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Spin Hall insulators beyond the helical Luttinger model

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We consider the interacting, spin-conserving, extended Kane-Mele-Hubbard model, and we rigorously establish the exact quantization of the edge spin conductance and the validity of the helical Luttinger liquid relations for Drude weights and susceptibilities. Our analysis takes fully into account lattice effects, typically neglected in the helical Luttinger model approximation, which play an essential role for universality. The analysis is based on exact renormalization-group methods and on a combination of lattice and emergent Ward identities, which enable the emergent chiral anomaly to be related with the finite renormalizations due to lattice corrections.

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I. INTRODUCTION

The remarkable edge transport properties of quantum spin Hall insulators (QSHI), predicted in [1–5] (see [6–8] for reviews), have been explained so far via topological arguments or effective quantum field theory (QFT) descriptions. In the absence of many-body interactions, and if the spin is conserved, topological arguments ensure the quantization of the spin Hall conductance. Many-body interactions, however, break the single-particle picture, and prevent the use of such methods. Nevertheless, experiments have shown values of spin conductances that are approximately quantized [9–13]. It is a challenge for theorists to understand a mechanism for universality, or to predict possible deviations from the quantized value.

Due to the reduced dimensionality and to the massless dispersion relation, the edge states form a strongly correlated system. To analytically understand its behavior, the *helical Luttinger* (HL) *model* [4], a QFT for relativistic one-dimensional fermions with locked spin and chirality, has been proposed as an effective field-theoretic description. This model can be studied via bosonization, see e.g., [14,15]; as a result, it exhibits anomalous decay of correlations, and the *chiral anomaly*. Also, nonuniversal anomalous exponents, velocities, and transport coefficients are related by *exact scaling relations*. Several generalizations of the HL model have been considered, see [16–27]. However, these effective QFT descriptions are insufficient to conclude whether many-body interactions break or not the quantization of the spin conductance, since they neglect important lattice effects; it is well known that nonlinear corrections to the dispersion relation and umklapp terms might produce finite corrections to the transport coefficients, as, for instance, in graphene [28,29].

In this paper, we establish the exact quantization of the edge spin conductance of a truly interacting lattice QSHI, going beyond the effective QFT description. Moreover, we establish the validity of the HL scaling relations by taking fully into account lattice effects and the nonlinearity of the energy bands. We use recently developed nonperturbative RG methods, introduced to prove rigorous universality results for nonsolvable statistical mechanics models [30].

II. THE KMH MODEL

A. The model

A basic model for interacting, time-reversal-invariant topological insulators is provided by the extended, spin-conserving *Kane-Mele-Hubbard (KMH)* model. The KMH model is a time-reversal-symmetric system, describing spinful fermions on the honeycomb lattice. The honeycomb lattice Λ can be represented as the superposition of two triangular sublattices Λ_A,Λ_B of side L, $\Lambda = \Lambda_A + \Lambda_B$. We denote by $\vec{\ell}_1$, $\vec{\ell}_2$ the normalized basis vectors of Λ_A, and we set $\Lambda_B = \Lambda_A + (1,0)$. We shall denote by x_1,x_2 the coordinates of the point $\vec{x} \in \Lambda_A$ in the $\vec{\ell}_1,\vec{\ell}_2$ basis. We introduce fermionic creation/annihilation operators $a^{\pm}_{\vec{x},\sigma}$ and $b^{\pm}_{\vec{y},\sigma}$, with spin labels $\sigma = \pm$, acting on the two triangular sublattices Λ_A and Λ_B. In the absence of interactions, the Hamiltonian is

$$\begin{aligned}\mathcal{H}_0 = -t_1 \sum_{\vec{x},j,\sigma} \big[a^+_{\vec{x},\sigma} b^-_{\vec{x}+\vec{\delta}_j,\sigma} + b^+_{\vec{x}+\vec{\delta}_j,\sigma} a^-_{\vec{x},\sigma}\big] \\ -it_2 \left[\sum_{\substack{\langle\langle \vec{x},\vec{y}\rangle\rangle \\ \sigma}} a^+_{\vec{x},\sigma}(\vec{\sigma}\,\vec{\nu}_{\vec{x},\vec{y}}) a^-_{\vec{y},\sigma} + \sum_{\substack{\langle\langle \vec{x},\vec{y}\rangle\rangle \\ \sigma}} b^+_{\vec{x},\sigma}(\vec{\sigma}\,\vec{\nu}_{\vec{x},\vec{y}}) b^-_{\vec{y},\sigma}\right] \\ -W \sum_{\vec{x},\sigma} \big[a^+_{\vec{x},\sigma} a^-_{\vec{x},\sigma} - b^+_{\vec{x}+\delta_1,\sigma} b^-_{\vec{x}+\delta_1}\big] - \mu\mathcal{N}, \qquad (1)\end{aligned}$$

where in the first sum $\vec{x} \in \Lambda_A$, and $j = 1,2,3$ labels one of its three nearest neighbors in Λ_B, connected by the vectors $\vec{\delta}_j$; see Fig. 1. The second and third sums run over next-to-nearest neighbors on the A,B sublattices, connected by the vectors $\pm\vec{\gamma}_j$, $j = 1,2,3$; we denote by $\vec{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ the vector of the Pauli matrices, and we set

$$\vec{\nu}_{\vec{x},\vec{y}} = (\vec{d}_{\vec{x},\vec{z}} \times \vec{d}_{\vec{z},\vec{y}})/|\vec{d}_{\vec{x},\vec{z}} \times \vec{d}_{\vec{z},\vec{y}}|, \qquad (2)$$

where $\vec{z}$ is the intermediate site between $\vec{x}$ and $\vec{y}$, and $\vec{d}_{\vec{x},\vec{y}} = \vec{x} - \vec{y}$. The third term includes a staggered potential

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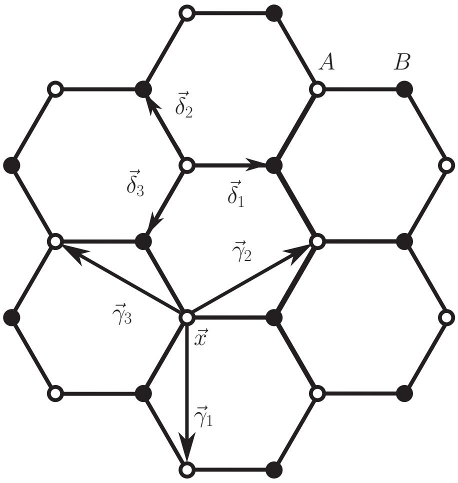

FIG. 1. The honeycomb lattice Λ: the empty dots belong to the A sublattice, while the black dots belong to the B sublattice.

$\pm W$ on the A,B sublattices, and the last term fixes the chemical potential μ ($\mathcal N$ is the number operator). The Hamiltonian of the model is the sum of two copies of the *Haldane model* [31]: $\mathcal H_0=\sum_\sigma \mathcal H_0^\sigma$, where $\mathcal H_0^\sigma$ acts on the σ-spin subsector. The connection between the different spin sectors is $\mathcal H_0^+ = C\mathcal H^- C$, with C the complex conjugation operator, which ensures the invariance under time-reversal symmetry of the full Hamiltonian.

To reduce the honeycomb lattice to a Bravais lattice, we collect the fermionic operators associated with the sites $\vec x,\vec x+\vec\delta_1$ in a single, two-component fermionic operator (see Fig. 1): $\phi^+_{\vec x,\sigma}=(a^+_{\vec x,\sigma},b^+_{\vec x+\vec\delta_1,\sigma})\equiv(\phi^+_{\vec x,A,\sigma},\phi^+_{\vec x,B,\sigma})$. With these notations, we rewrite the noninteracting Hamiltonian as

$$\mathcal H_0=\sum_{\vec x,\vec y}\sum_{\rho,\rho',\sigma}\phi^+_{\vec x,\rho,\sigma}H^\sigma_{\rho\rho'}(\vec x,\vec y)\phi^-_{\vec y,\rho',\sigma}, \tag{3}$$

with H^σ a one-particle Schrödinger operator, acting on $\Lambda_A\times\mathbb C^2$. Let us now define the density operator as $\rho_{\vec x,\sigma}=\sum_{\rho=A,B}\rho_{\vec x,\rho,\sigma}$, with $\rho_{\vec x,\rho,\sigma}=\phi^+_{\vec x,\rho,\sigma}\phi^-_{\vec x,\rho,\sigma}$. The *interacting* Hamiltonian is

$$\begin{aligned}\mathcal H&=\mathcal H_0+\lambda\mathcal V,\\ \mathcal V&=\sum_{\vec x,\vec y}\sum_{\rho,\rho'}\left[\rho_{\vec x,\rho,\sigma}-\frac12\right]\left[\rho_{\vec y,\rho',\sigma'}-\frac12\right]v_{\rho\rho'}(\vec x,\vec y)\end{aligned} \tag{4}$$

for $v_{\rho\rho'}(\vec x,\vec y)$ short-ranged, and where λ is the coupling constant.

B. Lattice currents and conservation laws

Let $A(t)=e^{i\mathcal Ht}Ae^{-i\mathcal Ht}$ be the time evolution of A. The density operator satisfies the following lattice continuity equation:

$$\begin{aligned}\partial_t\rho_{\vec x,\sigma}(t)&=i[\mathcal H,\rho_{\vec x,\sigma}(t)]=\sum_{\vec y}\sum_{\rho,\rho'}j^{\rho\rho';\sigma}_{\vec x,\vec y}(t),\\ j^{\rho\rho';\sigma}_{\vec x,\vec y}&=i\phi^+_{\vec y,\rho}H^\sigma_{\rho\rho'}(\vec y,\vec x)\phi^-_{\vec x,\rho'}+\text{H.c.}\end{aligned} \tag{5}$$

The operator $j^{\rho\rho';\sigma}_{\vec x,\vec y}$ is the *bond current operator*, corresponding to the pairs of honeycomb lattice sites labeled by $(\vec x,\rho;\vec y,\rho')$. Notice that, by the finite range of the hopping Hamiltonian, the only nonvanishing bond currents are those connecting $(\vec x,\vec x\pm\vec\ell_i)$, with $i=1,2$, and $(\vec x,\vec x\pm\vec\gamma_1)$, with $\vec\gamma_1=\vec\ell_1-\vec\ell_2$.

