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Existence and uniqueness of parallel transport in non-collapsed $\mathsf{RCD}(K,N)$ spaces

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Contents

Notations

- $\mathbb{R}^d$: Euclidean space of dimension d;
- d_e: Euclidean distance on $\mathbb{R}^d$;
- $\mathscr{L}^d$: d-dimensional Lebesgue measure;
- $\mathscr{B}(\mathrm{X})$: Borel sets of a topological space X;
- $\mathscr{M}(\mathbb{R}^d)$: locally finite signed measure;
- $\mathscr{M}^+(\mathbb{R}^d)$: locally finite non negative measure;
- $\mathscr{P}(\mathrm{X})$: non negative Borel probability measure on a topological space X;
- $\mathscr{P}_2(\mathrm{X})$: non negative Borel probability measure with finite second moment on a metric space (X, d);
- $C_c^0(\mathbb{R}^d)$: the space of continuous functions with compact support;
- $C_c^\infty(\mathbb{R}^d)$: the space of C^∞ functions on $\mathbb{R}^d$ with compact support;
- $C([0,1], \mathrm{X})$: the set of continuous curves with values in X;
- $\mathrm{Lip}(\mathrm{X})$: the space of Lipschitz functions $f \colon \mathrm{X} \to \mathbb{R}$;
- $\mathrm{Lip}_b(\mathrm{X})$: the space of Lipschitz and bounded functions $f \colon \mathrm{X} \to \mathbb{R}$;
- $\mathrm{Lip}_{bs}(\mathrm{X})$: the space of Lipschitz and bounded functions $f \colon \mathrm{X} \to \mathbb{R}$;
- $\mathrm{lip} f$: the slope of f;
- $\mathrm{lip}_a f$: the asymptotic Lipschitz constant of f;
- $\mathrm{Lip}(f)$: the Lipschitz constant of f;
- $L^p(\mathfrak{m}) = L^p(\mathrm{X}) = L^p(\mathrm{X}, \mathfrak{m})$: the Lebesgue space of p-integrable functions for $p \in [1, +\infty)$ defined on $(\mathrm{X}, \mathfrak{m})$;
- $L^\infty(\mathfrak{m}) = L^\infty(\mathrm{X}) = L^\infty(\mathrm{X}, \mathfrak{m})$: the space of essentially-bounded functions defined on $(\mathrm{X}, \mathfrak{m})$;
- $L^p_{\mathrm{loc}}([0,1] \times \mathrm{X})$, with $p \in [1, \infty]$: the space of functions that are in $L^p(B)$ for every bounded set $B \subseteq [0,1] \times \mathrm{X}$;

- $L^0(\mathfrak{m}) = L^0(\mathrm{X}) = L^0(\mathrm{X}, \mathfrak{m})$: the vector space of $\mathfrak{m}$-measurable functions, up to quotient $\mathfrak{m}$ a.e.;
- $\mathrm{supp}(f)$: support of a funtion $f \in L^0(\mathrm{X})$;
- χ_A: indicator function on the set A;
- M: Hardy-Littlewood maximal operator;
- M_λ: local Hardy-Littlewood maximal operator for some $0 < \lambda < +\infty$;
- h_t: heat flow at time $t > 0$;
- (TE): short-hand notation to denote the transport equation;
- (CE): short-hand notation to denote the continuity equation;
- $\mathbb{M}^2_k$: model space of curvature K, i.e. 2-dimensional Riemannian manifold with constant sectional curvature equal to K;
- Sec_x: sectional curvature of a smooth Riemannian manifold (M, g) at the point x;
- w_N: Lebesgue measure of the ball of radius 1 in $\mathbb{R}^N$;
- $v_{K,N}(r)$: model space with parameters K and N, i.e. Riemannian manifold with constant Ricci curvature K and dimension N.

Introduction

0.1 Non smooth spaces and parallel transport

The aim of my PhD thesis is to build parallel transport in the class of non collapsed $\mathsf{RCD}(K,N)$ spaces. Parallel transport is a widely used concept in Riemannian geometry. It can be stated as follows. Given a smooth Riemannian manifold and a smooth curve on it, we consider a vector belonging to the tangent space at the initial point of the curve. Then a parallel transport is a smooth vector field along the curve with vanishing covariant derivative at every time. The existence of such a vector field can be obtained by writing the condition of being a parallel transport in local coordinates and solving (locally) an ODE in $\mathbb{R}^d$ by taking into account the existence and uniqueness results given by the Cauchy-Lipschitz theory.
Uniqueness of parallel transport instead is a consequence of the uniqueness of the solution of the mentioned ODE in coordinates. Moreover, it is also a byproduct of the Leibniz formula, that we now describe.
We consider a smooth Riemannian manifold (M,g) and we denote by T_xM the tangent space at a point $x \in M$. Given a smooth curve $\gamma\colon [0,1] \to M$ and two smooth vector fields along the curve $[0,1] \ni t \mapsto V_t, W_t \in T_{\gamma_t}M$, we have that $[0,1] \ni t \mapsto g(V_t, W_t) \in \mathbb{R}$ belongs to $C^1([0,1])$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} g(V_t, W_t) = g(\nabla_{\dot\gamma_t} V_t, W_t) + g(V_t, \nabla_{\dot\gamma_t} W_t) \quad \text{for every } t \in [0,1].$$

Uniqueness can be retrieved as follows: consider two parallel transports $[0,1] \ni t \mapsto V_t^1, V_t^2 \in T_{\gamma_t}M$ of $\bar V$ along γ. Then, thanks to the Leibniz formula, we have that the function $[0,1] \ni t \mapsto |V_t^1 - V_t^2|^2 \in \mathbb{R}$ belongs to $C^1([0,1])$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} |V_t^1 - V_t^2|^2 = 2g(\nabla_{\dot\gamma_t} V_t^1 - \nabla_{\dot\gamma_t} V_t^2, V_t^1 - V_t^2) = 0 \quad \text{for every } t \in [0,1].$$

Since $|V_0^1 - V_0^2| = 0$, we get that $V_t^1 = V_t^2$ for every $t \in [0,1]$.

We are interested in the study of such properties when the underlying space is not anymore a smooth Riemannian manifold. When considering more general underlying spaces, the study of parallel transport encounters several difficulties. We have to take into account the regularity of functions and of vector fields (that are expected to have a lower regularity of 'one degree' with respect to functions) at our disposal and formulate a definition of parallel transport which is independent of solving an ODE in local coordinates. Before outlining the content of the thesis, let us mention two non smooth settings on which the problem is studied.

i) the Wasserstein space $(\mathscr{P}_2(M), W_2)$, that is a metric space defined in the following way. We consider a smooth Riemannian manifold (M,g), we denote by $\mathscr{P}(M)$ the set of Borel probability measures defined over M, we denote by d_g and Vol_g respectively the geodesic distance the volume measure induced by the metric and we define, given $x_0 \in M$,

$$\mathscr{P}_2(M) := \Big\{\mu \in \mathscr{P}(M) : \int \mathsf{d}_g^2(x, x_0)\, \mathrm{dVol}_g(x) < +\infty \\ \text{for some (and thus all) } x_0 \in M\Big\},$$

and

$$W_2^2(\mu, \nu) := \inf\Big\{\int \mathsf{d}_g^2(x, y)\, \mathrm{d}\boldsymbol{\gamma}(x, y) : \boldsymbol{\gamma} \in \mathscr{P}(M \times M), \\ \boldsymbol{\gamma}(A \times \mathrm{X}) = \mu(A),\ \boldsymbol{\gamma}(\mathrm{X} \times A) = \nu(A) \text{ for } A \text{ Borel.}\Big\}.$$

This setting can be considered a generalization of smooth Riemannian manifolds; indeed, M is isometrically embedded into $\mathscr{P}_2(M)$ via the map $x \mapsto \delta_x$; moreover, after the work of Otto in [85], $(\mathscr{P}_2(M), W_2)$ can be described as a sort of infinite dimensional Riemannian manifold and that W_2 can be interpreted as the geodesic distance induced by a suitably defined metric.
For the construction in this setting we refer to [5] for the case of $\mathscr{P}_2(\mathbb{R}^d)$, to [52] for the case of $\mathscr{P}_2(M)$, where (M, g) is a Riemannian manifold and to [82] for an alternative construction;

ii) the case of Alexandrov spaces, i.e. metric spaces having a lower bound on the sectional curvature, the so-called CBB(K) space (which stands for *curvature bounded from below by K*) whose systematic study has been initiated by Burago, Gromov and Perelman in [30]. The definition of such spaces takes inspiration from the following theorem due to Topogonov. We denote by Sec_x the sectional curvature of M at the point x and by $\mathbb{M}^2_K$ the 2-dimensional Riemannian manifold with sectional curvature constantly equal to K. We denote by d_K the distance on the model space $\mathbb{M}^2_K$.

Theorem 0.1.1 (Topogonov)**.** *Let M be a smooth, complete and connected Riemannian manifold and $K \in \mathbb{R}$. The following are equivalent:*

a) $\mathrm{Sec}_x(\sigma) \geq K$ *for every* $x \in M$ *and every 2-dimensional vector subspace* $\sigma \subseteq T_x M$*;*

b) for every $x, y, z \in M$ there is a minimal geodesic $[0, 1] \ni t \mapsto \gamma_t \in M$ with $\gamma_0 = y$ and $\gamma_1 = z$ such that, provided that $\mathsf{d}_g(x, y) + \mathsf{d}_g(y, z) + \mathsf{d}_g(x, z) \leq 2\frac{\pi}{\sqrt{K}}$ *in the case* $K > 0$*, if we consider $x', y', z' \in \mathbb{M}^2_K$ being such that*

$$\mathsf{d}_g(x, y) = \mathsf{d}_K(x, y), \quad \mathsf{d}_g(y, z) = \mathsf{d}_K(y, z), \quad \mathsf{d}_g(x, z) = \mathsf{d}_K(x, z)$$

(that always exists under these assumptions), there exists a minimal geodesic $[0, 1] \ni t \mapsto \tilde{\gamma}_t \in \mathbb{M}^2_K$ *such that $\tilde{\gamma}_0 = y'$ and $\tilde{\gamma}_1 = z'$*

$$\mathsf{d}_g(\gamma_t, x) \geq \mathsf{d}_K(\tilde{\gamma}_t, x') \quad \textit{for every } t \in [0, 1].$$

Notice that condition b) can be stated without relying on the existence of a smooth Riemannian metric, but just the concept of the distance. Therefore, this leads to the definition

of CBB(K) space, namely a metric space that is complete, geodesic and for which condition b) holds (obviously substituting the notion of geodesics with the one in Def. 1.1.3). For a survey introducing this topic, see [29, Chapter 10] and [1].
The construction of parallel transport in this setting is due to Petrunin in [87].

Before explaining the approach we followed in [32] in the case of non-collapsed $\mathsf{RCD}(K,N)$ spaces, we need an historical account of the theory of metric measure spaces with Riemannian Ricci curvature bounded from below.

As done for the case of the Alexandrov spaces, i.e. for lower bounds on the sectional curvature, there is a long story in defining what it means for a metric space to have a lower bound on the Ricci curvature. Since the Ricci curvature deals with distortion of volumes (see e.g. [67]), the right framework to look at in this case is that of metric measure spaces $(\mathrm{X},\mathsf{d},\mathfrak{m})$.

The first steps in this direction is through the works of Cheeger and Colding ([35], [36], [37], [38]) on Gromov-Hausdorff limits of Riemannian manifolds. The key step in this direction is the following consideration due to Gromov in [68]: fix $N \in \mathbb{N}$, $K \in \mathbb{R}$, $D \in (0,+\infty)$ and consider $\mathscr{A}_{K,N,D}$ the class of Riemannian manifolds with dimension bounded from above by N, Ricci curvature bounded from below (in this sense of symmetric bilinear forms) by K and diameter bounded from above by D. Then $\mathscr{A}_{K,N,D}$ is precompact in the Gromov-Hausdorff topology. A Ricci limit space is a limit of elements belonging to this class for some triple K, N, D.
One of the features of this theory is that it is extrinsic, meaning that the theory relies on the existence of a smooth approximating sequence of Riemannian manifolds to deduce properties of the Ricci limit space. We point out that a version of Cheeger-Gromoll splitting theorem ([39]) holds in this setting ([35]) and it is a crucial tool in order to study the structure theory of Ricci limit spaces.

What was really missing at that stage was an intrinsic treatment of Ricci curvature of metric measure spaces, i.e. a theory that does not rely on the approximation by a smooth sequence. The first steps in this direction were done in the smooth category in [86], [41] and [97]; in these works, optimal transportation is used to characterize the property of a Riemannian manifolds of having Ricci curvature bounded from below by K (in the sense of bilinear form) and dimension bounded from above by N in terms of convexity properties of entropy functionals along W_2-geodesics. For the adimensional case of the lower bound on K, the Shannon entropy functional is studied, defined as:

$$\mathrm{Ent}(\nu|\mathfrak{m}) = \int_X \rho \log(\rho)\, \mathrm{d}\mathfrak{m}$$

if $\nu \ll \mathfrak{m}$ with $\nu = \rho\mathfrak{m}$ and $\rho\log(\rho)$ is integrable with respect to $\mathfrak{m}$ and $+\infty$ otherwise. For the case of characterizing both the lower bound by K on the Ricci curvature and the upper bound by N on the dimension, other entropy functionals replace the Shannon's one, like for instance the Rényi entropy. These convexity properties do not involve the smooth structure and the Ricci tensor and they can be formulated even in the metric setting, definining a notion of lower bound on the Ricci curvature, thus leading to the definition of $\mathsf{CD}(K,\infty)$ (for the case of Shannon entropy) and of $\mathsf{CD}(K,N)$ spaces (for the case of Rényi one). The key results in this direction are due to independent works of Lott and Villani ([83]) and Sturm ([94],[95]). One of the key properties of this class of spaces is the compatibility with the case of (weighted) Riemmanian manifolds, the stability of the definition with respect to Gromov-Hausdorff convergence and the validity of geometric inequalities, such as the Brunn-Minkowski inequality and the Bishop-Gromov monotonicity formula. Bacher and Sturm introduced in [17] a weaker curvature dimension condition, called the reduced curvature dimension condition, called $\mathsf{CD}^*(K,N)$, having better globalization and tensorization properties.

The class of $\mathsf{CD}(K,N)$ spaces contains Finsler manifolds, namely manifolds whose norm on the tangent space does not satisfy the parallelogram rule, i.e. it does not come from a scalar product. Indeed, it is proved in the last theorem in [96] that $(\mathbb{R}^N, \|\cdot\|, \mathscr{L}^N)$, where $\|\cdot\|$ is any norm on $\mathbb{R}^N$, is a $\mathsf{CD}(0,N)$ space. In this direction, it was not possible to extend to an intrinsic setting theorems in differential geometry involving the Riemannian structure. Therefore, the idea due to Ambrosio, Gigli and Savaré in [10] is to enforce the $\mathsf{CD}(K,\infty)$ condition with the linearity of the heat flow, thus ruling out Finsler structures. They introduced the so called class of $\mathsf{RCD}(K,\infty)$ spaces, showing in particular stability of the condition under Gromov-Hausdorff convergence. In [54], Gigli proposed the definition of $\mathsf{RCD}(K,N)$ space enforcing the $\mathsf{CD}(K,N)$ condition with infinitesimal Hilbertianity (similarly, we can define $\mathsf{RCD}^*(K,N)$ spaces), giving a finite dimensional counterpart to the theory, strongly motivated by the proof the splitting theorem in the setting of $\mathsf{RCD}(0,N)$ spaces by the same author in [53]. The splitting theorem, that was already a key tool in the structure theory of Ricci limit spaces, motivates the definition providing a tool in order to study the regularity of such class of spaces (see for instance [57]). It has been proved in [33] that the definition of $\mathsf{RCD}(K,N)$ and $\mathsf{RCD}^*(K,N)$ spaces are equivalent in the case in which the reference measure is finite. The bound from below on the curvature and from above on the dimension in the smooth category can be also characterized by means of the so called dimensional Bochner inequality. Consider a smooth Riemannian manifold (M,g) and $f \in C^\infty(M)$; then we have

$$\Delta\left(\frac{|\nabla f|_g^2}{2}\right) \geq g(\nabla f, \nabla \Delta f) + \frac{(\Delta f)^2}{n} + \mathrm{Ric}(\nabla f, \nabla f).$$

A careful study of the heat flow that can be seen either as the gradient flow of the Dirichlet energy in L^2 or as the gradient flow of the Shannon entropy functional in the space $(\mathscr{P}_2(\mathrm{X}), W_2)$ (see [76]) provides a bridge between two ways of seeing the heat flow, as vertical displacement in the first case and as horizontal one in the second one (see [51], [56], [9]). This leads in [11] to the proof of the equivalence of the formulation of $\mathsf{RCD}(K,\infty)$ space with convexity properties of entropy functionals (connected to horizontal displacement) and with a distributional adimensional Bochner inequality (connected to vertical displacement). The same equivalence with a distributional dimensional Bochner inequality in the case of $\mathsf{RCD}(K,N)$ has been established in [49] (and indipendently in [14]).

In the works of Cheeger and Colding on Ricci limit spaces ([36], [37], [38]), the class of non collapsed Ricci limit spaces is presented. An intrinsic generalization of this class has been proposed by De Philippis and Gigli in [45] (after [77]), namely the class of non collapsed $\mathsf{RCD}(K,N)$ spaces.

Having at our disposal second order calculus tools and in particular a notion of covariant derivative, we can build parallel transport by borrowing and generalizing arguments coming from the smooth category. For what concerns our approach, it is important to identify three objects: the object we want to transport, along what we perform the construction and the notion of solution to the problem. We will present in the next section in details the case we studied in [32].

0.2 Major contributions

In this section, we describe the main contributions obtained in the works [32] and [31].

Parallel transport in non-collapsed $\mathsf{RCD}(K,N)$ spaces. We explain the strategy used in [32] to build a theory of existence and uniqueness of parallel transport in the class of non-collapsed $\mathsf{RCD}(K,N)$ spaces. By inspecting the regularity of the underlying space, we can quantify the regularity of objects at our disposal (functions, vector fields, tensors). In [55], Gigli developed a first order differentiable structure of metric measure spaces, without further regularity assumptions. Indeed, starting from the Sobolev space $W^{1,2}(\mathrm{X})$, it is possible to discuss what a L^2-integrable 1-form is and by duality what an L^2-integrable vector field is. To speak about gradients of Sobolev functions as single valued objects, a further regularity assumption on the underlying space is needed, that is the so-called infinitesimally Hilbertianity. Roughly speaking, this property amounts to require that the norm on cotangent objects (L^2-integrable 1-forms) is induced by a scalar product, thus asking a sort of Riemannian-like behaviour of $(\mathrm{X},\mathsf{d},\mathfrak{m})$. To discuss about second order objects, namely Hessian of functions and differentiation of vector fields, it is needed to ask that the underlying space satisfies a further regularity assumption, namely being an $\mathsf{RCD}(K,\infty)$ space, that gives extra regularity properties of the heat flow. In this class, it has been shown in [55] that there is a well defined notion of covariant derivative: this should be intended in an appropriate Sobolev sense, i.e. it is possible to define the concept of a vector field with covariant derivative in L^2. 'Covariant derivative' and 'parallel transport' are two extremely close concepts, and thus it is natural to expect that the latter also exists in the same generality, provided one pays due attention to the way to formulate the concepts in relation with distributional notions of covariant differentiation. An attempt in this direction has been made in [61]: there the problem of parallel transport is formulated not along a single given curve - as it is customary in the smooth setting - but rather along a test plan. A test plan is a probability measure on the metric space of continuous curves with values in X, concentrated on 'regular curves' (the class AC that will be presented later) and 'well distributed', in the sense that at every time the curves do not superpose too much (see Definition 1.2.1). One of the successes of these objects is that they can be used in duality with functions to define Sobolev functions over a metric measure space, i.e. the class $W^{1,2}(\mathrm{X},\mathsf{d},\mathfrak{m})$ (see [9]).
The well-posedness of covariant differentiation on $\mathsf{RCD}(K,\infty)$ spaces seems not sufficient to derive well-posedness of parallel transport and the main result in [61] can be roughly summarized as follows:

i) it makes sense to define what the 'covariant differentiation along a Lipschitz test plan $\boldsymbol{\pi}$' of 'smooth' vector fields is; the condition to be Lipschitz can be interpreted by saying that the speed of curves is essentially bounded in time and with respect to $\boldsymbol{\pi}$;

ii) to such differentiation operator, one can associate the 'W' and 'H' Sobolev spaces $\mathscr{W}^{1,2}(\boldsymbol{\pi})$ and $\mathscr{H}^{1,2}(\boldsymbol{\pi})$, defined respectively in duality or as closure of 'smooth' vector fields. In this generality, it is not known if the two spaces coincide (mainly due to the lack of a regularizing operator), but still it is proved that

$$\mathscr{H}^{1,2}(\boldsymbol{\pi}) \subseteq \mathscr{W}^{1,2}(\boldsymbol{\pi});$$

iii) in the Sobolev space $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ uniqueness of parallel transport can be proved (but not existence).

On top of these results, we shall see in Section 3.3 (taken from appendix A in [32]) that in $\mathscr{W}^{1,2}(\boldsymbol{\pi})$, by a general argument based on approximation by viscosity, existence of parallel transport can be established (but not uniqueness). It is therefore natural to look for some sort of intermediate space between $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ and $\mathscr{W}^{1,2}(\boldsymbol{\pi})$ where both existence and uniqueness of parallel transport can be obtained.

We didn't do that in the generality of $\mathsf{RCD}(K,\infty)$ spaces nor for Lipschitz test plans. Instead, we impose additional regularity on both the space and the test plan involved to get a better theory. We shall in fact work with only certain types of test plans on the class of non-collapsed $\mathsf{RCD}(K,N)$ spaces, obtaining

existence and uniqueness of parallel transport

in this setting. The actual statement of our main result requires a bit of terminology, so we postpone it to the main discussion. In Section 5.5 we shall compare the construction of [32] to those in [61] and prove that, in a very natural sense, the relevant space involved sits between $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ and $\mathscr{W}^{1,2}(\boldsymbol{\pi})$.

Let us give more details about our setting. Rather than investigating 'generic collections of smooth curves', as test plans can be thought of, we focus on flows of Sobolev vector fields, which are in some sense the most regular vector fields at disposal in such a non smooth setting. Specifically, in the setting of $\mathsf{RCD}(K,\infty)$ spaces the concept of Regular Lagrangian Flow ([15], see also [2], [47] for the Euclidean setting) provides a reasonable counterpart to the classical Cauchy–Lipschitz theory and it gives meaning to the flow (F_t^s) (with initial time t and final time s) of a family $(\boldsymbol{b}_t)$ of Sobolev vector fields with uniformly bounded divergence (see Theorem 2.2.8 for the precise statement). A non-trivial regularity result - established in [27] (see also [25]) - concerning such flows in the finite-dimensional case is that they are uniformly Lusin–Lipschitz, i.e. there is a Borel partition (E_i) of the space up to a $\mathfrak{m}$-negligible set such that the restriction of F_0^t to E_i is uniformly Lipschitz in t. Moreover, in the class of non-collapsed RCD spaces, sharper regularity estimates with respect to time are available at a.e. point x, that in particular allow to say that the norm of the differential of F_t^s at the point x deviates from 1 with a term of order $|s-t|$ (see Proposition 4.1.1 for the precise statement). This key regularity property, obtained in the recent [25], is needed if one wants to differentiate in time the differential of the flow and recover the covariant derivative of the underlying vector field $\boldsymbol{b}_t$. In this direction, Chapter 2 aims at being an informal introduction to the theory of wellposedness and regularity of such flows both in the Euclidean and in the metric setting.

Consider for a moment a Riemannian manifold (M,g) and assume that $(\boldsymbol{b}_t)$ is smooth. By direct computation, if $[0,1]\times M \ni (t,x) \mapsto v_t(x) \in T_xM$ is smooth in t,x and defining the vector field $V_t := \mathrm{d}F_0^t(v_t)$ we can compute the covariant derivative along the curve $t \mapsto F_0^t(x)$ by

$$\nabla_{\dot{F}_0^t(x)} V_t(F_0^t(x)) = (\mathrm{d}F_0^t(\dot{v}_t) + \nabla_{V_t}\boldsymbol{b}_t)(F_0^t(x)).$$

In the smooth setting a vector field $t \mapsto V_t \in T_{F_0^t(x)}M$ is a parallel transport along $t \mapsto F_0^t(x)$ if and only if for $v_t = \mathrm{d}F_t^0(V_t) \in T_xM$ we have

$$\dot{v}_t = -\mathrm{d}F_t^0(\nabla_{\mathrm{d}F_0^t(v_t)}\boldsymbol{b}_t), \qquad \text{for every } t. \tag{1}$$

In other words, the quantity $\mathrm{d}F_t^0(\nabla_{\mathrm{d}F_0^t(v_t)}\boldsymbol{b}_t)$ takes into account the time derivative of the distortion given by the differential of the flow. The advantage of working with the vectors v_t is that they all belong to the same tangent space, so that it is clear what the derivative $\dot{v}_t$ is and this advantage is particularly felt in our nonsmooth setting, where it is less understood how to define the covariant derivative of a vector field of the form $t \mapsto V_t \in T_{F_0^t(x)}M$.

Therefore, in the nonsmooth setting, we can measure the regularity of a vector fields along a family of integral curves of a regular Lagrangian flows in terms of regularity at initial point by means of the differential of the flow. We define Sobolev curves of vector fields with values in the 'tangent' at initial point and we call it $W^{1,2}([0,1], L^0(T\mathrm{X}))$ (for (v_t) belonging to this space, it makes sense to define $(\dot{v}_t)$). Then, we define Sobolev vector fields along integral curves as the image via the differential of the flow of the latter space. By analogy with the Riemannian case, we can define the convective derivative of a vector field $t \mapsto Z_t = \mathrm{d}F_0^t(z_t)$ with $(z_t) \in W^{1,2}([0,1], L^0(T\mathrm{X}))$ as

$$D_t Z_t := \mathrm{d}F_0^t(\dot{z}_t) + \nabla_{Z_t}\boldsymbol{b}_t.$$

Consistently with (1), a parallel transport is a vector field $t \mapsto Z_t = \mathrm{d}F_0^t(z_t)$ with $(z_t) \in W^{1,2}([0,1], L^0(T\mathrm{X}))$ for which

$$D_t Z_t = 0.$$

Using the strategy of solving an equivalent version of (1), we obtain the existence of a parallel transport (see Theorem 5.4.4).

Then, it is natural to study the problems of uniqueness of parallel transport and of the preservation of scalar products of vector fields along the transport. As we explained before, this is connected to the validity of a Leibniz formula, which can be stated in our setting as follows (see Theorem 5.3.4). Let $t \mapsto V_t = \mathrm{d}F_0^t(v_t), Z_t = \mathrm{d}F_0^t(z_t)$ with $(v_t), (z_t) \in W^{1,2}([0,1], L^0(T\mathrm{X}))$. Then for $\mathfrak{m}$-a.e. x, the map $t \mapsto \langle V_t, Z_t\rangle \circ F_0^t(x)$ belongs to $W^{1,2}([0,1])$ and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\langle V_t, Z_t\rangle \circ F_0^t(x)\big) = \langle D_t V_t, Z_t\rangle \circ F_0^t(x) + \langle V_t, D_t Z_t\rangle \circ F_0^t(x), \qquad \mathscr{L}^1\text{-a.e. } t.$$

When trying to prove the Leibniz formula, very soon we encounter the need to compute, given $V_t = \mathrm{d}F_0^t(v_t)$, the derivative in time of $\mathrm{d}f(V_t) \circ F_0^t = \mathrm{d}(f \circ F_0^t)(v_t)$ for f sufficiently regular. Therefore, a key step of our work is, given a sufficient regular f, showing $t \mapsto \mathrm{d}(f \circ F_0^t)$ is differentiable at a.e. time and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}(f \circ F_0^t) = \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_t) \circ F_0^t). \tag{2}$$

To actually make this plan work, in particular to prove (2), a few non-trivial technical obstacles have to be dealt with, in particular in relation with the need of interchanging differentiation in time and differentiation of functions/vector fields. Usually, this sort of issues are managed through closure properties of the differentiation operator considered in conjunction with Hille's theorem about exchanging (Bochner) integration and a closed operator. We stick to the philosophy of this general plan, although we need to revisit and adapt some crucial steps. Indeed, we have to work in two directions:

A) *definition of differential and the closure of such operator.* We shall indeed need to work with functions of the kind $f \circ F_t^s$ for $f \in W^{1,2}(\mathrm{X})$ and (F_t^s) Regular Lagrangian Flow as discussed above. It is well known that one should not expect Sobolev regularity for this kind of functions, but at least a notion of differentiation can naturally be given using the Lusin–Lipschitz property of F_t^s, that in turn ensures that also $f \circ F_t^s$ is Lusin–Lipschitz. The problem with the concept of differential defined by locality for Lusin–Lipschitz maps is that it is by no means a closed operator. What we need to do to get closure is to properly restrict the domain of the operator. The idea we use is inspired by the original work of Hajłasz [70] about Sobolev maps in metric measure spaces and the estimates by Crippa–De Lellis [43], revisited by Bruè–Semola [27]: for given $\phi \in L^0(\mathrm{X})$ non-negative and $R > 0$, we can quantify Lusin–Lipschitz regularity by considering the space $H_{\phi,R}(\mathrm{X}) \subset L^0(\mathrm{X})$ of functions $f \in L^2(\mathfrak{m})$ such that for some non-negative $G \in L^2(\mathrm{X})$ we have

$$|f(y) - f(x)| \leq \mathsf{d}(x,y)(G(x) + G(y))e^{\phi(x)+\phi(y)}$$

for every x, y outside some negligible subset of X and whose mutual distance does not exceed R. We shall denote by $A_{\phi,R}(f) \subset L^2(\mathrm{X})$ the set of G's as above and endow $H_{\phi,R}(\mathrm{X})$ with the natural norm

$$\|f\|_{H_{\phi,R}(\mathrm{X})} := \sqrt{\|f\|^2_{L^2(\mathfrak{m})} + \inf_{G \in A_{\phi,R}(f)} \|G\|^2_{L^2(\mathfrak{m})}} \quad \text{for every } f \in H_{\phi,R}(\mathrm{X})$$

(see Definition 4.3.1). It is clear that functions in $H_{\phi,R}(\mathrm{X})$ are Lusin–Lipschitz. Moreover, crucially, the differential restricted to bounded subsets of $H_{\phi,R}(\mathrm{X})$ is a closed operator on L^2 (see Proposition 4.3.5). Here the role of the estimates in [25] is to ensure that for $f \in W^{1,2}(\mathrm{X})$ the functions $f \circ F_0^t$ are uniformly bounded in $H_{\phi,R}(\mathrm{X})$ for some properly chosen ϕ, see Lemma 5.3.1;

B) *formulation of Hille's theorem and the underlying concept of integration.* Due to the low integrability properties of the differential of the flow, it is not natural to work with vector fields in L^p for $p \geq 1$, but rather it is better to deal with L^0 vector fields. In turn, since $L^0(T\mathrm{X})$ is not a Banach space (in fact, it is not even locally convex) we need to define what the integral of a map $t \mapsto v_t \in L^0(T\mathrm{X})$ is, or more generally of a map $t \mapsto v_t \in \mathscr{H}$ with $\mathscr{H}$ some L^0-Hilbert module: we do this in Section 4.4 and the approach that we choose might be described as a sort of 'pointwise Pettis integral'. More in detail, we first declare $t \mapsto f_t \in L^0(\mathrm{X})$ to be integrable provided so is the map $t \mapsto f_t(x) \in \mathbb{R}$ for $\mathfrak{m}$-a.e. x, and then we say that $t \mapsto v_t \in \mathscr{H}$ is integrable provided so is the function $t \mapsto |v_t| \in L^0(\mathrm{X})$. In this case, its integral can be defined noticing that for any $z \in \mathscr{H}$ the function $t \mapsto \langle z, v_t \rangle \in L^0(\mathrm{X})$ is integrable and its integral depends L^0-linearly and continuously on z, so that Riesz's theorem for L^0-Hilbert modules gives the desired notion.

To this concept of integration we need to attach a form of Hille's theorem. Its standard proof for closed operators $L : E \to F$, with E, F Banach spaces, uses integration in the product space and concludes using the fact that Bochner integral commutes with projections. In turn, this latter property follows from the more general fact that Bochner integration commutes with linear and continuous operators, a fact that trivially follows from the definition of Bochner integral. This exact line of thought does not work in our setting, partly because we cannot work with 'pointwise integration' as we are not assuming the closed operator L to be L^0-linear. Still, the general approach does, our idea being to first prove that our notion of integration can be realized as limit of Riemann sums in analogy with a classical statement by Hahn, so that its commutation with projections into factors can trivially be established. To the best of our knowledge, previous results in this direction required the additional assumptions on the operator L to be continuous, L^0-linear, and with uniformly bounded pointwise norm (see [50]).

It is worth to point out that our assumptions cover the case of geodesics in the following sense. Let (μ_t) be a W_2-geodesic so that μ_0, μ_1 have bounded supports and bounded densities. Also, let $\varepsilon \in (0, \frac{1}{2})$ and consider the restricted geodesic $\nu_t := \mu_{(1-t)\varepsilon+t(1-\varepsilon)}$. Then, there are vector fields $(\boldsymbol{b}_t)$ satisfying our regularity assumptions (see Proposition 4.1.1 for the precise set of these) such that the associated flow (F_t^s) satisfy $(F_t^s)_*\nu_t = \nu_s$ for every $t, s \in [0, 1]$. Moreover, for ν_0-a.e. $x \in \mathrm{X}$ the curve $t \mapsto F_0^t(x)$ is a geodesic and the map F_0^1 is the only optimal map from ν_0 to ν_1. The fact that these $(\boldsymbol{b}_t)$ exist is a consequence of the abstract Lewy–Stampacchia inequality [58] and the estimates in [55] (just let $\boldsymbol{b}_t := \nabla\eta_t$ with η_t obtained from Kantorovich potentials via the double obstacle problem, see [58, Theorem 3.13]). In particular, considering the (only, by [15]) lifting $\boldsymbol{\pi}$ of (ν_t), we have that $\boldsymbol{\pi} = (F_t^{\cdot})_*\nu_t$ for every $t \in [0, 1]$, where $F_t^{\cdot} : \mathrm{X} \to C([0, 1], \mathrm{X})$ is the map given by $x \mapsto \big(s \mapsto F_t^s(x)\big)$. Therefore our results can be read as existence and uniqueness of parallel transport along $\boldsymbol{\pi}$-a.e. geodesic. It is unclear to us whether the initial restriction from (μ_t) to (ν_t) is truly necessary, but let us point out that it is well known in geometric analysis and metric geometry that this sort of restriction of geodesics are much more regular than 'full' geodesics (see e.g. [40] and the already mentioned construction of parallel transport in item ii) from [87]).

Let us notice that the class of non-collapsed RCD spaces contains that of finite dimensional Alexandrov spaces with curvature bounded from below equipped with the appropriate Hausdorff measure (by the results in [80], [88], [98]), thus our construction provides a notion of parallel transport alternative to that in [87]. We obtain existence and uniqueness - in place of 'only' existence (plus a related second order differentiation formula that we do not explore) - at the price of describing parallel transport not along a single geodesic, but rather along a.e. geodesic, in a sense. It is certainly natural to try to compare the two notions, and while we expect them to agree, we do not investigate in this direction.

Local convergence in measure of differentials of flows associated to Sobolev vector fields. Let us conclude this introduction describing the content of the work in preparation [31]. The problem that we are going to present was motivated by the construction of a wellposed theory of parallel transport in $\mathsf{ncRCD}(K, N)$ spaces. Our original approach was to proceed by density. The reason is that, previously, we were able to prove the existence theorem of parallel transport and the Leibniz formula in the case of flows associated to piecewise-in-time autonomous vector

fields; then, the idea was to retrieve the mentioned results in the nonautonomous case for flows associated to vector fields that are the closure in a suitable sense of piecewise-in-time autonomous ones. Since existence theorem and the Leibniz formula rely on estimates on the differential of the flow, it is natural to study the problem of stability of differential of flows of converging vector fields; hence, we treat the problem in the Euclidean setting for Sobolev exponent $p > 1$ in [31]. However, we realized in [32] that a wellposed theory could be directly proved in the case of time dependent vector fields, without relying on the aforementioned strategy.
Let us explain the problem. We consider a sequence of vector fields, equibounded in $L^\infty_{t,x}$ with equibounded divergence in $L^1_t L^\infty_x$ and assume that they converge to a limit one in $L^1_t W^{1,p}_x$. We further assume equibounds of the distributional derivatives of the sequence of vector fields in $L^\infty_t L^p_x$. We prove that the approximate gradients (a.e. defined for Lusin-Lipschitz maps) of the associated flows converges locally in measure to the approximate gradient of the limit one (see Theorem 6.0.1). The strategy used is based on the study of the linearized equation

$$\partial_t \nabla F^t_0(x) = D\boldsymbol{b}_t(F^t_0(x))\, \nabla F^t_0(x), \tag{3}$$

that is satisfied by reading (2) in the Euclidean case and Theorem 6.0.1 can be proved by taking the limit in the distributional formulation of the linearized ODE. An important remark is due: from hyphothesis of order one on the converging sequence of vector fields, namely the assumptions that the limit vector field belongs to $L^1_t W^{1,p}_x$, and convergence in $L^1_{t,x}$ (plus equibounds on the vector fields and their divergence) the stability at order zero of flows, namely local convergence in measure, can be proved (see [43, Theorem 2.9]). However, for the local convergence in measure of the differentials of the flows, we don't need any condition at second order neither on the converging sequence nor on the limit vector field, thanks to the special form of (3). The validity of an analogous statement of Theorem 6.0.1 in the RCD setting is yet to be understood.

0.3 Organization of the thesis

The thesis is organized as follows. Chapter 1 has the general goal of presenting the calculus tools on metric measure spaces with the language of nonsmooth differential geometry developed by Gigli. Section 1.1 contains general preliminaries of measure theory and analysis on metric spaces. Then we present several calculus tools available at different hierarchies of regularity of the underlying space. Therefore Section 1.2 develops first order calculus on metric measure spaces; Section 1.3 contains definitions and some relevant properties for the scope of this manuscript of PI spaces and RCD spaces; Section 1.4 presents test functions on $\mathsf{RCD}(K,\infty)$ and the development of second order calculus in this setting. Chapter 2 has the goal of summarizing in an informal way the theory of flows associated to Sobolev vector fields, thus leading the reader to the key estimates we use in [32]. In particular, Section 2.1 is a recap on the classical theory of flows of Lipschitz vector fields, and the theory of flows of Sobolev vector fields according to the axiomatization of Ambrosio (after DiPerna-Lions) and their regularity; Section 2.2 presents such a theory in the context of $\mathsf{RCD}(K,\infty)$ spaces and the regularity of flows associated to Sobolev vector fields in the setting of $\mathsf{RCD}(K,N)$ spaces, plus the key refined estimates in the case of non-collapsed $\mathsf{RCD}(K,N)$ spaces. Chapter 3.1 has the goal of presenting the functional spaces of [61] to speak about vector fields along a test plan and existence and uniqueness of parallel transport respectively in the class $\mathscr{W}$ and $\mathscr{H}$ is discussed. Chapters 4 and 5 both rely on the results of [32]. Chapter

4 contains all the functional analytic tools needed to build our theory of wellposedness in the non–collapsed setting. Chapter 5 contains the theory of existence and uniqueness (thanks to the proof of a Leibniz formula) of parallel transport in the case of non–collapsed $\mathsf{RCD}(K,N)$ spaces. In particular, Section 5.5 acts as a comparison with the novel theory with the functional spaces of Chapter 3.1, showing that in a suitable sense our class of solutions fits in between the $\mathscr{H}$ and $\mathscr{W}$ space. Chapter 6 relies on the results of the forthcoming work [31] and we present a stability result in the Euclidean setting, under suitable convergence of the vector fields, for the gradients of associated flows.

Part I

Preliminaries

Chapter 1

Calculus on metric measure spaces

1.1 General preliminaries

A metric measure space, mms in short, is a triple $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, such that (X, d) is a complete and separable metric space, $\mathfrak{m}$ is a nonnegative Borel measure (where Borel sets are the one induced by the topology of d) that is finite on bounded sets. In particular, we denote by $\mathscr{B}(\mathrm{X})$ the set of Borel subsets of X and by $\mathscr{P}(\mathrm{X})$ the set of nonnegative Borel probability measure on X.
Given two metric spaces $(\mathrm{X}, \mathsf{d}_{\mathrm{X}})$ and $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}})$, a Borel map $f\colon \mathrm{X} \to \mathrm{Y}$ and $\mu \in \mathscr{P}(\mathrm{X})$, we define $f_*\mu \in \mathscr{P}(\mathrm{Y})$ as

$$f_*\mu(E) := \mu(f^{-1}(E))$$

for every Borel set E and we call it the pushforward measure of μ via the map f.
Consider a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m}_{\mathrm{X}})$ and a Borel map $\varphi\colon \mathrm{X} \to \mathrm{Y}$. We say that φ is $\mathfrak{m}_{\mathrm{X}}$-essentially invertible if there exists a Borel map $\psi\colon \mathrm{Y} \to \mathrm{X}$ such that $\psi \circ \varphi = \mathrm{Id}_{\mathrm{X}}$ $\mathfrak{m}_{\mathrm{X}}$-a.e. and $\varphi \circ \psi = \mathrm{Id}_{\mathrm{Y}}$ $\varphi_*\mathfrak{m}_{\mathrm{X}}$-a.e.. In this case, we call ψ a $\varphi_*\mathfrak{m}_{\mathrm{X}}$-essential inverse, which turns out to be unique up to $\varphi_*\mathfrak{m}_{\mathrm{X}}$-negligible sets.
Given two metric spaces $(\mathrm{X}, \mathsf{d}_{\mathrm{X}})$ and $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}})$, we denote by $\mathrm{X} \times \mathrm{Y}$ the cartesian product of X and Y and when is not furtherly specified we give it the structure of a metric space with the following distance

$$\mathsf{d}_{\mathrm{X}\times\mathrm{Y}}((x, y), (x', y'))^2 := \mathsf{d}_{\mathrm{X}}(x, x')^2 + \mathsf{d}_{\mathrm{Y}}(y, y')^2$$

for every $x, x' \in \mathrm{X}$ and $y, y' \in \mathrm{Y}$. We define the projection maps $\pi_1\colon \mathrm{X} \times \mathrm{Y} \to \mathrm{X}$ as $\pi_1(x, y) = x$ and $\pi_2\colon \mathrm{X} \times \mathrm{Y} \to \mathrm{Y}$ as $\pi_2(x, y) = y$ for every $x \in \mathrm{X}$ and $y \in \mathrm{Y}$. They are both linear and continuous, with the base space endowed with the topology induced by the distance $\mathsf{d}_{\mathrm{X}\times\mathrm{Y}}$.

Given a complete metric space (X, d), we consider $C([0, 1], \mathrm{X})$ the set of continuous curves with values in X. It is a complete metric space when endowed with the sup norm. If (X, d) is separable, then $C([0, 1], \mathrm{X})$ is separable. We refer to an element $\gamma\colon [0, 1] \to \mathrm{X}$ of this space as γ or (γ_t). We define, for every $t \in [0, 1]$, the continuous map

$$\mathrm{e}_t\colon C([0, 1], \mathrm{X}) \to \mathrm{X} \qquad \text{as} \qquad \mathrm{e}_t(\gamma) := \gamma_t$$

and we call it the evaluation map at time t.

Definition 1.1.1 (Absolutely continuous curves)**.** *Given $\gamma \in C([0,1],\mathrm{X})$, we say that $\gamma \in AC([0,1],\mathrm{X})$ if there exists $g \in L^1(0,1)$ such that for every $s < t$*

$$\mathsf{d}(\gamma_t,\gamma_s) \leq \int_s^t g_r \,\mathrm{d}r.$$

We will say without ambiguity that $\gamma \in AC$ or γ is an AC curve. An important property of AC curves is the following proposition.

Theorem 1.1.2 ([7, Theorem 1.1.2])**.** *Consider $\gamma\colon [0,1] \to \mathrm{X}$. Then the limit*

$$|\dot\gamma_t| := \lim_{s\to t} \frac{\mathsf{d}(\gamma_s,\gamma_t)}{|t-s|}$$

exists for $\mathscr{L}^1$-a.e. $t \in (0,1)$. Moreover, the function $(t \mapsto |\dot\gamma_t|) \in L^1(0,1)$, it is admissible as g in the definition of AC curves. It is minimal in the sense that given an admissible g $|\dot\gamma_\cdot| \leq g$ in the $\mathscr{L}^1$-a.e. sense.

A particular class of AC curves is that of (constant speed) geodesics.

Definition 1.1.3 (d-geodesic)**.** *We say that $\gamma\colon [0,1] \to \mathrm{X}$ is a (constant speed minimizing) d-geodesic if, for every $t,s \in [0,1]$, $\mathsf{d}(\gamma_t,\gamma_s) = |t-s|\mathsf{d}(\gamma_1,\gamma_0)$.*

We define some relevant weaker definitions of modulus of the gradient in the case of a Lipschitz function $f\colon \mathrm{X} \to \mathbb{R}$. We denote by $\mathrm{Lip}(\mathrm{X})$ the space of real valued Lipschitz continuous functions on (X,d) (usally there is no ambiguity about the distance), with $\mathrm{Lip}_b(\mathrm{X})$ and $\mathrm{Lip}_{bs}(\mathrm{X})$ respectively the space of Lipschitz and bounded functions and Lipschitz functions with bounded support. Given a Lipschitz function $f\colon \mathrm{X} \to \mathbb{R}$, we denote by $\mathrm{Lip}(f)$ its Lipschitz constant. Given $f\colon \mathrm{X} \to \mathbb{R}$ Lipschitz, we define the slope of f as $\mathrm{lip}f\colon \mathrm{X} \to [0,+\infty)$ defined as

$$\mathrm{lip}f(x) := \varlimsup_{y\to x} \frac{|f(y)-f(x)|}{\mathsf{d}(x,y)}$$

We define the asymptotic Lipschitz constant as

$$\mathrm{lip}_a f(x) := \lim_{r\to 0} \mathrm{Lip}(f|_{B_r(x)}).$$

In the last two definitions $\mathrm{lip}f(x) = \mathrm{lip}_a f(x) := 0$ if x is an isolated point.
We denote $L^p(\mathfrak{m})$ for $p \in [1,+\infty)$ the space of ($\mathfrak{m}$-a.e. equivalence class of) p-integrable functions and with $L^\infty(\mathfrak{m})$ the space of $\mathfrak{m}$-essentially bounded functions. We introduce here the vector space

$$L^0(\mathfrak{m}) := \{\text{Borel measurable functions } f\colon \mathrm{X} \to \mathbb{R} \text{ finite } \mathfrak{m}-\text{a.e.}\}\,/\sim$$

where $\sim$ denotes the equivalence relation given by $\mathfrak{m}$-a.e. equality. $L^0(\mathfrak{m})$ can be endowed by the notion of local convergence in measure that can be metrized by the following distance. Consider $\mathfrak{m}' \in \mathscr{P}(\mathrm{X})$ such that $\mathfrak{m}' \ll \mathfrak{m} \ll \mathfrak{m}'$ and define $\mathsf{d}_{L^0}\colon L^0(\mathfrak{m}) \times L^0(\mathfrak{m}) \to [0,+\infty)$ as

$$\mathsf{d}_{L^0}(f,g) := \int |f-g| \wedge 1 \,\mathrm{d}\mathfrak{m}'$$

We don't keep track in the notation d_{L^0} of the measure $\mathfrak{m}'$. It can be checked that d_{L^0} satisfies the axioms of a distance and that, endowed with this distance, $L^0(\mathfrak{m})$ is a complete and separable metric space. By explicit construction, it can be shown that such an $\mathfrak{m}'$ exists; for instance consider $\mathfrak{m}' := \sum_{n=0}^{\infty} 2^{-n}\mathfrak{m}(B_{n+1}(\mathsf{o}) \setminus B_n(\mathsf{o}))^{-1}\chi_{B_{n+1}(\mathsf{o})\setminus B_n(\mathsf{o})}\mathfrak{m}$ with the convention of neglecting the term in the sum when $\mathfrak{m}(B_{n+1}(\mathsf{o}) \setminus B_n(\mathsf{o})) = 0$ for some n; the choice of $\mathfrak{m}'$ satisfying the previous condition is clearly not unique. We mention the following classical proposition.

Proposition 1.1.4. *Consider $f \in L^0(\mathfrak{m})$ and $(f_n) \subseteq L^0(\mathfrak{m})$. The following are equivalent:*

i) $\mathsf{d}_{L^0}(f_n, f) \to 0$ *as* $n \to +\infty$*;*

ii) given any subsequence $(n_m)_m$, there exists a further subsequence $(n_{mk})_k$ such that for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ $\lim_k f_{n_{mk}}(x) = f(x)$;

iii) for every $\epsilon > 0$ and $E \subseteq \mathrm{X}$ with $\mathfrak{m}(E) < \infty$, $\mathfrak{m}(E \cap \{|f_n - f| > \epsilon\}) \to 0$ as $n \to +\infty$;

iv) for every $\epsilon > 0$, $\mathfrak{m}'(\{|f_n - f| > \epsilon\}) \to 0$ as $n \to +\infty$.

The last proposition implies the following property: different choice of $\mathfrak{m}'$ may change the distance d_{L^0}, but not the topology it induces. Moreover, the metric space $(L^0(\mathfrak{m}), \mathsf{d}_{L^0})$ is complete and separable. An important property that will be used in the following is that:

$$\text{the inclusion } L^p(\mathfrak{m}) \hookrightarrow L^0(\mathfrak{m}) \text{ is continuous for every } p \in [1,\infty]. \tag{1.1}$$

1.2 First order calculus on metric measure spaces

We consider in this section $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ to be a metric measure space. The literature on definitions of Sobolev functions on metric measure spaces is very broad, but it can summarized in at least three schools:

1) the original definition due to Cheeger, based on a relaxation procedure in [34]; the definition is reformulated in [9] in terms of slope of Lipschitz functions;

2) the definition due to Shanmugalingam (see [91]), based on p-modulus of curves (after being introduced in [79]);

3) the definition due to Ambrosio, Gigli and Savaré based on the notion of minimal weak upper gradient (see [9]).

Another distinct approach is the one due to Hailasz in [70], which provides a different notion with respect to the others. In this thesis, we will use in Section 4.3 a weaker notion of this space, that we call weighted Hailasz Sobolev space, which is inspired by the definition in [70]. We focus here on 3). All the approaches aims at identifying a function that is the counterpart of the 'modulus of distributional derivative'. All the notions of Sobolev spaces coincide and also all the functions playing the role of 'modulus of distributional derivative' (we refer to [91],[9] for the equivalence between 1) and 2) and to [8] for the equivalence of the previous ones with 3)). We explain now the approach in item 3), which is based on providing a nonsmooth counterpart to the fact that, given $f\colon \mathbb{R}^d \to \mathbb{R}$, $|\nabla f| \leq G$ everywhere if and only if for every smooth curve $\gamma\colon [0,1] \to \mathbb{R}^d$ we have $|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 G(\gamma_r)|\dot\gamma_r|\,\mathrm{d}r$.

We define the kinetic energy of a curve $\mathrm{KE}\colon C([0,1],\mathrm{X}) \to [0,+\infty]$ as $\mathrm{KE}(\gamma) = \int_0^1 |\dot\gamma_t|^2\,\mathrm{d}t$ if γ is an AC curve and $+\infty$ otherwise. For what concerns this approach, we refer to [9, Section 5]. We need the following definition.

Definition 1.2.1 (Test plan). *We say that $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],\mathrm{X}))$ is a test plan if the following two conditions hold:*

i) there exists a constant $C > 0$ such that $\mathrm{e}_{t*}\boldsymbol{\pi} \leq C\mathfrak{m}$ for every $t \in [0,1]$

ii) $\int \mathrm{KE}(\gamma)\,\mathrm{d}\boldsymbol{\pi}(\gamma) < +\infty$.

Definition 1.2.2. *We define the vector space $S^2(\mathrm{X},\mathsf{d},\mathfrak{m})$ ($S^2(\mathrm{X})$ in short, if there is no ambiguity) as the set of $f \in L^0(\mathrm{X})$ for which there exists $G \in L^2(\mathfrak{m})$ such that*

$$\int |f \circ \mathrm{e}_1 - f \circ \mathrm{e}_0|\,\mathrm{d}\boldsymbol{\pi} \leq \int_0^1 \int G(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t$$

for any test plan $\boldsymbol{\pi}$. In this case, we will say that G is a 2-weak upper gradient for f.

The set

$$\mathscr{A}_f := \{G : G \text{ is a weak upper gradient for } f\}$$

is trivially convex and closed in the $L^2(\mathfrak{m})$ topology, therefore it admits an element of minimal $L^2(\mathfrak{m})$-norm, that we denote by $|Df|$ and call the minimal weak upper gradient of f. It turns out to be also minimal $\mathfrak{m}$-a.e. We define

$$W^{1,2}(\mathrm{X},\mathsf{d},\mathfrak{m}) := L^2(\mathfrak{m}) \cap S^2(\mathrm{X},\mathsf{d},\mathfrak{m}).$$

We will write $W^{1,2}(\mathrm{X})$ when there is no ambiguity on the distance d and $\mathfrak{m}$. It is obvious that $W^{1,2}(\mathrm{X})$ is a vector space and it is a Banach space when endowed with the norm

$$\|f\|^2_{W^{1,2}(\mathrm{X})} := \|f\|^2_{L^2(\mathfrak{m})} + \||Df|\|^2_{L^2(\mathfrak{m})}.$$

The Sobolev spaces inherites some calculus rules, which are mainly inequalities since the minimal weak upper gradient estimates the modulus of the distributional derivative.
The following hold:

i) Locality: consider $f, g \in S^2(\mathrm{X})$. Then $|Df| = |Dg|$ holds $\mathfrak{m}$-a.e. on $\{f = g\}$;

ii) Chain rule: consider $f \in S^2(\mathrm{X})$ and a Borel set $N \subseteq \mathbb{R}$ with $\mathscr{L}^1(N) = 0$. Then $|Df| = 0$ $\mathfrak{m}$-a.e. on $f^{-1}(N)$; if $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, then $\varphi \circ f \in S^2(\mathrm{X})$ and $|D(\varphi \circ f)| = |\varphi'| \circ f\,|Df|$, where $|\varphi'| \circ f$ is arbitrarily defined on $f^{-1}(\{t \in \mathbb{R} : \nexists \varphi'(t)\})$;

iii) Leibniz rule: consider $f, g \in S^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$. Then $fg \in S^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ and $|D(fg)| \leq |f|\,|Dg| + |g|\,|Df|$ holds $\mathfrak{m}$-a.e.;

iv) Subadditivity: let $f, g \in S^2(\mathrm{X})$, then $f + g \in S^2(\mathrm{X})$ and $|D(f + g)| \leq |Df| + |Dg|$ holds $\mathfrak{m}$-a.e.

Definition 1.2.3 (The space $S^2_{\mathrm{loc}}(\mathrm{X},\mathsf{d},\mathfrak{m})$). *We define $S^2_{\mathrm{loc}}(\mathrm{X},\mathsf{d},\mathfrak{m})$ as the set of functions $f \in L^0(\mathrm{X})$ such that for any bounded Borel set $B \subseteq \mathrm{X}$ there exists a function $f_B \in S^2(\mathrm{X},\mathsf{d},\mathfrak{m})$ such that $f_B = f$ in the $\mathfrak{m}$-a.e. sense on B. For any $f \in S^2_{\mathrm{loc}}(\mathrm{X},\mathsf{d},\mathfrak{m})$, we define $|Df|$ as*

$$|Df| := |Df_B| \text{ in the } \mathfrak{m}\text{-a.e. sense on } B\text{, for any } f_B \text{ as before.}$$

$|Df|$ is well defined by the locality property of the minimal weak upper gradient. We define $W^{1,2}_{\rm loc}(\mathrm{X},\mathsf{d},\mathfrak{m}) := L^2_{\rm loc}(\mathfrak{m}) \cap S^2_{\rm loc}(\mathrm{X},\mathsf{d},\mathfrak{m})$.
Motivated by this theory, it is possible to define the differential d as a linear operator $\mathrm{d}\colon S^2(\mathrm{X}) \to A$ and to find a right ambient space A in order to speak about covectors, endowed with a structure that encodes all the properties we know about Sobolev functions on X, such as $|Df|$. For this reason, the language of $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules came into play, developed in connection with $W^{1,2}(\mathrm{X})$ in [55].

Definition 1.2.4 ($L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules, [55, Definitions 1.2.1, 1.2.10])**.** *Consider $p \in [1, +\infty]$. A $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules is a quadruplet $(\mathscr{M}, \|\cdot\|_{\mathscr{M}}, \cdot, |\cdot|)$, where $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$ is a Banach space. The multiplication by $L^\infty(\mathfrak{m})$ functions $L^\infty(\mathfrak{m}) \times \mathscr{M} \to \mathscr{M}$ $(f, v) \to f \cdot v$ is a bilinear map, satisfying $(fg)\cdot v = f \cdot (g \cdot v)$ and $1 \cdot v = v$ for every $f, g \in L^\infty(\mathfrak{m})$ and $v \in \mathscr{M}$ (we denote by 1 the function in $L^\infty(\mathfrak{m})$ identically equal to 1). The pointwise norm $|\cdot|\colon \mathscr{M} \to L^p(\mathfrak{m})$ satisfies $|v| \geq 0$ $\mathfrak{m}$-a.e. for every $v \in \mathscr{M}$, $|f \cdot v| = |f|\,|v|$ $\mathfrak{m}$-a.e. for every $f \in L^\infty(\mathfrak{m})$ and $v \in \mathscr{M}$. Moreover, $\|v\|^p_{\mathscr{M}} = \int |v|^p \,\mathrm{d}\mathfrak{m}$ for every $v \in \mathscr{M}$.*

We will say that $v \in \mathscr{M}$ is 0 on the $\mathfrak{m}$-measurable set E if $|v| = 0$ $\mathfrak{m}$-a.e. on E. As mentioned before, the connection with $W^{1,2}(\mathrm{X})$ is expressed in the following proposition.

Theorem 1.2.5 ([55, Section 2.2.1])**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Then there exists a unique couple $(L^2(T^*\mathrm{X}), \mathrm{d})$ where $L^2(T^*\mathrm{X})$ is an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module and $\mathrm{d}\colon S^2(\mathrm{X}) \to L^2(T^*\mathrm{X})$ is a linear map such that the following holds:*

i) $|\mathrm{d}f| = |Df|$ holds $\mathfrak{m}$-a.e. for every $f \in S^2(\mathrm{X})$;

ii) $\left\{\sum_{i=1}^n \chi_{A_i} \mathrm{d}f_i : A_i \in \mathscr{B}(\mathrm{X}), f_i \in S^2(\mathrm{X})\right\}$ is dense in $L^2(T^*\mathrm{X})$.

Uniqueness is intended up to unique isomorphism, i.e. given another couple $(M, \tilde{\mathrm{d}})$ satisfying i) and ii) there exists a module isomorphism $\Phi\colon L^2(T^*\mathrm{X}) \to M$ which preserves the pointwise norm such that $\Phi \circ \mathrm{d} = \tilde{\mathrm{d}}$.

Let us comment the items in Theorem 1.2.5: item i) states the consistency of d with the notion of minimal weak upper gradient and item ii) the fact that $L^2(T^*\mathrm{X})$ is the 'smallest' $L^2(\mathfrak{m})$-module containing the differential of elements in $S^2(\mathrm{X})$. We call $L^2(T^*\mathrm{X})$ the cotangent module associated to $(\mathrm{X},\mathsf{d},\mathfrak{m})$ and d the differential.
Usually, we refer to property ii) by saying that $L^2(T^*\mathrm{X})$ is *generated by differentials* of elements in $S^2(\mathrm{X})$.
Important consequences of the definition of differential are two following three properties:

i) Locality: let $f, g \in S^2(\mathrm{X})$. Then $\mathrm{d}f = \mathrm{d}g$ on the set $\{f = g\}$;

ii) Chain rule: consider $f \in S^2(\mathrm{X})$. Given a Borel set $N \subseteq \mathbb{R}$ such that $\mathscr{L}^1(N) = 0$, $\mathrm{d}f = 0$ on $f^{-1}(N)$; given $I \subseteq \mathbb{R}$ such that $(f_*\mathfrak{m})(\mathbb{R} \setminus I) = 0$ and $\varphi\colon I \to \mathbb{R}$ is a Lipschitz function, then $\varphi \circ f \in S^2(\mathrm{X})$ and $\mathrm{d}(\varphi \circ f) = \varphi' \circ f\, \mathrm{d}f$. The expression $\varphi' \circ f\, \mathrm{d}f$ is a well-defined element of $L^2(T^*\mathrm{X})$;

iii) Leibniz rule: given $f, g \in L^\infty(\mathfrak{m}) \cap S^2(\mathrm{X},\mathsf{d},\mathfrak{m})$ we have $fg \in L^\infty(\mathfrak{m}) \cap S^2(\mathrm{X},\mathsf{d},\mathfrak{m})$ and $\mathrm{d}(fg) = g\mathrm{d}f + f\mathrm{d}g$.

To speak about L^2-integrable vector fields, it is costumary to define the notion of dual module.

Definition 1.2.6 (Dual of an L^2-module)**.** *Let $\mathscr{M}$ be an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module. Then we define $\mathscr{M}^*$ as*

$$\mathscr{M}^* := \{L\colon \mathscr{M} \to L^1(\mathfrak{m}) \text{ such that } L \text{ is linear and continuous.}\}$$

and we endow it with the following operations:

- $(L+L')(v) := L(v) + L'(v)$ *for every* $L, L' \in \mathscr{M}^*$ *and* $v \in \mathscr{M}$*;*
- $(f \cdot L)(v) := L(f \cdot v)$ *for every* $L \in \mathscr{M}^*$*,* $v \in \mathscr{M}$ *and* $f \in L^\infty(\mathfrak{m})$*;*
- $|L| := \operatorname*{esssup}_{|v|\leq 1 \mathfrak{m}-a.e.} L(v)$ *for every* $L \in \mathscr{M}^*$*;*
- $\|L\|_{\mathscr{M}^*} := \||L|\|_{L^2(\mathfrak{m})}$ *for every* $L \in \mathscr{M}^*$*.*

Endowed with operations, $\mathscr{M}^*$ has the structure of a $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module.

Definition 1.2.7. *We define $L^2(T\mathrm{X}) := (L^2(T^*\mathrm{X}))^*$ and we call it the tangent module.*

It may happen that in some situation we don't care about the integrability of a vector field or we don't have it at our disposal; for this reason, the following language turns out to be very useful.

Definition 1.2.8 ($L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules)**.** *An $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module is a quadruplet $(\mathscr{M}, \tau, \cdot, |\cdot|)$, where $(\mathscr{M}, \tau)$ is a topological vector space. $\mathscr{M}$ is a module over the commutative ring $L^0(\mathfrak{m})$, namely the operator $\cdot\colon L^0(\mathfrak{m}) \times \mathscr{M} \to \mathscr{M}$ is bilinear, called the multiplication by $L^0(\mathfrak{m})$ functions satisfying $(fg) \cdot v = f \cdot (g \cdot v)$ and $1 \cdot v = v$ for every $f, g \in L^0(\mathfrak{m})$ and $v \in \mathscr{M}$ (we denote by 1 the function in $L^0(\mathfrak{m})$ identically equal to 1). The operator $|\cdot|\colon \mathscr{M} \to L^0(\mathfrak{m})$, called pointwise norm, satisfies $|v| \geq 0$ $\mathfrak{m}$-a.e. for every $v \in \mathscr{M}$ with equality if and only if $v = 0$ and $|v + w| \leq |v| + |w|$ $\mathfrak{m}$-a.e. for every $v, w \in \mathscr{M}$.*
For some Borel probability measure $\mathfrak{m}' \in \mathscr{P}(\mathrm{X})$ such that $\mathfrak{m}' \ll \mathfrak{m} \ll \mathfrak{m}'$ it holds that the distance $\mathsf{d}_{\mathscr{M}}$ defined for $v, w \in \mathscr{M}$ as

$$\mathsf{d}_{\mathscr{M}}(v, w) := \int |v - w| \wedge 1 \, \mathrm{d}\mathfrak{m}'$$

is complete and induces the topology τ.

We need to fix some notations, that will be repeatedly used in this manuscript.

Definition 1.2.9 (Support)**.** *Given an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module $\mathscr{M}$, we define for $v \in \mathscr{M}$ the support of v (and we denote it as $\operatorname{supp}(v)$) as $\operatorname{supp}(v) := \operatorname{supp}(|v|)$.*

Similarly as for the case of $L^0(\mathfrak{m})$ with Proposition 1.1.4, we have the following result in the case of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules.

Proposition 1.2.10. *Let $(\mathscr{M}, \tau, \cdot, |\cdot|)$ be an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. Consider $v \in \mathscr{M}$ and $(v_n) \subseteq \mathscr{M}$. The following are equivalent:*

i) $\mathsf{d}_{\mathscr{M}}(v_n, v) \to 0$ *as* $n \to +\infty$*;*

ii) for every $\epsilon > 0$ and $E \subseteq \mathrm{X}$ with $\mathfrak{m}(E) < \infty$, $\mathfrak{m}(E \cap \{|v_n - v| > \epsilon\}) \to 0$ as $n \to +\infty$;

iii) for every $\epsilon > 0$, $\mathfrak{m}'(\{|v_n - v| > \epsilon\}) \to 0$ as $n \to +\infty$.

It is important to point out that, as a consequence of the last proposition, in particular of the equivalence (i)-(iii) the choice of $\mathfrak{m}'$ affects only the value of $\mathsf{d}_{\mathscr{M}}$, but not its completness nor the fact that τ is the induced topology.
We want to recall now the notion of 'measurable sections of the tangent bundle' or 'measurable vector fields'.
In order to do so, it is important to understand two constructions in order to pass from an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module to an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module and viceversa.
The construction is similar to the case of functions. The Lebesgue spaces and the space $L^0(\mathfrak{m})$ are related in the following way. When considering $L^2(\mathfrak{m})$, the space $L^0(\mathfrak{m})$ can be seen as the completion of $L^2(\mathfrak{m})$ with respect to the distance $\mathsf{d}_{L^0(\mathfrak{m})}$. On the other side, $L^2(\mathfrak{m})$ is defined as $\{f \in L^0(\mathfrak{m}) : f \in L^2(\mathfrak{m})\}$.

The first link is given by the following proposition (see [55, Section 1.3]).

Proposition 1.2.11 (Restriction of an $L^0(\mathfrak{m})$-module)**.** *Let $(\mathscr{M}^0, \tau, \cdot, |\cdot|)$ be an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. Then*

$$\mathscr{M} := \{v \in \mathscr{M}^0 : |v| \in L^p(\mathfrak{m})\}$$

has the strcture of a $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module with $\cdot, |\cdot|$ inherited from $\mathscr{M}^0$ and $\|\cdot\|_{\mathscr{M}} := \||\cdot|\|_{L^p(\mathfrak{m})}$.

The following result is proved in [55, Section 1.3] and actually was the construction that motivated the axiomation in Definition 1.2.8.

Proposition 1.2.12 (Completion of an $L^0(\mathfrak{m})$-normed module)**.** *Consider $(\mathscr{M}, \|\cdot\|_{\mathscr{M}}, \cdot, |\cdot|)$ an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module. Consider $\mathfrak{m}' \in \mathscr{P}(\mathrm{X})$ with $\mathfrak{m}' \ll \mathfrak{m} \ll \mathfrak{m}'$ and define $\mathsf{d}_{\mathscr{M}}$ as above. Then consider the completion $\mathscr{M}^0$ of $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$ with respect to d_{L^0} and τ the induced topology of $\mathsf{d}_{\mathscr{M}}$. Then $(\mathscr{M}^0, \tau)$ has the structure of an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module with the operator $\cdot, |\cdot|$ extended from $\mathscr{M}$ to $\mathscr{M}^0$.*

Let us make a comment about the last statement in the proposition. The operator $\cdot : L^\infty(\mathfrak{m}) \times \mathscr{M} \to \mathscr{M}$ can be extended to an operator $\cdot : L^0(\mathfrak{m}) \times \mathscr{M}^0 \to \mathscr{M}$ in the following way: given $f \in L^0(\mathfrak{m})$, $v \in \mathscr{M}^0$, $(f_n)_n \subseteq L^2(\mathfrak{m})$, $(v_n)_n \subseteq \mathscr{M}$ such that $f_n \to f$ in $L^0(\mathfrak{m})$ and $v_n \to v$ in $(\mathscr{M}^0, \mathsf{d}_{\mathscr{M}})$, we have that $f_n \cdot v_n$ is Caucy in with respect to $\mathsf{d}_{\mathscr{M}}$ with limit that we call $f \cdot v \in \mathscr{M}^0$. It can be checked that $\cdot$ satisfies all the properties in Definition 1.2.8. We can argue similarly for $|\cdot|$.

Definition 1.2.13. *We define $L^0(T^*\mathrm{X}) := L^2(T^*\mathrm{X})^0$ and $L^0(T\mathrm{X}) := L^2(T\mathrm{X})^0$ and we call them respectively the measurable cotangent module and the measurable tangent module.*
Morever, we define for $p \in [1, +\infty]$

$$L^p(T^*\mathrm{X}) := \left\{v \in L^0(T^*\mathrm{X}) : |v| \in L^p(\mathfrak{m})\right\} \quad \textit{and} \quad L^p(T\mathrm{X}) := \left\{v \in L^0(T\mathrm{X}) : |v| \in L^p(\mathfrak{m})\right\}.$$

Another construction on $L^0(\mathfrak{m})$-normed modules is localization.

Definition 1.2.14 (Localization of a $L^0(\mathfrak{m})$-normed module)**.** *Let $\mathscr{M}^0$ be a $L^0(\mathfrak{m})$-normed module and consider $E \subseteq \mathrm{X}$. Then*

$$\mathscr{M}^0|_E := \left\{\chi_E v : v \in \mathscr{M}^0\right\} \subseteq \mathscr{M}^0.$$

It turns out that $\mathscr{M}^0|_E$ is stable under all the module operations of $\mathscr{M}^0$ and it is complete, thus has the structure of a $L^0(\mathfrak{m})$-normed module.

Definition 1.2.15 (Dual of an $L^0(\mathfrak{m})$-normed module)**.** *Let $\mathscr{M}^0$ be an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. Then we define $(\mathscr{M}^0)^*$ as*

$$(\mathscr{M}^0)^* := \{L \colon \mathscr{M}^0 \to L^0(\mathfrak{m}) \text{ such that } L \text{ is } L^0(\mathfrak{m})\text{-linear and continuous.}\}$$

and we endow it with the following operations:

- $(L + L')(v) := L(v) + L'(v)$ *for every* $L, L' \in (\mathscr{M}^0)^*$ *and* $v \in \mathscr{M}^0$*;*
- $(f \cdot L)(v) := L(f \cdot v)$ *for every* $L \in (\mathscr{M}^0)^*$*,* $v \in \mathscr{M}^0$ *and* $f \in L^0(\mathfrak{m})$*;*
- $|L| := \operatorname*{esssup}_{|v| \leq 1 \mathfrak{m}-a.e.} L(v)$ *for every* $L \in (\mathscr{M}^0)^*$*.*

Endowed with operations, $(\mathscr{M}^0)^*$ has the structure of a $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module.

Remark 1.2.16. *Fix an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{M}$. It can be checked that there is an isomorphism of modules which preserves the pointwise norm between the dual of its $L^0(\mathfrak{m})$-completion $(\mathscr{M}^0)^*$ and the completion of its dual $(\mathscr{M}^*)^0$.*

Still concerning the theory of normed modules, it is important to mention the class of Hilbert modules.

Definition 1.2.17. *Let $(\mathscr{H}, \|\cdot\|_{\mathscr{H}}, \cdot, |\cdot|)$ be an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module. We say that $\mathscr{H}$ is an Hilbert module if for every $v, w \in \mathscr{H}$*

$$|v+w|^2 + |v-w|^2 = 2|v|^2 + 2|w|^2 \qquad \mathfrak{m}\text{-}a.e. \tag{1.2}$$

The same definition holds for the case of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules.

The definition above is equivalent to say, in the case of $L^2(\mathfrak{m})$-modules, that $(\mathscr{H}, \|\cdot\|_{\mathscr{H}})$ is an Hilbert space. This condition states that $|\cdot|$ is 'pointwisely induced by a scalar product'. In particular, given an $L^2(\mathfrak{m})$-normed Hilbert module $\mathscr{H}$, defining $\langle \cdot, \cdot \rangle \colon \mathscr{H} \times \mathscr{H} \to L^1(\mathfrak{m})$ as

$$\langle v, w \rangle := \frac{|v+w|^2 - |v|^2 - |w|^2}{2} \qquad \text{for every } v, w \in \mathscr{H} \tag{1.3}$$

we have that it is $L^\infty(\mathfrak{m})$-bilinear. When $\mathscr{H}$ is an $L^0(\mathfrak{m})$-normed Hilbert module, we have that $\langle \cdot, \cdot \rangle \colon \mathscr{H} \times \mathscr{H} \to L^0(\mathfrak{m})$ defined as in (1.3) is $L^0(\mathfrak{m})$-bilinear.
Both in the case of $L^2(\mathfrak{m})$-normed Hilbert modules and of $L^0(\mathfrak{m})$-normed Hilbert modules we have that

$$\langle v, v \rangle = |v|^2, \qquad |\langle v, w \rangle| \leq |v||w|, \qquad \text{for every } v, w \in \mathscr{H}. \tag{1.4}$$

Remark 1.2.18. It follows that an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{H}$ is Hilbert if and only if its L^0-completion $\mathscr{H}^0$ is. Moreover, it can be readily checked that the scalar product in (1.3) can be extendend to a $L^0(\mathfrak{m})$-bilinear map on the L^0-completion $\mathscr{H}^0$, satisfying also (1.4). ■

Another important property for both $L^2(\mathfrak{m})$-normed and $L^0(\mathfrak{m})$-normed Hilbert module is the Riesz representation theorem.

Proposition 1.2.19 (Riesz theorem for $L^2(\mathfrak{m})$-normed Hilbert modules, [55, Theorem 1.2.24])**.** *Let $\mathscr{H}$ be a $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{H}$. Then the map $\mathscr{H} \ni v \mapsto L_v \in \mathscr{H}^*$, defined as*

$$L_v(w) := \langle v, w\rangle \quad \textit{for every } w \in \mathscr{H}$$

is an isomorphism of modules and is called Riesz isomorphism for $L^2(\mathfrak{m})$-normed modules.

Proposition 1.2.20 (Riesz theorem for $L^0(\mathfrak{m})$-normed Hilbert modules)**.** *Let $\mathscr{H}^0$ be a $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module $\mathscr{H}$. Then the map $\mathscr{H}^0 \ni v \mapsto L_v \in (\mathscr{H}^0)^*$, defined as*

$$L_v(w) := \langle v, w\rangle \quad \textit{for every } w \in \mathscr{H}^0$$

is an isomorphism of modules and is called Riesz isomorphism for $L^0(\mathfrak{m})$-normed modules.

