Periodic solutions of second order differential equations in Hilbert spaces

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Abstract

We prove the existence of periodic solutions of some infinite-dimensional systems by the use of the lower/upper solutions method. Both the well-ordered and non-well-ordered cases are treated, thus generalizing to systems some well established results for scalar equations.

1 Introduction

The use of lower and upper solutions in boundary value problems dates back to the pioneering papers of Peano [19] in 1885 and Picard [20] in 1893. The first results for the periodic problem were obtained by Knobloch [15] in 1963. There is nowadays a large literature on this subject, dealing with different types of boundary conditions for ordinary and partial differential equations of elliptic or parabolic type (see, e.g., [5, 7] and the references therein).

In this paper we consider the periodic problem

$$(P) \qquad \begin{cases} \ddot{x} = f(t,x)\,, \\ x(0) = x(T)\,, \quad \dot{x}(0) = \dot{x}(T)\,. \end{cases}$$

In the scalar case when $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous, the $\mathcal{C}^2$-functions $\alpha, \beta : [0,T] \to \mathbb{R}$ are said to be lower/upper solutions of problem (P), respectively, if

$$\ddot{\alpha}(t) \geq f(t,\alpha(t))\,, \qquad \ddot{\beta}(t) \leq f(t,\beta(t))\,.$$

for every $t \in [0,T]$, and

$$\alpha(0) = \alpha(T)\,, \quad \beta(0) = \beta(T)\,, \quad \dot{\alpha}(0) \geq \dot{\alpha}(T)\,, \quad \dot{\beta}(0) \leq \dot{\beta}(T)\,.$$

We say that (α, β) is a well-ordered pair of lower/upper solutions if $\alpha \leq \beta$. It is well known that, when such a pair exists, problem (P) has a solution x such that $\alpha \leq x \leq \beta$.

When the inequality $\alpha \leq \beta$ does not hold, we say that the lower and upper solutions are non-well-ordered. In this case, with the aim of obtaining existence results, some further conditions have to be added in order to avoid resonance with the positive eigenvalues of the differential operator $-\ddot{x}$ with T-periodic conditions (recall that 0 is an eigenvalue, and all the other eigenvalues are positive). See [1, 6, 11, 12, 13, 18] for results in this direction.

The aim of this paper is to extend those classical existence results for scalar equations to systems, both in a finite-dimensional and in an infinite-dimensional setting.

Bebernes and Schmitt [3] generalized the scalar well-ordered case to a system of type (P), with $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$. Their result is reported in Section 2 below, in a slightly more general version. We are not aware of any results for systems in the non-well-ordered case, not even in the finite-dimensional case.

In Section 3 we provide an existence result for a system in $\mathbb{R}^N$ when the components of the lower/upper solutions can be both well-ordered and non-well-ordered. In order to avoid resonance with the higher part of the spectrum, for simplicity we ask the function f to be globally bounded in the non-well-ordered components, even if such an assumption could certainly be weakened (see the remarks in Section 5).

The case of a system in an infinite-dimensional Banach space E has been analyzed by Schmitt and Thompson [21] in 1975 for boundary value problems of Dirichlet type. However, when facing the periodic problem, they needed to assume E to be finite-dimensional, concluding their paper by saying: "Whether the results of this section [...] remain true in case E is infinite dimensional is not known at this time". We are not aware of any progress in this direction till now. In this paper we will try to give a partial answer to this question.

In Section 4 we extend our existence result of Section 3 to an infinite-dimensional separable Hilbert space. The lack of compactness is recovered by assuming the lower and upper solutions to take their values in a Hilbert cube. Moreover, we ask the function f to be globally bounded and completely continuous in the non-well-ordered components. These assumptions are reminiscent of an infinite-dimensional version of the Poincaré–Miranda Theorem as given in [16].

The study of periodic solutions for infinite-dimensional Hamiltonian systems has been already faced by several authors, see, e.g., [2, 4, 8, 10, 14]. Our approach does not need a Hamiltonian structure, and could be applied also to systems with nonlinearity depending on the derivative of x, provided some Nagumo-type condition is assumed. Such kind of systems were studied, e.g., in [21]. In Section 5 we will discuss on these and other possible extensions and generalizations of our results, possibly also to partial differential equations of elliptic or parabolic type.

2 Well-ordered lower and upper solutions for systems

In this section and the next one we consider the problem

$$(P) \qquad \begin{cases} \ddot{x} = f(t,x)\,, \\ x(0) = x(T)\,, \quad \dot{x}(0) = \dot{x}(T)\,, \end{cases}$$

where $f : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function. We are thus in a finite-dimensional setting. Let us recall a standard procedure to reduce the search of solutions of (P) to a fixed point problem in Banach space. We define the set

$$\mathcal{C}^2_T = \{x \in \mathcal{C}^2([0,T], \mathbb{R}^N) : x(0) = x(T),\ \dot{x}(0) = \dot{x}(T)\}\,,$$

and the linear operator

$$\mathcal{L} : \mathcal{C}^2_T \to \mathcal{C}([0,T], \mathbb{R}^N)\,, \quad \mathcal{L}x = -\ddot{x} + x\,.$$

which is invertible and has a bounded inverse. We consider as well the Nemytskii operator

$$\mathcal{N} : \mathcal{C}([0,T], \mathbb{R}^N) \to \mathcal{C}([0,T], \mathbb{R}^N)\,, \quad (\mathcal{N}x)(t) = x(t) - f(t, x(t))\,.$$

Problem (P) is so equivalent to the fixed point problem in $\mathcal{C}([0,T],\mathbb{R}^N)$

$$x = \mathcal{L}^{-1}\mathcal{N}x\,.$$

Notice that $\mathcal{L}^{-1}\mathcal{N} : \mathcal{C}([0,T],\mathbb{R}^N) \to \mathcal{C}([0,T],\mathbb{R}^N)$ is completely continuous.

Here, we recall and slightly generalize [3, Theorem 4.1].

Definition 1. *Given two $\mathcal{C}^2$-functions $\alpha,\beta : [0,T] \to \mathbb{R}^N$, we say that (α,β) is a* well-ordered pair of lower/upper solutions *of problem (P) if, for every $j \in \{1,\dots,N\}$ and $t \in [0,T]$,*

$$\alpha_j(t) \le \beta_j(t)\,,$$

$$\alpha_j(0) = \alpha_j(T)\,, \quad \beta_j(0) = \beta_j(T)\,, \quad \dot\alpha_j(0) \ge \dot\alpha_j(T)\,, \quad \dot\beta_j(0) \le \dot\beta_j(T)\,,$$

and, for every $x \in \prod_{m=1}^N [\alpha_m(t),\beta_m(t)]$,

$$\begin{aligned}\ddot\alpha_j(t) &\ge f_j(t,x_1,\dots,x_{j-1},\alpha_j(t),x_{j+1},\dots,x_N)\,,\\ \ddot\beta_j(t) &\le f_j(t,x_1,\dots,x_{j-1},\beta_j(t),x_{j+1},\dots,x_N)\,.\end{aligned}$$

Theorem 2 (Bebernes–Schmitt). *If there exists a well-ordered pair of lower/upper solutions (α,β), then problem (P) has a solution $x(t)$ such that*

$$\alpha_j(t) \le x_j(t) \le \beta_j(t)\,, \quad \text{for every } j \in \{1,\dots,N\} \text{ and } t \in [0,T]\,. \tag{1}$$

Proof. *Step 1.* Define the functions $\gamma_j : [0,T]\times\mathbb{R} \to \mathbb{R}$ as

$$\gamma_j(t,s) = \begin{cases} \alpha_j(t) & \text{if } s < \alpha_j(t)\,,\\ s & \text{if } \alpha_j(t) \le s \le \beta_j(t)\,,\\ \beta_j(t) & \text{if } s > \beta_j(t)\,,\end{cases}$$

and the functions $\Gamma, \bar f : [0,T]\times\mathbb{R}^N \to \mathbb{R}^N$ as

$$\Gamma(t,x) = (\gamma_1(t,x_1),\dots,\gamma_N(t,x_N))\,, \quad \bar f(t,x) = f(t,\Gamma(t,x))\,.$$

Consider the auxiliary problem

$$(P') \qquad \begin{cases} \ddot x = \bar f(t,x) + x - \Gamma(t,x)\,,\\ x(0) = x(T)\,, \quad \dot x(0) = \dot x(T)\,.\end{cases}$$

and the corresponding Nemytskii operator

$$\widetilde{\mathcal{N}} : \mathcal{C}([0,T],\mathbb{R}^N) \to \mathcal{C}([0,T],\mathbb{R}^N)\,, \quad (\widetilde{\mathcal{N}}x)(t) = \Gamma(t,x(t)) - \bar f(t,x(t))\,.$$

Problem (P') can then be equivalently written as a fixed point problem in $\mathcal{C}([0,T],\mathbb{R}^N)$, namely

$$x = \mathcal{L}^{-1}\widetilde{\mathcal{N}}x\,.$$

By Schauder Theorem, since $\mathcal{L}^{-1}\widetilde{\mathcal{N}} : \mathcal{C}([0,T],\mathbb{R}^N) \to \mathcal{C}([0,T],\mathbb{R}^N)$ is completely continuous and has a bounded image, it has a fixed point, so that (P') has a solution $x(t)$.

Step 2. Let us show that (1) holds for every solution of (P'), thus proving the theorem. By contradiction, assume that there is a $j \in \{1,\dots,N\}$ and a $t_j \in [0,T]$ for which $x_j(t_j) \notin [\alpha_j(t_j),\beta_j(t_j)]$. For instance, let $x_j(t_j) < \alpha_j(t_j)$ (the case $x_j(t_j) > \beta_j(t_j)$ being similar). Set $v_j(t) = \alpha_j(t) - x_j(t)$, and let $\hat t_j \in [0,T]$ be such that $v_j(\hat t_j) = \max\{v_j(t) : t \in [0,T]\}$. We distinguish two cases.

Case 1: $\hat{t}_j \in]0,T[$. In this case, surely $\ddot{v}_j(\hat{t}_j) \le 0$. On the other hand,

$$\begin{aligned}
\ddot{v}_j(\hat{t}_j) &= \ddot{\alpha}_j(\hat{t}_j) - \ddot{x}_j(\hat{t}_j) \\
&= \ddot{\alpha}_j(\hat{t}_j) - \bar{f}_j(\hat{t}_j, x(\hat{t}_j)) - x_j(\hat{t}_j) + \gamma_j(\hat{t}_j, x_j(\hat{t}_j)) \\
&> \ddot{\alpha}_j(\hat{t}_j) - f_j(\hat{t}_j, \gamma_1(\hat{t}_j, x_1(\hat{t}_j)), \dots, \alpha_j(\hat{t}_j), \dots, \gamma_N(\hat{t}_j, x_N(\hat{t}_j))) \ge 0 \,,
\end{aligned}$$

leading to a contradiction.

