SISSA

Scuola
Internazionale
Superiore di
Studi Avanzati

Mathematics Area - PhD course in
Geometry and Mathematical Physics

On formal schemes and smoothings

Candidate:
Alessandro Nobile

Advisor:
Prof. Barbara Fantechi

Academic Year 2020-21

Contents

Chapter 1

Introduction

1.1 Motivation: moduli spaces

Classifying geometric objects has always been one of the aspiring goals of algebraic geometry which led to the idea of moduli spaces. A moduli space can be seen as a collection of geometric objects that are continuously parametrised by another geometric object. Examples of these moduli spaces are the moduli spaces of curves, the moduli space of surfaces and their higher dimensional analogues.

Of great interest in my research is the moduli space of surfaces of general type. A *surface of general type* is a projective variety of dimension two and maximal Kodaira dimension. In this case the correct moduli space of such varieties is a collection of *birational* equivalence classes. Since each birational equivalence class contains a unique *minimal model*, i.e. a smooth projective surface whose canonical class is NEF, in order to examine the moduli space, one only needs minimal models. However, with the works of Mumford [MFK65] and Artin [Art74] it becomes clear that for surfaces of general type the best way to study their moduli space is to consider *canonical models*, not minimal models. A canonical model is a projective surface with ample canonical class admitting mild singularities. Such singularities are known as *canonical singularities*.

As in the case of the moduli space of curves, these moduli spaces are not proper, therefore the compactification problem arises. Many mathematicians have worked on the problem (see for example Mumford [MFK65]), but after the key work of Kollár and Shepherd-Barron [KS88] it became clear that the right notion is that of *stable surfaces*. For a surface to be stable, it must satisfy two conditions, one local and one global. The local condition restricts the singularities admitted, namely to those which are *semi-log-canonical*. The global condition states that the dualizing sheaf must be ample as a $\mathbb{Q}$-Cartier divisor. By adding stable surfaces we can compactify the moduli space of surfaces of general type.

1.2 Smoothability and infinitesimal deformations

This leads to the natural question: given a singular surface, is it in the closure of the moduli space of smooth canonical surfaces? Various cases have already been studied: for example, if X has isolated singularities, then $\mathrm{H}^2(X, \mathcal{T}_X)$ is an obstruction space for globalising local deformations; hence, if X is locally smoothable and $\mathrm{H}^2(X, \mathcal{T}_X) = 0$, then X is smoothable [BW74]. The question now is: what can we say about non-isolated singularities? Some have already been considered: for example, normal crossing singularities were studied by Friedman in [Fri83], while Kawamata and Namikawa in [KN94] introduced and studied logarithmic deformations for a normal crossing reduced scheme.

Therefore, it is of interest to focus on non-isolated singularities. In [Tzi10], Tziolas introduces the notion of formal smoothing for formal deformations of formal schemes. He then uses deformation theoretical arguments to answer the following question: under what conditions is a scheme formally smoothable? Specifically, he provides sufficient conditions for the existence of a formal smoothing (see [Tzi10, Theorem 12.5]). Another important result Tziolas provides, [Tzi10, Proposition 11.8], is a link between formal smoothing and the smoothing of a fixed scheme. In particular, the existence of a formal smoothing is a necessary condition for the existence of geometric smoothings. In detail, he showed that if we have a (classical) deformation over the spectrum of DVR of a proper equidimensional scheme of finite type over a separable field K, then being a smoothing, i.e. having smooth generic fibre, is equivalent to being a formal smoothing.

However there are two questions left unaddressed in his work. The first is that verifying Tziolas' conditions that guarantees formal smoothing is not a trivial task and, as far as we know, has been done only in relatively few cases. The second one is, using Tziolas' words ([Tzi10, p. 66]):

> *"However, the methods of this paper are formal so produce only formal deformations of X. It is therefore of interest to know under which conditions a formal deformation is algebraic as well as which properties of an algebraic deformation can be read from the associated formal deformation."*

In other words, in order to apply those results to "real life" moduli problems one needs to pass from a formal deformation to a classical deformation and hope that the smoothing condition holds.

One of the main goals of this thesis is to address this second question.

1.3 Main results

Conventions 1.3.1. All schemes are defined over an algebraically closed field k of characteristic 0. We will assume that all schemes will be defined over k and we will denote by FTS_k (or simply by FTS) the category whose objects are separated k-schemes of finite type and whose morphisms are morphisms of k-schemes.

As mentioned above, in this thesis we want to build a bridge between formal deformations and (classical) deformations and study what properties are preserved when one travels in either directions.

More precisely, we answer the following two questions:

(i) suppose we have a formal smoothing of a scheme X, can we find sufficient conditions on X to deduce the existence of a deformation of X over a smooth curve which is a smoothing of X?

(ii) suppose we have a formal smoothing of X, is it possible to determine if it is a formal smoothing just by looking at a finite number of thickenings, i.e. infinitesimal deformations defying the formal deformation?

In other words, in point (i), we wonder what the conditions on X are that allow us to extend a formal smoothing to a whole deformation of X over a smooth curve maintaining the smoothing property. The answer we found and present in this work is given in the following:

Theorem A. *If X is a reduced, projective, equidimensional, scheme of finite type over a field k such that one of the following assumptions hold:*

1. $\mathrm{H}^2(X, \mathcal{O}_X) = 0$, *or*
2. *if X Gorenstein, then either the dualizing sheaf ω_X or its dual $\omega_X^\vee$ is ample,*

then X is formally smoothable if and only if X is geometrically smoothable.

By *geometrically smoothable* we mean that there exists a non-singular curve C together with a proper, flat morphism of finite type $g\colon \mathcal{Y} \to C$ and a closed point $c \in C$ such that

$$X \cong \mathcal{Y} \times_C \operatorname{Spec}\left(\frac{\mathcal{O}_{C,c}}{\mathfrak{m}_c}\right)$$

and the fibre over the generic point is smooth.

The result partially answers the question asked by Grothendieck in [Gro60, p. 207]

> *"On notera que le théorème 3, et la technique correspondante, n'est valable que pour un anneau de base (local pour fixer les idées) complet. Pour passer de résultats connus pour cet anneau local lui-même, il faudrait un quatrième "théorème fondamental" dont l'énoncé définitif reste à trouver."*

This result allows one to apply Tziolas' formal smoothing criterion to show the smoothability in explicit cases. An application to moduli theory of the results of Tziolas and this thesis is given in [FFP21]; therein the authors prove, among other things, that every point in the moduli space corresponding to a stable, semi-smooth Godeaux surface is in the closure of the locus of points representing smooth canonical surfaces.

The following result answer question (ii) above; i.e. it gives a finiteness condition for formal smoothability.

Theorem B. *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ be a flat, lci morphism of relative dimension d of LNFSs with X_0 reduced, proper and of finite type and $\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}}$ finitely generated. Then the following are equivalent:*

1. *there exists an $r \in \mathbb{N}$ such that $\mathfrak{I}^r \subset \operatorname{Fitt}_d\left(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}\right)$, for any ideal of definition $\mathfrak{I}$ of $\mathfrak{X}$ (equivalently, $\mathfrak{f}$ is a formal smoothing);*
2. *there exists an $r \in \mathbb{N}$ such that the natural morphism*
$$\mathcal{T}^1_{X_r/S_r} \to \mathcal{T}^1_{X_{r+1}/S_{r+1}}$$
is an isomorphism;
3. *there exists an $r \in \mathbb{N}$ such that for every natural $m \geq r$, the natural morphism*
$$\mathcal{T}^1_{X_m/S_m} \to \mathcal{T}^1_{X_r/S_r}$$
is an isomorphism.

This is a result in formal geometry and uses formal methods, some of them new, such as the notion of lci adic morphisms which we define and study here.

From the above finiteness condition theorem, if we have a formal deformation $\mathfrak{f}\colon \mathfrak{X} \to \operatorname{Spf} k[\![t]\!]$ of a k-scheme which is proper, reduced and of finite type and we happen to know that $\mathfrak{f}$ is a formal smoothing, then by the previous theorem there exists a natural number r such that $\mathcal{T}^1_{X_r/S_r} \cong \mathcal{T}^1_{X_{r+1}/S_{r+1}}$. Using Artin's approximation theorem [Ser07, Theorem 2.5.22], we can approximate the formal deformation $\mathfrak{f}$ with a deformation $g\colon \mathcal{Y} \to \operatorname{Spec} E$ of the same scheme X, with E a k-algebra (essentially) of finite type, at the order $r+1$. This tell us in particular that the deformation g is a smoothing.

1.4 Contents

This thesis is divided in 5 chapters.

Chapter 2 collects a self contained and expository presentation of known properties of formal schemes following classical texts of Grothendieck [EGA1] and Illusie [FGA-Ill] and the more recent work [AJP05] by Alonso Tarrío, Jeremías López and Pérez Rodríguez. It starts with an algebraic background on topological and adic rings, presenting some examples of such rings that will be important in the future. Then, following the treatment of classical schemes, we first introduce affine formal schemes and then formal schemes as locally topologically ringed spaces that admit an open cover made of affine formal spaces. Formal schemes generalise schemes since they encode in the structure sheaf a topological structure. This topological structure allows one to think of them as topological spaces whose schematic structure encodes infinitesimal information. This means that a formal scheme can be defined as a compatible collection of infinitesimal neighbourhoods, or thickenings.

The study of formal schemes continues with the definition of the stalk of a formal scheme at a closed point. Here we give a warning: in general, the localisation of an adic, hence complete, ring does not return a complete ring, but will always return a local ring.

We then define a morphism of formal schemes as a morphism of topological spaces that has a continuous homomorphism between the structure sheaves. We also define what a locally Noetherian formal scheme is. With these definitions we can introduce the category LNFSs of locally Noetherian formal schemes and their morphisms.

We define formal coherent sheaves starting from the affine formal case using the $(-)^\Delta$ functor; this functor is the equivalent of the $\widetilde{(-)}$ functor in the scheme case. There is major difference with the classical case of sheaves on schemes: all formal sheaves are coherent. As for formal schemes, formal coherent sheaves can also be defined as coherent sheaves on each thickening with a compatibility condition.

Next, we go back to study morphisms of formal schemes. In this work, a very relevant category of morphisms of formal schemes are the adic morphisms. An adic morphism is a morphism of formal schemes with the property that the topology on the structure sheaf of the source is induced by the one of the target. These morphisms can be understood as a compatible collection of morphisms of schemes, called thickenings. This behaviour is in line with the one of formal schemes and formal sheaves.
Examples of adic morphisms are closed immersions, of finite type, proper, flat, smooth, smooth of some relative algebraic dimension. All these properties of morphisms also enjoy a stronger property: they can be read at all thickenings. In other words, to check if an adic morphism satisfies one of the aforementioned properties is equivalent to check that all the thickenings, which exist since the morphism is assumed adic, enjoy the same property.
A feature of affine formal schemes is that they can be defined as the global formal completion along a closed subscheme. This property in general is very rare for arbitrary locally Noetherian schemes and goes by the name of algebraisability. In other terms, an algebraisable formal scheme is a formal schemes that can be globally described as the completion of a classical scheme along one of its closed subschemes.
The chapter ends with two useful results: one that relates the property of being an invertible formal sheaf with the same property on all the infinitesimal thickenings of the sheaf, while the other is due to Grothendieck and gives some conditions for algebraising a formal scheme.

Having developed the language of formal schemes, in Chapter 3 we introduce, for the reader's convenience, the part of Tziolas' work on formal smoothings in [Tzi10] that is relevant in this thesis.
The chapter starts by introducing the notion of formal deformations. Then we prove that, as in the case of formal schemes, a formal deformation can be defined by an infinite sequence of infinitesimal deformations, that we call thickenings, satisfying a compatibility condition. We then introduce the notions of algebraisability and the weaker effectivity of a formal deformation. Then we recall the notions of smoothing and formal smoothing following [Tzi10]. The main results of this sections are twofold, both due to Tziolas: the first shows that smoothing and formal smoothing are the same in the case of an algebraic formal deformation over the formal spectrum of a DVR. The second one is a sufficient condition for formal smoothability.
Since these concepts of smoothing, formal smoothing, algebraisation and effectivisation and their relationships play an extremely relevant role in this work, we

explain in depth their interplay in the section 3.2 at the end of the chapter.

Chapter 4 is divided into three sections: in the first one we recall the Gorenstein condition on schemes and morphisms and study its behaviour under infinitesimal deformations. A useful result of this section, which is well known but for which we were not able to find a reference, is that if X is a projective and Gorenstein scheme then we can always extend the dualising sheaf on X to an invertible formal sheaf on any formal deformation of X. This is one of the key ingredients of the main theorem of this chapter and we present it here because we were not able to find a reference.
Since being lci implies being Gorenstein, we have that the dualising sheaf of an lci scheme always extends to any formal deformations of the scheme. However, this result for lci schemes has been proved (by us) before the Gorenstein one and the proof uses an argument involving the cotangent complex and its naïve version. Since this result is interesting in itself, we devote the second section to introducing the lci property for morphisms of schemes and show that the properties of being lci and flat behaves well under any base change. Then we introduce the naïve cotangent complex, we explain its connection with the cotangent complex and show how to derive the dualising sheaf. The section ends with the proof of the property that the dualising sheaf of an lci scheme always extends to an invertible formal sheaf on any formal deformation.
The last section is devoted to showing that, for a reduced projective equidimensional Gorenstein scheme X that satisfies either $\mathrm{H}^2(X, \mathcal{O}_X) = 0$ or the dualising sheaf or its dual being ample, being formally smoothable is equivalent to being "geometrically smoothable". By "geometrically smoothable" we mean that there exists a flat family of schemes over a smooth curve such that the generic fibre is smooth and the central fibre is our scheme. This theorem covers the gap left by Tziolas to apply his criterion of formal smoothability, how to pass from the formal situation to the geometrical situation which is relevant, for example, in the study of moduli spaces. The above result also generalises the algebraisation result by Grothendieck since we provide a way to extend formal smoothing to a flat family of schemes over a smooth curve while Grothendieck's algebraisation result stops when the base of the deformation is the spectrum of an algebra essentially of finite type (see section 3.2 for more). This result was motivated by and has been used in a specific moduli problem: studying stable semi-smooth complex Godeaux surfaces, see [FFP21].

Chapter 5 contains new results. We start by generalising the notion of flat, lci morphism to the case of formal schemes by requiring that all the thickenings are flat and lci morphisms of schemes. Such definition is very natural for two reasons:

in the first place, it allows us to overcome the fact that pull-backs of lci morphisms are not lci. The second reason why this condition is not at all artificial is because of our setting; indeed, we study deformation problems and in that context having the flatness hypotheses is a very natural property to require.
Then we present two new results connected with formal schemes: the first one gives equivalent conditions of formal smoothness. To do so, we start by presenting a relationship between the Fitting sheaf of ideals of the formal sheaf of differentials and the first Schlessinger's relative cotangent formal sheaf. Then, supposing we have a flat lci adic morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ with reduced X_0, we find that the following conditions are equivalent: there exists a power of any ideal of definition contained in the Fitting ideals of the formal sheaf of relative differentials, there exists a power of any ideal of definition that annihilates the Schlessinger's first relative cotangent formal sheaf, and there exists a natural number after which all the infinitesimal Schlessinger's relative cotangent sheaves become isomorphic. When the scheme X_0 is also proper, the above equivalences imply that formal smoothability is equivalent to the existence of a natural number that as soon as we reach the infinitesimal thickening at that number we have formally smoothed our scheme; in this sense this condition can be expressed as "formal smoothability at finite steps". This result generalises to the case of non-isolated singularities a similar result that is clear in the case of isolated singularities, see [BW74] and references therein. The second and last result of the second section is in spirit an Artin approximation theorem (see [Ser07, Theorem 2.5.22]) involving formal smoothings. Indeed, it links the finiteness condition on formal smoothability stated in the previous theorem with the property that every formal deformation can be approximated at one's pleasure with an algebraic deformation, i.e. a deformation whose base is an affine k-scheme (essentially) of finite type. This allows us in particular to check whether a family of deformations of a reduced, projective lci scheme with base a smooth k-curve is a smoothing.

1.5 Future projects

We end this thesis with Chapter 6: there we list some of the main interesting research topics that we would like to investigate in the near future. In particular we would like to extend the results in chapter 4 to the case of $\mathbb{Q}$-Gorenstein schemes and DM stacks. Another research line that we would like to pursue is generalising the notion of cotangent complex to the case of formal schemes and prove that satisfies same properties as the one in the case of classical schemes and that it can be recovered, in some sense, as the limit of all the cotangent complexes of the thickenings. Finally, we have started working on computing the sheaves $\mathcal{T}_X$ and $\mathcal{T}^1_X$ in the case of transversal A_n, $n \geq 2$, transversal D_n and transversal E_n singularities, the simplest canonical non-isolated singularities, in terms of the associated non-singular DM algebraic stack.

Chapter 2

Formal schemes

This chapter contains an introduction to formal schemes and their morphisms, with particular focus on adic morphisms.

This section will follow Illusie's and Grothendieck's language and presentation that can be found in [FGA-Ill, Part 4] and [EGA1]. At points we will also refer to [AJP05].

Definition 2.0.1. We say that a topological ring A is *linearly topologized* if there exists a fundamental system of neighborhoods of 0 made by ideals of A.

By definition, these ideals must be open in the topology of A.

Definition 2.0.2. Let A be a linearly topologized ring. We say that an ideal I is an *ideal of definition* of A if it is open and for any neighborhood U of 0 there exists a natural number n such that $I^n \subset U$.

In general the ideal of definition is not unique. Indeed for another ideal J to be an ideal of definition it is necessary and sufficient that there are two non-negative integers n, m such that $J \supset I^m \supset J^n$.

Definition 2.0.3. An *adic ring* (sometimes also called I-adic ring) is a linearly topologized ring A that admits an ideal of definition I such that

- $\{I^n\}_{n \in \mathbb{N}}$ is a fundamental system of neighborhoods of 0 in A;
- the topology induced on A turns A into a separated and complete topological space.

Remark 2.0.4. We remark that A is an I-adic ring if and only if $A = \varprojlim_n A/I^n =: \hat{A}$, where $\hat{A}$ denotes the formal completion of A along the ideal I.

If A is a ring, I an ideal of A and $\hat{A}$ denotes the formal completion of A along I, then we have that, for every $n \in \mathbb{N}$:

$$\frac{\hat{A}}{I^{n+1}\hat{A}} \cong \frac{A}{I^{n+1}}.$$

Another condition hidden in the definition of I-adic ring is that the topology induced by the powers of the ideal of definition must be separated. This is equivalent to ask that

$$\bigcap_{n\geq 0} I^{n+1} = 0.$$

The prototype of adic ring is the ring of power series.

Example 2.0.5. Let k be a field and let t be an indeterminate. The *ring of formal power series* in one variable is the (t)-adic ring

$$k[\![t]\!] := \varprojlim_n \frac{k[t]}{(t^{n+1})}.$$

Another example of complete ring is the ring of restricted power series that we ware going to introduce now and list some of it properties that will be useful later.

Definition 2.0.6. Let A be a Noetherian I-adic ring, let $r \in \mathbb{N}$ and let $T_1, \dots, T_r$ be indeterminates. The *ring of restricted power series* is defined as follows:

$$A\{T_1, \dots, T_r\} := \varprojlim_n \frac{A[T_1, \dots, T_r]}{I^{n+1} \cdot A[T_1, \dots, T_r]} = \varprojlim_n \frac{A}{I^{n+1}}[T_1, \dots, T_r].$$

Remark 2.0.7. Let A be a Noetherian I-adic ring, $r \in \mathbb{N}$ and let $T_1, \dots, T_r$ be indeterminates.

1. For every $n \in \mathbb{N}$ we have

$$\begin{aligned}
\frac{A\{T_1,\dots,T_r\}}{I^n \cdot A\{T_1,\dots,T_r\}} &= \frac{\varprojlim_m \frac{A[T_1,\dots,T_r]}{I^{m+1}\cdot A[T_1,\dots,T_r]}}{I^n \cdot \varprojlim_m \frac{A[T_1,\dots,T_r]}{I^{m+1}\cdot A[T_1,\dots,T_r]}} \\
&= \frac{\varprojlim_m \frac{A[T_1,\dots,T_r]}{I^{m+1}\cdot A[T_1,\dots,T_r]}}{\varprojlim_m \frac{I^n+I^{m+1}\cdot A[T_1,\dots,T_r]}{I^{m+1}\cdot A[T_1,\dots,T_r]}} \\
&\cong \varprojlim_m \frac{A[T_1,\dots,T_r]}{I^n + I^{m+1}\cdot A[T_1,\dots,T_r]} && \text{[Eis95, Lemma 7.15]} \\
&= \varprojlim_m \frac{A[T_1,\dots,T_r]}{I^n \cdot A[T_1,\dots,T_r]} \\
&= \frac{A}{I^n}[T_1,\dots,T_r]
\end{aligned}$$

where the second to last equality holds since, if $m \geq n$, then $I^n + I^m = I^n$.

2. Since A is a Noetherian ring, then $A[T_1,\dots,T_r]$ is also Noetherian and by [Stacks, Tag 0316] also $A\{T_1,\dots,T_r\}$ is a Noethrian ring. Furthermore, since $A[T_1,\dots,T_r]$ is Noetherian ring, then $I \cdot A[T_1,\dots,T_r]$ is finitely generated and by [Stacks, Tag 0AL0] we have that $A\{T_1,\dots,T_r\}$ is $I\cdot A\{T_1,\dots,T_r\}$-adic ring.

