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Twisted Real Structures in Noncommutative Geometry

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TWISTED REAL STRUCTURES IN NONCOMMUTATIVE GEOMETRY

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SISSA

A thesis submitted in partial fulfilment of the requirement for the degree of Philosophiae Doctor in Geometry & Mathematical Physics, equivalent to the title of *Dottore di Ricerca in Matematica*, under the supervision of Ludwik Dąbrowski

Dedicated to my parents,
who always supported my education.

ABSTRACT

Twisted real structures are a generalisation of real structures for spectral triples which are motivated as a way to implement the conformal transformation of a Dirac operator without needing to twist the noncommutative 1-forms. Taking inspiration from this example, in this thesis, we study further applications of twisted real structures, in particular those pertaining to commutative or almost-commutative geometries. We investigate how a reality operator can implement a noncommutative Clifford algebra Morita equivalence bimodule and find that the corresponding real structure on a commutative spectral triple must be twisted. We also investigate how the presence of a twisted real structure affects the implementation of the C*-algebra self-Morita equivalence bimodule which gives the gauge transformations of a spectral triple and find that the twist operators must be tightly constrained to yield meaningful physical action functionals. The form of the resulting action functionals suggests that the twist operator may implement a Krein structure, which often appears in pseudo-Riemannian generalisations of spectral triples. Thus we further investigate if twisted real structures can implement Wick rotations, and though we do not find a fully satisfactory construction, our preliminary attempts are encouraging and suggest that the possibility cannot yet be ruled out. Lastly we identify from the literature that the twisted spectral triple for κ-Minkowski space admits a reality operator which gives a twisted real structure. This indicates that twisted real structures are compatible with twisted spectral triples as had been previously conjectured, opening up a whole new range of potential applications.

PUBLICATIONS

Some material has appeared previously in the following publications:

Francesco D'Andrea, Ludwik Dąbrowski, and Adam M. Magee. "Twisted reality and the second-order condition." In: *Mathematical Physics, Analysis and Geometry* (2021).

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Nothing will ever please me, no matter how excellent or beneficial,
if I must retain the knowledge of it to myself.
And if wisdom were given me under the express condition
that it must be kept hidden and not uttered, I should refuse it.
No good thing is pleasant to possess, without friends to share it.

— Seneca

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Quote from Lucius Annaeus Seneca the Younger. "Epistulae morales ad Lucilium." Trans. Latin by Richard M. Gummere. Harvard University Press, 1917. Letter VI.

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CONTENTS

SYMBOLS

A	$*$-algebra
$\mathcal{A}$	C*-algebra
$A^{\times}$	Invertible elements of A
A_{+}	Positive elements of A
A^{op}	Opposite algebra of A
$\bar{A}$	Norm-closure/completion of A
$\mathrm{Aut}(A)$	($*$-)Automorphisms of A
$\mathcal{B}(\mathcal{H})$	Bounded operators on a Hilbert space $\mathcal{H}$
$\mathbb{C}$	Complex numbers
$C(M)$	Continuous functions on a (compact) manifold M
$C^{\infty}(M)$	Smooth functions on a (compact) manifold M
$C_c(G, V)$	Continuous functions with compact support from G to V
$C^*(G)$	Group C*-algebra of a group G
$\det(T)$	Determinant of T
$\mathrm{diag}(t_1, \ldots, t_n)$	$n \times n$ diagonal matrix with diagonal entries $t_1, \ldots, t_n$
$\mathrm{Dom}(T)$	Domain of T
$\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}'$	$\mathcal{A}$-balanced tensor product of modules $\mathcal{E}$ and $\mathcal{E}'$
$\mathrm{End}(\mathcal{H})$	Linear endomorphisms of $\mathcal{H}$ with dense domain
$\mathrm{End}_{\mathcal{A}}(\mathcal{E})$	Adjointable maps on the Hilbert $\mathcal{A}$-module $\mathcal{E}$
$\Gamma(E)$	(Continuous or smooth) sections of the bundle $E \to M$
$G \ltimes G'$	Semidirect product of groups G and G'
$G \ltimes_{\alpha,r} \mathcal{A}$	Reduced crossed product C*-algebra
$\hbar$	Planck's constant
$\mathbb{H}$	Quaternions
$H^k(M, R)$	k^{th} cohomology group of M, valued in R
$\mathrm{Hom}(V, V')$	Linear maps between vector spaces V and V'

$\mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{E}')$	$\mathcal{A}$-linear homomorphisms (adjointable maps) from $\mathcal{E}$ to $\mathcal{E}'$
id	Identity map
$\mathbb{K}$	Field
$\mathcal{K}(\mathcal{H})$	Compact operators on $\mathcal{H}$
$L^p(X)$	Lebesgue space of functions on X with the p-norm
$M_n(V)$	V-valued $n \times n$ square matrices
$\mathrm{O}(V)$	Orthogonal group of V; denoted $\mathrm{O}(n)$ when $V = \mathbb{R}^n$
$\mathbb{N}$	Natural numbers (excluding 0)
$\mathbb{N}_0$	Natural numbers including 0; $\mathbb{N} \cup \{0\}$
$\mathbb{R}$	Real numbers
S_k	Symmetric group of k symbols
S^n	Sphere of dimension n
$\mathrm{SO}(V)$	Special orthogonal group of V; denoted $\mathrm{SO}(n)$ when $V = \mathbb{R}^n$
$\mathrm{SU}(V)$	Special unitary group of V; denoted $\mathrm{SU}(n)$ when $V = \mathbb{C}^n$
TM	Tangent bundle over M
T^*M	Cotangent bundle over M
T_xM	Tangent space at $x \in M$
T_x^*M	Cotangent space at $x \in M$
$\mathrm{tr}(T)$	Trace of T
$\mathrm{U}(V)$	Unitary group of V; denoted $\mathrm{U}(n)$ when $V = \mathbb{C}^n$
$\mathbb{Z}$	Integers
$\mathbb{Z}_n$	Integers modulo n
$Z(A)$	Centre of A

1 INTRODUCTION

1.1 NONCOMMUTATIVE GEOMETRY

The language of classical physics is that of functions on smooth manifolds, and represented the dominant mathematical paradigm within physics until the establishment of modern quantum mechanics in the 1920s and 1930s. This new quantum framework introduced noncommuting observables which led to the famous Heisenberg uncertainty relation

$$[X, P] = i\hbar \implies \sigma_x \sigma_p \geq \frac{\hbar}{2},$$

which meant it was no longer possible (even in principle) to resolve arbitrarily small regions in phase space.

Today, fundamental physics is almost entirely expressed within the language of quantum mechanics and quantum field theory. However, there remains the significant exception of gravity, which remains firmly placed within the realm of classical mechanics, in particular, in the description of (curved) Riemannian manifolds.

It is widely believed that for gravity in the most extreme circumstances (in regions of very high energy-momentum density, or at very small length/timescales), quantum effects will dominate and it will be necessary to supplant general relativity by a quantum theory of gravity. As the spirit of general relativity is that gravity is only a fictitious force, the effect of observers inhabiting a curved spacetime, a quantum theory of gravity may necessitate the definition of a new kind of *quantum space(time)* and correspondingly a new mathematical notion of quantum geometry to describe it.

A well-known heuristic argument goes that the energy required to make Planck-scale measurements would be great enough to in turn create a Planck-scale black hole; of course, other arguments have also been made.

As such, 'quantum geometry' can be understood as a project spanning multiple fields of mathematics and physics to define in a rigorous and meaningful way a suitable notion of 'quantum space'. Within quantum gravity, a wide variety of approaches have been made coming from *e.g.* string theory and holography, loop quantum gravity and spin foams, causal dynamical triangulation, and causal sets, for a very incomplete list. However, the challenge of developing new and exotic kinds of spaces carries its own inherent mathematical interest, and so attempts have also come from the realm of pure mathematics. Coming from this side, arguably the biggest field working on quantum geometry is *noncommutative geometry* introduced by A. Connes,[1] which is

Noncommutative geometry is not to be confused with 'algebraic geometry', which takes a similar philosophy of algebraicisation but instead works with polynomial rings inspired by Hilbert's Nullstellensatz.

1 Alain Connes. "Non-commutative differential geometry." In: *Publications Mathématiques de l'IHÉS* (1985).

characterised by the philosophy that geometry ought to be generalised through algebraicisation inspired by Gelfand duality.

Gelfand duality[2] (also known as Stone-Gelfand or Gelfand-Naimark duality) is the statement that the category of locally compact Hausdorff topological spaces is dually equivalent to the category of commutative C*-algebras. In less formal language, this means that we can forsake the traditional language of topological spaces and instead talk about the same mathematical objects in purely algebraic terms (algebraicisation). Commutative C*-algebras, all of which are (hence) each an algebra of continuous functions on some topological space, mesh neatly with the language of classical physics. However, the study of C*-algebras initially grew out of W. Heisenberg's matrix mechanics formalism of quantum physics, developed by P. Jordan and then J. von Neumann into a full axiomatisation of quantum mechanics, a program which has also since been continued (incompletely) into quantum field theory.

Def. *A* C*-algebra *$\mathcal{A}$ is a Banach $*$-algebra satisfying $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$.*

We can (formally) extend Gelfand duality such that, by definition, the dual of the category of noncommutative C*-algebras is the category of "noncommutative topological spaces".That quantum mechanics is intimately connected with noncommutative C*-algebras makes for a very compelling motivation to interpret these 'virtual' noncommutative spaces as quantum spaces. However, topology alone is not sufficient; in order to provide a meaningful notion of a quantum space, one also needs geometry, so the question then becomes: what extra structure needs to be added into the mix (and how) to recover geometric notions like distance?

The answer to the question of how is hinted at by considering the Gelfand-Naimark theorem, which states that any C*-algebra is isometrically $*$-isomorphic to a C*-subalgebra of the bounded operators on a Hilbert space. It makes sense, then, that the missing ingredient should also be a (possibly unbounded) operator on a Hilbert space. As for what, the winding path to the answer will be sketched in §1.2, but skipping ahead, what we are looking for is a *generalised Dirac operator*, an operator which generalises the elliptic differential operator of the Atiyah-Singer index theorem. Then the distance function on the state space of a C*-algebra $\mathcal{A}$ can be given as

Def. *A* state *$\phi\colon \mathcal{A} \to \mathbb{C}$ is a positive linear functional with $\|\phi\| = 1$.*

$$d(\phi, \phi') = \sup\left\{\left|\phi(a) - \phi'(a)\right| : a \in \mathcal{A}, \|[D, \pi(a)]\| \leq 1\right\}$$

for ϕ, ϕ' states on $\mathcal{A}$, $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$, and $D\colon \mathcal{H} \to \mathcal{H}$ the generalised Dirac operator [14]. This formula reduces to the ordinary distance function on a manifold when $\mathcal{A} = C(M)$ and so D encodes the information of the metric. As the name suggests, the Dirac operator of physics is an example of such an operator; the fact that D encodes distance can be understood in physical terms by heuristically considering D^{-1} as the fermion propagator, or more mathematically, as the (infinitesimal)

2 First shown in Israel Gelfand. "Normierte Ringe." In: *Recueil Mathématique* (1941), only later made categorical.

line element. The Dirac operator can also be used to give a smooth structure and notions of calculus in similarly algebraic terms, and so we arrive at a candidate quantum Riemannian manifold, the *spectral triple*

$$(\mathcal{A}, \mathcal{H}, D).$$

It should be noted that the story of quantum spaces is far from over. Spectral triples offer an excellent language for generalising Riemannian manifolds, but can at best be seen as only a first quantisation of geometry. Indeed, the action functionals derived from spectral triples are classical in nature, needing to be subsequently quantised in the usual fashion; the matter of finding a suitable second quantisation, by which one might obtain a true quantum space potentially suitable for understanding Planck-scale physics, is still very much an open question. That said, interestingly and promisingly, spectral triples find fruitful applications in settings far removed from physics as well, in particular to spaces which are badly behaved as point sets such as the space of Penrose tilings, the space of leaves of a foliation, and the space of irreducible unitary representations of a discrete group [14], and more recently there have been strong hints of applications to number theory as well.

Apart from spectral triples, as a field, 'noncommutative geometry' also encompasses a variety of related approaches. A second major research program treats the question of quantum space as secondary and takes as primary the perspective that what is more essential is not the underlying space itself but its symmetry properties, and hence presupposes that just as the symmetries of classical spaces are described by finite and Lie groups, quantum spaces should likewise have symmetries described by 'quantum groups' which generalise these notions. To avoid confusion or conflation, we will occasionally refer to the framework of spectral triples as 'spectral geometry', and it is spectral geometry which will be the focus and setting of this thesis. Even so, there is considerable overlap between spectral geometry and quantum groups and we will see an example of that overlap in Ch. 6.

1.2 REAL STRUCTURES FOR SPECTRAL TRIPLES

To a spectral triple one may be able to add extra structure, depending on the specific example under consideration. One particularly important kind of extra structure is a *real structure*. In this section, we go into more mathematical depth to give a concise explanation of what a real structure for a spectral triple is and where that definition comes from. The presentation is primarily based on Ref. [41].

1.2.1 *Spin geometry*

The first and main source of inspiration for (real) spectral triples comes from the field of spin geometry, and in order to speak that language we begin by introducing some important vector bundles over a (compact) oriented manifold M. The first bundle of interest is the orthonormal (tangent) frame bundle $F_{\mathrm{SO}}(M) \subset \mathrm{Hom}(M \times \mathbb{R}^n, TM)$, which is a principal $\mathrm{SO}(n)$-bundle over M.

An *oriented Riemannian manifold* is an oriented manifold M whose orthonormal frame bundle $F_{\mathrm{SO}}(M)$ satisfies the vector bundle isomorphism

To fix notation, to any principal G-bundle $(P \to M, G)$, for $\rho\colon G \to \mathrm{End}(V)$, one can define an associated vector bundle *over M which we denote by $P{\times_G}V := (P{\times}V)/G$.*

$$F_{\mathrm{SO}}(M) \times_{\mathrm{SO}(n)} \mathbb{R}^n \simeq TM.$$

The same recipe can be used for other groups. In particular, we will be interested in the spin group $\mathrm{Spin}(n)$ which is the double cover of $\mathrm{SO}(n)$, and the spin^c group which can be understood as the central extension of $\mathrm{SO}(n)$ by S^1, *i.e.* $\mathrm{Spin}^c(n) \simeq \mathrm{Spin}(n) {\times_{\mathbb{Z}_2}} \mathrm{U}(1)$. So, assuming our manifold M is oriented and Riemannian, a *spin$^{(c)}$ manifold* is an oriented Riemannian manifold M whose principal $\mathrm{Spin}^{(c)}(n)$-bundle $\mathrm{Spin}^{(c)}(M)$ satisfies the vector bundle isomorphism

$$\mathrm{Spin}^{(c)}(M) \times_{\mathrm{Spin}^{(c)}(n)} \mathbb{R}^n \simeq TM.$$

The main reason for considering spin^c manifolds in spectral geometry is that a manifold being spin^c is sufficient for the (local) existence of a Dirac operator, but they (and more so spin manifolds) have other very nice properties.

In the previous section we saw how the concept of a noncommutative topological space was inspired by Gelfand duality. We now pursue a similar notion of a noncommutative vector bundle, this time taking inspiration from the (topological) Serre-Swan theorem:[3] One has a complex vector bundle $E \to M$ if and only if there exists a finitely-generated projective $C(M)$-module $\mathcal{E}$, *i.e.*

$$\mathcal{E} \simeq p\Big(\bigoplus^n C(M)\Big), \quad p = p^\dagger = p^2 \in M_n(C(M)).$$

One then has $\mathcal{E} = \Gamma(E)$.

In order to make use of the notion of noncommutative spaces, we need to make contact with Hilbert spaces. This is done by considering *Hilbert C*-modules* (see Def. 3.2) which provide a good notion of a module over a C*-algebra coming from the theory of Hilbert spaces. Special cases include Hilbert spaces themselves, C*-algebras, and hermitian vector bundles (vector bundles whose fibres are equipped with an inner product).

A key role in going ahead will be played by Clifford algebras.

3 Richard Swan. "Vector bundles and projective modules," In: *Transactions of the American Mathematical Society* (1962).

Definition 1.1 (Clifford algebra). A *Clifford algebra* is the algebra given by $C\ell(V,g) := T(V)/\mathcal{I}_g$ where $T(V) := \bigoplus_{i=0}^{\infty} V^{\otimes i}$ and $\mathcal{I}_g$ is the ideal generated by $v \otimes w + w \otimes v - 2g(v,w)$ for $g\colon V \times V \to \mathbb{K}$ a symmetric bilinear form. ◇

The most important Clifford algebras are the real Clifford algebras and the complex Clifford algebras. Real Clifford algebras $C\ell_{p,q}$ have $V = \mathbb{R}^{p+q}$ with $g(x,y) = \sum_{j=1}^{p} x_i y_j - \sum_{j=p+1}^{p+q} x_j y_j$. Complex Clifford algebras $\mathbb{C}\mathrm{l}_n$ have $V = \mathbb{C}^n$ with $g(z,z') = \sum_{j=1}^{n} z_j z'_j$. In particular, when V is real, one can obtain a complex Clifford algebra by complexifying $C\ell(V^{\mathbb{C}}, g^{\mathbb{C}}) \equiv C\ell(V,g) \otimes_{\mathbb{R}} \mathbb{C} =: \mathbb{C}\mathrm{l}(V)$. Complex Clifford algebras are especially useful since they are C*-algebras.

One can define on $\mathbb{C}\mathrm{l}(V) = \mathbb{C}\mathrm{l}^+(V) \oplus \mathbb{C}\mathrm{l}^-(V)$ a $\mathbb{Z}_2$-grading χ such that $\chi(v_1 v_2 \ldots v_k) = (-1)^k v_1 v_2 \ldots v_k$, $v_j \in V$. Combining χ with complex conjugation $a \mapsto a^*$ gives the antilinear isomorphism

$$\kappa(a) := \chi(a^*)$$

called *charge conjugation*. Another useful map is the product-reversal map $(v_1 v_2 \ldots v_k)^{\intercal} = v_k \ldots v_2 v_1$ which can be used to define the involution $a^{\dagger} := (a^*)^{\intercal}$.

Let c be an irreducible representation of $\mathbb{C}\mathrm{l}^{(+)}(V)$ on $\mathcal{H}$. We can define an antiunitary operator $C\colon \mathcal{H} \to \mathcal{H}$ satisfying

We write $\mathbb{C}\mathrm{l}^{(+)}(V)$ to denote $\mathbb{C}\mathrm{l}(V)$ when $\dim V$ *is even and* $\mathbb{C}\mathrm{l}^+(V)$ *when* $\dim V$ *is odd.*

$$\langle C\varphi, C\psi \rangle = \langle \psi, \varphi \rangle, \quad C^2 = \pm 1.$$

This *charge conjugation operator* implements charge conjugation as an antilinear isomorphism

$$c(\kappa(a)) = C c(a) C^{-1}.$$

We can then consider (c, C) as a representation of $(\mathbb{C}\mathrm{l}^{(+)}(V), \kappa)$, taken as an involutive algebra. These representations can then be classified.

It is possible to determine the action of κ on $\mathbb{C}\mathrm{l}(\mathbb{R}^n)$ by its restriction to $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$ using its antilinearity. Hence we represent on the space $\mathbb{R}^p \oplus i\mathbb{R}^q$, $p + q = n$, such that the representations of $(\mathbb{C}\mathrm{l}(\mathbb{R}^n), \kappa)$ are given by those of $C\ell_{p,q}$. An important feature of $C\ell_{p,q}$ is that has the periodicities

$$C\ell_{p+1,q+1} \simeq C\ell_{p,q} \otimes_{\mathbb{R}} M_2(\mathbb{R}), \tag{1.1}$$

$$C\ell_{p+8,q} \simeq C\ell_{p,q} \otimes_{\mathbb{R}} M_{16}(\mathbb{R}) \simeq C\ell_{p,q+8}, \tag{1.2}$$

which will be important later. Note that this is considerably richer than the complex case, since $\mathbb{C}\mathrm{l}_n$ has the much smaller periodicity $\mathbb{C}\mathrm{l}_{n+2} \simeq \mathbb{C}\mathrm{l}_n \otimes_{\mathbb{C}} \mathbb{C}\mathrm{l}_2$.

The unitary elements ($u^{\dagger} u = u u^{\dagger} = 1$) of $\mathbb{C}\mathrm{l}(V)$ generate $\mathrm{O}(V)$ via the map $\Phi_u(a) = \chi(u) a u^{-1}$ (a reflection of a in the hyperplane $u_{\perp}$), and the subgroup $\mathrm{SO}(V)$ is formed by even compositions of Φ (rotations). The group $\mathrm{Spin}^c(V)$ is then generated by even products of unitaries acting by the same map Φ, which produces the short exact sequence

$$1 \to \mathrm{U}(1) \to \mathrm{Spin}^c(V) \xrightarrow{\Phi} \mathrm{SO}(V) \to 1.$$

If we denote by $\tau\colon u \mapsto u^{\intercal}u$ the map $\mathrm{Spin}^c(V) \to U(1)$, then we can define the spin group by $\mathrm{Spin}(V) = \ker(\tau) \subset \mathrm{Spin}^c(V)$, which in turn gives the short exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(V) \xrightarrow{\Phi} \mathrm{SO}(V) \to 1.$$

Note that this sequence demonstrates the well-known fact that $\mathrm{Spin}(V)$ is the double cover of $\mathrm{SO}(V)$.

Since the elements u of $\mathrm{Spin}(V)$ are unitary ($u^{\dagger} = u^{-1}$) and satisfy $u^{\intercal}u = 1$, they also satisfy $u^* = u$. Since, furthermore, they are even ($\chi(u) = u$) by definition, $\mathrm{Spin}(V)$ can be viewed as the charge conjugation-invariant subgroup of $\mathrm{Spin}^c(V)$.

In order to understand under what conditions a manifold is $\mathrm{spin}^{(c)}$, we first define the *Clifford bundle* as the algebra bundle coming from the associated vector bundle

In simpler terms, the Clifford bundle over (M, g) is the algebra bundle generated by T^*M equipped with the bilinear form g^{-1} on fibres.

$$\mathbb{C}\mathrm{l}(M) \equiv \mathbb{C}\mathrm{l}_n(M, g) := F_{\mathrm{SO}}(M) \times_{\mathrm{SO}(n)} \mathbb{C}\mathrm{l}_n.$$

This is a hermitian vector bundle, and so we can employ the Serre-Swan theorem to employ the language of modules. In particular, a (complex) *Clifford module* over (M, g) is a finitely generated and projective Hilbert $\Gamma(\mathbb{C}\mathrm{l}(M))$-$C(M)$-bimodule.

An interesting question to ask is, for a Clifford module which we will suggestively call $\mathcal{S}^c = \Gamma(S^c)$, when $\mathbb{C}\mathrm{l}_x(M)$ has an irreducible representation on the fibres Σ_x of $S^c \to M$. The answer is when $\mathcal{S}^c$ is a Morita equivalence bimodule, or equivalently, when $\mathbb{C}\mathrm{l}^{(+)}(M) \simeq \mathrm{End}(\mathcal{S}^c)$. When this happens, $\{\mathbb{C}\mathrm{l}_x^{(+)}(M)\}$ forms a locally trivial continuous field of elementary C*-algebras $\underline{\mathbb{C}\mathrm{l}}^{(+)}(M)$. In that case, one finds

Speaking roughly, continuous fields of C-algebras can be thought of as generalised bundles whose fibres are C*-algebras. Elementary C*-algebras are those isomorphic to $\mathcal{K}(\mathcal{H})$ for $\dim \mathcal{H}$ finite or $\dim \mathcal{H}$ infinite and $\mathcal{H}$ separable.*

$$C(M) \sim \Gamma(\mathbb{C}\mathrm{l}^{(+)}(M)) \text{ if and only if } \delta(\underline{\mathbb{C}\mathrm{l}}^{(+)}(M)) = 0 \text{ in } H^3(M, \mathbb{Z}), \quad (1.3)$$

where δ gives the Dixmier-Douady class, which in this case is the third integral Stiefel-Whitney class, the usual topological obstruction to a manifold being spin^c.

The fibres Σ can be identified with the *fermionic Fock space*, and the irreducible representation of $\mathbb{C}\mathrm{l}_x(M) \simeq \mathbb{C}\mathrm{l}_n$ restricts to the *spin representation c* for the subgroups $\mathrm{Spin}^{(c)}(n) \subset \mathbb{C}\mathrm{l}_n$. We have thus found that the bundle corresponding to the module $\mathcal{S}^c$ is the *charged spinor bundle*

$$S_n^c = \mathrm{Spin}^c(M) \times_{\mathrm{Spin}^c(n)} \Sigma_n,$$

and $\mathcal{S}^c$, along with an orientation on M, give a spin^c-structure, making M a spin^c-manifold. This presentation neatly motivates the reason spin$^{(c)}$ manifolds are so interesting: it is because they are especially rich in structure precisely because one has access to the spin representations.

The natural next step is to understand how to recover the *spinor bundle*

$$S_n = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \Sigma_n.$$

The way to do so is by recalling that $\mathrm{Spin}(V)$ can be viewed as the κ-invariant subgroup of $\mathrm{Spin}^c(V)$ and applying this observation to the bundle structure, which we can do using a charge conjugation operator acting on the module $\mathcal{S}^c$.

Let $C\colon \mathcal{S}^c \to \mathcal{S}^c$ be a bijective antilinear map satisfying

$$C(\psi f) = (C\psi)f^*, \quad C(a\psi) = \chi(a^*)C\psi, \quad \langle C\varphi, C\psi\rangle = \langle \psi, \varphi\rangle,$$

for $f \in C(M)$, $a \in \Gamma(\mathbb{C}\mathrm{l}^{(+)}(M))$, and $\psi, \varphi \in \mathcal{S}^c$. Then we have that M is spin if and only if such a $C\colon \mathcal{S}^c \to \mathcal{S}^c$ exists. For clarity, we denote by $\mathcal{S}$ a module $\mathcal{S}^c$ admitting such an antilinear map C. Then C along with $\mathcal{S}$ and an orientation on M provide a *spin structure*, making M a spin manifold, and the spinor bundle $S \to M$ is hence obtained from the module $\mathcal{S}$.

For $\mathcal{A} := \Gamma(\mathbb{C}\mathrm{l}^{(+)}(M))$, denote $\mathcal{S}^c_* = \mathrm{Hom}_{C(M)}(\mathcal{S}^c, C(M))$. The invariant giving the obstruction to M being spin is given by mod 2 reduction on $[\mathrm{Hom}_{\mathcal{A}}(\mathcal{S}^c_*, \mathcal{S}^c)] \in H^2(M, \mathbb{Z}_2)$, cf. the second Stiefel-Whitney class being the mod 2 reduction of the third integral Stiefel-Whitney class. For details on how this relates to C, see e.g. the proof of [41, Thm. 9.6].

We can now start to connect to spectral triples, which we will do via Fredholm modules. A *pre-Fredholm module* over a pre-C*-algebra A is a representation $\sigma\colon A \to \mathrm{End}(\mathcal{H})$ for $\mathcal{H}$ a Hilbert space along with an operator $F\colon \mathcal{H} \to \mathcal{H}$ such that

$$\sigma(a)(F - F^\dagger) \in \mathcal{K}(\mathcal{H}), \quad \sigma(a)(F^2 - 1) \in \mathcal{K}(\mathcal{H}), \quad [F, \sigma(a)] \in \mathcal{K}(\mathcal{H})$$

for all $a \in A$. Any pre-Fredholm module gives a *K-cycle over A* so that $[(A, \mathcal{H}, F)]$ is a class in K-homology which generates $K^0(A)$ or $K^1(A)$. The reason there are only two possible K-homology groups is related to the 2-periodicity of complex Clifford algebras.

Def. *A* pre-C*-algebra *is a $*$-algebra which satisfies the requirements to be a C*-algebra but is not norm-complete.*

If D is an unbounded, self-adjoint operator with compact resolvent satisfying $[D, \pi(a)] \in B(\mathcal{H})$ for all $a \in A$, $\pi\colon A \to \mathcal{B}(\mathcal{H})$ (*i.e.* a Dirac operator), the bounded transform

$$F' := D(1 + D^2)^{-1/2}$$

gives a pre-Fredholm module $(A, \mathcal{H}, F')$. All K-homology classes can be obtained in this way and thus the spectral triple $(A, \mathcal{H}, D)$ is motivated as an 'unbounded K-cycle'.

Recall that the representations of $(\mathbb{C}\mathrm{l}_n, \kappa)$ are given by those of $C\ell_{p,q}$. As such, if A carries an involution ι, we can incorporate a compatible action by $C\ell_{p,q}$ on an unbounded K-cycle to produce an *unreduced KR-cycle over* (A, ι).[4] This is an unbounded K-cycle over A with an antilinear isometry implementing ι, a grading (if even) and a representation of

Note that, if A is equipped with the standard involution $a \mapsto a^*$, ι need not coincide with $*$.

4 First motivated by Michael Atiyah. "K-theory and reality," In: *The Quarterly Journal of Mathematics* (1966); this was extended to homology (in the context of KK-theory) by Gennady Kasparov. "The operator K-functor and extensions of C^*-algebras," In: *Mathematics of the USSR-Izvestiya* (1981).

$C\ell_{p,q}$ on the Hilbert space compatible with the other operators.

We would like Dirac operators on spin manifolds to give examples of KR-cycles, but the supercommuting representation of $C\ell_{p,q}$ is incompatible with the irreducibility of spinor modules. Thankfully, any unreduced KR-cycle over (A,ι) can be reduced using the (1,1)-periodicity of real Clifford algebras (1.1) to a *reduced* KR^j*-cycle*: a representation of A on a Hilbert (sub)space $\mathcal{H}$ together with

1. $D\colon \mathcal{H} \to \mathcal{H}$ self-adjoint such that $(A,\mathcal{H},D)$ is an unbounded K-cycle;
2. $J\colon \mathcal{H} \to \mathcal{H}$ an antilinear isometry implementing ι, that is to say, $\pi(\iota(a)) = J\pi(a)J^{-1}$;
3. When j is even, a grading χ of $\mathcal{H}$ with $\chi D = -D\chi$;
4. J obeying the algebraic relations

$$J^2 = \pm 1, \quad JD = \pm DJ, \quad J\chi = \pm \chi J. \tag{1.4}$$

The 8-periodicity of real Clifford algebras (1.2) dictates that there should be at most 8 cases remaining, and indeed, these 8 cases are encoded by the three signs (1.4) dictated by j, which comes from the real Clifford algebra as $q-p \pmod 8$.

That a real spectral triple, with some additional axioms, is a fully algebraic characterisation of a spin manifold when $A = C^\infty(M)$ [18], commonly known as Connes' reconstruction theorem, is the theoretical cornerstone of spectral geometry.

1.2.2 *Other ingredients and the real structure*

Reduced KR-cycles provide a blueprint for a spectral triple equipped with a real structure, but being closely modelled on the commutative case, they are not quite powerful enough by themselves to handle all of the cases where A is not commutative. For that, we turn for inspiration to the setting of von Neumann algebras and, in particular, Tomita's theorem.[5]

Def. *For a Hilbert space $\mathcal{H}$, a* von Neumann algebra *is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in the weak operator topology.*

Def. *A vector $\xi \in \mathcal{H}$ is* cyclic *for $\mathcal{V}$ if $\mathcal{V}\xi$ is dense in $\mathcal{H}$. It is* separating *for $\mathcal{V}$ if $V\xi = 0$ implies $V = 0$ for all $V \in \mathcal{V}$.*

Let $\mathcal{V}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with $\xi \in \mathcal{H}$ a norm-1, cyclic and separating vector for $\mathcal{V}$. Define an unbounded antilinear operator S_0 on $\mathcal{H}$ such that $S_0 V\xi = V^\dagger \xi$ for all $V \in \mathcal{V}$. This extends to an operator S on a dense subset of $\mathcal{H}$ admitting the polar decomposition

$$S = J\Delta^{1/2} = \Delta^{-1/2}J$$

5 Minoru Tomita. "On canonical forms of von Neumann algebras," In: *Fifth Functional Analysis Symposium* (1967) in Japanese, and Minoru Tomita. "Quasi-standard von Neumann algebras," unpublished. The material was explained and much more widely disseminated in Masamichi Takesaki. *Tomita's Theory of Modular Hilbert Algebras and Its Applications*. Springer-Verlag, 1970.

where $J = J^{-1} = J^\dagger$ is an antilinear isometry often called the 'modular conjugation'. Then $J\mathcal{V}J^{-1} = \mathcal{V}'$.

It is not difficult to see how this can be applied to the setting of spectral triples; all one needs to do is replace the von Neumann algebra $\mathcal{V}$ by the (representation of the) pre-C*-algebra A (noting that all von Neumann algebras are C*-algebras) and require that the antilinear operator of the reduced KR-cycle implements the commutation

$$\left[\pi(a), J\pi(b)J^{-1}\right] = 0 \tag{1.5}$$

for all $a, b \in A$. This limited commutativity, often called the *reality* or *zeroth-order condition* (the latter name comes only from the visual similarity to the first-order condition below, and should not be taken literally), has a number of advantages, but an important one is it allows for the definition of the first-order condition.

In the commutative setting, the algebraic way to express that a differential operator D has order k is that it satisfies

$$[[[[D, f_0], f_1], \ldots], f_k] = 0$$

for $f_0, \ldots, f_k \in C^\infty(M)$. Unfortunately, this expression relies explicitly on the commutatitvity of the functions. However, the axiom inspired by Tomita's theorem (1.5) allows us to use the same technique to describe differential operators up to first order in the noncommutative setting as satisfying

Def. *For vector bundles E and E' over M, the* order *of a differential operator $D\colon \Gamma(E) \to \Gamma(E')$ is the smallest integer k such that D factors through the jet bundle $J^k(E)$.*

$$\left[[D, \pi(a_0)], J\pi(a_1)J^{-1}\right] = 0$$

for $a_0, a_1 \in A$, which is called the *first-order condition*. This is ideal since the Dirac operator is a differential operator which is (famously) first-order by construction, but it is also sufficient to allow a spectral triple to describe a noncommutative spin$^{(c)}$ manifold in a manner analogous to (1.3), as will be explained in Ch. 3.

Having now collected all of the necessary ingredients, one finally arrives at the definition of a spectral triple equipped with a real structure, Def. 2.2, first introduced in Ref. [15], as a reduced KR^j-cycle whose antilinear isometry implements the zeroth- and first-order conditions.

Examples of real spectral triples include some of the most important and well-understood spectral triples: spin manifolds, the noncommutative torus, and matrix geometries, to name a few. In particular, they are essential for describing the spectral Standard Model, arguably the crowning achievement of the field thus far.

1.3 TWISTED REAL STRUCTURES AND THE STRUCTURE OF THE THESIS

Real structures as laid out in the previous section are powerful and well-motivated auxiliary structures for spectral triples. However, they have two main drawbacks: the conditions in their definition are very restrictive, meaningful there are relatively few examples of real spectral triples, and related to this, not every spectral triple which we would expect to be real actually is. As such, it is not out of the question to consider weakening the definition of the real structure.

A proposal to do just this was made in Ref. [7], and the weakened real structure proposed therein was called a *twisted real structure*. The redefinition offered, Def. 2.3, implements the weakening by way of an additional operator ν, called a *twist operator*, which factors into the conditions on the antilinear isometry J to provide additional freedom in a relatively direct way.

The particular implementation of this 'twist' was inspired by two examples, which each neatly illustrate the two problems mentioned above. The first is the quantum cone,[6] which is a special case of a more general construction which we outline here. Let G be a group and let $A = \bigoplus_{g\in G} A_g$ be a G-graded $*$-algebra whose $*$-structure is compatible with the grading in the sense that $A_g^* \subseteq A_{g^{-1}}$ such that A_e is a $*$-subalgebra of A for e the identity element of G.

Furthermore, let $G_+ \subset G$ and set $G_- := \{g^{-1} : g \in G_+\}$. Then

$$\mathcal{H}_\pm = \bigoplus_{g\in G_\pm} A_g, \quad \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

noting that by construction $\mathcal{H}_\pm^* \subseteq \mathcal{H}_\mp$. The twist operator ν is then obtained from the degree-preserving algebra automorphism $\hat{\nu}$ of A which satisfies $\hat{\nu} \circ * \circ \hat{\nu} = *$.

Let $\partial_\pm \colon A \to A$ be linear maps satisfying

$$\begin{aligned} &\partial_\pm(ab) = \partial_\pm(a)\hat{\nu}^2(b) + a\partial_\pm(b), \quad \hat{\nu} \circ \partial_\pm \circ \hat{\nu}^{-1} = q^{\pm 2}\partial_\pm, \\ &\hat{\nu}(\partial_\pm(a)^*) = \hat{\nu}^{-1}(\partial_\mp(a^*)), \quad \partial_\pm(\mathcal{H}_\mp) \subset \mathcal{H}_\pm. \end{aligned}$$

Then the Dirac operator D is given by the map

$$D\colon (\eta_+, \eta_-) \mapsto \big(-q^{-1}\partial_+(\eta_-), q\partial_-(\eta_+)\big)$$

for $\eta_\pm \in \mathcal{H}_\pm$. Then, for $J\colon (\eta_+, \eta_-) \mapsto (-\eta_-^*, \eta_+^*)$, $\mathcal{H}$ viewed as a left A_e-module (with $\pi(a)\eta_\pm = \hat{\nu}^2(a)\eta_\pm$) equipped with D is a spectral triple with the twisted real structure (J, ν).

The other example, which is even more compelling, is that of a conformal transformation. In the commutative case, by 'conformal transformation' we mean rescaling the metric by a Weyl factor $g \mapsto e^{-4h}g = g'$ for $h = h^* \in C^\infty(M)$. Denoting by $\not{\partial}$ the Dirac operator associated to g, one can show that the Dirac operator associated to g' is

$$\not{\partial}' = e^h \not{\partial} e^h.$$

Generalising this to the noncommutative setting in the most straightforward way by taking

$$D \mapsto e^{\pi(a)} D e^{\pi(a)} = D' \tag{1.6}$$

6 Tomasz Brzeziński. "Complex geometry of quantum cones," In: *Fortschritte der Physik* (2014).

for $a \in A$, one finds that one no longer has a spectral triple since $[D', \pi(b)] \notin \mathcal{B}(\mathcal{H})$ for $b \in A$. One way to approach this problem is to weaken the axioms of the spectral triple to only require that twisted commutators of the Dirac operator with the algebra are bounded [21], which leads to one to the formalism of 'twisted spectral triples'. However, it is perhaps troubling that such a simple transformation of the metric should require such a dramatic change to the formalism to accommodate it.

Twisted real structures offer a way to handle conformal transformations while remaining within the scope of spectral triples. In particular, for $\pi(k)$, $k \in A_+^\times$, taking the role of $e^{\pi(a)}$, provided the original spectral triple $(A, \mathcal{H}, D)$ is real with real structure J, the conformal transformation

$$D \mapsto J\pi(k)J^{-1}DJ\pi(k)J^{-1} = D''$$

gives a spectral triple $(A, \mathcal{H}, D')$ with the twisted real structure (J, ν) for $\nu = \pi(k)^{-1}J\pi(k)J^{-1}$. Note that D'' works just as well as a conformally-transformed Dirac operator as the D' of (1.6) since in the commutative case, $Je^hJ^{-1} = e^h$.

Since Ref. [7], excluding the works which contribute to this thesis, publications on twisted real structures include Ref. [30], which shows that the presence of a twisted real structure enriches even the otherwise simple spectral triples coming from $A = \mathbb{C}^2$ acting on $\mathbb{C}^3$ and $\mathbb{C}^4$, Ref. [8], which draws a connection between certain kinds of twisted spectral triples and spectral triples with twisted real structures (and which directly or indirectly inspired much of the work in this thesis), and Ref. [32], which generalises twisted real structures to permit multiple twist operators associated to decompositions of the Dirac operator.

This thesis aims to continue the work investigating the potential applications of twisted real structures and the role they might play within noncommutative geometry. For what concerns the structure of the thesis, it is broken up into seven chapters based on theme, each of which is further subdivided into several sections. The first chapter, of which this is the third section, introduces and motivates noncommutative geometry, spectral triples, real structures and twisted real structures. Ch. 2 presents the key definitions and conventions that will be used throughout the text as well as some minor preliminary results; Ch. 3 is largely based on Ref. [24] and describes an application of twisted real structures to Hodge-de Rham spectral triples which describe Riemannian manifolds; Ch. 4 is largely based on Ref. [50] and adapts the construction of gauge transformations for real spectral triples to twisted real structures; Ch. 5 investigates whether the twists selected out as most relevant in Ch. 4 impart a Lorentzian structure; Ch. 6 discusses a twisted spectral triple from the literature which admits a twisted real structure; and finally, Ch. 7 mentions ongoing work and possible avenues of future research.

As a point of wording, previous papers on twisted real structures often refer to 'twisted reality' and 'twisted real spectral triples'. We will not use these terms, instead preferring the admittedly inelegant 'twisted real structure' and 'spectral triple with twisted real structure' instead, purely to be as clear as possible. We take special care to avoid confusion with 'real twisted spectral triples', which are a related but very separate concept which we will also encounter. And as a final point on the format of the thesis, throughout, margins will be used for small notes and comments, clarifications, and minor definitions, whilst footnotes will be used (sparingly) for minor technical points and informal or historical references.

2 PRELIMINARIES

2.1 DEFINITIONS

In this section we collate all the definitions and notations which will be used in multiple chapters of the thesis. All algebras will be assumed to be unital unless otherwise noted, as will all Hilbert spaces be assumed to be complex.

2.1.1 *Notation*

When working with local expressions, we will make use of the Einstein summation convention, where there is an implied summation over upstairs-downstairs pairs of indices. No summation is ever implied for repeated indices which are all upstairs or all downstairs. When considering spaces of dimension n where there is a time(like) direction, Greek indices will run from 0 to $n-1$ with 0 being associated with the time(like) direction (even when the spaces are of Euclidean signature).

Throughout the thesis, $\{\gamma^\mu\}$ denotes the representatives of the generators of a Clifford algebra; in particular, in 4 dimensions they are the familiar gamma (Dirac) matrices; $\{\sigma^j\}$ are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $i = \sqrt{-1}$ throughout, and $\sigma^0 := 1_2$ is the 2×2 identity matrix.

Except where otherwise noted, we will follow the physics convention of describing the Hilbert space adjoint by $(\bullet)^\dagger$ with $(\bullet)^*$ being complex conjugation or the abstract involution of an involutive ($*$-)algebra.

If Σ is any subset of $\mathcal{B}(\mathcal{H})$ we will denote by

$$\Sigma' := \left\{T \in \mathcal{B}(H) : [T, S] = 0, \forall S \in \Sigma\right\}$$

its *commutant*.

Involutive algebras are commonly known as $*$-algebras. We will use the names interchangeably, typically preferring the latter except in cases where the involution in question is not denoted by $*$, in which case we will use the former to avoid confusion.

2.1.2 *Real spectral triples*

Definition 2.1. A *spectral triple* is the collection of data $(A, \mathcal{H}, D)$ comprising

1. a Hilbert space $\mathcal{H}$;
2. a real or complex $*$-algebra A along with a $*$-representation $\pi\colon A \to \mathcal{B}(\mathcal{H})$;
3. a self-adjoint operator D on $\mathcal{H}$ with compact resolvent, *i.e.*

$$(D - z1)^{-1} \in \mathcal{K}(\mathcal{H}), \quad z \in \mathbb{C} \setminus \mathbb{R};$$

Some definitions of the Dirac operator require that D be essentially *self-adjoint, but since essentially self-adjoint operators are those with a unique self-adjoint extension, in practice this is a relatively minor technical distinction.*

such that $\pi(a)\,\mathrm{Dom}(D) \subset \mathrm{Dom}(D)$ for all $a \in A$ and $[D, \pi(a)] \in \mathcal{B}(\mathcal{H})$ for all $a \in A$. We call D a (generalised) Dirac operator. ◇

We call a spectral triple *commutative* if the algebra A is commutative and *finite* if $\dim \mathcal{H} < \infty$. We implicitly assume that $\pi(A)$ preserves the domain of D.

If the Hilbert space admits a $\mathbb{Z}_2$-grading operator $\chi\colon \mathcal{H} \to \mathcal{H}$ satisfying $\chi^2 = 1$, $\chi^\dagger = \chi$, $\chi D = -D\chi$ and $\chi\pi(a) = \pi(a)\chi$ for all $a \in A$, the spectral triple is called *even*. Spectral triples which are not even are referred to as *odd*.

Notation. When necessary, the representation π will also be included explicitly in a spectral triple as $((A, \pi, \mathcal{H}), D)$. However, wherever the particular representation is clear from context or unimportant, it will be omitted for brevity.

It is sometimes possible to supplement a spectral triple $(A, \mathcal{H}, D)$ with an antilinear map $J\colon \mathcal{H} \to \mathcal{H}$ satisfying a number of compatibility conditions with the algebra and Dirac operator. In this case, the antilinear operator is known as a 'real structure' and the spectral triple is called *real*:

Definition 2.2 ([15])**.** If $(A, \mathcal{H}, D)$ is a spectral triple, we call J a *real structure* for the spectral triple if

1. J is an antilinear map $J\colon \mathcal{H} \to \mathcal{H}$ such that $J^\dagger = J^{-1}$ and

$$J^2 = \varepsilon 1, \quad \varepsilon = \pm 1; \tag{2.1}$$

2. with respect to the Dirac operator D, the antilinear map J satisfies

$$JD = \varepsilon' DJ, \quad \varepsilon' = \pm 1, \tag{2.2}$$

assuming that J preserves the domain of D;

3. for all $a, b \in A$, we have

$$\left[\pi(a), J\pi(b)J^{-1}\right] = 0 \tag{2.3}$$

and

$$\left[[D, \pi(a)], J\pi(b)J^{-1}\right] = 0. \tag{2.4}$$

If the spectral triple is even, we further require

$$\chi J = \varepsilon'' J\chi, \quad \varepsilon'' = \pm 1. \tag{2.5}$$

◇

Commonly (2.3) is called the *reality* or *zeroth-order condition* and (2.4) the *first-order condition*. Note that according to the above definition, every real structure is an antilinear isometry.

The three signs $\varepsilon, \varepsilon', \varepsilon''$ determine what is called the *KO-dimension* of a real spectral triple according to Table 1. We will occasionally and informally refer to them as 'KO-signs' for brevity. Note that (2.5) is only relevant in the even case, and that in (2.2) we are implicitly assuming that J preserves the domain of D.

KO-dim	0	1	2	3	4	5	6	7
ε	$++$	$+$	$-+$	$-$	$--$	$-$	$+-$	$+$
ε'	$+-$	$-$	$+-$	$+$	$+-$	$-$	$+-$	$+$
ε''	$++$		$--$		$++$		$--$	

Table 1: KO-dimensions of a spectral triple.

Note that in the even cases there are two possible choices of antilinear isometry J related by the grading χ [27]. The standard choice is the one given on the left.

2.1.3 *Spectral triples with twisted real structure*

The above definitions encapsulate what can be considered the standard presentation of (real) (even) spectral triples. We now move on to the definitions related to twisted real structures. We begin with the definition of a spectral triple with twisted real structure introduced in Ref. [7].

Let ν be a bounded operator on $\mathcal{H}$ with bounded inverse such that there exists an algebra automorphism $\hat{\nu}\colon A \to A$ implemented by

$$\pi(\hat{\nu}(a)) := \nu\pi(a)\nu^{-1} \tag{2.6}$$

for all $a \in A$. We will call such an operator ν a *twist operator*. In the same vein, we define another algebra automorphism $\tilde{\nu}\colon A \to A$ using ν which is given by

$$\pi(\tilde{\nu}(a)) := \nu^{\dagger}\pi(a)\big(\nu^{\dagger}\big)^{-1}. \tag{2.7}$$

Note that we can express $\tilde{\nu}$ in terms of $\hat{\nu}$ by $\tilde{\nu}(a) = (\hat{\nu}^{-1}(a^*))^*$ for any $a \in \mathcal{A}$.

We then have the following definition of a spectral triple equipped with a "twisted real structure":

Definition 2.3 ([7]). A *spectral triple with twisted real structure* is a spectral triple $(A, \mathcal{H}, D)$ along with an antilinear operator $J\colon \mathcal{H} \to \mathcal{H}$, $J^{\dagger} = J^{-1}$, and twist operator ν such that (2.1), (2.3) and the following conditions are satisfied:

1. the '(ν-)twisted first-order condition'

$$[D,\pi(a)]J\pi(\hat{\nu}^2(b))J^{-1} = J\pi(b)J^{-1}[D,\pi(a)], \tag{2.8}$$

for all $a, b \in A$;

2. the '(ν-)twisted ε' condition'

$$DJ\nu = \varepsilon'\nu JD, \quad \varepsilon' = \pm 1, \tag{2.9}$$

assuming that $J\nu$ preserves the domain of D.

We will denote such spectral triples with twisted real structures by $(A, \mathcal{H}, D, (J, \nu))$. ◇

It is clear from this definition that the familiar, 'ordinary' real spectral triples $(A, \mathcal{H}, D, J)$ can be considered as special cases of spectral triples with twisted real structure with the trivial twist operator $\nu = 1$. In this thesis, we will occasionally refer to these spectral triples as 'trivially-twisted' to emphasise this point. In such cases, for the sake of ease, we will retain the familiar notation omitting ν rather than writing $(A, \mathcal{H}, D, (J, 1))$ for these trivially-twisted spectral triples.

To clarify nomenclature, we will call J a *reality operator* and the combination (J, ν) the 'twisted real structure'. When the twist is trivial, the reality operator and the real structure become synonymous.

An additional condition, which is not strictly necessary but which will be assumed to hold out of convenience unless otherwise specified, is the *regularity condition*

$$\nu J \nu = J. \tag{2.10}$$

A twisted real structure satisfying the regularity condition will be referred to as 'regular'. Note that the conditions (2.8) and (2.9) have clear analogues in the usual case of real spectral triples, namely (2.4) and (2.2) respectively.

Because $\hat{\nu}$ is an automorphism of A, we can equivalently express (2.8) in the 'balanced' form

$$[D,\pi(a)]J\pi(\hat{\nu}(b))J^{-1} = J\pi(\hat{\nu}^{-1}(b))J^{-1}[D,\pi(a)]. \tag{2.11}$$

Compatibility of (2.8) with the $*$-structure forces one to also require

$$[D,\pi(a)]J\pi(\tilde{\nu}(b))J^{-1} = J\pi(\tilde{\nu}^{-1}(b))J^{-1}[D,\pi(a)], \tag{2.12}$$

which, as $\tilde{\nu}$ is also an automorphism, can also be expressed in the 'unbalanced' form of (2.8). We note that (2.12) follows as a consequence of (2.8) (rather than being taken as an additional assumption) when $\nu = \nu^{\dagger}$.

Definition 2.4. A spectral triple with twisted real structure $(A, \mathcal{H}, D, (J, \nu))$ is called *even* if it admits a grading operator χ such that $\chi D = -D\chi$ and $\chi\pi(a) = \pi(a)\chi$ for all $a \in A$. Furthermore, χ is also required to satisfy

$$\chi\nu J = \varepsilon''\nu J\chi, \quad \varepsilon'' = \pm 1, \tag{2.13}$$

which we refer to as the '(ν-)twisted ε'' condition', and the commutation relation

$$\chi\nu^2 = \nu^2\chi. \tag{2.14}$$

⋄

Note that the condition (2.13) has a better-known analogue in the trivially-twisted case, namely (2.5).

Remark 2.5. In fact, the literature on twisted real structures (*e.g*. Refs. [7, 8]) requires (2.5) in the even case rather than (2.13). The motivation for using (2.13) in our definition comes from Prop. 4.7, and the fact that (2.13) is the weaker choice of the two constraints. This will be discussed in more detail in §4.2.2.2. ⋄

As a point of notation, such even spectral triples with twisted real structure will be denoted by $(A, \mathcal{H}, D, (J, \nu), \chi)$. To be consistent with the convention established earlier, (trivially-twisted) even real spectral triples will be denoted by $(A, \mathcal{H}, D, J, \chi)$ as per the standard notation.

A generalisation of twisted real structures was proposed in Ref. [32], motivated by anisotropic conformal rescalings and other complicated conformal transformations beyond what Def. 2.3 can implement:

Definition 2.6 ([32])**.** If $(A, \mathcal{H}, D)$ is a spectral triple, we call $(J, \{\nu_k\})$ a *multitwisted real structure* for the spectral triple if J is an antilinear map $J \colon \mathcal{H} \to \mathcal{H}$, $J^\dagger = J^{-1}$, satisfying (2.1) and

1. the Dirac operator D can be decomposed into

$$D = \sum_{k \in K} D_k,$$

$K = \{1, 2, \ldots, N\}$, where each D_k is a densely defined operator with $\mathrm{Dom}(D) \subseteq \mathrm{Dom}(D_k)$;

2. for each D_k there exists an associated *twist operator* $\nu_k \in \mathcal{B}(\mathcal{H})$ with bounded inverse such that

$$\nu_k J D_k = \varepsilon' D_k J \nu_k, \quad \varepsilon' = \pm 1; \tag{2.15}$$

3. conjugation by ν_k gives an automorphism of $\mathcal{B}(\mathcal{H})$ for all $k \in K$, and for all $a, b \in A$, we have

$$\left[\pi(a), J\nu_k\pi(b)\nu_k^{-1}J^{-1}\right] = 0 = \left[\pi(a), J\nu_k^{-1}\pi(b)\nu_k J^{-1}\right] \tag{2.16}$$

and

$$[D_k, \pi(a)]J\nu_k\pi(b)\nu_k^{-1}J^{-1} = J\nu_k^{-1}\pi(b)\nu_kJ^{-1}[D_k, \pi(a)] \quad (2.17)$$

for each $k \in K$.