Case 2: $\hat{t}_j = 0$ or $\hat{t}_j = T$. Assume for instance that $\hat{t}_j = 0$ (the other situation being similar). Then,

$$0 \ge \dot{v}_j(0) = \dot{\alpha}_j(0) - \dot{x}_j(0) \ge \dot{\alpha}_j(T) - \dot{x}_j(T) = \dot{v}_j(T) \,,$$

so that, being $v_j(T) = v_j(0)$ the maximum value of $v_j(t)$ over $[0,T]$, it has to be that $\dot{v}_j(T) = 0$, hence also $\dot{v}_j(0) = 0$. Now, since $v_j(0) > 0$, there is a small $\delta > 0$ such that $v_j(s) > 0$, for every $s \in [0,\delta]$. Then if $t \in [0,\delta]$, we have that $x_j(s) < \alpha_j(s)$, for every $s \in [0,t]$, hence

$$\begin{aligned}
\dot{v}_j(t) &= \dot{v}_j(0) + \int_0^t \ddot{v}_j(s)\,\mathrm{d}s \\
&= \int_0^t \Big(\ddot{\alpha}_j(s) - \ddot{x}_j(s)\Big)\,\mathrm{d}s \\
&= \int_0^t \Big(\ddot{\alpha}_j(s) - \bar{f}_j(s, x(s)) - x_j(s) + \gamma_j(s, x_j(s))\Big)\,\mathrm{d}s \\
&> \int_0^t \Big(\ddot{\alpha}_j(s) - f_j(s, \gamma_1(s, x_1(x)), \dots, \alpha_j(s), \dots, \gamma_N(s, x_N(x)))\Big)\,\mathrm{d}s \ge 0 \,,
\end{aligned}$$

a contradiction, since 0 is a maximum point for $v_j(t)$. The proof is thus completed. □

We now provide some illustrative examples.

Example 3. Let, for every $j \in \{1,\dots,N\}$,

$$f_j(t,x) = a_j x_j^3 + h_j(t,x) \,,$$

for some constants $a_j > 0$, and assume that there is a $c > 0$ such that

$$|h(t,x)| \le c \,, \qquad \text{for every } (t,x) \in [0,T] \times \mathbb{R}^N. \tag{2}$$

Then, taking the constant functions $\alpha_j = -\sqrt[3]{c/a_j}$, $\beta_j = \sqrt[3]{c/a_j}$, we see that Theorem 2 applies, hence (P) has a solution.

Example 4. Let us consider, for every $j \in \{1,\dots,N\}$,

$$f_j(t,x) = x_j^2 \sin x_j + h_j(t,x) \,,$$

and assume that there is a $c > 0$ such that (2) holds. Then, for every $\ell \in \mathbb{Z}$ with $|\ell|$ sufficiently large, taking the constant functions $\alpha_j = -\pi/2 + 2\ell\pi$, $\beta_j = \pi/2 + 2\ell\pi$, we see that Theorem 2 applies, and we conclude that (P) admits an infinite number of solutions.

In order to work with Leray-Schauder degree, we need to introduce the notions of *strict* lower/upper solutions.

Definition 5. *The well-ordered pair of lower/upper solutions (α,β) of problem (P) is said to be* strict *if $\alpha_j(t) < \beta_j(t)$ for every $j \in \{1,\dots,N\}$ and $t \in [0,T]$, and the following property holds:*

if $x(t)$ is a solution of (P) such that for a certain $j \in \{1, \dots, N\}$, $\alpha_j(t) \le x_j(t) \le \beta_j(t)$ holds for every $t \in [0, T]$, then one has $\alpha_j(t) < x_j(t) < \beta_j(t)$ for every $t \in [0, T]$.

When we have a well-ordered pair of strict lower/upper solutions, the previous theorem provides some additional information.

Theorem 6. *If (α, β) is a strict well-ordered pair of lower/upper solutions of problem (P), then*

$$d(I - \mathcal{L}^{-1}\mathcal{N}, \Omega) = 1\,,$$

where

$$\Omega := \big\{x \in \mathcal{C}([0, T], \mathbb{R}^N) \,:\, \alpha_j(t) < x_j(t) < \beta_j(t)\,,\ \textit{for every } j \in \{1, \dots, N\} \textit{ and } t \in [0, T]\big\}\,.$$

Proof. Arguing as in Step 1 of the proof of Theorem 2, we can introduce the modified problem (P') and we know, by Schauder Theorem, that

$$d(I - \mathcal{L}^{-1}\widetilde{\mathcal{N}}, B_R) = 1\,,$$

where B_R is an open ball in $\mathbb{R}^N$ centered at the origin with a sufficiently large radius $R > 0$. In particular, we may assume that $\Omega \subseteq B_R$. By the argument in Step 2 of the same proof and the fact that the pair of lower/upper solutions is strict, we have that all the solutions of (P') belong to Ω. In other words, there are no zeroes of $I - \mathcal{L}^{-1}\widetilde{\mathcal{N}}$ in the set $B_R \setminus \overline{\Omega}$. Then, by the excision property of the degree,

$$d(I - \mathcal{L}^{-1}\widetilde{\mathcal{N}}, \Omega) = 1\,.$$

Finally, since $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ coincide on the set Ω, the conclusion follows. □

3 Non-well-ordered lower and upper solutions for systems

In this section we still consider problem (P) in the finite-dimensional space $\mathbb{R}^N$. We will treat the case in which we can find lower and upper solutions which are not well-ordered. To this aim, we need to distinguish the components which are well-ordered from the others.

We will say that the couple $(\mathcal{J}, \mathcal{K})$ is a partition of the set of indices $\{1, \dots, N\}$ if and only if $\mathcal{J} \cap \mathcal{K} = \varnothing$ and $\mathcal{J} \cup \mathcal{K} = \{1, \dots, N\}$. Correspondingly we can decompose a vector

$$x = (x_1, \dots, x_N) = (x_n)_{n \in \{1, \dots, N\}} \in \mathbb{R}^N$$

as $x = (x_\mathcal{J}, x_\mathcal{K})$ where $x_\mathcal{J} = (x_j)_{j \in \mathcal{J}} \in \mathbb{R}^{\#\mathcal{J}}$ and $x_\mathcal{K} = (x_k)_{k \in \mathcal{K}} \in \mathbb{R}^{\#\mathcal{K}}$. Here $\#\mathcal{J}$ and $\#\mathcal{K}$ denote respectively the cardinality of the sets $\mathcal{J}$ and $\mathcal{K}$.

Similarly, every function $\mathcal{F} : \mathcal{A} \to \mathbb{R}^N$ can be written as $\mathcal{F}(x) = \big(\mathcal{F}_\mathcal{J}(x), \mathcal{F}_\mathcal{K}(x)\big)$ where $\mathcal{F}_\mathcal{J} : \mathcal{A} \to \mathbb{R}^{\#\mathcal{J}}$ and $\mathcal{F}_\mathcal{K} : \mathcal{A} \to \mathbb{R}^{\#\mathcal{K}}$.

Definition 7. *Given two $\mathcal{C}^2$-functions $\alpha, \beta : [0, T] \to \mathbb{R}^N$ we will say that (α, β) is a* pair of lower/upper solutions of (P) related to the partition $(\mathcal{J}, \mathcal{K})$ of $\{1, \dots, N\}$ *if the following four conditions hold:*

1. *for any $j \in \mathcal{J}$, $\alpha_j(t) \le \beta_j(t)$ for every $t \in [0, T]$;*
2. *for any $k \in \mathcal{K}$, there exists $t_k^0 \in [0, T]$ such that $\alpha_k(t_k^0) > \beta_k(t_k^0)$;*

3. *for any $n \in \{1, \dots, N\}$ we have*

$$\ddot{\alpha}_n(t) \geq f_n(t, x_1, \dots, x_{n-1}, \alpha_n(t), x_{n+1}, \dots, x_N)\,, \tag{3}$$
$$\ddot{\beta}_n(t) \leq f_n(t, x_1, \dots, x_{n-1}, \beta_n(t), x_{n+1}, \dots, x_N)\,, \tag{4}$$

for every $(t, x) \in \mathcal{E}$, where

$$\mathcal{E} := \Big\{(t, x) \in [0, T] \times \mathbb{R}^N \,:\, x = (x_{\mathcal{J}}, x_{\mathcal{K}})\,, x_{\mathcal{J}} \in \prod_{j \in \mathcal{J}} [\alpha_j(t), \beta_j(t)]\Big\}\,.$$

4. *for any $n \in \{1, \dots, N\}$,*

$$\alpha_n(0) = \alpha_n(T)\,, \qquad \beta_n(0) = \beta_n(T)\,,$$
$$\dot{\alpha}_n(0) \geq \dot{\alpha}_n(T)\,, \qquad \dot{\beta}_n(0) \leq \dot{\beta}_n(T)\,.$$

Definition 8. *The pair (α, β) of lower/upper solutions of (P) is said to be* strict with respect to the j-th component, *with $j \in \mathcal{J}$, if $\alpha_j(t) < \beta_j(t)$ for every $t \in [0, T]$, and for every solution x of (P) we have*

$$\big(\forall t \in [0, T]\,,\, \alpha_j(t) \leq x_j(t) \leq \beta_j(t)\big) \Rightarrow \big(\forall t \in [0, T]\,,\, \alpha_j(t) < x_j(t) < \beta_j(t)\big)\,; \tag{5}$$

it is said to be strict with respect to the k-th component, *with $k \in \mathcal{K}$, if for every solution x of (P) we have*

$$\big(\forall t \in [0, T]\,,\, x_k(t) \geq \alpha_k(t)\big) \Rightarrow \big(\forall t \in [0, T]\,,\, x_k(t) > \alpha_k(t)\big)\,, \tag{6}$$
$$\big(\forall t \in [0, T]\,,\, x_k(t) \leq \beta_k(t)\big) \Rightarrow \big(\forall t \in [0, T]\,,\, x_k(t) < \beta_k(t)\big)\,. \tag{7}$$

The following proposition provides a sufficient condition in order to guarantee the *strictness* property of a pair of lower/upper solutions of (P) with respect to a certain component.

