Limit of vanishing regulator in the functional renormalization group

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The nonperturbative functional renormalization group equation depends on the choice of a regulator function, whose main properties are a "coarse-graining scale" k and an overall dimensionless amplitude a. In this paper we shall discuss the limit $a \to 0$ with k fixed. This limit is closely related to the pseudoregulator that reproduces the beta functions of the $\overline{\mathrm{MS}}$ scheme that we studied in a previous paper. It is not suitable for precision calculations but it appears to be useful to eliminate the spurious breaking of symmetries by the regulator, both for nonlinear models and within the background field method.

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I. INTRODUCTION

The functional renormalization group (FRG) [1–4] is a powerful tool to study quantum and statistical field theories and their applications in statistical mechanics, condensed matter theory, and high energy physics [5]. It describes a continuous interpolation between an UV action describing some microscopic physics and the effective action (EA) where all the quantum/statistical fluctuations have been integrated out. The functional that provides this interpolation is called the effective average action (EAA) and is denoted $\Gamma_k[\phi]$, where ϕ are the fields, k is a coarse-graining scale, and $\Gamma_0 = \Gamma$ is the EA. The EAA can be defined by a functional integral with a cutoff suppressing the contribution of low-momentum modes, thus realizing Wilson's idea of integrating out high momentum modes first. The cutoff itself is implemented by adding to the action the term

$$\Delta S_k[\phi] = \frac{1}{2}\int d^d x \phi R_k(-\partial^2)\phi, \qquad (1.1)$$

leading to the functional differential equation

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1} k\frac{dR_k}{dk}. \qquad (1.2)$$

This provides a nonperturbative definition of RG that reduces to the perturbative one in the appropriate domain [6–13]. In this context, comparison with the results of dimensional regularization become meaningful. In Ref. [14] we have discussed a two-parameter family of regulators $R_k(a, \epsilon)$ that includes (for $\epsilon = 0$) a popular class of regulators used in the FRG literature. On the other hand, taking the limits $a \to 0$ and $\epsilon \to 0$ (in this order), it reproduces the beta functions of $\overline{\mathrm{MS}}$. In this paper we shall discuss what happens when the limits are taken in the opposite order (see Fig. 1).

We next discuss the motivation for this study. The notation $\Gamma_k[\phi]$ emphasizes the important dependence of this functional on the scale k, but Γ_k also depends on the shape of the cutoff function R_k. The notation $\Gamma[\phi, R_k]$ would thus be more appropriate, and Eq. (1.2) could be replaced by a functional equation where the derivatives with respect to k are replaced by functional derivatives with respect to R_k. As mentioned above, all the relevant physical information is contained in the EA and therefore *a priori*, all the dependence on R_k is unphysical, including the dependence on k. However, there are situations where k can be identified with a physical parameter that acts in the theory as an IR cutoff. In these cases, the dependence on k can assume a physical meaning.

Even though in such cases the dependence on k reproduces the dependence on physical parameters, the dependence on the shape of R_k still remains unphysical. Thus any observable must be independent of this shape. On the other hand, when one makes approximations, even physical observables will exhibit some spurious dependence on the shape of the cutoff. We will refer to this as "cutoff dependence."[1] For example, in statistical physics, the position of a fixed point is not universal, but the critical

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[1] It is distinct from, but closely related to the "scheme dependence" of renormalized perturbation theory.

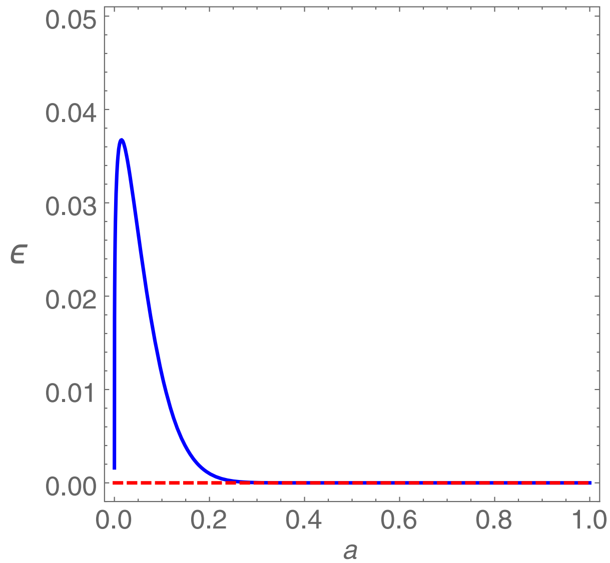

FIG. 1. Blue continuous curve: a path that reproduces the beta functions of dimensional regularization. Red dashed curve: the limit of a vanishing regulator. For a more detailed discussion see Sec. III.

exponents are. Still, when one calculates the critical exponents, one must use some approximation and the results always depend on the shape of the regulator. In a specific calculation, one can then try to exploit this cutoff dependence to optimize the cutoff, i.e., to find the cutoff that yields the best possible value for the observables. This is, in practice, an important aspect of FRG studies [15–20].

The main motivation for this study comes from another issue that arises in certain applications of the FRG. The central idea is simple and can be stated in great generality. Suppose that the action at the microscopic level is invariant under certain transformations. Since the symmetry reflects physical properties of the system, one would like to maintain it in the course of the RG flow. However, for technical reasons, it may be difficult to construct a regulator that has the symmetry, and in this case the EAA will not have it either. To be more precise, the classical symmetry of the bare action is translated into a "quantum" symmetry of the EAA, which is deformed by the presence of the regulator. The latter symmetry is only implicitly determined, as the corresponding regulator-dependent Ward identity cannot in general be analytically and exactly solved [21]. This will give rise to unpleasant complications. Intuitively, we may try to minimize the breaking of the symmetry by making the regulator as "small" as possible. Let us make this notion a bit more precise. For dimensional reasons, we can write the regulator as

$$R_k(z) = k^2 r_a(y) = k^2 a r_1(y), \tag{1.3}$$

r_1 is a dimensionless function of the dimensionless variable $y = z/k^2$ that is assumed to satisfy the normalization condition $r_1(0) = 1$, and a is a positive real number.[2] In many applications it is convenient to choose a shape function r_1 depending on some of the parameters appearing in the ansatz adopted for the EAA. The most common example is the insertion of an overall wave-function-renormalization factor Z_k. In this paper we shall mainly neglect these subtleties, as in most of our studies we will truncate the effective action to a scale-dependent local effective potential, and we will be concerned with the limit $a \to 0$, which we call the limit of vanishing cutoff.[3] One expects that in this limit the spurious effects due to the breaking of the symmetry by the regulator can be removed, or at least minimized. It may seem that this limit is trivial, because for $a = 0$ there is no cutoff, and the right-hand-side (RHS) of the exact FRG equation vanishes, but we shall see that some important physical information remains available even in this limit.

Even though many of the challenges and properties of the vanishing regulator limit can be expected to characterize large families of shape functions r_1, in this paper we mainly focus on the following regulator choice:

$$R_k(z) = a(k^2 - z)\theta(k^2 - z), \tag{1.4}$$

as in several interesting cases it is hardly feasible to study the vanishing regulator limit without having first specified a shape function. The reasons for this are explained in Sec. III C and further discussed in Sec. VI.

In order to better explain the problems arising from the use of vanishing regulators, and ways to circumvent them, it is best to focus on simple and well-understood systems. In Sec. II we consider the harmonic and anharmonic oscillator. Some of the features of vanishing regulators appear already in these cases. In Sec. III we deal with the $\mathbb{Z}_2$-invariant scalar field theory in $d \geq 2$ Euclidean dimensions and its RG fixed point (representing the Ising universality class). We find that the main features of the Wilson-Fisher (WF) fixed point remain accessible in the limit of a vanishing regulator, but the best approximation (after this limit is taken and among all possible polynomial truncations of the potential) for the correlation-length critical exponent ν is obtained with the simplest truncation that only involves relevant couplings (the mass and the quartic coupling). There we also discuss the relation between the vanishing-a limit of (1.4) and the constant (momentum-independent) regulator, as well as the subtleties concerning the application of vanishing regulators in an even number of dimensions.

[2]Consider a fixed shape function r_1, such that $r_1(y) = 0$ for $y > 1$. The limit $a \to \infty$ is expected to completely remove from the path integral all the fluctuations with momenta $q^2 < k^2$. This is often referred to as the sharp cutoff limit. Numerically optimal results are usually obtained for $a \approx 1$.

[3]Thus, the vanishing cutoff should not be misinterpreted as $k \to 0$.

In Sec. IV we address the $O(N+1)$ nonlinear sigma model, using a particular coordinate system on the sphere S^N. This is an example of a system where the regulator breaks the symmetry of the theory [respecting only the subgroup $O(N)$] but in the limit of vanishing regulators the symmetry is seen to be restored. In Sec. V we discuss a similar problem that arises in applications of the background field method. It is generally the case that the regulator breaks the symmetry of the classical action consisting of equal and opposite shifts of the background and fluctuation fields. Also this symmetry is seen to be restored in the limit of vanishing regulators. We conclude in Sec. VI with a brief discussion of our results and some outlooks. Some auxiliary formulas and analyses are provided in two appendixes.

II. QUANTUM OSCILLATORS

In this section we shall consider a very simple application of the FRG equation as a tool to compute the EA at $k=0$. This will allow us to investigate the effect of the vanishing regulators on the calculation of some physical observable. We shall consider first the simple harmonic oscillator and then the anharmonic one.

The general bare action we are interested in reads

$$S=\int dt\left(\frac{1}{2}\dot{x}^2+\frac{1}{2}\omega^2x^2+\frac{\lambda}{4!}x^4\right). \tag{2.1}$$

In the local potential approximation (LPA), which is the first term in a derivative expansion, the EAA is approximated by

$$\Gamma_k=\int dt\left(\frac{1}{2}\dot{x}^2+V_k(x)\right). \tag{2.2}$$

Using the regulator in Eq. (1.4) we get the following flow equation for the potential:

$$\partial_kV_k=\frac{1}{\pi}\left(\frac{a\,k\arctan\left(k\sqrt{\frac{1-a}{ak^2+V_k''}}\right)}{\sqrt{(1-a)(ak^2+V_k'')}}-\sqrt{\frac{a}{1-a}}\arctan\left(\sqrt{\frac{1-a}{a}}\right)\right). \tag{2.3}$$

The second term on the RHS is equal to the first term evaluated at $V_k=0$. This subtraction is not *ad hoc* and is actually always meant to be present in the FRG equation [22], although in most applications it is dropped since it only affects the ground-state energy.[4] This term is due to the regularization of the functional measure of the path integral, which ensures complete suppression of the functional integral in the $k\to\infty$ limit. As it is uniquely defined to reproduce the Weyl ordering prescription, it gives rise to the conventional ground-state energies of the nongravitating quantum/statistical mechanical models [23,24].

Expanding the potential into a Taylor series

$$V_k(x)=E_k+\frac{1}{2}\omega_k^2x^2+\frac{\lambda_k}{4!}x^4+\cdots, \tag{2.4}$$

the beta functions are

$$k\partial_kE_k=\frac{k}{\pi}\left[\frac{a\arctan\left(\sqrt{\frac{1-a}{a+\tilde{\omega}_k^2}}\right)}{\sqrt{(1-a)(a+\tilde{\omega}_k^2)}}-\sqrt{\frac{a}{1-a}}\arctan\left(\sqrt{\frac{1-a}{a}}\right)\right], \tag{2.5a}$$

$$k\partial_k\omega_k^2=-\frac{a\lambda_k}{2\pi k}\frac{\frac{\sqrt{1-a}\sqrt{a+\tilde{\omega}_k^2}}{1+\tilde{\omega}_k^2}+\arctan\left(\sqrt{\frac{1-a}{a+\tilde{\omega}_k^2}}\right)}{(a+\tilde{\omega}_k^2)^{3/2}\sqrt{1-a}}, \tag{2.5b}$$

$$k\partial_k\lambda_k=\frac{3a\lambda_k^2}{4\pi k^3}\left[\frac{3+2a+5\tilde{\omega}_k^2}{(1+\tilde{\omega}_k^2)^2(a+\tilde{\omega}_k^2)^2}+\frac{3\arctan\left(\sqrt{\frac{1-a}{a+\tilde{\omega}_k^2}}\right)}{(a+\tilde{\omega}_k^2)^{5/2}\sqrt{1-a}}\right], \tag{2.5c}$$

where $\tilde{\omega}_k=\omega_k/k$.

A. Harmonic oscillator

We start by addressing the computation of the vacuum energy of the harmonic oscillator. When $\lambda_k=0$, Eq. (2.5b) shows that ω_k is independent of k, thus we shall simply write $\omega_k=\omega$. The solution of (2.5a) is

$$E_k=\frac{\omega}{2}+\frac{k}{\pi}\left[\sqrt{\frac{a+\tilde{\omega}^2}{1-a}}\arctan\left(\sqrt{\frac{1-a}{a+\tilde{\omega}^2}}\right)-\sqrt{\frac{a}{1-a}}\arctan\left(\sqrt{\frac{1-a}{a}}\right)-\tilde{\omega}\arctan\left(\frac{1}{\tilde{\omega}}\right)\right]. \tag{2.6}$$

This function is plotted in Fig. 2 for various values of a. First of all we see that $E_0=\omega/2$ for any a. The a independence of the result is just an example of a more general phenomenon: while the k dependence of any quantity along an RG trajectory is sensitive to the functional form of R_k, the boundary values at $k\to+\infty$ and $k\to0$ are not.