Let $j^\sigma_{\vec x,\vec y}=\sum_{\rho,\rho'}j^{\rho\rho';\sigma}_{\vec x,\vec y}$. Let us define the discrete lattice derivative as $d_i f(\vec x)=f(\vec x)-f(\vec x-\vec\ell_i)$. Then, the continuity equation can be rewritten as

$$\begin{aligned}\partial_t\rho_{\vec x,\sigma}(t)&=-d_1 j_{\vec x,\vec x+\vec\ell_1}-d_2 j_{\vec x,\vec x+\vec\ell_2}-j_{\vec x,\vec x+\vec\ell_1-\vec\ell_2}-j_{-\vec\ell_1+\vec\ell_2+\vec x,\vec x}\\ &\equiv -d_1 j_{1,\vec x}-d_2 j_{2,\vec x},\end{aligned} \tag{6}$$

where we defined $j^\sigma_{1,\vec x}=j^\sigma_{\vec x,\vec x+\vec\ell_1}+j^\sigma_{\vec x,\vec x+\vec\ell_1-\vec\ell_2}$ and $j^\sigma_{2,\vec x}=j^\sigma_{\vec x,\vec x+\vec\ell_2}+j^\sigma_{\vec x,\vec x-\vec\ell_1+\vec\ell_2}$. We shall collect densities and currents in a single 3-current $j^\sigma_{\mu,\vec x}$, $\mu=0,1,2$. Also, we define the charge and spin 3-currents as $j^{\rm c}_{\mu,\vec x}=\sum_\sigma j^\sigma_{\mu,\vec x}$, $j^{\rm s}_{\mu,\vec x}=\sum_\sigma \sigma j^\sigma_{\mu,\vec x}$, which satisfy $\partial_0 j^\sharp_{0,\vec x}+\sum_i d_i j^\sharp_{i,\vec x}=0$, with $\sharp=\text{c,s}$.

We shall study the thermodynamic properties of the model in the grand-canonical ensemble. The Gibbs state of the model is $\langle\,\rangle_{\beta,L}=(1/\mathcal Z_{\beta,L})\text{Tr}\,e^{-\beta\mathcal H}$, with $\mathcal Z_{\beta,L}=\text{Tr}\,e^{-\beta\mathcal H}$ the partition function. We introduce the imaginary-time (or Euclidean) evolution of the fermionic operators as $\phi^\pm_{\mathbf x,\rho,\sigma}:=e^{x_0\mathcal H}\phi^\pm_{\vec x,\rho,\sigma}e^{-x_0\mathcal H}$, $\mathbf x=(x_0,\vec x)$, with $x_0\in[0,\beta)$, extended antiperiodically for all $x_0\in\mathbb R$.

A crucial ingredient in our analysis will be the use of *Ward identities*, implied by the charge and spin conservation laws. Let $d_0\equiv i\partial_{x_0}$. The lattice continuity equation can be rewritten in a compact form as $\sum_\mu d_\mu j^\sigma_{\mu,\mathbf x}=0$. This relation can be used to derive identities among correlations, such as

$$\sum_\mu d_{x_\mu}\big\langle\mathbf T j^\sigma_{\mu,\mathbf x}\,;j^\sigma_{\nu,\mathbf y}\big\rangle_{\beta,L}=i\delta(x_0-y_0)\big\langle\big[j^\sigma_{0,\vec x},j^\sigma_{\nu,\vec y}\big]\big\rangle_{\beta,L}. \tag{7}$$

In Eq. (7), $\mathbf T$ is the time-ordering operator, and the contact term on the right-hand side is called the *Schwinger term*. Equation (7) is the Ward identity for the current-current correlation functions. In the same way, one can also derive a Ward identity relating the vertex functions of the lattice model to the two-point correlation function:

$$\begin{aligned}\sum_\mu d_{z_\mu}\langle\mathbf T j^\sharp_{\mu,\mathbf z}\,;\phi^-_{\mathbf y,\sigma,\rho'}\phi^+_{\mathbf x,\sigma,\rho}\rangle_{\beta,L}&=i\sigma_\sharp[\langle\mathbf T\phi^-_{\mathbf y,\sigma,\rho'}\phi^+_{\mathbf x,\sigma,\rho}\rangle_{\beta,L}\delta_{\mathbf x,\mathbf z}\\ &\quad-\langle\mathbf T\phi^-_{\mathbf y,\sigma,\rho'}\phi^+_{\mathbf x,\rho}\rangle_{\beta,L}\delta_{\mathbf y,\mathbf z}],\end{aligned} \tag{8}$$

where $\delta_{\mathbf x,\mathbf z}=\delta(x_0-y_0)\delta_{\vec x,\vec y}$ and $\sigma_c=+$, $\sigma_s=\sigma$.

III. NONINTERACTING TOPOLOGICAL INSULATORS

In the absence of interactions, $\lambda=0$, the Hamiltonian reduces to the sum of two noninteracting Haldane Hamiltonians, $\mathcal H_0=\sum_{\sigma=\pm}\mathcal H_0^\sigma$. Suppose the model is equipped with periodic boundary conditions. Then [using the fact that the single-particle Hamiltonian is translation-invariant, $H^\sigma(z,z')\equiv H^\sigma(z-z')$], we can introduce the *Bloch Hamiltonian* as $\widehat H^\sigma(\vec k)=\sum_z e^{-i\vec z\cdot\vec k}H^\sigma(z)$ for $\vec k$ in the Brillouin zone $\mathcal B$. We have

$$\widehat H^\sigma(\vec k)=\begin{pmatrix} m_\sigma(k) & -t_1\Omega^*(k)\\ -t_1\Omega(k) & -m_\sigma(k)\end{pmatrix}, \tag{9}$$

where $m_\sigma(\vec{k}) = W - 2\sigma t_2 \alpha(\vec{k})$, $\alpha(\vec{k}) = \sum_{i=1}^{3} \sin \vec{k} \cdot \vec{\gamma}_i$, and $\Omega(\vec{k}) = 1 + e^{-i\vec{k}\cdot\vec{\ell}_1} + e^{-i\vec{k}\cdot\vec{\ell}_2}$. The corresponding energy bands are

$$E^\sigma_\pm(\vec{k}) = \pm\sqrt{m_\sigma(\vec{k})^2 + t^2|\Omega(\vec{k})|^2}.$$

To make sure that the energy bands do not overlap, we assume that $t_2/t_1 < 1/3$. The two bands can only touch at the *Fermi points* $\vec{k}^\pm_F = (\frac{2\pi}{3}, \pm\frac{2\pi}{3\sqrt{3}})$, which are the two zeros of $\Omega(\vec{k})$, around which $\Omega(\vec{k}^\pm_F + \vec{k}') \simeq \frac{3}{2}(ik'_1 \pm k'_2)$. The condition that the two bands touch at $\vec{k}^\omega_F$, with $\omega = +,-$, is that $m^\sigma_\omega = 0$, with

$$m^\sigma_\omega \equiv m_\sigma\big(\vec{k}^\omega_F\big) = W + \omega\sigma 3\sqrt{3}\, t_2.$$

Therefore, the unperturbed critical points are given by the values of W such that $W = \pm 3\sqrt{3}\, t_2$. Choosing the chemical potential $\mu = 0$, which lies halfway between the two energy bands, the condition $W \neq \pm 3\sqrt{3}\, t_2$ corresponds to the insulating phase for which the correlations decay exponentially fast. In the insulating phase, the system may or may not be in a topologically nontrivial phase, depending on the value of the *Hall conductivity*. This quantity is defined starting from the Kubo formula, which we use directly in its imaginary-time version (see [32] for a discussion of the Wick rotation):

$$\sigma^\sigma_{12} = \lim_{p_0\to 0}\lim_{\vec{p}\to 0}\lim_{\beta,L\to\infty}\frac{1}{p_0}\int_0^\beta dx_0 \sum_x (1 - e^{-i\mathbf{p}\cdot\mathbf{x}})\big\langle j^\sigma_{1,\mathbf{x}}\,;\, j^\sigma_{2,\mathbf{0}}\big\rangle_{\beta,L}. \tag{10}$$

In the absence of interactions, the Hall conductivity of the Haldane model can be computed explicitly. One finds

$$\sigma^\sigma_{12} = \frac{\nu^\sigma}{2\pi}, \qquad \nu^\sigma = \mathrm{sgn}(m^\sigma_-) - \mathrm{sgn}(m^\sigma_+). \tag{11}$$

Concerning the Kane-Mele model, its net Hall conductivity $\sigma^{\mathrm{c}}_{12} = \sigma^+_{12} + \sigma^-_{12}$ vanishes while the net *spin* conductivity $\sigma^{\mathrm{s}}_{12} = \sigma^+_{12} - \sigma^-_{12}$ is nonzero:

$$\sigma^{\mathrm{s}}_{12} = \sigma^+_{12} - \sigma^-_{12} = \frac{\nu^+}{\pi}. \tag{12}$$

This is the *quantum spin Hall effect*. In the spin-symmetric case, the quantization of σ^{s}_{12} follows from the quantization of σ^σ_{12}, which is ensured for topological reasons. In the absence of spin symmetry, for instance in the presence of Rashba couplings, one does not expect the spin conductivity to be quantized. Nevertheless, topology survives in the sense that the Hamiltonians are classified by a suitable $\mathbb{Z}_2$ invariant [1,2].

