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Topological stability obstructions and geometry of algebraic surfaces

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Abstract

In this thesis we study a few topics in the field of complex differential and algebraic geometry. In the first part it suggests a new interpretation of a vector bundle on a families of algebraic varieties (or any structures) as an object in the corresponding moduli spaces. Further, we find a new simple expression for the $(2,2)$-form $c_2(E)$, in terms of $\partial^2 h_{ij}$, for a vector bundle E of an arbitrary rank on the one-dimensional family of Riemann surfaces, and as a consequence show that $c_2(E) > 0$ (Bogomolov & Lukzen, 2022). Thus it gives a new way to prove the Chen-Donaldson-Sun theorem. Next, using the operation of elementary transformation along the curve, we show that $c_2(E)$ can be itself interpreted as a cycle in the moduli space of vector bundles on the arbitrary algebraic surface. In the case of absence of singularities, we show that $c_2(E) > 0$. When the bundle E inherits certain singularities (of the number N), we use the classical Bott-Baum formula and prove $c_2(E) = N^2 - F$. Moving the Lefschetz pencil of curves on a surface and observing how does the cycle , which corresponds to $c_2(E)$, change, we calculate the corresponding monodromy.

The next topic of the thesis is "curves on the algebraic surfaces". Brunebarbe-Klinger-Totaro theorem asserts that X has a nonzero symmetric differential if there is a finite-dimensional representation of $\pi_1(X)$ with infinite image. We give a proof of the bound $h_L(P) \leq Ad(P) + O(1)$ for such X.

The last topic concerns the deformation theory of surfaces of general type. We prove the Severi inequality $c_1(X)^2 < 3c_2(X)$ for a X^2 a complex projective surface with an ample canonical class K which generates $PicX^2 = \mathbb{Z}$.

Keywords— Vector bundle - Stability - Families of curves- Surface of general type-Bogomolov-Miyaoka-Yau inequality-Deformation theory-Families of curves on a surface with symmetric differential

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1 | Introduction

Stability of vector bundles on different geometric spaces has been an object of study for a long time deriving from work in algebraic geometry. The concept of stability arose naturally from the question of whether a space which could parameterize all bundles of a given rank exists, moreover, a subsequent question was if there is a special class of vector bundles which could be naturally parametrized in the same way.

André Weil was a first who has introduced a language of "matrix-valued divisors" which allowed to give a possible description of vector bundles on curves. Other results known at that time were the classification of vector bundles on curves of $g = 0$ by Grothedieck and $g = 1$ by Atiyah (Atiyah, 1957b).
One of the main achievements in the area was the notion of *stable vector bundle*, which originates with Hilbert. Tyurin (A. N. Tyurin, 1965) proved that, for bundles over a curve of fixed rank and fixed degree of their determinant, if the rank and the degree are coprime, the moduli space of these bundles determines the original curve uniquely. A little later, Mumford, Newstead, Ramanan and Seshadri have obtained the similar results independently (Ramanan, 1973; Newstead, 1968, 1967; Seshadri, 1967; Deligne & Mumford, 1969).

The general question was if it was possible to parametrize bundles in such way that the set of bundles formed an algebraic variety. It was shown that such bundles would satisfy the so-called *stability* condition which turned out to be a reasonable direction for a further exploration. As known, one of the most famous results belongs to (Narasimhan & Seshadri, 1965), which states that one can build any stable vector bundle on a Riemann surface using the unitary representation $\pi_1(C) \to U(n)$ of the fundamental group of the surface. Thus a set of stable vector bundles forms an algebraic family. Notable contribution to the notion of stability have been made by Takemoto (Takemoto, 1972), Maruyama (Maruyama, 1977, 1978, 1975) and Mumford (Mumford, Fogarty, & Kirwan, 1994; Mumford, 1977).

It was reasonable to study vector bundles on algebraic surfaces, restricting it to embedded curves and applying already discovered results afterwards. But first one had to understand the structure of the curves lying on algebraic surfaces.
In his work "Families of curves on a surface of general type" (Bogomolov, 1977) Fedor Bogomolov has proved that on any surface of general type with $c_1^2 > c_2$ there are only a finite number of curves of a given genus.
In work (Bogomolov, 1979) it was shown that the notion of T-stability,-polystability is equivalent to Takemoto's stability. T-stability allows one to use symmetric tensors on vector bundles and the theory of algebraic reductive groups (see [2] for more details) to make a conclusion if a particular bundle is stable or not.
Another important theorem obtained by F. Bogomolov is the effective stability restricton theorem of vector bundles: any stable vector bundle $E \to X$ on a smooth projective surface X restricts to a stable bundle on any ample curve belonging to sufficiently big class. Let Num denote numerical equivalence and K_+ a positive cone (see (Bogomolov, 1994; Huybrechts & Lehn, 2010)). Then

Theorem 1. *(Bogomolov's restriction theorem) Let F be a locally free sheaf of rank $r \geq 2$. Assume that F is μ-stable with respect to an ample class $H \in K^+ \cap Num$. Let $C \subset X$ be a smooth curve with $[C] = nH$. If $2n \geq \frac{R}{r}\Delta(F) + 1$, then $F|_C$ is a stable sheaf.*