The second point to notice is that the convergence of the flow toward the IR becomes faster for decreasing a. This can be understood as follows. The regulator term is

[4] The correct counterpart of this term in applications to gravity, especially within the background-field formulation, is still uncertain.

effective in suppressing the propagation when it becomes comparable to or larger than the kinetic term, i.e., for $R_k(q^2) > q^2$. For the regulator (1.4), this happens when

$$q^2 < k_{\rm eff}^2 \equiv \frac{a}{1+a} k^2. \tag{2.7}$$

Thus decreasing a has the same effect as decreasing $k_{\rm eff}$.

From this discussion, there seems to be no issue with the limit $a \to 0$. Some subtlety appears, however, when we try to construct the RG trajectory from the $a \to 0$ limit of Eq. (2.5a), which reads

$$k\partial_k E_k \sim -\frac{k}{2}\sqrt{a} + O(a). \tag{2.8}$$

This way of taking the limits is, of course, nonsensical, as the resulting beta functions would be identically vanishing. The ω independence is a consequence of the fact that the numerator of the RHS of the FRG equation is already proportional to a. However, we obtain a nontrivial equation if we rescale

$$\omega^2 = a\hat{\omega}^2, \qquad E_k = \sqrt{a}\hat{E}_k, \tag{2.9}$$

and then take the $a \to 0$ limit of Eq. (2.5a). This leads to the finite result

$$k\partial_k \hat{E}_k = \frac{1}{2}\left(\frac{k}{\sqrt{k^2+\hat{\omega}^2}} - 1\right) + O(\sqrt{a}), \tag{2.10}$$

which is the same flow equation one would find with a constant (often called a "Callan-Symanzik") regulator,

$$R_k = k^2. \tag{2.11}$$

The latter leads to the flow

$$E_k = \frac{1}{2}(\sqrt{k^2+\omega^2} - k), \tag{2.12}$$

which is plotted as the dashed curve in Fig. 2.

The rescaling (2.9) is formulated as an "active" transformation of the couplings E_k and ω_k through a factor of $\sqrt{a}$, while the RG coordinate k stays independent of a. Within the flow equations of the dimensionless couplings $\tilde{E}_k \equiv E_k/k$ and $\tilde{\omega}^2$, it is also possible to reinterpret it as a "passive" transformation, in which the dimensionful couplings E_k and ω_k are independent of a, while the RG coordinate changes by $k = \hat{k}/\sqrt{a}$. This second interpretation is consistent with the observation that $k_{\rm eff} \to \hat{k}$ for $a \to 0$ according to Eq. (2.7).

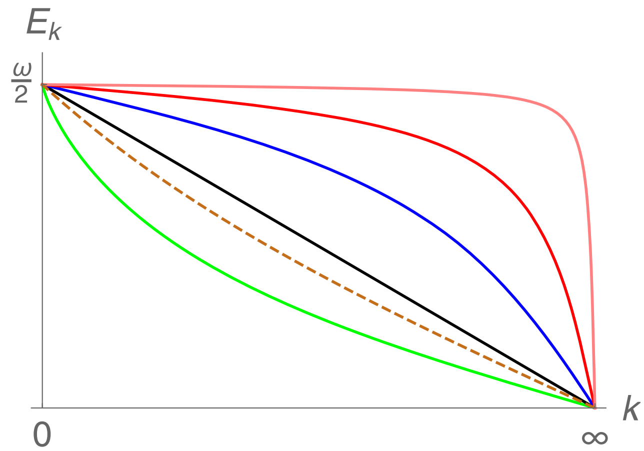

FIG. 2. The continuous curves are the RG trajectories (2.6) for the harmonic oscillator ground-state energy E_k for the regulator (1.4) and various values of a. From bottom to top: $a \to \infty$ (sharp cutoff, green), $a = 1$ (black), $a = 1/10$ (blue), $a = 1/100$ (red), $a = 1/10000$ (pink). The dashed curve is the flow (2.12). The horizontal axis has been rescaled by the function $k = \tan(\pi x/2)$.

B. Anharmonic oscillator

Next, we turn to the anharmonic oscillator with $\lambda \neq 0$. As our interest is in comparing different possible prescriptions for taking the $a \to 0$ limit with the available solutions for some interesting quantities, we do not address the numerical analyses needed to compute the energy levels, finding analytical expressions more instructive. We therefore consider only the first order of the expansion in λ. We see from (2.5c) that at this order the beta function of λ_k is zero. Therefore $\lambda_k = \lambda$ at all scales. Expanding the vacuum energy parameter to first order in λ,

$$E_k = E_k|_{\lambda=0} + \left.\frac{dE_k}{d\lambda}\right|_{\lambda=0} \lambda + \cdots, \tag{2.13}$$

and solving the flow equation with the initial condition that $\lim_{k\to\infty}\Gamma_k = S$ is the bare action (2.1), we find that the first order correction to the energy is

$$\begin{aligned}\left.\frac{dE_k}{d\lambda}\right|_{\lambda=0} = \frac{1}{32\omega^2} &+ \frac{\arctan(\frac{1}{\tilde{\omega}})[2\tilde{\omega}\sqrt{\frac{1-a}{a+\tilde{\omega}^2}}\arctan\left(\sqrt{\frac{1-a}{a+\tilde{\omega}^2}}\right) + (a-1)(\pi + \arctan(\frac{1}{\tilde{\omega}})^2)]}{8\pi^2(a-1)\omega^2} \\ &- \frac{\arctan\left(\sqrt{\frac{1-a}{a+\tilde{\omega}^2}}\right)\left(\pi\sqrt{\frac{1-a}{a+\tilde{\omega}^2}} + \frac{\tilde{\omega}}{a+\tilde{\omega}^2}\arctan\left(\sqrt{\frac{1-a}{a+\omega^2}}\right)\right)}{8\pi^2(a-1)k\omega}.\end{aligned} \tag{2.14}$$

On the other hand, for the frequency we find to first order in λ

$$\omega_k^2 = \omega^2 + \frac{\lambda}{4\omega}\left[1 + \frac{2\tilde{\omega}\sqrt{\frac{1-a}{a+\tilde{\omega}^2}}\arctan\left(\sqrt{\frac{1-a}{a+\tilde{\omega}^2}}\right)}{\pi(1-a)} - \frac{2}{\pi}\arctan\left(\frac{1}{\tilde{\omega}}\right)\right]. \tag{2.15}$$

The quantities $\frac{dE_k}{d\lambda}|_{\lambda=0}$ and ω_k given in (2.14) and (2.15) are the solutions of the flow equations at arbitrary k. They interpolate between the initial conditions $\lim_{k\to\infty}\frac{dE_k}{d\lambda}|_{\lambda=0}=0$ and $\lim_{k\to\infty}\omega_k = \omega$ and the corresponding parameters in the EA at $k = 0$.

Also in this case, it is not possible to directly take the limit $a \to 0$ in the flow equations, because then the beta functions simply vanish. However, having solved the flow equations one can take the limits $a \to 0$ and $k \to 0$ in any order obtaining

$$\omega_0^2 = \omega^2 + \frac{\lambda}{4\omega}, \tag{2.16a}$$

$$E_0 = \frac{\omega}{2} + \frac{\lambda}{32\omega^2}. \tag{2.16b}$$

One can take the limit $a \to 0$ in the flow equations provided the potential is rescaled to

$$V_k(x) = \sqrt{a}\hat{V}_k(\hat{x}), \tag{2.17a}$$

$$x = a^{-1/4}\hat{x}. \tag{2.17b}$$

Expanding around $a = 0$ the flow equation becomes

$$\partial_k\hat{V}_k = \frac{1}{2}\left(\frac{k}{\sqrt{k^2 + \hat{V}_k''}} - 1\right). \tag{2.18}$$

This is the flow equation of the potential for a constant regulator. Projecting the latter on a polynomial truncation of the potential as in Eq. (2.4), we deduce the beta functions

$$\partial_k E_k = \frac{1}{2}\left(\frac{k}{\sqrt{k^2 + \omega_k^2}} - 1\right), \tag{2.19a}$$

$$\partial_k\omega_k^2 = -\frac{1}{4}\frac{k}{(k^2 + \omega_k^2)^{3/2}}\lambda_k, \tag{2.19b}$$

$$\partial_k\lambda_k = \frac{9}{8}\frac{k}{(k^2 + \omega_k^2)^{5/2}}\lambda_k^2. \tag{2.19c}$$

Solving these equations one reobtains (2.16).

III. THE ISING UNIVERSALITY CLASS

In this section we shall consider the theory of a single, $\mathbb{Z}_2$-invariant scalar field ϕ in the LPA

$$\Gamma_k = \int d^dx\left[\frac{1}{2}(\partial\phi)^2 + V_k(\phi)\right]. \tag{3.1}$$

While the FRG equation allows us to treat the potential as a whole, it will be instructive to further expand

$$V_k(\phi) = \sum_{n=0}^{\infty}\frac{\lambda_{2n}}{(2n)!}\phi^{2n}. \tag{3.2}$$

The term $n = 0$ is the vacuum energy and can usually be ignored, but we shall need it later in our discussion. Then, from the FRG equation we can derive infinitely many beta functions $\beta_{2n} = k\frac{\partial\lambda_{2n}}{\partial k}$. For arbitrary regulator, and in any dimension, for the first few couplings this leads to

$$\beta_0 = \frac{1}{2(4\pi)^{d/2}}Q_{d/2}\left[\frac{\partial_t R_k}{P_k + \lambda_2}\right], \tag{3.3a}$$

$$\beta_2 = -\frac{1}{2(4\pi)^{d/2}}\lambda_4 Q_{d/2}\left[\frac{\partial_t R_k}{(P_k + \lambda_2)^2}\right], \tag{3.3b}$$

$$\beta_4 = \frac{1}{2(4\pi)^{d/2}}\left(6\lambda_4^2 Q_{d/2}\left[\frac{\partial_t R_k}{(P_k + \lambda_2)^3}\right] - \lambda_6 Q_{d/2}\left[\frac{\partial_t R_k}{(P_k + \lambda_2)^2}\right]\right), \tag{3.3c}$$

$$\beta_6 = \frac{1}{2(4\pi)^{d/2}}\left(-90\lambda_4^3 Q_{d/2}\left[\frac{\partial_t R_k}{(P_k + \lambda_2)^4}\right] + 30\lambda_4\lambda_6 Q_{d/2}\left[\frac{\partial_t R_k}{(P_k + \lambda_2)^3}\right] - \lambda_8 Q_{d/2}\left[\frac{\partial_t R_k}{(P_k + \lambda_2)^2}\right]\right), \tag{3.3d}$$

where

$$Q_n[W] = \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}W(z) \tag{3.4}$$

are momentum integrals ($z = q^2$). These integrals can be evaluated in closed forms by using the optimized regulator (1.4). The Q functionals are then given by hypergeometric functions