A remarkable feature of topological insulators is the presence of *gapless edge modes*. Suppose now the system is equipped with *cylindric boundary conditions*, say periodic in the $\vec{\ell}_1$ direction and Dirichlet in the $\vec{\ell}_2$ direction, on the boundaries at $x_2 = 0$, $x_2 = L$. By translation invariance in the $\vec{\ell}_1$ direction, we can introduce a partial Bloch transformation of the initial Hamiltonian, $\widehat{H}_{\rho\rho'}(k_1;x_2,y_2) = \sum_{z_1} e^{-iz_1k_1} H_{\rho\rho'}(z_1;x_2,y_2)$, with $k_1 \in S^1$. By construction, the Hamiltonian is symmetric under the action of the time-reversal operator, $T^*\widehat{H}(k_1)T \equiv T^*[\widehat{H}^+(k_1) + \widehat{H}^-(k_1)]T = \widehat{H}^-(-k_1) + \widehat{H}^+(-k_1) \equiv \widehat{H}(-k_1)$ since $\widehat{H}^\sigma(k_1) = \widehat{H}^{-\sigma}(-k_1)$. *Edge states* correspond to solutions of the Schrödinger equation $\widehat{H}(k_1)\xi(k_1) = \varepsilon(k_1)\xi(k_1)$ at the Fermi level μ, which are exponentially localized around one of the two edges:

$$\big|\xi_{x_2}(k_1)\big| \leqslant Ce^{-c|x_2|}, \qquad \big|\xi_{x_2}(k_1)\big| \leqslant Ce^{-c|L-x_2|}. \tag{13}$$

These 1D eigenfunctions of $\widehat{H}(k_1)$ correspond to 2D eigenfunctions for the Hamiltonian H, of the form $e^{-ik_1x_1}\xi_{x_2}(k_1)$; they are responsible for the transport of dissipationless *edge currents*. In the Haldane model, the edge eigenfunctions can be found explicitly [33]: each cylinder edge supports either a zero- or one-edge mode. Consequently, the Kane-Mele Hamiltonian $H = \sum_{\sigma=\pm} H^\sigma$ supports either zero- or two-edge states per edge. Let $\varepsilon_+,\varepsilon_-$ be their dispersion relations. By time-reversal symmetry, $\varepsilon_+(k_1) = \varepsilon_-(-k_1)$: the model displays two Fermi points $k^\pm_F$, $k^+_F = -k^-_F$, such that

$$\varepsilon_+(k^+_F) = \varepsilon_-(k^-_F) = \mu.$$

Time-reversal symmetry implies that the edge modes are counterpropagating: $v_+ = \partial_{k_1}\varepsilon_+(k^+_F) = -v_-$.

The edge transport of the system can be investigated by probing the variation of the density or of the current (of charge or of spin) after introducing an external perturbation supported in a strip of width a from the $x_2 = 0$ edge. We shall study these transport phenomena in the linear-response regime. To define the edge transport coefficients, let us introduce the following notations. Given a local operator $O_{\vec{x}}$, we define its partial space-time Fourier transform as $\widehat{O}_{\underline{p},x_2} = \int_0^\beta dx_0 \sum_{x_1} e^{-i\underline{p}\cdot\underline{x}} O_{\mathbf{x}}$, with $\underline{p} = (p_0,p_1)$, p_0 the Matsubara frequency, and $\underline{x} = (x_0,x_1)$. Let $\langle\,\cdot\,\rangle_\infty = \lim_{\beta,L\to\infty}(\beta L)^{-1}\langle\cdot\rangle_{\beta,L}$. We define, for $\sharp,\sharp' = \mathrm{c},\mathrm{s}$,

$$\begin{aligned}
G^{\sharp;a}_{\rho,\rho}(\underline{p}) &= \sum_{x_2=0}^{a}\sum_{y_2=0}^{\infty}\big\langle \mathbf{T}\rho^\sharp_{\underline{p},x_2}\,;\,\rho^{\sharp'}_{\underline{p},y_2}\big\rangle_\infty,\\
G^{\sharp;a}_{\rho,j}(\underline{p}) &= \sum_{x_2=0}^{a}\sum_{y_2=0}^{\infty}\big\langle \mathbf{T}\rho^\sharp_{\underline{p},x_2}\,;\,j^{\sharp'}_{1,-\underline{p},y_2}\big\rangle_\infty,\\
G^{\sharp;a}_{j,j}(\underline{p}) &= \sum_{x_2=0}^{a}\Bigg[\sum_{y_2=0}^{\infty}\big\langle \mathbf{T}j^\sharp_{1,\underline{p},x_2}\,;\,j^{\sharp'}_{1,-\underline{p},y_2}\big\rangle_\infty - i\,\Delta(x_2)\Bigg],
\end{aligned} \tag{14}$$

where $\Delta(x_2) = \lim_{\beta,L\to\infty}\langle\sum_\sigma[t^\sigma_{\vec{x},\vec{x}+\vec{\ell}_1} + t^\sigma_{\vec{x},\vec{x}+\vec{\ell}_1-\vec{\ell}_2}]\rangle_{\beta,L}$, with $t^\sigma_{\vec{x},\vec{y}} = \sum_{\rho\rho'} -i\phi^+_{\vec{y},\rho}H^\sigma_{\rho\rho'}(\vec{y},\vec{x})\phi^-_{\vec{x},\rho'}$ − H.c. As we shall see later, this function is related to the Schwinger term in Eq. (7). The *edge spin conductance* is

$$\sigma^{\mathrm{s}} = \lim_{a\to\infty}\lim_{p_0\to 0^+}\lim_{p_1\to 0} G^{\mathrm{c,s};a}_{\rho,j}(\underline{p}). \tag{15}$$

It measures the variation of the spin current after introducing a shift of the chemical potential supported in a region of width a from the $x_2 = 0$ edge. Similarly, the *edge charge conductance* is

$$\sigma^{\mathrm{c}} = \lim_{a\to\infty}\lim_{p_0\to 0^+}\lim_{p_1\to 0} G^{\mathrm{c,c};a}_{\rho,j}(\underline{p}). \tag{16}$$

Instead, the edge susceptibilities and Drude weights, of charge or of spin, are

$$\begin{aligned}
\kappa^\sharp &= \lim_{a\to\infty}\lim_{p_1\to 0}\lim_{p_0\to 0^+} G^{\sharp,\sharp;a}_{\rho,\rho}(\underline{p}),\\
D^\sharp &= -\lim_{a\to\infty}\lim_{p_0\to 0^+}\lim_{p_1\to 0} G^{\sharp,\sharp;a}_{j,j}(\underline{p}).
\end{aligned} \tag{17}$$

As we shall see, due to the lack of continuity at $\underline{p} = (0,0)$ of the expressions in (14), the order of the limits in the above definitions is crucial. It turns out that, in the absence of interactions, the edge transport coefficients can be computed. One has

$$\sigma^{\rm c} = 0, \qquad \sigma^{\rm s} = \sigma^{\rm s}_{12},$$
$$\kappa^{\sharp} = \frac{1}{\pi |v_+|}, \qquad D^{\sharp} = \frac{|v_+|}{\pi}. \tag{18}$$

The equivalence of the edge spin conductance with the bulk spin conductivity is a manifestation of the *bulk-edge correspondence*: namely, a duality between the presence of edge modes at the Fermi level with the value of the topologically invariant classifying bulk Hamiltonians (acting on infinite lattices, with no edges). For the IQHE [34–37], this duality implies that the sum of the chiralities of the edge states $\sum_e \omega_e$, with $\omega_e = \mathrm{sgn}[\partial_{k_1}\varepsilon_e(k_F^e)]$, equals the Chern number of the Bloch bundle, which fixes the value of the Hall conductivity. For time-reversal-invariant systems, instead, $\frac{1}{2}\sum_e |\omega_e|$ mod 2 turns out to be equal to the bulk $\mathbb{Z}_2$ invariant [38–40]; in particular, for the spin-conserving Kane-Mele model, this implies that the edge spin conductance equals the bulk spin conductivity. The bulk-edge correspondence has been rigorously established for single-particle Hamiltonians: there is no general argument ensuring its validity for interacting many-body systems. Finally, notice that in contrast to $\sigma^{\rm s}$, the edge susceptibility $\kappa^{\sharp}$ and the Drude weight $D^{\sharp}$ are nonuniversal quantities, depending on the velocity of the edge modes.

The goal of this paper is to understand the effect of many-body interactions of the edge transport coefficients: the natural question we address here is whether some form of universality persists, and in particular if the quantization of $\sigma^{\rm s}$ holds true.

IV. MAIN RESULT

Here we shall consider the edge transport properties of the Kane-Mele-Hubbard model, $\lambda \neq 0$. Our main result is the following theorem.

Theorem. Consider the KMH Hamiltonian (4) with cylindric boundary conditions. Let us choose the chemical potential μ in the gap of the bulk Hamiltonian. Suppose that the single-particle KM Hamiltonian supports a pair of edge modes, $\varepsilon_+(k_F^+) = \varepsilon_-(k_F^-) = \mu$, and that $v_+ \neq 0$. Then, there exists $\lambda_0 > 0$ such that, for $|\lambda| < \lambda_0$, the following is true. Let $\omega = \mathrm{sgn}(v_+)$. The edge spin conductance is universal:

$$\sigma^s = -\frac{\omega}{\pi}. \tag{19}$$

Moreover, the Drude weights and the susceptibilities satisfy the helical Luttinger liquid relations:

$$\kappa^c = \frac{K}{\pi v}, \quad D^c = \frac{vK}{\pi}, \quad \kappa^s = \frac{1}{\pi v K}, \quad D^s = \frac{v}{\pi K} \tag{20}$$

with $K = 1 + O(\lambda) \neq 1$, $v = v_+ + O(\lambda) \neq v_+$. Finally, the two-point function decays with an anomalous exponent, $\eta = (K + K^{-1} - 2)/2$.

As a corollary, our result combined with the universality of bulk transport, following from the analysis of [32,41], provides a rigorous example of bulk-edge correspondence for an interacting time-reversal-invariant topological insulator (see [42] for the analogous result for Hall systems). The lack of many-body corrections to the conductance is in agreement with experimental results [9,11]. Notice that, in contrast with the conductance, the susceptibilities and the Drude weights are interaction-dependent: nevertheless, if combined with the dressed Fermi velocity v, they verify a marginal form of universality, in the sense of the validity of the helical Luttinger liquid relation:

$$\frac{\kappa^{\sharp} v^2}{D^{\sharp}} = 1. \tag{21}$$

Moreover, the HL parameter K allows us to determine the anomalous exponent of the two-point function via the formula $\eta = (K + K^{-1} - 2)/2$.