In order to define the notion of formal affine scheme, needed for the definition of formal scheme, we first observe that for an I-adic ring A and for any non-negative integer n, if we denote by A_n the quotient A/I^{n+1} and by X_n the affine scheme $\operatorname{Spec} A_n$, then it is clear that we have a chain of closed subschemes

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$$

and all these subschemes have the same underlying topological space, namely $|\operatorname{Spec} A/I|$.

Definition 2.0.8. Let A be an adic ring with I an ideal of definition. The *affine formal scheme* of A is the ringed space $(\operatorname{Spf} A, \mathcal{O}_{\operatorname{Spf} A})$ where

- the topological space is defined as follows

$$\operatorname{Spf} A := \{\mathfrak{p} \in \operatorname{Spec} A \colon \mathfrak{p} \text{ is open and } I \subset \mathfrak{p}\}$$

  which is naturally homeomorphic to $|\operatorname{Spec} A/I|$;

- the structure sheaf is defined as

$$\mathcal{O}_{\operatorname{Spf} A} := \varprojlim_n \mathcal{O}_{X_n}.$$

It is a sheaf of topological rings and its topology is given by

$$\Gamma(U, \mathcal{O}_{\operatorname{Spf} A}) = \varprojlim_n \Gamma(U, \mathcal{O}_{X_n})$$

for every open subset U of $\operatorname{Spf} A$, where $\Gamma(U, \mathcal{O}_{X_n})$ has the discrete topology.

It is clear that the definition above of affine formal scheme does not depend on the ideal of definition. Indeed if a prime ideal $\mathfrak{p}$ of A contains I it contains also all of its powers, in particular it contains I^m and hence J^n. Since $\mathfrak{p}$ is prime, it follows that it contains also J.

We also point out that, since the topology of $\operatorname{Spf} A$ admits a base of neighborhoods made by quasi-compact open subsets, it is enough to require that for every quasi-compact open subset U of $\operatorname{Spf} A$

$$\Gamma(U, \mathcal{O}_{\operatorname{Spf} A}) = \varprojlim_n \Gamma(U, \mathcal{O}_{X_n})$$

where $\Gamma(U, \mathcal{O}_{X_n})$ has the discrete topology (see [EGA1, (**1**.10.1.1)]).

Remark 2.0.9. If A is an I-adic ring, then the canonical morphism $A \to \hat{A}$, which is an isomorphism, induces a morphism of ringed spaces $\operatorname{Spf} \hat{A} = \operatorname{Spf} A \to \operatorname{Spec} A$.

Example 2.0.10. An example of affine formal schemes is $\operatorname{Spf} k[\![t]\!]$.

For any Noetherian I-adic ring A, any $r \in \mathbb{N}$ and any indeterminates $T_1, \ldots, T_r$, the ring $\operatorname{Spf} A\{T_1, \ldots, T_r\}$ is another example of affine formal scheme; it is called the *formal affine r-space* and denoted by $\mathbb{A}^r_A$ or by $\mathbb{A}^r_{\operatorname{Spf} A}$. Its underlying topological space is $\operatorname{Spec} A/I[T_1, \ldots, T_r]$

Notation 2.0.11. In what follows, we will use the conventions $\mathfrak{S} := \operatorname{Spf} k[\![t]\!]$ and, for every $n \in \mathbb{N}$, $S_n := \operatorname{Spec} \frac{k[\![t]\!]}{(t^{n+1})} = \operatorname{Spec} \frac{k[t]}{(t^{n+1})}$.

Proposition 2.0.12 ([EGA1, (**1**.10.1.3)])**.** *Let A be an I-adic ring. Then topological ring $\Gamma(\operatorname{Spf} A, \mathcal{O}_{\operatorname{Spf} A})$ is homeomorphic to A.*

Now we investigate what are the principal affine formal subsets and the localizations at closed points of formal schemes.

Definition 2.0.13. For an I-adic ring A and $f \in A$, we define the *open principal formal subset* $\mathfrak{D}(f)$ to be $\mathrm{D}(f) \cap \operatorname{Spf} A$.

It is the open subset of $\operatorname{Spf} A$ where the imagine of f in A/I is invertible.

Proposition 2.0.14 ([EGA1, (**1**.10.1.4)])**.** *Suppose that A is an I-adic ring and take $f \in A$. Then $(\mathfrak{D}(f), \mathcal{O}_{\operatorname{Spf} A}|_{\mathfrak{D}(f)})$ is isomorphic to the affine formal scheme* $\operatorname{Spf} A_{\{f\}}$*, where*

$$A_{\{f\}} := \varprojlim_n \frac{S_f^{-1}A}{S_f^{-1}I^n}$$

is the completed ring of fractions and S_f denotes the multiplicative system made by powers of f.

Now we define the stalk of the structure sheaf of a affine formal scheme at a closed point.

Definition 2.0.15. Let A be an I-adic ring and consider the multiplicative system S of given by $A \setminus \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of A. We define the localization of A at $\mathfrak{p}$ as follows:

$$A_{\{\mathfrak{p}\}} := \varinjlim_{f \in S} A_{\{f\}}.$$

Definition 2.0.16. Let A be an I-adic ring and let $x \in \operatorname{Spf} A$ be a closed point defined, say, by the open maximal ideal $\mathfrak{m}_x$ of A. We define the *stalk of* $\operatorname{Spf} A$ *at* x by

$$\mathcal{O}_{\operatorname{Spf} A, x} := A_{\{\mathfrak{m}_x\}}.$$

Equivalently,

$$\mathcal{O}_{\operatorname{Spf} A, x} = \varinjlim_{x \in \mathfrak{D}(f)} \mathcal{O}_{\operatorname{Spf} A}|_{\mathfrak{D}(f)}(\mathfrak{D}(f)).$$

We want to emphasize that, even though every $A_{\{f\}}$ is a complete ring, the limit of all these needs not to be complete; in particular we get that the localization of the structure sheaf at a point of the formal scheme is a local ring but not in general a complete ring.

In general we have that the following statement holds (see [EGA1, (**0**.7.6.2)]): if A is a linearly topologized ring, $\{I_\lambda\}_{\lambda \in \Lambda}$ a fundamental system of neighborhoods of 0 made by ideals of A and S is a closed multiplicative subset of A, then the ring $A\{S^{-1}\}$ is homeomorphic to the completion separation of the ring $S^{-1}A$ for the topology induced by the fundamental system of neighborhoods of 0 made by the ideals $S^{-1}I_\lambda$.

We now give an example of the bad behavior under localization of adic rings which shows that the localization does not need to be separate.

Example 2.0.17. Let us consider the ring of formal power series in one variable $k[\![t]\!]$ and as localizing set the set S_t of all powers of the formal variable t. It is then clear that $S_t^{-1}k[\![t]\!]$ is not zero. Moreover, for every $n \in \mathbb{N}$, we have that $1 = t^n/t^n \in S_t^{-1}(t^n)$ hence $S_t^{-1}(t^n) = S_t^{-1}k[\![t]\!]$. Now, if V_0 is a neighborhood of 0 in the localized ring $S_t^{-1}k[\![t]\!]$, then by definition of ideal of definition, there exist a natural number n such that $(t^n) = (t)^n \subset V_0$. Since t^n is invertible in $S_t^{-1}k[\![t]\!]$, it follows that $1 = t^n/t^n \in (t^n) \subset V_0$, showing that separateness fails since every neighborhood of 0 also contains 1.

We can now define what are a locally Noetherian and a Noetherian formal scheme. We start by the affine case.

Definition 2.0.18. Let A be an I-adic ring. Then we say that $\operatorname{Spf} A$ is a *Noetherian affine formal scheme* if A is a Noetherian ring.

Definition 2.0.19. A *locally Noetherian formal scheme* is a topologically ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ such that every point has an open neighborhood which is isomorphic to an Noetherian affine formal scheme.

Since affine formal schemes are locally topologically ringed spaces, also locally Noetherian formal schemes are locally topologically ringed spaces.

As for usual, we denote the locally Noetherian formal scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ by $\mathfrak{X}$.

The examples mentioned in example 2.0.10 are example of Noetherian affine formal schemes.

Definition 2.0.20. We say that a formal scheme is a *Noetherian formal scheme* if it is locally Noetherian and the underlying topological space is quasi-compact (hence a Noetherian topological space).

Now we define what are the morphisms among formal schemes.

Definition 2.0.21. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two locally Noetherian formal schemes. A *morphism of locally Noetherian formal schemes* is a morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ of locally topologically ringed spaces such that for every open subset $\mathfrak{V}$ of $\mathfrak{Y}$ we have a continuous map

$$\Gamma(\mathfrak{V}, \mathcal{O}_{\mathfrak{Y}}) \to \Gamma(\mathfrak{f}^{-1}(\mathfrak{V}), \mathcal{O}_{\mathfrak{X}}).$$

Locally Noetherian formal schemes (resp. Noetherian formal schemes) together with their morphisms form a category.

Notation 2.0.22. In the rest of the notes we abbreviate "locally Noetherian formal scheme(s)" by LNFS(s).

As in the classical case of schemes, there is an equivalence of categories between adic Noetherian rings and Noetherian formal affine scheme.

Proposition 2.0.23 ([EGA1, (**1**.10.2.2)]). *There is an equivalence of categories*

$$\begin{tikzcd} \mathit{NFS}_{\mathit{aff}} \arrow[rr, bend left, "\Gamma(\,,\,)"] & & \mathit{aNRing} \arrow[ll, bend left, "\mathrm{Spf}()"] \end{tikzcd}$$

where $\mathit{NFS}_{\mathit{aff}}$ denotes the category of affine Noetherian formal schemes and the category of adic Noetherian rings is denoted by aNRing.

We can now define the notion of formal coherent sheaf on a LNFS. As in the classical case of schemes there was the $\widetilde{(-)}$ functor that associate to any module its sheaf, in the formal case there is the $(-)^{\Delta}$ functor that associate to any finitely generated module it coherent formal sheaf.

Remark 2.0.24. Suppose that A is a Noetherian ring, I and ideal of A, M a finitely generated A-module and let $\hat{A}$ be the completion of A for the I-adic topology. Then M inherits a topology, which we call IM-adic, with bases $\{I^n M\}_{n\in\mathbb{N}}$. We can then consider its completion by this topology: $\widehat{M} := \varprojlim_n M/(I^n \cdot M)$. By [EGA1, (**0**.7.3.3)], we have a canonical isomorphism $M \otimes_A \hat{A} \cong \widehat{M}$.

Moreover, by [EGA1, (**0**.7.3.6)], if A is a Noetherian I-adic ring and M is a finitely generated A-module, then it is separated and complete for the induced IM-adic topology; in other words $M = \widehat{M}$.

This last result has a nice consequence which lead us to the following notation.

Notation 2.0.25. Suppose that A is a Noetherian I-adic ring and let M and N be finitely generated A-modules that are separated and complete in the induced I-adic topology. Then, by [EGA1, (**0**.7.8.1)] it follows that every A-module homomorphism is automatically continuous. Therefore, in what follows, we will write $\mathrm{Hom}_A(M,N)$ in place of $\mathrm{Hom}_{A-\mathrm{cont}}(M,N)$. We trust that it will be clear from the context what we mean.

Remark 2.0.26. If A is a Noetherian I-adic ring and M and N are two finitely generated modules, by [EGA1, (**0**.7.8.2)] we have a canonical isomorphism

$$\mathrm{Hom}_A(M,N) \xrightarrow{\cong} \varprojlim_n \mathrm{Hom}_{\frac{A}{I^{n+1}}}\left(\frac{M}{I^{n+1}M}, \frac{N}{I^{n+1}N}\right).$$

From this we conclude that, if A is a Noetherian I-adic ring and M is a finitely generated A-module, then

$$M^\vee := \mathrm{Hom}_A(M,A) = \varprojlim_n \mathrm{Hom}_{\frac{A}{I^{n+1}}}\left(\frac{M}{I^{n+1}M}, \frac{A}{I^{n+1}}\right) = \varprojlim_n \left(\frac{M}{I^{n+1}M}\right)^\vee.$$

Definition 2.0.27. Let A be a Noetherian I-adic ring and let M be a finitely generated A-module. Then we define the *coherent formal sheaf* M^Δ on $\mathrm{Spf}\, A$ to be the completion of $\widetilde{M}$ along the sheaf of ideals $\widetilde{I}$ given by the closed embedding $\mathrm{Spec}\, A/I \hookrightarrow \mathrm{Spec}\, A$:

$$M^\Delta := \varprojlim_n \frac{\widetilde{M}}{\widetilde{I^n} \cdot \widetilde{M}}.$$

Under the hypothesis of the above definition, the functor $(-)^\Delta$ satisfy equivalent properties of the functor $\widetilde{(-)}$, namely:

1. the definition of M^Δ does not depend on the ideal of definition chosen, for any finitely generate A-module M;

2. the functor $(-)^\Delta$ from the category of finitely generated A-modules to the category of coherent formal sheaves of $\mathcal{O}_{\mathrm{Spf}\, A}$-modules is an equivalence of categories and an exact functor ([EGA1, (**1**.10.10.2)]);

3. we have a canonical, functorial isomorphism $\Gamma(\mathrm{Spf}\, A, M^\Delta) \cong M$[1] ([EGA1, (**1**.10.10.2)]);

4. if M and N are two finitely generated modules over the I-adic ring A, then we have the following canonical isomorphisms

$$(M \otimes_A N)^\Delta \cong M^\Delta \otimes_{\mathcal{O}_{\mathrm{Spf}\, A}} N^\Delta$$

and

$$(\mathrm{Hom}_A(M,N))^\Delta \cong \mathscr{H}om_{\mathcal{O}_{\mathrm{Spf}\, A}}(M^\Delta, N^\Delta);$$

[1] If we assume that A is an I-adic ring, we get that $M = M \otimes_A A \cong M \otimes_A \hat{A} \cong \widehat{M}$.

5. if $\iota\colon \operatorname{Spf} A \to \operatorname{Spec} A$ is the canonical inclusion, see remark 2.0.9, and M is a finitely generated A-module, then we have that
$$M^{\Delta} = \iota^{*}\widetilde{M};$$

6. if A is an I-adic ring, if M is a finitely generated A-module and if $f \in A$, then we have that
$$\Gamma(\mathfrak{D}(f), M^{\Delta}) = M_{\{f\}};$$
where $M_{\{f\}}$ is the completion of M_f for the I_f-adic topology; in symbols:
$$M_{\{f\}} := M\{S_f^{-1}\} = \varprojlim_n \frac{S_f^{-1}M}{(S_f^{-1}I^{n+1})M},$$
where $S_f := \{f^m \colon m \geq 0\}$.

Definition 2.0.28. Let $\mathfrak{X}$ be a LNFS. *An ideal of definition* of $\mathfrak{X}$ is a formal coherent sheaf of ideals $\mathfrak{I}$ of $\mathcal{O}_{\mathfrak{X}}$ such that for any point $x \in \mathfrak{X}$ there exist a formal affine neighborhood $\operatorname{Spf} A$ of x in $\mathfrak{X}$ and there exists an ideal of definition I of A such that $\mathfrak{I}|_{\operatorname{Spf} A} = I^{\Delta}$.

An ideal of definition is not unique: indeed, any other formal coherent sheaf of ideals $\mathfrak{J}$ on the LNFS $\mathfrak{X}$ is an ideal of definition if and only if there are positive integers m, n such that the chain of inclusions hold $\mathfrak{J} \supset \mathfrak{I}^m \supset \mathfrak{J}^n$.

Remark 2.0.29. As in the affine formal case, it is also possible to define LNFSs as a collection of all of their infinitesimal neighborhoods (or thickenings).

More precisely, let $\mathfrak{X}$ be a LNFS and let $\mathfrak{I}$ be an ideal of definition. For $n \in \mathbb{N}$, define $(X_n, \mathcal{O}_{X_n})$ to be the locally Noetherian scheme $(|\mathfrak{X}|, \mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^{n+1})$. Then we have a sequence of closed embeddings of schemes, which we will call thickenings:
$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_n \hookrightarrow \cdots$$
whose ideals of definition are nilpotent and all the maps on the underlying topological spaces are the identity. Then $\mathfrak{X}$ can be recovered by the above sequence of thickenings passing through the direct limit in the category of locally Noetherian topologically ringed spaces, i.e.
$$\mathfrak{X} = \varinjlim_n X_n.$$

In particular there are natural morphisms of locally ringed spaces

$$\psi_n \colon X_n \to \mathfrak{X},$$

where ψ_n is the identity on the underlying topological space and the map of sheaves of topological rings is the quotient map

$$\psi_n^\natural \colon \mathcal{O}_\mathfrak{X} \to \mathcal{O}_{X_n} = \frac{\mathcal{O}_\mathfrak{X}}{\mathfrak{I}^{n+1}}.$$

Conversely, see [EGA1, (**1**.10.6.3)], given a collection of locally Noetherian rings $\{X_n\}_{n\in\mathbb{N}}$, satisfying:

(i) for every $n \in \mathbb{N}$, there are morphisms of schemes $\psi_{n+1,n} \colon X_n \to X_{n+1}$ such that they are homeomorphisms on the underlying topological spaces and induces surjective morphisms of sheaves $\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n}$;

(ii) if $\mathcal{I}_n := \ker(\mathcal{O}_{X_n} \to \mathcal{O}_{X_0})$, then $\ker(\mathcal{O}_{X_n} \to \mathcal{O}_{X_m}) = \mathcal{I}_n^{m+1}$, for $m \leq n$;

(iii) $\mathcal{I}_1 \in \mathrm{Coh}(X_0)$;

then the topologically ringed space $\mathfrak{X} := \varinjlim_n X_n$ obtained by taking the direct limit is a locally Noetherian formal scheme. Moreover, if $\mathfrak{I} := \ker(\mathcal{O}_\mathfrak{X} \to \mathcal{O}_{X_0})$, then $\mathfrak{I}$ is an ideal of definition of $\mathfrak{X}$ and satisfies the following properties

$$\mathfrak{I} = \varprojlim_n \mathcal{I}_n \qquad \text{and} \qquad \mathfrak{I}^{n+1} = \ker(\mathcal{O}_\mathfrak{X} \to \mathcal{O}_{X_n}).$$

Notation 2.0.30. In what follows, if $\mathfrak{I}$ is an ideal of definition of a LNFS $\mathfrak{X}$, for every $n \in \mathbb{N}$, we will denote by X_n the n-th thickening $(|\mathfrak{X}|, \frac{\mathcal{O}_\mathfrak{X}}{\mathfrak{I}^{n+1}})$.

Definition 2.0.31. A *coherent formal sheaf* on a LNFS $\mathfrak{X}$ is a sheaf $\mathfrak{F}$ such that for every open formal affine subset $\mathfrak{U} = \mathrm{Spf}\, A$ of $\mathfrak{X}$ there exists a finitely generated A-module M with $\mathfrak{F}|_\mathfrak{U} = M^\Delta$.

Next we give an interpretation of coherent formal sheaves on a LNFS as limit of coherent sheaves on all thickenings.

Remark 2.0.32. Let $\mathfrak{X}$ be a LNFS, let $\mathfrak{I}$ be an ideal of definition and let $\mathfrak{F}$ be a coherent formal sheaf of $\mathcal{O}_\mathfrak{X}$-modules. If, for every $n \in \mathbb{N}$, we define

$$\mathscr{F}_n := \frac{\mathfrak{F}}{\mathfrak{I}^{n+1}\mathfrak{F}},$$

then we have that $\mathscr{F}_n \in \mathrm{Coh}(X_n)$ and we recover $\mathfrak{F}$ by considering $\varprojlim_n \mathscr{F}_n$. Conversely, see [EGA1, (**1**.10.11.3)], let $\mathfrak{X}$ be a LNFS, $\mathfrak{I}$ and ideal of definition of $\mathfrak{X}$, let $\{X_n\}_{n\in\mathbb{N}}$ be a collection of thickenings defining $\mathfrak{X}$ and for $m \leq n$, let $\psi_{n,m}\colon X_m \to X_n$ denotes the canonical maps. Suppose that for every $n \in \mathbb{N}$, $\mathscr{F}_n$ is a coherent sheaf on X_n and we are given with morphisms, for $m \leq n$

$$\phi_{n,m}\colon \mathscr{F}_m \to (\psi_{n,m})_*\mathscr{F}_n,$$

such that for every $l \geq m \geq n$ we have $\phi_{n,m} \circ \phi_{m,l} = \phi_{n,l}$[2]. Then we have that the limit $\mathfrak{F} := \varprojlim_n \mathscr{F}_n$ is a coherent formal sheaf on $\mathfrak{X}$.

Definition 2.0.33. A formal coherent sheaf $\mathfrak{F}$ on a LNFS is *locally free of finite rank* r if each sheaf $\mathscr{F}_n := \frac{\mathfrak{F}}{\mathfrak{I}^{n+1}\mathfrak{F}}$ is locally free of the same rank r, for all natural numbers n, where $\mathfrak{I}$ is an ideal of definition of the formal scheme.

In order to also give a description in terms of thickenings for morphisms of LNFSs, we need to restrict our interest to a particular kind of morphisms: the adic morphisms.

Definition 2.0.34. A morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ of LNFSs is called an *adic morphism* if there exists an ideal of definition $\mathfrak{K}$ of $\mathfrak{Y}$ such that $\mathfrak{f}^*\mathfrak{K} \cdot \mathcal{O}_{\mathfrak{X}}$ is an ideal of definition of $\mathfrak{X}$.