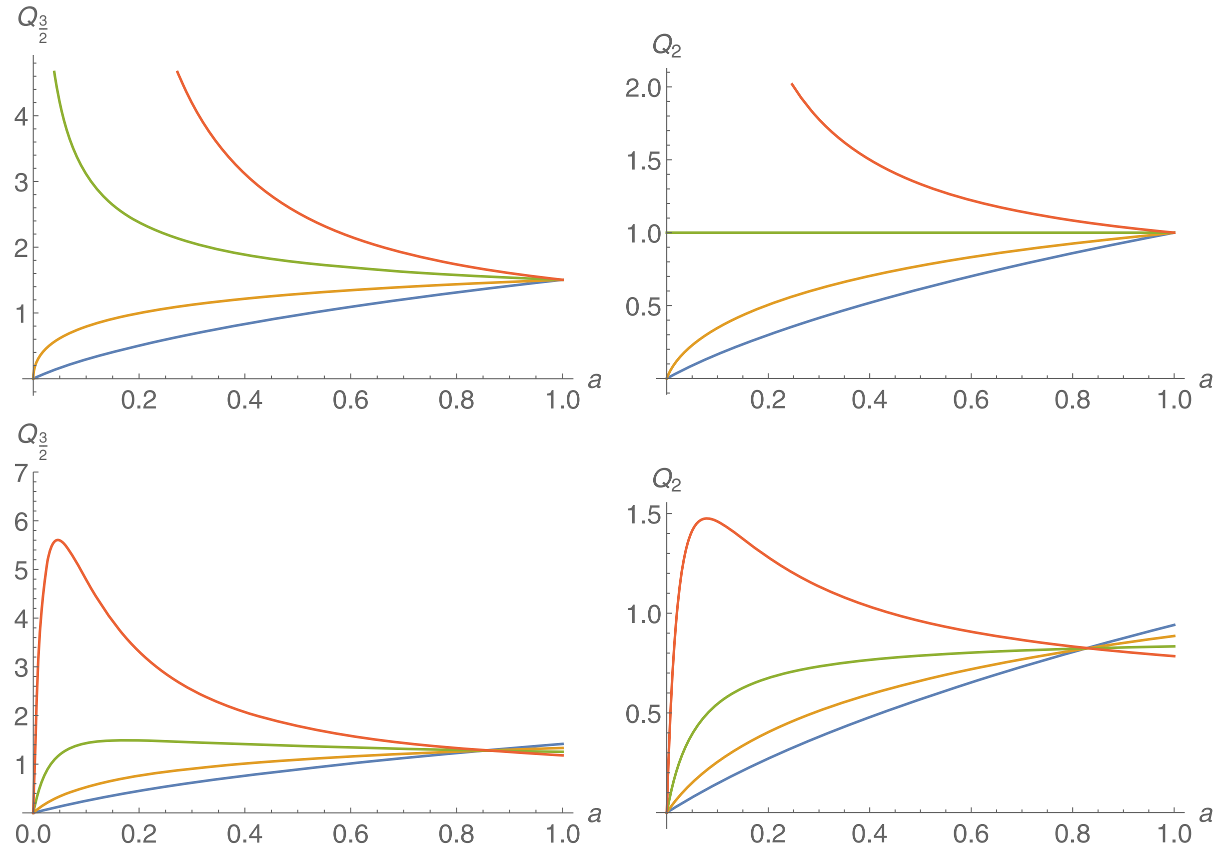

FIG. 3. The functionals (3.5) with $n = 3/2$ ($d = 3$, left) and $n = 2$ ($d = 4$, right), with $\tilde{m} = 0$ (top line) and $\tilde{m} = 0.25$ (bottom line). In each figure $\ell = 1, 2, 3, 4$, from bottom to top.

$$Q_n\left(\frac{\partial_t R}{(P+m^2)^\ell}\right) = \frac{2ak^{2(n+1-\ell)}}{\Gamma(n+1)(a+\tilde{m}^2)^\ell} \times {}_2F_1\left(\ell, n, 1+n, \frac{a-1}{a+\tilde{m}^2}\right). \tag{3.5}$$

These are plotted in Fig. 3 for $d = 3$ and $d = 4$.

In particular, in the massless case and in the limit of a vanishing regulator we obtain

$$\lim_{a\to 0} Q_n\left(\frac{\partial_t R}{P^\ell}\right) = \begin{cases} 0 & \text{for} \quad \ell < n+1, \\ 1 & \text{for} \quad \ell = n+1, \\ \infty & \text{for} \quad \ell > n+1. \end{cases} \tag{3.6}$$

Clearly, the beta functions will not be finite.[5] For this reason an additional regularizing device is needed to make sense of vanishing regulators. In Ref. [14] we have discussed a family of regulators depending on an additional parameter ϵ that, in the limit $a \to 0$ and $\epsilon \to 0$ (in this order) reproduces the results of dimensional regularization. In the $a - \epsilon$ plane the limit had to be taken along a curve of the general form shown in Fig. 1. In this paper we shall instead try to take the limits in the inverse order. In fact, we shall not even talk about the parameter ϵ and try to take the limit $a \to 0$ along the path $\epsilon = 0$ (red dashed curve) in Fig. 1.

Returning to the beta functions (3.3), we note that if we set $\lambda_2 = 0$, the beta functions of the relevant couplings go to zero, those of the marginal couplings are independent of a, and those of the irrelevant couplings diverge in the limit of a vanishing regulator. Given this rather singular behavior, one may fear that all physical information gets lost in this limit. Actually, this is not so, as we intend to show in $d = 3$, where the system is known to have a nontrivial fixed point.

A. The Wilson-Fisher fixed point: Relevant couplings

In order to make our point it will be enough, as a first step, to consider a truncation that contains only the relevant couplings (we are now in $d = 3$):

$$V_k = \frac{\lambda_2}{2}\phi^2 + \frac{\lambda_4}{24}\phi^4. \tag{3.7}$$

Defining the dimensionless variables

[5] Note that these are infrared divergences: in the massive case all Q functionals go to zero for $a \to 0$.

$$\tilde{\lambda}_{2n} = k^{-d+n(d-2)}\lambda_{2n}, \tag{3.8}$$

the beta functions are

$$\tilde{\beta}_2 = -2\tilde{\lambda}_2 - \frac{a\tilde{\lambda}_4}{6\pi^2(a+\tilde{\lambda}_2)^2}\,{}_2F_1\left(2,\frac{3}{2},\frac{5}{2};\frac{a-1}{a+\tilde{\lambda}_2}\right), \tag{3.9a}$$

$$\tilde{\beta}_4 = -\tilde{\lambda}_4 + \frac{a\tilde{\lambda}_4^2}{\pi^2(a+\tilde{\lambda}_2)^3}\,{}_2F_1\left(3,\frac{3}{2},\frac{5}{2};\frac{a-1}{a+\tilde{\lambda}_2}\right). \tag{3.9b}$$

Expanding in $\tilde{\lambda}_2$

$$\tilde{\beta}_2 = -2\tilde{\lambda}_2 - \frac{a\tilde{\lambda}_4}{4\pi^2(1-a)}\left[\left(\frac{\arctan\left(\frac{\sqrt{1-a}}{\sqrt{a}}\right)}{\sqrt{(1-a)a}} - 1\right) + \left(\frac{2a-1}{2a} - \frac{\arctan\left(\frac{\sqrt{1-a}}{\sqrt{a}}\right)}{2a\sqrt{a(1-a)}}\right)\tilde{\lambda}_2\right], \tag{3.10a}$$

$$\tilde{\beta}_4 = -\tilde{\lambda}_4 + \frac{\tilde{\lambda}_4^2}{\pi^2 a^2}\,{}_2F_1\left(\frac{3}{2},3;\frac{5}{2};\frac{a-1}{a}\right). \tag{3.10b}$$

The WF fixed point is now located at

$$\tilde{\lambda}_2^* = \frac{2a^3\left(1-\frac{\arctan(\frac{\sqrt{1-a}}{\sqrt{a}})}{\sqrt{(1-a)a}}\right)}{a^2\left(2a-1-\frac{\arctan(\frac{\sqrt{1-a}}{\sqrt{a}})}{\sqrt{(1-a)a}}\right)+16(1-a)\,{}_2F_1(\frac{3}{2},3;\frac{5}{2};\frac{a-1}{a})}, \tag{3.11a}$$

$$\tilde{\lambda}_4^* = \frac{\pi^2 a^2}{{}_2F_1(\frac{3}{2},3;\frac{5}{2};\frac{a-1}{a})}. \tag{3.11b}$$

If we expand the critical couplings around $a = 0$,

$$\tilde{\lambda}_2^* \underset{a\to 0}{\sim} -\frac{2a}{5} + o(a^{3/2}), \tag{3.12a}$$

$$\tilde{\lambda}_4^* \underset{a\to 0}{\sim} \frac{16\pi\sqrt{a}}{3} + o(a^{3/2}). \tag{3.12b}$$

Thus the WF fixed point merges with the Gaussian fixed point. Note that since $\tilde{\lambda}_2 = \tilde{m}^2$ is linear in a for $a \to 0$ at the WF fixed point, the Q functional (3.5) does not go to zero, and this entails that the quantum/statistical contribution to the critical exponents will be nontrivial for $a \to 0$.

Indeed, the position of the fixed point is not physically significant. If we consider the stability matrix at the nontrivial fixed point

$$M = \left(\frac{\partial\tilde{\beta}_i}{\partial\tilde{\lambda}_j}\right)_* = \begin{pmatrix} -\frac{5}{3} & \frac{4a\,{}_2F_1(\frac{3}{2},3;\frac{5}{2};\frac{a-1}{a})\left(1-\frac{\tan^{-1}(\frac{\sqrt{1-a}}{\sqrt{a}})}{\sqrt{(1-a)a}}\right)}{\pi^2a^2\left(2a-\frac{\tan^{-1}(\frac{\sqrt{1-a}}{\sqrt{a}})}{\sqrt{(1-a)a}}-1\right)+16(1-a)\,{}_2F_1(\frac{3}{2},3;\frac{5}{2};\frac{a-1}{a})} \\ 0 & 1 \end{pmatrix}, \tag{3.13}$$

we see that the component (1,2) of M goes to zero for $a \to 0$ and so the stability matrix becomes diagonal. The eigenvalues of M, that is, minus the critical exponents θ_i, are actually independent of a, in particular, $\nu = (\theta_1)^{-1} = 0.6$. We see that even though the WF fixed point collapses toward the Gaussian one, it keeps its distinct character in the limit $a \to 0$ and a different critical exponent ν. In fact, the numerical value is not very bad, considering the drastic approximation.

B. The Wilson-Fisher fixed point in the LPA

Let us now treat the potential as a whole [25]. Inserting (3.1) in the FRGE we obtain the "beta functional"

$$\partial_t V_k = \frac{1}{2(4\pi)^{d/2}} Q_{d/2}\left[\frac{\partial_t R_k}{P_k + V_k''}\right]. \tag{3.14}$$

Using the regulator (1.4), setting $d = 3$, and rescaling

$$\phi = \frac{k^{1/2}}{\pi\sqrt{6}}\tilde{\phi}, \tag{3.15a}$$

$$V_k(\phi) = \frac{k^3}{6\pi^2}v(\tilde{\phi}), \tag{3.15b}$$

the beta function of the dimensionless potential v becomes

$$\partial_t v = -3v + \frac{1}{2}\tilde{\phi}v' + \frac{a}{a+v''}\,{}_2F_1\left(1,\frac{3}{2},\frac{5}{2};\frac{a-1}{a+v''}\right). \tag{3.16}$$

We look for even scaling solutions shooting from the origin with initial conditions $v''(0)$ and $v'(0) = 0$. There are only two values of $v''(0)$ that can be identified as fixed-point solutions: $v''(0) = 0$, which corresponds to the Gaussian fixed point, and some negative value, which corresponds to the WF fixed point. As in the preceding section, for decreasing values of a, the WF fixed point moves toward the Gaussian one. We see that also in the functional treatment, the WF fixed point collapses into the Gaussian one.

This is confirmed by shooting from infinity. The potential for the WF solution has the following asymptotic behavior for large field

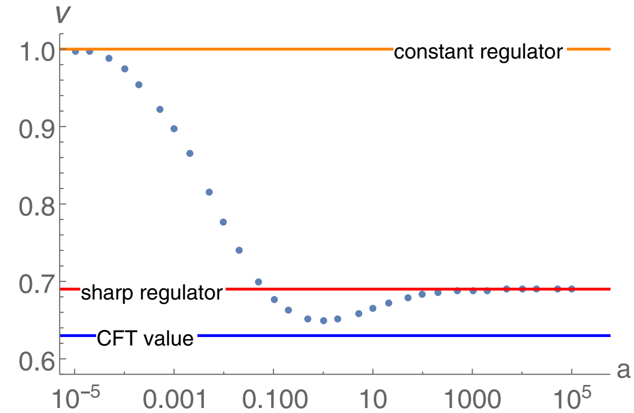

FIG. 4. The dots represent the values of the critical exponent ν as a function of a. For comparison we have drawn the values of ν for the sharp regulator and the constant (mass) regulator, as well as the conformal-bootstrap value [27]. This figure extends Fig. 12 in Ref. [16] to low values of a.

$$v = A\tilde{\phi}^6 + a\left(\frac{1}{150A\tilde{\phi}^4} - \frac{2a+3}{31500A^2\tilde{\phi}^8} + \frac{8a^2+12a+15}{8505000A^3\tilde{\phi}^{12}} - \frac{a}{67500A^3\tilde{\phi}^{14}} + O(A^{-4}\tilde{\phi}^{-16})\right). \tag{3.17}$$

The free parameter A can be fixed as a function of a by requiring $\mathbb{Z}_2$ symmetry at the origin of field space [26]. We find that in the limit $a \to 0$, A tends to $A \approx 0.0015$.[6]

The scaling exponents θ_i are obtained by linearizing the flow equation around the fixed point and calculating the spectrum of eigenperturbations. The analysis has to be done numerically. For the Gaussian fixed point the spectrum is independent of a. Figure 4 gives ν of the WF fixed point as a function of a for $10^{-5} < a < 10^5$. As expected, the best value is obtained for $a \approx 1$, while in the limit of vanishing regulator ν appears to approach $\nu = 1$. Besides the correlation-length exponent $\nu = (\theta_1)^{-1}$, we also find positive eigenvalues, as reported in Table I.

For vanishing a all the scaling exponents are odd integers. This coincides with the spectrum of the $O(N)$ model in the limit of large N, which is known, and we have, indeed, checked to be independent of a [28,29].

C. Vanishing regulators and constant regulators

At this point it is relevant to recall that the critical exponent $\nu = 1$ is known to result also from the LPA equations for a constant regulator (2.11) [16]. Together with the findings of Sec. II, this observation points at a more general result, which we detail in this section.