The rest of the paper is organized as follows. In Sec. V we introduce a Grassmann integral representation for the transport coefficients. We then integrate out the "bulk degrees of freedom" corresponding to the energy modes far from the Fermi level. As a result, we end up with an effective one-dimensional model, which is reminiscent of the helical Luttinger model up to some crucial differences: the fermionic fields are defined on a lattice, the interaction involves arbitrarily high monomials in the fields, the energy-dispersion relation is nonlinear, and the umklapp scattering process is present. Then, in Sec. VI we study this lattice QFT via exact RG, which allows us to represent the transport coefficients in terms of renormalized, convergent series. Such expansions can be reorganized by isolating the contributions corresponding to an emergent, effective chiral QFT theory with suitably fine-tuned bare parameters, from a remainder term, that depends on all lattice details. The advantage of this rewriting is that the current-current correlation functions of the emergent QFT can be computed exactly (see Sec. VII) thanks to the validity of extra chiral Ward identities. This allows us to compute the edge transport coefficients of the KMH model up to finite multiplicative and additive renormalizations, depending on all the microscopic details of the model. The values of these renormalizations are, however, severely constrained from one side by the validity of the Adler-Bardeen anomaly nonrenormalization property of the emergent chiral theory, and from the other side by the lattice WIs of the KMH model. As we show in Sec. VIII, these facts imply nonperturbative relations among all finite renormalizations, from which our theorem follows.

V. REDUCTION TO AN EFFECTIVE 1*D* THEORY

For simplicity, we shall directly consider the case $L = \infty$, which corresponds to having just one edge. It is useful to switch to a functional integral representation of the correlation functions of the lattice model. We define the generating functional of the correlations as

$$e^{\mathcal{W}(A)} = \int P(d\Psi)\, e^{-V(\Psi) + B(\Psi;A)}, \tag{22}$$

where $\Psi^{\pm}_{\mathbf{x},\sigma,\rho}$ are Grassmann variables, labeled by $\mathbf{x} = (x_0, \vec{x}) \in [0,\beta) \times \Lambda_A$, $\sigma = \pm$, $\rho = A, B$; $P(d\psi)$ is a Gaussian Grassmann integration with a propagator given by the noninteracting Euclidean two-point function,

$$g_{\sigma,\sigma'}(\mathbf{x},\mathbf{y}) = \delta_{\sigma\sigma'} \int \frac{d\underline{k}}{(2\pi)^2} \frac{e^{-i\underline{k}\cdot(\underline{x}-\underline{y})}}{-ik_0 + \widehat{H}^{\sigma}(k_1) - \mu}(\vec{x};\vec{y}), \tag{23}$$

where $\underline{k} = (k_0,k_1)$, with k_0 the fermionic Matsubara frequency and k_1 the quasimomentum associated with the translation-invariant direction $\vec{\ell}_1$. The Grassmann counterpart of the many-body interaction is

$$V(\Psi) = \lambda \sum_{\substack{\rho,\rho' \\ \sigma,\sigma'}} \int d\mathbf{x}\, d\mathbf{y}\, n_{\mathbf{x},\rho,\sigma} n_{\mathbf{y},\rho',\sigma'} v_{\rho\rho'}(\vec{x},\vec{y})\delta(x_0 - y_0),$$

where $\int d\mathbf{x} = \int_0^\beta dx_0 \sum_{\vec{x}}$, and $n_{\mathbf{x},\rho,\sigma}$ is the Grassmann counterpart of the density operator. Finally, $B(\Psi; A)$ is a source term of the form

$$B(\Psi; A) = \sum_{\mu,\sharp} \int d\mathbf{x}\, A^\sharp_{\mu,\mathbf{x}} J^\sharp_{\mu,\mathbf{x}} \tag{24}$$

with $J^\sharp_{\mu,\mathbf{x}}$ the Grassmann counterpart of $j^\sharp_{\mu,\mathbf{x}}$.

We now use the addition principle of the Grassmann variables to write $\Psi = \Psi^{(e)} + \Psi^{(b)}$, with $\Psi^{(e)}$, $\Psi^{(b)}$ independent Grassmann variables, with propagators $g^{(\mathrm{edge})}$ and $g^{(\mathrm{bulk})}$, where $g^{(e)}$ takes into account the energy modes close enough to the Fermi level. That is,

$$g^{(e)}_{\sigma\sigma'}(\mathbf{x},\mathbf{y}) = \delta_{\sigma\sigma'} \sum_e \int \frac{d\underline{k}}{(2\pi)^2} e^{-i\underline{k}\cdot(\underline{x}-\underline{y})} \times \frac{\chi_\sigma(k_1)}{-ik_0 + \varepsilon_\sigma(k_1) - \mu} P^\sigma_{k_1}(x_2; y_2), \tag{25}$$

with $P^\sigma_{k_1} = |\xi^\sigma\rangle\langle\xi^\sigma|$, where ξ^σ is the edge mode of $\widehat{H}^\sigma(k_1)$, with energy ε_σ, and $\chi_\sigma(k_1) \equiv \chi(|k_1 - k^\sigma_F| \leqslant \delta)$ is a compactly supported cutoff function. By construction, the propagator $g^{(\mathrm{bulk})}$ is gapped; it only depends on the energy modes that are at a distance at least $\sim\delta$ from the Fermi level. Thus, $|g^{(\mathrm{bulk})}(\mathbf{x},\mathbf{y})| \leqslant Ce^{-c|\mathbf{x}-\mathbf{y}|}$. Instead, due to the fact that, for $k_1 = k'_1 + k^\sigma_F$ and k'_1 small

$$\varepsilon_\sigma\big(k'_1 + k^\sigma_F\big) - \mu = \sigma v_+ k'_1 + O\big({k'_1}^2\big), \tag{26}$$

the edge propagator in Eq. (25) only decays as $|\underline{x} - \underline{y}|^{-1} e^{-c(|x_2|+|y_2|)}$.

The field $\Psi^{(b)}$ can be integrated out, expanding the integrand of (22) in the coupling λ and using the exponential decay of the bulk propagator together with fermionic cluster-expansion techniques [43]. We then get

$$e^{\mathcal{W}(A)} = e^{\mathcal{W}^{(b)}(A)} \int P_e(d\Psi^{(e)}) e^{-V^{(e)}(\Psi^{(e)}) + B^{(e)}(\Psi^{(e)};A)}, \tag{27}$$

where the new effective interaction $V^{(e)}(\Psi^{(e)})$ is a sum over monomials P in the fields $\Psi^{(e)}$ of any order $|P| = n$, with kernels $W^{(e)}_P(\mathbf{x}_1,\ldots,\mathbf{x}_n)$, exponentially decaying in $|\mathbf{x}_i - \mathbf{x}_j|$ for $i \neq j$. Graphically, a given kernel can be represented as a sum of Feynman diagrams with $|P|$ external lines, corresponding to the edge fields, and an arbitrary number of quartic vertices connected by the bulk propagators. This expansion turns out to be *convergent* for small λ, thanks to determinant bounds for fermionic field theories, combined with the good decay properties of the bulk propagators. The new effective source term $B^{(e)}$ admits a similar representation, where now external lines corresponding to the A fields are present as well.

Due to the special form of the edge propagator, given by Eq. (25), we now notice that the edge field can be represented as the convolution of a truly one-dimensional field with the edge mode eigenfunctions. That is,

$$\int P_e(d\Psi^{(e)}) e^{-V^{(e)}(\Psi^{(e)}) + B^{(e)}(\Psi^{(e)};A)} = \int P_{1\mathrm{D}}(d\psi) e^{-V^{(e)}(\psi * \check{\xi}) + B^{(e)}(\psi * \xi;A)}, \tag{28}$$

where $P_{1\mathrm{D}}$ is a Grassmann Gaussian integration for a one-dimensional field $\psi^\pm_{\vec{x},\sigma}$, with the propagator given, in momentum space, by

$$\widehat{g}_{\sigma,\sigma'}(\underline{k}) = \delta_{\sigma\sigma'} \frac{\chi_\sigma(\underline{k})}{-ik_0 + \varepsilon_\sigma(k_1) - \mu_0}, \tag{29}$$

where now $\chi_\sigma(|\underline{k}|) = \chi(|\underline{k} - \underline{k}^\sigma_F| \leqslant \delta)$ and $\mu - \mu_0 = \nu_0$, with $\nu_0 = O(\lambda)$ a counter term that is chosen so as to fix the value of the interacting chemical potential; and

$$(\psi^+ * \check{\xi})_{\mathbf{x},\rho} = \sum_{y_1} \psi^-_{(x_0,y_1),\sigma} \overline{\check{\xi}^\sigma_{x_2}(x_1 - y_1; \rho)}, \tag{30}$$

where $\check{\xi}^\sigma_{x_2}(x_1;\rho)$ is the Fourier transform of $\chi_\sigma(k_1)\xi^\sigma_{x_2}(k_1;\rho)$. This representation of the edge field allows us to decouple the x_2 variables from the remaining x_0,x_1 variables in the effective interaction. Summing over x_2 (recalling the exponential decay of the edge modes), one finally gets

$$e^{\mathcal{W}(A)} = e^{\mathcal{W}^{(b)}(A)} \int P_0(d\psi)\, e^{-V^{(0)}(\psi) + B^{(0)}(\psi;A)}, \tag{31}$$

where $P_0 \equiv P_{1\mathrm{D}}$ and for suitable new effective interaction and source terms, which can be again expressed as sums over monomials of arbitrary order in the 1D fields ψ. One has

$$V^{(0)}(\psi) = \int d\underline{x} \left[\lambda_0 \psi^+_{\underline{x},+} \psi^-_{\underline{x},+} \psi^+_{\underline{x},-} \psi^-_{\underline{x},-} + \sum_\sigma \nu_0 \psi^+_{\underline{x},\sigma} \psi^-_{\underline{x},\sigma}\right] + \mathcal{R}V^{(0)}(\psi), \tag{32}$$