Proposition 9. *Given a pair (α, β) of lower/upper solutions of (P),*

1. *if, for any $n \in \mathcal{J}$, both* (3) *and* (4) *hold with strict inequalities, then* (5) *holds for $n = j$;*
2. *if, for any $n \in \mathcal{K}$,* (3) *holds with strict inequality, then* (6) *holds for $n = k$;*
3. *if, for any $n \in \mathcal{K}$,* (4) *holds with strict inequality, then* (7) *holds for $n = k$.*

The proof can be easily adapted from the corresponding scalar result in [5, Proposition III-1.1] and is omitted.

We are able to prove the existence of a solution of (P) in presence of a pair of lower/upper solutions (α, β) provided that we ask the *strictness* property when the components α_k, β_k are non-well-ordered.

Theorem 10. *Let (α, β) be a pair of lower/upper solutions of (P) related to the partition $(\mathcal{J}, \mathcal{K})$ of $\{1, \dots, N\}$, and assume that it is strict with respect to the k-th component, for every $k \in \mathcal{K}$. Assume moreover the existence of a constant $C > 0$ such that*

$$|f_{\mathcal{K}}(t, x)| \leq C\,, \quad \text{for every } (t, x) \in \mathcal{E}\,.$$

Then, (P) has a solution x with the following property: for any $(j, k) \in \mathcal{J} \times \mathcal{K}$,

(W_j) $\alpha_j(t) \le x_j(t) \le \beta_j(t)$, *for every* $t \in [0,T]$ *;*

(NW_k) *there exist* $t_k^1, t_k^2 \in [0,T]$ *such that* $x_k(t_k^1) < \alpha_k(t_k^1)$ *and* $x_k(t_k^2) > \beta_k(t_k^2)$ *.*

In Section 3.2 we will provide a generalization of the above result, removing the strictness assumption on one of the components $\kappa \in \mathcal{K}$. Let us now present two illustrative examples.

Example 11. Assume $\mathcal{J} = \varnothing$ and let, for every $k \in \mathcal{K}$,

$$f_k(t,x) = -\frac{a_k x_k}{1+|x_k|} + h_k(t,x)\,,$$

for some $a_k > 0$, with

$$\|h_k\|_\infty := \sup\left\{|h_k(t,x)| : (t,x) \in [0,T] \times \mathbb{R}^N\right\} < a_k\,. \tag{8}$$

Then, taking the constant functions

$$\alpha_k = \frac{\|h_k\|_\infty}{a_k - \|h_k\|_\infty} + 1\,, \qquad \beta_k = -\frac{\|h_k\|_\infty}{a_k - \|h_k\|_\infty} - 1\,,$$

we see that Theorem 10 applies. The same would be true if $\mathcal{J} \neq \varnothing$, assuming for $j \in \mathcal{J}$, e.g., a situation like in Examples 3 and 4.

Example 12. Let

$$f_n(t,x) = -a_n \sin x_n + h_n(t,x)\,,$$

with $a_n > 0$ and h_n satisfying (8) with $k = n$. For every $n \in \{1,\dots,N\}$ we have constant lower and upper solutions

$$\alpha_n \in \left\{\frac{\pi}{2} + 2m\pi : m \in \mathbb{Z}\right\}, \qquad \beta_n \in \left\{-\frac{\pi}{2} + 2m\pi : m \in \mathbb{Z}\right\}.$$

Then, for each equation we have both well-ordered and non-well-ordered pairs of lower/upper solutions. Let us fix, e.g.,

$$\alpha_n = \frac{\pi}{2}\,, \qquad \beta_n^\iota = \frac{\pi}{2} + \iota\pi\,, \qquad \text{with } \iota \in \{-1,1\}\,.$$

Choosing $\vec{\iota} = (\iota_1,\dots,\iota_N) \in \{-1,1\}^N$, and defining (α,β) with $\beta_n = \beta_n^{\iota_n}$, by Theorem 10 we get the existence of at least 2^N solutions $x^{\vec{\iota}}$ of problem (P), whose components are such that

$$\begin{aligned} \iota_n = 1 \qquad &\Rightarrow \quad \forall t \in [0,T],\ x_n^{\vec{\iota}}(t) \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \\ \iota_n = -1 \qquad &\Rightarrow \quad \exists \bar{t}_n \in [0,T],\ x_n^{\vec{\iota}}(\bar{t}_n) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \end{aligned}$$

We notice that, even if the function $h(t,x_1,\dots,x_n)$ is 2π-periodic in each variable x_n, the solutions we find are indeed geometrically distinct. We thus get a generalization of a result obtained for the scalar equation in [17].

3.1 Proof of Theorem 10

Notice that the case $\mathcal{K} = \varnothing$ reduces to Theorem 2. We thus assume $\mathcal{K} \neq \varnothing$ and, without loss of generality, we take either $\mathcal{J} = \varnothing$, or $\mathcal{J} = \{1,\dots,M\}$ and $\mathcal{K} = \{M+1,\dots,N\}$ for a certain $M \in \{1,\dots,N\}$. Indeed, mixing the coordinates of $x = (x_1,\dots,x_N)$, we can always reduce to such a situation. We continue the proof in the case $\mathcal{J} \neq \varnothing$. (The case $\mathcal{J} = \varnothing$ can be treated essentially in the same way.)

We need to suitably modify problem (P). For every $r > 0$, we consider the problem

$$(P_r) \qquad \begin{cases} \ddot{x} = g_r(t,x)\,, \\ x(0) = x(T)\,, \quad \dot{x}(0) = \dot{x}(T)\,, \end{cases}$$

where $g_r : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$, with

$$g_r(t,x) = \big(g_{r,1}(t,x), \dots, g_{r,M}(t,x), g_{r,M+1}(t,x), \dots, g_{r,N}(t,x)\big)\,,$$

is defined as follows.

We first introduce the functions $\bar{f} : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ and $\Gamma : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ as

$$\begin{aligned} &\bar{f}(t,x) = f(t, \Gamma(t,x))\,, \\ &\Gamma(t,x) = \big(\gamma_1(t,x_1), \dots, \gamma_M(t,x_N), x_{M+1}, \dots, x_N\big)\,. \end{aligned}$$

where, for $j \in \mathcal{J}$,

$$\gamma_j(t,s) = \begin{cases} \alpha_j(t)\,, & \text{if } s < \alpha_j(t)\,, \\ s\,, & \text{if } \alpha_j(t) \le s \le \beta_j(t)\,, \\ \beta_j(t)\,, & \text{if } s > \beta_j(t)\,. \end{cases}$$

Now we define, for every index $j \in \mathcal{J}$,

$$g_{r,j}(t,x) = \bar{f}_j(t,x) + x_j - \gamma_j(t,x_j)\,,$$

and for every index $k \in \mathcal{K}$,

$$g_{r,k}(t,x) = \begin{cases} \bar{f}_k(t,x) & \text{if } |x_k| \le r\,, \\ (|x_k| - r)\, C\dfrac{x_k}{|x_k|} + (1 + r - |x_k|)\bar{f}_k(t,x) & \text{if } r < |x_k| < r+1\,, \\ C\dfrac{x_k}{|x_k|} & \text{if } |x_k| \ge r+1\,. \end{cases}$$

Notice that, for the indices $j \in \mathcal{J}$, the value $r > 0$ does not affect the definition of the components $g_{r,j}$.

Proposition 13. *If x is a solution of (P_r), then $\alpha_j(t) \le x_j(t) \le \beta_j(t)$ for every $j \in \mathcal{J}$ and $t \in [0,T]$.*

The proof follows from a classical reasoning and can be easily adapted from Step 2 of the proof of Theorem 2.

Proposition 14. *There is a constant $K > 0$ such that, if x is a solution of (P_r), for any $r > 0$, which satisfies (NW_k) for a certain index $k \in \mathcal{K}$, then $\|x_k\|_{\mathcal{C}^2} \le K$.*

Proof. Notice that

$$|g_{r,k}(t,x)| \le C\,, \qquad \text{for every } (t,x) \in [0,T] \times \mathbb{R}^N, k \in \mathcal{K} \text{ and } r > 0\,. \tag{9}$$

Fix any $k \in \mathcal{K}$. If $x(t)$ is a solution of (P_r), multiplying the k-th equation by $\tilde{x}_k$ and integrating, we have that

$$\|\tilde{x}_k\|_2^2 \le \left(\frac{T}{2\pi}\right)^2 \|\dot{x}_k\|_2^2 \le \left(\frac{T}{2\pi}\right)^2 C\sqrt{T}\|\tilde{x}_k\|_2\,.$$

So, by a classical reasoning, there is a constant $C_1 > 0$ such that $\|\tilde{x}_k\|_{H^1} \le C_1$, and there is a constant $C_0 > 0$ such that $\|\tilde{x}_k\|_\infty \le C_0$, for every solution x of (P_r). Define

$$u_k(t) = \min\{\alpha_k(t), \beta_k(t)\}\,, \quad \mathcal{U}_k(t) = \max\{\alpha_k(t), \beta_k(t)\}\,. \tag{10}$$

Since (NW_k) holds, there is a $\tau_0 \in [0,T]$ such that

$$u_k(\tau_0) \le x_k(\tau_0) \le \mathcal{U}_k(\tau_0)\,. \tag{11}$$

Then, if x is a solution of (P_r),

$$\begin{aligned} |x_k(t)| = \left| x_k(\tau_0) + \int_{\tau_0}^t \dot{x}_k(s)\,\mathrm{d}s \right| &\le |x_k(\tau_0)| + \int_0^T |\dot{x}_k(s)|\,\mathrm{d}s \le |x_k(\tau_0)| + \sqrt{T}\|\dot{x}_k\|_2 \\ &\le \max\{\|\alpha\|_\infty\,,\, \|\beta\|_\infty\} + \sqrt{T}C_1 =: K_0\,, \end{aligned}$$

hence $\|x_k\|_\infty \le K_0$. Moreover, by periodicity, there is a $\tau_1 \in [0,T]$ such that $\dot{x}_k(\tau_1) = 0$, hence by (9)

$$|\dot{x}_k(t)| = \left| \dot{x}_k(\tau_1) + \int_{\tau_1}^t \ddot{x}_k(s)\,\mathrm{d}s \right| = \left| \int_{\tau_1}^t g_{r,k}(s, x(s))\,\mathrm{d}s \right| \le \int_0^T |g_{r,k}(s,x(s))|\,\mathrm{d}s \le CT\,,$$

so that $\|\dot{x}_k\|_\infty \le CT$. Then,

$$\|x_k\|_{\mathcal{C}^2} = \|x_k\|_\infty + \|\dot{x}_k\|_\infty + \|\ddot{x}_k\|_\infty \le K_0 + CT + C =: K\,,$$

thus proving the proposition. □

From now on, we fix $r > \max\{K, \|\alpha\|_\infty\,,\, \|\beta\|_\infty\}$, where K is given by Lemma 14. Problem (P_r) is equivalent to the fixed point problem

$$x = \mathcal{L}^{-1}\mathcal{N}_r x\,, \qquad x \in \mathcal{C}([0,T], \mathbb{R}^N)\,,$$

where we have introduced the Nemytskii operator

$$\mathcal{N}_r : \mathcal{C}([0,T], \mathbb{R}^N) \to \mathcal{C}([0,T], \mathbb{R}^N)\,, \quad (\mathcal{N}_r x)(t) = x(t) - g_r(t, x(t))\,.$$

Since we are looking for zeros of

$$\mathcal{T}_r x := (I - \mathcal{L}^{-1}\mathcal{N}_r)(x)\,,$$

we are going to compute the Leray-Schauder degree on a family of open sets. Let us define the constant functions

$$\hat{\alpha} = -r - 1\,, \qquad \hat{\beta} = r + 1\,,$$

as well as the functions

$$\check{\alpha}_j(t) = \alpha_j(t) - 1\,, \quad \text{and} \quad \check{\beta}_j(t) = \beta_j(t) + 1\,,$$

for every $j \in \mathcal{J}$.