[6]The asymptotic parameter is $A = 0.001$ for $a = 1$, and it increases monotonically for $a \to 0$.

TABLE I. The first few critical exponents at the Wilson-Fisher fixed point computed in the local potential approximation for the regulator (1.4). We report the most common choice $a = 1$ and the limiting case of the vanishing regulator.

	θ_1	θ_2	θ_3	θ_4	θ_5
$a = 1$	1.539	−0.656	−3.180	−5.912	−8.796
$a = 0$	1	−1	−3	−5	−7

So far we have first solved the fixed point equations for generic a and then sent $a \to 0$. On the other hand, we are now going to argue that when the vanishing regulator limit is taken on the LPA beta functions, i.e., before integrating the flow, it results, in general noneven d, in the flow equations of the constant regulator.

The first way of reaching this conclusion is based on a redefinition of the RG scale k, which we have already introduced in Sec. II. Suppose that in addition to the parameter a we also introduce a parameter b rescaling the cutoff k,

$$R_k(z) = a(bk^2 - z)\theta(bk^2 - z). \tag{3.18}$$

This rescaling can be motivated as follows. First of all, it should not change the scaling solutions. Furthermore, as discussed in Sec. II A, we can define an "effective" cutoff scale $k_{\rm eff}$ as the momentum scale where the cutoff term R_k becomes comparable to the kinetic term. If we decrease a, the effective cutoff scale also decreases. It was suggested in Ref. [16] that the decrease of a should be compensated by choosing b so that at some conventional scale $z_0 < k^2$, the regulator is normalized: $R_k(z_0) = k^2$. This fixes $b = \frac{1}{a} + \frac{z_0}{k^2}$, leading to the regulator

$$R_k(z) = (k^2 - a(z - z_0))\theta(k^2 - a(z - z_0)). \tag{3.19}$$

Now we see that in the limit $a \to 0$, the regulator becomes a constant as in Eq. (2.11). The latter leads to the dimensionless flow equation

$$\partial_t v = -dv + \left(\frac{d}{2} - 1\right)\tilde{\phi}v' + \frac{\pi(1+v'')^{\frac{d}{2}-1}}{(4\pi)^{d/2}\Gamma(\frac{d}{2})\sin(\frac{d\pi}{2})}. \tag{3.20}$$

In $d = 3$ and after the rescaling $v \to v/(4\pi)$ and $\tilde{\phi} \to \tilde{\phi}/\sqrt{4\pi}$ this takes the simple form

$$\partial_t v = -3v + \frac{1}{2}\tilde{\phi}v' - \sqrt{1 + v''}. \tag{3.21}$$

This argument can actually easily be generalized to arbitrary shape functions r_1, as defined in Eq. (1.3). We first include the parameter b in the regulator, to account for the possibility to rescale k,

$$R_k(z) = bk^2 a r_1(y/b). \tag{3.22}$$

Then we choose $b = 1/a$ such that the regulator becomes

$$R_k(z) = k^2 r_1(ay). \tag{3.23}$$

Then the $a \to 0$ limit of Eq. (3.22) results in the constant regulator.[7]

An alternative way of arguing that the $a \to 0$ limit reduces the LPA flow equation for the regulator (1.4) to the constant regulator case (3.20) is by performing an a-dependent rescaling as in Sec. II. Namely, by redefining

$$\tilde{\phi} = a^{(d-2)/4}\hat{\phi}, \tag{3.24a}$$

$$v(\tilde{\phi}) = a^{d/2}\hat{v}(\hat{\phi}) + a\frac{1}{(d-2)(4\pi)^{d/2}\Gamma(1+d/2)}, \tag{3.24b}$$

in the flow equations for the regulator (1.4) and then taking the $a \to 0$ limit at fixed $\hat{\phi}$ and $\hat{v}$, we again find Eq. (3.20). For instance, in $d = 3$ this rescaling entails that the prefactor a in (3.16) goes away.

Both kind of arguments, however, are applicable only for nonexceptional d. In particular, in some cases removing the momentum dependence of the regulator by sending $a \to 0$, as in Eq. (3.19) and Eq. (3.23), is not possible, because the $a \to 0$ limit and the momentum integral cannot be exchanged. This happens whenever the integral corresponding to the constant regulator is divergent. In fact, the momentum integral leading to Eq. (3.20) is convergent only for $d < 2$.[8]

If in the scalar LPA we adopt the constant regulator in $d \geq 2$, using analytic continuation as a tool for the definition of the momentum integral, the result has a meromorphic structure with poles for even values of d. On the other hand, if we try to directly take the limit $a \to 0$ with the regulator (1.4) and expand the Q functionals (3.5), with $n = d/2$, d even, and $\tilde{m} = 0$, in a around $a = 0$, there appear terms with $\log a$. As a consequence, we expect that the vanishing regulator limit of the LPA flow equation will enjoy special properties in even dimensions. As a matter of fact, if analytic continuation is not adopted in the definition of the loop integrals, the arguments we just outlined point to the conclusion that the vanishing regulator limit does not need to reproduce the constant regulator case in the whole range $d \geq 4$.

[7]By comparing this with the original regulator in Eq. (1.3) we see that we have effectively cast the regulator as a function of $k_{\rm eff}^2 = ak^2$, rather than of k itself, and then considered $k_{\rm eff}$ as a independent.

[8]However, the UV divergence in $2 \leq d < 4$ affects only the field-independent part of the effective potential, and in these cases it could be removed by implementing the standard subtraction as in Sec. II [see, e.g., Eq. (2.18)]. Notice that this subtraction would introduce an IR divergence in $d = 2$. For some values of d the limit $a \to 0$ cannot be taken at the level of the integrands. In these cases we first of all have to compute the integrals, and this requires one to specify the shape function r_1. This is the main reason why in the present paper we focus on the special regulator choice of Eq. (1.4). More general results holding for arbitrary shape functions can be deduced once the field-theory model, the number of Euclidean dimensions d, and the truncation of the EAA is specified.

D. Beta functions in two and four dimensions

As we argued at the end of the previous section, in the case of even dimensions the limit of vanishing regulators has a more intricate structure. Therefore, in this section we analyze these special cases in more detail.

We start with $d = 2$, where the flow equation of the LPA reads

$$\partial_t v = -2v + \frac{a}{4\pi(1-a)}\log\left(\frac{1+v''}{a+v''}\right). \tag{3.25}$$

Defining

$$v(\tilde{\phi}) = a\hat{v}(\tilde{\phi}) \tag{3.26}$$

and simplifying a factor a from the flow equation, in the $a \to 0$ limit we are left with

$$\partial_t\hat{v} = -2\hat{v} - \frac{1}{4\pi}\log a - \frac{1}{4\pi}\log(1+\hat{v}''). \tag{3.27}$$

The potential must be shifted by a factor that contains $\log a$, i.e., $\hat{v} \to \hat{v} - \frac{1}{8\pi}\log a$, in order to eliminate this divergent term for the limit $a \to 0$. We observe that the coefficient of the $\log a$ term matches exactly the coefficient of the $1/\epsilon$ pole of the expansion of (3.20) for $d = 2 + \epsilon$.[9] The finite logarithmic contribution coincides with the one in Eq. (3.20). Therefore, up to a field-independent shift of the potential, in $d = 2$ the vanishing regulator limit agrees with the constant regulator.

We then turn to the LPA in $d = 4$. We first truncate the potential to a polynomial expansion around vanishing fields as in Eq. (3.2). For continuity with the previous sections, we also turn to the dimensionless couplings defined in Eq. (3.8). By considering the leading contributions to the beta functions $\tilde{\beta}_{2n}$ for vanishing a, we construct an ansatz based on the following scaling:

$$\hat{\lambda}_2 = a^{-1}\tilde{\lambda}_2, \tag{3.28a}$$

$$\hat{\lambda}_4 = \log(a)\tilde{\lambda}_4, \tag{3.28b}$$

[9]This correspondence between $\log a$ singularities of the flow equations for the regulator (1.4) and $1/\epsilon$ poles of (3.20) holds also in higher even dimensions.

$$\hat{\lambda}_{2n} = a^{n-2}(\log a)^n \tilde{\lambda}_{2n}, \quad n > 2. \tag{3.28c}$$

Assuming the $\hat{\lambda}_{2n}$ couplings can be kept fixed in the $a \to 0$ limit results in the following set of beta functions:

$$\partial_t \hat{\lambda}_2 = -2\hat{\lambda}_2 + \frac{\hat{\lambda}_4}{16\pi^2}\left[1 + \frac{1+\log(1+\hat{\lambda}_2)}{\log a}\right], \tag{3.29a}$$

$$\partial_t \hat{\lambda}_4 = \frac{1}{\log a}\left[\frac{3}{16\pi^2}\frac{\hat{\lambda}_4^2}{1+\hat{\lambda}_2} + \frac{1}{16\pi^2}\hat{\lambda}_6\right], \tag{3.29b}$$

$$\partial_t \hat{\lambda}_6 = 2\hat{\lambda}_6 - \frac{15}{16\pi^2}\frac{\hat{\lambda}_4^3}{(1+\hat{\lambda}_2)^2} + \frac{1}{16\pi^2}\hat{\lambda}_8 + \frac{\hat{\lambda}_8}{16\pi^2}\frac{1+\log(1+\hat{\lambda}_2)}{\log a} + \frac{15}{16\pi^2}\frac{\hat{\lambda}_4\hat{\lambda}_6}{(1+\hat{\lambda}_2)\log a}, \tag{3.29c}$$

and similar results for higher couplings. Notice that terms of order $(\log a)^{-1}$ could be neglected as subleading in all beta functions apart from the second one, where such a term is, in fact, the leading one.

In order to include the beta functions of all couplings in a functional treatment, we turn to the task of including the definitions (3.28) in a rescaling of the effective potential. It is impossible to achieve this goal by a two-parameters rescaling of the kind studied in the previous sections. However, Eq. (3.28c) trivially lends itself to a functional rescaling. Hence, we can treat the first two couplings on a special footing and embed the remaining ones in a functional which is related to higher derivatives of $v(\phi)$.

First, to simplify notations, it is convenient to define

$$\tilde{\rho} = \tilde{\phi}^2/2, \tag{3.30}$$

$$u(\tilde{\rho}) = v(\tilde{\phi}). \tag{3.31}$$

Next, we define

$$f(\tilde{\rho}) = u'(\tilde{\rho}) - \tilde{\lambda}_2 - \frac{\tilde{\lambda}_4}{3}\tilde{\rho}. \tag{3.32}$$

So by construction $f(0) = f'(0) = 0$, while $f^{(n)}(0) \propto \lambda_{2(n+1)}$. The functional flow equation for f can be obtained from the functional equation for u' by

$$\partial_t f(\tilde{\rho}) = \partial_t u'(\tilde{\rho}) - \tilde{\beta}_2 - \frac{\tilde{\beta}_4}{3}\tilde{\rho}, \tag{3.33}$$

and then replacing u'_k through the definition (3.32). The identities $\partial_t f_k(0) = \partial_t f'_k(0) = 0$ also follow from this definition. By the rescaling

$$f(\tilde{\rho}) = \frac{a}{\log a}\hat{f}(\hat{\rho}), \tag{3.34}$$

$$\tilde{\rho} = a \log a \hat{\rho}, \tag{3.35}$$

together with the previous definitions of $\hat{\lambda}_2$ and $\hat{\lambda}_4$, we recover the full tower of relations (3.28). By inserting the previous definitions in the flow equation for $u'(\rho)$ one can deduce the following functional flow equation:

$$\begin{aligned}\partial_t \hat{f}(\hat{\rho}) = &-2\hat{f}(\hat{\rho}) + 2\hat{\rho}\hat{f}'(\hat{\rho}) \\ &+ \frac{3}{16\pi^2}\hat{f}'(\hat{\rho}) + \frac{1}{8\pi^2}\hat{\rho}\hat{f}''(\hat{\rho}) + \frac{5}{16\pi^2}\hat{\rho}\hat{f}''(0) \\ &- \frac{1}{16\pi^2}\frac{\hat{\rho}\hat{\lambda}_4^2}{1+\hat{\lambda}_2} - \frac{1}{16\pi^2}\hat{\lambda}_4 \log(1+\hat{\lambda}_2) \\ &+ \frac{1}{16\pi^2}\hat{\lambda}_4 \log(1+\hat{\lambda}_2+\hat{\lambda}_4\hat{\rho}).\end{aligned} \tag{3.36}$$

This functional flow generates the leading terms in Eq. (3.29c), and similar beta functions for the higher-order couplings, upon truncating it to a polynomial ansatz regular at the origin. However, we stress again that Eq. (3.36) does not include Eqs. (3.29) and (3.29b), which therefore have to be supplemented to exhaust the LPA flow equations.