where the new coupling constant is

$$\lambda_0 = \lambda \sum_{\substack{x_2,y_2 \\ \rho,\rho'}} \widehat{v}_{\rho\rho'}(0; x_2, y_2) \overline{\xi^{(1,\sigma)}_{x_2}(k_F;\rho)} \xi^{(1,\sigma)}_{x_2}(k_F;\rho) \times \overline{\xi^{(1,\sigma)}_{y_2}(k_F;\rho')} \xi^{(1,\sigma)}_{y_2}(k_F;\rho') + O(\lambda^2), \tag{33}$$

and $\mathcal{R}V^{(0)}$ collects all the higher-order terms, together with nonlocal terms. All these contributions turn out to be *irrelevant* in the RG sense. Similarly,

$$B^{(0)}(\psi; A) = \sum_{\mu,\sharp} \int d\mathbf{x}\, Z^\sharp_\mu(x_2) A^\sharp_{\mu,\mathbf{x}} n^\sharp_{\mu,\underline{x}} + \mathcal{R}B^{(0)}(\psi; A), \tag{34}$$

where $Z^\sharp_\mu(x_2)$ is such that $|Z^\sharp_\mu(x_2)| \leqslant Ce^{-cx_2}$, and it is analytic in λ; and

$$n^{\mathrm{c}}_{0,\underline{x}} = \sum_\sigma \psi^+_{\underline{x},\sigma} \psi^-_{\underline{x},\sigma}, \qquad n^{\mathrm{c}}_{1,\underline{x}} = \sum_\sigma \sigma \psi^+_{\underline{x},\sigma} \psi^-_{\underline{x},\sigma},$$
$$n^{\mathrm{s}}_{0,\underline{x}} = n^{\mathrm{c}}_{1,\underline{x}}, \qquad n^{\mathrm{s}}_{1,\underline{x}} = n^{\mathrm{c}}_{0,\underline{x}}. \tag{35}$$

Let us give a quick proof of Eqs. (34) and (35). After the integration of $\Psi^{(b)}$ and the reduction to 1D theory, the effective source term has the following form, in momentum space:

$$B^{(0)}(\psi;A) = \sum_{\mu,\sharp,x_2} \int \frac{d\underline{k}}{(2\pi)^2}\frac{d\underline{p}}{(2\pi)^2} \times \widehat{A}^{\sharp}_{\mu,(\underline{p},x_2)} \widehat{\psi}^{+}_{\underline{k}+\underline{p},\sigma} \widehat{\psi}^{-}_{\underline{k},\sigma} \widehat{W}^{\sharp}_{\mu,\sigma}(\underline{p},\underline{k};x_2) + O(A^2) \tag{36}$$

for suitable kernels $\widehat{W}^{\sharp}_{\mu,\sigma}$. The higher orders in A turn out to be irrelevant in the RG sense. Let us localize the kernel by writing $\widehat{W}^{\sharp}_{\mu,\sigma}(\underline{p},\underline{k};x_2) = \widehat{W}^{\sharp}_{\mu,\sigma}(\underline{0},\underline{k}^{\sigma}_{F};x_2) + \mathcal{R}\widehat{W}^{\sharp}_{\mu,\sigma}$, where the $\mathcal{R}$ error terms are irrelevant. The effective 1D model is invariant under time-reversal symmetry [recall that $\overline{\xi^{\sigma}(k_1)} = \xi^{-\sigma}(-k_1)$, and that $\widehat{H}^{\sigma}(k_1) = \overline{\widehat{H}^{-\sigma}(-k_1)}$]:

$$\widehat{A}^{\sharp}_{\mu,\underline{p},x_2} \to \gamma_{\sharp}\gamma_{\mu}\widehat{A}^{\sharp}_{\mu,-\underline{p},x_2}, \quad \widehat{\psi}^{\varepsilon}_{\underline{k},\sigma} \to \widehat{\psi}^{\varepsilon}_{-\underline{k},-\sigma}, \quad c \to \bar{c}, \tag{37}$$

with c a generic constant in the action, $\gamma_{\mathrm{c}} = 1 = -\gamma_{\mathrm{s}}$, and $\gamma_0 = 1 = -\gamma_1$. This symmetry implies that $\widehat{W}^{\sharp}_{\sigma,\mu}(\underline{0},\underline{k}^{\sigma}_{F};x_2) = \gamma_{\sharp}\gamma_{\mu}\overline{\widehat{W}^{\sharp}_{-\sigma,\mu}(\underline{0},\underline{k}^{-\sigma}_{F};x_2)}$. Also, the model is invariant under complex conjugation:

$$\begin{aligned} \widehat{A}^{\sharp}_{\mu,\underline{p},x_2} &\to \widehat{A}^{\sharp}_{\mu,\underline{\widetilde{p}},x_2}, \qquad \widehat{\psi}^{+}_{\underline{k},\sigma} \to -\widehat{\psi}^{-}_{\underline{\widetilde{k}},\sigma}, \\ \widehat{\psi}^{-}_{\underline{k},\sigma} &\to \widehat{\psi}^{+}_{\underline{\widetilde{k}},\sigma}, \qquad c \to \bar{c} \end{aligned} \tag{38}$$

with $\widetilde{k} = (-k_0,k_1)$. This last symmetry implies that $\widehat{W}^{\sharp}_{\sigma,\mu}(\underline{0},\underline{k}^{\sigma}_{F};x_2)$ is real. Going back to configuration space, Eq. (35) follows.

Equation (31) is an *exact* (but very involved) representation of the generating functional of the KMH model in terms of an effective one-dimensional field. It differs from the HL model by the presence of nonlinear corrections in the dispersion and irrelevant terms in the effective interaction.

VI. MULTISCALE ANALYSIS OF THE EDGE MODES

Due to the absence of a mass gap, the field ψ cannot be integrated in a single step. Instead, we proceed in a multiscale fashion, exploiting a renormalization procedure at every step. We rewrite the ψ field in terms of single-scale quasiparticle fields as follows:

$$\psi^{\pm}_{\underline{x},\sigma} = e^{\pm i k^{\sigma}_{F} x_1} \sum_{h=h_{\beta}}^{0} \psi^{(h)}_{\underline{x},\sigma}, \tag{39}$$

where each field varies on a scale 2^{-h}, with $h \leqslant 0$. The last scale h_{β} is fixed by the inverse temperature, $h_{\beta} \sim |\log_2 \beta|$. The covariance of the fields is defined inductively. We integrate the fields in an iterative fashion. From a RG point of view, the $\psi^{+}_{\underline{x}}\psi^{-}_{\underline{x}}$ terms are relevant, while the $\psi^{+}_{\underline{x}}\partial_{\mu}\psi^{-}_{\underline{x}}$, $\psi^{+}_{\underline{x},\sigma}\psi^{-}_{\underline{x},\sigma}\psi^{+}_{\underline{x},\sigma'}\psi^{-}_{\underline{x},\sigma'}$ terms are marginal.

After the integration of the scales $h+1,\dots,0$, we obtain the following representation of the generating functional:

$$e^{\mathcal{W}(A)} = e^{\mathcal{W}^{(h)}(A)} \int P_h(d\psi^{(\leqslant h)}) e^{-V^{(h)}(\sqrt{Z_h}\psi) + B^{(h)}(\psi;A)}, \tag{40}$$

where the new Gaussian Grassmann integration has a propagator:

$$g^{(\leqslant h)}_{\sigma,\sigma'}(\underline{x},\underline{y}) = \frac{\delta_{\sigma\sigma'}}{Z_h} \int \frac{d\underline{k}'}{(2\pi)^2} \frac{e^{-i\underline{k}'\cdot(\underline{x}-\underline{y})}\chi_h(\underline{k}')}{-ik_0 + \sigma v_h k'_1}[1 + r_h(\underline{k}')],$$

where χ_h is a smooth cutoff function supported for $|\underline{k}'| \leqslant 2^{h+1}$; r_h is an error term, $|r_h(\underline{k}')| \leqslant C|\underline{k}'|$; and Z_h and v_h are, respectively, the wave-function renormalization and the effective Fermi velocity, whose RG flow, as a function of h, is marginal. Time-reversal symmetry (37) and complex conjugation (38) imply that these parameters are real and spin-independent.

The new effective interaction is a sum of Grassmann monomials of arbitrary order. We rewrite it as $V^{(h)} = \mathcal{L}V^{(h)} + \mathcal{R}V^{(h)}$, where $\mathcal{L}V^{(h)}$ takes into account all the relevant and marginal contributions:

$$\mathcal{L}V^{(h)}(\sqrt{Z_h}\psi) = \int d\underline{x} \left[\lambda_h Z_h^2 \psi^{+}_{\underline{x},+}\psi^{-}_{\underline{x},+}\psi^{+}_{\underline{x},-}\psi^{-}_{\underline{x},-} + \sum_{\sigma} 2^h Z_h \nu_h \psi^{+}_{\underline{x},\sigma}\psi^{-}_{\underline{x},\sigma}\right],$$

while $\mathcal{R}V^{(h)}$ takes into account all irrelevant terms. By the symmetries (37) and (38), the parameters λ_h and ν_h are again real and spin-independent. In the same spirit, we rewrite $B^{(h)} = \mathcal{L}B^{(h)} + \mathcal{R}B^{(h)}$, where $\mathcal{L}B^{(h)}$ collects all marginal terms (there are no relevant terms in the source term):

$$\mathcal{L}B^{(h)}(\psi;A) = \int d\mathbf{x}\, Z^{\sharp}_{h,\mu}(x_2) A^{\sharp}_{\mu,\mathbf{x}} n^{\sharp}_{\mu,\underline{x}} \tag{41}$$

for suitable (real) running coupling functions $Z^{\sharp}_{h,\mu}(x_2)$.