We define, for every multi-index $\mu = (\mu_{M+1}, \dots, \mu_N) \in \{1,2,3,4\}^{N-M}$, the open set

$$\Omega_\mu := \{x \in \mathcal{C}([0,T],\mathbb{R}^N) \,:\, (\mathcal{O}_j^0) \text{ and } (\mathcal{O}_k^{\mu_k}) \text{ hold for every } j \in \mathcal{J} \text{ and } k \in \mathcal{K}\}\,, \tag{12}$$

where the conditions $(\mathcal{O}_j^0)$ and $(\mathcal{O}_k^{\mu_k})$ read as

$(\mathcal{O}_j^0)$ $\check{\alpha}_j(t) < x_j(t) < \check{\beta}_j(t)$, for every $t \in [0,T]$,

$(\mathcal{O}_k^1)$ $\hat{\alpha} < x_k(t) < \hat{\beta}$, for every $t \in [0,T]$,

$(\mathcal{O}_k^2)$ $\hat{\alpha} < x_k(t) < \beta_k(t)$, for every $t \in [0,T]$,

$(\mathcal{O}_k^3)$ $\alpha_k(t) < x_k(t) < \hat{\beta}$, for every $t \in [0,T]$,

$(\mathcal{O}_k^4)$ $\hat{\alpha} < x_k(t) < \hat{\beta}$, for every $t \in [0,T]$, and there are $t_k^1, t_k^2 \in [0,T]$ such that $x(t_k^1) < \alpha_k(t_k^1)$ and $x(t_k^2) > \beta_k(t_k^2)$.

Proposition 15. *The Leray-Schauder degree $d(\mathcal{T}_r, \Omega_\mu)$ is well-defined for every $\mu \in \{1,2,3,4\}^{N-M}$.*

Proof. Assume by contradiction that there is $x \in \partial\Omega_\mu$ such that $\mathcal{T}_r x = 0$, i.e., x is a solution of (P_r). All the several different situations which may arise lead back to the following four cases.

Case A. For some index $j \in \mathcal{J}$, $\check{\alpha}_j(t) \le x_j(t) \le \check{\beta}_j(t)$, for every $t \in [0,T]$, and $\check{\alpha}_j(\tau) = x_j(\tau)$ for a certain $\tau \in [0,T]$ (the case when $x_j(\tau) = \check{\beta}_j(\tau)$ is similar). We can prove that

$$\ddot{\check{\alpha}}_j(t) > g_{r,j}(t, x_1(t), \dots, x_{j-1}(t), \check{\alpha}_j(t), x_{j+1}(t), \dots, x_N(t))\,, \qquad \text{for every } t \in [0,T]\,,$$

so that arguing as in Step 2 of the proof of Theorem 2 we obtain a contradiction.

Case B. For some index $k \in \mathcal{K}$, $\hat{\alpha} \le x_k(t) \le \hat{\beta}$, for every $t \in [0,T]$, and $\hat{\alpha} = x_k(\tau)$ for a certain $\tau \in [0,T]$ (the case when $x_k(\tau) = \hat{\beta}$ is similar). Since

$$g_{r,k}(t, x_1(t), \dots, x_{k-1}(t), \hat{\alpha}, x_{k+1}(t), \dots, x_N(t)) = -C < 0\,, \qquad \text{for every } t \in [0,T]\,,$$

we easily get a contradiction as before.

Case C. For some index $k \in \mathcal{K}$, $\hat{\alpha} < x_k(t) \le \beta_k(t)$, for every $t \in [0,T]$, and $x_k(\tau) = \beta_k(\tau)$ for a certain $\tau \in [0,T]$. Such a situation cannot arise since (7) holds by assumption.

Case D. For some index $k \in \mathcal{K}$, $\alpha_k(t) \le x_k(t) < \hat{\beta}$, for every $t \in [0,T]$, and $x_k(\tau) = \alpha_k(\tau)$ for a certain $\tau \in [0,T]$. Such a situation cannot arise since (6) holds by assumption. □

Proposition 16. *For every multi-index $\mu \in \{1,2,3\}^{N-M}$ we have $d(\mathcal{T}_r, \Omega_\mu) = 1$.*

Proof. In this case, it can be verified by the arguments of the previous proof, that the definition of the set Ω_μ provides us a well-ordered pair of strict lower/upper solutions of problem (P_r). The conclusion is then an immediate consequence of Theorem 6. □

For any multi-index $\hat{\mu} \in \{1,2,3\}^{N-M-1}$ we can consider, for every $\ell \in \{1,2,3,4\}$, the multi-index

$$(\ell, \hat{\mu}) = (\ell, \mu_{M+2}, \dots, \mu_N) \in \{1,2,3,4\}^{N-M}\,.$$

We can verify that $\Omega_{(2,\hat{\mu})}, \Omega_{(3,\hat{\mu})}, \Omega_{(4,\hat{\mu})}$ are pairwise disjoint and all contained in $\Omega_{(1,\hat{\mu})}$ so that

$$\Omega_{(4,\hat{\mu})} = \Omega_{(1,\hat{\mu})} \setminus \overline{\Omega_{(2,\hat{\mu})} \cup \Omega_{(3,\hat{\mu})}}\,. \tag{13}$$

Proposition 17. *For every multi-index $\hat{\mu} \in \{1,2,3\}^{N-M-1}$ we have $d(\mathcal{T}_r, \Omega_{(4,\hat{\mu})}) = -1$.*

Proof. By Proposition 16 and (13),

$$\begin{aligned} 1 &= d(\mathcal{T}_r, \Omega_{(1,\hat{\mu})}) \\ &= d(\mathcal{T}_r, \Omega_{(2,\hat{\mu})}) + d(\mathcal{T}_r, \Omega_{(3,\hat{\mu})}) + d(\mathcal{T}_r, \Omega_{(4,\hat{\mu})}) \\ &= 2 + d(\mathcal{T}_r, \Omega_{(4,\hat{\mu})}) \end{aligned}$$

and the conclusion follows. □

Arguing similarly we can prove by induction the following result.

Proposition 18. *For every $K \in \{1, \ldots, N-M\}$ and every multi-index $\mu \in \{4\}^K \times \{1,2,3\}^{N-M-K}$, we have*

$$d(\mathcal{T}_r, \Omega_\mu) = (-1)^K .$$

Proof. We proceed by induction. The validity of the statement for $K = 1$ follows by Proposition 17. So, we fix $K \geq 2$ and assume that

$$d(\mathcal{T}_r, \Omega_\mu) = (-1)^{K-1}, \text{ for every } \mu \in \{4\}^{K-1} \times \{1,2,3\}^{N-M-K+1} .$$

Consider the multi-index $\mu = (4, \ldots, 4, \mu_{M+K}, \mu_{M+K+1}, \ldots, \mu_N) \in \{4\}^{K-1} \times \{1,2,3\}^{N-M-K+1}$ and define for every $\ell \in \{1,2,3,4\}$, the multi-index

$$\bar{\mu}^\ell = (4, \ldots, 4, \ell, \mu_{M+K+1}, \ldots, \mu_N) .$$

We then see that

$$\begin{aligned} (-1)^{K-1} &= d(\mathcal{T}_r, \Omega_{\bar{\mu}^1}) \\ &= d(\mathcal{T}_r, \Omega_{\bar{\mu}^2}) + d(\mathcal{T}_r, \Omega_{\bar{\mu}^3}) + d(\mathcal{T}_r, \Omega_{\bar{\mu}^4}) \\ &= 2 \cdot (-1)^{K-1} + d(\mathcal{T}_r, \Omega_{\bar{\mu}^4}) , \end{aligned}$$

yielding $d(\mathcal{T}_r, \Omega_{\bar{\mu}^4}) = (-1)^K$. The proof is complete. □

By the previous proposition we conclude that

$$d(\mathcal{T}_r, \Omega_{(4,\ldots,4)}) = (-1)^{N-M} . \tag{14}$$

As a consequence, there is a solution x of problem (P_r) in the set $\Omega_{(4,\ldots,4)}$. Recalling the a priori bounds in Propositions 13 and 14, we see that the solution x is indeed a solution of problem (P) and satisfies (W_j) and (NW_k), for every $j \in \mathcal{J}$ and $k \in \mathcal{K}$. The proof is thus completed.

3.2 An extension of Theorem 10

The existence of a solution of (P) can be obtained also removing from the assumptions of Theorem 10 the *strictness* assumption on *one* of the components.

Theorem 19. *Let (α, β) be a pair of lower/upper solutions of (P) related to the partition $(\mathcal{J}, \mathcal{K})$ of $\{1, \ldots, N\}$. Fix $\kappa \in \mathcal{K}$ and assume that (α, β) is strict with respect to the k-th component, for every $k \in \mathcal{K} \setminus \{\kappa\}$. Assume moreover the existence of a constant $C > 0$ such that*

$$|f_{\mathcal{K}}(t,x)| \leq C , \quad \textit{for every } (t,x) \in \mathcal{E} .$$

Then, (P) has a solution x such that (W_j) and (NW_k) hold for every $(j,k) \in \mathcal{J} \times (K \setminus \{\kappa\})$, and

$(\widetilde{NW}_\kappa)$ *there exist* $t_\kappa^1, t_\kappa^2 \in [0,T]$ *such that* $x_\kappa(t_\kappa^1) \le \alpha_\kappa(t_\kappa^1)$ *and* $x_\kappa(t_\kappa^2) \ge \beta_\kappa(t_\kappa^2)$.