The definition of adic morphism does not depend on the choice of the ideal of definition; indeed one could equivalently ask that the condition $\mathfrak{f}^*\mathfrak{K} \cdot \mathcal{O}_{\mathfrak{X}}$ holds *for all* ideals of definition $\mathfrak{K}$ of $\mathfrak{Y}$, see [EGA1, (**1**.10.12.1)].

Furthermore, the condition that $\mathfrak{f}^*\mathfrak{K} \cdot \mathcal{O}_X$ is an ideal of definition of $\mathfrak{X}$, means that the topology on $\mathcal{O}_{\mathfrak{X}}$ is induced by the topology on $\mathcal{O}_{\mathfrak{Y}}$.

Remark 2.0.35. Suppose that $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ is an adic morphism of formal schemes, let $\mathfrak{K}$ and $\mathfrak{I} := \mathfrak{f}^*\mathfrak{K} \cdot \mathcal{O}_{\mathfrak{X}}$ be ideals of definition of $\mathfrak{Y}$ and $\mathfrak{X}$ respectively.

Then we can consider the sequences of thickenings

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots X_n \hookrightarrow \cdots \quad \text{and} \quad Y_0 \hookrightarrow Y_1 \hookrightarrow \cdots Y_n \hookrightarrow \cdots$$

as in remark 2.0.29. Since the morphism was supposed to be adic, we get that for every $n \in \mathbb{N}$, $\mathfrak{f}^*(\mathfrak{K}^{n+1}) \cdot \mathcal{O}_{\mathfrak{X}} = \mathfrak{I}^{n+1}$ and hence we have induced morphisms

$$f_n\colon X_n \to Y_n$$

[2]The cocycle condition is equivalent to say that the system $\{\mathscr{F}_n, \phi_{m,n}\}_{n,m\in\mathbb{N}}$ forms a projective system.

such that all squares

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, hook] & Y_n \arrow[d, hook] \\ X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1} \end{tikzcd} \tag{2.1}$$

are Cartesian. Then $\mathfrak{f}$ can be recovered by such a collection of morphisms $\{f_n\}_{n\in\mathbb{N}}$ by considering the colimit, i.e. $\mathfrak{f} = \varinjlim_n f_n$.
Conversely, see [EGA1, (**1**.10.12.3)], any injective system of morphisms of schemes $\{f_n\colon X_n \to Y_n\}_{n\in\mathbb{N}}$ such that all squares as in eq. (2.1) are Cartesian induces an adic morphism of LNFS by considering their colimit.

Notation 2.0.36. In what follows we will use systematically the following convention. If $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ is an adic morphism of LNFSs, if $\mathfrak{K}$ and $\mathfrak{I} := \mathfrak{f}^*\mathfrak{K}\cdot\mathcal{O}_{\mathfrak{X}}$ are ideals of definition of $\mathfrak{Y}$ and $\mathfrak{X}$ respectively, we will always denote by $\{f_n\colon X_n \to Y_n\}_n$ the collection of compatible thickenings $f_n\colon X_n \to Y_n$ of $\mathfrak{f}$ such that $\varinjlim_n f_n = \mathfrak{f}$ (notation 2.0.30 is in place).

We now investigate some properties of morphisms of formal schemes such as being closed immersion, of finite type, proper,flat, formally smooth and smooth of some relative dimension.

Definition 2.0.37. Let $\mathfrak{X}$ be a LNFS and let $\mathfrak{J}$ be a coherent formal sheaf of ideals on $\mathfrak{X}$. The *closed formal subscheme of $\mathfrak{X}$ defined by $\mathfrak{J}$* is the pair $(\mathfrak{X}', \mathcal{O}_{\mathfrak{X}'})$, where

$$\mathfrak{X}' := \mathrm{Supp}\left(\frac{\mathcal{O}_{\mathfrak{X}}}{\mathfrak{J}}\right) = \left\{x \in \mathfrak{X}\colon \left(\frac{\mathcal{O}_{\mathfrak{X}}}{\mathfrak{J}}\right)_x \neq 0\right\}$$

is a closed subset of $\mathfrak{X}$ endowed with the sheaf of topological rings $\mathcal{O}_{\mathfrak{X}'} := \frac{\mathcal{O}_{\mathfrak{X}}}{\mathfrak{J}}|_{\mathfrak{X}'}$.

Remark 2.0.38. If $\mathfrak{X}$ is a LNFS, $\mathfrak{J}$ is a coherent formal sheaf of ideals on $\mathfrak{X}$, $\mathfrak{I}$ is an ideal of definition of $\mathfrak{X}$ and $(\mathfrak{X}', \mathcal{O}_{\mathfrak{X}'})$ is the closed formal subscheme of $\mathfrak{X}$ defined by $\mathfrak{J}$, then we have

$$\mathfrak{X}' = \varinjlim_n X'_n,$$

where X'_n is the closed subscheme of $X_n = (|\mathfrak{X}|, \frac{\mathcal{O}_{\mathfrak{X}}}{\mathfrak{I}^{n+1}})$ defined by the sheaf of ideals

$$\frac{\mathfrak{J} + \mathfrak{I}^{n+1}}{\mathfrak{I}^{n+1}},$$

for every non-negative integer n, see [AJP05, (1.2.14)].

The following definition is taken from [AJP05, Definiton 1.2.16].

Definition 2.0.39. A morphism $\mathfrak{j}\colon \mathfrak{Z} \to \mathfrak{X}$ of LNFS is a *closed immersion* if and only if there exists a factorization of $\mathfrak{j}$

$$\begin{tikzcd} \mathfrak{Z} \arrow[r, "\mathfrak{g}"] \arrow[rr, bend left, "\mathfrak{j}"] & \mathfrak{X}' \arrow[r, hook] & \mathfrak{X}, \end{tikzcd}$$

where $\mathfrak{g}$ is an isomorphism and $\mathfrak{X}'$ is a closed formal subscheme of $\mathfrak{X}$.

The following proposition is a **leitmotiv** in studying adic morhpisms of formal schemes; it will be a consequence of all definitions of relevant properties of adic morphisms in this theses and it will be our guiding principle in our definition of flat lci adic morphisms. It can be summarised by the following statement: *a good definition of a property of adic morphisms of formal schemes is one that can be checked at all the thickenings of the morphism*.

Proposition 2.0.40 ([AJP05, Proposition 1.2.17]). *Let $\mathfrak{j}\colon \mathfrak{Z} \to \mathfrak{X}$ be an adic morphism of LNFSs and let $\{j_n\colon Z_n \to X_n\}_{n\in\mathbb{N}}$ be the compatible collection of thickenings. The following are equivalent:*

1. *$\mathfrak{j}$ is a closed immersion;*
2. *for every $n \in \mathbb{N}$, $j_n\colon Z_n \to X_n$ is a closed immersion.*

Definition 2.0.41. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be LNFSs. A morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ is said *of finite type* if $\mathfrak{f}$ is an adic morphism and the morphism $f_0\colon X_0 \to Y_0$ is of finite type.

Proposition 2.0.42. *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be an adic morphism in LNFSs and let $\{f_n\colon X - n \to Y_n\}_{n\in\mathbb{N}}$ be a compatible collection of thickenings. Then the following are equivalent:*

1. *$\mathfrak{f}$ is of finite type;*
2. *for every $n \in \mathbb{N}$, $f_n\colon X_n \to Y_n$ is a morphism of finite type;*
3. *there is an open covering of $\mathfrak{Y}$ by formal affine subschemes $\{\mathfrak{V}_i = \operatorname{Spf} A_i\}_{i\in I}$ such that, for every $i \in I$, $\mathfrak{f}^{-1}(\mathfrak{V}_i)$ can be covered by a finite number of formal affine open subschemes $\{\mathfrak{U}_{ij} = \operatorname{Spf} B_{ij}\}_{j\in J_i}$ of $\mathfrak{X}$ and there exists $r_{ij} \in \mathbb{N}$ such that for every $i \in I$ and $j \in J_i$ we have*

$$B_{ij} \cong \frac{A_i\{T_1, \dots, T_{r_{ij}}\}}{N_{ij}},$$

where N_{ij} is an open ideal of $A_i\{T_1, \dots, T_{r_{ij}}\}$;

4. *for every $y \in \mathfrak{Y}$ and $x \in \mathfrak{X}$ with $\mathfrak{f}(x) = y$, there are formal affine neighbourhoods $\mathfrak{V}$ of y and $\mathfrak{U}$ of x with $\mathfrak{f}(\mathfrak{U}) \subset \mathfrak{V}$ such that $\mathfrak{f}|_{\mathfrak{U}}$ factors as*

$$\mathfrak{U} \xhookrightarrow{\mathfrak{j}} \mathbb{A}^r_{\mathfrak{V}} \xrightarrow{\mathfrak{p}} \mathfrak{V},$$

where $r \in \mathbb{N}$, $\mathfrak{j}$ is a closed immersion and $\mathfrak{p}$ is the canonical projection.

Proof. The proof of the equivalence of the first three points is given in [EGA1, (**1**.10.13.1)]. For (1) $\Longleftrightarrow$ (4), see [AJP05, Proposition 1.3.2 (2)]. □

Definition 2.0.43. A morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ of LNFSs is *proper* if it is of finite type and $\mathfrak{f}_0\colon X_0 \to Y_0$ is proper.

Proposition 2.0.44 ([EGAIII$_1$, (3.4.1)])**.** *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be an adic morphism of LNFSs and consider a compatible collection of thickenings $\{f_n\colon X_n \to Y_n\}_{n\in\mathbb{N}}$ of $\mathfrak{f}$. Then the following conditions are equivalent:*

1. *$\mathfrak{f}$ is proper;*
2. *for every $n \in \mathbb{N}$, f_n is proper.*

Definition 2.0.45. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of LNFSs. We say that $\mathfrak{f}$ is *flat* if it is adic and for every $x \in \mathfrak{X}$, $\mathcal{O}_{\mathfrak{X},x}$ is a flat $\mathcal{O}_{\mathfrak{Y},\mathfrak{f}(x)}$-module.

Remark 2.0.46. We could have defined a flat morphism of LNFSs $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ without assuming it to be adic. However, with that choice, we could not deduce the flatness of $\mathfrak{f}$ from the flatness of all $\{f_n\}_{n\in\mathbb{N}}$ and vice versa. See [AJP05, (1.4.5)] for a counterexample.

Proposition 2.0.47 ([AJP05, Proposition 1.4.7])**.** *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be an adic morphism of LNFSs and consider a compatible collection of thickenings $\{f_n\colon X_n \to Y_n\}_{n\in\mathbb{N}}$ of $\mathfrak{f}$. Then the following conditions are equivalent:*

1. *$\mathfrak{f}$ is flat;*
2. *for every $n \in \mathbb{N}$, f_n is flat.*

Definition 2.0.48. A morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ of LNFSs is called *formally smooth* if and only if for any affine $\mathfrak{Y}$-scheme T and any closed subscheme Z of T given by a square-zero sheaf of ideals, the natural map

$$\mathrm{Hom}_{\mathfrak{Y}}(T, \mathfrak{X}) \to \mathrm{Hom}_{\mathfrak{Y}}(Z, \mathfrak{X})$$

is surjective.

The following definition is taken from [AJP05, Definition 2.4.8]. However, since we work only with adic morphisms, we will call "smooth morphisms of formal scheme" what is there called "adic smooth morphism of formal scheme".

Definition 2.0.49. A morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ of LNFSs is called *smooth* if it is of finite type and formally smooth.

We point out that the "adic" hypotheses in the above definition is hidden in that of "of finite type".

Proposition 2.0.50 ([AJP05, Corollary 3.2.2])**.** *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be an adic morphism of LNFSs and consider a compatible collection of thickenings $\{f_n\colon X_n \to Y_n\}_{n\in\mathbb{N}}$ of $\mathfrak{f}$. Then the following conditions are equivalent:*

1. *$\mathfrak{f}$ is smooth;*
2. *for every $n \in \mathbb{N}$, f_n is smooth.*

Definition 2.0.51. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of LNFSs and let $y \in \mathfrak{Y}$ be a point. The *fibre of $\mathfrak{f}$ over y* is defined as follows:

$$\mathfrak{f}^{-1}(y) := \mathfrak{X} \times_{\mathfrak{Y}} \operatorname{Spec}(\kappa(y)),$$

where $\kappa(y)$ denotes the residue field of $\mathcal{O}_{\mathfrak{Y},y}$.

Remark 2.0.52. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be an adic morphism of LNFSs and consider a compatible collection of thickenings $\{f_n\colon X_n \to Y_n\}_{n\in\mathbb{N}}$ of $\mathfrak{f}$. Let also $y \in \mathfrak{Y}$ be a closed point. Then $\mathfrak{f}^{-1}(y)$ is a scheme and, for every $n \in \mathbb{N}$, we have $\mathfrak{f}^{-1}(y) = f_n^{-1}(y)$.

Definition 2.0.53 ([AJP05, Definition 1.2.11])**.** [relative algebraic dimension of a morphism] Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism in LNFSs, let $x \in \mathfrak{X}$ a point and $y := \mathfrak{f}(x)$. The *relative algebraic dimension of $\mathfrak{f}$ at x* is

$$\dim_x \mathfrak{f} := \dim_x \mathfrak{f}^{-1}(y) = \dim\left(\mathcal{O}_{\mathfrak{X},x} \otimes_{\mathcal{O}_{\mathfrak{Y},y}} \kappa(y)\right)$$

The following lemma relates the relative algebraic dimension with the rank of the formal sheaf of differential. For the definition and properties of the formal sheaf of differentials see [AJP05, Chapter 2].

Lemma 2.0.54 ([AJP05, Corollary 3.2.10])**.** *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a smooth morphism of LNFSs, let $x \in \mathfrak{X}$ and $y := \mathfrak{f}(x)$. Then*

$$\dim_x \mathfrak{f} = \operatorname{rank}\left(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}\right).$$

The above lemma motivates the following definition.

Definition 2.0.55. Let $r \in \mathbb{N}$. We say that a smooth morphism of LNFSs $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{Y}$ is *smooth of relative (algebraic) dimension* r if $\operatorname{rank}\left(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}\right) = r$.

The property of being a smooth morphism of some relative (algebraic) dimension can be read at all thickenings, as explained in the following proposition.

Proposition 2.0.56. *Let $r \in \mathbb{N}$, $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{Y}$ be an adic morphism of LNFSs and consider a compatible collection of thickenings $\{f_n \colon X_n \to Y_n\}_{n\in\mathbb{N}}$ of $\mathfrak{f}$. Then the following statements are equivalent:*

1. *$\mathfrak{f}$ is smooth of relative (algebraic) dimension r;*

2. *for every $n \in \mathbb{N}$, f_n is smooth of relative dimension r.*

Proof. For every $x \in \mathfrak{X}$, let $y := \mathfrak{f}(x)$. Then for every integer $n \in \mathbb{N}$ we have

$$\begin{aligned} r &= \operatorname{rank}\left(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}\right) \\ &= \dim_x \mathfrak{f} && (2.0.53) \\ &= \dim_x \mathfrak{f}^{-1}(y) \\ &= \dim_x f_n^{-1}(y) && (2.0.52) \\ &= \operatorname{rank}\left(\Omega^1_{X_n/Y_n}\right) && \text{[Har77, III.Prop. 10.4]} \end{aligned}$$

□

Example 2.0.57. Another example of formal schemes, we can consider the *formal completion of a scheme along a closed subscheme*. Suppose that Y is a locally Noetherian scheme and consider a closed subscheme X of Y with sheaf of ideals given by $\mathcal{I}$. Then we can consider the schemes $X_n := (|X|, \mathcal{O}_Y/\mathcal{I}^{n+1})$, for every $n \in \mathbb{N}$, which gives rise to a sequence of thickenings

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots X_n \hookrightarrow \cdots$$

Taking now the colimit we get a LNFS, denoted by $\hat{Y}_{/X}$ and called the formal completion of Y along X.

We point out that, if $Y = X$, then $\hat{X}_{/X} = X$. Therefore the category LNFSs contains the category of locally Noetherian schemes.

However Hironaka and Mastumura in [HM68, Theorem (5.3.3)] and independently Hartshorne in [Har06, Example 3.3] constructed two examples showing that not all formal schemes appear globally as completion of a single scheme along a closed subscheme. This consideration motivates the following definition.

Definition 2.0.58. A LNFS $\mathfrak{X}$ is called *algebraizable* if there are a scheme Y and a closed subscheme X of Y such that $\mathfrak{X} = \hat{Y}_{/X}$.

If such X and Y exist, then X is unique up to unique isomorphism and we call the algebraization of $\mathfrak{X}$.

The next theorem will be useful in the proof of one of the two main results of this thesis: it gives an infinitesimal condition on a formal sheaf to be invertible.

Theorem 2.0.59 ([Har77, II-Ex. 9.6 (c)]). *Let $\mathfrak{X}$ be a LNFS, let $\mathfrak{I}$ be an ideal of definition of $\mathfrak{X}$ and let $\{X_n\}_{n\in\mathbb{N}}$ be a collection of infinitesimal thickenings. Suppose that, for every $n \in \mathbb{N}$, we are given invertible sheaves $\mathcal{L}_n$ on X_n with isomorphisms $\mathcal{L}_{n+1} \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_n} \cong \mathcal{L}_n$. Then the sheaf*

$$\mathfrak{L} := \varprojlim_n \mathcal{L}_n$$

is an invertible sheaf on $\mathfrak{X}$.

The next theorem gives a sufficient condition for a formal schemes to be algebraisable.

Theorem 2.0.60 ([EGAIII$_1$, (**3**.5.4.5)]). *Let A be a Noetherian I-adic ring and let $\mathfrak{f}\colon \mathfrak{X} \to \operatorname{Spf} A$ be a proper morphism of formal schemes. Suppose that $\mathfrak{L}$ is a locally free sheaf of rank one on $\mathfrak{X}$ such that $\mathcal{L}_0 := \mathfrak{L}/(I^{\Delta} \cdot \mathfrak{L})$ is ample. Then $\mathfrak{X}$ is algebrizable and its algebrization is projective over $\operatorname{Spec} A$.*

See section 3.2 for a very important remark on this theorem.

Chapter 3

On formal deformations and smoothings

Having developed the language of formal schemes, we now introduce, for the reader's convenience, the part of Tziolas' work on formal smoothing present in [Tzi10] that is relevant to us. In particular we introduce the notions of formal deformation and formal smoothing. We also present some algebraisation creteria for formal schemes.

Since these concepts of smoothing, formal smoothing, algebraisation and effectivisation and their relationship play an extremely relevant role in this work, we explain in depth their interplay in the section 3.2 at the end of the theses.

3.1 Formal deformations and smoothings

This section is an exposition of Tziolas' work, [Tzi10], with a notation compatible with this thesis. In particular we define what are formal deformations and how they can be seen as a collection of compatible infinitesimal deformations. Then we introduce the notion of formal smoothing and characterise them in particular cases. We end the section giving a sufficient condition to achieve formal smoothing.

We start with the definition of formal deformation.

Definition 3.1.1. Let X be a scheme and let $(R, \mathfrak{m})$ be a local complete Noetherian ring. A *formal deformation* of X over R is a Cartesian diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathfrak{X} \arrow[d, "\mathfrak{f}"] \\ \operatorname{Spf}(\frac{R}{\mathfrak{m}}) \arrow[r, hook] & \operatorname{Spf} R \end{tikzcd} \tag{3.1}$$

with $\mathfrak{f}$ a flat morphism.

Notation 3.1.2. In the future, in order to ease the notation, we will denote any deformation (either classical or formal) by its flat morphism. For example, we will refer to the formal deformation of eq. (3.1) only by $\mathfrak{f}\colon \mathfrak{X} \to \operatorname{Spf} R$.

As formal schemes can be defined by a sequence of closed subschemes, a formal deformation of a scheme over a complete local ring can also be defined by a sequence of compatible deformations of the given scheme as shown next.

Remark 3.1.3. Fix a formal deformation of a scheme X over a complete local Noetherian k-algebra R as in eq. (3.1) and, for any non-negative integer n, let us denote by R_n the quotient ring $R/\mathfrak{m}^{n+1}$. Then, for any $n \in \mathbb{N}$, we have diagrams

$$\begin{tikzcd} & \mathfrak{X} \arrow[d, "\mathfrak{f}"] \\ \operatorname{Spec} R_n \arrow[r, hook] & \operatorname{Spf} R. \end{tikzcd}$$

Pulling back $\mathfrak{f}$ along the closed immersion $\operatorname{Spec} R_n \hookrightarrow \operatorname{Spf} R$, we obtain a collection of deformations $\{f_n\colon \mathcal{X}_n \to \operatorname{Spec} R_n\}_{n\in\mathbb{N}}$ of X over $\operatorname{Spec} R_n$. Moreover, by construction, all these deformations of X are compatible, i.e. for every non-negative integer n, we have Cartesian diagrams

$$\begin{tikzcd} \mathcal{X}_n \arrow[r, hook] \arrow[d, "f_n"] & \mathcal{X}_{n+1} \arrow[d, "f_{n+1}"] \\ \operatorname{Spec} R_n \arrow[r, hook] & \operatorname{Spec} R_{n+1}. \end{tikzcd}$$

The converse also holds true, as stated in the following proposition.