These flow equations are different from those of a constant regulator. Indeed, the latter are formally UV divergent. More specifically, in $\tilde{\beta}_{2n}$ the contribution linear in $\tilde{\lambda}_{2n+2}$ corresponds to a momentum integral with dimension two, which is not regularized by the constant regulator (2.11). Similar discrepancies arise in $d = 6, 8, \ldots$. The flow equation for the constant regulator in $d = 4 - \epsilon$ reads

$$\begin{aligned}\partial_t v = &-4v + 2\tilde{\rho}v' \\ &+ \frac{(2\tilde{\rho}v'' + v' + 1)[\log(2\tilde{\rho}v'' + v' + 1) - 1]}{16\pi^2} \\ &+ \frac{(2\tilde{\rho}v'' + v' + 1)}{16\pi^2}\left[\gamma - \log(4\pi) - \frac{2}{\epsilon}\right].\end{aligned} \tag{3.37}$$

The third line in this equation arises from the expansion of the sine in the denominator of Eq. (3.20). It provides contributions to the $\tilde{\lambda}_{2n+2}$ term inside $\tilde{\beta}_{2n}$. Such terms would be absent in the $\overline{\mathrm{MS}}$ scheme. These $1/\epsilon$ contributions that are divergent in $d = 4$ are a typical product of the analytic continuation adopted in the definition of the integral. Similar contributions that diverge in $d = 4$ are expected also if any other alternative definition is chosen. For instance, if a sharp UV cutoff Λ is introduced, the third line of Eq. (3.37) would be replaced by a different expression that is ill-defined in the $\Lambda \to \infty$ limit.

If we perform an *ad hoc* subtraction of the third line, the flow equation (3.37) leads to the following beta functions:

$$\tilde{\beta}_2 = -2\tilde{\lambda}_2 + \frac{\tilde{\lambda}_4 \log(\tilde{\lambda}_2 + 1)}{16\pi^2}, \tag{3.38a}$$

$$\tilde{\beta}_4 = \frac{3\tilde{\lambda}_4^2}{16\pi^2(\tilde{\lambda}_2 + 1)} + \frac{\tilde{\lambda}_6 \log(\tilde{\lambda}_2 + 1)}{16\pi^2}, \tag{3.38b}$$

$$\tilde{\beta}_6 = 2\tilde{\lambda}_6 - \frac{15\tilde{\lambda}_4^3}{16\pi^2(\tilde{\lambda}_2 + 1)^2} + \frac{\tilde{\lambda}_8 \log(\tilde{\lambda}_2 + 1)}{16\pi^2} + \frac{15\tilde{\lambda}_6\tilde{\lambda}_4}{16\pi^2(\tilde{\lambda}_2 + 1)}. \tag{3.38c}$$

A comparison with Eq. (3.29) immediately reveals several differences. Apart from the scaling (classical) terms, the first two quantum/statistical terms are equal, up to the fact that the λ_2 dependence of the λ_{2n+2} term has been washed away in Eq. (3.29) by the $a \to 0$ limit, and up to the crucial $\log a$ dependence of Eq. (3.29b). However, all the additional quantum/statistical terms in Eq. (3.38) are absent in Eq. (3.29).

The peculiar simplicity that Eqs. (3.29) attain in the $a \to 0$ limit, together with the $1/\log a$ dependence of Eq. (3.29b), raises the question as to whether these beta functions retain enough physical information for being practically useful. As a first step toward addressing this question, we limit ourselves to a simple observation. Namely, as long as the subleading logarithmic a dependence is retained in Eq. (3.29), the ϕ^4-theory beta function and other universal physics is still present. For instance, we can study the WF fixed point in $d = 4 - \epsilon$. In order to employ Eq. (3.29) in this study, we need to prescribe that the $\epsilon \to 0$ limit be taken before the $a \to 0$ one. This means in practice that the vanishing regulator limit is taken on the $d = 4$ FRG equations. Had we sent $a \to 0$ in $d < 4$, we would have found different equations for $\hat{\lambda}_{2n}$ and precisely the constant regulator ones, as already mentioned in Sec. III C.

Within the simplest truncation corresponding to retaining only $\tilde{\lambda}_2$ and $\tilde{\lambda}_4$, where we add the classical scaling term $-\epsilon\lambda_4$ to $\hat{\beta}_4$ to account for the shift of dimensionality, the WF fixed point to first order in ϵ is located at

$$\hat{\lambda}_2 = \frac{1}{6}\epsilon(1 + \log a), \tag{3.39}$$

$$\hat{\lambda}_4 = \frac{16}{3}\pi^2\epsilon \log a. \tag{3.40}$$

These fixed-point couplings have to be interpreted as small, even if they seemingly blow up for $a \to 0$, because the limit $\epsilon \to 0$ has to be taken first. Notice that keeping the subleading order-$(\log a)^{-1}$ contribution to $\hat{\beta}_4$ is essential for revealing the fixed point. By computing the corresponding critical exponents, we find the universal one-loop result

$$\theta_1 = 2 - \frac{\epsilon}{3}, \qquad \theta_2 = -\epsilon. \tag{3.41}$$

IV. $O(N+1)$ SYMMETRY IN NONLINEAR MODELS

As a first example of a symmetry that is broken by the regulator, we shall consider here the two-dimensional $O(N+1)$ nonlinear sigma model in a particular coordinate system. We start from the order-∂^2 expansion of Γ_k for a $O(N)$-invariant multiplet of scalars,

$$\Gamma_k[\phi] = \int d^2x \left[U_k(\rho) + \frac{1}{2} Z_k(\rho)\partial_\mu\phi_a\partial^\mu\phi^a + \frac{1}{4} Y_k(\rho)\partial_\mu\rho\partial^\mu\rho \right], \tag{4.1}$$

where the N fields ϕ^a are in the fundamental representation of $O(N)$ and $\rho = \phi^a\phi^a/2$ is the corresponding local invariant. We further define the radial wave function renormalization

$$\tilde{Z}_k(\rho) = Z_k(\rho) + \rho Y_k(\rho), \tag{4.2}$$

The beta functions for Z_k, $\tilde{Z}_k$, and U_k are given in Appendix A.

If we make the assumptions

$$Z_k(\rho) = \frac{Z_k}{g_k^2}, \tag{4.3a}$$

$$\tilde{Z}_k(\rho) = \frac{1}{g_k^2}\left(\frac{1}{Z_k} - 2\rho\right)^{-1}, \tag{4.3b}$$

$$U_k = 0, \tag{4.3c}$$

the EAA becomes

$$\Gamma_k[\phi] = \int d^dx \frac{Z_k}{2g_k^2}\left(\delta_{ab} + \frac{\phi_a\phi_b}{\frac{1}{Z_k} - 2\rho}\right)\partial_\mu\phi^a\partial^\mu\phi^b. \tag{4.4}$$

The tensor in parentheses is the metric of the N-dimensional sphere of radius $Z_k^{-1/2}$, written in a coordinate system that consists of projecting a point of the sphere orthogonally on the equatorial plane. In this way the northern hemisphere is mapped to the domain $\phi^a\phi^a < 1/Z_k$. The symmetry group is extended to $O(N+1)$.

A standard cutoff

$$\Delta S_k(\phi) = \frac{Z_k}{2g_k^2}\int d^2x\phi_a R_k(-\partial^2)\phi^a$$

breaks $O(N+1)$ invariance, while preserving $O(N)$. Therefore, if we start at some scale k with an EAA of the form (4.4), the flow will immediately generate $O(N+1)$-violating terms, and thus it will take place in the larger theory space parametrized by (4.1).

This can be seen already by projecting the flow generated by the ansatz (4.4) on the local potential, i.e., by considering Eq. (A2). For nonvanishing a and for field-dependent wave function renormalizations, the choice $U_k = 0$ is not preserved by the RG flow. However, in the $a \to 0$ limit it, indeed, becomes a consistent ansatz, as in $\partial_t U_k$ the RHS behaves as $a \log a$ when $a \to 0$.

Let us then inspect the flow of the wave function renormalizations. Inserting the previous ansatz in the flow equation (A3) for $\tilde{Z}_k(\rho)$,[10] in the limit $a \to 0$ we obtain

$$-\frac{2Z_k \partial_t g_k}{g_k^3(1-2Z_k\rho)} - \frac{Z_k \eta_k}{g_k^2(1-2Z_k\rho)^2} = \frac{Z_k}{4\pi} \frac{(2\partial_t g_k + (\eta_k - 2)g_k)(2(N-1)Z_k\rho + 1)}{g_k(1-2Z_k\rho)^2}. \quad (4.5)$$

As the functional ρ dependence on each side of the equation is comparable, this equation can be algebraically solved for $\partial_t g_k$ and η_k, resulting in

$$\partial_t g_k = -\frac{(N-1)g_k^3}{4\pi + g_k^2}, \quad (4.6a)$$

$$\eta_k = -\partial_t \log Z_k = \frac{2N g_k^2}{4\pi + g_k^2}. \quad (4.6b)$$

These are the correct one-loop beta functions, augmented by RG resummations due to the dependence of the regulator on Z_k and g_k. The same result can be derived by considering the flow equation for $Z_k(\rho)$. Thus, within the present truncation the nonlinearly realized $O(N+1)/O(N)$ symmetry is preserved by taking the limit $a \to 0$. Essentially the same flow equations have been obtained in Ref. [14] with a pseudoregulator reproducing the $\overline{\mathrm{MS}}$ scheme.[11]

The assumption $U_k = 0$, although justified by the observation that only a trivial potential is compatible with the nonlinearly realized symmetry, can easily be relaxed as long as this explicit symmetry-breaking term is treated as an external source. The simplest of such terms is a linear coupling to the $O(N+1)/O(N)$ variation of ϕ^a, i.e., ϕ^{N+1},

$$U_k = -H\sqrt{\frac{1}{Z_k} - 2\rho}. \quad (4.7)$$

[10] Note that now $Z_k(\rho = 0) = Z_k/g_k^2$, so inside the formulas for the Q functionals we must send $Z_k \to Z_k/g_k^2$ and $\eta_k \to \eta_k + 2\partial_t g_k/g_k$.

[11] The present result is obtained by setting $\sigma = 1$ in the beta functions of Ref. [14].

This ansatz, comprehending an arbitrary source H, was observed to be compatible with the flow equation in the case of an $\overline{\mathrm{MS}}$ pseudoregulator [14]. This linear term can also be used to construct an exact FRG equation that manifestly preserves the full $O(N+1)$ symmetry for every regulator function R_k (see Appendix B). For the present standard FRG implementation and regularization scheme, the ansatz (4.7) is not compatible with the flow equation of the potential, neither for $a \neq 0$ nor in the $a \to 0$ limit. Only by assuming that H is a function of a vanishing faster than a itself, closure of the $O(\partial^2)$ RG flow on the ansatz (4.4) is recovered. To understand this phenomenon it is necessary to study how the modified master equation for the $O(N+1)/O(N)$ symmetry behaves in the $a \to 0$ limit, which is the topic of Appendix B. There we show how the construction of a nonvanishing potential term for the nonlinear sigma model is, indeed, a complicated problem that requires the simultaneous solution of both the flow equation and the modified master equation. As explained in Appendix B, in solving this problem the $a \to 0$ limit is of limited use.

V. BACKGROUND FIELD ISSUES

When one splits the field into a classical background and a quantum/statistical fluctuation

$$\phi = \phi_B + \varphi, \quad (5.1)$$

the action, being a function of ϕ, is invariant under the shift symmetry

$$\phi_B \mapsto \phi_B + \epsilon, \quad (5.2a)$$

$$\varphi \mapsto \varphi - \epsilon. \quad (5.2b)$$

This can be expressed by the identity

$$\frac{\delta S}{\delta \phi_B} - \frac{\delta S}{\delta \varphi} = 0. \quad (5.3)$$

On the other hand, the regulator depends only on the background field and is therefore not invariant under the split symmetry. In particular in gauge theories, in order to preserve background gauge invariance, the cutoff is usually written as a function of the background covariant derivative: $R_k(-\bar{D}^2)$. This effect can be mimicked in the scalar case by artificially introducing a dependence of R_k on ϕ_B. For example, Morris and collaborators considered regulators of the general form [26]

$$R_k(z) = (k^2 - k^2 h(\tilde{\phi}_B) - z)\theta(k^2 - k^2 h(\tilde{\phi}_B) - z). \quad (5.4)$$

The EA then becomes a functional $\Gamma_k[\varphi, \phi_B]$; i.e., it has a separate dependence on these two arguments. The breaking of the shift symmetry results in a modified Ward identity

$$\frac{\delta\Gamma_k}{\delta\phi_B}-\frac{\delta\Gamma_k}{\delta\varphi}=\frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2\Gamma_k}{\delta\varphi\delta\varphi}+R_k\right)^{-1}\frac{\delta R_k}{\delta\phi_B}\right]. \quad (5.5)$$

It has been shown that such a background dependence in the regulator can either destroy physical fixed points or create artificial ones [26]. On the other hand, when the FRG equation (1.2) is solved together with the Ward identity (5.5), the correct physical picture can be reconstructed. While this can be achieved in the scalar case [26], it is much harder in the case of gauge theories, and in particular for gravity [30]. It is therefore desirable to find other ways around this obstacle. The form of Eq. (5.5) suggests that in the limit of a vanishing regulator the shift symmetry is restored. One would therefore expect that in this limit the aforementioned pathologies should also disappear. In this section we will see how this actually happens in the scalar theory.