Let us briefly discuss the flow of the running coupling constants. The (relevant) flow of ν_h is controlled via a fixed point argument by properly choosing the initial shift of the chemical potential ν_0; see [42] for details in a similar case. Instead, the (marginal) flows of λ_h, v_h are controlled using a highly nontrivial cancellation in the renormalized expansions, the *vanishing of the beta function* [30], giving $\lambda_h = \lambda_0 + O(\lambda^2)$ and $v_h = v_0 + O(\lambda)$ *uniformly* in h. Instead, the flows of the wave function and vertex renormalizations *diverge* with anomalous exponents,

$$Z_h \sim 2^{-\eta h}, \qquad Z^{\sharp}_{h,\mu}(x_2) \sim 2^{-\eta h} Z^{\sharp}_{0,\mu}(x_2), \tag{42}$$

with $\eta = \frac{\lambda_0^2}{8\pi^2 v_0^2} + O(\lambda_0^4)$.

The outcome of this construction is a *convergent* expansion for the correlation functions in terms of the running coupling constants, which can be used to prove bounds for the decay of the current-current correlations. Convergence follows from the use fermionic cluster expansion at every step of integration, as in [42], and excludes nonperturbative effects. We have

$$\left| \lim_{\beta,L\to\infty} \langle \mathbf{T}\, j^{\sharp}_{\mu,\mathbf{x}}\,;\, j^{\sharp'}_{\nu,\mathbf{y}} \rangle_{\beta,L} \right| \leqslant C e^{-c|x_2-y_2|}/(1 + |\underline{x}-\underline{y}|^2). \tag{43}$$

This estimate, however, is not for the computation of the edge transport coefficients. In fact, it is not even enough to prove the boundedness of the Fourier transform of the current-current correlation, uniformly in $\underline{p}$. To improve on this, we need to exploit *cancellations* in the renormalized expansion, following from the emergent *chiral symmetry* of the theory.

VII. EMERGENT CHIRAL QFT

In this section we introduce an emergent effective chiral QFT theory, defined by the generating functional:

$$e^{\mathcal{W}^{\chi}(A)} = \int P_N(d\psi) e^{-\lambda^{\chi} Z^{\chi 2} \int d\underline{x}\, d\underline{y}\, v(\underline{x}-\underline{y}) n_{\underline{x},+} n_{\underline{y},-} + B(\psi;A)}, \quad (44)$$

where $P_N(d\psi)$ is a Gaussian Grassmann measure with a propagator:

$$g^{\chi}_{\sigma,\sigma'}(\underline{x},\underline{y}) = \frac{\delta_{\sigma\sigma'}}{Z^{\chi}} \int \frac{d\underline{k}}{(2\pi)^2} e^{-i\underline{k}\cdot(\underline{x}-\underline{y})} \frac{\chi_N(\underline{k})}{-ik_0 + \sigma v^{\chi} k_1}, \quad (45)$$

where χ_N is an ultraviolet cutoff, supported for $|\underline{k}| \leqslant 2^{N+1}$, for $N \gg 1$ (to be sent to infinity at the end). The source term is $B(\psi;A) = \sum_{x_2=0}^{\infty} \int d\underline{x}\, Z^{\sharp,\chi}_{\mu}(x_2) A^{\sharp}_{\mu,\mathbf{x}} n^{\sharp}_{\mu,\underline{x}}$. The interaction potential $v(\underline{x}-\underline{y})$ is nonlocal and short-ranged. The presence of the UV cutoff is crucial to give a nonperturbative meaning to Eq. (44). Its final removal is done through an ultraviolet multiscale analysis, in which the nonlocal, short-range nature of the interaction plays an essential role [30]. The infrared regime of this QFT can be studied as for the lattice model. Let us denote by λ^{χ}_h, Z^{χ}_h, v^{χ}_h, and $Z^{\sharp,\chi}_{\mu,h}(x_2)$ the running coupling constants of the emergent chiral model $\langle\,\rangle^{\chi}$.

The bare parameters Z^{χ}, v^{χ}, λ^{χ}, and $Z^{\sharp,\chi}_{\mu}$ will be chosen in such a way that the running coupling constants of lattice and chiral theory converge to the same limit as $h \to -\infty$. This fact, together with the convergence of the renormalized expansions for both models, implies that the correlations of the KMH model can be written in terms of the correlations of the emergent chiral model, up to finite multiplicative and additive renormalizations, depending on all the microscopic details of the KMH model:

$$\big\langle \mathbf{T}\, j^{\sharp}_{\mu,\underline{p},x_2} j^{\sharp'}_{\nu,-\underline{p},y_2} \big\rangle_{\infty} = Z^{\sharp,\chi}_{\mu}(x_2) Z^{\sharp,\chi}_{\nu}(y_2) \langle n^{\sharp}_{\mu,\underline{p}} n^{\sharp'}_{\nu,-\underline{p}} \rangle^{\chi} + \widehat{H}^{\sharp,\sharp'}_{\mu,\nu}(\underline{p},x_2,y_2), \quad (46)$$

where $\langle\cdot\rangle^{\chi}$ denotes the correlations of the emergent chiral model, and $\widehat{H}^{\sharp,\sharp'}_{\mu,\nu}(\underline{p};x_2,y_2)$ is an error term, *continuous* in $\underline{p}$, in contrast with the first term on the right-hand side of (46). The improved regularity of this contribution is due to the fact that it involves irrelevant terms in the RG sense, which all come with a dimensional gain: in configuration space, such a term decays as, for large distances, $e^{-c|x_2-y_2|}/(1+|\underline{x}-\underline{y}|^{2+\vartheta})$ for some $\vartheta > 0$. Thus, even though this term disappears pointwise in the scaling limit of the correlations, it gives a *finite* contribution to the Fourier transform of the lattice correlations. Concerning the multiplicative renormalizations, they verify the bound $|Z^{\sharp,\chi}_{\mu}(x_2)| \leqslant Ce^{-cx_2}$ as a consequence of the exponential decay of the edge states.

Similarly, up to subleading terms for small external momenta,

$$\big\langle \mathbf{T}\, \widehat{j}^{\sharp}_{\underline{p},z_2,\mu}\,; \widehat{\phi}^{-}_{\underline{k}+\underline{p},x_2,\rho,\sigma} \widehat{\phi}^{+}_{\underline{k},y_2,\rho,\sigma} \big\rangle_{\infty} = Z^{\sharp,\chi}_{\mu}(z_2) Q^{\sigma}_{x_2}\big(k^{\omega}_F;\rho\big) \overline{Q^{\sigma}_{y_2}\big(k^{\omega}_F;\rho\big)} \langle n^{\sharp}_{\mu,\underline{p}}\,; \widehat{\psi}^{-}_{\underline{k}+\underline{p},\sigma} \widehat{\psi}^{+}_{\underline{k},\sigma} \rangle^{\chi} \quad (47)$$

for some functions Q^{σ}, such that $Q^{\sigma} = [1 + O(\lambda)]\xi^{\sigma}$, which satisfy the exponential bound $|Q^{\sigma}_{x_2}| \leqslant Ce^{-c|x_2|}$. Moreover, up to subleading terms in the external momenta,

$$\big\langle \mathbf{T}\, \widehat{\phi}^{-}_{\underline{k},x_2,\rho,\sigma} \widehat{\phi}^{+}_{\underline{k},y_2,\rho',\sigma} \big\rangle_{\infty} = Q^{\sigma}_{x_2}\big(k^{\omega}_F;\rho\big) \overline{Q^{\sigma}_{y_2}\big(k^{\omega}_F;\rho'\big)} \langle \widehat{\psi}^{-}_{\underline{k},\sigma} \widehat{\psi}^{+}_{\underline{k},\sigma} \rangle^{\chi}. \quad (48)$$

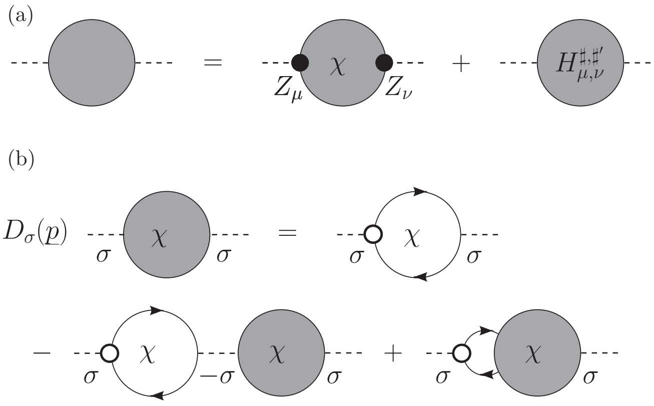

FIG. 2. (a) Graphical representation of Eq. (46). "χ" denotes the contributions due to the emergent chiral model. The full dots correspond to the vertex renormalizations, associated with the factors $Z^{\sharp,\chi}_{\mu}$ in Eq. (46). (b) Graphical representation of the first WI in Eq. (49) for a finite UV cutoff N. The small white circle denotes a *correction vertex*, corresponding to the insertion of $C_{\sigma}(\underline{p},\underline{k})\psi^{+}_{\underline{k}+\underline{p},\sigma}\psi^{-}_{\underline{k},\sigma}$. The empty bubble is a noninteracting diagram, whose $N \to \infty$ value is $-\frac{1}{4\pi|v|Z^2} D_{-\sigma}(\underline{p})$.

The advantage of comparing the lattice correlations with those of the emergent model is that the latter can be computed in a closed form, thanks to chiral Ward identities, following from $U(1)$ chiral gauge symmetry. Notice that this symmetry is only approximate, due to the presence of the ultraviolet cutoff. As a result, the UV regularization produces extra terms in the Ward identities of the emergent chiral theory, which do not vanish as $N \to \infty$, but rather produce *anomalies* breaking the conservation of the chiral current; see Fig. 2(b). In the figure, the white circle corresponds to the insertion of a *correction vertex* $C_{\sigma}(\underline{p},\underline{k}) = [\chi_N^{-1}(\underline{k}) - 1]D_{\sigma}(\underline{k}) - [\chi_N^{-1}(\underline{k}+\underline{p}) - 1]D_{\sigma}(\underline{k}+\underline{p})$, with $D_{\sigma}(\underline{p}) = -ip_0 + \sigma v^{\chi} p_1$. For $\underline{p} = O(1)$, this vertex insertion fixes the momentum of the incoming and outgoing fermionic lines on the scale of the ultraviolet cutoff. In the figure, we isolated the terms where the fermionic lines incident to the correction vertex meet at the same point; instead, the last term on the right-hand side of Fig. 2(b) collects all contributions corresponding to diagrams where the lines meet at different points. It turns out that this last term *vanishes* as $N \to \infty$, thanks to the nonlocality of the interaction, and to the support properties of the correction vertex. See [30,44] for a detailed proof of this statement, in a similar case.