Proof. Without loss of generality we can choose $\mathcal{J} = \{1,\dots,M\}$, $\mathcal{K} = \{M+1,\dots,N\}$ and $\kappa = N$. We can follow the proof of Theorem 10 step by step in the first part, noticing that Proposition 14 holds with the same constant when we assume $(\widetilde{NW}_N)$. Moreover, since we do not ask the strictness assumption with respect to the N-th component, when we introduce the sets Ω_μ as in (12), we can consider only multi-indices with the last component frozen to 1, i.e. $\mu = (\mu_{M+1},\dots,\mu_{N-1},1) \in \{1,2,3,4\}^{N-M-1} \times \{1\}$. Indeed, with this new choice of the multi-indices we can still guarantee that the Leray-Schauder degree is well-defined.

Then, arguing as in Propositions 16, 17 and 18 we have

- $d(\mathcal{T}_r, \Omega_\mu) = 1$ for every $\mu \in \{1,2,3\}^{N-M-1} \times \{1\}$,
- $d(\mathcal{T}_r, \Omega_\mu) = -1$ for every $\mu \in \{4\} \times \{1,2,3\}^{N-M-2} \times \{1\}$,
- for every $K \in \{1,\dots,N-M-1\}$, $d(\mathcal{T}_r, \Omega_\mu) = (-1)^K$ for every multi-index $\mu \in \{4\}^K \times \{1,2,3\}^{N-M-K-1} \times \{1\}$.

However, we cannot conclude the proof saying that the Leray-Schauder degree is different from zero in $\Omega_{(4,\dots,4)}$ as in (14), since we cannot ensure that it is well defined in the sets $\Omega_{(4,\dots,4,\ell)}$ with $\ell = 2,3,4$.

Anyhow, at this step of the proof, we can follow the classical reasoning adopted in the scalar case in presence of non-well-ordered lower/upper solutions, cf. [5, Theorem III-3.1]. If there exists $x \in \partial\Omega_{(4,\dots,4,2)}$ such that $\mathcal{T}_r x = 0$, then we can easily see that x must be a solution of (P_r) such that $x_N(t) \le \beta_N(t)$ for every $t \in [0,T]$ and $x_N(\tau) = \beta_N(\tau)$ for a certain $\tau \in [0,T]$. Since the components α_N, β_N are non-well-ordered, we have $\alpha_N(t_N^0) > \beta_N(t_N^0) \ge x_N(t_N^0)$ for some $t_0^N \in [0,T]$. So $(\widetilde{NW}_N)$ holds, thus giving us that x is a solution of (P_r) satisfying all the required assumptions.

We can argue similarly if there exists $x \in \partial\Omega_{(4,\dots,4,3)}$ such that $\mathcal{T}_r x = 0$.

If the previous situations do not occur, we can compute the degree both in $\Omega_{(4,\dots,4,2)}$ and $\Omega_{(4,\dots,4,3)}$. As in (13), we have

$$\Omega_{(4,\dots,4,4)} = \Omega_{(4,\dots,4,1)} \setminus \overline{\Omega_{(4,\dots,4,2)} \cup \Omega_{(4,\dots,4,3)}}\,. \tag{15}$$

so that the degree is well defined also for $\Omega_{(4,\dots,4,4)}$. Performing the same computation adopted in Propositions 17 and 18 we can conclude that $d(\mathcal{T}_r, \Omega_{(4,\dots,4)}) = (-1)^{N-M}$, thus finding also in this case a solution x with the desired properties. The proof is thus completed. $\square$

4 Lower and upper solutions for infinite-dimensional systems

We now focus our attention on a system defined in a separable Hilbert space H with scalar product $\langle\cdot,\cdot\rangle$ and corresponding norm $|\cdot|$. We study the problem

$$(P) \qquad \begin{cases} \ddot{x} = f(t,x)\,, \\ x(0) = x(T)\,, \quad \dot{x}(0) = \dot{x}(T)\,, \end{cases}$$

where $f : [0,T] \times H \to H$ is a continuous function. In what follows, we extend the results of Section 3 to an infinite-dimensional setting, trying to maintain similar notations.

Let $\mathbb{N}_+ = \{1, 2, 3, \dots\}$. Choosing a Hilbert basis $(e_n)_{n\in\mathbb{N}_+}$, every vector $x \in H$ can be written as $x = \sum_{n\in\mathbb{N}_+} x_n e_n$, or $x = (x_n)_{n\in\mathbb{N}_+} = (x_1, x_2, \dots)$. Similarly, for the function f, we will write

$$f(t, x) = (f_1(t, x), f_2(t, x), \dots)\,.$$

As in the finite-dimensional case, we will say that the couple $(\mathcal{J}, \mathcal{K})$ is a partition of $\mathbb{N}_+$ if and only if $\mathcal{J} \cap \mathcal{K} = \varnothing$ and $\mathcal{J} \cup \mathcal{K} = \mathbb{N}_+$. Correspondingly, we can decompose the Hilbert space as $H = H_{\mathcal{J}} \times H_{\mathcal{K}}$, where every $x \in H$ can be written as $x = (x_{\mathcal{J}}, x_{\mathcal{K}})$ with $x_{\mathcal{J}} = (x_j)_{j\in\mathcal{J}} \in H_{\mathcal{J}}$ and $x_{\mathcal{K}} = (x_k)_{k\in\mathcal{K}} \in H_{\mathcal{K}}$.

Similarly, every function $\mathcal{F} : \mathcal{A} \to H$ can be written as $\mathcal{F}(x) = \big(\mathcal{F}_{\mathcal{J}}(x), \mathcal{F}_{\mathcal{K}}(x)\big)$ where $\mathcal{F}_{\mathcal{J}} : \mathcal{A} \to H_{\mathcal{J}}$ and $\mathcal{F}_{\mathcal{K}} : \mathcal{A} \to H_{\mathcal{K}}$.

We rewrite Definition 7 in this context.

Definition 20. *Given two $\mathcal{C}^2$-functions $\alpha, \beta : [0, T] \to H$ we will say that (α, β) is a* pair of lower/upper solutions of (P) related to the partition $(\mathcal{J}, \mathcal{K})$ of $\mathbb{N}_+$ *if the four conditions of Definition 7 hold replacing $\{1, \dots, N\}$ by $\mathbb{N}_+$ and the inequalities* (3), (4) *by*

$$\ddot{\alpha}_n(t) \geq f_n(t, x_1, \dots, x_{n-1}, \alpha_n(t), x_{n+1}, \dots)\,, \tag{16}$$

$$\ddot{\beta}_n(t) \leq f_n(t, x_1, \dots, x_{n-1}, \beta_n(t), x_{n+1}, \dots)\,. \tag{17}$$

Moreover, it is said to be strict with respect to the n-th component, *with $n \in \mathbb{N}_+$, if the conditions of Definition 8 hold.*

We recall the definition of the set

$$\mathcal{E} := \Big\{(t, x) \in [0, T] \times \mathbb{R}^N \;:\; x = (x_{\mathcal{J}}, x_{\mathcal{K}})\,, x_{\mathcal{J}} \in \prod_{j\in\mathcal{J}} [\alpha_j(t), \beta_j(t)]\Big\}\,.$$

Here is our result in this infinite-dimensional setting.

Theorem 21. *Let (α, β) be a pair of lower/upper solutions of (P) related to the partition $(\mathcal{J}, \mathcal{K})$ of $\mathbb{N}_+$, and assume the following conditions:*

- *there exists a sequence $(d_n)_{n\in\mathbb{N}_+} \in \ell^2$ such that*

$$-d_n \leq \alpha_n(t) \leq d_n \quad \text{and} \quad -d_n \leq \beta_n(t) \leq d_n\,, \qquad \text{for every } n \in \mathbb{N}_+ \text{ and } t \in [0, T];$$

- *(α, β) is strict with respect to the k-th component, for every $k \in \mathcal{K}$;*
- *there exists a constant $C > 0$ such that*

$$|f_{\mathcal{K}}(t, x)| \leq C\,, \quad \text{for every } (t, x) \in \mathcal{E}\,;$$

- *for every bounded set $\mathcal{B} \subset \mathcal{E}$, the set $f_{\mathcal{K}}(\mathcal{B})$ is precompact.*

Then, (P) has a solution x with the following property: for any $(j, k) \in \mathcal{J} \times \mathcal{K}$,

(W_j) *$\alpha_j(t) \leq x_j(t) \leq \beta_j(t)$, for every $t \in [0, T]$;*

(NW_k) *there exist $t_k^1, t_k^2 \in [0, T]$ such that $x_k(t_k^1) < \alpha_k(t_k^1)$ and $x_k(t_k^2) > \beta_k(t_k^2)$.*

The proof of the theorem is carried out in Section 4.2.

Remark 22. *As in Theorem 19, we can drop the strictness assumption for a certain index $\kappa \in \mathcal{K}$. In that case, the so-found solution will satisfy the corresponding condition $(\widetilde{NW}_\kappa)$.*

As an immediate consequence of Theorem 21, taking α and β constant functions, we have the following.

Corollary 23. *Let there exist two sequences $(p_n)_{n\in\mathbb{N}_+}$ and $(q_n)_{n\in\mathbb{N}_+}$ in ℓ^2, with $p_n < q_n$ for every $n \in \mathbb{N}_+$, and a partition $(\mathcal{J}, \mathcal{K})$ of $\mathbb{N}_+$, such that, for every $(t,x) \in [0,T] \times \prod_{j\in\mathcal{J}}[p_j, q_j] \times H_\mathcal{K}$,*

$$j \in \mathcal{J} \quad \Rightarrow \quad f_j(t, x_1, \dots, x_{j-1}, p_j, x_{j+1}, \dots) \le 0 \le f_j(t, x_1, \dots, x_{j-1}, q_j, x_{j+1}, \dots)\,; \tag{18}$$

$$k \in \mathcal{K} \quad \Rightarrow \quad f_k(t, x_1, \dots, x_{k-1}, p_k, x_{k+1}, \dots) > 0 > f_k(t, x_1, \dots, x_{k-1}, q_k, x_{k+1}, \dots)\,. \tag{19}$$

Furthermore, let there exists a sequence $(C_k)_{k\in\mathcal{K}} \in \ell^2$ such that, for every $k \in \mathcal{K}$,

$$|f_k(t,x)| \le C_k\,, \quad \textit{for every } (t,x) \in [0,T] \times \prod_{j\in\mathcal{J}}[p_j, q_j] \times H_\mathcal{K}\,. \tag{20}$$

Then, (P) has a solution $x(t)$ such that, for every $j \in \mathcal{J}$, $k \in \mathcal{K}$,

$$\{x_j(t) : t \in [0,T]\} \subseteq [p_j, q_j]\,; \tag{21}$$

$$\{x_k(t) : t \in [0,T]\} \cap [p_k, q_k] \ne \varnothing\,. \tag{22}$$

We now give some examples of applications, where we implicitly assume all the functions to be continuous.