Proof. It is straightforward to check that $v \mapsto L_v$ is a $L^0(\mathfrak{m})$-linear isometry. It is enough to prove surjectivity (the proof is taken from [18]). We consider $L \in (\mathscr{H}^0)^*$ and we solve the problem of finding $v \in \mathscr{H}^0$ such that

$$\frac{1}{2}|v|^2 - L(v) = \operatorname{essinf}_{w\in\mathscr{H}^0}\frac{1}{2}|w|^2 - L(w). \tag{1.5}$$

We can choose a sequence $(v_n)_n \subseteq \mathscr{H}^0$ such that $\frac{1}{2}|v_n|^2 - L(v_n) \geq \frac{1}{2}|v_{n+1}|^2 - L(v_{n+1})$ and such that $\frac{1}{2}|v_n|^2 - L(v_n) \to \operatorname{essinf}_{w\in\mathscr{H}^0}\frac{1}{2}|w|^2 - L(w)$ $\mathfrak{m}$-a.e.. It can be checked that

$$\frac{1}{4}|v_n - v_m|^2 \leq \frac{1}{2}|v_n|^2 - L(v_n) + \frac{1}{2}|v_m|^2 - L(v_m) - 2\operatorname{essinf}_{w\in\mathscr{H}^0}\frac{1}{2}|w|^2 - L(w).$$

So $(v_n)_n$ is Cauchy in $\mathscr{H}^0$ and admits a limit v; by continuity of $\mathscr{H}^0 \ni v \mapsto \frac{1}{2}|v|^2 - L(v) \in L^0(\mathfrak{m})$, v solves (1.5). The strict subadditivity of $|\cdot|^2$ grants that the solution is unique. Consider v the solution of (1.5): we have that, for every $w \in \mathscr{H}^0$ $t \in \mathbb{R}$,

$$\frac{1}{2}|v + tw|^2 - L(v + tw) \geq \frac{1}{2}|v|^2 - L(v)$$

holds $\mathfrak{m}$-a.e.., which yields

$$\frac{1}{2}t^2|w|^2 + t\langle v, w\rangle - tL(w) \geq 0.$$

Dividing by t and taking the limit $t \searrow 0$ and $t \nearrow 0$, we get that $L(w) = \langle v, w\rangle$ for every $w \in \mathscr{H}^0$. □

We recall the definition of the pullback of an $L^0(\mathfrak{m})$-normed module. The original construction comes from [55] for the case of $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules.

Theorem 1.2.21 ([61, Theorem 2.7])**.** *Let* $(\mathrm{X}, \mathsf{d}_{\mathrm{X}}, \mathfrak{m}_{\mathrm{X}})$ *and* $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}})$ *be two metric measure spaces. Let* $\varphi\colon \mathrm{X} \to \mathrm{Y}$ *be a Borel map such that* $\varphi_*\mathfrak{m}_{\mathrm{X}} \ll \mathfrak{m}_{\mathrm{Y}}$*. Let* $\mathscr{M}^0$ *be an* $L^0(\mathfrak{m}_{\mathrm{Y}})$*-normed module. Then there exists a unique couple* $(\mathscr{N}^0, T)$ *(up to unique module isomorphism preserving the pointwise norm) such that* $\mathscr{N}^0$ *is a* $L^0(\mathfrak{m}_{\mathrm{X}})$*-normed module and* $T\colon \mathscr{M}^0 \to \mathscr{N}^0$ *is a linear map such that*

i) $|T(v)| = |v| \circ \varphi$ *holds in the* $\mathfrak{m}_{\mathrm{X}}$*-a.e. sense, for every* $v \in \mathscr{M}^0$*;*

ii) the set of elements of the form $\sum_{i=1}^{n} \chi_{A_i} T(v_i)$*, with* $(A_i)_i$ *partition of* X *and* $v_1, \dots, v_n \in \mathscr{M}^0$ *is dense in* $\mathscr{N}^0$*.*

In this case, we denote the couple $(\mathscr{N}^0, T) = (\varphi^*\mathscr{M}^0, \varphi^*)$ *and we call* $\varphi^*\mathscr{M}^0$ *the* pullback module of $\mathscr{M}^0$ via the map φ *and* φ^* *the* pullback map.

We now turn to recall the definition of some differential operators that are needed in this manuscript.

Definition 1.2.22 (Divergence, [55, Definition 2.3.11])**.** *Consider* $p \in [1, +\infty]$*. The space* $D(\mathrm{div}_p) \subseteq L^1(T\mathrm{X})$ *is the set of all vector fields* $X \in L^1(T\mathrm{X})$ *for which there exists* $f \in L^p(\mathfrak{m})$ *such that for every* $g \in \mathrm{Lip}_{\mathrm{bs}}(\mathrm{X})$

$$\int fg \,\mathrm{d}\mathfrak{m} = -\int \mathrm{d}g(X)\,\mathrm{d}\mathfrak{m}.$$

In this case, we call f *(uniquely determined by the density of* $\mathrm{Lip}_{\mathrm{bs}}(\mathrm{X})$ *in* $L^p(\mathfrak{m})$*) the divergence of* X *and denote it by* $\mathrm{div}(X)$*.*

As a consequence of Leibniz rule for differentials, we get that, given $X \in L^p(T\mathrm{X}) \cap D(\mathrm{div}_p)$ and $f \in L^\infty(\mathfrak{m}) \cap S^2(X)$ with $|\mathrm{d}f| \in L^\infty(\mathfrak{m})$ we have

$$fX \in D(\mathrm{div}_p) \quad \text{and} \quad \mathrm{div}(fX) = \mathrm{d}f(X) + f\mathrm{div}X.$$

We say that, given $f \in S^2(\mathrm{X})$, an element $L \in L^2(T\mathrm{X})$ belongs to $\mathrm{Grad}(f)$ if

$$\mathrm{d}f(L) = |L|^2 = |\mathrm{d}f|^2 \quad \text{holds } \mathfrak{m}\text{-a.e.}$$

In particular, when $\mathrm{Grad}(f)$ consists of only one element, we call it ∇f.

We say $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is infinitesimally strictly convex if for every $f \in S^2(\mathrm{X})$ $\mathrm{Grad}(f)$ consists of only one element.

An important definition which encodes the fact that the pointwise norm on elements of the cotangent and tangent modules come from a scalar product is that of infinitesimally Hilbertian metric measure spaces and the content of the next proposition.

Proposition 1.2.23 ([55, Proposition 4.22])**.** *The following are equivalent:*

i) $W^{1,2}(\mathrm{X})$ *is an Hilbert space;*

ii) $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is infinitesimally strictly convex and* $\mathrm{d}f(\nabla g) = \mathrm{d}g(\nabla f)$*,* $\mathfrak{m}$*-a.e.;*

iii) $L^2(T\mathrm{X})$ *and* $L^2(T^*\mathrm{X})$ *are Hilbert modules;*

iv) $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is infinitesimally strictly convex and* $\nabla\colon S^2(\mathrm{X}) \to L^2(T\mathrm{X})$ *is linear;*

v) $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is infinitesimally strictly convex and* $\nabla\colon S^2(\mathrm{X}) \to L^2(T\mathrm{X})$ *satisfies the Leibniz rule, i.e.* $\nabla(fg) = g\nabla f + f\nabla g$ *for* $f, g \in S^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$.

We say that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is infinitesimally Hilbertian if one (and thus all) of i)-v) hold.*

From now on, we work in the following setting:

we assume that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is *infinitesimally Hilbertian*

We stress out that, in the case of infinitesimally Hilbertian metric measure spaces, $(L^2(T\mathrm{X}), \nabla)$ can be characterized as the only couple satisfying:

i) $\nabla\colon S^2(\mathrm{X}) \to L^2(T\mathrm{X})$ is linear and $|\nabla f| = |Df|$ holds $\mathfrak{m}$-a.e.;

ii) $\left\{\sum_{i=1}^n \chi_{A_i} \nabla f_i : A_i \in \mathscr{B}(\mathrm{X}), f_i \in S^2(\mathrm{X})\right\}$ is dense in $L^2(T\mathrm{X})$.

As before, uniqueness is meant up to module isomorphism preserving the pointwise norm.

Remark 1.2.24. Notice that item i) and ii) are indeed the same properties satisfied by the couple $(L^2(T^*\mathrm{X}), \mathrm{d}))$. This is not surprising compared with Theorem 1.2.5, because there is an isomorphism of modules which preserves the pointwise norm between $L^2(T^*\mathrm{X})$ and $L^2(T\mathrm{X})$. ■

We now define the Laplacian operator with range in $L^2(\mathfrak{m})$ and in $L^2_{\mathrm{loc}}(\mathfrak{m})$ (in particular, for the case of a general metric measure space and of the measure valued Laplacian, possibly not single valued, we refer the reader to [54, Definition 4.4]).

Definition 1.2.25 (Laplacian in L^2)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an infinitesimally Hilbertian metric measure space. Given* $f \in W^{1,2}(\mathrm{X})$*, we say that* $f \in D(\Delta)$ *provided there exists* $g \in L^2(\mathfrak{m})$ *such that*

$$\int gh \,\mathrm{d}\mathfrak{m} = -\int \langle \nabla f, \nabla h \rangle \,\mathrm{d}\mathfrak{m} \tag{1.6}$$

for every $h \in W^{1,2}(\mathrm{X})$*. The function* g *(uniquely determined by the density of* $W^{1,2}(\mathrm{X})$ *in* $L^2(\mathfrak{m})$*) will be denoted by* Δf*.*

The infinitesimal Hilbertianity assumption, thanks to item iv) in Proposition 1.2.23 yields indeed that $\Delta\colon D(\Delta) \subseteq L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$ is linear.
We introduce a weaker notion of Laplacian as an element in $L^2_{\mathrm{loc}}(\mathfrak{m})$.

Definition 1.2.26 (Laplacian in L^2_{loc})**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an infinitesimally Hilbertian metric measure space. Given* $f \in W^{1,2}_{\mathrm{loc}}(\mathrm{X})$*, we say that* $f \in D_{\mathrm{loc}}(\Delta)$ *provided there exists* $g \in L^2_{\mathrm{loc}}(\mathfrak{m})$ *such that*

$$\int gh \,\mathrm{d}\mathfrak{m} = -\int \langle \nabla f, \nabla h \rangle \,\mathrm{d}\mathfrak{m} \tag{1.7}$$

for every $h \in \mathrm{Lip}_{bs}(\mathrm{X})$*. The function* g*, uniquely determined, is called the Laplacian of* f *and denoted by* Δf*.*

Notice that, given $f \in D(\Delta)$, we have that $f \in D_{\mathrm{loc}}(\Delta)$ and Δf coincides in the two definitions. In order to define the $\mathsf{RCD}(K,N)$ in a distributional way, it is important to produce some 'regular' functions. To this aim, the main tool in this setting that act as a regularization operator on functions is the heat flow, formalized as gradient flow on Hilbert spaces, according to the theory in [23],[78] (see also the monograph [24]). We need some preliminary definitions. Consider an Hilbert space H and denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ respectively its scalar product and the induced norm. Given a function $E\colon H \to [0,+\infty]$, we denote by

$$\mathrm{Dom}(E) := \{x \in H : E(x) < +\infty\}.$$

In particular, if E is convex and lower semicontinuous, we define the subdifferential of E at the point $x \in H$ and we denote it by $\partial^- E(x)$ as the set

$$\partial^- E(x) := \{v \in H : E(x) + \langle v, y - x\rangle \leq E(y) \text{ for every } y \in H\}.$$

Moreover, we define the slope of E as the functional $|\partial^- E|\colon H \to [0,+\infty]$ given by

$$|\partial^- E(x)| := \begin{cases} \sup_{y \neq x} \frac{(E(x)-E(y))^+}{|x-y|} & \text{if } x \in \mathrm{Dom}(E) \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that $|\partial^- E|(x) = 0$ if and only if x is a minimum point of E. It can be proved that

$$|\partial^- E|(x) = \min_{v \in \partial^- E(x)} |v|. \tag{1.8}$$

Theorem 1.2.27 (Gradient flow on Hilbert spaces)**.** *Let H be an Hilbert space and $E\colon H \to [0,+\infty]$ a convex and lower semicontinuous functional. Consider $x \in \overline{\mathrm{Dom}(E)}$. Then there exists a continuous curve $[0,+\infty) \ni t \mapsto x_t \in H$ such that $x_0 = x$, is locally AC in $(0,+\infty)$ and satisfies $x_t \in -\partial^- E(x_t)$ for a.e. $t \in [0,+\infty)$. Such a curve is called the gradient flow of E starting from x. Moreover:*

i) given two gradient flows (x_t) and (y_t) starting respectively from x and y, we have

$$|x_t - y_t| \leq |x - y| \quad \text{for every } t \geq 0.$$

Such a property is usually called contraction property and yields uniqueness of the gradient flow from a fixed initial point;

ii) the maps $t \mapsto x_t$ and $t \mapsto E(x_t)$ are locally Lipschitz in $(0,+\infty)$;

iii) the maps $t \mapsto E(x_t), |\partial^- E(x_t)|$ are non increasing for $t \in [0,+\infty)$;

iv) given $y \in H$, we have that $E(x_t) + \langle x'_t, x_t - y\rangle \leq E(y)$ holds for a.e. $t \in (0,+\infty)$;

v) we have that $-\frac{\mathrm{d}}{\mathrm{d}t}E(x_t) = |\dot{x}_t|^2 = |\partial^- E|^2(x_t)$ for a.e. $t \in [0,+\infty)$;

vi) Given $y \in H$ and $t \in [0,+\infty)$, the following estimates hold:

$$E(x_t) \leq E(y) + \frac{|x_0 - y|^2}{2t}, \quad |\partial^- E|^2(x_t) \leq |\partial^- E|^2(y) + \frac{|x_0 - y|^2}{t^2}. \tag{1.9}$$

vii) fiven $t > 0$*,* $\lim_{h\searrow 0}\frac{x_{t+h}-x_t}{h} =: v$*, where* v *is the element of minimal norm in* $\partial^- E(x_t)$*, where for* $t = 0$ *holds if* $\partial^- E(x_0) \neq \emptyset$*.*

We define the Cheeger energy functional $\mathrm{Ch}\colon L^2(\mathfrak{m}) \to [0, +\infty]$ as

$$\mathrm{Ch}(f) := \begin{cases} \frac{1}{2}\int |Df|^2 \,\mathrm{d}\mathfrak{m} & \text{if } f \in W^{1,2}(\mathrm{X}), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.10}$$

Definition 1.2.28 (Heat flow)**.** *We define the heat flow* $(h_t)_{t\geq 0}$ *as the gradient flow in* $L^2(\mathfrak{m})$ *of the Dirichlet energy* Ch.

Proposition 1.2.29. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an infinitesimally Hilbertian metric measure space. Consider* $\mathrm{Ch}\colon L^2(\mathfrak{m}) \to [0, +\infty]$ *be defined as in* (1.10)*. Then* $f \in W^{1,2}(\mathrm{X})$ *belongs to* $D(\Delta)$ *if and only if* $\partial^-\mathrm{Ch}(f) \neq 0$ *and in this case* $\partial^-\mathrm{Ch}(f) = \{-\Delta f\}$*.*

Thanks to the last proposition, Theorem 1.2.27 can be specified to the case of the Cheeger energy in the setting of infinitesimally Hilbertian metric measure spaces, by saying that for every $f \in L^2(\mathfrak{m})$ there exists a curve $[0, +\infty) \ni t \mapsto f_t \in L^2(\mathfrak{m})$ such that

i) $f_0 = f$;

i) $t \mapsto f_t$ belongs to $C([0, +\infty), L^2(\mathfrak{m}))$ and $AC(K, L^2(\mathfrak{m}))$ for every compact interval $K \subseteq (0, +\infty)$;

ii) for a.e. t, $f_t \in D(\Delta)$ and $\partial_t f = \Delta f$, where the limit in ∂_t is meant in the norm of $L^2(\mathfrak{m})$.

In particular, item vi) in Theorem 1.2.27 can be read in the case of the heat flow for $f, g \in W^{1,2}(\mathrm{X})$ as

$$\begin{aligned} \||Dh_t f|\|^2_{L^2(\mathfrak{m})} &\leq \||Dg|\|^2_{L^2(\mathfrak{m})} + \frac{\|f - g\|^2_{L^2(\mathfrak{m})}}{2t}, \\ \|\Delta h_t f\|^2_{L^2(\mathfrak{m})} &\leq \|\Delta g\|^2_{L^2(\mathfrak{m})} + \frac{\|f - g\|^2_{L^2(\mathfrak{m})}}{t^2}. \end{aligned} \tag{1.11}$$

Some relevant properties of the heat flow are the following ones.

Proposition 1.2.30 ([60, Proposition 5.2.14])**.** *Let* $f \in L^2(\mathfrak{m})$ *be fixed. Then the following properties hold:*

i) weak maximum principle: assume that $f \leq c$ $\mathfrak{m}$*-a.e. for* $c \in \mathbb{R}$*. Then* $h_t f \leq c$ $\mathfrak{m}$*-a.e. for every* $t > 0$*;*

ii) consider $p \in [1, +\infty]$*. Then* $\|h_t f\|_{L^p(\mathfrak{m})} \leq \|f\|_{L^p(\mathfrak{m})}$ *for every* $t > 0$*.*

The definition of $\mathsf{RCD}(K, N)$ space in Section 1.3 in terms of Bochner inequality is formulated in a 'distributional' way. We want to do some comment on how to obtain the function with the needed regularity in the definition. We will have the following two classes:

$$\begin{aligned} \mathscr{A}_1 &:= \{f \in D(\Delta),\, \Delta f \in W^{1,2}(\mathrm{X})\}, \\ \mathscr{A}_2 &:= \{g \in D(\Delta) \cap L^\infty(\mathfrak{m})^+,\, \Delta g \in L^\infty(\mathfrak{m})\}. \end{aligned} \tag{1.12}$$

Notice that by the properties of the heat flow, $\{h_t f : f \in L^2(\mathfrak{m})$ for some $t \geq 0\} \subseteq \mathscr{A}_1$, giving the information that $\mathscr{A}_1$ is not empty and contains 'a lot' of elements. For $\mathscr{A}_2$, we need to introduce the following object, the time regularized heat flow h_φ (see e.g. [55, Section 3.2]), defined as follows: consider a non-negative $\varphi \in C_c^\infty(0, +\infty)$ and $f \in L^2(\mathfrak{m}) \cap L^\infty(\mathfrak{m})$; we define $h_\varphi f \in L^2(\mathfrak{m}) \cap L^\infty(\mathfrak{m})$ as

$$h_\varphi f := \int_0^{+\infty} h_s f \, \varphi(s) \, \mathrm{d}s. \tag{1.13}$$

In particular, $h_\varphi f \in D(\Delta)$ and there exists $C(\varphi) > 0$ such that $\|\Delta h_\varphi f\|_{L^2(\mathfrak{m})} \leq C(\varphi)\|f\|_{L^2(\mathfrak{m})}$. Moreover, using the fact that $\Delta \colon D(\Delta) \subseteq L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$ is a closed operator and Hille's theorem, we have that for $f \in L^2(\mathfrak{m}) \cap L^\infty(\mathfrak{m})$

$$\Delta h_\varphi f = \int_0^\infty \Delta h_s f \, \varphi(s) \, \mathrm{d}s = \int_0^\infty \partial_s h_s f \, \varphi(s) \, \mathrm{d}s = -\int_0^\infty h_s f \, \varphi'(s) \, \mathrm{d}s \tag{1.14}$$

thus having that $\Delta h_\varphi f \in L^\infty(\mathfrak{m})$ and $\|\Delta h_\varphi f\|_{L^\infty(\mathfrak{m})} \leq \int_0^\infty |\varphi'(s)| \, \mathrm{d}s \|f\|_{L^\infty(\mathfrak{m})}$. Moreover, as a consequence of the definition of h_φ, both items in Proposition 1.2.30 and the linearity of the heat flow, it follows that $h_\varphi f \geq 0$ whenevert $f \geq 0$ and $h_\varphi f \in L^\infty(\mathfrak{m})$.
Hence, collecting the previous results, we have that

$$\{h_\varphi f : f \in L^2(\mathfrak{m}) \cap L^\infty(\mathfrak{m}), \, f \geq 0\} \subseteq \mathscr{A}_2$$

showing as before that $\mathscr{A}_2$ is not empty and producing 'a lot' of elements belonging to it.

1.3 Relevant properties of PI spaces and RCD spaces

Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. A very general theory that needs to be mentioned is that of PI spaces, firstly introduced by Heinonen and Koskela in [72]. In order to formulate it, we need the following definitions. We say that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is locally uniformly doubling if, for every $R > 0$, there exists $C_R > 0$ such that for every $x \in \mathrm{X}$ and $r \leq R$ we have

$$\mathfrak{m}(B_{2r}(x)) \leq C_R \mathfrak{m}(B_r(x)) \tag{1.15}$$

We refer to C_R as the *doubling constant up to scale R.*
An important consequence of the definition of the doubling assumption is that (X, d) is proper, i.e. closed and bounded sets are compact.
It is a well known fact (see e.g. [73, Section 3.4]) that, if $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is locally uniformly doubling, Lebesgue differentiation theorem holds. As a direct consequence, given a Borel set E, almost every point in E is a density point. Given $f : \mathrm{X} \to \mathbb{R} \cup \{\pm\infty\}$ Borel and $x \in \mathrm{X}$ we define the approximate limsup of f at x as

$$\operatorname{ap-}\overline{\lim_{y \to x}} f(y) := \inf \left\{ \lambda \in \mathbb{R} \cup \{\pm\infty\} \ : \ x \text{ is a density point of } \{f \leq \lambda\} \right\}$$

and analogously we define the approximate liminf $\operatorname{ap-}\underline{\lim}_{y \to x} f(y)$. Then the approximate Lipschitz constant at x of a Borel function $F : \mathrm{X} \to \mathrm{Y}$ with $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ metric space is defined as

$$\operatorname{ap-lip} F(x) := \operatorname{ap-}\overline{\lim_{y \to x}} \frac{\mathsf{d}_\mathrm{Y}(F(y), F(x))}{\mathsf{d}(y, x)}. \tag{1.16}$$

Finally, we recall that a Borel map $F : \mathrm{X} \to \mathrm{Y}$ is said Lusin–Lipschitz provided there exists a sequence (E_i) of Borel sets in X such that $\mathfrak{m}\big(\mathrm{X} \setminus \bigcup_i E_i\big) = 0$ and $F_{|E_i}$ is Lipschitz for all $i \in \mathbb{N}$.

We recall the definition of Hardy Littlewood maximal operator. Given $f \in L^1_{\mathrm{loc}}(\mathrm{X}, \mathfrak{m})$ and $\lambda > 0$, we define

$$M_\lambda f(x) := \sup_{0<r<\lambda} \frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |f| \, \mathrm{d}\mathfrak{m}. \tag{1.17}$$

that we call the *Hardy Littlewood maximal operator up to scale* λ. When $\lambda = +\infty$, we call it the *Hardy Littlewood maximal operator* and denote it by M. Then on a locally uniformly doubling space we have that: for every $1 < p \le \infty$ and $\lambda > 0$ there exists a constant $C_{p,\lambda}$ such that

$$\|M_\lambda f\|_{L^p} \le C_{p,\lambda} \|f\|_{L^p} \text{ for every } f \in L^p(\mathrm{X}, \mathfrak{m}). \tag{1.18}$$

See e.g. [73, Theorem 3.5.6] for a proof (notice that in such reference the measure is assumed to be doubling, i.e. with the constant in (1.15) to be independent of both x and R, however as the argument in the proof shows, since we are considering the restricted maximal function, this is not really an issue).

We say that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ satisfies a weak local (1-1) Poincaré inequality provided for every $R > 0$ there exists $C_P(R) > 0$ and $\lambda \ge 1$ such that for every $f \colon \mathrm{X} \to \mathbb{R}$ Lipschitz, $x \in \mathrm{X}$, $0 < r < R$ we have

$$⨍_{B_r(x)} |f - f_{B_r(x)}| \, \mathrm{d}\mathfrak{m} \le C_P r ⨍_{B_{\lambda r}(x)} \mathrm{lip}(f) \, \mathrm{d}\mathfrak{m},$$

where $f_{B_r(x)} := \frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} f \, \mathrm{d}\mathfrak{m}$.

Definition 1.3.1 (PI space)**.** *We say that* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is a PI space if it is locally uniformly doubling and satisfies a weak local (1-1) Poincaré inequality.*

For us, this class of spaces is relevant because the results in [34] apply and because $\mathsf{RCD}(K, N)$ spaces are PI spaces. This a consequence of the fact that they are locally uniformly doubling (see [95]) and satisfies a local Poincaré inequality (see [89]). Another relevant property of PI spaces is that, as proven in [34], given $f \in W^{1,2}(\mathrm{X}) \cap \mathrm{Lip}(\mathrm{X})$, we have

$$|Df| = \mathrm{lip} f \text{ in the } \mathfrak{m}\text{-a.e. sense}$$

(while the inequality $\le$ holds in general metric measure spaces).

On PI spaces, the following maximal estimate holds (see [71, Theorem 3.2]): given $f \in W^{1,2}(\mathrm{X})$, there exists a $\mathfrak{m}$-null set N such that

$$|f(x) - f(y)| \le C\big(M_{4R}(|Df|)(x) + M_{4R}(|Df|)(y)\big)\mathsf{d}(x, y) \qquad \forall x, y \in \mathrm{X} \setminus N, \ \mathsf{d}(x, y) \le R. \tag{1.19}$$

In particular, notice that (1.19) can naturally be used in the Euclidean setting when $(\mathrm{X}, \mathsf{d}, \mathfrak{m}) = (\mathbb{R}^d, \mathsf{d}_e, \mathscr{L}^d)$.

We now recall the definition of $\mathsf{RCD}(K, \infty)$ and of $\mathsf{RCD}(K, N)$ following the Eulerian characterization by means of the Bochner inequality in [11].

Definition 1.3.2 ($\mathsf{RCD}(K,\infty)$ spaces)**.** *Let $K \in \mathbb{R}$. Then a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is a $\mathsf{RCD}(K,\infty)$ space provided the following conditions hold:*

i) there exists $C>0$ and a point $x \in \mathrm{X}$ such that $\mathfrak{m}(B_r(x)) \le Ce^{Cr^2}$;

ii) for every $f \in W^{1,2}(\mathrm{X})$ with $|Df| \in L^\infty(\mathfrak{m})$, there exists a Lipschitz function $\tilde{f}$ which is a representative of f and $\||Df|\|_{L^\infty(\mathfrak{m})} = \mathrm{Lip}(\tilde{f})$;

iii) the space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian;

iv) for every $f \in D(\Delta)$ and positive $g \in D(\Delta) \cap L^\infty(\mathfrak{m})$ such that $\Delta f \in W^{1,2}(\mathrm{X})$ and $\Delta g \in L^\infty(\mathfrak{m})$, it holds that

$$\frac{1}{2}\int |Df|^2\,\Delta g\,\mathrm{d}\mathfrak{m} \ge \int \big(\langle \nabla f, \nabla \Delta f\rangle + K|Df|^2\big)\, g\,\mathrm{d}\mathfrak{m}.$$

The condition iv) is a distributional-like version of Bochner inequality. Still following [11], we have the following finite dimensional counterpart in terms of a distributional-like Bochner inequality.

Definition 1.3.3. *Let $K \in \mathbb{R}$ and $N \in (1,\infty)$. Then a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is a $\mathsf{RCD}(K,N)$ space if it is an $\mathsf{RCD}(K,\infty)$ space and for every $f \in D(\Delta)$ and positive $g \in D(\Delta) \cap L^\infty(\mathfrak{m})$ such that $\Delta f \in W^{1,2}(\mathrm{X})$ and $\Delta g \in L^\infty(\mathfrak{m})$ the following holds:*

$$\frac{1}{2}\int |Df|^2\,\Delta g\,\mathrm{d}\mathfrak{m} \ge \int \left(\frac{(\Delta f)^2}{N} + \langle \nabla f, \nabla \Delta f\rangle + K|Df|^2\right) g\,\mathrm{d}\mathfrak{m}.$$

Remark 1.3.4. We point out that the objects in (1.12) are the one needed in order to write down the Bochner inequality in Definition 1.3.2 and 1.3.3; we refer to the discussion there explaining how to produce objects in those class by means of the heat flow. ■

In the class of $\mathsf{RCD}(K,N)$ spaces, the Bishop-Gromov monotonicity formula holds (actually it holds in the more general class of $\mathsf{CD}(K,N)$ spaces, as proven in [95]). We denote by $v_{K,N}(r)$ the volume of the ball of radius r in the Riemannian manifold of Ricci curvature K and dimension N. Given $(\mathrm{X},\mathsf{d},\mathfrak{m})$ a $\mathsf{RCD}(K,N)$ space and a point $x \in \mathrm{X}$, we have that the function

$$r \mapsto \frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)} \quad \text{is non increasing.} \tag{1.20}$$

For the proof, we refer to [96, Theorem 30.11]. Thanks to the monotonicity property of Bishop-Gromov monotonicity formula (1.20), it makes sense to introduce the following definition.

Definition 1.3.5 (Bishop-Gromov density)**.** *Let $K \in \mathbb{R}$ and $N \in [1,\infty)$. Assume $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is a $\mathsf{CD}(K,N)$ space. Given $x \in \mathrm{X}$, we define the Bishop-Gromov density at x as*

$$\theta(x) = \lim_{r\to 0}\frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)} = \sup_{r>0}\frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)}$$

It follows from the definition of $v_{K,N}(r)$ that $\lim_{r\to 0}\frac{v_{K,N}(r)}{w_N r^N} = 1$, where w_N is the Lebesgue measure of the ball of radius 1 in $\mathbb{R}^N$; therefore, $\theta(x)$ can be equivalently characterized as

$$\theta(x) = \lim_{r\to 0}\frac{\mathfrak{m}(B_r(x))}{w_N r^N}.$$

We point out that the function $\theta\colon \mathrm{X} \to [0,+\infty]$, being the supremum of a family of continuous functions, namely $x \mapsto \frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)}$, is lower semicontinuous.
The notion of non-collapsed $\mathsf{RCD}(K,N)$ spaces ($\mathsf{ncRCD}(K,N)$ spaces in short) has been introduced in [45] (after [77]) as a synthetic counterpart of non-collapsed Ricci limit spaces studied in [36],[37] and [38].

Definition 1.3.6 (Non-collapsed $\mathsf{RCD}(K,N)$ spaces)**.** *We say that an* $\mathsf{RCD}(K,N)$ *space* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is a non collapsed* $\mathsf{RCD}(K,N)$ *space (*$\mathsf{ncRCD}(K,N)$ *space in short) if* $\mathfrak{m} = \mathscr{H}^N$*.*

In this setting, we have major information on the Bishop-Gromov density. Notice that, as a general theorem about differentiation of measures (see e.g. [45, Lemma 2.11]), we have that $\overline{\lim}_{r\to 0} \frac{\mathscr{H}^N(B_r(x))}{w_N r^N} \leq 1$ for $\mathscr{H}^N$-a.e. x, which together with the lower semicontinuity of θ yields that $\theta(x) \leq 1$ for every $x \in \mathrm{X}$.

Definition 1.3.7 (Regular point)**.** *Let* $K \in \mathbb{R}$ *and* $N \in [1,\infty)$*. Assume* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is a* $\mathsf{ncRCD}(K,N)$ *space. We say that* $x \in \mathrm{X}$ *is a regular point if* $\theta(x) = 1$*.*

1.4 Heat flow, test functions and second order calculus on $\mathsf{RCD}(K,\infty)$ spaces

Assume from now on that $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is an $\mathsf{RCD}(K,\infty)$ space. The heat flow sastisfies further regularizing property in such a setting.

Proposition 1.4.1 (L^∞ to Lipschitz regularization, [10, Theorem 6.8])**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathsf{RCD}(K,\infty)$ *space. Consider* $f \in L^\infty(\mathfrak{m})$ *and* $t > 0$*. Then* $|Dh_t f| \in L^\infty(\mathfrak{m})$ *and*

$$\||Dh_t f|\|_{L^\infty(\mathfrak{m})} \leq \frac{C(K)}{\sqrt{t}} \|f\|_{L^\infty(\mathfrak{m})} \tag{1.21}$$

We point out that (1.21), the first in (1.11) with $g = 0$, together with Marcinkiewicz interpolation theorem grants that, for every $f \in L^p(\mathfrak{m})$ with $p \in [2,+\infty]$, $|Dh_t f| \in L^p(\mathfrak{m})$ and:

$$\||Dh_t f|\|_{L^p(\mathfrak{m})} \leq \frac{C(K)}{\sqrt{t}} \|f\|_{L^p(\mathfrak{m})}.$$

Moreover, we can define (see [10], [6]) the dual semigroup $\bar{h}_t\colon \mathscr{P}_2(\mathrm{X}) \to \mathscr{P}_2(\mathrm{X})$, defined by

$$\int_{\mathrm{X}} f \, \mathrm{d}\bar{h}_t\mu := \int_{\mathrm{X}} h_t f \, \mathrm{d}\mu$$

for every $\mu \in \mathscr{P}_2(\mathrm{X})$ and for every $f \in \mathrm{Lip}_b(\mathrm{X})$. It can be proved that the image of the operator indeed lies in $\mathscr{P}_2(\mathrm{X})$ and that actually is contained in the set of absolutely continuous (with respect to $\mathfrak{m}$) measures. Therefore, we define the heat kernel $p\colon (0,+\infty) \times \mathrm{X} \times \mathrm{X} \to [0,+\infty)$ by defining for every $x \in \mathrm{X}$

$$p_t(x,\cdot)\mathfrak{m} := \bar{h}_t \delta_x.$$

In the case of $\mathsf{RCD}(K,N)$ spaces, Jiang, Li and Zhang in [75] (after the works of Sturm [92] and [93]) proved the following bound: there exists $C_p \geq 1$ and $c \geq 0$ such that

$$\frac{1}{C_p \mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{\mathsf{d}^2(x,y)}{3t} - ct\right) \leq p_t(x,y) \leq \frac{C_p}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{\mathsf{d}^2(x,y)}{5t} + ct\right) \tag{1.22}$$

for any $x, y \in \mathrm{X}$ and for any $t > 0$. We point out that, in the case of $\mathsf{RCD}(0, N)$, it is possible to take $c = 0$ in the estimate (1.22).

We introduce the notions of *test functions*, firstly introduced by Savaré in [90].

$$\mathrm{Test}(\mathrm{X}) := \{f \in \mathrm{LIP}(\mathrm{X}) \cap L^\infty(\mathfrak{m}) \cap D(\Delta) : \Delta f \in W^{1,2}(\mathrm{X}) \cap L^\infty(\mathfrak{m})\}.$$

We call elements belonging to this set *test functions.* Some important properties of this class are the following ones:

i) $\mathrm{Test}(\mathrm{X})$ is dense in $W^{1,2}(\mathrm{X})$ in the $W^{1,2}(\mathrm{X})$ norm;

ii) $\mathrm{Test}(\mathrm{X})$ is an algebra;

iii) given $f \in \mathrm{Test}(\mathrm{X})$, $|Df|^2 \in W^{1,2}(\mathrm{X})$ (see [90, Lemma 3.2]).

Notice that item i) implies the fact that test functions are 'a lot'. To produce them and also to prove item i), we need to use time regularized heat flow h_φ, defined in (1.13), applied to a function $f \in L^2(\mathfrak{m}) \cap L^\infty(\mathfrak{m})$. The main reson to use this operator is that it produces L^∞ bound on the Laplacian. The RCD assumption instead comes into play in the proof of item i) in the fact that $h_\varphi f \in \mathrm{Lip}(\mathrm{X})$ thanks to Proposition 1.4.1 and the definition of h_φ.
It will be very useful in our discussion to introduce the vector space of 'regular vector fields', that we call *test vector fields.* We define $\mathrm{TestV}(\mathrm{X}) \subseteq L^2(T\mathrm{X})$ as the vector space

$$\mathrm{TestV}(\mathrm{X}) := \left\{ \sum_{i=1}^n f_i \nabla g_i : f_i, g_i \in \mathrm{Test}(\mathrm{X}) \, \text{for every } i \text{ and } n \in \mathbb{N} \right\}.$$

We recall the tensorization property of the tangent module. Given a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. Let $\mathscr{M}$, $\mathscr{N}$ be $L^2(\mathfrak{m})$–normed $L^\infty(\mathfrak{m})$–modules. Then the product module $\mathscr{M} \times \mathscr{N}$ has a natural structure of $L^2(\mathfrak{m})$–normed $L^\infty(\mathfrak{m})$–module when endowed with the pointwise norm

$$(v, w) := |v|^2 + |w|^2 \, \mathfrak{m} \text{ a.e. for every } (v, w) \in M \times N.$$

Let $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$ and $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ be metric measure spaces. The product space $\mathrm{X} \times \mathrm{Y}$ is always implicitly endowed with the distance

$$\mathsf{d}_{\mathrm{X}\times\mathrm{Y}}((x, y), (x', y')) := \mathsf{d}_\mathrm{X}(x, x')^2 + \mathsf{d}_\mathrm{Y}(y, y')^2$$

for every $(x, y), (x', y') \in \mathrm{X} \times \mathrm{Y}$ and the product measure $\mathfrak{m}_\mathrm{X} \otimes \mathfrak{m}_\mathrm{Y}$. Let $\mathscr{M}$ be an $L^2(\mathfrak{m}_\mathrm{X})$–normed $L^\infty(\mathfrak{m}_\mathrm{X})$–module. Then the space $L^2(\mathrm{Y}, \mathscr{M})$ has a natural $L^2(\mathfrak{m}_\mathrm{X} \otimes \mathfrak{m}_\mathrm{Y})$–normed $L^\infty(\mathfrak{m}_\mathrm{X} \otimes \mathfrak{m}_\mathrm{Y})$–module structure: given any $v, w \in L^2(\mathrm{Y}, \mathscr{M})$ and $f \in L^\infty(\mathfrak{m}_\mathrm{X} \otimes \mathfrak{m}_\mathrm{Y})$, we put

$$\begin{aligned}
&(v + w)(y) := v(y) + w(y) \in \mathscr{M} \text{ for } \mathfrak{m}_\mathrm{Y}\text{-a.e. } y \in \mathrm{Y}, \\
&(f \cdot v)(y) := f(\cdot, y) \cdot v(y) \text{ for } \mathfrak{m}_\mathrm{Y}\text{-a.e. } y \in \mathrm{Y}, \\
&|v|(x, y) := |v(y)|(x) \text{ for } \mathfrak{m}_\mathrm{X} \otimes \mathfrak{m}_\mathrm{Y}\text{-a.e. } (x, y) \in \mathrm{X} \times \mathrm{Y}.
\end{aligned}$$

We review here the tensorization properties of the Cheeger energy and derive the tensorization properties of the tangent module. Let $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$,$(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ be metric measure spaces. Given $f \in L^2(\mathfrak{m}_\mathrm{X} \otimes \mathfrak{m}_\mathrm{Y})$, we define $f_{(x)} \in L^2(\mathfrak{m}_\mathrm{Y})$ and $f^{(y)} \in L^2(\mathfrak{m}_\mathrm{X})$ as

$$f_{(x)}(y) = f^{(y)}(x) := f(x, y) \text{ for } (\mathfrak{m}_\mathrm{X} \otimes \mathfrak{m}_\mathrm{Y})\text{–a.e. } (x, y) \in \mathrm{X} \times \mathrm{Y}.$$

Theorem 1.4.2 ([11, Theorem 5.1]). *Let* $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$,$(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ *be metric measure spaces. Consider* $f \in W^{1,2}(\mathrm{X} \times \mathrm{Y})$. *Then*

$$f_{(x)} \in W^{1,2}(\mathrm{Y}) \quad \textit{for } \mathfrak{m}_\mathrm{X}\textit{-a.e. } x \in \mathrm{X} \textit{ with } \int |Df_{(x)}|(y) \,\mathrm{d}(\mathfrak{m}_\mathrm{X} \otimes \mathfrak{m}_\mathrm{Y})(x,y) < \infty,$$
$$f^{(y)} \in W^{1,2}(\mathrm{X}) \quad \textit{for } \mathfrak{m}_\mathrm{Y}\textit{-a.e. } y \in \mathrm{Y} \textit{ with } \int |Df^{(y)}|(x) \,\mathrm{d}(\mathfrak{m}_\mathrm{X} \otimes \mathfrak{m}_\mathrm{Y})(x,y) < \infty,$$

Moreover, it holds that

$$|Df_{(x)}|^2(y) + |Df^{(y)}|^2(x) = |Df|^2(x,y).$$

As a consequence of the last proposition, we have the following theorem about the tensorization of the tangent module.

Theorem 1.4.3 ([63, Theorem 3.13]). *Let* $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$,$(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ *be metric measure spaces. Then the map*

$$\Phi \colon L^2(T(\mathrm{X} \times \mathrm{Y})) \to L^2(\mathrm{X}, L^2(T\mathrm{Y})) \times L^2(\mathrm{Y}, L^2(T\mathrm{X}))$$

defined for $W^{1,2}(\mathrm{X} \times \mathrm{Y})$ *as*

$$\Phi(\mathrm{d}f) := (x \mapsto \mathrm{d}f_{(x)}, y \mapsto \mathrm{d}f^{(y)})$$

extends to an isomorphism $L^2(\mathfrak{m}_\mathrm{X} \otimes \mathfrak{m}_\mathrm{Y})$*–normed* $L^\infty(\mathfrak{m}_\mathrm{X} \otimes \mathfrak{m}_\mathrm{Y})$*-modules preserving the pointwise norm.*

To define what the Hessian of a function is, we need to recall here the objects needed in order to speak about Hess$f$ as a 'tensor of type $(0,2)$'.
We outline here the construction of the tensor product of two Hilbert modules. Consider two Hilbert $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules $\mathscr{H}_1, \mathscr{H}_2$.
We consider $\mathscr{H}_1 \otimes^{\mathrm{Alg}}_{L^\infty(\mathfrak{m})} \mathscr{H}_2$ the tensor product between $\mathscr{H}_1$ and $\mathscr{H}_2$, consisting of formal finite sums of objects of the kind $w_1 \otimes w_2$ for $w_1 \in \mathscr{H}_1$ and $w_2 \in \mathscr{H}_2$, quotentied with respect to the subspace generated by the elements of the form

$$\begin{aligned}
&(\alpha_1 v_1 + \alpha_2 v_2) \otimes w - \alpha_1 (v_1 \otimes w) - \alpha_2 (v_2 \otimes w),\\
&v \otimes (\beta_1 w_1 + \beta_2 w_2) - \beta_1 (v \otimes w_1) - \beta_2 (v \otimes w_2),\\
&(fv) \otimes w - v \otimes (fw).
\end{aligned}$$

for $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, $v_1, v_2, v \in \mathscr{H}_1$, $w, w_1, w_2 \in \mathscr{H}_2$ and $f \in L^\infty(\mathfrak{m})$. We can endow it with the structure of algebraic module over $L^\infty(\mathfrak{m})$ by means of the following pointwise operation: given $f \in L^\infty(\mathfrak{m})$, $w_1 \in \mathscr{H}_1$ and $w_2 \in \mathscr{H}_2$ we have

$$f \cdot (w_1 \otimes w_2) := (f w_1) \otimes w_2 = w_1 \otimes (f w_2).$$

We can endow $\mathscr{H}_1 \otimes^{\mathrm{Alg}}_{L^\infty(\mathfrak{m})} \mathscr{H}_2$ with the operation : $(\mathscr{H}_1 \otimes^{\mathrm{Alg}}_{L^\infty(\mathfrak{m})} \mathscr{H}_2)^2 \ni (A, B) \to A : B \in L^0(\mathfrak{m})$, defined as

$$(w_1 \otimes w_2) : (\tilde{w}_1 \otimes \tilde{w}_2) := \langle w_1, \tilde{w}_1 \rangle \langle w_2, \tilde{w}_2 \rangle \quad \text{for every } w_1, w_2, \tilde{w}_1, \tilde{w}_2 \in L^2(T^*\mathrm{X})$$

and extending it by bilinearity. The map is $L^\infty(\mathfrak{m})$-bilinear. The poinwise norm on $\mathscr{H}_1 \otimes^{\mathrm{Alg}}_{L^\infty(\mathfrak{m})} \mathscr{H}_2$ is defined for $A \in \mathscr{H}_1 \otimes^{\mathrm{Alg}}_{L^\infty(\mathfrak{m})} \mathscr{H}_2$ as $|A| := \sqrt{A : A}$. Then $\mathscr{H}_1 \otimes \mathscr{H}_2$ is defined as the completion

of $\{A \in \mathscr{H}_1 \otimes^{\text{Alg}}_{L^\infty(\mathfrak{m})} \mathscr{H}_2 : |A| \in L^2(\mathfrak{m})\}$ with respect to the norm $\|A\|_{\mathscr{H}_1\otimes\mathscr{H}_2} := \sqrt{\int |A|^2 \, \mathrm{d}\mathfrak{m}}$. Therefore, constructed in this way, it is possible to check that $(\mathscr{H}_1 \otimes \mathscr{H}_2, \cdot, |\cdot|, \|\cdot\|_{\mathscr{H}_1\otimes\mathscr{H}_2})$ has the structure of a $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module.
When we consider $\mathscr{H}^1 = \mathscr{H}^2 = \mathscr{H}$, we denote the tensor product $\mathscr{H} \otimes \mathscr{H}$ by $\mathscr{H}^{\otimes 2}$.

We introduce the transpose operator and so also symmetric and antisymmetric tensors. Given $v, w \in \mathscr{H}$, $v \otimes w \in \mathscr{H} \otimes^{\text{Alg}}_{L^\infty(\mathfrak{m})} \mathscr{H}$, we define the transpose $(v \otimes w)^t \in \mathscr{H} \otimes^{\text{Alg}}_{L^\infty(\mathfrak{m})} \mathscr{H}$ as

$$(v \otimes w)^t := w \otimes v$$

and extending it by linearity. The map $\mathscr{H} \otimes^{\text{Alg}}_{L^\infty(\mathfrak{m})} \mathscr{H} \ni A \mapsto A^t \in \mathscr{H} \otimes^{\text{Alg}}_{L^\infty(\mathfrak{m})} \mathscr{H}$ is a linear involution, preserving the pointwise norm, being such that $(fA)^t = fA^t$ for every $f \in L^\infty(\mathfrak{m})$ and $A \in \mathscr{H} \otimes^{\text{Alg}}_{L^\infty(\mathfrak{m})} \mathscr{H}$. In particular, it can be extended to a linear involution preserving the pointwise norm and being such that $(fA)^t = fA^t$ for every $f \in L^\infty(\mathfrak{m})$ and $A \in \mathscr{H}^{\otimes 2}$.

Definition 1.4.4. *Given an Hilbert module $\mathscr{H}$, a tensor $A \in \mathscr{H}^{\otimes 2}$ is called symmetric if $A^t = A$ and antisymmetric if $A^t = -A$.*

Definition 1.4.5. *We define $L^2(T^{*\otimes 2}\mathrm{X}) := L^2(T^*\mathrm{X}) \otimes L^2(T^*\mathrm{X})$ and $L^2(T^{\otimes 2}\mathrm{X}) := L^2(T\mathrm{X}) \otimes L^2(T\mathrm{X})$. Moreover, we denote with $L^0(T^{*\otimes 2}\mathrm{X})$ and $L^0(T^{\otimes 2}\mathrm{X})$ the completion with respect to the distance d_{L^0} respectively of $L^2(T^{*\otimes 2}\mathrm{X})$ and $L^2(T^{\otimes 2}\mathrm{X})$.*

Since both $L^2(T^{*\otimes 2}\mathrm{X})$ and $L^2(T^{\otimes 2}\mathrm{X})$ are Hilbert modules, they are reflexive and canonically isomorphic to their dual. We can identify one with the dual of the other, via the duality mapping:

$$(w_1 \otimes w_2)(X_1 \otimes X_2) = w_1(X_1)\, w_2(X_2) \quad \mathfrak{m}\text{-a.e.}$$

for $w_1 \otimes w_2 \in L^2(T^{*\otimes 2}\mathrm{X})$ and $X_1 \otimes X_2 \in L^2(T^{\otimes 2}\mathrm{X})$ and then extended to both spaces by linearity and continuity. We can use the duality map together with the pointwise scalar product to build the two isomorphisms

$$T^\flat(S) := T : S, \qquad A^\# : T := A(T),$$

for $T, S \in L^2(T^{\otimes 2}\mathrm{X})$ and $A \in L^2(T^{*\otimes 2}\mathrm{X})$.
The definition of Hessian as an element 'L^2 integrable' goes together with the definition of the Sobolev space $W^{2,2}(\mathrm{X})$. The content of what follows is mainly taken from [55, Section 3.3]. The definition in the nonsmooth setting comes from the following computation in the Riemannian case: consider a smooth Riemannian manifold (M, g) and $f, g, h \in C^\infty(M)$; then we have

$$2\mathrm{Hess}(f)(\nabla g, \nabla h) = \langle g, \nabla\langle\nabla f, \nabla h\rangle\rangle + \langle h, \nabla\langle\nabla f, \nabla g\rangle\rangle - \langle\nabla f, \nabla\langle\nabla g, \nabla h\rangle\rangle.$$

Writing down 'distributionally' the last identity gives rise to the definition of $W^{2,2}(\mathrm{X})$.

Definition 1.4.6 (The space $W^{2,2}(\mathrm{X})$)**.** *The space $W^{2,2}(\mathrm{X}) \subseteq W^{1,2}(\mathrm{X})$ is the space of all functions $f \in W^{1,2}(\mathrm{X})$ for which there exists $A \in L^2(T^{*\otimes 2}\mathrm{X})$ such that for every $g_1, g_2, h \in \mathrm{Test}(\mathrm{X})$ the following holds*

$$\begin{aligned} &2\int hA(\nabla g_1, \nabla g_2)\, \mathrm{d}\mathfrak{m} \\ &= -\int \langle\nabla f, \nabla g_1\rangle \mathrm{div}(h\nabla g_2) + \langle\nabla f, \nabla g_2\rangle \mathrm{div}(h\nabla g_1) + h\langle\nabla f, \nabla\langle\nabla g_1, \nabla g_2\rangle\rangle\, \mathrm{d}\mathfrak{m}. \end{aligned}$$

In this case, $\mathrm{Hess}\, f := A$ *and is called the Hessian of* f*. We endow the vector space with the norm*

$$\|f\|^2_{W^{2,2}(\mathrm{X})} := \|f\|^2_{L^2(\mathfrak{m})} + \|\mathrm{d}f\|^2_{L^2(\mathfrak{m})} + \|\mathrm{Hess}\, f\|^2_{L^2(\mathfrak{m})}.$$

By the property that

$$\text{finite sums of } \{h\nabla g_1 \otimes \nabla g_2 :\ h, g_1, g_2 \in \mathrm{Test}(\mathrm{X})\} \text{ are dense in } L^2(T^{\otimes 2}\mathrm{X})$$

we have that element A defined as above is uniquely determined.

Remark 1.4.7. We point out that, to produce the last definition, all the elements tested against f and its gradient needs to be L^2 integrable; indeed it is straightfoward to check that, if $g_1, g_2, h \in \mathrm{Test}(\mathrm{X})$,

$$\nabla g_1 \,\mathrm{div}(h\nabla g_2), \nabla g_2 \,\mathrm{div}(h\nabla g_1), h\, \nabla\langle\nabla g_1, \nabla g_2\rangle\rangle \in L^2(T\mathrm{X}).$$

The last of the three terms belongs to $L^2(T\mathrm{X})$ thanks to item iii) in the properties of test functions, that gives by polarization that, if $f, g \in \mathrm{Test}(\mathrm{X})$, $\langle\nabla f, \nabla g\rangle \in W^{1,2}(\mathrm{X})$, hence it makes sense to compute $\nabla\langle\nabla g_1, \nabla g_1\rangle \in L^2(T\mathrm{X})$. ■

An important property of test functions is that [55, Theorem 3.3.8]

$$\mathrm{Test}(\mathrm{X}) \subseteq W^{2,2}(\mathrm{X})$$

so it makes sense to compute the Hessian of a test function.
Some remarkable properties of the space $W^{2,2}(\mathrm{X})$ are the following ones [55, Theorem 3.3.2, Propositions 3.3.20, 3.3.21]:

i) $W^{2,2}(\mathrm{X})$ is a separable Hilbert space;

ii) given $f_1, f_2 \in \mathrm{Test}(\mathrm{X})$, we have that

$$\mathrm{Hess}\,(f_1 f_2) = f_2 \,\mathrm{Hess}\, f_1 + f_1 \,\mathrm{Hess}\, f_2 + \mathrm{d}f_1 \otimes \mathrm{d}f_2 + \mathrm{d}f_2 \otimes \mathrm{d}f_1$$

as elements of $L^2(T^{*\otimes 2}\mathrm{X})$;

iii) given $f \in \mathrm{Test}(\mathrm{X})$ and $\varphi \in C^{1,1}(\mathbb{R})$ with bounded first and second derivative, we have that $\varphi \circ f \in W^{2,2}(\mathrm{X})$ and

$$\mathrm{Hess}\,(\varphi \circ f) = \varphi'' \circ f \,\mathrm{d}f \otimes \mathrm{d}f + \varphi' \circ f \,\mathrm{Hess}\, f$$

as elements of $L^2(T^{*\otimes 2}\mathrm{X})$.

With the definition of Hessian of a function in mind, we can differentiate vector field, thus defining what the covariant derivative of a vector field is. Similarly as for the Hessian, the definition of 'covariant derivative' that is L^2-integrable is tightly linked to the definition of the functional space $W^{1,2}_C(T\mathrm{X})$.
Similarly as we did for the Hessian, we can introduce the covariant derivative of a vector field, by making distributional-like the following smooth computation. We refer to [55, Section 3.4]. Consider a smooth Riemannian manifold (M, g), a smooth vector field X and $h, g_1, g_2 \in C^\infty(M)$, we have

$$h\langle\nabla_{\nabla g_1} X, \nabla g_2\rangle = \langle\nabla\langle X, \nabla g_2\rangle, \nabla g_1\rangle - \mathrm{Hess}\, g_2(\nabla g_1, X).$$

Definition 1.4.8 (The space $W^{1,2}_C(T\mathrm{X})$)**.** *The Sobolev space $W^{1,2}_C(T\mathrm{X}) \subseteq L^2(T\mathrm{X})$ is defined as the space of all $X \in L^2(T\mathrm{X})$ for which there exists an element $T \in L^2(T^{\otimes 2}\mathrm{X})$ such that for every $g_1, g_2, h \in \mathrm{Test}(\mathrm{X})$ the following holds*

$$\int h\, T : (\nabla g_1 \otimes \nabla g_2)\,\mathrm{d}\mathfrak{m} = -\int \langle X, \nabla g_2\rangle\,\mathrm{div}(h\nabla g_1) + h\,\mathrm{Hess}\, g_2\,(X, \nabla g_1)\,\mathrm{d}\mathfrak{m}.$$

In this case, we will call T the covariant derivative of X and will denote in ∇X. We endow $W^{1,2}_C(T\mathrm{X})$ with the norm $\|\cdot\|_{W^{1,2}_C(T\mathrm{X})}$ defined as follows:

$$\|X\|^2_{W^{1,2}_C(T\mathrm{X})} := \|X\|^2_{L^2(T\mathrm{X})} + \|\nabla X\|^2_{L^2(T^{\otimes 2}\mathrm{X})}.$$

The element $T \in L^2(T^{\otimes 2}\mathrm{X})$ is uniquely determined by the density of finite sums of elements in $\{h\nabla g_1 \otimes \nabla g_2 :\ h, g_1, g_2 \in \mathrm{Test}(\mathrm{X})\}$ in $L^2(T^{\otimes 2}\mathrm{X})$.

Remark 1.4.9. We point out that, to produce the last definition, as for the Hessian, all the elements tested against X needs to be L^2 integrable; indeed it is straightfoward to check that, if $g_1, g_2, h \in \mathrm{Test}(\mathrm{X})$,

$$\nabla g_2 \mathrm{div}(h\nabla g_1) \in L^2(T\mathrm{X}), \quad h\,\mathrm{Hess}\, g_2(\cdot, \nabla g_1) \in L^2(T^*\mathrm{X}).$$

■

Some remarkable properties of the space $W^{1,2}_C(T\mathrm{X})$ are the following ones [55, Theorem 3.4.2]:

i) $W^{1,2}_C(T\mathrm{X})$ is a separable Hilbert space;

ii) for $f \in W^{2,2}(\mathrm{X})$, we have $\nabla f \in W^{1,2}_C(T\mathrm{X})$ with $\nabla(\nabla f) = (\mathrm{Hess}\, f)^{\#}$;

iii) $\mathrm{Test}V(\mathrm{X}) \subseteq W^{1,2}_C(T\mathrm{X})$ and given $v = \sum_{i=1}^n f_i \nabla g_i$ for $f_i, g_i \in \mathrm{Test}(\mathrm{X})$

$$\nabla v = \sum_{i=1}^n \nabla f_i \otimes \nabla g_i + f_i(\mathrm{Hess}\, g_i)^{\#}.$$

It can be readily checked that ∇v is well posed, i.e. depends only on v and not in the way it is written. The following notation is very useful in the following manuscript, in analogy with Riemannian geometry. Given $X \in W^{1,2}_C(T\mathrm{X})$ and $Z \in L^0(T\mathrm{X})$, we denote by $\nabla_Z X$ the element in $L^0(T\mathrm{X})$ defined as:

$$\langle \nabla_Z X, Y\rangle := \nabla X : (Z \otimes Y),\ \mathfrak{m}\text{-a.e., for every } Y \in L^0(T\mathrm{X}).$$

An important calculus rule is the following one, that is a suitable restatement of [55, Prop. 3.46].

Proposition 1.4.10. *Let $f \in \mathrm{Test}(\mathrm{X})$ and $Z \in L^\infty(\mathfrak{m}) \cap W^{1,2}_C(T\mathrm{X})$. Then $\mathrm{d}f(Z) \in W^{1,2}(\mathrm{X})$ and*

$$\mathrm{d}(\mathrm{d}f(Z)) = \mathrm{Hess}\, f(\cdot \otimes Z) + \nabla Z : (\cdot \otimes \nabla f). \tag{1.23}$$

When dealing with flow of nonsmooth vector field (see Section 2), to have a well-posedness theory it is enough to have a control of the integrability of the symmetric part of the covariant derivative and not of all of it. Therefore, we recall the definition of vector fields with symmetric covariant derivative in L^2, according to the axiomatization given in [27, Definition 1.18].

Definition 1.4.11 (The space $W^{1,2}_{C,s}(T\mathrm{X})$)**.** *The Sobolev space $W^{1,2}_{C,s}(T\mathrm{X}) \subseteq L^2(T\mathrm{X})$ is the space of all elements $X \in L^2(T\mathrm{X})$ with $\mathrm{div}X \in L^2(\mathfrak{m})$ for which there exists $S \in L^2(T^{\otimes 2}\mathrm{X})$ such that, for every $h, g_1, g_2 \in \mathrm{Test}(\mathrm{X})$, the following holds*

$$\begin{aligned}&\int hS(\nabla g_1, \nabla g_2)\,\mathrm{d}\mathfrak{m}\\ = -\frac{1}{2}&\int \langle X, \nabla g_2\rangle\,\mathrm{div}(h\nabla g_1) + \langle X, \nabla g_1\rangle\,\mathrm{div}(h\nabla g_2) + \mathrm{div}(hX)\langle \nabla g_1, \nabla g_2\rangle.\end{aligned}$$

In this case, we will call S the symmetric covariant derivative of X and will denoted by $\nabla_{\mathrm{sym}}X$. We endow the space $W^{1,2}_{C,s}(T\mathrm{X})$ with the norm

$$\|X\|^2_{W^{1,2}_{C,s}(T\mathrm{X})} := \|X\|^2_{L^2(T\mathrm{X})} + \|\nabla_{\mathrm{sym}}X\|^2_{L^2(T^{\otimes 2}\mathrm{X}}.$$

Notice that $\nabla_{\mathrm{sym}}X$ defined above is a symmetric tensor in the sense of Definition 1.4.4. It is remarked in [27] that, from the definition, it follows that

$$X \in W^{1,2}_C(T\mathrm{X}) \text{ with } \mathrm{div}X \in L^2(\mathfrak{m}) \Rightarrow X \in W^{1,2}_{C,s}(T\mathrm{X})$$

and $\nabla_{\mathrm{sym}}X = \frac{1}{2}(\nabla X + (\nabla X)^t)$.

Chapter 2

Flows associated to Sobolev vector fields

We want to summarize in this section the developments of the theory of flows of nonsmooth vector fields in the Euclidean space (in Section 2.1) and then in the case of nonsmooth underlying space (in Section 2.2).
From now on, the setting is $\mathbb{R}^d$ and we consider a vector field $\boldsymbol{b} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$. We write in short $\boldsymbol{b}_t(x) = \boldsymbol{b}(t, x)$. Similarly for a function $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$, we shall write $u_t(x) = u(t, x)$. We don't specify for the moment the regularity assumptions on the objects involved. We consider two related problems:

- **Lagrangian problem**. Given $x \in \mathbb{R}^d$, we consider $\gamma\colon [0, T] \to \mathbb{R}^d$ such that

$$\begin{cases} \dot\gamma_t = \boldsymbol{b}_t(\gamma_t) \\ \gamma_0 = x \end{cases} \tag{2.1}$$

- **Eulerian problem**. Given $\bar\rho\colon \mathbb{R}^d \to \mathbb{R}$, we consider a function $\rho\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ which solves the continuity equation (denoted with short notation (CE)), namely

$$\begin{cases} \partial_t \rho_t + \operatorname{div}(\boldsymbol{b}_t \rho_t) = 0, \\ \rho_0 = \bar\rho \end{cases} \tag{2.2}$$

  Another PDE related to the ODE problem is the transport equation (denoted with short notation (TE)): given $\bar u$, a function $u\colon [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d$ which solves the transport equation, i.e. that solves

$$\begin{cases} \partial_t u_t + \boldsymbol{b}_t \cdot \nabla u_t = 0, \\ u_0 = \bar u. \end{cases}$$

The two PDEs are related and in the case of divergence free vector fields the solutions of the two PDEs coincide. The interpretation underlying the last two problems can be seen as follows:

- **Lagrangian problem**. The first problem aims at following the trajectories of a single particle up to time t that moves according to the time dependent velocity field $\boldsymbol{b}_t$;

- **Eulerian problem**. The second problem in the case of (CE) aims at considering an initial distribution of pollutant $\bar{\rho}$ in a liquid where $\bar{\rho}(x)$ stands for the density of pollutant at the point x; then we let the pollutant evolving along a velocity field $\boldsymbol{b}_t$ in the liquid and the distribution of pollutant at freezed time t at the point x is $\rho_t(x)$.

2.1 The Euclidean setting: wellposedness and regularity

We start considering the Lagrangian problem (2.1). Existence of solutions to this problem for fixed initial data x can be proved under continuity assumptions on $\boldsymbol{b}$, while uniqueness needs some regularity assumptions at the first order of differentiability for $\boldsymbol{b}$, known in literature as the Cauchy-Lipschitz theory. This is the content of the following classical theorem.

Theorem 2.1.1. *Let $\boldsymbol{b}\colon [0,1]\times\mathbb{R}^d \to \mathbb{R}^d$ be a continuous a vector field. Assume that there exists $L>0$ such that*

$$|\boldsymbol{b}(t,x)-\boldsymbol{b}(t,y)| \le L|x-y| \quad \textit{for every } t\in[0,1], x,y\in\mathbb{R}^d.$$

Then for every $x\in\mathbb{R}^d$ there exists a unique solution $\gamma\in C^1([0,1])$ of (2.1) starting from x. Moreover, the following stability estimate holds: fixed $t\in[0,1]$

$$\textit{the map } x\mapsto \gamma_x(t)$$

where $\gamma_x(t)$ is the solution of (2.1) starting from $x\in\mathbb{R}^d$ at time $t\in[0,1]$ is Lipschitz.

Existence can be proved by means of Banach-Caccioppoli theorem, while uniqueness and stability is a consequence of Gronwall lemma, as follows. Consider respectively a solution $\gamma_x(\cdot)$ of (2.1) starting from x and another one $\gamma_y(\cdot)$ starting from y; then the map $t\mapsto|\gamma_x(t)-\gamma_y(t)|$ is Lipschitz (since it is the composition of the Lipschitz function $|\cdot|$ with a C^1 function), hence a.e. differentiable, with derivative given by

$$\partial_t|\gamma_x(t)-\gamma_y(t)| \le |\boldsymbol{b}(t,\gamma_x(t))-\boldsymbol{b}(t,\gamma_y(t))| \le L|\gamma_x(t)-\gamma_y(t)| \quad \text{for a.e. } t$$

Hence Gronwall lemma grants that

$$|\gamma_x(t)-\gamma_y(t)| \le e^{Lt}|x-y| \quad \text{for every } t\in[0,1]. \tag{2.3}$$

This computation will be crucial when passing to the Sobolev setting in the Euclidean space (see Proposition 2.1.16 below). Once we know the wellposedness theorem in the Lipschitz setting, it will be convenient to bundle the trajectories together, defining the flow map $\boldsymbol{X}: [0,1]\times\mathbb{R}^d\to\mathbb{R}^d$, where $\boldsymbol{X}(\cdot,x)$ is the solution of (2.1) starting from x, hence having

$$\begin{cases} \partial_t\boldsymbol{X}(t,x) = \boldsymbol{b}(t,\boldsymbol{X}(t,x)), \\ \boldsymbol{X}(0,x) = x. \end{cases} \tag{2.4}$$

It is convenient to define $\boldsymbol{X}^{-1}(t,\cdot)$ as the inverse of the map $\boldsymbol{X}(t,\cdot)$, i.e. as a map such that

$$\boldsymbol{X}(t,\boldsymbol{X}^{-1}(t,x)) = \boldsymbol{X}^{-1}(t,\boldsymbol{X}(t,x)) = x \quad \text{for every } t,x.$$

Under the standing assumptions on $\boldsymbol{b}$, $\boldsymbol{X}(t,\cdot)$ is a biLipschitz homeomorphism of $\mathbb{R}^d$. If $\boldsymbol{b}\in C^\infty([0,1]\times\mathbb{R}^d)$, then $\boldsymbol{X}$ belongs $C^\infty([0,1]\times\mathbb{R}^d)$ and so does its inverse.

Remark 2.1.2. The uniqueness of solutions in $C^1([0,1])$ stems, with the same arguments of the proof of Theorem 2.1.1, also from the assumptions that $\boldsymbol{b}(t,\cdot)$ is Lipschitz for a.e. $t \in [0,1]$ and

$$\int_0^1 \mathrm{Lip}(\boldsymbol{b}_t)\,\mathrm{d}t < \infty.$$

In this case, the stability estimates holds with Lipschitz constant in (2.3) $e^{\int_0^t \mathrm{Lip}(\boldsymbol{b}_s)\,\mathrm{d}s}$ instead of e^{Lt}. ■

We consider now the Eulerian problem. Consider $\boldsymbol{b}$ verifying the hyphothesis of Theorem 2.1.1. The Cauchy-Lipschitz theory of flows yields wellposedness of smooth solutions of the transport and continuity equations together with a representation formula for both of them. We assume that $\boldsymbol{b} \in C^\infty([0,1]\times\mathbb{R}^d)$.
Wellposedness of smooth solutions of the transport equation follows by an application of the theory of characteristic lines. Consider $\bar{u} \in C^\infty(\mathbb{R}^d)$. We consider the flow $\boldsymbol{X}$ defined as before. For every solution $u \in C^\infty([0,1]\times\mathbb{R}^d)$ of the transport equation, we have that for every x, $t \mapsto u(t,\boldsymbol{X}(t,x))$ is $C^1([0,1])$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t,\boldsymbol{X}(t,x)) = (\partial_t u_t)(\boldsymbol{X}(t,x)) + (\boldsymbol{b}_t\cdot\nabla u_t)(\boldsymbol{X}(t,x)) = 0 \tag{2.5}$$

for every t, thus having that, for every $x \in \mathbb{R}^d$, $u(t,\boldsymbol{X}(t,x)) = \bar{u}(x)$. Therefore, the following formula gives an expression of a solution of the transport equation in terms of the flow associated to $\boldsymbol{b}$:

$$u(t,x) := \bar{u}(\boldsymbol{X}(t,\cdot)^{-1}(x)) \tag{2.6}$$

thus showing existence and uniqueness of solutions in $C^\infty([0,1]\times\mathbb{R}^d)$ to (TE) for a given $\bar{u} \in C^\infty(\mathbb{R}^d)$.
We now turn to the case of the continuity equation (2.2). We discuss wellposedness of solution of (CE) in the class $C^\infty([0,1]\times\mathbb{R}^d)$. We fix $\bar{\rho} \in C^\infty(\mathbb{R}^d)$.
For this case, we can also derive an explicit formula for a solution of the continuity equation in terms of the initial distribution $\bar{\rho}$ and the flow map $\boldsymbol{X}$, which is due to Liouville. We start noticing that, for fixed $x \in \mathbb{R}^d$, the map $t \mapsto \det(\nabla_x \boldsymbol{X}_t(x))$ belongs to $C^1([0,1])$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(\nabla_x \boldsymbol{X}_t(x)) = \mathrm{div}\boldsymbol{b}_t(\boldsymbol{X}_t(x))\det(\nabla_x \boldsymbol{X}_t(x)) \quad \text{for every } t.$$

using Gronwall's lemma we get for every x and $t \in [0,1]$:

$$e^{-\int_0^t \|\mathrm{div}\boldsymbol{b}_s\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s} \le \det(\nabla_x \boldsymbol{X}_t(x)) \le e^{\int_0^t \|\mathrm{div}\boldsymbol{b}_s\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s}. \tag{2.7}$$

Therefore, it is easy to check that

$$\partial_t\Big(\rho(t,\boldsymbol{X}_t(x))\det(\nabla_x \boldsymbol{X}_t(x))\Big) = 0;$$

therefore, given a smooth solution ρ of (CE) the following formula holds for every $(t,x) \in [0,1]\times\mathbb{R}^d$

$$\rho(t,\boldsymbol{X}(t,x)) = \frac{\bar{\rho}(x)}{\det(\nabla_x \boldsymbol{X}_t(x))}.$$

We point out that the formula above is well defined as a consequence of the fact that X_t is a diffeomorphism. A much more elegant and coincise way of writing down the solution can be expressed via measure theory and the definition of pushforward. We have that the function ρ defined as above is given by

$$\rho_t \mathscr{L}^d := \boldsymbol{X}(t,\cdot)_*(\rho_0 \mathscr{L}^d).$$

The aim of this chapter is to go beyond the Cauchy-Lipschitz theory and study vector fields with less regularity than the Lipschitz ones. When the vector field $\boldsymbol{b}$ does not have a Lipschitz regularity in space uniformly in time or doesn't satisfy the hyphothesis of Remark 2.1.2, we don't have in general uniqueness of solutions of the Lagrangian problem for fixed initial point. Let us present this problem with the following famous textbook example. From now on, with a little abuse of notations we denote the Lebesgue and Sobolev spaces of vector fields with $L^p(\mathbb{R}^d)$ and $W^{1,p}(\mathbb{R}^d)$, as done for functions.

Example 2.1.3. *A classical example of non uniqueness of integral curves in $\mathbb{R}$ is the following one. Consider the (autonomous) vector $\boldsymbol{b}^1(x) := \sqrt{x}$. In this case, fixed as initial point 0 we have that the one parameter family of curves $(\gamma^c)_{c\in\mathbb{R}^+}$*

$$\gamma^c(t) := \begin{cases} 0 & t \leq c \\ \frac{1}{4}(t-c)^2 & t \geq c \end{cases} \tag{2.8}$$

solves (2.1) with $\gamma^c(0) = 0$ and the choice of $\boldsymbol{b}$ as above.

In this case, the vector field is indeed not Lipschitz, but belongs to $W^{1,p}_{\mathrm{loc}}(\mathbb{R})$ for every $1 \leq p < 2$.

We are interested in the case in which $\boldsymbol{b}$ has Sobolev regularity in space. Apart from the problem expressed in the above example, there is another (more technical) problem: we need to find a notion of flow which is independent of the choice of the representative of the vector field. Both problems together impose the necessity of a way of selecting integral curves of a vector field, that leads to the following definition. Notice that we think $\boldsymbol{b}$ in the following definition as a pointwise defined object.

Definition 2.1.4 (Regular Lagrangian flow)**.** *Let $\boldsymbol{b}\colon [0,1]\times\mathbb{R}^d \to \mathbb{R}^d$ be Borel and in $L^1([0,1], L^2(\mathbb{R}^d))$. Then we say that $F\colon [0,1]\times\mathbb{R}^d \to \mathbb{R}^d$ (we denote $F(t,x) = F_t(x)$) is a regular Lagrangian flow associated to $\boldsymbol{b}$ provided it is Borel and*

i) for every $x \in \mathbb{R}^d$, the curve $F_\cdot(x)$ is continuous and $F_0(x) = x$;

ii) there exists $C > 0$ such that $F_{t*}\mathscr{L}^d \leq C\mathscr{L}^d$ for every $t \in [0,1]$;

iii) it solves the ODE: for a.e. x the curve $t \mapsto F_t(x)$ is absolutely continuous and

$$\partial_t F_t(x) = \boldsymbol{b}_t(F_t(x)) \quad \textit{for a.e. } s.$$

Item ii) imposes that the trajectories do not overlap too much at any time $t \in [0,1]$. The constant C is usually called a compressibility constant of the flow. Consider $\boldsymbol{b} \in L^1([0,1], L^2(\mathbb{R}^d)) \subseteq L^0(\mathscr{L}^{d+1})$ and two Borel representatives $\boldsymbol{b}^1$ and $\boldsymbol{b}^2$ of $\boldsymbol{b}$. In particular, it holds that

$$\mathscr{L}^{d+1}\left(\left\{(t,x) \in \mathbb{R}^{d+1} : \boldsymbol{b}^1(t,x) \neq \boldsymbol{b}^2(t,x)\right\}\right) = 0.$$

Assume for a moment that a regular Lagrangian flow F associated to $\boldsymbol{b}^1$ exists; then F is also a regular Lagrangian flow associated to $\boldsymbol{b}^2$. Moreover, if we consider a Borel $\tilde{F}\colon [0,1]\times\mathbb{R}^d\to\mathbb{R}^d$ such that $\tilde{F}_\cdot(x)\in C^0([0,1],\mathbb{R}^d)$ for every x and $\tilde{F}_\cdot(x)=F_\cdot(x)$ for $\mathscr{L}^d$-a.e. x, then $\tilde{F}$ is a regular Lagrangian flow associated to $\boldsymbol{b}^1$. In a reverse order to how we proceeded in the Lipschitz setting, in the Sobolev one wellposedness of regular Lagrangian flows stems from wellposedness of distributional solutions of the PDEs at the Eulerian level (transport and continuity equations) in the class of non-negative bounded solutions. The technique, due to DiPerna and Lions in [47], is to introduce the concept of renormalized solutions for which well posedness holds and then show that all the nonnegative bounded solutions are renormalized. The notations are taken from [21]. We denote by $\mathscr{M}(\mathbb{R}^d)$ the vector space of locally finite signed measure on $\mathbb{R}^d$, normed with total variant variation norm (possibly taking the value $+\infty$). To avoid confusion, we denote with ∇_x the gradient operator defined in $C^\infty([0,1]\times\mathbb{R}^d)$ just with respect to the last d variables.