Setting $D_{\sigma}(\underline{p}) = -ip_0 + \sigma v^{\chi} p_1$, we have

$$\begin{aligned} D_{\sigma}(\underline{p})\langle \widehat{\rho}_{\underline{p},\sigma}\,; \widehat{\rho}_{-\underline{p},\sigma} \rangle^{\chi} &= -\frac{D_{-\sigma}(\underline{p})}{4\pi|v^{\chi}|Z^{\chi 2}} + \tau D_{-\sigma}(\underline{p}) \widehat{v}(\underline{p}) \langle \widehat{\rho}_{\underline{p},-\sigma}\,; \widehat{\rho}_{-\underline{p},\sigma} \rangle^{\chi}, \\ \langle \widehat{\rho}_{\underline{p},\sigma}\,; \widehat{\rho}_{-\underline{p},-\sigma} \rangle^{\chi} &= \tau \frac{D_{-\sigma}(\underline{p})}{D_{\sigma}(\underline{p})} \widehat{v}(\underline{p}) \langle \widehat{\rho}_{\underline{p},-\sigma}\,; \widehat{\rho}_{-\underline{p},-\sigma} \rangle^{\chi}, \end{aligned} \quad (49)$$

where $\tau = \frac{\lambda^\chi}{4\pi|v^\chi|}$ is the *chiral anomaly*. The linearity of the anomaly in the bare coupling constant is a highly nontrivial fact, known as *Adler-Bardeen anomaly nonrenormalization*. The explicit value of the anomaly can be used to determine the critical exponents of the emergent chiral model. For instance, the anomalous exponent of the two-point Schwinger function is $\eta = K + K^{-1} - 2$ with $K = \frac{1-\tau}{1+\tau}$.

Thus, supposing that $\widehat{v}(\underline{0}) = 1$, we have, up to subleading terms in $\underline{p}$,

$$\begin{aligned}\langle \widehat{\rho}_{\underline{p},\sigma} \widehat{\rho}_{-\underline{p},\sigma} \rangle^\chi &= \frac{-1}{4\pi|v^\chi|Z^{\chi 2}} \frac{1}{1-\tau^2} \frac{D_{-\sigma}(\underline{p})}{D_\sigma(\underline{p})}, \\ \langle \widehat{\rho}_{\underline{p},-\sigma} \widehat{\rho}_{-\underline{p},\sigma} \rangle^\chi &= \frac{-1}{4\pi|v^\chi|Z^{\chi 2}} \frac{\tau}{1-\tau^2}.\end{aligned} \tag{50}$$

These expressions can be plugged in the representation for the lattice current-current correlation function, (46). All we have left to do is to determine the unknown multiplicative and additive renormalizations.

VIII. UNIVERSALITY

To fix the values of the finite multiplicative and additive renormalizations, we use again Ward identities, this time for the lattice model. These identities introduce nonperturbative relations between the renormalization coefficients, which, as we shall see, imply a dramatic cancellation in the final expression of the edge transport coefficients. To begin, it is convenient to rewrite the Schwinger term of the lattice WI (7) in the following more explicit way:

$$\begin{aligned}\big\langle \big[j^\sigma_{0,\vec{x}}\, , j^\sigma_{1,\vec{y}} \big] \big\rangle &= \big(\delta_{\vec{x},\vec{y}} - \delta_{\vec{x},\vec{y}+\vec{\ell}_1}\big)\big\langle t^\sigma_{\vec{y},\vec{y}+\vec{\ell}_1} + t^\sigma_{\vec{y},\vec{y}+\vec{\ell}_1-\vec{\ell}_2}\big\rangle \\ &\quad + \big(\delta_{\vec{x},\vec{y}+\vec{\ell}_1} - \delta_{\vec{x},\vec{y}+\vec{\ell}_1-\vec{\ell}_2}\big)\big\langle t^\sigma_{\vec{y},\vec{y}+\vec{\ell}_1-\vec{\ell}_2}\big\rangle \big\langle \big[j^\sigma_{0,\vec{x}}\, , j^\sigma_{2,\vec{y}} \big] \big\rangle \\ &= \big(\delta_{\vec{x},\vec{y}} - \delta_{\vec{x},\vec{y}+\vec{\ell}_2}\big)\big\langle t^\sigma_{\vec{y},\vec{y}+\vec{\ell}_2} + t^\sigma_{\vec{y},\vec{y}-\vec{\ell}_1+\vec{\ell}_2}\big\rangle \\ &\quad + \big(\delta_{\vec{x},\vec{y}+\vec{\ell}_2} - \delta_{\vec{x},\vec{y}-\vec{\ell}_1+\vec{\ell}_2}\big)\big\langle t^\sigma_{\vec{y},\vec{y}-\vec{\ell}_1+\vec{\ell}_2}\big\rangle\end{aligned} \tag{51}$$

with $t^\sigma_{\vec{x},\vec{y}}$ defined after (14). Summing up (7) over y_2, one gets

$$\begin{aligned} & d_{y_0} \sum_{y_2} \langle \mathbf{T}\, j^\sharp_{1,\vec{x}}\, ; j^\sharp_{0,\vec{y}} \rangle + d_{y_1} \sum_{y_2} \langle \mathbf{T}\, j^\sharp_{1,\vec{x}}\, ; j^\sharp_{1,\vec{y}} \rangle \\ &\quad = i\delta(x_0 - y_0)\big(\delta_{x_1,y_1} - \delta_{x_1,y_1+1}\big)\Delta(x_2), \\ & d_{y_0} \sum_{y_2} \langle \mathbf{T}\, j^\sharp_{0,\vec{x}}\, ; j^{\sharp'}_{0,\vec{y}} \rangle + d_{y_1} \sum_{y_2} \langle \mathbf{T}\, j^\sharp_{0,\vec{x}}\, ; j^{\sharp'}_{1,\vec{y}} \rangle = 0. \end{aligned} \tag{52}$$

To get these relations, we crucially used that $\sum_{y_2} d_{y_2}(\cdots) = 0$, which is implied by the Dirichlet boundary conditions. By going into Fourier space, we can use the relations (52) to prove identities for the edge transport coefficients:

$$\begin{aligned} -\, i p_0 G^{\sharp,\sharp;a}_{j,\rho}(\underline{p}) + p_1 \eta(p_1) G^{\sharp,\sharp;a}_{j,j}(\underline{p}) &= 0, \\ -i p_0 G^{\sharp,\sharp';a}_{\rho,\rho}(\underline{p}) + p_1 \eta(p_1) G^{\sharp,\sharp';a}_{\rho,j}(\underline{p}) &= 0, \end{aligned} \tag{53}$$

with $p_1\eta(p_1) = p_1 + O(p_1^2)$ the Fourier symbol associated with the lattice derivative d_{y_1}. Equations (53) can be used to determine the $\underline{p} \to \underline{0}$ limit of the additive renormalization $\sum_{x_2=0}^{a} \sum_{y_2=0}^{\infty} \widehat{H}^{\sharp,\sharp'}_{\mu,\nu}(\underline{p}; x_2, y_2)$ (which exists by continuity in $\underline{p}$). For instance, consider the edge charge conductance, $G^{\mathrm{c,s};a}_{\rho,j}(\underline{p})$. We can rewrite the second of Eqs. (53) as $G^{\mathrm{c,s};a}_{\rho,j}(\underline{p}) = [i p_0 / p_1 \eta(p_1)] G^{\mathrm{c,s};a}_{\rho,\rho}(\underline{p})$; thus, this relation implies that $\lim_{p_1\to 0} \lim_{p_0 \to 0} G^{\mathrm{c,s};a}_{\rho,j}(\underline{p}) = 0$. This identity, together with the representation (46) of the current-current correlation function, allows us to compute the $\underline{p} \to 0$ limit of $\sum_{x_2=0}^{a} \sum_{y_2=0}^{\infty} \widehat{H}^{\mathrm{c,s}}_{0,1}(\underline{p}; x_2, y_2)$ in terms of the other unknown renormalized parameters. A similar strategy can be followed for the other transport coefficients.

For simplicity, let us drop the χ label, and let us set $Z^\sharp_\mu \equiv \sum_{z_2} Z^{\sharp,\chi}_\mu(z_2)$. The above-mentioned strategy allows us to compute, up to subleading terms in $\underline{p}$,

$$\begin{aligned} \lim_{a\to\infty} G^{\mathrm{c,s};a}_{\rho,j}(\underline{p}) &= -\frac{Z^c_0 Z^s_1}{Z^2(1-\tau^2)} \frac{1}{\pi|v|} \frac{p_0^2}{p_0^2 + v^2 p_1^2}, \\ \lim_{a\to\infty} G^{\sharp,\sharp;a}_{j,j}(\underline{p}) &= -\frac{Z^\sharp_1 Z^\sharp_1}{Z^2(1-\tau^2)} \frac{1}{\pi|v|} \frac{p_0^2}{p_0^2 + v^2 p_1^2}, \\ \lim_{a\to\infty} G^{\sharp,\sharp;a}_{\rho,\rho}(\underline{p}) &= \frac{Z^\sharp_0 Z^\sharp_0}{Z^2(1-\tau^2)} \frac{1}{\pi|v|} \frac{v^2 p_1^2}{p_0^2 + v^2 p_1^2}. \end{aligned} \tag{54}$$