Example 24. Let, for every $j \in \mathbb{N}_+$,

$$f_j(t,x) = x_j^3 + h_j(t,x)\,,$$

and assume that there is a $c > 0$ such that

$$|h_j(t,x)| \le \frac{c}{j^3}\,, \quad \text{for every } (t,x) \in [0,T] \times H\,. \tag{23}$$

Then, $f : [0,T] \times \ell^2 \to \ell^2$ is well-defined and taking $q_j = -p_j = \sqrt[3]{c}/j$, we see that both $(p_j)_j, (q_j)_j$ belong to ℓ^2, and (18) is satisfied, so that Corollary 23 applies with $\mathcal{K} = \varnothing$.

Example 25. Let us consider, for every $j \in \mathbb{N}_+$,

$$f_j(t,x) = x_j^2 \sin x_j + h_j(t,x)\,,$$

and assume that there is a $c > 0$ such that (23) holds. Then, $f : [0,T] \times \ell^2 \to \ell^2$ is well-defined. Since $x^2 \sin x \ge \frac{1}{2}x^3$ in the interval $[0, \pi/2]$, taking $q_j = -p_j = \sqrt[3]{2c}/j$, we see that both $(p_j)_j, (q_j)_j$ belong to ℓ^2, and (18) is satisfied, so that Corollary 23 applies with $\mathcal{K} = \varnothing$.

Furthermore, for every $\ell \in \mathbb{Z}$ with $|\ell|$ sufficiently large, we can see that the constants $p^\ell = -\pi/2 + 2\ell\pi$, $q^\ell = \pi/2 + 2\ell\pi$ satisfy (18), for every $j \in \mathbb{N}_+$. Thus, we can replace a finite number of couples (p_j, q_j) with some couples (p^ℓ, q^ℓ). Such a replacement must be performed only for a *finite* number of indices $j \in \mathbb{N}_+$ since we need to guarantee that the new sequences $(p_j)_j$ and $(q_j)_j$ remain in ℓ^2. Recalling that the so found solution of problem (P) must satisfy (22) then we conclude that (P) admits an infinite number of solutions.

Example 26. Let, for every $k \in \mathbb{N}_+$,

$$f_k(t,x) = -\frac{x_k}{1+k|x_k|} + h_k(t,x)\,,$$

and assume that there is a $c \in \,]0,1[$ such that

$$|h_k(t,x)| \le \frac{c}{k}\,, \qquad \text{for every } (t,x) \in [0,T] \times H\,.$$

Then, $f : [0,T] \times \ell^2 \to \ell^2$ is well-defined and taking $q_k = -p_k = \frac{c}{(1-c)k}$, we see that both $(p_k)_k, (q_k)_k$ belong to ℓ^2, and (19) is verified, so that Corollary 23 applies with $\mathcal{J} = \varnothing$.

Example 27. Let $(a_n)_n$ and $(\sigma_n)_n$ be sequences of positive numbers in ℓ^2 and let, for every $n \in \mathbb{N}_+$,

$$f_n(t,x) = -a_n \sin\Big(\frac{2\pi x_n}{\sigma_n}\Big) + h_n(t,x)\,.$$

If h_n satisfies

$$\sup\big\{|h_n(t,x)| : (t,x) \in [0,T] \times H\big\} < a_n\,, \tag{24}$$

we see that, for every $n \in \{1,\dots,N\}$, it is possible to find pairs of constant lower and upper solutions

$$\alpha_n \in \Big\{\frac{\sigma_n}{4} + m\sigma_n : m \in \mathbb{Z}\Big\}\,, \qquad \beta_n \in \Big\{-\frac{\sigma_n}{4} + m\sigma_n : m \in \mathbb{Z}\Big\}\,.$$

Then, for each equation we have both well-ordered and non-well-ordered pairs of lower/upper solutions. Applying Corollary 23 we thus get the existence of infinitely many solutions of problem (P). By the same argument in Example 12 we notice that, even if the function $h(t,x_1,x_2,\dots)$ is σ_n-periodic in each variable x_n, the solutions we find are indeed geometrically distinct.

Remark 28. *This result should be compared with the ones in [4, 10], where one or two geometrically distinct solutions were found assuming a Hamiltonian structure of the problem, i.e.,*

$$h_n(t,x) = \frac{\partial \mathcal{V}}{\partial x_n}(t,x)\,,$$

for some function $\mathcal{V}(t,x_1,x_2,\dots)$ which is σ_n-periodic in each variable x_n. It was said in the final section of [10] that it remained an open problem to know if the existence of more than two T-periodic solutions could be proved, and in [4] that "it would be natural to conjecture the existence of infinitely many T-periodic solutions". It is interesting to notice that even in [4, 10], in order to recover some compactness, it was assumed that the sequence of the periods $(\sigma_n)_n$ belong to ℓ^2.

Remark 29. *For any choice of a partition $(\mathcal{J},\mathcal{K})$ of $\mathbb{N}_+$, we can consider functions f satisfying the requirements of Examples 24, 25 or 27 for every $j \in \mathcal{J}$ and of Examples 26 or 27 for every $k \in \mathcal{K}$. Corollary 23 applies also in this case.*

In the next section we provide some preliminary lemmas, which will be used in order to prove Theorem 21.

4.1 Some compactness lemmas

For every sequence $\tau = (\tau_n)_{n\in\mathbb{N}_+}$ contained in $[0,T]$ and every function $u \in \mathcal{C}([0,T],H)$, define the function $P_\tau u : [0,T] \to H$ as

$$(P_\tau u)_n(t) = \int_{\tau_n}^t u_n(s)\,\mathrm{d}s\,, \quad n \in \mathbb{N}_+\,.$$

We will need the following extension of [10, Lemma 3.2].

Lemma 30. *Let $E \subseteq \mathcal{C}([0,T],H)$ be such that the set*

$$A = \{u(t) : u \in E,\, t \in [0,T]\}$$

is precompact in H. Then the set

$$\Sigma = \left\{P_\tau u : \tau \in [0,T]^{\mathbb{N}_+}, u \in E\right\}$$

is precompact in $\mathcal{C}([0,T],H)$. As a consequence, the set

$$\Xi = \left\{P_\tau u(t) : \tau \in [0,T]^{\mathbb{N}_+}, u \in E,\, t \in [0,T]\right\}$$

is precompact in H.

Proof. Fix $\varepsilon > 0$. Since A is precompact, there exist $v_1, \ldots, v_m$ in H such that

$$A \subseteq \bigcup_{\iota=1}^{m} B(v_\iota, \varepsilon)\,. \tag{25}$$

Let $V = \operatorname{Span}(v_1, \ldots, v_m)$, and denote by $Q : H \to V$ the corresponding orthogonal projection. We first prove that the set

$$\mathcal{R} = \left\{P_\tau(Qu) : u \in E,\ \tau \in [0,T]^{\mathbb{N}_+}\right\}$$

is precompact in $\mathcal{C}([0,T],V)$.

The set $Q(A)$ is precompact in V and hence bounded; there exists a real constant D such that

$$|Qu(t)| < D\,, \quad \text{for all } u \in E \text{ and } t \in [0,T]\,. \tag{26}$$

Moreover, for every $u \in E$, $\tau \in [0,T]^{\mathbb{N}_+}$ and $t \in [0,T]$,

$$|(P_\tau(Qu))_n(t)| = \left|\int_{\tau_n}^{t} (Qu)_n(s)\,\mathrm{d}s\right| \le \left|\int_{\tau_n}^{t} |(Qu)_n(s)|\,\mathrm{d}s\right|\,, \quad n \in \mathbb{N}_+$$

and consequently

$$|P_\tau(Qu)(t)|^2 = \sum_{n=1}^{\infty}|(P_\tau(Qu))_n(t)|^2 \le \sum_{n=1}^{\infty}\left|\int_{\tau_n}^{t} |(Qu)_n(s)|\,\mathrm{d}s\right|^2 \le \sum_{n=1}^{\infty}\Big(\int_0^T |(Qu)_n(s)|\,\mathrm{d}s\Big)^2;$$

by the Hölder Inequality and the use of the Monotone Convergence Theorem, recalling (26),

$$\begin{aligned}
\sum_{n=1}^{\infty}\Big(\int_0^T |(Qu)_n(s)|\,\mathrm{d}s\Big)^2 &\le T\sum_{n=1}^{\infty}\int_0^T |(Qu)_n(s)|^2\,\mathrm{d}s \\
&= T\int_0^T \sum_{n=1}^{\infty}|(Qu)_n(s)|^2\,\mathrm{d}s \\
&= T\int_0^T |Qu(s)|^2\,\mathrm{d}s < T^2D^2,
\end{aligned}$$

and then

$$|P_\tau(Qu)(t)| \le TD\,.$$

Since V is finite dimensional, the set $\mathcal{S} = \{w(t) : w \in \mathcal{R}\} \subseteq V$ is precompact. On the other hand, for every $u \in E$, $\tau \in [0,T]^{\mathbb{N}_+}$ and every $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, we have

$$|P_\tau(Qu)(t_1) - P_\tau(Qu)(t_2)| = \left|\int_{t_1}^{t_2} (Qu)(s)\,\mathrm{d}s\right| \le \int_{t_1}^{t_2} |(Qu)(s)|\,\mathrm{d}s \le D(t_1 - t_2)\,,$$

so that $\mathcal{R}$ is equi-uniformly continuous as a subset of $\mathcal{C}([0,T],V)$. By the Ascoli–Arzelà Theorem, the set $\mathcal{R}$ is precompact in $\mathcal{C}([0,T],V)$.