Definition 2.1.5 (Distributional solution of (CE))**.** *Consider a Borel vector field $\boldsymbol{b}\colon[0,1]\times\mathbb{R}^d\to\mathbb{R}^d$. A weakly Borel function $[0,1]\ni t\mapsto\mu_t\in\mathscr{M}(\mathbb{R}^d)$ such that $\boldsymbol{b}_t\mu_t$ is a locally finite signed measure is a distributional solution of (CE) associated to $\boldsymbol{b}$ with initial distribution $\bar{\mu}$ if*

$$\int_0^T\int[\partial_t\varphi(t,x)+\boldsymbol{b}(t,x)\cdot\nabla_x\varphi(t,x)]\,\mathrm{d}\mu_t\,\mathrm{d}t+\int\varphi(0,x)\,\mathrm{d}\bar{\mu}(x)=0$$

for every $\varphi\in C_c^\infty([0,1)\times\mathbb{R}^d)$.

We denote the curve $(t\mapsto\mu_t)$ sometimes with μ. We say that $t\mapsto\mu_t$ is weak* continuous in the sense of measures if for every $f\in C_c^0(\mathbb{R}^d)$ $t\mapsto\int f\,\mathrm{d}\mu_t$ is continuous. When $\mu_t=\rho_t\mathscr{L}^d$ for every t with $L^\infty(\mathbb{R}^d)$, we recall that $t\mapsto\mu_t$ is weak* continuous in $L^\infty(\mathbb{R}^d)$ if for every $f\in L^1(\mathbb{R}^d)$ $t\mapsto\int f\,\mathrm{d}\mu_t$ is continuous. It is worth pointing out the regularity of the curve $[0,1]\ni t\mapsto\mu_t\in\mathscr{M}(\mathbb{R}^d)$ by knowing that is solves the continuity equation in the sense of Definition 2.1.5. Indeed, we have the following results, whose proof can be found in [21, Proposition 1.7, 1.8]:

$$\begin{aligned}&t\mapsto\mu_t \text{ solves (CE) and } \operatorname{ess\,sup}|\mu_t|(\mathbb{R}^d)<\infty\\&\quad\Rightarrow\exists\,t\mapsto\tilde{\mu}_t \text{ such that } \mu_t=\tilde{\mu}_t \text{ for a.e. } t,\ t\mapsto\tilde{\mu}_t \text{ is weak* cont.in the sense of measures}\end{aligned}$$

and

$$\begin{aligned}t\mapsto\mu_t \text{ solves (CE) and } \mu_t=\rho_t\mathscr{L}^d \text{ for a.e. } t,\ \rho\in L^\infty([0,1]\times\mathbb{R}^d)\\ \Rightarrow\exists\,t\mapsto\tilde{\mu}_t=\tilde{\rho}_t\mathscr{L}^d \text{ such that } \tilde{\rho}_t\in L^\infty(\mathbb{R}^d) \text{ for every } t,\\ \tilde{\rho}_t=\rho_t \text{ for a.e. } t,\ t\mapsto\tilde{\mu}_t \text{ is weak* continuous in } L^\infty(\mathbb{R}^d).\end{aligned}$$

We formulate distributionally the transport equation with linear term:

$$\partial_t u_t+\boldsymbol{b}_t\cdot\nabla u_t=c_t u_t,\quad u_0=\bar{u}.$$

The situation here is more delicate: since we are interested in $u\in L^\infty([0,1]\times\mathbb{R}^d)$, the term $\boldsymbol{b}\cdot\nabla u$ is not in general a distribution. Nevertheless, in the case in which the distribution $\mathrm{div}\boldsymbol{b}$ is represented by a function in $L^1_{\mathrm{loc}}([0,1]\times\mathbb{R}^d)$, it makes sense to define the distribution $\boldsymbol{b}\cdot\nabla u$ as follows: given $\varphi\in C_c^\infty((0,1)\times\mathbb{R}^d)$, we define $\langle\boldsymbol{b}_t\cdot\nabla u_t,\varphi\rangle:=-\langle\boldsymbol{b}_t\,u_t,\nabla\varphi\rangle-\langle u_t\,\mathrm{div}\boldsymbol{b}_t,\varphi\rangle$.

Definition 2.1.6 (Distributional solution of (TE))**.** *Assume that* $\boldsymbol{b} \in L^1_{\text{loc}}([0,1] \times \mathbb{R}^d)$*,* $\text{div}\boldsymbol{b} \in L^1_{\text{loc}}([0,1] \times \mathbb{R}^d)$ *and* $c \in L^1_{\text{loc}}([0,1] \times \mathbb{R}^d)$*. Consider* $\bar{u} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$*. Then we say that* $u \in L^\infty_{\text{loc}}([0,1] \times \mathbb{R}^d)$ *is a distributional solution to the (TE) if for every* $\varphi \in C^1_c([0,T) \times \mathbb{R}^d)$

$$\int_0^1 \int [u\partial_t\varphi + u\boldsymbol{b}_t \cdot \nabla_x\varphi + u \,\text{div}\boldsymbol{b}_t \,\varphi + c_t u_t \varphi] \,\mathrm{d}x \,\mathrm{d}t + \int \bar{u}(x)\varphi(0,x)\,\mathrm{d}x = 0.$$

We have in the case of (TE) the following regularity of a distributional solution.

$$\begin{aligned} &t \mapsto u_t \text{ solves (TE) and } u \in L^\infty([0,1] \times \mathbb{R}^d) \\ &\quad \Rightarrow \exists\, t \mapsto \tilde{u}_t \text{ such that } \tilde{u}_t \in L^\infty(\mathbb{R}^d) \text{ for every } t, \\ &\quad u_t = \tilde{u}_t \text{ for a.e. } t \text{ and is weak* continuous in } L^\infty(\mathbb{R}^d). \end{aligned}$$

This formulation will include the case of the continuity equation when $c_t(x) = -\text{div}\boldsymbol{b}_t(x) \in L^1_{\text{loc}}([0,1] \times \mathbb{R}^d)$. Therefore, we study here only (TE) and retrieve the results for (CE) by means of the choice of c_t as above. Before discussing the wellposedness, we define the concept of renormalized solutions.

Definition 2.1.7 (Renormalized solution of (TE))**.** *Let* $\boldsymbol{b} \in L^1_{\text{loc}}([0,1] \times \mathbb{R}^d, \mathbb{R}^d)$ *be such that* $\text{div}\boldsymbol{b} \in L^1_{\text{loc}}([0,1] \times \mathbb{R}^d, \mathbb{R}^d)$ *and* $c \in L^1_{\text{loc}}([0,1] \times \mathbb{R}^d)$*. We say that* $u \in L^\infty_{\text{loc}}([0,1] \times \mathbb{R}^d)$ *is a renormalized solution of (TE) with initial datum* $\bar{u}$*, if for every* $\beta \in C^1(\mathbb{R})$*,* $\beta(u)$ *solves (TE) with initial datum* $\beta(\bar{u})$ *in the sense of Definition 2.1.6.*

The general philosophy of the concept of renormalized solutions is that, fixed $\bar{u}$ and $\boldsymbol{b}$ belonging to some vector space $\mathscr{A}$, if all the solutions of (CE) in a vector space $\mathscr{A}$ are rernormalized, then, if they exist, they are unique in $\mathscr{A}$. Indeed, uniqueness of solutions of (TE) in $\mathscr{A}$ follows from the fact that the transport equation is linear and the comparison principle holds for renormalized solution, namely given two renormalized solutions to (TE) u^1 and u^2

$$u^1_0 \leq u^2_0 \quad \Rightarrow \quad u^1_t \leq u^2_t \quad \text{for every } t \in [0,1].$$

We can retrieve the same results for (CE), choosing $c_t = -\text{div}\boldsymbol{b}_t$.

The wellposedness of solution to the transport equation with linear term consists of two results:

i) the existence result can be easily achieved via the following technique for a given $\boldsymbol{b} \in L^1([0,1], L^p(\mathbb{R}^d))$, $c \in L^1([0,1], L^\infty(\mathbb{R}^d))$, $\bar{u} \in L^\infty(\mathfrak{m})$: we fix a parameter $\epsilon > 0$, we regularize the vector field $\boldsymbol{b}$, c and the initial datum $\bar{u}$ with a mollifier of size ϵ ρ_ε, by defining

$$\begin{aligned} &\boldsymbol{b}^\varepsilon_t := \boldsymbol{b}_t \star \rho_\varepsilon, \quad c^\varepsilon_t := c_t \star \rho_\varepsilon \quad \text{for every } t, \\ &\bar{u}^\varepsilon := \bar{u} \star \rho_\varepsilon. \end{aligned}$$

We call u^ε the solution of (TE) associated to $\boldsymbol{b}^\varepsilon$ (and associated flow to it $\boldsymbol{X}^\varepsilon$) and linear term c with initial datum $\bar{u}$. For fixed $\epsilon > 0$, it holds that for every x

$$\frac{\mathrm{d}}{\mathrm{d}t} u^\epsilon(t, \boldsymbol{X}^\epsilon_t(x)) = (\partial_t u^\epsilon + \boldsymbol{b}^\epsilon \cdot \nabla u^\epsilon)(t, \boldsymbol{X}^\epsilon_t(x)) = c^\epsilon(t, \boldsymbol{X}^\epsilon_t(x)) u^\epsilon(t, \boldsymbol{X}^\epsilon_t(x)).$$

Then, applying Gronwall lemma and using $\|\bar{u}^\epsilon\|_{L^\infty} \leq \|\bar{u}\|_{L^\infty}$, $\|c_t^\epsilon(\boldsymbol{X}_t^\epsilon(\cdot))\|_{L^\infty} \leq \|c_t^\epsilon\|_{L^\infty} \leq \|c_t\|_{L^\infty}$ for every t

$$|u^\varepsilon(t, \boldsymbol{X}^\varepsilon(t,x))| \leq \|\bar{u}\|_{L^\infty(\mathfrak{m})} e^{\int_0^1 \|c_t\|_{L^\infty(\mathfrak{m})} \,\mathrm{d}t}.$$

We retrieve L^∞ bound on the approximate solutions u^ε; by weak*-compactness in $L^\infty([0,1]\times\mathbb{R}^d)$, we pass to the limit in the distributional formulation, obtaining a solution to (TE);

ii) for uniqueness result, it is important to know some regularity properties about the distributional derivative $D\boldsymbol{b}_t$ for a.e. t (for the literature about it, far from being complete, we refer to [47], [2], [22], [19]) and to the survey [4]. The case of our interest here is that of Sobolev vector fields. Thus consider $\boldsymbol{b} \in L^1([0,1], W^{1,p}(\mathbb{R}^d))$ for some $p \geq 1$ and $\mathrm{div}\boldsymbol{b} \in L^1([0,1], L^\infty(\mathbb{R}^d))$. What is proved in [47] is that all the solutions in $L^\infty([0,1]\times\mathbb{R}^d))$ are renormalized. Therefore, this discussion can be read in a concise way in the following theorem of uniqueness.

Theorem 2.1.8 (DiPerna-Lions, '89)**.** *Let* $\boldsymbol{b} \in L^1([0,1], W^{1,p}(\mathbb{R}^d))$, $u_0 \in L^\infty(\mathbb{R}^d)$ *with* $p \geq 1$. *Assume moreover that* $\mathrm{div}\boldsymbol{b} \in L^1([0,1], L^\infty(\mathbb{R}^d))$*; then there exists a unique solution* $u \in L^\infty([0,1]\times\mathbb{R}^d)$ *to (TE).*

Remark 2.1.9. The last theorem holds in a larger class for the initial datum and the solution. Given $\boldsymbol{b} \in L^1([0,1], W^{1,p}(\mathbb{R}^d))$, $\bar{u} \in L^q(\mathbb{R}^d)$ with $\frac{1}{q} + \frac{1}{p} \leq 1$, uniqueness holds for $u \in L^\infty([0,1], L^q(\mathbb{R}^d))$. ■

The wellposedness of regular Lagrangian flows stems from the wellposedness of (non negative) solutions in $L^\infty([0,1]\times\mathbb{R}^d)$ of the continuity equation (CE). To do so, as a connection between the Eulerian and the Lagrangian theory, we have the following result due to Ambrosio in [2] that acts as a probabilistic theory of characteristic lines. We recall that we denote with $\mathscr{M}^+(\mathbb{R}^d)$ the set of non-negative, locally finite measure.

Theorem 2.1.10 (Ambrosio's superposition principle)**.** *Consider a bounded, Borel* $\boldsymbol{b}\colon [0,1]\times\mathbb{R}^d \to \mathbb{R}^d$ *and* $[0,1] \ni t \mapsto \mu_t \in \mathscr{M}^+(\mathbb{R}^d)$ *a distributional solution of (CE) starting from* $\bar{\mu} \in \mathscr{M}^+(\mathbb{R}^d)$ *in the sense of Definition 2.1.5. Then there exists a (weakly Borel) family* $\{\boldsymbol{\eta}_x\}_{x\in\mathbb{R}^d}$, *with* $\boldsymbol{\eta}_x \in \mathscr{P}(C([0,1],\mathbb{R}^d))$ *such that* $\mathrm{e}_{0*}\boldsymbol{\eta}_x = \delta_x$ *and for every* $t \in [0,1]$

$$\mu_t = \int_{\mathbb{R}^d} (\mathrm{e}_{t*}\boldsymbol{\eta}_x)\mathrm{d}\bar{\mu}(x). \tag{2.9}$$

Moreover, for $\bar{\mu}$*-a.e.* x, $\boldsymbol{\eta}_x$ *is concentrated on absolutely continuous curve solving the ODE associated to* $\boldsymbol{b}$, *namely for* $\bar{\mu}$*-a.e.* x

$$\int |\gamma(t) - x - \int_0^t \boldsymbol{b}(s,\gamma(s))\,\mathrm{d}s|\,\mathrm{d}\boldsymbol{\eta}_x(\gamma) = 0, \quad \textit{for every } t \in [0,T].$$

The integral in (2.9) has to be intended in the sense of Appendix A.1 because since the family $\{\boldsymbol{\eta}_x\}_x$ is weakly Borel and, given t, e_t is continuous, then also $\{\mathrm{e}_{t*}\boldsymbol{\eta}_x\}_x$ is also weakly Borel. The following theorem transfers wellposedness of the continuity equation into wellposedness of regular Lagrangian flows.

Theorem 2.1.11 ([42, Proposition 6.4.3]). *Consider a vector field $\boldsymbol{b} \in L^\infty([0,1]\times\mathbb{R}^d)$. Assume that, given $\bar\mu = \bar\rho\mathscr{L}$ with $\bar\rho \in L^\infty(\mathbb{R}^d)$, the solution to the (CE) in the class $L^\infty([0,1]\times\mathbb{R}^d)$ is unique. Consider a (weakly Borel) family $\{\boldsymbol{\eta}_x\}_{x\in\mathbb{R}^d}$ with $\boldsymbol{\eta}_x \in \mathscr{P}(C([0,1],\mathbb{R}^d))$ concentrated on absolutely continuous solutions of the ODE starting from x, for $\mathscr{L}^d$-a.e. x. Consider $\boldsymbol{\eta} := \int \boldsymbol{\eta}_x \,\mathrm{d}\bar\mu(x)$ and assume $\mathrm{e}_{t*}\boldsymbol{\eta} = \rho(t,\cdot)\mathscr{L}^d$ for $\rho \in L^\infty([0,1]\times\mathbb{R}^d)$. Then $\boldsymbol{\eta}_x$ is a Dirac mass for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$.*

Let us outline the strategy for the proof of the theorem. Assuming that the conclusion of the theorem is false, we can find $C \in \mathscr{B}(\mathbb{R}^d)$ with $\bar\mu(C) > 0$, $\bar t \in [0,1]$, $E, E' \in \mathscr{B}(\mathbb{R}^d)$ with $E \cap E' = \emptyset$ such that for some $M > 0$ for every $x \in C$

$$0 < \boldsymbol{\eta}_x(\{\gamma : \gamma_{\bar t} \in E\}) \le M\boldsymbol{\eta}_x(\{\gamma : \gamma_{\bar t} \in E'\}).$$

Starting from this observation, we can build two different non-negative solutions of the (CE) in the class $L^\infty([0,1]\times\mathbb{R}^d)$ as follows. We define

$$\boldsymbol{\eta}^1 := \int_C \boldsymbol{\eta}_x|_{\{\gamma:\gamma_{\bar t}\in E\}} \,\mathrm{d}\bar\mu(x), \quad \boldsymbol{\eta}^2 := \int_C \boldsymbol{\eta}_x|_{\{\gamma:\gamma_{\bar t}\in E'\}} \,\mathrm{d}\bar\mu(x).$$

Then we define the curve $t \mapsto \mu_t^i := \mathrm{e}_{t*}\boldsymbol{\eta}^i$ for $i = 1,2$, that are both distributional solutions to (CE). The crucial fact is that by the fact that E and E' are disjoint we have $\mu^1_{\bar t} \perp \mu^2_{\bar t}$ and $\mu^1_0 \le M\mu^2_0$. The conclusion follows by 'renormalizing' the measure $\boldsymbol{\eta}^2$ in order to have, without renaming the measures, $\mu^1_0 = \mu^2_0$ and still preserving orthogonality at time $\bar t$. As a consequence of this general theorem, we have the following corollary.

Proposition 2.1.12 ([42, Proposition 6.4.1]). *Consider a vector field $\boldsymbol{b} \in L^\infty([0,1]\times\mathbb{R}^d)$. Assume that, given $\bar\mu = \bar\rho\mathscr{L}^d$ with $\bar\rho \in L^\infty(\mathbb{R}^d)$, the solution to the (CE) in the class $L^\infty([0,1]\times\mathbb{R}^d)$ is unique. Then the regular Lagrangian flow associated to $\boldsymbol{b}$, if it exists, is unique. Assume furthermore that the (CE) with initial datum $\bar\mu = \mathscr{L}^d$ has a non-negative solution in $L^\infty([0,1]\times\mathbb{R}^d)$. Then we have existence of a regular Lagrangian flow associated to $\boldsymbol{b}$.*

Coupling the last general result with Theorem 2.1.8, we obtain the following.

Theorem 2.1.13 (Existence and uniqueness of regular Lagrangian flows for $\boldsymbol{b} \in W^{1,p}$). *Consider a vector field $\boldsymbol{b} \in L^\infty([0,1]\times\mathbb{R}^d)$. Assume $\boldsymbol{b} \in L^1([0,1], W^{1,p}(\mathbb{R}^d))$, $u_0 \in L^\infty(\mathbb{R}^d)$ with $p \ge 1$ and that $\mathrm{div}\boldsymbol{b} \in L^1([0,1], L^\infty(\mathbb{R}^d))$. Then the regular Lagrangian flow F associated to $\boldsymbol{b}$ exists and is unique. Uniqueness is meant in the following sense: consider two regular Lagrangian flows F^1 and F^2 associated to $\boldsymbol{b}$. Then, for $\mathscr{L}^d$-a.e. x, $F^1_\cdot(x) = F^2_\cdot(x)$ in $C([0,1],\mathbb{R}^d)$.*

Remark 2.1.14. It is convenient to define the flow starting from time t up to time s. We can define $F\colon [0,1]^2 \times \mathbb{R}^d \to \mathbb{R}^d$ as follows. We define $F_t^\cdot\colon [t,1]\times\mathbb{R}^d \to \mathbb{R}^d$ as

$$F_t^s = \bar F_{s-t},$$

where $\bar F$ is the regular Lagrangian flow in the time interval $[0,1-t]$ associated to the vector field $\bar{\boldsymbol{b}}_s := \boldsymbol{b}_{s+t}$ and we point out $\bar{\boldsymbol{b}}$ satisfies the hyphothesis of Theorem 2.1.13 yielding existence and uniqueness of the flow. Similarly, we define $F_t^\cdot\colon [0,t]\times\mathbb{R}^d \to \mathbb{R}^d$ as

$$F_t^s = \bar F_{t-s},$$

where $\bar F$ is the regular Lagrangian flow in the time interval $[0,t]$ associated to the vector field $\bar{\boldsymbol{b}}_s := -\boldsymbol{b}_{t-s}$. Similarly, Theorem 2.1.13 applies, since $\bar{\boldsymbol{b}}$ satisfies all the hyphothesis. ■

Therefore, existence and uniqueness of regular Lagrangian flows in Theorem 2.1.13 can be proved according to the following axiomatization.

Definition 2.1.15 (Regular Lagrangian flow, axiomatization 2)**.** *Let $\boldsymbol{b}\colon [0,1]\times\mathbb{R}^d \to \mathbb{R}^d$ be Borel and in $L^1([0,1],L^2(\mathbb{R}^d))$. Then we say that $F\colon [0,1]^2\times\mathbb{R}^d \to \mathbb{R}^d$ is a regular Lagrangian flow associated to $\boldsymbol{b}$ provided it is Borel and*

i) for every $x\in\mathbb{R}^d$ and $t\in[0,1]$, the curve $F_t^{\cdot}(x)$ is continuous and satisfies $F_t^t(x)=x$;

ii) there exists $C>0$ such that for $F_{t\,*}^{s}\mathscr{L}^d \le C\mathscr{L}^d$ for every $t,s\in[0,1]$;

iii) it solves the ODE: for every t, for a.e. x, the curve $s\mapsto F_t^s(x)$ is absolutely continuous and

$$\partial_s F_t^s(x) = \boldsymbol{b}_s(F_t^s(x)) \quad \textit{for a.e. } s\in[0,1].$$

The uniqueness statement is meant in this case by saing that, given two regular Lagrangian flows F and $\tilde F$ associated to $\boldsymbol{b}$ satisfying the hyphothesis of Theorem 2.2.8, we have that for every $t\in[0,1]$

$$\text{for } \mathscr{L}^d\text{-a.e. point } x\in \mathrm{X} \text{ it holds } \tilde F_t^s(x) = F_t^s(x) \text{ for every } s\in[0,1].$$

We now turn to the problem of regularity with respect to the initial condition of regular Lagrangian flows. As remarked before, in the Lipschitz setting, (2.3) grants that the maps F_t are Lipschitz. In the case of Sobolev vector fields, this is not the case; however, we have that F_t satisfies a Lusin Lipschitz property uniform in time, namely we can find a partition in Borel sets $(E_i)_i$ which covers a.a. $\mathbb{R}^d$ such that $F_t|_{E_i}$ is Lipschitz for every t. The first result in this direction is in [12], using the continuity equation in order to transfer the information at a Lagrangian level. In [43], the authors provided a proof of Lusin Lipschitz regularity of flows by introducing a functional directly at the Lagrangian level, measuring the incremental ratio of flow maps F_t; the same technique grants stability estimates and wellposedness results of flows. We present here a short proof of Lusin–Lipschitz regularity, presented in [26, Proposition 2.9].

Proposition 2.1.16. *Consider $\boldsymbol{b}\in L^1([0,1],W^{1,p}(\mathbb{R}^d))$ for some $p>1$, and F a regular Lagrangian flow associated to $\boldsymbol{b}$ with compressibility constant L. Then for every $t\in[0,1]$, there exists a measurable function $g_t(x)\colon\mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$ such that $\|g_t\|_{L^p(\mathbb{R}^d)} \le C(p,d)L\int_0^t \|\nabla\boldsymbol{b}_s\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}s$ for every $t\in[0,1]$ and*

$$e^{-g_t(x)-g_t(y)} \le \frac{|F_t^s(x)-F_t^s(y)|}{|x-y|} \le e^{g_t(x)+g_t(y)} \quad \textit{for every } x,y\in\mathbb{R}^d,\ s\in[0,1].$$

Proof. We consider N the null set in item iii) of Definition 2.1.15, $\varepsilon>0$, $x,y\in\mathbb{R}^d\setminus N$ and $t\in[0,1]$. We have that $s\mapsto\log(\varepsilon+|F_t^s(x)-F_t^s(y)|)$ is absolutely continuous, hence a.e. differentiable and

$$\begin{aligned}\left|\log\left(\frac{\varepsilon+|F_t^s(x)-F_t^s(y)|}{\varepsilon+|x-y|}\right)\right| &= \left|\int_t^s \frac{\mathrm{d}}{\mathrm{d}r}\log\left(\varepsilon+|F_t^r(x)-F_t^r(y)|\right)\mathrm{d}r\right| \\ &\le \int_t^s \frac{|\boldsymbol{b}_r(F_t^r(x))-\boldsymbol{b}_r(F_t^r(y))|}{|F_t^r(x)-F_t^r(y)|}\,\mathrm{d}r.\end{aligned}$$

Using the estimate for maximal functions in (1.19) (recalling $M_R f \le Mf$ for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$), and taking the limit as $\varepsilon \to 0$, we get

$$\left|\log\left(\frac{|F_t^s(x)-F_t^s(y)|}{|x-y|}\right)\right| \le C_d \int_t^s (M|\nabla \boldsymbol{b}_r|)(F_t^r(x))\,\mathrm{d}r + C_d \int_t^s (M|\nabla \boldsymbol{b}_r|)(F_t^r(y))\,\mathrm{d}r.$$

We set $g_t(x) := C_d \int_0^1 (M|\nabla \boldsymbol{b}_s|)(F_t^s(x))\,\mathrm{d}s$ for $x \in \mathbb{R}^d \setminus N$ and $+\infty$ otherwise. The fact that $\|g_t\|_{L^p(\mathbb{R}^d)} \le C(p,d) L \int_0^1 \|\nabla \boldsymbol{b}_s\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}s$ follows by applying the strong $L^p - L^p$ estimate for $p > 1$ and item ii) in Definition 2.1.4. □

The last statement provides a sort of quantitative Lusin-Lipschitz estimate. Indeed, the maps (F_t^s) are uniformly Lipschitz with respect to s on the sublevels of the function g_t.

2.2 The RCD setting: wellposedness and regularity

Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{RCD}(K,\infty)$ space. In this section, we discuss the wellposedness and the regularity theory of flows of nonsmooth vector field in such a nonsmooth setting. A vector field has to be interpreted in the sense of normed modules, as in Section 1. Since we don't have at our disposal Lipschitz vector fields in such a setting, the theory that we export from the Euclidean space is the one of flows induced by Sobolev vector fields. The philosophy pursued in [15] is the same of the one in the Euclidean setting. The arguments work in the context of theory of Dirichlet forms and Γ-calculus; however, for this presentation, we tailor and discuss it in the case of $\mathsf{RCD}(K,\infty)$ spaces. In particular, as observed in [15] and using the notation of the authors, we can use as the algebra of functions $\mathscr{A}$ the space $\mathrm{Lip}_{\mathrm{bs}}(\mathrm{X})$, which plays the role of the space $C_c^\infty(\mathbb{R}^d)$ in such a setting.

Definition 2.2.1 (Time dependent vector fields)**.** *Consider $\boldsymbol{b}\colon [0,1] \to L^2(T\mathrm{X})$ is a time dependent vector field if, for every $f \in W^{1,2}(\mathrm{X})$, the map $(t,x) \mapsto \boldsymbol{b}_t \cdot \nabla f(x)$ is measurable with respect to the product sigma-algebra $\mathscr{L}^1 \otimes \mathscr{B}(\mathrm{X})$.*

Definition 2.2.2 (Distributional formulation of (CE))**.** *Given $\boldsymbol{b}$ a time dependent vector field with $|\boldsymbol{b}| \in L^1_{\mathrm{loc}}([0,1]\times\mathrm{X})$. Consider $\bar{u} \in L^\infty_{\mathrm{loc}}(\mathfrak{m})$. Then we say that $u \in L^\infty_{\mathrm{loc}}([0,1]\times\mathrm{X})$ is a distributional solution to the (CE) if for every $\varphi \in C_c^1([0,T))$ and $f \in \mathrm{Lip}_{bs}(\mathrm{X})$*

$$\int_0^T \int \left[\varphi'(t) u_t f + \varphi(t)(u_t \boldsymbol{b}_t \cdot \nabla f)\right]\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t = \varphi(0)\int \bar{u} f\,\mathrm{d}\mathfrak{m}. \tag{2.10}$$

Remark 2.2.3. Given u that solves (CE) in the sense of Definition 2.2.2, we have that, for every $f \in \mathrm{Lip}_{\mathrm{bs}}(\mathrm{X})$ $(t \mapsto \int u_t f\,\mathrm{d}\mathfrak{m}) \in W^{1,1}(0,1)$, so it admits a representative in $C([0,1])$, being such that

$$\lim_{t\to 0}\int u_t f\,\mathrm{d}\mathfrak{m} = \int \bar{u} f\,\mathrm{d}\mathfrak{m}.$$

■

For the meaning of the class $L^p([0,1],\mathbb{B})$ with $\mathbb{B}$ being a Banach space, we refer to Appendix A.2; moreover, given $\boldsymbol{b} \in L^1([0,1], L^2(T\mathrm{X}))$, we have that it is a time dependent vector field in the sense given by Definition 2.2.1.

Theorem 2.2.4 (Existence of solution to (CE), [16, Thm. 6.1])**.** *Consider $\bar{u} \in L^\infty(\mathfrak{m})$ and a time dependent vector field $\boldsymbol{b}$ with bounded support such that $\boldsymbol{b} \in L^1([0,1], L^2(T\mathrm{X}))$ with $\mathrm{div}\boldsymbol{b} \in L^1([0,1], L^\infty(\mathfrak{m}))$. Then there exists a solution $u \in L^\infty([0,1] \times \mathrm{X})$ to (CE), namely* (2.10)*. Moreover, it is possible to build $u \geq 0$ in the case in which $\bar{u} \geq 0$.*

Let us sketch the proof of the theorem and let us highlight the main differences with the Euclidean case. While in the Euclidean setting, we can mollify the vector field via convolution and take into account the theory of characteristic lines in order to derive uniform estimates on the approximations, here this is no more possible. Therefore, the strategy is to introduce a regularizing term in the equation, namely we solve 'distributionally'

$$\partial_t u_t^\sigma + \mathrm{div}(\boldsymbol{b}_t u_t^\sigma) = \sigma \Delta u_t^\sigma, \quad u_0^\sigma = \bar{u}, \tag{2.11}$$

by using a variant of Lax-Milgram's lemma due to Lions, which is Lemma 3.3.1 in this manuscript (indeed the proof of Theorem 3.3.3 follows the same argument). Then we get uniform bounds (where uniform means in the parameter σ of the approximation) of the norms $\|u^\sigma\|_{L^\infty([0,1]\times \mathrm{X})}$ which allow to pass to the limit in the distributional formulation and to obtain a solution of (2.10). As discussed before, the uniqueness of solutions to (CE) follows by assumptions at first order of differentiability of $\boldsymbol{b}$.

Theorem 2.2.5. *Consider a time dependent vector field $\boldsymbol{b}$ such that $|\boldsymbol{b}| \in L^\infty([0,1] \times \mathrm{X})$ with bounded support. Moreover, assume that $\boldsymbol{b} \in L^1([0,1], W^{1,2}_{C,s}(T\mathrm{X}))$ and $\mathrm{div}\boldsymbol{b} \in L^1([0,1], L^\infty(\mathfrak{m}))$. Then, given $\bar{u} \in L^\infty(\mathfrak{m})$, there exists a unique solution to (CE) in $u \in L^\infty([0,1] \times \mathrm{X})$.*

Let us be informal about the proof of the result, the aim of the following discussion being clearifying why we need some regularity on $\boldsymbol{b}$. The philosophy also here is to show that, under the standing assumptions on $\boldsymbol{b}$, all the solutions in $L^\infty([0,1] \times \mathrm{X})$ are renormalized, which means that, 'distributionally', solve

$$\partial_t \beta(u_t) + \boldsymbol{b}_t \cdot \nabla \beta(u_t) = -u_t \beta'(u_t) \mathrm{div}\boldsymbol{b}_t \tag{2.12}$$

for every $\beta \in C^1(\mathbb{R})$. To do so, the strategy is to regularize the solution u by means of the heat flow $(h_t)_{t\geq 0}$ (again the difference with the Euclidean setting is that the convolution is not available); therefore, the curve $t \mapsto h_\alpha u_t =: u_t^\alpha$ solves 'distributionally' the following equation:

$$\partial_t u_t^\alpha + \mathrm{div}(\boldsymbol{b}_t u_t^\alpha) = \mathscr{C}^\alpha(\boldsymbol{b}_t, u_t).$$

where for every $f \in \mathrm{Lip_{bs}}(\mathrm{X})$ $\langle \mathscr{C}^\alpha(\boldsymbol{c}, v), f\rangle := \int \mathrm{div}(\boldsymbol{c} h_\alpha v) f - \mathrm{div}(\boldsymbol{c} v) h_\alpha f \, \mathrm{d}\mathfrak{m}$. Then we need to pass the limit as $\alpha \to 0$ in the following equation for $\beta \in C^1(\mathbb{R})$:

$$\partial_t \beta(u_t^\alpha) + \boldsymbol{b}_t \cdot \nabla(\beta(u_t^\alpha)) = -u_t^\alpha \beta'(u_t^\alpha) \mathrm{div}\boldsymbol{b}_t + \beta'(u_t^\alpha) \mathscr{C}^\alpha(\boldsymbol{b}_t, u_t).$$

The commutator $\mathscr{C}^\alpha(\boldsymbol{c}, v)$ can be manipulated and expressed as follows:

$$\begin{aligned} \langle \mathscr{C}^\alpha(\boldsymbol{c}, v), f\rangle = \int_0^\alpha \int & 2\langle D^{\mathrm{sym}}\boldsymbol{c} : \nabla h_s v \otimes \nabla h_{\alpha-s} f\rangle \\ & + \mathrm{div}\boldsymbol{c}\,(-h_s v\, \Delta h_{\alpha-s} f + \langle \nabla h_s v, \nabla h_{\alpha-s} f\rangle)\, \mathrm{d}\mathfrak{m}\, \mathrm{d}\alpha. \end{aligned}$$

Therefore, the standing assumptions on $\boldsymbol{b}$ and the L^∞-Lip regularization in (1.21) (which holds in $\mathsf{RCD}(K,\infty)$ space) used on the first term yields that the commutator converges to zero, obtaining a solution to (2.12). We introduce the concept of regular Lagrangian flows in such a nonsmooth setting.

Definition 2.2.6. *Consider $\boldsymbol{b} \in L^1([0,1], L^2(T\mathrm{X}))$. We say that $F\colon [0,1] \times \mathrm{X} \to \mathrm{X}$ is a regular Lagrangian flow associated to $\boldsymbol{b}$ provided it is Borel and the following properties are verified:*

i) For some $C > 0$ we have

$$(F_t)_*\mathfrak{m} \leq C\mathfrak{m} \qquad \forall t \in [0,1].$$

ii) *For every $x \in \mathrm{X}$, $t \in [0,1]$ the curve $[0,1] \ni t \mapsto F_t(x) \in \mathrm{X}$ is continuous and satisfies $F_0(x) = x$.*

iii) *Given $f \in W^{1,2}(\mathrm{X})$ and $t \in [0,1]$, one has that for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ the map $[0,1] \ni t \mapsto (f \circ F_t)(x) \in \mathbb{R}$ belongs to $W^{1,1}(0,1)$ and satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ F_t)(x) = \mathrm{d}f(\boldsymbol{b}_t)\big(F_t(x)\big) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,1].$$

Notice that X in general is just a metric space, without any vector space structure, so we give the definition of solving the ODE in item iii) in a weak way by postcomposition with Sobolev functions. We turn the Lagrangian part of the problem. The authors in [15] developed therein a superposition principle, which allows to turn from the Eulerian formulation into the Lagrangian formulation, obtaining wellposedness at the Lagrangian level.

Theorem 2.2.7 (Superposition principle)**.** *Consider a time dependent vector field $\boldsymbol{b}$ such that $|\boldsymbol{b}| \in L^\infty([0,1] \times \mathrm{X})$ and $t \mapsto u_t$ is a nonnegative solution to (CE) (in the sense of Definition 2.2.2), starting from $\bar{u}$ with $\|u_t\|_{L^1(\mathfrak{m})} = 1$ for every t. Then there exists a (weakly Borel) family $\{\boldsymbol{\eta}_x\}_{x\in\mathrm{X}}$, with $\boldsymbol{\eta}_x \in \mathscr{P}(C([0,1],\mathrm{X}))$ such that $\mathrm{e}_{0*}\boldsymbol{\eta}_x = \delta_x$ and for every $t \in [0,1]$*

$$\mu_t = \int_{\mathbb{R}^d} (\mathrm{e}_{t*}\boldsymbol{\eta}_x)\mathrm{d}\mu(x).$$

Moreover, for $\bar{u}\mathfrak{m}$-a.e. x, $\boldsymbol{\eta}_x$ is concentrated on absolutely continuous curves solving the ODE associated to $\boldsymbol{b}$, namely for every $f \in \mathrm{Test}(\mathrm{X})$ for $\bar{u}\mathfrak{m}$-a.e. x

$$\int \left| f(\gamma(t)) - f(x) - \int_0^t \mathrm{d}f(\boldsymbol{b}_s)(\gamma(s))\,\mathrm{d}s \right| \mathrm{d}\boldsymbol{\eta}_x(\gamma) = 0, \quad \text{for every } t \in [0,T].$$

With a similar argument as in the Euclidean setting, the following theorem about wellposedness of Lagrangian flows in the setting of RCD spaces follows.

Theorem 2.2.8. *Consider a time dependent vector field $\boldsymbol{b}$ with bounded support such that $|\boldsymbol{b}| \in L^\infty([0,1] \times \mathrm{X})$. Moreover, assume that $\boldsymbol{b} \in L^1([0,1], W^{1,2}_{C,s}(T\mathrm{X}))$ and $\mathrm{div}\boldsymbol{b} \in L^1([0,1], L^\infty(\mathfrak{m}))$. Then there exists a unique regular Lagrangian flow (F_t) of $\boldsymbol{b}$. Uniqueness has to be intended as: if $(\tilde{F}_t)$ is another Regular Lagrangian Flow, then for every $t \in [0,1]$ we have that for $\mathfrak{m}$-a.e. point $x \in \mathrm{X}$ it holds $\tilde{F}_t(x) = F_t(x)$ for every $t \in [0,1]$.*

Notice that, as done in the Euclidean case, we can define the flow F_s^t starting from $s \in [0,1]$ up to time $t \in [0,1]$. The vector field $\boldsymbol{b}$, suitably translated still satisfies the hyphothesis of wellposedness of regular Lagrangian flows, therefore the same arguments of Remark 2.1.14 can be repeated verbatim in this case. Therefore, existence and uniqueness in Theorem 2.2.8 may be interpreted with the following axiomatization of regular Lagrangian flow.

Definition 2.2.9 (Regular Lagrangian Flow, axiomatization 2)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K,\infty)$ *space, for some* $K \in \mathbb{R}$ *and* $(\boldsymbol{b}_t) \in L^1([0,1], L^2(T\mathrm{X}))$*. Then* $F : [0,1]^2 \times \mathrm{X} \to \mathrm{X}$ *is said to be a* Regular Lagrangian Flow *for* w *provided it is Borel and the following properties are verified:*

i) For some $C > 0$ *we have*

$$(F_t^s)_*\mathfrak{m} \le C\mathfrak{m} \qquad \forall t, s \in [0,1]. \tag{2.13}$$

ii) *For every* $x \in \mathrm{X}$, $t \in [0,1]$ *the curve* $[0,1] \ni s \mapsto F_t^s(x) \in \mathrm{X}$ *is continuous and satisfies* $F_t^t(x) = x$.

iii) *Given* $f \in W^{1,2}(\mathrm{X})$ *and* $t \in [0,1]$, *one has that for* $\mathfrak{m}$*-a.e.* $x \in \mathrm{X}$ *the map* $[0,1] \ni s \mapsto (f \circ F_t^s)(x) \in \mathbb{R}$ *belongs to* $W^{1,1}(0,1)$ *and satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}s}(f \circ F_t^s)(x) = \mathrm{d}f(\boldsymbol{b}_s)\big(F_t^s(x)\big) \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in [0,1]. \tag{2.14}$$

We shall typically write (F_t^s) in place of F for Regular Lagrangian Flows.
The uniqueness statement is meant in this case by saing that, given two regular Lagrangian flows F and $\tilde{F}$ associated to $\boldsymbol{b}$ satisfying the hyphothesis of Theorem 2.2.8, we have that for every $t \in [0,1]$

$$\text{for } \mathfrak{m}\text{-a.e. point } x \in \mathrm{X} \text{ it holds } \tilde{F}_t^s(x) = F_t^s(x) \text{ for every } s \in [0,1].$$

The regularity theory in $\mathsf{RCD}(K,N)$ setting was studied by Bruè and Semola in [28] in the case of compact N-Ahlfors regular metric measure spaces and then extended to the setting of $\mathsf{RCD}(K,N)$ spaces in order to prove that the dimension of an $\mathsf{RCD}(K,N)$ space is constant in a suitable sense in [27]. The last part of the discussion concerning the case of non-collapsed $\mathsf{RCD}(K,N)$ comes from [25]. We point out that, as it will be clearer later, the estimates in the collapsed setting are fundamental in order to obtain uniqueness of parallel transport and the validity of a Leibniz formula.
The main idea in [27] is to try to reproduce the scheme of Crippa-DeLellis in [43] of differentiating the distance of two different flow lines $\mathsf{d}(F_t^s(x), F_t^s(y))$ with respect to s.
In the setting of metric measure spaces under integral assumptions on the vector fields, it is not possible using the following computation to quantify the rate of change of $\mathsf{d}(F_t^s(x), F_t^s(y))$ in time. Let us be more specific in the case of a Riemannian manifold (M, g) and for simplicity we assume that $\boldsymbol{b} \in L^\infty([0,1] \times M)$. It can be readily checked that this assumptions grants in particular that the map $s \mapsto (F_t^s(x), F_t^s(y)) \in M \times M$ is Lipschitz for every $x, y \in M \setminus N$, where N is Vol_g–negiglible. Therefore, the map $s \mapsto \mathsf{d}_g^2(F_t^s(x), F_t^s(y))$ is Lipschitz, hence $\mathscr{L}^1$-a.e. differentiable, with derivative given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathsf{d}_g^2(F_t^s(x), F_t^s(y)) = (\nabla \mathsf{d}^2_{g_{F_t^s(x)}} \cdot \boldsymbol{b}_s)(F_t^s(y)) + (\nabla \mathsf{d}^2_{g_{F_t^s(y)}} \cdot \boldsymbol{b}_s)(F_t^s(x)) \quad \text{for a.e. } s.$$

We recall the following fact: if the function d_{g_x} is differentiable at $y \in M$, then there exists a unique geodesic $\gamma \colon [0,1] \to M$ such that $\gamma_0 = y$ and $\gamma_1 = x$ and $\gamma_0' = -\nabla(\mathsf{d}_{g_x}^2/2)(y)$. We interpolate the last term, by using the compatibility of the Riemannian metric with the Levi Civita connection. Indeed, consider a smooth curve $\gamma \colon [0,1] \to M$ and a vector field V; we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle \dot{\gamma}_s, V\rangle_{\gamma_s} = \langle \nabla_{\dot{\gamma}_s}\dot{\gamma}_s, V\rangle_{\gamma_s} + \langle \nabla_{\dot{\gamma}_s} V, \dot{\gamma}_s\rangle_{\gamma_s} \quad \text{for every } s \in [0,1].$$

Integrating the last formula in the case in which $V = \boldsymbol{b}_t$ and γ is a geodesic such that $\gamma_0 = F_t^s(x)$ and $\gamma_1 = F_t^s(y)$ we have

$$\begin{aligned}(\nabla(\mathsf{d}^2_{g\,F_t^s(x)}/2)\cdot \boldsymbol{b}_s)(F_t^s(y)) + (\nabla(\mathsf{d}^2_{g\,F_t^s(y)}/2)\cdot \boldsymbol{b}_s)(F_t^s(x)) &= \langle \dot\gamma_1, \boldsymbol{b}_s\rangle_{\gamma_1} - \langle \dot\gamma_0, \boldsymbol{b}_s\rangle_{\gamma_0} \\ &= \int_0^1 \nabla \boldsymbol{b}_s(\dot\gamma_r, \dot\gamma_r)_{\gamma_r}\,\mathrm{d}r = \int_0^1 \nabla_{\mathrm{sym}}\boldsymbol{b}_s(\dot\gamma_r,\dot\gamma_r)_{\gamma_r}\,\mathrm{d}r\end{aligned}$$

where $\nabla_{\mathrm{sym}}\boldsymbol{b}_t := \frac{\nabla \boldsymbol{b}_t + (\nabla \boldsymbol{b}_t)^T}{2}$. The problem here is that it is not known whether it is possible to find a Borel function H for which we can estimate the last term as follows

$$\int_0^1 \nabla_{\mathrm{sym}}\boldsymbol{b}_s(\dot\gamma_r,\dot\gamma_r)_{\gamma_r}\,\mathrm{d}r \le \big(H(F_0^t(x)) + H(F_0^t(y))\big)\, \mathsf{d}_g^2(F_0^t(x), F_0^t(y))$$

thus providing by means of Gronwall lemma a quantitative Lusin-Lipschitz estimate as in Proposition 2.1.16.

Coming back to the RCD setting, to solve the problem, we deal with another quasi-metric d_G, comparable to d and whose derivative involves the tensors above in integrated form with respect to $\mathfrak{m}$. The quasi-metric is defined by noticing that in $\mathbb{R}^d$ (with $d \ge 3$) the Green function of the Laplacian has the form $G(x,y) = c_d\,\mathsf{d}(x,y)^{2-d}$. Taking this into account, we can define the quasi-metric $\mathsf{d}_G(x,y) := \frac{1}{G(x,y)}$ whenever $x \ne y$ and 0 otherwise. Therefore, since we want to compute the rate of change of d_G along the flow lines, it is enough to do it before for the function G. When differentiating the map $s \mapsto G(F_t^s(x), F_t^s(y))$, we obtain the quantity $\boldsymbol{b}_t \cdot \nabla G_x(y) + \boldsymbol{b}_t \cdot \nabla G_y(x)$, that can be estimated formally (or distributionally in the Euclidean setting) using that, for every $x \in \mathrm{X}$, $-\Delta G_x = \delta_x$, having:

$$\begin{aligned}\boldsymbol{b}_t\cdot\nabla G_x(y) + \boldsymbol{b}_t\cdot\nabla G_y(x) &= \int \boldsymbol{b}_t\cdot\nabla G_x\,\mathrm{d}\delta_y(z) + \int \boldsymbol{b}_t\cdot\nabla G_y\,\mathrm{d}\delta_x(z) \\ &= -\int \boldsymbol{b}_t\cdot\nabla G_x\,\mathrm{d}\Delta G_y(z) - \int \boldsymbol{b}_t\cdot\nabla G_y\,\mathrm{d}\Delta G_x(z) \\ &= 2\int \nabla_{\mathrm{sym}}\boldsymbol{b}_t(\nabla G_x, \nabla G_y)\,\mathrm{d}\mathfrak{m} - \int \mathrm{div}\boldsymbol{b}_t\langle \nabla G_x, \nabla G_y\rangle\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Provided the bound on $\nabla_{\mathrm{sym}}\boldsymbol{b}$ is of integral type with respect to $\mathfrak{m}$, the last estimate allows to obtain a bound on $\boldsymbol{b}_t\cdot\nabla G_x(y) + \boldsymbol{b}_t\cdot\nabla G_y(x)$. The last result is used together with the the following formula: we have that there exists $C > 0$ such that, given a Borel function $f\colon \mathbb{R}^d \to \mathbb{R}$,

$$\int f|\nabla G_x||\nabla G_y|\,\mathrm{d}\mathfrak{m} \le CG(x,y)\,(Mf(x) + Mf(y))$$

where M is the Hardy-Littlewood maximal function. The last formula will be applied, for fixed $t \in [0,1]$, to $f = |\nabla_{\mathrm{sym}}\boldsymbol{b}_t|$.

We now underline which hyphothesis are needed on the base space X in order to perform the above argument. We consider for a moment a complete Riemannian manifold (M,g) with $\mathrm{Ric}_g \ge 0$. Due to a characterization by Varopoulos (we refer for instance to [66, Corollary 9.9]), we have that there exists a positive Green function of the Laplacian, namely a smooth function $G\colon M \times M \setminus \{(x,y) \in M \times M : x = y\} \to (0,+\infty)$ (we denote $G(x,y) = G_x(y)$) such that $-\Delta G_x = \delta_x$, if and only if for some (and thus all) $x \in M$

$$\int_1^\infty \frac{s}{\mathrm{Vol}_g(B_s(x))}\,\mathrm{d}s < \infty.$$

In this case, we say that (M, g) is non–parabolic. The idea is to work in the non smooth setting under assumptions on the base space which ensures existence and finiteness properties of a positive Green function.

Definition 2.2.10. *We say that a metric measure space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is non–parabolic if for some (and thus all)* $x \in M$

$$\int_1^\infty \frac{s}{\mathfrak{m}(B_s(x))} \,\mathrm{d}s < \infty.$$

We point out that the arguments here cover the case of negative K. The idea is to work in the setting of $\mathsf{RCD}(K, N)$ spaces satisfying the following assumption, which implies non parabolicity of the metric measure space. Then the final estimate (2.17) can be derived for a general $\mathsf{RCD}(K, N)$ space using the calculus tools on tensor products of metric measure spaces, as it will be mentioned in Remark 2.2.18.

Assumption 1. *We assume that* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is the tensor product between an* $\mathsf{RCD}(K, N-3)$ *space for some* $K \in \mathbb{R}$ *and* $4 < N < +\infty$ *and the Euclidean factor* $(\mathbb{R}^3, \mathsf{d}_e, \mathscr{L}^3)$.

For the case of $\mathsf{RCD}(0, N)$ spaces satisfying Assumption 1, the authors in [27] used the Green function of the Laplacian, defined as

$$G(x, y) := \int_0^\infty p_t(x, y) \,\mathrm{d}t.$$

Since in this treatment, we want to directly review the case for general K, we point out that they had to slightly change the definition, the reason being to compensate the term in the upperbound of (1.22) of the form e^{ct}. Therefore, this leads to the definition

$$\bar{G}(x, y) := \int_0^\infty e^{-ct} p_t(x, y) \,\mathrm{d}t, \qquad \bar{G}^\epsilon(x, y) := \int_\epsilon^\infty e^{-ct} p_t(x, y) \,\mathrm{d}t \quad \text{for } \epsilon > 0.$$

To stress the fact that one variable is freezed, we denote $\bar{G}^\epsilon(x, y) = \bar{G}^\epsilon_x(y) = \bar{G}^\epsilon_y(x)$ for $\epsilon > 0$ and $\bar{G}(x, y) = \bar{G}_x(y) = \bar{G}_y(x)$. Assumption 1 grants finiteness at every point, with the exception of x, of the function G_x and of $\bar{G}_x$. As mentioned before, we introduce the Green quasi-metric $\mathsf{d}_{\bar{G}}$

$$\mathsf{d}_{\bar{G}}(x, y) := \begin{cases} \frac{1}{\bar{G}(x,y)} & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

We introduce the function $F\colon \mathrm{X} \times (0, +\infty) \to (0, +\infty)$ by $F(x, r) := \int_r^\infty \frac{s}{\mathfrak{m}(B_s(x))} \,\mathrm{d}s$, useful in the following computations. The function $\bar{G}$ formally solves the PDE $\Delta \bar{G}_x = -\delta_x + c\bar{G}_x$. An instance of this fact can be seen in the following proposition (from [27, Lemma 2.5]).

Proposition 2.2.11. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *space satisfying the Assumption 1. Then the function* $\bar{G}^\epsilon_x$ *defined as above belongs to* $\mathrm{Lip}_b(\mathrm{X}) \cap D_{\mathrm{loc}}(\Delta)$ *and*

$$\Delta \bar{G}^\epsilon_x = -e^{-c\epsilon} p_\epsilon(x, y) + c\bar{G}^\epsilon_x \quad \mathfrak{m} - a.e.$$

Moreover, $\bar{G}_x \in W^{1,2}_{\mathrm{loc}}(\mathrm{X})$ *and* $\bar{G}^\epsilon_x \to \bar{G}_x$ *in* $W^{1,2}_{\mathrm{loc}}(\mathrm{X})$.

By using an approximation procedure based on the funciton $\bar{G}_x^\epsilon$, the following theorem holds.

Theorem 2.2.12 ([27, Proposition 2.27])**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathsf{RCD}(K,N)$ *space satisfying Assumption 1 and consider* $B_R(p)\subseteq \mathrm{X}$. *Then for any bounded* $\boldsymbol{b}\in W^{1,2}_{C,s}(T\mathrm{X})$ *such that* $\mathrm{div}\boldsymbol{b}\in L^2(\mathfrak{m})$, *with support contained in* $B_R(p)$, *there exists a positive function* $h\in L^2(B_R(p),\mathfrak{m})$ *such that*

$$\boldsymbol{b}\cdot\nabla\bar{G}_x(y)+\boldsymbol{b}\cdot\nabla\bar{G}_y(x)\le \bar{G}(x,y)(h(x)+h(y))$$

for $\mathfrak{m}\times\mathfrak{m}$*-a.e.* $(x,y)\in B_R(p)\times B_R(p)$ *and* $\|h\|_{L^2(B_R(p),\mathfrak{m})}\le C_V\left(\|\nabla_{\mathrm{sym}}\boldsymbol{b}\|_{L^2(\mathfrak{m})}+\|\mathrm{div}\boldsymbol{b}\|_{L^2(\mathfrak{m})}\right)$, *where* $C_V=C_V(B_R(p))>0$.

This result allows to estimate the difference of the Green function along flow maps.

Proposition 2.2.13. *Let* $\boldsymbol{b}$ *be a time dependent vector field supported in* $B_R(p)$ *for some* $R>0$ *and* $p\in\mathrm{X}$, *such that* $|\boldsymbol{b}|\in L^\infty([0,1]\times\mathrm{X})$, $\boldsymbol{b}\in L^1([0,1],W^{1,2}_{C,s}(T\mathrm{X}))$ *and* $\mathrm{div}\,\boldsymbol{b}\in L^1([0,1],L^\infty(\mathfrak{m}))$. *Then for every* t, *for* $\mathfrak{m}\times\mathfrak{m}$*-a.e.* $(x,y)\in B_R(p)\times B_R(p)$

$$\mathsf{d}_{\bar{G}}(F_t^s(x),F_t^s(y))\le \mathsf{d}_{\bar{G}}(x,y)\,e^{\int_{t\wedge s}^{t\vee s}h_r(F_t^r(x))\,\mathrm{d}r+\int_{t\wedge s}^{t\vee s}h_r(F_t^r(y))\,\mathrm{d}r}\quad \textit{for every } s. \tag{2.15}$$

Proof. We have that, for $\mathfrak{m}\times\mathfrak{m}$-a.e. $(x,y)\in B_R(p)\times B_R(p)$ using Theorem 2.2.12 with $\boldsymbol{b}_s$ for a.e. s

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathsf{d}_{\bar{G}}(F_t^s(x),F_t^s(y))&=\frac{\mathrm{d}}{\mathrm{d}s}\bar{G}(F_t^s(x),F_t^s(y))^{-1}\\&=-\bar{G}(F_t^s(x),F_t^s(y))^{-2}\left(\boldsymbol{b}_s\cdot\nabla\bar{G}_{F_t^s(x)}(F_t^s(y))+\boldsymbol{b}_s\cdot\nabla\bar{G}_{F_t^s(y)}(F_t^s(x))\right)\\&\le\bar{G}(F_t^s(x),F_t^s(y))^{-1}(h_s(F_t^s(x))+h_s(F_t^s(y)))\\&=\mathsf{d}_{\bar{G}}(F_t^s(x),F_t^s(y))(h_s(F_t^s(x))+h_s(F_t^s(y)))\end{aligned}$$

where h_t are the functions defined in Theorem 2.2.12 for fixed t. An application of Gronwall lemma yields for $\mathfrak{m}\times\mathfrak{m}$-a.e. $(x,y)\in B_R(p)\times B_R(p)$ for every $s\in[t,1]$

$$\mathsf{d}_{\bar{G}}(F_t^s(x),F_t^s(y))\le\mathsf{d}_{\bar{G}}(x,y)\,e^{\int_t^s h_r(F_t^r(x))\,\mathrm{d}r+\int_t^s h_r(F_t^r(y))\,\mathrm{d}r}.$$

and similarly for $s\in[0,t]$, concluding the proof. □

To obtain a similar estimate for d, we need to compare the quasi-metric $\mathsf{d}_{\bar{G}}$ with d.

Proposition 2.2.14 ([27, Proposition 2.21])**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathsf{RCD}(K,N)$ *space satisfying Assumption 1. Then, given* $B_R(p)\subseteq\mathrm{X}$, *there exists a constant* $\bar{C}=\bar{C}(B_R(p))\ge 1$ *such that for any* $x,y\in B_R(p)$

$$\frac{1}{\bar{C}}F_x(\mathsf{d}(x,y))\le\mathsf{d}_{\bar{G}}(x,y)\le\bar{C}\,F_x(\mathsf{d}(x,y)).$$

Corollary 2.2.15. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathsf{RCD}(K,N)$ *space satisfying Assumption 1. Then, given* $B_R(p)\subseteq\mathrm{X}$, *there exists a constant* $C_1=C_1(B_R(p))\ge 1$ *such that for any* $x,y\in B_R(p)$

$$\frac{1}{C_1}\mathsf{d}(x,y)\le\mathsf{d}_{\bar{G}}(x,y)\le C_1\,\mathsf{d}(x,y). \tag{2.16}$$

Proof. Indeed, from Proposition 2.2.14, it is enough to find $C_2 = C_2(B_R(p)) > 0$ such that $\frac{1}{C_2}\,\mathsf{d}(x,y) \le F_x(\mathsf{d}(x,y)) \le C_2\,\mathsf{d}(x,y)$. The second inequality is a consequence of the fact that since there exists $C > 0$ such that, for every $x \in B_R(p)$, $\mathfrak{m}(B_r(x)) \ge Cr^3$. Hence $\int_{\mathsf{d}(x,y)} \frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s \le C^{-1}\mathsf{d}(x,y)$ for every $x, y \in B_R(p)$. The first inequality is a consequence of Bishop-Gromov inequality. We observe that $\mathsf{d}(x,y) \le 2R$ for every $x, y \in B_R(p)$. Moreover, since by Bishop-Gromov inequality we have that, for every $s \ge 2R$,

$$\frac{s}{\mathfrak{m}(B_s(x))} \ge \frac{(2R)^N}{\mathfrak{m}(B_{2R}(x))\,s^{N-1}}$$

we can infer that

$$\int_{\mathsf{d}(x,y)}^{+\infty} \frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s \ge \int_{2R}^{+\infty} \frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s \ge \int_{2R}^{+\infty} \frac{(2R)^N}{\mathfrak{m}(B_{2R}(x))\,s^{N-1}}\,\mathrm{d}s \ge \frac{(2R)}{\mathfrak{m}(B_{2R}(x))}\mathsf{d}(x,y).$$

□

Corollary 2.2.16. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K,N)$ *space satisfying Assumption 1. Let* $\boldsymbol{b}$ *be a time dependent vector field supported in* $B_R(p)$ *for some* $R > 0$ *and* $p \in \mathrm{X}$*, such that* $|\boldsymbol{b}| \in L^\infty([0,1] \times \mathrm{X})$*,* $\boldsymbol{b} \in L^1([0,1], W^{1,2}_{C,s}(T\mathrm{X}))$ *and* $\operatorname{div}\boldsymbol{b} \in L^1([0,1], L^\infty(\mathfrak{m}))$*. Then there exists* $C_1 > 0$ *such that, for every* $t \in [0,1]$*, for* $\mathfrak{m} \times \mathfrak{m}$*-a.e.* $(x,y) \in B_R(p) \times B_R(p)$

$$\mathsf{d}(F_t^s(x), F_t^s(y)) \le C_1 \mathsf{d}(x,y)\, e^{\int_{t\wedge s}^{t\vee s} h_r(F_t^r(x))\,\mathrm{d}r + \int_{t\wedge s}^{t\vee s} h_r(F_t^r(y))\,\mathrm{d}r} \quad \textit{for every } s \in [0,1]. \tag{2.17}$$

Proof. An application of (2.16), together with (2.15) yields the conclusion. □

We define the function $\bar{h}_t(x) := \int_0^1 h_r(F_t^r(x))\,\mathrm{d}t$. To turn this estimate into an estimate for every $x, y \in B_R(p)$, we may proceed as in the next remark.

Remark 2.2.17. Let us comment how to obtain the result in (2.17) for every $x, y \in B_R(p)$. Let us fix $t \in [0,1]$ and let C_1 be the constant in Corollary 2.2.16. We call $S := \{x \in B_R(p) : \mathsf{d}(F_t^s(x), F_t^s(y)) \le C_1 \mathsf{d}(x,y) \exp\left(\bar{h}_t(x) + \bar{h}_t(y)\right)$ for every $s \in [0,1]$ for $\mathfrak{m}$-a.e. $y\}$. In particular, it follows from (2.17) that $\mathfrak{m}(B_R(p) \setminus S) = 0$. For every $x, y \in S$, we have that, setting $\bar{r} := \mathsf{d}(x,y)$

$$\log\left(1 + \frac{\mathsf{d}(F_t^s(x), F_t^s(y))}{\mathsf{d}(x,y)}\right) \le \fint_{B_{\bar{r}}(x) \cap B_{\bar{r}}(y)} \log\left(1 + \frac{\mathsf{d}(F_t^s(x), F_t^s(z))}{\mathsf{d}(x,z)}\right) \mathrm{d}\mathfrak{m}(z)$$
$$+ \fint_{B_{\bar{r}}(x) \cap B_{\bar{r}}(y)} \log\left(1 + \frac{\mathsf{d}(F_t^s(y), F_t^s(z))}{\mathsf{d}(y,z)}\right) \mathrm{d}\mathfrak{m}(z).$$

It is enough to estimate the first term in the last expression, thus having

$$\fint_{B_{\bar{r}}(x) \cap B_{\bar{r}}(y)} \log\left(1 + \frac{\mathsf{d}(F_t^s(x), F_t^s(z))}{\mathsf{d}(x,z)}\right) \mathrm{d}\mathfrak{m}(z) \le \log(1 + C_1^2) + \bar{h}_t(x) + \fint_{B_{\bar{r}}(x) \cap B_{\bar{r}}(y)} \bar{h}_t(z)\,\mathrm{d}\mathfrak{m}(z)$$
$$\le \log(1 + C_1^2) + \bar{h}_t(x) + C_R \fint_{B_{\bar{r}}(x)} \bar{h}_t(z)\,\mathrm{d}\mathfrak{m}(z)$$

where C_R depends on the local doubling costant up to scale R. The second term can be estimated accordingly. Therefore, we can define $\bar{g}_t(x) := \bar{h}_t(x) + C_R\, M_{2R}\bar{h}_t(x)$ for $x \in S$, $\bar{g}_t(x) = +\infty$ for every $x \in B_R(p) \setminus S$ and $C_2 = 1 + C_1^2$. With this choice, we have for every $x, y \in B_R(p)$, for every $t, s \in [0,1]$

$$\mathsf{d}(F_t^s(x), F_t^s(y)) \leq C_2 \mathsf{d}(x,y)\, e^{\bar{g}_t(x)+\bar{g}_t(y)}. \tag{2.18}$$

■

Remark 2.2.18. We remark that the estimate in (2.18) holds in the more general setting of $\mathsf{RCD}(K,N)$ space, namely without Assumption 1. Consider an $\mathsf{RCD}(K,N)$ space and a time dependent vector field $\boldsymbol{b}$ such that $|\boldsymbol{b}| \in L^\infty([0,1] \times \mathrm{X})$ supported in $B_R(p)$ and such that $\boldsymbol{b} \in L^1([0,1], W^{1,2}_{C,s}(T\mathrm{X}))$ and $\operatorname{div} \boldsymbol{b} \in L^1([0,1], L^\infty(\mathfrak{m}))$. We can define a metric measure space $(\bar{\mathrm{X}}, \bar{\mathsf{d}}, \bar{\mathfrak{m}})$ as follows: $\bar{\mathrm{X}} := \mathrm{X} \times \mathbb{R}^3$,

$$\bar{\mathsf{d}}((x,x'),(y,y'))^2 := \mathsf{d}(x,y)^2 + \mathsf{d}_e(x',y')^2$$

for every $x, y \in \mathrm{X}$ and $x', y' \in \mathbb{R}^3$ and $\bar{\mathfrak{m}} := \mathfrak{m} \times \mathbb{R}^3$. Then we can define a vector field $\bar{\boldsymbol{b}}$ on $\bar{\mathrm{X}}$ (which satisfyies Assumption 1), as follows. We consider the map

$$\Phi \colon L^2(T(\mathrm{X} \times \mathbb{R}^3)) \to L^2(\mathrm{X}, L^2(T\mathbb{R}^3)) \times L^2(\mathbb{R}^3, L^2(T\mathrm{X}))$$

defined as in Theorem 1.4.3.
Then we consider $\varphi \in C_c^\infty(\mathbb{R}^3)$ such that $\varphi = 1$ on $B_{R/2}(0)$ and $\operatorname{supp} \varphi \subseteq B_R(0)$ and define

$$\bar{\boldsymbol{b}}_t := \Phi^{-1}((0, (y \mapsto \varphi(y)\, \boldsymbol{b}_t))) \quad \text{for a.e. } t.$$

It is easy to check that $\bar{\boldsymbol{b}}$ is supported in $B_{2R}((p,0))$ and $|\bar{\boldsymbol{b}}| \in L^\infty([0,1] \times \bar{\mathrm{X}})$. Moreover, we can check that (see [64]) $\operatorname{div} \bar{\boldsymbol{b}} \in L^1([0,1], L^\infty(\bar{\mathfrak{m}}))$ and $\bar{\boldsymbol{b}} \in L^1([0,1], W^{1,2}_{C,s}(T\bar{\mathrm{X}}))$. Therefore, the estimates in (2.17) hold for the regular Lagrangian flow $\bar{F}$ associated to $\bar{\boldsymbol{b}}$ for every $(x,y) \in B_{2R}((p,0))$; since, for every $(x,c) \in \mathrm{X} \times \mathbb{R}^3$ and $t \in [0,1]$, $F_t^\cdot((x,c))$ lies in $\{(x,y) \in \mathrm{X} \times \mathbb{R}^d : y = c\}$, we obtain the estimate (2.17) on X. ■

We can collect the considerations in Corollary 2.2.16, Remarks 2.2.17 and 2.2.18 in the following theorem. We present here the result with more restrictive assumptions on $\boldsymbol{b}$ (the one used in [32]) with respect to the presentation up to now.

Theorem 2.2.19. *Let* $(\mathrm{X}, \mathrm{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K,N)$ *space with* $K \in \mathbb{R}$, $N < \infty$, *and* $(\boldsymbol{b}_t) \in L^2([0,1], W^{1,2}_C(T\mathrm{X}))$ *be such that* $|\boldsymbol{b}_t|, \operatorname{div}(\boldsymbol{b}_t) \in L^\infty([0,1] \times \mathrm{X})$ *and for some* $\bar{x} \in \mathrm{X}$ *and* $R > 0$ *we have* $\operatorname{supp}(\boldsymbol{b}_t) \subset B_R(\bar{x})$ *for a.e.* $t \in [0,1]$.

Then, for every $t \in [0,1]$, *there exists a nonnegative function* $\bar{g}_t \in L^2_{loc}(\mathrm{X}, \mathfrak{m})$ *such that for any* $s \in [0,1]$ *we have*

$$\frac{\mathsf{d}(F_t^s(x), F_t^s(y))}{\mathsf{d}(x,y)} \leq e^{\bar{g}_t(x)+\bar{g}_t(y)}, \qquad \forall x, y \in \mathrm{X}. \tag{2.19}$$

We now briefly review what the main content of [25] is, where better estimates in the case of non-collapsed $\mathsf{RCD}(K,N)$ spaces are retrieved. We recall that an $\mathsf{RCD}(K,N)$ space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is said to be non collapsed ($\mathsf{ncRCD}(K,N)$ in short) if $\mathfrak{m} = \mathscr{H}^N$. The reason of restricting to noncollapsed setting is that this class allow a comparison of the distance function and the Green function introduced before at an infinitesimal scale, as stated in the following proposition. We recall that, when dealing with the Green function and to have finite properties on it, we have to consider Assumption 1.

Proposition 2.2.20 ([25, Corollary 2.4])**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathscr{H}^N)$ *be an* $\mathsf{RCD}(K,N)$ *space satisfying Assumption 1. Then for every* $x \in \mathrm{X}$ *we have*

$$\lim_{y \to x} \mathsf{d}(x,y)^{N-2}\, \bar{G}(x,y) = \frac{1}{\theta(x) w_N N(N-2)}.$$

When using the last estimate we come up to the following computation. Fixed $t \in [0,1]$, $x \in \mathrm{X}$ and a Borel set E of density one at x such that $F_t^s|_E$ is Lipschitz for every s (that can be found taking the sublevels of $\bar{g}_t$ in Theorem 2.2.19) we have

$$\limsup_{y \in E, y \to x} \frac{\mathsf{d}_{\bar{G}}(F_t^s(x), F_t^s(y))}{\mathsf{d}_{\bar{G}}(x,y)} = \left(\limsup_{y \in E, y \to x} \frac{\mathsf{d}(F_t^s(x), F_t^s(y))}{\mathsf{d}(x,y)} \right)^{N-2} \frac{\theta(F_t^s(x))}{\theta(x)}.$$

To get rid of the term $\frac{\theta(F_t^s(x))}{\theta(x)}$, the authors in [25] proved that for $\mathscr{H}^N$-a.e. initial point x the curve $F_t^{\cdot}(x)$ passes through regular points for every $s \in [t,1]$. Therefore, the last proposition, suitably used in conjunction with (2.15) allows to obtain the following result.

We recall the definition of $\mathrm{ap}-\mathrm{lip}$ in (1.16).

Theorem 2.2.21 ([25, Theorem 1.6])**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathscr{H}^N)$ *be an* $\mathsf{RCD}(K,N)$ *space. Let* $\boldsymbol{b} \in L^2([0,1], W^{1,2}_{C,s}(T\mathrm{X}))$*, supported in* $B_R(p)$ *for some* $p \in \mathrm{X}$ *and* $R > 0$*. Assume that* $|\boldsymbol{b}| \in L^\infty([0,1] \times \mathrm{X})$ *and* $\operatorname{div} \boldsymbol{b} \in L^2([0,1], L^\infty(\mathfrak{m}))$*. Then the regular Lagrangian flow* $(F_s^t)_{s,t}$ *associated to* $\boldsymbol{b}$ *satisfies the following property. For every* $0 \le s \le 1$*, for* $\mathscr{H}^N$*-a.e.* $x \in B_R(p)$*, for any* $t \in [s,1]$ $F_s^t(x)$ *is a regular point (see Definition 1.3.7) and*

$$\mathrm{ap}-\mathrm{lip} F_s^t(x) \le e^{\int_s^t g_r(F_s^r(x))\, \mathrm{d}r}$$

for $\mathscr{H}^N$*-a.e.* x*, where* $g \in L^2([0,1], L^2(\mathfrak{m}))$ *and* g_r *is non negative for a.e.* r*.*

Part II

Main contributions

Chapter 3

Existence in $\mathscr{W}^{1,2}(\boldsymbol{\pi})$ and uniqueness in $\mathscr{H}^{1,2}(\boldsymbol{\pi})$

The goal of this chapter is to define in Section 3.1 the functional spaces of Sobolev vector fields along a test plan. As for the theory of Sobolev functions in the Euclidean setting, building upon 'smooth objects', two definitions of Sobolev vector fields along a test plan: the space $\mathscr{W}^{1,2}(\boldsymbol{\pi})$ defined by integration by parts against 'smooth objects' and the space $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ as the closure of 'smooth objects' with respect to the norm of $\mathscr{W}^{1,2}(\boldsymbol{\pi})$.
Then, we propose two notions of parallel transport, one in the class $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ (in Section 3.2, taken from [61]) and we prove uniqueness and one in the class $\mathscr{W}^{1,2}(\boldsymbol{\pi})$ and we prove existence (in Section 3.3, taken from Appendix A in [32]).

3.1 Functional spaces of vector fields along a test plan

Let us assume that throughout all the following sections $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}(K, \infty)$ space. We introduce here the right functional spaces in order to speak about vector fields along a test plan. The aim of this section is twofold:

i) we review the theory of [61] in the case of L^0-normed modules;

ii) we recast the theory built along general test plan in the case of test plan $\boldsymbol{\pi}_\mu = F^{\cdot}_{0*}\mu$, where F^s_t is the regular Lagrangian flow associated to a vector field $\boldsymbol{b}$ and $\mu \in \mathscr{P}(\mathrm{X})$ with $\mu \leq C\mathfrak{m}$.

We recall that the space $\mathrm{VF}(\boldsymbol{\pi})$ of vector fields along $\boldsymbol{\pi}$ has been defined in [61] as

$$\mathrm{VF}(\boldsymbol{\pi}) := \prod_{t\in[0,1]} \mathrm{e}_t^* L^0(T\mathrm{X})$$

(actually in [61] the pullbacks of $L^2(T\mathrm{X})$ are considered). Notice that the product of $(\mathsf{v}_t) \in \mathrm{VF}(\boldsymbol{\pi})$ by a function in $L^0(\boldsymbol{\pi})$ can be defined componentwise and similarly for $(\mathsf{v}_t), (\mathsf{z}_t) \in \mathrm{VF}(\boldsymbol{\pi})$ the function $\langle (\mathsf{v}_t), (\mathsf{z}_t) \rangle$ is defined as $t \mapsto \langle \mathsf{v}_t, \mathsf{z}_t \rangle \in L^0(\boldsymbol{\pi})$. We discuss the regularity of elements in $\mathrm{VF}(\boldsymbol{\pi})$ with respect to time. We thus define the subspace $\mathrm{TestVF}(\boldsymbol{\pi}) \subseteq \mathrm{VF}(\boldsymbol{\pi})$ of *test vector fields* along $\boldsymbol{\pi}$ as

$$\mathrm{TestVF}(\boldsymbol{\pi}) := \left\{ t \mapsto \sum_{i=0}^n \varphi_i(t) \chi_{\Gamma_i} \mathrm{e}_t^* v_i \,:\, n \in \mathbb{N},\, \Gamma_i \in \mathscr{B}(C([0,1],\mathrm{X}))\, \varphi_i \in \mathrm{LIP}([0,1]), v_i \in \mathrm{TestV}(\mathrm{X}) \right\}.$$

Definition 3.1.1 (Borel vector fields in $\mathrm{VF}(\boldsymbol{\pi})$)**.** *An element* (v_t) *of* $\mathrm{VF}(\boldsymbol{\pi})$ *is Borel provided* $t \mapsto \langle \mathsf{v}_t, \mathsf{z}_t \rangle \in L^0(\boldsymbol{\pi})$ *is Borel for every* $(\mathsf{z}_t) \in \mathrm{TestVF}(\boldsymbol{\pi})$*.*

It is not hard to check that for $(\mathsf{v}_t) \in \mathrm{VF}(\boldsymbol{\pi})$ Borel the map

$$[0,1] \ni t \mapsto \|(\mathsf{v}_s)\|_t := \|\mathsf{v}_t\|_{\mathrm{e}_t^* L^2(T\mathrm{X})} = \||\mathsf{v}_t|\|_{L^2(\boldsymbol{\pi})} = \sqrt{\int |\mathsf{v}_t|^2 \,\mathrm{d}\boldsymbol{\pi}} \in [0, +\infty]$$

is Borel. To see this, consider $\varphi_n(y) = (y \wedge n) \vee 0$ for $y \in \mathbb{R}$, notice $L^0(\boldsymbol{\pi}) \ni f \mapsto T_n(f) \in [0, +\infty)$ is Borel where $T_n(f) = \int \varphi_n(f)^2 \,\mathrm{d}\boldsymbol{\pi}_\mu$ and that for every t $\mathrm{e}_t^* L^0(T\mathrm{X}) \ni v \mapsto \int |v|^2 \,\mathrm{d}\boldsymbol{\pi} = \sup_n T_n(|v|)$.