Proposition 3.1.4 ([Har09, Proposition 21.1])**.** *Consider $(R, \mathfrak{m})$ a local complete Noetherian ring with residue field k, let X be a scheme and define $R_n := R/\mathfrak{m}^{n+1}$*

for every non-negative integer n. Suppose that for every $n \in \mathbb{N}$ we are given a family $\{f_n \colon \mathcal{X}_n \to \operatorname{Spec} R_n\}_{n\in\mathbb{N}}$ of deformations such that $\mathcal{X}_0 = X$, the morphisms f_n are flat, of finite type and the following compatibility condition holds: for all $n \in \mathbb{N}$, the diagrams

$$\begin{tikzcd} \mathcal{X}_n \arrow[r, hook] \arrow[d, "f_n"] & \mathcal{X}_{n+1} \arrow[d, "f_{n+1}"] \\ \operatorname{Spec} R_n \arrow[r, hook] & \operatorname{Spec} R_{n+1} \end{tikzcd} \tag{3.2}$$

are all Cartesian.

Then there exists a (Noetherian) formal scheme $\mathfrak{X}$, flat over $\operatorname{Spf} R$, such that $\mathcal{X}_n \cong \mathfrak{X} \times_{\operatorname{Spf} R} \operatorname{Spec} R_n$, for every natural number n.

Concluding, remark 3.1.3 together with proposition 3.1.4 imply that a formal deformation $\mathfrak{f} \colon \mathfrak{X} \to \operatorname{Spf} R$ is uniquely determined by a family of deformations $\{f_n \colon \mathcal{X}_n \to \operatorname{Spec} R_n\}_{n\in\mathbb{N}}$ satisfying the compatibility condition expressed by asking that all diagrams of eq. (3.2) must be Cartesian.

We collect here some definitions, from [Ser07], that we will need to describe Tziolas' results.

Remark/definition 3.1.5. Let X be a scheme, let $(A, \mathfrak{m})$ be a k-algebra essentially of finite type, i.e. a localisation of a k-algebra of finite type. Consider a deformation of X over A

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} A. \end{tikzcd}$$

Let $\hat{A}$ be the formal completion of A at $\mathfrak{m}$; for every $n \in \mathbb{N}$, define A_n to be the quotient ring $A/\mathfrak{m}^{n+1}$ and note that (see [Eis95, Theorem 7.1 b)]) we have canonical isomorphisms $\hat{A}/\mathfrak{m}^{n+1}\hat{A} \cong A_n$. Now, for every natural number n, consider the following diagram of solid arrows

$$\begin{tikzcd} \mathcal{X}_n \arrow[r, hook, dashed] \arrow[d, dashed, "f_n"] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} A_n \arrow[r, hook] & \operatorname{Spec} A \end{tikzcd}$$

and complete it to a Cartesian one. Then we have that $f_n \colon \mathcal{X}_n \to \operatorname{Spec} A_n$ is a deformation of X and all these deformations satisfy the compatibility condition of eq. (3.2). By applying proposition 3.1.4 we have constructed a formal deformation $\mathfrak{f}\colon \mathfrak{X} \to \operatorname{Spf} \hat{A}$.

We call the formal deformation $\mathfrak{f}$ the *formal deformation associated* to f.

Definition 3.1.6. Let X be a scheme and let $(R, \mathfrak{m})$ be a complete local Noetherian ring. A formal deformation $\mathfrak{f}\colon \mathfrak{X} \to \operatorname{Spf} R$ is called *algebraisable* if there exist

- a k-algebra essentially of finite type $(A, \mathfrak{n})$,
- a deformation $g\colon \mathcal{Y} \to \operatorname{Spec} A$ of X,
- an isomorphism $R \cong \hat{A}_{\mathfrak{n}}$,
- an isomorphism between the formal deformation $\mathfrak{g}\colon \mathfrak{Y} \to \operatorname{Spf} \hat{A}$ associated to g and $\mathfrak{f}$.

The deformation $g\colon \mathcal{Y} \to \operatorname{Spec} A$ is called an *algebraisation* of $\mathfrak{f}$.

The existence of an algebraisation is a very difficult problem, see for more [Ser07, page 80]. However not all hope is lost: indeed, Artin in [Art69] introduced a weaker condition then algebraicity, namely effectivity of a formal deformation, that together with the versality of the deformation functor impiles algebraisability. This important result is called the Artin algebraisation theorem, see [Ser07, Theorem 2.5.14].

For an in depth explanation of algebraicity and effectivity of a formal deformation and their relationship in this work see section 3.2.

Definition 3.1.7. Let X be a scheme and let $(R, \mathfrak{m})$ be a complete local Noetherian ring. A formal deformation as in eq. (3.1) is called *effective* if there exists a deformation

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \\ \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} R \end{array}$$

with f a flat morphism of finite type such that $\mathfrak{X} = \hat{\mathcal{X}}_{/X}$ (see example 2.0.57 for the notation).

Example 3.1.8. An example of a non-effective deformation is given by the universal formal deformation of a K3 surface, see [Ser07, Example 2.5.12].

Next we introduce two different notion of smoothing of a deformation: in one we require that the generic fibre of the deformation is smooth, in the other we give a condition involving the Fitting ideal. We start from the first. See section 3.2 for a more clear explanation of the differences among the various definitions.

Definition 3.1.9. Let Y be a proper, equidimensional scheme and let A be a k-algebra which is a DVR. We say that a deformation $g\colon \mathcal{Y} \to \operatorname{Spec} A$ of Y over A is a *smoothing* if the generic fibre $\mathcal{Y}_{\text{gen}} := \mathcal{Y} \times_{\operatorname{Spec} A} \operatorname{Spec} \kappa(A)$ is smooth.

Following [Tzi10], we now recall the notion of formal smoothing introduced at the beginning. The definition of formal smoothing will require the definition of the Fitting ideal, which can be found in [Eis95, Chapter 20.2] and [Stacks, TAG 0C3C]. The a^{th} Fitting ideal sheaf of a (formal) sheaf measures the obstructions for the (formal) sheaf to be locally generated by a elements. Indeed, a (formal) sheaf on a (formal) schemes is locally generated by a elements if and only if the a^{th} Fitting ideal equals the structure sheaf of the (formal) scheme.

Remember that from notation 2.0.11 we have that $\mathfrak{S} := \operatorname{Spf} k[\![t]\!]$ and, for every $n \in \mathbb{N}$, $S_n = \operatorname{Spec} k[\![t]\!]/(t^{n+1})$.

Definition 3.1.10. Let X be a proper, equidimensional scheme. A formal deformation of X over $\mathfrak{S}$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathfrak{X} \arrow[d, "\pi"] \\ \operatorname{Spf} k \arrow[r, hook] & \mathfrak{S} \end{tikzcd}$$

is called a *formal smoothing* of X if and only if there exists a natural number a such that $\mathfrak{I}^a \subset \operatorname{Fitt}_{\dim X}(\Omega^1_{\mathfrak{X}/\mathfrak{S}})$, where $\mathfrak{I}$ is an ideal of definition of $\mathfrak{X}$ and $\operatorname{Fitt}_{\dim X}(\Omega^1_{\mathfrak{X}/\mathfrak{S}})$ is the Fitting sheaf of ideals.

We say that X is *formally smoothable* if it admits a formal smoothing.

The following is a key result to our argument; Tziolas in [Tzi10] shows that given a deformation over the spectrum of a DVR, this is a smoothing (in the sense of definition 3.1.9) if and only if the associated algebraic formal deformation is a formal smoothing.

Proposition 3.1.11 ([Tzi10, Proposition 11.8])**.** *Let Y be a proper, equidimensional scheme and let A be a k-algebra which is a DVR. Let $g\colon \mathcal{Y} \to \operatorname{Spec} A$ also be a deformation of Y over A and let $\mathfrak{g}\colon \mathfrak{Y} \to \operatorname{Spf} \hat{A}$ be the associated formal deformation. Then g is a smoothing if and only if $\mathfrak{g}$ is a formal smoothing.*

To end the chapter we present a theorem of Tziolas which gives a sufficient condition for the existence of formal smoothings. We start by introducing the following notation.

Notation 3.1.12. Let $f\colon X \to Y$ be a morphism of schemes. We denote the relative tangent sheaf by $\mathcal{T}_{X/Y} := \mathscr{H}om_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{O}_X)$ and for $i \in \mathbb{N}$, the i^{th} *relative cotangent sheaf* in the sense of Schelessinger, see [LS67], by $\mathcal{T}^i_{X/Y} := \mathscr{E}xt^i_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{O}_X)$. In case Y is the spectrum of the ground field k, we let $\mathcal{T}_X := \mathcal{T}_{X/k}$ and $\mathcal{T}^i_X := \mathcal{T}^i_{X/k}$ be the tangent sheaf and the i^{th} cotangent sheaf respectively.

Theorem 3.1.13 ([Tzi10, Theorem 12.5])**.** *Let X be a proper, reduced, pure dimensional scheme. If the following conditions hold*

(a) *X has complete intersection singularities;*

(b) $\mathrm{H}^2(X, \mathcal{T}_X) = 0$*;*

(c) $\mathrm{H}^1(X, \mathcal{T}^1_X) = 0$*;*

(d) *$\mathcal{T}^1_X$ is finitely generated by its global sections;*

then X is formally smoothable, i.e. it admits a formal smoothing.

As a corollary we would like to mention the following result that can be found in [Tzi10, Corollary 12.9]. For a discussion on lci schemes, see section 4.2.

Corollary 3.1.14. *Let X be a projective, lci scheme such that there exists a regular embedding in a smooth scheme Y. If the normal sheaf $\mathcal{N}_{X/Y}$ is finitely generated by its global sections, $\mathrm{H}^1(X, \mathcal{T}^1_X) = \mathrm{H}^2(X, \mathcal{T}_X) = 0$, then X admits a formal smoothing.*

3.2 Remarks on deformations and smoothings

In this section we give two remarks that could help the reader to understand better what it has been done up until now and clarify some definitions. We start by explaining the differences among the various definition of deformations introduced, then we motivate why the definition of formal smoothing is, in a sense, the only one possible.

3.2.1 Different kind of deformations

We are going to introduce the various notions of deformations and explain their connections.

Definition 3.2.1. Let X be a proper scheme of finite type over an algebraically closed field k and consider the following Cartesian diagram of schemes

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[r, hook, "b"'] & B \end{tikzcd} \tag{3.3}$$

with f flat, proper and surjective morphism, $b \in B$ a closed point inducing the closed embedding $b\colon \operatorname{Spec} k \hookrightarrow B$. We say eq. (3.3) is

(a) a *family of deformations* of X iff B is a connected k-scheme;

(b) an *algebraic deformation* of X iff B is a k-scheme (essentially) of finite type;

(c) a *local deformation* of X iff B is the affine spectrum of a local Noetherian k-algebra with residue field k;

(d) an *infinitesimal deformation* of X iff $B = \operatorname{Spec} A$ with A a local Artinian k-algebra with residue field k;

(e) a *first-order deformation* of X iff $B = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$.

(f) We say that a Cartesian diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathfrak{X} \arrow[d, "\mathfrak{f}"] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spf} A \end{tikzcd}$$

of formal schemes is a *formal deformation* iff A is a local complete Noetherian k-algebra with residue field k and $\mathfrak{f}$ is a flat proper morphism of finite type of formal schemes. This is equivalent to give a collection of infinitesimal deformations $\{f_n\colon X_n \to B_n\}_{n\in\mathbb{N}}$, where $B_n := \operatorname{Spec} A/\mathfrak{m}_A^{n+1}$, such that the following diagram is Cartesian

$$\begin{tikzcd} X_n \arrow[r, hook] \arrow[d, "f_n"] & X_{n+1} \arrow[d, "f_{n+1}"] \\ B_n \arrow[r, hook] & B_{n+1}. \end{tikzcd}$$

We remark that properties (c), (d), (e) and (f) does not depend on the topology in the sense that the underlying topological spaces of X and $\mathcal{X}$ (respectively $\mathfrak{X}$) are the same and what is changing is the schematic structure. In particular it follows that the condition that X being proper is equivalent of to say that f (respectively $\mathfrak{f}$) is proper.

The following relations always hold

$$\begin{tikzcd}
\text{(b)} \arrow[r, Rightarrow, "(1)"] & \text{(c)} \arrow[d, Rightarrow, "(4)"'] & \text{(d)} \arrow[l, Rightarrow, "(2)"'] & \text{(e)} \arrow[l, Rightarrow, "(3)"'] \\
& \text{(f)} \arrow[ur, Rightarrow, "(5)"'] & &
\end{tikzcd}$$

(1) is given by taking any point $b \in B$ and considering the pull-back of $f\colon \mathcal{X} \to B$ along $\operatorname{Spec} \mathcal{O}_{B,b} \hookrightarrow B$ obtaining the following tetrahedron in which all lateral faces are Cartesian diagrams

$$\begin{tikzcd}
& X \arrow[dl, hook'] \arrow[dr, hook] \arrow[dd] & \\
\mathcal{X} \times_B \operatorname{Spec} \mathcal{O}_{B,b} \arrow[rr, hook] \arrow[dd] & & \mathcal{X} \arrow[dd, "\pi"] \\
& \operatorname{Spec} k \arrow[dl, hook'] \arrow[dr, hook] & \\
\operatorname{Spec} \mathcal{O}_{B,b} \arrow[rr, hook] & & B
\end{tikzcd}$$

(2) is obvious since every Artinian ring is in particular a Noetherian one. Similarly (3) follows from the fact that the ring of dual numbers $k[\varepsilon]/(\varepsilon^2)$, is an example of Artinian ring. To see (4), let $\mathfrak{m}_b$ denotes the maximal ideal of the local ring $\mathcal{O}_{B,b}$

and, for any $n \in \mathbb{N}$, consider the diagram made by Cartesian faces

$$\begin{tikzcd}
& X \arrow[dl, hook'] \arrow[dr, hook] \arrow[dd] & \\
\mathcal{X}_n \arrow[rr, hook] \arrow[dd, "\pi_n"'] & & \mathcal{X} \arrow[dd, "\pi"] \\
& \operatorname{Spec} k \arrow[dl, hook'] \arrow[dr, hook] & \\
\operatorname{Spec} \frac{\mathcal{O}_{B,b}}{\mathfrak{m}_b^{n+1}} \arrow[rr, hook] & & \operatorname{Spec} \mathcal{O}_{B,b}
\end{tikzcd}$$

Doing this for every $n \in \mathbb{N}$ we get a collection of compatible deformations of X, which defines a formal deformation. (5) follows from observing that a formal deformation over a complete Noetherian local ring can be equivalently written as a compatible collection of infinitesimal deformations over the spectrum of the quotients of the complete local ring by the powers of its maximal ideal.

Unfortunately in this work things aren't so easy: indeed there are more steps to be aware of which are particular cases of the above definitions but nevertheless play an important role in this document and in general in deformation theory. To explain them, let us consider the following diagram of deformations of X (this simply means that each vertical arrow is a deformation of X):

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[d, "l"'] & \mathfrak{X} \arrow[r, hook] \arrow[d, "\mathfrak{f}"'] & \mathcal{T} \arrow[d, "f"'] \arrow[ddr, dashed] \arrow[ddrr, bend left] & & & \\
\operatorname{Spec} C \arrow[r, hook] & \operatorname{Spf} A \arrow[r, hook] & \operatorname{Spec} A \arrow[dddr, dashed] \arrow[dddrr] & & & \\
& & & & & \\
& & & \mathcal{Z} \arrow[r] \arrow[d, "g"] & \mathcal{Y} \arrow[r, hook] \arrow[d, "h"] & \mathcal{X} \arrow[d, "w"] \\
& & & \operatorname{Spec} D \arrow[r] & \operatorname{Spec} E \arrow[r, hook] & B
\end{tikzcd}$$

where B is a k-scheme of finite type, E is a k-algebra (essentially) of finite type (essentially of finite type means that it is the localization of a k-algebra of finite type), D is a k algebra which is also a DVR, A is a local complete Noetherian k-algebra and C is a local Artinian k-algebra. We point out that, in general, there is not a natural arrow from f to g, hence the dashed arrow, unless A is taken to be the completion of the DVR D along its maximal ideal.

We will say that a morphism defining a deformation is induced by another if the second deformation is isomorphic (as deformations) to the pull-back of the first along the closed embedding on the base.

Now, h is induced by w since the closed embedding $b\colon \operatorname{Spec} k \to B$ factors through the spectrum of a k-algebra (essentially) of finite type. The relationship among g and h has been explained in (1). Passing from h to f is considering first construction (1) above and then (4). From g we can deduce f by considering the formal completion of a DVR along its maximal ideal. f induces $\mathfrak{f}$ since the formal spectrum factors through the affine spectrum. Since the quotient of a DVR by powers of its maximal ideal is an Artinian ring, it follows that g induces l, hence also f induces l. Now the formal deformation $\mathfrak{f}$ induces the infinitesimal deformation l since the quotient of a local complete Noetherian ring is an Artinian ring.

However, reversing the constructions mentioned above is usually a hard problem and without further hypotheses on the scheme X is almost never possible. For example, passing from a formal deformation over an affine spectrum of a k-algebra (essentially) of finite type means to find an algebraisation of the formal deformation $\mathfrak{f}$. To solve this problem, Artin introduced a weaker notion of algebraisation: effectivity of a formal deformation (see [Ser07, Definition 5.1.7]). The idea of Artin was to split the problem of algebraisation in two subproblems:

(i) effectivity of the formal deformation: in other words find conditions on X to extend the formal deformation $\mathfrak{f}$ to the deformation f;

(ii) the second step is to find hypotheses on the formal deformation to extend it to the spectrum of an (essentially) of finite type k-algebra.

For step (ii), a sufficient criterion was given by Artin in [Art69]. There, under the hypotheses that the central fibre X is a projective scheme, he showed that if the formal deformation is versal, see [Ser07, Definition 2.2.6], and effective then it is algebraisable.

In order to present Grothendieck's classical results about sufficient conditions to achieve step (i), we will introduce some remarks on the notion of algebraisation and effectivity.

Definition 3.2.2. A LNFS $\mathfrak{X}$ is called *algebraizable* if there are a scheme Y and a closed subscheme X of Y such that $\mathfrak{X} = \hat{Y}_{/X}$.

It also makes sense to define algebraisable schemes in the relative setting. Suppose we have a formal scheme $\mathfrak{X}$ over the formal scheme $\operatorname{Spf} A$, with A a local adic Noetherian k-algebra A with residue field k. We say that $\mathfrak{X}$ is algebraisable over $\operatorname{Spf} A$ if there exists a scheme $\mathcal{X}$ over $\operatorname{Spec} A$ and a closed subscheme X of $\mathcal{X}$ such that $\mathfrak{X}$ is isomorphic to the formal completion $\widehat{\mathcal{X}}_{/X}$ of $\mathcal{X}$ along the closed subscheme X. Note that the ring A is left fixed but we are changing the locally ringed space structure on it.

Definition 3.2.3. Let X be a scheme and let $(A, \mathfrak{m})$ be a complete local Noetherian ring. A formal deformation $\mathfrak{f} \colon \mathfrak{X} \to \operatorname{Spf} A$ is called

(a) *algebraisable* if there exist

- a k-algebra essentially of finite type $(R, \mathfrak{n})$,
- a deformation $g \colon \mathcal{Y} \to \operatorname{Spec} R$ of X,
- an isomorphism $A \cong \hat{R}_{\mathfrak{n}}$,
- an isomorphism of formal deformations between $\mathfrak{f}$ and $\mathfrak{g} \colon \mathfrak{Y} \to \operatorname{Spf} \hat{R}$ associated to g.

(b) *effective* if there exists a deformation

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} A \end{tikzcd}$$

with f a flat morphism of finite type such that $\mathfrak{X} = \hat{\mathcal{X}}_{/X}$.

We recall here two fundamental classical results on algebraisation of formal schemes:

Theorem 3.2.4 ([EGAIII$_1$, Théorème (3.5.4.5)])**.** *Let A be a Noetherian I-adic ring, $T = \operatorname{Spec} A$, $\mathfrak{T} := \operatorname{Spf} A = \widehat{T}$, let $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{T}$ be a proper morphism of formal schemes. Let, for any $l \in \mathbb{N}$, $T_l := \operatorname{Spec}(A/I^{l+1})$, $X_l := \mathfrak{X} \times_{\mathfrak{T}} T_l$. Let $\mathfrak{L}$ be an invertible formal sheaf such that $\mathfrak{L}_0 := \mathfrak{L}/I\mathfrak{L}$ is an ample invertible sheaf on X_0. Then $\mathfrak{X}$ is algebraisable and if its algebraisation is projective then there exists an ample sheaf $\mathcal{M}$ on X such that $\mathfrak{L} \cong \widehat{\mathcal{M}}$.*

A corollary of the above theorem, is the following:

Theorem 3.2.5 ([Gro60, Théorème 4]). *Let A be a local complete Noetherian ring with residue field k, let $\mathfrak{X}$ be a proper formal scheme over $\operatorname{Spf} A$ and suppose that*

1. *the local rings of $\mathcal{O}_{\mathfrak{X}}$ are flat over A (in other words $\mathfrak{f}$ is flat);*
2. *$X_0 := \mathfrak{X} \otimes_A k$ satisfies $\mathrm{H}^2(X_0, \mathcal{O}_{X_0}) = 0$;*
3. *X_0 is projective.*

Then $\mathfrak{X}$ is algebraisable and its algebraisation is projective over $\operatorname{Spec} A$.