We begin by briefly reviewing some results of Ref. [26]. We consider the same system as in Sec. III B, in $d=3$, but we use the regulator (5.4). In a single-field approximation one identifies $\phi_B=\varphi$. The corresponding flow equation for the potential reads

$$\partial_t v=-3v+\frac{1}{2}\tilde{\varphi}v' +\frac{(1-h)^{3/2}}{1-h+v''}\left(1-h-\frac{1}{2}\partial_t h+\frac{1}{4}\tilde{\varphi}h'\right)\theta(1-h). \quad (5.6)$$

Two special cases for h have been considered. The first case is $h=\alpha\tilde{\varphi}^2$. In this case, for $\alpha<0$ the Heaviside theta on the RHS of Eq. (5.6) is equal to one. Solving the fixed point equation, one finds that the Gaussian fixed point becomes interacting and an increasing number of fake fixed points appear, as α becomes more negative. For example, Table II presents the nontrivial fixed points and the associated relevant critical exponents for two negative values of α. In both cases FP_2 is the deformed Gaussian fixed point. For $\alpha>0$ because of the Heaviside theta function on the RHS of Eq. (5.6), $v=A\tilde{\varphi}^6$ for $\tilde{\varphi}>1/\sqrt{\alpha}$. The Gaussian fixed point is always absent, and for $\alpha>0.08$ also the WF fixed point disappears.

The second case is $h=\alpha v''$. The Gaussian[12] and the WF fixed points always exist, but when α is increased, new fixed points appear near the Gaussian one[13] and move away from it as α becomes bigger: for example, for $\alpha=1$ there is a spurious fixed point and for $\alpha=2$ there are three of them.

In Ref. [26] the authors solve the anomalous Ward identity for shift symmetry and show how to recover the physical results. Instead, we shall discuss here the effect of taking the limit of a vanishing regulator. To this end, we first introduce the parameter a in (5.4),

$$R_k(z)=a(k^2-k^2h(\tilde{\phi}_B)-z)\theta(k^2-k^2h(\tilde{\phi}_B)-z). \quad (5.7)$$

Within a single-field LPA truncation this leads to the flow equation

$$\partial_t v=-3v+\frac{1}{2}\tilde{\varphi}v'+\theta(1-h)\frac{a(1-h)^{3/2}}{a(1-h)+v''}\times\left(1-h-\frac{1}{2}\partial_t h+\frac{1}{4}\tilde{\varphi}h'\right)\times{}_2F_1\left(1,\frac{3}{2},\frac{5}{2};\frac{(a-1)(1-h)}{a(1-h)+v''}\right). \quad (5.8)$$

Again, we discuss separately the two choices for the function h.

a. *First case:* $h=\alpha\tilde{\varphi}^2$.—Following Ref. [26] we start with a quadratically field-dependent regulator. However, we slightly depart from that reference in that we find it more convenient to portray the landscape of fixed points by a different numerical method, a shooting from the origin. This consists in constructing numerical solutions for each possible value of the boundary condition $v''(0)$. The generic solutions, however, terminate at a finite value $\tilde{\varphi}_S$ of the field, which corresponds to a movable singularity of the fixed-point equation. In this process one obtains a plot of $\tilde{\varphi}_S$ as a function of $v''(0)$ (also known as the spike plot). Sharp maxima in the latter variable are in one-to-one correspondence with global fixed points. The result is presented in Fig. 5. We can see that by decreasing a the spurious fixed points disappear and the physical fixed points converge to the origin. This is the same phenomenon that we observed in Secs. III A and III B.

TABLE II. The nontrivial fixed-point solutions of Eq. (5.6) with $h=\alpha\tilde{\phi}^2$ and the corresponding relevant critical exponents for $\alpha=-1/2$ (upper panel) and $\alpha=-2$ (lower panel). The entries that are left blank correspond to irrelevant deformations. FP_1 is the Wilson-Fisher fixed point, while FP_2 is a "deformed Gaussian" fixed point as it possesses two relevant directions.

FP	θ_1	θ_2
1	1.17	⋯
2	2.11	0.82
	$\alpha=-1/2$	

FP	θ_1	θ_2	θ_3	θ_4
1	0.89	⋯	⋯	⋯
2	2.35	0.76	⋯	⋯
3	2.02	1.43	0.60	⋯
4	2.10	1.69	1.08	0.39
		$\alpha=-2$		

[12]Note that the Gaussian fixed point corresponds to the point $(v(0),v'(0))=(1/3,0)$.

[13]In particular for $\alpha\geq 0.85$ a first additional fixed point appears.

At these fixed points, we compute the spectrum of critical exponents with the same method used in Ref. [26], namely by shooting from infinity, as we did in Sec. III B. This means that we first construct an asymptotic expansion of the fixed-point potential as well as of the eigenfunction of the linearized flow around the fixed point. For $\alpha < 0$ the Heaviside theta on the RHS of Eq. (5.8) is equal to one, and the potential has the following behavior at infinity:

$$v = A\tilde{\varphi}^6 + \frac{a|\alpha|^{5/2}}{150A}|\tilde{\varphi}| + \frac{a|\alpha|^{3/2}(525A - (3+2a)\alpha^2)}{31500A^2|\tilde{\varphi}|} + \frac{a\sqrt{-\alpha}(212625A^2 - \alpha^2(3780aA + 5670A) + (16a^2 + 24a + 30)\alpha^4)}{17010000A^3|\tilde{\varphi}|^3} + O(|\tilde{\varphi}|^{-5}). \tag{5.9}$$

Shooting on A and on a corresponding asymptotic parameter for the perturbation, and by demanding $\mathbb{Z}_2$ parity at the origin, we determine the location of the fixed point as well as the quantized values of the critical exponents. In the $a \to 0$ limit the latter become independent of α and agree with the spectrum discussed in Sec. III B.

For $\alpha > 0$, because of the Heaviside theta one the RHS of Eq. (5.8) $v = A\tilde{\varphi}^6$ for $\tilde{\varphi} > 1/\sqrt{\alpha}$. Therefore for $\tilde{\varphi} < 1/\sqrt{\alpha}$ the potential as a function of $\delta = (\frac{1}{\sqrt{\alpha}} - \tilde{\varphi})^{1/2}$ has the following asymptotic behavior:

$$v = \frac{A}{\alpha^3} - \frac{6A}{\alpha^{5/2}}\delta^2 + \frac{15A}{\alpha^2}\delta^4 + \frac{2\sqrt{2}a\alpha^{13/4}}{75A}\delta^5 - \frac{135000A^4 + a^2\alpha^{10}}{6750\alpha^{3/2}A^3}\delta^6 + o(\delta^7). \tag{5.10}$$

Shooting from infinity and decreasing a we recover the Gaussian and the WF fixed points. In particular, for $\alpha = 1/25$ the Gaussian fixed point reappears for $a \lesssim 10^{-2}$, while for $\alpha = 1/9$ the WF fixed point reappears for $a \lesssim 0.35$ and the Gaussian one for $a \lesssim 4 \times 10^{-3}$. Also in this case the critical exponents of the Gaussian and WF fixed points approach the values found for vanishing a in Sec. III B.

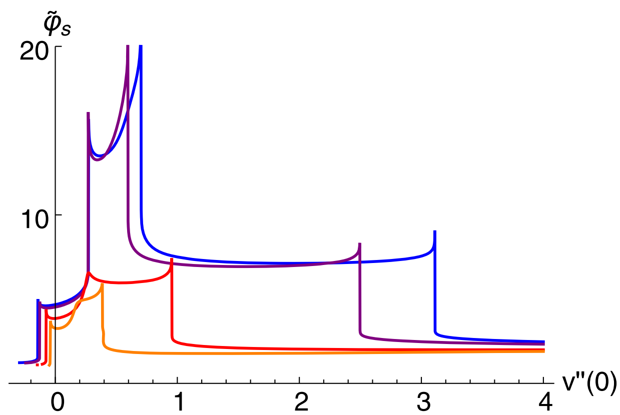

FIG. 5. Spike plots with $h = -2\tilde{\varphi}^2$ and with different values of a: $a = 0.5$ (blue curve), $a = 0.4$ (purple curve), $a = 0.16$ (red curve), and $a = 0.07$ (orange curve). For each curve, the rightmost spike is the deformed Gaussian fixed point, and the leftmost one is the Wilson-Fisher fixed point. Decreasing a further, both these fixed points move toward the origin. In the red and in the orange curves one and two of the fake fixed points have disappeared correspondingly.

b. **Second case:** $h = \alpha v''$.—We then move on to consider a regulator that depends on the second derivative of the effective potential, through a constant $\alpha > 0$. In this particular case shooting from the origin is not convenient for technical reasons; therefore we shoot from large field values.

This time $v = A\tilde{\varphi}^6$ for $\tilde{\varphi} > \tilde{\varphi}_c \equiv (30A\alpha)^{-1/4}$ provided $v'' > 1/\sqrt{\alpha}$. Below $\tilde{\varphi}_c$ the potential can be expanded in $\delta = \tilde{\varphi}_c - \tilde{\varphi}$ as follows:

$$v = \frac{A}{(30A\alpha)^{3/2}} - \frac{6A}{(30A\alpha)^{5/4}}\delta + \frac{1}{2\alpha^2}\delta^2 + F(\delta), \tag{5.11}$$

$$F = \delta^{16/5}\left(-\frac{25\sqrt{5}A^{1/10}\alpha^{-17/10}}{88\sqrt{23}^{3/10}a^{2/5}} + \frac{1255^{3/4}A^{-1/20}\alpha^{-53/20}}{5984\sqrt{423}^{17/20}a^{4/5}}\delta^{1/5} - \frac{71875A^{-1/5}\alpha^{-18/5}}{2464450563^{2/5}a^{6/5}}\delta^{2/5} + o(\delta^{3/5})\right). \tag{5.12}$$

Shooting on A and searching for values that correspond to a vanishing $v'(0)$ one can reveal several spurious fixed points at nonvanishing α and a. More and more of them are generated from the Gaussian fixed point for bigger and bigger values of α. We find that decreasing a at fixed $\alpha > 0$ reduces the number of spurious fixed points, and in the $a \to 0$ limit all of them disappear while the Gaussian and the WF fixed points merge. We verify that also in this case the critical exponents tend to the values obtained in Sec. III B for $a \to 0$.

A. Ward identity for the shift symmetry

Going beyond a single-field approximation, i.e., keeping both φ and ϕ_B as distinct, the LPA truncation becomes[14]

[14]The mixing term $\partial_\mu\phi_B\partial^\mu\phi$ is ruled out by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry on the arguments of the EAA.

$$\Gamma_k[\varphi, \phi_B] = \int d^dx \left(\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}(\partial_\mu \phi_B)^2 + V_k(\varphi, \phi_B)\right). \tag{5.13}$$

Using the regulator (5.7) the modified Ward identity (5.5) and the flow equation become

$$\partial_{\tilde\varphi} v - \partial_{\tilde\phi_B} v = c_d \frac{h'}{2} \frac{a(1-h)^{d/2}}{a(1-h) + \partial_{\tilde\varphi}^2 v}\, {}_2F_1\left(1, \frac{d}{2}, \frac{d}{2}+1; -\frac{(1-a)(1-h))}{a(1-h)+\partial_{\tilde\varphi}^2 v}\right), \tag{5.14}$$

$$\begin{aligned} &\partial_t v + dv - \frac{(d-2)}{2}(\tilde\varphi \partial_{\tilde\varphi} v + \tilde\phi_B \partial_{\tilde\phi_B} v) \\ &\quad = c_d \frac{a(1-h)^{d/2}}{a(1-h)+\partial_{\tilde\varphi}^2 v}\left(1 - h - \frac{1}{2}\partial_t h + \frac{1}{4}(d-2)\tilde\varphi h'\right) {}_2F_1\left(1, \frac{d}{2}, \frac{d}{2}+1; -\frac{(1-a)(1-h))}{a(1-h)+\partial_{\tilde\varphi}^2 v}\right), \end{aligned} \tag{5.15}$$

where $c_d = ((4\pi)^{d/2}\Gamma(\frac{d}{2}+1))^{-1}$. We rescale all the quantities in the following way:

$$\tilde\varphi = a^{(d-2)/4}\hat\varphi, \tag{5.16}$$

$$\tilde\phi_B = a^{(d-2)/4}\hat\phi_B, \tag{5.17}$$

$$v(\tilde\varphi) = a^{d/2}\hat v(\hat\varphi) + a\frac{1}{(4\pi)^{d/2}(d-2)\Gamma(\frac{d}{2}+1)}, \tag{5.18}$$

$$h = a^\gamma \hat h. \tag{5.19}$$

This set of definitions agrees with the one in Eq. (III.24). Here γ depends on the choice of h: for example, $\gamma = 1$ for both $h = \alpha\tilde\phi_B^2$ and $h = \alpha v''$. For the sake of generality we shall keep γ free for the time being. Expanding for small a and assuming $2 < d < 4$, the Ward identity and the flow equation become

$$\partial_{\hat\varphi}\hat v - \partial_{\hat\phi_B}\hat v = \frac{a^{\gamma+1-d/2}}{d(4\pi)^{d/2}\Gamma(\frac{d}{2})}\hat h' + \cdots, \tag{5.20}$$

$$\begin{aligned} &\partial_t \hat v + d\hat v - \frac{(d-2)}{2}(\hat\varphi\partial_{\hat\varphi}\hat v + \hat\phi_B \partial_{\hat\phi_B}\hat v) \\ &\quad = -\frac{a^{\gamma+1-d/2}}{d(4\pi)^{d/2}\Gamma(\frac{d}{2})}\left(\partial_t \hat h + d\hat h - \frac{(d-2)}{2}\hat\phi_B \hat h'\right) \\ &\qquad + \frac{\Gamma(\frac{d}{2}-1)}{(4\pi)^{d/2}}(1+\partial_{\hat\varphi}^2\hat v)^{d/2-1} + \cdots, \end{aligned} \tag{5.21}$$

where the dots denote quantities that go to zero for $a \to 0$.