Definition 3.1.2 (The space $\mathscr{L}^2(\boldsymbol{\pi})$)**.** *We define* $\mathscr{L}^2(\boldsymbol{\pi}) \subset \mathrm{VF}(\boldsymbol{\pi})$ *as the space of those Borel* (v_t)*'s such that*

$$\|(\mathsf{v}_t)\|^2_{\mathscr{L}^2(\boldsymbol{\pi})} := \int_0^1 \|(\mathsf{v}_s)\|_t^2 \,\mathrm{d}t < \infty,$$

with the usual identification up to equality for a.e. t.

It can be checked that $(\mathscr{L}^2(\boldsymbol{\pi}_\mu), \|\cdot\|_{\mathscr{L}^2(\boldsymbol{\pi}_\mu)})$ is an Hilbert space.

Definition 3.1.3 (The space $\mathscr{C}(\boldsymbol{\pi})$)**.** *Let* $\mathsf{v} \in \mathrm{VF}(\boldsymbol{\pi})$*. Then we say that* v *is a* continuous vector field *provided*

$$[0,1] \ni t \quad \mapsto \quad \int \langle \mathsf{v}_t, \mathsf{w}_t \rangle \,\mathrm{d}\boldsymbol{\pi} \quad \textit{is continuous} \tag{3.1}$$

for every $\mathsf{w} \in \mathrm{TestVF}(\boldsymbol{\pi})$ *and*

$$[0,1] \ni t \quad \mapsto \quad \|\mathsf{v}_t\|_{\mathrm{e}_t^* L^2(T\mathrm{X})} \qquad \textit{is continuous.} \tag{3.2}$$

We denote the family of all continuous vector fields by $\mathscr{C}(\boldsymbol{\pi})$ *and, for every* $\mathsf{v} \in \mathscr{C}(\boldsymbol{\pi})$*, we put*

$$\|\mathsf{v}\|_{\mathscr{C}(\boldsymbol{\pi})} := \max_{t \in [0,1]} \|\mathsf{v}_t\|_{\mathrm{e}_t^* L^2(T\mathrm{X})}.$$

It can be checked that $(\mathscr{C}(\boldsymbol{\pi}), \|\cdot\|_{\mathscr{C}(\boldsymbol{\pi})})$ is a Banach space (see [61, Proposition 3.12]). Moreover, given $\mathsf{v} \in \mathscr{C}(\boldsymbol{\pi})$, we have that (see [61, Corollary 3.13]

$$\text{the map } [0,1] \ni t \mapsto |\mathsf{v}_t|^2 \in L^1(\boldsymbol{\pi}) \text{ is continuous.} \tag{3.3}$$

In order to define the convective derivative as in [61], it is necessary the notion of speed of a test plan, namely to overcome the fact that for a general test plan $\boldsymbol{\pi}$ the map e_t could be not $\boldsymbol{\pi}$-essentially injective. Therefore, we restrict to the following class, to tailor the discussion to the relevant plans for the scope of the thesis.

Definition 3.1.4 (Test plan induced by a regular Lagrangian flow)**.** *Given* $\boldsymbol{\pi} \in \mathscr{P}(C([0,1], \mathrm{X}))$*, we say that it is a test plan induced by a regular Lagrangian flow if there exists* $\boldsymbol{b} \in L^2([0,1], L^2(T\mathrm{X}))$ *with* $|\boldsymbol{b}| \in L^\infty([0,1] \times \mathrm{X})$ *for which there exists a regular Lagrangian flow* F *and* $\mu \in \mathscr{P}(\mathrm{X})$ *with* $\mu \leq C\mathfrak{m}$ *such that*

$$F^{\cdot}_{0*}\mu = \boldsymbol{\pi}.$$

In this case, to shorten the notation, we say that $\boldsymbol{\pi}$ *is a regular plan,* μ *is the initial distribution of* $\boldsymbol{\pi}$ *and* $\boldsymbol{\pi}$ *is drifted by* $\boldsymbol{b}$*.*

To stress the dependence on μ, we denote $\boldsymbol{\pi}$ by $\boldsymbol{\pi}_\mu$. It is straightforward to check by the properties of F_t^s and the fact $\boldsymbol{b} \in L^2([0,1], L^2(T\mathrm{X}))$ that $\boldsymbol{\pi}_\mu$ is a test plan. Let us fix one such test plan $\boldsymbol{\pi}_\mu$.

Definition 3.1.5 (Convective derivative along a test plan). *We define the* convective derivative *operator* $\mathrm{D}_{\boldsymbol{\pi}_\mu} : \mathrm{TestVF}(\boldsymbol{\pi}_\mu) \to \mathscr{L}^2(\boldsymbol{\pi}_\mu)$ *as follows: to the element* $\mathsf{v} \in \mathrm{TestVF}(\boldsymbol{\pi}_\mu)$*, of the form* $\mathsf{v}_t = \varphi(t)\,\chi_A\,\mathrm{e}_t^* v$*, for* $\varphi \in \mathrm{Lip}([0,1])$*,* $A \in \mathscr{B}(C([0,1],\mathrm{X}))$*,* $v \in \mathrm{TestV}(\mathrm{X})$*, we associate the vector field* $\mathsf{D}_{\boldsymbol{\pi}_\mu}\mathsf{v} \in \mathscr{L}^2(\boldsymbol{\pi}_\mu)$ *given by*

$$(\mathsf{D}_{\boldsymbol{\pi}_\mu}\mathsf{v})_t := \chi_A\Big(\varphi_i'(t)\,\mathrm{e}_t^* v_i + \varphi_i(t)\,\mathrm{e}_t^* \nabla_{\boldsymbol{b}_t} v\Big) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,1]. \tag{3.4}$$

and then extended by linearity. For the sake of simplicity, we will briefly write $\mathsf{D}_{\boldsymbol{\pi}_\mu}\mathsf{v}_t$ *instead of* $(\mathsf{D}_{\boldsymbol{\pi}_\mu}\mathsf{v})_t$.

Remark 3.1.6. The hyphothesis $|\boldsymbol{b}| \in L^\infty([0,1]\times\mathrm{X})$ is related to the fact that, under this assumption, $(\mathsf{D}_{\boldsymbol{\pi}_\mu}\mathsf{v}) \in \mathscr{L}^2(\boldsymbol{\pi}_\mu)$.

We introduce two spaces of Sobolev vector fields along $\boldsymbol{\pi}_\mu$, the $\mathscr{W}$ and the $\mathscr{H}$ space.

Definition 3.1.7 (The space $\mathscr{W}^{1,2}(\boldsymbol{\pi}_\mu)$). *The space* $\mathscr{W}^{1,2}(\boldsymbol{\pi}_\mu) \subset \mathscr{L}^2(\boldsymbol{\pi}_\mu)$ *is defined as the collection of* (v_t)*'s for which there is* (v_t') *such that*

$$\int_0^1 \int \langle \mathsf{v}_t, \mathsf{D}_t \mathsf{z}_t \rangle \, \mathrm{d}\boldsymbol{\pi}_\mu \, \mathrm{d}t = -\int_0^1 \int \langle \mathsf{v}_t', \mathsf{z}_t \rangle \, \mathrm{d}\boldsymbol{\pi}_\mu \, \mathrm{d}t \tag{3.5}$$

holds for any $(\mathsf{z}_t) \in \mathrm{TestVF}(\boldsymbol{\pi}_\mu)$ *with 'compact support', i.e. such that*

$$\mathsf{z}_t = 0 \text{ for every } t \text{ in a neighbourhood of } 0 \text{ and } 1. \tag{3.6}$$

The vector field (v_t') *is uniquely defined by the above, called* convective derivative *of* (v_t) *along* $\boldsymbol{\pi}_\mu$ *and denoted* $(\mathsf{D}_t\mathsf{v}_t)$.

Then $\mathscr{W}^{1,2}(\boldsymbol{\pi}_\mu)$ is endowed with the norm

$$\|(\mathsf{v}_t)\|_{\mathscr{W}^{1,2}}^2 := \|(\mathsf{v}_t)\|_{\mathscr{L}^2}^2 + \|(\mathsf{D}_t\mathsf{v}_t)\|_{\mathscr{L}^2}^2$$

and can - easily - be proved to be a Hilbert space.

It turns out that this latter definition of D_t is compatible with the previous one, i.e. $\mathrm{TestVF}(\boldsymbol{\pi}_\mu) \subseteq \mathscr{W}^{1,2}(\boldsymbol{\pi}_\mu)$ and for a vector field in $\mathrm{TestVF}(\boldsymbol{\pi}_\mu)$ the convective derivative defined by formula (3.4) coincides with the one defined by (3.5). In particular, it makes sense to define $\mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$ as the closure of $\mathrm{TestVF}(\boldsymbol{\pi}_\mu)$ in $\mathscr{W}^{1,2}(\boldsymbol{\pi}_\mu)$. By definition, it follows that $(\mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu), \|\cdot\|_{\mathscr{W}^{1,2}(\boldsymbol{\pi}_\mu)})$ is an Hilbert space and

$$\mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu) \subseteq \mathscr{W}^{1,2}(\boldsymbol{\pi}_\mu).$$

An important property of vector fields in $\mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$ is that they admit a *continuous representative.*

Theorem 3.1.8 ([61, Theorem 3.23]). *The inclusion* $\mathrm{TestVF}(\boldsymbol{\pi}_\mu) \hookrightarrow \mathscr{C}(\boldsymbol{\pi}_\mu)$ *uniquely extends to a linear continuous and injective operator* $\iota : \mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu) \to \mathscr{C}(\boldsymbol{\pi}_\mu)$.

Another crucial property of the space $\mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$ is the Leibniz rule.

Proposition 3.1.9 (Leibniz formula for $\mathsf{D}_{\boldsymbol{\pi}}$). *Let* $\mathsf{v} \in \mathscr{W}^{1,2}(\boldsymbol{\pi}_\mu)$ *and* $\mathsf{w} \in \mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$*. Then the function* $t \mapsto \langle \mathsf{v}_t, \mathsf{w}_t \rangle$ *is in* $W^{1,1}([0,1], L^1(\boldsymbol{\pi}_\mu))$ *and its derivative is given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathsf{v}_t, \mathsf{w}_t \rangle = \langle \mathsf{D}_{\boldsymbol{\pi}_\mu}\mathsf{v}_t, \mathsf{w}_t \rangle + \langle \mathsf{v}_t, \mathsf{D}_{\boldsymbol{\pi}_\mu}\mathsf{w}_t \rangle \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,1].$$

3.2 Uniqueness of parallel transport in $\mathscr{H}^{1,2}(\boldsymbol{\pi})$

In this section, we review the results of [61, Section 4.1.1], in which uniqueness of parallel transport in $\mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$, as a consequence of Proposition 3.1.9.
Again, we assume that $\boldsymbol{\pi}$ is a regular plan, in the sense of Definition 3.1.4, with initial distribution μ and drifted by $\boldsymbol{b}$. The map ι used in the following is the one introduced in Theorem 3.1.8.

Definition 3.2.1 (Parallel transport in $\mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$)**.** *Let $K \in \mathbb{R}$, $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ an $\mathsf{RCD}(K,\infty)$ space and $\boldsymbol{\pi}$ be a regular plan on* X. *A* parallel transport *along $\boldsymbol{\pi}$ starting from $\bar{\mathsf{v}} \in \mathrm{e}_0^* L^2(T\mathrm{X})$ is an element $\mathsf{v} \in \mathscr{H}^{1,2}(\boldsymbol{\pi})$ such that $\mathsf{D}_{\boldsymbol{\pi}}\mathsf{v} = 0$ and $\iota(\mathsf{v})_0 = \bar{\mathsf{v}}$.*

The linearity of the requirement $\mathrm{D}_{\boldsymbol{\pi}}\mathsf{v} = 0$ ensures that the set of parallel transports along $\boldsymbol{\pi}$ is a vector space. From Proposition 3.1.9 we deduce the following simple but crucial result:

Proposition 3.2.2 (Norm preservation)**.** *Let v be a parallel transport along the regular plan $\boldsymbol{\pi}$. Then $t \mapsto |\mathsf{v}_t|^2 \in L^1(\boldsymbol{\pi})$ is constant.*

Proof. We know from (3.3) that $t \mapsto |\mathsf{v}_t|^2 \in L^1(\boldsymbol{\pi})$ is continuous. Hence the choice $\mathsf{w} = \mathsf{v}$ in Proposition 3.1.9 tells that such map is absolutely continuous with derivative given by

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathsf{v}_t|^2 = 2\langle \mathrm{D}_{\boldsymbol{\pi}}\mathsf{v}_t, \mathsf{v}_t\rangle = 0, \qquad \textit{a.e. } t.$$

This is sufficient to conclude. □

Linearity and norm preservation imply uniqueness:

Corollary 3.2.3 (Uniqueness of parallel transport)**.** *Let $\boldsymbol{\pi}$ be a regular plan and $\mathsf{v}_1, \mathsf{v}_2$ two parallel transports along it such that for some $t_0 \in [0,1]$ it holds $\mathsf{v}_{1,t_0} = \mathsf{v}_{2,t_0}$. Then $\mathsf{v}_1 = \mathsf{v}_2$. In particular, there is at most one parallel transport along $\boldsymbol{\pi}$ starting from any $\bar{\mathsf{v}} \in \mathrm{e}_0^* L^2(T\mathrm{X})$.*

Proof. Since $\mathrm{D}_{\boldsymbol{\pi}}(\mathsf{v}_1 - \mathsf{v}_2) = \mathrm{D}_{\boldsymbol{\pi}}\mathsf{v}_1 - \mathrm{D}_{\boldsymbol{\pi}}\mathsf{v}_2 = 0$, we have that $\mathsf{v}_1 - \mathsf{v}_2$ is a parallel transport and by assumption we know that $|\mathsf{v}_{1,t_0} - \mathsf{v}_{2,t_0}| = 0$ $\boldsymbol{\pi}$-a.e.. Thus Proposition 3.2.2 above grants that for every $t \in [0,1]$ it holds $|\mathsf{v}_{1,t} - \mathsf{v}_{2,t}| = 0$ $\boldsymbol{\pi}$-a.e., i.e. that $\mathsf{v}_{1,t} = \mathsf{v}_{2,t}$. □

3.3 Existence of the parallel transport in $\mathscr{W}^{1,2}(\boldsymbol{\pi})$

In this section, we show an existence result of the parallel transport of an initial vector field along a regular plan in the class $\mathscr{W}^{1,2}(\boldsymbol{\pi})$. In particular, we can show, by an abstract argument of functional analysis, that for a given initial vector field $\bar{V} \in \mathrm{e}_0^* L^2(T\mathrm{X})$ we can find a Borel vector field $t \to V_t$ along a regular plan $\boldsymbol{\pi}$ such that $V \in \mathscr{W}^{1,2}(\boldsymbol{\pi})$, $(D_{\boldsymbol{\pi}}V)_t = 0$ for a.e. t, and satisfies the initial condition in an appropriate sense. We use in this section a vanishing viscosity approach: we approximate our problem with a sequence of problems that are coercive; on this class of problems we can apply a variant of Lax–Milgram lemma; thanks to compactness, we can pass to the limit and obtain a solution to our problem. More precisely, the form of Lax–Milgram lemma is the following one (taken from [15]).

Lemma 3.3.1 (Lions)**.** *Let E and H be a normed and a Hilbert space, respectively. Assume that E is continuously embedded in H, with $\|v\|_H \leq \|v\|_E$ for every $v \in E$. Let $B\colon H \times E \to \mathbb{R}$ be a bilinear form such that $B(\cdot, v)$ is continuous for every $v \in E$. If B is coercive, namely there exists $c > 0$ such that $B(v,v) \geq c\|v\|_E^2$ for every $v \in E$, then for all $l \in E'$ there exists $h \in H$ such that*

$$B(h, v) = l(v) \quad \text{for every } v \in E$$

and

$$\|h\|_H \leq \frac{\|l\|_{E'}}{c}. \tag{3.7}$$

We introduce the following class of approximations. For a given ε, we solve in a distributional sense the partial differential equation:

$$(\mathrm{D}_{\boldsymbol{\pi}} V)_t = \varepsilon(-V_t + (\mathrm{D}^2_{\boldsymbol{\pi}} V)_t), \tag{3.8}$$

looking for a solution in $\mathscr{H}^{1,2}(\boldsymbol{\pi})$.

Definition 3.3.2 (Parallel transport in $\mathscr{W}^{1,2}(\boldsymbol{\pi})$)**.** *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K,\infty)$ space and $\boldsymbol{\pi}$ a regular plan on X. Given $\bar{V} \in \mathrm{e}_0^* L^2(T\mathrm{X})$ we say that $V \in \mathscr{W}^{1,2}(\boldsymbol{\pi})$ is a parallel transport in $\mathscr{W}^{1,2}(\boldsymbol{\pi})$ of $\bar{V}$ along $\boldsymbol{\pi}$ if $D_{\boldsymbol{\pi}} V = 0$ and for every $Z \in \mathrm{TestVF}(\boldsymbol{\pi})$*

$$R(Z)_0 = \int \langle \bar{V}, Z_0 \rangle \, \mathrm{d}\boldsymbol{\pi}, \tag{3.9}$$

where we denote by $t \mapsto R(Z)_t$ the absolutely continuous representative of $t \mapsto \int \langle V_t, Z_t \rangle \, \mathrm{d}\boldsymbol{\pi}$.

Theorem 3.3.3 (Existence of PT in $\mathscr{W}^{1,2}(\boldsymbol{\pi})$)**.** *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K,\infty)$ space and $\boldsymbol{\pi}$ a regular plan on X. Let $\bar{V} \in \mathrm{e}_0^* L^2(T\mathrm{X})$ be given. Then there exists $V \in \mathscr{W}^{1,2}(\boldsymbol{\pi})$ that is a parallel transport in $\mathscr{W}^{1,2}(\boldsymbol{\pi})$ of $\bar{V}$ along $\boldsymbol{\pi}$.*

Proof. Fix $\varepsilon \in (0, 1/2)$. Consider the Hilbert space $H \coloneqq \big(\mathscr{H}^{1,2}(\boldsymbol{\pi}), \|\cdot\|_{\mathscr{H}^{1,2}(\boldsymbol{\pi})}\big)$. Define also

$$E \coloneqq \big\{ Z \in \mathrm{TestVF}(\boldsymbol{\pi}) \,\big|\, \mathrm{spt}(Z) \subseteq [0,1) \big\} \subseteq \mathscr{H}^{1,2}(\boldsymbol{\pi}),$$
$$\|Z\|_E \coloneqq \big(\|Z_0\|^2_{\mathrm{e}_0^* L^2(T\mathrm{X})} + \|Z\|_H^2 \big)^{1/2} \quad \text{for every } Z \in E.$$

Clearly, $\|Z\|_H \leq \|Z\|_E$ for every $Z \in E$. Now let us define $B\colon H \times E \to \mathbb{R}$ and $\ell\colon E \to \mathbb{R}$ as

$$B(V, Z) \coloneqq \int_0^1 \!\! \int -\langle V_t, \mathrm{D}_{\boldsymbol{\pi}} Z_t \rangle + \varepsilon \, \langle V_t, Z_t \rangle + \varepsilon \, \langle \mathrm{D}_{\boldsymbol{\pi}} V_t, \mathrm{D}_{\boldsymbol{\pi}} Z_t \rangle \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t,$$
$$\ell(Z) \coloneqq \int \langle \bar{V}, Z_0 \rangle \, \mathrm{d}\boldsymbol{\pi},$$

respectively. The map B is bilinear by construction. Moreover, for some constant $C > 0$ we have that $\big|B(V,Z)\big| \leq C \, \|V\|_H \, \|Z\|_H$ for every $V \in H$ and $Z \in E$, thus in particular $B(\cdot, Z)$ is continuous for any $Z \in E$. The Leibniz rule grants that $-2 \int_0^1\!\!\int \langle Z_t, \mathrm{D}_{\boldsymbol{\pi}} Z_t \rangle \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t = \int |Z_0|^2 \, \mathrm{d}\boldsymbol{\pi}$ holds for every $Z \in E$, whence coercivity of the map B follows: given any $Z \in E$, we have

$$B(Z, Z) = \frac{1}{2} \int |Z_0|^2 \, \mathrm{d}\boldsymbol{\pi} + \varepsilon \int_0^1 \!\! \int |Z_t|^2 + |\mathrm{D}_{\boldsymbol{\pi}} Z_t|^2 \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t \geq \varepsilon \, \|Z\|_E^2.$$

Furthermore, it holds that $\ell \in E'$ and $\|\ell\|_{E'} \leq \|\bar{V}\|_{\mathrm{e}_0^* L^2(T\mathrm{X})}$. Therefore, Lemma 3.3.1 yields the existence of an element $V^\varepsilon \in H$ such that $\|V^\varepsilon\|_H \leq \|\bar{V}\|_{\mathrm{e}_0^* L^2(T\mathrm{X})}/\varepsilon$ and $B(V^\varepsilon, Z) = \ell(Z)$ for every $Z \in E$, which explicitly reads as

$$\int_0^1\!\!\int -\langle V_t^\varepsilon, \mathrm{D}_{\boldsymbol{\pi}} Z_t\rangle + \varepsilon\,\langle V_t^\varepsilon, Z_t\rangle + \varepsilon\,\langle \mathrm{D}_{\boldsymbol{\pi}} V_t^\varepsilon, \mathrm{D}_{\boldsymbol{\pi}} Z_t\rangle\,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t = \int \langle \bar{V}, Z_0\rangle\,\mathrm{d}\boldsymbol{\pi} \tag{3.10}$$

for every $Z \in E$. Given any $Z \in \mathrm{TestVF}(\boldsymbol{\pi})$ and $\varphi \in \mathrm{LIP}([0,1])$ with $\mathrm{spt}(\varphi) \subseteq [0,1)$, it holds that $t \mapsto \varphi(t) Z_t$ belongs to E and $\mathrm{D}_{\boldsymbol{\pi}}(\varphi Z)_t = \varphi'(t) Z_t + \varphi(t)\,\mathrm{D}_{\boldsymbol{\pi}} Z_t$ for a.e. $t \in [0,1]$. Then

$$\begin{aligned}\varphi(0)\int \langle \bar{V}, Z_0\rangle\,\mathrm{d}\boldsymbol{\pi} = &\int_0^1 \varphi(t)\int -\langle V_t^\varepsilon, \mathrm{D}_{\boldsymbol{\pi}} Z_t\rangle + \varepsilon\,\langle V_t^\varepsilon, Z_t\rangle + \varepsilon\,\langle \mathrm{D}_{\boldsymbol{\pi}} V_t^\varepsilon, \mathrm{D}_{\boldsymbol{\pi}} Z_t\rangle\,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t \\ &+ \int_0^1 \varphi'(t)\int -\langle V_t^\varepsilon, Z_t\rangle + \varepsilon\,\langle \mathrm{D}_{\boldsymbol{\pi}} V_t^\varepsilon, Z_t\rangle\,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t.\end{aligned} \tag{3.11}$$

Fix a Lebesgue point $s \in (0,1)$ of $t \mapsto \int -\langle V_t^\varepsilon, Z_t\rangle + \varepsilon\,\langle \mathrm{D}_{\boldsymbol{\pi}} V_t^\varepsilon, Z_t\rangle\,\mathrm{d}\boldsymbol{\pi}$. Define φ_n as

$$\varphi_n(t) := \begin{cases} 1 & \text{if } t \in [0,s), \\ -n(t-s)+1 & \text{if } t \in [s, s+1/n), \\ 0 & \text{if } t \in [s+1/n, 1], \end{cases}$$

for all $n \in \mathbb{N}$, $n > 1/(1-s)$. Note that $(\varphi_n)_n \subseteq \mathrm{LIP}([0,1])$ is a bounded sequence in $L^\infty(0,1)$, $\mathrm{spt}(\varphi_n) \subseteq [0,1)$ for all n, and $\varphi_n \to \chi_{[0,s]}$ pointwise as $n \to \infty$. Moreover, it holds that

$$\begin{aligned}\int_0^1 \varphi_n'(t)\int -\langle V_t^\varepsilon, Z_t\rangle + \varepsilon\,\langle \mathrm{D}_{\boldsymbol{\pi}} V_t^\varepsilon, Z_t\rangle\,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t = & n\int_s^{s+1/n}\int \langle V_t^\varepsilon, Z_t\rangle - \varepsilon\,\langle \mathrm{D}_{\boldsymbol{\pi}} V_t^\varepsilon, Z_t\rangle\,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t \\ &\to \int \langle V_s^\varepsilon, Z_s\rangle - \varepsilon\,\langle \mathrm{D}_{\boldsymbol{\pi}} V_s^\varepsilon, Z_s\rangle\,\mathrm{d}\boldsymbol{\pi} \quad \text{as } n \to \infty.\end{aligned}$$

Therefore, by plugging $\varphi = \varphi_n$ into (3.11) and letting $n \to \infty$, we deduce that

$$\begin{aligned}\int \langle \bar{V}, Z_0\rangle\,\mathrm{d}\boldsymbol{\pi} = &\int_0^s\!\!\int -\langle V_t^\varepsilon, \mathrm{D}_{\boldsymbol{\pi}} Z_t\rangle + \varepsilon\,\langle V_t^\varepsilon, Z_t\rangle + \varepsilon\,\langle \mathrm{D}_{\boldsymbol{\pi}} V_t^\varepsilon, \mathrm{D}_{\boldsymbol{\pi}} Z_t\rangle\,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t \\ &+ \int \langle V_s^\varepsilon, Z_s\rangle - \varepsilon\,\langle \mathrm{D}_{\boldsymbol{\pi}} V_s^\varepsilon, Z_s\rangle\,\mathrm{d}\boldsymbol{\pi}.\end{aligned} \tag{3.12}$$

Given that $V^\varepsilon \in \mathscr{H}^{1,2}(\boldsymbol{\pi})$, we can find a sequence $(Z^n)_n \subseteq \mathrm{TestVF}(\boldsymbol{\pi})$ that $\mathscr{W}^{1,2}(\boldsymbol{\pi})$-converges to V^ε. We start noticing that from Theorem 3.1.8 there exists a continuous injection $i : \mathscr{H}^{1,2}(\boldsymbol{\pi}) \to \mathscr{C}(\boldsymbol{\pi})$ such that $\|V^\varepsilon - Z^n\|_{\mathscr{C}(\boldsymbol{\pi})} \leq \sqrt{2}\|V^\varepsilon - Z^n\|_{\mathscr{W}^{1,2}(\boldsymbol{\pi})}$, which grants that

$$\lim_{n\to\infty} \|V_0^\varepsilon - Z_0^n\|_{\mathrm{e}_0^* L^2(T\mathrm{X})} = 0.$$

Therefore, it follows that

$$\lim_{n\to\infty}\int \langle \bar{V}, Z_0^n\rangle\,\mathrm{d}\boldsymbol{\pi} = \int \langle \bar{V}, V_0^\varepsilon\rangle\,\mathrm{d}\boldsymbol{\pi}.$$

By plugging $Z = Z^n$ into (3.12), letting $n \to \infty$, and using the Leibniz rule in $\mathscr{H}^{1,2}(\boldsymbol{\pi})$, we get

$$\begin{aligned}\int \langle \bar{V}, V_0^\varepsilon \rangle \, \mathrm{d}\boldsymbol{\pi} &= \int_0^s\!\!\int -\langle V_t^\varepsilon, \mathrm{D}_{\boldsymbol{\pi}} V_t^\varepsilon \rangle + \varepsilon \, |V_t^\varepsilon|^2 + \varepsilon \, |\mathrm{D}_{\boldsymbol{\pi}} V_t^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t + \int |V_s^\varepsilon|^2 - \varepsilon \, \langle \mathrm{D}_{\boldsymbol{\pi}} V_s^\varepsilon, V_s^\varepsilon \rangle \, \mathrm{d}\boldsymbol{\pi} \\ &\geq \int_0^s\!\!\int -\langle V_t^\varepsilon, \mathrm{D}_{\boldsymbol{\pi}} V_t^\varepsilon \rangle \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t + \int |V_s^\varepsilon|^2 - \varepsilon \, \langle \mathrm{D}_{\boldsymbol{\pi}} V_s^\varepsilon, V_s^\varepsilon \rangle \, \mathrm{d}\boldsymbol{\pi} \\ &= -\frac{1}{2} \int |V_s^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} + \frac{1}{2} \int |V_0^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} + \int |V_s^\varepsilon|^2 - \varepsilon \, \langle \mathrm{D}_{\boldsymbol{\pi}} V_s^\varepsilon, V_s^\varepsilon \rangle \, \mathrm{d}\boldsymbol{\pi} \\ &= \frac{1}{2} \int |V_s^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} + \frac{1}{2} \int |V_0^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} - \varepsilon \, \langle \mathrm{D}_{\boldsymbol{\pi}} V_s^\varepsilon, V_s^\varepsilon \rangle \, \mathrm{d}\boldsymbol{\pi}.\end{aligned}$$

Since $\frac{1}{2} \int |V_0^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} - \int \langle \bar{V}, V_0^\varepsilon \rangle \, \mathrm{d}\boldsymbol{\pi} = \frac{1}{2} \int |\bar{V} - V_0^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} - \frac{1}{2} \int |\bar{V}|^2 \, \mathrm{d}\boldsymbol{\pi}$, we can rewrite the former expression as

$$\frac{1}{2} \int |\bar{V} - V_0^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} - \frac{1}{2} \int |\bar{V}|^2 \, \mathrm{d}\boldsymbol{\pi} \leq \varepsilon \int \langle \mathrm{D}_{\boldsymbol{\pi}} V_s^\varepsilon, V_s^\varepsilon \rangle \, \mathrm{d}\boldsymbol{\pi} - \frac{1}{2} \int |V_s^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi}.$$

Therefore, we obtain that

$$\frac{1}{2} \int |V_s^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} \leq \frac{1}{2} \int |V_s^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} + \frac{1}{2} \int |\bar{V} - V_0^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} \leq \frac{1}{2} \int |\bar{V}|^2 \, \mathrm{d}\boldsymbol{\pi} + \varepsilon \int \langle \mathrm{D}_{\boldsymbol{\pi}} V_s^\varepsilon, V_s^\varepsilon \rangle \, \mathrm{d}\boldsymbol{\pi}.$$

By integrating the above inequality over the interval $[0, 1]$, multiplying by 2, and then applying Young's inequality $ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$, we infer that

$$\begin{aligned}\int_0^1\!\!\int |V_s^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}s &\leq \int |\bar{V}|^2 \, \mathrm{d}\boldsymbol{\pi} + 2\varepsilon \int_0^1\!\!\int \langle \mathrm{D}_{\boldsymbol{\pi}} V_s^\varepsilon, V_s^\varepsilon \rangle \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}s \\ &\leq \int |\bar{V}|^2 \, \mathrm{d}\boldsymbol{\pi} + 2\varepsilon^2 \int_0^1\!\!\int |\mathrm{D}_{\boldsymbol{\pi}} V_s^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}s + \frac{1}{2} \int_0^1\!\!\int |V_s^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}s,\end{aligned}$$

whence accordingly $\frac{1}{2} \int_0^1\!\!\int |V_s^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}s \leq \int |\bar{V}|^2 \, \mathrm{d}\boldsymbol{\pi} + 2\varepsilon^2 \int_0^1\!\!\int |\mathrm{D}_{\boldsymbol{\pi}} V_s^\varepsilon|^2 \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}s \leq 3 \, \|\bar{V}\|^2_{\mathrm{e}_0^* L^2(T\mathrm{X})}$. Observe also that $\{\varepsilon V^\varepsilon\}_{\varepsilon \in (0,1/2)}$ is bounded in H. Therefore, there exist $V \in \mathscr{L}^2(\boldsymbol{\pi})$, $W \in H$, and a sequence $\varepsilon_n \searrow 0$, such that $V^{\varepsilon_n} \rightharpoonup V$ weakly in $\mathscr{L}^2(\boldsymbol{\pi})$ and $\varepsilon_n V^{\varepsilon_n} \rightharpoonup W$ weakly in H. In particular, it must hold that $W = 0$. Hence, by letting $n \to \infty$ in the identity

$$\int_0^1\!\!\int -\langle V_t^{\varepsilon_n}, \mathrm{D}_{\boldsymbol{\pi}} Z_t \rangle + \langle \varepsilon_n V_t^{\varepsilon_n}, Z_t \rangle + \langle \mathrm{D}_{\boldsymbol{\pi}} (\varepsilon_n V^{\varepsilon_n})_t, \mathrm{D}_{\boldsymbol{\pi}} Z_t \rangle \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t = \int \langle \bar{V}, Z_0 \rangle \, \mathrm{d}\boldsymbol{\pi},$$

which holds for every $n \in \mathbb{N}$ and $Z \in \mathrm{TestVF}_c(\boldsymbol{\pi})$ by (3.10), we can finally conclude that $\int_0^1\!\!\int \langle V_t, \mathrm{D}_{\boldsymbol{\pi}} Z_t \rangle \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t = 0$ is satisfied for every $Z \in \mathrm{TestVF}_c(\boldsymbol{\pi})$. This grants that $V \in \mathscr{W}^{1,2}(\boldsymbol{\pi})$ and $\mathrm{D}_{\boldsymbol{\pi}} V = 0$. Finally, let us prove (3.9). Fix any $Z \in \mathrm{TestVF}(\boldsymbol{\pi})$ and denote by $t \mapsto R(Z)_t$ the absolutely continuous representative of $t \mapsto \int \langle V_t, Z_t \rangle \, \mathrm{d}\boldsymbol{\pi}$ (that belongs to $W^{1,1}(0, 1)$). Write (3.12) with $\varepsilon = \varepsilon_n$, integrate over $s \in [0, 1]$, and then let $n \to \infty$: by exploiting the weak convergence $V^{\varepsilon_n} \rightharpoonup V$ in $\mathscr{L}^2(\boldsymbol{\pi})$ (and by using the dominated convergence theorem), we obtain that

$$\begin{aligned}\int \langle \bar{V}, Z_0 \rangle \, \mathrm{d}\boldsymbol{\pi} &= -\int_0^1 \int_0^s \int \langle V_t, \mathrm{D}_{\boldsymbol{\pi}} Z_t \rangle \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t \, \mathrm{d}s + \int_0^1 \int \langle V_s, Z_s \rangle \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}s \\ &= -\int_0^1 \int_0^s \left(\frac{\mathrm{d}}{\mathrm{d}t} R(Z)_t \right) \mathrm{d}t \, \mathrm{d}s + \int_0^1 R(Z)_s \, \mathrm{d}s \\ &= -\int_0^1 R(Z)_s - R(Z)_0 \, \mathrm{d}s + \int_0^1 R(Z)_s \, \mathrm{d}s = R(Z)_0,\end{aligned}$$

where we applied the Leibniz rule with one vector field in $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ and the other in $\mathscr{W}^{1,2}(\boldsymbol{\pi})$. Hence, the statement is achieved. □

Chapter 4

Functional analytic tools

In this section, we outline all the functional analytic tools developed in [32] in order to build the theory of existence and uniqueness of parallel transport along 'a.e. integral curve' in non collapsed $\mathsf{RCD}(K,N)$ spaces.

4.1 Useful properties of Lagrangian flows

We continue the discussion about RLF associated to a time dependent vector field $\boldsymbol{b}$ and their properties. Notice that the uniqueness statement and the very definition of RLF, imply the following group(oid) property:

$$\text{for every } t,s,r \in [0,1] \text{ we have} \qquad F_r^s \circ F_t^r = F_t^s \qquad \mathfrak{m} - a.e.. \tag{4.1}$$

It can be proved that for $\mathfrak{m}$-a.e. x the metric speed of the curve $s \mapsto F_t^s(x)$ is exactly $|\boldsymbol{b}_s|(F_t^s(x))$, in particular, we have the uniform Lipschitz estimate: for every $t, s_1, s_2 \in [0,1]$ we have

$$\mathsf{d}(F_t^{s_2}(x), F_t^{s_1}(x)) \leq |s_2 - s_1| \| |\boldsymbol{b}| \|_{L^\infty([0,1]\times \mathrm{X})}, \qquad \mathfrak{m} - a.e.\ x \in \mathrm{X}. \tag{4.2}$$

A consequence of the continuity of $s \mapsto F_t^s(x)$ and of the bounded compression property (2.13) is that

$$f \in L^p(\mathrm{X}) \qquad \Rightarrow \qquad [0,1]^2 \ni (t,s) \mapsto f \circ F_t^s \in L^p(\mathrm{X}) \text{ is continuous}, \qquad \forall p \in [1,\infty). \tag{4.3}$$

Indeed, the bounded compression property gives the uniform estimate

$$\|f \circ F_t^s\|_{L^p}^p \leq C \|f\|_{L^p}^p \qquad \forall t,s \in [0,1], \tag{4.4}$$

and the continuity of $s \mapsto F_t^s(x)$ ensures that for f Lipschitz with bounded support $s \mapsto f \circ F_t^s \in L^p$ is continuous for every $t \in [0,1]$. From the density of such Lipschitz functions in L^p and (4.4) we deduce that $s \mapsto f \circ F_t^s \in L^p$ is continuous for every $t \in [0,1]$ and $f \in L^p(\mathrm{X})$. Now notice that

$$\begin{aligned}
\|f \circ F_{t'}^{s'} - f \circ F_t^s\|_{L^p} &\leq \|f \circ F_{t'}^{s'} - f \circ F_{t'}^s\|_{L^p} + \|f \circ F_{t'}^s - f \circ F_t^s\|_{L^p} \\
\text{(by (4.4), (4.1))} \qquad &\leq C^{\frac{1}{p}} \|f \circ F_s^{s'} - f\|_{L^p} + \|(f \circ F_t^s) \circ F_{t'}^t - f \circ F_t^s\|_{L^p} \\
\text{(by (4.4), (4.1))} \qquad &\leq C^{\frac{1}{p}} \|f \circ F_s^{s'} - f\|_{L^p} + C^{\frac{1}{p}} \|f \circ F_t^s - (f \circ F_t^s) \circ F_t^{t'}\|_{L^p}
\end{aligned}$$

so that the claim (4.3) follows from what already proved.

In a similar way, if for some bounded set $B \subset \mathrm{X}$ we have that $\mathrm{supp}(\boldsymbol{b}_t) \subset B$ for a.e. t we have that

$$f \in L^0(\mathrm{X}) \qquad \Rightarrow \qquad [0,1]^2 \ni (t,s) \mapsto f \circ F_t^s \in L^0(\mathrm{X}) \text{ is continuous.} \tag{4.5}$$

Indeed, recall that the topology of $L^0(\mathrm{X})$ is metrized by the distance

$$\mathsf{d}_{L^0}(f,g) := \int 1 \wedge |f-g| \, \mathrm{d}\mathfrak{m}',$$

where $\mathfrak{m}'$ is any Borel probability measure with $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$. We thus pick $\mathfrak{m}'$ so that $\mathfrak{m}'|_B = c\mathfrak{m}|_B$ for some $c > 0$ and notice that the assumption on w_t ensures that F_t^s is the identity outside B for every $t, s \in [0,1]$. Hence the bounded compression property gives $(F_t^s)_*\mathfrak{m}' \leq C\mathfrak{m}'$ for any t, s and therefore, in analogy with (4.4), we have

$$\mathsf{d}_{L^0}(f \circ F_t^s, g \circ F_t^s) = \int 1 \wedge |f-g| \circ F_t^s \, \mathrm{d}\mathfrak{m}' = \int 1 \wedge |f-g| \, \mathrm{d}(F_t^s)_*\mathfrak{m}' \leq C \int 1 \wedge |f-g| \, \mathrm{d}\mathfrak{m}' = C\mathsf{d}_{L^0}(f,g)$$

and the claim (4.5) follows along the same lines as (4.3).

From (4.3) for $p = 2$ and by integrating (2.14) we see that

$$f \circ F_t^{s_2} - f \circ F_t^{s_1} = \int_{s_1}^{s_2} \mathrm{d}f(\boldsymbol{b}_r) \circ F_t^r \, \mathrm{d}r \qquad \mathfrak{m} - a.e., \qquad \forall f \in W^{1,2}(\mathrm{X}) \tag{4.6}$$

for any $t, s_1, s_2 \in [0,1]$ with $s_1 < s_2$, where the integral is intended in the pointwise a.e. sense (or, equivalently, in the Bochner sense). It is then also clear that for a.e. $s \in [0,1]$ we have

$$\lim_{h \to 0} \frac{f \circ F_t^{s+h} - f \circ F_t^s}{h} = \mathrm{d}f(\boldsymbol{b}_s) \circ F_t^s, \qquad \mathfrak{m} - a.e., \qquad \forall f \in W^{1,2}(\mathrm{X}), \tag{4.7}$$

the limit being in $L^2(\mathrm{X})$ and by usual maximal-type arguments it is not hard to see that the exceptional set of times may be chosen independent of $f \in W^{1,2}(\mathrm{X})$ (see e.g. the arguments in Proposition 5.3.2).

We now turn to the main regularity estimates on Regular Lagrangian Flows as established in [25] (see also the earlier [27]). We recall the definition of ap $-$ lip in (1.16). We summarize here in a more compact way the main estimates we need, that we already presented in the discussion of Section 2.2 (extracted mainly from [25, Theorem 1.6, Proposition 3.3]).

Proposition 4.1.1. *Let* $(\mathrm{X}, \mathrm{d}, \mathfrak{m})$ *be a* $\mathsf{ncRCD}(K,N)$ *space with* $K \in \mathbb{R}$, $N < \infty$, *and* $(\boldsymbol{b}_t) \in L^2([0,1], W_C^{1,2}(T\mathrm{X}))$ *be such that* $|\boldsymbol{b}_t|, \mathrm{div}(\boldsymbol{b}_t) \in L^\infty([0,1] \times \mathrm{X})$ *and for some* $\bar{x} \in \mathrm{X}$ *and* $R > 0$ *we have* $\mathrm{supp}(\boldsymbol{b}_t) \subset B_R(\bar{x})$ *for a.e.* $t \in [0,1]$.

Then there exists a nonnegative function $(g_t) \in L^2([0,1], L^2(\mathrm{X}, \mathfrak{m}))$ *such that for any* $t, s \in [0,1]$ *we have*

$$\mathrm{ap}\text{-}\mathrm{lip}\, F_t^s(x) \leq e^{\int_{t \wedge s}^{t \vee s} g_r(F_t^r(x)) \, \mathrm{d}r} \qquad \mathfrak{m} - a.e. \ x \in \mathrm{X} \tag{4.8}$$

and, for every $t \in [0,1]$, *a nonnegative function* $\bar{g}_t \in L^2_{loc}(\mathrm{X}, \mathfrak{m})$ *such that for any* $s \in [0,1]$ *we have*

$$\frac{\mathsf{d}(F_t^s(x), F_t^s(y))}{\mathsf{d}(x,y)} \leq e^{\bar{g}_t(x) + \bar{g}_t(y)}, \qquad \forall x, y \in \mathrm{X}. \tag{4.9}$$

Proof. The estimate (4.9) for $t \le s$ is the content of [25, Proposition 3.3], the case $t \ge s$ then also follows noticing that $s \mapsto F_t^{1-s}$ is the Regular Lagrangian Flow of $(-\boldsymbol{b}_t)$.

We pass to (4.8) and notice that [25, Theorem 1.6] states that for $t, s \in [0,1]$, $t \le s$, for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ we have

$$e^{-\int_t^s g_r(F_t^r(x))\,\mathrm{d}r} \le \operatorname{ap-}\varliminf_{y \to x} \frac{\mathsf{d}(F_t^s(y), F_t^s(x))}{\mathsf{d}(y,x)} \le \operatorname{ap-}\varlimsup_{y \to x} \frac{\mathsf{d}(F_t^s(y), F_t^s(x))}{\mathsf{d}(y,x)} \le e^{\int_t^s g_r(F_t^r(x))\,\mathrm{d}r}, \tag{4.10}$$

where the functions g_r satisfy the bound $\|g_r\|_{L^2} \le C(K,N,R)(\||\nabla \boldsymbol{b}_r|\|_{L^2} + \|\mathrm{div}(\boldsymbol{b}_r)\|_{L^\infty})$ (thus from the integrability assumptions on $(\boldsymbol{b}_t)$ the integrability of (g_t) as in the statement follows). In particular, this gives (4.8) for $t \le s$. For the case $t \ge s$ we assume for the moment that for $\mathfrak{m}$-a.e. x we can establish the 'change of variable formula' marked with a star in the following computation

$$\operatorname{ap-}\varliminf_{y \to x} \frac{\mathsf{d}(F_s^t(y), F_s^t(x))}{\mathsf{d}(y,x)} \overset{*}{=} \operatorname{ap-}\varliminf_{w \to F_s^t(x)} \frac{\mathsf{d}(w, F_s^t(x))}{\mathsf{d}(F_t^s(w), x)} = \frac{1}{\operatorname{ap-}\varlimsup_{w \to F_s^t(x)} \frac{\mathsf{d}(F_t^s(w),x)}{\mathsf{d}(w,F_s^t(x))}}.$$

Then (4.10) (that we apply with t, s swapped) gives

$$\operatorname{ap-}\varlimsup_{w \to F_s^t(x)} \frac{\mathsf{d}(F_t^s(w), x)}{\mathsf{d}(w, F_s^t(x))} \le e^{\int_s^t g_r(F_s^r(x))\,\mathrm{d}r} \qquad \mathfrak{m} - a.e.\ x.$$

Writing $x = F_t^s(z)$ and keeping in mind the bounded compression property (2.13) and the group property (4.1) we deduce that

$$\operatorname{ap-}\varlimsup_{w \to z} \frac{\mathsf{d}(F_t^s(w), F_t^s(z))}{\mathsf{d}(w,z)} \le e^{\int_s^t g_r(F_t^r(z))\,\mathrm{d}r} \qquad \mathfrak{m} - a.e.\ z,$$

which is (4.8) in the case $t \ge s$. Thus it remains to prove that the starred identity in the above holds for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. By the very definition of $\operatorname{ap-}\varliminf$, this will follow if we show that for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ we have that: for any $A \subset \mathrm{X}$ Borel we have that $F_s^t(x)$ is a Lebesgue point of A if and only if x is a Lebesgue point of $(F_s^t)^{-1}(A)$. Since (4.9) ensures that F_t^s and F_s^t are Lusin–Lipschitz, (4.1) that they are one the essential inverse of the other and recalling (2.13), the conclusion follows from Lemma 4.1.2 below. □

Lemma 4.1.2. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a locally doubling space,* $T, S : \mathrm{X} \to \mathrm{X}$ *Borel maps, Lusin–Lipschitz, such that* $T \circ S = \mathrm{Id}$ *and* $S \circ T = \mathrm{Id}$ $\mathfrak{m}$*-a.e. and finally so that* $T_*\mathfrak{m} \le C\mathfrak{m}$ *and* $S_*\mathfrak{m} \le C\mathfrak{m}$ *for some* $C > 0$.

Then for $\mathfrak{m}$*-a.e.* $x \in \mathrm{X}$ *the following holds: for any* $A \subset \mathrm{X}$ *Borel we have that* $T(x)$ *is a Lebesgue point of* A *if and only if* x *is a Lebesgue point of* $T^{-1}(A)$.

Proof. The assumptions about essential invertibility and bounded compression give the existence of $X_1, X_2 \subset \mathrm{X}$ Borel of full measure such that T, S are invertible bijections from X_1 to X_2 and vice versa, respectively. Also, from the Lusin–Lipschitz regularity we see that $\mathfrak{m}$-a.a. X_1 can be covered by a countable number of Borel sets B such that $T|_B$ is Lipschitz and $S|_{T(B)}$ is also Lipschitz. Fix such B, let L be a bound on the Lipschitz constants of $T|_B$ and $S|_{T(B)}$ and let $B' \subset B$ be the set of x's such that x is a Lebesgue point of B and $T(x)$ is a Lebesgue point of $T(B) = S^{-1}(B) \cap X_2$. Clearly, $\mathfrak{m}(B \setminus B') = 0$, so the conclusion will follow if we show that any $x \in B'$ satisfies the required conditions. Thus fix $x \in B'$, let $A \subset \mathrm{X}$ be Borel and $r \in (0,1)$. Assume that x is a Lebesgue point of $T^{-1}(A)$. Since $T_{|B}$ is L-Lipschitz, we have that $T(B_{r/L}(x) \cap B) \subset B_r(T(x)) \cap T(B)$ and therefore

$$\begin{aligned}\mathfrak{m}\big(B_r(T(x))\big) &\geq \mathfrak{m}\big(B_r(T(x)) \cap T(B)\big) \geq \mathfrak{m}\big(T(B_{r/L}(x) \cap B)\big) = \mathfrak{m}\big(S^{-1}(B_{r/L}(x) \cap B)\big) \\ &= S_*\mathfrak{m}\big(B_{r/L}(x) \cap B\big) \geq C^{-1}\mathfrak{m}\big(B_{r/L}(x) \cap B\big) \geq C^{-1}D^{-(2\log_2(L)+1)}\mathfrak{m}\big(B_{rL}(x) \cap B\big),\end{aligned} \tag{4.11}$$

where D is the local doubling constant at x and we used the bound $S_*\mathfrak{m} \geq C^{-1}\mathfrak{m}$ (which follows taking S_* on both sides of $T_*\mathfrak{m} \leq C\mathfrak{m}$). On the other hand we have

$$\mathfrak{m}\big(B_r(T(x)) \setminus A\big) \leq \mathfrak{m}\big(B_r(T(x)) \setminus (A \cap T(B))\big) = \mathfrak{m}\big(B_r(T(x)) \setminus T(B)\big) + \mathfrak{m}_{|T(B)}\big(B_r(T(x)) \setminus A\big)$$

and since we assumed $T(x)$ to be a Lebesgue point for $T(B)$, when we divide the first addend in the rightmost side by $\mathfrak{m}(B_r(T(x)))$ and let $r \downarrow 0$ it converges to 0. For the other addend we have the estimate

$$\begin{aligned}\mathfrak{m}_{|T(B)}\big(B_r(T(x)) \setminus A\big) &\leq C(T_*\mathfrak{m})_{|T(B)}\big(B_r(T(x)) \setminus A\big) \\ &= C\mathfrak{m}\big(T^{-1}\big(T(B) \cap B_r(T(x))\big) \setminus T^{-1}(A)\big) \\ &= C\mathfrak{m}\big(S\big(T(B) \cap B_r(T(x))\big) \setminus T^{-1}(A)\big) \\ &\leq C\mathfrak{m}\big(B_{Lr}(x) \setminus T^{-1}(A)\big).\end{aligned}$$

Thus recalling (4.11) we get

$$\overline{\lim_{r\downarrow 0}} \frac{\mathfrak{m}\big(B_r(T(x)) \setminus A\big)}{\mathfrak{m}\big(B_r(T(x))\big)} \leq C^2 D^{2\log_2(L)+1} \overline{\lim_{r\downarrow 0}} \frac{\mathfrak{m}\big(B_{Lr}(x) \setminus T^{-1}(A)\big)}{\mathfrak{m}(B_{Lr}(x))} \frac{\mathfrak{m}(B_{Lr}(x))}{\mathfrak{m}\big(B_{Lr}(x) \cap B\big)} = 0,$$

having used the assumption that x is a Lebesgue point of B and $T^{-1}(A)$. This proves that $T(x)$ is a Lebesgue point of A. The converse implication is proved analogously. □

Remark 4.1.3. The results in [25] are based on the slightly less stringent assumption that there is an L^2 control over only the symmetric part of the covariant derivative. We phrased the result in this weaker formulation because in any case we will need a control on the full covariant derivative later on (when discussing the properties of the convective derivative introduced in Definition 5.2.5).

Notice also that in Proposition 4.1.1 one needs to assume L^2 integrability in time, rather than the L^1 integrability which is sufficient in Theorem 2.2.8. We shall therefore make this assumption throughout the manuscript. In any case, notice that by a simple reparametrization argument one can always reduce to the case of vector fields bounded in $W^{1,2}_C$ at the only price of reparametrizing the flow in time. ■

It is clear from (4.9) that

$$\text{for any } c > 0 \text{ and } t \in [0,1] \text{ the restriction of } F_t^s \text{ to } \{\bar{g}_t \leq c\} \text{ is Lipschitz, uniformly in } s \in [0,1]. \tag{4.12}$$

4.2 Differential of Lusin–Lipschitz maps

In this section, inspired by some discussions in [55], [59] we develop a language for the differential of a Lusin–Lipschitz and invertible map $\varphi\colon \mathrm{X} \to \mathrm{X}$ such that $\varphi_*\mathfrak{m} \le C\mathfrak{m}$ for some $C > 0$. The results will be applied to the flow maps (F_t^s).

We start noticing that if $f : \mathrm{X} \to \mathbb{R}$ is Lusin–Lipschitz and (E_j) is a Borel partition of $\mathfrak{m}$-a.a. X such that $f|_{E_j}$ is Lipschitz for every j, then the formula

$$\mathrm{d}f := \sum_{j\in\mathbb{N}} \chi_{E_j}\mathrm{d}g_j, \qquad \text{where } g_j \in \mathrm{LIP}(\mathrm{X}) \text{ is equal to } f \text{ on } E_j \text{ for every } j$$

defines an element of $L^0(T\mathrm{X})$ that, by locality of the differential, is independent of the particular functions g_j and Borel sets E_j as above. We shall refer to $\mathrm{d}f$ as the differential of f and notice that this definition poses no ambiguity as, again by locality, for f Sobolev and Lusin–Lipschitz the definition above produces the same differential of f as defined in [55].

We then notice the following simple lemma.

Lemma 4.2.1. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be locally uniformly doubling. Let* $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ *be a complete space and* $\varphi : \mathrm{X} \to \mathrm{Y}$ *a Lusin–Lipschitz map. Let* $(E_i)_{i\in\mathbb{N}}$ *be a Borel partition of* X *up to* $\mathfrak{m}$*-null sets such that* $\varphi|_{E_i}$ *is a Lipschitz map for every* $i \in \mathbb{N}$. *Then*

$$\text{ap-lip}\,\varphi = \sum_{i=1}^{\infty} \chi_{E_i}\mathrm{lip}(\varphi|_{E_i}) \qquad \mathfrak{m} - a.e.. \tag{4.13}$$

Proof. Let $i \in \mathbb{N}$ be fixed. Let $x \in E_i$ be a density point of E_i; recall that $\mathfrak{m}$-a.e. point of E_i has this property. As observed, e.g., in the paragraph following [65, Eq. (2.6)], the quantity $\text{ap-lip}\,\varphi(x)$ is independent of the behaviour of φ outside E_i. Therefore, [65, Proposition 2.5] grants that $\text{ap-lip}\,\varphi(x) = \mathrm{lip}(\varphi|_{E_i})(x)$, whence (4.13) follows. □

We come to the definition of differential of a Lusin–Lipschitz map. In what follows, it will be useful to notice that if $\varphi : \mathrm{X} \to \mathrm{Y}$ is Lusin–Lipschitz with $\varphi_*\mathfrak{m}_\mathrm{X} \ll \mathfrak{m}_\mathrm{Y}$ and $f : \mathrm{Y} \to \mathbb{R}$ is also Lusin–Lipschitz, then $f \circ \varphi : \mathrm{X} \to \mathbb{R}$ is Lusin–Lipschitz as well. Indeed, let (F_i) (resp. (E_j)) be a Borel partition of $\mathfrak{m}_\mathrm{Y}$-a.a. Y (resp. $\mathfrak{m}_\mathrm{X}$-a.a. X) such that $f|_{F_i}$ (resp. $\varphi|_{E_j}$) is Lipschitz for every i (resp. j). Then since $\varphi_*\mathfrak{m}_\mathrm{X} \ll \mathfrak{m}_\mathrm{Y}$, we have that $(E_j \cap \varphi^{-1}(F_i))$ is a Borel partition of $\mathfrak{m}_\mathrm{X}$-a.a. X such that $f \circ \varphi|_{E_j\cap\varphi^{-1}(F_i)}$ is Lipschitz for every i, j.

In particular, the right hand side in formula (4.14) below makes sense.

Theorem 4.2.2 (Differential of a Lusin–Lipschitz map)**.** *Let* $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$, $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ *be metric measure spaces. Suppose* $(\mathrm{X}, \mathsf{d}_\mathrm{X})$ *is geodesic,* $\mathfrak{m}_\mathrm{X}$ *is locally uniformly doubling, and* $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ *is a PI space. Let* $\varphi\colon \mathrm{X} \to \mathrm{Y}$ *be an essentially invertible Lusin–Lipschitz map such that* $\varphi_*\mathfrak{m}_\mathrm{X} \ll \mathfrak{m}_\mathrm{Y}$ *and* $\varphi^{-1}_*\mathfrak{m}_\mathrm{Y} \ll \mathfrak{m}_\mathrm{X}$, *where* φ^{-1} *is an essential inverse of* φ *(that is uniquely defined up to* $\mathfrak{m}_\mathrm{Y}$*-negligible sets).*

Then there exists a unique linear and continuous operator $\mathrm{d}\varphi\colon L^0(T\mathrm{X}) \to L^0(T\mathrm{Y})$ *such that*

$$\mathrm{d}f\big(\mathrm{d}\varphi(v)\big) \circ \varphi = \mathrm{d}(f \circ \varphi)(v) \quad \textit{holds } \mathfrak{m}_\mathrm{X}\textit{-a.e. on } \mathrm{X}, \tag{4.14}$$

for every $v \in L^0(T\mathrm{X})$ *and* $f : \mathrm{Y} \to \mathbb{R}$ *Lusin–Lipschitz.*

Moreover, it holds that

$$\big|\mathrm{d}\varphi(v)\big|\circ\varphi\le \mathrm{ap\text{-}lip}\,\varphi\,|v|\quad \mathfrak{m}_{\mathrm{X}}\text{-}a.e.\ on\ \mathrm{X},\qquad for\ every\ v\in L^0(T\mathrm{X}). \tag{4.15}$$

Proof. For $f:\mathrm{Y}\to\mathbb{R}$ Lusin–Lipschitz and $v\in L^0(T\mathrm{X})$ we define $T_v(f)\in L^0(T\mathrm{Y})$ as

$$T_v(f):=\big(\mathrm{d}(f\circ\varphi)(v)\big)\circ\varphi^{-1}$$

(notice that $\mathrm{d}(f\circ\varphi)(v)$ is in $L^0(\mathrm{X})$, so the above defines a function in $L^0(\mathrm{Y})$ thanks to the assumption $\varphi_*^{-1}\mathfrak{m}_{\mathrm{Y}}\ll\mathfrak{m}_{\mathrm{X}}$). Notice that since, as previously remarked, $f\circ\varphi$ is Lusin–Lipschitz, the right hand side of the above is well defined. We claim that for any Borel partition $(E_j)_j$ of $\mathfrak{m}_{\mathrm{X}}$-a.a. X such that $\varphi|_{E_j}$ is Lipschitz for all $j\in\mathbb{N}$ we have

$$|T_v(f)|\le\Big(|v|\sum_{j\in\mathbb{N}}\mathrm{lip}(\varphi|_{E_j})\chi_{E_j}\Big)\circ\varphi^{-1}|\mathrm{d}f|\qquad \mathfrak{m}_{\mathrm{X}}-a.e.. \tag{4.16}$$

To see this we start noticing that $|T_v(f)|\le|v|\circ\varphi^{-1}|\mathrm{d}(f\circ\varphi)|\circ\varphi^{-1}$, so the claim will follow if we show that

$$|\mathrm{d}(f\circ\varphi)|\le|\mathrm{d}f|\circ\varphi\Big(\sum_{j\in\mathbb{N}}\mathrm{lip}(\varphi|_{E_j})\chi_{E_j}\Big). \tag{4.17}$$

To see this, let (F_i) be a Borel partition of $\mathfrak{m}_{\mathrm{Y}}$-a.a. Y such that $f|_{F_i}$ is Lipschitz for every i and recall that $(E_j\cap\varphi^{-1}(F_i))$ is a Borel partition of $\mathfrak{m}_{\mathrm{X}}$-a.a. X such that $f\circ\varphi|_{E_j\cap\varphi^{-1}(F_i)}$ is Lipschitz for every i,j. For every $i,j\in\mathbb{N}$ let $h_i:\mathrm{Y}\to\mathbb{R}$ be Lipschitz and equal to f on F_i and $g_{i,j}:\mathrm{X}\to\mathbb{R}$ be Lipschitz and equal to $f\circ\varphi$ on $E_j\cap\varphi^{-1}(F_i)$.

Note that since $\mathfrak{m}_{\mathrm{X}}$ is locally doubling we have $\mathrm{lip}(g_{i,j})=\mathrm{lip}(g_{i,j}|_{E_j\cap\varphi^{-1}(F_i)})$ $\mathfrak{m}_{\mathrm{X}}$-a.e. on $E_j\cap\varphi^{-1}(F_i)$ (see e.g. [65]) and since $(\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\mathfrak{m}_{\mathrm{Y}})$ is PI we have $\mathrm{lip}(h_i)=|\mathrm{d}h_i|$ $\mathfrak{m}_{\mathrm{Y}}$-a.e. on Y (see [34]). Then $\mathfrak{m}_{\mathrm{X}}$-a.e. on $E_j\cap\varphi^{-1}(F_i)$ we have

$$\begin{aligned}|\mathrm{d}(f\circ\varphi)|&=|\mathrm{d}g_{i,j}|\le\mathrm{lip}(g_{i,j})=\mathrm{lip}(g_{i,j}|_{E_j\cap\varphi^{-1}(F_i)})=\mathrm{lip}\big((f\circ\varphi)|_{E_j\cap\varphi^{-1}(F_i)}\big)\\&\le\mathrm{lip}(\varphi|_{E_j\cap\varphi^{-1}(F_i)})\mathrm{lip}(f|_{\varphi(E_j\cap\varphi^{-1}(F_i))})\circ\varphi\le\mathrm{lip}(\varphi|_{E_j})\mathrm{lip}(f|_{F_i})\circ\varphi\\&\le\mathrm{lip}(\varphi|_{E_j})\mathrm{lip}(h_i)\circ\varphi=\mathrm{lip}(\varphi|_{E_j})|\mathrm{d}h_i|\circ\varphi=\mathrm{lip}(\varphi|_{E_j})|\mathrm{d}f|\circ\varphi\end{aligned}$$

whence (4.17) - and thus also (4.16) - follows. From (4.16) and the linearity of T_v it follows that if $\mathrm{d}f=\mathrm{d}f'$ on some Borel set $E\subset\mathrm{X}$, then $T_v(f)=T_v(f')$ on E as well. Therefore the operator L_v : {differentials of Lusin–Lipschitz functions on Y} $\to L^0(\mathrm{Y})$ defined by

$$L_v(\mathrm{d}f):=T_v(f)$$

is well defined and satisfies $L_v(\chi_E\mathrm{d}f)=\chi_E L_v(\mathrm{d}f)$. Also, since (4.16) holds for any partition (E_j) as above, from (4.13) we see that

$$|L_v(\mathrm{d}f)|\le\big(|v|\,\mathrm{ap\text{-}lip}\,\varphi\big)\circ\varphi^{-1}\,|\mathrm{d}f|. \tag{4.18}$$

Now notice that $W^{1,2}(\mathrm{Y})$ is reflexive (see [34]), thus $\mathrm{LIP}(\mathrm{Y})\cap W^{1,2}(\mathrm{Y})$ is strongly dense in $W^{1,2}(\mathrm{Y})$ (see [3]), therefore $L^0(T^*\mathrm{Y})$ is generated by $\big\{\mathrm{d}f\ :\ f\in\mathrm{LIP}(\mathrm{Y})\big\}$ and so the set {differentials of Lusin–Lipschitz functions on Y} is dense in $L^0(T^*\mathrm{Y})$. Also, from (4.18) it is easy to see (see also the arguments in Section 5.1 that lead to (4.60)) that L_v is uniformly continuous, hence it can be uniquely extended to a continuous map, still denoted L_v, from $L^0(T^*\mathrm{Y})$ to $L^0(\mathrm{Y})$ and this extension satisfies

$$|L_v(\omega)| \le \big(|v|\,\mathrm{ap\text{-}lip}\,\varphi\big)\circ\varphi^{-1}\,|\omega| \qquad \mathfrak{m}_{\mathrm{Y}} - a.e.,\ \forall\omega\in L^0(T^*\mathrm{Y}). \tag{4.19}$$

It is clear from the previous considerations that L_v is also $L^0(\mathrm{Y})$-linear, and thus an element of $L^0(T^*\mathrm{Y})^*$. We denote by $\mathrm{d}\varphi(v)$ the element of $L^0(T\mathrm{Y})$ corresponding to $L_v \in L^0(T^*\mathrm{Y})^*$, by applying Riesz representation theorem for $L^0(\mathfrak{m})$-normed modules (Proposition 1.2.20).

It is clear that the map $L^0(T\mathrm{X}) \ni v \mapsto \mathrm{d}\varphi(v) \in L^0(T\mathrm{Y})$ is linear and that this assignment is the only one satisfying (4.14). Finally, (4.15) follows from (4.19). □

The pointwise norm $|\mathrm{d}\varphi| \in L^0(\mathrm{X})$ of the differential $\mathrm{d}\varphi$ introduced in Theorem 4.2.2 is defined as follows:

$$|\mathrm{d}\varphi| := \mathfrak{m} - \operatorname*{ess\,sup}_{\substack{v\in L^0(T\mathrm{X}):\\ |v|\le 1\ \mathfrak{m}_{\mathrm{X}}\text{-a.e.}}} \big|\mathrm{d}\varphi(v)\big|\circ\varphi. \tag{4.20}$$

Proposition 4.2.3 (Basic properties of the differential)**.** *With the same assumptions on* X, Y, φ *as in Theorem 4.2.2 the following holds.*

We have

$$|\mathrm{d}\varphi| \le \mathrm{ap\text{-}lip}\ \varphi \quad \mathfrak{m}_{\mathrm{X}} - a.e. \tag{4.21}$$

and for $v \in L^0(T\mathrm{X})$ *the bound*

$$|\mathrm{d}\varphi(v)|\circ\varphi \le |\mathrm{d}\varphi||v| \quad \mathfrak{m}_{\mathrm{X}} - a.e.. \tag{4.22}$$

Moreover, for $f : \mathrm{Y} \to \mathbb{R}$ *Lusin–Lipschitz, the function* $f\circ\varphi$ *is also Lusin–Lipschitz and we have*

$$\big|\mathrm{d}(f\circ\varphi)\big| \le |\mathrm{d}f|\circ\varphi\,|\mathrm{d}\varphi|, \qquad \mathfrak{m}_{\mathrm{X}} - a.e.. \tag{4.23}$$

Also, for $v \in L^0(T\mathrm{X})$ *and* $h\in L^0(\mathrm{X})$ *we have the identity*

$$\mathrm{d}\varphi(hv) = h\circ\varphi^{-1}\,\mathrm{d}\varphi(v). \tag{4.24}$$

Finally, if $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$ *is another PI space and* $\psi : \mathrm{Y}\to\mathrm{Z}$ *is Lusin–Lipschitz, essentially invertible and such that* $\psi_*\mathfrak{m}_{\mathrm{Y}} \ll \mathfrak{m}_{\mathrm{Z}}$, $\psi_*^{-1}\mathfrak{m}_{\mathrm{Z}} \ll \mathfrak{m}_{\mathrm{Y}}$, *then* $\psi\circ\varphi\colon \mathrm{X}\to\mathrm{Z}$ *is Lusin–Lipschitz and satisfies the identity*

$$\mathrm{d}(\psi\circ\varphi) = \mathrm{d}\psi\circ\mathrm{d}\varphi \tag{4.25}$$

and the bound

$$\big|\mathrm{d}(\psi\circ\varphi)\big| \le |\mathrm{d}\psi|\circ\varphi\,|\mathrm{d}\varphi| \quad \mathfrak{m}_{\mathrm{X}} - a.e.. \tag{4.26}$$

Proof. The bound (4.21) follows directly from the definition and the estimate (4.15), while (4.24) is a direct consequence of the defining property (4.14). For (4.22) we put $v' := |v|^{-1}v$, where $|v|^{-1}$ is intended to be 0 on $\{v = 0\}$, notice that $|v'| \le 1$ $\mathfrak{m}_{\mathrm{X}}$-a.e. and $v = |v|v'$, thus taking (4.24) into account we get

$$|\mathrm{d}\varphi(v)|\circ\varphi = \big|\mathrm{d}\varphi(|v|v')\big|\circ\varphi = |v|\,|\mathrm{d}\varphi(v')|\circ\varphi \le |v|\,|\mathrm{d}\varphi| \qquad \mathfrak{m}_{\mathrm{X}} - a.e.,$$

having used the definition of $|\mathrm{d}\varphi|$ in the last inequality.

To prove (4.23) we notice that the Lusin–Lipschitz regularity of $f\circ\varphi$ has already been established before Theorem 4.2.2, then for every $v\in L^0(T\mathrm{X})$ we have

$$|\mathrm{d}(f\circ\varphi)(v)|\overset{(4.14)}{=}|\mathrm{d}f(\mathrm{d}\varphi(v))|\circ\varphi\le|\mathrm{d}f|\circ\varphi|\mathrm{d}\varphi(v)|\circ\varphi\overset{(4.22)}{\le}|\mathrm{d}f|\circ\varphi\,|\mathrm{d}\varphi|\,|v|$$

and (4.23) follows from the arbitrariness of v.

For the last claim, we notice that the fact that $\psi\circ\varphi$ is Lusin–Lipschitz can be proved as we did for the function $f\circ\varphi$ before Theorem 4.2.2. Now notice that for $f:\mathrm{Z}\to\mathbb{R}$ Lusin–Lipschitz, the maps $f\circ\psi$ and $f\circ\psi\circ\varphi$ are Lusin–Lipschitz, therefore for any $v\in L^0(T\mathrm{X})$ we have

$$\mathrm{d}f(\mathrm{d}\psi(\mathrm{d}\varphi(v)))\circ(\psi\circ\varphi)\overset{(4.14)}{=}\mathrm{d}(f\circ\psi)(\mathrm{d}\varphi(v))\circ\varphi\overset{(4.14)}{=}\mathrm{d}(f\circ\psi\circ\varphi)(v).$$

According to Theorem 4.2.2, this is sufficient to prove (4.25). Finally, let $v\in L^0(T\mathrm{X})$ and notice that

$$|\mathrm{d}(\psi\circ\varphi)(v)|\circ\psi\circ\varphi\overset{(4.25)}{=}|\mathrm{d}\psi(\mathrm{d}\varphi(v))|\circ\psi\circ\varphi\overset{(4.22)}{\le}\big(|\mathrm{d}\psi||\mathrm{d}\varphi(v)|\big)\circ\varphi\overset{(4.22)}{\le}|\mathrm{d}\psi|\circ\varphi|\mathrm{d}\varphi||v|,$$

thus (4.26) follows from the very definition (4.20). □

Remark 4.2.4. In [55] the concept of differential for a map φ between metric measure spaces has been introduced under the assumptions that φ is Lipschitz, essentially invertible and such that $\varphi_*\mathfrak{m}_\mathrm{X}\le C\mathfrak{m}_\mathrm{Y}$ and $\varphi_*^{-1}\mathfrak{m}_\mathrm{Y}\le C\mathfrak{m}_\mathrm{X}$ for some $C>0$ (in fact, the existence of the inverse was not really needed in [55], but in the general case one has to work with pullback modules). In this case, $\mathrm{d}\varphi:L^2(T\mathrm{X})\to L^2(T\mathrm{Y})$ was defined as the only linear continuous operator such that

$$\mathrm{d}f\big(\mathrm{d}\varphi(v)\big)=\mathrm{d}(f\circ\varphi)(v)\circ\varphi^{-1}\quad\mathfrak{m}_\mathrm{Y}\text{-a.e.},\qquad\text{for every }f\in W^{1,2}(\mathrm{Y})\text{ and }v\in L^2(T\mathrm{X}).\tag{4.27}$$

Then under the assumptions of Theorem 4.2.2 it is clear that the above defines the same object as the one given by (4.14). Indeed, from (4.27) and the fact that Lipschitz functions are locally Sobolev we see that (4.14) holds for f Lipschitz, and then by locality for f Lusin–Lipschitz. Conversely, once we know (4.14) we have that (4.27) holds at least for f Lipschitz, and the fact that it holds for f Sobolev follows from the same density arguments used in proving Theorem 4.2.2.

Finally, the fact that under the current assumptions $\mathrm{d}\varphi$ maps $L^2(T\mathrm{X})$ to $L^2(T\mathrm{Y})$ follows from the bound (4.15) and $\varphi_*\mathfrak{m}_\mathrm{X}\le C\mathfrak{m}_\mathrm{Y}$. ■

Notice that under the assumptions of Proposition 4.1.1, we know from (4.12) that F_t^s is Lusin–Lipschitz, from (4.1) that it is essentially invertible, so that keeping into account the bounded compression property (2.13) we see by Theorem 4.2.2 that the differential $\mathrm{d}F_t^s$ of F_t^s is a well defined map from $L^0(T\mathrm{X})$ into itself.

For us, the following estimate is of crucial importance:

Proposition 4.2.5. *With the same assumptions and notation as in Proposition 4.1.1, we have*

$$|\mathrm{d}F_t^s|\circ F_{t'}^t\le\exp\Big(\int_{t\wedge s}^{t\vee s}g_r\circ F_{t'}^r\,\mathrm{d}r\Big),\qquad\mathfrak{m}-a.e.\quad\forall t',t,s\in[0,1].\tag{4.28}$$

Proof. We have

$$|\mathrm{d}F_t^s|(F_{t'}^t(x)) \overset{(4.15)}{\leq} (\text{ap-lip}\, F_t^s)(F_{t'}^t(x)) \overset{(4.8)}{\leq} e^{\int_{t\wedge s}^{t\vee s} g_r \circ F_t^r \,\mathrm{d}r}(F_{t'}^t(x)) \overset{(4.1)}{=} e^{\int_{t\wedge s}^{t\vee s} g_r(F_{t'}^r(x))\,\mathrm{d}r}$$

for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$, having used the fact that $(F_{t'}^t)_*\mathfrak{m} \ll \mathfrak{m}$ to justify the precomposition with $F_{t'}^t$ in the above. □

4.3 Weighted Hajłasz–Sobolev space

In [70], a notion of Sobolev function defined over a metric measure space (in fact the first one) was studied. We study here, with a similar approach, a space of functions which satisfy a weaker condition.

Definition 4.3.1 (The space $H_{\phi,R}(\mathrm{X})$)**.** *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space, $R>0$ and $\phi \in L^0(\mathrm{X})$ non-negative be fixed. Put for brevity $F_\phi(x,y) := e^{\phi(x)+\phi(y)}$. Given any $f \in L^2(\mathrm{X})$, we say that a function $G \in L^2(\mathrm{X})$, $G \geq 0$ is* admissible for f *provided there exists a $\mathfrak{m}$-negligible Borel set $N \subseteq \mathrm{X}$ such that*

$$\big|f(x)-f(y)\big| \leq F_\phi(x,y)\big(G(x)+G(y)\big)\mathsf{d}(x,y) \quad \text{for every } x,y \in \mathrm{X}\setminus N,\ \mathsf{d}(x,y) \leq R. \tag{4.29}$$

We call $A_{\phi,R}(f)$ the family of all admissible functions for f. Then the space $H_{\phi,R}(\mathrm{X})$ is given by

$$H_{\phi,R}(\mathrm{X}) := \big\{f \in L^2(\mathrm{X}) \;\big|\; A_{\phi,R}(f) \neq \emptyset\big\}.$$

*We say that $H_{\phi,R}(\mathrm{X})$ is the ϕ-*weighted Hajłasz–Sobolev space *on* X *at scale R and we endow it with the norm*

$$\|f\|_{H_{\phi,R}(\mathrm{X})} := \sqrt{\|f\|_{L^2}^2 + \inf_{G \in A_{\phi,R}(f)} \|G\|_{L^2}^2} \quad \text{for every } f \in H_{\phi,R}(\mathrm{X}). \tag{4.30}$$

Given any function $f \in H_{\phi,R}(\mathrm{X})$, it is easy to see that $A_{\phi,R}(f)$ is convex and closed in $L^2(\mathrm{X})$, thus it admits a unique element of minimal L^2-norm, that we call the optimal function for f.