If the spectral triple is even, we further assume

$$\chi\nu_k^2 = \nu_k^2\chi \quad (2.18)$$

and

$$\nu_kJ\chi = \varepsilon''\chi\nu_kJ, \quad \varepsilon'' = \pm 1 \quad (2.19)$$

for all $k \in K$. We call the multitwisted real structure *regular* if

$$\nu_kJ\nu_k = J$$

for all $k \in K$. ⋄

We call (2.16) the *multitwisted zeroth-order condition* and (2.17) the *multitwisted first-order condition*. It should be noted that the relaxation of the requirement that the twist operators implement an automorphism of the algebra is a substantial difference. Not only does it mean that Def. 2.6 is weaker than Def. 2.3 even in the $N = 1$ case, but one also (notably) loses the ordinary zeroth-order condition, and (2.17) is forced to take the 'balanced' form given; the twist can no longer be freely shifted from one side of the equation to the other. We will call (2.16) and (2.17) "multitwisted" even when $N = 1$ to avoid confusion with other terminology (see §2.1.4). Note that, as for twisted real structures, it is still the case that when the twist operator(s) ν_k are trivial, Def. 2.6 reduces to the standard Def. 2.2.

2.1.4 *Real twisted spectral triples*

We will denote by $[X, Y]_{\bar{\sigma}}$ the *twisted commutator* $XY - \bar{\sigma}(Y)X$ for $X, Y \in \mathrm{End}(\mathcal{H})$ and $\bar{\sigma}\colon \mathrm{End}(\mathcal{H}) \to \mathrm{End}(\mathcal{H})$.

Putting aside the matter of reality for a moment, we now turn to 'twisted spectral triples', a generalisation of spectral triples where commutators of the Dirac operator and algebra are replaced by twisted commutators. Twisted spectral triples were introduced in Ref. [21] to describe 'spectral triples' arising from type III algebras, but the framework has since been adapted to a wide variety of situations beyond what 'ordinary' spectral triples encompass.

Def. *A C*-algebra $\mathcal{A}$ is of* type I/II/III *if $\pi(\mathcal{A})'' \subset \mathcal{B}(\mathcal{H})$ is a type I/II/III factor. A* factor *is a von Neumann algebra $\mathcal{V}$ with $Z(\mathcal{V}) = \mathbb{C}1$.*

Definition 2.7 ([21]). A *twisted spectral triple* is the collection of spectral data $(A, \mathcal{H}, D)_\rho$ comprising

1. a Hilbert space $\mathcal{H}$;

2. a real or complex $*$-algebra A along with a $*$-representation $\pi\colon A \longrightarrow \mathcal{B}(\mathcal{H})$;

3. an automorphism ρ of A which is compatible with the $*$-structure (called *regular*)
$$\rho \circ * = * \circ \rho^{-1}, \tag{2.20}$$
and lifts to an automorphism of $\mathcal{B}(\mathcal{H})$,
$$\pi(\rho(a)) = \bar{\rho}(\pi(a));$$

4. a self-adjoint operator D on $\mathcal{H}$ with compact resolvent (a generalised Dirac operator);

such that $\pi(A)$ preserves the domain of D and $[D, \pi(a)]_{\bar{\rho}} \in \mathcal{B}(\mathcal{H})$ for all $a \in A$. ◇

A notion of reality for twisted real structures mirroring Def. 2.2 was given in Ref. [46]:

Definition 2.8 ([46]). If $(A, \mathcal{H}, D)_\rho$ is a twisted spectral triple, we call J a *real structure* for the spectral triple if

1. J is an antilinear map $J\colon \mathcal{H} \longrightarrow \mathcal{H}$ such that $J^\dagger = J^{-1}$ and
$$J^2 = \varepsilon 1, \quad \varepsilon = \pm 1; \tag{2.21}$$

2. with respect to the Dirac operator D, the antilinear map J satisfies
$$JD = \varepsilon' DJ, \quad \varepsilon' = \pm 1, \tag{2.22}$$
assuming that J preserves the domain of D;

3. we have an automorphism of the (representation of the) opposite algebra $\bar{\rho}^\circ$ implemented by
$$\bar{\rho}^\circ(J\pi(c)J^{-1}) := J\bar{\rho}(\pi(c))J^{-1}, \quad \text{for all } c \in A; \tag{2.23}$$

Regularity for twist automorphisms and for twist operators can be seen as very roughly analogous in the sense that $\bar{\rho}^\circ(\pi^\circ(a)) = \pi^\circ(\rho^{-1}(a))$ and $\bar{\nu}(\pi_J(a)) = \pi_J(\hat{\nu}^{-1}(a))$.

4. for all $a, b \in A$, we have
$$\left[\pi(a), J\pi(b)J^{-1}\right] = 0 \tag{2.24}$$
and
$$\left[[D, \pi(a)]_{\bar{\rho}}, J\pi(b)J^{-1}\right]_{\bar{\rho}^\circ} = 0; \tag{2.25}$$
are satisfied.

If the twisted spectral triple is even (the meaning is the same as for spectral triples), we further require

$$\chi J = \varepsilon'' J \chi, \quad \varepsilon'' = \pm 1. \tag{2.26}$$

◇

Such a twisted spectral triple is called a *real twisted spectral triple*, and is denoted by $(A, \mathcal{H}, D, J)_\rho$. Equation (2.25) is referred to as the *twisted first-order condition*, although we will not do so here except where there is no chance of confusion with (2.8). Where there is the possibility for confusion, we will call (2.8) the twisted first-order condition and (2.25) the fully-twisted first-order condition, since it involves a twist on both commutators.

Remark 2.9. It is important to note that spectral triples with twisted real structure are different to real twisted spectral triples, despite the similar names, and one should take special care not confuse the two. However, it is possible in many cases to draw equivalences between the two frameworks, and some looser parallels can be drawn too, as we shall see in subsequent chapters of the thesis. ◇

2.2 PRECURSORY RESULTS

In this section, we collect some results which are preliminary to the results of the remainder of the thesis, or which are relatively minor or not strongly connected to the material to follow. We begin by establishing some notation and terminology.

Notation. It will frequently occur that we will take the twist operator to be self-adjoint or involutive up to sign. When it is necessary to keep track of these signs, we will consistently refer to them as α_1 and α_2 respectively, *i.e.*, if ν is self-adjoint up to sign we will say $\nu = \alpha_1 \nu^\dagger$ and if ν is involutive up to sign, we will say that $\nu = \alpha_2 \nu^{-1}$.

Regarding the case of $\nu = \pm\nu^{-1}$ in particular, we will refer to such spectral triples with twisted real structures as being *mildly-twisted*, because these spectral triples satisfy the ordinary first-order condition (2.4) since $\nu^2 = 1$ (and $\hat{\nu}^2 = \mathrm{id}$).

As a further point of notation, with respect to a $*$-algebra A we additionally define the conjugate $*$-representation $\pi_J(a) := J\pi(a)J^{-1}$ for any $a \in A$ and the $*$-antirepresentation $\pi_J^*(a) := J\pi(a)^\dagger J^{-1} \equiv \pi^\circ(a)$ for any $a \in A$. These will make it more convenient to use the following notation for twisted commutators:

$$[T, a]_\sigma^\pi := T\pi(a) - \pi(\sigma(a))T,$$

for $a \in A, T \in \mathrm{End}(\mathcal{H}), \pi \colon A \to \mathcal{B}(\mathcal{H})$ and σ an algebra automorphism of A. Of course, in this notation we might write ordinary commutators of operators with algebra elements as $[T,a]^{\pi}_{\mathrm{id}}$ although in this thesis we will favour the standard notation $[T,\pi(a)]$ for simplicity. With this notation, we can make use of the following results from Ref. [8].

Lemma 2.10 ([8])**.** *The ν-twisted first-order condition (2.8) can be equivalently written as*

$$[[D,\pi(a)],b]^{\pi_J}_{\hat{\nu}^{-2}} = 0$$

for any $a,b \in A$.

Lemma 2.11 ([8])**.** *Let T be an operator on $\mathcal{H}$, with the algebra A represented on $\mathcal{H}$ by $\pi\colon A \to \mathcal{B}(\mathcal{H})$ and $\pi_J\colon A \to \mathcal{B}(\mathcal{H})$, with the algebra automorphisms $\sigma,\rho \in \mathrm{Aut}(A)$. Then if (2.3) holds, we have*

$$[[T,a]^{\pi}_{\sigma},b]^{\pi_J}_{\rho} = [[T,b]^{\pi_J}_{\rho},a]^{\pi}_{\sigma}$$

for all $a,b \in A$.

2.2.1 *Triviality and KO-dimension*

At this point, we introduce another sign,

$$\nu J \nu = \varepsilon''' J. \tag{2.27}$$

When $\varepsilon''' = +1$ this is nothing but the regularity condition (2.10), but we will relax this condition in Ch. 5, so it is worthwhile to consider it here now as well. Having done so, we present another result that will be of some relevance later:

Proposition 2.12. *Suppose that $(A,\mathcal{H},D,(J,\nu),\chi)$ is an even spectral triple with twisted real structure with KO-signs $(\varepsilon,\varepsilon',\varepsilon'')$ whose twist operator satisfies $\nu = \alpha\nu^{\dagger} = \alpha\nu^{-1}$ for $\alpha \in \{+1,-1\}$. Then $(A,\mathcal{H},D,\nu J,\chi)$ is an even (trivially-twisted) real spectral triple with KO-signs $(\varepsilon\varepsilon''',\alpha\varepsilon'\varepsilon''',\varepsilon'')$ provided that ν is a linear operator.*

Alternatively, suppose that $(A,\mathcal{H},D,(J,\nu),\chi)$ is an even spectral triple with twisted real structure with KO-signs $(\varepsilon,\varepsilon',\varepsilon'')$ whose twist operator satisfies $\nu = \alpha_2\nu^{-1}$ for $\alpha_2 \in \{+1,-1\}$ and $\nu D = \alpha' D\nu$, $\nu\chi = \alpha''\chi\nu$ for $\alpha',\alpha'' \in \{+1,-1\}$. Then this spectral triple with twisted real structure is equivalent to the even (trivially-twisted) real spectral triple $(A,\mathcal{H},D,J,\chi)$ with KO-signs $(\varepsilon,\alpha_2\alpha'\varepsilon'\varepsilon''',\alpha''\varepsilon'')$.

The question of what happens when ν is an antilinear operator, both in the context of Prop. 2.12 and more broadly, is an interesting one which merits further investigation but which will not be addressed in this thesis.

Proof. We begin with the first claim. Let us call $\nu J =: \mathcal{J}$ and require that $\nu = \alpha_1\nu^{\dagger} = \alpha_2\nu^{-1}$ for $\alpha_1,\alpha_2 \in \{+1,-1\}$. We first check that $\mathcal{J}$ is a valid real structure. First of all, as it is the product of a linear and an antilinear operator, it is itself antilinear. It is straightforward to find

$$\mathcal{J}^{\dagger} = J^{\dagger}\nu^{\dagger} = \alpha_1 J^{-1}\nu = \alpha_1\alpha_2 J^{-1}\nu^{-1} = \alpha_1\alpha_2\mathcal{J}^{-1}$$

and so it is antiunitary provided $\alpha_1\alpha_2 = 1$. This implies that $\alpha_1 = \alpha_2$, and so we call $\alpha = \alpha_1 = \alpha_2$. Furthermore,

$$\mathcal{J}^2 = \nu J \nu J = \varepsilon''' J^2 = \varepsilon\varepsilon''' 1$$

by (2.27).

It is equally straightforward to see that

$$\begin{aligned} DJ\nu &= \varepsilon'\nu JD \\ \alpha\varepsilon'''D\nu J &= \varepsilon'\nu JD \\ D\mathcal{J} &= \alpha\varepsilon'\varepsilon'''\mathcal{J}D \end{aligned}$$

and so (2.9) reduces to (2.1) with the sign $\alpha\varepsilon'\varepsilon'''$. Similar reasoning shows that the ν-twisted first-order condition (2.8) reduces to the first-order condition (2.4), which is particularly clear if one considers the 'balanced' form

$$\begin{aligned} [D,\pi(a)]J\nu\pi(b)\nu^{-1}J^{-1} &= J\nu^{-1}\pi(b)\nu J^{-1}[D,\pi(a)] \\ \alpha^2(\varepsilon''')^2[D,\pi(a)]\mathcal{J}\pi(b)\mathcal{J}^{-1} &= (\varepsilon''')^2\mathcal{J}\pi(b)\mathcal{J}^{-1}[D,\pi(a)] \\ [D,\pi(a)]\mathcal{J}\pi(b)\mathcal{J}^{-1} &= \mathcal{J}\pi(b)\mathcal{J}^{-1}[D,\pi(a)]. \end{aligned}$$

In the even case, all of the above applies unchanged. The requirement that $\left[\chi,\nu^2\right] = 0$ is satisfied trivially as $\nu^2 \propto 1$ and $\chi\nu J = \varepsilon''\nu J\chi$ immediately becomes $\chi\mathcal{J} = \varepsilon''\mathcal{J}\chi$ by the definition of $\mathcal{J}$, and so (2.13) reduces to (2.5).

For the second claim, $\nu^2 \propto 1$ immediately reduces (2.8) to (2.4). The reduction of (2.9) to (2.1) and (2.13) to (2.5) comes quite immediately from the fact that (2.27) gives $\nu J = \alpha_2\varepsilon'''J\nu$ combined with the commutation relations of ν with D and χ respectively. □

The results of Prop. 2.12 appear very strong, but they follow from the equally strong assumptions that a given (even) spectral triple with twisted real structure and (even) real spectral triple are related by having the same Dirac operator and grading. As such, the most reasonable interpretation is to take it as a kind of consistency result, since twisted real structures are proposed to be a genuine generalisation of ordinary real structures. In this light, it makes sense to consider KO-dimension in a little more detail. KO-dimension is rigorously defined for ordinary real structures, but there is no guarantee the concept still makes sense in the same way when considering twisted real structures. Even so, if it is to have any significance, one would expect that it should reduce to the standard situation in contexts such as Prop. 2.12.

While it would be ideal to remedy this, it is not clear that KO-dimension even should *make sense for twisted real structures – the connection to Clifford algebras that one has in the ordinary case becomes much more tenuous, for example.*

So, regarding Prop. 2.12, in the first case, the requirement that the spectral triple with twisted real structure and the real spectral triple have the same KO-dimension is given by $\alpha = \varepsilon''' = +1$. The situation appears not altogether different in the second case, where there are two requirements, $\alpha_2\alpha'\varepsilon''' = +1$ and $\alpha'' = +1$. However, we should note that there is a great deal more structure in the assumptions of the second case, and an example may be helpful to tease out the implications of this extra structure.

Example 2.13. Let us assume that the twist ν is equal to the grading operator χ (while this seems an odd choice, it is not precluded by

Def. 2.6). In that case, denoting the KO-signs of the real spectral triple by $(\varepsilon_0, \varepsilon_0', \varepsilon_0'')$, we have the commutation signs $\alpha = +1$, $\varepsilon''' = \varepsilon_0''$, $\alpha' = -1$, $\alpha'' = +1$. In this case, the consistency relations simplify to $\varepsilon_0'' = -1$, which occurs only for KO-dimension 2 or 6 $\pmod 8$. ⋄

The grading-twist example makes clear that not every twist (even when mild) will necessarily be fully consistent with the notion of KO-dimension for every possible dimension. However, this is not necessarily a fatal problem.

Example 2.14. Let us return to the case of $\nu = \chi$, noting that for this choice, the first case of Prop. 2.12 is also applicable. In this case, the real spectral triple $(A, \mathcal{H}, D, \chi J, \chi)$ has KO-signs $(\varepsilon\varepsilon_0'', \varepsilon'\varepsilon_0'', \varepsilon'')$. Clearly the same KO-dimension is maintained with $\varepsilon_0'' = +1$, but this does not mean $\varepsilon_0'' = -1$ is excluded; on the contrary, this is precisely consistent with the 'alternative' table of KO-dimensions[1] given by taking $J \mapsto \chi J$, and as such is, if anything, the *more* consistent choice. ⋄

Clearly at least part of the cause of the (apparent) inconsistency in KO-dimension came from the specific choice of χ as the twist, noting that χ is already assumed to satisfy various axioms in the general case. Even so, it may be interesting to consider the construction of an even more general table of KO-dimensions which more naturally includes the grading-twist (perhaps according to some alternative choice $J \mapsto \nu J$ in analogy to what is done with the grading in the ordinary case). However, we will not do so in this thesis, especially since we are primarily interested in *avoiding* the application of Prop. 2.12, since in those cases the real structure is trivialised.

Remark 2.15. Choosing to use the grading as a twist is an interesting example for another reason unrelated to KO-dimension: even though the twist is not trivial, it does correspond to a trivial automorphism on the algebra because $[\chi, \pi(a)] = 0$ for all $a \in A$, and thus $\hat{\chi}(a) = a$ for all $a \in A$. Thus, given that twists which satisfy $\nu^2 = 1$ are referred to as 'mild', we might consider such mild twists with trivial algebra action as 'ultra-mild'. ⋄

2.2.2 *Products for twisted real structures*

Yet another preliminary consistency result comes from examining products. In this subsection we consider the standard product of spectral triples [27, 59], although in later sections we will allow for some flexibility. In particular, here we will consider only the products between even spectral triples, and between even and odd spectral triples; the case of products between odd spectral triples requires some extra technical consideration but follows the standard treatment [27] – we exclude it for brevity.

---

1 The map for an even real spectral triple is given by $(\varepsilon, \varepsilon', \varepsilon'') \mapsto (\varepsilon\varepsilon'', -\varepsilon', \varepsilon'')$.

The simplest possible way to obtain a product structure for two spectral triples with twisted real structures is to simply take the 'standard' product spectral triple with the simplest product twist operator $\nu_1 \otimes \nu_2$ and enforce the conditions of Def. 2.3.

Lemma 2.16. *Let $(A_1, \mathcal{H}_1, D_1, (J_1, \nu_1), \chi_1)$ be an even spectral triple with twisted real structure, and $(A_2, \mathcal{H}_2, D_2, (J_2, \nu_2), \chi_2)$ be an even spectral triple with twisted real structure, with respective KO-signs $(\varepsilon_j, \varepsilon'_j, \varepsilon''_j)$, $j = 1, 2$. Then the usual product of spectral triples*

$$\begin{aligned}&\big(A, \mathcal{H}, D, (J, \nu), \chi\big) = \\ &\qquad (A_1 \otimes A_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D_1 \otimes 1 + \chi_1 \otimes D_2, (J_1 \otimes J_2, \nu_1 \otimes \nu_2), \chi_1 \otimes \chi_2)\end{aligned}$$

is an even spectral triple with twisted real structure with KO-signs

$$(\varepsilon_1\varepsilon_2, \varepsilon'_1\alpha_2, \varepsilon''_1\varepsilon''_2)$$

if $\nu_j = \alpha_j \nu_j^{-1}$ for $\alpha_j \in \{+1, -1\}$ and $\varepsilon'_1\alpha_2 = \alpha_1\varepsilon''_1\varepsilon'_2$.

Proof. The proof consists of checking that the conditions of Def. 2.3 are satisfied, which are straightforward computations. However, (2.9) is not automatically satisfied, and so we give its computation explicitly:

$$\begin{aligned} DJ\nu &= D_1J_1\nu_1 \otimes J_2\nu_2 + \chi_1 J_1\nu_1 \otimes D_2J_2\nu_2 \\ &= \varepsilon'_1\nu_1J_1D_1 \otimes J_2\nu_2 + \varepsilon''_1\varepsilon'_2 J_1\nu_1\chi_1 \otimes \nu_2J_2D_2, \end{aligned} \tag{2.28}$$

making use of (2.9) and (2.13); likewise

$$\varepsilon'\nu JD = \varepsilon'\nu_1J_1D_1 \otimes \nu_2J_2 + \varepsilon'\nu_1J_1\chi_1 \otimes \nu_2J_2D_2. \tag{2.29}$$

Clearly, the right-hand sides of (2.28) and (2.29) can only be equal if J_j commutes with ν_j up to sign for both $j = 1, 2$. Assuming the regularity condition (2.10), this only happens when

$$\nu_j = \alpha_j \nu_j^{-1}$$

for $\alpha_j \in \{-1, +1\}$, in which case $\nu_jJ_j = \alpha_jJ_j\nu_j$. Enforcing this requirement, we have from (2.28) and (2.29) that $DJ\nu = \varepsilon'\nu JD$ holds when $\varepsilon' = \varepsilon'_1\alpha_2 = \alpha_1^{-1}\varepsilon''_1\varepsilon'_2$. □

Lem. 2.16 equally holds when the second initial spectral triple is odd with the only change being that the product spectral triple is also taken to be odd, and so there is no product grading operator. In the even-even and odd-even cases, one can also use the alternative product $D' = D_1 \otimes \chi_2 + 1 \otimes D_2$ to obtain an analogous result, except in this case the product KO-signs are $(\varepsilon_1\varepsilon_2, \alpha_1\varepsilon'_2, \varepsilon''_1\varepsilon''_2)$ satisfying $\varepsilon'_1\alpha_2\varepsilon''_2 = \alpha_1\varepsilon'_2$. In all cases, the product twist operator ν is involutive (mild) up to the sign $\alpha = \alpha_1\alpha_2$.

When both initial spectral triples are even, the Dirac operators D and D' are related by the unitary transformation $D' = U'DU'^{\dagger}$ for $U' := \frac{1}{2}(1 \otimes 1 + 1 \otimes \chi_2 + \chi_1 \otimes 1 - \chi_1 \otimes \chi_2)$.

Remark 2.17. A very similar result to Lem. 2.16 was found in Ref. [8, Lem. 3.4]. The difference is that said paper assumes (2.5) rather than (2.13), and so additionally requires $\nu\chi = \alpha''\chi\nu$, $\alpha'' \in \{-1, +1\}$ to be satisfied. Hence, their result includes the additional sign α''. ◇

It is important to note that Lem. 2.16 is fundamentally due to the fact that the relationship between the twist operator and the antiunitary operator in a twisted real structure is typically governed by the regularity condition (unless stricter conditions are imposed). As such, re-computing product spectral triples using graded tensor products, or alternative definitions of the product twist operator and product reality operator which include on a factor of the grading of a sign will not be sufficient to alleviate the result that the twists must be mild.

An interesting observation is that $D_1 \otimes \nu_2$ satisfies (2.9) with respect to the product twist operator and reality operator with $\varepsilon' = \varepsilon'_1$, which suggests that perhaps the form of the product Dirac operator could be modified to something like

$$D_1 \otimes \nu_2 + \chi'_1 \otimes D_2$$

where χ'_1 is some operator chosen so that the whole Dirac operator now satisfies (2.9). The trouble with this is that the square of the Dirac operator should have the form $D_1^2 \otimes 1 + 1 \otimes D_2^2$, which guarantees that the KO-dimension of the product spectral triple is the sum of the dimensions of the factors (mod 8). However, the square of the proposed modification is

$$D^2 = D_1^2 \otimes \nu_2^2 + D_1\chi'_1 \otimes \nu_2 D_2 + \chi'_1 D_1 \otimes D_2\nu_2 + (\chi'_1)^2 \otimes D_2^2,$$

whose cross-terms are not easy to eliminate even under quite strong assumptions on ν_2 and χ'_1, which suggests this idea is a non-starter.

Other alternative solutions are possible, of course. It could be that it is necessary to (also) modify the product twist operator and/or antiunitary operator, *etc*. It could also be possible that the tensor product also picks up some kind of 'twisting', although this would be a rather more radical proposal. These possibilities deserve further consideration, but for the time being it seems best to accept that only mild twists are compatible with products, especially since in all of the relevant examples we will come across later in the thesis, the twists will be mild anyway.

When one considers the product of real (trivially-twisted) spectral triples, there are similar limitations on the KO-signs in order for the products to be well-defined when compared to Lem. 2.16. Indeed, for the usual product one has the restriction $\varepsilon'_1 = \varepsilon''_1\varepsilon'_2$ and the restriction for the 'alternative' product is $\varepsilon'_1\varepsilon''_2 = \varepsilon'_2$. In both cases, these are identical to when there is a non-trivial twist provided that $\alpha_1 = \alpha_2$, in other words, this is when the table of KO-dimensions for twisted real structures can be taken to be the same as for trivially-twisted real structures, an interesting result with respect to the previous subsection.

An elegant explanation of the sign structure of products of real spectral triples was given in Ref. [37] in terms of graded tensor products. In brief, the initial Hilbert spaces are explicitly considered in terms of their ($\mathbb{Z}_2$ or trivial) grading, and the compatibility of the action of operators with the grading is interpreted in terms of (anti)commutation relations with the grading operator. All of the sign restrictions on products then come from the requirement that the (graded) tensor product respect these gradings. If we choose to view the product in these terms, we would expect that ν should necessarily have some compatibility with the grading, in which case we ultimately return to the result of Ref. [8, Lem. 3.4].

3 TWISTED REALITY AND THE SECOND-ORDER CONDITION

3.1 INTRODUCTION

The material in this chapter is based on Ref. [24].

As was mentioned in §1.2.2, one of the great applications of real spectral triples has been to the Standard Model of particle physics. The spectral triple of the (spectral) Standard Model is the product of the canonical real spectral triple $(A_M, \mathcal{H}_M, D_M, J_M)$ of a spin manifold M and a finite-dimensional noncommutative real spectral triple $(A_F, \mathcal{H}_F, D_F, J_F)$ encoding the internal degrees of freedom of elementary particles. An interesting feature of the spectral model is that J_F also implements a "second-order" condition [38]: conjugation by J_F maps the Clifford algebra $\mathcal{C}\ell_{D_F}(A_F)$ into its commutant. In fact an even stronger property holds: the commutant of $\mathcal{C}\ell_{D_F}(A_F)$ is isomorphic to $\mathcal{C}\ell_{D_F}(A_F)$ itself, with $\mathcal{H}_F$ a self-Morita equivalence $\mathcal{C}\ell_{D_F}(A_F)$-bimodule (we call this the *Hodge* property, *cf*. Def. 3.5). These features and their consequences for the specific example of the Standard Model were studied in Refs. [22, 26, 31] in the context of finite-dimensional spectral triples.

Similar such $\mathcal{C}\ell_D(A)$-bimodules were investigated in great detail in the context of spectral triples of closed oriented Riemannian manifolds in Ref. [49]. These spectral triples, which we refer to as "Hodge-de Rham" spectral triples, are built on the space of complex exterior forms rather than spinors. The Dirac operator is built from the exterior derivative either as $d + d^\dagger$ or $-i(d - d^\dagger)$, and there is a natural reality operator associated to the Hodge star operator that intertwines these two Dirac operators, as observed already in Ref. [39] (see also Refs. [18, 41, 49]). Another natural antilinear involution is given by the modular conjugation (Tomita-Takesaki) operator, and although it does not give a real spectral triple *sensu stricto* (per Def. 2.2), it does implement a $\mathcal{C}\ell_D(A)$ self-Morita equivalence (*cf*. [49]).

Motivated by the work in Ref. [38] and by the properties of the spectral triple describing the internal degrees of freedom of particles in the spectral geometry approach to the Standard Model of particle physics, we are interested in spectral triples satisfying the second-order condition. Such a condition should in some sense characterise the difference between differential forms and Dirac spinors.

In §3.2, we recall some background material about spectral triples and give a gentle review of the canonical spectral triple of a closed oriented Riemannian manifold, built from differential forms. We are particularly interested in reality operators and observe that of the two natural reality operators, the one coming from the Hodge star does not satisfy the second-order condition, whilst the one coming from the involution does not (anti)commute with the Dirac operator. The observation in Ref. [39] that the former real structure intertwines the two natural Dirac operators is re-interpreted in the framework of twisted real structures.

A natural question is then whether there exists an alternative antilinear involution giving both a real spectral triple *sensu stricto* and satisfying the second-order condition. In §3.3 we consider the simple example of the 2-torus and prove that such an operator doesn't exist (*cf*. Thm. 3.19). The use of twisted real structures is the best one can aim for if one is interested in the second-order condition.

In light of the fact that the spectral triple of the Standard Model is a product of two factors, in §3.4 we study how the above mentioned conditions behave under products of spectral triples. In particular we will argue in §3.4.3 that, if one defines the tensor product of real structures in the correct way, then the Hodge property is preserved under products of real spectral triples.

3.2 SPECTRAL TRIPLES AND RIEMANNIAN MANIFOLDS

3.2.1 *Spectral triples*

Here we collect some preliminary definitions needed for this chapter. The presentation of real spectral triples in terms of 1-forms and the Clifford algebra can be found in Ref. [39]. The Clifford algebra of a spectral triple was also introduced in Ref. [39] (see also Ref. [49]). Given a unital spectral triple we define

- $\Omega^1_D(A)$ to be the *complex* vector subspace of $\mathcal{B}(\mathcal{H})$ spanned by $\pi(a)[D, \pi(b)]$, for $a, b \in A$; and
- $\mathcal{C}\ell_D(A)$ to be the *complex* C*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by A and $\Omega^1_D(A)$.

We think of $\Omega^1_D(A)$ as the analogue of (smooth) differential 1-forms and $\mathcal{C}\ell_D(A)$ as the analogue of (continuous) sections of the Clifford algebra bundle on a compact Riemannian manifold.

Given an antilinear isometry J on $\mathcal{H}$, for all $T \in \mathcal{B}(\mathcal{H})$ and $\Sigma \subset \mathcal{B}(\mathcal{H})$ we will denote

$$T^\circ := JT^\dagger J^{-1} \quad \text{and} \quad \Sigma^\circ := \left\{S^\circ : S \in \Sigma\right\}.$$

By design, when the operator comes from the algebra, $\pi(a)^\circ \equiv \pi^\circ(a)$. We also remark that π°, which is an antirepresentation of A, can also be viewed as a representation of A^{op}, and so there is no confusion when representations are omitted; the meanings of a° and A° in this notation are unambiguous.

Notice that the map $T \mapsto T^\circ$ is complex-linear and antimultiplicative:

$$(TS)^\circ = S^\circ T^\circ,$$

for all $T, S \in \mathcal{B}(\mathcal{H})$. If Σ is a subalgebra of $\mathcal{B}(\mathcal{H})$, sometimes it will be useful to think of the above map as defining a right action $\triangleleft$ of Σ on $\mathcal{H}$ by

$$\psi \triangleleft S := S^\circ \psi, \tag{3.1}$$

for all $S \in \Sigma$ and $\psi \in \mathcal{H}$.

Given a unital spectral triple and an antilinear isometry J we formulate the following set of conditions:

$$A^\circ \subset A', \tag{3.2a}$$
$$\Omega^1_D(A)^\circ \subset A', \tag{3.2b}$$
$$\Omega^1_D(A)^\circ \subset \Omega^1_D(A)'. \tag{3.2c}$$

Remark 3.1. The above conditions can be recast in more familiar forms: condition (3.2a) means that $[a, JbJ^{-1}] = 0$ for all $a, b \in A$, *i.e.* the zeroth-order condition (2.3); (3.2b) that $\big[[D, a], JbJ^{-1}\big] = 0$ for all $a, b \in A$, *i.e.* the first-order condition (2.4); and (3.2c) that $\big[[D, a], J[D, b]J^{-1}\big] = 0$ for all $a, b \in A$, which is called the *second-order condition* in Ref. [38]. ⋄

Notice that (3.2a) and (3.2b) are together equivalent to the condition

$$\mathcal{C}\ell_D(A)^\circ \subset A', \tag{3.3}$$

and the three conditions (3.2) are together equivalent to

$$\mathcal{C}\ell_D(A)^\circ \subset \mathcal{C}\ell_D(A)'. \tag{3.4}$$

In view of the above considerations, we can interpret (3.3) by saying that $\mathcal{H}$ is an A-$\mathcal{C}\ell_D(A)$-bimodule, where the left action of A is given by its inclusion as a subalgebra of $\mathcal{B}(\mathcal{H})$ and the right action of $\mathcal{C}\ell_D(A)$ is given by (3.1). Since (3.3) is also equivalent to $A^\circ \subset \mathcal{C}\ell_D(A)'$, we can also interpret it by saying that $\mathcal{H}$ is a $\mathcal{C}\ell_D(A)$-A-bimodule, where now the left action of $\mathcal{C}\ell_D(A)$ is given by its inclusion as a subalgebra of $\mathcal{B}(\mathcal{H})$ and the right action of A is given by (3.1). Finally, (3.4) can be interpreted as saying that $\mathcal{H}$ is a $\mathcal{C}\ell_D(A)$-$\mathcal{C}\ell_D(A)$-bimodule.

In this chapter, for what concerns the twisted real structures, the twist operator ν will satisfy $\nu = \nu^\dagger = \nu^{-1}$ and will commute with both J and A, and so will give an ultra-mild twist.

3.2.2 *Morita equivalence*

Before continuing, it will be useful to first describe Hilbert C*-modules.

Left pre-Hilbert C-modules can also be defined, making the obvious replacements.*

Definition 3.2. Let $\mathcal{A}$ be a C*-algebra. A right *pre-Hilbert $\mathcal{A}$-module* is a complex vector space V equipped with a right $\mathcal{A}$-module structure along with an inner product $\langle \bullet, \bullet \rangle : V \times V \longrightarrow \mathcal{A}$ satisfying, for all $v, w, w_1, w_2 \in V$, $a \in A$ and $\lambda_1, \lambda_2 \in \mathbb{C}$,

1. $\langle v, w_1\lambda_1 + w_2\lambda_2 \rangle = \langle v, w_1 \rangle \lambda_1 + \langle v, w_2 \rangle \lambda_2$;
2. $\langle v, wa \rangle = \langle v, w \rangle a$;
3. $\langle v, w \rangle = \langle w, v \rangle^*$;
4. $\langle v, v \rangle \geq 0$;
5. $\langle v, v \rangle = 0$ if and only if $v = 0$.

One can define a norm on the pre-Hilbert $\mathcal{A}$-module V by taking $\|v\| = \|\langle v, v \rangle\|^{\frac{1}{2}}$. One then obtains a *Hilbert $\mathcal{A}$-module* by taking the norm-completion of V with respect to this norm. ⋄

We can further define a *pre-Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule* by considering the complex vector space V as both a left pre-Hilbert $\mathcal{B}$-module and a right pre-Hilbert $\mathcal{A}$-module, such that the inner products are compatible:

$$v_1 \langle v_2, v_3 \rangle_{\mathcal{A}} = {}_{\mathcal{B}}\langle v_1, v_2 \rangle v_3 \text{ for all } v_1, v_2, v_3 \in V.$$

The two norms coming from the two inner products coincide and so we can obtain a *Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule* by completing with respect to the norm $\|v\| = \left\|\langle v, v \rangle_{\mathcal{A}}\right\|^{\frac{1}{2}} = \left\|{}_{\mathcal{B}}\langle v, v \rangle\right\|^{\frac{1}{2}}$.

Loosely speaking, two C*-algebras are Morita equivalent[1] when there exists a categorical equivalence of their (left or right) module structures. Morita equivalence will play a central role both in this chapter and the next. In this subsection, based on Ref. [55], we will give a brief, high-level overview of Morita equivalence for C*-algebras to understand its topological meaning. In §4.2.2, we will take a more detailed, computational perspective to understand the interplay between Morita equivalence and the Dirac operator in the context of (real) spectral triples.

Def. *A right or left Hilbert C*-module $\mathcal{E}$ is (right- or left-)*full *if* span $\{\langle e_1, e_2 \rangle_{\mathcal{A}} : e_1, e_2 \in \mathcal{E}_{\mathcal{A}}\}$ *is dense in $\mathcal{A}$ or* span $\{{}_{\mathcal{B}}\langle e_1, e_2 \rangle : e_1, e_2 \in {}_{\mathcal{B}}\mathcal{E}\}$ *is dense in $\mathcal{B}$ respectively.*

Def. *The C*-algebra of $\mathcal{A}$-compact operators $\mathcal{K}_{\mathcal{A}}(\mathcal{E})$ is the norm-closure of the two-sided ideal of finite sums of $\mathcal{A}$-linear $\mathcal{E}$-valued ketbras inside* $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$.

Given two (complex) C*-algebras $\mathcal{A}$ and $\mathcal{B}$, an *$\mathcal{A}$-$\mathcal{B}$ Morita equivalence bimodule* (also known as an *imprimitivity bimodule*) is a pair $(\mathcal{E}_{\mathcal{B}}, \Psi)$ of a full right Hilbert $\mathcal{B}$-module $\mathcal{E}_{\mathcal{B}}$ and an isomorphism $\Psi : \mathcal{A} \longrightarrow \mathcal{K}_{\mathcal{B}}(\mathcal{E})$ of $\mathcal{E}_{\mathcal{B}}$. Two C*-algebras $\mathcal{A}$ and $\mathcal{B}$ are called *Morita equivalent* if such an $\mathcal{A}$-$\mathcal{B}$ Morita equivalence bimodule exists. An $\mathcal{A}$-$\mathcal{A}$ Morita equivalence bimodule is also called a *self-Morita equivalence bimodule*.

For a more concrete example, if $E \longrightarrow X$ is a complex hermitian vector bundle over a compact Hausdorff space X, $C(X)$ is the C*-algebra

1 Initially, M.A. Rieffel used the name 'strong Morita equivalence' when considering specifically C*-algebras, but it is now customary to omit the word 'strong'. Indeed, if $\mathcal{A}$ and $\mathcal{B}$ are unital C*-algebras, it is now known that they are strongly Morita equivalent if and only if they are Morita equivalent as rings [2].

of continuous functions on X, and $\Gamma(\mathrm{End}(E))$ is the C*-algebra of continuous sections of the endomorphism bundle of E, then the set of continuous sections of E, $\Gamma(E)$, is a $\Gamma(\mathrm{End}(E))$-$C(X)$ Morita equivalence bimodule.

Every C*-algebra $\mathcal{B}$ is a self-Morita equivalence $\mathcal{B}$-bimodule. If $\mathcal{B}$ is unital, every finitely-generated and projective (hereafter, 'finite projective') right B-module is a full right Hilbert $\mathcal{B}$-module (with a canonical $\mathcal{B}$-valued inner product). If $\mathcal{A}$ and $\mathcal{B}$ are two unital C*-algebras, every $\mathcal{A}$-$\mathcal{B}$ Morita equivalence bimodule is finite projective (both as a left $\mathcal{A}$-module and right $\mathcal{B}$-module) *cf.* [41, Ex. 4.20]. In particular, if $\mathcal{B}$ is unital, every self-Morita equivalence $\mathcal{B}$-bimodule is finite projective.

If $\mathcal{B}$ is finite-dimensional (and therefore unital), every finite projective right $\mathcal{B}$-module is a finite-dimensional complex vector space. Using the structure theorem for finite-dimensional complex C*-algebras it is easy to show that, conversely, every finite-dimensional complex vector space V carrying a right action of $\mathcal{B}$ is finite projective as a right $\mathcal{B}$-module. If $\mathcal{A}$ and $\mathcal{B}$ are finite-dimensional, an $\mathcal{A}$-$\mathcal{B}$ Morita equivalence bimodule is then just a pair (V, Ψ) of a finite-dimensional complex vector space V carrying a right action of $\mathcal{B}$ and an isomorphism Ψ such that $\Psi(\mathcal{A})$ commutes with the right action of $\mathcal{B}$ (note that every right $\mathcal{B}$-linear endomorphism is adjointable and also compact in the finite-dimensional case).

Definition 3.3. Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{A}, \mathcal{B} \subset \mathcal{B}(\mathcal{H})$ be two C*-subalgebras, and J an antilinear isometry on $\mathcal{H}$. Regarding $\mathcal{H}$ as a right $\mathcal{B}$-module with right action given by (3.1),

$$\psi \triangleleft b = b^\circ \psi \quad \text{for all } b \in \mathcal{B}, \psi \in \mathcal{H},$$

we say that J implements a Morita equivalence between $\mathcal{A}$ and $\mathcal{B}$, and write

$$\mathcal{A} \sim_J \mathcal{B}$$

if there exists a dense vector subspace $\mathcal{E} \subset \mathcal{H}$ such that the pair $(\mathcal{E}, \Psi = \mathrm{id}_{\mathcal{A}})$ is an $\mathcal{A}$-$\mathcal{B}$ Morita equivalence bimodule. Concretely, this means that $\mathcal{A} = \mathcal{K}_{\mathcal{B}}(\mathcal{E})$. ⋄

Here we think of $\mathcal{A}$ and $\mathcal{B}$ as concrete C-algebras of bounded operators on $\mathcal{H}$ and use J to define a right action of one of the two algebras, while the isomorphism $\Psi\colon \mathcal{A} \to \mathcal{K}_{\mathcal{B}}(\mathcal{E})$ is just the identity. Notice that $\mathcal{A}$ must commute with the right action of $\mathcal{B}$, given by $\mathcal{B}^\circ$, and not with the left action of $\mathcal{B}$. We implicitly assume that $\mathcal{A}$ and $\mathcal{B}^\circ$ preserve the subspace $\mathcal{E}$.*

Remark 3.4. If $\mathcal{H}$ is finite-dimensional, one has $\mathcal{A} \sim_J \mathcal{B}$ if and only if $\mathcal{A} = (\mathcal{B}^\circ)'$. This can be equivalently rephrased as $\mathcal{A}' = \mathcal{B}^\circ$ (by the double commutant theorem), as $\mathcal{A}^\circ = \mathcal{B}'$ (since $[a, b^\circ] = 0$ if and only if $[a^\circ, b] = [b^\circ, a]^\circ = 0$), or as $(\mathcal{A}^\circ)' = \mathcal{B}$. ⋄

We can now strengthen the conditions (3.3) and (3.4) by requiring that the inclusions are equalities.

Definition 3.5. Let $(A, \mathcal{H}, D, (J, \nu), \chi)$ be a unital, possibly even, spectral triple with (possibly ν-twisted) real structure, $\bar{A}$ the norm-closure of A, $C\ell_D^\chi(A)$ the C*-algebra generated by $C\ell_D(A)$ and χ the grading (in the even case). We will call the spectral triple:

$$\begin{aligned} \textit{spin} &\iff C\ell_D(A) \sim_J \bar{A}, && \text{(3.5a)}\\ \textit{even-spin} &\iff C\ell_D^\chi(A) \sim_J \bar{A}, && \text{(3.5b)}\\ \textit{Hodge} &\iff C\ell_D(A) \sim_J C\ell_D(A). && \text{(3.5c)} \end{aligned}$$

◇

Notice that (3.5c) implies (3.4), while (3.5a) and (3.5b) both imply (3.3). Less obvious is that (3.5a) implies (3.5b), as shown in the first part of the next proposition.

Proposition 3.6. *Let $(A, \mathcal{H}, D, (J, \nu), \chi)$ be a unital even spectral triple with (possibly ν-twisted) real structure.*

1. *If (3.5a) is satisfied, then $\chi \in C\ell_D(A)$ and (3.5b) is satisfied as well.*
2. *If (3.4) is satisfied and $\chi \in C\ell_D(A)$, then $\Omega^1_D(A) = 0$.*

Proof. 1. Since $\chi \in A'$, one has $\chi^\circ \in (A^\circ)'$. But $\chi^\circ = \pm\chi$ due to (2.5), hence the restriction of χ to $\mathcal{E}$ commutes with the right action of $\bar{A}$. If (3.5a) is satisfied, there must exist an element $\xi \in C\ell_D(A)$ such that $\chi - \xi$ is zero on $\mathcal{E}$, but since it is a bounded operator, it must be zero on the whole of $\mathcal{H}$.

2. If $\chi \in C\ell_D(A)$ and (3.4) is satisfied, one has $\chi = \pm\chi^\circ \in C\ell_D(A)'$ as well, and it follows that every 1-form ω commutes with χ. However, all 1-forms must also anticommute with χ, since χ commutes with A and anticommutes with D. Therefore using $\omega = \frac{1}{2}\chi\big([\chi, \omega] + \{\chi, \omega\}\big)$ we hence find $\omega = 0$. □

A special class of spectral triples with $\chi \in C\ell_D(A)$ is given by the so-called orientable spectral triples. Recall that a spectral triple $(A, \mathcal{H}, D)$ is called *orientable* if there is a Hochschild n-cycle

According to the original terminology of Ref. [16].

$$c = \sum a_0 \otimes a_1 \otimes \ldots \otimes a_n \in Z_n(A, A)$$

for the sum finite and $a_0, \ldots, a_n \in A$ such that $\sum a_0[D, a_1] \cdots [D, a_n]$ is equal to either 1 or, in the even case, to χ. In particular, an even orientable spectral triple has $\chi \in C\ell_D(A)$. Prop. 3.6(2) shows that on an orientable spectral triple satisfying the second-order (or Hodge) condition, all 1-forms are zero.

For real spectral triples there is another notion of orientation. A real spectral triple is called *real-orientable* if there is a Hochschild n-cycle

In Ref. [20, §18.1] another version of real-orientability is presented where the elements $a_1, \ldots, a_n$ are required to commute with J.

$$c = \sum (a_0 \otimes b_0) \otimes a_1 \otimes \ldots \otimes a_n \in Z_n(A, A \otimes A^{\mathrm{op}})$$

for $a_0, \ldots, a_n \in A$ and $b_0 \in A^{\mathrm{op}}$ such that $\sum a_0 b_0 [D, a_1] \cdots [D, a_n]$ is equal to either 1 or, in the even case, to χ [60].

In general, real-orientability is a weaker notion than orientability. In the example of the Hodge-de Rham spectral triple (which will be described in the subsections to follow), one has $JaJ^{-1} = a^*$ for all $a \in A$ so that $A = A^\circ$ and the two notions coincide. In particular, the Hodge-de Rham spectral triple reviewed in §3.2.3 is orientable if one takes the appropriate grading operator.

3.2.3 *Closed oriented Riemannian manifolds*

There are two main classes of spectral triples $(A, \mathcal{H}, D)$ that one can associate to a closed oriented Riemannian manifold M. For one, the Hilbert space is given by $\mathcal{H} = L^2(\bigwedge^\bullet_{\mathbb{C}} T^*M)$, the square-integrable (complex-valued) forms on M, and the Dirac operator D is the Hodge-de Rham operator. If M is a spin manifold, the other is obtained by using as the Hilbert space $\mathcal{H}$ the space of square-integrable spinors on M, and $D = i\nabla\!\!\!/$ is the Dirac operator corresponding to the spin structure. In both cases, $A = C^\infty(M)$ is the algebra of smooth functions on M acting on $\mathcal{H}$ by pointwise multiplication.

Any commutative unital spectral triple, satisfying a suitable additional set of axioms (listed for example in Ref. [18]), turns out to be of one of these two types. More precisely, depending on the axioms, from a commutative unital spectral triple one can reconstruct either a closed oriented Riemannian manifold or a spin^c manifold (see Thms. 1.1 and 1.2 in Ref. [18]), and in the latter case the zeroth-order condition selects spin manifolds among spin^c manifolds. This last step follows from an algebraic characterisation of spin^c manifolds in terms of Morita equivalence, sketched in §1.2.1: such an equivalence is implemented by a real structure J exactly when the manifold is spin. This characterisation is recalled for example at the beginning of Ref. [22] and motivates the first part of Def. 3.5. In this subsection we spell out the construction of the spectral triple given by the Hodge-de Rham operator on differential forms and discuss some aspects related to the self-Morita equivalence of the Clifford algebra and how to implement it by means of an antiunitary operator. We will adopt the notations and conventions of Refs. [41, 48].

Let M be a closed oriented n-dimensional Riemannian manifold with metric tensor g. In the following, we let

$$A := C^\infty(M) \tag{3.6}$$

be the algebra of complex-valued smooth functions on M, $\bar{A} = C(M)$ the algebra of continuous functions, $\Omega^\bullet_{\mathbb{C}}(M)$ the space of smooth sections of the complexified bundle of forms $\bigwedge^\bullet_{\mathbb{C}} T^*M \to M$, and

$$\mathcal{E} := \Gamma(\textstyle\bigwedge^\bullet_{\mathbb{C}} T^*M) \tag{3.7}$$

the $C(M)$-module of continuous sections. The Riemannian metric g induces a hermitian product on the fibres of the bundle $\bigwedge^\bullet_{\mathbb{C}} T^*M$, and a $C(M)$-valued hermitian product on $\mathcal{E}$, given by (*e.g.* [41, §9.B])

$$(\eta, \xi) := \det\left(g^{-1}(\eta_i^*, \xi_j)\right)$$

for all products of 1-forms $\eta = \bigwedge_{i=1}^{k} \eta_i$ and $\xi = \bigwedge_{j=1}^{k} \xi_j$, which is extended to $\mathcal{E}$ by linearity and by declaring that forms with different degree are mutually orthogonal. With the above Hermitian structure, $\mathcal{E}$ becomes a full right Hilbert $\bar{A}$-module (like any module of continuous sections of a hermitian vector bundle on M).

We let $\mathcal{H} := L^2(\bigwedge_{\mathbb{C}}^{\bullet} T^*M)$ be the Hilbert space completion of $\mathcal{E}$ with respect to the inner product

$$\langle \eta, \xi \rangle := \int_M (\eta, \xi)\, \omega_g,$$

where ω_g is the Riemannian volume form, given on any positively oriented chart by $\omega_g = \sqrt{\det(g)}\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$.

The Hodge star operator $\star$ on real-valued k-forms is implicitly defined by the equality

$$\eta \wedge (\star\xi) = (\eta, \xi)\, \omega_g$$

for all real k-forms η and ξ, and satisfies the well-known relations

$$\star^2 = (-1)^{k(n-k)}$$

on k-forms, and

$$\star \circ d \circ \star = (-1)^{n(k+1)+1} d^\dagger,$$

where d is the exterior derivative on (smooth) forms and $d^\dagger$ is its formal adjoint. The Hodge star operator can be extended to complex-valued forms linearly, *e.g.* as in Ref. [61], or antilinearly, *e.g.* as in Ref. [57]. We adopt the former convention.

Proposition 3.7 ([41, 48]). *With $A = C^\infty(M)$ and $\mathcal{H} = L^2(\bigwedge_{\mathbb{C}}^{\bullet} T^*M)$ as above, and with the closure of the (essentially self-adjoint elliptic [48]) operator $d + d^\dagger$ (the Hodge-de Rham operator of (M, g)), we get a unital even spectral triple $(A, \mathcal{H}, d + d^\dagger)$.*

Def. *A linear (pseudo-)differential operator is* elliptic *if its principal symbol is invertible.*

Two natural gradings χ_{deg} and $\chi_\star$ are given on k-forms by

$$\chi_{\mathrm{deg}} := (-1)^k$$

and (see *e.g.* the proof of [18, Thm. 11.4])

$$\chi_\star := i^{-\frac{n(n+1)}{2}} (-1)^{k(n-k)+\frac{k(k+1)}{2}} \star .$$

Evidently $d + d^\dagger$ anticommutes with χ_{deg} and, since we have that $\chi_\star \circ d \circ \chi_\star = (-1)^{n+1} d^\dagger$, if n is even $\chi_\star$ also anticommutes with $d + d^\dagger$.

If n is odd, since $d + d^\dagger$ and $\chi_\star$ commute one can use this grading to reduce the Hilbert space and build a spectral triple on the eigenspace of $\chi_\star$ with eigenvalue $+1$ (which is what is done *e.g.* in Ref. [18]). We will not follow this approach, since we want $\mathcal{H}$ to be isomorphic to the space of sections of the Clifford algebra bundle (up to a completion) in both the odd- and even-dimensional cases.

In order to talk about the Hodge condition, we must first recall the (geometric) definition of the Clifford algebra bundle $\mathbb{C}\mathrm{l}(M, g) \to M$. The fibre at a point $x \in M$ is the unital associative complex algebra generated by $v_1, v_2 \in T_x^*M$ satisfying $v_1 v_2 + v_2 v_1 = 2g^{-1}(v_1, v_2)$. Denoting by $\lrcorner$ the interior product, which is the adjoint of the (left) exterior product, a left action λ and a right action ρ (an antirepresentation) of the algebra $\Gamma(\mathbb{C}\mathrm{l}(M, g))$ on forms are given on each fibre by

$$\lambda(v)w := v \wedge w + v \lrcorner w \quad \text{and} \quad \rho(v)w := (-1)^k(v \wedge w - v \lrcorner w)$$

for all $v \in T_x^*M$ and $w \in \bigwedge_{\mathbb{C}}^k T_x^*M$. We will refer to λ and ρ as *left* and *right Clifford multiplication*. They turn the space $\mathcal{E}$ in (3.7) into a $\Gamma(\mathbb{C}\mathrm{l}(M, g))$-bimodule. One can show that (see Ref. [18] or Ref. [48]) the grading $\chi_\star$ is given at each point by

$$\chi_\star = i^{-\frac{n(n-1)}{2}} \lambda(e^1 e^2 \cdots e^n), \tag{3.8}$$

where $\{e^i\}$, $i = 1, \dots, n$ is a positively oriented orthonormal basis of T_x^*M.