Consequently, there exist $f_1, \dots, f_\ell$ in $\mathcal{C}([0,T], V)$ such that

$$\mathcal{R} \subseteq \bigcup_{\iota=1}^{\ell} B(f_\iota, \varepsilon)\,. \tag{27}$$

Now, for every $u \in E$, $\tau \in [0,T]^{\mathbb{N}_+}$ and $t \in [0,T]$, by (25),

$$\begin{aligned}
|P_\tau u(t) - P_\tau(Qu)(t)|^2 &= \sum_{n=1}^{\infty} |(P_\tau u)_n(t) - (P_\tau(Qu))_n(t)|^2 \\
&\le \sum_{n=1}^{\infty} \left| \int_{\tau_n}^{t} |u_n(s) - (Qu)_n(s)| \,\mathrm{d}s \right|^2 \\
&\le \sum_{n=1}^{\infty} T \int_0^T |u_n(s) - (Qu)_n(s)|^2 \,\mathrm{d}s \\
&= T \int_0^T \sum_{n=1}^{\infty} |u_n(s) - (Qu)_n(s)|^2 \,\mathrm{d}s \\
&= T \int_0^T |u(s) - (Qu)(s)|^2 \,\mathrm{d}s \le T^2 \varepsilon^2,
\end{aligned}$$

and so

$$|P_\tau u(t) - P_\tau(Qu)(t)| \le T\varepsilon.$$

On the other hand, since $P_\tau(Qu) \in \mathcal{R}$, by (27) there exists $\bar{\iota}$ such that

$$\|P_\tau(Qu) - f_{\bar{\iota}}\|_\infty < \varepsilon\,,$$

hence

$$|P_\tau u(t) - f_{\bar{\iota}}(t)| \le |P_\tau u(t) - P_\tau(Qu)(t)| + |P_\tau(Qu)(t) - f_{\bar{\iota}}(t)| \le \varepsilon T + \varepsilon = \varepsilon(T+1).$$

We have thus shown that, given $\varepsilon > 0$, there are $f_1, \dots, f_\ell$ in $\mathcal{C}([0,T], H)$ such that

$$\Sigma \subseteq \bigcup_{\iota=1}^{\ell} B(f_\iota, (T+1)\varepsilon)\,,$$

hence proving that Σ is precompact.

The fact that Ξ is precompact in H now follows again from the Ascoli–Arzelà Theorem, recalling that this theorem gives a necessary and sufficient condition for precompactness. $\square$

Let us denote by $\Pi_N : H \to H$ the projection

$$\Pi_N(x) = (x_1, \dots, x_N, 0, 0, \dots)\,. \tag{28}$$

Lemma 31. *Let A be a compact subset of H. Then, for every $\varepsilon > 0$, there is a $M \ge 1$ such that, for every $a = (a_n)_{n \in \mathbb{N}_+}$ in A,*

$$\sum_{n=M}^{\infty} |a_n|^2 \le \varepsilon^2.$$

In particular $\lim_{N\to\infty}(\Pi_N - \mathrm{Id})x = 0$ uniformly for $x \in A$.

Proof. By contradiction, let there exist an $\varepsilon > 0$ such that, for every $M \geq 1$, there is $a^M = (a_n^M)_{n\in\mathbb{N}_+} \in A$ such that $\sum_{n=M}^\infty |a_n^M|^2 > \varepsilon^2$. By compactness, the sequence $(a^M)_{M\in\mathbb{N}_+}$ has a subsequence, for which we keep the same notation, such that $a^M \to a^*$, for some $a^* \in A$. Let M_* be any positive integer. Then, taking $M \geq M_*$ sufficiently large,

$$\begin{aligned}
\Big(\sum_{n=M_*}^\infty |a_n^*|^2\Big)^{1/2} &\geq \Big(\sum_{n=M}^\infty |a_n^*|^2\Big)^{1/2} \\
&\geq \Big(\sum_{n=M}^\infty |a_n^M|^2\Big)^{1/2} - \Big(\sum_{n=M}^\infty |a_n^M - a_n^*|^2\Big)^{1/2} \\
&\geq \varepsilon - \|a^M - a^*\|_{\ell^2} \geq \frac{\varepsilon}{2}\,.
\end{aligned}$$

We thus get a contradiction with the fact that $a^* \in H$. $\square$

As an immediate consequence we find the following compactness property.

Lemma 32. *Let A be a compact subset of H. Then, the set*

$$A^{\mathcal{P}} := \bigcup_{N\in\mathbb{N}_+} \Pi_N A$$

is precompact in H.

Proof. Let us consider a sequence $(x_n)_{n\in\mathbb{N}_+}$ contained in $A^{\mathcal{P}}$.

If there exists $N_0 \in \mathbb{N}_+$ and a subsequence $(x_{n_\ell})_\ell$ such that $x_{n_\ell} \in \Pi_{N_0} A$ for every ℓ, then the conclusion is reached since $\Pi_{N_0} A$ is compact.

If the previous situation does not arise, then we can find a diverging sequence $(N_\ell)_\ell \subset \mathbb{N}_+$ and a subsequence $(x_{n_\ell})_\ell$ such that $x_{n_\ell} \in \Pi_{N_\ell} A$ for every ℓ. So, there is a sequence $(y_{n_\ell})_\ell \subseteq A$ such that $x_{n_\ell} = \Pi_{N_\ell} y_{n_\ell}$. Since A is compact, then, up to a subsequence, we have $y_{n_\ell} \to \bar{y} \in A$. Hence,

$$|x_{n_\ell} - \bar{y}| \leq |x_{n_\ell} - y_{n_\ell}| + |y_{n_\ell} - \bar{y}| \leq |(\Pi_{N_\ell} - \mathrm{Id}) y_{n_\ell}| + |y_{n_\ell} - \bar{y}| \to 0\,,$$

where Lemma 31 has been applied. $\square$

Remark 33. *The above statements have been formulated for a Hilbert space H. We will apply them also treating the previously introduced Hilbert spaces $H_{\mathcal{K}}$ and $H_{\mathcal{J}}$.*

4.2 Proof of Theorem 21

We consider, for every $N \in \mathbb{N}_+$, the auxiliary system

$$\begin{cases}
\ddot{x}_1 = f_1(t, x_1, \dots, x_N, \alpha_{N+1}(t), \alpha_{N+2}(t), \dots) \\
\quad\vdots \\
\ddot{x}_N = f_N(t, x_1, \dots, x_N, \alpha_{N+1}(t), \alpha_{N+2}(t), \dots) \\
\ddot{x}_{N+1} = 0 \\
\ddot{x}_{N+2} = 0 \\
\quad\vdots
\end{cases}$$

We recall the projections Π_N, introduced in (28), and define the function

$$\widehat{\Pi}_N : \mathcal{C}([0,T],H) \to \mathcal{C}([0,T],H) \tag{29}$$

$$\widehat{\Pi}_N x(t) = (x_1(t), \dots, x_N(t), \alpha_{N+1}(t), \alpha_{N+2}(t), \dots)\,. \tag{30}$$

The auxiliary problem can then be written as

$$(\widehat{P}_N) \qquad \begin{cases} \ddot{x} = \Pi_N f(t, \widehat{\Pi}_N x(t))\,, \\ x(0) = x(T)\,, \quad \dot{x}(0) = \dot{x}(T)\,. \end{cases}$$

Notice that

$$(t, \widehat{\Pi}_N x^N(t)) \in \mathcal{E}\,, \qquad \text{for every } N \in \mathbb{N}_+ \text{ and } t \in [0,T]\,. \tag{31}$$

By Theorem 10, for every $N \in \mathbb{N}_+$, there is a solution $x^N(t)$ of $(\widehat{P}_N)$ such that (W_j) and (NW_k) hold for every $j \in \mathcal{J} \cap [1,N]$ and $k \in \mathcal{K} \cap [1,N]$. We impose

$$x_n^N(t) = 0\,, \quad \text{for every } n > N \text{ and } t \in [0,T]\,. \tag{32}$$

Arguing as in the proof of Proposition 14, cf. (10) and (11), we conclude that x^N satisfies

$$\{x_j^N(t) : t \in [0,T]\} \subseteq [-d_j, d_j]\,,$$
$$\{x_k^N(t) : t \in [0,T]\} \cap [-d_k, d_k] \neq \varnothing\,,$$

for every $k \in \mathcal{K}$ and $j \in \mathcal{J}$. Concerning the indices $j \in \mathcal{J}$ we thus have

$$x_{\mathcal{J}}^N(t) \in \mathcal{D}_{\mathcal{J}} := \prod_{j \in \mathcal{J}} [-d_j, d_j]\,, \tag{33}$$

for every $N \in \mathbb{N}_+$ and $t \in [0,T]$.

Now, we repeat the arguments of Proposition 14 with a slight modification. Given the solution x^N of $(\widehat{P}_N)$, we can compute

$$\|\tilde{x}_{\mathcal{K}}^N\|_2^2 \leq \left(\frac{T}{2\pi}\right)^2 \|\dot{x}_{\mathcal{K}}^N\|_2^2 \leq \left(\frac{T}{2\pi}\right)^2 C\sqrt{T} \|\tilde{x}_{\mathcal{K}}^N\|_2\,,$$

so that $\|\tilde{x}_{\mathcal{K}}^N\|_{H^1} \leq C_1$ and $\|\tilde{x}_{\mathcal{K}}^N\|_\infty \leq C_0$ for some constants C_1 and C_0.