Proposition 4.3.2. *$(H_{\phi,R}(\mathrm{X}),\|\cdot\|_{H_{\phi,R}(\mathrm{X})})$ is a Banach space. Moreover, if $f_n \rightharpoonup f$, $G_n \rightharpoonup G$ in $L^2(\mathrm{X})$ and $G_n \in A_{\phi,R}(f_n)$ for every $n \in \mathbb{N}$, then $G \in A_{\phi,R}(f)$. Finally, we have*

$$f_n \overset{L^2}{\rightharpoonup} f \qquad \Rightarrow \qquad \|f\|_{H_{\phi,R}(\mathrm{X})} \leq \varliminf_{n\to\infty} \|f_n\|_{H_{\phi,R}(\mathrm{X})} \tag{4.31}$$

(where as customary $\|f\|_{H_{\phi,R}}$ is set to be $+\infty$ if $f \notin H_{\phi,R}(\mathrm{X})$).

Proof. The triangle inequality follows from the implication '$F \in A_{\phi,R}(f)$, $G \in A_{\phi,R}(g)$ imply $F+G \in A_{\phi,R}(f+g)$', which is easy to prove. The other properties of the norm are trivial.

We turn to the second statement and start noticing that by Mazur's lemma we can find, for every $n \in \mathbb{N}$, non-negative coefficients $\{\alpha_k^n\}_{k=n}^{N(n)}$ with $\sum_{k=n}^{N(n)} \alpha_k^n = 1$ such that $\tilde f_n := \sum_{k=n}^{N(n)} \alpha_k^n f_k \to f$ and $\tilde G_n := \sum_{k=n}^{N(n)} \alpha_k^n G_k \to G$ strongly in $L^2(\mathrm{X})$. It is clear from our first claim that $\sum_{k=n}^{N(n)} \alpha_k^n G_k \in A_{\phi,R}\big(\sum_{k=n}^{N(n)} \alpha_k^n f_k\big)$ for all $n \in \mathbb{N}$. Possibly taking a further subsequence, we also have that $\tilde f_n \to f$ and $\tilde G_n \to G$ pointwise $\mathfrak{m}$-a.e. respectively, whence by letting $n \to \infty$ for every x,y outside of a

$\mathfrak{m}$-null set in the inequality $\big|\tilde f_n(x)-\tilde f_n(y)\big|\leq F_\phi(x,y)\big(\tilde G_n(x)+\tilde G_n(y)\big)\mathsf{d}(x,y)$, we get $G\in A_{\phi,R}(f)$, as claimed.

Now the L^2-lower semicontinuity of the $H_{\phi,R}$-norm stated in (4.31) is clear: we can assume that the $\underline{\lim}$ is a finite limit, then we pick $G_n\in A_{\phi,R}(f_n)$ such that $\|f_n\|^2_{L^2}+\|G_n\|^2_{L^2}\leq\|f_n\|^2_{H_{\phi,R}}+\frac{1}{n}$ and observe that up to passing to a subsequence, we have $G_n\rightharpoonup G$ in L^2 for some $G\in L^2$. Then (4.31) follows by what already proved.

Finally, the completeness of $H_{\phi,R}(\mathrm{X})$ is now a standard consequence of (4.31): let (f_n) be $H_{\phi,R}(\mathrm{X})$-Cauchy and f its L^2-limit (which exists because the $H_{\phi,R}$-norm is bigger than the L^2-norm and $L^2(\mathfrak{m})$ is complete). Then we have

$$\varlimsup_{n\to\infty}\|f-f_n\|_{H_{\phi,R}(\mathrm{X})}\overset{(4.31)}{\leq}\varlimsup_{n\to\infty}\varliminf_{m\to\infty}\|f_m-f_n\|_{H_{\phi,R}(\mathrm{X})}=0,$$

where in the last step we used the fact that (f_n) is $H_{\phi,R}(\mathrm{X})$-Cauchy. □

The following is easily verified:

Proposition 4.3.3. *Every $f\in H_{\phi,R}(\mathrm{X})$ has the Lusin–Lipschitz property and*

$$|\mathrm{d}f|\leq 2e^{2\phi}G\qquad \mathfrak{m}-a.e.. \tag{4.32}$$

Proof. Let $f\in H_{\phi,R}(\mathrm{X})$ and $G\in A_{\phi,R}(f)$ be fixed. Pick any $\mathfrak{m}$-null set N satisfying (4.29). Given any $a,b\in\mathbb{Q}\cap(0,\infty)$, we define $E_{a,b}:=\{\phi\leq a\}\cap\{G\leq b\}\setminus N$. Then from (4.29) we see that

$$\big|f(x)-f(y)\big|\leq 2be^{2a}\mathsf{d}(x,y)\quad\text{ for every } x,y\in E_{a,b}\text{ with }\mathsf{d}(x,y)\leq R,$$

proving that $f|_{E_{a,b}}$ is locally Lipschitz and that $|\mathrm{d}f|\leq 2be^{2a}$ $\mathfrak{m}$-a.e. on $E_{a,b}$. The conclusion follows by the arbitrariness of a,b. □

Remark 4.3.4 (Weighted normed modules)**.** Fix a Radon measure μ on (X,d) such that $\mu\ll\mathfrak{m}$. Denote by $\pi_\mu\colon L^0(\mathfrak{m})\to L^0(\mu)$ the the canonical projection map sending the $\mathfrak{m}$-a.e. equivalence class of a Borel function to its μ-a.e. equivalence class. Given a $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module $\mathscr{M}^0$, we define

$$\mathscr{M}^0_\mu:=\mathscr{M}^0/\sim_\mu,\quad\text{ where } v\sim_\mu w\text{ if and only if }\pi_\mu(|v-w|)=0\text{ holds }\mu\text{-a.e. on X.} \tag{4.33}$$

The resulting set $\mathscr{M}^0_\mu$ can be endowed with a natural structure of $L^0(\mu)$-normed $L^0(\mu)$-module.

Moreover, given a $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{M}$, we define

$$\mathscr{M}_\mu:=\big\{v\in\mathscr{M}^0_\mu\;:\;|v|\in L^2(\mu)\big\},$$

where $\mathscr{M}^0$ stands for the $L^0(\mathfrak{m})$-completion of $\mathscr{M}$. The space $\mathscr{M}_\mu$ inherits a natural structure of $L^2(\mu)$-normed $L^\infty(\mu)$-module. One can readily check that

$$\mathscr{M}\text{ is Hilbert}\qquad\Longrightarrow\qquad\mathscr{M}_\mu\text{ is Hilbert.} \tag{4.34}$$

When $\mathscr{M}$ is the cotangent module $L^2(T^*\mathrm{X})$, we write $L^2(T^*\mathrm{X},\mu)$ in place of $L^2(T^*\mathrm{X})_\mu$. ■

The following technical proposition will be of crucial importance in the study of the regularity properties of $s\mapsto f\circ F^s_t$ that will be performed in Section 5.3:

Proposition 4.3.5 (Closure of d on bounded subsets of $H_{\phi,R}(\mathrm{X})$)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an infinitesimally Hilbertian metric measure space,* $\phi\in L^0(\mathrm{X})$ *non-negative, and* $R>0$*. Let* $(f_n)_n\subset H_{\phi,R}(\mathrm{X})$ *be a bounded sequence such that* $f_n\to f$ *in* $L^2(\mathrm{X})$ *and* $\mathrm{d}f_n\to\omega$ *in* $L^0(T^*\mathrm{X})$*, for some* $f\in L^2(\mathrm{X})$ *and* $\omega\in L^0(T^*\mathrm{X})$ *respectively.*

Then $f\in H_{\phi,R}(\mathrm{X})$ *and* $\omega=\mathrm{d}f$*.*

Proof. The fact that $f\in H_{\phi,R}(\mathrm{X})$ (with $\|f\|_{H_{\phi,R}(\mathrm{X})}\le\underline{\lim}_n\|f_n\|_{H_{\phi,R}(\mathrm{X})}$) follows from Proposition 4.3.2. To prove that $\omega=\mathrm{d}f$, we start picking $(G_n)_n$ in $L^2(\mathrm{X})$ such that $G_n\in A_{\phi,R}(f_n)$ and $\|f_n\|^2_{L^2(\mathrm{X})}+\|G_n\|^2_{L^2(\mathrm{X})}\le\|f_n\|^2_{H_{\phi,R}(\mathrm{X})}+1/n$ for every $n\in\mathbb{N}$. Then up to a not relabeled subsequence, we have that $G_n\rightharpoonup G$ weakly in $L^2(\mathfrak{m})$, for some $G\in L^2(\mathrm{X})$. We apply Mazur's lemma once again to find, for every $n\in\mathbb{N}$, non-negative coefficients $\{\alpha^n_k\}_{k=n}^{N(n)}$ with $\sum_{k=n}^{N(n)}\alpha^n_k=1$ such that $\tilde G_n:=\sum_{k=n}^{N(n)}\alpha^n_kG_k\to G$ strongly in $L^2(\mathrm{X})$. Putting $\tilde f_n:=\sum_{k=n}^{N(n)}\alpha^n_kf_k$ it is clear that $\tilde f_n\to f$ in $L^2(\mathrm{X})$ and $\tilde G_n\in A_{\phi,R}(\tilde f_n)$ for all $n\in\mathbb{N}$. We claim that $\mathrm{d}\tilde f_n\rightharpoonup\omega$ in $L^2(T^*\mathrm{X},\tilde{\mathfrak{m}})$, where $\tilde{\mathfrak{m}}:=e^{-4\phi}\mathfrak{m}$. To see this observe that (4.32) tells that

$$\|\mathrm{d}f\|_{L^2(T^*\mathrm{X},\tilde{\mathfrak{m}})}\le 2\|G\|_{L^2(\mathrm{X},\mathfrak{m})}\qquad\text{if }G\in A_{\phi,R}(f).$$

Thus from our assumptions it follows that $(\mathrm{d}f_n)$ is a bounded sequence in $L^2(T^*\mathrm{X},\tilde{\mathfrak{m}})$, hence up to a non-relabeled subsequence it converges weakly in such space to some $\tilde\omega$; here, we are using the fact that $L^2(T^*\mathrm{X},\tilde{\mathfrak{m}})$ is Hilbert and thus reflexive, cf. (4.34).

Since we also have $\mathrm{d}f_n\to\omega$ in $L^0(T^*\mathrm{X})$ it is clear that $\tilde\omega=\omega$, showing in particular that the weak limit $\tilde\omega$ does not depend on the subsequence chosen. To get the claim notice that since the sequence $(\mathrm{d}\tilde f_n)$ is made of convex combinations of the $\mathrm{d}f_n$'s, we also have that

$$\mathrm{d}\tilde f_n\rightharpoonup\omega\qquad\text{in }L^2(T^*\mathrm{X},\tilde{\mathfrak{m}}).\tag{4.35}$$

Possibly taking a further subsequence, we also have that $\sum_{n=1}^\infty\|\tilde G_{n+1}-\tilde G_n\|_{L^2(\mathrm{X})}<\infty$, whence $H:=\tilde G_1+\sum_{n=1}^\infty|\tilde G_{n+1}-\tilde G_n|$ belongs to $L^2(\mathrm{X},\mathfrak{m})$. Since clearly $\tilde G_n\le H$ $\mathfrak{m}$-a.e. for any $n\in\mathbb{N}$, we deduce that $H\in A_{\phi,R}(\tilde f_n)$ for every $n\in\mathbb{N}$ and thus we can find a $\mathfrak{m}$-null Borel set $N\subset\mathrm{X}$ such that

$$\big|\tilde f_n(x)-\tilde f_n(y)\big|\le F_\phi(x,y)\big(H(x)+H(y)\big)\mathsf{d}(x,y)\quad\text{for all }n\in\mathbb{N}\text{ and }x,y\in\mathrm{X}\setminus N,\text{ with }\mathsf{d}(x,y)\le R.\tag{4.36}$$

Let $(x_j)\subset\mathrm{X}$ be countable and dense and for $j,k\in\mathbb{N}$ let

$$E_{j,k}:=\big(B_{R/2}(x_j)\cap\{H\le k\}\cap\{\phi\le k\}\big)\setminus N.$$

Fix $j,k\in\mathbb{N}$ and notice that the bound (4.36) ensures that the functions $\tilde f_n$ are uniformly Lipschitz on the bounded set $E_{j,k}$. Therefore, we can find a sequence $(g^{j,k}_n)\subset\mathrm{LIP}(\mathrm{X})$ made of functions with uniformly bounded support such that $g^{j,k}_n=\tilde f_n|_{E_{j,k}}$ for every $n\in\mathbb{N}$, and $\sup_n\mathrm{Lip}(g^{j,k}_n)<+\infty$. This grants that $(g^{j,k}_n)$ is bounded in $W^{1,2}(\mathrm{X},\mathsf{d},\mathfrak{m})$, so that (up to a not relabeled subsequence) by the continuity of $\mathrm{d}:W^{1,2}(\mathrm{X})\to L^2(T^*\mathrm{X},\mathfrak{m})\hookrightarrow L^2(T^*\mathrm{X},\tilde{\mathfrak{m}})$ we have $g^{j,k}_n\rightharpoonup g^{j,k}$ weakly in $L^2(\mathrm{X},\mathfrak{m})$ and $\mathrm{d}g^{j,k}_n\rightharpoonup\mathrm{d}g^{j,k}$ weakly in $L^2(T^*\mathrm{X},\tilde{\mathfrak{m}})$, for some $g^{j,k}\in W^{1,2}(\mathrm{X},\mathsf{d},\mathfrak{m})$. In particular, $\chi_{E_{j,k}}\mathrm{d}g^{j,k}_n\rightharpoonup\chi_{E_{j,k}}\mathrm{d}g^{j,k}$ weakly in $L^2(T^*\mathrm{X},\tilde{\mathfrak{m}})$ and since the construction ensures that $g^{j,k}=f$ $\mathfrak{m}$-a.e. on $E_{j,k}$, we also know that $\mathrm{d}g^{j,k}=\mathrm{d}f$ on $E_{j,k}$ and, similarly, that $\mathrm{d}g^{j,k}_n=\mathrm{d}\tilde f_n$ on $E_{j,k}$ for every $n\in\mathbb{N}$. We thus proved that $\chi_{E_{j,k}}\mathrm{d}\tilde f_n\rightharpoonup\chi_{E_{j,k}}\mathrm{d}f$ weakly in $L^2(T^*\mathrm{X},\tilde{\mathfrak{m}})$, which coupled with (4.35) implies $\chi_{E_{j,k}}\mathrm{d}f=\chi_{E_{j,k}}\omega$. Since the sets $E_{j,k}$ cover $\mathfrak{m}$-a.a. X, by the arbitrariness of j,k this is sufficient to conclude that $\mathrm{d}f=\omega$, as desired. □

4.4 Integration of module-valued maps and related topics

The goal of this section is to study integration (and differentiation) of maps with values in a Hilbert module $\mathscr{H}$. We shall mostly apply this theory to the case $\mathscr{H} = L^0(T\mathrm{X})$ in order to study the module $W^{1,2}_{fix}([0,1], L^0(T\mathrm{X}))$ (see Definition 5.2.4). From the conceptual point of view, the most important result here is perhaps the Hille-like Theorem 4.4.9 below, that we will use in conjunction with the closure result for the differential of functions in $H_{\phi,R}(\mathrm{X})$ in Proposition 4.3.5.

We shall work with maps on $[0,1]$ with values in Hilbert modules, but several parts of the discussion below can be adapted to more general settings.

Before coming to general module-valued maps, we consider the case of maps taking values into $L^0(\mathrm{X})$. Recall that the topology of $L^0(\mathrm{X})$ is metrized by the complete and separable distance

$$\mathsf{d}_{L^0}(f,g) := \int 1 \wedge |f-g| \, \mathrm{d}\mathfrak{m}',$$

where $\mathfrak{m}' \in \mathscr{P}(\mathrm{X})$ is any Borel probability measure having the same negligible sets of $\mathfrak{m}$. Let us fix such $\mathfrak{m}'$, and thus the distance d_{L^0}: the actual choice of $\mathfrak{m}'$ does not matter, but in establishing some inequalities it is useful to have it fixed (and for convenience we shall add some further requirement to $\mathfrak{m}'$ in Section 5.1).

Definition 4.4.1 (Some spaces of functions)**.** *We shall consider:*

i) For $p \in [1,\infty]$ the space $L^p([0,1], L^0(\mathrm{X})) \subset L^0([0,1], L^0(\mathrm{X}))$ is the collection of functions (f_t) such that for $\mathfrak{m}$-a.e. x the function $t \mapsto f_t(x)$ is in $L^p(0,1)$.

ii) The space $W^{1,2}([0,1], L^0(\mathrm{X})) \subset L^0([0,1], L^0(\mathrm{X}))$ is the collection of functions (f_t) such that for $\mathfrak{m}$-a.e. x the function $t \mapsto f_t(x)$ is in $W^{1,2}(0,1)$.

iii) The space $AC^2([0,1], L^0(\mathrm{X})) \subset C([0,1], L^0(\mathrm{X}))$ is the collection of functions (f_t) such that for $\mathfrak{m}$-a.e. x the function $t \mapsto f_t(x)$ is in $W^{1,2}(0,1)$.

Remark 4.4.2 (Comments on the notation)**.** The notation $L^0(\mathrm{X}, L^p([0,1]))$ and $L^0(\mathrm{X}, W^{1,2}([0,1]))$ would be more in line, as opposed to $L^p([0,1], L^0(\mathrm{X}))$ and $W^{1,2}([0,1], L^0(\mathrm{X}))$, with the standard notation for Banach-valued maps: our choice is motivated by convenience in dealing with module-valued curves, where we will speak of $L^p([0,1], \mathscr{H})$ and $W^{1,2}([0,1], \mathscr{H})$.

Also, notice that the way we defined it makes $AC^2([0,1], L^0(\mathrm{X}))$ different from the usual space of absolutely continuous curves with values in the metric space $L^0(\mathrm{X})$ (and the same holds for the space $AC^2([0,1], \mathscr{H})$ defined below).

Finally, let us stress that by $C([0,1], L^0(\mathrm{X}))$ (and $C([0,1], \mathscr{H})$ below) we intend the standard space of continuous curves from $[0,1]$ to $L^0(\mathrm{X})$ equipped with the usual 'sup' distance. In particular, $C([0,1], L^0(\mathrm{X}))$ has nothing to do with Borel collections of continuous functions $t \mapsto f_t(x)$ and is not contained in $L^\infty([0,1], L^0(\mathrm{X}))$. ■

For $(f_t) \in L^0([0,1], L^0(\mathrm{X}))$ and $p \in [1,\infty]$ we define (up to equality for $\mathfrak{m}$-a.e. x) the map

$$|(f_t)|_{L^p}(x) := \|f_\cdot(x)\|_{L^p(0,1)}$$

and similarly

$$|(f_t)|_{W^{1,2}}(x) := \|f_\cdot(x)\|_{W^{1,2}(0,1)}.$$

Then clearly $L^p([0,1],L^0(\mathrm{X}))$ (resp. $W^{1,2}([0,1],L^0(\mathrm{X}))$) is the subspace of $L^0([0,1],L^0(\mathrm{X}))$ made of those functions for which $|(f_t)|_{L^p}$ (resp. $|(f_t)|_{W^{1,2}}$) is finite $\mathfrak{m}$-a.e.. In particular, the distances

$$\mathsf{d}_{L^p}\big((f_t),(g_t)\big) := \mathsf{d}_{L^0}\big(|(f_t-g_t)|_{L^p},0\big) \qquad \text{and} \qquad \mathsf{d}_{W^{1,2}}\big((f_t),(g_t)\big) := \mathsf{d}_{L^0}\big(|(f_t-g_t)|_{W^{1,2}},0\big)$$

are well defined on $L^p([0,1],L^0(\mathrm{X}))$ and $W^{1,2}([0,1],L^0(\mathrm{X}))$ respectively. It is then easy to see that $L^p([0,1],L^0(\mathrm{X}))$ and $W^{1,2}([0,1],L^0(\mathrm{X}))$ are $L^0(\mathrm{X})$-normed modules when equipped with the above pointwise norms and with the product $g(t\mapsto f_t) := (t\mapsto gf_t)$. Here the only possibly non-trivial claim is completeness: this follows from the completeness of $L^0([0,1],L^0(\mathrm{X}))\sim L^0([0,1]\times\mathrm{X})$ and the lower semicontinuity of $L^p/W^{1,2}$-norms w.r.t. convergence a.e.. Indeed, the inequality

$$\iint_0^1 |f_t-g_t|(x)\wedge 1\,\mathrm{d}t\,\mathrm{d}\mathfrak{m}(x) = \int 1\wedge\Big(\int_0^1 1\wedge|f_t-g_t|(x)\,\mathrm{d}t\Big)\,\mathrm{d}\mathfrak{m}(x) \leq \int 1\wedge|(f_t-g_t)|_{L^p}\,\mathrm{d}\mathfrak{m}$$

shows that if $(f^n_t)\subset L^p([0,1],L^0(\mathrm{X}))$ is d_{L^p}-Cauchy, then it is also Cauchy in $L^0([0,1]\times\mathrm{X})$ and thus converges to some (f_t) in such space. Thus some subsequence $(f^{n_k}_t)$ converges $(\mathfrak{m}\times\mathcal{L}^1)$-a.e., and thus for $\mathfrak{m}$-a.e. $x\in\mathrm{X}$ we have that $f^{n_k}_t(x)\to f_t(x)$ for a.e. $t\in[0,1]$. Then Fatou's theorem implies that $|(f^m-f)|_{L^p}\leq\underline{\lim}_k|(f^m-f^{n_k})|_{L^p}$ $\mathfrak{m}$-a.e. and thus

$$\mathsf{d}_{L^p}\big((f_t),(f^m_t)\big) = \int 1\wedge|(f_t-f^m_t)|_{L^p}\,\mathrm{d}\mathfrak{m} \leq \varliminf_{k\to\infty}\int 1\wedge|(f^{n_k}_t-f^m_t)|_{L^p}\,\mathrm{d}\mathfrak{m} = \varliminf_{k\to\infty}\mathsf{d}_{L^p}\big((f^{n_k}_t),(f^m_t)\big),$$

so that completeness follows letting $m\to\infty$ and recalling that (f^n_t) is d_{L^p}-Cauchy. The argument for $W^{1,2}([0,1],L^0(\mathrm{X}))$ is analogous.

The space $AC^2([0,1],L^0(\mathrm{X}))$ is complete w.r.t. the distance

$$\mathsf{d}_{AC^2}\big((f_t),(g_t)\big) := \mathsf{d}_{W^{1,2}}\big((f_t),(g_t)\big) + \sup_{t\in[0,1]}\mathsf{d}_{L^0}(f_t,g_t),$$

as it is trivial to check. It is an algebraic module over $L^0(\mathrm{X})$ with respect to the operation $g(t\mapsto f_t) = t\mapsto gf_t$, but it does not have the structure of a normed $L^0(\mathrm{X})$ module. The same holds for $C([0,1],L^0(\mathrm{X}))$.

For $(f_t)\in L^1([0,1],L^0(\mathrm{X}))$ and $A\subset[0,1]$ Borel, Fubini's theorem ensures that the function

$$\Big(\int_A f_t\,\mathrm{d}t\Big)(x) := \int_A f_t(x)\,\mathrm{d}t$$

is a well-defined element of $L^0(\mathrm{X})$ and it is clear that

$$\Big|\int_A f_t\,\mathrm{d}t\Big| \leq \int_A |f_t|\,\mathrm{d}t \qquad \mathfrak{m}-a.e..$$

In particular, for any $(f_t)\in L^1([0,1],L^0(\mathrm{X}))$ and $t,s\in[0,1]$, $t<s$ we have

$$\Big|\int_0^s f_r\,\mathrm{d}r - \int_0^t f_r\,\mathrm{d}r\Big| \leq \int_t^s |f_r|\,\mathrm{d}r, \qquad \mathfrak{m}-a.e.,$$

showing that $t \mapsto \int_0^t f_r \, \mathrm{d}r$ is continuous w.r.t. $\mathfrak{m}$-a.e. convergence and thus also w.r.t. $L^0(\mathrm{X})$-convergence.

Now notice that since the map assigning to a function in the classical space $W^{1,2}(0,1)$ its distributional derivative in $L^2([0,1])$ is continuous, we have that for $(f_t) \in W^{1,2}([0,1], L^0(\mathrm{X}))$ the pointwisely defined distributional derivative, that we shall denote by $(\dot{f}_t)$, is an element of $L^2([0,1], L^0(\mathrm{X}))$. It is also clear by comparison with the classical case that

$$\begin{aligned} &(f_t) \in W^{1,2}([0,1], L^0(\mathrm{X})) \quad \Leftrightarrow \quad \exists (g_t) \in L^2([0,1], L^0(\mathrm{X})) \text{ such that } \forall h \in (0,1) \text{ we have} \\ &f_{t+h} - f_t = \int_t^{t+h} g_r \, \mathrm{d}r \quad \textit{for a.e. } t \in [0, 1-h] \text{ and in this case } \dot{f}_t = g_t, \quad \textit{a.e. } t, \end{aligned} \tag{4.37}$$

where the identity between functions are intended $\mathfrak{m}$-a.e.. Similarly, the continuity in t, s of $\int_t^s g_r \, \mathrm{d}r$ gives

$$\begin{aligned} &(f_t) \in AC^2([0,1], L^0(\mathrm{X})) \quad \Leftrightarrow \quad \exists (g_t) \in L^2([0,1], L^0(\mathrm{X})) \text{ such that} \\ &f_s - f_t = \int_t^s g_r \, \mathrm{d}r \qquad \forall t, s \in [0,1],\ t < s \quad \text{and in this case } \dot{f}_t = g_t, \quad \textit{a.e. } t. \end{aligned} \tag{4.38}$$

Also, still by looking at the classical one dimensional case, we have the following characterization of functions in $W^{1,2}([0,1], L^0(\mathrm{X}))$:

$$\begin{aligned} &(f_t) \in W^{1,2}([0,1], L^0(\mathrm{X})) \quad \Leftrightarrow \quad \exists (g_t) \in L^2([0,1], L^0(\mathrm{X})) \text{ such that} \\ &|f_s - f_t| \le \int_t^s g_r \, \mathrm{d}r \quad \textit{for a.e. } t, s \in [0,1],\ t < s \text{ and in this case } |\dot{f}_t| \le g_t, \quad \textit{a.e. } t, \end{aligned} \tag{4.39}$$

where again the inequalities between functions are intended $\mathfrak{m}$-a.e.. We also notice the existence of a unique *continuous representative* of elements in $W^{1,2}([0,1], L^0(\mathrm{X}))$:

$$\begin{aligned} &\text{for any } (f_t) \in W^{1,2}([0,1], L^0(\mathrm{X})) \text{ there is a unique } (\bar{f}_t) \in AC^2([0,1], L^0(\mathrm{X})) \\ &\text{such that } f_t = \bar{f}_t \text{ for a.e. } t,\ \mathfrak{m} - \textit{a.e.}. \end{aligned} \tag{4.40}$$

Indeed, uniqueness is clear. For existence, we simply pick for $\mathfrak{m}$-a.e. x the continuous representative $\bar{f}_\cdot(x)$ of $f_\cdot(x) \in W^{1,2}(0,1)$: the fact that $x \mapsto \bar{f}_t(x)$ is Borel can be proved by building upon the fact that the map from $W^{1,2}(0,1)$ to $C([0,1])$ sending a Sobolev function to its continuous representative is continuous. Then it is clear that $t \mapsto \bar{f}_t \in L^0(\mathrm{X})$ is continuous w.r.t. a.e. convergence, and thus w.r.t. the L^0-topology.

The existence of such representatives that are absolutely continuous for $\mathfrak{m}$-a.e. x can also be used to prove that

$$(f_t) \in AC^2([0,1], L^0(\mathrm{X})) \quad \Rightarrow \quad \lim_{h \to 0} \frac{f_{t+h} - f_t}{h} = \dot{f}_t, \qquad \textit{in } L^0(\mathrm{X}),\ \textit{for a.e. } t \in [0,1]. \tag{4.41}$$

Indeed, for $\bar{f}_t$ as in the proof of (4.40), the differentiability of functions in $AC^2([0,1], \mathbb{R})$ and Fubini's theorem ensure that $\frac{\bar{f}_{t+h} - \bar{f}_t}{h} \to \dot{f}_t$ $\mathfrak{m}$-a.e. for a.e. t. Also, again Fubini's theorem grants that $\bar{f}_t = f_t$ $\mathfrak{m}$-a.e. for a.e. t, and since $(f_t), (\bar{f}_t)$ are both in $C([0,1], L^0(\mathrm{X}))$ (the first by assumption, the second because by construction it is continuous w.r.t. $\mathfrak{m}$-a.e. convergence), we deduce that $\bar{f}_t = f_t$ $\mathfrak{m}$-a.e. for every $t \in [0,1]$, thus (4.41) follows.

We conclude the discussion about the space $W^{1,2}([0,1], L^0(\mathrm{X}))$ with two simple results: the first concerns stability property and the second is a sort of density criterion.

Proposition 4.4.3. *Let $(f_t^n) \in W^{1,2}([0,1], L^0(\mathrm{X}))$, $n \in \mathbb{N}$, be such that $(f_t^n) \to (f_t)$ and $(\dot{f}_t^n) \to (g_t)$ in $L^0([0,1], L^0(\mathrm{X}))$ for some $(f_t), (g_t) \in L^0([0,1], L^0(\mathrm{X}))$. Assume that*

$$\lim_{C \to +\infty} m(C) = 0, \qquad \textit{where} \qquad m(C) := \sup_{n \in \mathbb{N}} \mathfrak{m}'\big(\{|(f_t^n)|_{W^{1,2}} \geq C\}\big). \tag{4.42}$$

Then $(f_t) \in W^{1,2}([0,1], L^0(\mathrm{X}))$ and $\dot{f}_t = g_t$ $\mathfrak{m}$-a.e. for a.e. t.

Notice that in particular, condition (4.42) holds provided $|(f_t^n)|_{W^{1,2}} \leq g_n$ $\mathfrak{m}$-a.e. for some sequence (g_n) having a limit in $L^0(\mathrm{X})$.

Proof. Up to pass to a non-relabeled subsequence we have that for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ the functions $t \mapsto f_t^n(x), \dot{f}_t^n(x)$ converge to $t \mapsto f_t(x), g_t(x)$ for a.e. $t \in [0,1]$. By standard results about Sobolev functions on $(0,1)$, to conclude that $t \mapsto f_t(x)$ belongs to $W^{1,2}(0,1)$ with derivative $t \mapsto g_t(x)$ it is sufficient to prove that $\underline{\lim}_{n\to\infty} |(f_t^n)|_{W^{1,2}}(x) < \infty$. The fact that this holds for $\mathfrak{m}$-a.e. x is a consequence of Borel–Cantelli's lemma and the assumption (4.42). Indeed $\underline{\lim}_{n\to\infty} |(f_t^n)|_{W^{1,2}}(x) < C$ if and only if $x \in \cap_{n\in\mathbb{N}} \cup_{i \geq n} \{|(f_t^i)|_{W^{1,2}} < C\}$ and since the sequence of sets $n \mapsto \cup_{i\geq n}\{|(f_t^n)|_{W^{1,2}} < C\}$ is decreasing we have

$$\begin{aligned}
\mathfrak{m}'\Big(\bigcap_{n\in\mathbb{N}} \bigcup_{i\geq n} \{|(f_t^i)|_{W^{1,2}} < C\}\Big) &= \inf_{n\in\mathbb{N}} \mathfrak{m}'\Big(\bigcup_{i\geq n} \{|(f_t^i)|_{W^{1,2}} < C\}\Big) \\
&\geq \inf_{n\in\mathbb{N}} \mathfrak{m}'\big(\{|(f_t^n)|_{W^{1,2}} < C\}\big) \geq 1 - m(C).
\end{aligned}$$

Thus (4.42) ensures a.e. finiteness of the $\underline{\lim}$, as claimed.

For the last statement observe that condition (4.42) is satisfied by the sequence (g_n) (rather trivially by the definition of local convergence in measure and/or of distance d_{L^0}). □

Proposition 4.4.4. *Let $\mathcal{A} \subset W^{1,2}([0,1], L^0(\mathrm{X}))$. Assume that $\mathcal{A}$:*

- *o) is a vector space,*
- *i) is stable by the 'restriction' operation, i.e. $(f_t) \in \mathcal{A}$ and $E \subset \mathrm{X}$ Borel imply that $t \mapsto \chi_E f_t$ is in $\mathcal{A}$,*
- *ii) is stable by multiplication by a C^1 function, i.e. $(f_t) \in \mathcal{A}$ and $\varphi \in C^1([0,1])$ imply that $t \mapsto \varphi(t) f_t$ is in $\mathcal{A}$,*
- *iii) is closed in the $W^{1,2}([0,1], L^0(\mathrm{X}))$-topology,*
- *iv) contains the constant functions, i.e. for any $f \in L^0(\mathrm{X})$ the map $t \mapsto f$ is in $\mathcal{A}$.*

Then $\mathcal{A} = W^{1,2}([0,1], L^0(\mathrm{X}))$.

Proof. Let $(\varphi_n) \subset C^1([0,1])$ be countable and dense in $W^{1,2}(0,1)$. Fix $\varepsilon > 0$ and let $T : W^{1,2}(0,1) \to W^{1,2}(0,1)$ be the map sending f to φ_n, where $n \in \mathbb{N}$ is the least index $j \in \mathbb{N}$ such that $\|f - \varphi_j\|_{W^{1,2}} \leq \varepsilon$. It is clear that T is well defined and Borel. Thus for $(f_t) \in W^{1,2}([0,1], L^0(\mathrm{X}))$ the curve $t \mapsto T(f)_t$ defined by $T(f)_t(x) := T(f_\cdot(x))(t)$ is in $W^{1,2}([0,1], L^0(\mathrm{X}))$ and, by construction, its $\mathsf{d}_{W^{1,2}}$-distance from (f_t) is $\leq \varepsilon$. As $\varepsilon > 0$ is arbitrary, by the closure property (iii) to conclude it is sufficient to show that $(T(f)_t) \in \mathcal{A}$. To see this, for every $n \in \mathbb{N}$ let $E_n \subset \mathrm{X}$ be the set of x's such that $T(f_\cdot)(x) = \varphi_n$. Then we have $T(f)_t = \sum_n \chi_{E_n} \varphi_n(t)$ (where the convergence of the partial sums is intended in the $W^{1,2}([0,1], L^0(\mathrm{X}))$-topology) and the properties $(o), (i), (ii), (iv)$ trivially ensure that $(T(f)_t) \in \mathcal{A}$, as desired. □

We pass to the vector case. Fix a separable, Hilbertian L^0-module $\mathscr{H}$. Recall that the space $L^0([0,1],\mathscr{H})$, that we shall sometimes abbreviate in $L^0_{\mathscr{H}}$, is the space of Borel maps from $[0,1]$ to $\mathscr{H}$ identified up to equality for a.e. t. The space $L^0_{\mathscr{H}}$ is complete w.r.t. the distance

$$\mathsf{d}_{L^0_{\mathscr{H}}}\big((v_t),(z_t)\big) := \int_0^1 1 \wedge \mathsf{d}_{L^0}(|v_t - z_t|, 0)\,\mathrm{d}t.$$

The space $C([0,1],\mathscr{H})$ denotes, as usual, the space of continuous curves with values in $\mathscr{H}$ equipped with the 'sup' distance.

We pass to the 'vector versions' of the spaces introduced in Definition 4.4.1:

Definition 4.4.5 (Some spaces of vectors)**.** *We shall denote by:*

i) For $p \in [1,\infty]$ the space $L^p([0,1],\mathscr{H}) \subset L^0([0,1],\mathscr{H})$, that we shall abbreviate in $L^p_{\mathscr{H}}$, is the collection of vector fields (v_t) such that the quantity

$$|(v_t)|_{L^p_{\mathscr{H}}}(x) := |(|v_t|)|_{L^p}(x) = \||v_\cdot|(x)\|_{L^p(0,1)}$$

(which is well defined up to equality $\mathfrak{m}$-a.e.) is finite $\mathfrak{m}$-a.e..

ii) The space $W^{1,2}([0,1],\mathscr{H}) \subset L^0([0,1],\mathscr{H})$ is the collection of vector fields (v_t) for which there is $(\dot v_t) \in L^2([0,1],\mathscr{H})$ such that for any $z \in \mathscr{H}$ the curve $t \mapsto \langle v_t, z\rangle$ belongs to $W^{1,2}([0,1],L^0(\mathrm{X}))$ with

$$\partial_t \langle v_t, z\rangle = \langle \dot v_t, z\rangle \qquad \mathfrak{m}-a.e.,\ a.e.\ t.$$

For $(v_t) \in W^{1,2}([0,1],\mathscr{H})$ we define

$$|(v_t)|^2_{W^{1,2}_{\mathscr{H}}} := |(v_t)|^2_{L^2_{\mathscr{H}}} + |(\dot v_t)|^2_{L^2_{\mathscr{H}}} \in L^0(\mathrm{X}).$$

iii) The space $AC^2([0,1],\mathscr{H}) \subset C([0,1],\mathscr{H})$ is the collection of vector fields (v_t) for which there is $(\dot v_t) \in L^2([0,1],\mathscr{H})$ such that for any $z \in \mathscr{H}$ the curve $t \mapsto \langle v_t, z\rangle$ is in $W^{1,2}([0,1],L^0(\mathrm{X}))$ (and thus in $AC^2([0,1],L^0(\mathrm{X}))$) with

$$\partial_t \langle v_t, z\rangle = \langle \dot v_t, z\rangle \qquad \mathfrak{m}-a.e.,\ a.e.\ t.$$

The three spaces defined above are naturally endowed with the respective distances

$$\begin{aligned}
\mathsf{d}_{L^p_{\mathscr{H}}}\big((v_t),(z_t)\big) &:= \mathsf{d}_{L^0}(|(v_t - z_t)|_{L^p_{\mathscr{H}}}, 0),\\
\mathsf{d}_{W^{1,2}_{\mathscr{H}}}\big((v_t),(z_t)\big) &:= \mathsf{d}_{L^0}(|(v_t - z_t)|_{W^{1,2}_{\mathscr{H}}}, 0),\\
\mathsf{d}_{AC^2_{\mathscr{H}}}\big((v_t),(z_t)\big) &:= \mathsf{d}_{L^0}(|(v_t - z_t)|_{W^{1,2}_{\mathscr{H}}}, 0) + \sup_{t\in[0,1]} \mathsf{d}_{\mathscr{H}}(v_t, z_t).
\end{aligned}$$

These are complete distances, as can be seen arguing as for the respective spaces of functions (to show that the derivative of the limit is the limit of the derivative in considering the spaces $W^{1,2}_{\mathscr{H}}$ and $AC^2_{\mathscr{H}}$ we use Proposition 4.4.3).

The spaces $L^p_{\mathscr{H}}, W^{1,2}_{\mathscr{H}}, AC^2_{\mathscr{H}}$ are also endowed with the product with $L^0(\mathrm{X})$ functions defined as $g(t \mapsto v_t) := (t \mapsto g v_t)$ and it is clear that $L^p_{\mathscr{H}}, W^{1,2}_{\mathscr{H}}$ are L^0-normed modules.

Let us turn to the definition of integral of a vector field (v_t) in $L^1_{\mathscr{H}}$: for $A \subset [0,1]$ Borel we want to define $\int_A v_t \, \mathrm{d}t$ as element of $\mathscr{H}$. To this aim, notice that for any $z \in \mathscr{H}$ the function $\langle z, v_t\rangle$ satisfies $|\langle z, v_t\rangle| \leq |z||v_t|$ $(\mathfrak{m} \times \mathcal{L}^1)$-a.e., thus for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ we have that $t \mapsto \langle z, v_t\rangle(x)$ is in $L^1(0,1)$. Hence $\int_A \langle z, v_t\rangle \, \mathrm{d}t$ is a well defined function in $L^0(\mathrm{X})$ and it is clear that the assignment $z \mapsto \int_A \langle z, v_t\rangle \, \mathrm{d}t$ is linear and satisfies

$$\Big| \int_A \langle z, v_t\rangle \, \mathrm{d}t \Big| \leq \int_A |\langle z, v_t\rangle| \, \mathrm{d}t \leq |z| \int_A |v_t| \, \mathrm{d}t \leq |z| |(v_t)|_{L^1_{\mathscr{H}}}. \tag{4.43}$$

This is sufficient to establish that $\mathscr{H} \ni z \mapsto \int_A \langle z, v_t\rangle \, \mathrm{d}t \in L^0(\mathrm{X})$ is $L^0(\mathrm{X})$-linear and continuous and thus represented by - thanks to Riesz's theorem for $L^0(\mathfrak{m})$-normed Hilbert modules (Proposition 1.2.20)- an element of $\mathscr{H}$ that we shall denote by $\int_A v_t \, \mathrm{d}t$. Notice that the bound (4.43) gives

$$\Big| \int_A v_t \, \mathrm{d}t \Big| \leq \int_A |v_t| \, \mathrm{d}t, \qquad \mathfrak{m} - a.e.. \tag{4.44}$$

This bound is sufficient to prove that

$$\text{for } (v_t) \in L^1_{\mathscr{H}} \text{ the map } t \mapsto \int_0^t v_s \, \mathrm{d}s \in \mathscr{H} \text{ is in } C([0,1], \mathscr{H}) \tag{4.45}$$

(because $| \int_0^s v_r \, \mathrm{d}r - \int_0^t v_r \, \mathrm{d}r| \leq \int_t^s |v_r| \, \mathrm{d}r \to 0$ $\mathfrak{m}$-a.e. as $s \to t$).

Observe that a direct consequence of the definitions and of (4.37) is that for $(v_t) \in W^{1,2}_{\mathscr{H}}$ and $h \in (0,1)$ it holds

$$v_{t+h} - v_t = \int_t^{t+h} \dot v_r \, \mathrm{d}r, \qquad \mathfrak{m} - a.e., \ a.e. \ t \in [0, 1-h], \tag{4.46}$$

and thus by (4.44) that $|v_t| \leq |v_s| + \int_0^1 |\dot v_r| \, \mathrm{d}r$ is valid $\mathfrak{m}$-a.e. for a.e. t, s. Integrating this in s we deduce that

$$|(v_t)|_{L^\infty_{\mathscr{H}}} \leq |(v_t)|_{L^2_{\mathscr{H}}} + |(\dot v_t)|_{L^1_{\mathscr{H}}} \leq \sqrt{2} |(v_t)|_{W^{1,2}_{\mathscr{H}}} \qquad \mathfrak{m} - a.e., \tag{4.47}$$

showing in particular that $W^{1,2}_{\mathscr{H}} \subset L^\infty_{\mathscr{H}}$. Notice that by Fubini's theorem, an equivalent way of stating this bound is by saying that for a.e. $s \in [0,1]$ we have $|v_s| \leq 2 |(v_t)|_{W^{1,2}_{\mathscr{H}}}$ $\mathfrak{m}$-a.e.. Now observe that if $(v_t) \in AC^2_{\mathscr{H}}$, from the continuity of the pointwise norm as map from $\mathscr{H}$ to $L^0(\mathrm{X})$ we see that $t \mapsto |v_t| \in L^0(\mathrm{X})$ is continuous, and thus

$$(v_t) \in AC^2_{\mathscr{H}} \qquad \Rightarrow \qquad |v_t| \leq \sqrt{2} |(v_s)|_{W^{1,2}_{\mathscr{H}}} \qquad \mathfrak{m} - a.e., \ \forall t \in [0,1]. \tag{4.48}$$

Another direct consequence of the definitions and of (4.38) is

$$\begin{aligned} &(v_t) \in AC^2([0,1], \mathscr{H}) \quad \Leftrightarrow \quad \exists (z_t) \in L^2([0,1], \mathscr{H}) \text{ such that} \\ &v_s - v_t = \int_t^s z_r \, \mathrm{d}r \qquad \forall t, s \in [0,1], \ t < s \quad \text{and in this case } \dot v_t = z_t, \quad \mathfrak{m} - a.e., \ a.e. \ t. \end{aligned} \tag{4.49}$$

To further investigate the properties of $W^{1,2}_{\mathscr{H}}$ it will be convenient to notice the following fact (reminiscent of the classical statement 'weak convergence+convergence of norms $\Rightarrow$ strong convergence'):

$$\left. \begin{array}{l} \langle v^n_t, z\rangle \to \langle v_t, z\rangle, \quad \text{in } L^0([0,1], L^0(\mathrm{X})), \quad \forall z \in \mathscr{H} \\ |(v^n_t)|_{L^2_{\mathscr{H}}} \leq |(v_t)|_{L^2_{\mathscr{H}}} < \infty, \quad \mathfrak{m} - a.e. \end{array} \right\} \qquad \Rightarrow \qquad (v^n_t) \overset{L^2_{\mathscr{H}}}{\to} (v_t). \tag{4.50}$$

To see this, notice that for $t \mapsto z_t \in \mathscr{H}$ piecewise constant, i.e. of the form $z_t = \sum_{i=1}^N \chi_{A_i}(t) z^i$ with $A_i \subseteq [0,1]$ Borel and $z^i \in \mathscr{H}$, the first in (4.50) easily gives $\langle v_t^n, z_t \rangle \to \langle v_t, z_t \rangle$ in $L^0([0,1], L^0(\mathrm{X}))$. Moreover, the second in (4.50) gives $|\langle v_t^n, z_t\rangle|_{L^2} \le |(v_t)|_{L^2_{\mathscr{H}}} |(z_t)|_{L^\infty_{\mathscr{H}}}$ $\mathfrak{m}$-a.e. for every $n \in \mathbb{N}$. Notice that if $f^n(t) \to f(t)$ for a.e. t and $\sup_n \|f^n\|_{L^2} < \infty$, then we have $\int_0^1 f^n(t)\,\mathrm{d}t \to \int_0^1 f(t)\,\mathrm{d}t$: thanks to the reflexivity of $L^2(0,1)$, we have that $(f^n)_n$ has a $L^2(0,1)$-weakly converging subsequence, with limit g; the pointwise a.e. convergence $f^n \to f$ ensures that $g = f$, thus the original sequence converges to f weakly in $L^2(0,1)$; in particular (by testing against the constant function $1 \in L^2(0,1)$) we conclude that $\int_0^1 f^n(t)\,\mathrm{d}t \to \int_0^1 f(t)\,\mathrm{d}t$, as desired.

Using this fact in conjunction with what already mentioned and recalling that a sequence converges in L^0 if and only if any subsequence has a further sub-subsequence converging a.e., we deduce that $\int_0^1 \langle v_t^n, z_t\rangle\,\mathrm{d}t \to \int_0^1 \langle v_t, z_t\rangle\,\mathrm{d}t$ in $L^0(\mathrm{X})$. Hence

$$\begin{aligned}
\varlimsup_{n\to\infty} \mathsf{d}_{L^2_{\mathscr{H}}}\big((v_t^n),(z_t)\big) &= \varlimsup_{n\to\infty} \mathsf{d}_{L^0}\Big(\sqrt{|(v_t^n)|^2_{L^2_{\mathscr{H}}} + |(z_t)|^2_{L^2_{\mathscr{H}}} - 2\int_0^1 \langle v_t^n, z_t\rangle\,\mathrm{d}t}, 0\Big)\\
&\le \varlimsup_{n\to\infty} \mathsf{d}_{L^0}\Big(\sqrt{|(v_t)|^2_{L^2_{\mathscr{H}}} + |(z_t)|^2_{L^2_{\mathscr{H}}} - 2\int_0^1 \langle v_t^n, z_t\rangle\,\mathrm{d}t}, 0\Big)\\
&= \mathsf{d}_{L^0}\Big(\sqrt{|(v_t)|^2_{L^2_{\mathscr{H}}} + |(z_t)|^2_{L^2_{\mathscr{H}}} - 2\int_0^1 \langle v_t, z_t\rangle\,\mathrm{d}t}, 0\Big)\\
&= \mathsf{d}_{L^2_{\mathscr{H}}}\big((v_t),(z_t)\big).
\end{aligned}$$

Now notice that the set of (z_t)'s considered is dense in $L^2_{\mathscr{H}}$ (as it is easy to establish from the definition), thus from

$$\varlimsup_{n\to\infty} \mathsf{d}_{L^2_{\mathscr{H}}}\big((v_t^n),(v_t)\big) \le \varlimsup_{n\to\infty} \mathsf{d}_{L^2_{\mathscr{H}}}\big((v_t^n),(z_t)\big) + \mathsf{d}_{L^2_{\mathscr{H}}}\big((v_t),(z_t)\big) \le 2\mathsf{d}_{L^2_{\mathscr{H}}}\big((v_t),(z_t)\big)$$

we conclude letting $(z_t) \to (v_t)$ in $L^2_{\mathscr{H}}$.

Now let $(v_t) \in L^2_{\mathscr{H}}$ and for $t, \varepsilon \in (0,1)$ define $v_t^\varepsilon := \varepsilon^{-1}\int_t^{t+\varepsilon} v_s\,\mathrm{d}s$, where v_s is intended to be 0 if $s > 1$. We claim that

$$(v_t^\varepsilon) \quad \to \quad (v_t) \qquad \text{in } L^2_{\mathscr{H}}, \qquad \text{as } \varepsilon \downarrow 0 \tag{4.51}$$

and we shall prove this using (4.50). Let $z \in \mathscr{H}$ and for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ apply Lebesgue's theorem to the $L^1(0,1)$-function $t \mapsto \langle z, v_t\rangle(x)$ to conclude that for a.e. t we have $\langle z, v_t^\varepsilon\rangle(x) \to \langle z, v_t\rangle(x)$: this proves the first condition in (4.50). The second follows from the inequality $|v_t^\varepsilon| \le \varepsilon^{-1}\int_t^{t+\varepsilon} |v_s|\,\mathrm{d}s$ as it implies $|(v_t^\varepsilon)|_{L^2_{\mathscr{H}}} \le |(v_t)|_{L^2_{\mathscr{H}}}$ $\mathfrak{m}$-a.e. for every ε, so the conclusion follows from (4.50).

We also claim that

$$(v_t^\varepsilon) \in W^{1,2}_{\mathscr{H}} \qquad \text{with} \qquad \dot v_t^\varepsilon = \frac{v_{t+\varepsilon} - v_t}{\varepsilon}, \quad \textit{a.e. } t, \tag{4.52}$$

where again $v_{t+\varepsilon}$ is intended to be 0 if $t + \varepsilon > 1$. To prove this, just notice that by the very definition of $W^{1,2}_{\mathscr{H}}$ and of the corresponding notion of derivative, it is sufficient to consider the scalar case $\mathscr{H} = L^0(\mathrm{X})$. In turn, in this setting the conclusion is a direct consequence of the analogous well-known result for real valued functions on $[0,1]$. Notice that, in particular, from (4.52) and (4.51) we deduce that

$$W^{1,2}_{\mathscr{H}} \quad \text{is dense in} \quad L^2_{\mathscr{H}}. \tag{4.53}$$

Applying (4.51) to the derivative of a vector field in $W^{1,2}_{\mathscr{H}}$ by keeping (4.46) in mind we obtain that

$$\frac{v_{t+h}-v_t}{h} \quad \to \quad \dot v_t \qquad in\ L^2_{\mathscr{H}} \qquad \text{as } h\to 0, \tag{4.54}$$

where $\frac{v_{t+h}-v_t}{h}$ is intended to be 0 if $t+h\notin[0,1]$.

A further consequence is that

$$(v_t),(z_t)\in W^{1,2}_{\mathscr{H}}\ (resp.\ AC^2_{\mathscr{H}}) \quad \Rightarrow \quad \begin{array}{c}(\langle v_t,z_t\rangle)\in W^{1,2}([0,1],L^0(\mathrm{X}))\\ (resp.\ AC^2([0,1],L^0(\mathrm{X})))\\ \text{with } \frac{\mathrm{d}}{\mathrm{d}t}\langle v_t,z_t\rangle=\langle\dot v_t,z_t\rangle+\langle v_t,\dot z_t\rangle.\end{array} \tag{4.55}$$

Indeed, the inequality

$$\begin{aligned}\big|\langle v_{t+h},z_{t+h}\rangle-\langle v_t,z_t\rangle\big| &\le \big|\langle v_{t+h},z_{t+h}-z_t\rangle\big|+\big|\langle v_{t+h}-v_t,z_t\rangle\big|\\ \text{(by (4.44), (4.46), (4.47))}\qquad &\le |(v_s)|_{L^\infty_{\mathscr{H}}}\int_t^{t+h}|\dot z_r|\,\mathrm{d}r+|(z_s)|_{L^\infty_{\mathscr{H}}}\int_t^{t+h}|\dot v_r|\,\mathrm{d}r \qquad \mathfrak{m}-a.e.,\end{aligned}$$

valid for every $h\in(0,1)$ and a.e. $t\in[0,1-h]$, shows that $(\langle v_t,z_t\rangle)\in W^{1,2}([0,1],L^0(\mathrm{X}))$ (recall (4.39)). Then letting $h\to0$ in

$$\frac{\langle v_{t+h},z_{t+h}\rangle-\langle v_t,z_t\rangle}{h}=\langle v_t,\tfrac{z_{t+h}-z_t}{h}\rangle+\langle\tfrac{v_{t+h}-v_t}{h},z_t\rangle+h\langle\tfrac{v_{t+h}-v_t}{h},\tfrac{z_{t+h}-z_t}{h}\rangle$$

and using (4.54) we see that the right-hand side goes to $(\langle\dot v_t,z_t\rangle+\langle v_t,\dot z_t\rangle)$ in $L^0([0,1],L^0(\mathrm{X}))$. On the other hand, applying (4.54) with $\mathscr{H}:=L^0(\mathrm{X})$ and $(\langle v_t,z_t\rangle)$ in place of (v_t) we see that the left-hand side converges to $(\partial_t\langle v_t,z_t\rangle)$ in $L^2([0,1],L^0(\mathrm{X}))$, thus also in $L^0([0,1],L^0(\mathrm{X}))$, so that (4.55) is proved.

Arguing along the same lines one uses to prove that $W^{1,2}(0,1)\hookrightarrow C([0,1])$, we can get existence of continuous representatives of elements of $W^{1,2}_{\mathscr{H}}$:

Proposition 4.4.6. *Let $(v_t)\in W^{1,2}_{\mathscr{H}}$. Then there is a unique $(\bar v_t)\in C([0,1],\mathscr{H})$ with $\bar v_t=v_t$ for a.e. $t\in[0,1]$.*

Proof. For $\varepsilon\in(0,1)$ we define $v^\varepsilon_t:=\varepsilon^{-1}\int_t^{t+\varepsilon}v_s\,\mathrm{d}s$ as before (here v_s is intended to be 0 if $s>1$). Notice that

$$\dot v^\varepsilon_t \overset{(4.52)}{=} \frac{v_{t+\varepsilon}-v_t}{\varepsilon} \overset{(4.46)}{=} \varepsilon^{-1}\int_t^{t+\varepsilon}\dot v_s\,\mathrm{d}s,$$

therefore by (4.51) we deduce that $(v^\varepsilon_t),(\dot v^\varepsilon_t)$ converge to $(v_t),(\dot v_t)$ respectively in $L^2_{\mathscr{H}}$ as $\varepsilon\downarrow0$. In other words, $(v^\varepsilon_t)\to(v_t)$ in $W^{1,2}_{\mathscr{H}}$ as $\varepsilon\downarrow0$.

Now we claim that for every $\varepsilon\in(0,1)$ the curve $t\mapsto v^\varepsilon_t\in\mathscr{H}$ is continuous. Indeed, for any $t,s\in[0,1]$ with $0\le s-t\le\varepsilon$, it is clear that we have

$$|v^\varepsilon_s-v^\varepsilon_t|=\varepsilon^{-1}\Big|\int_{t+\varepsilon}^{s+\varepsilon}v_r\,\mathrm{d}r+\int_t^s v_r\,\mathrm{d}r\Big|\le\varepsilon^{-1}\Big(\int_{t+\varepsilon}^{s+\varepsilon}|v_r|\,\mathrm{d}r+\int_t^s|v_r|\,\mathrm{d}r\Big) \qquad \mathfrak{m}-a.e.,$$

and that the right-hand side goes to 0 $\mathfrak{m}$-a.e. both as $t\uparrow s$ and as $s\downarrow t$. Hence $|v^\varepsilon_s-v^\varepsilon_t|\to0$ in the $\mathfrak{m}$-a.e. sense as $s\to t$, and thus a fortiori $v^\varepsilon_s\to v^\varepsilon_t$ in $\mathscr{H}$.

We now claim that the family of curves $\{(v_t^\varepsilon)\}_{\varepsilon\in(0,1)}$ is Cauchy in $C([0,1],\mathscr{H})$ as $\varepsilon\downarrow 0$. Indeed, for $\varepsilon,\eta\in(0,1)$ we have

$$\sup_{t\in[0,1]} \mathsf{d}_{\mathscr{H}}(v_t^\varepsilon,v_t^\eta)=\sup_{t\in[0,1]} \mathsf{d}_{L^0}(|v_t^\varepsilon-v_t^\eta|,0)\overset{(4.48)}{\leq} \mathsf{d}_{L^0}\big(|(v_t^\varepsilon-v_t^\eta)|_{W^{1,2}_{\mathscr{H}}},0\big)=\mathsf{d}_{W^{1,2}_{\mathscr{H}}}\big((v_t^\varepsilon),(v_t^\eta)\big).$$

Since we proved that $(v_t^\varepsilon)\overset{W^{1,2}_{\mathscr{H}}}{\to}(v_t)$, we know that the right-hand side of the above goes to 0 as $\varepsilon,\eta\downarrow 0$, hence our claim is proved. Let $(\bar v_t)$ be the limit of (v_t^ε) in $C([0,1],\mathscr{H})$ as $\varepsilon\downarrow 0$: since we also know that $(v_t^\varepsilon)\overset{L^2_{\mathscr{H}}}{\to}(v_t)$ and it is clear that $C([0,1],\mathscr{H})$ continuously embeds in $L^2_{\mathscr{H}}$, we conclude that $(\bar v_t)$ and (v_t) agree as elements of $L^2_{\mathscr{H}}$, i.e. that $\bar v_t=v_t$ for a.e. t, as desired. □

We shall make use of the following simple density-like result:

Proposition 4.4.7. *Let $\mathcal{A}\subset W^{1,2}([0,1],\mathscr{H})$. Assume that $\mathcal{A}$:*

- *o) is a vector space,*
- *i) is stable under 'restriction', i.e. if $(v_t)\in\mathcal{A}$ and $E\subset \mathrm{X}$ is Borel, then $t\mapsto\chi_E v_t$ belongs to $\mathcal{A}$,*
- *ii) is stable under multiplication by functions in $C^1([0,1])$, i.e. if $(v_t)\in\mathcal{A}$ and $\varphi\in C^1([0,1])$ the map $t\mapsto\varphi(t)v_t$ belongs to $\mathcal{A}$,*
- *iii) is closed in the $W^{1,2}_{\mathscr{H}}$ topology,*
- *iv) contains the constant vector fields $t\mapsto\bar v$ for any given $\bar v\in\mathscr{H}$.*

Then $\mathcal{A}=W^{1,2}([0,1],\mathscr{H})$.

Proof. Let $(v_t)\in W^{1,2}_{\mathscr{H}}$ be arbitrary, $(e^n)\subset\mathscr{H}$ be a local Hilbert base (see [55, Theorem 1.4.11]) and for every $N\in\mathbb{N}$ let $v_t^N:=\sum_{n\leq N}\langle v_t,e^n\rangle e^n$ and $\dot v_t^N:=\sum_{n\leq N}\langle \dot v_t,e^n\rangle e^n$ for a.e. $t\in[0,1]$. By the properties of local Hilbert bases we have that $|v_t^N|\leq|v_t|$ $\mathfrak{m}$-a.e. and $v_t^N\to v_t$ in $\mathscr{H}$ as $N\to\infty$ for a.e. t and similarly for $\dot v_t^N$. By (4.50) this is sufficient to deduce that $(v_t^N)\overset{L^2_{\mathscr{H}}}{\to}(v_t)$ and $(\dot v_t^N)\overset{L^2_{\mathscr{H}}}{\to}(\dot v_t)$ as $N\to\infty$. Also, from the definition of $W^{1,2}_{\mathscr{H}}$ it is clear that $(v_t^N)\in W^{1,2}_{\mathscr{H}}$ with derivative $(\dot v_t^N)$ for every $N\in\mathbb{N}$, thus $(v_t^N)\overset{W^{1,2}_{\mathscr{H}}}{\to}(v_t)$.

Therefore to conclude that $(v_t)\in\mathcal{A}$ it is sufficient to prove that $(v_t^N)\in\mathcal{A}$ for every $N\in\mathbb{N}$. Since we assumed $\mathcal{A}$ to be a vector space, to prove this it is sufficient to show that for any $v\in\mathscr{H}$, the collection of those (f_t)'s in $W^{1,2}_{L^0}$ such that $t\mapsto f_tv$ is in $\mathcal{A}$ coincides with the whole $W^{1,2}_{L^0}$. This is a direct consequence of Proposition 4.4.4 and our assumptions. □

Our last goal for the section is the study of an analogue of Hille's theorem in this context. The classical proof of this fact for Bochner integral of Banach-valued maps uses the fact that if v_t belongs to some convex closed set for a.e. t, then so does its integral over $[0,1]$. In our setting, this is also true and the proof is based on the possibility of obtaining integration as limit of properly chosen Riemann sums. For real valued functions on $[0,1]$, this last property is a classical statement of Hahn [69] (see also [74]); the proof we give is closely related to the ones in

[48, Theorem I-2.8] and [84, Lemma 4.4]. In the course of the proof we shall use properties like additivity of the integral over disjoint sets and change-of-variable formulas that can be trivially obtained from the very definition of integration.

Proposition 4.4.8. *Let $(v_t) \in L^1([0,1], \mathscr{H})$. Then for every sequence $\tau_k \downarrow 0$ there is a subsequence, not relabeled, such that for a.e. $t \in [0,1]$ we have*

$$\lim_{k\to\infty} \tau_k \sum_{i=0}^{[\tau_k^{-1}]-1} v_{\tau_k(t+i)} = \int_0^1 v_s \,\mathrm{d}s \qquad \text{in } \mathscr{H},$$

where $[\cdot]$ denotes the integer part.

In particular, if $\mathrm{C} \subset \mathscr{H}$ is convex and closed and $v_t \in \mathrm{C}$ for a.e. t, then $\int_0^1 v_t \,\mathrm{d}t \in \mathrm{C}$ as well.

Proof. Fix a Borel representative of (v_t) such that $v_t \in \mathrm{C}$ for every $t \in [0,1]$ and notice that $\frac{1}{[\tau^{-1}]} \sum_{i=0}^{[\tau^{-1}]-1} v_{\tau(t+i)}$ is a convex combination of the v_t's, and thus belongs to C. Since $\tau[\tau^{-1}] \to 1$ as $\tau \downarrow 0$, it is clear that the second statement is a consequence of the first one, so we focus on this.

Put $v_t = 0$ for $t > 1$ and notice that (4.45) gives that $\int_{\tau t}^{1+\tau t} v_s \,\mathrm{d}s \to \int_0^1 v_s \,\mathrm{d}s$ in $\mathscr{H}$ as $\tau \downarrow 0$ for every $t \in [0,1]$. Thus to conclude it is sufficient to prove that

$$\lim_{\tau\downarrow 0} \int_0^1 \mathsf{d}_{\mathscr{H}}(z_{\tau,t}, 0) \,\mathrm{d}t = 0, \qquad \text{where} \qquad z_{\tau,t} := \tau \sum_{i=0}^{[\tau^{-1}]-1} v_{\tau(t+i)} - \int_{\tau t}^{1+\tau t} v_s \,\mathrm{d}s. \tag{4.56}$$

To this aim we start claiming that

$$\lim_{\tau\downarrow 0} \int_0^1 \int_0^1 \left|v_t - v_{t+\tau s}\right| \mathrm{d}t \,\mathrm{d}s \quad \to \quad 0, \qquad \text{in } L^0(\mathrm{X}) \text{ as } \tau \downarrow 0. \tag{4.57}$$

To prove this, we first notice that the truncations $t \mapsto \chi_{\{|v_t| \le n\}} v_t = \varphi_n(|v_t|) v_t$, where $\varphi_n : \mathbb{R} \to \mathbb{R}$ is given by $\varphi_n := \chi_{[0,n]}$, belong to $L^\infty_{\mathscr{H}} \subset L^2_{\mathscr{H}}$ and trivially converge to (v_t) in $L^1_{\mathscr{H}}$. Thus recalling (4.53) we can find $(w_t^n) \subset W^{1,2}_{\mathscr{H}}$ converging to (v_t) in $L^1_{\mathscr{H}}$. Now for every $n \in \mathbb{N}$ put $w_t^n = 0$ for $t > 1$ and notice that

$$\begin{aligned}\int_0^1 \int_0^1 \left|v_t - v_{t+\tau s}\right| \mathrm{d}t \,\mathrm{d}s &\le \int_0^1 \int_0^1 \left|w_t^n - w_{t+\tau s}^n\right| \mathrm{d}t \,\mathrm{d}s + \int_0^1 \int_0^1 \left|v_t - w_t^n\right| + \left|v_{t+\tau s} - w_{t+\tau s}^n\right| \mathrm{d}t \,\mathrm{d}s \\ &\le \underbrace{\int_0^1 \int_0^1 \int_t^{t+\tau s} |\dot{w}_r| \,\mathrm{d}r \,\mathrm{d}t \,\mathrm{d}s}_{\to 0 \quad \mathfrak{m}-a.e.\ as\ \tau\downarrow 0} + 2|(v_t - w_t^n)|_{L^1_{\mathscr{H}}}.\end{aligned}$$

Thus (4.57) follows by first letting $\tau \downarrow 0$ and then $n \to \infty$ in the above (in fact the argument easily gives that there is convergence $\mathfrak{m}$-a.e. in (4.57)).

Now recall that $z_{\tau,t}$ is defined in (4.56) and notice that

$$\begin{aligned}\int_0^1 |z_{\tau,t}|\,\mathrm{d}t &= \int_0^1 \Big|\tau \sum_{i=0}^{[\tau^{-1}]-1} \Big(v_{\tau(t+i)} - \tau^{-1}\int_{\tau(t+i)}^{\tau(t+i+1)} v_s\,\mathrm{d}s\Big)\Big|\,\mathrm{d}t \\ \big(s=\tau(t+i+s')\big) \qquad &= \int_0^1 \Big|\tau \sum_{i=0}^{[\tau^{-1}]-1} \Big(v_{\tau(t+i)} - \int_0^1 v_{\tau(t+s'+i)}\,\mathrm{d}s'\Big)\Big|\,\mathrm{d}t \\ \big(\text{by } (4.44)\big) \qquad &\le \int_0^1\int_0^1 \tau \sum_{i=0}^{[\tau^{-1}]-1} \big|v_{\tau(t+i)} - v_{\tau(t+s'+i)}\big|\,\mathrm{d}t\,\mathrm{d}s' \\ \big(t'=\tau(t+i)\big) \qquad &= \int_0^1\int_0^1 \big|v_{t'} - v_{t'+\tau s'}\big|\,\mathrm{d}t'\,\mathrm{d}s' \qquad \mathfrak{m}-a.e..\end{aligned} \tag{4.58}$$

Therefore

$$\int_0^1 \mathsf{d}_{L^0}(|z_{\tau,t}|,0)\,\mathrm{d}t = \int_0^1\int 1\wedge|z_{\tau,t}|\,\mathrm{d}\mathfrak{m}'\,\mathrm{d}t \le \int 1\wedge\Big(\int_0^1 |z_{\tau,t}|\,\mathrm{d}t\Big)\,\mathrm{d}\mathfrak{m}' \quad \overset{(4.57),(4.58)}{\to} \quad 0,$$

which is (4.56). □

It is now easy to establish the desired version of Hille's theorem:

Theorem 4.4.9 (A Hille-type result)**.** *Let $\mathscr{H}_1,\mathscr{H}_2$ be separable Hilbert $L^0(\mathfrak{m})$-modules, $V\subset\mathscr{H}_1$ a vector subspace, $\|\cdot\|_V : V\to\mathbb{R}^+$ a norm on V and $L : V\to\mathscr{H}_2$ a linear map. Let $(v_t)\in L^1([0,1],\mathscr{H}_1)$ and assume that:*

i) $v_t\in V$ for a.e. $t\in[0,1]$ and $t\mapsto L(v_t)$ is in $L^1([0,1],\mathscr{H}_2)$,

ii) for any $C>0$ the set $\mathrm{Graph}(L)\cap(\{v\in V:\|v\|_V\le C\}\times\mathscr{H}_2)\subset\mathscr{H}_1\times\mathscr{H}_2$ *is closed,*

iii) $\int_0^1\|v_t\|_V\,\mathrm{d}t<\infty$.

Then $\int_0^1 v_t\,\mathrm{d}t\in V$ and $L(\int_0^1 v_t\,\mathrm{d}t)=\int_0^1 L(v_t)\,\mathrm{d}t$.

Proof. Start noticing that for any $C>0$ we have

$$\{t\ :\ \|v_t\|_V\le C\} = \big\{t\ :\ (v_t,L(v_t))\in\mathrm{Graph}(L)\cap(\{v\in V:\|v\|_V\le C\}\times\mathscr{H}_2)\big\},$$

thus from (*ii*) and the assumed Borel regularity of $t\mapsto v_t, L(v_t)$ we deduce that $t\mapsto\|v_t\|_V$ is Borel, hence assumption (*iii*) makes sense.

Let $\mathscr{H}:=\mathscr{H}_1\times\mathscr{H}_2$ be equipped with the pointwise norm $|(v,z)|^2:=|v|^2+|z|^2$ $\mathfrak{m}$-a.e. for every $v\in\mathscr{H}_1$, $z\in\mathscr{H}_2$. It is clear that $\mathscr{H}$ is a Hilbert $L^0(\mathrm{X})$-normed module, when equipped with the obvious distance and product with $L^0(\mathrm{X})$ functions defined componentwise. Also, directly by definition of integral, testing with elements of the form $(v,0)$ and $(0,z)$ we see that $\int_0^1(v_t,z_t)\,\mathrm{d}t = (\int_0^1 v_t\,\mathrm{d}t,\int_0^1 z_t\,\mathrm{d}t)$.

Now observe that the assumptions $(v_t) \in L^1_{\mathscr{H}_1}$ and $(L(v_t)) \in L^1_{\mathscr{H}_2}$ together with the trivial bound $|(v,z)| \leq |v| + |z|$ show that $t \mapsto (v_t, L(v_t))$ is in $L^1_{\mathscr{H}}$. We can therefore apply Proposition 4.4.8 above to $t \mapsto (v_t, L(v_t)) \in \mathscr{H}$ to find $\tau_k \downarrow 0$ so that for a.e. $t \in [0,1]$ we have

$$\lim_{k\to\infty} \tau_k \sum_{i=0}^{[\tau_k^{-1}]-1} \big(v_{\tau_k(t+i)}, L(v_{\tau_k(t+i)})\big) = \Big(\int_0^1 v_s \,\mathrm{d}s, \int_0^1 L(v_s)\,\mathrm{d}s\Big) \qquad \text{in } \mathscr{H}. \tag{4.59}$$

By the original result of Hahn (or, which is the same, by Proposition 4.4.8 with $\mathscr{H} := \mathbb{R}$, that is a L^0-normed module over a Dirac mass) we also know that for a.e. $t \in [0,1]$ we have

$$\varlimsup_{k\to\infty} \Big\|\tau_k \sum_{i=0}^{[\tau_k^{-1}]-1} v_{\tau_k(t+i)}\Big\|_V \leq \lim_{k\to\infty} \tau_k \sum_{i=0}^{[\tau_k^{-1}]-1} \|v_{\tau_k(t+i)}\|_V = \int_0^1 \|v_s\|_V \,\mathrm{d}s.$$

Hence there is $t \in [0,1]$ such that $C := \sup_k \Big\|\tau_k \sum_{i=0}^{[\tau_k^{-1}]-1} v_{\tau_k(t+i)}\Big\|_V < \infty$ and (4.59) holds. Thus the argument of the limit in the left-hand side of (4.59) is in $\mathrm{Graph}(L) \cap (\{v \in V : \|v\|_V \leq C\} \times \mathscr{H}_2)$, and since by assumption this set is closed, it contains also the limit. In particular, this means that the right-hand side is in $\mathrm{Graph}(L)$, which is the conclusion. □

4.5 Linear and continuous operators on topological vector spaces

The topological vector spaces we are dealing with are metrized by translation invariant distances, therefore linear and continuous operators are uniformly continuous. To see this, let $\mathsf{d}_1, \mathsf{d}_2$ be the distances on two such spaces V_1, V_2 and $T : V_1 \to V_2$ be the linear operator, then if $\mathsf{d}_1(v_n, z_n) \to 0$ we have $\mathsf{d}_1(v_n - z_n, 0) \to 0$ and thus $\mathsf{d}_2(T(v_n) - T(z_n), 0) = \mathsf{d}_2(T(v_n - z_n), 0) \to 0$ and therefore $\mathsf{d}_2(T(v_n), T(z_n)) \to 0$ (in fact this is not really due to distances, but to the fact that topological vector spaces, much like topological groups, are uniform spaces).

This can be seen as a weaker counterpart to the fact that a linear operator between Banach spaces is continuous if and only if it is Lipschitz.

We can apply this general principle to the product by a given function $G \in L^0(\mathrm{X})$, which is a linear continuous map from $L^0(\mathrm{X})$ to itself, to deduce

$$\mathsf{d}_{L^0}(Gf, Gg) \leq \Omega\big(\mathsf{d}_{L^0}(f,g)\big), \qquad \forall f, g \in L^0(\mathrm{X}). \tag{4.60}$$

The same principle applied to the linear and continuous embedding of $L^p(\mathrm{X})$ into $L^0(\mathrm{X})$ gives

$$\mathsf{d}_{L^0}(f,g) \leq \Omega\big(\|f-g\|_{L^p}\big), \qquad \forall f, g \in L^p(\mathrm{X}). \tag{4.61}$$

Clearly, the map Ω appearing in the last inequality depends on the fixed function G and the chosen exponent p, but since such dependence is not important in our discussion, to keep the notation simpler we will not emphasize this fact. We shall often use the following simple lemma in conjunction with the bound in (5.1) in the next section.