A vector bundle isomorphism $\mathbb{C}\mathrm{l}(M, g) \to \bigwedge_{\mathbb{C}}^\bullet T^*M$ is given on each fibre by

$$e^{i_1} e^{i_2} \cdots e^{i_k} \mapsto e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_k}, \tag{3.9}$$

for $1 \le i_1 < i_2 < \ldots < i_k \le n$, or more generally by

$$\sigma \colon v_1 v_2 \cdots v_k \mapsto \lambda(v_1)\lambda(v_2) \cdots \lambda(v_k)1$$

for all $v_1, \dots, v_k \in T_x^*M$. The inverse map [41, Eqn. (5.4)]

$$Q(v_1 \wedge \ldots \wedge v_k) = \frac{1}{k!} \sum_{s \in S_k} (-1)^{\|s\|} v_{s(1)} \cdots v_{s(k)} \tag{3.10}$$

will be useful later on. The maps σ and Q are called the *symbol map* and the *quantisation map* respectively in Ref. [41].

These maps give a vector space isomorphism $\Gamma(\mathbb{C}\mathrm{l}(M, g)) \to \mathcal{E}$ on sections intertwining the left/right multiplication of the algebra $\Gamma(\mathbb{C}\mathrm{l}(M, g))$ on itself with the left/right Clifford multiplication on $\mathcal{E}$. The next proposition then follows.

We use a different sign convention than in §3.3 to adapt to the convention in Ref. [41]. The choice of sign in front of g is in any case irrelevant after complexifying.

For λ and ρ we follow the conventions of Ref. [41, §5.1]. In particular, $\lambda(v)^2 = +g^{-1}(v, v)$ in contrast to the $-$ sign in the proof of [18, Thm. 11.4].

Globally, $\chi_\star$ is proportional to left Clifford multiplication by the Riemannian volume form. Note also the different phase in (3.8) compared to Ref. [18], due to our different sign conventions.

Proposition 3.8. *$\mathcal{E}$ is a self-Morita equivalence* $\Gamma(\mathbb{C}\mathrm{l}(M, g))$*-bimodule.*

An involution on sections of the Clifford algebra bundle is defined as follows. At each point x, on products of real cotangent vectors we have

$$(v_1 v_2 \cdots v_k)^\mathsf{T} := v_k \cdots v_2 v_1$$

for all $v_1, \dots, v_k \in T_x^*M$. The map is then extended antilinearly to the fibre of $\mathbb{C}\mathrm{l}(M, g)$ at x, and pointwise to the algebra of continuous sections. The left Clifford action then transforms this involution into the adjoint operation $\dagger$.

Lemma 3.9. *One has*

$$\lambda(\xi)^\dagger = \lambda(\xi^\dagger)$$

for all $\xi \in \Gamma(\mathbb{C}l(M, g))$.

Proof. Both sides of the equality are antilinear antihomomorphisms. It is enough to prove the equality for generators, which means that $\lambda(v)^\dagger = \lambda(v)$ for all real cotangent vectors $v \in T_x^*M$ and all $x \in M$. This immediately follows from the definition of the interior product as the adjoint of the left wedge product: $\langle v \wedge w_1, w_2 \rangle = \langle w_1, v \lrcorner w_2 \rangle$ for all $v \in T_x^*M$ and $w_1 \in \bigwedge_{\mathbb{C}}^{k-1} T_x^*M$, $w_2 \in \bigwedge_{\mathbb{C}}^{k} T_x^*M$. □

Let $\omega = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_k$ be a product of k 1-forms. Two natural antilinear isometries on forms C_1 and C_2 are given by pointwise complex conjugation

$$C_1\omega = \omega^*, \tag{3.11}$$

and its composition with the canonical anti-involution, explicitly,

$$\begin{aligned} C_2\omega &= \omega_k^* \wedge \omega_{k-1}^* \wedge \ldots \wedge \omega_1^* \\ &= (-1)^{k(k-1)/2}\omega^*. \end{aligned} \tag{3.12}$$

We will adopt the notation $\omega^\dagger := C_2(\omega)$. We can define an antilinear isomorphism $\xi \mapsto \xi^\dagger$ on the Clifford algebra as well, by declaring it to be the identity on real cotangent vectors [41, Ex. 5.5]. The symbol map and quantisation map intertwine the complex conjugations as $Q(\omega^*) = Q(\omega)^*$ for all forms ω. One can also check on a basis (3.9) that $\sigma(\xi^\dagger) = \sigma(\xi)^\dagger$, so that the symbol map and quantisation map intertwine the main anti-involutions as well.

Remark 3.10. It is well-known and straightforward to check that, when equipped with the antilinear isometry C_1, $(A, \mathcal{H}, D, C_1)$ is a real spectral triple. ◇

The closely related charge conjugation operator is obtained from C_1 by $\chi_{\deg}C_1$ [41, Ex. 5.6].

Lemma 3.11. *Let J be the antilinear isometry on $\mathcal{H}$ given by*

$$J\omega := \omega^\dagger. \tag{3.13}$$

Then

1. *for all sections ξ of the Clifford algebra bundle,*

$$J\lambda(\xi)J^{-1} = \rho(\xi);$$

2. $J \circ d \circ J^{-1} = d \circ \chi_{\deg}$.

Proof. Note that $J^{-1} = J$ and that $J := \sigma \circ \dagger \circ Q$. Since the main anti-involution on the Clifford algebra exchanges left and right multiplication, the corresponding operator on forms intertwines the left and right Clifford actions.

Let ω be a k-form. Then

$$\begin{aligned} dJ\omega &= (-1)^{k(k-1)/2} d\omega^* = (-1)^{k(k-1)/2}(d\omega)^* \\ &= (-1)^{k(k-1)/2}(-1)^{k(k+1)/2} J(d\omega) = (-1)^{k^2} J(d\omega) = J(d(\chi_{\deg}\omega)), \end{aligned}$$

where last equality follows from the observation that k^2 and k have the same parity. □

Since d and $\chi_{\deg}$ anticommute, it follows from previous lemma that

$$J \circ d^\dagger \circ J^{-1} = (d \circ \chi_{\deg})^\dagger = -d^\dagger \circ \chi_{\deg}.$$

If we denote by D the closure of the operator $-i(d - d^\dagger)$ (called the Hodge-Dirac operator in Ref. [41, Def. 9.24]), then D and the Hodge-de Rham operator $d + d^\dagger$ of Prop. 3.7 are related by the operator J:

$$J(d + d^\dagger)J^{-1} = iD\chi_{\deg}.$$

Note that the definition of a noncommutative manifold in Ref. [39] is motivated by a similar observation, see Ref. [39, p.102].

We now adopt D as our Dirac operator and relate the geometric and algebraic definitions of Clifford algebras. We know that D is a Dirac-type operator, given on smooth sections by [41, p.426]

$$D = -i(d - d^\dagger) = -i\lambda \circ \nabla,$$

Similarly, $(d+d^\dagger)\chi_{\deg} = \rho \circ \nabla$.

where $\nabla\colon \Omega^\bullet_{\mathbb{C}}(M) \longrightarrow \Omega^{\bullet+1}_{\mathbb{C}}(M)$ is the Levi-Civita connection. In particular, it follows from the Leibniz rule that

$$i[D, f] = \lambda(df) \tag{3.14}$$

for all $f \in C^\infty(M)$.

Proposition 3.12. *When $A = C^\infty(M)$, $C\ell_D(A) = \Gamma(\mathbb{C}\mathrm{l}(M, g))$, which acts on $\mathcal{H}$ via λ.*

Proof. It follows from (3.14) that the (algebraic) Clifford algebra is the C*-algebra of bounded operators on $\mathcal{H}$ generated by smooth functions and Clifford multiplication by 1-forms. In other words, $C\ell_D(A)$ is generated by smooth sections of the Clifford algebra bundle. The algebra $\Gamma(\mathbb{C}\mathrm{l}(M, g))$ on the other hand is generated by continuous functions and continuous sections of T^*M (as one can show by using a partition of unity subordinated to a finite open cover of M, which exists since M is compact). Every continuous function is the norm limit of a sequence of smooth functions. Also every continuous section ξ of T^*M is the norm limit of a sequence of smooth sections, where the norm $\|\lambda(\xi)\|$ is the operator norm on $\mathcal{H}$ composed with λ. Indeed, for $\xi \in \Gamma(T^*M)$ one has $\lambda(\xi)^\dagger\lambda(\xi) = g^{-1}(\xi, \xi)$ and then

$$\|\lambda(\xi)\|^2 = \left\|\lambda(\xi)^\dagger\lambda(\xi)\right\| = \sup_{x \in M} |\xi(x)|^2, \tag{3.15}$$

where the norm on the right hand side is the one on T_x^*M coming from the Riemannian metric.

Now, any continuous ξ can be written as a finite sum of continuous sections each supported on a chart (by using a partition of unity). To conclude the proof it is then enough to show that continuous sections supported on a chart are the norm limit of smooth sections supported on a chart. But this follows trivially from (3.15) and the fact that in a chart, sections of the Clifford algebra bundle look like matrices of (continuous/smooth) functions. □

It follows from Prop. 3.12 that, for the spectral triple considered here, the dense subspace $\mathcal{E}$ of $\mathcal{H}$ in (3.7) is a self-Morita equivalence $C\ell_D(A)$-bimodule. By Lemma 3.11 the main anti-involution J exchanges the left and right Clifford multiplication, so that we can finally make the following claim.

Proposition 3.13. *The data*

$$\big(A, \mathcal{H}, D, (J, \nu), \chi\big) = \big(C^\infty(M), L^2(\wedge_{\mathbb{C}}^{\bullet} T^*M), -i(d-d^\dagger), (J, \nu), \chi_{\deg}\big),$$

with J given by (3.13) *and ν given by $(-1)^{k(k+1)/2}$ times the identity operator on k-forms, comprise an even spectral triple with twisted real structure satisfying the Hodge condition* (3.5c)*. The KO-dimension of this spectral triple is* $0 \pmod 8$*.*

Proof. The statement about the Hodge condition follows from the discussion above. Clearly $J^2 = 1$, so that (2.1) is satisfied with sign $\varepsilon = +1$. Since J doesn't change the degree of a form, (2.5) is also satisfied with sign $\varepsilon'' = +1$. Finally, D anticommutes with C_1 (since $d(\omega^*) = (d\omega)^*$ for all forms ω). If ω has degree k, then

$$JDJ^{-1}\omega = (-1)^{\frac{k(k-1)}{2}} JDC_1\omega = (-1)^{\frac{k(k-1)}{2}+\frac{k(k+1)}{2}} C_1DC_1\omega = (-1)^{k+1}D\omega,$$

where we used the fact that k^2 and k have the same parity. It follows that

$$\begin{aligned}\nu JD\omega = (-1)^{\frac{(k+1)(k+2)}{2}} JD\omega &= (-1)^{\frac{(k+1)(k+2)}{2}+(k+1)} DJ\omega \\ &= (-1)^{2(k+1)^2} DJ\nu\omega = DJ\nu\omega.\end{aligned}$$

Thus, (2.9) is satisfied with sign $\varepsilon' = +1$. □

It is worth mentioning that, although in the reconstruction theorem (*cf.* [18]) the role of J is to pass from spin^c structures to spin, the presence of a reality operator is crucial even for spectral triples that are not built from Dirac spinors. In the noncommutative case, it is necessary to make sense of the condition which makes the Dirac operator a first-order differential operator, and in the spectral approach to gauge theories, it is required in order to define the adjoint action of the gauge group, the real part of the spectral triple, and more. Furthermore, differential forms can be constructed from spinors by the use of twisted modules [60, §2.5]. It is therefore not unreasonable to investigate real structures even in the absence of (explicit) spinors.

The interest in self-Morita equivalences of a Clifford algebra (which we note are implemented by J) is mainly motivated by the fact that this is what happens for the finite-dimensional part of the spectral triple of the Standard Model [26, 31]. It is an interesting observation, coming from SU(5) grand unified theories, that the Hilbert space of such a spectral triple (when considering only one generation of particles) is isomorphic to the exterior algebra $\bigwedge^5\mathbb{C}$ (with the representation of the gauge group the restriction of the natural representation of SU(5) on such a space). It is tempting then to speculate that the Hilbert space of the spectral triple of the Standard Model might have as much to do with differential forms as it does Dirac spinors, though any further investigation lies outside the scope of this thesis.

3.3 THE HODGE-DE RHAM OPERATOR ON THE TORUS

In this section we will focus our discussion to the Hodge-de Rham spectral triple on the 2-torus $T^2 := \mathbb{R}^2/\mathbb{Z}^2$. We use this example to argue that, when considering the Hodge-de Rham spectral triple of a closed oriented Riemannian manifold, the second-order condition (3.4) is incompatible with (2.2) and, if one wants to enforce (3.4), then (2.2) must be replaced by (2.9).

We think of functions/forms on T^2 as $\mathbb{Z}^2$-invariant functions/forms on $\mathbb{R}^2$ respectively. Consider the complex vector space isomorphism $\mu\colon \Omega^\bullet_{\mathbb{C}}(T^2) \to M_2(C^\infty(T^2))$ given by

$$f_0 + f_1\,\mathrm{d}x + f_2\,\mathrm{d}y + f_3\,\mathrm{d}x \wedge \mathrm{d}y \mapsto \sigma^0 f_0 + i\sigma^1 f_1 + i\sigma^2 f_2 - i\sigma^3 f_3 \tag{3.16}$$

for $f_0, \ldots, f_3 \in C^\infty(T^2)$. Under this isomorphism, the natural inner product of forms (associated to the flat metric on T^2) becomes the natural inner product on matrices of functions

$$\langle \psi, \varphi \rangle = \frac{1}{2}\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, \mathrm{tr}\Big(\psi^\dagger \varphi\Big),$$

and the Hilbert space completion is $\mathcal{H} := M_2(L^2(T^2))$. The Hodge-de Rham operator is mapped to the operator

$$D := \mu \circ (d + d^\dagger) \circ \mu^{-1} = iL_{\sigma^1}\frac{\partial}{\partial x} + iL_{\sigma^2}\frac{\partial}{\partial y},$$

where L and R denote respectively left and right pointwise matrix multiplication:

$$L_a\psi := a\psi \quad \text{and} \quad R_a\psi := \psi a,$$

for all $a \in M_2(L^\infty(T^2))$ and $\psi \in M_2(L^2(T^2))$. If f is a scalar function, we identify f with $f\sigma^0$ and $L_f = R_f$ will be denoted simply by f. The natural grading given by the degree of forms is transformed by μ into the operator $\chi_{\deg}$ on $\mathcal{H}$ given by

$$\chi_{\deg}\psi := \sigma^3\psi\sigma^3,$$

for all $\psi \in \mathcal{H}$. Lastly, we set $A := C^\infty(T^2)$.

Remark 3.14. With the notation above, $(A, H, D, \chi_{\deg})$ is a spectral triple unitarily equivalent to the Hodge-de Rham spectral triple of T^2. The equivalence is given by the L^2-closure of the map μ in (3.16). ⋄

For all f we have

$$[D, f] = L_{\mu(df)},$$

so that under the isomorphism μ, Clifford multiplication becomes left matrix multiplication. One easily computes $C\ell_D(A)$, which is given by the full matrix algebra $M_2(C(T^2))$ acting via left multiplication on $\mathcal{H}$. Indeed, let us denote by

$$u(x, y) := e^{ix} \qquad \text{and} \qquad v(x, y) := e^{iy} \tag{3.17}$$

the unitary generators of A. Since

$$-u^*[D, u] = L_{\sigma^1} \qquad \text{and} \qquad -v^*[D, v] = L_{\sigma^2},$$

the elements σ^1 and σ^2 belong to $C\ell_D(A)$, and as is well-known they generate $M_2(\mathbb{R})$ as an algebra. Elements in the norm-closure of A belong to $C\ell_D(A)$ as well, which therefore contains the algebra generated by $C(T^2)$ and $M_2(\mathbb{R})$, *i.e.* all of $M_2(C(T^2))$. We therefore have the following results.

Lemma 3.15. *When $A = C^\infty(T^2)$, $C\ell_D(A) = M_2(C(T^2))$, which acts on $\mathcal{H}$ by left matrix multiplication.*

Proposition 3.16. *When $A = C^\infty(T^2)$, $C\ell_D(A)' = M_2(L^\infty(T^2))$, which acts on $\mathcal{H}$ by right matrix multiplication.*

Proof. Up to a natural identification, we have $\mathcal{H} = L^2(T^2) \otimes M_2(\mathbb{R})$ and $C\ell_D(A) = C(T^2) \otimes M_2(\mathbb{R})$ where $M_2(\mathbb{R})$ acts by left matrix multiplication. Continuous functions on T^2 are dense (in the strong operator topology) in the von Neumann algebra $L^\infty(T^2)$ of essentially bounded measurable functions (with respect to the Lebesgue measure), and $L^\infty(T^2)$ is its own commutant. Using the commutation theorem for tensor products of von Neumann algebras [56] one finds $C\ell_D(A)' = (L^\infty(T^2) \otimes M_2(\mathbb{R}))' = L^\infty(T^2)' \otimes M_2(\mathbb{R})'$ and the thesis follows. □

Recall (3.11) and (3.12), the definitions of the the antilinear maps C_1 and C_2 on forms, the former multiplicative and the latter antimultiplicative, with $C_1(\omega) = \omega^*$ the pointwise complex conjugate of a form and $C_2(\omega) = (-1)^{k(k-1)/2}\omega^*$ on forms of degree k. On $\mathcal{H}$ two corresponding antilinear isometries are given by $J_j := \chi_{\deg}\mu \circ C_j \circ \mu^{-1}$,

$j = 1, 2$ (where $\chi_{\deg}$ is included to simplify the expressions). One easily checks that, for all $\psi_{ij} \in L^2(T^2)$

$$J_1 \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} = \begin{pmatrix} \psi_{22}^* & \psi_{21}^* \\ \psi_{12}^* & \psi_{11}^* \end{pmatrix}, \quad \text{and} \quad J_2 \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} = \begin{pmatrix} \psi_{11}^* & \psi_{21}^* \\ \psi_{12}^* & \psi_{22}^* \end{pmatrix}. \tag{3.18}$$

Note that $J_2 = \dagger$ is just matrix hermitian conjugation. It is useful in the computations to recognise that $J_1 = L_{\sigma^1} R_{\sigma^1} J_0$, where $J_0 = *$ is entrywise pointwise complex conjugation,

$$J_0 \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} = \begin{pmatrix} \psi_{11}^* & \psi_{12}^* \\ \psi_{21}^* & \psi_{22}^* \end{pmatrix}. \tag{3.19}$$

In particular, it follows that J_1 is multiplicative (since J_0 is multiplicative and $(\sigma^1)^2 = 1_2$):

$$J_1(a\psi) = J_1(a)J_1(\psi)$$

for all $a \in M_2(L^\infty(T^2))$ and $\psi \in M_2(L^2(T^2))$. By contrast, J_2 is antimultiplicative:

$$J_2(\psi a) = J_2(a)J_2(\psi)$$

for all $\psi \in M_2(L^2(T^2))$ and $a \in M_2(L^\infty(T^2))$. In particular, one finds that for all $a \in M_2(L^\infty(T^2))$ we have

$$J_0 L_a J_0 = L_{a^*}, \qquad J_1 L_a J_1 = L_{J_1(a)}, \qquad J_2 L_a J_2 = R_{a^\dagger}, \tag{3.20}$$

along with the analogous relations with L and R interchanged.

Proposition 3.17. *$(A, \mathcal{H}, D, J_1, \chi_{\deg})$ is a unital even real spectral triple.*

The proof is given by a straightforward computation.

Proposition 3.18. *$(A, \mathcal{H}, D, (J_2, \nu), \chi_{\deg})$ is a unital even spectral triple with twisted real structure, where $\nu = J_1 J_2$, or explicitly,*

$$\nu \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} := \begin{pmatrix} \psi_{22} & \psi_{12} \\ \psi_{21} & \psi_{11} \end{pmatrix}. \tag{3.21}$$

Furthermore, this spectral triple satisfies the second-order condition and the Hodge condition (3.5c).

Proof. The condition (3.4) follows from the fact that conjugation by J_2 transforms left into right matrix multiplication, *cf.* (3.20). Since J_1 satisfies (2.2), clearly $J_2 = \nu J_1 = J_1 \nu$ satisfies (2.9). The subspace $\mathcal{E} := M_2(C(T^2)) = C\ell_D(A)$ is dense in $\mathcal{H}$ and a self-Morita equivalence $C\ell_D(A)$-bimodule, with the left/right action of $C\ell_D(A)$ on $\mathcal{E}$ given by left/right multiplication, as requested. The map $L_a \mapsto J_2 L_{a^\dagger} J_2^{-1} = R_a$ transforms the left into the right action and vice versa, *cf.* (3.20), so that the self-Morita equivalence is implemented by J_2. □

The spectral triples in Props. 3.17 and 3.18 both have $\varepsilon = \varepsilon' = \varepsilon'' = +1$, *i.e.* KO-dimension 0 (mod 8). Notice that, firstly, $\nu = \mu \circ C_1 \circ C_2 \circ \mu^{-1}$, and $C_1 \circ C_2$ is the canonical anti-involution of the Clifford algebra, given on k-forms by $(-1)^{k(k-1)/2}$ times the identity; secondly, the spectral triple in Prop. 3.17 does *not* satisfy the second-order condition, because, for example, $J_1 L_{\sigma^1} J_1 = L_{\sigma^1}$ does not commute with L_{σ^2}; and thirdly, the spectral triple in Prop. 3.18 does *not* satisfy condition (2.2), since

$$-J_2 D J_2 = R_{\sigma^1} i \frac{\partial}{\partial x} + R_{\sigma^2} i \frac{\partial}{\partial y} \neq \pm D.$$

The next theorem shows that (2.2) and (3.4) are incompatible, so that if one wants the second-order condition to be satisfied, one is forced to introduce a twist.

Theorem 3.19. *The spectral triple $(A, \mathcal{H}, D)$ admits no antilinear isometry J satisfying both (2.2) and (3.4).*

Proof. Assume that both (2.2) and (3.4) are satisfied. Since conjugation by J maps $a \in \mathcal{C}\ell_D(A)$ into its commutant, it follows from (3.4) that

$$J L_{a^\dagger} J^{-1} = R_{\eta(a)} \tag{3.22}$$

for some $\eta(a) \in M_2(L^\infty(T^2))$. This defines a homomorphism

$$\eta \colon M_2(C(T^2)) \to M_2(L^\infty(T^2)).$$

If $f \in C^\infty(T^2)$ is a smooth scalar function, it follows from (2.2) that, since both J and $L_{f^\dagger}$ preserve the domain of D, $R_{\eta(f)}$ preserves the domain of D as well and

$$\left[D, R_{\eta(f)}\right] = \varepsilon' J \left[D, L_{f^\dagger}\right] J^{-1} \tag{3.23}$$

extends to a bounded operator on $\mathcal{H}$. Moreover, if we apply $R_{\eta(f)}$ to $1 \in \mathcal{H}$ we deduce that $R_{\eta(f)}(1) = \eta(f)$ is in the domain of self-adjointness of D, *i.e.* a matrix of functions in the Sobolev space $W^{1,2}(T^2)$. Now let u be the unitary element in (3.17) and write

Def. *The* Sobolev *space $W^{k,p}(X)$ on $X \subseteq \mathbb{R}^n$, $k, n \in \mathbb{N}$, $1 \le p \le \infty$, is given by $W^{k,p}(X) := \{f \in L^p(X) : \partial_x^I f \in L^p(X)\ \forall |I| \le k\}$ where $I = (I_1, \ldots, I_n)$ is a multi-index and $\partial_x^I = \partial_{x^1}^{I_1} \ldots \partial_{x^n}^{I_n}$ are weak derivatives. Sobolev spaces can be endowed with a norm and when $p = 2$ they are Hilbert spaces.*

$$\eta(u) = f_0 \sigma^0 + f_1 \sigma^1 + f_2 \sigma^2 + f_3 \sigma^3$$

for some $f_0, \ldots, f_3 \in W^{1,2}(T^2)$. Then

$$\begin{aligned} 0 \overset{(2.2)}{=} \left[L_{\sigma^1}, J[D, L_{u^\dagger}]J^{-1}\right] &= \varepsilon' \left[L_{\sigma^1}, \left[D, R_{\eta(u)}\right]\right] \\ &= \varepsilon' \sum_{\alpha=0}^{3} [L_{\sigma^1}, [D, f_\alpha]] R_{\sigma^\alpha} \\ &= -\varepsilon' \sum_{\alpha=0}^{3} L_{\sigma^3} \frac{\partial f_\alpha}{\partial y} R_{\sigma^\alpha}. \end{aligned}$$

From the independence of the linear maps R_{σ^α} we get $\partial_y f_\alpha = 0$ for all α (where the derivative is in the sense of distributions), and in a similar way one proves that $\partial_x f_\alpha = 0$ for all α. Thus, $\eta(u)$ is a constant matrix and

$$L_{\sigma^1} = [D, L_{u^\dagger}] = \varepsilon' \big[D, R_{\eta(u)}\big] J^{-1} = 0,$$

which is a contradiction. □

One may wonder how unique the example in (3.17) is, and how unique a J satisfying the second-order condition is. A partial answer is provided by the following proposition.

Proposition 3.20. *Let $U \in M_2(L^\infty(T^2))$ be a unitary operator and let ν be given by* (3.21)*. Then*:

1. *The operator*

   $$J_U := L_U R_U J_2$$

   is an antilinear isometry satisfying $J_U^2 = 1$ and the second-order condition.

2. *If $\nu(U) = U^\dagger$ almost everywhere, then J_U satisfies* (2.9) *with $\varepsilon' = +1$ and twist given by*

   $$\nu_U := L_U R_U \nu.$$

   Moreover, in such a case J_U and ν_U commute, and $\nu_U^2 = 1$.

3. *If $\nu(U) = U^\dagger$ almost everywhere, then J_U also commutes with $\chi_{\deg}$.*

Def. *A property holds* almost everywhere *if it holds everywhere except on a subset of measure* 0.

Thus if $\nu(U) = U^\dagger$, the data $(A, D, \mathcal{H}, (J_U, \nu_U), \chi_{\deg})$ form a unital even spectral triple with twisted real structure.

Proof. 1. From (3.20) we deduce that $L_U R_U J_2 = J_2 L_{U^\dagger} R_{U^\dagger}$ and from the unitarity of U it follows that $J_U^2 = 1$. It also follows that

$$J_U L_a J_U = J_2 L_{U^\dagger a U} J_2 = R_{U^\dagger a U} \in \mathcal{C}\ell_D(A)'$$

for all $a \in M_2(L^\infty(T^2))$.

2. Consider some $\psi \in \mathcal{H}$. Notice that $\nu(\psi) = \sigma^1 \psi^\intercal \sigma^1$, so that ν is antimultiplicative and

$$\nu L_U R_U(\psi) = \nu(U\psi U) = \nu(U)\nu(\psi)\nu(U)$$

for all $\psi \in \mathcal{H}$, which means that

$$\nu L_U R_U = L_{\nu(U)} R_{\nu(U)} \nu.$$

From the proof of part 1, we have $J_U \nu_U = J_2 \nu$. We now compute

$$\nu_U J_U = L_U R_U \nu L_U R_U J_2 = L_{U\nu(U)} R_{U\nu(U)} \nu J_2.$$

If $\nu(U) = U^\dagger$ then $J_U \nu_U = J_2 \nu = \nu J_2 = \nu_U J_U$ and

$$\nu_U J_U D = \nu J_2 D = D J_2 \nu = D J_U \nu_U .$$

One similarly checks that $\nu_U^2 = 1$. Since $J_U \nu_U = J_2 \nu$, the operator clearly preserves the domain of D.

This holds even if the matrix entries of U are not necessarily in the domain of D.

3. For almost all $x \in T^2$, $U(x)$ is a constant unitary matrix and the compatibility condition with ν implies that

$$U(x) = \begin{pmatrix} u_{11} & u_{12} \\ u_{12}^* & u_{11}^* \end{pmatrix}$$

for some $u_{ij} \in \mathbb{C}$. Such a matrix is unitary if and only if it is of the form

$$U(x) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \quad \text{or} \quad U(x) = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix} \tag{3.24}$$

for some $\theta \in \mathbb{R}$. In the first case $U(x)$ commutes with σ^3, while in the second they anticommute. In both cases $L_{U(x)} R_{U(x)}$ commutes with $\chi_{\deg}$. □

In Prop. 3.20(2) the condition $\nu(U) = -U^\dagger$ would work as well, but notice that it implies $\nu(iU) = (iU)^\dagger$ and the rescaling $U \mapsto iU$ simply changes J_U and ν_U by a sign. The condition $\nu(U) = U^\dagger$ implies that, at almost every point x, U is of one of the two types in (3.24), though both θ and the type may depend on x (U being a constant matrix is a special case).

Let us close this section with a comment on orientability. Note that $\Omega^1_D(A) \neq 0$, since it is isomorphic to the A-module of de Rham forms on the torus. The spectral triple in Prop. 3.18 is then not orientable, since it satisfies the second-order condition (see the remark at the end of §3.2.2). But the triple in Prop. 3.17 is also not orientable: $\chi_{\deg} \notin C\ell_D(A)$ since, for example, it doesn't commute with $R_{\sigma^1} \in C\ell_D(A)'$. To get an orientable spectral triple we need to choose a different grading operator.

Proposition 3.21. *The unital even real spectral triple $(A, H, D, J_1, L_{\sigma^3})$ is orientable.*

Proof. A straightforward computation. A possible choice of Hochschild 2-cycle giving the orientation is

$$c = -2^{-1} i u^\dagger v^\dagger \otimes (u \otimes v - v \otimes u).$$

Clearly

$$-2^{-1} i u^\dagger v^\dagger \big([D,u][D,v] - [D,v][D,u] \big) = L_{\sigma^3}$$

is the grading, and one can check that the Hochschild boundary of c is zero with a simple computation. □

One may wonder what this new grading is. Let $\chi := \mu^{-1} \circ L_{\sigma^3} \circ \mu$. Then one finds

$$\chi(1) = i\,\mathrm{d}x \wedge \mathrm{d}y\,, \quad \chi(\mathrm{d}x) = i\,\mathrm{d}y\,, \quad \chi(\mathrm{d}y) = -i\mathrm{d}x, \quad \chi(\mathrm{d}x \wedge \mathrm{d}y) = -i.$$

Evidently the map χ is the grading coming from the Hodge star operator.

3.4 PRODUCTS OF SPECTRAL TRIPLES

Let $(A_1, \mathcal{H}_1, D_1, J_1, \chi_1)$ and $(A_2, \mathcal{H}_2, D_2, J_2)$ be two unital real spectral triples, the former even. Their product $(A, \mathcal{H}, D, J)$ is given by

$$A = A_1 \otimes A_2, \quad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \quad D = D_1 \otimes 1 + \chi_1 \otimes D_2, \quad J = J_1 \otimes J_2. \tag{3.25}$$

Here we assume that A_1 and A_2 are both complex, so that $\otimes$ is everywhere the tensor product over $\mathbb{C}$ (algebraic, minimal or of Hilbert spaces depending on the type of object we are considering). If both spectral triples are even, a grading on the product is given by:

$$\chi = \chi_1 \otimes \chi_2.$$

We will not consider the case where both spectral triples are odd. It is not very different, but for the sake of brevity we will always assume that at least one of the spectral triples is even. Notice that the example we are interested in, the Hodge-de Rham spectral triple of a closed oriented Riemannian manifold, is always even.

If J_1 and J_2 satisfy (2.9) for twist operators ν_1 and ν_2, then J satisfies (2.9) with twist operator $\nu = \nu_1 \otimes \nu_2$.

For suitable values of the signs $\varepsilon, \varepsilon', \varepsilon''$, which will not be discussed here. See the comments in §3.4.2.

Lemma 3.22. $\Omega^1_D(A) = \Omega^1_{D_1}(A_1) \otimes A_2 + \chi_1 A_1 \otimes \Omega^1_{D_2}(A_2)$.

Proof. Elements of the algebraic tensor product $A_1 \otimes A_2$ are finite sums of decomposable tensors. From the fact that

$$(a_1 \otimes a_2)[D, b_1 \otimes b_2] = a_1[D_1, b_1] \otimes a_2 b_2 + \chi_1 a_1 b_1 \otimes a_2[D_2, b_2] \tag{3.26}$$

for all $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$, we get the inclusion

$$\Omega^1_D(A) \subset \Omega^1_{D_1}(A_1) \otimes A_2 + \chi_1 A_1 \otimes \Omega^1_{D_2}(A_2).$$

Since A_1 and A_2 are unital, if we choose $b_1 = 1$ in (3.26) we find that $\chi_1 A_1 \otimes \Omega^1_{D_2}(A_2)$ is a vector subspace of $\Omega^1_D(A)$; if we choose $b_2 = 1$ (and b_1 arbitrary), we find that $\Omega^1_{D_1}(A_1) \otimes A_2$ is a vector subspace of $\Omega^1_D(A)$. □

It then follows that

$$\mathcal{C}\ell_D(A) \subset \mathcal{C}\ell^{\chi_1}_{D_1}(A_1) \otimes \mathcal{C}\ell_{D_2}(A_2). \tag{3.27}$$

Lemma 3.23. *If $\chi_1 \in C\ell_{D_1}(A_1)$, then*

$$C\ell_D(A) = C\ell_{D_1}(A_1) \otimes C\ell_{D_2}(A_2). \tag{3.28}$$

Proof. Since $C\ell^{\chi_1}_{D_1}(A_1) = C\ell_{D_1}(A_1)$, the inclusion "$\subset$" follows from (3.27).

We saw in the proof of Lemma 3.22 that both $\Omega^1_{D_1}(A_1) \otimes A_2$ and $\chi_1 A_1 \otimes \Omega^1_{D_2}(A_2)$ are contained in $\Omega^1_D(A)$, and therefore in $C\ell_D(A)$.

Since A_2 is unital, both $\Omega^1_{D_1}(A_1) \otimes 1$ and $A_1 \otimes 1 \subset A$ are in $C\ell_D(A)$. Thus $C\ell_D(A) \supset C\ell_{D_1}(A_1) \otimes 1$.

Since A_1 is unital, both $\chi_1 \otimes \Omega^1_{D_2}(A_2)$ and $1 \otimes A_2$ are in $C\ell_D(A)$. But $\chi_1 \otimes 1 \in C\ell_{D_1}(A_1) \otimes 1 \subset C\ell_D(A)$ as well. Therefore $1 \otimes \Omega^1_{D_2}(A_2) = (\chi_1 \otimes 1)\big(\chi_1 \otimes \Omega^1_{D_2}(A_2)\big)$ is contained in $C\ell_D(A)$ as well. This proves that $C\ell_D(A) \supset 1 \otimes C\ell_{D_2}(A_2)$ and hence $C\ell_D(A) \supset C\ell_{D_1}(A_1) \otimes C\ell_{D_2}(A_2)$. □

3.4.1 *Products and the spin condition*

Given two spectral triples satisfying one of the conditions in Def. 3.5, one wonders if the product satisfies such a condition as well. The answer is affirmative for condition (3.5a).

Proposition 3.24. *If two unital real spectral triples satisfy (3.5a), then their product satisfies (3.5a) as well.*

Proof. Using the notation above, suppose $\mathcal{E}_1 \subset \mathcal{H}_1$ and $\mathcal{E}_2 \subset \mathcal{H}_2$ are dense subspaces, with $\mathcal{E}_1$ and $\mathcal{E}_2$ full right Hilbert $\bar{A}_1$- and $\bar{A}_2$-modules respectively, where the right action of $a \in \bar{A}_i$ is given by $J_j a^\dagger J_j^{-1}$, and suppose for $j = 1, 2$ one has

$$C\ell_{D_j}(A_j) = \mathcal{K}_{\bar{A}_j}(\mathcal{E}_j).$$

Let $\mathcal{E} := \mathcal{E}_1 \otimes \mathcal{E}_2$ be the exterior tensor product of Hilbert C*-modules. This is a full right Hilbert $\bar{A}_1 \otimes \bar{A}_2$-module, where here the tensor product is the minimal tensor product of C*-algebras. Note that therefore $\bar{A}_1 \otimes \bar{A}_2 = \bar{A}$. The right action of a decomposable tensor $a = a_1 \otimes a_2 \in \bar{A}$ is given by $J a^\dagger J^{-1} = J_1 a_1^\dagger J_1^{-1} \otimes J_2 a_2^\dagger J_2^{-1}$, so by linearity and continuity the right action of any element in $\bar{A}$ is implemented by $J = J_1 \otimes J_2$. One also has

$$C\ell_{D_1}(A_1) \otimes C\ell_{D_2}(A_2) = \mathcal{K}_{\bar{A}_1}(\mathcal{E}_1) \otimes \mathcal{K}_{\bar{A}_2}(\mathcal{E}_2).$$

But $\mathcal{K}_{\bar{A}_1}(\mathcal{E}_1) \otimes \mathcal{K}_{\bar{A}_2}(\mathcal{E}_2) = \mathcal{K}_{\bar{A}}(\mathcal{E})$ (see *e.g.* Ref. [44, p.45]). From Prop. 3.6(1) it follows that $\chi_1 \in C\ell_D(A)$. From Lem. 3.23 it follows that $C\ell_{D_1}(A_1) \otimes C\ell_{D_2}(A_2) = C\ell_D(A)$. Hence $C\ell_D(A) = \mathcal{K}_{\bar{A}}(\mathcal{E})$ as requested, and the product spectral triple satisfies (3.5a). □

Def. *The* exterior tensor product *is the completion of the algebraic tensor product of Hilbert $\mathcal{A}_1$- and $\mathcal{A}_2$-modules to the Hilbert C*-module over the minimal tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$.*

Def. *For *-representations $\pi_j \colon \mathcal{A}_j \to \mathcal{B}(\mathcal{H}_j)$, denote by $\underline{\pi}$ the unique rep'n of $\mathcal{A}_1 \otimes \mathcal{A}_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that $\underline{\pi}(\sum_j a_{1,j} \otimes a_{2,j}) = \sum_j \pi_1(a_{1,j})\pi_2(a_{2,j})$. Then $\|a\|_{\min} := \sup\{\|\underline{\pi}(a)\|\}$. The* minimal tensor product *is the completion of $\mathcal{A}_1 \otimes \mathcal{A}_2$ with respect to $\|\bullet\|_{\min}$.*

Recall that if an even spectral triple is spin then it is also even-spin. In the next example we present two even-spin spectral triples whose product is not even-spin (and then also not spin).

Example 3.25. Let $A_1 = M_2(\mathbb{C})$ be represented on $\mathcal{H}_1 = M_2(\mathbb{C}) \otimes \mathbb{C}^2$ by left multiplication on the first factor (we think of elements of $\mathbb{C}^2$ as column vectors), and $J_1(m \otimes v) := m^\dagger \otimes v^*$ where $m^\dagger$ is the hermitian conjugate and v^* the componentwise conjugation. Also let

$$D_1(m \otimes v) := [\sigma^1, m] \otimes \sigma^1 v, \quad \chi_1 := 1 \otimes \sigma^3.$$

Since for $a = -i\sigma^3$ and $b = \sigma^2 \in A_1$ one has

$$a[D_1, b] = 1_2 \otimes \sigma^1 =: \omega, \tag{3.29}$$

1-forms are freely generated (as an A_1-module) by ω. The Clifford algebra $C\ell_{D_1}^{\chi_1}(A_1)$ is then generated by $M_2(\mathbb{C}) \otimes 1$, $\omega = 1 \otimes \sigma^1$, and $\chi_1 = 1 \otimes \sigma^3$. Hence

$$C\ell_{D_1}^{\chi_1}(A_1) = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$$

with its action on $\mathcal{H}_1$ given by left multiplication. The commutant is $C\ell_{D_1}^{\chi_1}(A_1)' = J_1 A_1 J_1^{-1}$ and (3.5b) is satisfied. Note that $\chi_1 \notin C\ell_{D_1}(A_1)$, so (3.5a) is not satisfied.

Let (A, H, D, J, χ) be a product of two copies of the above spectral triple. $C\ell_D^\chi(A)$ is generated by A_1, A_2, $\chi_1 \otimes \chi_2 = 1 \otimes \sigma^3 \otimes 1 \otimes \sigma^3$, and the 1-forms

$$\omega \otimes 1 = 1 \otimes \sigma^1 \otimes 1 \otimes 1, \quad \text{and} \quad \chi_1 \otimes \omega = 1 \otimes \sigma^3 \otimes 1 \otimes \sigma^1.$$

The element

$$1 \otimes \sigma^1 \otimes 1 \otimes \sigma^2$$

is in the commutant of $C\ell_D^\chi(A)$, but does not belong to JAJ^{-1}. ⋄

Let us make note of this result.

Remark 3.26. There exist spectral triples satisfying (3.5b) whose product does not satisfy (3.5b) (nor (3.5a)). ⋄

Finally, let us consider a mixed case. In the following we present an even-spin spectral triple and a spin spectral triple (which is in fact also Hodge), whose product is not even-spin (and indeed also not spin).

Example 3.27. Take the first spectral triple to be as in Example 3.25 and let the second one be given by $A_2 = \mathcal{H}_2 = M_2(\mathbb{C})$, $D_2(a) = \left[\sigma^1, a\right]$, and $J_2(a) = a^\dagger$ for all $a \in A_2$. This spectral triple satisfies both (3.5a) and (3.5c) (which is only possible in the finite-dimensional case when 1-forms are contained in the algebra, $\Omega^1_{D_2}(A_2) \subset A_2$, which implies $A_2 = C\ell_{D_2}(A)$).

The Clifford algebra $C\ell_D^\chi(A)$ of the product triple is generated by $A_1 \otimes 1, 1 \otimes A_2$ and the 1-form $\omega \otimes 1 = 1 \otimes \sigma^1 \otimes 1$, where ω is as in (3.29); $\omega \otimes 1$ belongs to the commutant of $C\ell_D^\chi(A)$, but not to $A_1 \otimes A_2$, hence the product spectral triple does not satisfy (3.5b). ⋄

Mixed Hodge-spin cases are not particularly interesting. There is no reason to expect that such a product is either Hodge or spin – there is also no reason to expect in general that a product of two Hodge spectral triples is spin, or that a product of two (even-)spin spectral triples is Hodge – and in fact it is quite easy to produce counterexamples.

3.4.2 *Products and the second-order condition*

Given two real spectral triples, one may wonder how the KO-dimension of their product is related to those of its two factors. It would be nice if the KO-dimension were multiplicative, but unfortunately if the real structure is defined by $J = J_1 \otimes J_2$ this is not true. This was first noticed in Ref. [59], where a modified definition of J was proposed in order to fix this problem (taking either $J = J_1 \otimes J_2\chi_2$ or $J = J_1\chi_1 \otimes J_2$ depending on the dimension of the factors). This study was completed in Ref. [27], where the odd-odd case was considered as well (in Ref. [59] one of the spectral triples is always assumed to be even), along with several possible choices of Dirac operators and real structures. The modified definition of J, which perhaps seems somewhat artificial, was reinterpreted in Ref. [37] as a graded tensor product.

Here we follow this idea in spirit, but we will find that the "correct" definition of J is not the one in Refs. [27, 37, 59]. Our motivation here is to have the second-order property (3.4) be preserved by products, and this will lead to yet another different definition of J. Although the natural way to study products of real spectral triples is in the category of graded vector spaces, we will argue that in terms of "ungraded" objects and operations this amounts to merely changing the reality operator.

We will use the same notation as the previous subsection and assume that we have two unital real spectral triples $(A_j, \mathcal{H}_j, D_j, J_j, \chi_j)$, $j = 1, 2$. For simplicity we will also assume that both spectral triples are even. If $\psi \in \mathcal{H}_1$ is an eigenvector of χ_1 with eigenvalue $(-1)^{|\psi|}$ we will say that ψ is homogeneous of degree $|\psi|$. Explicitly,

Recall that the Hodge-de Rham spectral triple of a closed oriented Riemannian manifold is always even, regardless of the dimension of the manifold, whether even or odd.

$$|\psi| = \begin{cases} 0 & \text{if } \chi_1(\psi) = +\psi, \\ 1 & \text{if } \chi_1(\psi) = -\psi. \end{cases}$$

A bounded operator $T \in \mathcal{B}(\mathcal{H}_1)$ has degree 0 (called *even*) if it commutes with χ_1, and degree 1 (called *odd*) if it anticommutes with it. This notion extends to unbounded operators (such as D_1), provided that χ_1 preserves their domain.

The same definitions apply to the second spectral triple. According to Koszul's rule of signs, the graded tensor product $T_1 \hat{\otimes} T_2 \in \mathcal{B}(\mathcal{H})$ of two (homogeneous) bounded operators is now defined by

$$(T_1 \hat{\otimes} T_2)(\psi_1 \otimes \psi_2) := (-1)^{|T_2||\psi_1|} T_1\psi_1 \otimes T_2\psi_2$$

for all homogeneous vectors $\psi_j \in \mathcal{H}_j$. This definition also makes sense when one of the two operators is unbounded. For example, if T_1 is unbounded, then $T_1 \hat{\otimes} T_2$ will be unbounded with its domain given by the algebraic tensor product of $\mathrm{Dom}(T_1)$ and $\mathcal{H}_2$.

If T_2 is odd, $T_1 \hat{\otimes} T_2 = T_1\chi_1 \otimes T_2$. Thus, the Dirac operator in a product of spectral triples can be written as (the closure of)

$$D = D_1 \hat{\otimes} 1 + 1 \hat{\otimes} D_2.$$

Since we are considering unital spectral triples, the following lemma now becomes evident.

Lemma 3.28. $C\ell_D(A) = C\ell_{D_1}(A_1) \hat{\otimes} C\ell_{D_2}(A_2)$.

Here if $\mathcal{B}_1 \subset \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}_2 \subset \mathcal{B}(\mathcal{H}_2)$ are C*-subalgebras, we define $\mathcal{B}_1 \hat{\otimes} \mathcal{B}_2$ as the norm closure in $\mathcal{B}(\mathcal{H})$ of the vector subspace spanned by elements $T_1 \hat{\otimes} T_2$, with $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$.

Proof. The inclusion "$\subset$" is given by (3.27). The opposite inclusion is analogous to the proof of Lemma 3.23: one shows that the elements $A_1 \otimes 1 = A_1 \hat{\otimes} 1$, $1 \otimes A_2 = 1 \hat{\otimes} A_2$, $\Omega^1_{D_1}(A_1) \otimes 1 = \Omega^1_{D_1}(A_1) \hat{\otimes} 1$ and $\chi_1 \otimes \Omega^1_{D_2}(A_2) = \chi_1 \hat{\otimes} \Omega^1_{D_2}(A_2)$ are contained in $C\ell_D(A)$, hence the thesis. □

The idea is now to modify the definition of the product reality operator. Since this should in principle change the order of factors in a (tensor) product, what is suggested again by Koszul's rule of signs is to define J by

$$J(\psi_1 \otimes \psi_2) := (-1)^{|\psi_1||\psi_2|} J_1(\psi_1) \otimes J_2(\psi_2) \tag{3.30}$$

Note that this is exactly what happens in a product of Hodge-de Rham spectral triples of a manifold, if the real structure is the one coming from the main anti-involution of the Clifford algebra.

for all homogeneous $\psi_1 \in \mathcal{H}_1$ and $\psi_2 \in \mathcal{H}_2$. With this choice, we have the following proposition.

Proposition 3.29. *Consider two unital even spectral triples with (possibly twisted) real structure, both satisfying the second-order condition. Then their product $(A, \mathcal{H}, D, \chi)$, equipped with the antilinear map in (3.30), satisfies the second-order condition as well.*

Proof. For $j = 1, 2$, let $a_j \in A_j$ and $\omega_j \in \Omega^1_{D_j}(A_j)$. The algebra $C\ell_D(A)$ is generated by $A_1 \otimes 1$, $1 \otimes A_2$ and elements of the form $\omega_1 \otimes 1$ and $\chi_1 \otimes \omega_2$. Thus $C\ell_D(A)^\circ$ is generated by

$$\begin{aligned} &J(a_1^\dagger \otimes 1)J^{-1} = a_1^\circ \otimes 1, \quad J(\omega_1^\dagger \otimes 1)J^{-1} = \omega_1^\circ \otimes \chi_2, \\ &J(1 \otimes a_2)J^{-1} = 1 \otimes a_2^\circ, \quad J(\chi_1 \otimes \omega_2)J^{-1} = 1 \otimes \omega_2^\circ. \end{aligned} \tag{3.31}$$

Note the presence of χ_2, while χ_1 has disappeared. If the two factors satisfy (3.4), then the above four elements commute with $C\ell_D(A)$, and in particular,

$$[\omega_1^\circ \otimes \chi_2, \chi_1 \otimes \omega_2] = 0$$

because ω_1° anticommutes with χ_1 and χ_2 anticommutes with ω_2. Thus the product spectral triple satisfies $C\ell_D(A)^\circ \subset C\ell_D(A)'$. □

We stress again that (3.30) is *not* the product real structure in Refs. [27, 37, 59], and that Prop. 3.29 does *not* hold if the product real structure is defined as in Refs. [27, 37, 59].

Notice that in Prop. 3.29 we do not claim that the product spectral triple is real. If one checks the conditions for J one finds that there is a problem with (2.1) and (2.2). Since here we are mainly interested in the second-order and Hodge condition, we will not investigate how to modify (3.30) so that (2.1) and (2.2) are also satisfied. We will merely make the following observation.

Proposition 3.30. *If the spectral triples $(A_j, \mathcal{H}_j, D_j, J_j, \chi_j)$, $j = 1, 2$, satisfy (2.5) with signs ε''_j, then their product – with the product reality operator given by (3.30) – satisfies (2.5) with sign $\varepsilon'' = \varepsilon''_1 \varepsilon''_2$.*

If in addition the factors satisfy (2.1) with signs ε_j and $\varepsilon''_1 = \varepsilon''_2 = +1$, then their product will satisfy (2.1) with sign $\varepsilon = \varepsilon_1 \varepsilon_2$.

Notice that this is the case for Hodge-de Rham spectral triples. It happens when the KO-dimension is a multiple of 4 (mod 8).

Proof. On decomposable homogeneous tensors

$$J\chi(\psi_1 \otimes \psi_2) = (-1)^{|\psi_1||\psi_2| + |\psi_1| + |\psi_2|} J_1(\psi_1) \otimes J_2(\psi_2),$$
$$\chi J(\psi_1 \otimes \psi_2) = \varepsilon''_1 \varepsilon''_2 (-1)^{|\psi_1||\psi_2| + |\psi_1| + |\psi_2|} J_1(\psi_1) \otimes J_2(\psi_2),$$

which proves the first part of the statement.

If $\varepsilon''_1 = \varepsilon''_2 = +1$, the operators J_j do not change the degree of a vector, and one easily verifies that

$$J^2(\psi_1 \otimes \psi_2) = (-1)^{2|\psi_1||\psi_2|} J_1^2(\psi_1) \otimes J_2^2(\psi_2)$$

on decomposable homogeneous tensors. Hence $J^2 = J_1^2 \otimes J_2^2$ and we get the second part of the theorem. □

The problem with condition (2.2) is not surprising, since in the example of a closed oriented Riemaniann manifold we are forced to introduce a twist to make it work. Note that we can introduce another graded product $\hat{\otimes}'$ via the rule

$$(T_1 \hat{\otimes}' T_2)(\psi_1 \otimes \psi_2) = (-1)^{|T_1||\psi_2|} T_1\psi_1 \otimes T_2\psi_2$$

for all homogeneous $\psi_1 \in \mathcal{H}_1$ and $\psi_2 \in \mathcal{H}_2$ and all homogeneous operators T_j on $\mathcal{H}_j$. With this convention, $T_1 \hat{\otimes}' T_2 = T_1 \otimes T_2\chi_2$ for all T_1 of degree 1. This is the natural convention for right modules, *i.e.* if we imagine that endomorphisms act from the right on vectors. This graded product gives an alternative Dirac operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$:

$$D' := D_1 \hat{\otimes}' 1 + 1 \hat{\otimes}' D_2 = D_1 \otimes \chi_2 + 1 \otimes D_2. \tag{3.32}$$

It turns out that the modified real structure J transforms the 'left' into the 'right' Dirac operator.