It remains to determine the multiplicative renormalization in Eqs. (54). This is done by comparing the vertex WIs of lattice and emergent models. From Eq. (8) we have, setting $\eta_0(p_1) = -i$,

$$\begin{aligned} & \sum_{\mu=0}^{1} \eta_\mu(p_1) \sum_{z_2} \big\langle \mathbf{T}\, \widehat{j}^\sharp_{\underline{p},z_2,\mu}\, ; \widehat{\phi}^-_{\underline{k}+\underline{p},x_2,\sigma} \widehat{\phi}^+_{\underline{k},y_2,\sigma} \big\rangle_{\beta,L} \\ &\quad = \sigma_\sharp \big[\big\langle \mathbf{T}\, \widehat{\phi}^-_{\underline{k},x_2,\sigma} \widehat{\phi}^+_{\underline{k},y_2,\sigma} \big\rangle_{\beta,L} - \big\langle \mathbf{T}\, \widehat{\phi}^-_{\underline{k}+\underline{p},x_2,\sigma} \phi^+_{\underline{k}+\underline{p},y_2,\sigma} \big\rangle_{\beta,L} \big] \end{aligned} \tag{55}$$

with $\sigma_c = 1$ and $\sigma_s = \sigma$. On the other hand, the WIs for the emergent chiral model are

$$\begin{aligned} & -i p_0 \langle \widehat{n}^\sharp_{0,\underline{p}}\, ; \widehat{\psi}^-_{\underline{k}+\underline{p},\sigma} \widehat{\psi}^+_{\underline{k},\sigma} \rangle + p_1 v \langle n^\sharp_{1,\underline{p}}\, ; \widehat{\psi}^-_{\underline{k}+\underline{p},\sigma} \widehat{\psi}^+_{\underline{k},\sigma} \rangle \\ &\quad = \frac{\sigma_\sharp}{Z(1-\eta_\sharp \tau)} [\langle \widehat{\psi}^-_{\underline{k},\sigma} \widehat{\psi}^+_{\underline{k},\sigma} \rangle - \langle \widehat{\psi}^-_{\underline{k}+\underline{p},\sigma} \widehat{\psi}^+_{\underline{k}+\underline{p},\sigma} \rangle] \end{aligned} \tag{56}$$

with $\eta_c = +$, $\eta_s = -$. As before, we now express the lattice correlation functions appearing in the lattice WI in terms of those of the emerging chiral model, using Eqs. (47) and (48); we therefore get *two* identities for the correlations of the emergent chiral model, one involving the $Z^\sharp_\mu$ parameters, the other involving Z, v, τ. Therefore, we can use these identities to prove relations among these coefficients; we get

$$\frac{v Z^\sharp_0}{Z^\sharp_1} = 1, \qquad \frac{Z^\sharp_0}{Z(1-\eta_\sharp \tau)} = 1. \tag{57}$$

Remarkably, Eq. (57) provides a link between the emergent chiral anomaly and the finite lattice renormalizations. We can now use Eq. (57) to simplify the expressions in Eqs. (54). Setting $K^c = K$, $K^s = K^{-1}$, we get

$$\begin{aligned} \frac{Z^\sharp_0 Z^\sharp_1}{Z^2(1-\tau^2)v} &= K^\sharp, \quad \frac{Z^c_0 Z^s_1}{Z^2(1-\tau^2)v} = 1, \\ \frac{Z^\sharp_1 Z^\sharp_1}{Z^2(1-\tau^2)v} &= K^\sharp v, \quad \frac{Z^\sharp_0 Z^\sharp_0}{Z^2(1-\tau^2)v} = \frac{K^\sharp}{v}. \end{aligned} \tag{58}$$

The second relation implies the quantization of σ^s [for λ small, sgn(v) is independent of λ]. The last two imply the nonuniversality of $D^\sharp$, $\kappa^\sharp$, and the helical Luttinger liquid relation $D^\sharp = v^2\kappa^\sharp$.

IX. CONCLUSIONS

We have established the exact quantization of the edge spin conductance for the spin-conserving Kane-Mele-Hubbard model. As a corollary, our result provides an example of bulk-edge correspondence for a nonsolvable, interacting time-reversal-invariant system. In addition, we proved a marginal form of universality for the susceptibilities and the Drude weights, showing the validity of the helical Luttinger liquid scaling relations for the KMH model. Our strategy is based on an exact RG construction of the lattice model, and on the combination of lattice Ward identities, following from lattice conservation laws, with relativistic Ward identities, following from the emergent chiral gauge symmetry of the system. Even though they break the integrability of the interacting system, lattice effects and bulk degrees of freedom play a crucial role for universality.

As an open problem, it would be interesting to include spin-nonconserving terms in the Hamiltonian, and to quantify the possible breaking of universality of the edge spin conductance.

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[1] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005).
[2] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005).
[3] B. A. Bernevig and S.-C. Zhang, Phys. Rev. Lett. **96**, 106802 (2006).
[4] C. Wu, B. A. Bernevig, and S.-C. Zhang, Phys. Rev. Lett. **96**, 106401 (2006).
[5] C. Xu and J. E. Moore, Phys. Rev. B **73**, 045322 (2006).
[6] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[7] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[8] M. Hohenadler and F. F. Assaad, J. Phys.: Condens. Matter **25**, 143201 (2013).
[9] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang, Science **318**, 766 (2007).
[10] A. Roth, C. Brüne, H. Buhmann, L. W. Molenkamp, J. Maciejko, X.-L. Qi, and S.-C. Zhang, Science **325**, 294 (2009).
[11] K. C. Nowack, E. M. Spanton, M. Baenninger, M. König, J. R. Kirtley, B. Kalisky, C. Ames, P. Leubner, C. Brüne, H. Buhmann, L. W. Molenkamp, D. Goldhaber-Gordon, and K. A. Moler, Nat. Mater. **12**, 787 (2013).
[12] I. Knez, R.-R. Du, and G. Sullivan, Phys. Rev. Lett. **107**, 136603 (2011).
[13] T. Li, P. Wang, H. Fu, L. Du, K. A. Schreiber, X. Mu, X. Liu, G. Sullivan, G. A. Csáthy, X. Lin, and R.-R. Du, Phys. Rev. Lett. **115**, 136804 (2015).
[14] D. C. Mattis, *The Many-Body Problem: An Encyclopedia of Exactly Solved Models in One Dimension* (World Scientific, Singapore, 1993).
[15] D. C. Mattis and V. Mastropietro. *The Luttinger Model: The First Fifty Years and Some New Directions* (World Scientific, Singapore, 2015).
[16] T. L. Schmidt, S. Rachel, F. von Oppen, and L. I. Glazman, Phys. Rev. Lett. **108**, 156402 (2012).
[17] N. Lezmy, Y. Oreg, and M. Berkooz, Phys. Rev. B **85**, 235304 (2012).
[18] N. Kainaris, I. V. Gornyi, S. T. Carr, and A. D. Mirlin, Phys. Rev. B **90**, 075118 (2014).
[19] Y.-Z. Chou, A. Levchenko, and M. S. Foster, Phys. Rev. Lett. **115**, 186404 (2015).
[20] A. Strom, H. Johannesson, and G. I. Japaridze, Phys. Rev. Lett. **104**, 256804 (2010).
[21] J. Maciejko, C. Liu, Y. Oreg, X.-L. Qi, C. Wu, and S.-C. Zhang, Phys. Rev. Lett. **102**, 256803 (2009).
[22] M. Hohenadler and F. F. Assaad, Phys. Rev. B **85**, 081106 (2012); **86**, 199901(E) (2012).
[23] B. L. Altshuler, I. L. Aleiner, and V. I. Yudson, Phys. Rev. Lett. **111**, 086401 (2013).
[24] J. I. Väyrynen, M. Goldstein, and L. I. Glazman, Phys. Rev. Lett. **110**, 216402 (2013).
[25] N. Traverso Ziani, C. Fleckenstein, F. Crépin, and B. Trauzettel, Europhys. Lett. **113**, 37002 (2016).
[26] H.-Y. Xie, H. Li, Y.-Z. Chou, and M. S. Foster, Phys. Rev. Lett. **116**, 086603 (2016).
[27] J. C. Budich, F. Dolcini, P. Recher, and B. Trauzettel, Phys. Rev. Lett. **108**, 086602 (2012).
[28] I. F. Herbut, V. Juričić, and O. Vafek, Phys. Rev. Lett. **100**, 046403 (2008).
[29] A. Giuliani, V. Mastropietro, and M. Porta, Phys. Rev. B **83**, 195401 (2011); Commun. Math. Phys. **311**, 317 (2012).
[30] G. Benfatto, P. Falco, and V. Mastropietro, Phys. Rev. Lett. **104**, 075701 (2010); Commun. Math. Phys. **292**, 569 (2009); **330**, 153 (2014); **330**, 217 (2014).
[31] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).
[32] A. Giuliani, V. Mastropietro, and M. Porta, Commun. Math. Phys. **349**, 1107 (2016).
[33] N. Hao, P. Zhang, Z. Wang, W. Zhang, and Y. Wang, Phys. Rev. B **78**, 075438 (2008).
[34] B. I. Halperin, Phys. Rev. B **25**, 2185 (1982).
[35] Y. Hatsugai, Phys. Rev. Lett. **71**, 3697 (1993).
[36] H. Schulz-Baldes, J. Kellendonk, and T. Richter, J. Phys. A **33**, L27 (2000).

[37] P. Elbau and G. M. Graf, Commun. Math. Phys. **229**, 415 (2002).

[38] X.-L. Qi, Y.-S. Wu, and S.-C. Zhang, Phys. Rev. B **74**, 085308 (2006).

[39] J. C. Avila, H. Schulz-Baldes, and C. Villegas-Blas, Math. Phys. Anal. Geom. **16**, 137 (2013).

[40] G. M. Graf and M. Porta, Commun. Math. Phys. **324**, 851 (2013).

[41] A. Giuliani, I. Jauslin, V. Mastropietro, and M. Porta, Phys. Rev. B **94**, 205139 (2016).

[42] G. Antinucci, V. Mastropietro, and M. Porta, arXiv:1708.08517.

[43] D. C. Brydges, *Phénomènes Critiques, Systèmes Aléatoires, Théories de Jauge* (North-Holland, Amsterdam, 1984), pp. 129–183.

[44] V. Mastropietro, J. Math. Phys. **48**, 022302 (2007).