the first one which we restate for convenience.

Proposition 5.7.1. *Let $S \subset \mathbb{R}^d$ and $n \in \mathbb{N}$ with $n < d$. Then*

$$\mathsf{e}(x, r)^2 \leq C_n \varepsilon(x, r), \quad \text{for every } x \in S, r > 0.$$

Proof. Set $\mathsf{e} := \mathsf{e}(x, r)$. Up to rescaling we can suppose $r = 1$. There exists a map $f : B_1^{\mathbb{R}^n}(0) \to S \cap B_1^{\mathbb{R}^d}(0)$ that is a $2\mathsf{e}$−GH-approximation. Moreover, since $|f(0) - 0| \leq 7\mathsf{e}$ (recall Proposition 5.1.10), up to modifying the image of the point 0, we can suppose that $f(0) = 0$ and that f is a $10\mathsf{e}$−GH-approximation. Then from Lemma 5.1.11 we have that there exists an isometry $I : \mathbb{R}^k \to \mathbb{R}^n$ such that $|I - f| \leq C(n)\sqrt{\mathsf{e}}$, for some constant $C(n)$ depending only on n. We consider the n-dimensional affine plane given by $\Gamma = I(\mathbb{R}^n)$. Then $\Gamma \cap B_1^{\mathbb{R}^d}(0) = I(B_1^{\mathbb{R}^n}(0))$ and this clearly implies that

$$\Gamma \cap B_1^{\mathbb{R}^d}(0) \subset (B_1(0) \cap S)_{C(n)\sqrt{\mathsf{e}}}.$$

Moreover take any $s \in S \cap B_1^{\mathbb{R}^d}(0)$, then since f is a GH-approximation, there exists $y \in B_1^{\mathbb{R}^n}(0)$ such that $|x - f(y)| \leq 10\mathsf{e}$, hence $|x - I(y)| \leq C(n)\sqrt{\mathsf{e}} + 10\mathsf{e} \leq (C(n) + 10)\sqrt{\mathsf{e}}$. This proves that

$$B_1(0) \cap S \subset (\Gamma \cap B_1^{\mathbb{R}^n}(0))_{(C(n)+10)\sqrt{\mathsf{e}}},$$

that completes the proof. □

We will now prove a stronger version of Proposition 5.5.10. We first need to introduce some notations. Given a set of $n + 1$ points $x_0, ..., x_n \in \mathbb{R}^d$ we denote by $\mathsf{Vol}_n(x_0, ..., x_n)$ the volume of the n-dimensional simplex with vertices $x_0, ..., x_n$. It is well know that for every $n \in \mathbb{N}$ there exists a polynomial $P_n : \mathbb{R}^{(n+1)n/2} \to \mathbb{R}$ such that

$$\mathsf{Vol}_n(x_0, ..., x_n)^2 = P_n(\{|x_i - x_j|^2\}_{0 \leq i < j \leq n}), \tag{5.7.1}$$

see for example [**51**, Sec. 40] for a proof.

The following results states that $\mathsf{b}(x, r) \leq C\sqrt{\beta(x, r)}$, provided $B_r(x) \cap S$ contains n points which are 'sufficiently independent' in the sense that they span a simplex with large volume.

Proposition 5.7.2. *Let $S \subset \mathbb{R}^d$ and $n \in \mathbb{N}$ with $n < d$. Then*

$$\mathsf{b}(x, r) \leq C\left(\frac{\sqrt{\beta(x, r)}}{V_n} \wedge V_n^{\frac{1}{n}}\right), \quad \text{for every } x \in S, r > 0,$$

where $V_n = \sup_{\{x_i\}_{i=0}^n \subset S \cap B_r(x)} r^{-n}\mathsf{Vol}_n(x_0, ..., x_n)$, $\mathsf{b}(x, r)$ is as in (5.5.2) and $C = C(n) > 0$. In particular, if $\mathsf{e}(x, r) < 1/8$ ($\mathsf{e}(x, r)$ is as in (5.5.1)), then

$$\mathsf{b}(x, r) \leq 100C\sqrt{\beta(x, r)}, \quad \text{for every } x \in S, r > 0,$$

PROOF. After a rescaling we can consider only the case $r = 1$. We can also suppose that $V_n > 0$, otherwise there is nothing to prove. Fix $\varepsilon > 0$. There exists a map $f : S \cap B_1(0) \to B_1^{\mathbb{R}^n}(0)$ that is a $(\beta(x,1)+\varepsilon)$-isometry. Moreover there exist $\{x_i\}_{i=0}^n \subset S \cap B_1(x)$ such that $\mathsf{Vol}_n(x_0, ..., x_n) > V_n - \varepsilon$. Let $x_{n+1} \in S \cap B_1(x)$ be arbitrary and observer that, since $f(x_0), ..., f(x_n), f(x_{n+1}) \in \mathbb{R}^n$, we must have $\mathsf{Vol}_{n+1}(f(x_0), ..., f(x_n), f(x_{n+1})) = 0$. From (5.7.1) and the fact that P_{n+1} is locally Lipschitz, it follows that

$$\mathsf{Vol}_{n+1}(x_0, ..., x_n, \bar{x})^2 \le C(n) \sup_{0 \le i < j \le n} ||f(x_i) - f(x_j)|^2 - |x_i - x_j|^2| \le 4C(n)(\beta(x,1)+\varepsilon).$$

Therefore, denoted by Γ the n-dimensional plane spanned by $x_0, ..., x_n$, it holds

$$\mathsf{d}(x_{n+1}, \Gamma) = \frac{\mathsf{Vol}_{n+1}(x_0, ..., x_n, x_{n+1})}{\mathsf{Vol}_n(x_0, ..., x_n)} \le C(n) \frac{\sqrt{\beta(x,1)+\varepsilon}}{V_n - \varepsilon}.$$

Moreover it is clear that there exists a constant $C'(n) > 0$ such that $\mathsf{Vol}_{n+1}(x_0, ..., x_n, x_{n+1}) \le C' V_n^{\frac{n+1}{n}}$. From the arbitrariness of $x_{n+1} \in S \cap B_1(x)$ and $\varepsilon > 0$ the conclusion follows. $\square$

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