Proposition 3.31. *If the spectral triples $(A_j, \mathcal{H}_j, D_j, \chi_j, J_j)$, $j = 1, 2$, satisfy Def. 2.2 with signs $(\varepsilon_j, \varepsilon'_j, \varepsilon''_j)$ and*

$$\varepsilon'_1 \varepsilon''_1 = \varepsilon'_2$$

then their product – with J given by (3.30) – satisfies

$$JD = \varepsilon' D' J$$

with D' as in (3.32) and $\varepsilon' = \varepsilon'_1$.

The proof is given by a straightforward computation.

The following observation will be useful later on, and holds regardless of the KO-signs. If $\mathcal{B}_1 \subset \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}_2 \subset \mathcal{B}(\mathcal{H}_2)$ are C*-subalgebras, denote by $\mathcal{B}_1 \hat{\otimes}' \mathcal{B}_2$ the norm closure in $\mathcal{B}(\mathcal{H})$ of the vector subspace spanned by elements $T_1 \hat{\otimes}' T_2$, with $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$.

Lemma 3.32. *In a product of unital real even spectral triples, and with J given by* (3.30)*, one has $J\mathcal{C}\ell_D(A)J^{-1} = \mathcal{C}\ell_{D_1}(A_1) \hat{\otimes}' \mathcal{C}\ell_{D_2}(A_2)$.*

Proof. We see from (3.31) that conjugation by J sends generators of $\mathcal{C}\ell_D(A)$ into generators of $\mathcal{C}\ell_{D_1}(A_1) \hat{\otimes}' \mathcal{C}\ell_{D_2}(A_2)$. □

3.4.3 *Products and the Hodge condition*

The behaviour of the Hodge condition under products is more technical and to simplify the discussion we will study it only in the finite-dimensional case. We want to prove the following proposition.

Proposition 3.33. *Let $(A_j, \mathcal{H}_j, D_j, \chi_j, J_j)$ be two unital finite-dimensional even real spectral triples satisfying the Hodge condition* (3.5c)*. Define $(A, \mathcal{H}, D)$ as in* (3.25) *and J as in* (3.30)*. Then $J\mathcal{C}\ell_D(A)J^{-1} = \mathcal{C}\ell_D(A)'$.*

That is, that the product spectral triple satisfies the Hodge condition as well. In fact, in light of Lem. 3.28 and Lem. 3.32, Prop. 3.33 is a corollary of the following theorem.

Theorem 3.34. *Let $B_j \subset \mathrm{End}_{\mathbb{C}}(\mathcal{H}_j)$ be two unital subalgebras, $j = 1, 2$. Then*

$$(B_1 \hat{\otimes} B_2)' = B_1' \hat{\otimes}' B_2'.$$

Proof. For all homogeneous elements $b_1 \in B_1$, $b_2 \in B_2$, $c_1 \in B_1'$, $c_2 \in B_2'$, $\psi_1 \in \mathcal{H}_1$, and $\psi_2 \in \mathcal{H}_2$ one has:

$$\begin{aligned}(b_1 \hat{\otimes} b_2)(c_1 \hat{\otimes}' c_2)(\psi_1 \otimes \psi_2) &= (-1)^{|b_2||\psi_1|+|c_1||\psi_2|+|b_2||c_1|} b_1 c_1 \psi_1 \otimes b_2 c_2 \psi_2 \\ &= (c_1 \hat{\otimes}' c_2)(b_1 \hat{\otimes} b_2)(\psi_1 \otimes \psi_2).\end{aligned}$$

This proves the inclusion $B_1' \hat{\otimes}' B_2' \subseteq (B_1 \hat{\otimes} B_2)'$.

For the opposite inclusion, since all spaces are finite-dimensional and χ_2 is invertible, every element $T \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$ can be written as a finite sum

$$\begin{aligned}T &= \sum_j (R_{1,j} \otimes R_{2,j} + S_{1,j} \otimes S_{2,j}\chi_2) \\ &= \sum_j (R_{1,j} \hat{\otimes}' R_{2,j} + S_{1,j} \hat{\otimes}' S_{2,j})\end{aligned}$$

for some $R_{1,j}, S_{1,j} \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}_1)$, $R_{2,j}, S_{2,j} \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}_2)$, with $R_{1,j}$ even and $S_{1,j}$ odd. The elements $\{R_{2,j}, S_{2,j}\}$ can be chosen to be linearly independent. Assume $T \in (B_1 \hat{\otimes} B_2)'$. Since B_2 is unital, for all $b_1 \in B_1$ one has $b_1 \otimes 1 = b_1 \hat{\otimes} 1 \in B_1 \hat{\otimes} B_2$ and

$$[T, b_1 \otimes 1] = \sum_j \Big([R_{1,j}, b_1] \otimes R_{2,j} + \big[S_{1,j}, b_1\big] \otimes S_{2,j}\chi_2 \Big)$$

must be zero. From the linear independence of the elements in the second factor, we deduce $\big[R_{1,j}, b_1\big] = \big[S_{1,j}, b_1\big] = 0$, and hence that $R_{1,j}, S_{1,j} \in B_1'$ and $T \in B_1' \hat{\otimes}' \mathrm{End}_{\mathbb{C}}(\mathcal{H}_2)$. It follows that T can be written as a finite sum $T = \sum_i (\tilde{R}_{1,j} \hat{\otimes}' \tilde{R}_{2,j} + \tilde{S}_{1,j} \hat{\otimes}' \tilde{S}_{2,j})$ where $\tilde{R}_{1,j}, \tilde{S}_{1,j} \in B_1'$ and $\tilde{R}_{2,j}, \tilde{S}_{2,j} \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}_2)$, with $\tilde{R}_{1,j}$ even, $\tilde{S}_{1,j}$ odd and now $\{\tilde{R}_{1,j}, \tilde{S}_{1,j}\}$ are chosen to be linearly independent. Since B_1 is unital, for all even $b_2 \in B_2$ one has $1 \hat{\otimes} b_2 = 1 \otimes b_2 \in B_1 \hat{\otimes} B_2$ and for all even $b_2 \in B_2$ one has $1 \hat{\otimes} b_2 = \chi_1 \otimes b_2 \in B_1 \hat{\otimes} B_2$. If b_2 is even,

$$\big[T, 1 \hat{\otimes} b_2\big] = \sum_j \Big(\tilde{R}_{1,j} \otimes \big[\tilde{R}_{2,j}, b_2\big] + \tilde{S}_{1,j} \otimes \big[\tilde{S}_{2,j}, b_2\big]\chi_2 \Big),$$

while if b_2 is odd,

$$\big[T, 1 \hat{\otimes} b_2\big] = \sum_j \Big(\tilde{R}_{1,j}\chi_1 \otimes \big[\tilde{R}_{2,j}, b_2\big] - \tilde{S}_{1,j}\chi_1 \otimes \big[\tilde{S}_{2,j}, b_2\big]\chi_2 \Big).$$

From the linear independence of the elements in the first factor we deduce that in both cases $\big[\tilde{R}_{2,j}, b_2\big] = \big[\tilde{S}_{2,j}, b_2\big] = 0$, and hence that $\tilde{R}_{2,j}, \tilde{S}_{2,j} \in B_2'$ and $T \in B_1' \hat{\otimes}' B_2'$. □

4 SPECTRAL GAUGE THEORY WITH TWISTED REAL STRUCTURES

4.1 INTRODUCTION

The material in this chapter is based on Ref. [50].

As has been previously hinted at, real spectral triples appear well-suited to providing a natural mathematical framework for expressing certain gauge theories, including that of the Standard Model of particle physics, which accurately describes the results of all current high energy physics experiments.

A detailed treatment covering the noncommutative geometric formulation of the Standard Model is provided in Ref. [20], but for our purposes it is sufficient to say that the real even spectral triple describing the Standard Model comes from the product of the canonical spin manifold spectral triple

$$(C^\infty(M), L^2(M,S), i\nabla\!\!\!\!/\,, J_M, \chi_M),$$

which describes the spatial degrees of freedom, with the finite spectral triple

$$(A_{\mathrm{SM}}, \mathbb{C}^{96}, D_{\mathrm{SM}}, J_F, \chi_F),$$

which describes the internal degrees of freedom of the theory. Here, for a manifold M, $C^\infty(M)$ is the algebra of smooth complex functions on M, $L^2(M,S)$ is the space of square-integrable spinors on M, $i\nabla\!\!\!\!/$ is the Dirac operator associated to the spinor bundle $S \longrightarrow M$, A_{SM} is the real $*$-algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ where $\mathbb{H}$ denotes the quaternions, D_{SM} is the fermionic mass matrix, $J_M \otimes J_F$ is the charge conjugation operator, and $\chi_M \otimes \chi_F$ is the chirality operator.

In noncommutative geometry, the spinorial Dirac operator is often denoted by $D\!\!\!\!/$ or $iD\!\!\!\!/$ according to convention.

As was discussed in §§1.2 and 2.1, real spectral triples have a natural notion of dimension coming from K-theoretic concepts known as KO-dimension, which coincides with the dimension of the manifold (modulo 8) in the commutative case. It is well-known that the KO-dimension of the finite part of the spectral triple for the Standard Model must be 6 (mod 8) [1, 17]. Assuming that the Hilbert space admits a symplectic structure,[1] the smallest irreducible representation of a matrix algebra on a finite-dimensional Hilbert space, whose grading is compatible with the grading on the algebra, and which is of KO-dimension 6 (mod 8), is $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ [9]. The form of the grading breaks this algebra down to the 'left-right symmetric algebra'[2]

A more principled justification for considering algebras of the form $M_k(\mathbb{H}) \oplus M_{2k}(\mathbb{C})$ for applications to quantum physics is offered in Ref. [10].

1 In other words, $\mathcal{H}$ can carry an irreducible representation of (A, π, J) with $J^2 = -1$.
2 This name is sometimes given to the algebra $\mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$.

$A_{\text{LR}} = \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_4(\mathbb{C})$, and the fulfilment of the first-order condition breaks A_{LR} down to A_{SM}.

As the gauge group of a spectral triple comes from the choice of finite algebra, and extensions of the Standard Model coming from enlargements of the gauge group are of ongoing physical interest, it is natural to ask if the first-order condition can be jettisoned such that A_{LR} can be taken as the algebra of the spectral triple, and if so, what gauge theory this spectral triple would correspond to. The first of these questions was answered in the affirmative by Ref. [12]. The second question was answered by Refs. [11, 13], and it was found that this spectral triple corresponds to a family of $\text{SU}(2)_L \times \text{SU}(2)_R \times \text{SU}(4)$ Pati-Salam-type models.

One should not be too hasty in discarding the first-order condition, though. For one thing, it is the noncommutative equivalent of the requirement that a generalised Dirac operator be a first-order differential operator. Furthermore, it was introduced (along with the real structure J which implements it) in Ref. [15] at least partly to better define the notion of gauge theories in spectral geometry. It would seem advantageous, then, to search for a less radical solution. A generalised notion of real structure is given by twisted real structures, the range of applicability of which was subsequently further extended by multitwisted real structures. We hence investigate the possibility that such twisted real structures might offer a route to implementing the left-right symmetric spectral triple with only a weakening, rather than a complete discarding, of the first-order condition, or otherwise if the reduction to the Standard Model is unavoidable, as occurs when imposing the (untwisted) first-order condition.

In order to do so, in §4.2 (culminating in Thm. 4.10) we present in great detail a construction of Morita (self-)equivalence bimodules for spectral triples with twisted real structures that gives the expected form of inner fluctuations of the Dirac operator (*cf.* [7, §2.2]). In §4.3 we use this construction to develop a notion of gauge transformations for spectral triples with twisted real structure, given in Thm. 4.16. The necessary alterations to the spectral action are then described in §4.4 and finally, in §4.5 we attempt to apply the formalism to the toy model based on the algebra $\mathbb{C}_L \oplus \mathbb{C}_R \oplus M_2(\mathbb{C})$, which takes the role of a simplified version of the spectral Pati-Salam model. In the course of doing so we discuss various issues and limitations we encounter both for the toy model and for the full physical model.

To be precise, in Ref. [12] it is argued that the finite algebra $\mathbb{C}_L \oplus \mathbb{C}_R \oplus M_2(\mathbb{C})$ gives rise to a $\text{U}(1)_L \times \text{U}(1)_R \times \text{U}(2)$ gauge theory in the absence of the first-order condition.

4.2 MORITA EQUIVALENCE WITH TWISTED REAL STRUCTURES

Before we can talk about the applications of spectral triples with twisted real structure to gauge theories, we should understand how the changes to the usual definitions discussed in the previous subsection affect the definition of gauge transformations, and in order to do this we must discuss how the notion of Morita equivalence has been changed.

4.2.1 *Inner fluctuations*

The space of noncommutative 1-forms associated to a spectral triple $(A, \mathcal{H}, D)$ is generated by (the representation of) the algebra A and the derivation $[D, \bullet]$, and is denoted by

$$\Omega^1_D(A) := \left\{ \sum_i \pi(a_i)[D, \pi(b_i)] : a_i, b_i \in A \right\}.$$

For what follows, we would like to maintain this conventional 1-form structure as much as possible.

If the spectral triple is equipped with a (trivially-twisted) real structure J, an *inner fluctuation* of the Dirac operator is given by

$$D_\omega = D + \omega + \varepsilon' J \omega J^{-1} \tag{4.1}$$

for $\omega^\dagger = \omega \in \Omega^1_D(A)$ a self-adjoint 1-form. However, if D_ω is to satisfy (2.9), it should instead be of the form

$$D_\omega = D + \omega + \varepsilon' \nu J \omega J^{-1} \nu. \tag{4.2}$$

The relevant question is then whether or not we can sensibly implement such a fluctuation as (4.2).

Remark 4.1. Equation (4.2) is not the only possible inner fluctuation of D which satisfies (2.9). One could also define

$$D'_\omega = D + \nu\omega\nu + \varepsilon' J \omega J^{-1},$$

now with $\nu\omega\nu \in \Omega^1_D(A)$. The difficulty with this choice is that it would necessitate modifying the structure of noncommutative 1-forms, which we would like to avoid, but instead requiring $\nu[D, \pi(a)]\nu = [D, \pi(a')]$ for $a, a' \in A$ then places constraints on the Dirac operator and the twist. ◇

4.2.2 *Morita equivalence*

Applying (self-)Morita equivalence to a C*-algebra which forms part of a spectral triple necessarily impacts the geometric structure. In particular, when one considers a C*-algebra as Morita equivalent to itself, the algebra and associated Hilbert C*-module are only 'the same' up to isomorphism, and so the action of the Dirac operator on the Hilbert space is only defined up to a connection 1-form. This is the ultimate source of inner fluctuations.

Furthermore, as we saw in the previous chapter, when considering real spectral triples, the left and right module structures are related by the real structure. Finding an inner fluctuation of the Dirac operator which is compatible with the real structure is a two-step process which involves taking Morita self-equivalences of the left and the right module structures, and imposing self-consistency.

As we will demonstrate in this subsection, this procedure is more complicated in the case of twisted real structures, as the connection will be an ordinary 1-form for the right module case and a twisted 1-form for the left module case. We will largely follow Refs. [45] and [47] for the right and left module cases respectively, but the modifications necessary to combine the two approaches and adapt them to the twisted real structure formalism are original.

Rather than following the approach of §3.2.2, where we understood Morita equivalence bimodules as coming from a right Hilbert C*-module equipped with a commuting left action of another C*-algebra, we will instead (equivalently) consider Hilbert C*-bimodules from the beginning; doing so, a $\mathcal{B}$-$\mathcal{A}$ Morita equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a Hilbert $\mathcal{B}$-$\mathcal{A}$-bimodule which is full as a bimodule, *i.e.* it is both left-full with respect to ${}_{\mathcal{B}}\langle \bullet, \bullet \rangle$ and right-full with respect to $\langle \bullet, \bullet \rangle_{\mathcal{A}}$.

In this section, we will be primarily interested in the case of Morita self-equivalence, where we take $\mathcal{B} = \mathcal{A}$ and consider ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}}$ as the bimodule of $\mathcal{A}$ over itself. A standard result which we will later make use of is that if $\mathcal{E}$ is a full and finite projective (left or right) Hilbert $\mathcal{A}$-module, then $\mathcal{A}$ is Morita equivalent to $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$.

If two C*-algebras are Morita equivalent, they can be said to have equivalent representation theories. To see this, suppose two C*-algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent via the bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$, with $\mathcal{A}$ represented on the Hilbert space $\mathcal{H}$ as bounded operators by the map $\pi_{\mathcal{A}}$. This allows us to define the new Hilbert space

$$\mathcal{H}' := {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{H}$$

such that

$$ea \otimes \psi = e \otimes \pi_{\mathcal{A}}(a)\psi$$

for all $e \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$, $a \in \mathcal{A}$ and $\psi \in \mathcal{H}$, equipped with the inner product

$$\langle e_1 \otimes \psi_1, e_2 \otimes \psi_2 \rangle_{\mathcal{H}'} := \big\langle \psi_1, \langle e_1, e_2 \rangle_{\mathcal{A}} \psi_2 \big\rangle_{\mathcal{H}}$$

for all $e_1, e_2 \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and $\psi_1, \psi_2 \in \mathcal{H}$. One can then construct a representation of $\mathcal{B}$ on $\mathcal{H}'$ by

$$\pi_{\mathcal{B}}(b)(e \otimes \psi) := (be) \otimes \psi$$

for all $b \in \mathcal{B}$ and $e \otimes \psi \in \mathcal{H}'$. One then finds that the two representations $(\mathcal{A}, \pi_{\mathcal{A}}, \mathcal{H})$ and $(\mathcal{B}, \pi_{\mathcal{B}}, \mathcal{H}')$ are equivalent.

Since the above construction produces the representation $(\mathcal{B}, \pi_{\mathcal{B}}, \mathcal{H}')$ using the fact that ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a right $\mathcal{A}$-module, we refer to it as Morita equivalence by right module. Of course, if $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent, then there also exists an $\mathcal{A}$-$\mathcal{B}$-bimodule, the conjugate bimodule ${}_{\mathcal{A}}\bar{\mathcal{E}}_{\mathcal{B}}$, which allows one to start with a representation of $\mathcal{B}$ and construct a unitarily equivalent representation of $\mathcal{A}$ using the fact that the conjugate bimodule is a left $\mathcal{A}$-module. We refer to this construction as Morita equivalence by left module.

4.2.2.1 *Morita self-equivalence by right module*

For this subsection, we will not need the real structure, and the twist will play no role, so we simply summarise the standard construction. Consider a spectral triple $((\mathcal{A},\pi,\mathcal{H}),D)$, and a representation $(\mathcal{B},\pi_{\mathcal{B}},\mathcal{H}_R)$ equivalent to $(\mathcal{A},\pi,\mathcal{H})$ by Morita *self*-equivalence by right module. We begin by attempting to construct a Dirac operator on $\mathcal{H}_R$ in the simplest way, which is by just taking the naïve action of D given by $D_r(a\otimes\psi) := a\otimes D\psi$. However, this does not respect the module structure as $D_r(e\otimes\pi(a)\psi)\neq D_r(ea\otimes\psi)$ for all $a\in\mathcal{A}, e\in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ since D is not assumed to commute with $\pi(\mathcal{A})$. Instead we find

$$\begin{aligned}D_r(ea\otimes\psi) &= ea\otimes D\psi\\ &= e\otimes\pi(a)D\psi\\ &= e\otimes(D\pi(a)-[D,\pi(a)])\psi\\ &= D_r(e\otimes\pi(a)\psi)-e\otimes[D,\pi(a)]\psi.\end{aligned}$$

Therefore one instead considers a connection on the bimodule

$$\nabla\colon {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}\to {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}\otimes_{\mathcal{A}}\Omega^1(\mathcal{A})$$

satisfying the Leibniz rule

$$\nabla(ea) = (\nabla(e))a + e\otimes\delta(a),$$

where $\mathrm{Der}(\mathcal{A})\ni\delta\colon\mathcal{A}\to\Omega^1(\mathcal{A})$. When ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is finite projective *i.e.* ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} = p\mathcal{A}^N$ for $p=p^\dagger=p^2\in M_N(\mathcal{A})$, we can define any such connection as the Grassmann connection $p\delta$ up to some $\omega=p\omega=\omega p=p\omega p\in\mathrm{End}_{\mathcal{A}}({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}, {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}\otimes_{\mathcal{A}}\Omega^1(\mathcal{A}))$.

We now impose the Morita self-equivalence by taking $\mathcal{B}=\mathcal{A}$ and treating the Morita equivalence bimodule $\mathcal{E}=p\mathcal{A}^N$ as the module of $\mathcal{A}$ over itself (*i.e.* take $p=1$ and $N=1$). We have that the Hilbert spaces are isomorphic $\mathcal{H}_R=\mathcal{A}\otimes_{\mathcal{A}}\mathcal{H}\overset{\iota}{\simeq}\mathcal{H}$ under the isomorphism

$$\iota\colon a\otimes\psi\mapsto\pi(a)\psi$$

with inverse $\iota^{-1}\colon\psi\mapsto 1\otimes\psi$. Then the representation of the algebra $\mathcal{B}=\mathcal{A}$ is simply given by

$$\iota\circ\pi_{\mathcal{B}}(b)\circ\iota^{-1}=\pi(b)$$

for all $b \in \mathcal{A}$. Working in the Dirac calculus, the derivation δ is given by

$$\delta(a) := [D, \pi(a)]$$

for all $a \in \mathcal{A}$, which generates the space of noncommutative 1-forms $\Omega^1_D(\mathcal{A})$, which is an $\mathcal{A}$-bimodule with bimodule product

$$a \cdot \omega \cdot a' := \pi(a)\omega\pi(a')$$

for $a, a' \in \mathcal{A}$ and $\omega \in \Omega^1_D(\mathcal{A})$. We then have that any $\Omega^1_D(\mathcal{A})$-valued connection on the right module $\mathcal{A}$ reads

We understand $\omega\colon \mathcal{A} \to \Omega^1_D(\mathcal{A})$ to act via the module product, i.e. $\omega(a) = \omega \cdot a = \omega\pi(a)$ *as an operator.*

$$\nabla = \delta + \omega$$

for $\omega \in \Omega^1_D(\mathcal{A})$.

This suggests that the appropriate construction for the Dirac operator comes from making D_r compatible with the module structure. Denoting the new candidate Dirac operator by D_R, we achieve this with the addition of a connection like so:

$$\begin{aligned} D_R(a \otimes \psi) &:= a \otimes D\psi + \nabla(a)\psi \\ &= a \otimes D\psi + 1 \otimes \delta(a)\psi + 1 \otimes (\omega \cdot a)\psi \\ &= 1 \otimes D\pi(a)\psi + 1 \otimes \omega\pi(a)\psi \\ &= D_R(1 \otimes \pi(a)\psi), \end{aligned}$$

where the last line comes from observing that $\nabla(1) = \delta(1) + (\omega \cdot 1) = \omega$ as an operator. Lastly, via the isomorphism ι we find the compatible Dirac operator

$$D_R = D + \omega$$

on $\mathcal{H}$ with $\omega \in \Omega^1_D(\mathcal{A})$.

Thus we took the representation $(\mathcal{A}, \pi, \mathcal{H})$ with Dirac operator D on $\mathcal{H}$ and found the Morita self-equivalent representation $(\mathcal{A}, \pi, \mathcal{H})$ with Dirac operator $D_R = D + \omega$ on $\mathcal{H}$ for $\omega \in \Omega^1_D(\mathcal{A})$. However, if our original spectral triple is equipped with a twisted real structure (J, ν), then it should necessarily obey the twisted ε'-condition (2.9). However, $(\mathcal{A}, \mathcal{H}, D + \omega, (J, \nu))$ obeys (2.9) if and only if $\omega = \varepsilon'\nu J\omega J^{-1}\nu$. But it is not difficult to show that, for $\omega = \sum_j \pi(a_j)\big[D, \pi(b_j)\big] \in \Omega^1_D(\mathcal{A})$,

$$\begin{aligned} \varepsilon'\nu J\omega J^{-1}\nu &= \sum_j \nu J\pi(a_j)J^{-1}\nu^{-1}(D\nu^{-1}J\pi(b_j)J^{-1}\nu - \nu J\pi(b_j)J^{-1}\nu^{-1}D) \\ &= \sum_j \pi_J(\hat{\nu}^{-1}(a_j))\big[D, \hat{\nu}(b_j)\big]^{\pi_J}_{\hat{\nu}^{-2}}, \end{aligned} \tag{4.3}$$

which has no reason to be equal to ω, which we will address in the next subsection.

4.2.2.2 *Morita self-equivalence by left module*

The significance of (4.3) is that if ω is a 1-form, then $\nu J\omega J^{-1}\nu$ is a twisted 1-form. We have already seen that the standard Morita self-equivalence by right module can obtain ω, so we expect that some changes should need to be made to the construction in the left module case to obtain the twisted 1-form $\nu J\omega J^{-1}\nu$. For said changes, we look to the construction offered by Ref. [47] for obtaining inner fluctuations of the Dirac operator for real twisted spectral triples in the spirit of Ref. [21, Prop. 3.4].

As telegraphed, given an $\mathcal{A}$-$\mathcal{B}$ Morita equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$, there also exists a conjugate bimodule ${}_{\mathcal{A}}\bar{\mathcal{E}}_{\mathcal{B}}$ which has the canonical product $a\bar{e}b = \overline{b^*ea^*}$ where $\bar{\mathcal{E}} = \{\bar{e} : e \in \mathcal{E}\}$. If we replace $a_j \mapsto a_j^*$ and $b_j \mapsto b_j^*$ in (4.3) then we find that we can write

$$\varepsilon'\nu J\omega J^{-1}\nu = \sum_j \pi_J^*(\tilde{\nu}(a_j))\big[D, \tilde{\nu}^{-1}(b_j)\big]_{\tilde{\nu}^2}^{\pi_J^*}.$$

Motivated by this equality, we define the space of "twisted-opposite" 1-forms as

$$\tilde{\Omega}_D^1(A^{\mathrm{op}}) := \left\{ \sum_j \pi_J^*(\tilde{\nu}(a_j))\big[D, \tilde{\nu}^{-1}(b_j)\big]_{\tilde{\nu}^2}^{\pi_J^*} : a_j, b_j \in A \right\}. \tag{4.4}$$

In light of this, it will prove convenient to define two "twisted-opposite" maps

$$a^{\oplus} := \pi_J^*\big(\tilde{\nu}(a)\big) = J\nu^{-1}\pi(a)^*\nu J^{-1}, \tag{4.5a}$$
$$a^{\ominus} := \pi_J^*\big(\tilde{\nu}^{-1}(a)\big) = J\nu\pi(a)^*\nu^{-1}J^{-1}, \tag{4.5b}$$

noting that both $a^{\oplus}$ and $a^{\ominus}$ are (still) elements of $\pi_J^*(\mathcal{A}) = \mathcal{A}^\circ$ for $a \in \mathcal{A}$, and that each twisted-opposite map separately preserves the algebra product. This notation allows us to write twisted-opposite 1-forms in a more compact fashion:

Lemma 4.2. *The operator $\varepsilon'\nu J\omega J^{-1}\nu$, for $\omega = \sum_j \pi(a_j^*)\Big[D, \pi(b_j^*)\Big] \in \Omega_D^1(\mathcal{A})$, can be rewritten in the form $\sum_j a_j^{\oplus}(Db_j^{\ominus} - b_j^{\oplus}D)$.*

Proof. The proof is simply by computation. For the sake of simplicity and without loss of generality, we will omit summations and the representation π.

$$\begin{aligned}
\varepsilon'\nu J\omega J^{-1}\nu &= \varepsilon'\nu Ja^*[D, b^*]J^{-1}\nu \\
&= \varepsilon'\nu J(a^*Db^* - a^*b^*D)J^{-1}\nu \\
&= \varepsilon'(\nu Ja^*Db^*J^{-1}\nu - \nu Ja^*b^*DJ^{-1}\nu) \\
&= \nu Ja^*\nu JDJ^{-1}\nu b^*J^{-1}\nu - \nu J^{-1}a^*b^*\nu JDJ^{-1}\nu J\nu \text{ by (2.9)}, \\
&= J\nu^{-1}a^*\nu J^{-1}DJ\nu b^*\nu^{-1}J^{-1} - J\nu^{-1}a^*b^*\nu J^{-1}D \\
&= a^{\oplus}Db^{\ominus} - a^{\oplus}b^{\oplus}D.
\end{aligned}$$

□

Thus the space of twisted-opposite 1-forms (4.4) could equivalently be defined

$$\tilde{\Omega}^1_D(A^{\text{op}}) = \left\{ \sum_j a_j^{\oplus}(Db_j^{\ominus} - b_j^{\oplus}D) : a_j^{\oplus}, b_j^{\ominus}, b_j^{\oplus} \in A^{\circ} \right\},$$

which the reader may find easier to parse. We denote the elements of this space by $\omega^{\odot}$ to contrast with the more familiar $\omega^{\circ} := \varepsilon' J\omega J^{-1}$. This space can be considered an $\mathcal{A}$-bimodule with bimodule action defined by

$$a \cdot \omega^{\odot} \cdot b := b^{\oplus}\omega^{\odot}a^{\ominus}$$

for all $a, b \in \mathcal{A}$ and $\omega^{\odot} \in \tilde{\Omega}^1_D(\mathcal{A}^{\text{op}})$, and is generated by the derivation

$$\delta^{\odot}(a) := Da^{\ominus} - a^{\oplus}D \tag{4.6}$$

for $a \in \mathcal{A}$. This is a derivation in the sense that:

$$\begin{aligned}
\delta^{\odot}(ab) &= D(ab)^{\ominus} - (ab)^{\oplus}D \\
&= Db^{\ominus}a^{\ominus} - b^{\oplus}a^{\oplus}D \\
&= Db^{\ominus}a^{\ominus} - b^{\oplus}Da^{\ominus} + b^{\oplus}Da^{\ominus} - b^{\oplus}a^{\oplus}D \\
&= \delta^{\odot}(b)a^{\ominus} + b^{\oplus}\delta^{\odot}(a) \\
&= a \cdot \delta^{\odot}(b) + \delta^{\odot}(a) \cdot b.
\end{aligned}$$

Of course, rather than taking $\delta^{\odot}$ to be a derivation with respect to an unusual bimodule action, one could take it to be a twisted derivation with respect to the usual bimodule action, obeying the twisted Leibniz rule for $\bar{\nu}^2$ the twist: $\delta^{\odot}(ab) = \delta^{\odot}(b)a^{\ominus} + (\bar{\nu}(b)^2)^{\ominus}\delta^{\odot}(a)$. We choose instead to keep close to the approach of Ref. [47], as this also keeps us in close contact with the typical construction of a connection on a Hilbert C-module.*

We now return to the matter of Morita equivalence. Finding a Morita equivalent representation of $(\mathcal{A}, \mathcal{H})$ by left module (using now the conjugate module) follows similarly to the right module construction given above. We denote the resultant representation as $(\mathcal{B}, \mathcal{H}_L)$, where

$$\mathcal{H}_L := \mathcal{H} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\bar{\mathcal{E}}_{\mathcal{B}}$$

is the Hilbert space with the inner product

$$\langle \psi_1 \otimes \bar{e}_1, \psi_2 \otimes \bar{e}_2 \rangle_{\mathcal{H}_L} = \left\langle \psi_1 \, {}_{\mathcal{A}}\langle \bar{e}_1, \bar{e}_2 \rangle, \psi_2 \right\rangle_{\mathcal{H}}$$

for all $\psi_i \in \mathcal{H}$ and $\bar{e}_i \in {}_{\mathcal{A}}\bar{\mathcal{E}}_{\mathcal{B}}$ (as a left $\mathcal{A}$-module), where ${}_{\mathcal{A}}\langle \bar{e}_1, \bar{e}_2 \rangle := \langle e_1, e_2 \rangle_{\mathcal{A}}$. Furthermore, the right action of $\mathcal{B} \simeq \text{End}_{\mathcal{A}}({}_{\mathcal{A}}\mathcal{E})$ on ${}_{\mathcal{A}}\mathcal{E}$ is extended to $\mathcal{H}_L$ by

$$(\psi \otimes e)b := \psi \otimes eb.$$

In the standard construction, we would use the standard algebra bijection

$$a^{\circ} := \pi_J^*(a) = J\pi(a)^{\dagger}J^{-1} \tag{4.7}$$

for the right action of $\mathcal{A}$ on $\mathcal{H}$, such that $\psi \otimes a\bar{e} = \psi a \otimes \bar{e} := \pi_J^*(a)\psi \otimes \bar{e}$. However, this is where we instead choose to start making changes.

The first change we will make is to use the twisted-opposite maps to require that the right action of $\mathcal{A}$ on the original Hilbert space be now given by

$$\psi a := a^{\ominus}\psi = J\nu\pi(a)^{\dagger}\nu^{-1}J^{-1}\psi \tag{4.8}$$

for all $a \in \mathcal{A}$ and $\psi \in \mathcal{H}$, which extends to the module structure in the obvious way. This choice of right action is of course not unique, but its motivation will soon become clear.

As in the right module case, the naïve implementation of the action of the Dirac operator on $\mathcal{H}_L$ given by

$$D_\ell(\psi \otimes \bar{e}) := D\psi \otimes \bar{e}$$

fails to be compatible with the module structure since D does not commute with the algebra and we require the tensor product to be balanced for $\mathcal{A}$, not just $\mathbb{C}$.

Note that, as ${}_{\mathcal{A}}\bar{\mathcal{E}}$ is finite projective by assumption, we have that ${}_{\mathcal{A}}\bar{\mathcal{E}} \simeq \mathcal{A}^N p$, $p = p^\dagger = p^2 \in M_N(\mathcal{A})$. Thus we introduce an invertible linear module map

$$\tilde{\nu}_{\mathcal{E}} \colon {}_{\mathcal{A}}\bar{\mathcal{E}} \to {}_{\mathcal{A}}\bar{\mathcal{E}} \tag{4.9}$$

whose action is given elementwise by $\tilde{\nu} \in \operatorname{Aut}(\mathcal{A})$, the twist automorphism of (2.7), and under which we assume p is invariant. This allows us to make our second change, which is to the construction of the candidate Dirac operator

$$\big((\mathrm{id} \otimes \tilde{\nu}_{\mathcal{E}}^2) \circ D_\ell\big)(\psi \otimes \bar{e}) = D\psi \otimes \tilde{\nu}_{\mathcal{E}}^2(\bar{e})$$

for all $\psi \in \mathcal{H}$ and $\bar{e} \in {}_{\mathcal{A}}\bar{\mathcal{E}}$. Of course, this operator is still not compatible with the tensor product:

$$\big((\mathrm{id} \otimes \tilde{\nu}_{\mathcal{E}}^2) \circ D_\ell\big)(\psi \otimes a\bar{e}) - \big((\mathrm{id} \otimes \tilde{\nu}_{\mathcal{E}}^2) \circ D_\ell\big)(\psi a \otimes \bar{e}) = -\delta^{\odot}(a)\psi \otimes \tilde{\nu}_{\mathcal{E}}^2(\bar{e}) \neq 0. \tag{4.10}$$

However, just as in the right module case, the presence of a derivation suggests that we try to introduce a connection.

Thus, working from the derivation (4.6), we give the following definition.

Definition 4.3. An $\tilde{\Omega}^1_D(\mathcal{A}^{\mathrm{op}})$-*valued connection* on the left $\mathcal{A}$-module ${}_{\mathcal{A}}\bar{\mathcal{E}}$ is a map $\nabla^{\odot} \colon {}_{\mathcal{A}}\bar{\mathcal{E}} \to \tilde{\Omega}^1_D(\mathcal{A}^{\mathrm{op}}) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\bar{\mathcal{E}}$ such that

$$\nabla^{\odot}(a\bar{e}) - a \cdot \nabla^{\odot}(\bar{e}) = \delta^{\odot}(a) \otimes \bar{e}$$

for all $a \in \mathcal{A}$ and $\bar{e} \in {}_{\mathcal{A}}\bar{\mathcal{E}}$, with the left multiplication by $\mathcal{A}$ on $\tilde{\Omega}^1_D(\mathcal{A}^{\mathrm{op}}) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\bar{\mathcal{E}}$ given by the left module structure of $\tilde{\Omega}^1_D(\mathcal{A}^{\mathrm{op}})$. ⋄

Using the action of $\tilde{\Omega}^1_D(\mathcal{A}^{\mathrm{op}})$ on $\mathcal{H}$ we can therefore define the map $\nabla^{\odot} \colon \mathcal{H} \otimes_{\mathbb{C}} {}_{\mathcal{A}}\bar{\mathcal{E}} \to \mathcal{H} \otimes_{\mathbb{C}} {}_{\mathcal{A}}\bar{\mathcal{E}}$ by

$$\nabla^{\odot}(\psi \otimes \bar{e}) \coloneqq \psi \nabla^{\odot}(\bar{e}) \tag{4.11}$$

for all $\bar{e} \in {}_{\mathcal{A}}\bar{\mathcal{E}}$ and $\psi \in \mathcal{H}$. We cannot factorise this into a map on $\mathcal{H} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\bar{\mathcal{E}}$ because $\psi\nabla^{\odot}(a\bar{e}) - (\psi a)\nabla^{\odot}(\bar{e})$ need not vanish. However, the obstruction is captured by the derivation $\delta^{\odot}$ because the actions of $\mathcal{A}$ and $\tilde{\Omega}^1_D(\mathcal{A}^{\mathrm{op}})$ are compatible, *i.e.*

$$(a \cdot \omega^{\odot})\psi = \omega^{\odot} a^{\ominus} \psi = \omega^{\odot}(\psi a). \tag{4.12}$$

In short, this means that

$$\psi\nabla^{\odot}(a\bar{e}) - \psi a \nabla^{\odot}(\bar{e}) = \delta^{\odot}(a)\psi \otimes \bar{e}. \tag{4.13}$$

Remark 4.4. One notes that, though of course the meanings and rules established are distinct, much of what we have stated (and will state) for objects like $\nabla^{\odot}$, $\omega^{\odot}$ and $\delta^{\odot}$ are analogous to equivalent statements in the familiar case with ∇°, ω° and δ° instead. Indeed, many proofs carry over analogously with only the need to substitute different symbols, though to some extent this comes as a result of deliberate choices of notation. ◇

By (4.11) we therefore have

$$\begin{aligned}\big((\mathrm{id} \otimes \tilde{v}^2_{\mathcal{E}}) \circ \nabla^{\odot}\big)(\psi \otimes a\bar{e}) &- \big((\mathrm{id} \otimes \tilde{v}^2_{\mathcal{E}}) \circ \nabla^{\odot}\big)(\psi a \otimes \bar{e}) \\ &= (\mathrm{id} \otimes \tilde{v}^2_{\mathcal{E}})(\delta^{\odot}(a)\psi \otimes \bar{e}) \\ &= \delta^{\odot}(a)\psi \otimes \tilde{v}^2_{\mathcal{E}}(\bar{e}),\end{aligned} \tag{4.14}$$

and so combining (4.10) and (4.14) we find that the correct construction for the Dirac operator on $\mathcal{H}_L$ is given by

$$D_L \coloneqq (\mathrm{id} \otimes \tilde{v}^2_{\mathcal{E}}) \circ (D_\ell + \nabla^{\odot}),$$

which is compatible with the module structure since $D_L(\psi \otimes a\bar{e}) - D_L(\psi a \otimes \bar{e}) = 0$ as desired.

As before, we take the left $\mathcal{A}$-module ${}_{\mathcal{A}}\bar{\mathcal{E}}$ to be finite projective, we know that ${}_{\mathcal{A}}\bar{\mathcal{E}} \simeq \mathcal{A}^N p$ with $p = p^\dagger = p^2 \in M_N(\mathcal{A})$. In these terms, the connection decomposes as

$$\nabla^{\odot} = \nabla^{\odot}_0 + \vec{\omega}^{\odot}$$

with 'twisted' Grassmann connection

$$\nabla^{\odot}_0(\bar{e}) = (\delta^{\odot}(e_1), \ldots, \delta^{\odot}(e_N))p$$

for all $\bar{e} = (e_1, \ldots, e_N) \in {}_{\mathcal{A}}\bar{\mathcal{E}}$ with $e_j \in \mathcal{A}$. Meanwhile, $\vec{\omega}^{\odot}$ is a map ${}_{\mathcal{A}}\bar{\mathcal{E}} \to \tilde{\Omega}^1_D(\mathcal{A}^{\mathrm{op}}) \otimes_{\mathcal{A}} {}_{\mathcal{A}}\bar{\mathcal{E}}$ which is $\mathcal{A}$-linear in the sense that

$$\vec{\omega}^{\odot}(a\bar{e}) = a \cdot \vec{\omega}^{\odot}(\bar{e}).$$

We can now impose the self-equivalence by taking $\mathcal{B} = \mathcal{A}$ and ${}_{\mathcal{A}}\bar{\mathcal{E}}_{\mathcal{B}} = \mathcal{A}$ as a module (*i.e.* taking $p = 1$, $N = 1$ considering ${}_{\mathcal{A}}\bar{\mathcal{E}}_{\mathcal{B}}$ as finite projective over $\mathcal{A}$) such that $\mathcal{H}_L = \mathcal{H} \otimes_{\mathcal{A}} \mathcal{A} \simeq \mathcal{H}$.

Proposition 4.5. *In the case of Morita self-equivalence, with $\mathcal{B} = \mathcal{A}$ and ${}_{\mathcal{A}}\bar{\mathcal{E}}_{\mathcal{B}} = \mathcal{A}$, the form of the Dirac operator on $\mathcal{H}_L$ is nothing but the bounded perturbation*

$$D_L = D + \omega^{\odot} = D + \varepsilon' \nu J \omega J^{-1} \nu$$

for some $\omega^{\odot} = \varepsilon' \nu J \omega J^{-1} \nu \in \tilde{\Omega}^1_D(\mathcal{A}^{\mathrm{op}})$ with $\omega \in \Omega^1_D(\mathcal{A})$.

Proof. Because ${}_{\mathcal{A}}\bar{\mathcal{E}}_{\mathcal{B}} \simeq \mathcal{A}^N p$ with $p = 1$ and $N = 1$, we have $\nabla^{\odot} : \mathcal{A} \to \tilde{\Omega}^1_D(\mathcal{A}^{\mathrm{op}}) \otimes_{\mathcal{A}} \mathcal{A}$ such that $\nabla^{\odot} = \nabla^{\odot}_0 + \vec{\omega}^{\odot}$ with

$$\begin{aligned} \nabla^{\odot}_0(a) &= \delta^{\odot}(a) \otimes 1, \\ \vec{\omega}^{\odot}(a) &= (\omega^{\odot} a^{\ominus}) \otimes 1, \end{aligned}$$

where $\omega^{\odot} \in \tilde{\Omega}^1_D(\mathcal{A}^{\mathrm{op}})$. We therefore find

$$\begin{aligned} D_L(\psi \otimes a) &= (\mathrm{id} \otimes \tilde{\nu}^2_{\mathcal{E}}) \circ (D_\ell + \nabla^{\odot})(\psi \otimes a) \\ &= \big((\mathrm{id} \otimes \tilde{\nu}^2_{\mathcal{E}}) \circ D_\ell\big)(\psi \otimes a) + \big((\mathrm{id} \otimes \tilde{\nu}^2_{\mathcal{E}}) \circ \delta^{\odot}\big)(\psi \otimes a) + \big((\mathrm{id} \otimes \tilde{\nu}^2_{\mathcal{E}}) \circ \omega^{\odot}\big)(\psi \otimes a) \\ &= D\psi \otimes \tilde{\nu}^2(a) + (\mathrm{id} \otimes \tilde{\nu}^2_{\mathcal{E}})(\delta^{\odot}(a)\psi \otimes 1) + (\mathrm{id} \otimes \tilde{\nu}^2_{\mathcal{E}})(\omega^{\odot} a^{\ominus} \psi \otimes 1) \\ &= (D\psi)\tilde{\nu}^2(a) \otimes 1 + \delta^{\odot}(a)\psi \otimes 1 + \omega^{\odot} a^{\ominus} \psi \otimes 1 \\ &= a^{\oplus} D\psi \otimes 1 + D a^{\ominus} \psi \otimes 1 - a^{\oplus} D\psi \otimes 1 + \omega^{\odot} a^{\ominus} \psi \otimes 1 \\ &= D a^{\ominus} \psi \otimes 1 + \omega^{\odot} a^{\ominus} \psi \otimes 1, \end{aligned}$$

where we have used the fact that $(\tilde{\nu}^2(a))^{\ominus} = (\tilde{\nu}^{-1}(\tilde{\nu}^2(a)))^{\circ} = (\tilde{\nu}(a))^{\circ} = a^{\oplus}$. By making the identification $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{A} \simeq \mathcal{H}$ via the identification of $\psi \otimes a = a^{\ominus} \psi \otimes 1$ with ψ, one immediately finds that $D_L = D + \omega^{\odot}$. The final result then follows as a consequence of Lem. 4.2. □

Something to take note of before moving on is what happens when the original spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is even. In that case, we have the following lemma.

Lemma 4.6. *If a grading operator χ anticommutes with the Dirac operator D and commutes with (representations of) all $a \in A$, then χ also anticommutes with all $\omega \in \Omega^1_D(A)$.*

As a result of this lemma, the grading operator will automatically anticommute with $D + \omega$ for $\omega \in \Omega^1_D(A)$. We now show that it also anticommutes with $D + \varepsilon' \nu J \omega J^{-1} \nu$ as well.

Proposition 4.7. *Let χ be a grading operator which anticommutes with the Dirac operator D and commutes with (representations of) any $a \in A$. Then χ also anticommutes with $D + \varepsilon' \nu J \omega J^{-1} \nu$ for any $\omega \in \Omega^1_D(A)$ provided*

$$\chi \nu J = \varepsilon'' \nu J \chi \tag{4.15}$$

for $\varepsilon'' = \pm 1$, and

$$\chi \nu^2 = \nu^2 \chi. \tag{4.16}$$

Proof. We first focus on the second term of the fluctuated Dirac operator. By (4.15), we have that

$$\chi(\varepsilon'\nu J\omega J^{-1}\nu) = \varepsilon'\varepsilon''\nu J\chi\omega J^{-1}\nu = -\varepsilon'\varepsilon''\nu J\omega\chi J^{-1}\nu,$$

where the second equality is due to Lem. 4.6. We now have that

$$\begin{aligned}\chi J^{-1}\nu &= \varepsilon\chi J\nu \\ &= \varepsilon\chi\nu J\nu^2 \text{ using (2.10)}, \\ &= \varepsilon\varepsilon''\nu J\chi\nu^2 \text{ by (4.15)}, \\ &= \varepsilon\varepsilon''\nu J\nu^2\chi \text{ by (4.16)}, \\ &= \varepsilon\varepsilon'' J\nu\chi \\ &= \varepsilon'' J^{-1}\nu\chi,\end{aligned}$$

and therefore

$$\chi(\varepsilon'\nu J\omega J^{-1}\nu) = -\varepsilon'\varepsilon''\nu J\omega\chi J^{-1}\nu = -\varepsilon'\varepsilon''\nu J\omega\varepsilon'' J^{-1}\nu\chi = -(\varepsilon'\nu J\omega J^{-1}\nu)\chi.$$

As $\chi D = -D\chi$ by assumption, this is sufficient to establish the result. □

Remark 4.8. Indeed, that $\varepsilon'\nu J\omega J^{-1}\nu$ should anticommute with χ is what motivates Def. 2.4. Contrast this with what is taken in the literature, $\chi J = \varepsilon'' J\chi$ [7, 8], which is insufficient to establish the anticommutation in general, even assuming $\chi\nu^2 = \nu^2\chi$. Note that, alternatively to $\chi\nu J = \varepsilon''\nu J\chi$, one could instead take $\chi J\nu = \varepsilon'' J\nu\chi$. Both only hold simultaneously if $J\nu = \nu J$, which assuming regularity (2.10) only happens when $\nu^2 = 1$. ⋄

Thus we find that for a 1-form ω, one has that $(\mathcal{A}, \mathcal{H}, D+\varepsilon'\nu J\omega J^{-1}\nu, \chi)$ is an even spectral triple. However, it fails to admit (J, ν) as a twisted real structure for more or less the same reason as in the right module case. We will resolve this problem for both left and right module cases in the next subsection.

4.2.2.3 *Bimodule and twisted real structure*

To ensure the compatibility of the Dirac operator constructed from Morita self-equivalences with the twisted real structure, one needs to combine the above two left and right module constructions. First, one fluctuates the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ using the bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} = \mathcal{A}$, and then one fluctuates the resulting triple by the conjugate bimodule ${}_{\mathcal{A}}\bar{\mathcal{E}}_{\mathcal{B}} = \mathcal{A}$. This yields the triple $(\mathcal{A}, \mathcal{H}, D')$ where

$$D' = D + \omega_R + \varepsilon'\nu J\omega_L J^{-1}\nu \tag{4.17}$$

with ω_L and ω_R two *a priori* distinct elements of $\Omega^1_D(\mathcal{A})$. The compatibility of (4.17) with the twisted ε'-condition (2.9) for the reality operator J and twist operator ν can then always be demanded, as the following proposition guarantees.

Proposition 4.9. *The Dirac operator D' satisfies (2.9) if and only if there exists an element $\omega \in \Omega^1_D(\mathcal{A})$ such that*

$$D' = D + \omega + \varepsilon' \nu J \omega J^{-1} \nu.$$

Proof. We have that $D' = D + \omega_R + \varepsilon' \nu J \omega_L J^{-1} \nu$. For D' to satisfy (2.9), we must have $D' J \nu = \varepsilon' \nu J D'$, which is the case if and only if

$$(\omega_L - \omega_R) - \varepsilon' \nu J (\omega_L - \omega_R) J^{-1} \nu = 0.$$

Adding half of the left hand side of this equation (which is still equal to 0) to the right hand side of (4.17), one gets

$$D' = D + \frac{1}{2}(\omega_R + \omega_L) + \varepsilon' \nu J \frac{1}{2}(\omega_R + \omega_L) J^{-1} \nu.$$

This gives the claimed result for $\omega = \frac{1}{2}(\omega_R + \omega_L)$. □

The sum total of these past three subsections can thus be expressed as follows.

Theorem 4.10. *For a spectral triple with twisted real structure $(\mathcal{A}, \mathcal{H}, D, (J, \nu))$, the inner-fluctuated Dirac operator*

$$D_\omega := D + \omega + \varepsilon' \nu J \omega J^{-1} \nu$$

for $\omega = \omega^\dagger \in \Omega^1_D(\mathcal{A})$ arises from D by implementing the bimodule Morita self-equivalence of $(\mathcal{A}, \mathcal{H}, D, (J, \nu))$ and requiring that the resulting Dirac operator satisfies (2.9) with respect to J and ν. Thus the data $(\mathcal{A}, \mathcal{H}, D_\omega, (J, \nu))$ form a spectral triple with twisted real structure.

Remark 4.11. As noted in Ref. [7], Dirac operators are closed with respect to these inner fluctuations, in the sense that

$$(D_\omega)_{\omega'} = D_{\omega''}$$

for $D_\omega := D + \omega + \varepsilon' \nu J \omega J^{-1} \nu$ with $\omega, \omega'' \in \Omega^1_D(\mathcal{A})$ and $\omega' \in \Omega^1_{D_\omega}(\mathcal{A})$. Explicitly, if $\omega = \sum_i \pi(a_i)[D, \pi(b_i)]$ and $\omega' = \sum_j \pi(c_j)\big[D_\omega, \pi(d_j)\big]$ for $a_i, b_i, c_i, d_i \in \mathcal{A}$, then one finds (suppressing representations and summations/indices) that ω'' is given by

$$\omega'' = (a - cda)[D, b] + (c - cab)[D, d] + cd[D, bd].$$

However, ω'' will not necessarily be self-adjoint, even if ω and ω' both are. This is no worse than in the standard trivially-twisted case, though. ◇

All that remains is to verify that the inner-fluctuated Dirac operator D_ω satisfies the twisted first-order condition (assuming the original Dirac operator D does as well).

Proposition 4.12. *If $(A, \mathcal{H}, D, (J, \nu))$ is a spectral triple with twisted real structure, then the fluctuated Dirac operator $D_\omega = D + \omega + \varepsilon' \nu J \omega J^{-1} \nu$ for $\omega \in \Omega^1_D(A)$ satisfies the twisted ε'-condition (2.8) with respect to the reality operator J and twist operator ν.*

Proof. In order for D_ω to satisfy (2.8), it is sufficient for each of the three summands to individually satisfy (2.8). As $(A, \mathcal{H}, D, (J, \nu))$ is a spectral triple with twisted real structure, D satisfies (2.8) by assumption.

Since $\omega \in \Omega^1_D(A)$, we can write it in the form $\omega = \sum_j \pi(c_j)\big[D, \pi(d_j)\big]$ for $c_j, d_j \in A$. In that case, for $a \in A$ we have (suppressing summations and representations)

$$\begin{aligned}[\omega, a] &= [c[D,d], a] \\ &= c[D, da] - cd[D,a] - ac[D,d].\end{aligned}$$

Each three of these terms satisfies (2.8) because D does by assumption. As an example calculation, D satisfying (2.8) and $a, d \in A$ implies that $[D, da]J\nu^2 b\nu^{-2}J^{-1} = JbJ^{-1}[D, da]$. Since (2.3) implies that $cJbJ^{-1} = JbJ^{-1}c$, we thus have that $c[D,da]J\nu^2 b\nu^{-2}J^{-1} = JbJ^{-1}c[D,da]$, *i.e.* that $c[D, da]$ satisfies (2.8). The computation for the other terms is carried out in the same way.