Lemma 4.5.1. *Let $G_i \in L^0(\mathrm{X})$, $i \in I$, be such that*

$$\lim_{C\to+\infty} \sup_{i\in I} \mathfrak{m}'\big(\{|G_i| \geq C\}\big) = 0. \tag{4.62}$$

Then the operators $L^0(\mathrm{X}) \ni f \mapsto G_i f \in L^0(\mathrm{X})$ are uniformly continuous, i.e. for some modulus of continuity Ω independent of $i \in I$ we have

$$\mathsf{d}_{L^0}(G_i f, G_i g) \leq \Omega\big(\mathsf{d}_{L^0}(f,g)\big), \qquad \forall f,g \in L^0(\mathrm{X}),\ i \in I. \tag{4.63}$$

Proof. Fix $\varepsilon \in (0,1)$, let $C_\varepsilon > 1$ be such that $\mathfrak{m}'\big(\{|G_i| \geq C_\varepsilon\}\big) \leq \varepsilon$ for every $i \in I$ and put $\delta := \frac{\varepsilon^2}{C_\varepsilon}$. We shall prove that

$$\mathsf{d}_{L^0}(f_1,f_2) \leq \delta \qquad \Rightarrow \qquad \mathsf{d}_{L^0}(G_i f_1, G_i f_2) \leq 3\varepsilon, \qquad \forall i \in I, \tag{4.64}$$

that, by the arbitrariness of ε, gives the claim. Put for brevity $f := |f_1 - f_2|$ and use Cavalieri's formula to get

$$\delta \geq \mathsf{d}_{L^0}(f_1,f_2) = \int_0^1 \mathfrak{m}'(\{f>t\})\,\mathrm{d}t \geq \int_0^{\frac{\delta}{\varepsilon}} \mathfrak{m}'(\{f>t\})\,\mathrm{d}t \geq \tfrac{\delta}{\varepsilon}\mathfrak{m}'\big(\{f > \tfrac{\delta}{\varepsilon}\}\big). \tag{4.65}$$

On the other hand, the trivial bound

$$\int_0^1 \mathfrak{m}'(\{|G_i|f>t\})\,\mathrm{d}t \leq \mathfrak{m}'(\{|G_i|f>\varepsilon\}) + \int_0^\varepsilon \mathfrak{m}'(\{|G_i|f>t\})\,\mathrm{d}t \leq \mathfrak{m}'(\{|G_i|f>\varepsilon\}) + \varepsilon$$

and the inclusion $\{|G_i|f > \varepsilon\} \subset \{|G_i| > C_\varepsilon\} \cup \{f > \frac{\varepsilon}{C_\varepsilon}\}$ give

$$\mathsf{d}_{L^0}(G_i f_1, G_i f_2) = \int_0^1 \mathfrak{m}'(\{|G_i|f>t\})\,\mathrm{d}t \leq \mathfrak{m}'(\{|G_i| > C_\varepsilon\}) + \mathfrak{m}'(\{f > \tfrac{\varepsilon}{C_\varepsilon}\}) + \varepsilon.$$

Since $\frac{\varepsilon}{C_\varepsilon} = \frac{\delta}{\varepsilon}$, the choice of C_ε and (4.65) give the claim (4.64) and thus (4.63). □

Chapter 5

An existence and uniqueness theory in $\mathsf{ncRCD}(K, N)$ spaces

5.1 The setting

Let us fix once and for all the assumptions and the notations that we shall use in the next sections: unless otherwise specified, our theorems will all be based on these:

- $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is a $\mathsf{ncRCD}(K, N)$ space, $K \in \mathbb{R}$, $N \in \mathbb{N}$.
- $(\boldsymbol{b}_t) \in L^2([0,1], W^{1,2}_C(T\mathrm{X}))$ is such that $|\boldsymbol{b}_t|, |\mathrm{div}(\boldsymbol{b}_t)| \in L^\infty([0,1] \times \mathrm{X})$ and for some $\bar{x} \in \mathrm{X}$ and $R > 0$ we have $\mathrm{supp}(\boldsymbol{b}_t) \subset B_R(\bar{x})$ for a.e. $t \in [0,1]$.
- (F^s_t) is the Regular Lagrangian Flow of $(\boldsymbol{b}_t)$.
- $(g_t) \in L^2([0,1], L^2(\mathrm{X}))$ is the function associated to $(\boldsymbol{b}_t)$ as in Proposition 4.1.1.
- $\mathfrak{m}'$ is a Borel probability measure on X such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$ and coinciding with $c\mathfrak{m}$ on $B_R(\bar{x})$ for some $c > 0$.
- The distance d_{L^0} on the space $L^0(\mathrm{X})$ is given by

  $$\mathsf{d}_{L^0}(f, g) := \int 1 \wedge |f - g| \, \mathrm{d}\mathfrak{m}', \qquad \forall f, g \in L^0(\mathrm{X}).$$

  Similarly the distance $\mathsf{d}_{L^0(T^*\mathrm{X})}$ on $L^0(T^*\mathrm{X})$ is defined as

  $$\mathsf{d}_{L^0(T^*\mathrm{X})}(\omega, \eta) := \int 1 \wedge |\omega - \eta| \, \mathrm{d}\mathfrak{m}', \qquad \forall \omega, \eta \in L^0(T^*\mathrm{X})$$

  and analogously for the distance $\mathsf{d}_{L^0(T\mathrm{X})}$ on $L^0(T\mathrm{X})$.
- G is a non-negative function in $L^0(\mathrm{X})$ acting as 'generic constant'. Its actual value may change in the various instances where it appears, but it depends only on the structural data (i.e. the space and the vector field $(\boldsymbol{b}_t)$), but not on the specific vector fields $(v_t), (V_t)$ we shall work with later on.

Occasionally, we will need to work with such 'generic function' depending on some additional parameter (often a time variable). In this case we shall write G_i, $i \in I$, to emphasize such dependence. In any case, whenever we do so we tacitly (or explicitly) assume that the G_i's are 'uniformly small in L^0', i.e. it will always be intended that they satisfy

$$\lim_{C\to+\infty}\sup_{i\in I}\mathfrak{m}'\big(\{|G_i|\geq C\}\big)=0. \tag{5.1}$$

- $\Omega:\mathbb{R}^+\to\mathbb{R}^+$ is a generic modulus of continuity, i.e. a non-decreasing continuous function such that $\Omega(z)\downarrow 0$ as $z\downarrow 0$. Much like for the function G its actual value may change in the various instances where it appears, but it depends only on the structural data of the problem.

Notice that the choice of $\mathfrak{m}'$, the fact that the flow (F_t^s) has bounded compression, and the fact that F_t^s is the identity on the complement of $\mathrm{supp}(\boldsymbol{b})$, ensure that

$$(F_t^s)_*\mathfrak{m}'\leq C\mathfrak{m}',\qquad \forall t,s\in[0,1], \tag{5.2}$$

for some $C>0$. In particular, this implies

$$C^{-1}\mathsf{d}_{L^0}(f,g)\leq \mathsf{d}_{L^0}(f\circ F_t^s,g\circ F_t^s)\leq C\mathsf{d}_{L^0}(f,g),\qquad \forall t,s\in[0,1],\ \forall f,g\in L^0(\mathrm{X}). \tag{5.3}$$

We point out that the considerations and the statements of Section 4.5 apply.

It will be useful to notice that for $(f_s)\in L^2([0,1],L^2(\mathrm{X}))$, from the bounded compression property of the flow we deduce that $(f_s\circ F_t^s)\in L^2([0,1],L^2(\mathrm{X}))$ as well for any $t\in[0,1]$. Thus from Fubini's theorem we deduce that

$$(f_s)\in L^2([0,1],L^2(\mathrm{X}))\qquad\Rightarrow\qquad \int_0^1|f_r|^2\circ F_t^r\,\mathrm{d}r<\infty,\quad \mathfrak{m}-a.e.,\ \forall t\in[0,1]. \tag{5.4}$$

In particular, applying this to $f_s:=|\nabla\boldsymbol{b}_s|$ we deduce that

$$\int_0^1|\nabla\boldsymbol{b}_t|^2\circ F_0^t\,\mathrm{d}t\leq G\qquad \mathfrak{m}-a.e., \tag{5.5}$$

according to our convention on G described above. Similarly, from (5.4) applied to (g_s), the trivial inequality $e^a\leq 1+ae^b$ valid for any $0\leq a\leq b$, and the estimate (4.28), we deduce the key bound

$$|\mathrm{d}F_t^s|\circ F_{t'}^t\leq 1+G_{t'}\int_{t\wedge s}^{t\vee s}g_r\circ F_{t'}^r\,\mathrm{d}r\qquad \mathfrak{m}-a.e.,\ \forall t',t,s\in[0,1], \tag{5.6}$$

where $G_{t'}:=e^{\int_0^1 g_r\circ F_{t'}^r\,\mathrm{d}r}$. We claim that the functions $G_{t'}$ so defined satisfy the uniform bound (5.1), so that our notation is justified according to what we declared at the beginning of the section. To see this, notice that we have the uniform estimate

$$\int\Big|\int_0^1 g_r\circ F_{t'}^r\,\mathrm{d}r\Big|^2\mathrm{d}\mathfrak{m}\leq\int_0^1\int|g_r|^2\circ F_{t'}^r\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r\overset{(2.13)}{\leq}C\iint_0^1|g_r|^2\,\mathrm{d}r\,\mathrm{d}\mathfrak{m},\qquad\forall t'\in[0,1],$$

where C is the compressibility constant of F_t^s; thus Chebyshev's inequality gives for every $M > 0$

$$\mathfrak{m}(\{G_{t'} > M\}) = \mathfrak{m}\Big(\Big\{\big|\int_0^1 g_r \circ F_{t'}^r \,\mathrm{d}r\big|^2 > |\log M|^2\Big\}\Big) \le \frac{C \iint_0^1 |g_r|^2 \,\mathrm{d}r \,\mathrm{d}\mathfrak{m}}{|\log M|^2}, \qquad \forall t' \in [0,1]$$

and by the absolute continuity of the integral it follows that

$$\lim_{M \to +\infty} \sup_{t' \in [0,1]} \mathfrak{m}'\big(\{G_{t'} \ge M\}\big) = 0. \tag{5.7}$$

Notice that (5.6) trivially implies the weaker, but still useful, bound

$$|\mathrm{d}F_t^s| \circ F_{t'}^t \le G_{t'} \qquad \mathfrak{m} - a.e., \ \forall t', t, s \in [0,1], \tag{5.8}$$

for some $G_{t'}$'s that still satisfy (5.1).

5.2 Time dependent vector fields at 'fixed' and 'variable' points

We have seen in Section 4.2 that for every $t \in \mathbb{R}$ the map F_0^t admits a differential $\mathrm{d}F_0^t : L^0(T\mathrm{X}) \to L^0(T\mathrm{X})$. We now want to consider all these maps as t varies in $[0,1]$.

In dealing with composition with the flow maps, the following simple lemma will be useful:

Lemma 5.2.1. $[0,1]^2 \ni (t,s) \mapsto f_t^s \in L^0(\mathrm{X})$ *is continuous if and only if* $[0,1]^2 \ni (t,s) \mapsto f_t^s \circ F_t^s \in L^0(\mathrm{X})$ *is continuous.*

Proof. Since $(F_t^s)^{-1} = F_s^t$, it is sufficient to prove the 'only if'. Thus suppose that $t, s \mapsto f_t^s$ is continuous in $L^0(\mathrm{X})$ and notice that

$$\begin{aligned}\mathsf{d}_{L^0}\big(f_{t'}^{s'} \circ F_{t'}^{s'}, f_t^s \circ F_t^s\big) &\le \mathsf{d}_{L^0}\big(f_{t'}^{s'} \circ F_{t'}^{s'}, f_t^s \circ F_{t'}^{s'}\big) + \mathsf{d}_{L^0}\big(f_t^s \circ F_{t'}^{s'}, f_t^s \circ F_t^s\big) \\ &\le C\, \mathsf{d}_{L^0}\big(f_{t'}^{s'}, f_t^s\big) + \mathsf{d}_{L^0}\big(f_t^s \circ F_{t'}^{s'}, f_t^s \circ F_t^s\big)\end{aligned}$$

having used the uniform bound (5.3). The conclusion follows recalling (4.5). □

The regularity in time of $\mathrm{d}F_t^s$ will be obtained by duality starting from the following result:

Lemma 5.2.2. *Let* $f : \mathrm{X} \to \mathbb{R}$ *be Lusin–Lipschitz. Then* $[0,1]^2 \ni (t,s) \mapsto \mathrm{d}(f \circ F_t^s) \in L^0(T^*\mathrm{X})$ *is continuous.*

Proof. We start claiming that for any f Lusin–Lipschitz we have

$$\lim_{s' \to s} \mathrm{d}(f \circ F_s^{s'})(v) = \mathrm{d}f(v) = \lim_{t' \to t} \mathrm{d}(f \circ F_{t'}^t)(v), \qquad in\ L^0(\mathrm{X}),\ \forall v \in L^0(T\mathrm{X}),\ \forall t, s \in [0,1]. \tag{5.9}$$

We start with the first limit and notice that by definition of differential for Lusin–Lipschitz maps we have $\mathrm{d}((\chi_E f) \circ F_s^{s'}) = \chi_E \circ F_s^{s'}\, \mathrm{d}(f \circ F_s^{s'})$, thus taking into account (4.5) and using the fact that the flow is the identity outside a bounded set, it is easy to see that the first in (5.9) will follow in the general case if we prove it just for f Lipschitz. Thus let this be the case and recall from (4.12) that there is a Borel partition (E_i) of $\mathfrak{m}$-a.a. X made of bounded sets such that $\sup_{s' \in [0,1]} \mathrm{Lip}(F_s^{s'}|_{E_i}) < \infty$ for every $i \in \mathbb{N}$. Thus $\sup_{s' \in [0,1]} \mathrm{Lip}((f \circ F_s^{s'})|_{E_i}) < \infty$ and by

L^0-linearity and continuity, the first in (5.9) will follow if we show that for any $i \in \mathbb{N}$ we have $\chi_{E_i} \mathrm{d}(f \circ F_s^{s'}) \rightharpoonup \mathrm{d}f$ in the weak topology of $L^2(T^*\mathrm{X})$ as $s' \to s$.

Let $(s'_n) \subset [0,1]$ be converging to s and, for every $n \in \mathbb{N} \cup \{\infty\}$, let g_n be uniformly Lipschitz with uniformly bounded support such that $g_n|_{E_i} = (f \circ F_s^{s'_n})|_{E_i}$ (it is easy to see that they can be found). Then (g_n) is a bounded sequence in $W^{1,2}(\mathrm{X})$, hence up to pass to a non-relabeled subsequence we can assume that it weakly converges to some $g \in W^{1,2}(\mathrm{X})$. Clearly, $g = f$ on E_i, thus from the locality of the differential we see that

$$\chi_{E_i} \mathrm{d}(f \circ F_s^{s'_n}) = \chi_{E_i} \mathrm{d}g_n \overset{L^2(T^*\mathrm{X})}{\rightharpoonup} \chi_{E_i} \mathrm{d}g = \chi_{E_i} \mathrm{d}f.$$

As this result does not depend on the subsequence chosen, the first in (5.9) follows.

We turn to the second and start noticing that arguing as for Lemma 5.2.1 above it is sufficient to prove that

$$\mathrm{d}(f \circ F_{t'}^t)(v) \circ F_t^{t'} \to \mathrm{d}f(v) \qquad \textit{in } L^0(\mathrm{X}),\ \forall v \in L^0(T\mathrm{X}),\ \forall t \in [0,1]. \tag{5.10}$$

Then taking into account the uniform bound

$$|\mathrm{d}(f \circ F_{t'}^t)(v) \circ F_t^{t'}| \overset{(4.23)}{\leq} |v| \circ F_t^{t'} |\mathrm{d}f| |\mathrm{d}F_{t'}^t| \circ F_t^{t'} \overset{(5.8)}{\leq} G_t |v| \circ F_t^{t'} |\mathrm{d}f|,$$

the fact that $|v| \circ F_t^{t'} \to |v|$ in $L^0(\mathrm{X})$ as $t' \to t$ (recall (4.5)), and the density of differentials of Lusin–Lipschitz functions in $L^0(T^*\mathrm{X})$ (already noticed in the proof of Theorem 4.2.2), we see that to prove (5.10), and thus the second in (5.9), it is sufficient to prove that

$$\langle \mathrm{d}(f \circ F_{t'}^t), \mathrm{d}h \rangle \circ F_t^{t'} \to \langle \mathrm{d}f, \mathrm{d}h \rangle \qquad \textit{in } L^0(\mathrm{X}),\ \forall h : \mathrm{X} \to \mathbb{R} \text{ Lusin–Lipschitz},\ \forall t \in [0,1]. \tag{5.11}$$

Now we put for brevity $\hat{g}_t^{t'} := \int_{t \vee t'}^{t \wedge t'} g_r \circ F_t^r \, \mathrm{d}r \in L^0(\mathrm{X})$ and notice that, rather trivially, we have

$$\lim_{t' \to t} \hat{g}_t^{t'} = 0, \qquad \textit{in } L^0(\mathrm{X}) \tag{5.12}$$

(as for any sequence $t'_n \to t$ we have $\mathfrak{m}$-a.e. convergence). The fact that $(1+a)^{-2} \geq (1-a)^2$ for any $a \in \mathbb{R}$ grants, together with (5.6), that

$$-|\mathrm{d}F_t^{t'}|^{-2} \leq -(1 + G_t \hat{g}_t^{t'})^{-2} \leq -(1 - G_t \hat{g}_t^{t'})^2. \tag{5.13}$$

Also notice that (4.23) with $f \circ F_{t'}^t$ in place of f and $F_t^{t'}$ in place of φ gives $|\mathrm{d}(f \circ F_{t'}^t)| \circ F_t^{t'} \geq |\mathrm{d}f| |\mathrm{d}F_t^{t'}|^{-1}$. Similarly, one can show that $|\mathrm{d}h| \circ F_t^{t'} \geq |\mathrm{d}(h \circ F_t^{t'})| |\mathrm{d}F_t^{t'}|^{-1}$. Using these bound in conjunction with (4.23) gives

$$\begin{aligned}
\langle \mathrm{d}(f \circ F_{t'}^t), \mathrm{d}h \rangle \circ F_t^{t'} &= \tfrac{1}{2} \Big[|\mathrm{d}(f \circ F_{t'}^t + h)|^2 \circ F_t^{t'} - (|\mathrm{d}(f \circ F_{t'}^t)|^2 + |\mathrm{d}h|^2) \circ F_t^{t'} \Big] \\
&\leq \tfrac{1}{2} \Big[|\mathrm{d}(f + h \circ F_t^{t'})|^2 \, |\mathrm{d}F_{t'}^t|^2 \circ F_t^{t'} - |\mathrm{d}F_t^{t'}|^{-2} (|\mathrm{d}f|^2 + |\mathrm{d}(h \circ F_t^{t'})|^2) \Big] \\
\text{(by (5.6) and (5.13))} \qquad &\leq \tfrac{1}{2} \Big[\big(1 + G_t \hat{g}_t^{t'}\big)^2 |\mathrm{d}(f + h \circ F_t^{t'})|^2 - (1 - G_t \hat{g}_t^{t'})^2 (|\mathrm{d}f|^2 + |\mathrm{d}(h \circ F_t^{t'})|^2) \Big] \\
&= \big(1 + G_t \hat{g}_t^{t'}\big)^2 \langle \mathrm{d}f, \mathrm{d}(h \circ F_t^{t'}) \rangle + 2(|\mathrm{d}f|^2 + |\mathrm{d}(h \circ F_t^{t'})|^2) \, G_t \hat{g}_t^{t'} \\
&\leq \big(1 + G_t \hat{g}_t^{t'}\big)^2 \langle \mathrm{d}f, \mathrm{d}(h \circ F_t^{t'}) \rangle + 2(|\mathrm{d}f|^2 + |\mathrm{d}h|^2 \circ F_t^{t'} G_t^2) \, G_t \hat{g}_t^{t'}.
\end{aligned}$$

Now notice that since we already proved the first in (5.9), as $t' \to t$ we have that $\langle \mathrm{d}f, \mathrm{d}(h \circ F_t^{t'})\rangle \to \langle \mathrm{d}f, \mathrm{d}h\rangle$ in $L^0(\mathrm{X})$. Moreover, from (4.5), we have that $|\mathrm{d}h|^2 \circ F_t^{t'} \to |\mathrm{d}h|^2$ in $L^0(\mathrm{X})$ as $t' \to t$. Thus taking (5.12) into account we just proved that

$$\big(\langle \mathrm{d}(f \circ F_{t'}^t), \mathrm{d}h\rangle \circ F_t^{t'} - \langle \mathrm{d}f, \mathrm{d}h\rangle\big)^+ \to 0 \qquad \text{in } L^0(\mathrm{X}).$$

A very similar argument gives that the negative part of the same quantity as above goes to 0 in $L^0(\mathrm{X})$, hence the claim (5.11), and thus the second in (5.9), follows.

Now we claim that

$$\lim_{s' \to s} \mathrm{d}(f \circ F_s^{s'}) = \mathrm{d}f = \lim_{t' \to t} \mathrm{d}(f \circ F_{t'}^t), \qquad \text{in } L^0(T^*\mathrm{X}),\ \forall t, s \in [0,1]. \tag{5.14}$$

Indeed

$$\begin{aligned} |\mathrm{d}(f \circ F_{t'}^t) - \mathrm{d}f|^2 \circ F_t^{t'} &\le |\mathrm{d}f|^2 |\mathrm{d}F_{t'}^t|^2 \circ F_t^{t'} + |\mathrm{d}f|^2 \circ F_t^{t'} - 2\langle \mathrm{d}(f \circ F_{t'}^t), \mathrm{d}f\rangle \circ F_t^{t'} \\ \text{(by (5.6))} \qquad &\le |\mathrm{d}f|^2 \big(1 + G_t\, \hat{g}_t^{t'}\big)^2 + |\mathrm{d}f|^2 \circ F_t^{t'} - 2\langle \mathrm{d}(f \circ F_{t'}^t), \mathrm{d}f\rangle \circ F_t^{t'} \\ \text{(by Lemma 5.2.1, (5.11), and (5.12))} \qquad &\to \quad |\mathrm{d}f|^2 + |\mathrm{d}f|^2 - 2|\mathrm{d}f|^2 = 0, \end{aligned}$$

the convergence being in $L^0(\mathrm{X})$ as $t' \to t$. The second in (5.14) follows using again Lemma 5.2.1 to conclude that $|\mathrm{d}(f \circ F_{t'}^t) - \mathrm{d}f|^2 \to 0$ in $L^0(\mathrm{X})$. The first in (5.14) is proved analogously.

Now observe that from the group property (4.1) and the chain rule (4.25) we get

$$\begin{aligned} |\mathrm{d}(f \circ F_{t'}^{s'}) - \mathrm{d}(f \circ F_t^s)| &\le |\mathrm{d}(f \circ F_s^{s'} \circ F_{t'}^s) - \mathrm{d}(f \circ F_{t'}^s)| + |\mathrm{d}(f \circ F_t^s \circ F_{t'}^t) - \mathrm{d}(f \circ F_t^s)| \\ &\le |\mathrm{d}(f \circ F_s^{s'}) - \mathrm{d}f| \circ F_{t'}^s\, |\mathrm{d}F_{t'}^s| + |\mathrm{d}(f \circ F_t^s \circ F_{t'}^t) - \mathrm{d}(f \circ F_t^s)| \\ \text{(by (5.8))} \qquad &\le G_{t'}\, |\mathrm{d}(f \circ F_s^{s'}) - \mathrm{d}f| \circ F_{t'}^s + |\mathrm{d}(f \circ F_t^s \circ F_{t'}^t) - \mathrm{d}(f \circ F_t^s)|. \end{aligned}$$

Hence

$$\begin{aligned} \mathsf{d}_{L^0(T^*\mathrm{X})}(\mathrm{d}(f \circ F_{t'}^{s'}), \mathrm{d}(f \circ F_t^s)) &= \mathsf{d}_{L^0}(|\mathrm{d}(f \circ F_{t'}^{s'}) - \mathrm{d}(f \circ F_t^s)|, 0) \\ &\le \mathsf{d}_{L^0}(G_{t'}\, |\mathrm{d}(f \circ F_s^{s'}) - \mathrm{d}f| \circ F_{t'}^s, 0) \\ &\qquad + \mathsf{d}_{L^0}(|\mathrm{d}(f \circ F_t^s \circ F_{t'}^t) - \mathrm{d}(f \circ F_t^s)|, 0) \\ \text{(by (5.7), Lemma 4.5.1 and (5.3))} \qquad &\le \Omega\big(\mathsf{d}_{L^0}(|\mathrm{d}(f \circ F_s^{s'}) - \mathrm{d}f|, 0)\big) \\ &\qquad + \mathsf{d}_{L^0}(|\mathrm{d}(f \circ F_t^s \circ F_{t'}^t) - \mathrm{d}(f \circ F_t^s)|, 0) \\ &= \Omega\big(\mathsf{d}_{L^0(T^*\mathrm{X})}(\mathrm{d}(f \circ F_s^{s'}), \mathrm{d}f)\big) \\ &\qquad + \mathsf{d}_{L^0(T^*\mathrm{X})}(\mathrm{d}((f \circ F_t^s) \circ F_{t'}^t), \mathrm{d}(f \circ F_t^s)) \end{aligned}$$

and the conclusion follows from (5.14) and the fact that $f \circ F_t^s$ is Lusin–Lipschitz. □

We then have the following basic regularity result:

Proposition 5.2.3. *Let $(v_t) \in C([0,1], L^0(T\mathrm{X}))$. Then $t \mapsto V_t := \mathrm{d}F_0^t(v_t) \in L^0(T\mathrm{X})$ is also continuous. Moreover, the assignment $(v_t) \mapsto (V_t)$ from $C([0,1], L^0(T\mathrm{X}))$ to itself is continuous, invertible, with continuous inverse.*

Similarly, if $(v_t) \in L^0([0,1], L^0(T\mathrm{X}))$ the map $t \mapsto V_t := \mathrm{d}F_0^t(v_t) \in L^0(T\mathrm{X})$ is an element of $L^0([0,1], L^0(T\mathrm{X}))$ and the assignment $(v_t) \mapsto (V_t)$ from $L^0([0,1], L^0(T\mathrm{X}))$ to itself is continuous, invertible, with continuous inverse.

Proof.
STEP 1. We claim that

$$\forall v \in L^0(T\mathrm{X}) \text{ the map } [0,1]^2 \ni (t,s) \mapsto \mathrm{d}F_t^s(v) \in L^0(T\mathrm{X}) \text{ is continuous} \tag{5.15}$$

and we shall prove this by arguing as in Lemma 5.2.2 above. Fix $v \in L^0(T\mathrm{X})$. From Lemma 5.2.2 we see that for f Lusin–Lipschitz the map $[0,1]^2 \ni (t,s) \mapsto \mathrm{d}(f \circ F_t^s)(v) = \mathrm{d}f(\mathrm{d}F_t^s(v)) \circ F_t^s \in L^0(\mathrm{X})$ is continuous. Taking into account Lemma 5.2.1 we deduce that also $[0,1]^2 \ni (t,s) \mapsto \mathrm{d}f(\mathrm{d}F_t^s(v)) \in L^0(\mathrm{X})$ is continuous. We now claim that

$$[0,1]^2 \ni (t,s) \mapsto \langle \mathrm{d}F_t^s(v), z\rangle \in L^0(\mathrm{X}) \quad \text{ is continuous for every } z \in L^0(T\mathrm{X}) \tag{5.16}$$

and since differentials of Lusin–Lipschitz maps are dense in $L^0(T^*\mathrm{X})$, this follows by what already established and the uniform estimate

$$\begin{aligned}
\mathsf{d}_{L^0}\big(\langle \mathrm{d}F_t^s(v), z\rangle\,,\, \langle \mathrm{d}F_t^s(v), z'\rangle\big) &\leq \mathsf{d}_{L^0}\big(|\mathrm{d}F_t^s(v)||z-z'|, 0\big)\\
\text{(by (4.22),(5.8))} \qquad &\leq \mathsf{d}_{L^0}\big(G_t|v| \circ F_s^t|z-z'|, 0\big)\\
\text{(by Lemma 4.5.1 and (5.7))} \qquad &\leq \Omega\big(\mathsf{d}_{L^0}\big(|v| \circ F_s^t|z-z'|, 0\big)\big)\\
\text{(by Lemma 4.5.1)} \qquad &\leq \Omega\big(\mathsf{d}_{L^0}\big(|z-z'|, 0\big)\big),
\end{aligned}$$

valid for any $t, s \in [0,1]$ (the resulting modulus of continuity will depend on $|v|$, but this is not an issue to get (5.16) - notice also that in applying Lemma 4.5.1 in the last step we used the fact that $\{|v| \circ F_s^t > c\} = (F_s^t)^{-1}(\{|v| > c\})$ and (5.2)). Now we claim that

$$\lim_{s' \to s} \mathrm{d}F_s^{s'}(v) = \lim_{t' \to t} \mathrm{d}F_{t'}^t(v) = v \qquad \textit{in } L^0(T\mathrm{X}). \tag{5.17}$$

For $t, t' \in [0,1]$ we put, as in Lemma 5.2.2 above, $\hat{g}_t^{t'} := \int_{t \vee t'}^{t \wedge t'} g_r \circ F_t^r \,\mathrm{d}r \in L^0(\mathrm{X})$ and recall that (5.12) holds. Thus

$$\begin{aligned}
|\mathrm{d}F_{t'}^t(v) - v|^2 &\leq \big(|v|\,|\mathrm{d}F_{t'}^t|\big)^2 \circ F_t^{t'} + |v|^2 - 2\langle v, \mathrm{d}F_{t'}^t(v)\rangle\\
\text{(by (5.6))} \qquad &\leq |v|^2 \circ F_t^{t'}\,(1 + G_t\hat{g}_t^{t'})^2 + |v|^2 - 2\langle v, \mathrm{d}F_{t'}^t(v)\rangle\\
\text{(by Lemma 5.2.1 and (5.12), (5.16))} \qquad &\to \quad |v|^2 + |v|^2 - 2|v|^2 = 0.
\end{aligned}$$

This proves the second in (5.17). The first follows by similar arguments.

Now recall the chain rule (4.25) to get that

$$\begin{aligned}
|\mathrm{d}F_{t'}^{s'}(v) - \mathrm{d}F_t^s(v)| &\leq |\mathrm{d}F_t^{s'}(\mathrm{d}F_{t'}^t(v) - v)| + |\mathrm{d}F_s^{s'}(\mathrm{d}F_t^s(v)) - \mathrm{d}F_t^s(v)|\\
\text{(by (4.22),(5.8))} \qquad &\leq \big(G_t|\mathrm{d}F_{t'}^t(v) - v|\big) \circ F_{s'}^t + |\mathrm{d}F_s^{s'}(\mathrm{d}F_t^s(v)) - \mathrm{d}F_t^s(v)|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathsf{d}_{L^0(T^*\mathrm{X})}(\mathrm{d}F_{t'}^{s'}(v), \mathrm{d}F_t^s(v)) &= \mathsf{d}_{L^0}(|\mathrm{d}F_{t'}^{s'}(v) - \mathrm{d}F_t^s(v)|, 0) \leq \mathsf{d}_{L^0}\big(\big(G_t|\mathrm{d}F_{t'}^t(v) - v|\big) \circ F_{s'}^t, 0\big)\\
&\quad + \mathsf{d}_{L^0}\big(|\mathrm{d}F_s^{s'}(\mathrm{d}F_t^s(v)) - \mathrm{d}F_t^s(v)|, 0\big)\\
\text{(by (5.7), Lemma 4.5.1 and (5.3))} \qquad &\leq \Omega\big(\mathsf{d}_{L^0}(|\mathrm{d}F_{t'}^t(v) - v|, 0)\big) + \mathsf{d}_{L^0}\big(|\mathrm{d}F_s^{s'}(\mathrm{d}F_t^s(v)) - \mathrm{d}F_t^s(v)|, 0\big)\\
&= \Omega\big(\mathsf{d}_{L^0(T\mathrm{X})}(\mathrm{d}F_{t'}^t(v), v)\big) + \mathsf{d}_{L^0(T\mathrm{X})}\big(\mathrm{d}F_s^{s'}(\mathrm{d}F_t^s(v)), \mathrm{d}F_t^s(v)\big).
\end{aligned}$$

Thus our claim (5.15) follows from (5.17).

STEP 2. Let $(v_t) \in C([0,1], L^0(T\mathrm{X}))$ and notice that

$$\begin{aligned}|\mathrm{d}F_0^s(v_s) - \mathrm{d}F_0^t(v_t)| &\leq |\mathrm{d}F_0^s(v_s - v_t)| + |\mathrm{d}F_0^s(v_t) - \mathrm{d}F_0^t(v_t)| \\ \text{(by (5.8))} \qquad &\leq (G|v_s - v_t|) \circ F_s^0 + |\mathrm{d}F_0^s(v_t) - \mathrm{d}F_0^t(v_t)|, \qquad \mathfrak{m} - a.e.\end{aligned}$$

for every $t, s \in [0,1]$. Thus, as before, (5.3), (4.60), and the continuity of (v_t) yield $(G|v_s - v_t|) \circ F_s^0 \to 0$ in $L^0(\mathrm{X})$ as $s \to t$, hence recalling (5.15) the continuity of $t \mapsto \mathrm{d}F_0^t(v_t) \in L^0(T\mathrm{X})$ follows.

The continuity of the assignment $(v_t) \mapsto (\mathrm{d}F_0^t(v_t))$ will follow if we show that for some modulus of continuity Ω we have

$$\mathsf{d}_{L^0(T\mathrm{X})}\big(\mathrm{d}F_0^t(v), \mathrm{d}F_0^t(z)\big) \leq \Omega\big(\mathsf{d}_{L^0(T\mathrm{X})}(v,z)\big), \qquad \forall t \in [0,1],\ v, z \in L^0(T\mathrm{X}). \tag{5.18}$$

To see this, notice that (5.3) and (5.8) give

$$\begin{aligned}\mathsf{d}_{L^0(T\mathrm{X})}(\mathrm{d}F_0^t(v), \mathrm{d}F_0^t(z)) &= \mathsf{d}_{L^0}(|\mathrm{d}F_0^t(v-z)|, 0) \leq \mathsf{d}_{L^0}((G|v-z|) \circ F_t^0, 0) \\ \text{(by (5.3),(4.60))} \qquad &\leq \Omega\big(\mathsf{d}_{L^0}(|v-z|, 0)\big) = \Omega\big(\mathsf{d}_{L^0(T\mathrm{X})}(v,z)\big).\end{aligned}$$

Observe now that the estimate (5.18) also tells that the map $(v_t) \mapsto (\mathrm{d}F_0^t(v_t))$ is uniformly continuous as a map from $C([0,1], L^0(T\mathrm{X}))$ with the $L^0([0,1], L^0(T\mathrm{X}))$-topology to $L^0([0,1], L^0(T\mathrm{X}))$. By the density of $C([0,1], L^0(T\mathrm{X}))$ in $L^0([0,1], L^0(T\mathrm{X}))$ we deduce that $(v_t) \mapsto (\mathrm{d}F_0^t(v_t))$ can be (uniquely) extended to a linear and continuous map from $L^0([0,1], L^0(T\mathrm{X}))$ to itself, and again (5.18) also ensures that such extension is still given by $(v_t) \mapsto (\mathrm{d}F_0^t(v_t))$.

Using the result (5.15) obtained in Step 1, we can prove that $(V_t) \mapsto (\mathrm{d}F_t^0(V_t))$ has the same continuity properties we established for $(v_t) \mapsto (\mathrm{d}F_0^t(v_t))$ and since these two maps are one the inverse of the other (by the chain rule (4.25)), the proof is finished. □

The source and target spaces of the map $(v_t) \mapsto (\mathrm{d}F_0^t(v_t))$, albeit formally equal, have different roles in the theory and will be treated differently: we shall think at the source space as a family of vector fields defined at fixed base points, i.e. (v_t) in the source space will be thought of as a collection of maps of the form $t \mapsto v_t(x)$"$\in T_x\mathrm{X}$". On the other hand, we shall think at elements (V_t) of the target space as a family of vector fields defined along flow lines, i.e. (V_t) in the target space will be thought of as a collection of maps of the form $t \mapsto V_t(F_0^t(x))$"$\in T_{F_0^t(x)}\mathrm{X}$".

This distinction is particularly relevant in the case of the spaces L^0 and to help keeping this in mind we shall denote the source space as $L^0_{fix}([0,1], L^0(T\mathrm{X}))$ (in short L^0_{fix}) and the target as $L^0_{var}([0,1], L^0(T\mathrm{X}))$ (in short L^0_{var}). The spaces L^0_{fix} and L^0_{var} are equipped with two different structures as modules over $L^0(\mathrm{X})$:

$$\text{for } (v_t) \in L^0_{fix} \text{ and } f \in L^0(\mathrm{X}) \text{ we define } f(v_t) \in L^0_{fix} \text{ as } t \mapsto f v_t \tag{5.19}$$

and

$$\text{for } (V_t) \in L^0_{var} \text{ and } f \in L^0(\mathrm{X}) \text{ we define } f \times (V_t) \in L^0_{var} \text{ as } t \mapsto f \circ F_t^0 V_t. \tag{5.20}$$

Notice that by (4.24) we see that these two products are conjugated via $\mathrm{d}F$ (here intended as the map sending $t \mapsto v_t$ to $t \mapsto \mathrm{d}F_0^t(v_t)$), i.e.

$$\mathrm{d}F\big(f(v_t)\big) = f \times \mathrm{d}F\big((v_t)\big).$$

Notice also that L^0_{fix} and L^0_{var} are *not* $L^0(\mathrm{X})$-normed modules, as there is not really a $L^0(\mathrm{X})$-valued pointwise norm on them (they could be endowed with the structure of $L^0([0,1]\times\mathrm{X})$-normed modules, but we are not interested in doing this).

In what follows we shall typically use lowercase letters (v_t) to denote an element of L^0_{fix} and uppercase ones (V_t) for elements of L^0_{var}. Also, typically (v_t) and (V_t) are related by

$$V_t = \mathrm{d}F_0^t(v_t) \qquad \text{and equivalently} \qquad v_t = \mathrm{d}F_t^0(V_t). \tag{5.21}$$

We are now ready to discuss time integrability/regularity for vector fields in L^0_{fix} and the related concept of derivative in time. In fact, given the discussion made in Section 4.4, and in particular Definition 4.4.5, the following definitions are quite natural:

Definition 5.2.4 (Some spaces of vectors at 'fixed points')**.** *We shall denote by*

i) $L^p_{fix}([0,1],L^0(T\mathrm{X}))\subset L^0_{fix}([0,1],L^0(T\mathrm{X}))$ *(or simply* L^p_{fix}*),* $p\in[1,\infty]$*, the space* $L^p_{\mathscr{H}}$ *for* $\mathscr{H}:=L^0(T\mathrm{X})$*.*

ii) $W^{1,2}_{fix}([0,1],L^0(T\mathrm{X}))\subset L^0_{fix}([0,1],L^0(T\mathrm{X}))$ *(or simply* $W^{1,2}_{fix}$*) the space* $W^{1,2}_{\mathscr{H}}$ *for* $\mathscr{H}:=L^0(T\mathrm{X})$*.*

iii) $AC^2_{fix}([0,1],L^0(T\mathrm{X}))\subset C([0,1],L^0(T\mathrm{X}))$ *(or simply* AC^2_{fix}*) the space* $AC^2_{\mathscr{H}}$ *for* $\mathscr{H}:=L^0(T\mathrm{X})$*.*

Recall from Section 4.4 that the elements (v_t) of $W^{1,2}_{fix}$ come with a natural notion of derivative $(\dot v_t)\in L^2_{fix}$, that the space L^p_{fix} can be characterized as the subspace of L^0_{fix} of those (v_t)'s for which the pointwise norm

$$|(v_t)|_{L^p_{fix}}(x) := |(|v_t|)|_{L^p}(x) = \||v_\cdot|(x)\|_{L^p(0,1)}$$

is finite $\mathfrak{m}$-a.e., and that $W^{1,2}_{fix}$ comes with the pointwise norm

$$|(v_t)|^2_{W^{1,2}_{fix}} := |(v_t)|^2_{L^2_{fix}} + |(\dot v_t)|^2_{L^2_{fix}}.$$

Also, $L^p_{fix}, W^{1,2}_{fix}, AC^2_{fix}$ are all complete w.r.t. the corresponding distances

$$\begin{aligned}
\mathsf{d}_{L^p_{fix}}\big((v_t),(z_t)\big) &:= \mathsf{d}_{L^0}(|(v_t-z_t)|_{L^p_{fix}},0),\\
\mathsf{d}_{W^{1,2}_{fix}}\big((v_t),(z_t)\big) &:= \mathsf{d}_{L^0}(|(v_t-z_t)|_{W^{1,2}_{fix}},0),\\
\mathsf{d}_{AC^2_{fix}}\big((v_t),(z_t)\big) &:= \mathsf{d}_{L^0}(|(v_t-z_t)|_{W^{1,2}_{fix}},0) + \sup_{t\in[0,1]} \mathsf{d}_{L^0(T\mathrm{X})}(v_t,z_t).
\end{aligned}$$

Finally, it is clear that the product defined in (5.19) gives $L^p_{fix}, W^{1,2}_{fix}, AC^2_{fix}$ the structure of module over $L^0(\mathrm{X})$ and that with the pointwise norms defined above the spaces L^p_{fix} and $W^{1,2}_{fix}$ are $L^0(\mathrm{X})$-normed modules.

We now turn to the corresponding notions for vector fields in L^0_{var}. In order to justify the definitions we are going to give in a moment, let us illustrate the situation in the case of a smooth manifold M and flow of a smooth vector field. In this case, a vector field $(v_t) \in L^0_{fix}$ belongs to L^p_{fix} if and only if for a.e. x the curve $t \mapsto v_t(x) \in T_xM \sim \mathbb{R}^d$ is in $L^p([0,1], \mathbb{R}^d)$. Equivalently, this is the same as to ask that $t \mapsto |v_t|(x) \in \mathbb{R}$ is in $L^p(0,1)$.

Now, we have said that we want to think of elements $(V_t) \in L^0_{var}$ as collections of vector fields defined along the flow lines, i.e. as the collection, for a.e. $x \in M$, of the vector fields $t \mapsto V_t(F^t_0(x)) \in T_{F^t_0(x)}M$. Therefore, and by analogy with the 'fixed' case, their natural pointwise L^p-norm should be given by the L^p norm of $t \mapsto |V_t|(F_t(x))$.

For the case of Sobolev vector fields, the derivative $\dot{v}_t(x)$, which is computed in the fixed tangent space T_xM, should be replaced by the covariant derivative of the vector field $t \mapsto V_t(F^t_0(x)) \in T_{F_t(x)}M$ along the curve $t \mapsto F_t(x)$. By direct computation, if $(t,x) \mapsto v_t(x)$ is smooth in t, x and $V_t(x) := \mathrm{d}F^t_0(v_t)(x)$ it is not hard to check that such covariant derivative is given by

$$\nabla_{\dot{F}^t_0(x)} V_t(F^t_0(x)) = (\mathrm{d}F^t_0(\dot{v}_t) + \nabla_{V_t}\boldsymbol{b}_t)(F^t_0(x)).$$

Notice also that when dealing with vector fields defined on the whole manifold (rather than along a single flow line) one typically speaks of 'convective' derivative, rather than 'covariant' one (think e.g. to the setting of fluid dynamics).

We now turn to the actual definitions; the subsequent discussion will make clear the link with what just said.

Definition 5.2.5 (Some spaces of vectors at 'variable points')**.** *We shall denote by*

i) $L^p_{var}([0,1], L^0(T\mathrm{X})) \subset L^0_{var}([0,1], L^0(T\mathrm{X}))$ *(or simply* L^p_{var}*),* $p \in [1,\infty]$*, the space of vector fields* (V_t) *of the form* $V_t = \mathrm{d}F^t_0(v_t)$ *for a.e.* t *for some* $(v_t) \in L^p_{fix}$*. We also define* $\mathfrak{m}$*-a.e. the quantity*

$$|(V_t)|_{L^p_{var}} := \| |V_t| \circ F^t_0 \|_{L^p(0,1)}.$$

ii) $W^{1,2}_{var}([0,1], L^0(T\mathrm{X})) \subset L^0_{var}([0,1], L^0(T\mathrm{X}))$ *(or simply* $W^{1,2}_{var}$*) the space of vector fields* (V_t) *of the form* $V_t = \mathrm{d}F^t_0(v_t)$ *for a.e.* t *for some* $(v_t) \in W^{1,2}_{fix}$*. In this case we also define the convective derivative* $(D_tV_t) \in L^0_{var}([0,1], L^0(T\mathrm{X}))$ *as*

$$D_tV_t := \mathrm{d}F^t_0(\dot{v}_t) + \nabla_{V_t}\boldsymbol{b}_t, \qquad \textit{a.e. } t \in [0,1]. \tag{5.22}$$

Also, we consider the $\mathfrak{m}$*-a.e. defined quantity*

$$|(V_t)|^2_{W^{1,2}_{var}} := |(V_t)|^2_{L^2_{var}} + |(D_tV_t)|^2_{L^2_{var}} = \int_0^1 |V_t|^2 \circ F^t_0 + |D_tV_t|^2 \circ F^t_0 \,\mathrm{d}t.$$

iii) $AC^2_{var}([0,1], L^0(T\mathrm{X})) \subset C([0,1], L^0(T\mathrm{X}))$ *(or simply* AC^2_{var}*) the space of vector fields* (V_t) *of the form* $V_t = \mathrm{d}F^t_0(v_t)$ *for every* $t \in [0,1]$ *for some* $(v_t) \in AC^2_{fix}$*.*

Notice that Proposition 5.2.3 above ensures that (D_tV_t) is indeed an element of $L^0_{var}([0,1], L^0(T\mathrm{X}))$ and that if $(v_t) \in AC^2_{fix} \subset C([0,1], L^0(T\mathrm{X}))$, then $t \mapsto \mathrm{d}F^t_0(v_t)$ is also an element of $C([0,1], L^0(T\mathrm{X}))$, so that AC^2_{var} is actually a subset of $C([0,1], L^0(T\mathrm{X}))$. It is also clear that the spaces defined above are vector spaces and that $(V_t) \mapsto (D_tV_t)$ is linear.

Let us remark that by (4.22), (5.8) it follows that for any $v \in L^0(T\mathrm{X})$ and $t \in [0,1]$, putting $V := \mathrm{d}F_0^t(v) \in L^0(T\mathrm{X})$, so that $v = \mathrm{d}F_t^0(V)$ by the group property (4.1) and the chain rule (4.25), we have

$$|v| \le G|V| \circ F_0^t \qquad \text{and} \qquad |V| \circ F_0^t \le G|v| \qquad \mathfrak{m} - a.e., \tag{5.23}$$

for some non-negative function $G \in L^0(\mathrm{X})$ depending only on X and $(\boldsymbol{b}_t)$ (and in particular independent of v,t). It follows that $(V_t) \in L^0_{var}$ belongs to L^p_{var} if and only if the quantity $|(V_t)|_{L^p_{var}}$ is finite $\mathfrak{m}$-a.e.. Then arguing as in the 'fixed' case, it is easy to see that the distance

$$\mathsf{d}_{L^p_{var}}\big((V_t),(Z_t)\big) := \mathsf{d}_{L^0}(|(V_t - Z_t)|_{L^p_{var}}, 0)$$

is lower semicontinuous w.r.t. L^0_{var}-convergence and thus - by the completeness of L^0_{var} - that it is a complete distance on L^p_{var}. Alternatively, completeness of L^p_{var} can be established noticing that for (v_t) and (V_t) as in (5.21), the uniform bound (5.23) gives

$$|(v_t)|_{L^p_{fix}} \le G|(V_t)|_{L^p_{var}} \qquad \text{and} \qquad |(V_t)|_{L^p_{var}} \le G|(v_t)|_{L^p_{fix}} \tag{5.24}$$

$\mathfrak{m}$-a.e.. Thus for $(v_t),(z_t) \in L^p_{fix}$ and the corresponding $(V_t),(Z_t) \in L^p_{var}$ as in (5.21) we have

$$\begin{aligned} \mathsf{d}_{L^p_{fix}}\big((v_t),(z_t)\big) &= \mathsf{d}_{L^0}\big(|(v_t - z_t)|_{L^p_{fix}}, 0\big) \\ \text{(by (5.24))} \qquad &\le \mathsf{d}_{L^0}\big(G|(V_t - Z_t)|_{L^p_{var}}, 0\big) \\ \text{(by (4.60))} \qquad &\le \Omega\big(\mathsf{d}_{L^0}\big(|(V_t - Z_t)|_{L^p_{var}}, 0\big)\big) = \Omega\big(\mathsf{d}_{L^p_{var}}\big((V_t),(Z_t)\big)\big) \end{aligned} \tag{5.25}$$

and analogously it holds $\mathsf{d}_{L^p_{var}}\big((V_t),(Z_t)\big) \le \Omega\big(\mathsf{d}_{L^p_{fix}}\big((v_t),(z_t)\big)\big)$. This proves that $(v_t^n) \subset L^p_{fix}$ is Cauchy if and only if the corresponding sequence $(V_t^n) \subset L^p_{var}$ is Cauchy, thus the completeness of L^p_{var} follows from that of L^p_{fix}.

Analogous considerations are in place for $W^{1,2}_{var}$, but are a bit harder to prove. The key point is an analogue of (5.24) in the Sobolev case, which is established in the following lemma:

Lemma 5.2.6. *Let $(v_t) \in W^{1,2}_{fix}$ and $(V_t) \in W^{1,2}_{var}$ be as in* (5.21). *Then*

$$|(V_t)|_{W^{1,2}_{var}} \le G|(v_t)|_{W^{1,2}_{fix}} \qquad \text{and} \qquad |(v_t)|_{W^{1,2}_{fix}} \le G|(V_t)|_{W^{1,2}_{var}} \qquad \mathfrak{m} - a.e. \tag{5.26}$$

and

$$|(V_t)|_{L^\infty_{var}} \le G|(V_t)|_{W^{1,2}_{var}} \qquad \mathfrak{m} - a.e.. \tag{5.27}$$

Proof. From the very definition (5.22) we see that $\mathfrak{m}$-a.e. and for a.e. t it holds

$$|D_t V_t| \circ F_0^t \overset{(5.8)}{\le} G|\dot v_t| + |\nabla \boldsymbol{b}_t| \circ F_0^t |V_t| \overset{(5.23)}{\le} G\big(|\dot v_t| + |\nabla \boldsymbol{b}_t| \circ F_0^t |v_t|\big) \overset{(4.47)}{\le} G(|\dot v_t| + |\nabla \boldsymbol{b}_t| \circ F_0^t |(v_s)|_{W^{1,2}_{fix}}).$$

Squaring and integrating in t we obtain

$$|(D_t V_t)|^2_{L^2_{var}} \le G\Big(|(\dot v_t)|^2_{L^2_{fix}} + |(v_t)|^2_{W^{1,2}_{fix}} |(|\nabla \boldsymbol{b}_t| \circ F_0^t)|^2_{L^2}\Big) \overset{(5.5)}{\le} G|(v_t)|^2_{W^{1,2}_{fix}},$$

so that the first in (5.26) follows taking also (5.24) into account.

For the second in (5.26) we notice that (5.22) gives $\dot v_t = \mathrm{d}F_t^0(D_tV_t - \nabla_{V_t}\boldsymbol{b}_t)$, thus

$$|\dot v_t| \overset{(4.22)}{\leq} \Big(|\mathrm{d}F_t^0|\big(|D_tV_t| + |V_t|\,|\nabla\boldsymbol{b}_t|\big)\Big)\circ F_0^t \overset{(5.8)}{\leq} G\Big(|D_tV_t|\circ F_0^t + |V_t|\circ F_0^t\,|\nabla\boldsymbol{b}_t|\circ F_0^t\Big) \tag{5.28}$$

$\mathfrak{m}$-a.e. for a.e. $t\in[0,1]$. Integrating in t we deduce that $\mathfrak{m}$-a.e. we have

$$|(\dot v_t)|_{L^1_{fix}} \leq G\Big(|(D_tV_t)|_{L^2_{var}} + |(V_t)|_{L^2_{var}}\,|(|\nabla\boldsymbol{b}_t|\circ F_0^t)|_{L^2}\Big) \overset{(5.5)}{\leq} G|(V_t)|_{W^{1,2}_{var}}.$$

Using this bound in conjunction with the first inequality in (4.47) and (5.24) we obtain

$$|(V_t)|_{L^\infty_{var}} \leq G|(v_t)|_{L^\infty_{fix}} \leq G\big(|(V_t)|_{L^2_{var}} + |(V_t)|_{W^{1,2}_{var}}\big) \leq G|(V_t)|_{W^{1,2}_{var}} \qquad \mathfrak{m}-a.e.,$$

which is (5.27). Plugging this inequality in (5.28) we see that

$$|\dot v_t| \leq G\big(|D_tV_t|\circ F_0^t + |(V_s)|_{W^{1,2}_{var}}|\nabla\boldsymbol{b}_t|\circ F_0^t\big) \qquad \mathfrak{m}\text{-a.e. for a.e. } t\in[0,1].$$

Squaring, integrating in t and recalling (5.5) we see that $|(\dot v_t)|^2_{L^2_{fix}} \leq G|(V_t)|^2_{W^{1,2}_{var}}$ holds $\mathfrak{m}$-a.e.. Then the second in (5.26) follows taking (5.24) into account. □

This lemma ensures in particular that $|(V_t)|_{W^{1,2}_{var}} < \infty$ $\mathfrak{m}$-a.e. for any $(V_t)\in W^{1,2}_{var}$. Hence the following is a well defined distance on $W^{1,2}_{var}$:

$$\mathsf{d}_{W^{1,2}_{var}}\big((V_t),(Z_t)\big) := \mathsf{d}_{L^0}(|(V_t - Z_t)|_{W^{1,2}_{var}}, 0).$$

Then starting from (5.26), arguing as we did in (5.25) from (5.24), we see that for $(v_t),(z_t)\in W^{1,2}_{fix}$ and the corresponding $(V_t),(Z_t)\in W^{1,2}_{var}$ as in (5.21) we have

$$\mathsf{d}_{W^{1,2}_{fix}}\big((v_t),(z_t)\big) \leq \Omega\big(\mathsf{d}_{W^{1,2}_{var}}\big((V_t),(Z_t)\big)\big) \qquad \text{and} \qquad \mathsf{d}_{W^{1,2}_{var}}\big((V_t),(Z_t)\big) \leq \Omega\big(\mathsf{d}_{W^{1,2}_{fix}}\big((v_t),(z_t)\big)\big).$$

Therefore, much like in the L^p case, $(v_t^n)\subset W^{1,2}_{fix}$ is Cauchy if and only if the corresponding sequence $(V_t^n)\subset W^{1,2}_{var}$ is Cauchy. Hence completeness of $W^{1,2}_{var}$ follows from that of $W^{1,2}_{fix}$.

It is then clear that AC^2_{var} is also a complete space.

5.3 Calculus with the convective derivative

The main goal of this section is to establish appropriate calculus rules for the convective derivative that mimic those available in the smooth setting.

In particular, we shall ultimately prove that for $(V_t),(W_t)\in AC^2_{var}$ we have $\big(\langle V_t, W_t\rangle\circ F_0^t\big)\in AC^2_{L^0(\mathrm{X})}$ with derivative $\langle D_tV_t, W_t\rangle\circ F_0^t + \langle V_t, D_tW_t\rangle\circ F_0^t$. This can be read as *compatibility with the metric* of our convective/covariant derivative and will be crucial to obtain uniqueness of parallel transport and preservation of norm.

The proof of this fact will be obtained following roughly the ideas in [44], but due to much lower regularity we have at disposal now, things are now more involved. The idea is to establish regularity for $t\mapsto\langle V_t, W_t\rangle\circ F_0^t$ by duality, i.e. we first study the regularity of $t\mapsto \mathrm{d}(f\circ F_0^t)$ (Proposition 5.3.2) and later that of $t\mapsto\langle V_t, W_t\rangle\circ F_0^t$ (Theorem 5.3.4).

To pursue this program, and in particular in the first step, we shall use the closure of the differential on bounded subsets of the Hajłasz–Sobolev space as discussed in Proposition 4.3.5. In turn, this will be possible thanks to the following lemma:

Lemma 5.3.1. *Let $\bar{g} := \bar{g}_0 \in L^2_{loc}(\mathrm{X})$ be given by Proposition 4.1.1 for $t = 0$, $f \in W^{1,2}(\mathrm{X})$, and $R > 0$. Then $f \circ F_0^t \in H_{\bar{g},R}(\mathrm{X})$ for every $t \in [0,1]$ and there is $C > 0$ such that*

$$\|f \circ F_0^t\|_{H_{\bar{g},R}} \leq C\|f\|_{W^{1,2}}, \qquad \forall t \in [0,1]. \tag{5.29}$$

Proof. Since $\mathsf{RCD}(K,N)$ spaces are PI spaces, we know from (1.19) that for every $R' > 0$ there exists a $\mathfrak{m}$-negligible set N and $C > 0$ such that

$$|f(x) - f(y)| \leq C\big(M_{4R'}(|Df|)(x) + M_{4R'}(|Df|)(y)\big)\mathsf{d}(x,y) \tag{5.30}$$

holds for every $x, y \in \mathrm{X} \setminus N$ with $\mathsf{d}(x,y) \leq R'$.

Also, recalling (4.2) we see that possibly enlarging N, keeping it $\mathfrak{m}$-negligible, for $L := \|\boldsymbol{b}_t\|_{L^\infty_{t,x}}$ we have

$$\mathsf{d}(F_0^t(x), F_0^t(y)) \leq \mathsf{d}(x,y) + 2tL, \qquad \forall x,y \in \mathrm{X} \setminus N,\ \forall t \in [0,1].$$

Put $R' := R + 2L$ and $H := M_{4R'}(|Df|)$ (notice that $H \in L^2(\mathrm{X},\mathfrak{m})$ by (1.18)). Then for every $x, y \in \mathrm{X} \setminus (F_0^t)^{-1}(N)$ with $\mathsf{d}(x,y) \leq R$, from (5.30) and (4.9) we have that

$$\begin{aligned}|f(F_0^t(x)) - f(F_0^t(y))| &\leq C\big(H(F_0^t(x)) + H(F_0^t(y))\big)\mathsf{d}(F_0^t(x), F_0^t(y))\\ &\leq C\big(H(F_0^t(x)) + H(F_0^t(y))\big)e^{\bar{g}(x)+\bar{g}(y)}\mathsf{d}(x,y)\end{aligned}$$

holds for any $t \in [0,1]$. In other words, since $(F_0^t)^{-1}(N)$ is $\mathfrak{m}$-negligible, we proved that $H \circ F_0^t \in A_{\bar{g},R}(f)$. Then the conclusion follows recalling that since (F_0^t) has bounded compression, we have that $\|H \circ F_0^t\|_{L^2} \leq C'\|H\|_{L^2}$ for some $C' > 0$ and every $t \in [0,1]$. □

We can now study the regularity of $t \mapsto \mathrm{d}(f \circ F_0^t)$:

Proposition 5.3.2. *Let $f \in \mathrm{Test}(\mathrm{X})$. Then for every $t \in [0,1]$ the map $s \mapsto \mathrm{d}(f \circ F_t^s)$ is in $AC^2([0,1], L^0(T^*\mathrm{X}))$ and there is a Borel negligible set $N \subset [0,1]$, independent of f and $t \in [0,1]$, such that for every $s \in [0,1] \setminus N$ we have*

$$\lim_{h \to 0} \frac{\mathrm{d}(f \circ F_t^{s+h}) - \mathrm{d}(f \circ F_t^s)}{h} = \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_t) \circ F_t^s) \qquad \text{in } L^0(T^*\mathrm{X}). \tag{5.31}$$

Proof.
Step 1: basic integrability estimates. We know from Lemma 5.2.2 that $s \mapsto \mathrm{d}(f \circ F_t^s) \in L^0(T^*\mathrm{X})$ is continuous for any $t \in [0,1]$ and since $\mathrm{d}f(\boldsymbol{b}) \in W^{1,2}(\mathrm{X})$ for any $\boldsymbol{b} \in L^\infty(\mathfrak{m}) \cap W^{1,2}_C(T\mathrm{X})$ (recall Prop. (1.4.10)), the same lemma ensures that for any such w we have that $s \mapsto \mathrm{d}(\mathrm{d}f(\boldsymbol{b}) \circ F_t^s) \in L^0(T^*\mathrm{X})$ is continuous. Then a simple approximation argument shows that $s \mapsto \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_s) \circ F_t^s)$ is Borel.

Now observe that the bound $|\mathrm{d}f(\boldsymbol{b}_t)| \leq \|\mathrm{d}f\|_{L^\infty}|\boldsymbol{b}_t|$, the assumption $(\boldsymbol{b}_t) \in L^2([0,1], L^2(T\mathrm{X}))$ and (5.4) ensure that $s \mapsto \mathrm{d}f(\boldsymbol{b}_s) \circ F_t^s$ is in $L^2([0,1], L^0(\mathrm{X}))$. Similarly, (4.23) and (5.8) give

$$|\mathrm{d}(\mathrm{d}f(\boldsymbol{b}_s) \circ F_t^s)| \leq G_t\,|\mathrm{d}(\mathrm{d}f(\boldsymbol{b}_s))| \circ F_t^s \overset{(1.23)}{\leq} G_t\,\big(|\mathrm{Hess}(f)|\|\boldsymbol{b}_s\|_{L^\infty} + |\nabla \boldsymbol{b}_s||\mathrm{d}f|\big) \circ F_t^s, \tag{5.32}$$

$\mathfrak{m}$-a.e. for every $t, s \in [0,1]$.
It is clear from our assumptions on $(\boldsymbol{b}_t)$ and f that $s \mapsto (|\mathrm{Hess}(f)|\|\boldsymbol{b}_s\|_{L^\infty} + |\nabla \boldsymbol{b}_s||\mathrm{d}f|)$ is in $L^2([0,1], L^2(\mathrm{X}))$, thus from (5.4) and the above we see that $s \mapsto \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_s) \circ F_t^s)$ is in $L^2([0,1], L^0(T^*\mathrm{X}))$.

Step 2: Sobolev regularity and computation of the derivative. We now claim that for any $s_1, s_2 \in [0,1]$, $s_1 < s_2$ we have

$$\mathrm{d}(f \circ F_t^{s_2}) - \mathrm{d}(f \circ F_t^{s_1}) = \int_{s_1}^{s_2} \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_r) \circ F_t^r)\,\mathrm{d}r. \tag{5.33}$$

Notice that by (4.38) this proves that $s \mapsto \mathrm{d}(f \circ F_t^s)$ is in $AC^2([0,1], L^0(T^*\mathrm{X}))$ with derivative given by $\mathrm{d}(\mathrm{d}f(\boldsymbol{b}_s) \circ F_t^s)$. Recalling (4.6), we see that (5.33) above will follow if we show that for any s_1, s_2 as before it holds

$$\mathrm{d}\Big(\int_{s_1}^{s_2} \mathrm{d}f(\boldsymbol{b}_r) \circ F_t^r\,\mathrm{d}r\Big) = \int_{s_1}^{s_2} \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_r) \circ F_t^r)\,\mathrm{d}r. \tag{5.34}$$

To prove this latter identity we apply Theorem 4.4.9 with $\mathscr{H}_1 := L^0(\mathrm{X})$, $\mathscr{H}_2 := L^0(T^*\mathrm{X})$, $V := H_{\bar g,1}(\mathrm{X})$ equipped with its norm as defined in (4.30) (here $\bar g = \bar g_0$ is given by Proposition 4.1.1), $L := \mathrm{d}$, and $v_s := \mathrm{d}f(\boldsymbol{b}_s) \circ F_t^s$. We already observed that $(v_s) \in L^2([0,1], \mathscr{H}_1)$ and $(L(v_s)) \in L^2([0,1], \mathscr{H}_2)$. Also, the trivial bound $\|\mathrm{d}f(\boldsymbol{b}_s)\|_{L^2} \le \|\boldsymbol{b}_s\|_{L^\infty_{s,x}} \|\mathrm{d}f\|_{L^2}$, the inequality

$$\|\mathrm{d}(\mathrm{d}f(\boldsymbol{b}_s))\|_{L^2} \le \|\boldsymbol{b}_s\|_{L^\infty_{s,x}} \|\mathrm{Hess}(f)\|_{L^2} + \|\mathrm{d}f\|_{L^\infty} \||\nabla \boldsymbol{b}_s|\|_{L^2},$$

and again the assumption $(\boldsymbol{b}_t) \in L^2([0,1], W^{1,2}_C(T\mathrm{X}))$ with $|\boldsymbol{b}_t| \in L^\infty([0,1] \times \mathrm{X})$, give that $s \mapsto \mathrm{d}f(\boldsymbol{b}_s)$ is in $L^1([0,1], W^{1,2}(\mathrm{X}))$. Therefore by Lemma 5.3.1 above we deduce that $\int_0^1 \|v_s\|_{H_{\bar g,1}}\,\mathrm{d}s < \infty$. Finally, Proposition 4.3.5 ensures that assumption *(ii)* in Theorem 4.4.9 is satisfied, thus we can apply this proposition and deduce that (5.34), and thus (5.33), holds.

Now for $f \in \mathrm{Test}(\mathrm{X})$ and $t \in [0,1]$ let $N(f,t) \subset [0,1]$ be the set of $s \in [0,1]$ for which the limiting relation

$$\lim_{h \to 0} \frac{\mathrm{d}(f \circ F_t^{s+h}) - \mathrm{d}(f \circ F_t^s)}{h} = \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_s) \circ F_t^s), \qquad \text{in } L^0(T^*\mathrm{X})$$

does *not* hold. We claim that $N(f,t)$ does not depend on t. To see this, let $t, t' \in [0,1]$ and observe that from the bound $|\mathrm{d}(g \circ F_t^{t'})| \le G_t |\mathrm{d}g| \circ F_t^{t'}$ (which follows from (5.8)) and Lemma 4.5.1 we deduce

$$\mathsf{d}_{L^0}(|\mathrm{d}(g \circ F_t^{t'})|, 0) \le \Omega\big(\mathsf{d}_{L^0}(|\mathrm{d}g| \circ F_t^{t'}, 0)\big) \overset{(5.3)}{\le} \Omega\big(\mathsf{d}_{L^0}(|\mathrm{d}g|, 0)\big).$$

Applying this to $g := \frac{f \circ F_{t'}^{s+h} - f \circ F_{t'}^s - \mathrm{d}f(\boldsymbol{b}_s) \circ F_{t'}^s}{h}$ we deduce that $N(f,t') \supset N(f,t)$ and the claim follows by the arbitrariness of t, t'.

We shall put $N(f) := N(f,0) = N(f,t)$ for any $t \in [0,1]$. Notice that $N(f)$ is a Borel negligible subset of $[0,1]$ and that for every $t \notin N(f)$ we have $t \notin N(f,t)$ and thus (5.31) holds.

Step 3: the exceptional set can be chosen independent of f. To conclude it is sufficient to prove that for some $N \subset [0,1]$ Borel and negligible we have $N(f) \subset N$ for every $f \in \mathrm{Test}(\mathrm{X})$. To this aim we employ a standard idea based on uniform continuity of difference quotients; specifically, we start claiming that for a.e. $t \in [0,1]$ the linear operators $T_t^s : \mathrm{Test}(\mathrm{X}) \to L^0(T^*\mathrm{X})$ defined as

$$T_t^s(f) := \frac{1}{|s-t|} \int_{t \wedge s}^{t \vee s} \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_r) \circ F_0^r)\,\mathrm{d}r$$

are uniformly continuous in $s \in [0,1] \setminus \{t\}$ if we equip Test(X) with the $W^{2,2}$-norm. To see this notice that

$$
\begin{aligned}
|T_t^s(f)| &\le \frac{1}{|s-t|}\int_{t\wedge s}^{t\vee s} |\mathrm{d}(\mathrm{d}f(\boldsymbol{b}_r)\circ F_0^r)|\,\mathrm{d}r \\
\text{(by (5.32))} \qquad &\le G\,\frac{1}{|s-t|}\int_{t\wedge s}^{t\vee s} \big(|\mathrm{Hess}(f)|\|\boldsymbol{b}_r\|_{L^\infty} + |\nabla \boldsymbol{b}_r||\mathrm{d}f|\big)\circ F_0^r\,\mathrm{d}r \qquad \mathfrak{m}-a.e.
\end{aligned}
\tag{5.35}
$$

and that

$$
\begin{aligned}
&\mathsf{d}_{L^0}\Big(\frac{1}{|s-t|}\int_{t\wedge s}^{t\vee s} \big(|\mathrm{Hess}(f)|\|\boldsymbol{b}_r\|_{L^\infty} + |\nabla \boldsymbol{b}_r||\mathrm{d}f|\big)\circ F_0^r\,\mathrm{d}r, 0\Big) \\
&\le \mathsf{d}_{L^0}\Big(\frac{\|\boldsymbol{b}_r\|_{L^\infty_{r,x}}}{|s-t|}\int_{t\wedge s}^{t\vee s} |\mathrm{Hess}(f)|\circ F_0^r\,\mathrm{d}r, 0\Big) + \mathsf{d}_{L^0}\Big(\frac{1}{|s-t|}\int_{t\wedge s}^{t\vee s} \big(|\nabla \boldsymbol{b}_r||\mathrm{d}f|\big)\circ F_0^r\,\mathrm{d}r, 0\Big) \\
&\overset{*}{\le} \Omega\Big(\Big\|\frac{\|\boldsymbol{b}_r\|_{L^\infty_{r,x}}}{|s-t|}\int_{t\wedge s}^{t\vee s} |\mathrm{Hess}(f)|\circ F_0^r\,\mathrm{d}r\Big\|_{L^2} + \Big\|\frac{1}{|s-t|}\int_{t\wedge s}^{t\vee s} \big(|\nabla \boldsymbol{b}_r||\mathrm{d}f|\big)\circ F_0^r\,\mathrm{d}r\Big\|_{L^1}\Big) \\
&\overset{**}{\le} \Omega\Big(\frac{\|\boldsymbol{b}_r\|_{L^\infty_{r,x}}}{|s-t|}\int_{t\wedge s}^{t\vee s} \big\||\mathrm{Hess}(f)|\big\|_{L^2}\,\mathrm{d}r + \frac{1}{|s-t|}\int_{t\wedge s}^{t\vee s} \big\||\nabla \boldsymbol{b}_r||\mathrm{d}f|\big\|_{L^1}\,\mathrm{d}r\Big) \\
&\le \Omega\Big(\|\boldsymbol{b}_r\|_{L^\infty_{r,x}}\||\mathrm{Hess}(f)|\|_{L^2} + \||\mathrm{d}f|\|_{L^2}\sqrt{\frac{1}{|s-t|}\int_{t\wedge s}^{t\vee s} \||\nabla \boldsymbol{b}_r|\|_{L^2}^2\,\mathrm{d}r}\Big),
\end{aligned}
$$

having used (4.61) in $*$ and (4.4) in $**$.

Now observe that since by assumption we have that $t \mapsto \||\nabla \boldsymbol{b}_t|\|_{L^2}$ is in $L^2(0,1)$, the Hardy–Littlewood maximal inequality grants that $M(t) := \sup_{s\neq t}\sqrt{\frac{1}{|s-t|}\int_{t\wedge s}^{t\vee s}\||\nabla \boldsymbol{b}_r|\|_{L^2}^2\,\mathrm{d}r} < \infty$ for a.e. t, therefore for $f, f' \in \mathrm{Test(X)}$, putting for brevity $g := f - f'$ the above gives

$$
\begin{aligned}
\mathsf{d}_{L^0(T^*\mathrm{X})}(T_t^s(f), T_t^s(f')) &= \mathsf{d}_{L^0}(|T_t^s(g)|, 0) \\
\text{(by (5.35))} \qquad &\le \mathsf{d}_{L^0}\Big(G\,\frac{1}{|s-t|}\int_{t\wedge s}^{t\vee s} \big(|\mathrm{Hess}(g)|\|\boldsymbol{b}_r\|_{L^\infty} + |\nabla \boldsymbol{b}_r||\mathrm{d}g|\big)\circ F_0^r\,\mathrm{d}r, 0\Big) \\
\text{(by (4.60))} \qquad &\le \Omega\Big(\mathsf{d}_{L^0}\Big(\frac{1}{|s-t|}\int_{t\wedge s}^{t\vee s} \big(|\mathrm{Hess}(g)|\|\boldsymbol{b}_r\|_{L^\infty} + |\nabla \boldsymbol{b}_r||\mathrm{d}g|\big)\circ F_0^r\,\mathrm{d}r, 0\Big)\Big) \\
&\le \Omega\Big(\|\boldsymbol{b}_r\|_{L^\infty_{r,x}}\||\mathrm{Hess}(g)|\|_{L^2} + \||\mathrm{d}g|\|_{L^2}M(t)\Big) \\
&\le \Omega\Big(\big(M(t)+1\big)\|f - f'\|_{W^{2,2}}\Big),
\end{aligned}
\tag{5.36}
$$

proving the desired uniform continuity for a.e. $t \in [0,1]$. Similarly, for $f, f' \in \mathrm{Test(X)}$ and $t \in [0,1]$ such that $w_t \in L^\infty \cap W^{1,2}_C(T\mathrm{X})$, starting from

$$
\begin{aligned}
|\mathrm{d}(\mathrm{d}f(\boldsymbol{b}_t)\circ F_0^t) - \mathrm{d}(\mathrm{d}f'(\boldsymbol{b}_t)\circ F_0^t)| &= |\mathrm{d}(\mathrm{d}(f-f')(\boldsymbol{b}_t)\circ F_0^t)| \\
&\le G\big(|\mathrm{Hess}(f-f')|\|\boldsymbol{b}_t\|_{L^\infty} + |\nabla \boldsymbol{b}_t|\,|\mathrm{d}(f-f')|\big),
\end{aligned}
$$

and arguing as in (5.36), we get

$$
\mathsf{d}_{L^0(T^*\mathrm{X})}\big(\mathrm{d}(\mathrm{d}f(\boldsymbol{b}_t)\circ F_0^t), \mathrm{d}(\mathrm{d}f'(\boldsymbol{b}_t)\circ F_0^t)\big) \le \Omega\big((\|\boldsymbol{b}_t\|_{L^\infty} + \|\boldsymbol{b}_t\|_{W^{1,2}_C})\|f - f'\|_{W^{2,2}}\big). \tag{5.37}
$$

Now let $(f_n) \subset \text{Test}(\mathrm{X})$ be dense in $\text{Test}(\mathrm{X})$ with respect to the separable norm of $W^{2,2}(\mathrm{X})$. Define $N \subset [0,1]$ as

$$N := \bigcup_n N(f_n) \cup \{t \ : \ M(t) = +\infty \text{ or } w_t \notin L^\infty \cap W^{1,2}_C(T\mathrm{X})\}$$

and notice that it is Borel and negligible. Let $t \notin N$, $f \in \text{Test}(\mathrm{X})$ and notice that (5.36) and (5.37) ensure that for any $n \in \mathbb{N}$ we have

$$\begin{aligned}
&\mathsf{d}_{L^0(T^*\mathrm{X})}\big(T^s_t(f), \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_t) \circ F^t_0)\big) \\
&\leq \Omega\Big(\big(M(t) + 1 + \|\boldsymbol{b}_t\|_{L^\infty} + \|\boldsymbol{b}_t\|_{W^{1,2}_C}\big)\|f - f_n\|_{W^{2,2}}\Big) + \mathsf{d}_{L^0(T^*\mathrm{X})}\big(T^s_t(f_n), \mathrm{d}(\mathrm{d}f_n(\boldsymbol{b}_t) \circ F^t_0)\big),
\end{aligned}$$

thus letting first $s \to t$ and then taking the infimum over $n \in \mathbb{N}$ we conclude that

$$\lim_{s \to t} T^s_t(f) = \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_t) \circ F^t_0), \qquad \text{in } L^0(T^*\mathrm{X}).$$

By the definition of $T^s_t(f)$ and (5.33), this proves that $t \notin N(f,t) = N(f)$, i.e. concludes the proof. $\square$

Before coming to the main result of the section, we need the following statement about equicontinuity of quadratic forms on $L^0(T\mathrm{X})$:

Proposition 5.3.3. *Let $B_i : L^0(T\mathrm{X})^2 \to L^0(\mathrm{X})$ be a family of $L^0(\mathrm{X})$-bilinear, symmetric, and continuous maps indexed by $i \in I$, and let $Q_i : L^0(T\mathrm{X}) \to L^0(\mathrm{X})$ be the associated quadratic forms, i.e. $Q_i(v) := B_i(v,v)$. Let $(G_i)_{i \in I} \subset L^0(\mathrm{X})$ be a family of non-negative functions satisfying* (4.62) *and assume that*

$$|Q_i(v)| \leq G_i|v|^2, \qquad \mathfrak{m}-a.e., \ \forall v \in L^0(T\mathrm{X}), \ \forall i \in I.$$

Then the Q_i's are locally equicontinuous, i.e.

$$\lim_{v' \xrightarrow{L^0} v} \sup_{i \in I} \mathsf{d}_{L^0}(Q_i(v'), Q_i(v)) = 0, \qquad \forall v \in L^0(T\mathrm{X}).$$

Proof. Let $\tilde{B}_i(v,z) := B_i(v,z) + G_i\langle v, z\rangle$ and $\tilde{Q}_i(v) := \tilde{B}_i(v,v)$. Then

$$0 \leq \tilde{Q}_i(v,v) \leq 2G_i|v|^2, \qquad \mathfrak{m}-a.e., \ \forall v \in L^0(T\mathrm{X}), \ \forall i \in I.$$

By the pointwise Cauchy–Schwarz inequality (that is easily seen to hold also in this setting), we deduce $|B_i(v,z)| \leq \sqrt{Q_i(v)Q_i(z)}$ $\mathfrak{m}$-a.e. for every $v, z \in L^0(T\mathrm{X})$ and $i \in I$. Thus for arbitrary $v, v' \in L^0(T\mathrm{X})$, putting $z := v' - v$ we have

$$|Q_i(v') - Q_i(v)| = |Q_i(z) + 2B_i(z,v)| \leq 2G_i(|z|^2 + |z||v|), \qquad \mathfrak{m}-a.e., \ \forall i \in I$$

and therefore, recalling that we are in a position to apply Lemma 4.5.1, we get

$$\mathsf{d}_{L^0}(Q_i(v'), Q_i(v)) \leq \mathsf{d}_{L^0}\big(2G_i(|z|^2 + |z||v|)\big) \overset{(4.63)}{\leq} \Omega\big(\mathsf{d}_{L^0}\big(|z|^2 + |z||v|\big)\big).$$

To conclude, notice that $1 \wedge |z|^2 \leq 1 \wedge |z|$ and thus

$$\mathsf{d}_{L^0}\big(|z|^2 + |z||v|, 0\big) \leq \mathsf{d}_{L^0}(|z|^2, 0) + \mathsf{d}_{L^0}(|z||v|, 0) \leq \mathsf{d}_{L^0}(|z|, 0) + \Omega(\mathsf{d}_{L^0}(|z|, 0)) = \Omega(\mathsf{d}_{L^0(T\mathrm{X})}(v', v)),$$

where the Ω appearing here depends on v (it is the one in (4.60) for $G := |v|$), but does not depend on v', z, i. The conclusion follows. $\square$

Theorem 5.3.4. *Let* $(V_t),(Z_t) \in W^{1,2}_{var}([0,1],L^0(T\mathrm{X}))$. *Then* $t \mapsto \langle V_t, Z_t\rangle \circ F_0^t$ *belongs to* $W^{1,2}([0,1],L^0(\mathrm{X}))$ *and for a.e.* $t\in[0,1]$ *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\langle V_t, Z_t\rangle \circ F_0^t\big) = \langle D_tV_t, Z_t\rangle \circ F_0^t + \langle V_t, D_tZ_t\rangle \circ F_0^t, \qquad \mathfrak{m}-a.e.. \tag{5.38}$$

If moreover $(V_t),(Z_t) \in AC^2_{var}([0,1],L^0(T\mathrm{X}))$, *then* $t\mapsto \langle V_t,Z_t\rangle\circ F_0^t$ *belongs to* $AC^2([0,1],L^0(\mathrm{X}))$.