The difference between algebraisation of a formal schemes over a formal affine scheme, say $\operatorname{Spf} A$ with A as above, setting and algebraisation of a formal deformation over $\operatorname{Spf} A$ lies in the base affine scheme: in the algebraisation of the formal scheme, the k-algebra is required to be complete, while in the algebraisation of the formal deformation the k-algebra is required to be (essentially) of finite type.

If we interpret theorem 3.2.5 as a theorem on deformations, it says that, using the same notations as above, if we have a formal deformation $\mathfrak{f}\colon \mathfrak{X} \to \operatorname{Spf} A$ of a projective scheme X_0 with $\mathrm{H}^2(X_0, \mathcal{O}_{X_0}) = 0$, then the formal deformation $\mathfrak{f}$ is effective.

3.2.2 Motivating the definition of formal smoothing

We now explain why the definition of formal smoothing definition 3.1.10, given in [Tzi10], is a good definition and, in a sense, the only possible one. For this, observe that any DVR which is a k-algebra is a local Noetherian ring; in particular we have that its completion with respect to the adic topology induced by its maximal ideal is isomorphic to the formal power series in one variable $k[\![t]\!]$. We point out also that the spectrum of a DVR contains the closed and the generic point, while the formal spectrum of the formal power series ring is made of only one point.

Therefore it is natural to define the notion of smoothing of a scheme over a DVR as a deformation of X whose general fibre, i.e. the fibre over the open generic point, is smooth. On the other hand, in the case of formal deformation over $\operatorname{Spf} k[\![t]\!]$ such idea it is not possible.

We are going to explain what is the idea behind Tziolas' definition of formal smoothing. Let us suppose that $\pi\colon X \to B$ is a flat of relative dimension r morphism locally of finite type. Define

$$U_r = \{x \in X\colon \pi \text{ is smooth at } x \text{ of relative dimension } r\}.$$

By [Stacks, TAG 02G2], it is an open subset of X and by [Eis95, p. 407] or [Stacks, TAG 0C3K] we have that

$$U_r = X \setminus \mathrm{V}(\mathrm{Fitt}_r(\Omega^1_\pi)) \quad \text{and} \quad \mathrm{Sing}_r(\pi) = \mathrm{V}(\mathrm{Fitt}_r(\Omega^1_\pi)).$$

If we assume that π is proper, then $\pi(U_r) \subset B$ is open too and $\pi|_{U_r}\colon U_r \to A_r$ is smooth of relative dimension r, where $A_r := B \setminus \pi(\mathrm{V}(\mathrm{Fitt}_r(\Omega^1_\pi)))$. By base change I can always find a smoothing from the family over B if and only if A_r is not empty.

Assume now that B is affine, say $B = \operatorname{Spec} R$.

Remark 3.2.6. Suppose that B is a smooth curve over an algebraically closed field k and let $p \in B$ be a closed point and $R := \mathcal{O}_{B,p}$; it is well knows that R is a DVR with residue field is k. Then $\pi\colon X \to B$ is a geometric smoothing (according to definition definition 4.3.1) if and only if the pullback deformation $X_R \to \operatorname{Spec} R$ along the localization morphism $\operatorname{Spec} R \to B$ is a smoothing (in the sense of definition definition 3.1.9).

Summing up, we have following situation:

$$\begin{array}{ccccccc} X_n & \hookrightarrow & X_{\widehat{R}} & \xhookrightarrow{\beta} & X_R & \hookrightarrow & X \\ \downarrow{\scriptstyle \pi|_{X_n}} & & \downarrow{\scriptstyle \pi|_{X_{\widehat{R}}}} & & \downarrow{\scriptstyle \pi|_{X_R}} & & \downarrow{\scriptstyle \pi} \\ S_n & \hookrightarrow & \operatorname{Spec} \widehat{R} & \xhookrightarrow[\alpha]{} & \operatorname{Spec} R & \hookrightarrow & B \end{array}$$

where all squares are Cartesian, $R = \mathcal{O}_{B,p}$ with $p \in B(k)$ a closed point, $\widehat{R}$ denotes the completion of $\mathcal{O}_{B,p}$ along its maximal ideal $\mathfrak{m}_p$ and $R_n := \frac{k[\![t]\!]}{(t^{n+1})} = \frac{k[t]}{(t^{n+1})}$ and $S_n := \operatorname{Spec} R_n$, for every $n \in \mathbb{N}$. Observe that the completion of $\mathcal{O}_{B,p}$ along the maximal ideal is isomorphic to $k[\![t]\!]$.

In order to lighten the notation, let us denote $\pi_n := \pi|_{X_n}$, $\widehat{\pi} := \pi|_{X_{\widehat{R}}}$, $\widetilde{\pi} := \pi|_{X_R}$.

Observe now that α is a homeomorphism, hence β is at least a bijective function on the sets, and by [Eis95, Corollary 20.5] we have that

$$\mathrm{Sing}_r(\widehat{\pi}) = \beta^{-1}(\mathrm{Sing}_r(\widetilde{\pi})).$$

Therefore, $\widetilde{\pi}$ is smooth of relative dimension r along $\pi^{-1}(\eta)$ if and only if $\widehat{\pi}$ is smooth of relative dimension r along $\widehat{\pi}^{-1}(\widehat{\eta})$, where η and $\widehat{\eta}$ are the generic points of $\operatorname{Spec} R$ and $\operatorname{Spec} \widehat{R}$ respectively.

Now $\operatorname{Spec} R$ has only two points: the closed one, Y with ideal sheaf $\mathcal{I}_Y = (t)$, and the open one, η. Let also $C := \mathrm{Sing}_r(\widetilde{\pi})$.

When is that $\widetilde{\pi}(C) \subset Y$ as sets? The answer is if and only if $C \subset \widetilde{\pi}^{-1}(\widetilde{Y})$.

However $\pi(C)$ carries a schematic structure. Therefore, $\pi(C) \subset Y$ as schemes if and only if there exists a structure of closed subscheme $\widetilde{Y}$ on Y with $\widetilde{Y}_{\text{red}} = Y$, such that $\pi(C) \subset \widetilde{Y}$ as sets.

Hence we are reduced to classify all such closed subscheme structures on $\operatorname{Spec} k[\![t]\!]$. These are given by $Y_k := \mathrm{V}((t^{k+1}))$, for every $k \in \mathbb{N}$. In particular we have a sequence of closed subschemes

$$Y = (t) = Y_0 \subset Y_1 = (t^2) \subset Y_3 \subset \cdots .$$

Summing up the last paragraphs, $\widetilde{Y}$ is closed with $\widetilde{Y}_{\text{red}} = Y$ if and only if there exists a non-negative integer k such that $\widetilde{Y} = Y_k$. Going back in the argument we find that $\pi(C) \subset Y_k$ as closed subschemes if and only if $C \subset \widetilde{\pi}^{-1}((t^{k+1}))$ as closed subschemes which is equivalent to the existence of a non-negative integer k such that

$$\operatorname{Fitt}_r(\Omega^1_{\widetilde{\pi}}) = \mathcal{I}_{C/X} \supseteq \widetilde{\pi}^{-1}((t^{k+1})) = \widetilde{\pi}^{-1}(\mathcal{I}_{Y_k}).$$

Applying this argument when $B = \operatorname{Spf} k[\![t]\!]$ and observing that if the condition we have found works for one ideal of definition it works for all ideals of definitions, we have deduced the definition of formal smoothing as given in [Tzi10, Definition 11.6].

Chapter 4

From formal to geometric smoothings

This chapter is divided in three sections: in the first one we introduce the Gorenstein condition on schemes and morphisms and study its behaviour under infinitesimal deformations. The main result of this section is that if X is a projective and Gorenstein scheme then we can always extend the dualising sheaf on X to an invertible formal sheaf on any formal deformation of X. Since lci morphisms are Gorenstein, it follows that the results proven in the first chapter are valid also for lci morphisms. However, in the section section we present the same result (the dualising sheaf for lci morphisms always extends to any formal deformation) using different different arguments that involve cotangent complex, naïve cotangent comeplex and determinants of complexes among others. This fact that dualising complexes always extends is one of the key ingredient of the main theorem of this chapter which is presented in the last section.

4.1 Gorenstein morphisms and deformations

In this section we review, following [Stacks], the notions of dualising complexes and of Gorenstein morphisms. We then discuss how the Gorenstein property behaves under infinitesimal deformations. The main results of this section are two: a deformation of a Gorenstein morphism is still Gorenstein, which is a result already known but for which we weren't able to find any reference, and the extension property of the relative dualising sheaf.

Definition 4.1.1. Let A be a Noetherian ring. A *dualising complex* is a complex of A modules $\omega_A^\bullet$ such that

1. $\omega_A^\bullet$ has finite injective dimension;

2. $\mathrm{H}^i(\omega_A^\bullet)$ is a finite A-module, for every i;

3. $A \to \mathbf{R}\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$ is a quasi-isomorphism in the derived category of A-modules.

Definition 4.1.2. Let A be a local Noetherian ring. We say that A is a *Gorenstein local ring* if $A[0]$ is a dualising complex.

Definition 4.1.3. A scheme X is called *Gorenstein* if it is locally Noetherian and for every $x \in X$, $\mathcal{O}_{x,X}$ is a Gorenstein local ring according to definition 4.1.2.

Definition 4.1.4. Let $f\colon X \to Y$ be a morphism of schemes such that for every $y \in Y$, the fibre X_y is a locally Noetherian scheme.

1. Let $x \in X$ and $y := f(x)$. We say that f is *Gorenstein at x* if f is flat at x and $\mathcal{O}_{X_y,x}$ is a Gorenstein local ring.

2. We say that f is *Gorenstein* if it is Gorenstein at x, for all $x \in X$.

Lemma 4.1.5 ([Stacks, Tag 0C12])**.** *Let $f\colon X \to Y$ be a flat morphism of locally Noetherian schemes. If X is Gorenstein, then f is Gorenstein.*

Proposition 4.1.6 ([Stacks, Tag 0C07])**.** *Let $f\colon X \to Y$ be a morphism of schemes such that for every $y \in Y$ the fiber X_y is locally Noetherian and let $g\colon Y' \to Y$ be a locally of finite type morphism of schemes. Consider the following Cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array} \tag{4.1}$$

If f' is Gorenstein at $x' \in X'$ and f is flat at $g'(x')$, then f is Gorenstein at $g'(x')$.

The next topic is a review of the right adjoint functor of the right derived push-forward functor; we will also explain its relation with the upper shriek functor and with the Gorenstein property. In particular, we will define the relative dualising complex and show that it behaves well under pullbacks.

We recall that, see [Stacks, Tag 0A9E], every morphism of schemes $f\colon X \to Y$, with Y quasi-compact, admits a right adjoint for the right derived functor $\mathbf{R}f_*$ and we denote it by $\Psi\colon \mathrm{D}_{\mathrm{QCoh}}(Y) \to \mathrm{D}_{\mathrm{QCoh}}(X)$.

Definition 4.1.7. Let Y be a quasi-compact scheme, let $f\colon X \to Y$ be a proper, flat morphism of finite presentation and let Ψ be the right adjoint for $\mathbf{R}f_*$. We define the *relative dualising complex* $\omega_f^\bullet$ of f (or of X over Y) as follows

$$\omega_f^\bullet := \Psi(\mathcal{O}_Y).$$

The next proposition explains the behaviour of the relative dualising complex under base change.

Proposition 4.1.8 ([Stacks, Tag 0AAB])**.** *Let X be a scheme, let Y and Y' be quasi-compact schemes, let $g\colon Y' \to Y$ also be any morphism and let $f\colon X \to Y$ be a proper, flat morphism of finite presentation. If we have a fibre diagram as in eq.* (4.1)*, then we have a canonical isomorphism*

$$\omega_{f'}^\bullet \cong \mathbf{L}(g')^*\omega_f^\bullet.$$

We now move to the second half of this section, namely introducing the upper shriek functor and understanding its relationships with the right adjoint functor for the pushforward functor and the Gorenstein condition.

Remember from 1.3.1 that FTS is the category whose objects are separated, algebraic schemes over the field k and whose morphisms are morphisms of k-schemes.

Definition 4.1.9. Let $f\colon X \to Y$ be a morphism in the category of FTS schemes. We define the *upper shriek functor*

$$f^!\colon \mathrm{D}^+_{\mathrm{QCoh}}(\mathcal{O}_Y) \to \mathrm{D}^+_{\mathrm{QCoh}}(\mathcal{O}_X)$$

as follows: we choose a compactification $X \to \bar{X}$ of X over Y,[1] $\bar{f}\colon \bar{X} \to Y$ denotes the structure morphism and we consider its right adjoint functor $\bar{\Psi}$; we then let $f^!K := \bar{\Psi}(K)|_X$ for $K \in \mathrm{D}^+_{\mathrm{QCoh}}(\mathcal{O}_Y)$.

According to [Stacks, Tag 0AA0], the definition of the upper shriek functor is, up to canonical isomorphism, independent of the choice of the compactification of X.

Remark 4.1.10. We point out that if $f\colon X \to Y$ is a proper morphism in the category FTS, then $\bar{\Psi} = \Psi$, implying that the upper shriek functor is the restriction to $\mathrm{D}_{\mathrm{QCoh}}(\mathcal{O}_Y)$ of Ψ, the right adjoint functor of $\mathbf{R}f_*$ (see [Stacks, Tag 0AU3]).

[1] Such a compactification always exists by [Stacks, Tag 0F41] and [Stacks, Tag 0A9Z].

We are now ready to present the link between the Gorenstein condition and the upper shriek functor.

Proposition 4.1.11 ([Stacks, Tag 0C08]). *Let $f\colon X \to Y$ be a flat morphism of schemes in FTS and let $x \in X$. Then the following conditions are equivalent:*

1. *f is Gorenstein at x;*

2. *$f^!\mathcal{O}_Y$ is isomorphic to an invertible object (of the derived category) in a neighbourhood of x.*

In particular the set $\{x \in X\colon f \text{ is Gorenstein at } x\}$ is open in X.

We point out that the condition of f being Gorenstein is an open condition on the source and if we assumed the morphism f to be proper, then the condition is also an open condition on the target.

The following proposition introduces the notion of the relative dualising sheaf for a morphism in the category FTS and exploits its relationship with the relative dualising complex under the Gorenstein hypotheses on the morphism f.

Proposition 4.1.12 ([Stacks, Tag 0BV8]). *Let X and Y be separated schemes and let $f\colon X \to Y$ be a Gorenstein morphism of schemes. Then there exists a coherent, invertible sheaf, called the relative dualising sheaf of f and denoted by ω_f, which is flat over Y and satisfies*

$$f^!\mathcal{O}_Y \cong \omega_f[-d],$$

where d is the locally constant function on X which gives the relative dimension of X over Y.

If f is also proper, flat and of finite presentation, then $\omega_f^\bullet = \omega_f[-d]$.

If $Y = \operatorname{Spec} k$, then we denote the *relative dualising sheaf* of X over k by ω_X.

Proposition 4.1.13. *Let X be a Gorenstein scheme and let A be an Artinian local k-algebra with residue field k. Consider now a deformation of X over A; that is a Cartesian diagram*

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} A \end{tikzcd}$$

with f flat. Then f is a Gorenstein morphism.

Proof. Since X is Gorenstein and $X \to \operatorname{Spec} k$ is flat, by lemma 4.1.5 it follows that $X \to \operatorname{Spec} k$ is Gorenstein. Applying now proposition 4.1.6, we deduce that $f\colon \mathcal{X} \to \operatorname{Spec} A$ is Gorenstein. □

Now we present the first result that will help us to deduce the existence of a geometric smoothing. Remember from notation 2.0.11 that $\mathfrak{S} = \operatorname{Spf} k[\![t]\!]$ and, for every $n \in \mathbb{N}$, $S_n = \operatorname{Spec} k[\![t]\!]/(t^{n+1})$.

Proposition 4.1.14. *Let X be a projective, Gorenstein scheme. If $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ is a formal deformation of X, then the relative dualising sheaf $\omega_{X/k} = \omega_X$ always extends to an invertible formal sheaf on $\mathfrak{X}$. In other words, there exists a unique invertible formal sheaf $\mathfrak{L}$ on $\mathfrak{X}$ such that $\mathfrak{L} \otimes_{k[\![t]\!]} k \cong \omega_X$.*

Proof. By proposition 3.1.4, the formal deformation $\mathfrak{f}$ is equivalent to a collection of deformations $\{f_n\colon \mathcal{X}_n \to S_n\}_{n\in\mathbb{N}}$ satisfying the compatibility condition of eq. (3.2), with f_n flat, proper morphisms. Since X is Gorenstein, applying proposition 4.1.13 we deduce that for every natural number n, the morphism f_n is Gorenstein. Now consider the following Cartesian diagram

$$\begin{tikzcd} \mathcal{X}_n \arrow[r, hook, "j_n"] \arrow[d, "f_n"] & \mathcal{X}_{n+1} \arrow[d, "f_{n+1}"] \\ S_n \arrow[r, hook] & S_{n+1}; \end{tikzcd}$$

we have, for every natural number n, the following chain of equalities and natural isomorphisms

$$\begin{aligned} j_n^*\omega_{f_{n+1}} &= \mathrm{H}^{-\dim X}(j_n^*\omega_{f_{n+1}}[-\dim X]) && \text{(proposition 4.1.12)} \\ &= \mathrm{H}^{-\dim X}(\mathbf{L}j_n^*\omega^\bullet_{f_{n+1}}) \\ &\cong \mathrm{H}^{-\dim X}(\omega^\bullet_{f_n}) && \text{(proposition 4.1.8)} \\ &= \mathrm{H}^{-\dim X}(\omega_{f_n}[-\dim X]) && \text{(proposition 4.1.12)} \\ &= \omega_{f_n}. \end{aligned}$$

Theorem 2.0.59 then implies that there exists an invertible formal sheaf $\mathfrak{L}$ on $\mathfrak{X}$ such that $\mathfrak{L} \otimes_{k[\![t]\!]} k \cong \omega_X$. □

As a consequence of this last proposition, we get that if X is a projective, lci scheme over a field k and we have a formal deformation $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$, then the relative dualising sheaf ω_X always extends to the formal deformation $\mathfrak{f}$. To see this, it is enough to observe that lci schemes/morphisms are in particular Gorenstein schemes/morphisms and then apply the previous proposition.

4.2 Lci morphisms, cotanget complex & deformations

This section contains a review of the definitions of lci morphism of schemes. We also study the behaviour of the lci property under infinitesimal deformations. Furthermore, we introduce the notions of naïve cotangent complex and cotangent complex relating them to the property of being local complete intersection. We will follow [Stacks].

4.2.1 Lci morphisms of schemes

We start with a review of lci condition and mention some of its properties.

Definition 4.2.1. Let R be a ring. A sequence of elements $f_1, \dots, f_c \in R$ is called a *regular sequence* if and only if the two conditions are satisfied

1. $(f_1, \dots, f_r)$ is a proper ideal of R;

2. for every $j = 1, \dots, r$, f_j is not a zero-divisor in $R/(f_1, \dots, f_{j-1})$. Equivalently, the ring homomorphism
$$- \cdot f_j \colon \frac{R}{(f_1, \dots, f_{j-1})} \to \frac{R}{(f_1, \dots, f_{j-1})}$$
is injective.

We point out that the notion of regular sequence depends on the order of the elements. An example that shows this behaviour can be found in [Stacks, TAG 00LG]. However, if the ring R is a local Noetherian ring, then any permutation of a regular sequence is regular, see [Stacks, TAG 00LJ].

Definition 4.2.2. Let X be a ringed space. A sequence $f_1, \dots, f_r$ of elements of $\Gamma(X, \mathcal{O}_X)$ is called a *regular sequence of length r* if and only if for every $j = 1, \dots, r$ the morphism of sheaves
$$- \cdot f_j \colon \frac{\mathcal{O}_X}{(f_1, \dots, f_{j-1})} \to \frac{\mathcal{O}_X}{(f_1, \dots, f_{j-1})}$$
is injective.

Since injectivity of a morphism of sheaves can be checked on the stalks, the above condition is equivalent to the following one: for every $x \in X$ and for every $j = 1, \dots, r$, the morphism

$$- \cdot f_{jx} \colon \frac{\mathcal{O}_{X,x}}{(f_{1x}, \dots, f_{j-1x})} \to \frac{\mathcal{O}_{X,x}}{(f_{1x}, \dots, f_{j-1x})},$$

where g_x means the image of $g \in \Gamma(X, \mathcal{O}_X)$ in $\mathcal{O}_{X,x}$.

Definition 4.2.3. Let X be a ringed space, $r \in \mathbb{N}$ and let $\mathcal{J}$ be a sheaf of ideals. We say that $\mathcal{J}$ is a *r-regular sheaf of ideals* if and only if for every $x \in \operatorname{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists a neighbourhood U of x and there is a regular sequence of length r $f_1, \dots, f_r \in \Gamma(U, \mathcal{O}_X)$ such that $\mathcal{J}|_U = (f_1, \dots, f_r)$.