From the modified Ward identity we see that to have a well-defined vanishing regulator limit we must demand $\gamma \geq \frac{d}{2} - 1$. If $\gamma > \frac{d}{2} - 1$, $\partial_{\hat\varphi}\hat v = \partial_{\hat\phi_B}\hat v$: this implies that $\hat v(\hat\varphi, \hat\phi_B) = \hat v(\hat\varphi + \hat\phi_B)$ and so we recover the shift symmetry and the flow equation without background. If $\gamma = \frac{d}{2} - 1$, the modified Ward identity gives

$$\hat v(\hat\varphi, \hat\phi_B) = \hat v_s(\hat\varphi + \hat\phi_B) - \frac{1}{d(4\pi)^{d/2}\Gamma(\frac{d}{2})}\hat h(\hat\phi_B). \tag{5.22}$$

Inserting this result into the flow equation, we recover again the equation without background.

VI. DISCUSSION

We have discussed the effect of an overall suppression of the regulator with a constant factor a, and in particular, the limit $a \to 0$, that we called the limit of a vanishing regulator. Let us summarize the main results.

As is clearly seen already in the case of the quantum mechanical oscillator, decreasing a has the effect of accelerating the flow, in the sense that already a small decrease of k leads very fast to the effective action. Thereafter things remain nearly constant with k. However, the quantum-mechanical study shows that in general different results are obtained depending on whether the $a \to 0$ limit is performed on the solutions of the flow equations, or on the beta functions themselves (see Fig. 2). While the former way of taking the limit is rather straightforward, obtaining meaningful results from the latter process requires a suitable a-dependent rescaling of the couplings.

In the case of the Wilson-Fisher fixed point, we have first studied the first form of the vanishing regulator limit by analyzing the a dependence of the fixed-point solution. Decreasing a has the effect of shifting the fixed points toward the Gaussian one, but the scaling exponents remain distinct even in the limit $a \to 0$. Here we have limited our analysis to the leading order of the derivative expansion.

In a polynomial approximation of the potential, the values of the scaling exponents become progressively worse as one increases the order of the polynomial. This is in agreement with the statement in Ref. [16] that the radius of convergence of the Taylor expansion of V is proportional to a. We have avoided this problem by also considering the functional treatment (LPA), but in this case one gets the exponent $\nu = 1$, which is worse than for any polynomial and coincides with the upper boundary conjectured in [16].

We have then analyzed the second form of the vanishing regulator limit, taking it on the LPA beta functional of scalar field theory, finding agreement with the first kind of limit as far as the critical exponents are concerned, although the locations of the fixed point differ. Even though some naive arguments suggest that the limit of a vanishing regulator might generally reproduce the results of a constant (momentum independent) masslike regulator, we have observed that in the LPA this is the case only when the constant regulator momentum integrals are convergent. As we adopted analytic continuation in the definition of the integrals, this excludes even integer values of $d \geq 4$ (the $d = 2$ case can, indeed, be reduced to the constant regulator case by a field-independent shift in the potential). As a consequence, the vanishing regulator limit remains different from the constant regulator in $d = 4$. We expect this conclusion to hold also in higher even dimensions, if analytic continuation is used, or in the whole range $d \geq 4$ without analytic continuation. It remains to be seen whether these conclusions are robust against enlargements of the truncation. For instance, at the second order of the derivative expansion, there might be a nontrivial interplay between the momentum derivatives of the regulator and the $a \to 0$ limit, resulting in further differences between the constant and the vanishing regulators.

For all these reasons, it will be quite interesting to systematically study the next order of the derivative expansion, including a field-dependent wave function renormalization $Z(\phi)$. In this paper, this level of approximation has been analyzed only for the two-dimensional nonlinear sigma model, as in this case it is the first nontrivial order of the derivative expansion. It is also known that in the case of quantum critical points the convergence of this expansion requires an increasingly accurate tuning of a. For the three-dimensional Wilson-Fisher fixed point, this tuning process is expected to converge to optimal values within the range $0.5 < a < 1$ [19]. Hence, it appears very unlikely that at the special point $a \to 0$ the derivative expansion might be convergent.

We should mention, however, that the amplitude a is only one of an infinite series of free parameters within the regulator R_k. In this work we have not allowed for such residual freedom, having fixed the regulator to a piecewise linear form. This choice has been justified as follows. In some circumstances, depending on the theory (or approximation) under study, as well as on the number of Euclidean dimensions d, the argument of the momentum integral might be nonintegrable in the $a \to 0$ limit. Nonetheless the integral might allow for a finite $a \to 0$ limit; i.e., the limit and the integral cannot be exchanged. Whenever this happens, one must first clearly define the momentum integrals by choosing a specific shape function and when applicable a unique analytic continuation, and then investigate the possible behavior of these integrals in the parametric $a \to 0$ limit. In all other cases, namely when the $a \to 0$ limit can be brought inside the momentum integrals, one can easily generalize the discussion to arbitrary shape functions r_1, as done in Sec. III C. Still, optimization criteria over the remaining parameters might be essential to obtain accurate results in the vanishing regulator limit. It might also be possible to take advantage of these additional parameters, with their associated free limiting behavior, to construct alternative flow equations resulting from the vanishing regulator limit. For instance, in the so-called LPA$'$ truncation, this kind of additional freedom allowed us to construct a one-parameter family of $\overline{\text{MS}}$ -like schemes within the FRG [14].

Indeed, as we explained in Sec. I the limit of a vanishing regulator shares several features with the more specific case of the $\overline{\text{MS}}$-like pseudoregulators discussed in Ref. [14]. In that reference, and in particular in Sec. VI, we observed that the best way of capturing the effect of quantum/statistical fluctuations beyond one loop is not adopting the derivative expansion, but rather accounting for the momentum dependence of vertices as in a vertex expansion. Because of their similarities, it is reasonable to expect that this behavior of $\overline{\text{MS}}$-like pseudoregulators against the choice of truncation scheme might be shared by the larger class of vanishing regulators.

In spite of the poor results of the $a \to 0$ limit of the LPA for the benchmark case of the Wilson-Fisher fixed point, we think that this limit may be useful in simple approximations, in problems where a symmetry is broken by the regulator. As a first example we have discussed the $O(N+1)$-nonlinear sigma model, in a formulation where the regulator breaks the global symmetry to $O(N)$. In this case we have shown that in the limit of a vanishing regulator the beta functions converge to those of the $O(N+1)$-symmetric theory.

We have then considered the shift symmetry arising in the background field treatment of a scalar theory. When this symmetry is broken by the regulator, this can either generate unphysical fixed points or, what is worse, destroy a physical fixed point. We have verified that the Ward identities of the shift symmetry are restored in the limit of a vanishing regulator, and that all the unphysical features of the flow disappear when a becomes sufficiently small.

It is important to stress the difference between this logic and the following one that is sometimes found in the FRG literature: the RG flow equations are solved first (and independently of the Ward identities) for a parametric family of regulators; then the latter parameters are tuned such that the violation of some finite-dimensional subset of the Ward identities is minimized. This procedure, when applied to the parameter a of Eq. (1.3), typically results in some nonvanishing value which is close to the value maximizing the rate of convergence of the chosen truncation scheme ($a \sim 1$). This approach has been studied, for instance, in the case of conformal symmetry [31]. In this reference the Ward identities for special conformal transformation, in either their quantum or their classical form (i.e., regulator dependent or independent, respectively), are not solved as functional constraints.[15]

[15]The truncated modified Ward identity is cast in the form $f(\tilde{\phi}) = 0$ for a certain function f. This equation is not fulfilled for any value of a. However, it is possible to tune a such that the function f is minimized in an almost $\tilde{\phi}$-independent sense.

By contrast, in the studies we presented in Secs. IV and V, the ansätze for the EAA included exact solutions of the classical Ward identities for $O(N+1)$ and shift symmetry, respectively, which are easy to solve independently from the RG equations. It is thus not surprising that the symmetry breaking induced by the RG flow is minimized for $a \to 0$. In fact, one might expect that the quantum Ward identities reduce to their classical counterparts when $a \to 0$. Thus, because of the different strategy followed in the choice of the initial ansatz for the EAA, the authors of Ref. [31] could only minimize the unavoidable symmetry breaking, whereas in this work we could tune it to zero by taking the limit of vanishing regulators.

It is interesting that a study similar to the one of Ref. [31] was performed in Ref. [32], where the symmetry expected to emerge at the RG fixed point is supersymmetry rather than conformal symmetry. In this latter work the ansatz for the EAA does indeed include an exact solution of the classical supersymmetric Ward identity. The minimization of the breaking of supersymmetry at the fixed point by means of the optimization of the regulator was also studied, but unfortunately the limit of a vanishing regulator was not within the parametric space considered in this reference. In fact, we expect the application of the vanishing regulator limit to supersymmetric models to be interesting and useful.

The main motivation of this work was the hope that vanishing regulators, or perhaps just "sufficiently small regulators," may be useful also in the application of the FRG to gauge theories and gravity, where the background field method is almost always adopted. Our results suggest that this may be possible, but that the usefulness of this idea may be restricted to the simplest truncations.

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APPENDIX A: FLOW EQUATIONS AT THE ORDER $O(\partial^2)$ OF THE DERIVATIVE EXPANSION

We introduce the following notations:

$$G_0 = (Z_k(\rho)q^2 + R_k(q^2) + U_k'(\rho))^{-1}, \tag{A1a}$$

$$G_1 = (\tilde{Z}_k(\rho)q^2 + R_k(q^2) + U_k'(\rho) + 2\rho U_k''(\rho))^{-1}, \tag{A1b}$$

for the Goldstone bosons and radial-mode propagators. The flow equations for U_k and $\tilde{Z}_k$, which is defined in Eq. (4.2), are

$$\partial_t U_k = \frac{(Q_{\frac{d}{2}}[G_1\partial_t R_k] + (N-1)Q_{\frac{d}{2}}[G_0\partial_t R_k])}{2(4\pi)^{d/2}}, \tag{A2}$$

$$\begin{aligned}
\partial_t \tilde{Z}_k = &-\frac{(\tilde{Z}_k{}' + 2\rho\tilde{Z}_k{}'')}{2(4\pi)^{d/2}} Q_{\frac{d}{2}}[G_1^2\partial_t R_k] - (N-1)\frac{(Z_k' + \rho Y_k')}{2(4\pi)^{d/2}} Q_{\frac{d}{2}}[G_0^2\partial_t R_k] \\
&+ \frac{2\rho(\tilde{Z}_k{}')^2}{(4\pi)^{d/2}}\left[\frac{2d+1}{2} Q_{\frac{d}{2}+1}[G_1^3\partial_t R_k] + \frac{(d+2)(d+4)}{4}(Q_{\frac{d}{2}+2}[G_1^2 G_1'\partial_t R_k] + Q_{\frac{d}{2}+3}[G_1^2 G_1''\partial_t R_k])\right] \\
&+ \frac{2\rho(3U_k'' + 2\rho U_k''')^2}{(4\pi)^{d/2}}(Q_{\frac{d}{2}}[G_1^2 G_1'\partial_t R_k] + Q_{\frac{d}{2}+1}[G_1^2 G_1''\partial_t R_k]) \\
&+ \frac{2\rho\tilde{Z}_k{}'(3U_k'' + 2\rho U_k''')}{(4\pi)^{d/2}}[(d+2)(Q_{\frac{d}{2}+1}[G_1^2 G_1'\partial_t R_k] + Q_{\frac{d}{2}+2}[G_1^2 G_1''\partial_t R_k]) + 2Q_{\frac{d}{2}}[G_1^3\partial_t R_k]] \\
&+ (N-1)\frac{\rho Y_k}{(4\pi)^{d/2}}(2U_k'' Q_{\frac{d}{2}}[G_0^3\partial_t R_k] + dZ_k' Q_{\frac{d}{2}+1}[G_0^3\partial_t R_k]) \\
&+ (N-1)\frac{2\rho(Z_k')^2}{(4\pi)^{d/2}}\left[\frac{(d+2)(d+4)}{4}(Q_{\frac{d}{2}+2}[G_0^2 G_0'\partial_t R_k] + Q_{\frac{d}{2}+3}[G_0^2 G_0''\partial_t R_k]) + \frac{1}{2} Q_{\frac{d}{2}+1}[G_0^3\partial_t R_k]\right] \\
&+ (N-1)\frac{2\rho(U_k'')^2}{(4\pi)^{d/2}}(Q_{\frac{d}{2}}[G_0^2 G_0'\partial_t R_k] + Q_{\frac{d}{2}+1}[G_0^2 G_0''\partial_t R_k]) \\
&+ (N-1)\frac{2\rho Z_k' U_k''}{(4\pi)^{d/2}}(d+2)(Q_{\frac{d}{2}+1}[G_0^2 G_0'\partial_t R_k] + Q_{\frac{d}{2}+2}[G_0^2 G_0''\partial_t R_k]).
\end{aligned} \tag{A3}$$