Recalling the validity of (33), we can find a sequence $\tau_{\mathcal{K}}^N = (\tau_k^N)_{k \in \mathcal{K}} \subset [0,T]$ such that

$$|x_k^N(\tau_k^N)| \leq d_k\,, \qquad \text{for every } k \in \mathcal{K}\,. \tag{34}$$

Then, we can prove that the sequence $(x_{\mathcal{K}}^N)_{N \in \mathbb{N}_+}$ is uniformly bounded. Indeed,

$$\begin{aligned} |x_{\mathcal{K}}^N(t)|^2 &= \sum_{k \in \mathcal{K}} |x_k^N(t)|^2 = \sum_{k \in \mathcal{K}} \left| x_k^N(\tau_k^N) + \int_{\tau_k^N}^t \dot{x}_k^N(s)\, ds \right|^2 \\ &\leq 2 \sum_{k \in \mathcal{K}} \left(|x_k^N(\tau_k^N)|^2 + \left| \int_{\tau_k^N}^t \dot{x}_k^N(s)\, ds \right|^2 \right) \\ &\leq 2 \sum_{k \in \mathcal{K}} d_k^2 + 2T \|\dot{x}_{\mathcal{K}}^N\|_2^2 \leq 2 \sum_{k \in \mathcal{K}} d_k^2 + 2TC_1^2 =: \varrho^2\,, \end{aligned}$$

Then, choosing $\mathcal{B} = \{(t,x) \in \mathcal{E} \,:\, |x_\mathcal{K}| \le \varrho\}$ and recalling (31) and that $f_\mathcal{K}$ is completely continuous in $\mathcal{E}$, we notice that the set $A = \{f_\mathcal{K}(t, \widehat{\Pi}_N x^N(t)) \,:\, N \in \mathbb{N}_+, t \in [0,T]\} \subseteq f_\mathcal{K}(\mathcal{B})$ is precompact. Then, using Lemma 32, we deduce that the set $\{\ddot{x}^N_\mathcal{K}(t) \,:\, N \in \mathbb{N}_+, t \in [0,T]\}$ is precompact. By periodicity, there exists a sequence $t^N_\mathcal{K} = (t^N_k)_{k\in\mathcal{K}}$ such that $\dot{x}^N_k(t^N_k) = 0$ for every $k \in \mathcal{K}$. Writing

$$\dot{x}^N_k(t) = \dot{x}^N_k(t^N_k) + \int_{t^N_k}^t \ddot{x}^N_k(s)\,\mathrm{d}s = \int_{t^N_k}^t \ddot{x}^N_k(s)\,\mathrm{d}s = \Big(P_{t^N_\mathcal{K}} \ddot{x}^N_\mathcal{K}\Big)(t)\,,$$

we deduce from Lemma 30 that the set $\{\dot{x}^N_\mathcal{K}(t) : N \in \mathbb{N}_+, t \in [0,T]\}$ is precompact.

Finally we prove that also the set $\{x^N_\mathcal{K}(t) \,:\, N \in \mathbb{N}_+, t \in [0,T]\}$ is precompact. Recalling the sequence $\tau^N_\mathcal{K} = (\tau^N_k)_{k\in\mathcal{K}}$ in (34), we can write using the notation of Section 4.1,

$$x^N_\mathcal{K}(t) = \xi^N_\mathcal{K} + \Big(P_{\tau^N_\mathcal{K}} \dot{x}^N_\mathcal{K}\Big)(t)\,, \qquad \text{where } \xi^N_\mathcal{K} := (x^N_k(\tau^N_k))_{k\in\mathcal{K}}\,.$$

By construction $\xi^N_\mathcal{K} \in \mathcal{D}_\mathcal{K} := \prod_{k\in\mathcal{K}}[-d_k, d_k]$, so that, by Lemma 30, we conclude that both the addenda are in a compact set. Hence there is a compact set $\widehat{\mathcal{D}}_\mathcal{K}$ such that

$$x^N_\mathcal{K}(t) \in \widehat{\mathcal{D}}_\mathcal{K}\,, \qquad \text{for every } N \in \mathbb{N}_+ \text{ and } t \in [0,T]\,. \tag{35}$$

We can now prove similar properties for the components of $x^N(t)$, and their derivatives, with indices $j \in \mathcal{J}$. At this step, the continuity of $f_\mathcal{J}$ is sufficient. Indeed, from (33) and (35), the compactness of $\{f_\mathcal{J}(t, \widehat{\Pi}_N x^N(t)) \,:\, N \in \mathbb{N}_+, t \in [0,T]\}$ follows. Then, arguing as above, we can prove that both $\{\ddot{x}^N_\mathcal{J}(t) \,:\, N \in \mathbb{N}_+, t \in [0,T]\}$ and $\{\dot{x}^N_\mathcal{J}(t) \,:\, N \in \mathbb{N}_+, t \in [0,T]\}$ are precompact.

Consider now the sequence $(u^N)_{N\in\mathbb{N}_+}$ of functions $u^N : [0,T] \to H \times H$ defined by

$$u^N(t) = (x^N(t), \dot{x}^N(t))\,.$$

By the above arguments, the sequence $(u^N)_{N\in\mathbb{N}_+}$ takes its values in a compact set, and it is equi-uniformly continuous. By the Ascoli–Arzelà Theorem there exists a subsequence, for which we keep the same notation, which uniformly converges to some $u^* : [0,T] \to H \times H$. Writing $u^*(t) = (x^*(t), y^*(t))$, we have that $(x^N, \dot{x}^N)$ uniformly converges to (x^*, y^*). In particular $x^*(0) = x^*(T)$, $y^*(0) = y^*(T)$. Rewriting the differential equation in $(\widehat{P}_N)$ as a planar system, we have

$$(\widehat{Q}_N) \qquad \begin{cases} \dot{x} = y, \\ \dot{y} = \Pi_N f(t, \widehat{\Pi}_N x(t))\,, \end{cases}$$

or equivalently

$$\dot{u} = F^N(t,u)\,,$$

where $F^N(t,x,y) = (y, \Pi_N f(t, \widehat{\Pi}_N x(t)))$. The corresponding integral formulation is then

$$u(t) = u(0) + \int_0^t F^N(s, u(s))\,\mathrm{d}s\,. \tag{36}$$

System $(\widehat{Q}_N)$ has a solution $u^N = (x^N, \dot{x}^N)$ such that $u^N(0) = u^N(T)$ for every $N \in \mathbb{N}_+$. We want to show that

$$F^N(t, u^N(t)) \to F(t, u^*(t))\,, \quad \text{uniformly in } t \in [0,T]\,, \tag{37}$$

where $F(t,x,y) = (y, f(t,x))$. Fix $\varepsilon > 0$; for N sufficiently large, we have

$$\begin{aligned}|F^N(t,u^N(t)) - F(t,u^*(t))| &\le |y^N(t) - y^*(t)| + |\Pi_N f(t,\widehat{\Pi}_N x^N(t)) - f(t,x^*(t))| \\ &\le \varepsilon + |\Pi_N f(t,\widehat{\Pi}_N x^N(t)) - f(t,\widehat{\Pi}_N x^N(t))| + |f(t,\widehat{\Pi}_N x^N(t)) - f(t,x^*(t))| \,.\end{aligned}$$

Since $\{\widehat{\Pi}_N x^N(t) \,:\, N \in \mathbb{N}_+ \,, t \in [0,T]\}$ is precompact, cf. (33) and (35), then by continuity $\{f(t,\widehat{\Pi}_N x^N(t)) \,:\, N \in \mathbb{N}_+ \,, t \in [0,T]\}$ is precompact, too. So, by Lemma 31, for N sufficiently large,

$$|\Pi_N f(t,\widehat{\Pi}_N x^N(t)) - f(t,\widehat{\Pi}_N x^N(t))| = |(\Pi_N - \mathrm{Id}) f(t,\widehat{\Pi}_N x^N(t))| \le \varepsilon \,.$$

Moreover,

$$|\widehat{\Pi}_N x^N(t) - \Pi_N x^N(t)| = |(0,\dots,0,\alpha_{N+1}(t),\alpha_{N+2}(t),\dots)| \tag{38}$$

$$\le \sum_{n=N}^{\infty} d_n^2 \to 0 \,, \qquad \text{as } N \to \infty \,. \tag{39}$$

Then, applying Lemma 31,

$$|\widehat{\Pi}_N x^N(t) - x^*| \le |\widehat{\Pi}_N x^N(t) - \Pi_N x^N(t)| + |\Pi_N x^N(t) - x^N(t)| + |x^N(t) - x^*(t)| \to 0 \,,$$

as $N \to \infty$, so that by continuity, for N large enough,

$$|f(t,\widehat{\Pi}_N x^N(t)) - f(t,x^*(t))| \le \varepsilon \,.$$

Summing up, if N is large, then

$$|F^N(t,u^N(t)) - F(t,u^*(t))| \le 3\varepsilon \,, \qquad \text{for every } t \in [0,T] \,,$$

thus proving (37). Passing to the limit in (36), we get

$$u^*(t) = u^*(0) + \int_0^t F(s,u^*(s))\,\mathrm{d}s \,,$$

and so $x^*(t)$ is a solution of (P). The proof is thus completed. □

5 Final remarks

In this final section, we briefly outline some possible extensions of the previous results.

1. The boundedness assumption on the function $f_{\mathcal{K}}(t,x)$ could be replaced by a nonresonance condition with respect to the higher part of the spectrum of the differential operator $-\ddot{x}$ with T-periodic conditions. For instance, denoting by λ_2 the first positive eigenvalue $(2\pi/T)^2$, one could assume that

$$-f_{\mathcal{K}}(t,x) = \gamma_{\mathcal{K}}(t,x)x + r_{\mathcal{K}}(t,x) \,,$$

where $\gamma_{\mathcal{K}}(t,x) \le c < \lambda_2$ and $r_{\mathcal{K}}(t,x)$ is bounded. Or, more generally, one could assume an asymmetric behaviour of the type

$$-f_{\mathcal{K}}(t,x) = \mu_{\mathcal{K}}(t,x)x^+ - \nu_{\mathcal{K}}(t,x)x^- + r_{\mathcal{K}}(t,x) \,,$$

where $(\mu_{\mathcal{K}}(t,x), \nu_{\mathcal{K}}(t,x))$ lie below the first curve of the Fučík spectrum (here, as usual, $x^+ = \max\{x,0\}$ and $x^- = \max\{-x,0\}$).

2. One could deal with nonlinearities of the type $f(t,x,\dot{x})$, depending also on the derivative of x, assuming some type of Nagumo growth condition (see [5]). Such a situation has already been studied in the infinite-dimensional setting, e.g., in [21].

3. In this paper we defined the lower and upper solutions as C^2-functions. However, this regularity could be weakened, and different definitions could be adopted. We do not enter into the details, for briefness, and we refer to the book [5] for further possible developments.

4. The results of this paper hold the same for the Neumann problem

$$\begin{cases} \ddot{x} = f(t,x)\,, \\ \dot{x}(0) = 0 = \dot{x}(T)\,, \end{cases}$$

with almost identical proofs. Concerning the Dirichlet problem

$$\begin{cases} \ddot{x} = f(t,x)\,, \\ x(0) = 0 = x(T)\,, \end{cases}$$

some modifications are needed in the non-well-ordered case. Both problems have their partial differential equations analogues. We will provide in [9] an extension of Theorem 10 in a finite-dimensional abstract setting including the case of elliptic and parabolic type systems with different types of boundary conditions, thus generalizing the results in [6, 7, 11]. However, an infinite-dimensional extension in the PDE case remains an open problem.

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