Proposition 4.2.4 ([Stacks, TAG 063I])**.** *Let X be a locally Noetherian scheme, let $r \in \mathbb{N}$ and let $\mathcal{J}$ be a sheaf of ideals. The following are equivalent:*

1. *$\mathcal{J}$ is a r-regular sheaf of ideals;*
2. *for every $x \in \operatorname{Supp}(\mathcal{O}_X/\mathcal{J})$, $\mathcal{J}_x$ is generated by a regular sequence of length r in $\mathcal{O}_{X,x}$;*
3. *for every $x \in \operatorname{Supp}(\mathcal{O}_X/\mathcal{J})$, there exist an affine open neighbourhood $\operatorname{Spec} A$ of x and a r-regular ideal J of A such that $\mathcal{J}|_U = \widetilde{J}$.*

Lemma 4.2.5 ([Stacks, TAG 063H])**.** *Let X be a locally ringed space and let $\mathcal{J}$ be a quasi-coherent sheaf of ideals. If $\mathcal{J}$ is a r-regular sheaf then $\mathcal{J}/\mathcal{J}^2$ is a locally free $\mathcal{O}_X/\mathcal{J}$-module of rank r.*

Definition 4.2.6. Let $i \colon Z \to X$ be an immersion of schemes, i.e. there are an open subset U of X and a closed subset C of U such that $i(Z)$ is isomorphic to C. We say that that i is a *regular immersion* if the sheaf of ideals $\mathcal{I}_{C/U}$ associated to the closed immersion $C \hookrightarrow U$ is a regular sheaf of ideals.

The definition does not depend on the choice of U and C. Indeed, if U, C and U', C' are two pair of open subsets of X and closed subsets of U and U' respectively and if $\mathcal{I}_{C/U}$ and $\mathcal{I}_{C'/U'}$ are the induced sheaf of ideals, then $\mathcal{I}_{C/U}|_{U \cap U'} = \mathcal{I}_{C'/U'}|_{U \cap U'}$. Since $\operatorname{Supp}(\mathcal{O}_U/\mathcal{I}_{C/U}) = Z = \operatorname{Supp}(\mathcal{O}_{U'}/\mathcal{I}_{C'/U'})$, it follows that $\mathcal{I}_{C/U}$ is a regular sheaf of ideals if and only if $\mathcal{I}_{C'/U'}$ is.

Lemma 4.2.7 ([Stacks, TAG 063M])**.** *Let $i \colon Z \to X$ be a regular immersion. Then the conormal sheaf $\mathcal{C}_{Z/X} := i^*\mathcal{I}_{C/U}$ is a locally free sheaf of finite rank, where C and U are as in definition definition 4.2.6.*

Definition 4.2.8. Let $f\colon X \to Y$ be a morphism of schemes of finite type over a field k and let $d \in \mathbb{Z}$. We say that f is a *local complete intersection morphism of relative dimension* d (lci morphism for short) if and only if for every open subset U of X and every subset V of Y with $f(U) \subset V$ there exist a natural number r and a factorization $U \hookrightarrow \mathbb{A}^r_V \to V$ of $f|_U$ such that $U \hookrightarrow \mathbb{A}^r_V$ is a closed regular embedding of codimension $c = r - d$.

Remark 4.2.9. The above definition does not depend on the factorization chosen. Indeed, suppose that we are given two such factorizations of $f|_U$:

$$\begin{tikzcd}
& \mathbb{A}^{r_1}_V \arrow[dr] & \\
U \arrow[ur, hook, "i"] \arrow[dr, hook, "j"'] & & V \\
& \mathbb{A}^{r_2} \arrow[ur] &
\end{tikzcd}$$

and consider the fibre product $\mathbb{A}^{r_1}_V \times_V \mathbb{A}^{r_2}_V \cong \mathbb{A}^{r_1+r_2}_V$. By the universal property of fibre product there exists a unique morphism $\iota\colon U \to \mathbb{A}^{r_1+r_2}_V$, $\iota(u) := (i(u), j(u))$. By [Stacks, TAG 067S] and [Stacks, TAG 0693] we conclude that i is a regular embedding if and only if ι is a regular embedding if and only if j is a regular embedding.

Furthermore, the property of being a lci morphism is local both on the target and on the source.

We remark that this definition is different from the one given in [Ful84]; there, he also assumes that the morphism has a global factorization, i.e. it can be written as $p \circ i$ where $i\colon X \to P$ is a closed embedding and $p\colon P \to Y$ is a smooth (hence, locally of finite presentation) morphism.

We also point out that if $f\colon X \to Y$ is any morphism of schemes, then the locus X_{lci} of points of X such that f is a lci morphism at x, is open in X. If we further assume that f is proper, then the locus of points

$$Y_{\text{lci}} := \{y \in Y\colon f \text{ is lci at } x, \forall x \in f^{-1}(y)\}$$

is open in Y.

4.2.2 Cotangent complex and deformation of lci morphisms

In this subsection we introduce the cotangent complex and we show that in case of lci morphisms cotangent complex and naïve cotangent complex are quasi isomorphic. We then prove that the dualising sheaf of a lci scheme extends to any formal deformation.

Definition 4.2.10. Let $f\colon X \to Y$ be a factorizable morphism of schemes and choose a global factorization $p \circ i$ of f, where $i\colon X \to P$ is a closed embedding and $p : P \to Y$ is smooth. We define the *naïve contangent complex associate to f* to be the complex

$$\mathrm{N}\mathbb{L}_f := \left[\frac{\mathcal{I}}{\mathcal{I}^2} \to \left(\Omega^1_{P/Y}\right)|_X\right] = \left[i^*\mathcal{I} \to i^*\Omega^1_{P/Y}\right] \in \mathrm{D}^{[-1,0]}(X),$$

where $\mathcal{I}$ denotes the sheaf of ideals associated to the closed embedding $i\colon X \hookrightarrow P$.

The definition does not depend on the factorization chosen, since any two choices lead to complexes which are canonically isomorphic in $\mathrm{D}^{[-1,0]}(X)$. To see this, confront two factorizations $X \xrightarrow{i_a} P_a \xrightarrow{p_a} Y$ for $a = 1, 2$ with $X \to P_1 \times_Y P_2 \to Y$ (see [Ful84] for details).

If $f\colon X \to Y$ is a lci morphism, then the naïve cotangent complex of f can locally be described as follows: choose a local factorization of f as $U \hookrightarrow \mathbb{A}^r_V \to V$, with U and V open subsets of X and Y respectively such that $f(U) \subset V$ and $U \hookrightarrow \mathbb{A}^r_V$ is a regular embedding with sheaf of ideals given by $\mathcal{I} := \mathcal{I}_{U/\mathbb{A}^r_V}$. Then there is a quasi isomorphisms in $\mathrm{D}^{[-1,0]}(U)$:

$$\mathrm{N}\mathbb{L}_f|_U \cong \left[\frac{\mathcal{I}}{\mathcal{I}^2} \to \left(\Omega^1_{\mathbb{A}^r_V/V}\right)|_U\right]$$

The next proposition shows that any infinitesimal deformation of a lci scheme is still lci.

Proposition 4.2.11 ([Ser07, Example A.12])**.** *Let X be a lci, algebraic k-scheme and let A be a local Artinian k-algebra with residue field k. Then every infinitesimal deformation $f\colon \mathcal{X} \to \operatorname{Spec} A$ of X over A is lci, i.e. the morphism f is lci.*

Next proposition presents the relationship among the naïve cotangent complex and the cotangent complex for lci scheme.

Proposition 4.2.12. *Let $f\colon X \to Y$ be a lci morphism. Then the cotangent complex $\mathbb{L}_f$ exists, it is a perfect complex in $D(X)$, the derived category of quasi-coherent sheaves on X, of Tor-amptitude in $[-1, 0]$ and it is quasi-isomorphic in $D(X)$ to the naïve cotangent complex $\mathrm{N}\mathbb{L}_f$. Moreover if we fix a local factorization $s \circ j$*

of $f|_U$ as regular immersion $j\colon U \to P$ with sheaf of ideals $\mathcal{I}$ and smooth map $s\colon P \to Y$, then we have a quasi-isomorphism in $D^{[-1,0]}(U)$

$$\mathbb{L}_f|_U \cong \left[\frac{\mathcal{I}}{\mathcal{I}^2} \to \Omega^1_{P/Y}|_U\right].$$

Proof. By [Stacks, TAG 08T3][2], we can assume X and Y affine schemes, say $\operatorname{Spec} B$ and $\operatorname{Spec} A$ respectively, since we would get a quasi-isomorphism $\widetilde{L_{B/A}} \cong \mathbb{L}_f$ in $D(X)$, for any two open affine subschemes $\operatorname{Spec} B$ of X, $\operatorname{Spec} A$ of Y that satisfies $f(\operatorname{Spec} B) \subset \operatorname{Spec} A$. Moreover, thanks to [Stacks, TAG 07DB], we deduce that the morphism $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is lci if and only if the morphism $A \to B$ is. In particular, given a factorization $\operatorname{Spec} B \hookrightarrow P = \operatorname{Spec} R \to \operatorname{Spec} A$ of f with $\operatorname{Spec} B \hookrightarrow P$ regular closed embedding, we deduce that $P \to B$ is a ring surjection with kernel I generated by a regular sequence (with $\tilde{I} = \mathcal{I}|_{\operatorname{Spec} B}$). By [Stacks, TAG 08SL] we deduce that the cotangent complex of a lci ring homomorphism $A \to B$ is a perfect complex in the derived category of finitely generated B-modules of Tor-amptitude in $[-1, 0]$ and

$$\begin{aligned}\mathbb{L}_f \cong \widetilde{L_{B/A}} \cong \left[\frac{I}{I^2} \to \Omega^1_{R/A} \otimes_R B\right]^{\sim} = \left[\frac{\widetilde{I}}{I^2} \to (\Omega^1_{R/A} \otimes_R B)^{\sim}\right] \\ = \widetilde{NL_{B/A}},\end{aligned}$$

where the last equality is the definition of naive cotangent complex for a morphism of rings (see [Stacks, TAG 07BN]). Since by [Stacks, TAG 08UW] we have a global natural map $\mathbb{L}_f \to \mathrm{N}\mathbb{L}_f$ which identifies the naïve cotangent complex with the truncation of the cotangent complex in degree less than -1; it then follows that the global natural map above is an isomorphism. □

Proposition 4.2.13 ([Ill72, Proposition 1.3.3]). *Suppose it is given a Cartesian diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

such that $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_{Y'}) = 0$, for $i > 0$. Then the natural map of complexes $\mathbf{L}(g')^*\mathbb{L}_f \to \mathbb{L}_{f'}$ is a quasi-isomorphism in $D(X')$.

[2] We point out that in the reference, the (naïve) cotangent complex of a lci morphism $f\colon X \to Y$ is denoted by $L_{X/Y}$ (by $NL_{X/Y}$ respectively). We decided to use the notation $\mathrm{N}\mathbb{L}_f$ for the same object in order lighten the notation.

Using notations of proposition 4.2.13, we remark that the condition of the vanishing of the Tor is satisfied whenever either f or g are flat morphism.

Proposition 4.2.14 ([Stacks, TAG 0FJX]). *Let X be a scheme. There is a functor*

$$\det\colon \left\{\begin{matrix}\textit{category of perfect complexes}\\ \textit{with Tor-amplitude in } [-1,0]\\ \textit{morphism are isomorphism}\end{matrix}\right\} \to \left\{\begin{matrix}\textit{category of invertible sheaves}\\ \textit{morphisms are isomorphisms}\end{matrix}\right\},$$

such that if $L^\bullet = [L^{-1} \to L^0]$, with L^{-1} and L^0 locally free of finite rank, then $\det L^\bullet = \det L^0 \otimes \det(L^{-1})^{\otimes -1}$ and such that the functor $\det$ commutes with restriction to open subsets.

Proposition 4.2.15 ([Stacks, TAG 0FJY]). *Given a morphism $f\colon X \to Y$ of schemes and an object $L^\bullet \in D^{[-1,0]}(\mathcal{O}_Y)$ which is perfect of Tor-amplitude in $[-1,0]$, then we conclude that $\mathbf{L}f^*L^\bullet \in D^{[-1,0]}(\mathcal{O}_X)$ and there is a canonical identification*

$$f^*(\det L^\bullet) \to \det(\mathbf{L}f^*L^\bullet).$$

We now state the promised lemma: the dualising complex of an lci schemes always extends to an invertible sheaf on the formal deformation.

Lemma 4.2.16. *Suppose that X is a projective, lci k-scheme. If $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ is a formal deformation of X, then the dualizing sheaf ω_X° extends to an invertible sheaf on $\mathfrak{X}$, i.e. there exists an invertible sheaf $\mathfrak{L}$ on $\mathfrak{X}$ such that $\mathfrak{L} \otimes_{k[\![t]\!]} k \cong \omega_X^\circ$.*

Proof. By proposition 3.1.4, the formal deformation $\mathfrak{f}$ is equivalent to a collection of deformations $\{f_n\colon X_n \to S_n\}_{n\in\mathbb{N}}$ satisfying the compatibility condition of eq. (3.2), with f_n flat, proper morphisms. Since X is lci, applying proposition 4.2.11 we deduce that, for every non-negative integer n, f_n is a lci morphism. Now lci morphisms admit cotangent complex $\mathbb{L}_{f_n}$ which is perfect of Tor-amptitude $[-1,0]$ and it is quasi-isomorphic to the naive cotangent complex, see proposition 4.2.12. Since f_n are flat maps, by proposition 4.2.13, it follows that the natural map in the derived category $D(X_n)$ of coherent sheaves of $\mathcal{O}_{X_n}$-modules

$$\mathbf{L}j_n^*\mathbb{L}_{f_{n+1}} \to \mathbb{L}_{f_n}$$

is a quasi-isomorphism, where we used the following notation:

$$\begin{tikzcd} X_n \arrow[r, hook, "j_n"] \arrow[d, "f_n"] & X_{n+1} \arrow[d, "f_{n+1}"] \\ S_n \arrow[r, hook] & S_{n+1} \end{tikzcd}.$$

We conclude that for every integer $n \geq 0$ there is an invertible sheaf $\omega^\circ_{X_n}$ on X_n together with isomorphisms $\omega^\circ_{X_n} \otimes_{\mathcal{O}_{S_n}} k \cong \omega^\circ_X$ and

$$\begin{aligned}
\omega^\circ_{X_n} &\cong \det\left(\Omega^1_{P_n/S_n}\right) \otimes \det\left(\frac{\mathcal{I}_n}{\mathcal{I}_n^2}\right)^{\otimes -1} && \text{([Har77, III-Theorem 7.11])}\\
&= \det\left(\mathbb{L}_{f_n}\right) && \text{(proposition 4.2.14)}\\
&\cong \det\left(\mathbf{L}j_n^*\mathbb{L}_{f_{n+1}}\right) && \text{(proposition 4.2.13)}\\
&\cong j_n^*\left(\det \mathbb{L}_{f_{n+1}}\right) && \text{(proposition 4.2.15)}\\
&\cong j_n^*\omega^\circ_{X_{n+1}}\\
&= \omega^\circ_{X_{n+1}} \otimes_{\mathcal{O}_{S_n}} \mathcal{O}_{X_n}
\end{aligned}$$

where the second to last isomorphism is motivated by applying the previous three steps in reverse order and $\mathcal{I}_n$ is the sheaf of ideals associated to the regular embedding $\mathcal{X}_n \hookrightarrow P_n$. Theorem 2.0.59 then implies that there exists an invertible sheaf $\mathfrak{L}$ on $\mathfrak{X}$, that can be described by $\varprojlim_n \omega^\circ_{X_n}$, such that $\mathfrak{L} \otimes_{k[\![t]\!]} k \cong \omega^\circ_X$. $\square$

Under the assumptions and notations of lemma 4.2.16, the same argument also shows that the dual $(\omega^\circ_X)^\vee$ of the dualizing sheaf extends to an invertible sheaf $\mathfrak{L}^\vee$ on the formal scheme $\mathfrak{X}$.

4.3 From formal smoothing to geometric smoothing

In this last section we use all the previous results to pass from a formal smoothing to a geometric one. We start by recalling the definition of geometric smoothing.

Definition 4.3.1. Let X be a proper scheme. A *geometric smoothing* is a Cartesian diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\
\operatorname{Spec} k = \operatorname{Spec} \frac{\mathcal{O}_{c,C}}{\mathfrak{m}_c} \arrow[r, hook] & C
\end{tikzcd} \tag{4.2}$$

where C is a smooth curve, $c \in C$ is a closed point and π is a flat and proper morphism, such that $\pi^{-1}(\eta_C) =: \mathcal{X}_{\text{gen}}$ is smooth, where η_C is the generic point of C. We say that X is *geometrically smoothable* if it has a geometric smoothing.

We remark that, if X is smooth over $\operatorname{Spec} k$, then X is geometrically smoothable in a trivial way by considering the trivial family $\mathrm{pr}_2\colon X \times_k C \to C$ of deformations.

We now present some results that will be needed in the proof of the main theorem.

Lemma 4.3.2 ([Kem93, Lemma 7.2.1 page 87])**.** *Let X be a scheme, let U be an open, dense subset of X and let $p \in X$ be a closed point. Then there exists an affine curve C in X such that C intersects U and passes through p.*

Remark 4.3.3. Let C be a smooth curve over k and let $c \in C(k)$ be a closed point. Denote by l a local parameter of the maximal ideal $\mathfrak{m}_c$ in $\mathcal{O}_{C,c}$. Then there is a isomorphism of topological rings

$$\widehat{\mathcal{O}_{C,c}} \cong k[\![t]\!]$$

such that l is sent to t.

Proposition 4.3.4. *Let $f\colon X \to Y$ be a morphism of schemes such that X is reduced and irreducible. Then there exists an irreducible and reduced component Y' of Y such that f factors trough Y', i.e. the following diagram commutes*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & Y' \arrow[ur, hook] & \end{tikzcd}.$$

Proof. Since X is irreducible, by [Stacks, Tag 0379], $f(X)$ is an irreducible subset of Y. Then $Y' := \overline{f(X)}$ is an irreducible component of Y and f factors through Y' by construction. By [Har77, II-Ex. 2.3(c)], we can always assume Y' to be a reduced scheme. □

Remember from notation 2.0.11 that $\mathfrak{S} = \operatorname{Spf} k[\![t]\!]$ and that, for any natural number n, $S_n = \operatorname{Spec} k[\![t]\!]/(t^{n+1})$.

The next lemma shows that being geometrically smooth implies being formally smooth.

Lemma 4.3.5. *Let X be a projective, equidimensional scheme. If X is geometrically smoothable, then it is also formally smoothable.*

Proof. Suppose X has a geometric smoothing like eq. (4.2), where c is the closed point of C such that the fibre of π over c is X. Consider the pullback $\tilde{\pi}$ of π along the composite morphism $\operatorname{Spec}\widehat{\mathcal{O}_{C,c}} \to \operatorname{Spec}\mathcal{O}_{C,c} \to C$; since π is a smoothing of X, so is $\tilde{\pi}$. By Remark 4.3.3 we have that the completion of the regular local ring $\mathcal{O}_{C,c}$ is continuously isomorphic to $\mathfrak{S}$. Now using Remark/definition 3.1.5, we can construct the associated formal deformation $\mathfrak{p}\colon \mathfrak{X} \to \mathfrak{S}$. We end the argument by invoking Proposition 3.1.11. □

At this point we are ready to restate and prove the main result of this chapter.

Theorem 4.3.6. *Let X be a reduced, projective, equidimensional scheme such that one of the following hypotheses hold:*

1. $\mathrm{H}^2(X, \mathcal{O}_X) = 0$,
2. *if X Gorenstein, then either the dualising sheaf ω_X or its dual $\omega_X^\vee$ is ample.*

Then X is formally smoothable if and only if X is geometrically smoothable.

Proof. One implication is proved in Lemma 4.3.5

Suppose we are given a formal smoothing $\mathfrak{p}\colon \mathfrak{X} \to \mathfrak{S}$. Now,

1. if $\mathrm{H}^2(X, \mathcal{O}_X) = 0$, then by [Ser07, Theorem 2.5.13], we get that every formal deformation of X is effective; that is to say that there exists a deformation of schemes $p\colon \mathcal{X} \to S$ such that $\mathfrak{X} \cong \hat{\mathcal{X}}_{/X}$. In particular, from the proof, we also deduce that the morphism p is projective.
2. By proposition 4.1.14 the dualising sheaf ω_X (or $\omega_X^\vee$) extends to an invertible formal sheaf $\mathfrak{L}$ on the formal scheme $\mathfrak{X}$. Theorem 2.0.60 allows us to effectivise the formal deformation, i.e. it gives us a deformation $p\colon \mathcal{X} \to S$ of X such that the completion of $\mathcal{X}$ along the central fibre is $\mathfrak{X}$. Moreover, as bonus point of the aforementioned theorem, we deduce that $\mathcal{X}$ is projective over S.

Concluding, from either hypothesis, if we start with a formal deformation

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathfrak{X} \arrow[d, "\mathfrak{p}"] \\ \operatorname{Spf} k \arrow[r, hook] & \mathfrak{S} \end{tikzcd}$$

then we can construct a deformation of schemes

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "p"] \\ \operatorname{Spec} k \arrow[r, hook] & S \end{tikzcd} \tag{4.3}$$

such that $\mathfrak{X} \cong \hat{\mathcal{X}}_{/X}$. Since $\mathfrak{p}$ is assumed to be a formal smoothing and since $k[\![t]\!]$ is a DVR, we use proposition 3.1.11 to conclude that eq. (4.3) is a smoothing of X. Moreover, in eq. (4.3), the scheme $\mathcal{X}$ is projective over S; i.e. there is a non-negative integer d such that p factors as a closed embedding $\iota\colon \mathcal{X} \hookrightarrow \mathbb{P}^d_S = S \times_k \mathbb{P}^d_k$ followed by the first projection $\mathrm{pr}_1\colon \mathbb{P}^d_S \to S$.