For the final term, we can simply make use of Lem. 4.2 to note that $\nu J\omega J^{-1}\nu \in \tilde{\Omega}^1_D(A^{\mathrm{op}})$ for $\omega \in \Omega^1_D(A)$. By the definition of $\tilde{\Omega}^1_D(A^{\mathrm{op}})$, any element of $\tilde{\Omega}^1_D(A^{\mathrm{op}})$ can be written in the form $\sum_j \pi_J^*(\tilde{\nu}(c_j))\big[D, \tilde{\nu}^{-1}(d_j)\big]^{\pi_J^*}_{\tilde{\nu}^2}$ or, equivalently, $\sum_j \pi_J(\hat{\nu}^{-1}(c_j))\big[D, \hat{\nu}(d_j)\big]^{\pi_J}_{\hat{\nu}^{-2}}$ for $c_j, d_j \in A$. We now compute (again suppressing summations and representations) that

$$\Big[J\hat{\nu}^{-1}(c)J^{-1}[D, \hat{\nu}(d)]^{\pi_J}_{\hat{\nu}^{-2}}, a\Big] = J\hat{\nu}^{-1}(c)J^{-1}\Big[[D, \hat{\nu}(d)]^{\pi_J}_{\hat{\nu}^{-2}}, a\Big]$$

by (2.3). However, from Lem. 2.11 we have that $\Big[[D, \hat{\nu}(d)]^{\pi_J}_{\hat{\nu}^{-2}}, a\Big] = [[D,a], \hat{\nu}(d)]^{\pi_J}_{\hat{\nu}^{-2}}$ and since D satisfies (2.8), we find $[[D,a], \hat{\nu}(d)]^{\pi_J}_{\hat{\nu}^{-2}} = 0$ by Lem. 2.10, and thus (2.8) is satisfied by $\nu J\omega J^{-1}\nu$ trivially. □

The general procedure described in this subsection relies on defining connections on projective modules. In the setting of Hopf-Galois extensions ('quantum principal bundles'), strong connections similarly induce connections on associated projective modules [28]. It would be interesting to investigate the link between fluctuations of the Dirac operator in the various approaches to Connes' noncommutative geometry (including spectral triples with (twisted) real structure, real twisted spectral triples, real spectral triples without the first-order condition, *etc.*) and strong connections for Hopf-Galois extensions in scenarios where the formalisms are compatible. Such an investigation, being beyond the scope of this thesis, we leave for future research.

4.3 GAUGE TRANSFORMATIONS WITH TWISTED REAL STRUCTURES

4.3.1 *Gauge transformations as a Morita self-equivalence*

In the context of spectral geometry, gauge transformations can be viewed as a special case of Morita self-equivalences, which we will describe briefly as follows. Considering a Morita equivalence between two algebras $\mathcal{A}$ (acting on $\mathcal{H}$) and $\mathcal{B}$, we have a Morita equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and the conjugate bimodule ${}_{\mathcal{A}}\bar{\mathcal{E}}_{\mathcal{B}}$ (such that $\mathcal{B}$ acts on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\bar{\mathcal{E}}_{\mathcal{B}}$ as a Hilbert space). As mentioned in the previous subsection, when ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is finite projective as a right $\mathcal{A}$-module (which we will assume it to be), we have $\mathcal{B} \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}})$ (and similarly in the left module case). Consider the group of unitary endomorphisms $\mathcal{U}(\mathcal{E}) := \{u \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}) : uu^* = u^*u = \mathrm{id}_{\mathcal{E}}\}$ (with $\mathcal{U}(\bar{\mathcal{E}})$ defined likewise). Then for $u \in \mathcal{U}(\mathcal{E}) \simeq \mathcal{U}(\bar{\mathcal{E}})$ we call a *gauge transformation* the action of u on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ ($u(e) = ue$) and on ${}_{\mathcal{A}}\bar{\mathcal{E}}_{\mathcal{B}}$ ($u(\bar{e}) = \overline{eu^*}$).

In the case of self-equivalence, ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} = {}_{\mathcal{A}}\bar{\mathcal{E}}_{\mathcal{B}} = \mathcal{A}$ and $u \in \mathcal{U}(\mathcal{A})$. Then gauge transformations on the algebra can be interpreted as inner automorphisms $a \mapsto uau^*$ for all $a \in \mathcal{A}$. This in turn gives gauge transformations on the Hilbert space $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \mathcal{A} \simeq \mathcal{H}$ as $\psi \mapsto u\psi u^*$ for all $\psi \in \mathcal{H}$. Here, we already see a divergence from the standard construction due to the modified choice of right action (4.8), as

$$u\psi u^* =: \mathrm{Ad}(u)\psi = \pi(u)(u^*)^{\ominus}\psi = \pi(u)J\nu\pi(u)\nu^{-1}J^{-1}\psi \tag{4.18}$$

for all $\psi \in \mathcal{H}$, which contrasts with the usual case where $u\psi u^* = \pi(u)J\pi(u)J^{-1}\psi$. Note that in terms of the notation we have developed, this has the inverse

$$\mathrm{Ad}(u)^{-1}\psi = u^*\psi u = \pi(u^*)u^{\ominus}\psi = \pi(u^*)J\nu\pi(u^*)\nu^{-1}J^{-1}\psi.$$

In addition to the adjoint action above, by $\widetilde{\mathrm{Ad}}(u)$ we denote the complementary 'twisted' adjoint action

$$\widetilde{\mathrm{Ad}}(u)\psi := u\psi\bar{\nu}^2(u^*) = \pi(u)(u^*)^{\oplus}\psi = \pi(u)J\nu^{-1}\pi(u)\nu J^{-1}\psi \tag{4.19}$$

for all $\psi \in \mathcal{H}$, with inverse

$$\widetilde{\mathrm{Ad}}(u)^{-1}\psi = \pi(u^*)J\nu^{-1}\pi(u^*)\nu J^{-1}\psi.$$

A peculiarity of (4.18) (and (4.19)) is that in general, neither $\mathrm{Ad}(u)$ nor $\widetilde{\mathrm{Ad}}(u)$ is a unitary operator despite being generated by a unitary element, due to the presence of the twist. This means that in the context of twisted real structures, Morita equivalences do not generally give unitary equivalences of spectral triples (which will be established explicitly in Prop. 4.18).

Gauge transformations are also defined on (right $\mathcal{A}$-module) connections $\nabla \colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ as maps

$$\nabla \mapsto \nabla^u := u\nabla u^* \tag{4.20}$$

for $u \in \mathcal{U}(\mathcal{E})$, where $\mathcal{U}(\mathcal{E})$ acts on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ by $u \otimes \mathrm{id}$. Note that the gauge-transformed connection ∇^u is itself a connection for any $u \in \mathcal{U}(\mathcal{E})$ and connection ∇.

For the case of right module Morita *self*-equivalence, we have $\mathcal{E}_{\mathcal{A}} = \mathcal{A}$ and the connection is given by

$$\nabla = \delta + \omega$$

for $\omega \in \Omega^1_D(\mathcal{A})$ self-adjoint. In this context, the connection 1-form ω is referred to as a *gauge potential*, and a gauge transformation maps the connection to

$$\nabla^u = \delta + \omega^u$$

For $a \in \mathcal{A}$, $\nabla^u(a) = u \cdot \delta(u^* a) + u \cdot \omega(u^* a) = (\delta + (u \cdot \omega \cdot u^*) + (u \cdot \delta(u^*)))(a) \equiv (\delta + \omega^u)(a)$.

where the gauge transformation is fully encoded in the transformation law for the gauge potential

$$\omega \mapsto \omega^u = \pi(u)\omega\pi(u^*) + \pi(u)[D, \pi(u^*)]. \tag{4.21}$$

In the case of right module self-equivalence, the implementation of gauge transformations on the Dirac operator amounts to substituting the gauge transformed connection $\nabla \mapsto \nabla^u$ into the definition of D_R. Thus for a gauge transformation by the unitary $u \in \mathcal{A}$, $D_R = D + \omega$ is mapped to

$$D^u_R = D + \omega^u = D + \pi(u)\omega\pi(u^*) + \pi(u)[D, \pi(u^*)].$$

As we have already seen in Lem. 4.2, for a given twisted real structure (J, ν) it is possible to find a 'twisted-opposite' 1-form $\omega^{\odot}$ associated to the 1-form ω using the map $\omega \mapsto \varepsilon' \nu J \omega J^{-1} \nu$. However, it is not clear that this map is compatible with the gauge transformation of gauge potentials (4.21), or in other words, it is not clear if $(\omega^{\odot})^u$ coming from the action of u on the connection $\nabla^{\odot}$ is actually equal to $(\omega^u)^{\odot}$. In what follows, we show that this is indeed the case.

Proposition 4.13. *Let $\omega \in \Omega^1_D(\mathcal{A})$ be a gauge potential. Under the gauge transformation $\omega \mapsto \omega^u$, given in (4.21) for $u \in \mathcal{U}(\mathcal{A})$, the corresponding twisted 1-form under the map $\omega^u \mapsto (\omega^u)^{\odot}$ is given by*

$$(\omega^u)^{\odot} = (u^*)^{\oplus} \omega^{\odot} u^{\ominus} + (u^*)^{\oplus} \delta^{\odot}(u).$$

Proof. The proof is by a straightforward computation. We will omit representations for brevity.

$$\begin{aligned}
(\omega^u)^{\odot} &= \varepsilon' \nu J \omega^u J^{-1} \nu \\
&= \varepsilon' \nu J u \omega u^* J^{-1} \nu + \varepsilon' \nu J u [D, u^*] J^{-1} \nu \\
&= \varepsilon' J \nu^{-1} (u^*)^* \nu J^{-1} \nu J \omega J^{-1} \nu J \nu u^* \nu^{-1} J^{-1} \\
&\qquad + \varepsilon' J \nu^{-1} (u^*)^* \nu J^{-1} \nu J [D, u^*] J^{-1} \nu \\
&= (u^*)^{\oplus} \omega^{\odot} u^{\ominus} + (u^*)^{\oplus} (\varepsilon' \nu J [D, u^*] J^{-1} \nu) \\
&= (u^*)^{\oplus} \omega^{\odot} u^{\ominus} + (u^*)^{\oplus} \delta^{\odot}(u),
\end{aligned}$$

where in going to the last line we have made use of (4.6) and Lem. 4.2 (there taking $a = 1, b = u$). □

To compare with the result of Prop. 4.13, we now derive a transformation law from Morita self-equivalence as we did in the (standard) right module case above. For the case of left module self-equivalence, we have that the conjugate module is ${}_{\mathcal{A}}\bar{\mathcal{E}}_{\mathcal{B}} = \mathcal{A}$ with $\mathcal{B} = \mathcal{A}$, and the unitary endomorphisms are just the unitary elements of the algebra acting by

$$u(a) := au^* \tag{4.22}$$

for $a \in \mathcal{A}$, $u \in \mathcal{U}(\mathcal{A})$.

Given the derivation $\delta^{\odot}$ as defined in (4.6), the gauge transformation for the connection $\nabla^{\odot} = \delta^{\odot} + \omega^{\odot}$ on the module ${}_{\mathcal{A}}\bar{\mathcal{E}}_{\mathcal{B}} = \mathcal{A}$, $\mathcal{B} = \mathcal{A}$ is given by $\nabla^{\odot} \mapsto (\nabla^{\odot})^u := u\nabla^{\odot}u^*$, analogous to (4.20).

Lemma 4.14. *Let $\nabla^{\odot}$ be an $\tilde{\Omega}^1_D(\mathcal{A}^{\mathrm{op}})$-valued connection on $\mathcal{A}$ as an $\mathcal{A}$-bimodule with unitary endomorphisms u acting via (4.22). Then, for any $u \in \mathcal{U}(\mathcal{A})$ we have*

$$(\mathrm{id} \otimes \tilde{v}^2_{\mathcal{E}}) \circ (\nabla^{\odot})^u(a) = \delta^{\odot}(au) \otimes \tilde{v}^2(u^*) + \omega^{\odot}(au)^{\ominus} \otimes \tilde{v}^2(u^*)$$

for all $a \in \mathcal{A}$ with $(\nabla^{\odot})^u$ the gauge transformation $(\nabla^{\odot})^u := u\nabla^{\odot}u^*$ and $\tilde{v}_{\mathcal{E}} \in \mathrm{End}_{\mathcal{A}}(\mathcal{A})$ as given in (4.9).

Proof. First, we have $\nabla^{\odot} = \nabla^{\odot}_0 + \vec{\omega}^{\odot}$ given by $\nabla^{\odot}_0(a) = \delta^{\odot}(a) \otimes 1$ and $\vec{\omega}^{\odot}(a) = (\omega^{\odot}a^{\ominus}) \otimes 1$ for any $a \in \mathcal{A}$. We also have $u(a) = au^*$, and thus we find

$$\begin{aligned}(\nabla^{\odot})^u(a) := (u\nabla^{\odot}u^*)(a) &= (u\nabla^{\odot})(au) = u\big(\nabla^{\odot}_0(au) + \vec{\omega}^{\odot}(au)\big)\\ &= u\big(\delta^{\odot}(au) \otimes 1 + \omega^{\odot}(au)^{\ominus} \otimes 1\big)\\ &= \delta^{\odot}(au) \cdot u^* \otimes 1 + (\omega^{\odot}(au)^{\ominus}) \cdot u^* \otimes 1\\ &= \delta^{\odot}(au) \otimes u^* + \omega^{\odot}(au)^{\ominus} \otimes u^*.\end{aligned}$$

Therefore, we have

$$\begin{aligned}(\mathrm{id} \otimes \tilde{v}^2_{\mathcal{E}}) \circ (\nabla^{\odot})^u(a) &= (\mathrm{id} \otimes \tilde{v}^2_{\mathcal{E}})(\delta^{\odot}(au) \otimes u^* + \omega^{\odot}(au)^{\ominus} \otimes u^*)\\ &= \delta^{\odot}(au) \otimes \tilde{v}^2(u^*) + \omega^{\odot}(au)^{\ominus} \otimes \tilde{v}^2(u^*).\end{aligned}$$

□

Just like in the right module case, we implement the gauge transformation for the left module case by replacing $\nabla^{\odot}$ with $(\nabla^{\odot})^u$ in the definition of D_L. In the case of self-equivalence, we obtain the following explicit formula:

Proposition 4.15. *For a gauge transformation with $u \in \mathcal{U}(\mathcal{A})$, the operator $D_L = D + \omega^{\odot}$ is mapped to $D^u_L = D + (\omega^{\odot})^u$ where the transformed 1-form is given by*

$$(\omega^{\odot})^u = (u^*)^{\oplus}\omega^{\odot}u^{\ominus} + (u^*)^{\oplus}\delta^{\odot}(u).$$

Proof. We have from the previous lemma

$$\begin{aligned}
D_L^u(\psi \otimes a) &= ((\mathrm{id} \otimes \tilde{\nu}_{\mathcal{E}}^2) \circ (D_L + (\nabla^{\odot})^u))(\psi \otimes a) \\
&= D\psi \otimes \tilde{\nu}^2(a) + \delta^{\odot}(au)\psi \otimes \tilde{\nu}^2(u^*) + \omega^{\odot}(au)^{\ominus}\psi \otimes \tilde{\nu}^2(u^*) \\
&= \left(a^{\oplus}D + (u^*)^{\oplus}D(au)^{\ominus} - (u^*)^{\oplus}(au)^{\oplus}D\right)\psi \otimes 1 \\
&\quad + (u^*)^{\oplus}\omega^{\odot}u^{\ominus}a^{\ominus}\psi \otimes 1 \\
&= (u^*)^{\oplus}Du^{\ominus}a^{\ominus}\psi \otimes 1 + (u^*)^{\oplus}\omega^{\odot}u^{\ominus}a^{\ominus}\psi \otimes 1 \\
&= \left(D + (u^*)^{\oplus}(Du^{\ominus} - u^{\oplus}D)\right)a^{\ominus}\psi \otimes 1 + (u^*)^{\oplus}\omega^{\odot}u^{\ominus}a^{\ominus}\psi \otimes 1.
\end{aligned}$$

By definition, $Du^{\ominus} - u^{\oplus}D = \delta^{\odot}(u)$. Identifying $\psi \otimes a = a^{\ominus}\psi \otimes 1$ with $\psi \in \mathcal{H} \otimes_{\mathcal{A}} \mathcal{A} \simeq \mathcal{H}$ therefore gives the proposition. □

Comparing the results of Prop. 4.13 and Prop. 4.15, we find that indeed, $(\omega^u)^{\odot} = (\omega^{\odot})^u$.

4.3.2 *Gauge transformations for a spectral triple*

The results of the previous subsection, along with Thm. 4.10, give the following result:

Theorem 4.16. *Let $(A, \mathcal{H}, D_\omega, (J, \nu))$ be a spectral triple with twisted real structure obtained by bimodule Morita self-equivalence from the spectral triple with twisted real structure $(A, \mathcal{H}, D, (J, \nu))$, where D_ω is the Dirac operator fluctuated from D by the 1-form $\omega \in \Omega^1_D(A)$.*

Then the law for the gauge transformation of the Dirac operator D_ω by $u \in \mathcal{U}(A)$ is given by

$$D_\omega \mapsto D_\omega^u = D + \omega^u + \varepsilon' \nu J \omega^u J^{-1} \nu \equiv D_{\omega^u},$$

where

$$\omega \mapsto \omega^u = \pi(u)\omega\pi(u^*) + \pi(u)[D, \pi(u^*)]$$

is the gauge transformation of a gauge potential.

The above expression is found by mapping $\omega \mapsto \omega^u$ on the operator D_ω after ω_L and ω_R have been identified. For the sake of consistency, one should check that the same result applies for gauge transforming both left and right gauge potentials separately, and indeed this does prove to be the case, though we will not give the proof here.

Similar to the standard case, the gauge transformation of the Dirac operator can be implemented using the operator which implements gauge transformations of the Hilbert space (4.18). The difference is that we require not just the operator for the adjoint action, but also the corresponding operator for the twisted adjoint action (4.19).

Dirac operators obtained as the gauge transformations of a non-fluctuated Dirac operator are often referred to as being pure gauge.

Lemma 4.17. *Let $(A, \mathcal{H}, D, (J, \nu))$ be a spectral triple with twisted real structure. For any $u \in \mathcal{U}(A)$ it holds that*

$$\widetilde{\mathrm{Ad}}(u) D\, \mathrm{Ad}(u)^{-1} = D + \pi(u)[D, \pi(u^*)] + \varepsilon' \nu J \pi(u)[D, \pi(u^*)] J^{-1} \nu.$$

Proof. The proof is given by a straightforward computation (we suppress representations for brevity):

$$\begin{aligned}\widetilde{\mathrm{Ad}}(u)D\,\mathrm{Ad}(u)^{-1} &= uJ\nu^{-1}u\nu J^{-1}DJ\nu u^*\nu^{-1}J^{-1}u^* \\ &= \varepsilon' u\nu JuDu^*J^{-1}\nu u^* \\ &= \varepsilon' u\nu J(D+u[D,u^*])J^{-1}\nu u^* \\ &= uDu^* + \varepsilon' u\nu Ju[D,u^*]J^{-1}\nu u^* \\ &= D + u[D,u^*] + \varepsilon' u\nu Ju[D,u^*]J^{-1}\nu u^*.\end{aligned}$$

Now all that remains is to massage the final term on the right hand side:

$$\begin{aligned}\varepsilon' u\nu Ju[D,u^*]J^{-1}\nu u^* &= \varepsilon'\nu J\nu J^{-1}u\nu Ju[D,u^*]J^{-1}\nu u^*J\nu J^{-1}\nu \\ &= \varepsilon'\nu J(J\nu^{-1}u\nu J^{-1})u[D,u^*](J\nu u^*\nu^{-1}J^{-1})J^{-1}\nu \\ &= \varepsilon'\nu Ju[D,u^*](J\nu u\nu^{-1}J^{-1})(J\nu u^*\nu^{-1}J^{-1})J^{-1}\nu \\ &= \varepsilon'\nu Ju[D,u^*]J^{-1}\nu,\end{aligned}$$

thus giving the claimed result. Note that on the third line, we have used (2.3) and (2.8). □

Proposition 4.18. *Let $(A,\mathcal{H},D,(J,\nu))$ be a spectral triple with twisted real structure, and consider a fluctuated Dirac operator $D_\omega = D+\omega+\varepsilon'\nu J\omega J^{-1}\nu$. Then for any $u\in\mathcal{U}(A)$, one has*

$$\widetilde{\mathrm{Ad}}(u)D_\omega\,\mathrm{Ad}(u)^{-1} = D + \omega^u + \varepsilon'\nu J\omega^u J^{-1}\nu,$$

with the gauge transformed ω^u given as above.

Proof. Without loss of generality, we can take $\omega = a[D,b]$, where we are again suppressing representations for brevity. In that case, we have

$$\begin{aligned}\widetilde{\mathrm{Ad}}(u)\omega\,\mathrm{Ad}(u)^{-1} &= uJ\nu^{-1}u\nu J^{-1}a[D,b]J\nu u^*\nu^{-1}J^{-1}u^* \\ &= ua[D,b](J\nu u\nu^{-1}J^{-1})(J\nu u^*\nu^{-1}J^{-1})u^* \\ &= ua[D,b]u^* = u\omega u^*,\end{aligned}$$

and a slightly more involved (but not qualitatively different) calculation gives

$$\begin{aligned}\widetilde{\mathrm{Ad}}(u)(\varepsilon'\nu J\omega J^{-1}\nu)\,\mathrm{Ad}(u)^{-1} &= \varepsilon' uJ\nu^{-1}u\nu J^{-1}\nu Ja[D,b]J^{-1}\nu J\nu u^*\nu^{-1}J^{-1}u^* \\ &= \varepsilon' u\nu Jua[D,b]u^*J^{-1}\nu u^* \\ &= \varepsilon'\nu J\nu J^{-1}u\nu Jua[D,b]u^*J^{-1}\nu u^*J\nu J^{-1}\nu \\ &= \varepsilon'\nu J(J\nu^{-1}u\nu J^{-1})ua[D,b]u^*(J\nu u^*\nu^{-1}J^{-1})J^{-1}\nu \\ &= \varepsilon'\nu Jua[D,b]u^*(J\nu u\nu^{-1}J^{-1})(J\nu u^*\nu^{-1}J^{-1})J^{-1}\nu \\ &= \varepsilon'\nu Jua[D,b]u^*J^{-1}\nu = \varepsilon'\nu Ju\omega u^*J^{-1}\nu.\end{aligned}$$

Collecting these results and combining with Lem. 4.17, one finds the claimed result. □

Proposition 4.19. *Let $(A, \mathcal{H}, D, (J, \nu), \chi)$ be an even spectral triple with twisted real structure. Then for any $u \in \mathcal{U}(A)$, the gauge transformation of the spectral triple by u is characterised by the following operator actions:*

$$\tilde{V}\pi(a)\tilde{V}^{-1} = V\pi(a)V^{-1} = \pi(uau^*), \tag{4.23}$$

$$V\psi = \psi^u \tag{4.24}$$

$$\tilde{V}DV^{-1} = D^u, \tag{4.25}$$

$$\tilde{V}\chi\tilde{V}^{-1} = V\chi V^{-1} = \chi, \tag{4.26}$$

$$\tilde{V}\nu J\tilde{V}^{-1} = \nu J, \tag{4.27}$$

$$VJ\nu V^{-1} = J\nu, \tag{4.28}$$

for all $a \in A$ and $\tilde{V} := \widetilde{\mathrm{Ad}}(u) = \pi(u)J\nu^{-1}\pi(u)\nu J^{-1}$ and $V := \mathrm{Ad}(u) = \pi(u)J\nu\pi(u)\nu^{-1}J^{-1}$.

Proof. We have that $V\psi = \psi^u$ from the definition of gauge transformations. This is also true of $\pi(a)^u = \pi(uau^*)$, which one checks is given by conjugation by both V and $\tilde{V}$ by computation. That $\tilde{V}DV^{-1} = D^u$ is given by Prop. 4.18. All of the other relations are obtained by straightforward computations. □

The invariance of χ under both V and $\tilde{V}$ nicely dovetails with the gauge transformations of A and D, which follows from the earlier constructions (specifically, χ must commute with A and anticommute with D). However, it is curious that the Dirac operator does not generally respect gauge transformations on the Hilbert space, *e.g.* $D^u\psi^u = \tilde{V}DV^{-1}V\psi = \tilde{V}D\psi \neq (D\psi)^u$ when $\tilde{V} \neq V$. The meaning of this is not entirely clear, although given that the choice of right A-action (4.8) was not unique (we could equally have chosen $\psi a := a^{\oplus}\psi$), it may be a hint that our construction of Morita equivalence is in some way half-complete, and that we should have two gauge transformations, one for each twist (or perhaps even for an infinite number of twists coming from $\tilde{\nu}^n$, $n \in \mathbb{Z}$). However, as the construction presented in this thesis is sufficient for our purposes, we leave this possibility for further investigation.

Another interesting point is that the reality operator J (and hence the twist operator ν) does not individually obey any invariance with respect to V, $\tilde{V}$, or their inverses, as we might hope. However, the pair of equations (4.27) and (4.28) do make sense as the weakest way of ensuring (4.25) is compatible with (2.9). The most natural inference from the pair (4.27) and (4.28) is to accept that there is a covariance instead of an invariance, in which case we should define $J^u = VJ\tilde{V}^{-1}$ and $\nu^u = \tilde{V}\nu V^{-1}$. But in that case J^u can only possibly be a reality operator if ν is both self-adjoint and unitary (up to sign). This should be borne in mind for the following subsections.

It is of course also important to note that, for $U := \pi(u)J\pi(u)J^{-1}$, when $\nu = 1$ (the trivially-twisted case) or $\nu\pi(u)\nu^{-1} = \pi(u)$ for all

$u \in \mathcal{U}(A)$, we have that $\tilde{V} = V = U$ and $V^{-1} = \tilde{V}^{-1} = U^\dagger$ and all of the above relations reduce to the familiar ones for unitary equivalences of spectral triples.

Remark 4.20. A curious observation is that when u is permitted to remain invariant under $\hat{\nu}$, or equivalently, we request that $[\nu, \pi(u)] = 0$, we see that $\tilde{V}$ reduces to $\pi(u)J\pi(u)J^{-1}$ and V^{-1} to $\pi(u^*)J\pi(u^*)J^{-1}$, which are simply the familiar unitary operators which implement gauge transformations in the trivially-twisted case. This is interesting because in principle the remainder of the twisted real structure remains intact, unlike in the case of Prop. 2.12, for example. That said, it is likely not practical to impose such invariance, as it would almost surely be far too restrictive on either the available twists or the usable unitary elements. ◇

In the case of Prop. 2.12, the spectral triple with twisted real structure $(J, \nu = \pm\nu^* = \pm\nu^{-1})$ is equivalent to the trivially-twisted spectral triple with real structure $\mathcal{J} = \nu J$. Unitary equivalence would then be restored automatically with $V = \mathcal{U}$ for $\mathcal{U} = \pi(u)\mathcal{J}\pi(u)\mathcal{J}^{-1}$.

4.3.3 *Self-adjointness of the Dirac operator*

In the trivially-twisted case, for $U := \pi(u)J\pi(u)J^{-1}$, a gauge transformation preserves the self-adjointness of the Dirac operator automatically. The transformed operator $D_{\omega^u} = UD_\omega U^\dagger$ is self-adjoint if and only if D_ω is, since U is unitary, and so a gauge transformation yields a spectral triple which is unitarily equivalent to the former. This is no longer necessarily true when the real structure is twisted. We now investigate the cases in which it is.

Lemma 4.21. *If $(A, \mathcal{H}, D, (J, \nu))$ is a spectral triple with twisted real structure, and D^u is the Dirac operator obtained from D by a gauge transformation by the unitary element $u \in \mathcal{U}(A)$ then*

$$\nu = \pm\nu^\dagger$$

is a sufficient condition for $D^u = (D^u)^\dagger$.

Proof. From Lem. 4.17 we know that $D^u = \tilde{V}DV^{-1}$ for $\tilde{V} = \pi(u)J\nu^{-1}\pi(u)\nu J^{-1}$ and $V^{-1} = \pi(u^*)J\nu\pi(u^*)\nu^{-1}J^{-1}$. Since D is self-adjoint by the definition of a Dirac operator, we therefore have that $D^u = (D^u)^\dagger$ is given by $\tilde{V}DV^{-1} = (V^{-1})^\dagger D\tilde{V}^\dagger$ or, expressed more fully,

$$uJ\nu^{-1}u\nu J^{-1}DJ\nu u^*\nu^{-1}J^{-1}u^* = uJ(\nu^{-1})^\dagger u\nu^\dagger J^{-1}DJ\nu^\dagger u^*(\nu^{-1})^\dagger J^{-1}u^*.$$

It is clear from simple substitution that $\nu = \pm\nu^\dagger$ satisfies this equation. □

Note that when $\nu = \pm\nu^\dagger$, one has that $\tilde{V}^\dagger = V^{-1}$ and $V^\dagger = \tilde{V}^{-1}$.

Remark 4.22. The requirement that the twist operator be self-adjoint (up to sign) is well-motivated by comparison to the literature on (real) twisted spectral triples, where it is equivalent to the common requirement (going back to Ref. [21]) that $\rho(a^*) = (\rho^{-1}(a))^*$, where ρ is the twisting algebra automorphism (ρ can be seen as very roughly analogous to $\hat{\nu}^2$ or $\tilde{\nu}^2$ in the twisted real structure context, though of course the frameworks differ). ◇

Recall that $\tilde{\nu}(a^*) = (\hat{\nu}^{-1}(a))^*$ and $\tilde{\nu} \equiv \hat{\nu}$ exactly when $\nu^\dagger = \nu$ (up to sign), see also the comment on p.19 for a different interpretation.

Proposition 4.23. *Let $(A, \mathcal{H}, D, (J, \nu))$ be a spectral triple with twisted real structure, with D_ω the Dirac operator obtained by fluctuating D by a 1-form $\omega \in \Omega^1_D(A)$ and D^u_ω the Dirac operator obtained from D_ω by a gauge transformation by the unitary element $u \in \mathcal{U}(A)$. Then, for $\nu = \pm\nu^\dagger$,*

$$\omega = \omega^\dagger$$

is a sufficient condition for $D^u_\omega = (D^u_\omega)^\dagger$.

Proof. We begin by considering the Dirac operator $D_\omega = D + \omega + \varepsilon' \nu J \omega J^{-1} \nu$. As D is self-adjoint by assumption, the self-adjointness of D_ω is guaranteed by the condition that

$$\omega + \varepsilon' \nu J \omega J^{-1} \nu = \omega^\dagger + \varepsilon' \nu^\dagger J \omega^\dagger J^{-1} \nu^\dagger.$$

Taking $\nu = \pm\nu^\dagger$, this can be rewritten

$$(\omega - \omega^\dagger) + \varepsilon' \nu J (\omega - \omega^\dagger) J^{-1} \nu = 0,$$

which is clearly satisfied when $\omega = \omega^\dagger$.

By Thm. 4.16, $D^u_\omega = D_{\omega^u}$ where $\omega^u = \pi(u)\omega\pi(u^*) + \pi(u)[D, \pi(u^*)]$. The same reasoning therefore applies with ω^u replacing ω, and so D^u_ω will be self-adjoint when $\nu = \pm\nu^\dagger$ and $\omega^u = (\omega^u)^\dagger$. However, $\omega = \omega^\dagger$ immediately implies $\omega^u = (\omega^u)^\dagger$. □

Strictly speaking, it is not necessary to take $\nu = \pm\nu^\dagger$ for the twisted real structure formulation to be fully self-consistent. For example, in the case of the trivial Dirac operator $D = 0$, which also has trivial fluctuations, one is free to take a non-self-adjoint twist operator. However, as we are interested in the general case, assuming self-adjointness (up to sign) proves to be the only really practical option to guarantee everything will work.

4.4 THE FERMIONIC AND BOSONIC ACTION FUNCTIONALS

In light of the results of the previous subsection, from here on we will take $\nu = \alpha_1 \nu^\dagger$ for $\alpha_1 \in \{-1, +1\}$. When considering twisted real structures, we have no reason to expect that the standard bilinear form which gives the fermionic action $\mathfrak{A}_D(\psi, \varphi) := \langle J\psi, D\varphi \rangle$, where $\psi, \varphi \in \mathrm{Dom}(D)$, should still hold unchanged. Indeed, in the setting of twisted real structures, this bilinear form fails to be suitably symmetric or antisymmetric; we find

$$\begin{aligned} \mathfrak{A}_D(\psi, \varphi) &= \langle J\psi, D\varphi \rangle \\ &= \varepsilon \left\langle J\psi, J^2 D\varphi \right\rangle \\ &= \varepsilon \langle JD\varphi, \psi \rangle \\ &= \varepsilon\varepsilon' \left\langle \nu^{-1} D J \nu \varphi, \psi \right\rangle \\ &= \varepsilon\varepsilon' \left\langle J\varphi, {\nu^{-1}}^\dagger D^\dagger {\nu^{-1}}^\dagger \psi \right\rangle \end{aligned} \tag{4.29}$$

which is not equal to $\mathfrak{A}_D(\varphi,\psi)$ (up to sign) unless $D = \pm v^{-1}D^\dagger v^{-1}$, which is not true in general.

Therefore our first task is to see if we can construct an alternative bilinear form which is gauge-covariant, and indeed, this can be done:

Lemma 4.24. *Let v be a linear operator. Then the bilinear form*

$$\tilde{\mathfrak{A}}_D(\psi,\varphi) := \langle Jv\psi, D\varphi\rangle, \tag{4.30}$$

where $\langle\bullet,\bullet\rangle$ is the inner product on the Hilbert space $\mathcal{H}$, is gauge-covariant and is correctly (anti)symmetric for the appropriate KO-dimension.

The (anti)symmetry of the form is important when interpreting the bracket in the physical context of the Grassmann nature of fermion fields.

Proof. That the bilinear form be gauge-covariant is equivalent to requiring that it satisfies

$$\tilde{\mathfrak{A}}_D(\psi,\varphi) = \tilde{\mathfrak{A}}_{D^u}\big(\mathrm{Ad}(u)(\psi), \mathrm{Ad}(u)(\varphi)\big).$$

Using the notation $V := \mathrm{Ad}(u) = \pi(u)Jv\pi(u)v^{-1}J^{-1}$ and $\tilde{V} := \widetilde{\mathrm{Ad}}(u) = \pi(u)Jv^{-1}\pi(u)vJ^{-1}$, one finds this is equivalent to the requirement that

$$\begin{aligned}\langle Jv\psi, D\varphi\rangle &= \langle JvV\psi, D^uV\varphi\rangle\\ &= \big\langle JvV\psi, \tilde{V}DV^{-1}V\varphi\big\rangle\\ &= \big\langle JV\psi, \tilde{V}D\varphi\big\rangle,\end{aligned}$$

which in turn is equivalent to requiring that

$$\tilde{V}^\dagger JvV = Jv,$$

which one can easily compute to be true. The (anti)symmetry of the bilinear form can also be verified by computation. Assuming that D is self-adjoint, we have

$$\begin{aligned}\tilde{\mathfrak{A}}_D(\psi,\varphi) &= \langle Jv\psi, D\varphi\rangle\\ &= \varepsilon\big\langle Jv\psi, J^2D\varphi\big\rangle\\ &= \varepsilon\langle JD\varphi, v\psi\rangle\\ &= \alpha_1\varepsilon\langle vJD\varphi,\psi\rangle\\ &= \alpha_1\varepsilon\varepsilon'\langle DJv\varphi,\psi\rangle\\ &= \alpha_1\varepsilon\varepsilon'\langle Jv\varphi, D\psi\rangle\\ &= \alpha_1\varepsilon\varepsilon'\tilde{\mathfrak{A}}_D(\varphi,\psi),\end{aligned}$$

which can always be made to have the same sign as in the untwisted case (for example, by setting $\alpha_1 = +1$). $\square$

Remark 4.25. Equation (4.30) can be written in the form $\tilde{\mathfrak{A}}_D(\psi,\varphi) = \alpha_1\big\langle J\psi, v^{-1}D\varphi\big\rangle$, which allows one to draw an analogy to the work presented in Ref. [33], wherein the authors define the sesquilinear form $\langle\psi,\varphi\rangle_\rho := \langle\psi, R\varphi\rangle$ for a linear operator $R = R^\dagger = R^{-1}$ implementing the algebra automorphism $\rho(\bullet) := R(\bullet)R^\dagger$. However, for present purposes it is preferable to keep the reality operator and twist together, so we choose not to use a similarly modified bracket in this chapter. $\diamond$

The main consequence of Lem. 4.24 is the following:

Proposition 4.26. *The appropriate form for the fermionic action functional for spectral triples with twisted real structure is*

$$S_F[D,\psi] := \langle J\nu\tilde{\psi}, D\tilde{\psi}\rangle, \tag{4.31}$$

where $\tilde{\psi}$ is the Grassmann variable corresponding to $\psi \in \{\varphi \in \mathcal{H} : \chi\varphi = \varphi\}$, which is well-defined and gauge-covariant when $D = D^\dagger$ and $\nu = \alpha_1\nu^\dagger$ and is antisymmetric in KO-dimension $2 \pmod 8$ when $\alpha_1 = +1$.

The case for the bosonic action

$$S_B[D] = \mathrm{Tr}\left(f\left(\frac{D}{\Lambda}\right)\right)$$

for $\Lambda \in \mathbb{R}$ is less neat. The action should be invariant with respect to gauge transformations of the Dirac operator, but it is not difficult to see that D^u need not have the same spectrum as D in general, which is a problem. The simplest fix for this is to require $\nu = \pm\nu^{-1}$, which ensures D and D^u have the same spectrum (as then $\tilde{V} = V$), but one must be careful not to run afoul of Prop. 2.12. This also means that, unfortunately, we cannot in general take the gauge transformation of a conformally transformed real spectral triple if we wish to compute a meaningful bosonic action.

If $D\psi = \lambda\psi$, we have $D^u\psi^u = \tilde{V}DV^{-1}V\psi = \lambda\tilde{V}\psi \neq \lambda\psi^u$ (see Prop. 4.19).

4.5 THE STANDARD MODEL AND BEYOND

Having gone to the effort of constructing a consistent formulation of gauge transformations using twisted real structures, the natural thing is to try to put it to work. As we found in §4.3.3, we require $\nu = \pm\nu^\dagger$ to ensure that fluctuations of the Dirac operator are self-adjoint, and doing so then means we require $\nu = \pm\nu^\dagger$ to ensure the bosonic action is well defined. Unfortunately, the requirement that $\nu = \alpha_1\nu^\dagger = \alpha_2\nu^{-1}$ for $\alpha_1, \alpha_2 \in \{-1, +1\}$ is very restrictive. Certainly, at the very least, it forces the twist to be mild, such that (2.8) simplifies to (2.4). Even so, it is worth exploring the realm of applicability.

The first place to look for something new is the Standard Model, or at least, the finite part of its spectral triple. We would like to keep the spectral data $(A_{\mathrm{SM}}, \mathbb{C}^{96}, D_{\mathrm{SM}}, J_F, \chi_F)$ unchanged, as they all carry quite neat physical interpretations, but it is not difficult to see that this does not leave much room for adding a twist that does anything. For one thing, consider how one should simultaneously satisfy $\nu J_F D_{\mathrm{SM}} = D_{\mathrm{SM}} J_F \nu$ and $J_F D_{\mathrm{SM}} = D_{\mathrm{SM}} J_F$ for a nontrivial ν.

The next place to look, as suggested in §4.1, is the Pati-Salam model of Ref. [11], which takes for its spectral data

$$(A_{\mathrm{LR}}, \mathbb{C}^{96}, D_{\mathrm{SM}}, J_F, \chi_F). \tag{4.32}$$

It is well known that A_{LR} does not respect the first-order condition with respect to D_{SM} and J_F, so there might be room for a twisted first-order condition to hold instead. But first we will briefly summarise the approach of Ref. [11].

4.5.1 *Discarding the first-order condition*

The solution to the problem that the first-order condition is not satisfied which is offered by Ref. [11] is to discard the first-order condition altogether. This can be done, as explained in Ref. [12], by changing the way the Dirac operator fluctuates, so that if one wants to fluctuate a Dirac operator D by a 1-form $\omega = \sum_i \pi(a_i)[D, \pi(b_i)]$, $a_i, b_i \in \mathcal{A}$, rather than use (4.1), one instead takes

$$\begin{aligned} D_\omega = \sum_i \pi(a_i)[D, \pi(b_i)] + \sum_i J\pi(a_i)J^{-1}\big[D, J\pi(b_i)J^{-1}\big] \\ + \sum_{i,j} J\pi(a_i)J^{-1}\pi(a_j)\big[\big[D, \pi(b_j)\big], J\pi(b_i)J^{-1}\big], \end{aligned} \tag{4.33}$$

the substantial difference being the inclusion of the final non-linear 'quadratic' correction term, which of course goes to 0 when one assumes (2.4) holds.

This quadratic term, which we will denote $\omega_{(2)}$, then gauge transforms under the rule

$$\omega^u_{(2)} = J\pi(u)J^{-1}\omega_{(2)}J\pi(u^*)J^{-1} + J\pi(u)J^{-1}\big[\pi(u)[D, \pi(u^*)], J\pi(u^*)J^{-1}\big].$$

Though Ref. [11] demonstrates that the presence of the quadratic term does not lead to anything radical in terms of the particle content of the theory, that is not to say that we should not tread carefully; the sacrifice of the first-order condition means the Dirac operator can no longer be considered a (noncommutative) first-order differential operator. This in particular may be considered too high a price to pay to push the bounds of applicability of the noncommutative approach to gauge theory, and so we investigate as an alternative if the above described *weakening* of the first-order condition provided by a twisted real structure might serve to obtain some alternative noncommutative description of the Pati-Salam model.

4.5.2 *Spectral triples with multitwisted real structure*

The trouble with this approach is that, as we have seen, for the bosonic spectral action to make sense, we need $\nu = \pm\nu^{-1}$ which reduces the twisted first-order condition to the ordinary first-order condition, which we already know does not hold for (4.32). A loophole is provided by the proposed spectral triples with multitwisted real structure of Ref. [32]. To briefly recapitulate, one decomposes the Dirac operator such that

$$D = \sum_{\ell} D_{\ell}, \quad \ell \in \{1, 2, \ldots, N\},$$

and to each component D_ℓ one associates a twist operator ν_ℓ. Therefore, practically speaking, one replaces ν in all of the definitions for a twisted real structure with ν_ℓ and D with D_ℓ, additionally replacing the zeroth-order condition (2.3) with the multitwisted zeroth-order condition (2.16)

$$\left[\pi(a), J\nu_\ell \pi(b)\nu_\ell^{-1}J^{-1}\right] = 0 = \left[\pi(a), J\nu_\ell^{-1}\pi(b)\nu_\ell J^{-1}\right] \tag{4.34}$$

for all ℓ, and replacing the twisted first-order condition (2.8) with the multitwisted first-order condition (2.17)

$$[D_\ell, \pi(a)]J\nu_\ell\pi(b)\nu_\ell^{-1}J^{-1} = J\nu_\ell^{-1}\pi(b)\nu_\ell J^{-1}[D_\ell, \pi(a)] \tag{4.35}$$

for all ℓ. These last two changes are not trivial (even when $N = 1$) because we also now no longer require that $\nu_\ell \pi(A)\nu_\ell^{-1} \simeq A$, and instead only require that conjugation by ν_ℓ is an automorphism of $\mathcal{B}(\mathcal{H})$.

It is clear that one cannot start from the multitwisted perspective and proceed via the path we have carved out above. For one thing, the formulation of Morita equivalences provided in §4.2 clearly will not carry over directly, as this would require some abstract decomposition of connections which would be hard to account for, amongst other difficulties. However, for our purposes, Morita equivalence is only necessary for developing a framework of gauge transformations for spectral triples as given in §§4.3–4.4, which can be expressed wholly in terms of operators. Once these definitions have been laid out, they can be extended in an entirely analogous fashion to that of extending spectral triples with twisted real structures to spectral triples with multitwisted real structures described above, *i.e.* everywhere replacing $D \mapsto D_\ell$ and $\nu \mapsto \nu_\ell$.

While it is reasonably straightforward to write down new gauge transformations for the multitwist, it should be remarked upon that this process is not always trivial. For example, suppose

$$\Omega^1_D(\mathcal{A}) \ni \omega = \pi(a)[D, \pi(b)] = \pi(a)\left[\sum_\ell D_\ell, \pi(b)\right] = \sum_\ell \pi(a)[D_\ell, \pi(b)].$$

The only way to make sense of an equivalent version of the map $\omega \mapsto \omega^{\odot}$ is by defining

$$\omega^{\odot} := \sum_\ell \nu_\ell J\pi(a)[D_\ell, \pi(b)]J^{-1}\nu_\ell,$$

but unlike ω, this $\omega^{\odot}$ cannot be derived from any object built from the complete Dirac operator D; for example, the fluctuation $D \mapsto$

$D' = \sum_\ell (D_\ell + a[D_\ell, b] \pm \sum_k \nu_k J a[D_\ell, b] J^{-1} \nu_k)$ would not satisfy $\nu_\ell J D'_\ell = \pm D'_\ell J \nu_\ell$ except when $k = \ell$.

Thus there are two ways in which the spectral Pati-Salam model might be (re)constructed using multitwisted real structures: one is that even for a single twist ($N = 1$), we have a broader selection of twists to work with than earlier assumed, and of course the other is the possibility to try to use multiple twists ($N > 1$).

4.5.3 *The Pati-Salam case (shown with a toy model)*

The matrices involved in the (finite part of the) spectral Pati-Salam case are quite large and unwieldy, and thus difficult to express written out in full. However, many of the issues that come up when working with them also arise in the simpler toy model given in Ref. [12], and so for demonstrative purposes we will largely present that case instead. We take for the algebra $A_{\text{toy}} = \mathbb{C}_L \oplus \mathbb{C}_R \oplus M_2(\mathbb{C})$, represented as matrices acting on the Hilbert space $\mathbb{C}^8$ by

$$\pi \colon (\lambda_L, \lambda_R, M) \mapsto (\operatorname{diag}(\lambda_L, \lambda_R) \otimes 1_2) \oplus (1_2 \otimes M).$$

The real structure is then given by $J = \left(\begin{smallmatrix} 0 & 1_4 \\ 1_4 & 0 \end{smallmatrix}\right) \circ *$. Next we consider the Dirac operator D, which is given by

$$D = \begin{pmatrix} S & T^\dagger \\ T & S^* \end{pmatrix}$$

where $S = \left(\begin{smallmatrix} 0 & k_x \\ k_x^* & 0 \end{smallmatrix}\right) \otimes 1_2$ and $T = \operatorname{diag}(k_y, 0, 0, 0)$. With respect to this Dirac operator, the ordinary first-order condition (2.4) is only satisfied for the 'symmetry-broken' subalgebra $\mathbb{C}_L \oplus \mathbb{C}_R \oplus \mathbb{C}_0 \subset A_{\text{toy}}$. First we will briefly investigate if it is possible to satisfy the multitwisted first-order condition (2.17) for the unbroken algebra A_{toy} instead.

We start by considering matrices which, for the sake of notational ease, we call $B_\ell^+ := J\nu_\ell \pi(b) \nu_\ell^{-1} J^{-1}$ and $B_\ell^- := J\nu_\ell^{-1} \pi(b) \nu_\ell J^{-1}$ for $b \in A_{\text{toy}}$, all of which must be elements of $\pi(A_{\text{toy}})'$ in order to satisfy (2.16). As such, $B_\ell^\pm$ must take the form $m \oplus n \oplus \operatorname{diag}(\mu_1, \mu_1, \mu_2, \mu_2)$ for $m, n \in M_2(\mathbb{C})$, $\mu_1, \mu_2 \in \mathbb{C}$. The question then is whether we can identify some operator(s) ν_ℓ which would allow us to obtain $B_\ell^\pm$ from a given $b \in A_{\text{toy}}$. Assuming we have $N = 1$ twists, it is not difficult to compute the 1-forms $[D, \pi(a)]$, $a \in A_{\text{toy}}$, and it is with respect to such 1-forms we can try to impose the usual twisted first-order condition (2.8), *i.e.* we demand that $[D, \pi(a)]B^+ = B^-[D, \pi(a)]$. Doing so further restricts $B^\pm$ to be of the form

$$\begin{aligned} B^+ &= \begin{pmatrix} m_{11}^+ & 0 \\ m_{21}^+ & m_{22}^+ \end{pmatrix} \oplus \begin{pmatrix} \mu_1^+ & n_{12}^+ \\ 0 & n_{22}^+ \end{pmatrix} \oplus \operatorname{diag}(\mu_1^+, \mu_1^+, \mu_2^+, \mu_2^+), \\ B^- &= \begin{pmatrix} \mu_1^+ & n_{12}^+ \\ 0 & n_{22}^+ \end{pmatrix} \oplus \begin{pmatrix} m_{11}^+ & 0 \\ m_{21}^+ & m_{22}^+ \end{pmatrix} \oplus \operatorname{diag}(m_{11}^+, m_{11}^+, \mu_2^-, \mu_2^-), \end{aligned} \tag{4.36}$$

where we have tried to express everything in terms of elements of B^+.

Even relaxing the requirement that $\nu^2 = 1$ (in which case $B^- = B^+$), reading off (4.36) there is a strong suggestion that ν^2 be given by a pair of blockwise flips (on the first and last pairs of 2×2 blocks respectively), but in that case one must still make the identifications $\mu_2^+ = m_{11}^+$ and $\mu_2^- = \mu_1^+$, which necessarily breaks the algebra A_{toy} down to a subalgebra. Not imposing those identifications, it is not at all clear what (or if any) ν can be found to relate B^- to B^+. When $\nu^2 = 1$ however, it is immediately clear that A_{toy} breaks to $\mathbb{C}^3$.

The ultimate reason for this breaking of the algebra is T being nonzero; thus, if multitwists are to be applicable, it would make sense to decompose D into $D_1 = S \oplus S^*$ and either

$$D_2 = \begin{pmatrix} 0 & T^\dagger \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad D_3 = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix},$$

or $D_{23} := D_2 + D_3$. Using the same method as before we find that, as expected, the $\ell = 1$ case is well behaved, but the same cannot be said for the other components.

Proposition 4.27. *The requirement that* $(A_{\text{toy}}, \mathbb{C}^8, D_1 + D_{23}, (J, \{\nu_1, \nu_{23}\}))$, *with twist operators satisfying* $\nu_1^2 = \nu_{23}^2 = 1$, *be a multitwisted spectral triple breaks the algebra* A_{toy} *to* $\mathbb{C}^3$.

Proof. Since we are taking $\nu_1^2 = \nu_{23}^2 = 1$, all twisted commutators become ordinary commutators. This means we can directly apply [22, Prop. 4.1] to each component of the Dirac operator (with the associated twist), but instead taking the map $(\bullet)^\circ$ of Ref. [22] to mean $\nu_\ell J\pi(\bullet)J^{-1}\nu_\ell$ for a given $\ell \in \{1, 23\}$ (which takes the algebra to its commutant thanks to the twisted zeroth-order condition (2.16)).

Thus, focusing on the second component, we have that D_{23} satisfies the twisted first-order condition if and only if it decomposes into

$$D_{23} = D_{23,0} + D_{23,1}$$

for $D_{23,0} \in (\nu_{23}J\pi(A_{\text{toy}})J^{-1}\nu_{23})'$ and $D_{23,1} \in \pi(A_{\text{toy}})'$. However, we know the shape of D_{23}, and so we know that no nonzero part of it lies within $\pi(A_{\text{toy}})'$, and so we must have $D_{23,1} = 0$. This means we must have

$$\left[D_{23}, \nu_{23}J\pi(a)J^{-1}\nu_{23}\right] = 0 \tag{4.37}$$

for all $a \in A_{\text{toy}}$.

Now, by the definition of a spectral triple with multitwisted real structure we should have $\nu_{23}JD_{23} = \varepsilon' D_{23}J\nu_{23}$. However, we know that $JD_{23} = D_{23}J$, which implies that $D_{23} = \pm\nu_{23}D_{23}\nu_{23}$. Substituting this into (4.37) gives

$$\nu_{23}\left[D_{23}, J\pi(a)J^{-1}\right]\nu_{23} = 0,$$

but this is only true when a lies within the symmetry-broken subalgebra $\mathbb{C}^3$. □

Indeed, the above argument carries over in exactly the same manner for the full Pati-Salam case, making the appropriate replacements, *i.e.* replacing D_{23} by the block off-diagonal part of D_{SM}, A_{toy} by A_{LR} (the symmetry-broken subalgebra of A_{LR} being A_{SM}), and the other data by their higher-dimensional equivalents.