Using the regulator $R_k = aZ_k(k^2 - z)\theta(k^2 - z)$[16] we have

[16] We defined $Z_k = Z_k(\rho = 0)$.

$$Q_n[G^\ell \partial_t R_k] = \frac{1}{\Gamma(n)}((2-\eta_k)k^2 q_{n,\ell}(a,\omega,\zeta) + \eta_k q_{n+1,\ell}(a,\omega,\zeta)), \tag{A4}$$

$$Q_n[G^\ell G' \partial_t R_k] = -\frac{(\zeta - aZ_k)}{\Gamma(n)}((2-\eta_k)k^2 q_{n,\ell+2}(a,\omega,\zeta) + \eta_k q_{n+1,\ell+2}(a,\omega,\zeta)), \tag{A5}$$

$$Q_n[G^\ell G'' \partial_t R_k] = \frac{2(\zeta - aZ_k)^2}{\Gamma(n)}((2-\eta_k)k^2 q_{n,\ell+3}(a,\omega,\zeta) + \eta_k q_{n+1,\ell+3}(a,\omega,\zeta)) - \frac{a^2 Z_k^2 k^{2(n-\ell-2)}}{\Gamma(n)(\zeta+\omega/k^2)^{\ell+2}}, \tag{A6}$$

where

$$G = (\zeta z + R_k + \omega)^{-1},$$

$$q_{n,\ell}(a,\omega,\zeta) = \frac{1}{n}\frac{aZ_k k^{2(n-\ell)}}{(aZ_k+\omega/k^2)^\ell} \tag{A7}$$

$$\times {}_2F_1\left(\ell, n, n+1; \frac{aZ_k - \zeta}{aZ_k + \omega/k^2}\right), \tag{A8}$$

and G can be G_0 or G_1 depending on the choice of ζ and ω, in particular,

$$G = G_0 \quad \text{if} \begin{cases} \omega = U_k'(\rho), \\ \zeta = Z_k(\rho), \end{cases} \tag{A9}$$

$$G = G_1 \quad \text{if} \begin{cases} \omega = U_k'(\rho) + 2\rho U_k''(\rho), \\ \zeta = \tilde{Z}_k(\rho). \end{cases} \tag{A10}$$

For the constant regulator (2.11) one finds

$$Q_n[G^\ell \partial_t R_k] = k^{2(n-\ell+1)} Z_k (2-\eta_k)\frac{\Gamma(\ell-n)}{\Gamma(n)\Gamma(\ell)} \times \zeta^{-n}(Z_k + \omega/k^2)^{n-\ell}, \tag{A11}$$

$$Q_n[G^\ell G' \partial_t R_k] = -\zeta Q_n[G^{\ell+2}\partial_t R_k], \tag{A12}$$

$$Q_n[G^\ell G'' \partial_t R_k] = 2\zeta^2 Q_n[G^{\ell+3}\partial_t R_k]. \tag{A13}$$

APPENDIX B: MASTER EQUATION FOR THE NONLINEAR $O(N+1)$ MODEL

In Sec. IV the use of a vanishing regulator for the two-dimensional nonlinear $O(N+1)$ model has been discussed. We have observed that a nonvanishing potential term of the form (4.7) is not preserved by the flow equation in the $a \to 0$ limit. In this section we provide more details about the constraints on a general local potential $U_k(\rho, H)$. Here we show how the noncompatibility of the ansatz (4.7) with the flow equation is encoded in the modified master equation for the $O(N+1)/O(N)$ symmetry.

Our starting point is, indeed, the following modified master equation:

$$\frac{\delta\Gamma_k}{\delta\phi^a}\frac{\delta\Gamma_k}{\delta H} + H\phi_a = \mathrm{Tr}\left\{R_k(\Gamma_k^{(2)} + R_k)^{-1}_{ab}\frac{\delta^2\Gamma_k}{\delta H \delta\phi^b}\right\}. \tag{B1}$$

This identity, which differs from the standard master equation for a nonvanishing RHS, can be derived, for instance, from a functional integral representation of Γ_k, in the presence of a linear source term of the form (4.7) in the bare action, by performing a change of the integration variable corresponding to a $O(N+1)/O(N)$ infinitesimal transformation. It is straightforward to prove that this functional identity is compatible with the exact RG flow equation [33], meaning that it defines an RG-invariant hypersurface in theory space. However, truncations of the theory space often spoil this property, such that the truncated master equation becomes an additional requirement on the RG flow, to be enforced at every k.

Whenever the regularization preserves the (unmodified) nonlinear $O(N+1)/O(N)$ symmetry, the one-loop regulator-dependent term on the RHS of Eq. (B1) vanishes identically. The modified master equation then reduces to the standard master equation, which is a tree-level identity. In this case the equation is straightforward to solve, independently from and prior to the analysis of the RG flow equation. For an introduction to the role played by this identity in the construction of a renormalized perturbation theory in two dimensions see, for instance, Ref. [34]. Before analyzing in details the shape that this constraint takes for vanishing a, one can already apply its form of Eq. (B1) to the truncation we assumed in Sec. IV. There we took $U = 0$ and $H = 0$. This combination trivially fulfills the modified master equation. It should, however, be noted that Eq. (B1) represents the constraint of nonlinear $O(N+1)/O(N)$ symmetry only in the theory space of generic functionals of ϕ^a and H. If a nonvanishing H is never introduced in the effective action, i.e., on the subspace where $H = 0$, there nevertheless is a functional constraint encoding the nonlinear $O(N+1)/O(N)$ symmetry, and it can be obtained from Eq. (B1) by replacing derivatives involving H with the expectation values of the corresponding composite operators. The analysis of this kind of modified master equation is therefore highly nontrivial and will not be addressed in this work.

We then address the constraints that Eq. (B1) imposes on a truncation similar to the one in Eq. (4.4), but with an arbitrary nonvanishing $U_k(\rho, H)$.[17] For this truncation Eq. (B1) becomes

$$\partial_\rho U_k \partial_H U_k + H = \partial_\rho \partial_H U_k \frac{1}{4\pi} \int_0^\infty dz \frac{R_k(z)}{\tilde{Z}_k z + R_k(z) + \partial_\rho U_k + 2\rho \partial_\rho^2 U_k}. \tag{B2}$$

For the Litim regulator the loop integral is readily evaluated leading to

$$\begin{aligned} 4\pi \frac{\partial_\rho U_k \partial_H U_k + H}{\partial_\rho \partial_H U_k} &= -\frac{a k^2}{a - \frac{g_k^2}{Z_k} \tilde{Z}_k} \\ &- \frac{a}{(a - \frac{g_k^2}{Z_k}\tilde{Z}_k)^2} \frac{g_k^2}{Z_k} (\tilde{Z}_k k^2 + \partial_\rho U_k + 2\rho \partial_\rho^2 U_k) \\ &\times \log\left(\frac{\tilde{Z}_k k^2 + \partial_\rho U_k + 2\rho\partial_\rho^2 U_k}{a \frac{Z_k}{g_k^2} k^2 + \partial_\rho U_k + 2\rho \partial_\rho^2 U_k} \right). \end{aligned} \tag{B3}$$

The loop contribution to the modified master equation is a nonlinear function of derivatives of U_k up to second order. Therefore solving this equation for U_k is a difficult task. Even more so, as this solution must be required to also obey the RG flow equation. As the LPA projection breaks compatibility of the modified master equation with the RG flow equation, the latter is an independent nonlinear second order partial differential equation for U_k. This illustrates the difficulty of dealing with modified Ward identities in the FRG framework. For a discussion of these issues in the context of gauge theories, see, for instance, [35,36].

Can the limit $a \to 0$ be of any help in solving this complex problem? In addressing this question we need to specify the behavior of the functions U_k and $\tilde{Z}_k$ for $a \to 0$. For definiteness, we assume the scaling

$$\partial_\rho U \sim a, \qquad \tilde{Z}_k \sim a^0, \qquad \phi \sim a^0. \tag{B4}$$

Considering then Eq. (B3), it is natural to assume

$$H \sim a, \tag{B5}$$

which allows the linear source term to be interpreted as being part of the potential. However, inspection of the RHS of Eq. (B3) reveals that the leading behavior of the one-loop contribution is, in fact, $a \log a$. As a consequence we provide an ansatz encoded in the following definitions:

$$H = a\hat{H}, \tag{B6}$$

$$\begin{aligned} U_k(\rho, H) = a\hat{U}_k(\rho, \hat{H}) - a \log a F_0(\rho) \\ - a \log(-\log a) F_1(\rho). \end{aligned} \tag{B7}$$

Notice that we choose an ansatz with F_0 and F_1 independent of H. This might lead us to a particular solution of the modified master equation. The modified master equation then can be projected on three distinct equations, each showing a different small a asymptotic behavior. The $O(a \log(a))$, $O(a \log(-\log a))$, and $O(a)$ terms in this equation, respectively, lead to

$$F_0'(\rho) \hat{U}^{(0,1)}(\rho, H) = -\frac{Z_k k^2}{4\pi g_k^2} \frac{\hat{U}^{(1,1)}(\rho, H)}{\tilde{Z}_k(\rho)}, \tag{B8}$$

$$F_1'(\rho) \hat{U}^{(0,1)}(\rho, H) = -\frac{Z_k k^2}{4\pi g_k^2} \frac{\hat{U}^{(1,1)}(\rho, H)}{\tilde{Z}_k(\rho)}, \tag{B9}$$

$$\begin{aligned} H + \hat{U}^{(0,1)}(\rho, H) \hat{U}^{(1,0)}(\rho, H) \\ = -\frac{Z_k k^2}{4\pi g_k^2} \frac{\hat{U}^{(1,1)}(\rho, H)}{\tilde{Z}_k(\rho)} \left(\log\left(\frac{\tilde{Z}_k k^2}{F_0'(\rho) + 2\rho F_0''(\rho)} \right) - 1 \right). \end{aligned} \tag{B10}$$

It is evident how the $a \to 0$ limit does not relieve the nonlinearity of the modified master equation. While the first two equations can be straightforwardly solved for F_0 and F_1, once $U(\rho, H)$ is known, the third equation is highly nontrivial. In fact, Eq. (B8) can be replaced inside Eq. (B10) to obtain a second order nonlinear partial differential equation for $\hat{U}$. While the construction of the most general solution is a very complex task, which we expect in general to be possible only by numerical methods, a particular solution can be found by assuming the ansatz

$$\hat{U}(\rho, H) = \pm \hat{H} \sqrt{\frac{1}{Z_k} - 2\rho} + F_2(\rho). \tag{B11}$$

This leads to a first order ordinary differential equation for F_2, which can easily be solved. The determination of the corresponding F_0 and F_1 results in the following particular solution:

$$F_{0,1}(\rho) = c_{0,1} + \frac{Z_k k^2 \rho}{4\pi}, \tag{B12}$$

[17] This general ansatz can be made compatible with the assumed linear H dependence of the bare action, by requiring the linearity of the potential at the UV cutoff scale $k = \Lambda$.

$$F_2(\rho) = c_2 - \frac{Z_k k^2 \rho}{4\pi} \log\left(\frac{g^2}{4\pi}\right) + \frac{k^2}{8\pi}(1 - 2Z_k\rho)\log(1 - 2Z_k\rho), \tag{B13}$$

where $c_{0,1,2}$ are integration constants that can depend on k. Having an analytic formula for a particular solution of the master equation is, of course, a nice result, which is possible only thanks to the simplifications brought by the $a \to 0$ limit. However, in itself this result is of limited use, for two main reasons. First, in general there is no reason to expect that this anstaz be closed under the RG flow. Given the compatibility of the flow equation with the master equation, any particular solution is free to flow into the most general solution during an infinitesimal RG step. Second, in the LPA case even this compatibility is lost. As a consequence, the solution of Eqs. (B7), (B12), and (B13) will flow into a potential that does not fulfill the modified master equation. Therefore, this solution would be useful only if accompanied by a prescription for projecting the latter potential back onto a functional of the same form as the particular solution itself. We do not explore possible prescriptions for this projection in this work.

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