Now we use the fact that the Hilbert functor $\mathfrak{Hilb}_{\mathbb{P}^d}$ is representable to deduce the existence of an isomorphism

$$\alpha_S\colon \mathfrak{Hilb}_{\mathbb{P}^d}(S) \to \mathrm{Hom}_{(\mathsf{Sch})}(S, \mathrm{Hilb}_{\mathbb{P}^d}) := \mathrm{h}_{\mathrm{Hilb}_{\mathbb{P}^d}}(S).$$

Therefore there exists a unique morphism $\psi\colon S \to \mathrm{Hilb}_{\mathbb{P}^d}$ such that both the following diagrams are Cartesian

$$\begin{tikzcd}
\mathcal{X} \arrow[r, hook, "\iota"] \arrow[d, "(\mathrm{Id}\times\psi)|_{\mathcal{X}}"] \arrow[rr, bend left, "p"] & S \times_k \mathbb{P}^d_k \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\psi\times\mathrm{Id}"] & S \arrow[d, "\psi"] \\
\mathrm{Univ}_{\mathbb{P}^d} \arrow[r, hook] \arrow[rr, bend right, "\mathrm{pr}_1"'] & \mathrm{Hilb}_{\mathbb{P}^d} \times_k \mathbb{P}^d_k \arrow[r, "\mathrm{pr}_1"] & \mathrm{Hilb}_{\mathbb{P}^d}
\end{tikzcd}.$$

Recall that $\mathrm{Univ}_{\mathbb{P}^d}$ is by definition a closed subscheme of $\mathbb{P}^d_k \times_k \mathrm{Hilb}_{\mathbb{P}^d}$. Inside the Hilbert scheme we consider the smooth locus, defined as follows

$$\mathrm{H}_{\text{smooth}} := \{[Z] \in \mathrm{Hilb}_{\mathbb{P}^d}(\operatorname{Spec} k)\colon Z \text{ is smooth }\}$$

By [Stacks, Tag 01V5], $\mathrm{H}_{\text{smooth}}$ is an open subset of the Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^d}$.

Now we study the map $\psi\colon S \to \mathrm{Hilb}_{\mathbb{P}^d}$. To do so, we first observe that, since $k[\![t]\!]$ is a DVR, its spectrum S is made of two points: the closed point, q, and the generic point, η. According to our results so far we have that

- $\psi(\eta) = [\mathcal{X}_{\text{gen}}] \in \mathrm{H}_{\text{smooth}}$, since (eq. (4.3)) is a smoothing;
- $\psi(q) = [X] \in \mathrm{Hilb}_{\mathbb{P}^d} \setminus \mathrm{H}_{\text{smooth}}$, since X was singular.

Since S is connected, there exists a polynomial $\Phi \in \mathbb{Q}[m]$ such that the image of ψ is contained in the connected component $\mathrm{Hilb}^{\Phi}_{\mathbb{P}^d}$ of the Hilbert scheme. By

Proposition 4.3.4 there exists a reduced, irreducible component Y of $\mathrm{Hilb}^{\Phi}_{\mathbb{P}^d}$ such that ψ factors through it:

$$\begin{tikzcd} S \arrow[rr, "\psi"] \arrow[dr, "\widetilde{\psi}"'] & & \mathrm{Hilb}^{\Phi}_{\mathbb{P}^d} \\ & Y \arrow[ur, hook, "i"'] & \end{tikzcd}.$$

Observe now that if we let $Y_{\text{smooth}} := Y \cap \mathrm{H}_{\text{smooth}}$ and if we denote by $\overline{Y_{\text{smooth}}}$ the schematic closure of Y_{smooth}, then $\widetilde{\psi}(\eta) \in Y_{\text{smooth}}$ and $\widetilde{\psi}(q) \in \overline{Y_{\text{smooth}}}$. Since Y_{smooth} is a non-empty open, and therefore dense, subset of $\overline{Y_{\text{smooth}}}$ and $\widetilde{\psi}(q) \in \overline{Y_{\text{smooth}}}$, then we can apply lemma 4.3.2 concluding that there exists a curve C inside $\overline{Y_{\text{smooth}}}$ such that $\widetilde{\psi}(q) \in C$ and $C \cap Y_{\text{smooth}} \neq \emptyset$.

Now let $\nu\colon \tilde{C} \to C$ be the normalisation morphism, and $\tilde{p}\colon \widetilde{\mathcal{X}} \to \tilde{C}$ be the pullback under the normalisation morphism ν of the universal family over Y. Since ν is surjective, let $\tilde{c} \in \tilde{C}$ be such that $\nu(\tilde{c}) = \widetilde{\psi}(q)$. This completes the proof since we have that the fibre $\tilde{p}^{-1}(\tilde{c})$ is isomorphic to X and $\widetilde{\mathcal{X}}$ is smooth. □

As an application we have geometric smoothability of lci schemes.

Proposition 4.3.7. *Let X be a singular, projective, l.c.i. variety*[3] *over k satisfying conditions (a), (b) and (c) of Theorem 3.1.13 and such that either its dualising sheaf or its dual is ample. Then X is geometrically smoothable.*

To get the result, consider corollary 3.1.14 and theorem 4.3.6. An application of the above proposition can be found in [FFP21]. There the authors verified the hypotheses of Tziolas' theorem 3.1.13 and then apply our result to show, among other things, that stable semi-smooth complex Godeaux surfaces appear as smooth points in the moduli stack of stable surfaces.

[3] A variety is an integral Noetherian scheme of finite type over a field.

Chapter 5

Finite conditions on formal smoothings

In this chapter we generalise the notion of flat, lci morphism to the case of adic morphisms of formal schemes requiring that both properties can be read at all thickenings. Then we present two new results connected with formal schemes: the first one shows that formal smoothness can be read at a finite number of thickenings. To reach this result we relate the Fitting ideal of the sheaf of formal differentials and the Fitting ideal of the first Schlessinger's relative cotangent formal sheaf; we then state and prove some equivalent conditions on formal smoothigs. The second is in spirit an Artin approximation theorem, see [Ser07, Theorem 2.5.22], involving formal smoothings.

Remember notation 2.0.11: there we denoted $\mathfrak{S} = \operatorname{Spf} k[\![t]\!]$ and, for $n \in \mathbb{N}$, $S_n = \operatorname{Spec} k[\![t]\!]/(t^{n+1}) = \operatorname{Spec} k[t]/(t^{n+1})$.

Definition 5.0.1. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ be a morphism of finite type of LNFSs and consider a collection of compatible thickenings $\{f_n\colon X_n \to S_n\}_{n\geq 0}$. Then $\mathfrak{f}$ is lci and flat of relative dimension d if and only if all f_n are lci and flat of relative dimension d.

Lemma 5.0.2. *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ be a flat lci morphism of relative dimension d among LNFSs with X_0 reduced and, for an open formal affine subset $\mathfrak{U}$ of $\mathfrak{X}$, let us fix a factorization of $\mathfrak{f}$ as $\mathfrak{U} \hookrightarrow \mathbb{A}^r_{\mathfrak{S}} \to \mathfrak{S}$. Then the sequence*

$$\mathcal{C}_{\mathfrak{U}/\mathbb{A}^r_{\mathfrak{S}}} \to \widehat{\Omega}^1_{\mathbb{A}^r_{\mathfrak{S}}/\mathfrak{S}} \otimes_{\mathcal{O}_{\mathbb{A}^r_{\mathfrak{S}}}} \mathcal{O}_{\mathfrak{U}} \to \widehat{\Omega}^1_{\mathfrak{U}/\mathfrak{S}} \to 0$$

is also left exact and $\mathcal{C}_{\mathfrak{U}/\mathbb{A}^r_{\mathfrak{S}}}$ is locally free.

Proof. Since the statement is local, we can assume that $\mathfrak{X} = \operatorname{Spf} B$ with B a $K := (t)B$-adic ring.

To easy the notation, let us denote $R := k[\![t]\!]$, $\tilde{A} := R\{T_1, \ldots, T_r\}$ and, for $n \in \mathbb{N}$, let $R_n := k[\![t]\!]/(t^{n+1})$, $\tilde{A}_n := R_n[T_1, \ldots, T_r]$ and $B_n := \tilde{A}_n/(a_1^n, \ldots, a_c^n)$, where a_j^n denotes the image of a_j in B_n, for every $j = 1, \ldots, c$.

By definition 5.0.1 and [Ser07, Proposition C.12)], for every $n \in \mathbb{N}$ we have short exact sequences

$$0 \to \mathcal{C}_{B_n/\tilde{A}_n} \to \Omega^1_{\tilde{A}_n/R_n} \otimes_{\tilde{A}_n} B_n \to \Omega^1_{B_n/R_n} \to 0.$$

Since for every pair of natural numbers $m \geq n$ we have a surjective morphisms

$$\mathcal{C}_{B_m/\tilde{A}_m} \to \mathcal{C}_{B_n/\tilde{A}_n} \to 0,$$

its inverse system satisfies the Mittag-Leffler condition (see [Stacks, TAG 0594]), and by [Stacks, TAG 0598] we deduce that the sequence

$$0 \to \varinjlim_n \mathcal{C}_{B_n/\tilde{A}_n} \to \varinjlim_n \left(\Omega^1_{\tilde{A}_n/R_n} \otimes_{\tilde{A}_n} B_n\right) \to \varinjlim_n \Omega^1_{B_n/R_n} \to 0,$$

is also right exact, as claimed. □

Notation 5.0.3. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ be a flat lci morphism of relative dimension d of LNFSs with X_0 reduced. Define for every $n, j \in \mathbb{N}$, the j^{th} *Schlessinger's relative cotangent formal sheaf* as

$$\mathcal{T}^j_{\mathfrak{X}/\mathfrak{S}} := \mathscr{E}xt^j_{\mathfrak{X}}(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}, \mathcal{O}_{\mathfrak{X}})$$

and the j^{th} Schlessinger cotangent sheaf relative to the n^{th} thickening

$$\mathcal{T}^j_{X_n/S_n} := \mathscr{E}xt^j_{X_n}(\Omega^1_{X_n/S_n}, \mathcal{O}_{X_n}).$$

Remark 5.0.4. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ be a flat lci morphism of relative dimension d of LNFSs with X_0 reduced. Then for every $n, j \in \mathbb{N}$, $j \geq 2$ we have that

$$\mathcal{T}^j_{\mathfrak{X}/\mathfrak{S}} = 0 \quad \text{and} \quad \mathcal{T}^j_{X_n/S_n} = 0.$$

Indeed, this is a local statement so we can work with a local factorisation as in lemma 5.0.2. Now the result is a consequence of the theorem: from the left exactness of the above sequence and the fact that the conormal formal sheaf is locally free, it follows that we have a two-terms free resolution of $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}$ or $\Omega^1_{X_n/S_n}$ respectively.

Proposition 5.0.5. *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ be a flat lci morphism of relative dimension d of LNFSs with X_0 reduced. Then*

$$\mathrm{Fitt}_d(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}) = \mathrm{Fitt}_0(\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}}).$$

Proof. Since the statement is local, we can assume that $\mathfrak{X} = \operatorname{Spf} B$ with B a $K :=$ $(t)B$-adic ring.

To easy the notation, let us denote $R := k[\![t]\!]$, $\tilde{A} := R\{T_1, \dots, T_r\}$ and, for $n \in \mathbb{N}$, let $R_n := k[\![t]\!]/(t^{n+1})$ and $\tilde{A}_n := R_n[T_1, \dots, T_r]$. By lemma 5.0.2 we have that the following sequence is exact:

$$0 \to \mathcal{C}_{B/\tilde{A}} \xrightarrow{\alpha} \widehat{\Omega}^1_{\tilde{A}/R} \otimes_{\tilde{A}} B \longrightarrow \widehat{\Omega}^1_{B/R} \longrightarrow 0. \tag{5.1}$$

Since each $\mathcal{C}_{B_n/\tilde{A}_n}$ is a free B_n-module of rank c, then $\mathcal{C}_{B/\tilde{A}}$ is a free B-module of rank c and $\widehat{\Omega}^1_{\tilde{A}/R} \otimes_{\tilde{A}} B$ is a free B-module of rank r. Applying now $\operatorname{Hom}_B(-, B)$ to the above exact sequence we get

$$\begin{aligned}0 \to \operatorname{Hom}_B\left(\widehat{\Omega}^1_{B/R}, B\right) \to \operatorname{Hom}_B\left(\widehat{\Omega}^1_{\tilde{A}/R} \otimes_{\tilde{A}} B, B\right) \xrightarrow{\alpha^{\mathrm{t}}} \operatorname{Hom}_B\left(\mathcal{C}_{B/\tilde{A}}, B\right) \to \\ \to \operatorname{Ext}^1_B\left(\widehat{\Omega}^1_{B/R}, B\right) \to \operatorname{Ext}^1_B\left(\widehat{\Omega}^1_{\tilde{A}/R} \otimes_{\tilde{A}} B, B\right) \to \cdots\end{aligned}$$

Now, since $\widehat{\Omega}^1_{\tilde{A}/R} \otimes_{\tilde{A}} B$ is a free B-module, in particular projective, by [Har77, III.6.10.A(a)] it follows that

$$\operatorname{Ext}^1_B\left(\widehat{\Omega}^1_{\tilde{A}/R} \otimes_{\tilde{A}} B, B\right) = 0.$$

Since $\mathcal{C}_{B/\tilde{A}}$ and $\widehat{\Omega}^1_{\tilde{A}/R} \otimes_{\tilde{A}} B$ are free modules and remark 2.0.26 holds, we can use α and α^{t} to compute Fitting ideals of $\widehat{\Omega}^1_{B/R}$ and $\mathrm{T}^1_{B/A} := \operatorname{Ext}^1_B\left(\widehat{\Omega}^1_{B/R}, B\right)$. We therefore got the following chain of equalities:

$$\begin{aligned}\operatorname{Fitt}_0\left(\mathrm{T}^1_{B/A}\right) = \operatorname{Fitt}_0\left(\operatorname{Ext}^1_B\left(\widehat{\Omega}^1_{B/R}, B\right)\right) = \mathrm{I}_{c-0}(\alpha^{\mathrm{t}}) = \mathrm{I}_c(\alpha) = \mathrm{I}_{r-d}(\alpha) = \\ = \operatorname{Fitt}_d\left(\widehat{\Omega}^1_{B/R}\right),\end{aligned}$$

where $\mathrm{I}_l(\psi)$ denotes the determinantal ideal of the linear map ψ, see [Eis95, p. 496] for the notation. $\square$

Proposition 5.0.6. *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ be a flat, lci of relative dimension $d \in \mathbb{N}$ morphism of LNFSs with X_0 reduced and let $\mathfrak{I} := \mathfrak{f}^*(t)\mathcal{O}_{\mathfrak{X}}$ be an ideal of definition. Then the following conditions are equivalent:*

1. *there exists an $k \in \mathbb{N}$ such that $\mathfrak{I}^k \subset \operatorname{Fitt}_d(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}})$;*

2. *there exists an* $k \in \mathbb{N}$ *such that* $t^k \in \mathrm{Fitt}_d(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}})$*;*

3. *there exists a* $k \in \mathbb{N}$ *such that* $t^k \in \mathrm{Fitt}_0(\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}})$*;*

4. *there exists an* $l \in \mathbb{N}$ *such that* $t^l \in \mathrm{Ann}(\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}})$*;*

5. *there exists an* $m \in \mathbb{N}$ *such that* $t^m \cdot \mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}} = 0$*;*

6. *there exists an* $m \in \mathbb{N}$ *such that* $\mathfrak{I}^m \cdot \mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}} = 0$*;*

Proof. (1) $\iff$ (2): clear, since $\mathfrak{I} = \mathfrak{f}^*(t)$.

(2) $\iff$ (3): follows from proposition 5.0.5.

(3)$\Rightarrow$(4): this is a local statement; by [Eis95, Proposition 20.7(a)], we have that $\mathrm{Fitt}_0(\mathrm{T}^1_{\mathfrak{X}/\mathfrak{S}}) \subset \mathrm{Ann}(\mathrm{T}^1_{\mathfrak{X}/\mathfrak{S}})$.

(4)$\Rightarrow$(3): since $\mathfrak{f}$ is lci, it follows that it is of finite type. In particular, for every open affine $\mathfrak{V}$ of $\mathfrak{S}$, we can cover $\mathfrak{f}^{-1}(\mathfrak{V})$ by a finite number of affine subsets, say $\mathfrak{U}_i$. Since $\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}}$ is coherent, for every i, the module $\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}}(\mathfrak{U}_i)$ is generated by u_i elements. By [Eis95, Proposition 20.7(b)], we deduce that

$$\mathrm{Ann}(\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}}(\mathfrak{U}_i)^{u_i}) \subset \mathrm{Fitt}_0(\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}}(\mathfrak{U}_i)).$$

Since the number of i is finite, we can take $u_{\mathfrak{V}}$ to be the maximum. Since $\mathfrak{f}$ is of finite type and $\mathfrak{S}$ is Noetherian, by [EGA1, Corollaire (**1**.10.13.2)] we deduce that $\mathfrak{X}$ is Noetherian too. To get the result we take l to be the maximum among all $u_{\mathfrak{V}}$, as $\mathfrak{V}$ varies in a (finite) formal affine open cover of $\mathfrak{S}$.

(4) $\iff$ (5): clear.

(5) $\iff$ (6): clear, since $\mathfrak{I} = \mathfrak{f}^*(t)$ holds. □

In what follows, we call $\psi_{n,n-1} \colon X_{n-1} \to X_n$ and $\psi_n \colon X_n \to \mathfrak{X}$

Lemma 5.0.7. *Let* k *be a field, let* $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{S}$ *be a flat, lci of relative dimension* $d \in \mathbb{N}$ *morphism of LNFSs with* X_0 *reduced and lci over* $S_0 = \operatorname{Spec} k$ *and let* $\mathfrak{I} := \mathfrak{f}^*(t)\mathcal{O}_{\mathfrak{X}}$ *be an ideal of definition. Then we have a canonical isomorphism*

$$\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}} \cong \varinjlim_n \mathcal{T}^1_{X_n/S_n}.$$

It follows that, for every natural number $n \in \mathbb{N}$*, the natural maps:*

$$(\psi_{n,n-1})^* \, \mathcal{T}^1_{X_n/S_n} \to \mathcal{T}^1_{X_{n-1}/S_{n-1}} \tag{5.2}$$

and

$$(\psi_n)^* \, \mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}} \to \mathcal{T}^1_{X_n/S_n}$$

are all surjective. Furthermore, we have that the above map is an isomorphism

$$(\psi_n)^* \mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}} \cong \mathcal{T}^1_{X_n/S_n}.$$

Proof. We do this in two steps: first we prove that exists a global homomorphism of sheaves and then we show that it is an isomorphism.

For the first part, by adjunction it is enough to prove that there is a natural homomorphism of sheaves

$$\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}} \to (\psi_n)_* \mathcal{T}^1_{X_n/S_n}.$$

Again by adjunction, there is a functorial natural isomorphism:

$$\mathbf{R}(\psi_n)_*(\mathbf{R}\,\mathscr{H}om_{X_n}(\psi_n^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}, \mathcal{O}_{X_n})) \cong \mathbf{R}\,\mathscr{H}om_{\mathfrak{X}}(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}, \mathbf{R}(\psi_n)_*\mathcal{O}_n).$$

Since $\psi_n^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}} = \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}} \otimes_{k[\![t]\!]} k[\![t]\!]/(t^{n+1}) \cong \Omega^1_{X_n/S_n}$ and $(\psi_n)_*$ is an exact functor (ψ_n is a closed embedding), we have an induced natural isomorphism in

$$\mathbf{R}(\psi_n)_*(\mathbf{R}\,\mathscr{H}om_{X_n}(\Omega^1_{X_n/S_n}, \mathcal{O}_{X_n})) \cong \mathbf{R}\,\mathscr{H}om_{\mathfrak{X}}(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}, (\psi_n)_*\mathcal{O}_n).$$

Therefore we have an induced isomorphism on cohomology sheaves

$$\mathrm{h}^1\left(\mathbf{R}(\psi_n)_*(\mathbf{R}\,\mathscr{H}om_{X_n}(\Omega^1_{X_n/S_n}, \mathcal{O}_{X_n}))\right) \cong \mathrm{h}^1\left(\mathbf{R}\,\mathscr{H}om_{\mathfrak{X}}(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}, (\psi_n)_*\mathcal{O}_n)\right).$$

Again, by exactness of $(\psi_n)_*$, we have:

$$\begin{aligned}\mathrm{h}^1\left(\mathbf{R}(\psi_n)_*(\mathbf{R}\,\mathscr{H}om_{X_n}(\Omega^1_{X_n/S_n}, \mathcal{O}_{X_n}))\right) &= (\psi_n)_*\mathrm{h}^1(\mathbf{R}\,\mathscr{H}om_{X_n}(\Omega^1_{X_n/S_n}, \mathcal{O}_{X_n})) \\ &\cong (\psi_n)_* \mathcal{T}^1_{X_n/S_n}.\end{aligned}$$

On the other hand, we also have:

$$\mathrm{h}^1\left(\mathbf{R}\,\mathscr{H}om_{\mathfrak{X}}(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}, (\psi_n)_*\mathcal{O}_n)\right) \cong \mathscr{E}xt^1_{\mathfrak{X}}(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}, (\psi_n)_*\mathcal{O}_n).$$

We then deduce that there is a natural homomorphism of sheaves

$$\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}} = \mathscr{E}xt^1_{\mathfrak{X}}(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}, \mathcal{O}_{\mathfrak{X}}) \to \mathscr{E}xt^1_{\mathfrak{X}}(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}, (\psi_n)_*\mathcal{O}_n) \cong (\psi_n)_* \mathcal{T}^1_{X_n/S_n}$$

induced by the structure morphism $\mathcal{O}_{\mathfrak{X}} \to (\psi_n)_*\mathcal{O}_{X_n}$. The first step is proved.