While this doesn't in principle rule out the 2-twisted case (since the decomposition of the Dirac operator is not unique), it does eliminate the most promising candidate. For the 3-twisted case with the decomposition we described before, Prop. 4.27 carries over with only minor modifications.

Proposition 4.28. *The requirement that* $(A_{\mathrm{toy}}, \mathbb{C}^8, D_1{+}D_2{+}D_3, J, \{\nu_1, \nu_2, \nu_3\})$, *with twist operators satisfying* $\nu_1^2 = \nu_2^2 = \nu_3^2 = 1$, *be a spectral triple with multitwisted real structure breaks the algebra* A_{toy} *to* $\mathbb{C}^3$.

Proof. The preliminaries carry over exactly as in Prop. 4.27. Now, we focus on D_2 and D_3, beginning with D_2. By [22, Prop. 4.1], we have that D_2 satisfies the twisted first-order condition if and only if it decomposes into

$$D_2 = D_{2,0} + D_{2,1}$$

for $D_{2,0} \in (\nu_2 J\pi(A_{\mathrm{toy}})J^{-1}\nu_2)'$ and $D_{2,1} \in \pi(A_{\mathrm{toy}})'$. However, as before we know that $D_{2,1} = 0$. This means we must have

$$\left[D_2, \nu_2 J\pi(a)J^{-1}\nu_2\right] = 0 \tag{4.38}$$

for all $a \in A_{\mathrm{toy}}$.

Now, by the definition of a multitwisted real structure, we should have $\nu_2 J D_2 = \varepsilon' D_2 J \nu_2$. However, we know that $JD_2J^{-1} = D_3$, which implies that $D_2 = \pm\nu_2 D_3 \nu_2$. Substituting this into (4.38) gives

$$\nu_2\left[D_3, J\pi(a)J^{-1}\right]\nu_2 = 0. \tag{4.39}$$

Going through the same procedure for D_3 yields

$$\nu_3\left[D_2, J\pi(a)J^{-1}\right]\nu_3 = 0, \tag{4.40}$$

and the pair of equations (4.39) and (4.40) can only be satisfied when a lies within the symmetry-broken subalgebra $\mathbb{C}^3$ as before. □

One further point which is worth remarking upon is that the Dirac operator for this toy example is not simpler than the Standard Model/Pati-Salam case only due to the lower dimensionality. With respect to a given choice of basis, one has $D_{\mathrm{SM}} = \begin{pmatrix} S_{\mathrm{SM}} & T_{\mathrm{SM}}^\dagger \\ T_{\mathrm{SM}} & S_{\mathrm{SM}}^* \end{pmatrix}$ where

$$S_{\mathrm{SM}} = \begin{pmatrix} 0 & 0 & k_\nu^\dagger & 0 \\ 0 & 0 & 0 & k_e^\dagger \\ k_\nu & 0 & 0 & 0 \\ 0 & k_e & 0 & 0 \end{pmatrix} \oplus \bigoplus_{i=1}^{3} \begin{pmatrix} 0 & 0 & k_u^\dagger & 0 \\ 0 & 0 & 0 & k_d^\dagger \\ k_u & 0 & 0 & 0 \\ 0 & k_d & 0 & 0 \end{pmatrix}, \; T_{\mathrm{SM}} = \begin{pmatrix} k_{\nu_R} & 0_{1\times 15} \\ 0_{15\times 1} & 0_{15\times 15} \end{pmatrix},$$

Note that the matrices k_i $(i = \nu, e, u, d, \nu_R)$ are not arbitrary elements of $M_3(\mathbb{C})$, but are subject to further constraints which we will not go into here. See Ref. [20] for more.

and all entries are in $M_3(\mathbb{C})$. Taking $k_\nu = k_u$ and $k_e = k_d$ is called *quark-lepton coupling unification*, and this simplifies the mathematics significantly. For example, if one takes $T_{\mathrm{SM}} = 0$ and assumes quark-lepton coupling unification, then it is not particularly difficult to find twists (not dissimilar to the toy model case). However, even taking $T_{\mathrm{SM}} = 0$, without quark-lepton coupling unification this task becomes much more difficult. This is unfortunate because the model is defined at the gauge coupling unification scale, and so making such simplifying assumptions is likely to impose strong constraints on the physics up to that scale, and so ought to be avoided unless absolutely necessary.

4.5.4 *Other issues and future directions*

It is worth mentioning here that even apart from the above discussion, there are other issues worth mentioning in this context. Even if we had found twist(s) which recovered some twisted first-order condition, such twists would likely not be of much physical interest. The reason comes from (4.31) and the fact that D_{SM} is the fermionic mass matrix. Ordinarily, the Dirac/Majorana mass terms in the action come from the fermionic spectral action of the unfluctuated Dirac operator

$$\langle J_F \psi, D_{\mathrm{SM}} \psi \rangle,$$

but now in the multitwisted case it seems that this should be replaced by

$$\sum_\ell \langle J_F \nu_\ell \psi, D_\ell \psi \rangle,$$

where here $\sum_\ell D_\ell = D_{\mathrm{SM}}$ specifically.

If we would like to maintain the physical relevance of the model, it would likely be necessary to instead use some

$$J'_\ell := \nu_\ell J_F \text{ or } D'_\ell := \alpha_1 \nu_\ell D_\ell$$

Note that Prop. 2.12 does not hold in the multitwisted formalism, but it is worth keeping track of signs nevertheless.

instead of J_F or D_ℓ respectively (where $\nu_\ell = \alpha_1 \nu_\ell^\dagger$ for all ℓ). These both have problems though.

The choice of J'_ℓ seems initially preferable to D'_ℓ, as $\nu_\ell J'_\ell D_\ell = D_\ell J'_\ell \nu_\ell$ automatically gives $J_F D_{\mathrm{SM}} = D_{\mathrm{SM}} J_F$ provided that $\nu_\ell = \nu_\ell^{-1}$ (for all ℓ). However, this neatness is telling, and indeed one finds that using J'_ℓ reduces the multitwisted first-order condition (2.17) to the first-order condition (2.4) with J_F in this case, and we already know that (2.4) does not hold for A_{LR}.

If we use D'_ℓ instead, we run into the different issue that we already require (by definition) that $\sum_\ell D_\ell = D_{\mathrm{SM}}$, but in order to have the correct action we would also need $\sum_\ell D'_\ell = \alpha_1 \sum_\ell \nu_\ell D_\ell = D_{\mathrm{SM}}$, which needless to say also makes it difficult to have nontrivial twists.

The above investigation seems to leave little space for the application of twisted real structures to the spectral formulation of the left-right symmetric extension to the Standard Model, as they simply result in a reduction to the Standard Model. Is there any more that could be done? We suggest two possible avenues.

One approach could be to try to marry the (multi)twisted real structure formalism to the twisted spectral triple approach to the Standard Model and its extensions, which is an active area of research (see *e.g.* Refs. [35, 51] and subsequent papers). While the cited papers focus on twisting the (doubled) commutative part of the spectral triple, it might be worthwhile in this possible 'hybrid twisted' setting to investigate twists on the finite part, or even on both. We leave this long-term endeavour for future investigation.

Another idea which shifts perspective away from the finite part of the spectral triple and towards the commutative part instead is to attempt to follow more closely in the direction of Ref. [33] and investigate if twisted real structures could have any applications to Lorentzian spectral triples, for example, by using the 'untwisting' procedure described in Ref. [8]. This is a line of inquiry which we will follow in the next chapter.

5 KREIN STRUCTURE FROM TWISTED REAL STRUCTURES

5.1 INTRODUCTION

Pseudo-Riemannian manifolds, also known as semi-Riemannian manifolds, are smooth manifolds whose metric is nondegenerate but not necessarily positive-definite, as is the case for Riemannian manifolds. A useful concept in this context is *metric signature*, the number of positive and negative eigenvalues of the metric tensor with respect to an orthonormal basis. For an n-dimensional manifold, the most interesting signatures are *Euclidean* $(n, 0) \equiv (0, n)$ (when the manifold is Riemannian) and *Lorentzian* $(1, n-1) \equiv (n-1, 1)$, especially in physical contexts. As such, we will focus exclusively on these two cases, and hence use "pseudo-Riemannian" and "Lorentzian" (and "Riemannian" and "Euclidean") interchangeably, with the understanding that this is just a synecdoche.

The metric signatures (p, q) and (q, p) are not always strictly equivalent, but they typically are for practical purposes and so the choice comes down to convention.

In quantum mechanics and quantum field theory, one often finds that the equations of motion are much more tractable (or even only well-defined) in Euclidean space compared to Minkowski space ($\mathbb{R}^n$, usually $n = 4$, equipped with a metric of Lorentzian signature). In order to solve the equations of motion, then, one analytically continues the time coordinate to 'imaginary time' $t \mapsto it_{\Im}$ and solves the equations in the Euclidean space $(t_{\Im}, x^j)$ before substituting real time back in to return to the Minkowski space setting. This procedure is known as *Wick rotation*.

It was explained in Ch. 4 that with the framework of spectral triples, a physically relevant class of gauge theories (which includes the Standard Model) is expressed very naturally as a kind of generalised Kaluza-Klein theory, where the (classical) gauge theory on curved space is expressed as a gravitational theory on the product of a Riemannian manifold and a discrete space in a certain sense [21]. As described, the mathematical object for this product is known as an almost-commutative spectral triple.

This provides a strong motivation for broadening the noncommutative geometric framework to include not merely Riemannian but pseudo-Riemannian geometry, as since the advent of relativity, separate notions of (Euclidean) space and time have been superseded by spacetimes, modelled as pseudo-Riemannian manifolds of Lorentzian signature which are essential for describing gravity. However, there are a number of barriers to making this leap, both technical and philosophical – for example, should one treat Euclidean signature as fundamental and try to obtain Lorentzian equations of motion emergently from spectral triples, or instead attempt to generalise the notion of a spectral triple to describe pseudo-Riemannian geometries?

A good sketch of the many different approaches that have been made towards pseudo-Riemannian noncommutative geometry can be found in the references of Ref. [64].

We will not give a comprehensive overview of the literature, but there have been a number of attempts made at the latter approach, beginning with Ref. [58] which proposed a systematic approach to Lorentzian spectral geometry based on replacing Hilbert spaces with Krein spaces, that is, inner product spaces with nondegenerate (but not necessarily positive-definite) inner product. Other attempts have since been made, with varying degrees of difference to Connes' spectral triples, but the general schema based on Krein spaces remains the same.

Since no proposal for pseudo-Riemannian spectral geometry has reached universal consensus, a number of attempts have been made to link Krein space-based approaches to the mathematically sound footing of spectral triples by performing a kind of algebraic Wick rotation, typically by starting with the pseudo-Riemannian generalisation although occasionally going the other way (sometimes called 'reverse Wick rotation', although more commonly just referred to as Wick rotation for brevity, which we will also do here). This has been attempted at the very abstract level of KK-theory [65, 66], the very concrete level of vielbeins [25], and at the level of operators in between.

Vielbeins *are more commonly known in geometry as 'solder forms'. In the $n = 4$ case, they are usually called* vierbeins *or* tetrads.

This context is is why the results of §4.4 are especially interesting when considering pseudo-Riemannian geometry; if one considers an almost-commutative spectral triple equipped with a twisted real structure whose twist operator is self-adjoint and unitary up to sign, the requirement that its Hilbert space's inner product is suitable for giving the fermionic action for a gauge theory is the Krein-self-adjointness of the Dirac operator, and if the Dirac operator is self-adjoint, then the inner product needs to be changed into a Krein space inner product. Since the initial data are Euclidean by assumption, there is a hint that it may be possible to obtain some notion of Wick rotation to a pseudo-Riemannian spectral triple of some description automatically.

Such a possibility is not outlandish to consider; one of the original motivations behind twisted real structures is in being a way to implement conformal transformations of the metric in a way that could be extended to the noncommutative case [7]. In that context, one starts with a spectral triple $(A, \mathcal{H}, D)$ equipped with a real structure J, and defines its conformal transformation to be the spectral triple

$$\left(A, \mathcal{H}, JkJ^{-1}DJkJ^{-1}, (J, k^{-1}JkJ^{-1})\right)$$

for $k \in A$ positive and invertible, equipped with the *twisted* real structure $(J, k^{-1}JkJ^{-1})$. Thus, twisting the real structure allows one to translate into the noncommutative setting a transformation which corresponds to a particular modification of the metric in the commutative setting. Wick rotations, which change the metric signature, are not transformations of the metric in the same way that conformal transformations are, but given they both modify the metric in some way, it is not unreasonable to consider that twisted real structures may likewise provide a pathway to implementing noncommutative Wick rotations.

We hypothesise that the right starting point is to consider a twist operator based on a fundamental symmetry, a self-adjoint and unitary operator which can be used to turn a Hilbert space into a Krein space; the mathematical preliminaries will be introduced in §5.2, and an initial discussion will follow in §5.3. In §5.4 we take a different tack and attempt to obtain a twisted real structure from the minimally-twisted spectral triple before reverting in §5.5 to attempting to work from first principles.

5.2 DEFINITIONS

In this section we collect the various definitions and notations we will need for this chapter.

Where there is a distinction between the Euclidean and Lorentzian version of certain objects, we will distinguish them by 'E' and 'M' subscripts respectively. The choice of 'M' (from 'Minkowski') for the Lorentzian case is to match the literature, where an 'L' subscript could be confused with *e.g.* 'left parity'. We will use the convention that the zeroth gamma matrix γ^0 is identical in the Euclidean and Lorentzian cases and so will not decorate it with a subscript.

5.2.1 *Spectral triples with multitwisted real structure*

In the commutative case, the algebra of coordinate functions being unital implies that the manifold given by the spectral triple is compact. This should generally be avoided when considering pseudo-Riemannian geometries on physical grounds to avoid the presence of closed timelike curves. However, as the work presented here is only a preliminary investigation focussing primarily on algebraic aspects, we overlook this complication, and work with unital algebras to keep the presentation as tractable as possible.

A major point of departure from the previous chapters of the thesis is that we will no longer assume that (multi)twisted real structures will be regular. Rather, we will require a weaker 'signed' version of regularity which we here introduce,

We express signed regularity in the language of multitwisted real structures since when there is only a single twist the relevant equations are equally suitable for twisted real structures.

$$\nu_k J \nu_k = \varepsilon_k''' J, \quad \varepsilon_k''' = \pm 1. \tag{5.1}$$

In this case, (2.15) implies that

$$D_k = \varepsilon' \varepsilon_k''' \nu_k J D_k J^{-1} \nu_k \tag{5.2}$$

which means that fluctuations of the Dirac operator will take the form

$$D \mapsto D + \omega + \varepsilon' \sum_{k=1}^{N} \varepsilon_k''' \nu_k J \omega_k J^{-1} \nu_k \tag{5.3}$$

for $\omega_k = \sum_j a_j [D_k, b_j]$ and $\omega = \sum_{k \in I} \omega_k$. Note that, when $\nu_k = \alpha_k \nu_k^{-1}$, $\alpha_k = \pm 1$, it may be more convenient to work with the equivalent relation $\nu_k J = \tilde{\varepsilon}_k''' J \nu_k$ where $\tilde{\varepsilon}_k''' := \alpha_k \varepsilon_k'''$.

As we will focus on twist operators satisfying $\nu_k = \nu_k^{-1} = \nu_k^\dagger$ (up to sign), there is scope to consider other generalisations, such as (for example) $\nu_k J \nu_k^\dagger = \pm J$, which would remain compatible with the literature, but we leave this possibility for future investigation.

5.2.2 *Krein spaces*

For more details, the standard text on Krein spaces is Ref. [6].

Definition 5.1. Let $\mathcal{K}$ be a complex inner product space with nondegenerate inner product $(\bullet, \bullet)$. Suppose $\mathcal{K}$ admits a decomposition into orthogonal subspaces $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ such that $(\bullet, \bullet)$ is positive-definite on $\mathcal{K}_+$ and negative-definite on $\mathcal{K}_-$. If $\mathcal{K}_+$ and $\mathcal{K}_-$ are complete with respect to the respective norms coming from the inner product on each subspace, then $\mathcal{K}$ is called a *Krein space*. ⋄

Krein spaces are closely related to Hilbert spaces. For one, any Krein space $\mathcal{K}$ with $\mathcal{K}_- = 0$ is a Hilbert space, and so Krein spaces can be seen as a generalisation of Hilbert spaces. However, the connection between the two objects is better elucidated using the fundamental symmetry.

Definition 5.2. Let $\mathcal{K}$ be a Krein space decomposing into $\mathcal{K}_+ \oplus \mathcal{K}_-$. For P_+ and P_- projectors onto the respective subspaces, we call

$$\beta = P_+ - P_-$$

the *fundamental symmetry* associated to the given decomposition. ⋄

The fundamental symmetry is involutive, symmetric, and isometric with respect to the inner product $(\bullet, \bullet)$ on $\mathcal{K}$. Because of this, the fundamental symmetry can be used to define a new inner product $(\bullet, \bullet)_\beta := (\bullet, \beta \bullet)$, and when equipped with this new inner product $\mathcal{K}$ becomes a Hilbert space. Furthermore, if $(\bullet)^+$ denotes the Krein space adjoint associated to $(\bullet, \bullet)$, then the Hilbert space adjoint associated to $(\bullet, \bullet)_\beta$ can be expressed as $(\bullet)^\dagger = \beta (\bullet)^+ \beta$.

The description given above is the usual presentation of Krein spaces, but it is also possible to 'go in the other direction' (see *e.g.* Ref. [33]). Consider a Hilbert space $\mathcal{H}$ with inner product $\langle \bullet, \bullet \rangle$. If there exists a selfadjoint unitary operator β on $\mathcal{H}$, it can be used to decompose $\mathcal{H}$ into positive and negative eigenspaces $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. On these eigenspaces, the inner product

$$\langle \bullet, \bullet \rangle_\beta := \langle \bullet, \beta \bullet \rangle$$

is positive-definite and negative-definite on each respective eigenspace, and so equipping $\mathcal{H}$ with the inner product $\langle\bullet,\bullet\rangle_\beta$ turns it into a Krein space. We likewise denote the adjoint associated to $\langle\bullet,\bullet\rangle_\beta$ by $(\bullet)^+ = \beta(\bullet)^\dagger\beta$ for $(\bullet)^\dagger$ the adjoint from $\langle\bullet,\bullet\rangle$.

Example 5.3. One of the key motivating examples for Krein spaces comes from the inner product of spinors. On 4-dimensional Euclidean space, one has the inner product

$$\langle\psi,\psi'\rangle = \int \mathrm{d}^4x\, \psi^\dagger\psi',$$

but this inner product is not invariant under Lorentz transformations and so is unsuitable for describing spinors on 3 + 1-dimensional Minkowski spacetime. The 'fix' is to instead use

$$\int \mathrm{d}^4x\, \bar{\psi}\psi' =: \int \mathrm{d}^4x\, \psi^\dagger\gamma^0\psi' = \left\langle\psi,\gamma^0\psi'\right\rangle,$$

where the bar $\bar{\psi} = \psi^\dagger\gamma^0$ denotes the Dirac adjoint and γ^0 is the zeroth gamma matrix. It is immediate to see that γ^0 plays the role of the fundamental symmetry β described above, and does indeed meet all the requirements. ⋄

5.3 KREIN STRUCTURES FROM TWISTED REAL STRUCTURES

5.3.1 *Twisted real structures and gauge theories*

The primary motivation for this work comes from the results of the previous chapter of the thesis, and so as such we will briefly recall them here. From Def. 2.6, condition (2.15) combined with signed regularity demands that fluctuations of the Dirac operator take the form of (5.3). In the context of gauge theory in spectral geometry, when the spectral triple is almost-commutative and real, fluctuations of the Dirac operator are generated by 'gauge transformations', that is, via an action of $\mathcal{U}(A \otimes A^{\mathrm{op}})$ on D. Since the representation of A^{op} is constructed using the real structure, generalising to the setting of twisted real structures modifies this gauge transformation procedure in such a way as to incorporate the twist operator(s).

To be more precise, the gauge transformation of the Dirac operator D by $u \in \mathcal{U}(A)$ for the multitwisted real structure $(J,\{\nu_k\})$ is given by

$$D \mapsto D^u := \sum_{k\in I} uJ\nu_k^{-1}u\nu_kJ^{-1}D_kJ\nu_ku^\dagger\nu_k^{-1}J^{-1}u^\dagger.$$

For the case of a single twist, the self-adjointness of D^u can be ensured by requiring that the twist operator is also self-adjoint (up to sign). Following the physical requirement that D has the same spectrum as

D^u then imposes that the twist operator be unitary (up to sign), *i.e.* the twist operator must satisfy

$$\nu = \alpha\nu^{-1} = \alpha'\nu^{\dagger}$$

where $\alpha, \alpha' = \pm 1$ and the two signs may be distinct.

Continuing on from this, it was found that the quadratic form which gives the fermionic action must be modified when one has a twist to incorporate the twist operator, specifically taking the form

$$\mathfrak{A}(\psi, \phi) := \left\langle \nu J\psi, D\phi \right\rangle,$$

where $\langle \bullet, \bullet \rangle$ is the Hilbert space inner product. When $\alpha = \alpha' = +1$, the twist operator can be used to define an alternative inner product

$$\langle \bullet, \bullet \rangle_{\nu} = \langle \nu \bullet, \bullet \rangle$$

such that the quadratic form giving the fermionic action is equivalently given by

$$\mathfrak{A}(\psi, \phi) = \left\langle J\psi, D\phi \right\rangle_{\nu}. \tag{5.4}$$

One can see from the discussion in §5.2.2 that the Hilbert space $\mathcal{H}$ equipped with the inner product $\langle \bullet, \bullet \rangle_{\nu}$ is a Krein space, with the twist operator taking on the role of fundamental symmetry.

5.3.2 *Wick rotation*

In light of Ex. 5.3, there is a tantalising interpretation of the above result, which is that one can take an ordinary Riemannian (real) spectral triple, twist the real structure, and get a pseudo-Riemannian spectral triple as a result. Thus the twisting of the real structure would act like a 'noncommutative' version of Wick rotation.

In this chapter of the thesis, we use the term 'Riemannian' to contrast 'pseudo-Riemannian' qua *relating to signature. This should not be confused with the use of 'Riemannian spectral triples' to refer to spectral triples of Riemannian manifolds as opposed to spin manifolds, as was done in Ch. 3, and which is common in the literature. Of course, in that case the signature is generally Euclidean regardless.*

We start with the canonical spectral triple for a closed 4-dimensional spin manifold M:

$$\left(C^{\infty}(M), L^2(M, S), i\not{\nabla}_E, \gamma_E^5\right), \tag{5.5}$$

where $C^{\infty}(M)$ is the $*$-algebra of complex smooth functions on M, $L^2(M, S)$ is the Hilbert space of square-integrable spinors on M, and $i\not{\nabla}_E$ is the Dirac operator associated to the spinor bundle S. The Hilbert space $L^2(M, S)$ is equipped with the canonical inner product denoted by $\langle \bullet, \bullet \rangle$. In 4 dimensions this spectral triple is even and its grading is given by γ_E^5, the chirality operator, so-called because the (Euclidean) left/right-handed chirality projection operators are given by

$$P_L := \frac{1}{2}(1 - \gamma_E^5) \quad \text{and} \quad P_R := \frac{1}{2}(1 + \gamma_E^5).$$

The grading is explicitly given in terms of the gamma matrices by their ordered product

$$\gamma_E^5 = \gamma^0 \gamma_E^1 \gamma_E^2 \gamma_E^3.$$

In Euclidean signature, the real structure is given by the charge conjugation operator

$$C_E = i\gamma^0\gamma_E^2 \circ *. \tag{5.6}$$

For the most naïve attempt at using a twisted real structure to implement a Wick rotation, we take the charge conjugation unchanged as the reality operator (as happens in conformal transformations) and γ^0 as the twist operator. This clearly cannot work since $\gamma^0 C_E i\not{\nabla}_E \neq \pm i\not{\nabla}_E C_E \gamma^0$. The simplest 'fix' is to take the reality operator $J = i\gamma_E^2 \circ *$ which will (by construction) satisfy

$$\gamma^0 J i\not{\nabla}_E = -i\not{\nabla}_E J \gamma^0.$$

This does not really accomplish anything, though, since $\gamma^0 J = C_E$ and so $\mathfrak{A}(\psi, \phi)$ only gives the usual Euclidean signature fermionic action; clearly implementing a true Wick rotation requires something more sophisticated.

It is apparent that there is only so much that can be achieved by taking the canonical spectral triple and modifying the real structure alone, but this is not surprising. In the conformal case, the real structure is twisted but the Dirac operator is also modified, and so we expect something else will have to be changed as well. We will pursue two approaches to finding a working method: using an 'untwisting' procedure in §5.4 and from first principles in §5.5.

5.4 UNTWISTING THE MINIMALLY TWISTED SPECTRAL TRIPLE

Putting aside twisted real structures for a moment, a different approach to modifying the canonical spectral triple in the context of twisted spectral triples is given in Ref. [33] using a 'minimal twist'. We describe this minimal twist as it applies to our interests below.

For the canonical spectral triple (5.5), the algebra of smooth functions $C^\infty(M)$, represented by π_0 on the space of square integrable spinors $L^2(M, S)$, acts by pointwise multiplication of functions $(\pi_0(f)\psi)(x) = f(x)\psi(x)$. To implement the minimal twist, one first doubles the algebra to $C^\infty(M) \otimes \mathbb{C}^2 \simeq C^\infty(M) \oplus C^\infty(M)$, which one then represents on $L^2(M, S)$ by

$$\pi(f, g) = \pi_0(f)P_R + \pi_0(g)P_L$$

for $f, g \in C^\infty(M)$. For the twist automorphism, one takes the flip automorphism

$$\sigma(f, g) = (g, f)$$

for all $(f, g) \in C^\infty(M) \oplus C^\infty(M)$. With these choices, the spectral data

$$\left(C^\infty(M) \otimes \mathbb{C}^2, L^2(M, S), i\not{\nabla}_E, C_E, \gamma_E^5\right)_\sigma$$

give an even real twisted spectral triple (per Def. 2.8). Intriguingly, there is evidence to suggest [33, 34, 52] that the canonical spectral triple, when twisted by the minimal twist, leads to Lorentzian equations of motion, which is what motivates our consideration of it here.

The approach of doubling the algebra cannot be translated directly to the setting of twisted real structures, because $\left[i\not{\nabla}_E, \pi(f,g)\right]$ for $(f,g) \in C^\infty(M) \oplus C^\infty(M)$ is not bounded and Ch. 4 deals strictly with spectral triples which are not twisted (in the sense of Def. 2.8). However, in Ref. [8] the authors present a method of 'untwisting' real twisted spectral triples, essentially a procedure for turning a real twisted spectral triple into a spectral triple with twisted real structure. We present the part of the theorem in question which is relevant to this thesis as follows.

Theorem 5.4 ([8]). *Let A be a $*$-algebra and let $\pi\colon A \to \mathcal{B}(\mathcal{H})$ be a $*$-representation of A on a Hilbert space $\mathcal{H}$. Let $J\colon \mathcal{H} \to \mathcal{H}$ be an anti-linear isometry such that $J^2 = \pm 1$ and such that the zeroth-order condition is satisfied. Let ρ be an algebra automorphism satisfying (2.20) and let ϱ be a bounded operator on $\mathcal{H}$ with bounded inverse such that ϱ is a unitary operator implementing ρ via*

$$\pi(\rho(a)) = \varrho^{-2}\pi(a)\varrho^2$$

for all $a \in A$.

Furthermore, let

$$\pi_\varrho\colon A \to \mathcal{B}(\mathcal{H}), \quad a \mapsto \varrho^{-1}\pi(a)\varrho$$

be the induced representation of A and further assume that $\varrho J \varrho = J$. For an operator D on $\mathcal{H}$, set

$$D_\varrho = \varrho D \varrho.$$

Then $((A, \pi_\varrho, \mathcal{H}), D_\varrho, (J, \varrho^2))$ is a spectral triple with ϱ^2-twisted real structure per Def. 2.6 if and only if $((A, \pi, \mathcal{H}), D, J)_\rho$ is a real ρ-twisted spectral triple per Def. 2.8.

Given the twisted real structure given by the theorem has twist operator ϱ^2, the requirement that $\varrho J \varrho = J$ is quite strict, even stricter than regularity. Indeed, as mentioned in §5.2.1, we are interested in weakening regularity to signed regularity (5.2), and so we must replace this assumption with something weaker.

Proposition 5.5. *Let A be a $*$-algebra and $\pi\colon A \to \mathcal{B}(\mathcal{H})$ be a $*$-representation of A on a Hilbert space $\mathcal{H}$. Let $J, J_\varrho\colon \mathcal{H} \to \mathcal{H}$ be a pair of antilinear isometries with $J^2 = \pm 1$ and J implementing the zeroth-order condition (2.3). Let ρ be an algebra automorphism satisfying regularity (2.20) and let ϱ be a bounded operator on $\mathcal{H}$ with bounded inverse such that ϱ is a unitary operator implementing ρ via*

$$\pi(\rho(a)) = \varrho^{-2}\pi(a)\varrho^2$$

for all $a \in A$.

Furthermore, let

$$\pi_\varrho \colon A \longrightarrow \mathcal{B}(\mathcal{H}), \quad a \mapsto \varrho^{-1}\pi(a)\varrho \tag{5.7}$$

be the induced (∗-)representation of A and further assume that

$$\varrho J \varrho = J_\varrho \quad \text{and} \quad \varrho J_\varrho \varrho = \varepsilon'''_\varrho J, \tag{5.8}$$

where $\varepsilon'''_\varrho = \pm 1$ is determined by the choice of ϱ and J. For an operator D on $\mathcal{H}$, set

$$D_\varrho = \varrho D \varrho.$$

Then $((A, \pi_\varrho, \mathcal{H}), D_\varrho, (J_\varrho, \varrho^2))$ satisfy the conditions for a spectral triple with ϱ-twisted real structure if and only if $((A, \pi, \mathcal{H}), D, J)_\rho$ satisfy the conditions for a real ρ-twisted spectral triple.

Proof. By assumption,

$$\bar{\rho}(\pi(a)) = \varrho^{-2}\pi(a)\varrho^2$$

for all $a \in A$. Then for $D_\varrho = \varrho D \varrho$, a straightforward calculation gives

$$\left[D_\varrho, \pi_\varrho(a)\right] = \varrho[D, \pi(a)]_{\bar{\rho}}\varrho, \tag{5.9}$$

and since ϱ and ϱ^{-1} are bounded, $\left[D_\varrho, \pi_\varrho(a)\right]$ is bounded whenever $[D, \pi(a)]_{\bar{\rho}}$ is bounded. Furthermore,

$$\begin{aligned}
[D, \pi(a)]_{\bar{\rho}} J\pi(b)J^{-1} &= \varrho^{-1}\left[D_\varrho, \pi_\varrho(a)\right]\varrho^{-1} J\pi(b)J^{-1}\varrho\varrho^{-1} \\
&= \varrho^{-1}\left[D_\varrho, \pi_\varrho(a)\right]\varrho^{-1} J\varrho\pi_\varrho(b)\varrho^{-1}J^{-1}\varrho\varrho^{-1} \\
&= \varepsilon'''_\varrho \varrho^{-1}\left(\left[D_\varrho, \pi_\varrho(a)\right] J_\varrho \varrho^2 \pi_\varrho(b)\varrho^{-2}J_\varrho^{-1}\right)\varrho^{-1},
\end{aligned}$$

where we have used (5.7), (5.8) and (5.9). However, we also have

$$\begin{aligned}
\bar{\rho}^\circ(J\pi(b)J^{-1})[D, \pi(a)]_{\bar{\rho}} &= J\varrho^{-2}\pi(b)\varrho^2 J^{-1}\varrho^{-1}\left[D_\varrho, \pi_\varrho(a)\right]\varrho^{-1} \\
&= \varrho^{-1}\varrho J \varrho^{-1}\pi_\varrho(b)\varrho J^{-1}\varrho^{-1}\left[D_\varrho, \pi_\varrho(a)\right]\varrho^{-1} \\
&= \varepsilon'''_\varrho \varrho^{-1}\left(J_\varrho \varrho^{-2}\pi_\varrho(b)\varrho^2 J_\varrho^{-1})\left[D_\varrho, \pi_\varrho(a)\right]\right)\varrho^{-1}
\end{aligned}$$

where we have used (2.23), (5.7), (5.8) and (5.9). Thus D satisfies (2.25) with respect to ρ if and only if D_ϱ satisfies (2.17) with respect to ϱ^2.

If $DJ = \varepsilon' JD$, then

$$D_\varrho J_\varrho \varrho^2 = \varrho D \varrho J_\varrho \varrho^2 = \varepsilon'''_\varrho \varrho D J \varrho = \varepsilon' \varepsilon'''_\varrho \varrho J D \varrho = \varepsilon' \varrho^2 J_\varrho \varrho D \varrho = \varepsilon' \varrho^2 J_\varrho D_\varrho,$$

where we have used (5.8). By the same reasoning, assuming $D_\varrho J_\varrho \varrho^2 = \varepsilon' \varrho^2 J_\varrho D_\varrho$, one can show that $DJ = \varepsilon' JD$.

If $J^2 = \varepsilon 1$, then one has as a straightforward consequence of (5.8) that $J_\varrho^2 = \varepsilon\varepsilon_\varrho''' 1$. Then lastly, assuming $J\pi(b)J^{-1}\pi(a) = \pi(a)J\pi(b)J^{-1}$ for $a, b \in A$, we have

$$\begin{aligned} J_\varrho \pi_\varrho(b) J_\varrho^{-1} \pi_\varrho(a) &= \varrho J \pi(b) J^{-1} \pi(\rho(a)) \varrho^{-1} \\ &= \varrho \pi(\rho(a)) J \pi(b) J^{-1} \varrho^{-1} = \pi_\varrho(a) J_\varrho \pi_\varrho(b) J_\varrho^{-1} \end{aligned}$$

such that J_ϱ implements the zeroth-order condition, where we have used (5.7), (5.8) and the fact that $\rho \in \mathrm{Aut}(A)$. As above, one can use the same reasoning to show that J_ϱ implementing the zeroth-order condition implies that so does J. □

Note that (5.8) implies the twist ϱ^2 is regular or anti-regular depending on the sign ε_ϱ'''. If one has $J_\varrho = J$, then $\varepsilon_\varrho''' = +1$ necessarily, and one recovers Thm. 5.4.

When ϱ is unitary, there is no guarantee that $\varrho D \varrho$ is self-adjoint. The solution offered in Ref. [8] is, provided $\varrho^{\dagger 2} = \varrho^2$, to use the following 'doubling procedure'. Rather than take the spectral triple $((A, \pi_\varrho, \mathcal{H}), D_\varrho, (J_\varrho, \varrho^2))$, one instead takes the 'doubled' spectral triple $((A, \boldsymbol{\pi}, \mathcal{H} \oplus \mathcal{H}), \mathbf{D}, (\mathbf{J}, \boldsymbol{\varrho}), \boldsymbol{\chi})$ where $\boldsymbol{\pi}$ is the diagonal extension of π_ϱ and

$$\mathbf{D} = \begin{pmatrix} 0 & D_\varrho \\ D_\varrho^\dagger & 0 \end{pmatrix}, \quad \boldsymbol{\varrho} = \begin{pmatrix} \varrho^2 & 0 \\ 0 & \varrho^2 \end{pmatrix}, \quad \boldsymbol{\chi} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that regardless of whether the initial spectral triple was odd or even, the new one is even with a natural choice of grading coming from the doubling.

The argument for the form of $\boldsymbol{\varrho}$ can be summarised as follows: if the original spectral triple admitted the twisted real structure (J_ϱ, ϱ^2) with respect to D_ϱ, by taking "conjugate" conditions and using that $J_\varrho^\dagger = \pm J_\varrho$, one can find that $D_\varrho^\dagger$ will admit the twisted real structure $(J_\varrho, (\varrho^2)^\dagger)$. Clearly these twisted real structures will coincide when $\varrho^2 = (\varrho^2)^\dagger$, and so therefore the new (doubled) spectral triple can be equipped with the twist $\mathrm{diag}(\varrho^2, \varrho^2)$.

The third KO-sign is determined by the choice of $\mathbf{J}$ to be block-diagonal

$$\mathbf{J} = \begin{pmatrix} J_\varrho & 0 \\ 0 & J_\varrho \end{pmatrix} \qquad (\varepsilon'' = +1) \tag{5.10}$$

or block-anti-diagonal

$$\mathbf{J} = \begin{pmatrix} 0 & J_\varrho \\ J_\varrho & 0 \end{pmatrix} \qquad (\varepsilon'' = -1). \tag{5.11}$$

Again, because the new grading does not come from any old one, the sign ε'' here is not related to that of the original spectral triple.

Remark 5.6. In order for the doubled spectral triple to have a valid twisted real structure, it should hold that $\boldsymbol{\varrho}\mathbf{J}\mathbf{D} = \pm\mathbf{D}\mathbf{J}\boldsymbol{\varrho}$. This is always true if one takes (5.10) (in which case the sign is ε' coming from the original spectral data), but is *not* always true if one takes (5.11). In that case, $\boldsymbol{\varrho}\mathbf{J}\mathbf{D} = \pm\mathbf{D}\mathbf{J}\boldsymbol{\varrho}$ requires $\varrho^2 J_\varrho D_\varrho = \pm D_\varrho^\dagger J_\varrho \varrho^2$ (and likewise for D_ϱ and $D_\varrho^\dagger$ interchanged), which is not necessarily true. It may be incidentally true depending on the choice of D, J and ϱ, but does not follow for arbitrary choices. ◇

As noted, the use of the doubling procedure renders the original choice of grading irrelevant. However, since we are interested in what Wick rotation means in the context of spectral triples in any case, it is worth discussing what the untwisting procedure has to say about the grading for the sake of completeness.

Lemma 5.7. *Assuming that ϱ is unitary, and χ is the grading for the even real twisted spectral triple $((A, \pi, \mathcal{H}), D, J, \chi)_\varrho$, then the spectral triple with twisted real structure $((A, \pi_\varrho, \mathcal{H}), D_\varrho, (J_\varrho, \varrho^2))$ is even with the grading $\chi_\varrho := \varrho^{-1}\chi\varrho$ provided $\varrho^2\chi = \chi\varrho^2$.*

Proof. The grading χ_ϱ should satisfy the conditions of Def. 2.1 along with (2.18) and (2.19). For what concerns (2.19), using $J\chi = \varepsilon''\chi J$ we find

$$\varrho^2 J_\varrho \chi_\varrho = \varrho^3 J\chi\varrho = \varepsilon''\varrho^3\chi J\varrho = \varepsilon''\varrho^4\chi_\varrho\varrho^{-2}J_\varrho = \varepsilon''\chi_\varrho\varrho^2 J_\varrho.$$

Equation (2.18) is satisfied trivially, and $\chi_\varrho^2 = 1$ and $\chi_\varrho^\dagger = \chi_\varrho$ are straightforward to check. It is also not hard to see that $\pi_\varrho(A)$ commutes with χ_ϱ if $\pi(A)$ commutes with χ. For the final requirement, we have

$$D_\varrho\chi_\varrho = \varrho D\chi\varrho = -\varrho\chi D\varrho = -\varrho^2\chi_\varrho\varrho^{-2}D_\varrho = -\chi_\varrho D_\varrho.$$

□

The choice of $\chi_\varrho = \varrho^{-1}\chi\varrho$ is not unique, but seems a natural choice. Taking something like $\chi_\varrho = \varrho^{-1}\chi\varrho^{-1}$ won't satisfy the correct anticommutation with D_ϱ unless some addition commutation relation is imposed between ϱ^2 and D_ϱ, which would be very restrictive. A similar issue occurs if $\chi_\varrho = \chi$, since anticommutation with D_ϱ will require (anti)commutation between χ and ϱ, in which case one basically lands in a special case of Lem. 5.7, just without needing the unitarity requirement.

5.4.1 *Application to the minimally twisted spectral triple*

Let us now apply the untwisting procedure to the minimally twisted spectral triple. In four dimensions, the minimal twist σ is implemented by the zeroth gamma matrix, *i.e.*

$$\bar{\sigma}(\pi(f, g)) = \gamma^0\pi(f, g)\gamma^0 = \pi(g, f) = \pi(\sigma(f, g)).$$

From here, we will work in the chiral (Weyl) representation on spinor space, in which case one has

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}.$$

In order to pass to spectral triples via the untwisting procedure, it is necessary to take a square root of γ^0; in line with Ref. [8], we will take

$$\varsigma = \frac{1}{2}\big((1+i)1_4 + (1-i)\gamma^0\big),$$

which will play the role of ϱ in the previous subsection and which we note is appropriately a unitary operator.

The forms of

$$\pi_\varsigma(f,g) = \frac{1}{2}\begin{pmatrix} (f+g)1_2 & -i(f-g)1_2 \\ i(f-g)1_2 & (f+g)1_2) \end{pmatrix}$$

and

$$D_\varsigma = \frac{1}{2}\begin{pmatrix} D_- + D_+ & i(D_- - D_+) \\ -i(D_- - D_+) & D_- + D_+ \end{pmatrix} \quad \text{for} \quad i\partial\!\!\!/_E = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

likewise follow Ref. [8]. The point of deviation for us is the real structure, given by the (Euclidean) charge conjugation operator

$$J = i\gamma^0\gamma_E^2 \circ *,$$

whose form is prohibited in Ref. [8] by the regularity requirement. The antiunitary J is in fact antiregular with respect to the twist, $\varsigma^2 J \varsigma^2 = -J$, which is easily established by the anticommutation of gamma matrices. The untwisted antiunitary is thus given by

$$J_\varsigma = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \circ * = \gamma_E^2 \circ *.$$

For what concerns the grading, we have $\chi = \gamma_E^5$ coming from the minimally twisted spectral triple. This means that Lem. 5.7 does not apply, since $\varsigma^2 = \gamma^0$ by construction and $\gamma_E^5\gamma^0 = -\gamma^0\gamma_E^5$. Indeed, no suitable choice of χ_ς based on ς and χ is apparent, and so since the grading is not essential for the untwisting, we will ignore it for the remainder of this subsection.

The untwisted spectral triple

$$\big((C^\infty(M)\otimes\mathbb{C}^2, \pi_\varsigma, L^2(M,S)), D_\varsigma, (J_\varsigma, \varsigma^2)\big)$$

is not immediately recognisable. However, it is unitarily equivalent (via conjugation by $e^{i\pi/4}\varsigma$) to

$$\big((C^\infty(M)\otimes\mathbb{C}^2, \pi, L^2(M,S)), i\gamma^0\partial\!\!\!/_E, (i\gamma^0 C_E, \gamma^0)\big). \tag{5.12}$$

The geometrical interpretation of these data is far from straightforward. For one thing, the antiunitary is precisely the Lorentzian charge conjugation operator

$$i\gamma^0 C_E = -i\gamma_M^2 \circ * \equiv C_M. \tag{5.13}$$

Different sources use different conventions for the Lorentzian charge conjugation operator in the chiral basis, in particular $C_M = \pm i\gamma_M^2 \circ *$. The particular choice of sign is unimportant, but for consistency we will choose $-$.

Interestingly, the "Dirac operator" $i\gamma^0 \not{\nabla}_E$ is Krein-self-adjoint with respect to γ^0 as the fundamental symmetry, but is unrelated to the true Lorentzian Dirac operator $i\not{\nabla}_M$. The situation is not as trivial as that of §5.3.2, but from the perspective of the action, things remain confused:

$$\left\langle \varsigma^2 J_\varsigma \psi, D_\varsigma \varphi \right\rangle = \left\langle \gamma^0 i\gamma^0 C_E \psi, i\gamma^0 \not{\nabla}_E \varphi \right\rangle = \left\langle iC_E \psi \middle| i\not{\nabla}_E \varphi \right\rangle,$$

i.e. we have the Krein structure for the inner product, but the operators remain the same as in the Euclidean case.

Remark 5.8. The Dirac operator $i\gamma^0 \not{\nabla}_E$ can be understood as the so-called 'Krein-shift' of $i\not{\nabla}_E$, introduced in Ref. [5]. However, in that paper the set-up is rather different to here, though the motivations are similar; in said paper the Krein-shift is defined for the Lorentzian Dirac operator rather than the Euclidean one, in order to express the Lorentzian fermionic action in terms of $\psi^\dagger$ instead of $\bar{\psi}$. ⋄

In order to attempt to resolve this situation, we will aim to follow Ref. [8] and utilise the doubling procedure to produce a genuine (Euclidean) spectral triple with twisted real structure in the hope that, much like in the twisted spectral triple case, some Lorentzian structure emerges at the level of the action.

5.4.2 *Doubling*

Before examining the doubled spectral triple, we note that we have the following result:

Lemma 5.9 ([50])**.** *A spectral triple with twisted real structure (J, ν) and with KO-signs $(\varepsilon, \varepsilon', \varepsilon'')$, whose twist operator satisfies $\nu = \alpha\nu^\dagger = \alpha\nu^{-1}$ for $\alpha = \pm 1$, is equivalent to a real spectral triple with real structure νJ and KO-signs $(\varepsilon\varepsilon''', \alpha\varepsilon'\varepsilon''', \varepsilon'')$.*

The proof follows that of Prop. 2.12, but modified as appropriate to take signed regularity into account (as the original proposition assumed only regularity), *i.e.* inserting signs where appropriate.

Applying the doubling procedure to (5.12), we obtain a well-defined spectral triple with twisted real structure. However, the twist for the doubled spectral triple satisfies $\varrho = \varrho^\dagger = \varrho^{-1}$, and so Lem. 5.9 applies. In other words, we are really dealing with the even real spectral triple

$$\Big(\big(C^\infty(M) \otimes \mathbb{C}^2, \pi, L^2(M, S) \otimes \mathbb{C}^2 \big), \mathbf{D}, \varrho \mathbf{J}, \chi \Big) \tag{5.14}$$

with KO-signs $(-\varepsilon, -\varepsilon', \varepsilon'')$. The fact that we have $C^\infty(M) \otimes \mathbb{C}^2$ acting on $L^2(M, S) \otimes \mathbb{C}^2$ suggests the possibility that this spectral triple could be understood as the product of a manifold spectral triple with a two-point space, a possibility which was also investigated in Ref. [8].

We first note that the doubling is built around making the "Dirac operator" self-adjoint, and so that provides a fixed point for our investigation. The simplest Dirac operator for the 2-point space is simply 0, which means that, if $\mathbf{D}$ is equivalent to a product Dirac operator, the simplest form the product Dirac operator can take is $i\nabla\!\!\!/_E \otimes 1_2$.

Lemma 5.10. *The Dirac operator*

$$\mathbf{D} = \begin{pmatrix} 0 & i\gamma^0\nabla\!\!\!/_E \\ i\nabla\!\!\!/_E\gamma^0 & 0 \end{pmatrix}$$

is unitarily equivalent to the operator $i\nabla\!\!\!/_E \otimes 1_2$.

Proof. The unitary which implements the transformation (from the product to the doubled operator) is the permutation matrix

$$S_{13} = \begin{pmatrix} 0 & 0 & 1_2 & 0 \\ 0 & 1_2 & 0 & 0 \\ 1_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_2 \end{pmatrix}, \tag{5.15}$$

acting on $L^2(M,S) \otimes \mathbb{C}^2$, still working with $L^2(M,S)$ in the chiral basis. Confirming that $S_{13}(i\nabla\!\!\!/_E \otimes 1_2)S_{13}^{-1} = \mathbf{D}$ is then a straightforward computation. □

It is not difficult to show that, regardless of whether one takes $\mathbf{J}$ to have the form of (5.10) or (5.11), it is not possible to find any J_1 or J_2 such that $S_{13}^{-1}\boldsymbol{\varrho}\mathbf{J}S_{13} = J_1 \otimes J_2$. This result differs from that of Ref. [8], precisely because we have relaxed the regularity condition.

It is tempting to read into this that even the "ordinary" real structure is not so trivial for the doubled spectral triple when originally arising from a twisted real structure. However, drawing conclusions is premature, because the doubling procedure described in Ref. [8] and recapitulated above is not the unique process by which can produce a genuine spectral triple by doubling. Most relevant to the current discussion, one can also take

$$\boldsymbol{\varrho}' = \begin{pmatrix} 0 & \varrho^2 \\ -\varrho^2 & 0 \end{pmatrix},$$

noting that $\varrho^2 = (\varrho^2)^\dagger$ remains a sufficient assumption as the sign plays no part in the "conjugate" conditions; since the twist always appears in pairs, the sign will always cancel with itself.

As for the antiunitary, one can still take (5.10) or (5.11), the choice of which will still affect the ε''-sign in the same way, although the situation with respect to Rmk. 5.6 is reversed; now (5.11) is always permissible, whilst (5.10) will only be viable in certain special cases.

Applying this alternative doubling procedure to (5.12), and taking (5.11) for the form of $\mathbf{J}$ to maintain the greatest generality, one has

$$\varrho'\mathbf{J} = \begin{pmatrix} iC_E & 0 \\ 0 & -iC_E \end{pmatrix}.$$

Now applying the unitary S_{13}, one finds

$$S_{13}^{-1}\varrho'\mathbf{J}S_{13} = \begin{pmatrix} -i\sigma^2 & 0 & 0 & 0 \\ 0 & -i\sigma^2 & 0 & 0 \\ 0 & 0 & i\sigma^2 & 0 \\ 0 & 0 & 0 & i\sigma^2 \end{pmatrix} \circ * = -iC_E \otimes 1_2 \circ *,$$

and so one is able to recover the product structure. Applying S_{13} to the grading, one thus ultimately recovers

$$(C^\infty(M) \otimes \mathbb{C}^2, L^2(M,S) \otimes \mathbb{C}^2, \mathbf{D}, \varrho'\mathbf{J}, \chi) =$$
$$\left(C^\infty(M), L^2(M,S), i\nabla\!\!\!/_E, iC_E, \left(\begin{smallmatrix} -1_2 & 0 \\ 0 & 1_2 \end{smallmatrix}\right)\right) \times \left(\mathbb{C}^2, \mathbb{C}^2, 0, \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right),$$

i.e. the doubled spectral triple is equivalent to the product of the canonical spectral triple (the factor of i in the real structure and -1 in the grading make no practical difference) with the two-point space of KO-dimension 0. Since all the structures here are the familiar Euclidean ones, there is no hope of recovering Lorentzian structure at the level of the action, as the minimally twisted spectral triple does.

5.5 THE BOTTOM-UP APPROACH

We saw in §5.3 is that it is not sufficient to modify the real structure alone to recover Lorentzian metric structure. The approach of the minimally twisted spectral triple was to enlarge the algebra in order to accommodate a nontrivial twist automorphism which gives the Krein structure, but as we saw in the previous subsection, when translated into the language of spectral triples with twisted real structure, the result was poorly defined (pre-doubling) or trivial (post-doubling). As such, one must take a different approach.

In the approaches we have looked at thus far, the Dirac operator has been a recurring issue, and so it is natural to begin there. A sensible option in the commutative case is to to implement the Wick rotation at the level of the gamma matrices, *i.e.* to implement the Wick rotation for the Dirac operator by replacing γ_E^μ with γ_M^μ. To do so, we take inspiration from *e.g.* Ref. [63] and define the Wick rotation of a Dirac operator D by

$$D_W := \frac{1}{2}(D + D^+) - \frac{i}{2}(D - D^+), \tag{5.16}$$

where $(\bullet)^+ = \beta(\bullet)^\dagger\beta$ is the Krein adjoint as described in §5.2.2. A natural advantage of this option is that, as the implementation is algebraic, it is also applicable in noncommutative cases. Note that, by definition, $D = D^\dagger$, and that consequently $D_W^+ = D_W$ by construction, *i.e.* it is Krein-self-adjoint (but not self-adjoint, in the sense that $D_W^\dagger \neq D_W$). It is not hard to verify that, for example, taking $D = i\gamma_E^\mu\partial_\mu = i\partial\!\!\!/_E$, the Wick rotation given above yields $D_W = i\gamma_M^\mu\partial_\mu = \partial\!\!\!/_M$ as expected, where as mentioned, $\gamma_M^0 = \gamma_E^0$ and $\gamma_M^j = -i\gamma_E^j$.

Remark 5.11. Apart from the form given in (5.16), we could also have chosen

$$D_W := \frac{1}{2}(D + D^+) + \frac{i}{2}(D - D^+).$$

The reason for this comes from the choice of implementing the Wick rotation at the level of the gamma matrices. By definition, $(\gamma_E^j)^2 = 1$ whilst $(\gamma_M^j)^2 = -1$, and this can be obtained by setting $\gamma_M^j = \pm i\gamma_E^j$. Neither choice of sign is *a priori* preferable, so taking $+$ or $-$ is simply a matter of convention. ◇

In some sense, we are free to define the other key operators as we wish, but as much as possible we would like to avoid tailoring our choices to a desired outcome as possible. As such, since we have already made use of a map in obtaining the Wick rotated Dirac operator (5.16), it seems reasonable as a first attempt to apply the same map to the other relevant operators in the same way. We make an exception only for the algebra, since we may wish to consider real algebras. Regardless, if we think of the algebra as being related to topology, Wick rotation should not make any difference, as it should be a purely geometric transformation.