Proof.
Step 1: structure of the proof. The case of absolutely continuous vector fields follows from the Sobolev one provided we show that for $(V_t),(Z_t) \in C([0,1],L^0(T\mathrm{X}))$ the map $t\mapsto \langle V_t, Z_t\rangle \circ F_0^t \in L^0(\mathrm{X})$ is continuous. This follows noticing that, rather trivially, the map $t\mapsto \langle V_t, Z_t\rangle \in L^0(\mathrm{X})$ is continuous, then recalling Lemma 5.2.1. We thus focus on the Sobolev case.

For $(z_t) \in W^{1,2}_{fix}$ denote by $\mathcal{A}_{(z_t)} \subset W^{1,2}_{fix}$ the collection of those (v_t)'s for which the conclusion of the theorem holds for $V_t := \mathrm{d}F_0^t(v_t)$ and $Z_t := \mathrm{d}F_0^t(z_t)$. Our goal is to prove that $\mathcal{A}_{(z_t)} = W^{1,2}_{fix}$ for any $(z_t)\in W^{1,2}_{fix}$ and we shall do so by applying Proposition 4.4.7.

It is clear that $\mathcal{A}_{(z_t)}$ is a vector space, i.e. (o) holds, and - recalling (4.24) - that the 'restriction' property (i) holds as well. Also, by direct computation we can verify that (ii) holds. We pass to the stability property (iii), thus let $(v_t^n) \overset{W^{1,2}_{fix}}{\to} (v_t^\infty)$, and assume that the conclusion of the theorem holds for $n<\infty$ and the choices $V_t = V_t^n := \mathrm{d}F_0^t(v_t^n)$ and $Z_t := \mathrm{d}F_0^t(z_t)$. Our goal is to prove that the conclusion also holds for $V_t = V_t^\infty := \mathrm{d}F_0^t(v_t^\infty)$.

Since $(v_t^n)\to(v_t^\infty)$ and $(\dot v_t^n)\to(\dot v_t^\infty)$ in $L^2_{fix}\hookrightarrow L^0_{fix}$, Proposition 5.2.3 and the very definition (5.22) tell that $(V_t^n)\to(V_t^\infty)$ and $(D_tV_t^n)\to(D_tV_t^\infty)$ in L^0_{var}. It follows that

$$\begin{array}{ccc} \langle V_t^n, Z_t\rangle \circ F_0^t & \to & \langle V_t^\infty, Z_t\rangle\circ F_0^t \\ \langle D_tV_t^n, Z_t\rangle \circ F_0^t + \langle V_t^n, D_tZ_t\rangle\circ F_0^t & \to & \langle D_tV_t^\infty, Z_t\rangle \circ F_0^t + \langle V_t^\infty, D_tZ_t\rangle\circ F_0^t \end{array} \tag{5.39}$$

in $L^0([0,1],L^0(\mathrm{X}))$. Now notice that

$$|(\langle V_t^n, Z_t\rangle\circ F_0^t)|_{L^2} \le |(V_t^n)|_{L^2_{var}}|(Z_t)|_{L^\infty_{var}} \overset{(5.27)}{\le} G|(V_t^n)|_{W^{1,2}_{var}}|(Z_t)|_{W^{1,2}_{var}}, \qquad \mathfrak{m}-a.e.$$

and

$$\begin{aligned} |(\langle D_tV_t^n, Z_t\rangle \circ F_0^t + \langle V_t^n, D_tZ_t\rangle\circ F_0^t)|_{L^2} &\le |(D_tV_t^n)|_{L^2_{var}}|(Z_t)|_{L^\infty_{var}} + |(V_t^n)|_{L^\infty_{var}}|(D_tZ_t)|_{L^2_{var}} \\ \text{(by (5.27))} \qquad &\le G|(V_t^n)|_{W^{1,2}_{var}}|(Z_t)|_{W^{1,2}_{var}}, \qquad \mathfrak{m}-a.e., \end{aligned}$$

i.e., recalling also (5.26), we have

$$|(\langle V_t^n, Z_t\rangle\circ F_0^t)|_{W^{1,2}} \le G|(v_t^n)|_{W^{1,2}_{fix}}|(Z_t)|_{W^{1,2}_{var}}, \qquad \mathfrak{m}-a.e..$$

Since $|(v_t^n)|_{W^{1,2}_{fix}} \to |(v_t^\infty)|_{W^{1,2}_{fix}}$ (as a simple consequence of $(v_t^n) \overset{W^{1,2}_{fix}}{\to} (v_t^\infty)$), thanks to Proposition 4.4.3 the conclusion follows from this latter estimate and (5.39).

It remains to prove that $\mathcal{A}_{(z_t)}$ has property (iv) in Proposition 4.4.7. Suppose that we know that this is the case for (z_t) constant. Then the argument above shows that for (z_t) constant we have $\mathcal{A}_{(z_t)} = W^{1,2}_{fix}$. Therefore - by the symmetry in $(V_t),(Z_t)$ of the statement - we know that for

$(z_t) \in W^{1,2}_{fix}$ arbitrary, the set $\mathcal{A}_{(z_t)}$ contains the constant vector fields. Hence again the argument above shows that $\mathcal{A}_{(z_t)} = W^{1,2}_{fix}$, which is the conclusion.

We thus showed that it is sufficient to prove that for $v, z \in L^0(T\mathrm{X})$, the conclusion of the theorem holds for $V_t := \mathrm{d}F_0^t(v)$ and $Z_t := \mathrm{d}F_0^t(z)$. By polarization, it is actually sufficient to consider the case $v = z$.

Step 2: key point. To conclude the proof we need to prove that for $v \in L^0(T\mathrm{X})$, putting $V_t := \mathrm{d}F_0^t(v)$ we have that $t \mapsto |V_t|^2 \circ F_0^t$ is in $W^{1,2}([0,1], L^0(\mathrm{X}))$ and for a.e. $t \in [0,1]$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}|V_t|^2 \circ F_0^t = \langle V_t, D_t V_t\rangle \circ F_0^t, \qquad \mathfrak{m} - a.e.. \tag{5.40}$$

Step 2a: Sobolev regularity. For the Sobolev regularity we start recalling from (5.24) that $|(V_t)|_{L^\infty_{var}} < \infty$ $\mathfrak{m}$-a.e. and that the group property (4.1) and the chain rule (4.25) give $V_s = \mathrm{d}F_t^s(V_t)$ for any $t, s \in [0,1]$. Then for any $t, s \in [0,1]$ we have

$$\begin{aligned}
|V_s|^2 \circ F_0^s - |V_t|^2 \circ F_0^t &= (|V_s| \circ F_0^s + |V_t| \circ F_0^t)(|V_s| \circ F_0^s - |V_t| \circ F_0^t)\\
&\leq 2|(V_r)|_{L^\infty_{var}}\big(|\mathrm{d}F_t^s(V_t)| \circ F_0^s - |V_t| \circ F_0^t\big)\\
\text{(by (4.22), (5.6))} \qquad &\leq 2|(V_r)|^2_{L^\infty_{var}} G \int_{t\wedge s}^{t\vee s} g_r \circ F_0^r \,\mathrm{d}r
\end{aligned}$$

$\mathfrak{m}$-a.e.. Swapping the roles of t, s and recalling (5.24) we get

$$\big||V_s|^2 \circ F_0^s - |V_t|^2 \circ F_0^t\big| \leq |v|^2 G \int_t^s g_r \circ F_0^r \,\mathrm{d}r, \qquad \mathfrak{m} - a.e., \ \forall t, s \in [0,1], \ t \leq s. \tag{5.41}$$

Since $(t \mapsto g_t \circ F_0^t) \in L^2([0,1], L^0(\mathrm{X}))$ as a consequence of the fact that $(g_t) \in L^2([0,1], L^0(\mathrm{X}))$ (by Proposition 4.1.1) and (5.4), this last estimate, thanks to the characterization in (4.39), is sufficient to deduce that $t \mapsto |V_t|^2 \circ F_0^t$ is in $W^{1,2}([0,1], L^0(\mathrm{X}))$.

Now recall that (4.41) gives that $\frac{|V_{t+h}|^2 \circ F_0^{t+h} - |V_t|^2 \circ F_0^t}{h} \to \frac{\mathrm{d}}{\mathrm{d}t}(|V_t|^2 \circ F_0^t)$ in $L^0(\mathrm{X})$ as $h \to 0$ for a.e. $t \in [0,1]$. Therefore to conclude that (5.40) holds it is sufficient to prove that for a.e. $t \in [0,1]$ we have

$$\begin{aligned}
\varlimsup_{h\downarrow 0} \mathsf{d}_{L^0}\Big(\Big(\frac{|V_{t+h}|^2 \circ F_0^{t+h} - |V_t|^2 \circ F_0^t}{2h} - \langle D_t V_t, V_t\rangle \circ F_0^t\Big)^-, 0\Big) &= 0,\\
\varlimsup_{h\uparrow 0} \mathsf{d}_{L^0}\Big(\Big(\frac{|V_{t+h}|^2 \circ F_0^{t+h} - |V_t|^2 \circ F_0^t}{2h} - \langle D_t V_t, V_t\rangle \circ F_0^t\Big)^+, 0\Big) &= 0,
\end{aligned} \tag{5.42}$$

as these would imply that for a.e. t we have $\mathsf{d}_{L^0}\big(\big(\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}|V_t|^2 \circ F_0^t) - \langle D_t V_t, V_t\rangle \circ F_0^t\big)^\mp, 0\big) = 0$, i.e. that $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}|V_t|^2 \circ F_0^t) = \langle D_t V_t, V_t\rangle \circ F_0^t$.

Now define

$$Q_t^s(v) := \frac{|\mathrm{d}F_t^s(v)|^2 \circ F_t^s - |v|^2}{2(s-t)}$$

and observe that the identity $V_s = \mathrm{d}F_t^s(V_t)$ that we already noticed gives $Q_t^s(V_t) = \frac{|V_s|^2 \circ F_t^s - |V_t|^2}{2(s-t)}$. Moreover, since $D_t V_t = \nabla_{V_t} \boldsymbol{b}_t$ by the very definition (5.22) of convective derivative, we have

$\langle D_t V_t, V_t\rangle = \langle \nabla \boldsymbol{b}_t, V_t\otimes V_t\rangle$. Therefore recalling also (5.3), the claim (5.42) will follow if we show that for a.e. $t\in[0,1]$ we have that for any $v\in L^0(T\mathrm{X})$ it holds

$$\varlimsup_{s\uparrow t}\mathsf{d}_{L^0}\big(\big(Q_t^s(v)-\langle\nabla\boldsymbol{b}_t,v\otimes v\rangle\big)^+,0\big)=0=\varlimsup_{s\downarrow t}\mathsf{d}_{L^0}\big(\big(Q_t^s(v)-\langle\nabla\boldsymbol{b}_t,v\otimes v\rangle\big)^-,0\big). \tag{5.43}$$

STEP 2B: CONVERGENCE PROPERTIES ON A DENSE SET. Let $N\subset[0,1]$ be the Borel negligible set given by Proposition 5.3.2 and up to enlarge it, by keeping it negligible, assume that it contains the exceptional set of s's for which (4.7) does not hold. We claim that there is a dense set $\mathcal{D}\subset L^0(T\mathrm{X})$ for which the claim (5.43) holds for any $v\in\mathcal{D}$ and $t\in[0,1]\setminus N$.

Fix $t\in[0,1]\setminus N$ and let $v=\nabla f$ for $f\in\mathrm{Test}(\mathrm{X})$. Then, since for any $s\in[0,1]$ we have

$$\tfrac12|\mathrm{d}F_t^s(\nabla f)|^2\circ F_t^s\ge \mathrm{d}f(\mathrm{d}F_t^s(\nabla f))\circ F_t^s-\tfrac12|\mathrm{d}f|^2\circ F_t^s,\qquad \mathfrak{m}-a.e.$$

with equality for $s=t$, we deduce that

$$Q_t^s(\nabla f)\ge\underbrace{\frac{\mathrm{d}f(\mathrm{d}F_t^s(\nabla f))\circ F_t^s-|\mathrm{d}f|^2}{s-t}-\frac{\frac12|\mathrm{d}f|^2\circ F_t^s-\frac12|\mathrm{d}f|^2}{s-t}}_{=:RHS(s)},\qquad \mathfrak{m}-a.e.\ for\ s>t,$$

with opposite inequality for $s<t$. Using the identity $\mathrm{d}f(\mathrm{d}F_t^s(\nabla f))\circ F_t^s=\mathrm{d}(f\circ F_t^s)(\nabla f)$ and Proposition 5.3.2 to handle the first addend, and (4.7) and the fact that $|\mathrm{d}f|^2\in W^{1,2}(\mathrm{X})$ for the second, we see that

$$\lim_{s\to t}RHS(s)=\mathrm{d}(\mathrm{d}f(\boldsymbol{b}_t))(\nabla f)-\mathrm{d}(\tfrac12|\mathrm{d}f|^2)(\boldsymbol{b}_t)=\langle\nabla\boldsymbol{b}_t,\nabla f\otimes\nabla f\rangle,\qquad \text{in } L^0(\mathrm{X}).$$

Thus from the trivial implication $a\ge b\Rightarrow(a-c)^-\le|b-c|$ valid for real numbers a,b,c, we get

$$\varlimsup_{s\downarrow t}\mathsf{d}_{L^0}\big(\big(Q_t^s(v)-\langle\nabla\boldsymbol{b}_t,v\otimes v\rangle\big)^-,0\big)\le\varlimsup_{s\downarrow t}\mathsf{d}_{L^0}\big(\big|RHS(s)-\langle\nabla\boldsymbol{b}_t,v\otimes v\rangle\big|,0\big)=0,$$

which is the second in (5.43). The first follows from the same arguments starting from the implication $a\le b\Rightarrow(a-c)^+\le|b-c|$.

Now observe that if (E_i) is a finite Borel partition of X and $(v_i)\subset L^0(T\mathrm{X})$, for $v:=\sum_i\chi_{E_i}v_i$ we have $Q_t^s(v)=\sum_i\chi_{E_i}Q_t^s(v_i)$ (recall (4.24)). Similarly, $\langle\nabla\boldsymbol{b}_t,v\otimes v\rangle=\sum_i\chi_{E_i}\langle\nabla\boldsymbol{b}_t,v_i\otimes v_i\rangle$. Thus if $v_i=\nabla f_i$ for $f_i\in\mathrm{Test}(\mathrm{X})$ and every i, using what already established we get

$$\begin{aligned}\varlimsup_{s\downarrow t}\mathsf{d}_{L^0}\big(\big(Q_t^s(v)-\langle\nabla\boldsymbol{b}_t,v\otimes v\rangle\big)^-,0\big)&=\varlimsup_{s\downarrow t}\mathsf{d}_{L^0}\Big(\sum_i\chi_{E_i}\big(Q_t^s(\nabla f_i)-\langle\nabla\boldsymbol{b}_t,\nabla f_i\otimes\nabla f_i\rangle\big)^-,0\Big)\\&\le\sum_i\varlimsup_{s\downarrow t}\mathsf{d}_{L^0}\Big(\chi_{E_i}\big(Q_t^s(\nabla f_i)-\langle\nabla\boldsymbol{b}_t,\nabla f_i\otimes\nabla f_i\rangle\big)^-,0\Big)\\&\le\sum_i\varlimsup_{s\downarrow t}\mathsf{d}_{L^0}\Big(\big(Q_t^s(\nabla f_i)-\langle\nabla\boldsymbol{b}_t,\nabla f_i\otimes\nabla f_i\rangle\big)^-,0\Big)=0,\end{aligned}$$

which is the second in (5.43). The first follows along similar lines.

In summary, we proved that (5.43) holds for any v in the dense set $\mathcal{D}\subset L^0(T\mathrm{X})$ made of those vectors of the form $v:=\sum_i\chi_{E_i}\nabla f_i$, where the sum is finite and E_i,f_i are as above.

STEP 2C: EQUICONTINUITY AND CONCLUSION. Put for brevity

$$\tilde Q_t^s(v):=Q_t^s(v)-\langle\nabla\boldsymbol{b}_t,v\otimes v\rangle\qquad\forall v\in L^0(T\mathrm{X}),t,s\in[0,1],\ t\ne s.$$

We claim that for a.e. $t \in [0,1]$ we have that for any given $v \in L^0(T\mathrm{X})$ the $\tilde{Q}_t^s$'s are equicontinuous at v, i.e. that

$$\lim_{v' \xrightarrow{L^0} v} \sup_{s\in[0,1]\setminus\{t\}} \mathsf{d}_{L^0}\big(\tilde{Q}_t^s(v'), \tilde{Q}_t^s(v)\big) = 0, \tag{5.44}$$

and to prove this we are going to apply Proposition 5.3.3 above. Start observing that arguing as for (5.41) we get

$$|Q_t^s(v)| \le |v|^2 G_t \int_{t\wedge s}^{t\vee s} g_r \circ F_t^r \,\mathrm{d}r, \qquad \mathfrak{m}-a.e.,\ \forall t,s \in [0,1],\ s\neq t,\ v \in L^0(T\mathrm{X}), \tag{5.45}$$

then notice that for any $s \in [0,1] \setminus \{t\}$, $\tilde{Q}_t^s$ is a quadratic form induced by a bilinear form that we shall denote by $B_t^s : L^0(T\mathrm{X})^2 \to L^0(\mathrm{X})$. By (4.24) we see that B_t^s is $L^0(\mathrm{X})$-bilinear and (5.45) tells that

$$|\tilde{Q}_t^s(v)| \le |\nabla \boldsymbol{b}_t||v|^2 + G_t|v|^2 \underbrace{\frac{1}{|s-t|}\int_{s\wedge t}^{s\vee t} g_r \circ F_t^r\,\mathrm{d}r}_{=:\hat{g}_t^s}, \qquad \mathfrak{m}-a.e.,\ \forall t,s\in[0,1],\ s\neq t,\ v\in L^0(T\mathrm{X}). \tag{5.46}$$

Now put

$$M(t) := \sup_{s\in[0,1]\setminus\{t\}} \frac{1}{|s-t|}\int_{s\wedge t}^{s\vee t} \|g_r^2\|_{L^2}^2\,\mathrm{d}r$$

and notice that since $\int_0^1 \|g_r^2\|_{L^2}^2\,\mathrm{d}r < \infty$ (by Proposition 4.1.1), the Hardy–Littlewood maximal inequality ensures that $M(t) < \infty$ for a.e. $t \in [0,1]$. Then for every $s \in [0,1] \setminus \{t\}$ we have

$$\|\hat{g}_t^s\|_{L^2}^2 \le \frac{1}{|s-t|}\int_{s\wedge t}^{s\vee t}\int g_r^2 \circ F_t^r \,\mathrm{d}\mathfrak{m}\,\mathrm{d}r \overset{(2.13)}{\le} \frac{C}{|s-t|}\int_{s\wedge t}^{s\vee t}\int g_r^2\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r \le CM(t), \tag{5.47}$$

thus Chebyshev's inequality gives the uniform control

$$\mathfrak{m}(\{|\hat{g}_t^s| \ge c\}) = \mathfrak{m}(\{|\hat{g}_t^s|^2 \ge c^2\}) \le \frac{\|\hat{g}_t^s\|_{L^2}^2}{c^2} \le \frac{CM(t)}{c^2} \qquad \forall s \in [0,1]\setminus\{t\}. \tag{5.48}$$

Now notice that the set $N' := N \cup \{t : M(t) = \infty\} \cup \{t : \boldsymbol{b}_t \notin W_C^{1,2}(T\mathrm{X})\}$ is Borel and negligible and fix $t \in [0,1] \setminus N'$. From (5.48), $\mathfrak{m}' \ll \mathfrak{m}$, and the absolute continuity of the integral, we see that the functions $G_t^s := |\nabla \boldsymbol{b}_t| + G_t \hat{g}_t^s$, parametrized by $s \in [0,1]\setminus\{t\}$, satisfy (4.62) (with $i = s$). Thus the claim (5.44) follows from Proposition 5.3.3 and the bound (5.46).

Now let $v \in L^0(T\mathrm{X})$ be arbitrary and $v' \in \mathcal{D}$. Using the trivial bound $a^- \le b^- + |a-b|$ valid for any $a, b \in \mathbb{R}$, we get

$$\begin{aligned} \varlimsup_{s\downarrow t} \mathsf{d}_{L^0}\big((\tilde{Q}_t^s(v))^-, 0\big) &\le \varlimsup_{s\downarrow t} \mathsf{d}_{L^0}\big((\tilde{Q}_t^s(v'))^-, 0\big) + \varlimsup_{s\downarrow t} \mathsf{d}_{L^0}\big(|\tilde{Q}_t^s(v) - \tilde{Q}_t^s(v')|, 0\big) \\ \text{(by Step 2b and } v' \in \mathcal{D}) \qquad &\le \sup_{s\in[0,1]\setminus\{t\}} \mathsf{d}_{L^0}\big(\tilde{Q}_t^s(v), \tilde{Q}_t^s(v')\big). \end{aligned}$$

Taking the limit as $v' \to v$, $v' \in \mathcal{D}$ and using (5.44), we get the second in (5.43). The first is proved analogously, by exploiting the fact that, for every $a, b \in \mathbb{R}$, $a^+ \le b^+ + |a-b|$. □

Remark 5.3.5. In the last step of the proof we showed that the functions $\hat{g}_t^s$ defined in (5.46) satisfy the bound (5.47). Although not needed for our purposes, it is worth to point out that in fact the stronger estimate

$$\iint_0^1 H_t^2 \,\mathrm{d}t\,\mathrm{d}\mathfrak{m} \le C \iint_0^1 g_t^2 \,\mathrm{d}t\,\mathrm{d}\mathfrak{m} \qquad \text{for} \qquad H_t := \sup_{s\neq t} \hat{g}_t^s, \tag{5.49}$$

holds, for some universal constant C depending only on the constant in (2.13). Indeed, putting $H_t' := \sup_{s\neq t} \hat{g}_t^s \circ F_0^t = H_t \circ F_0^t$, from (2.13) we have $\iint_0^1 H_t^2 \,\mathrm{d}t\,\mathrm{d}\mathfrak{m} \le C \iint_0^1 H_t'^2 \,\mathrm{d}t\,\mathrm{d}\mathfrak{m}$. Now observe that the Hardy–Littlewood maximal inequality gives

$$\int_0^1 H_t'^2 \,\mathrm{d}t = \int_0^1 \Big| \sup_{s\neq t} \frac{1}{|s-t|} \int_{s\wedge t}^{s\vee t} g_r \circ F_0^r \,\mathrm{d}r \Big|^2 \,\mathrm{d}t \le C \int_0^1 g_t^2 \circ F_0^t \,\mathrm{d}t$$

so that the claim follows by integration in $\mathfrak{m}$ and using again (2.13). ■

5.4 Existence and uniqueness of Parallel Transport

We introduce the notion of parallel transport for what concerns our setting.

Definition 5.4.1 (Parallel Transport)**.** *A Parallel Transport along the flow (F_t^s) of the vector field $(\boldsymbol{b}_t)$ is a vector field $(V_t) \in AC^2_{var}([0,1], L^0(T\mathrm{X}))$ such that*

$$D_t V_t = 0, \qquad \mathfrak{m}-a.e.,\ for\ a.e.\ t \in [0,1].$$

We say that the Parallel Transport (V_t) starts *from $\bar{V} \in L^0(T\mathrm{X})$ provided $V_0 = \bar{V}$.*

The linearity of the condition of being a Parallel Transport together with the Leibniz rule (5.38) easily gives:

Proposition 5.4.2 (Uniqueness and preservation of norm)**.** *Let (V_t) be a Parallel Transport. Then the map $t \mapsto |V_t| \circ F_0^t \in L^0(\mathrm{X})$ is constant.*

Moreover, for any $\bar{V} \in L^0(T\mathrm{X})$ there exists at most one Parallel Transport starting from $\bar{V}$.

Proof. By Theorem 5.3.4, if (V_t) is a Parallel Transport then $t \mapsto |V_t|^2 \circ F_0^t \in L^0(\mathrm{X})$ is in $AC^2([0,1], L^0(\mathrm{X}))$ with null derivative. By (4.38) such map is constant.

Now let $(V_t^1), (V_t^2)$ be two Parallel Transports starting from $\bar{V}$. Then $t \mapsto V_t := V_t^1 - V_t^2$ is a Parallel Transport starting from 0. By what already proved we deduce that $|V_t| \circ F_0^t = |V_0| = 0$ $\mathfrak{m}$-a.e. for every $t \in [0,1]$, i.e. $V_t^1 = V_t^2$ for every $t \in [0,1]$, as claimed. □

We turn to the problem of existence, that will be addressed by transforming it into an appropriate ODE-like problem in L^2_{fix}. Let us fix some terminology. A family $(\ell_t)_{t\in[0,1]}$ of L^0-linear and continuous maps from $L^0(T\mathrm{X})$ into itself is said *Borel* provided $t \mapsto \ell_t(v_t)$ is in $L^0_{fix}([0,1], L^0(T\mathrm{X}))$ for any $(v_t) \in L^0_{fix}([0,1], L^0(T\mathrm{X}))$ (arguing as in the proof of Proposition 5.2.3 this is equivalent to asking that $t \mapsto \ell_t(v)$ is in $L^0_{fix}([0,1], L^0(T\mathrm{X}))$ for any $v \in L^0(T\mathrm{X})$). Recalling that for any L^0-linear and continuous map $\ell : L^0(T\mathrm{X}) \to L^0(T\mathrm{X})$ we have $|\ell| = \sup_n \ell(v_n)$ $\mathfrak{m}$-a.e. for an appropriate choice of the countable set $\{v_n\}_{n\in\mathbb{N}} \subset L^0(T\mathrm{X})$, if (ℓ_t) is a Borel family then the map $t \mapsto |\ell_t| \in L^0(\mathrm{X})$ is also Borel.

Assume now that the Borel family (ℓ_t) satisfies

$$\int_0^1 |\ell_t|^2 \,\mathrm{d}t < \infty, \qquad \mathfrak{m} - a.e. \tag{5.50}$$

and notice that in this case for any $(v_t) \in L^2_{fix}([0,1], L^0(T\mathrm{X}))$ we have

$$\int_0^1 |\ell_t(v_t)| \,\mathrm{d}t \le \int_0^1 |\ell_t|\,|v_t| \,\mathrm{d}t \le |(v_t)|_{L^2_{fix}} \sqrt{\int_0^1 |\ell_t|^2 \,\mathrm{d}t} \qquad \mathfrak{m} - a.e. \tag{5.51}$$

and thus $(\ell_t(v_t)) \in L^1_{fix}$. Hence, recalling the concept of integral of elements in L^1_{fix} discussed in Section 4.4, for any $t \in [0,1]$ we can define

$$L_t((v_r)) := \int_0^t \ell_r(v_r) \,\mathrm{d}r \in L^0(T\mathrm{X}) \tag{5.52}$$

and it is clear that $L_t : L^2_{fix} \to L^0(T\mathrm{X})$ is $L^0(\mathrm{X})$-linear and continuous. We shall write $L_t := \int_0^t \ell_r \,\mathrm{d}r$ for the operator defined by the above formula. Notice that from

$$|L_s((v_r)) - L_t((v_r))| = \Big| \int_t^s \ell_r(v_r) \,\mathrm{d}r \Big| \le \int_t^s |\ell_r(v_r)| \,\mathrm{d}r \qquad \mathfrak{m} - a.e.$$

it follows that $t \mapsto L_t((v_r)) \in L^0(T\mathrm{X})$ is continuous w.r.t. the $L^0(T\mathrm{X})$-topology.

We call $L : L^2_{fix} \to C([0,1], L^0(T\mathrm{X})) \subset L^0_{fix}$ the resulting map, i.e.

$$L((v_r))_t := L_t((v_r)), \qquad \forall t \in [0,1]. \tag{5.53}$$

Notice that (5.51) ensures that $L((v_r)) \in L^\infty_{fix} \subset L^2_{fix}$ for any $(v_t) \in L^2_{fix}$. Below, to simplify the notation, we shall write $L(v)$ in place of $L((v_t))$, as well as $L_t(v)$ in place of $L_t((v_r))$.

Proposition 5.4.3. *Let $\ell_t : L^0(T\mathrm{X}) \to L^0(T\mathrm{X})$ be a Borel family of L^0-linear and continuous maps from $L^0(T\mathrm{X})$ into itself satisfying* (5.50) *and define $L : L^2_{fix}([0,1], L^0(T\mathrm{X})) \to L^2_{fix}([0,1], L^0(T\mathrm{X}))$ as in* (5.53), (5.52). *Then:*

i) for every $(z_t) \in L^2_{fix}([0,1], L^0(T\mathrm{X}))$ there is a unique $(v_t) \in L^2_{fix}([0,1], L^0(T\mathrm{X}))$ such that

$$v_t = L_t(v) + z_t, \qquad a.e.\ t,\ \mathfrak{m} - a.e.. \tag{5.54}$$

ii) if $(z_t) \in AC^2_{fix}([0,1], L^0(T\mathrm{X}))$, then the unique $(v_t) \in L^2_{fix}([0,1], L^0(T\mathrm{X}))$ solving (5.54) *admits a continuous representative in $AC^2_{fix}([0,1], L^0(T\mathrm{X}))$ for which* (5.54) *holds for every $t \in [0,1]$ and moreover*

$$\begin{cases} \dot v_t &= \ell_t(v_t) + \dot z_t, \qquad a.e.\ t, \\ v_0 &= z_0. \end{cases} \tag{5.55}$$

Proof.
(I) For $(v_t) \in L^2_{fix}$ and $s \in [0,1]$ we shall define $|(v_t)|_{L^2_{fix}([0,s])} \in L^0(\mathrm{X})$ as

$$|(v_t)|^2_{L^2_{fix}([0,s])} := \int_0^s |v_r|^2 \,\mathrm{d}r,$$

so that $|(v_t)|_{L^2_{fix}([0,1])} = |(v_t)|_{L^2_{fix}}$. We claim that for every $n \in \mathbb{N}$ and $(v_t) \in L^2_{fix}$ we have

$$|L^n(v)|^2_{L^2_{fix}([0,t])} \leq \frac{t^n \bar{G}^n}{n!} |(v_t)|^2_{L^2_{fix}([0,t])}, \tag{5.56}$$

where $\bar{G} := \int_0^1 |l_t|^2 \, \mathrm{d}t$. The case $n = 0$ is obvious. Assume we proved the claim for n and let us prove it for $n+1$. We have

$$\begin{aligned}
|L^{n+1}(v)|^2_{L^2_{fix}([0,t])} &= \int_0^t |L_s(L^n(v))|^2 \, \mathrm{d}s = \int_0^t \Big| \int_0^s \ell_r(L^n(v)_r) \, \mathrm{d}r \Big|^2 \, \mathrm{d}s \\
&\leq \int_0^t \Big(\int_0^s |\ell_r|^2 \, \mathrm{d}r \Big) |L^n(v)|^2_{L^2_{fix}([0,s])} \, \mathrm{d}s \\
&\overset{*}{\leq} \frac{\bar{G}^{n+1}}{n!} \int_0^t s^n |(v_t)|^2_{L^2_{fix}([0,s])} \, \mathrm{d}s \\
&\leq \frac{t^{n+1} \bar{G}^{n+1}}{(n+1)!} |(v_t)|^2_{L^2_{fix}([0,t])},
\end{aligned}$$

where in the starred inequality we used (5.50) and the induction assumption. From (5.56) it follows that $\sum_n |L^n(v)|_{L^2_{fix}} < \infty$ $\mathfrak{m}$-a.e., hence the series $\sum_{n \in \mathbb{N}} L^n(v)$ is a well defined element of L^2_{fix} (meaning that the partial sums form a Cauchy sequence in L^2_{fix}). It is clear that the operator $(v_t) \mapsto \sum_{n \in \mathbb{N}} L^n(v)$ is the inverse of $\mathrm{Id} - L$, indeed

$$(\mathrm{Id} - L) \sum_{n \in \mathbb{N}} L^n(v) = \lim_N (\mathrm{Id} - L) \sum_{n=0}^N L^n(v) = \lim_N v - L^{N+1}(v) = v$$

and similarly $\sum_{n \in \mathbb{N}} L^n((\mathrm{Id} - L)(v)) = v$. Thus $(v_t) \in L^2_{fix}$ solves (5.54) if and only if $v = \sum_{n \in \mathbb{N}} L^n(z)$, proving existence and uniqueness.
(II) By (4.49) we know that $L(v) \in AC^2_{fix}$ for any $(v_t) \in L^2_{fix}$, thus if $(z_t) \in AC^2_{fix}$ as well, we have that the right-hand side of (5.54) is in AC^2_{fix}. By the previous step, such right-hand side is the required representative of (v_t) in AC^2_{fix} for which (5.54) holds for every $t \in [0,1]$. Then (5.55) follows from the identity $L_t(v) + z_t = z_0 + \int_0^t \ell_s(v_s) + \dot{z}_s \, \mathrm{d}s$ (recall (4.46)) and the general property (4.49). □

Theorem 5.4.4 (Existence and uniqueness of Parallel Transport)**.** *Let $\bar{V} \in L^0(T\mathrm{X})$ and $(Z_t) \in L^2_{var}$. Then there is a unique $(V_t) \in AC^2_{var}$ such that*

$$\begin{cases} D_t V_t = Z_t & \text{a.e. } t \in [0,1], \\ V_0 = \bar{V}. \end{cases} \tag{5.57}$$

In particular, there is a unique Parallel Transport (V_t) starting from $\bar{V}$.

Proof. Recalling the definition of $AC^2_{var}([0,1], L^0(T\mathrm{X}))$ and that of convective derivative, we see that (V_t) satisfies (5.57) if and only if for $v_t := \mathrm{d}F^0_t(V_t)$ we have $(v_t) \in AC^2_{fix}([0,1], L^0(T\mathrm{X}))$ with

$$\begin{cases} \dot{v}_t = \mathrm{d}F^0_t(Z_t - \nabla_{\mathrm{d}F^t_0(v_t)} \boldsymbol{b}_t) & \text{a.e. } t \in [0,1], \\ v_0 = \bar{V}. \end{cases} \tag{5.58}$$

Thus for a.e. $t \in [0,1]$ we define $\ell_t : L^0(T\mathrm{X}) \to L^0(T\mathrm{X})$ as

$$\ell_t(v) := -\mathrm{d}F_t^0(\nabla_{\mathrm{d}F_0^t(v)}\boldsymbol{b}_t)$$

and notice that the bound

$$|\ell_t(v)| \overset{(4.22)}{\leq} \big(|\mathrm{d}F_t^0|\,|\nabla_{\mathrm{d}F_0^t(v)}\boldsymbol{b}_t|\big) \circ F_0^t \leq \big(|\mathrm{d}F_t^0|\,|\mathrm{d}F_0^t(v)|\,|\nabla \boldsymbol{b}_t|\big) \circ F_0^t \overset{(4.22),(5.8)}{\leq} G|v|\,|\nabla \boldsymbol{b}_t| \circ F_0^t,$$

valid $\mathfrak{m}$-a.e. for every $v \in L^0(T\mathrm{X})$, yields $|\ell_t| \leq G|\nabla \boldsymbol{b}_t| \circ F_0^t$, so that (5.5) ensures that the bound (5.50) holds. Also, we put

$$z_t := \bar{V} + \int_0^t \mathrm{d}F_r^0(Z_r)\,\mathrm{d}r.$$

Notice that the bound $|\mathrm{d}F_t^0(Z_t)| \leq (|\mathrm{d}F_t^0|\,|Z_t|) \circ F_0^t \leq G|Z_t| \circ F_0^t$ (having used (5.8)) ensures that the definition is well-posed. It is then clear from (4.49) that $(z_t) \in AC^2_{fix}$.

We can therefore apply (ii) of Proposition 5.4.3 above with this choice of (ℓ_t) and (z_t): we thus obtain the existence of $(v_t) \in AC^2_{fix}([0,1], L^0(T\mathrm{X}))$ satisfying (5.58), so that $t \mapsto V_t := \mathrm{d}F_0^t(v_t)$ is the only solution of (5.57), as desired. □

5.5 $W^{1,2}_{var}([0,1], L^0(T\mathrm{X}))$ as intermediate space between $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ and $\mathscr{W}^{1,2}(\boldsymbol{\pi})$

In this section we compare the main definitions given in this manuscript with those provided in the earlier paper [61]: we shall prove that in a very natural sense - see (5.67) below - the space $W^{1,2}_{var}([0,1], L^0(T\mathrm{X}))$ lies between the spaces $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ and $\mathscr{W}^{1,2}(\boldsymbol{\pi})$ for relevant $\boldsymbol{\pi}$'s. For this reason, we refer the reader to the definitions given in Section 3.1.

Let us fix now a regular plan $\boldsymbol{\pi}_\mu$ with initial distribution μ (associated to $\boldsymbol{b}$). The situation is different now: from the uniqueness of Regular Lagrangian Flows it easily follows that e_t is $\boldsymbol{\pi}_\mu$-essentially injective for any $t \in [0,1]$, the left inverse being $F_0^{\cdot} \circ F_t^0$. It is clear that for any $t \in [0,1]$ we have

$$\begin{aligned} &(F_0^{\cdot} \circ F_t^0) \circ \mathrm{e}_t = \mathrm{Id}_{C([0,1],\mathrm{X})} \quad \boldsymbol{\pi}_\mu - a.e., \\ &\mathrm{e}_t \circ (F_0^{\cdot} \circ F_t^0) = \mathrm{Id}_{\mathrm{X}} \quad (\mathrm{e}_t)_*\boldsymbol{\pi}_\mu - a.e.. \end{aligned} \tag{5.59}$$

Now notice that since $(\mathrm{e}_t)_*\boldsymbol{\pi}_\mu \ll \mathfrak{m}$, we have a well-defined pullback map $\mathrm{e}_t^* : L^0(T\mathrm{X}) \to \mathrm{e}_t^* L^0(T\mathrm{X})$; we shall often denote this map Ψ_t rather than e_t^*, i.e.

$$\begin{array}{cccc} \Psi_t : & L^0(T\mathrm{X}) & \to & \mathrm{e}_t^* L^0(T\mathrm{X}) \\ & V & \mapsto & \mathrm{e}_t^*(V). \end{array}$$

Similarly, defining up to $\mathfrak{m}$-negligible sets the Borel set $\mathsf{A}_t := \{\frac{\mathrm{d}(\mathrm{e}_t)_*\boldsymbol{\pi}_\mu}{\mathrm{d}\mathfrak{m}} > 0\}$, we have that $(F_0^{\cdot} \circ F_t^0)_*\mathfrak{m}_{|\mathsf{A}_t} \ll (F_0^{\cdot} \circ F_t^0)_*(\mathrm{e}_t)_*\boldsymbol{\pi}_\mu = \boldsymbol{\pi}_\mu$ and therefore we have a natural pullback map $(F_0^{\cdot} \circ F_t^0)^* : \mathrm{e}_t^* L^0(T\mathrm{X}) \to \mathscr{M}$, where for brevity we denoted by $\mathscr{M}$ the $L^0(\mathrm{X}, \mathfrak{m}_{|\mathsf{A}_t})$-normed module $(F_0^{\cdot} \circ F_t^0)^* \mathrm{e}_t^* L^0(T\mathrm{X}, \boldsymbol{\pi}_\mu)$. Then from (5.59) and the functoriality of the pullback we see that $\mathscr{M}$ can, and will, be canonically identified with the restriction $L^0(T\mathrm{X})_{|\mathsf{A}_t}$ of $L^0(T\mathrm{X})$ to the set A_t. In turn, such restriction can naturally be seen as the subset of $L^0(T\mathrm{X})$ made of vector fields which are $\mathfrak{m}$-a.e. 0 on the complement of A_t: we shall denote by $\mathscr{M} \ni v \mapsto \chi_{\mathsf{A}_t} v \in L^0(T\mathrm{X})$ the inclusion map of $\mathscr{M}$ in $L^0(TX)$ (a similar construction can be made using the 'extension' functor defined in [62]). In summary, we have a natural map Φ_t from the $L^0(\boldsymbol{\pi}_\mu)$-normed module $e_t^* L^0(TX)$ to the $L^0(\mathfrak{m})$-normed module $L^0(TX)$ defined as

$$\begin{array}{rccc}\Phi_t: & e_t^* L^0(TX) & \to & L^0(TX)\\ & \mathsf{v} & \mapsto & \chi_{\mathsf{A}_t}(F_0^{\cdot}\circ F_t^0)^*(\mathsf{v}).\end{array}$$

From (5.59) and the functoriality of the pullback we see that

$$\begin{aligned}\Psi_t(\Phi_t(\mathsf{v})) &= \mathsf{v}, &\qquad \forall \mathsf{v}\in e_t^* L^0(TX),\\ \Phi_t(\Psi_t(V)) &= \chi_{\mathsf{A}_t}V, &\qquad \forall V\in L^0(TX).\end{aligned} \tag{5.60}$$

The maps Ψ_t, Φ_t naturally induce maps Ψ, Φ by acting componentwise:

$$\begin{array}{rccc}\Psi: & \prod_{t\in[0,1]} L^0(TX) & \to & \mathrm{VF}(\boldsymbol{\pi}_\mu)\\ & (t\mapsto V_t) & \mapsto & (t\mapsto \Psi_t(V_t)),\end{array}\qquad \begin{array}{rccc}\Phi: & \mathrm{VF}(\boldsymbol{\pi}_\mu) & \to & \prod_{t\in[0,1]} L^0(TX)\\ & (t\mapsto \mathsf{v}_t) & \mapsto & (t\mapsto \Phi_t(\mathsf{v}_t)).\end{array}$$

We have already noticed that $\mathrm{VF}(\boldsymbol{\pi}_\mu)$ comes with a natural structure of module over $L^0(\boldsymbol{\pi}_\mu)$ (but not of $L^0(\boldsymbol{\pi}_\mu)$-normed module) given by componentwise product. We also define a structure of $L^0(X)$ module on $\prod_{t\in[0,1]} L^0(TX)$ by putting, in analogy with (5.20):

$$f\times(V_t) := (f\circ F_t^0\, V_t) \qquad \forall f\in L^0(X),\ (V_t)\in\prod_{t\in[0,1]} L^0(TX).$$

These two module structures are related, indeed for $f,(V_t)$ as above, the identity $F_t^0\circ e_t = e_0$ valid $\boldsymbol{\pi}_\mu$-a.e. gives

$$e_t^*(f\circ F_t^0\, V_t) = f\circ F_t^0\circ e_t\, e_t^*(V_t) = f\circ e_0\, e_t^*(V_t)$$

and therefore

$$\Psi(f\times(V_t)) = f\circ e_0\,\Psi(V_\cdot). \tag{5.61}$$

A Borel time regularity of $\mathrm{VF}(\boldsymbol{\pi}_\mu)$ is expressed in terms of Definition 3.1.1. The time regularity $\prod_{t\in[0,1]} L^0(TX)$ can easily be defined using the fact that $\prod_{t\in[0,1]} L^0(TX)$ is the set of maps from $[0,1]$ to $L^0(TX)$.
We define the set $\mathrm{Test}\Pi\subseteq\prod_{t\in[0,1]} L^0(TX)$ as the vector space

$$\mathrm{Test}\Pi := \left\{ t\mapsto \sum_{i=1}^n \chi_{A^i}\times(\varphi_i(t)V_i)\ \text{ for } n\in\mathbb{N}, A^i\in\mathscr{B}(X), V_i\in\mathrm{TestV}(X), \varphi_i\in\mathrm{Lip}([0,1])\right\}.$$

Now notice that for $\Gamma\subset C([0,1],X)$ Borel, the set $\Gamma_0 := (F_0^{\cdot})^{-1}(\Gamma)\subset X$ satisfies $\chi_{\Gamma_0}\circ e_0 = \chi_\Gamma$ $\boldsymbol{\pi}_\mu$-a.e., therefore (5.61) gives

$$\Psi\big(\chi_{\Gamma_0}\times(V_t)\big) = \chi_\Gamma\,\Psi(V_\cdot) \tag{5.62}$$

and thus

$$\mathrm{TestVF}(\boldsymbol{\pi}_\mu) = \Psi\big(\mathrm{Test}\Pi\big).$$

Notice that arguing as in Proposition 5.2.3 it is easy to check that $(V_t)\in\prod_{t\in[0,1]} L^0(TX)$ is Borel (as a map from $[0,1]$ to $L^0(TX)$) if and only if $t\mapsto\langle V_t, Z_t\rangle\in L^0(X)$ is Borel for every

$(Z_t) \in \mathrm{Test}\Pi$. Then keeping in mind the simple identity $\langle \Psi_t(V), \Psi_t(Z)\rangle = \langle V, Z\rangle \circ F_0^t \circ \mathrm{e}_0$ valid $\boldsymbol{\pi}_\mu$-a.e. for every $V, Z \in L^0(T\mathrm{X})$ and the fact that e_0 is $\boldsymbol{\pi}_\mu$-essentially injective, it is easy to see that $(\mathsf{v}_t) \in \mathrm{VF}(\boldsymbol{\pi}_\mu)$ is Borel (in the sense of Def. 3.1.1) if and only if it is of the form $\Psi(V_\cdot)$ for some $(V_t) \in \prod_{t\in[0,1]} L^0(T\mathrm{X})$ Borel.

Having in mind the definition of $\mathscr{L}^2(\boldsymbol{\pi}_\mu)$, it is then clear that

$$\mathscr{L}^2(\boldsymbol{\pi}_\mu) = \Psi\Big(\Big\{(V_t) \in L^0_{var} \ : \ \iint_0^1 |V_t|^2 \circ F_0^t \,\mathrm{d}t\,\mathrm{d}\mu = \int_0^1 \int |V_t|^2 \circ \mathrm{e}_t \,\mathrm{d}\boldsymbol{\pi}_\mu\,\mathrm{d}t < \infty\Big\}\Big).$$

To define the spaces $\mathscr{W}^{1,2}(\boldsymbol{\pi}_\mu), \mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$, we first described differentiation of test vector fields along $\boldsymbol{\pi}_\mu$, by means of the operator D (see (3.4)). We do the same with $\mathrm{Test}\Pi$. We start defining $\tilde{\mathsf{D}}_t : \mathrm{Test}\Pi \to L^0_{var}$ as the linear extension of

$$\tilde{\mathsf{D}}_t(\chi_A \times (\varphi(t)V)) := \chi_A \times (\varphi'(t)V + \varphi(t)\nabla_{\boldsymbol{b}_t} V), \qquad a.e.\ t \in [0,1], \tag{5.63}$$

for $A \subset \mathrm{X}$ Borel, $\varphi \in \mathrm{LIP}([0,1])$, and $V \in \mathrm{TestV}(\mathrm{X})$, and thus having for the convective derivative $\mathsf{D}_t : \mathrm{TestVF}(\boldsymbol{\pi}_\mu) \to \mathscr{L}^2(\boldsymbol{\pi}_\mu)$ that

$$\mathsf{D}_t\Psi(V_\cdot) := \Psi(\tilde{\mathsf{D}}_t V_\cdot) \qquad \forall (V_t) \in \mathrm{Test}\Pi. \tag{5.64}$$

An important property of vector fields in $\mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$ is that they admit a *continuous representative.* Another way to phrase this property is by saying that

$$\begin{aligned} &\text{for every } (\mathsf{v}_t) \in \mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu) \text{ there is a unique } (\tilde{\mathsf{v}}_t) \in \mathrm{VF}(\boldsymbol{\pi}_\mu)\\ &\text{with } \mathsf{v}_t = \tilde{\mathsf{v}}_t \text{ for a.e. } t \text{ such that } \Phi(\tilde{\mathsf{v}}_\cdot) \text{ is in } C([0,1], L^0(T\mathrm{X})) \end{aligned} \tag{5.65}$$

(from Proposition 5.2.3 it is easy to see that the statement [61, Theorem 3.23] implies this property). In what follows we shall systematically identify elements of $\mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$ with their continuous representatives. Another crucial property of elements of $\mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$ is the absolute continuity of the norms:

$$(\mathsf{v}_t) \in \mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu) \quad \Rightarrow \quad (t \mapsto |\mathsf{v}_t|^2) \in AC^2([0,1], L^0(\boldsymbol{\pi}_\mu)) \quad \text{with} \quad \frac{\mathrm{d}}{\mathrm{d}t}|\mathsf{v}_t|^2 = 2\langle \mathsf{v}_t, \mathsf{D}_t\mathsf{v}_t\rangle, \quad a.e.\ t. \tag{5.66}$$

As mentioned, our goal in this section is to prove that the space $W^{1,2}_{var}$ is 'intermediate' between $\mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$ and $\mathscr{W}^{1,2}(\boldsymbol{\pi}_\mu)$. To make this statement rigorous we should actually consider $\Psi(W^{1,2}_{var})$ and enforce L^2-integrability of both the vector field into consideration and its convective derivative. We are therefore led to define the 'intermediate' space

$$\mathrm{Interm}(\boldsymbol{\pi}_\mu) := \big\{\Psi(V_\cdot) \ : \ (V_t) \in W^{1,2}_{var} \ \text{ and } \Psi(V_\cdot), \Psi(D_\cdot V_\cdot) \in \mathscr{L}^2(\boldsymbol{\pi}_\mu)\big\},$$

so that the main result of the section can be stated as

$$\mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu) \quad \subset \quad \mathrm{Interm}(\boldsymbol{\pi}_\mu) \quad \subset \quad \mathscr{W}^{1,2}(\boldsymbol{\pi}_\mu) \tag{5.67}$$

with compatible convective derivatives, see Proposition 5.5.3 below for the precise statement. It is worth to point out that once these inclusions are proved, from the completeness of $W^{1,2}_{var}$ it is not hard to prove that $\mathrm{Interm}(\boldsymbol{\pi}_\mu)$ equipped with the $\mathscr{W}^{1,2}(\boldsymbol{\pi}_\mu)$-norm is a Hilbert space.

The proof of (5.67) is based on the following lemma:

Lemma 5.5.1. *Let* $(V_t) \in W^{1,2}_{var}([0,1], L^0(T\mathrm{X}))$ *and* $f \in \mathrm{Test}(\mathrm{X})$*. Then* $t \mapsto \mathrm{d}f(V_t) \circ F_0^t$ *belongs to* $W^{1,2}([0,1], L^0(\mathrm{X}))$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{d}f(V_t) \circ F_0^t\big) = \mathrm{d}f(D_t V_t) \circ F_0^t + \mathrm{Hess}f(V_t, \boldsymbol{b}_t) \circ F_0^t, \qquad a.e.\ t,\ \mathfrak{m}-a.e.. \tag{5.68}$$

If moreover $(V_t) \in AC^2_{var}([0,1], L^0(T\mathrm{X}))$*, then we also have that* $t \mapsto \mathrm{d}f(V_t) \circ F_0^t$ *belongs to* $AC^2([0,1], L^0(\mathrm{X}))$*.*

Proof. It is clear that for $(V_t) \in C([0,1], L^0(T\mathrm{X}))$ the map $t \mapsto \mathrm{d}f(V_t) \in L^0(\mathrm{X})$ is continuous, so that by Lemma 5.2.1 also $t \mapsto \mathrm{d}f(V_t) \circ F_0^t$ is continuous. Thus the statement for absolutely continuous vector fields follows from the Sobolev case and in what follows we focus on the latter.

Let $\mathcal{A} \subset W^{1,2}_{fix}([0,1], L^0(T\mathrm{X}))$ be the collection of vector fields (v_t) for which the conclusion holds for $V_t := \mathrm{d}F_0^t(v_t)$. We shall prove that $\mathcal{A}$ has the properties $(o), \cdots, (iv)$ in Proposition 4.4.7, so that such proposition gives the conclusion.

By the linearity in (V_t) of the claim and of $\mathrm{d}F_0^t$, it is clear that $\mathcal{A}$ is a vector space, i.e. (o) holds. (i) follows from property (4.24) of the differential $\mathrm{d}F_0^t$ and (ii) by direct computation based on the identity $D_t(\varphi(t)V_t) = \varphi'(t)V_t + \varphi(t)D_tV_t$, which in turn is a direct consequence of the definition.

For (iii), let $(v_t^n) \subset \mathcal{A}$ be $W^{1,2}_{fix}$-converging to some $(v_t^\infty) \in W^{1,2}_{fix}([0,1], L^0(T\mathrm{X}))$. Put $V_t^n := \mathrm{d}F_0^t(v_t^n)$ for every $n \in \mathbb{N}$ and $t \in [0,1]$. We want to prove that $V_t^\infty := \mathrm{d}F_0^t(v_t^\infty)$ satisfies the conclusions in the statement. Since $(v_t^n) \to (v_t^\infty)$ and $(\dot v_t^n) \to (\dot v_t^\infty)$ in $L^2_{fix}([0,1], L^0(T\mathrm{X})) \hookrightarrow L^0_{fix}([0,1], L^0(T\mathrm{X}))$, Proposition 5.2.3 and the very definition (5.22) tell that $(V_t^n) \to (V_t^\infty)$ and $(D_tV_t^n) \to (D_tV_t^\infty)$ in $L^0_{var}([0,1], L^0(T\mathrm{X}))$. It follows that

$$\begin{array}{ccc} \mathrm{d}f(V_t^n) \circ F_0^t & \to & \mathrm{d}f(V_t^\infty) \circ F_0^t \\ \mathrm{d}f(D_tV_t^n) \circ F_0^t + \mathrm{Hess}f(V_t^n, \boldsymbol{b}_t) \circ F_0^t & \to & \mathrm{d}f(D_tV_t^\infty) \circ F_0^t + \mathrm{Hess}f(V_t^\infty, \boldsymbol{b}_t) \circ F_0^t \end{array}$$

in $L^0([0,1], L^0(\mathrm{X}))$. Since (5.68) holds for (V_t^n) and $|(v_t^n)|_{W^{1,2}_{fix}} \to |(v_t^\infty)|_{W^{1,2}_{fix}}$ in $L^0(\mathrm{X})$, by Proposition 4.4.3 the claim will be proved if we show that

$$|(\mathrm{d}f(V_t^n) \circ F_0^t)|_{W^{1,2}} \leq G|(v_t^n)|_{W^{1,2}_{fix}}, \qquad \mathfrak{m}-a.e. \tag{5.69}$$

for every $n \in \mathbb{N}$ for some $G \in L^0(\mathrm{X})$ possibly depending also on f. Since $|\mathrm{d}f| \in L^\infty(\mathrm{X})$ we have

$$|\mathrm{d}f(V_t^n) \circ F_0^t|_{L^2} \leq C|(V_t^n)|_{L^2_{var}} \overset{(5.24)}{\leq} G|(v_t^n)|_{L^2_{fix}}, \qquad \mathfrak{m}-a.e.$$

and since also $|\boldsymbol{b}_t| \in L^\infty(\mathrm{X} \times [0,1])$ we also have

$$\begin{aligned} |\mathrm{d}f(D_tV_t^n) \circ F_0^t + \mathrm{Hess}f(V_t^n, \boldsymbol{b}_t) \circ F_0^t|_{L^2} &\leq C|(D_tV_t^n)|_{L^2_{var}} + C|(V_t^n)|_{L^\infty_{var}}|(|\mathrm{Hess}f| \circ F_0^t)|_{L^2} \\ \text{(by (5.27), (5.26))} \qquad &\leq G|(v_t^n)|_{W^{1,2}_{fix}}(1 + |(|\mathrm{Hess}f| \circ F_0^t)|_{L^2}) \\ \text{(by (5.4) for } |\mathrm{Hess}f| \in L^2(\mathrm{X})) \qquad &\leq G|(v_t^n)|_{W^{1,2}_{fix}}, \qquad \mathfrak{m}-a.e.. \end{aligned}$$

The claim (5.69) follows.

To prove (iv) we use Proposition 5.3.2: let $\bar v \in L^0(T\mathrm{X})$ and put $V_t := \mathrm{d}F_0^t(\bar v)$ for $t \in [0,1]$. Then it is clear from Proposition 5.3.2 that $t \mapsto \mathrm{d}f(V_t)\circ F_0^t = \mathrm{d}(f\circ F_0^t)(\bar v)$ belongs to $W^{1,2}([0,1],L^0(\mathrm{X}))$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}f(V_t)\circ F_0^t) = \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_t)\circ F_0^t)(\bar v) = \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_t))(V_t)\circ F_0^t = \big(\mathrm{Hess}f(V_t,\boldsymbol{b}_t) + \mathrm{d}f(D_tV_t)\big)\circ F_0^t,$$

for a.e. t, having used the definition of D_t in the last step. This proves (iv) and gives the conclusion. □

From this lemma and the language recalled in this section we easily obtain the following:

Corollary 5.5.2. *Let* $(\mathsf{z}_t) \in \mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$ *and* $(V_t) \in W^{1,2}_{var}$*. Put* $\mathsf{v}_t := \Psi_t(V_t)$*. Then* $t \mapsto \langle \mathsf{z}_t,\mathsf{v}_t\rangle \in L^0(\boldsymbol{\pi}_\mu)$ *belongs to* $W^{1,2}([0,1],L^0(\boldsymbol{\pi}_\mu))$ *with*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathsf{z}_t,\mathsf{v}_t\rangle = \langle \mathsf{D}_t\mathsf{z}_t,\mathsf{v}_t\rangle + \langle \mathsf{z}_t,\Psi_t(D_tV_t)\rangle, \qquad \boldsymbol{\pi}_\mu - a.e.,\ a.e.\ t\in[0,1].$$

If (z_t) *is identified with its continuous representative and* $(V_t) \in AC^2_{var}$*, then* $t \mapsto \langle \mathsf{z}_t,\mathsf{v}_t\rangle \in L^0(\boldsymbol{\pi}_\mu)$ *belongs to* $AC^2([0,1],L^0(\boldsymbol{\pi}_\mu))$.

Proof.
Step 0. We prove the last statement assuming the first claim. Notice that

$$\langle \mathsf{z}_t,\Psi_t(V_t)\rangle \overset{(5.60)}{=} \langle \Psi_t(\Phi_t(\mathsf{z}_t)),\Psi_t(V_t)\rangle = \langle \Phi_t(\mathsf{z}_t),V_t\rangle\circ \mathrm{e}_t = \langle \Phi_t(\mathsf{z}_t),V_t\rangle\circ F_0^t\circ \mathrm{e}_0 \qquad \boldsymbol{\pi}_\mu - a.e..$$

Hence if (z_t) is continuous (recall (5.65)) and so is (V_t), Lemma 5.2.1 ensures that $t\mapsto \langle \Phi_t(\mathsf{z}_t),V_t\rangle\circ F_0^t \in L^0(\mathrm{X})$ is continuous, thus the above shows that so is $t\mapsto \langle \mathsf{z}_t,\mathsf{v}_t\rangle \in L^0(\boldsymbol{\pi}_\mu)$.
Step 1. We assume that $(\mathsf{z}_t) \in \mathrm{TestVF}(\boldsymbol{\pi}_\mu)$, say $(\mathsf{z}_t) = \Psi(Z_\cdot)$ for $(Z_t) := \sum_{i=0}^n \chi_{A_i}\times(\varphi_i(t)g_i\nabla f_i) \in \mathrm{Test}\Pi$, with A_i Borel, φ_i Lipschitz and $f_i,g_i \in \mathrm{Test}(\mathrm{X})$. Notice that by the very definition (5.20) we have

$$\langle Z_t,V\rangle\circ F_0^t = \sum_{i=0}^n \chi_{A_i}\varphi_i(t)\,g_i\circ F_0^t\,\langle \nabla f_i,V\rangle\circ F_0^t, \qquad \forall t\in[0,1],\ V\in L^0(T\mathrm{X}). \tag{5.70}$$

Noticing that $(t\mapsto g_i\circ F_0^t) \in W^{1,2}([0,1],L^0(\mathrm{X}))$ with derivative $\frac{\mathrm{d}}{\mathrm{d}t}g_i\circ F_0^t = \mathrm{d}g_i(w_t)\circ F_0^t$ and using Lemma 5.5.1 above, it follows, by the very definition of $W^{1,2}([0,1],L^0(\mathrm{X}))$, that $t\mapsto \langle Z_t,V_t\rangle\circ F_0^t \in L^0(\mathrm{X})$ is in $W^{1,2}([0,1],L^0(\mathrm{X}))$ with

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle Z_t,V_t\rangle\circ F_0^t &= \sum_{i=0}^n \chi_{A_i}\Big(\varphi_i(t)\,g_i\,(\langle \nabla f_i,D_tV_t\rangle + \mathrm{Hess}f_i(V_t,w_t))\\ &\qquad + \varphi_i'(t)\,g_i\,\langle \nabla f_i,V_t\rangle + \varphi_i(t)\mathrm{d}g_i(w_t)\langle \nabla f_i,V_t\rangle\Big)\circ F_0^t\\ \text{(by (5.63))}\qquad &= \langle Z_t,D_tV_t\rangle\circ F_0^t + \langle \tilde{\mathsf{D}}_tZ_t,V_t\rangle\circ F_0^t\end{aligned}$$

and thus (recall also (4.37)) for $(\mathcal{L}^1)^2$-a.e. $t,s\in[0,1]$ with $t<s$ we have

$$\langle Z_s,V_s\rangle\circ F_0^s - \langle Z_t,V_t\rangle\circ F_0^t = \int_t^s \langle Z_r,D_rV_r\rangle\circ F_0^r + \langle \tilde{\mathsf{D}}_rZ_r,V_r\rangle\circ F_0^r\,\mathrm{d}r, \qquad \mathfrak{m} - a.e.. \tag{5.71}$$

Now observe that pre-composing the defining identity $|\Psi_t(V)| = |V| \circ \mathrm{e}_t$ valid $\boldsymbol{\pi}_\mu$-a.e. by $F_0^{\cdot}$ and using the second in (5.59) we get

$$|\Psi_t(V)| \circ F_0^{\cdot} = |V| \circ F_0^t, \qquad \mathfrak{m} - a.e.\ on\ \mathsf{A}_0,\ \forall V \in L^0(T\mathrm{X}) \tag{5.72}$$

(recall that $\mathsf{A}_t := \{\frac{\mathrm{d}(\mathrm{e}_t)_*\boldsymbol{\pi}_\mu}{\mathrm{dm}} > 0\}$). We thus multiply (5.71) by χ_{A_0} and pre-compose it by e_0, then use (5.72) and the first in (5.59) for $t = 0$ and observe that $\chi_{\mathsf{A}_0} \circ \mathrm{e}_0 = 1$ $\boldsymbol{\pi}_\mu$-a.e. to get that for $(\mathcal{L}^1)^2$-a.e. $t, s \in [0,1]$ with $t < s$ we have

$$\langle \mathsf{z}_s, \mathsf{v}_s\rangle - \langle \mathsf{z}_t, \mathsf{v}_t\rangle = \int_t^s \langle \mathsf{z}_r, \Psi_r(D_r V_r)\rangle + \langle \mathsf{D}_r \mathsf{z}_r, \mathsf{v}_r\rangle\,\mathrm{d}r, \qquad \boldsymbol{\pi}_\mu - a.e., \tag{5.73}$$

having recalled (5.64). By (4.39), this proves our claim in the case $(\mathsf{z}_t) \in \mathrm{TestVF}(\boldsymbol{\pi}_\mu)$.
Step 2. For the general case, we start claiming that for $(\mathsf{z}_t) \in \mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$ we have

$$|(\mathsf{z}_t)|_{L^\infty} \le 2\big(|(\mathsf{z}_t)|_{L^2} + |(\mathsf{D}_t\mathsf{z}_t)|_{L^2}\big), \tag{5.74}$$

where here and below we adopt the notation $|\cdot|_{L^p}$ introduced in Section 4.4, referring this time to the space $L^0(\boldsymbol{\pi}_\mu)$ in place of $L^0(\mathfrak{m})$. Indeed, from (5.66) we see that $|\frac{\mathrm{d}}{\mathrm{dt}}|\mathsf{z}_t|^2| \le |\mathsf{z}_t|^2 + |\mathsf{D}_t\mathsf{z}_t|^2$ for a.e. t and thus after integration we obtain

$$|\mathsf{z}_t|^2 \le |\mathsf{z}_s|^2 + \int_{t\wedge s}^{t\vee s} |\mathsf{z}_r|^2 + |\mathsf{D}_r\mathsf{z}_r|^2\,\mathrm{d}r \le |\mathsf{z}_s|^2 + \int_0^1 |\mathsf{z}_r|^2 + |\mathsf{D}_r\mathsf{z}_r|^2\,\mathrm{d}r \qquad \boldsymbol{\pi}_\mu - a.e.,$$

for $(\mathcal{L}^1)^2$-a.e. $t, s \in [0,1]$. Integrating in s we deduce (5.74).

We shall use (5.74) in conjunction with Proposition 4.4.3 to conclude. Thus let $(\mathsf{z}_t) \in \mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$ and $(\mathsf{z}_t^n) \subset \mathrm{TestVF}(\boldsymbol{\pi}_\mu)$ be $\mathscr{W}^{1,2}(\boldsymbol{\pi}_\mu)$-converging to it. From the definition of $\mathscr{W}^{1,2}(\boldsymbol{\pi}_\mu)$-norm it is clear that

$$\begin{aligned} \langle \mathsf{z}_t^n, \mathsf{v}_t\rangle \quad &\to \quad \langle \mathsf{z}_t, \mathsf{v}_t\rangle \\ \langle \mathsf{z}_t^n, \Psi_t(D_tV_t)\rangle + \langle \mathsf{D}_t\mathsf{z}_t^n, \mathsf{v}_t\rangle \quad &\to \quad \langle \mathsf{z}_t, \Psi_t(D_tV_t)\rangle + \langle \mathsf{D}_t\mathsf{z}_t, \mathsf{v}_t\rangle \end{aligned}$$

in $L^0([0,1], L^0(\boldsymbol{\pi}_\mu))$ and that

$$|(\mathsf{z}_t^n)|_{L^2} \quad \to \quad |(\mathsf{z}_t)|_{L^2}, \qquad\qquad |(\mathsf{D}_t\mathsf{z}_t^n)|_{L^2} \quad \to \quad |(\mathsf{D}_t\mathsf{z}_t)|_{L^2}$$

in $L^0(\boldsymbol{\pi}_\mu)$. Also, from the definition of Ψ we have that $|(\Psi(V_\cdot))|_{L^p} = |(V_t)|_{L^p_{var}} \circ \mathrm{e}_0$, thus

$$\begin{aligned} |\langle \mathsf{z}_t^n, \mathsf{v}_t\rangle|_{L^2} \quad &\le \quad |(\mathsf{z}_t^n)|_{L^2}|(\mathsf{v}_t)|_{L^\infty} = |(\mathsf{z}_t^n)|_{L^2}|(V_t)|_{L^\infty_{var}} \circ \mathrm{e}_0 \\ \text{(by (5.27))} \qquad &\le \quad G|(\mathsf{z}_t^n)|_{L^2}|(V_t)|_{W^{1,2}_{var}} \circ \mathrm{e}_0, \end{aligned}$$

and

$$\begin{aligned} |\langle \mathsf{z}_t^n, \Psi_t(D_tV_t)\rangle + \langle \mathsf{D}_t\mathsf{z}_t^n, \mathsf{v}_t\rangle|_{L^2} \quad &\le \quad |(\mathsf{z}_t^n)|_{L^\infty}|(\Psi_t(D_tV_t))|_{L^2} + |(\mathsf{D}_t\mathsf{z}_t^n)|_{L^2}|(\mathsf{v}_t)|_{L^\infty} \\ \text{(by (5.27))} \qquad &\le \quad G\,|(V_t)|_{W^{1,2}_{var}} \circ \mathrm{e}_0\,\big(|(\mathsf{z}_t^n)|_{L^2} + |(\mathsf{D}_t\mathsf{z}_t^n)|_{L^2}\big). \end{aligned}$$

Hence Proposition 4.4.3 gives the conclusion. □

The main result of the section can now be proved rather easily:

Proposition 5.5.3. *The inclusions in* (5.67) *hold and the underlying notions of convective derivatives agree, i.e. for* $(V_t) \in W^{1,2}_{var}$ *such that* $\mathsf{v}_\cdot := \Psi(V_\cdot) \in \mathscr{L}^2(\boldsymbol{\pi}_\mu)$ *we have* $\Psi_t(D_t V_t) = \mathsf{D}_t \mathsf{v}_t$ *for a.e.* $t \in [0,1]$.

Proof.
STEP 1. We prove the second inclusion in (5.67) and the identification of convective derivatives. Let $(\mathsf{v}_t) \in \mathrm{Interm}(\boldsymbol{\pi}_\mu)$, say $(\mathsf{v_t}) = \Psi(V_\cdot)$ for $(V_t) \in W^{1,2}_{var}$. Also, let $(\mathsf{z}_t) \in \mathrm{TestVF}(\boldsymbol{\pi}_\mu)$ be with compact support in time in the sense of (3.6). Then from Corollary 5.5.2 above, since $\mathrm{TestVF}(\boldsymbol{\pi}_\mu) \subseteq \mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$, we obtain that

$$\int_0^1 \langle \mathsf{z}_r, \Psi_r(D_r V_r)\rangle + \langle \mathsf{D}_r \mathsf{z}_r, \mathsf{v}_r\rangle \,\mathrm{d}r = 0 \qquad \boldsymbol{\pi}_\mu - a.e..$$

The integrability assumptions coming from the hypothesis $\Psi(V_\cdot) \in \mathrm{Interm}(\boldsymbol{\pi}_\mu)$ and $(\mathsf{z}_t) \in \mathrm{TestVF}(\boldsymbol{\pi}_\mu)$ ensure that the left-hand side of the above is in $L^1(\boldsymbol{\pi}_\mu)$, thus upon integration we see that (3.5) holds with $\mathsf{v}'_t = \Psi_t(D_t V_t)$ for a.e. $t \in [0,1]$. By the arbitrariness of $(\mathsf{z}_t) \in \mathrm{TestVF}(\boldsymbol{\pi}_\mu)$, this is the claim.
STEP 2. We prove the first inclusion in (5.67). Let $(\mathsf{z}_t) \in \mathscr{H}^{1,2}(\boldsymbol{\pi}_\mu)$ be identified with its continuous representative as in (5.65). Put $Z_t := \Phi_t(\mathsf{D}_t \mathsf{z}_t)$ and $\bar V := \Phi_0(\mathsf{z}_0)$ and notice that the identity $|\Phi_t(\mathsf{z})| = \chi_{\mathsf{A}_t}|\mathsf{z}| \circ F_0^{\cdot} \circ F_t^0$ gives $|\Phi_t(\mathsf{z})| \circ F_0^t = \chi_{\mathsf{A}_0}|\mathsf{z}| \circ F_0^{\cdot}$, which in turn implies, thanks to $(\mathsf{D}_t \mathsf{z}_t) \in \mathscr{L}^2(\boldsymbol{\pi}_\mu)$, that $(Z_t) \in L^2_{var}$.