Since the second step is a local statement, we can assume that $\mathfrak{X} = \operatorname{Spf} B$ with B a $K := (t)B$-adic ring.

To easy the notation, let us denote $R := k[\![t]\!]$, $\tilde{A} := R\{T_1, \dots, T_r\}$ and, for $n \in \mathbb{N}$, let $R_n := k[\![t]\!]/(t^{n+1})$, $\tilde{A}_n := R_n[T_1, \dots, T_r]$ and $B_n := \tilde{A}_n/(a_1^n, \dots, a_c^n)$, where a_j^n denotes the image of a_j in B_n, for every $j = 1, \dots, c$.

From lemma 5.0.2 we have a free presentations of $\mathrm{Ext}^1_B(\widehat{\Omega}^1_{B/R}, B)$ and, for every $n \in \mathbb{N}$, $\mathrm{Ext}^1_{B_n}(\widehat{\Omega}^1_{B_n/R_n}, B_n)$ that fit in the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_B(\widehat{\Omega}^1_{\tilde{A}/R} \otimes_{\tilde{A}} B, B) \arrow[r, "\alpha^{\mathrm{t}}"] \arrow[d, two heads] & \mathrm{Hom}_B(\mathcal{C}_{B/\tilde{A}}, B) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Ext}^1_B(\widehat{\Omega}^1_{B/R}, B) \\
\mathrm{Hom}_{B_n}(\Omega^1_{\tilde{A}_n/R_n} \otimes_{\tilde{A}_n} B_n, B_n) \arrow[r, "\alpha^{\mathrm{t}}_n"] & \mathrm{Hom}_{B_n}(\mathcal{C}_{B_n/\tilde{A}_n}, B_n) \arrow[r, two heads] & \mathrm{Ext}^1_{B_n}(\Omega^1_{B_n/R_n}, B_n)
\end{tikzcd}$$

where the vertical arrows are surjective since we have isomorphisms

$$\mathrm{Hom}_B(\widehat{\Omega}^1_{\tilde{A}/R} \otimes_{\tilde{A}} B, B) \otimes_B B_n \cong \mathrm{Hom}_{B_n}(\widehat{\Omega}^1_{\tilde{A}_n/R_n} \otimes_{\tilde{A}_n} B_n, B_n)$$

and

$$\mathrm{Hom}_B(\mathcal{C}_{B/\tilde{A}} B) \otimes_B B_n \cong \mathrm{Hom}_{B_n}(\mathcal{C}_{B_n/\tilde{A}_n} B_n).$$

From the above isomorphisms it also follots that $\alpha^{\mathrm{t}} \otimes_B B_n \cong \alpha^{\mathrm{t}}_n$, for every $n \in \mathbb{N}$, and by the right exactness of the functor $- \otimes_R R_n$ we conclude that

$$\mathrm{Ext}^1_B(\widehat{\Omega}^1_{B/R}, B) \otimes_B B_n \cong \mathrm{Ext}^1_{B_n}(\Omega^1_{B_n/R_n}, B_n)$$

as desired. $\square$

Proposition 5.0.8. *Let k be a field, let $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{S}$ be a flat, lci of relative dimension $d \in \mathbb{N}$ morphism of LNFSs with X_0 reduced and let $\mathfrak{I} := \mathfrak{f}^*(t) \cdot \mathcal{O}_{\mathfrak{X}}$ be an ideal of definition. Then the following conditions are equivalent*

1. *there exists $r \in \mathbb{N}$ with $t^r \cdot \mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}} = 0$;*
2. *there exists $r \in \mathbb{N}$ such that $\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}} \cong \mathcal{T}^1_{X_r/S_r}$;*
3. *there exists $r \in \mathbb{N}$ such that, for every natural number $m \geq r$, the natural map*
$$\mathcal{T}^1_{X_r/S_r} \to \mathcal{T}^1_{X_m/S_m}$$
derived from eq. (5.2) is an isomorphism.

Proof. (1) $\iff$ (2): for $n \in \mathbb{N}$, let us consider the exact sequence

$$0 \to k[\![t]\!] \xrightarrow{-\cdot t^{n+1}} k[\![t]\!] \to \frac{k[t]}{(t^{n+1})} \to 0$$

and, since $\mathfrak{f}$ is flat, tensoring it by $- \otimes_{k[\![t]\!]} \mathcal{O}_{\mathfrak{X}}$ gives the exact sequence

$$0 \to \mathcal{O}_{\mathfrak{X}} \xrightarrow{-\cdot t^{n+1}} \mathcal{O}_{\mathfrak{X}} \to \frac{k[t]}{(t^{n+1})} \otimes_{k[\![t]\!]} \mathcal{O}_{\mathfrak{X}} \cong \mathcal{O}_{X_n} \to 0.$$

Tensoring it by $- \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}}$, we obtain

$$\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}} \xrightarrow{-\cdot t^{n+1}} \mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}} \to \mathcal{O}_{X_n} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}} \cong \mathcal{T}^1_{X_n/S_n} \to 0,$$

by lemma 5.0.7. Then we have that there exists a $r \in \mathbb{N}$ such that $t^{r+1}\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}} = 0$ if and only if there exists a $r \in \mathbb{N}$ such that the image of the multiplication by $- \cdot t^{r+1} \colon \mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}} \to \mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}}$ is 0, which is in turn equivalent to the existence of $r \in \mathbb{N}$ for which

$$\mathcal{T}^1_{X_r/S_r} \cong \frac{\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}}}{\mathrm{im}(- \cdot t^{r+1})} = \mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}}\,.$$

(2)⇒(3): suppose that for any $n \geq r$ and consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{T}^1_{\mathfrak{X}/\mathfrak{S}} \arrow[r, two heads, "\alpha_n"] \arrow[dr, "\alpha_r"', "\simeq"] & \mathcal{T}^1_{X_n/S_n} \arrow[d, two heads, "\alpha_{n,r}"] \\
 & \mathcal{T}^1_{X_r/S_r}\,.
\end{tikzcd}$$

Since α_n is surjective and α_r is an isomorphism, it follows that $\alpha_{n,r}$ is injective. Since it was already surjective, it is an isomorphism.

(3)⇒(2): thanks to the hypotheses we have that, after a certain index $r \in \mathbb{N}$, the collection of sheaves stabilizes and, by lemma 5.0.7, we conclude. □

Putting together the result so far proved, we deduce the following theorem that tell us that formal smoothness can be verified checking only a finite number of infinitesimal thickenings.

Theorem 5.0.9. *Let $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{S}$ be a flat, lci morphism of relative dimension d of LNFSs with X_0 reduced. Then the following are equivalent:*

1. *there exists a* $r \in \mathbb{N}$ *such that* $\mathfrak{I}^r \subset \operatorname{Fitt}_d\left(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}\right)$*, for any ideal of definition* $\mathfrak{I}$ *of* $\mathfrak{X}$*;*

2. *there exists a* $r \in \mathbb{N}$ *such that the natural morphism*

$$\mathcal{T}^1_{X_r/S_r} \to \mathcal{T}^1_{X_{r-1}/S_{r-1}}$$

   is an isomorphism;

3. *there exists a* $r \in \mathbb{N}$ *such that for every* $m \geq r$*, the natural morphism*

$$\mathcal{T}^1_{X_m/S_m} \to \mathcal{T}^1_{X_r/S_r}$$

   is an isomorphism.

Proof. Use the equivalences of proposition 5.0.6 and proposition 5.0.8. □

Remark 5.0.10. We point out that if X_0 is proper, the hypotheses 1. above means that the morphism $\mathfrak{f}$ is a formal smoothing of X_0 and the above theorem is an equivalent version of [Tzi10, Proposition 11.10] in the case of formal schemes without assuming the algebraicity of the deformation.

The next theorem connects the idea that formal smoothness can be checked at a finite number of thickenings of the formal deformation, with Artin approximation theorem.

We start by presenting a construction needed to formulate the result. Let X be a reduced proper lci scheme over k and suppose that we have a deformation $g\colon \mathcal{Y} \to B$ of X, with B a smooth curve and let $b \in B$ be the closed point such that $g^{-1}(b) = X$. then we can consider the following diagram with Cartesian squares:

$$\begin{tikzcd} \mathcal{Y}_n \arrow[r, hook] \arrow[d, "g_n"'] & \mathcal{Y}_b \arrow[r] \arrow[d, "g_b"] & \mathcal{Y} \arrow[d, "g"] \\ B_n \arrow[r, hook] & \operatorname{Spec} \mathcal{O}_{B,b} \arrow[r] & B \end{tikzcd}$$

where $g_b\colon \mathcal{Y}_b \to \operatorname{Spec} \mathcal{O}_{B,b}$ and, for $n \in \mathbb{N}$, $g_n\colon \mathcal{Y}_n \to B_n$, with $B_n := \operatorname{Spec} \frac{\mathcal{O}_{B,b}}{\mathfrak{m}_b^{n+1}}$, are the pull-backs of g.

Theorem 5.0.11. *Let* X *be a reduced projective lci scheme over* k*, let* $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ *be a flat, lci morphism of relative dimension* d *of LNFSs which is a formal smoothing of* X*, let* r *be the minimum natural number for which all conditions of theorem 5.0.9* *hold and let $g\colon \mathcal{Y} \to B$ be a deformation of X with base B a smooth curve. Suppose that $\mathfrak{f}$ is defined by the compatible collection of thickenings $\{f_n\colon X_n \to S_n\}_{n\in\mathbb{N}}$ and that there exists a non-negative integer n, with $n \geq r$ such that for every $j \in \mathbb{N}$, $j \leq n$, we have isomorphisms of deformations*

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ \mathcal{Y}_j \arrow[rr, "\simeq", "\phi_j"'] \arrow[dr, "g_j"'] & & X_j \arrow[dl, "f_j"] \\ & B_j & \end{tikzcd} .$$

Then also g is a smoothing, i.e. the generic fibre is smooth.

Proof. Observe first that, by remark 2.0.4 applied twice, we have that for any $n \in \mathbb{N}$

$$\frac{\mathcal{O}_{B,b}}{\mathfrak{m}_b^{n+1}} \cong \frac{\widehat{\mathcal{O}}_{B,b}}{\mathfrak{m}_b^{n+1}\,\widehat{\mathcal{O}}_{B,b}} \cong \frac{k[\![t]\!]}{(t^{n+1})} \cong \frac{k[t]}{(t^{n+1})},$$

hence $B_n \cong S_n$. By hypotheses, there exists $n \in \mathbb{N}$ such that for every $j \in \mathbb{N}$ with $j \leq n$, we have isomorphisms of B_j-schemes $\phi_j\colon \mathcal{Y}_j \to X_j$ inducing the identity on X. Then we also have isomorphisms of sheaves $\phi_j^* \mathcal{T}^1_{X_j/B_j} \cong \mathcal{T}^1_{\mathcal{Y}_j/B_j}$; by theorem 5.0.9 we have that

$$\mathcal{T}^1_{\mathcal{Y}_{r+1}/B_{r+1}} \cong \phi_{r+1}^* \mathcal{T}^1_{X_{r+1}/B_{r+1}} \cong \phi_r^* \mathcal{T}^1_{X_r/B_r} \cong \mathcal{T}^1_{\mathcal{Y}_r/B_r},$$

which ends the proof by theorem 4.3.6. □

Chapter 6

Future projects

We now list few ideas that can be considered for future research projects.

1. It is possible to define lci morphism of formal schemes by requiring that the morphism is of finite type, i.e. there exists a local factorisation of the adic morphism by closed embedding in the formal relative affine space followed by the projection, and the closed embedding in such factorisation is a regular closed embedding in the sense that its formal sheaf of ideals is generated by a regular sequence. We then believe that it is possible to prove that being flat, lci morphism can be read at all infinitesimal thickenings. In general, it is not believable that the property of being lci morphism of formal schemes can be read at all infinitesimal thickenings since, already in the case of schemes, the pull back of a lci morphism is not lci in general.

2. A research line I feel is worthy of investigation is the extension of Tziolas's criterion on formal smoothability to the case where X is a DM stack, not a simple scheme. While this is expected to be true (many results in infinitesimal deformations can be extended to DM stacks without any changes), the technical demands are nontrivial. First of all, one must develop an appropriate language to discuss formal stacks. Then if one wants to extend the formal to geometrical smoothability criterion, one has to find a way to connect the deformations of the stacks with those of its coarse moduli space. In particular, this result can be used to provide a common formulation of Tziolas's results on formal smoothability of Gorenstein and $\mathbb{Q}$-Gorenstein varities.

3. Back to Tziolas's impressive work, one can not help but feel that the main obstacle to its more frequent use is the lack of tools for explicitly calculating the sheaves $\mathcal{T}_X$ and $\mathcal{T}^1_X$ for X, say, a Gorenstein variety. A few cases are

known, such as transversal A_1 singularities in [Fan95] and [FM14], normal crossing singularities [Fri83] and semi-smooth singularities [FFP20] in terms of the normalisation and the gluing data.
Therefore I would like to compute the case of transversal A_n singularities, for $n \geq 2$. To maximize usability, they should be formulated in terms of the associated smooth DM stack, then specialised to the case of a global quotient by a finite group. If at all possible, I would then want to describe the transversal D_n and transversal E_n cases, and the relationship with deformations of the minimal resolution of singularities of X.

4. In Chapter 5, the property of being local complete intersection for an adic morphism was introduced. Motivated by the classical case of local complete intersection, I would like to generalise the notion of cotangent complex in the framework of formal schemes. Then it is of interest to check if its properties behave similarly to the case of an lci morphism of schemes. For example, if its definition does not depend on the choice of the factorisation, if it has a natural quasi-isomorphism with the first coherent formal sheaf of differentials, if it is perfect of Tor-amptitude in $[-1, 0]$, or if the "formal cotangent complex" is quasi-isomorphic to the "formal naïve cotangent complex". Also, assuming flatness, we can add to this list: if the "formal cotangent complex" respects the leitmotiv of formal schemes, i.e. the "formal cotangent complex" of a flat lci morphism is, in some sense, the limit of the "infinitesimal cotangent complexes" of the thickenings associated to the formal morphism. With this notion of "formal cotangent complex" developed, one can also try to approach the deformation theory of formal schemes. In particular, one could answer the question in [AJP05, Question 3]; there they relate deformation theory problems of formal schemes, such as the existence of liftings of formal deformations for semi-small extensions, to this "formal cotangent complex" as in the classical case.

5. In Chapter 5 all base schemes were supposed to be the formal power series ring in one variable, $k[\![t]\!]$. What can be said if we substitute $k[\![t]\!]$ with another local complete Noetherian ring?

6. Both in this thesis and in [Tzi10], the singularities considered are always complete intersections. A natural question is then: what can be said if one drops the complete intersection hypotheses?

7. A last idea for future research is to check if it is possible to generalise Tziolas' criterion of formal smoothabilty to the case of schemes that are not in general

projective but have projective smooth locus.

Bibliography

[EGAIII$_1$] A. Grothendieck and J. A. Dieudonné. “Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents, Premièr partie”. In: *Inst. Hautes Études Sci. Publ. Math.* 11 (1961), p. 167. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1961__11__5_0.

[AJP05] L. Alonso Tarrío, A. Jeremías López, and M. Pérez Rodríguez. “Infinitesimal local study of formal schemes”. In: *arXiv preprint math/0504256* (2005).

[Art69] M. Artin. “Algebraization of formal moduli: I”. In: *Global Analisys (papers in honor K. Kodaira)*. University of Tokyo Press, 1969, pp. 21–71.

[Art74] M. Artin. “Algebraic construction of Brieskorn’s resolutions”. In: *Journal of Algebra* 29 (1974), pp. 330–348.

[BW74] D.M. Burns and J. M. Wahl. “Local contributions to global deformations of surfaces”. In: *Inventiones mathematicae* 26.1 (1974), pp. 67–88.

[EGA1] A. Grothendieck and J. A. Dieudonné. “Éléments de géométrie algébrique. I. Le langage des schémas”. In: *Inst. Hautes Études Sci. Publ. Math.* 4 (1960), p. 228. ISSN: 0073-8301. URL: http://www.numdam.org/item/?id=PMIHES_1960__4__5_0.

[Eis95] D. Eisenbud. *Commutative Algebra: With a View Toward Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1995. ISBN: 9780387942698. URL: https://books.google.it/books?id=Fm%5C_yPgZBucMC.

[Fan95] B. Fantechi. “Deformation of Hilbert schemes of points on a surface”. en. In: *Compositio Mathematica* 98.2 (1995), pp. 205–217. URL: http://www.numdam.org/item/CM_1995__98_2_205_0/.

[FFP20] B. Fantechi, M. Franciosi, and R. Pardini. “Deformations of semi-smooth varieties”. In: *arXiv preprint arXiv:2010.02296* (2020).

[FFP21] B. Fantechi, M. Franciosi, and R. Pardini. *Smoothing semi-smooth Stable Godeaux surfaces*. 2021. arXiv: `2105.00786 [math.AG]`.

[FGA-Ill] B. Fantechi et al. "Fundamental algebraic geometry with a letter of Jean-Pierre Serre". In: ed. by Luc Illusie. Vol. 123. Mathematical Surveys and Monographs. Grothendieck's FGA explained. American Mathematical Society, Providence, RI, 2005, pp. 179–233. ISBN: 0-8218-3541-6. DOI: `10.1090/surv/123`. URL: `https://doi.org/10.1090/surv/123`.

[FM14] B. Fantechi and A. Massarenti. "On the rigidity of moduli of curves in arbitrary characteristic". In: *International Mathematics Research Notices* 2017 (2014), pp. 2431–2463.

[Fri83] R. Friedman. "Global Smoothings of Varieties with Normal Crossings". In: *Annals of Mathematics* 118.1 (1983), pp. 75–114. ISSN: 0003486X. URL: `http://www.jstor.org/stable/2006955`.

[Ful84] W. Fulton. *Intersection Theory:* Ergebnisse der Mathematik und ihrer Grenzgebiete : a series of modern surveys in mathematics. Folge 3. Springer-Verlag, 1984. ISBN: 9783540121763. URL: `https://books.google.it/books?id=cmoPAQAAMAAJ`.

[Gro60] A. Grothendieck. "Géométrie formelle et géométrie algébrique". fr. In: *Séminaire Bourbaki : années 1958/59 - 1959/60, exposés 169-204*. Séminaire Bourbaki 5. talk:182. Société mathématique de France, 1960. URL: `http://www.numdam.org/item/SB_1958-1960__5__193_0/`.

[Har06] R. Hartshorne. *Ample subvarieties of algebraic varieties*. Vol. 156. Springer, 2006.

[Har09] R. Hartshorne. *Deformation Theory*. Graduate Texts in Mathematics. Springer New York, 2009. ISBN: 9781441915955. URL: `https://books.google.it/books?id=bwhEX01JlXkC`.

[Har77] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1977. ISBN: 9780387902449. URL: `https://books.google.it/books?id=3rtX9t-nnvwC`.

[HM68] H. Hironaka and H. Matsumura. "Formal functions and formal embeddings". In: *Journal of the Mathematical Society of Japan* 20.1-2 (1968), pp. 52–82.

[Ill72] L. Illusie. "Cotangent complex and deformations of torsors and group schemes". In: *Toposes, Algebraic Geometry and Logic*. Ed. by F. W. Lawvere. Berlin, Heidelberg: Springer Berlin Heidelberg, 1972, pp. 159–189. ISBN: 978-3-540-37609-5.

[Kem93] G. H. Kempf. "Algebraic Varieties". In: London Mathematical Society Lecture Note Series. Cambridge University Press, 1993. Chap. Curves, pp. 85–97. DOI: `10.1017/CBO9781107359956.008`.

[KN94] Y. Kawamata and Y. Namikawa. "Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi-Yau varieties". In: *Inventiones mathematicae* 118 (1994), pp. 395–409.

[KS88] J. Kollár and N. I. Shepherd-Barron. "Threefolds and deformations of surface singularities". In: *Inventiones mathematicae* 91.2 (1988), pp. 299–338.

[LS67] S. Lichtenbaum and M. Schlessinger. "The Cotangent Complex of a Morphism". In: *Transactions of the American Mathematical Society* 128.1 (1967), pp. 41–70. ISSN: 00029947. URL: `http://www.jstor.org/stable/1994516`.

[MFK65] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*. Vol. 34. Springer Science & Business Media, 1965.

[Ser07] E. Sernesi. *Deformations of Algebraic Schemes*. Vol. 334. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2007. ISBN: 9783642067877. URL: `https://books.google.it/books?id=nQYFkgAACAAJ`.

[Stacks] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`. 2018.

[Tzi10] N. Tziolas. "Smoothings of schemes with nonisolated singularities". In: *Michigan Mathematical Journal* 59.1 (2010), pp. 25–84.

Index