For the sake of ease, let us call the Wick rotation map W. Beginning with the grading operator, taking $\chi_W = \frac{1}{2}(\chi + \chi^+) - \frac{i}{2}(\chi - \chi^+)$, we find that $W(\gamma_E^5) = -i\gamma_E^5$, which is perhaps a rather undesirable result given the Lorentzian chirality operator is $\gamma_M^5 = -\gamma_E^5$ (in the convention we are using). We will return to this point in the next subsection.

In the same way we define J_W using the map W, and subsequently, for $J^2 = \varepsilon 1$ and $J\beta = \varepsilon_\beta'''\beta J$, we find

$$J_W = \begin{cases} J & \text{if} \quad \varepsilon\varepsilon_\beta''' = +1, \\ -iJ & \text{if} \quad \varepsilon\varepsilon_\beta''' = -1. \end{cases} \tag{5.17}$$

In order for the conditions $J_W^+ = J_W^{-1}$ and $J_W^2 = \pm 1$ to be simultaneously satisfied, one requires that $\varepsilon_\beta''' = +1$. However, $J_W^\dagger = J_W^{-1}$ and $J_W^2 = \pm 1$ are always compatible. This means that we can always take J_W to be unitary but we can only take it to be Krein-unitary when $\varepsilon_\beta''' = +1$. As such, when $\varepsilon_\beta''' = -1$, J_W can only make sense as a (twisted) real structure with respect to the Hilbert inner product $\langle \bullet, \bullet \rangle$ and *not* the Krein inner product $\langle \bullet, \bullet \rangle_\beta$.

For simplicity, we assume that the representation of the algebra commutes with the fundamental symmetry. This is not a necessary assumption on mathematical grounds, but happens to be true for the 4-dimensional commutative case as well as proposed Lorentzian finite spectral triples in the literature [4, 63].

It should not be lost that this is quite a strong assumption, and even if it is satisfied in examples, many of the abstract results obtained in this section would not be true otherwise.

An important point to make is that, if we work taking $\mathcal{H}$ as a Hilbert space (and not a Krein space), then D_W will not be a self-adjoint operator on our space, leaving us in a similar spot to the pre-doubled untwisting of the minimally twisted spectral triple (5.12). Even so, apart from losing touch with the strict definition of a spectral triple, if we assume J_W forms part of a twisted real structure (J_W, ν), the non-self-adjointness of D_W means *we no longer require ν to be self-adjoint*, and imposing Krein-self-adjointness on D_W only requires $\nu = \pm\nu^+$. Recall that the twist operator being self-adjoint and unitary in order to make sense of the action in Ch. 4 was a large part of the motivation for treating ν as a fundamental symmetry and this now appears to be lost.

The situation is not as grim as it would appear, though. For one, $D_W J_W \neq J_W D_W$ but rather

$$D_W J_W \beta = \varepsilon'_W \beta J_W D_W \tag{5.18}$$

assuming $DJ = \varepsilon' JD$ and where $\varepsilon'_W = \varepsilon' \varepsilon'''_\beta$. In other words, the fundamental symmetry automatically takes the role of a twist operator independent of other considerations. In the case of four dimensions, where $J = C_E$, $\beta = \gamma^0$ and the two anticommute, it is straightforward to see that all the other requirements for a twisted real structure are satisfied since $-i\gamma^0 C_E = -C_M$.

Recalling the results of §4.4, the fact that D_W is not self-adjoint means that Lem. 4.24 does not hold and so there is no justification for introducing the Krein inner product $\langle \bullet, \bullet \rangle_\beta$ coming from the twist in order to get the correct fermionic action functional. However, as noted earlier in §4.4, there is no need to change the bilinear form when $D = \nu^{-1\dagger} D^\dagger \nu^{-1\dagger}$, and that is precisely the regime we are in; this is just the requirement that $D = D^+$ with respect to the fundamental symmetry ν. To formulate this result precisely:

Proposition 5.12. *Let $\nu = \nu^\dagger = \nu^{-1}$ be a linear operator, and $D_W = \nu D_W^\dagger \nu$. Then the bilinear form*

$$\mathfrak{A}_D(\psi, \phi) := \left\langle J_W \psi, D_W \phi \right\rangle, \tag{5.19}$$

where $\langle \bullet, \bullet \rangle$ is the inner product on the Hilbert space $\mathcal{H}$, is gauge-covariant and is correctly (anti)symmetric for the appropriate KO-dimension.

Proof. Proving the symmetry of the bracket is a straightforward computation, *cf.* (4.29):

$$\begin{aligned}\langle J_W\psi, D_W\varphi\rangle &= \varepsilon_W\langle J_W D_W\varphi, \psi\rangle \\ &= \varepsilon_W\left\langle J_W\beta D_W^\dagger\beta\varphi, \psi\right\rangle \\ &= \varepsilon_W\varepsilon_\beta'''\left\langle \beta J_W D_W^\dagger\beta\varphi, \psi\right\rangle \\ &= \varepsilon_W\varepsilon_\beta'''\varepsilon_W'\left\langle D_W^\dagger J_W\varphi, \psi\right\rangle \\ &= \varepsilon_W\varepsilon_\beta'''\varepsilon_W'\langle J_W\varphi, D_W\psi\rangle,\end{aligned}$$

where we have used $J_W^2 = \varepsilon_W 1$, the unitarity of J_W, $\beta J = \varepsilon_\beta''' J\beta$ which implies $\beta J_W = \varepsilon_\beta''' J_W\beta$, and (5.18) which implies $D_W^\dagger J_W\beta = \varepsilon_W'\beta J_W D_W^\dagger$.

The proof of gauge-invariance largely follows the proof of Lem 4.24; the gauge-invariance of the bilinear form is expressed as

$$\mathfrak{A}_D(\psi,\varphi) = \mathfrak{A}_{D^u}\left(\mathrm{Ad}(u)(\phi), \mathrm{Ad}(u)(\varphi)\right),$$

noting that the form of the gauge transformations does not rely on the self-adjointness of D and so remains the same as in the previous chapter. This is equivalent to

$$\begin{aligned}\langle J_W\psi, D_W\varphi\rangle &= \left\langle J_W V\psi, D_W^u V\varphi\right\rangle \\ &= \left\langle J_W V\psi, \tilde V D_W V^{-1} V\varphi\right\rangle \\ &= \left\langle J_W V\psi, \tilde V D_W\varphi\right\rangle\end{aligned}$$

which is equivalent to requiring $\tilde V^\dagger J_W V = J_W$. Provided that ν commutes with the algebra, as was earlier assumed, then this can easily be computed to be true for both cases of (5.17). □

Remark 5.13. The Krein-self-adjointness of D does not pose serious issues with respect to the results of §4.3; in particular, 1-forms ω should now be Krein-self-adjoint rather than self-adjoint, but as they should also be generated by commutators of the Dirac operator with the algebra, and as the algebra is assumed to commute with the fundamental symmetry, this is automatically satisfied, and something similar can be said for the opposite-1-forms as well. Gauge transformations are unaffected for the same reason; moreover, the operators $\tilde V$ and V both reduce to the standard unitary $U = \pi(u)J\pi(u)J^{-1}$. ⋄

Given that $J_W \propto J$, it might appear that the bilinear form $\langle J_W\psi, D_W\varphi\rangle$ is physically meaningless, since J_W is Euclidean but D_W is Lorentzian, not to mention that the inner product is the Hilbert space inner product. However, all these issues can be resolved simultaneously by noting that

$$\langle J_W\psi, D_W\varphi\rangle = \left\langle \nu^2 J_W\psi, D_W\varphi\right\rangle = \langle \nu J_W\psi, D_W\varphi\rangle_\nu.$$

This might seem like a meaningless change, but $\langle\bullet,\bullet\rangle_\nu = \langle\bullet|\bullet\rangle$ fixes the issue of the inner product, and for what concerns the antiunitary operator, in the commutative case with $J_W = -iJ$, then

$$\nu J_W = \gamma^0(-i)C_E = \gamma^0(-i)i\gamma^0\gamma_E^2\circ * = i\gamma_M^2\circ * = -C_M, \tag{5.20}$$

and we recover the Lorentzian charge conjugation. What's more, for either possibility of J_W, we have

$$(D_W)_\omega = D_W + \omega + \varepsilon'_W \varepsilon'''_\beta \nu J_W \omega J_W^{-1} \nu = D_W + \omega + C_M \omega C_M^{-1},$$

which is precisely the fluctuation of the Dirac operator which we would expect.

The trouble here is that in 4 dimensions, $\varepsilon = \varepsilon'''_\beta = -1$ for $J = C_E$ and $\beta = \gamma^0$, which means $J_W = J$ and not $-iJ$, meaning the bilinear form would have an extra factor of i compared to what we expect the Lorentzian fermionic action to give. A possible resolution to this discrepancy comes from recognising that in the noncommutative framework, what is physically important is not the commutative geometry by itself but the almost-commutative geometry which includes the internal degrees of freedom.

5.5.1 *The almost-commutative case*

We hence expand our focus to almost-commutative spectral triples *qua* products of commutative and finite spectral triples. We therefore begin by considering the almost-commutative spectral triple

$$(A, \mathcal{H}, D) := \big(C^\infty(M) \hat{\otimes} A_F, L^2(M,S) \hat{\otimes} \mathcal{H}_F, i\not{\nabla}_E \hat{\otimes} 1 + 1 \hat{\otimes} D_F\big),$$

where $(A_F, \mathcal{H}_F, D_F)$ is a (Riemannian) finite spectral triple. We assume both commutative and finite spectral triples are even and the product grading operator is given by $\chi = \gamma_E^5 \hat{\otimes} \chi_F$.

Regarding the products themselves, in Ref. [4] the authors advocate for the use of graded tensor products (*cf.* [37]). We used graded tensor products in §3.4, but to briefly recapitulate, the graded tensor product of operators can be given in terms of the ungraded one by

$$T_1 \hat{\otimes} T_2 = T_1 \chi_1^{|T_2|} \otimes T_2. \tag{5.21}$$

The graded tensor product could alternatively be given by $T_1 \hat{\otimes}' T_2 = T_1 \otimes T_2 \chi_2^{|T_1|}$, but for the sake of consistency we will strictly use the convention of (5.21) in this chapter.

Here $|T| \in \mathbb{Z}_2$ denotes the degree of an operator T with respect to the relevant grading; for example, considering (5.21), $|T_2| = 0$ if $T_2\chi_2 = \chi_2 T_2$ and $|T_2| = 1$ if $T_2\chi_2 = -\chi_2 T_2$.

The authors of Ref. [4] further propose the following definitions for product operators, obtained from the consideration of Clifford algebras:

$$J = J_1 \chi_1^{|J_2|} \hat{\otimes} J_2 \chi_2^{|J_1|} = J_1 \otimes J_2 \chi_2^{|J_1|}, \tag{5.22}$$

$$\beta = i^{|\beta_1||\beta_2|} \beta_1 \chi_1^{|\beta_2|} \hat{\otimes} \beta_2 \chi_2^{|\beta_1|} = i^{|\beta_1||\beta_2|} \beta_1 \otimes \beta_2 \chi_2^{|\beta_1|}. \tag{5.23}$$

The question of how to implement Wick rotations on the product is delicate. The simpler solution is to take the product first and then Wick rotate; it might be possible to make this equivalent to Wick rotating first and then taking the product, but even if one is able to tame the various signs and phase factors, one must then deal with the tensor product being graded with respect to the Wick-rotated grading operator, which sacrifices some conceptual clarity.

The Wick rotated form of the product reality operator is similar to (5.17), albeit involving rather more complicated signs due to the added number of operators involved. To be precise,

$$J_W = \frac{1}{2}\left(J + \beta J^\dagger \beta\right) - \frac{i}{2}\left(J - \beta J^\dagger \beta\right)$$

where J is given by (5.22) and

$$\beta J^\dagger \beta = (-1)^{|J_1||J_2|+|J_1||\beta_2|+|\beta_1||\beta_2|}\varepsilon_1 \varepsilon'''_{\beta_1}\varepsilon_2 \varepsilon''^{|\beta_1|}_2 \varepsilon'''_{\beta_2} J \tag{5.24}$$

where $J_j^2 = \varepsilon_j 1$, $\beta_j J_j = \varepsilon'''_{\beta_j} J_j \beta_j$, and $\chi_j J_j = \varepsilon''_j J_j \chi_j$, $j = 1, 2$, since the initial data are assumed to be Euclidean (*i.e.* from an ordinary spectral triple). Note that the grading is assumed to commute or anticommute with the fundamental symmetry and indeed, in obtaining the above result we have used that $\chi_j^{|T|}\beta_j = (-1)^{|T||\beta_j|}\beta_j \chi_j^{|T|}$. Since the sequence of signs on the right-hand side of (5.24) is rather unwieldy, we denote it by $\varepsilon_{J^+} := (-1)^{|J_1||J_2|+|J_1||\beta_2|+|\beta_1||\beta_2|}\varepsilon_1 \varepsilon'''_{\beta_1}\varepsilon_2 \varepsilon''^{|\beta_1|}_2 \varepsilon'''_{\beta_2}$ such that the end result of this computation is then

$$J_W = \begin{cases} J & \text{if} \quad \varepsilon_{J^+} = +1, \\ -iJ & \text{if} \quad \varepsilon_{J^+} = -1. \end{cases} \tag{5.25}$$

As a sanity check, we now examine the case starting with an almost-commutative spectral triple and Wick rotating only the commutative part, *i.e.* setting $\beta_2 = 1$. Assuming the commutative spectral triple is of dimension 4, taking $\beta_1 = \gamma^0$, $J_1 = C_E$ and $\chi_1 = \gamma_E^5$, we find

$$\varepsilon_{J^+} = \varepsilon_2 \varepsilon''_2.$$

With an eye towards the action and the discussion around the bilinear form in the previous subsection, it is not difficult to show that

$$\beta J_W = -i\varepsilon''_2 \gamma^0 C_E \otimes J_2 \chi_2 = -\varepsilon''_2 C_M \otimes J_2 \chi_2$$

if $\varepsilon_{J^+} = -1$. This can only occur if $\varepsilon_2 = -1$, $\varepsilon''_2 = +1$ (the finite spectral triple has KO-dimension 4) or if $\varepsilon_2 = +1$, $\varepsilon''_2 = -1$ (the finite spectral triple has KO-dimension 6), and indeed, the literature around almost-commutative spectral triples as applied to physics generally identifies the KO-dimension of the finite spectral triple to be 6 both when the commutative spectral triple is Euclidean (see *e.g.* Refs. [21, 62] for standard texts) and when it is Lorentzian (*e.g.* [1, 33]), although the meaning of KO-dimension is generally less clear in the latter case, *cf.* [3].

5.5.2 *The fermionic action*

Continuing the example of the previous subsection, where the commutative spectral triple has KO-dimension 4 and the finite spectral triple has KO-dimension 6, taking $\beta = \gamma^0 \otimes \chi_F$ and hence $J_W = -iC_E \otimes J_F$, with $D = i\not{\nabla}_E \otimes 1 + \gamma_E^5 \otimes D_F$, we compute the Wick rotated product Dirac operator to be

$$D_W = i\not{\nabla}_M \otimes 1 - \gamma_M^5 \otimes D_F,$$

where the − sign comes from $\gamma_E^5 = -\gamma_M^5$. This sign is a minor discrepancy with the literature, but should not affect the model physically as it can always be absorbed into D_F.

The bilinear form giving the full fermionic action can then be given by

$$\mathfrak{A}_{(D_W)_\omega}(\psi, \varphi) = \langle J_W \psi, (D_W)_\omega \varphi \rangle = \langle \beta J_W \psi | (D_W)_\omega \varphi \rangle \, .$$

The overcounting of degrees of freedom inherent in this approach (the so-called 'fermion doubling problem') means that we would like to reduce the size of the Hilbert space. Using Ref. [1] as a reference, J.W. Barrett suggests imposing Majorana and Weyl conditions $J\psi = \psi$ and $\chi\psi = i\psi$.

As far as the grading is concerned, the peculiar factor of i which appeared in the previous subsection here fits perfectly; we find that $\chi_W = -i\chi = -\gamma_E^5 \otimes i\chi_F = \gamma_M^5 \otimes i\chi_F$, which is exactly the grading used by Barrett. This is especially interesting because the grading used in Ref. [1] is somewhat non-standard; it is much more common to choose a grading on the finite space with eigenvalues ± 1 rather than $\pm i$.

The situation with the Majorana condition is less straightforward. Barrett's almost-commutative spectral triple consists of a commutative spectral triple which is initially Lorentzian, such that $J = C_M \otimes J_F$ and whose real structure is, needless to say, not twisted. The natural choice in the framework of this section in order to get as close to Barrett's Majorana condition as possible is to impose $\beta J_W \psi = \psi$, noting that $\beta J_W = C_M \otimes J_F \chi_F$. We hence define the 'physical subspace' to be

It makes sense to define the Majorana and Weyl conditions like so since $(\beta J_W)^2 = 1$ and $[\beta J_W, \chi_W] = 0$, at least in the cases we look at.

$$\mathcal{H}_{\text{phys}} \coloneqq \{\psi \in \mathcal{H} : \chi_W \psi = i\psi, \beta J_W \psi = \psi\}.$$

The bilinear form then reduces to

$$\mathfrak{A}_{(D_W)_\omega}(\psi, \varphi) = \langle \psi | (D_W)_\omega \varphi \rangle \, , \quad \psi, \varphi \in \mathcal{H}_{\text{phys}},$$

which is the same fermionic action as in Ref. [1] and which is in good agreement with physical models for appropriate choices of finite spectral triple.

5.5.3 *A fully Lorentzian model*

These results are very encouraging, but the charge can be made that this approach to Wick rotation simply exploits the coincidence that $\gamma^0 C_E \propto C_M$ in the 4-dimensional commutative setting. To investigate if there is more to it, it is worthwhile to compare the schema outlined in this section to proposed models from the literature which include a Lorentzian finite spectral triple, since such spectral triples do not come from classical Clifford algebras and so the efficacy of our approach will come down to the algebraic implementation, rather than by playing with gamma matrices.

To do this, we will consider the standard Euclidean formulation of electrodynamics in terms of spectral triples [67], Wick rotate it according to the procedure proposed in this section, and compare it to the fully Lorentzian proposal of Ref. [4] (a so-called 'indefinite spectral triple'). The advantage of this comparison is that we already use the same definition of products as Ref. [4], and their proposal also includes a real structure, making drawing comparisons relatively simple.

Take care when comparing products though; the graded tensor products of Ref. [4] use a different grading to the Euclidean, pre-Wick rotation ones we use.

We first recapitulate the Euclidean model. For the finite spectral triple, we take $A_F = \mathbb{C} \oplus \mathbb{C}$ acting on $\mathcal{H}_F = \mathbb{C}^4$ with basis $\{e_R, e_L, e_R^c, e_L^c\}$. We will not need the explicit representation, but for completeness, for $A_F \ni a = (z, w)$, $z, w \in \mathbb{C}$, the representation π_F is given by $\pi_F(a) = \begin{pmatrix} z1_2 & 0 \\ 0 & w1_2 \end{pmatrix}$.

We use the notation of e for electrons and e^c for positrons, with subscripts denoting parity. The notation $\bar{e}$ is more common for positrons, but we avoid it to prevent confusion with the Dirac adjoint.

Since the grading should be related to chirality, it is chosen to be the operator with eigenvalues of +1 on left-handed particles and −1 on right-handed particles, or explicitly,

$$\chi_F = \begin{pmatrix} -\sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}.$$

Since J_F should be related to charge conjugation such that $J_F e_P = e_P^c$ and $J_F e_P^c = e_P$, $P = L, R$, it takes the form

$$J_F = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \circ *.$$

For the Dirac operator, anticommutation with χ_F, commutation with J_F, and the first-order condition force it to have the form

$$D_F = \begin{pmatrix} 0 & d & 0 & 0 \\ d^* & 0 & 0 & 0 \\ 0 & 0 & 0 & d^* \\ 0 & 0 & d & 0 \end{pmatrix},$$

where $d \in \mathbb{C}$ is a complex parameter which is taken to be $d = -im$ to obtain the correct Euclidean action.

The fundamental symmetry used in Ref. [4] is

$$\beta_F = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix},$$

and we will likewise use it here. The product fundamental symmetry is then

It should be noted that Ref. [4] uses a differently ordered basis of $\mathcal{H}_F$; the matrices we present here have been adjusted to match the basis of Ref. [67].

$$\beta = \gamma^0 \otimes \beta_F \chi_F,$$

and for the reality operator, the sign $\varepsilon_{J^+} = -1$ means that

$$J_W = -iJ = -iC_E \otimes J_F.$$

The Dirac operator is Wick rotated to

$$\begin{aligned} D_W &= i\not{\nabla}_M \otimes 1 + \frac{1}{2}\gamma_E^5 \otimes (D_F + \beta_F D_F \beta_F) - \frac{i}{2}\gamma_E^5 \otimes (D_F - \beta_F D_F \beta_F) \\ &= i\not{\nabla}_M \otimes 1 + \gamma_M^5 \otimes iD_F \\ &= i\not{\nabla}_M \otimes 1 + \gamma_M^5 \otimes \begin{pmatrix} 0 & m & 0 & 0 \\ -m & 0 & 0 & 0 \\ 0 & 0 & 0 & -m \\ 0 & 0 & m & 0 \end{pmatrix}. \end{aligned}$$

Comparing this to the Dirac operator of Ref. [4],

$$D_{F,M} = \begin{pmatrix} 0 & m & 0 & 0 \\ -m & 0 & 0 & 0 \\ 0 & 0 & 0 & m \\ 0 & 0 & -m & 0 \end{pmatrix},$$

we see that even though we did obtain the needed factor of i, the placement of the signs is slightly off.

There are other differences, but they are relatively minor. The gradings differ by a factor of $-i$, but the Weyl conditions are equivalent and the grading doesn't otherwise impact the action. The real structure of Ref. [4] differs from βJ_W by a factor of $-1 \otimes \beta_F$, but this difference does not account for the difference in Dirac operators. We are thus forced to conclude that the fermionic action obtained by Wick rotating the Euclidean formulation of electrodynamics does not yield the correct Lorentzian fermionic action.

Another example in the literature of a Lorentzian finite space comes from the Lorentzian electroweak model with Majorana particles of Ref. [63, §6.1], but the 'Krein spectral triples' formulation therein is another step further removed from Connes' spectral triples compared to the 'indefinite spectral triples' of Ref. [4]. Even though a comparison can still be drawn, it is perhaps unsurprising that the Wick rotation procedure also does not recover the equivalent action in that case. We will not present the working here, but the main impediment was once again the finite Dirac operator.

5.6 DISCUSSION

It is interesting that the Wick rotation procedure failed to reproduce the finite Dirac operator. The map W was constructed to give $W(\gamma_E^\mu) = \gamma_M^\mu$, which is why it makes sense when applied to the commutative Dirac operator, which consists of sums of gamma matrices. One would expect that applying it to the charge conjugation and chirality operators, which consist of products of gamma matrices, would be therefore considerably less well-motivated and so less successful, and yet in the finite setting the opposite was the case.

That said, the fact that the procedure as described worked as well as it did largely relied on the fact that the operators being transformed (D, J, $\chi\ldots$) had their own prior commutation relations, and the fact that the fundamental symmetry commuted or anticommuted with all the operators in question; in that context, applying the same transformation to the operators in question is sensible. Indeed, even if the end result for the reality operator and grading was to only pick up a factor of $-i$, it is still interesting that this change was geometrically meaningful in the almost-commutative case, to the extent that several signs worked out in the correct way. Indeed, the simplicity of the attempt should not be considered a count against it; if anything, it is an encouraging starting point for further investigation.

One direction such further investigation might go is the use of multitwists. A compelling reason to consider this possibility is that the Wick rotated Dirac operator (5.16) naturally decomposes into two pieces,

$$D_t = \frac{1}{2}(D + \beta D \beta) \quad \text{and} \quad D_s = -\frac{i}{2}(D - \beta D \beta), \tag{5.26}$$

these two pieces being self-adjoint and skew-self-adjoint respectively. We use subscripts t and s because, for $D = i\nabla\!\!\!/_E$, one finds $D_t = i\gamma^0\nabla_0$ and $D_s = i\gamma_M^j\nabla_j$, splitting the Wick-rotated Dirac operator into time and space components. This decomposition suggests the use of a multitwist instead of only a single twist. In that case, we expect the multitwisted real structure to satisfy

The names 't' (time) and 's' (space) should not be taken overly literally, at least from a physical perspective; when β is trivial, $D_t = D$ and $D_s = 0$.

$$\nu_k J_W D_k = \pm D_k J_W \nu_k, \quad k = t, s, \tag{5.27}$$

where one of the ν_k should be β. The choice is arbitrary, and so, for example, we might select $\nu_t = \beta$ for which we have $\nu_t D_t = D_t \nu_t$ and $\nu_t D_s = -D_s \nu_t$ by construction. This gives the sign of (5.27) as $\varepsilon' \varepsilon_\beta'''$ for $k = t$, where $DJ = \varepsilon' JD$ and $\beta J = \varepsilon_\beta''' J\beta$. What remains is the question of what to choose for ν_s.

Remark 5.14. It is not difficult to see that for almost-commutative spectral triples, for the different choices of finite space fundamental symmetry, the Wick rotation is able to place the D_F term wholly within $\hat{D}_t$, for example, or break it into block-diagonal and block-anti-diagonal parts which are split between $\hat{D}_t$ and $\hat{D}_s$. This leaves the door open to inferring metric signature from the finite space Dirac operator, an intriguing possibility. ⋄

If we use the same naïve Wick rotation map W to obtain J_W, a conservative but promising approach would be to construct ν_s from β. For example, one ansatz might be, assuming the spectral triple in question is even, $\nu_s = \pm\beta\chi$.In that case, the sign of (5.27) is $-\varepsilon'\varepsilon''\varepsilon'''_\beta$ for $k = s$, where $\chi J = \varepsilon'' J\chi$. This means the two signs would be consistent when $\varepsilon'' = -1$, *i.e.* when the original Euclidean product spectral triple was of KO-dimension 2 or 6. While this is an intriguing idea, it is unlikely to be enough by itself to recover the finite Dirac operators of the examples we looked at.

Incidentally, in the commutative case one can use $\gamma^0\gamma^5_M$ to define a Lorentz-invariant inner product $\langle\bullet, \gamma^0\gamma^5_M\bullet\rangle$. However, unlike $\langle\bullet, \gamma^0\bullet\rangle$, said inner product is pseudoscalar, i.e. *it picks up a sign under parity transformations.*

Staying with this conservative multitwist approach where ν_s is assumed to contain β, one could also generalise the Wick rotation map slightly to take advantage of the multiple twist operators, for example,

$$W' \colon T \mapsto \frac{1}{2}(T + \nu_t T^\dagger \nu_t^{-1}) - \frac{i}{2}(T - \nu_s T^\dagger \nu_s^{-1}),$$

assuming ν_s is still self-adjoint and unitary, although W' could easily be modified if ν_s is only self-adjoint (or unitary) up to sign. One would need to take care to ensure that Prop. 5.12 still held, perhaps with signs incorporated as necessary, but this should not be difficult.

Another idea, perhaps to be employed in concert with a multitwist, and somewhat less ambitious than the above, is to simply choose a different fundamental symmetry on the finite space, one which may not commute or anticommute with the finite Dirac operator, which would allow for more complicated Wick rotations, although this by itself would not been enough to recover the examples from §5.5.3. Similarly, one could also choose different Wick rotation maps for the different operators, perhaps one which by itself obtained the Lorentzian charge conjugation from the Euclidean one without reference to the twist, or which did not introduce the factor of i into the grading, more in line with the conventions common in the literature. Again, this would not solve all (or perhaps any) problems by itself, but could eventually be a worthy refinement regardless.

In addition to the practical issues around the specific implementation of Wick rotations using twisted real structures, there are various more conceptual issues which should ultimately be addressed as well. It was already mentioned in 5.2.1 that the ideal setting for physical applications of pseudo-Riemannian geometry is non-compact manifolds, and so it would be desirable to extend the formalism to non-unital algebras. On the topic of algebras, it should also be investigated how essential the assumption that $\beta\pi(a) = \pi(a)\beta$ is, and if (or to what extent) it can be avoided in greater generality. Another technical point to consider is that in §5.5 we relied upon the definition of the product of fundamental symmetries (5.23) to describe the twist operator in order to retain all the necessary properties, this definition is not the same as the one supposed in §3.4.

In brief, the Lorentzian Dirac operator $i\not{\nabla}_M$ is neither symmetric nor elliptic, and so the smooth domain may be singular, the spectrum is more complicated, . . .

And of course, there are the ever-present analytic issues surrounding the Dirac operator and the (bosonic) spectral action, which remain open questions more broadly within the study of Lorentzian noncommutative geometry. This also touches on a point we have danced around so far, namely that abandoning self-adjointness of the Dirac operator means any spectral triple with twisted real structure we obtain will not be a true spectral triple; it will no longer satisfy Def. 2.1. This is not an inherently negative thing; almost all approaches to spectral geometry use some more general framework than spectral triples to accommodate non-self-adjoint Dirac operators, Krein spaces, *etc*. Even so, one should take special care in arriving at this destination using twisted real structures, since they were originally motivated to stay as close to true spectral triples as possible, in direct opposition to the philosophy of approaches which begin by assuming a more general framework on first principles. And lastly, if one embraces the notion of twisted real structures for generalised spectral triples (of whatever flavour), does it make sense to have twists which are totally unrelated to the fundamental symmetry? What results still hold and which need to be changed or abandoned? These are interesting questions, but to be clear, a very long way from being answered – there are still many unknowns about generalisations of spectral triples that should be settled before speculating on such things.

6 TWISTED REAL STRUCTURES FOR TWISTED SPECTRAL TRIPLES

6.1 INTRODUCTION

Just as it is true that not all spectral triples admit a real structure, so it is true that not all twisted spectral triples admit a real structure (in the sense of Ref. [46]). Apart from the fact that this means that not all twisted spectral triples can be untwisted (in the sense of Ref. [8]) into spectral triples, it also suggests the possibility that twisted spectral triples could admit twisted real structures, just as weakening the conditions for a spectral triple to be real to expand the domain of applicability of 'real structures' motivates the consideration of twisted real structures for spectral triples. Indeed, the possibility of just such a twisted spectral triple with twisted real structure was briefly speculated upon in Ref. [8]. Their suggested definition can be formally expressed as follows:

Definition 6.1. A *twisted spectral triple with twisted real structure* is a twisted spectral triple $(A, \mathcal{H}, D)_\rho$ along with a reality operator $J : \mathcal{H} \to \mathcal{H}$, $J^\dagger = J^{-1}$, and twist operator ν such that (2.1), (2.3), (2.9), (2.10) and the following '(ν, ρ)-twisted first-order condition'

$$\Big[[D, a]^\pi_\rho, b\Big]^{\pi^\circ}_{\rho^{-1}\circ\hat{\nu}^2} = 0,$$

for all $a, b \in A$, are satisfied. ◇

By 'modular spectral triple', we take the same loose attitude as the papers cited, namely, broadly taking the framework of Ref. [43] – that a modular spectral triple is effectively a twisted spectral triple where the growth of the resolvent of the Dirac operator is measured with respect to a weight instead of a trace – sufficiently relaxed to incorporate the (non-unital) examples of interest.

This chapter of the thesis describes an example of just such a twisted spectral triple with twisted real structure found in the pre-existing literature, though not previously identified as such, with the notable difference to Def. 6.1 that the twist operator is unbounded. To be more precise, the example found is the modular spectral triple describing κ-Minkowski space found in Ref. [54], which will be described in some detail in §6.3, as well as an alternative construction which will be described in §6.6.1. §6.4 will show that the example cannot be untwisted, and §6.5 will show that the unorthodox "real structure" described in Ref. [54] is in fact a twisted real structure.

The constructions of κ-Minkowski space described in Refs. [42, 54] are especially interesting in the context of this thesis because the constructions are very different from the (twisted) spectral triples discussed up to this point. To be specific, Ref. [54] uses Hopf algebras acting on a $*$-algebra represented on a Hilbert space to construct a modular spectral triple, which also touches on interesting aspects of modular theory, whilst Ref. [42] arrives at the same end by taking a crossed product extension of a spectral triple, that is to say, by building the twisted spectral triple from the action of a C*-dynamical system on a spectral triple. That a twisted real structure is found to arise from such disparate techniques is itself a notable fact which suggests the concept may be quite widely applicable.

6.2 DEFINITIONS

In contrast to the previous chapters, we will take κ-Minkowski space to be non-compact, which, from the algebraic perspective, means that the algebra of the associated (twisted) spectral triple A should be non-unital. One necessary change to the definition Def. 2.1 to take account of the algebra being non-unital is that instead of requiring that the Dirac operator D have a compact resolvent, we instead impose the weaker requirement

$$\pi(a)(D^2 + 1)^{-1/2} \in \mathcal{K}(\mathcal{H})$$

for all $a \in A$. Further, to be clear, whilst κ-Minkowski space can be defined in arbitrarily many dimensions, for simplicity we will follow Ref. [54] and work in only two dimensions (one space and one time).

As a point of notation, in this chapter we will be working with a noncommutative algebra of functions endowed with a nonstandard involution. The Hilbert space will also be obtained from this algebra and involution. As such, the Hilbert space adjoint will still be denoted by $\dagger$, but $*$ will *not* represent complex conjugation. To avoid overloading notation causing confusion, we will switch to the mathematics convention and denote the complex conjugate of a function f by $\bar{f}$, and denote the complex conjugation map by c.c., or in other words, we write $\bar{f} \equiv \text{c.c.}(f)$.

As was mentioned in the previous section, κ-Minkowski space is understood as the noncommutative space whose symmetries are given by the κ-Poincaré quantum group (Hopf algebra). Since a minimal understanding of Hopf algebras is necessary for this chapter, we give the definition here, beginning first with the definition of a coalgebra.

Definition 6.2. A *coalgebra* C is a vector space over a field $\mathbb{K}$ endowed with a coproduct map $\Delta\colon C \longrightarrow C \otimes C$ which is coassociative, *i.e.* it satisfies $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$, and a counit map $\epsilon\colon C \longrightarrow \mathbb{K}$ obeying $(\epsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \epsilon) \circ \Delta$. ◇

For $c \in C$, the coproduct can be expressed as $\Delta c = \sum_j c_j^{(1)} \otimes c_j^{(2)}$, for $c_j^{(k)} \in C$. For what follows, we will use sumless Sweedler notation, where instead, for brevity, we write $\Delta c = c^{(1)} \otimes c^{(2)}$. For what concerns the field, in this chapter we will always take $\mathbb{K} = \mathbb{C}$.

Having introduced coalgebras, we can now define Hopf algebras as a particular kind of bialgebra equipped with an additional 'invertibility' structure.

Definition 6.3. A *bialgebra* H is a unital algebra which is also a coalgebra, such that Δ and ϵ are algebra homomorphisms and $H \otimes H$ has the tensor product structure $(h \otimes g)(h' \otimes g') = hh' \otimes gg'$ for all $h, h', g, g' \in H$. ⋄

Definition 6.4. A *Hopf algebra* H is a bialgebra equipped with an antipode map $S\colon H \to H$ satisfying $S(h^{(1)})h^{(2)} = \epsilon(h) = h^{(1)}S(h^{(2)})$ for all $h \in H$. ⋄

A Hopf algebra H can be endowed with a $*$-structure, making it a Hopf $*$-algebra, if H is a $*$-algebra and

$$\Delta(h^*) = (h^{(1)})^* \otimes (h^{(2)})^*, \quad \epsilon(h^*) = \overline{\epsilon(h)}, \quad (S \circ *)^2 = \mathrm{id}$$

for all $h \in H$. Furthermore, two Hopf $*$-algebras H, G are dual if they are dual as Hopf algebras and $\langle g^*, h\rangle = \overline{\langle g, S(h)^*\rangle}$ for all $h \in H, g \in G$.

6.3 CONSTRUCTING THE TWISTED SPECTRAL TRIPLE

In this section we give a summary of the construction of a twisted spectral triple $(A, \mathcal{H}, D\!\!\!/)_\sigma$ for κ-Minkowski space following the presentation given in Ref. [54], along with the associated grading χ and antiunitary operator J. None of the material in this section is original.

6.3.1 *The $*$-algebra*

The construction of the $*$-algebra A can be divided into roughly three parts. The first part is to define the κ-Poincaré Hopf algebra $\mathcal{P}_\kappa$, identify the (extended) momentum Hopf subalgebra $\mathcal{T}_\kappa$, and then define κ-Minkowski space $\mathcal{M}_\kappa$ as the Hopf algebra obtained via duality with $\mathcal{T}_\kappa$.

First we give the standard presentation. We define the two-dimensional κ-Poincaré Hopf algebra $\mathcal{P}_\kappa$ in terms of the generators P_0, P_1 and N satisfying the commutation relations

$$[P_0, P_1] = 0, \quad [N, P_0] = P_1, \quad [N, P_1] = \frac{\kappa}{2}(1 - e^{-2P_0/\kappa}) - \frac{1}{2\kappa}P_1^2;$$

at the physical level, these generators are associated to time translation (energy), space translation (momentum) and the Lorentz boost respectively. The coproduct $\Delta\colon \mathcal{P}_\kappa \to \mathcal{P}_\kappa \otimes \mathcal{P}_\kappa$ is given on the generators by

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0, \quad \Delta(P_1) = P_1 \otimes 1 + e^{-P_0/\kappa} \otimes P_1, \quad \Delta(N) = N \otimes 1 + e^{-P_0/\kappa} \otimes N,$$

and the counit $\epsilon\colon \mathcal{P}_\kappa \to \mathbb{C}$ and antipode $S\colon \mathcal{P}_\kappa \to \mathcal{P}_\kappa$ are given by

$$\epsilon(P_0) = \epsilon(P_1) = \epsilon(N) = 0, \quad \text{and}$$

$$S(P_0) = -P_0, \quad S(P_1) = -e^{P_0/\kappa}P_1, \quad S(N) = -e^{P_0/\kappa}N.$$

In order to make sense of terms like $e^{P_0/\kappa}$ as power series, κ^{-1} should be understood as a formal parameter such that the tensor product is defined over the ring of formal power series $\mathbb{C}[[\kappa^{-1}]]$.

We denote by $\mathcal{T}_\kappa$ the Hopf subalgebra which is generated by the translation generators P_0 and P_1. Then κ-Minkowski space is a Hopf algebra $\mathcal{M}_\kappa$ in a non-degenerate dual pairing with $\mathcal{T}_\kappa$. Denoting this pairing by $\langle \bullet, \bullet \rangle\colon \mathcal{T}_\kappa \times \mathcal{M}_\kappa \to \mathbb{C}$, the structure of $\mathcal{M}_\kappa$ is then given by the relations

$$\langle t, xy \rangle = \langle t^{(1)}, x \rangle \langle t^{(2)}, y \rangle, \quad \langle ts, x \rangle = \langle t, x^{(1)} \rangle \langle s, x^{(2)} \rangle,$$

for $t, s \in \mathcal{T}_\kappa$ and $x, y \in \mathcal{M}_\kappa$. We see from this pairing that $\mathcal{M}_\kappa$ is noncommutative and cocommutative (contrary to $\mathcal{T}_\kappa$, which is commutative and noncocommutative). Thus we have that the algebraic relations for $\mathcal{M}_\kappa$ are given by

$$[X_0, X_1] = -\kappa^{-1}X_1 \quad \text{and} \quad \Delta(X_\mu) = X_\mu \otimes 1 + 1 \otimes X_\mu.$$

There is an alternative formulation, however, which treats κ^{-1} as a number [36] such that the tensor product is the usual one over $\mathbb{C}$. In this case, one can reconsider $e^{-P_0/\kappa}$ not as a formal power series but an invertible element in its own right $\mathcal{E}$, in terms of which the defining relations can be rewritten like so:

$$[P_0, P_1] = 0, \quad [P_0, Q] = [P_1, Q] = 0;$$

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0, \quad \Delta(P_1) = P_1 \otimes 1 + Q \otimes P_1, \quad \Delta(Q) = Q \otimes Q;$$

$$\epsilon(P_0) = \epsilon(P_1) = 0, \quad \epsilon(Q) = 1;$$

$$S(P_0) = -P_0, \quad S(P_1) = -Q^{-1}P_1, \quad S(Q) = Q^{-1}.$$

The role of $\mathcal{T}_\kappa$ is then played by the so-called extended momentum algebra, the Hopf subalgebra generated by P_μ and Q which is also denoted by $\mathcal{T}_\kappa$. One once again then obtains $\mathcal{M}_\kappa$ by a pairing with $\mathcal{T}_\kappa$, and one further obtains a $*$-structure on $\mathcal{T}_\kappa$ by defining $P_\mu^* = P_\mu$ and $Q^* = Q$.

Technically, the computations to follow are carried out in Euclidean signature, in which case the "κ-Poincaré algebra" is really the quantum Euclidean group, but since we will soon discard the boost generator N, this is not such an important distinction here. Another point is that we will work with $\lambda := \kappa^{-1}$ rather than κ directly, since this allows us to define the classical limit as $\lambda \to 0$, analogous to the situation in physics with $\hbar \to 0$.

The second part of the construction identifies the Lie algebra underpinning $\mathcal{M}_\kappa$, from which one obtains the associated Lie group G, and hence the convolution algebra $L^1(G)$.

The underlying algebra of $\mathcal{M}_\kappa$ is the enveloping algebra of the Lie algebra generated by iX_0, iX_1 with $[X_0, X_1] = i\lambda X_1$, which has the faithful representation

There is only one nonabelian Lie algebra of dimension 2 up to isomorphism.

$$\phi(iX_0) = \begin{pmatrix} -\lambda & 0 \\ 0 & 0 \end{pmatrix}, \quad \phi(iX_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This Lie algebra exponentiates to a Lie group G with elements of the form

$$M(a) = M(a_0, a_1) = \begin{pmatrix} e^{-\lambda a_0} & a_1 \\ 0 & 1 \end{pmatrix},$$

with group multiplication

$$M(a_0, a_1)M(b_0, b_1) = M(a_0 + b_0, a_1 + e^{-\lambda a_0} b_1)$$

and inverse

$$M(a_0, a_1)^{-1} = M(-a_0, -e^{\lambda a_0} a_1).$$

The group G is not unimodular because the Lebesgue measure $\mathrm{d}^2 a$ is right-invariant but not left-invariant. We denote by $L^1(G)$ the convolution algebra of G with respect to the right-invariant measure. Functions on G are identified with functions on $\mathbb{R}^2$ using the parametrisation in terms of (a_0, a_1). Then $L^1(G)$ is an involutive Banach algebra of integrable functions on $\mathbb{R}^2$ with product $\hat{\star}$ and involution $\hat{*}$ given by

$$(f \hat{\star} g)(a) = \int \mathrm{d}^2 b\, f(a_0 - b_0, a_1 - e^{-\lambda(a_0 - b_0)} b_1) g(b_0, b_1),$$
$$f^{\hat{*}}(a) = e^{\lambda a_0} \bar{f}(-a_0, -e^{\lambda a_0} a_1).$$

Any unitary representation π_u of G gives a representation $\tilde{\pi}$ of $L^1(G)$

$$\tilde{\pi}(f) = \int \mathrm{d}^2 a\, f(a) \pi_u(M(a)),$$

which is a $\hat{*}$-representation since $\tilde{\pi}(f \hat{\star} g) = \tilde{\pi}(f)\tilde{\pi}(g)$ and $\tilde{\pi}(f^{\hat{*}}) = \tilde{\pi}(f)^\dagger$.

The final part of the construction comes from applying Weyl quantisation to $L^1(\mathbb{R}^2) \simeq L^1(G)$ to obtain the algebra A.

We now introduce the Fourier transform $\mathcal{F}$ on $\mathbb{R}^2$ (using the unitary convention) and Weyl-quantise. The Weyl transform is given by

$$W_{\tilde{\pi}}(f) := \tilde{\pi}(\mathcal{F} f)$$

where $f \in L^1(\mathbb{R}^2) \cap \mathcal{F}^{-1}(L^1(\mathbb{R}^2))$. With this we introduce a new product $\star$ (a Moyal product) and involution $*$ given by

$$f \star g = \mathcal{F}^{-1}(\mathcal{F} f \hat{\star} \mathcal{F} g), \quad f^* = \mathcal{F}^{-1}(\mathcal{F} f)^{\hat{*}}$$

such that $W_{\tilde{\pi}}(f \star g) = W_{\tilde{\pi}}(f)W_{\tilde{\pi}}(g)$ and $W_{\tilde{\pi}}(f^*) = W_{\tilde{\pi}}(f)^\dagger$.

Def. Schwartz space *is defined as the space of rapidly decreasing functions* $\mathcal{S}(\mathbb{R}^n, \mathbb{C}) := \{C^\infty(\mathbb{R}^n, \mathbb{C}) : \sup_{x \in \mathbb{R}^n} \left| x^I (\partial_x^J f)(x) \right| < \infty \, \forall I, J \in \mathbb{N}^n\}$ *where* I, J *are multi-indices and* $\partial_x^I = \partial_{x^1}^{I_1} \dots \partial_{x^n}^{I_n}$.

Now denote by $\mathcal{S}_c$ the Schwartz functions on $\mathbb{R}^2$ with compact support in the first variable (*i.e.* for $f \in \mathcal{S}_c$, we have $\operatorname{supp}(f) \subseteq K \times \mathbb{R}$ for $K \subset \mathbb{R}$ compact). We restrict ourselves to $\mathcal{S}_c$ to ensure that $\star$ and $*$ are well defined. We then define A as the involutive algebra given by the set $\mathcal{F}(\mathcal{S}_c)$ equipped with the product $\star$ and involution $*$. To be explicit,

$$(f \star g)(x) = \int \frac{\mathrm{d}p_0}{2\pi} e^{ip_0 x_0} (\mathcal{F}_0 f)(p_0, x_1) g(x_0, e^{-\lambda p_0} x_1),$$
$$f^*(x) = \int \frac{\mathrm{d}p_0}{2\pi} e^{-p_0 x_0} (\mathcal{F}_0 \bar{f})(p_0, e^{-\lambda p_0} x_1),$$

where $\mathcal{F}_0$ is the Fourier transform on the first variable, given explicitly by $(\mathcal{F}_0 f)(p_0, x_1) = \int \mathrm{d}y_0 \, e^{-ip_0 y_0} f(y_0, x_1)$.

6.3.1.1 *Action of the momentum Hopf algebra on A*

The extended momentum algebra $\mathcal{T}_\kappa$ has a natural action on A. To understand this, consider that a Hopf algebra H and $*$-algebra A can be considered compatible if A is an H-module $*$-algebra. By definition, A is a left H-module $*$-algebra if

1. A is a left H-module such that the representation respects the algebra structure of A, *i.e.* $h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$;
2. the representation respects the $*$-structure of A, *i.e.* $(h \triangleright a)^* = S(h)^* \triangleright a^*$.

One finds that indeed, A is a left $\mathcal{T}_\kappa$-module $*$-algebra, with the representation given by

$$(P_\mu \triangleright f)(x) = -i(\partial_\mu f)(x), \quad (Q \triangleright f)(x) = f(x_0 + i\lambda, x_1).$$

6.3.2 *The Hilbert space*

Def. *A* weight *on a* $*$*-algebra* A *is a linear map* $\omega \colon A_+ \to [0, \infty]$. *A weight is* finite *if* $\omega(a) < \infty$ *for all* $a \in A_+$.

To obtain the Hilbert space, we use the GNS construction coming from A. Since A is non-unital, we begin by choosing a finite weight ω and taking the ideal $\mathcal{I} = \{f \in A : \omega(f^* \star f) = 0\}$. We take the quotient $A/\mathcal{I}$ and define the inner product $(f, g) := \omega(f^* \star g)$. We then have that $\mathfrak{H} := \overline{A/\mathcal{I}}$ is a Hilbert space, completing with respect to the norm coming from the inner product.

A natural choice of weight is

$$\omega(f) := \int \mathrm{d}^2 x \, f(x),$$

where $\mathrm{d}^2 x$ is the Lebesgue measure, chosen because this weight is invariant under the action of $\mathcal{P}_\kappa$, *i.e.* $\omega(h \triangleright f) = \epsilon(h)\omega(f)$. This weight also satisfies the twisted-trace property,

$$\omega(f \star g) = \omega((Q \triangleright g) \star f),$$

and in particular, one has

$$\omega(f^* \star g) = \int \mathrm{d}^2x\, f^*(x)\overline{g^*}(x). \tag{6.1}$$

The Hilbert space obtained using (6.1) for the inner product and completing with respect to the norm is unitarily equivalent to $L^2(\mathbb{R}^2)$ via the map

$$(Uf)(x) = \int \frac{\mathrm{d}p_0}{2\pi} e^{ip_0x_0}(\mathcal{F}_0 f)(p_0, e^{\lambda p_0}x_1).$$

The unitary U passes to an antiunitary isometry $J := \text{c.c.} \circ U$ which gives $J\psi = \psi^*$ for $*$ the involution on A.

For the purposes of constructing the spectral triple, we define the $*$-representation $\pi(f)\psi = f \star \psi$ for $f \in A$ and $\psi \in \mathfrak{H}$. The operator $\pi(f)$ is bounded for any $f \in A$.

6.3.2.1 *The representation of* $\mathcal{T}_\kappa$

In §6.3.1.1, $\mathcal{T}_\kappa$ was given an action on A. We want this action to extend to a representation on $\mathfrak{H}$ as unbounded operators. Let $\mathscr{H}$ be a dense linear subspace of $\mathfrak{H}$ with the inner product $(\bullet, \bullet)$, and H a Hopf algebra. An unbounded $*$-representation of H on $\mathscr{H}$ is a homomorphism $\rho\colon H \to \mathrm{End}(\mathscr{H})$ such that

$$(\rho(h)\psi, \phi) = (\psi, \rho(h^*)\phi)$$

for all $\psi, \phi \in \mathscr{H}$ and $h \in H$. Further, let A be a left H-module $*$-algebra with $*$-representation π. Then π on $\mathscr{H}$ is H-equivariant/covariant if there exists a ρ such that

$$\rho(h)\pi(a)\psi = \pi(h_{(1)} \triangleright a)\rho(h_{(2)})\psi.$$

Hence one defines equivariance for operators T on $\mathfrak{H}$ more generally as $T\rho(h)\psi = \rho(h)T\psi$ for all $h \in H$ and $\psi \in \mathscr{H}$.

A natural choice of the dense subspace $\mathscr{H}$ is A, which is dense in $\mathfrak{H}$ by construction. Since A is a left $\mathcal{T}_\kappa$-module $*$-algebra, we obtain ρ from the action of $\mathcal{T}_\kappa$ on A via $\rho(h)\psi := h \triangleright \psi$ for every $h \in \mathcal{T}_\kappa$ and $\psi \in A$. Note that A is invariant under the action of $\mathcal{T}_\kappa$ and π is automatically equivariant. Note that $\rho(P_\mu)$ and $\rho(Q)$ are essentially self-adjoint on $\mathfrak{H}$, and we will use the same symbols to refer to their closures. Furthermore, as a point of notation we will write

$$\hat{P}_\mu := \rho(P_\mu).$$

6.3.2.2 *Modular theory*

At this point, it will be useful to state a useful result relating to modular theory. For any $t \in \mathbb{R}$ and $f \in A$, define $(\sigma_t^\omega f)(x) := f(x_0 - \lambda t, x_1)$.

Then σ^ω is a one-parameter group of $*$-automorphisms of A. Since ω satisfies the KMS condition at inverse temperature $\beta = 1$ with respect to σ^ω, σ^ω is the modular automorphism group which is implemented by the modular operator $\Delta_\omega = e^{-\lambda \hat{P}_0}$, *i.e.* for $f \in A$ we have $\pi(\sigma_t^\omega(f)) = \Delta_\omega^{it}\pi(f)\Delta_\omega^{-it}$.

6.3.3 *The Dirac operator*

For some background, we review the classical case of the Dirac operator $\not{D}$ on $\mathbb{R}^2$. Let $A = \mathcal{S}(\mathbb{R}^2)$ be represented on the Hilbert space $L^2(\mathbb{R}^2) =: \mathcal{H}_r$ by pointwise multiplication and let the representation of the Clifford algebra given by

$$\gamma^0 := \sigma^1 \quad \text{and} \quad \gamma^1 := \sigma^2,$$

where σ^1, σ^2 are Pauli matrices. The Hilbert space corresponding to the trivial spinor bundle is given by $\mathcal{H}_r \otimes \mathbb{C}^2 =: \mathcal{H}$ and the representation is extended onto the doubling diagonally. The inner product on $\mathcal{H}$ is given by

$$\langle \psi, \phi \rangle_{\mathcal{H}} = \int \mathrm{d}^2 x \left(\bar{\psi}_1(x)\phi_1(x) + \bar{\psi}_2(x)\phi_2(x) \right),$$

where ψ_i and ϕ_j are the spinor components of ψ and ϕ respectively. The Dirac operator is built from γ^μ and $\hat{P}_\mu = -i\partial_\mu$ as

$$\not{D} = \gamma^\mu \hat{P}_\mu = -\begin{pmatrix} 0 & i\partial_0 + \partial_1 \\ i\partial_0 - \partial_1 & 0 \end{pmatrix}.$$

Note that the Dirac operator anticommutes with the grading $\chi = -i\gamma^0\gamma^1$ as required.