By Theorem 5.4.4 there is (a unique) $(V_t) \in AC^2_{var}$ satisfying (5.57) with these choices of $(Z_t), \bar V$. To conclude the proof it is therefore sufficient to show that $\Psi_t(V_t) = \mathsf{z}_t$ for every $t \in [0,1]$. Notice that the identity $|\Psi_t(V)| = |V| \circ \mathrm{e}_t$ gives that $t \mapsto |\Psi_t(V_t)|^2$ is in $AC^2([0,1], L^0(\boldsymbol{\pi}_\mu))$ with $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_t(V_t)|^2 = \langle \Psi_t(V_t), \Psi_t(D_t V_t)\rangle$. Thus taking into account (5.66), Corollary 5.5.2, and Theorem 5.3.4, we see that $t \mapsto |\mathsf{z}_t - \Psi_t(V_t)|^2 = |\mathsf{z}_t|^2 + |\Psi_t(V_t)|^2 - 2\langle \mathsf{z}_t, \Psi_t(V_t)\rangle$ is in $AC^2([0,1], L^0(\boldsymbol{\pi}_\mu))$ with derivative given by

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathsf{z}_t - \Psi_t(V_t)|^2 = \langle \mathsf{z}_t, \mathsf{D}_t \mathsf{z}_t\rangle + \langle \Psi_t(V_t), \Psi_t(D_t V_t)\rangle - \langle \mathsf{z}_t, \Psi_t(D_t V_t)\rangle - \langle \mathsf{D}_t \mathsf{z}_t, \Psi_t(V_t)\rangle = 0, \quad a.e.\ t$$

having used the fact that $\Psi_t(D_t V_t) = \Psi_t(\Phi_t(\mathsf{D}_t \mathsf{z}_t)) = \mathsf{D}_t \mathsf{z}_t$. Since $\Psi_0(V_0) = \Psi_0(\Phi_0(\mathsf{z}_0)) = \mathsf{z}_0$, the conclusion follows. □

Chapter 6

Local convergence in measure of gradients of flows associated to Sobolev vector fields

The main content of this chapter, that will be a part of a forthcoming work with N. Gigli, is to show the following theorem in the Euclidean setting.

Theorem 6.0.1. *Let $p > 1$ and $\boldsymbol{b}^n, \boldsymbol{b}\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be such that there exists $C_0 > 0$ such that $\|\boldsymbol{b}^n\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq C_0$, for every $M > 0$ there exists $C_M > 0$ such that*

$$\sup_n \Big(\mathrm{esssup}_{t\in[0,T]} \|D_x \boldsymbol{b}^n_t\|_{L^p(B_M(0))}\Big) \leq C_M.$$

Assume $\boldsymbol{b}^n \to \boldsymbol{b}$ in $L^1([0,T], W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d))$ and $\sup_n \int_0^T \|\mathrm{div}\boldsymbol{b}^n_t\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}t < \infty$. Consider the associated regular Lagrangian flows ${F^n}_s^t$ and F_s^t. Then there exists a subsequence such that

$$\nabla {F^n}_s^t \to \nabla F_s^t \quad \textit{locally in measure.} \tag{6.1}$$

We want to compare the assumptions on the Sobolev regularity of vector fields which ensure local convergence in measure of the associated flows with the ones ensuring (6.1). Under the assumptions that the limit vector field belongs to $L^1_t W^{1,p}_x$ and the sequence converges in $L^1_{t,x}$, plus $L^\infty_{t,x}$ equibounds on the sequence and $L^1_t L^\infty_x$ equibounds on the divergence, the stability at order zero of flows, namely local convergence in measure, can be proved (see e.g. [43, Theorem 2.9]). For the local convergence in measure of the differentials of the flows, we don't need any condition at second order neither on the converging sequence nor on the limit vector field, by taking advantage of to the fact the gradient of the flow satisfies a linearized ODE; however, we assume stronger conditions at the first order, namely convergence of the sequence of vector fields to the limit one in $L^1_t W^{1,p}_x$ and it turns out to be necessary.

The reason for which we got interested in such a problem is that we originally proved the theorem of existence and uniqueness of parallel transport in $\mathsf{ncRCD}(K,N)$ spaces (Theorem 5.4.4) along flows of an autonomous Sobolev vector field; moreover, the same trivially holds for flows of piecewise-in-time autonomous vector fields. Then our goal was to study time dependent vector fields as limits of piecewise-in-time autonomous ones. For this reason, our strategy was based

on the understanding of an analogue of (6.1) in the nonsmooth setting. Later, we realized that existence and uniqueness could be proved directly in the case of time dependent vector fields satisfying the conditions at the beginning of Section 5.1.
The presentation is organized as follows. We recall in Section 6.1 in which sense a regular Lagrangian flow is differentiable with respect to the initial conditions and solve the linearized ODE

$$\partial_t \nabla F_s^t(x) = D\boldsymbol{b}_t(F_s^t(x))\,\nabla F_s^t(x).$$

We provide in Section 6.2 a counterexample in which (6.1) does not hold. In this example, we consider in $\mathbb{R}$ a sequence of vector fields (v^n) for which $\sup_n \big(\|v^n\|_{L^\infty([0,T]\times\mathbb{R})} + \|\partial_x v_t^n\|_{L^\infty([0,T]\times\mathbb{R})}\big) < \infty$, $v^n \to v \in L^1([0,T], L^p_{\mathrm{loc}}(\mathbb{R}))$ for every $p \in [1,+\infty]$, but it is not true that $\partial_x v^n \to \partial_x v \in L^1([0,T], L^1_{\mathrm{loc}}(\mathbb{R}))$. Then, in Section 6.2.1, we exploit that $\nabla F_s^t(x)$ solves the linearized ODE in order to pass to pass to the limit along a sequence of vector fields and prove the stability result in Theorem 6.0.1.

6.1 The ODE solved by $\nabla F_s^t(x)$

We explain in this section in which sense a weak notion of the gradient of the flow map solves for fixed x the linearized ODE

$$\partial_s \nabla F_t^s(x) = D\boldsymbol{b}_s(F_t^s(x))\nabla F_t^s(x). \tag{6.2}$$

The idea is to tailor Proposition 5.3.2 in the Euclidean setting; however, the last proposition is proved under the assumptions that $(\boldsymbol{b}_t) \in L^2([0,1], W^{1,2}_C(T\mathrm{X}))$, $|\boldsymbol{b}_t|, |\mathrm{div}(\boldsymbol{b}_t)| \in L^\infty([0,1]\times\mathrm{X})$ and for some $\bar{x} \in \mathrm{X}$ and $R > 0$ we have $\mathrm{supp}(\boldsymbol{b}_t) \subset B_R(\bar{x})$ for a.e. $t \in [0,1]$. To this aim, we prove that the following version of Proposition 5.3.2 holds in the setting of $\mathsf{ncRCD}(K,N)$ spaces with more natural assumptions on the vector fields, the same under which Theorem 6.0.1 holds. The results will be used later in the Euclidean setting.

Proposition 6.1.1. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{ncRCD}(K,N)$ *space. Let* F_t^s *be the regular Lagrangian flow associated to* $(\boldsymbol{b}_t) \in L^1([0,T], W^{1,2}_C(T\mathrm{X}))$ *such that* $\mathrm{div}(\boldsymbol{b}_t) \in L^1([0,T], L^\infty(\mathfrak{m}))$, $|\boldsymbol{b}_t| \in L^\infty([0,T]\times \mathrm{X})$. *Then for every* $f \in \mathrm{Test}(\mathrm{X})$ *and* $t \in [0,T]$, *the map* $s \mapsto \mathrm{d}(f\circ F_t^s)$ *is in* $AC^2([0,T], L^0(T^*\mathrm{X}))$ *and we have*

$$\lim_{h\to 0} \frac{\mathrm{d}(f\circ F_t^{s+h}) - \mathrm{d}(f\circ F_t^s)}{h} = \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_s)\circ F_t^s) \qquad \text{in } L^0(T^*\mathrm{X}) \text{ for a.e. } s. \tag{6.3}$$

Proof. Step 1: assume that $\boldsymbol{b}$ has bounded support, namely for some $\bar{x} \in \mathrm{X}$ and $R > 0$ $\mathrm{supp}(\boldsymbol{b}_t) \subseteq B_R(\bar{x})$ for a.e. $t \in [0,T]$. We define

$$G(t) := \|\mathrm{div}\boldsymbol{b}_t\|_{L^\infty(\mathfrak{m})} + \|\boldsymbol{b}_t\|_{W^{1,2}_C(T\mathrm{X})}$$

and $h(s) := \int_0^s \epsilon + G(r)\,\mathrm{d}r$, where $h\colon [0,1] \to [0, T_\epsilon]$, where $T_\epsilon = \epsilon + \int_0^1 G(r)\,\mathrm{d}r$ for some $\epsilon > 0$. h is strictly increasing and absolutely continuous, with inverse g. We have that

$$g'(t) = \frac{1}{h'(g(t))} = \frac{1}{\epsilon + G(g(t))} \quad \text{for a.e. } t \in [0, T_\epsilon].$$

Therefore, $g'\,G\circ g \in L^\infty([0,T_\epsilon])$. We define $\boldsymbol{v}_t := \boldsymbol{b}_{g(t)}\,g'(t)$ for a.e. $t \in [0,T_\epsilon]$. We check that $\boldsymbol{v}$ satisfies all the hyphothesis of Proposition 5.3.2. We check that $|\boldsymbol{v}_t| \in L^\infty([0,T]\times\mathrm{X})$. We call for simplicity $a(t) := \|\boldsymbol{b}_t\|_{L^\infty(\mathfrak{m})}$. We check $\|a\circ g\|_{L^\infty([0,T_\epsilon])} \le \|a\|_{L^\infty([0,T])}$. Indeed, this follows from the fact that for every $M>0$ for which $\mathscr{L}^1(\{a>M\})=0$ the change of variable

$$0=\int_{\{a>M\}} 1\,\mathrm{d}t=\int_{g^{-1}(\{a>M\})} g'(t)\,\mathrm{d}t$$

and the fact that $g'>0$ for a.e. t yield $\mathscr{L}^1(\{g^{-1}(\{a>M\})\})=0$. Therefore, $\|\boldsymbol{v}_t\|_{L^\infty(\mathfrak{m})} = \|\boldsymbol{b}_{g(t)}\|_{L^\infty(\mathfrak{m})}|g'(t)| \le \frac{1}{\epsilon}\|\boldsymbol{b}_{g(t)}\|_{L^\infty(\mathfrak{m})} \le \frac{1}{\epsilon}\|\boldsymbol{v}\|_{L^\infty([0,T_\epsilon]\times \mathrm{X})}$ for a.e. t. Similarly, it can be readily checked that $\mathrm{supp}(\boldsymbol{v}_t)\subset B_R(\bar{x})$ for a.e. $t\in[0,T_\epsilon]$. Moreover, $(\boldsymbol{v}_t)\in L^2([0,T_\epsilon],W^{1,2}_C(T\mathrm{X}))$ and $|\mathrm{div}(\boldsymbol{v}_t)|\in L^\infty([0,T_\epsilon]\times\mathrm{X})$, as a consequence of the fact that $\|\mathrm{div}\boldsymbol{v}_t\|_{L^\infty(\mathfrak{m})}+\|\boldsymbol{v}_t\|_{W^{1,2}_C(T\mathrm{X})} = g'(t)\,G(g(t))\in L^\infty([0,T_\epsilon])$. Let $\bar{F}$ be the regular Lagrangian flows associated to $\boldsymbol{v}$. Therefore, Proposition 5.3.2 applies, having that there exists a $\mathfrak{m}$-negligible $N_0\subseteq[0,T_\epsilon]$ such that, for every $t\in[0,T_\epsilon]$ and $s\in[0,T_\epsilon]\setminus N_0$ and $f\in\mathrm{Test}(\mathrm{X})$, we have

$$\lim_{h\to 0}\frac{\mathrm{d}(f\circ\bar{F}^{s+h}_t)-\mathrm{d}(f\circ\bar{F}^s_t)}{h}=\mathrm{d}(\mathrm{d}f(\boldsymbol{v}_s)\circ\bar{F}^s_t)\qquad\text{in } L^0(T^*\mathrm{X}). \tag{6.4}$$

Let F the regular Lagrangian flows associated to $\boldsymbol{b}$ and by its very definition, for every $t\in[0,T]$ we have that for every $t\in[0,T]$, $\mathfrak{m}$-a.e.

$$\partial_s f\circ F^{g(s)}_{g(t)}=\mathrm{d}f(\boldsymbol{b}_{g(s)})\circ F^{g(s)}_{g(t)}g'(s)=\mathrm{d}f(\boldsymbol{v}_s)\circ F^{g(s)}_{g(t)}\quad\text{for a.e. } s\in[0,T_\epsilon].$$

By uniqueness of regular Lagrangian flow associated to $\boldsymbol{v}$ it follows that (the other properties are trivially verified)

$$\bar{F}^s_t=F^{g(s)}_{g(t)}\quad\text{for every } t,s\in[0,T_\epsilon].$$

We define $N_1:=N_0\cup\{\text{g is not differentiable or } g'=0\}$ and $\mathscr{L}^1(N_1)=0$. Therefore, from (6.4) and the last consideration for every $s\in[0,T]\setminus N_1$

$$\begin{aligned}\lim_{h\to 0}\frac{\mathrm{d}(f\circ F^{g(s)+h}_{g(t)})-\mathrm{d}(f\circ F^{g(s)}_{g(t)})}{h}&=\frac{1}{g'(s)}\lim_{h\to 0}\frac{\mathrm{d}(f\circ F^{g(s+h)}_{g(t)})-\mathrm{d}(f\circ F^{g(s)}_{g(t)})}{h}\\&=\mathrm{d}(\mathrm{d}f(\boldsymbol{b}_{g(s)})\circ F^{g(s)}_{g(t)})\qquad\text{in } L^0(T^*\mathrm{X}).\end{aligned}$$

Using that g is bijective and that $\mathscr{L}^1(g(N_1))\le\frac{1}{\varepsilon}\mathscr{L}^1(N_1)$ being g $1/\epsilon$-Lipschitz, we get (6.3) for every $s\in[0,T]\setminus N$ with $N=g(N_1)$.
Step 2: we prove that (6.3) holds withouth assumption on boundedness of the support of $\boldsymbol{b}$. We fix $\mathsf{o}\in\mathrm{X}$, $R>0$ and we call $M:=\|\boldsymbol{b}\|_{L^\infty_{t,x}}$. We consider $\varphi\in\mathrm{Lip}_{bs}(\mathrm{X})$ such that $\varphi=1$ on $B_{R+M}(\mathsf{o})$ and $\varphi\le 1$, define $\bar{\boldsymbol{b}}_t=\varphi\,\boldsymbol{b}_t$ for a.e. t and call it its associated regular Lagrangian flow $\bar{F}$. Notice that $F^s_t(x)\in B_{R+M}(\mathsf{o})$ for every t,s for every $x\in B_R(\mathsf{o})$, whence $\bar{F}^s_t=F^s_t$ on $B_R(\mathsf{o})$. By locality of the differential of Lusin-Lipschitz maps and the fact that $\varphi\circ\bar{F}^s_t=1$ on $B_R(\mathsf{o})$, it follows that

$$\begin{aligned}&\mathrm{d}(f\circ F^s_t)=\mathrm{d}(f\circ\bar{F}^s_t)\quad\text{on } B_R(\mathsf{o}),\\&\mathrm{d}(\mathrm{d}f(\bar{\boldsymbol{b}}_s)\circ\bar{F}^s_t)=\mathrm{d}(\varphi\circ\bar{F}^s_t\,\mathrm{d}f(\boldsymbol{b}_s)\circ\bar{F}^s_t)=\mathrm{d}(\mathrm{d}f(\boldsymbol{b}_s)\circ\bar{F}^s_t)=\mathrm{d}(\mathrm{d}f(\boldsymbol{b}_s)\circ F^s_t)\quad\text{on } B_R(\mathsf{o}).\end{aligned}$$

The previous considerations and Step 1 gives that for every s_1, s_2 with $s_1 < s_2$ we have

$$\chi_{B_R(\mathsf{o})}\mathrm{d}(f \circ F_t^{s_2}) - \chi_{B_R(\mathsf{o})}\mathrm{d}(f \circ F_t^{s_1}) = \chi_{B_R(\mathsf{o})}\mathrm{d}(f \circ \bar{F}_t^{s_2}) - \chi_{B_R(\mathsf{o})}\mathrm{d}(f \circ \bar{F}_t^{s_1})$$
$$= \chi_{B_R(\mathsf{o})} \int_{s_1}^{s_2} \mathrm{d}(\mathrm{d}f(\bar{\boldsymbol{b}}_r) \circ \bar{F}_t^r)\,\mathrm{d}r = \chi_{B_R(\mathsf{o})} \int_{s_1}^{s_2} \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_r) \circ F_t^r)\,\mathrm{d}r$$

thus having that by arbitrariness of R that for every s_1, s_2 with $s_1 < s_2$ we have

$$\mathrm{d}(f \circ F_t^{s_2}) - \mathrm{d}(f \circ F_t^{s_1}) = \int_{s_1}^{s_2} \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_r) \circ F_t^r)\,\mathrm{d}r$$

whence $s \mapsto \mathrm{d}(f \circ F_t^s)$ belongs to $AC^2([0,T], L^0(T^*\mathrm{X}))$ and (6.3) holds. □

We recall the notion of approximate differentiability (from [43, Appendix B]).

Definition 6.1.2 (Approximate gradient)**.** *We say that a Borel map $f\colon \mathbb{R}^d \to \mathbb{R}^k$ is approximately differentiable at the point x if there exists a map $\tilde{f}$, differentiable in the classical sense at x, such that $\tilde{f}(x) = f(x)$ and such that $\{y\colon \tilde{f}(y) = f(y)\}$ has density 1 at x. Moreover, we denote by $\nabla_a f(x) = \nabla \tilde{f}(x)$ and we call it the approximate gradient at the point x.*

Remark 6.1.3. We recall that, if $f\colon \mathbb{R}^d \to \mathbb{R}^k$ has the Lusin-Lispchitz property, then it is approximately differentiable at $\mathscr{L}^d$–a.e. $x \in \mathbb{R}^d$.

The connection between the differential $\mathrm{d}f$ and the approximate gradient $\nabla_a f$ is given the following proposition. We denote, for every i, by π_i the coordinate maps $\pi_i(x)\colon = x_i$ and we recall that $\{\mathrm{d}\pi_i\}_i$ is a local base of $L^0(T^*\mathbb{R}^d)$.

Lemma 6.1.4. *The linear map $\Phi\colon L^0(T^*\mathbb{R}^d) \to L^0(\mathbb{R}^d, \mathbb{R}^d)$ defined for $v = \sum_{i=1}^d a^i \mathrm{d}\boldsymbol{\pi}_i \in L^0(T^*\mathbb{R}^d)$ with $a^i \in L^0(\mathbb{R}^d)$ for every i as*

$$\Phi(v) = (a^1, \ldots, a^d) \tag{6.5}$$

is an isomorphism of $L^0(\mathbb{R}^d)$-modules which preserves the pointwise norm. Moreover, it holds that, given a Lusin-Lipschitz function f,

$$\Phi(\mathrm{d}f) = \nabla_a f. \tag{6.6}$$

Proof. The claim that Φ is an isomorphism of $L^0(\mathbb{R}^d)$-normed modules which preserves the pointwise norm is proved straightforwardly, noticing the fact that $\langle \mathrm{d}\pi_i, \mathrm{d}\pi_j \rangle = \delta_{i,j}$ a.e. on $\mathbb{R}^d$.
It remains to prove (6.6). We claim that it holds for f Lipschitz. By definition of Φ, it follows that $\Phi(\mathrm{d}\pi_i) = e_i = \nabla \pi_i$ for every i. By linearity of both members, given $g(x) = c \cdot x$, we have that $\Phi(\mathrm{d}g) = c = \nabla g$. Consider an enumeration $\{c^i\}$ of $\mathbb{Q}^d$. We fix $\epsilon > 0$ and we define $E_i\colon = \{x\colon f$ is differentiable at x and $|\nabla f(x) - c^i| < \epsilon\}$ for every $i \in \mathbb{N}$. By Rademacher theorem, we have that $(E_i)_i$ is a partition of a.a. $\mathbb{R}^d$. We fix one such index i. We define $h(x) = c^i \cdot x$, we have that $|\nabla(f-h)(x)| < \epsilon$ for every $x \in E_i$. Moreover, in every point of differentiability of f (hence also of $f-h$), it holds that $\mathrm{lip}(f-h)(x) = |\nabla(f-h)(x)|$, which together with the fact that $|\mathrm{d}(f-h)| \le \mathrm{lip}(f-h)$ holds a.e. on E_i yields that $|\mathrm{d}(f-h)| < r$ a.e. on E_i. By the triangular inequality, we have

$$|\Phi(\mathrm{d}f) - \nabla f| \le |\Phi(\mathrm{d}(f-h))| + |\Phi(\mathrm{d}h) - \nabla h| + |\nabla(f-h)| \text{ holds } \mathfrak{m}\text{-a.e. on } E_i,$$

hence we notice that the second term is zero and that Φ preserves the pointwise norm. Therefore, this gives that $|\Phi(\mathrm{d}f) - \nabla f| \leq 2\epsilon$ on E_i. By repeating the argument for every i, we get that $|\Phi(\mathrm{d}f) - \nabla f| \leq 2\epsilon$ a.e. on $\mathbb{R}^d$. By the arbitrariness of ϵ, we prove the claim.
Given a Lusin-Lipschitz f, consider an a.a. partition of X on which $f_{|A_i}$ is Lispchitz and, for every i, $f^i \in \mathrm{Lip}(\mathbb{R}^d)$ such that $f = f^i$ on A_i. By the definition of $\mathrm{d}f$, the $L^0(\mathrm{X})$-linearity of Φ and the fact it preserves the pointwise norm, we get that $\Phi(\mathrm{d}f) = \Phi(\sum_{i=1}^{\infty} \chi_{A_i} \mathrm{d}f^i) = \sum_{i=1}^{\infty} \chi_{A_i} \Phi(\mathrm{d}f^i) = \sum_{i=1}^{\infty} \chi_{A_i} \nabla f^i$ holds a.e.. To conclude, we notice that for every $x \in A_i$ which is a differentiability point of f^i and a density point for A_i it holds that $\nabla f^i(x) = \nabla_a f(x)$. □

Due to consistency of notation, we denote $\nabla_a = \nabla$.

Proposition 6.1.5. *Let $(\boldsymbol{b}_t) \in L^1([0,T], W^{1,2}(\mathbb{R}^d))$ be such that $\operatorname{div} \boldsymbol{b} \in L^1([0,T], L^\infty(\mathbb{R}^d))$, $|\boldsymbol{b}_t| \in L^\infty([0,T] \times \mathbb{R}^d)$ and (F_t^s) the associated regular Lagrangian flow. Then for every $t \in [0,T]$, for $\mathfrak{m}$-a.e. x the map $(s \mapsto \nabla F_t^s(x)) \in W^{1,1}([0,T], \mathbb{R}^{d \times d})$ and*

$$\partial_s \nabla F_t^s(x) = D\boldsymbol{b}_s(F_t^s(x))\, \nabla F_t^s(x).$$

Proof. We call $(F_t^s(x))_i$ the i-th component of $F_t^s(x)$. We fix $t \in [0,T]$. We apply Proposition 6.1.1 in $(\mathbb{R}^d, \mathsf{d}_e, \mathscr{L}^d)$ as follows. We fix i and $R > 0$ and we call $M := \|\boldsymbol{b}\|_{L^\infty_{t,x}}$. We consider $f \in C_c^\infty(\mathbb{R}^d)$ such that $f(x) = x_i$ on $B_{R+TM}(0)$. We recall that, by definition of d, for Lusin Lipschitz g and Borel E it holds that $\chi_E \mathrm{d}g = \mathrm{d}(\chi_E g)$. Applying Proposition 6.1.1 and the last consideration, we have that for a.e. $s \in [0,T]$

$$\begin{aligned}
\chi_{B_R(0)} \lim_{h \to 0} \frac{\mathrm{d}((F_t^{s+h})_i) - \mathrm{d}(F_t^s)_i)}{h} &= \lim_{h \to 0} \frac{\mathrm{d}(\chi_{B_R(0)}\,(F_t^{s+h})_i) - \mathrm{d}(\chi_{B_R(0)}\,(F_t^s)_i)}{h} \\
&= \lim_{h \to 0} \frac{\mathrm{d}(\chi_{B_R(0)}\, f \circ F_t^{s+h}) - \mathrm{d}(\chi_{B_R(0)}\, f \circ F_t^s)}{h} \\
&= \chi_{B_R(0)} \lim_{h \to 0} \frac{\mathrm{d}(f \circ F_t^{s+h}) - \mathrm{d}(f \circ F_t^s)}{h} = \chi_{B_R(0)} \mathrm{d}(\mathrm{d}f(\boldsymbol{b}_s) \circ F_t^s) \\
&= \mathrm{d}(\chi_{B_R(0)}\, \mathrm{d}f(\boldsymbol{b}_s) \circ F_t^s) = \mathrm{d}(\chi_{B_R(0)}\, \boldsymbol{b}_s^i \circ F_t^s) = \chi_{B_R(0)}\, \mathrm{d}(\boldsymbol{b}_s^i \circ F_t^s).
\end{aligned}$$

the limit intended in $L^0(T^*\mathbb{R}^d)$. By arbitrariness of R and repeating the argument for every i, we have that there exists a negligible set $N \subseteq [0,T]$ such that, for every $i = 1, \ldots, d$ and for every $s \in [0,T] \setminus N$

$$\lim_{h \to 0} \frac{\mathrm{d}((F_t^{s+h})_i) - \mathrm{d}(F_t^s)_i)}{h} = \mathrm{d}(\boldsymbol{b}_s^i \circ F_t^s). \tag{6.7}$$

We apply Lemma 6.1.4 to (6.7), thus having that $[0,T] \ni s \mapsto \nabla(F_t^s)_i \in L^0(\mathbb{R}^d, \mathbb{R}^d)$ belongs to $AC^2([0,T], L^0(\mathbb{R}^d, \mathbb{R}^d))$ and for every $s \in [0,T] \setminus N$

$$\lim_{h \to 0} \frac{\nabla((F_t^{s+h})_i) - \nabla((F_t^s)_i)}{h} = \nabla(\boldsymbol{b}_s^i \circ F_t^s)$$

the limit intended in $L^0(\mathbb{R}^d, \mathbb{R}^d)$. Collecting the results for every i we get

$$\partial_s \nabla F_t^s = \lim_{h \to 0} \frac{\nabla F_t^{s+h} - \nabla F_t^s}{h} = \nabla(\boldsymbol{b}_s \circ F_t^s) = D\boldsymbol{b}_s \circ F_t^s\, \nabla F_t^s,$$

the limit intended in $L^0(\mathbb{R}^d, \mathbb{R}^{d\times d})$. Since $(s \mapsto \nabla(F_t^s)_i) \in AC^2([0,T], L^0(\mathbb{R}^d, \mathbb{R}^d))$, there exists $(s \mapsto \dot v_s) \in L^2([0,T], L^0(\mathbb{R}^d, \mathbb{R}^d))$ such that for every $z \in L^0(\mathbb{R}^d, \mathbb{R}^d)$ we have that for $\mathfrak{m}$-a.e. x the map $(s \mapsto \langle \nabla(F_t^s)_i, z\rangle(x)) \in W^{1,2}([0,T])$ and

$$\partial_s \langle \nabla(F_t^s)_i, z\rangle(x) = \langle \dot v_s, z\rangle(x).$$

It can be readily checked that $\dot v_s = D\boldsymbol{b}_s^i(F_t^s(\cdot))\nabla F_t^s$. Considering $z = e_k$ for every k in the last equality we obtain that for $\mathfrak{m}$-a.e. x the map $(s \mapsto \nabla(F_t^s)_i(x)) \in W^{1,2}([0,T], \mathbb{R}^d)$ and

$$\partial_s \nabla(F_t^s)_i(x) = D\boldsymbol{b}_s^i(F_t^s(x))\nabla F_t^s.$$

Putting all together, we get that, for fixed $t \in [0,T]$, for $\mathfrak{m}$-a.e. x the map $(s \mapsto \nabla F_t^s(x))$ is in $W^{1,2}([0,T], \mathbb{R}^{d\times d})$ and

$$\partial_s \nabla F_t^s(x) = D\boldsymbol{b}_s(F_t^s(x))\nabla F_t^s(x).$$

□

Remark 6.1.6. We point out that the conclusion of Proposition 6.1.5 is already known in literature and can be derived at least in the following two ways under the same assumptions on $\boldsymbol{b}$:

i) the first proof can be derived combining the following results. Le Bris and Lions treated in [81] the vector field in $\mathbb{R}^{2d}$ $B_t(x,h) := (\boldsymbol{b}_t(x), D\boldsymbol{b}_t(x)h)$, proving existence and uniqueness of regular Lagrangian flows (notice that B does not satisfy the hypthesis of DiPerna-Lions theory). By looking at the last d components of the regular Lagrangian flows $W(t,x,h)$, we have that for $\mathscr{L}^{2d}$-a.e. (x,h)

$$\partial_t W(t,x,h) = D\boldsymbol{b}_t(F_0^t(x))W(t,x,h) \text{ for a.e. } t, \quad W(0,x,h) = h.$$

They also proved that $W(t,x,h)$ is the differential in measure of the flow $F_0^t(x)$ (in the sense of [13]); moreover, in [13] it is proved that $W(t,x,h) = \bar W(t,x)h$, where $\bar W\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times d}$. By linearity in h it is easy to obtain the last equation for $\mathscr{L}^d$-a.e. x, for every h. To conclude, the Lusin–Lipschitz property for $p > 1$ grants that for every $t, s \in [0,T]$ F_s^t is $\mathscr{L}^d$ a.e. approximate differentiable (see [43]), with approximate gradient $\nabla F_0^t(\cdot)$ equal $\mathscr{L}^d$-a.e. to $\bar W(t,\cdot)$ (see [13, Theorem 4.2]);

ii) a direct proof in [20, Section 2.1], later extended in the same work to the more general class of nearly incompressible BV vector fields.

6.2 Local convergence in measure of approximate gradients

We start providing an example in which we don't have the desired convergence of gradients of the flow maps. As mentioned before, we build in $\mathbb{R}$ a sequence of vector fields (v^n) for which $\sup_n \left(\|v^n\|_{L^\infty([0,T]\times\mathbb{R})} + \|\partial_x v_t^n\|_{L^\infty([0,T]\times\mathbb{R})}\right) < \infty$, $v^n \to v \in L^1([0,T], L^p_{\mathrm{loc}}(\mathbb{R}))$ for every $p \in [1,+\infty]$, but it is not true that $\partial_x v^n \to \partial_x v \in L^1([0,T], L^1_{\mathrm{loc}}(\mathbb{R}))$. Before introducing it, we need to recall a weak notion of convergence of vector fields we will use.

Definition 6.2.1. *Let* $(\boldsymbol{b}_n) \subseteq L^1([0,T], L^1_{\rm loc}(\mathbb{R}^d))$, $n \in \mathbb{N} \cup \{\infty\}$. *We say that* $\boldsymbol{b}_n \to \boldsymbol{b}_\infty$ *weakly in time and (locally) strongly in space provided for any* $\varphi \in C_c(\mathbb{R})$ *we have* $\boldsymbol{b}_n^\varphi \to \boldsymbol{b}_\infty^\varphi$ *strongly in* $L^1([0,T], L^1_{\rm loc}(\mathbb{R}^d))$, *where we put*

$$\boldsymbol{b}_{n,t}^\varphi := \int_{\mathbb{R}} \varphi(t-s)\boldsymbol{b}_{n,s}\,\mathrm{d}s \qquad \forall t \in [0,T],\ n \in \mathbb{N} \cup \{\infty\},$$

and it is intended that $\boldsymbol{b}_{n,s} = 0$ *for* $s \notin [0,T]$.

Example 6.2.2. *We define for every* n *the following flow maps, given* $T \le \frac{1}{2}$, *for* $t < T$:

$$X^n(t,x) = x + th_n(x) \quad \text{for } x \in \mathbb{R}$$

where $h_n(x) := \frac{h(nx)}{n}$ *and*

$$h(x) := \begin{cases} x - 2j & x \in [2j, 2j+1], \\ 2j - x & x \in [2j-1, 2j]. \end{cases} \tag{6.8}$$

By a straightforward computation we can rewrite the flow maps as

$$X^n(t,x) := \begin{cases} x + t(\frac{2j}{n} - x) & x \in [\frac{2j}{n} - \frac{1}{n}, \frac{2j}{n}], \\ x + t(x - \frac{2j}{n}) & x \in [\frac{2j}{n}, \frac{2j}{n} + \frac{1}{n}]. \end{cases}$$

and their inverse maps $Y^n(t,x)$, *i.e. such that* $Y^n(t, X^n(t,x)) = x$

$$Y^n(t,x) := \begin{cases} \frac{x}{1+t} + \frac{t}{1+t}\frac{2j}{n} & x \in [\frac{2j}{n}, \frac{2j}{n} + (1+t)\frac{1}{n}], \\ \frac{x}{1-t} - \frac{t}{1-t}\frac{2j}{n} & x \in [\frac{2(j-1)}{n} + (1+t)\frac{1}{n}, \frac{2j}{n}]. \end{cases}$$

We define $v_t^n(x) := \partial_t X^n(t, Y^n(t,x)) = h_n(Y^n(t,x))$. *We have that* $|v_t^n|(x) = \frac{1}{n}|h|(nY^n(t,x))$, *hence:*

$$\|v_t^n(x)\|_{C^0([0,T]\times\mathbb{R})} \le \frac{1}{n}.$$

Therefore, $v_t^n(x)$ *converges to zero uniformly in* $[0,T]\times\mathbb{R}$ *and strongly in* $L^1([0,T]\times K)$ *for every compact set* $K \subset \mathbb{R}$. *Moreover, we compute for fixed* $t \in [0,T]$:

$$|\partial_x v_t^n|(x) = |\partial_x h_n|(Y^n(t,x))|\partial_x Y^n(t,x)|.$$

Therefore, since $|\partial_x Y^n(t,x)| \le \frac{1}{1-T}$, *we have that for* $\mathscr{L}^1$*-a.e.* x, $|\partial_x v_t^n|(x) \le \frac{1}{1-T}$; *therefore* v_t^n *is Lipschitz with* $\mathrm{Lip}(v_t^n) \le \frac{1}{1-T}$, *which yields*

$$\sup_n \sup_{t\in[0,T]} \mathrm{Lip}(v_t^n) \le \frac{1}{1-T}.$$

This grants that $X^n(t,x)$ *is the classical flow map given by bundling the trajectories (unique by the Cauchy Lipschitz theory) associated to the vector field* v_t^n; *in particular, it is also the regular Lagrangian flow associated to* v_t^n. *We claim that* $\partial_x v_t^n$ *does not converge weakly in time and strongly in space to* 0. *To do so, it is enough to find* $\varphi \in C_c(\mathbb{R})$ *such that:*

$$\int_0^T \int_{[0,1]} \left| \int_{\mathbb{R}} \varphi(t-s)\partial_x v_s^n(x)\,\mathrm{d}s \right| \mathrm{d}x\,\mathrm{d}t \tag{6.9}$$

doesn't converge to zero. Indeed, consider a positive function $\varphi \in C_c^\infty(\mathbb{R})$ with $\int \varphi = 1$ and $\operatorname{supp}\varphi \subset B_\epsilon(0)$*. We recall that $\partial_x v_s^n(x) = h'(nY^n(s,x))\partial_x Y^n(s,x)$. We can explicitily compute*

$$\partial_x v_t^n(x) := \begin{cases} \frac{1}{1+t} & x \in [\frac{2j}{n}, \frac{2j}{n} + (1+t)\frac{1}{n}], \\ \frac{1}{t-1} & x \in [\frac{2(j-1)}{n} + (1+t)\frac{1}{n}, \frac{2j}{n}]. \end{cases}$$

We define $K^n := \cup_m [\frac{2m}{n}, \frac{2m}{n} + \frac{1}{n}]$. We have:

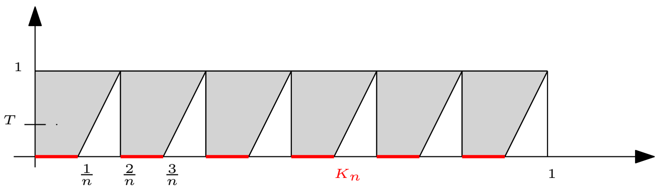

Figure 6.1: The function $\partial_x v_t^n(x)$ is constant in x for fixed t on the grey and white zones.

$$\int_\epsilon^{T-\epsilon} \int_{K^n\cap[0,1]} \left| \int_{\mathbb{R}} \varphi(t-s)\partial_x v_s^n(x)\,\mathrm{d}s \right| \mathrm{d}x\,\mathrm{d}t = \int_\epsilon^{T-\epsilon} \int_{K^n\cap[0,1]} \left| \int_{\mathbb{R}} \varphi(t-s)\frac{1}{1+s}\,\mathrm{d}s \right| \mathrm{d}x\,\mathrm{d}t$$
$$\geq \frac{1}{1+T}\,|K^n \cap [0,1]|(T-2\epsilon).$$

yielding the conclusion, since $|K^n \cap [0,1]| = C > 0$, with C independent of n. Moreover, we define $X(t,x) = x$ which is the classical flow associated to $v_t(x) := 0$. It is straightforward to check that for every t $X^n(t,\cdot) \to X(t,\cdot)$ in $L^0(\mathscr{L}^1)$, while it is not true that $\partial_x X^n(t,\cdot) \to \partial_x X(t,\cdot)$ in $L^0(\mathscr{L}^1)$. Indeed, take $\mu \in \mathscr{P}(\mathbb{R})$ such that $\mu \leq \mathscr{L}^1 \ll \mu$; we have that:

$$\int |\partial_x X^n(t,\cdot) - \partial_x X(t,\cdot)| \wedge 1\,\mathrm{d}\mu = \int |th'(nx)| \wedge 1\,\mathrm{d}\mu(x) = t$$

which doesn't converge to zero as $n \to 0$.

6.2.1 Proof of the main theorem

This section is devoted to the proof of Theorem 6.0.1. Before doing that, we introduce some notations and preliminary lemmas. We define $B(t,s,x) := D\boldsymbol{b}_t(F_s^t(x))$. Given x, we consider an absolutely continuous matrix-valued curve $[0,T] \ni t \mapsto A(t,s,x) \in \mathbb{R}^{d\times d}$ such that

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} A(t,s,x) = B(t,s,x)A(t,s,x) \quad \text{for } \mathscr{L}^1\text{-a.e. } t, \\ A(s,s,x) = x \end{cases} \tag{6.10}$$

and we consider accordingly, for every n, given x, an absolutely continuous matrix-valued curve $[0,T] \ni t \mapsto A^n(t,s,x) \in \mathbb{R}^{d\times d}$ such that (6.10) is solved with $B^n(t,s,x) := D\boldsymbol{b}_t^n({F^n}_s^t(x))$. We have, that, for fixed n, one solution of the problem (6.10) is given by $t \mapsto \nabla {F^n}_s^t(x)$, and we prove that for a.e x this is the only one. We need the following lemma.

Lemma 6.2.3. *Let $\boldsymbol{b}$ a bounded vector field such that $D_x\boldsymbol{b} \in L^1_{loc}([0,T]\times\mathbb{R}^d)$ for which there exists a regular Lagrangian flow $F^t_s(x)$; in particular, we denote by $L>0$ the constant such that $F^t_{s*}\mathscr{L}^d \le L\mathscr{L}^d$. Then there exists a measurable set $\bar{N}\subseteq\mathbb{R}^d$ such that $\mathscr{L}^d(\bar{N})=0$ and for every $x\in\mathbb{R}^d\setminus\bar{N}$*

$$\int_0^T |D_x\boldsymbol{b}(t,F^t_s(x))|\,\mathrm{d}t < \infty. \tag{6.11}$$

Proof. Fix $M\in\mathbb{N}$. We compute

$$\begin{aligned}
&\int_{B_M(0)}\int_0^T |D_x\boldsymbol{b}(t,F^t_s(x))|\,\mathrm{d}t\,\mathrm{d}x = \int_0^T\int_{B_M(0)} |D_x\boldsymbol{b}(t,F^t_s(x))|\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_0^T\int |D_x\boldsymbol{b}(t,x)|\,\mathrm{d}F^t_{s*}\left(\chi_{B_M(0)}\mathscr{L}^d\right) \le L\int_0^T\int_{F^t_s(B_M(0))} |D_x\boldsymbol{b}(t,x)|\,\mathrm{d}x\\
&\le L\int_0^T\int_{B_{M+T\|\boldsymbol{b}\|_{L^\infty_{t,x}}}(0)} |D_x\boldsymbol{b}(t,x)|\,\mathrm{d}x.
\end{aligned}$$

This gives in particular that for $\mathscr{L}^d$-a.e. $x\in B_M(0)$ (6.16) holds. By doing it for every $M\in\mathbb{N}$ we conclude. □

We will use from now on the notation $\bar{N}$ for the set given by Lemma 6.2.3. We now prove that for a.e. x the solution to 6.10 is unique. We consider the following problem for fixed $x\in\mathbb{R}^d$: given $y\in\mathbb{R}^d$, find an absolutely continuous curve $y_x\colon[0,T]\to\mathbb{R}^d$ such that

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}y_x(t) = B(t,s,x)y_x(t) & \text{for } \mathscr{L}^1\text{-a.e. } t,\\ y_x(0)=y \end{cases} \tag{6.12}$$

Proposition 6.2.4. *Let $\boldsymbol{b}$ a bounded vector field such that $D_x\boldsymbol{b}\in L^1_{loc}([0,T]\times\mathbb{R}^d)$ and for which there exists a regular Lagrangian flow $F^t_s(x)$; in particular, there exists $L>0$ such that $F^t_{s*}\mathscr{L}^d \le L\mathscr{L}^d$. Then for every $x\in\mathbb{R}^d\setminus\bar{N}$ there exists at most one solution to 6.10, with $B(t,s,x):=D_x\boldsymbol{b}_t(F^t_s(x))$*

Proof. Step 1. We show that, for every $x\in\mathbb{R}^d\setminus\bar{N}$ and $s\in[0,T]$, for every $y\in\mathbb{R}^d$ there exists at most one solution to (6.12). We call $B(t):=B(t,s,x)$. Let $y(t)$ be a solution starting from y. Uniqueness is a consequence of linearity of the problem and Gronwall inequality. Indeed, $[0,T]\ni t\mapsto|y(t)|\in\mathbb{R}$ is an absolutely continuous curve, so it is differentiable for a.e. t and $\frac{\mathrm{d}}{\mathrm{d}t}|y(t)|\le |B(t)|\,|y(t)|$ for a.e. $t\in[0,T]$. Notice that for every $x\in\mathbb{R}^n\setminus\bar{N}$, the curve $[0,T]\ni t\mapsto e^{-\int_0^t|B(r)|\,\mathrm{d}r}$ is absolutely continuous and the distributional derivative $\frac{\mathrm{d}}{\mathrm{d}t}\left(|y(t)|e^{-\int_0^t|B(r)|\,\mathrm{d}r}\right)\le 0$. Therefore $[0,T]\ni t\mapsto |y(t)|e^{-\int_0^t|B(r)|\,\mathrm{d}r}\in\mathbb{R}$ is non increasing, which gives that $|y(t)|\le|y|e^{\int_0^t|B(r)|\,\mathrm{d}r}$ yielding uniqueness.
Step 2. To conclude, consider two AC solutions $A_1(t,x),A_2(t,x)$ of 6.10 such that $A_1(0,x)=A_2(0,x)=\mathrm{Id}$, define $C(t,x):=A_1(t,x)-A_2(t,x)$ and use Step 1 by arguing componentwise. □

Before stating the main result, we need some preliminary lemmas.

Lemma 6.2.5. *Let $f^n, f \in L^0(\mathbb{R}^d)$ be such that $f^n \to f$ locally in measure. Let $\boldsymbol{b}^n : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ a sequence of vector fields such that there exists $C_0 > 0$ such that $\|\boldsymbol{b}^n(t,x)\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq C_0$. Assume that there exists for every $n \in \mathbb{N}$ a regular Lagrangian flow ${F^n}^t_s(x)$ associated to $\boldsymbol{b}^n$ and a regular Lagrangian flow $F^t_s(x)$ associated to $\boldsymbol{b}$. We further assume that for every $t, s \in [0,T]$ ${F^n}^t_s \to F^t_s$ locally in measure and that there exists $L > 0$ such that for every n, t, s*

$$ {F^n}^t_{s*}\mathscr{L}^d \leq L\,\mathscr{L}^d. $$

Then for every $t \in [0,T]$

$$ f^n({F^n}^t_s(\cdot))) \to f(F^t_s(\cdot)) \text{ locally in measure.} $$

Proof. Fix $r > 0$ and set $\lambda := r + TC_0$. We fix $\gamma > 0$. We fix $s \in [0,T]$ and we estimate

$$ \begin{aligned} &\mathscr{L}^d(B_r \cap \{|f^n \circ {F^n}^t_s - f \circ F^t_s| > \gamma\}) \\ &\quad \leq \mathscr{L}^d(B_r \cap \{|f^n \circ {F^n}^t_s - f \circ {F^n}^t_s| > \frac{\gamma}{2}\}) + \mathscr{L}^d(B_r \cap \{|f \circ {F^n}^t_s - f \circ F^t_s| > \frac{\gamma}{2}\}) \\ &\quad \leq \mathscr{L}^d(B_\lambda \cap \{|f^n - f| > \frac{\gamma}{2}\}) + \mathscr{L}^d(B_r \cap \{|f \circ {F^n}^t_s - f \circ F^t_s| > \frac{\gamma}{2}\}) = I + II. \end{aligned} $$

Since f_n converges to f locally in measure, we get that for n large $\mathscr{L}^d(B_\lambda \cap \{|f^n - f| > \frac{\gamma}{2}\}) \leq \epsilon$, so $I \leq \epsilon$ for n large. We need to show that $II \leq \epsilon$ for n large. We know by Lusin's theorem that there exists $G(\cdot) \in C(\overline{B_\lambda})$ such that $\mathscr{L}^d(\overline{B_\lambda} \cap \{f \neq G\}) \leq \epsilon$. Since $G(\cdot)$ is uniformly continuous, we know that there exists $\delta(\gamma)$ such that, if $|x-y| \leq \delta$, $|G(x) - G(y)| \leq \frac{\gamma}{2}$. We define $S^n := \{x \in B_r : |{F^n}^t_s(x) - F^t_s(x)| \leq \delta\}$. By the local convergence in measure of the flow maps, we get $\mathscr{L}^d(B_r \setminus S^n) \leq \epsilon$ for n large.
We will use the following fact: given a measurable function $u : \mathbb{R}^d \to \mathbb{R}$ and an invertible measurable function $h : \mathbb{R}^d \to \mathbb{R}^d$ such that there exists $c, L > 0$ such that $h_*\mathscr{L}^d \leq L\mathscr{L}^d$ and, for some $r > 0$, $h(B_r(0)) \subset B_{r+c}(0)$

$$ \mathscr{L}^d(B_r(0) \cap \{u \circ h \neq 0\}) \leq L\mathscr{L}^d(B_{r+c}(0) \cap \{u \neq 0\}). $$

Since h is invertible $\chi_{h(A)} \circ h = \chi_A$. The proof follows by using change of variable formula in the following computation

$$ \begin{aligned} \mathscr{L}^d(B_r(0) \cap \{u \circ h \neq 0\}) &= \int \chi_{B_r(0)} \left(\chi_{\{u\circ h>0\}} + \chi_{\{u \circ h<0\}}\right) \mathrm{d}\mathscr{L}^d \\ &= \int \chi_{h(B_r(0))} \left(\chi_{\{u>0\}} + \chi_{\{u<0\}}\right) \mathrm{d}h_*\mathscr{L}^d \\ &\leq L \int \chi_{B_{r+c}(0)} \left(\chi_{\{u>0\}} + \chi_{\{u<0\}}\right) \mathrm{d}\mathscr{L}^d = L\mathscr{L}^d(B_{r+c}(0) \cap \{u \neq 0\}). \end{aligned} $$

We apply this result twice considering $u = f - G$ and $h = {F^n}^t_s$ and $h = F^t_s$ the second time. We estimate II as follows, thus concluding the proof

$$ \begin{aligned} II &= \mathscr{L}^d(B_r \cap \{|f({F^n}^t_s) - f(F^t_s)| > \frac{\gamma}{2}\}) \\ &\leq \mathscr{L}^d(B_r \cap S^n \cap \{|G \circ {F^n}^t_s - G \circ F^t_s| > \frac{\gamma}{2}\}) + \mathscr{L}^d(B_r \setminus S^n) \\ &+ \mathscr{L}^d(B_r \cap \{G \circ {F^n}^t_s \neq f \circ {F^n}^t_s\}) + \mathscr{L}^d(B_r \cap \{G \circ F^t_s \neq f \circ F^t_s\}) \\ &\leq \epsilon + 2L\mathscr{L}^d(B_\lambda \cap \{G \neq f\}) \leq \epsilon + 2L\epsilon. \end{aligned} $$

□

Remark 6.2.6. Local convergence in measure of the flow maps follows from equibounds on the $L^\infty([0,T]\times\mathbb{R}^d)$ norm in time and space of the sequence of vector fields, equibounds on the compressibility constants, the fact that the limit vector fields belongs to $L^1([0,T],W^{1,p}_{loc}(\mathbb{R}^d))$ for $p>1$ and local convergence of the vector fields to the limit one in $L^1([0,T]\times\mathbb{R}^d)$ (see [43]).

Remark 6.2.7. We recall that, given a measure space (X,μ) with $\mu(\mathrm{X})<\infty$, $1\le p<\infty$ and $f_n, f\in L^p(\mu)$, $f_n\to f$ in $L^p(\mu)$ if and only if

1) for every γ, $\lim_{n\to+\infty}\mu(\{|f_n-f|>\gamma\})=0$;

2) given $A_n\in\mathscr{B}(\mathrm{X})$ such that $\mu(A_n)\to 0$, $\int_{A_n}|f_n|^p\,\mathrm{d}\mu\to 0$.

We recall the proof of the if part. We estimate:

$$\begin{aligned}\int|f_n-f|^p\,\mathrm{d}\mu &= \int_{\{|f_n-f|\le 1\}}|f_n-f|^p\,\mathrm{d}\mu+\int_{\{|f_n-f|>1\}}|f_n-f|^p\,\mathrm{d}\mu\\ &\le\int 1\wedge|f_n-f|\,\mathrm{d}\mu+C_p\int_{\{|f_n-f|>1\}}|f_n|^p\,\mathrm{d}\mu+C_p\int_{\{|f_n-f|>1\}}|f|^p\,\mathrm{d}\mu.\end{aligned}$$

By taking the limit as $n\to+\infty$ and recalling that $\lim_{n\to+\infty}\mu(\{|f_n-f|>1\})=0$ we conclude.

Lemma 6.2.8. *Let $p\ge 1$. Let $\boldsymbol{b}^n\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ a sequence of vector fields such that there exists $C_0>0$ such that $\|\boldsymbol{b}^n\|_{L^\infty([0,T]\times\mathbb{R}^n)}\le C_0$ and for every M there exists $C_M>0$ such that*

$$\sup_n\Big(\mathrm{esssup}_{t\in[0,T]}\|D\boldsymbol{b}^n_t\|_{L^p(B_M(0))}\Big)\le C_M. \tag{6.13}$$

Assume that there exists for every $n\in\mathbb{N}$ a regular Lagrangian flow ${F^n}^t_s(x)$ associated to $\boldsymbol{b}^n$ and a regular Lagrangian flow $F^t_s(x)$ associated to $\boldsymbol{b}$. We assume that, for every $t\in[0,T]$, ${F^n}^t_s\to F^t_s$ locally in measure and that for every n,t,s ${F^n}^t_s(\cdot)$ is invertible. We assume $D\boldsymbol{b}^n\to D\boldsymbol{b}$ in $L^1([0,T],L^p_{loc}(\mathbb{R}^d))$ and that there exists $L>0$ independent of n,t,s such that

$$\frac{1}{L}\mathscr{L}^d\le {F^n}^t_{s*}\mathscr{L}^d\le L\,\mathscr{L}^d. \tag{6.14}$$

Then, we have that, for every s, there exists a subsequence $\{n_k\}_k$ and a measurable set N with $\mathscr{L}^d(N)=0$ such that for every $x\in\mathbb{R}^d\setminus N$

$$\lim_{k\to+\infty}\int_0^T|D\boldsymbol{b}^{n_k}(t,{F^{n_k}}^t_s(x))-D\boldsymbol{b}(t,F^t_s(x))|^p\,\mathrm{d}t=0. \tag{6.15}$$

As a consequence, we get that

$$x\mapsto\sup_k\int_0^T|D\boldsymbol{b}^{n_k}_t|^p({F^{n_k}}^t_s(x))\,\mathrm{d}t \tag{6.16}$$

is finite for every $x\in\mathbb{R}^d\setminus N$.

Proof. Step 1. Since, for every $M > 0$, $D\boldsymbol{b}^n \to D\boldsymbol{b}$ in $L^1([0,T], L^p(B_M(0)))$, we have that, by using a diagunal argument, there exists $N_0 \subset [0,T]$ with $\mathscr{L}^1(N_0) = 0$ and a subsequence $\{n_k\}_k$ such that, for every $t \in [0,T] \setminus N_0$

$$D\boldsymbol{b}^{n_k}(t,\cdot) \to D\boldsymbol{b}(t,\cdot) \quad \text{in } L^p(B_M(0))$$

for every $M > 0$. We don' t relabel this subsequence. As a consequence, we have that, for fixed $t \in [0,T] \setminus N_0$, $\{|D\boldsymbol{b}^n(t,\cdot)|^p\}_n$ are equiintegrable, i.e. given a sequence $A_n \in \mathscr{B}(\mathbb{R}^d)$, such that for some $M > 0$ $A_n \subset B_M(0)$ for every n and $\lim_n \mathscr{L}^d(A_n) = 0$

$$\lim_{n\to+\infty} \int_{A_n} |D\boldsymbol{b}^n(t,x)|^p \, \mathrm{d}x = 0.$$

Step 2. We fix $s > 0$. We claim that for every $M > 0$ and every $t \in [0,T] \setminus N_0$

$$\lim_{n\to+\infty} \int_{B_M(0)} |D\boldsymbol{b}^n(t, F^{n}{}^t_s(x)) - D\boldsymbol{b}(t, F^t_s(x))|^p \, \mathrm{d}x = 0. \tag{6.17}$$

We can apply Lemma 6.2.5, taking $f_n(\cdot) = \partial_i \boldsymbol{b}^n(t,\cdot)$ and $f(\cdot) = \partial_i \boldsymbol{b}(t,\cdot)$, thus having that

$$D\boldsymbol{b}^n(t, F^{n}{}^t_s(\cdot)) \to D\boldsymbol{b}(t, F(t,\cdot)) \quad \text{locally in measure.}$$

Therefore, it is enough to prove item 2) of Remark 6.2.7 with the sequence $\{|D\boldsymbol{b}^n(t, F^{n}{}^t_s(\cdot))|^p\}_n$. We consider a sequence $A_n \in \mathscr{B}(\mathbb{R}^d)$ such that $\mathscr{L}^d(A_n) \to 0$ and we compute

$$\begin{aligned}
\int_{A_n} |D\boldsymbol{b}^n(t, F^{n}{}^t_s(\cdot))|^p \, \mathrm{d}\mathscr{L}^d &= \int \chi_{F^{n}{}^t_s(A_n)} \, |D\boldsymbol{b}^n(t,x)|^p \, \mathrm{d}F^{n}{}^t_{s*}\mathscr{L}^d \\
&\leq L \int_{F^{n}{}^t_s(A_n)} |D\boldsymbol{b}^n(t,x)|^p \, \mathrm{d}\mathscr{L}^d(x).
\end{aligned}$$

Since from (6.14) $\mathscr{L}^d(F^{n}{}^t_s(A_n)) = \mathscr{L}^d((F^{n}{}^s_t)^{-1}(A_n)) \leq L\mathscr{L}^d(A_n)$ and $\{|D\boldsymbol{b}_n(t,\cdot)|^p\}_n$ are equiintegrable

$$\lim_{n\to+\infty} \int_{A_n} |D\boldsymbol{b}^n(t, F^{n}{}^t_s(\cdot))|^p \, \mathrm{d}\mathscr{L}^d = 0$$

yielding (6.17) in view of Remark 6.2.7.
Step 3. We claim that there exists a (not relabeled) subsequence of the original sequence for which

$$\lim_{n\to+\infty} \int_0^T |D\boldsymbol{b}^n(t, F^{n}{}^t_s(x)) - D\boldsymbol{b}(t, F^t_s(x))|^p \, \mathrm{d}t = 0 \tag{6.18}$$

for every $x \in \mathbb{R}^d \setminus N_1$ where $\mathscr{L}^d(N_1) = 0$. We start computing

$$\begin{aligned}
&\int_{B_M(0)} |D\boldsymbol{b}^n(t, F^{n}{}^t_s(x)) - D\boldsymbol{b}(t, F^t_s(x))|^p \, \mathrm{d}\mathscr{L}^d(x) \\
&\qquad \leq C_p \int_{B_M(0)} |D\boldsymbol{b}^n(t, F^{n}{}^t_s(x))|^p + |D\boldsymbol{b}(t, F^t_s(x))|^p \, \mathrm{d}\mathscr{L}^d(x) \\
&\qquad \leq C_p \, L \int_{B_{M+TC_0}(0)} (|D\boldsymbol{b}^n(t,x)|^p + |D\boldsymbol{b}(t,x)|^p) \, \mathrm{d}\mathscr{L}^d(x) \\
&\qquad \leq 2C_p L \, C^p_{M+TC_0}
\end{aligned}$$

where the last term is bounded independently of n, t because of (6.13). Thus, we have, by an application of dominated convergence theorem, that, for the subsequence given by Step 1

$$\lim_{n\to+\infty}\int_0^T\int_{B_M(0)}|D\boldsymbol{b}^n(t,{F^n}_s^t(x))-D\boldsymbol{b}(t,F_s^t(x))|^p\,\mathrm{d}\mathscr{L}^d(x)\,\mathrm{d}t=0. \tag{6.19}$$

If we start Step 1 from a subsequence, following verbatim the proof up to this point we get that up to an extraction of subsequence, we get that (6.19) holds for this subsubsequence. Therefore (6.19) holds for the original sequence in the statement. An application of Fubini theorem and the fact that L^1 convergence implies convergence a.e. up to a subsequence yields the claim.
Step 4. We found a subsequence for which (6.15) holds for every $x \in \mathbb{R}^d \setminus N$. (6.16) is a consequence of (6.15). □

We state now the main theorem of this section.

Theorem 6.2.9. *Let $p > 1$. Let $\boldsymbol{b}^n, \boldsymbol{b}\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be a sequence of vector fields for which there exist associated regular Lagrangian flows ${F^n}_s^t(x)$ and $F_s^t(x)$ (possibly not unique) that verify all the hyphothesis of Lemma 6.2.8. Fixed $s \in [0,T]$, consider $B_n(t,s,x) := D_x\boldsymbol{b}^n(t,{F^n}_s^t(x))$ and $B(t,s,x) := D_x\boldsymbol{b}(t,F_s^t(x))$. Assume that there exists N_0 with $\mathscr{L}^d(N_0)=0$ such that for every $x \in \mathbb{R}^d \setminus N_0$ there exists a solution $A_n(t,s,x)$ of (6.10) for every n and a solution $A(t,s,x)$ of (6.10). Then, there exists a subsequence such that for every $t,s\in[0,T]$*

$$A_n(t,s,\cdot)\to A(t,s,\cdot) \qquad \textit{locally in measure.}$$

Proof. We apply Lemma 6.2.8 and we consider the subsequence $\{n_k\}$ given by the lemma and we call the negligible set in the statement N_1. As a consequence of (6.15), we get that for every $x \in \mathbb{R}^d \setminus N_1$, up to a not relabeled subsequence

$$B_n(\cdot,s,x)\to B(\cdot,s,x) \qquad \text{in } L^1(0,1).$$

We fix $x \in \mathbb{R}^d \setminus N_0 \cup N_1$; for every $t \in [0,T]$ and n, applying Gronwall lemma, we have

$$|A_n(t,s,x)| \le e^{\sup_n \int_0^T |D_x\boldsymbol{b}^n(t,\cdot)|({F^n}_s^t(x))\,\mathrm{d}t} = c(x) < \infty.$$

This in particular gives that $\|A(\cdot,s,x)\|_{L^\infty(0,T)} \le c(x)$ and $\|A(\cdot,s,x)\|_{L^p(0,T)} \le T^{\frac{1}{p}}c(x)$.
We can compute

$$\begin{aligned}\int_0^T\left|\frac{\mathrm{d}}{\mathrm{d}t}A_n(t,s,x)\right|^p\,\mathrm{d}t &\le \int_0^T |B_n(t,s,x)|^p\,|A_n(t,s,x)|^p\,\mathrm{d}t\\ &\le \sup_n\int_0^T |D_x\boldsymbol{b}_t^n|^p({F^n}_s^t(x))\,\mathrm{d}s\,e^{p\sup_n\int_0^T|D_x\boldsymbol{b}_t^n|({F^n}_s^t(x))\,\mathrm{d}s} =: d(x) < \infty.\end{aligned}$$

Fix $x \in \mathbb{R}^d \setminus (N_0 \cup N_1 \cup \bar{N})$ and the subsequence $\{n_k\}$. We have that for every subsequence of it, there exists a further subsequence (that we do not relabel) and $\bar{A}(\cdot,s,x) \in C^0([0,T]))$ such that

$$A_n(\cdot,s,x)\to\bar{A}(\cdot,s,x) \qquad \text{in } C^0([0,T]).$$

We claim that $A(\cdot,s,x) = \bar{A}(\cdot,s,x)$. Consider a function $\varphi \in C_c^\infty([0,T))$; being, for fixed n, a solution of (6.10) means that

$$\int_0^T \left(B_{n_k}(t,s,x)A_{n_k}(t,s,x)\varphi(t) + A_{n_k}(t,s,x)\varphi'(t)\right)\,\mathrm{d}t + \mathrm{Id}\,\varphi(0) = 0. \tag{6.20}$$

Thanks to what we just proved we can pass to the limit in (6.20), thus having that

$$\int_0^T \left(B(t,s,x)\bar{A}(t,s,x)\varphi(t) + \bar{A}(t,s,x)\varphi'(t)\right)\,\mathrm{d}t + \mathrm{Id}\,\varphi(0) = 0.$$

We recall that, if $x \in \mathbb{R}^d \setminus (N_0 \cup N_1 \cup \bar{N})$, $B(\cdot,s,x)\bar{A}(\cdot,s,x) \in L^1(0,T)$ and $[0,T] \ni t \mapsto \bar{A}(t,s,x) \in W^{1,1}(0,T)$. Therefore, it admits an absolutely continuous representative that solves (6.10). Proposition 6.2.4 grants that for every $x \in \mathbb{R}^d \setminus \bar{N}$ this is the only absolutely continuous curve that solves (6.10). Therefore, the claim is proved. Moreover, this implies that the original sequence $A_{n_k}(\cdot,s,x)$ converges in $C^0([0,T])$ to $A(\cdot,s,x)$. The simple estimate $\sup_{t\in[0,T]}\int_{B_M(0)} 1 \wedge |A_n(t,s,x) - A(t,s,x)|\,\mathrm{d}x \le \int_{B_M(0)} 1 \wedge \|A_n(\cdot,s,x) - A(\cdot,s,x)\|_{C^0([0,T])}\,\mathrm{d}x$ yields the conclusion. □

Proof of Theorem 6.0.1. We apply Proposition 6.1.5 with $\boldsymbol{b}_n$ for every n and $\boldsymbol{b}$. For fixed $s \in [0,T]$, we apply Theorem 6.2.9 with $\nabla {F^n}_s^t(x) = A_n(t,s,x)$ for every n and $\nabla F_s^t(x) = A(t,s,x)$ (the hyphothesis of local convergence in measure of flows is satisfied as a consequence of Remark 6.2.6). □

Appendix A

Miscellanea on measure theory

A.1 Integration of measures

We consider two complete and separable metric spaces $(\mathrm{X}, \mathsf{d}_{\mathrm{X}})$ and $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}})$. Suppose to have a family of positive probability measures $\mu_y \in \mathscr{P}(\mathrm{X})$ depending on a parameter $y \in \mathrm{Y}$ and ν a nonnegative measure finite on bounded sets of Y. We want to define a new measure

$$\mu(B) := \int_{\mathrm{Y}} \mu_y(B)\, \mathrm{d}\nu(y) \quad \text{for every } B \in \mathcal{B}(\mathrm{X}),$$

that is Borel and satisfies a generalization of the Fubini theorem

$$\int_{\mathrm{X}} f\, \mathrm{d}\mu = \int_{\mathrm{Y}} \left(\int_{\mathrm{X}} f\, \mathrm{d}\mu_y \right) \mathrm{d}\nu(y).$$

Before going into details, the following theorem clarifies when the function that we integrate is Borel.

Theorem A.1.1. *Let $(\mu_y)_{y\in\mathrm{Y}} \in \mathscr{P}(\mathrm{X})$. The following statements are equivalent:*

1) *for every $\varphi \in C_b(\mathrm{X})$, the map $y \mapsto \int_{\mathrm{X}} \varphi\, \mathrm{d}\mu_y$ is Borel;*

2) *for every $B \in \mathscr{B}(\mathrm{X})$, the map $y \mapsto \mu_y(B)$ is Borel;*

3) *for every $f : \mathrm{X} \mapsto [0, \infty]$ bounded and Borel, the map $y \mapsto \int_{\mathrm{X}} f\, \mathrm{d}\mu_y$ is Borel.*

If one of the previous conditions holds we say that the family $\mu_y \in \mathscr{P}(\mathrm{Y})$, $y \in \mathrm{Y}$ is weakly Borel.

Proof. 1) $\Rightarrow$ 2). Given an open set $U \subseteq \mathrm{X}$, we can pointwisely approximate χ_U from below by the sequence $(\varphi_n)_n \subseteq C_b(\mathrm{X})$ defined as $\varphi_n(x) := 1 \wedge (n\mathsf{d}(x, U^c))$, being such that $\varphi_n \nearrow \chi_U$. Using item 1), we have that $y \mapsto \mu_y(U)$ is Borel. We define

$$\mathscr{F} := \{B \in \mathscr{B}(\mathrm{X}) : \, y \mapsto \mu_y(B) \quad \text{is Borel }\}.$$

In particular, we know that $\{\text{open sets }\} \subseteq \mathscr{F}$. Moreover, given $(B_n)_n$ a sequence such that $B_n \subseteq B_{n+1}$, $B := \cup_{n=1}^{\infty} B_n$; since for every $y \in \mathrm{Y}$ $\mu_y(B_n) \nearrow \mu_y(B)$, we get that $y \mapsto \mu_y(B)$ is Borel. A similar argument holds for a decreasing sequence of sets. This shows that $\mathscr{F}$ is a

monotone class containing open sets, therefore $\mathscr{F} = \mathscr{B}(\mathrm{X})$, proving 2).
2) $\Rightarrow$ 3). It is trivial to check that, as a consequence of 2), $y \mapsto \int_X f \, \mathrm{d}\mu_y$ is Borel for every simple function with finite range. Then 3) follows by monotone approximation.
3) $\Rightarrow$ 1). It is obvious. □

Theorem A.1.2. *Let $\mu_y \in \mathscr{P}(\mathrm{X})$, $y \in \mathrm{Y}$ be a weakly Borel family. Define*

$$\mu(B) := \int \mu_y(B) \, \mathrm{d}\nu(y) \in [0, +\infty] \quad \text{for every } B \in \mathcal{B}(\mathrm{X}).$$

Then μ is a positive Borel measure.

Proof. We start by proving that μ is additive. Consider $B_1, B_2 \in \mathscr{B}(\mathrm{X})$ such that $B_1 \cap B_2 = \emptyset$. We have

$$\begin{aligned}\mu(B_1 \cup B_2) &= \int_{\mathrm{Y}} \mu_y(B_1 \cup B_2) \, \mathrm{d}\nu(y) = \int_{\mathrm{Y}} \mu_y(B_1) \, \mathrm{d}\nu(y) + \int_{\mathrm{Y}} \mu_y(B_2) \, \mathrm{d}\nu(y) \\ &= \mu(B_1) + \mu(B_2).\end{aligned}$$

To pass from additivity to countable additivity, it is enough to prove the continuity of the measure under an increasing sequence of Borel sets. Let $A_n \nearrow A$, with $A_n, A \in \mathscr{B}(\mathrm{X})$ for every n, and we claim that $\mu(A) = \lim_{n \to +\infty} \mu(A_n)$. Hence by monotone convergence

$$\mu(A) = \int_{\mathrm{Y}} \mu_y(A) \, \mathrm{d}\nu(y) = \lim_{n \to +\infty} \int_{\mathrm{Y}} \mu_y(A_n) \, \mathrm{d}\nu(y) = \lim_{n \to +\infty} \mu(A_n).$$

Therefore μ is positive Borel measure. □

Theorem A.1.3. *Let $\mu_y \in \mathscr{P}(\mathrm{X})$, $y \in \mathrm{Y}$ be a weakly Borel family. Let $f \colon \mathrm{X} \mapsto [0, +\infty]$ be a Borel function and let μ be defined as in Theorem A.1.2. Then*

$$\int_\Omega f \, \mathrm{d}\mu = \int_{\mathrm{Y}} \left(\int_{\mathrm{X}} f \, \mathrm{d}\mu_y \right) \mathrm{d}\nu(y).$$

Proof. If we take $f = \chi_B$, where $B \in \mathscr{B}(\Omega)$, then the statement is true by definition. Therefore by linearity the statement is true for every positive simple function. Finally notice that any positive Borel function f is the pointwise limit of a sequence of increasing positive simple functions f_n. Then

$$\begin{aligned}\int_{\mathrm{X}} f \, \mathrm{d}\mu &= \lim_{n \to +\infty} \int_{\mathrm{X}} f_n \, \mathrm{d}\mu = \lim_{n \to +\infty} \int_{\mathrm{Y}} \left(\int_{\mathrm{X}} f_n \, \mathrm{d}\mu_y \right) \mathrm{d}\nu(y) \\ &= \int_{\mathrm{Y}} \lim_{n \to +\infty} \left(\int_{\mathrm{X}} f_n \, \mathrm{d}\mu_y \right) \mathrm{d}\nu(y) = \int_{\mathrm{Y}} \left(\int_{\mathrm{X}} f \, \mathrm{d}\mu_y \right) \mathrm{d}\nu(y),\end{aligned}$$

since also the sequence of functions $y \mapsto \int_{\mathrm{X}} f_n \, \mathrm{d}\mu_y$ is monotone increasing. □

A.2 Bochner integral

We introduce the concept of Bochner integral in this manuscript for the following reason: we need to justify the notation used throughout the paper $L^1([0,1], L^2(\mathfrak{m}))$, $L^1([0,1], W^{1,2}_{C,s}(T\mathrm{X}))$ and $L^1([0,1], L^\infty(\mathfrak{m}))$. We suggest as an introduction to the topic [46] or [60, Section 1.3].
We fix a banach space $\mathbb{B}$ and a metric measure space $(\mathrm{X}, \mathsf{d}, \mu)$ with $\mu \in \mathscr{P}(\mathrm{X})$. Notice that, given a Borel function $f\colon \mathrm{X} \to \mathbb{B}$, since the map $\mathbb{B} \ni v \mapsto \|v\|_{\mathbb{B}} \in \mathbb{B}$ is continuous, we have that $\mathrm{X} \ni x \mapsto \|f(x)\|_{\mathbb{B}} \in \mathbb{R}$ is Borel.

Definition A.2.1. *A function $f\colon \mathrm{X} \to \mathbb{B}$ is said to be simple if there exists $v_1, \dots, v_n \in \mathbb{B}$ and $E_1, \dots, E_n \in \mathscr{B}(\mathrm{X})$, which is a Borel partition of X, such that $f = \sum_{i=1}^n \chi_{E_i} v_i$.*

Definition A.2.2 (Strongly Borel maps)**.** *A function $f\colon \mathrm{X} \to \mathbb{B}$ is said to be strongly Borel (resp. strongly μ-measurable) if it is Borel (resp. μ-measurable) and there exists a vector subspace $V \subseteq \mathbb{B}$ which is separable and $\mu(\{x \in \mathrm{X} : f(x) \notin V\}) = 0$.*

The interest for this class of function is that they can be approximated by simple functions.

Lemma A.2.3 ([60, Lemma 1.3.2])**.** *Given a function $f\colon \mathrm{X} \to \mathbb{B}$, we have that f is strongly Borel if and only if there exists a sequence $f_n\colon \mathrm{X} \to \mathbb{B}$ of simple functions such that $\lim_n \|f_n(x) - f(x)\|_{\mathbb{B}} = 0$ is satisfied for μ-a.e. $x \in \mathrm{X}$.*

We can define $\int f \, \mathrm{d}\mu \in \mathbb{B}$ as follows in the class of simple functions. Let $f = \sum_{i=1}^n \chi_{E_i} v_i$, we define

$$\int f \, \mathrm{d}\mu = \sum_{i=1}^n \mu(E_i) v_i \in \mathbb{B}.$$

It can be checked the definition is well-posed, namely it does not depend on the way we write the simple function f.

Definition A.2.4 (Bochner integrable maps)**.** *A function $f\colon \mathrm{X} \to \mathbb{B}$ is Bochner integrable provided there exists a sequence $(f_n)_n$ of simple maps such that $x \mapsto \|f_n(x) - f(x)\|_{\mathbb{B}}$ is μ-measurable for every n and $\lim_{n\to+\infty} \int \|f_n - f\|_{\mathbb{B}} \, \mathrm{d}\mu = 0$.*

Given a Bochner integrable maps and a sequence $(f_n)_n$ of simple functions as in Definition A.2.4, it can be readily checked that

$$\text{the sequence } \int f_n \, \mathrm{d}\mu \text{ is Cauchy.}$$

Therefore, given a Bochner integrable function $f\colon \mathrm{X} \to \mathbb{B}$, we can define

$$\int f \, \mathrm{d}\mu = \lim_{n\to+\infty} \int f_n \, \mathrm{d}\mu.$$

It can be checked the definition of $\int f \, \mathrm{d}\mu$ does not depend on the approximating sequence $(f_n)_n$ and that

$$\left\| \int f \, \mathrm{d}\mu \right\|_{\mathbb{B}} \leq \int \|f\|_{\mathbb{B}} \, \mathrm{d}\mu.$$

A useful theorem, characterizing Bochner integrable functions, is the following one due to Bochner.

Proposition A.2.5 (Bochner, (see [60, Proposition 1.3.6]))**.** *Given $f\colon \mathrm{X} \to \mathbb{B}$, f is Bochner integrable if and only if it is strongly μ-measurable and $\int \|f\|_{\mathbb{B}}\, \mathrm{d}\mu < +\infty$.*

We restrict now to the particular case in which $(\mathrm{X}, \mathsf{d}, \mu) = ([0,1], |\cdot|, \mathscr{L}^1|_{[0,1]})$ (the one needed in our applications). It is natural to introduce the following class of vector valued Lebesgue spaces. Given $p \in [1,\infty]$, we define $L^p([0,1],\mathbb{B})$ as the space of all (equivalence classes up to $\mathscr{L}^1|_{[0,1]}$-equality of) strongly measurable maps $f\colon [0,1] \to \mathbb{B}$ for which the quantity $\|f\|_{L^p([0,1],\mathbb{B})}$ is finite, where $\|f\|_{L^p([0,1],\mathbb{B})}$ is defined as

$$\|f\|_{L^p([0,1],\mathbb{B})} := \begin{cases} \left(\int_0^1 \|f\|_{\mathbb{B}}^p \, \mathrm{d}t\right)^{\frac{1}{p}} & p < \infty, \\ \operatorname{esssup}_{t\in[0,1]} \|f(t)\|_{\mathbb{B}} & p = \infty. \end{cases}$$

It can be checked that $(L^p([0,1],\mathbb{B}), \|\cdot\|_{L^p([0,1],\mathbb{B})})$ is a Banach space for every $p \in [1,\infty]$.

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