We now move on to the deformed case following the same template as the classical case. We now denote by $\mathcal{H}_r$ the Hilbert space $\mathfrak{H}$ described in §6.3.2, constructed with respect to the algebra $A = \mathcal{S}(\mathbb{R}^2)$, which is represented on $\mathcal{H}_r$ by the multiplication $(\pi(f)\psi)(x) = (f \star \psi)(x)$. We again use the Hilbert space coming from the trivial spinor bundle $\mathcal{H} = \mathcal{H}_r \otimes \mathbb{C}^2$ with the $*$-representation π extended diagonally. The inner product on the doubled Hilbert space $\mathcal{H}$ is

$$\langle \psi, \phi \rangle_{\mathcal{H}} = \int \mathrm{d}^2 x \left((\psi_1^* \star \phi_1)(x) + (\psi_2^* \star \phi_2)(x) \right).$$

Using the classical Dirac operator, we run into troubles with the left multiplication:

$$\hat{P}_1 \pi(f)\psi = \rho(P_1)\pi(f)\psi = \pi(P_1 \triangleright f)\psi + \pi(Q \triangleright f)\rho(P_1)\psi.$$

P_1 does not obey the Leibniz rule due to the nontrivial coproduct of $\mathcal{P}_\kappa$, and the commutator is not bounded since $\rho(P_1)$ is unbounded. Explicitly,

$$\left[\not{D}, \pi(f)\right] = \gamma^\mu \pi(P_\mu \triangleright f) + \gamma^1 \pi((Q-1) \triangleright f)\rho(P_1).$$

It is necessary then to modify the Dirac operator and instead look for the boundedness of the *twisted* commutator with respect to some automorphism.

We want the new Dirac operator to be self-adjoint and anticommute with the grading, so it must be of the form

$$\not{D} = \gamma^\mu \hat{D}_\mu$$

where the $\hat{D}_\mu$ are self-adjoint on $\mathcal{H}_r$. It should reduce to the classical Dirac operator in the classical limit $\lambda \to 0$ and from symmetry arguments we assume that $\hat{D}_\mu = \rho(D_\mu)$ for some $D_\mu \in \mathcal{T}_\kappa$, with the twisting automorphism σ having the action $\sigma(f) = \sigma \triangleright f$, for $\sigma \in \mathcal{T}_\kappa$ (note that σ giving an automorphism requires that $\Delta(\sigma) = \sigma \otimes \sigma$). As no topology has been specified on $\mathcal{T}_\kappa$, any element can be written as a finite sum of products of generators, and since $\mathcal{T}_\kappa$ is commutative, that means any such element is a sum of terms of the form $P_0^i P_1^j Q^k$ with $i, j \in \mathbb{N}_0$ and $k \in \mathbb{Z}$.

If $T, \sigma \in \mathcal{T}_\kappa$, then $[\rho(T), \pi(f)]_\sigma = \rho(T)\pi(f) - \pi(\sigma \triangleright f)\rho(T)$ is bounded if and only if $\Delta(T) = T' \otimes 1 + \sigma \otimes T$ for some $T' \in \mathcal{T}_\kappa$. The question then is which elements of $\mathcal{T}_\kappa$ have a coproduct of this form. In short, the answer is that we must have $\sigma = Q^m$ for $m \in \mathbb{Z}$ and for coefficients $c_i \in \mathbb{C}$ we have

$$T = \begin{cases} c_1 1 + c_2 Q^m & \text{if } m < 0, \\ c_1 1 + c_2 P_0 & \text{if } m = 0, \\ c_1 1 + c_2 Q + c_3 P_1 & \text{if } m = 1, \\ c_1 1 + c_2 Q^m & \text{if } m > 1. \end{cases}$$

Now, we consider $\not{D}$ to reduce to the classical case in the classical limit if the length-dimension of $\not{D}$ is -1 (as in the classical case) and $\lim_{\lambda \to 0} \hat{D}_\mu \psi = \hat{P}_\mu \psi$. Satisfying this requirement while ensuring the boundedness of the twisted commutator (with $\not{D} = \gamma^\mu \rho(D_\mu)$ and σ the twisting automorphism) imposes that

Recall that λ is a physical parameter with length-dimension 1, and that coordinates x^μ also have length-dimension 1.

$$D_0 = \frac{1}{\lambda}(1 - Q), \quad D_1 = P_1, \quad \sigma = Q.$$

To be explicit, we therefore have

$$\begin{aligned} \not{D}\psi(x) &= \frac{1}{\lambda}\gamma^0\big((1 \triangleright \psi)(x) - (Q \triangleright \psi)(x)\big) + \gamma^1(P_1 \triangleright \psi)(x) \\ &= \frac{1}{\lambda}\gamma^0\big(\psi(x) - \psi(x_0 + i\lambda, x_1)\big) - i\gamma^1(\partial_1\psi)(x), \end{aligned}$$

with the slight abuse of notation that ψ stands in for the spinor components ψ_i. Note that this does indeed reduce to the classical Dirac operator in the classical limit $\lambda \to 0$, since

$$\lim_{\lambda \to 0} -i\gamma^0 \frac{1}{(i\lambda)}\big(\psi(x_0 + i\lambda, x_1) - \psi(x_0, x_1)\big) = -i\gamma^0\partial_0\psi(x) = \gamma^0\hat{P}_0\psi(x).$$

Note further that

$$
\begin{aligned}
\rho(Q)\psi(x) = \Delta_\omega\psi(x) = e^{-\lambda\hat{P}_0}\psi(x) \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}(-\lambda)^k(-i)^k(\partial_0^k\psi)(x) \\
&= \psi(x_0 + i\lambda, x_1) = (Q \triangleright \psi)(x).
\end{aligned}
$$

There is a nice connection between the twisting automorphism of the commutator, σ, and the modular theory of §6.3.2.2. In particular,

$$\pi(\sigma(f)) = \Delta_\omega\pi(f)\Delta_\omega^{-1}, \tag{6.2}$$

that is to say that σ is implemented by the modular operator. In fact, σ is the analytic continuation at $t = -i$ of the modular group σ_t^ω (see Ref. [21, §3.4]).

6.3.4 *Antiunitary operator*

The real structure in the classical case is given by

$$J = i\gamma^0 \circ \text{c.c.}$$

acting on $\mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$. This operator is not suitable for the deformed case without some changes, however.

First, we define

$$\tilde{J}f := \sigma_{i/2}^\omega(f^*)$$

for $f \in A$. Because ω satisfies the KMS condition with respect to σ^ω, the $\sigma_{i/2}^\omega$ term compensates for the lack of a trace, which allows $\tilde{J}$ to be an antilinear isometry with respect to the inner product on $\mathcal{H}_r$. To extend $\tilde{J}$ from $\mathcal{H}_r$ to $\mathcal{H} = \mathcal{H}_r \otimes \mathbb{C}^2$, we define

$$J = i\gamma^0\tilde{J} = i\gamma^0\sigma_{i/2}^\omega \circ *.$$

Ref. [54] refers to J as a 'real structure', but to avoid confusion we will not do so here.

This antiunitary operator satisfies some modified properties with respect to a typical real structure, in particular,

$$J^2 = 1 \tag{6.3a}$$

$$J\not{D} = -\Delta_\omega^{-1}\not{D}J \tag{6.3b}$$

$$\left[\pi(f), J\pi(g^*)J^{-1}\right] = 0 \tag{6.3c}$$

$$J\chi = -\chi J \tag{6.3d}$$

$$\left[\left[\not{D}, \pi(f)\right]_\sigma, J\pi(g)J^{-1}\right] = 0 \tag{6.3e}$$

where $\chi = -i\gamma^0\gamma^1$ is the grading and the KO-signs are chosen by assuming the classical case has KO-dimension 2. What is interesting about (6.3) is that the conditions are very reminiscent of those of a twisted real structure, though (6.3b) and (6.3e) are slightly off compared to what one would expect. The motivating question of this chapter is then: Is this similarity coincidental, or is J a twisted real structure 'in disguise'?

6.4 RESISTANCE TO UNTWISTING

In order to answer the question at the end of the preceding section, we must first reckon with the fact that κ-Minkowski space, as constructed above, is a twisted spectral triple. In particular, it is essential to know whether the untwisting procedure of [8] can be used to 'shift' the twist from the commutator with the Dirac operator onto the real structure.

Strictly speaking, this possibility is precluded *prima facie* since the modular spectral triple presented above does not satisfy either of (2.10) or (2.11) in [8], which are requirements of [8, Thm. 4.1]. That is to say, κ-Minkowski space is a twisted spectral triple with an antiunitary operator J, but not a *real* twisted spectral triple in the sense of [46].

Of course, this fact is what makes the example interesting in the first place, so we will proceed unabated. The key to untwisting is to turn a twisted commutator into a product of operators, at least one of which is a bounded commutator, and the natural starting point is commutator identities. Ref. [54] briefly looks at the commutator identity $[D, \pi(a)]_{\hat{\nu}^2} = \nu\left[\nu^{-1}D, \pi(a)\right]_{\hat{\nu}}$, but this does not completely remove the twist on the commutator. The untwisting of Ref. [8] is based on the commutator identity

$$[D, \pi(a)]_{\hat{\nu}^2} = \nu\left[\nu^{-1}D\nu^{-1}, \nu\pi(a)\nu^{-1}\right]\nu,$$

which seems the most promising lead, identifying the twist as $\nu^2 = \Delta_\omega$. We then have that the (ordinary) commutator on the right-hand side is

$$\left[\Delta_\omega^{-1/2}\not{D}\Delta_\omega^{-1/2}, \Delta_\omega^{1/2}\pi(f)\Delta_\omega^{-1/2}\right]. \tag{6.4}$$

There is no problem defining $\nu = \Delta_\omega^{1/2}$ since $\rho(Q)$ is self-adjoint, so we can define a unique positive square root via the spectral theorem.

There are no complications coming from the requirement that conjugation by $\Delta_\omega^{1/2}$ should implement an isomorphism since, using the (analytic continuation of the) modular automorphism group σ_t^ω, we find that $\hat{\nu}$ is an automorphism by identifying $\hat{\nu} = \sigma_t^\omega|_{t=-i/2}$.

We note that one can simplify $\Delta_\omega^{-1/2}\not{D}\Delta_\omega^{-1/2}$:

$$\begin{aligned}
\Delta_\omega^{-1/2}\not{D}\Delta_\omega^{-1/2} &= \gamma^0\frac{1}{\lambda}\rho(Q)^{-1/2}\left(\rho(1) - \rho(Q)\right)\rho(Q)^{-1/2} + \gamma^1\rho(Q)^{-1/2}\rho(P_1)\rho(Q)^{-1/2} \\
&= \gamma^0\frac{1}{\lambda}\left(\rho(Q)^{-1} - 1\right) + \gamma^1\rho(Q)^{-1}\rho(P_1) \\
&= \gamma^0\frac{1}{\lambda}\rho(Q)^{-1}\left(\rho(1) - \rho(Q)\right) + \gamma^1\rho(Q)^{-1}\rho(P_1) \\
&= \Delta_\omega^{-1}\not{D}
\end{aligned} \tag{6.5}$$

This motivates consideration of the alternative commutator identity

$$[D, \pi(a)]_{\hat{\nu}} = \nu\left[\nu^{-1}D, \pi(a)\right].$$

If we then identify $\nu = \Delta_\omega$, we have that the (ordinary) commutator on the right-hand side is simply

$$\left[\Delta_\omega^{-1}\not{D}, \pi(f)\right]. \tag{6.6}$$

The most important question to answer for the putative untwisting procedure is, however, whether (6.4) (or (6.6)) is bounded, which is absolutely necessary to recover an ordinary spectral triple. The easiest way to answer this question is to defer to [54, Lem. 22], which states that $[\rho(T), \pi(f)]_\sigma$, $T, \sigma \in \mathcal{T}_\kappa$, is bounded if and only if $\Delta(T) = T' \otimes 1 + \sigma \otimes T$ for some $T' \in \mathcal{T}_\kappa$. Here we have the case of $\sigma = 1$, and the commutator in question (taking (6.6) for simplicity, as there is no practical difference between the two) is

$$\left[\Delta_\omega^{-1} \not{D}, \pi(f)\right] = \gamma^0 \frac{1}{\lambda} \left[\rho(Q^{-1} - 1), \pi(f)\right] + \gamma^1 \left[\rho(Q^{-1} P_1), \pi(f)\right].$$

We have that $\Delta(Q^{-1} - 1) = Q^{-1} \otimes Q^{-1} - 1 \otimes 1$ and $\Delta(Q^{-1} P_1) = Q^{-1} P_1 \otimes Q^{-1} + 1 \otimes Q^{-1} P_1$, neither of which is of the correct form, and so neither commutator is bounded, and thus neither is the sum.

The commutator identities we have looked at are the most obvious choices, and are instructive, but we can generalise. In general, the 'untwisting' commutator identity is given by

$$[D, \pi(a)]_{\hat{\nu}^n} = \nu^m [\nu^{-m} D \nu^{m-n}, \pi(\hat{\nu}^{n-m}(a))] \nu^{n-m}$$

for some $n, m \in \mathbb{R}$. When ν commutes with D, this simplifies to

$$[D, \pi(a)]_{\hat{\nu}^n} = \nu^m [\nu^{-n} D, \pi(\hat{\nu}^{n-m}(a))] \nu^{n-m},$$

and we assume $\nu^{-n} D$ is a valid Dirac operator. It is straightforward to see that neither $\Delta(Q^{-n} - Q^{1-n})$ nor $\Delta(Q^{-n} P_1)$ will have the form $T' \otimes 1 + 1 \otimes T$ for any $n \in \mathbb{R}$ regardless of any other considerations, and thus we find that the twisted spectral triple of κ-Minkowski space cannot be untwisted into a spectral triple with twisted real structure.

6.5 RECONCILING FORMALISMS

The main result of the previous section forces us to concede that the twisted spectral triple cannot be untwisted, and so we must take said twist seriously. With that in mind, it is argued in Ref. [46] that the definition of a *real* twisted spectral triple, that is, a twisted spectral triple equipped with a real structure, ought to be as per Def. 2.8

However, as previously noted, if we compare Def. 2.8 to (6.3), the relations satisfied by $(A, \mathcal{H}, \not{D})_\sigma$ with the antiunitary J, we notice two important differences. Firstly, the commutation relation (6.3b) between J and $\not{D}$ involves the operator Δ_ω^{-1}, and secondly, rather than the twisted first-order condition (2.25), we have a kind of 'semi-twisted' first-order condition (6.3e). These alterations call to mind what happens when one twists the real structure of a real spectral triple, and so it is natural to ask if twisting the real structure of a twisted real structure can obtain the same result.

This kind of twisted spectral triple with twisted real structure was briefly speculated about in Ref. [8]. Therein, the authors proposed, for a real twisted spectral triple $(A, \mathcal{H}, D, J)_\rho$, the real structure J could be replaced by a twisted real structure (J, ν) satisfying $\nu J \nu = J$ with $\pi(\hat{\nu}(a)) := \nu \pi(a) \nu^{-1}$, $\hat{\nu} \in \mathrm{Aut}(A)$. Then the commutation between D and J would be replaced by

$$DJ\nu = \varepsilon' \nu J D \tag{6.7}$$

whilst the twisted first-order condition would be replaced by

$$\left[[D, a]^{\pi}_{\rho}, b\right]^{\pi^\circ}_{\rho^{-1}\circ\hat{\nu}^2} = 0, \tag{6.8}$$

for all $a, b \in A$.

To be precise, Ref. [8] proposes $[[D, a]^{\pi}_{\rho}, b]^{\pi_J}_{\rho\circ\hat{\nu}^{-2}} = 0$ rather than (6.8). The difference is not a problem, since if the condition holds for $b^* \in A$ it must also hold for $(b^*)^*$. For simplicity we have chosen to match Ref. [46] as closely as possible using π° instead of π_J, for which it is reasonable to replace $\hat{\nu}^{-2}$ with $\hat{\nu}^2$ assuming that $\nu^* = \nu$.

Following this template, clearly we require that ν be chosen such that $\hat{\nu}^2 = \sigma$. From (6.2), we can see fairly immediately that this means we require $\nu = \Delta_\omega^{1/2}$. With this in mind, we note that (6.3b) is equivalent to

$$\Delta_\omega^{1/2} J \not{D} = -\Delta_\omega^{-1/2} \not{D} J,$$

and the question of if (6.3b) is equivalent to (6.7) reduces to a question of the commutation of $\Delta_\omega^{-1/2}$ with $\not{D}$ and J. We have already explicitly shown in (6.5) that $\Delta_\omega^{-1/2}$ commutes with $\not{D}$. For what concerns J, since ν is diagonal on $\mathcal{H}$ we need only consider the action on $\mathcal{H}_r$. Mirroring a similar calculation from Ref. [54], we have

$$\tilde{J} \Delta_\omega^{1/2} \tilde{J} \psi = \sigma^\omega_{i/2} \left(\rho(e^{-\lambda P_0/2}) \sigma^\omega_{i/2}(\psi^*) \right)^* = \left(\sigma^\omega_{-i/2} \rho(e^{-\lambda P_0/2}) \sigma^\omega_{i/2}(\psi^*) \right)^*.$$

Since $\sigma^\omega_{i/2}$ commutes with $\rho(P_0)$, this simplifies to $\left(\rho(e^{-\lambda P_0/2})\psi^*\right)^*$. As for any $T \in \mathcal{T}_\kappa$ we have $\rho(T)\psi = T \triangleright \psi$ and compatibility with the $*$-structure requires $T \triangleright \psi^* = (S(T)^* \triangleright \psi)^*$, this further simplifies to

$$\left(\rho(e^{-\lambda P_0/2})\psi^*\right)^* = \left(e^{-\lambda P_0/2} \triangleright \psi^*\right)^* = S(e^{-\lambda P_0/2})^* \triangleright \psi = \rho\left(S(e^{-\lambda P_0/2})^*\right)\psi.$$

It is not hard to see that $S(e^{-\lambda P_0/2}) = e^{\lambda P_0/2}$, which is self-adjoint since P_0 is self-adjoint, and therefore

$$\tilde{J} \Delta_\omega^{1/2} \tilde{J} = \Delta_\omega^{-1/2}.$$

Using the fact that $\tilde{J}^2 = 1$ and passing to $\mathcal{H}$, we find

$$\nu J = J \nu^{-1},$$

which not only gives us that

$$\Delta_\omega^{1/2} J \not{D} = -\not{D} J \Delta_\omega^{1/2},$$

which has the form of (6.7), but also shows that the twist operator satisfies the regularity condition $\nu J \nu = J$, as we would expect given that σ is a regular twist automorphism in the sense of (2.20). Thus we conclude that $(J, \Delta_\omega^{1/2})$ is a twisted real structure for the twisted spectral triple $(A, \mathcal{H}, \not{D})_\sigma$.

Remark 6.5. In Ref. [8] and other literature on twisted real structures (including the previous chapters of this thesis), the twist is typically taken to the bounded. That is not the case here. However, for practical purposes, this is not a problem, since $\pi \circ \hat{\nu}$ still gives a bounded representation of A on account of ν being defined on $A \subset \mathcal{H}$. ⋄

6.6 FUTURE DIRECTIONS

6.6.1 *Crossed product extensions*

It was shown in Ref. [42] that the example of Ref. [54] fits into a more general framework. The idea is that a twisted spectral triple $(B, \mathcal{H}, D)_\rho$ can be constructed as the crossed product extension of a spectral triple $(A, \mathfrak{H}, \mathscr{D})$ by a C*-dynamical system[1] $(\mathcal{A}, G, \alpha)$. We give an extremely compressed summary of the construction below; see Ref. [42] for a much more detailed exposition. None of the material in this section is original and we caution the reader that the notation does not necessarily align with that used in the previous sections.

We first take A to be a dense $*$-subalgebra of the C*-algebra $\mathcal{A}$. We take G to be a second-countable locally compact Hausdorff group with Haar measure μ_G and modular function Δ_G. Then $\alpha\colon G \to \mathrm{Aut}(\mathcal{A})$ is a continuous homomorphism and the $*$-automorphisms α_g, $g \in G$, are assumed to preserve A. If A is represented on $\mathfrak{H}$ by a faithful nondegenerate map π, then (π, U) is a faithful nondegenerate covariant representation of $(\mathcal{A}, G, \alpha)$ on $\mathfrak{H}$, *i.e.* $G \ni g \mapsto U_g \in \mathcal{B}(\mathfrak{H})$ is a strongly continuous unitary representation of G and

$$U_g \pi(a) U_g^* = \pi(\alpha_g(a))$$

Def. *The* integrated form $\hat{\pi}$ *of a covariant representation* (π, U) *of* $G \ltimes_{\alpha,r} \mathcal{A}$ *is given by* $\hat{\pi} := \int_G \pi(f(g)) U_g \,\mathrm{d}\mu_G(g)$ *for any* $f \in L^1(G, A)$.

for all $a \in A$. Then $\hat{\pi}$ is the integrated form of (π, U) of $G \ltimes_{\alpha,r} \mathcal{A}$ on

$$\hat{\mathfrak{H}} = L^2(G, \mathrm{d}\mu_G) \otimes \mathfrak{H}.$$

We take $B \subset C_c(G, A)$ to be a (not uniquely determined) dense $*$-subalgebra of the C*-algebra $G \ltimes_{\alpha,r} \mathcal{A}$. We further suppose there exists a map $z\colon G \to \mathcal{B}(\mathfrak{H})$ such that

$$U_g \mathscr{D} U_g^* = z(g)^{*-1} \mathscr{D} z(g)^{-1},$$

and that there exists a positive continuous $Z(A)^\times$-valued α-1-cocycle $g \mapsto p(g)$ such that

$$\pi(p(g)) = z(g) z(g)^*$$

and $G \ni g \mapsto p(g)^{\pm 1} f(g) \in A$ is in B for any $f \in B$.

1 Standard references for C*-dynamical systems, covariant representations and (reduced) crossed products include Ola Bratteli and Derek W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 1*. Springer Science+Business Media, 1979, and Bruce Blackadar. *K-Theory for Operator Algebras*. Springer-Verlag, 1986.

The unbounded operator $\hat{\theta}\colon \hat{\mathfrak{H}} \to \hat{\mathfrak{H}}$ defined by

$$(\hat{\theta}\hat{\psi})(g) := U_g^* \pi(p(g)) U_g \hat{\psi}(g)$$

implements the map $\rho \in \mathrm{Aut}(C_c(G, \mathcal{A}))$ in the sense that

$$\hat{\pi}(\rho(f))\hat{\psi} = \hat{\theta}\hat{\pi}(f)\hat{\theta}^{-1}\hat{\psi}$$

for all $\hat{\psi} \in C_c(G, \mathfrak{H}) \subset \hat{\mathfrak{H}}$ and $f \in C_c(G, \mathcal{A})$. One can show that ρ is also an automorphism of B.

We now define the unbounded operator

$$(\hat{\mathscr{D}}\hat{\psi})(g) := U_g^* \mathscr{D} U_g \hat{\psi}(g),$$

which gives that $\left[\hat{\mathscr{D}}, \hat{\pi}(f)\right]_\rho$ is bounded for any $f \in B$. In addition to this, we define another unbounded operator

$$\lambda_1 1 + \lambda_2 \hat{\theta}$$

for $\lambda_1, \lambda_2 \in \mathbb{R}$ real parameters with $\lambda_2 \neq 0$. This operator also has bounded twisted commutator with $\hat{\pi}(B)$ for the twist automorphism ρ.

Having defined on $\hat{\mathfrak{H}}$ all the necessary operators, we can now describe the desired twisted spectral triple. The Hilbert space is given by the doubling

$$\mathcal{H} = \hat{\mathfrak{H}} \otimes \mathbb{C}^2,$$

on which the $*$-algebra B is represented by $\Pi = \hat{\pi} \otimes 1$, *i.e.* $\hat{\pi}$ acting diagonally. The Dirac operator is given by

$$D := \hat{\mathscr{D}} \otimes \sigma^1 + (\lambda_1 1 + \lambda_2 \hat{\theta}) \otimes \sigma^2$$

for σ^1, σ^2 Pauli matrices. Similarly to the representation, the twist automorphism ρ is simply implemented by $\Theta = \hat{\theta} \otimes 1$, *i.e.* by $\hat{\theta}$ acting diagonally.

In the case of $z(g) = z_g 1$ for $z_g \in \mathbb{C}^\times$, Ref. [42] defines an antiunitary operator modelled on that of Ref. [54]. That is, we suppose $(A, \mathfrak{H}, \mathscr{D})$ is equipped with a real structure $\tilde{J}$ of KO-dimension $(\varepsilon, \varepsilon')$ (not considering any grading). Then, assuming $\tilde{J}U_g = U_g\tilde{J}$ for all $g \in G$ and $\lambda_1 = \varepsilon'\lambda_2$, one defines the antilinear isometry

Note that when $z(g) = z_g 1$ we have $(\hat{\mathscr{D}}\hat{\psi})(g) = \left|z_g\right|^2 D\hat{\psi}(g)$ and $(\hat{\theta}\hat{\psi})(g) = \left|z_g\right|^2 \hat{\psi}(g)$.

$$(\hat{J}\hat{\psi})(g) := \Delta_G(g)^{-1/2} U_{g^{-1}} \tilde{J} \hat{\psi}(g^{-1})$$

for $\hat{\psi} \in C_c(G, \mathfrak{H})$. This is then taken to act diagonally $J = \hat{J} \otimes 1$ such that J satisfies

$$J^2 = \varepsilon, \quad JD = \varepsilon' \Theta^{-1} DJ, \tag{6.9}$$

$$\left[\Pi(f_1), \Pi_J(f_2)\right] = 0, \quad \left[[D, \Pi(f_1)]_\rho, \Pi_J(f_2)\right] = 0 \tag{6.10}$$

for all $f_1, f_2 \in B$, where $\Pi_J(f) := J\Pi(f)J$.

The specific example of Ref. [54] can be obtained by starting from the spectral triple $A = C_c(\mathbb{R})$ represented on $H = L^2(\mathbb{R}, \mathrm{d}p^1)$ by the left regular representation of $\mathbb{R}$, with Dirac operator $(D\psi)(p^1) := p^1\psi(p^1)$ for any $\psi \in C_c(\mathbb{R}) \subset \mathfrak{H}$. Identifying the Hilbert space with $L^2(\hat{\mathbb{R}}, \mathrm{d}\hat{p}^1)$ via the Fourier transform (where $\hat{\mathbb{R}}$ is the Pontryagin dual of $\mathbb{R}$), $\mathscr{D}$ then becomes $-i\partial_{\hat{p}^1}$.

As is clear from Ch. 6.3, in this case the left regular representation coincides with the GNS representation.

For the C*-dynamical system, one takes $\mathcal{A} = C^*(\mathbb{R})$ and $G = \mathbb{R}$ with the action

$$\alpha_{p^0}(f_{\mathbb{R}})(p^1) := e^{p^0} f_{\mathbb{R}}(e^{p^0} p^1)$$

for $p^0, p^1 \in \mathbb{R}$ and $f_{\mathbb{R}} \in A$. Then B should be dense inside the C*-algebra $\mathbb{R} \ltimes_{\alpha,r} C^*(\mathbb{R}) \simeq C^*(\mathbb{R} \ltimes \mathbb{R})$, and in order to match Ref. [54] it is chosen to be $\mathcal{S}_c(\mathbb{R}^2)$. For the Hilbert space, we have $\hat{\mathfrak{H}} = L^2(\mathbb{R}, \mathrm{d}p^0) \otimes \mathfrak{H} \simeq L^2(\mathbb{R}^2, \mathrm{d}^2 p)$. On this, we have the unitary rep given by

$$(U_{p^0}\psi)(p^1) = e^{p^0/2}\psi(e^{p^0}p^1)$$

for $\psi \in C_c(\mathbb{R}) \subset \mathfrak{H}$, and so $z(p^0) = e^{-p^0/2}1$. Thus

$$(\hat{\mathscr{D}}\hat{\psi})(p^0)(p^1) = e^{-p^0} p^1 \hat{\psi}(p^0)(p^1).$$

We have $(\hat{\theta}\hat{\psi})(p^0) = e^{-p^0}\hat{\psi}(p^0)$, and for the other part of the Dirac operator D, the assumed existence of a suitable operator $\hat{J}$ restricts $\lambda_1 = -\lambda_2$ so that it takes the form $(\lambda_1 1 + \lambda_2\hat{\theta})\hat{\psi}(p^0)(p^1) = \lambda_1(1 - e^{-p^0})\hat{\psi}(p^0)(p^1)$.

In order to obtain the presentation given in Ref. [54], one takes the Hilbert space $L^2(\mathbb{R}^2, \mathrm{d}^2x) \otimes \mathbb{C}^2$, which is obtained from $\mathcal{H}$ by taking the Fourier transform (with the representation obtained by intertwining Π with the Fourier transform map). This also recovers the correct expression for the Dirac operator from D.

6.6.2 *Open questions*

Interestingly, Ref. [42] also considers the case of conformal transformations of a spin manifold, though not with respect to an antiunitary operator. This construction differs slightly from that of Ref. [21] (in particular it uses a different Hilbert space) and so it could be interesting to see what the antiunitary operator built in the mould of J looks like. Irrespective of this particular example, though, it is also of interest to know how generally this construction of the antiunitary operator presented in Ref. [42] takes the form of a twisted real structure. In particular: is $\Theta^{1/2}$ always well defined, and if so, does it always satisfy the regularity condition $\Theta^{1/2}J = J\Theta^{-1/2}$?

Apart from these questions, another potential point of interest is the source of $\hat{\theta}$. It was not stressed in the above summary, but this operator is by definition intimately related to cocycles (perhaps unsurprisingly given the construction relies on a group action). Another aspect not stressed in the above summary is connections to the GNS construction, which provide an alternative perspective to the hypotheses listed in the above summary; these could also give some insight into Θ and J.

However, these are fairly subtle points. Indeed, though the construction of Ref. [42] is quite broad, and interesting in its own right from an abstract mathematical perspective, it is perhaps not broad enough with respect to the existence and form of the reality structure to say much about twisted real structures in general, that is to say, it perhaps leans too heavily on the model of Ref. [54] to provide additional insights.

Unrelated to Ref. [42], it was noted in Rem. 6.5 that the twist operator considered in this chapter is unbounded, in contrast to the literature which typically takes the twist operator to be bounded. This was possible for the example in question because of the way the twist was obtained, but it is a worthwhile question to ask under what general conditions the twist can be taken to be unbounded.

Lastly, it should not go without note that even though the example of κ-Minkowski space can be understood as a twisted spectral triple with twisted real structure which is neither a spectral triple with twisted real structure nor a real twisted spectral triple, it is nevertheless a very simple example. In particular, the twist on the commutator and the twist operator undo one another, in the sense that $\rho^{-1} \circ \hat{\nu}^2 = \text{id}$. It would be extremely desirable to find an example where this was not the case, especially one where ρ and $\hat{\nu}$ were unrelated, in which case one would expect to find very rich mathematical structures.

7 FINAL COMMENTS

7.1 ONGOING WORK

7.1.1 *Composition of conformal transformations*

One of the initial motivating examples for twisted real structures is the translation of the conformal transformation of the metric into the language of spectral triples [7]. However, there is an issue at the core of this transformation, and that is that it does not compose with itself. To be more precise, for

$$D \mapsto J\pi(k)J^{-1}DJ^{-1}\pi(k)J^{-1} =: D_{JkJ^{-1}}$$

the conformal transformation of the Dirac operator by an element $k \in A_+^\times$ outlined in §1.3, we can imagine conformally transforming $D_{JkJ^{-1}}$ by another element $k' \in A_+^\times$ such that

$$D_{JkJ^{-1}} \mapsto J\pi(k')J^{-1}D_kJ^{-1}\pi(k')J^{-1} = J\pi(k'k)J^{-1}DJ^{-1}\pi(kk')J^{-1}.$$

The problem is that when A is not commutative, $k'k \neq kk'$, which means $J\pi(k')J^{-1}D_kJ^{-1}\pi(k')J^{-1}$ is *not* a conformal transformation of D.

Of course, one could choose to accept that conformal transformations may simply not be defined in such a way that composition makes sense when A is not commutative, but this is, of course, a rather unsatisfying tack to take. Much more interesting is to attempt to reformulate what it means to perform a conformal transformation in such a way that the transformations compose. This is not at all a simple task, which can be demonstrated by a simple example (from here onwards we suppress π).

Lemma 7.1. *Let $(A, \mathcal{H}, D, (J, \nu))$ be a spectral triple with twisted real structure. Then the conformal transformation by $k \in A_+^\times$ can be defined as*

$$D \mapsto JkJ^{-1}DJkJ^{-1}, \quad J \mapsto \nu J, \quad \nu \mapsto \hat{\nu}(k^{-1})JkJ^{-1};$$

or alternatively,

$$D \mapsto J\hat{\nu}(k)J^{-1}DJ\hat{\nu}(k)J^{-1}, \quad J \mapsto \nu J, \quad \nu \mapsto k^{-1}J\hat{\nu}(k)J^{-1},$$

where in both cases $\nu = \nu^{-1}$.

We do not claim that these proposed transformations are unique (or definitive), but it is difficult to imagine other possibilities since in any case they must give the original transformation of Ref. [7] when $\nu = 1$. Indeed, even between the two possibilities described, the former seems conceptually preferable, since otherwise the transformation being *conformal* might be lost unless one imposes that, for example, $\hat{\nu}$ leaves $A_+^\times$ invariant, which is even more restrictive. And of course, the main thrust of this example is that even without demanding that ν be related to conformal transformations itself, we are still forced to assume $\nu = \nu^{-1}$ (which, of course, conformal transformations do not themselves satisfy).

Furthermore, it is worthwhile to point out that even in the commutative case, where $(D_k)_{k'} = (D_{k'})_k = D_{kk'}$ holds, conformal transformations do not lend themselves to being expressed as fluctuations. We have, for example,

$$D_{kk'} = k^2 D_{k'} + k[D_{k'}, k],$$

which does not have the desired form unless $k^2 = 1$, which does not rescale the metric. This problem can be partially resolved, even in the noncommutative case, by using a twisted fluctuation with the automorphism $\sigma(a) = (a^*)^{-1}$. This resolves the issue of the pre-factor, since

The twist automorphism $\rho_k(a) := k^2 a k^{-2}$ which gives a bounded twisted commutator with D_k does not arise from a fluctuation.

$$(D_{k'})_k = D_{k'} + k[D_{k'}, k]_\sigma,$$

but comes at the cost that the twisted commutator is not bounded, and so is not a (twisted) 1-form as is usually understood. Understanding if this fluctuation is well-defined in any meaningful sense is a matter of ongoing investigation.[1]

In any case, as should be clear, this is predicated on working within the (extended) framework of twisted spectral triples, but supposing a real structure J is reintroduced and we have the map $D \mapsto D_{JkJ^{-1}}$, we should expect some simplifications to come from using the framework of twisted real structures coming from the conformal parts. While this will likely not be sufficient to banish all twisted commutators, we might instead end up with something like (generalised) twisted spectral triples with twisted real structures à la Ch. 6.

7.1.2 *Twisted real structures without the twisted first-order condition*

As was discussed in §4.5.1, it is possible to talk about fluctuations of the Dirac operator without reference to any first-order condition. Thereafter, we offered the twisted first-order condition as a possible way to weaken the first-order condition without abandoning it altogether. Of course, one should be precise here: whilst the twisted first-order condition does weaken the first-order condition, it is not true that the twisted first-order condition can then be thought of as a strengthening of the the case 'no first-order condition' case presented in Ref. [12].

The reason for this is that Ref. [12] employs an ordinary (trivially-twisted) real structure, or, in the language of Ch. 4, the fluctuations are

1 Joint work with L. Dąbrowski, A. Sitarz and Y. Liu.

build from the derivations δ and δ°, rather than δ and $\delta^{\odot}$, which are the appropriate choices for a twisted real structure. This means that the fluctuations in the absence of any first-order condition ought to be different when one is using a twisted real structure, which is not difficult to check.

Some alteration to the form of the fluctuation is to be expected due to the twisted ε'-condition (2.9). The fact that we should recover the fluctuation (4.2) when the twisted first-order condition is imposed means we can take the ansatz

$$\omega_{(2,\nu)} = \sum_{j,k} \pi_J\big(\hat{\nu}^{-1}(a_j)\big)\pi(a_k)\big[[D,\pi(b_k)],\hat{\nu}(b_j)\big]^{\pi_J}_{\hat{\nu}^{-2}}$$

for the quadratic term, using that the twisted first-order condition (2.11) can be written as $[[D,\pi(a)],\hat{\nu}(b)]^{\pi_J}_{\hat{\nu}^{-2}} = 0$ for all $a, b \in A$. A lengthy but straightforward computation then confirms that

$$\begin{aligned}\varepsilon'\nu J\omega_{(2,\nu)}J^{-1}\nu &= \sum_{i,j} \pi(a_j)\pi_J\big(\hat{\nu}^{-1}(a_k)\big)\Big[[D,\hat{\nu}(b_k)]^{\pi_J}_{\hat{\nu}^{-2}},\pi(b_j)\Big] \\ &= \omega_{(2,\nu)},\end{aligned}$$

where the second equality is obtained using Lem. 2.11 and the zeroth-order condition, and relabelling the summation indices. This result makes sense since it includes an ordinary commutator (as in δ) and a $\hat{\nu}^{-2}$-twisted commutator (as in δ°). As such, in the absence of the twisted first-order condition, we expect fluctuations (in the presence of a twisted real structure) to have the form

$$D \mapsto D + \omega + \varepsilon'\nu J\omega J^{-1}\nu + \omega_{(2,\nu)} = D_\omega.$$

It is possible to compute the transformation of D_ω in terms of the operators $\widetilde{\mathrm{Ad}}(u)$ and $\mathrm{Ad}(u)^{-1}$ as in Prop. 4.18. Term by term, this yields (suppressing π)

$$\begin{aligned} D &\mapsto D + u[D,u^*] + \varepsilon'\nu Ju[D,u^*]J^{-1}\nu \\ &\qquad + \pi_J\big(\hat{\nu}^{-1}(u)\big)u[[D,u^*],\hat{\nu}(u^*)]^{\pi_J}_{\hat{\nu}^{-2}}, \\ \omega &\mapsto u\omega u^* + \pi_J\big(\hat{\nu}^{-1}(u)\big)[u\omega u^*,\hat{\nu}(u^*)]^{\pi_J}_{\hat{\nu}^{-2}}, \\ \varepsilon'\nu J\omega J^{-1}\nu &\mapsto \varepsilon'\nu Ju\omega u^*J^{-1}\nu + \varepsilon' u\big[\nu Ju\omega u^*J^{-1}\nu, u^*\big], \\ \omega_{(2,\nu)} &\mapsto u\omega_{(2,\nu)}u^* + \pi_J\big(\hat{\nu}^{-1}(u)\big)\big[u\omega_{(2,\nu)}u^*,\hat{\nu}(u)\big]^{\pi_J}_{\hat{\nu}^{-2}}. \end{aligned}$$

Supposing that $D^u_\omega = \widetilde{\mathrm{Ad}}(u)D_\omega\,\mathrm{Ad}(u)^{-1}$, we obtain

$$\begin{aligned} D^u_\omega = D &+ \omega^u + \varepsilon'\nu J\omega^u J^{-1}\nu + u\omega_{(2,\nu)}u^* \\ &+ \pi_J\big(\hat{\nu}^{-1}(u)\big)\big[u\omega_{(2,\nu)}u^*,\hat{\nu}(u)\big]^{\pi_J}_{\hat{\nu}^{-2}} \\ &+ \pi_J\big(\hat{\nu}^{-1}(u)\big)[u\omega u^*,\hat{\nu}(u^*)]^{\pi_J}_{\hat{\nu}^{-2}} + \varepsilon' u\big[\nu Ju\omega u^*J^{-1}\nu, u^*\big] \\ &+ \pi_J\big(\hat{\nu}^{-1}(u)\big)u[[D,u^*],\hat{\nu}(u^*)]^{\pi_J}_{\hat{\nu}^{-2}}. \end{aligned}$$

This is equivalent to

$$\omega^u_{(2,\nu)} = \pi_J\big(\hat{\nu}^{-1}(u)\big)[\omega^u, \hat{\nu}(u^*)]^{\pi_J}_{\hat{\nu}^{-2}} \\ + \pi_J\big(\hat{\nu}^{-1}(u)\big)\left(\sum_j \pi_J\big(\hat{\nu}^{-1}(a_j)\big)\left[\omega^u, \hat{\nu}(b_j)\right]^{\pi_J}_{\hat{\nu}^{-2}}\right)\pi_J\big(\hat{\nu}(u^*)\big),$$

which is precisely the result one would expect working in analogy to Ref. [12]. The proof of an equivalent to Ref. [12, Lem. 3] has not yet been fully and rigorously worked out, but this result strongly suggests the result can be obtained by the strategic replacement of certain commutators with twisted commutators, following the schema for gauge transformations described in §4.3.

The purpose of abandoning the first-order condition in Ref. [12] is that it allows the fluctuations of the Dirac operator to form a semigroup, which is not ordinarily the case (*cf.* Rmk. 4.11). This extends fluctuations beyond being produced by gauge transformations or otherwise inserted by hand, and assuming this result can be extended to the twisted setting (for twisted real structures, or possibly even further to twisted spectral triple with twisted real structures building on Ref. [53]), they could be not only interesting in the abstract but useful for practical application to the project of the subsection above, although in that case the normalisation of the algebra action would surely need to be abandoned.

7.2 CONCLUDING REMARKS

In this thesis, we have presented novel advancements on the theory of twisted real structures for spectral triples in noncommutative geometry. Our approach has been primarily based in the search for applications, and in particular, we have examined applications of twisted real structures to Hodge-de Rham (Riemannian) spectral triples in Ch. 3, and generalised the approach to gauge theory for spectral triples with real structures to those with twisted real structures in Ch. 4. The twist operators which emerged as most meaningful in the context of gauge theory hinted at a possible implementation of Wick rotation, which was investigated in Ch. 5, and in Ch. 6 we explored an example of a twisted spectral triple equipped with a twisted real structure. These results, combined with the existing literature, demonstrate that twisted real structures show potential and may yet play an important role in the broader theory of spectral triples and related topics in noncommutative geometry. This potential is underpinned by the promise of the ongoing work described in the previous section, and to further punctuate the point, in the remainder of this final section, we will present open questions, loose ends, and various other possible directions for future work which have not yet already been mentioned.

7.2.1 *(Almost-)commutative spectral geometry*

The main contention of Ch. 3 is that twisted real structures allow one to extend the notion of 'reality' to Riemannian spectral triples, but we stopped short of attempting to interpret this result. As was mentioned, there are already hints from spectral approaches to particle physics that there may be some connection between differential forms and Dirac spinors on the finite side, so understanding the (noncommutative) algebraic equivalents of de Rham forms and spinors on the commutative side may help clarify matters, since there we have classical de Rham forms and spinors to compare against.

On the topic of the second-order/Hodge condition, it is interesting to ponder if this should also be appropriately twisted when considering twisted real structures, just as the first-order condition may be twisted. If not, it would be interested to know why not, and if so, it would be interesting to know how, since this is itself not necessarily obvious: the simplest approach is to replace $\big[[D,\pi(a)],\big[D,J\pi(b)J^{-1}\big]\big]=0$ with $\Big[[D,\pi(a)],[D,b]^{\pi_J}_{\hat{\nu}^{-2}}\Big]=0$, but preliminary computations suggest

$$\Big[[D,\pi(a)],[D,b]^{\pi_J}_{\hat{\nu}^{-2}}\Big]_{\bar{\nu}^2}=0$$

may be more appropriate. In either case, how a twisted second-order condition should relate to *e.g.* the Hodge condition would need to be determined.

Taking inspiration from Ch. 4, another possibility worth investigating (closely related to the earlier matter of noncommutative de Rham forms and spinors) is that if one equips the Hodge-de Rham spectral triple with a twisted real structure, then the twisted real structure may allow for nontrivial inner fluctuations of the Dirac operator $-i(d-d^\dagger)$, which cannot happen for the canonical real spectral triple of a spin manifold.[2] This could in turn lead to some interesting geometrical features (in isolation or when coupled with a finite spectral triple).

One final idea related to Riemannian spectral triples[3] comes from the observation that, as was duly noted in Ch. 3, the Hodge-de Rham spectral triple can be equipped with a real structure C_1 or a twisted real structure (C_2,ν). Both of these spectral triples are even with respect to the grading χ_{deg}, but both reality operators exhibit a strange commutation relation with $\chi_\star$. For example,

$$C_1\chi_\star=(-1)^{n/2}\chi_\star C_1$$

for n the dimension of the manifold. As a result, whether the spectral triple has KO-dimension 0 or 6 (mod 8) depends on whether or not $n/2$ is even or odd, which is intriguing behaviour. It would be interesting to know if this behaviour could be regulated by a twist

2 A similar result holds for real twisted spectral triples, *cf.* [46, Prop. 5.3].
3 We thank A. Rubin for this suggestion.

operator, *i.e.* if there exists a twisted real structure which satisfies the twisted ε'-condition (2.9) and the twisted ε''-condition (2.13) (and if not, whether this points towards a refinement of twists).

For what concerns the material of Ch. 4, it would be interesting to see if the construction of the twisted-opposite 1-forms could be done abstractly (perhaps in terms of twisted derivations), only introducing the representation in terms of operators at the end, similar to the construction of the noncommutative 1-forms from the universal 1-forms, although this is contrary to the approach needed for multitwists, which we only treated very lightly and is especially deserving of a more detailed analysis.

Regarding gauge transformations specifically, we attempted to use twist operators satisfying $\nu = \nu^\dagger = \nu^{-1}$ to maintain the maximum compatibility with the fermionic and bosonic action functionals. However, if we disregard the bosonic action functional, what is really essential for all other purposes is $\nu = \nu^\dagger$. This is much less restrictive on what twist operators can be selected; indeed, a natural example to investigate in this case would be the gauge transformation of a conformal transformation. It may even be possible to recover a workable bosonic action functional using the Connes-Lott action [19] instead of the spectral action, although this is just speculation.

The main suggestion at the end of Ch. 4 was that the twist may play the role of a Krein structure, a possibility which was interrogated in Ch. 5. This investigation was only very preliminary, and though the formalism developed did not recover something which one could reliably identify as 'noncommutative pseudo-Riemannian geometry', some tantalising hints were found that merit a more systematic investigation. For example, one thing which we did not fully investigate was the more indirect approach of attempting to recover a Lorentzian action whilst remaining within the formalism of (notionally Riemannian) spectral triples, which is the spirit of the approach taken in Ref. [33], the paper which inspired §5.4. An immediate advantage of taking this approach is that it neatly circumvents the problem that the adoption of the Wick-rotated Dirac operator left us no longer working with a true spectral triple.

7.2.2 *Products*

On a somewhat more technical point nevertheless closely related to the above is the matter of products. Firstly, one issue left open from Ch. 3 was the (re)definition of the product reality operator in such a way as to preserve the second-order condition. As was noted, in general this product reality operator does not satisfy $J^2 = \pm 1$ (except for certain KO-dimensions) or $JD = \pm DJ$. This latter failure is strongly suggestive that the 'real structure' may need to be twisted, especially since a twisted real structure was already used to ensure the Hodge-de Rham spectral triple satisfied the Hodge condition. Given that the product reality operator appears to relate the product Dirac operators D and D' (given respectively by the graded products $\hat{\otimes}$ and $\hat{\otimes}'$), this twist could be related to the unitary operator U' in the comment on p.24. That there could be a twist connecting the second-order/Hodge condition and the implementation of the (graded) tensor product is an especially intriguing possibility.

Indeed, the precise relationship between twisted real structures and products has remained messy throughout. As mentioned, a proposal to redefine the product of the reality operator was put forward in Ch. 3, although its precise relationship to twists or the product of twists was not investigated. The standard product was used in Chs. 2 and 4, with the simplest (naïve) product of twist operators used, and in both cases this placed strong limits on the form of the twist operator. Ch. 5 then employed graded tensor products coming from Clifford algebraic considerations. Even where these products coincide with the graded tensor product interpretation of the standard product for spectral triples, the product used for the twist operators was that of the fundamental symmetry, which differs from the naïve product, even when graded. It would be worthwhile to investigate the relationship between products and twisted real structures in a deliberate and systematic way to understand if all of these disparate threads can be brought together into a single unified framework.

7.2.3 *New directions*

Apart from the questions and problems branching off directly from what was covered in this thesis, it is worthwhile to take a step back and identify other, totally new directions that research into twisted real structures could go into, which we ourselves have not yet been able to pursue. We present two suggestions here.

Firstly, it was mentioned in §1.3 that the two examples which inspired twisted real structures were conformal transformations and quantum cones. Our research has primarily built upon the former example, and we have focused almost entirely on commutative and almost-commutative spectral triples. However, There are compelling reasons to believe that a wealth of research could be undertaken on so-called 'quantum spaces', those spaces which are obtained by q-deformations. In particular, a number of such spaces which admit a description as spectral triples, when equipped with a reality operator, do not satisfy the first-order condition, but instead, satisfy something weaker like

$$\big[[D, \pi(a)], J\pi(b)J^{-1}\big] \in \mathcal{K}(\mathcal{H}).$$

Especially well-known quantum groups with this behaviour include $\mathrm{SU}_q(2)$ [29] and the Podleś spheres [23]. It is well known that this is a somewhat generic feature of quantum groups, and so they should provide a wealth of new examples of spectral triples with twisted real structures. This would also provide another connection to Hopf algebras and perhaps Hopf-Galois extensions, which would be especially interesting in light of the link to strong connections mentioned at the end of §4.2.2.3.

Totally unrelated to q-deformations, in this thesis, a number of times we used (or took inspiration from) the untwisting procedure of Ref. [8], where a spectral triple with twisted real structure could be obtained from a real twisted spectral triple, loosely suggesting some notion of 'equivalence', where we are careful to speak in imprecise terms. A very different notion of 'untwisting' is presented in Ref. [40], wherein certain types of twisted spectral triples are found to correspond to spectral triples of the same K-homology class via a logarithmic dampening procedure. The motivation for this work comes primarily from index-theoretic considerations, and while the result in no way relates to the real structure (which not considered at all), it is curious that a second, totally unrelated method of translating twisted spectral triples into spectral triples exists, and it would be interesting to know if a meaningful comparison could be drawn by extending the result to KR-homology.

7.2.4 *Origins and interpretation*

Lastly, as was mentioned at the beginning of this section, our approach to the research presented in this thesis has been very practically-minded, mostly focussing on finding examples and applications. However, this has left certain philosophical questions about the research program open, questions which, though not addressed in our research, we still feel are deeply interesting and worthy of consideration.

First and foremost, the ultimate (theoretical) source of twisted real structures – and explanations for where they come from, why and when they work, what their geometric meaning is, *etc*. – is still not clear. Consider, for example, the Clifford/von Neumann algebraic construction for reality operators which was presented in §1.2; we lack any sort of parallel construction for twist operators. Indeed, the reason we presented the results of Refs. [42, 54] in Ch. 6 is that in using Hopf algebras and crossed product extensions respectively, they both touch on areas of noncommutative geometry which the literature on twisted real structures has not yet brushed up against, which produces novel perspectives like the already mentioned connection to cocycles. By examining a wealth of different approaches which each yield similar structures, our hope is that we may yet gain some insight into the answers to these questions, or at least a meaningful starting point to further prosecute the issue.

Finally, if we may be granted the luxury of wild speculation, it may be that the concept of twisted real structures is not quite the 'correct' one, and that some slightly different perspective is the better way forward. For one idea, it could be that we are over-emphasising the role of twists, and taking the fact that νJ and $J\nu$ appear so frequently as a hint, it could be that what we are really looking for is some notion of 'bi-reality', where instead of replacing the real structure with a twisted real structure, it should instead be replaced with a pair of generalised reality operators, taking the roles of νJ and $J\nu$ respectively and coinciding in the trivial case. Alternatively, at the other end of the spectrum, it could be that our focus on mild twists thus far has hidden the importance of the twist operator, and focussing more on unbounded or antilinear twists, or on multitwists which aggressively depart from the nearest single-twist case, would yield a deeper understanding.

Whatever the case may be, we dearly hope that the work presented in this thesis has helped to sow the fertile ground of this subject and that we in the future reap fruit rich in mathematical structure and physical applications, fulfilling the exciting promise we have described.

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colophon

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