As it follows from the theorem, the restriction of a bundle is stable on the singular fibers with relatively small number of singularities and on normalizations of singular curves. This result is the strongest and we show that Mehta-Ramanathan's theorem [4] follows from it (see [30] for the detailed proof). After Yau's proof (Yau, 1978) of Calabi (Calabi, 2015) conjecture it was proved that Hermitian-Yang-Mills connections are related to an algebraic property of the holomorphic vector bundle of being stable which is known as Donaldson-Uhlenbeck-Yau theorem, see (Uhlenbeck & Yau, 1986; S. K. Donaldson, 1985). Later on the significant contributions have been made by Hitchin (Hitchin, 1987), Corlette (Corlette, 1988), Simpson (Simpson, 1992) and others.

We should note that nowadays there exist many different notions of stability. We give an overview in [5]. We also provide a reader with a comprehensive Algebro-Analytic dictionary [7].

There are several results concerning the existence of special metrics on stable bundles over complex projective manifolds. One of the recent improvements in this question is Chen-Donaldson-Sun theorem (X. Chen, Donaldson, & Sun, 2015a, 2015b, 2015c) which claims the existence of flat unitary metric on a stable bundle with Chern classes c_1, c_2 equal zero. It was obtained by considering a special Futaki-type functional on the space of Hermitian metrics and proving the existence of a metric with a minimal value of the functional. The resulting metric happens to be flat in case of $c_2 \geq 0$ which implied the result, see (X. Chen et al., 2015a, 2015b, 2015c) for more details.

Recall that all Kähler metrics in class $[\omega]$ can be parametrized by smooth functions in the space of Kähler potentials $\mathcal{H}_\omega = \{\varphi \in C^\infty(M) : \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}$. Then, for $\varphi \in \mathcal{H}_\omega$, the scalar curvature is $R_\varphi = -g_\varphi^{i\bar{j}}\partial_i\partial_{\bar{j}} log(det(g_{i\bar{j}} + \varphi_{i\bar{j}}))$ and the constant scalar curvature equation reads as $R_\varphi = R'$ (S. K. Donaldson, 2005), where R' is a constant depending only on manifold, class and a complex structure $(M, [\omega], J)$.

Conjecture 2. *(Yau-Tian-Donaldson conjecture) YTD conjecture is that (X,L) ad-*

mits a CSCK metric if and only if it is K-stable.

YTD conjecture is proven for Fanos and Tropical varieties.

1.1 Summary of results

In our thesis, we show how one-dimensional cycles in the moduli space of vector bundles on curves are associated to bundles on families of curves which are stable on the restriction to every curve of the family. We present a connection between existence of a special metric on a class of surfaces formed by the families of curves and its' natural relation to one-dimensional cycles in the moduli space of stable bundles on curves.

Suppose that B is a family of curves, $B = \{C_\lambda\}_{\lambda\in\Lambda}$, where Λ is a parameter set. Assume that vector bundle E on B has a stable restriction on any curve C_λ. If, in addition, the first Chern class of the restriction E_λ of E on any curve C_λ is trivial, then by (Narasimhan & Seshadri, 1965) there is a flat hermitian metric $h_{i,j}$ on E_λ which is unique modulo multiplication by a constant. Thus, it defines the unique flat hermitian connection on E_λ and the unique modulo conjugation irreducible unitary representation $\rho_\lambda : \pi_1(C_\lambda) \to U(n)$.

The choice of the flat hermitian metrics $h_{i,j}$ on E_λ for any $\lambda \in B$ defines hermitian metric h on E over B. The metric h is not unique, but depends on a real-valued invertible function of Λ corresponding to the choice of actual hermitian metric on E_λ. Hence we obtain a family of such metrics on E and corresponding Hermitian connections Θ_h. The latter defines Chern forms representing all Chern classes of E. We are particularly interested in the local properties of $(2, 2)$-form on B representing the second Chern class $c_2(E)$ obtained from Θ_h when B is a two-dimensional complex surface. We can express the second Chern class through the curvature of the cycle $\mathscr{S}$ in the moduli space of vector bundles, which corresponds to a particular vector bundle E on the base B.

$$\boxed{c_2(E) = \frac{tr(\Theta(\mathscr{S})^2) - tr^2(\Theta(\mathscr{S}))}{8\pi^2}}$$

The following result can be obtained by a direct calculation [8.11.2]. The second Chern class is represented locally by the formula (53)

$$\boxed{c_2(E) = \sum_{i,j} -\frac{det(D_{ij})det(\bar{D}_{ij})}{det(h_{ij})} dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2}$$

Corollary 1. *The form which represents a second chern class $c_2(E)$ on B is non-negative since $\frac{det(D_{ij})det(\bar{D}_{ij})}{det(h_{ij})} \geq 0$.*

Corollary 2. *Assume that $(-det\partial^2 h_{i,j})$ is nonzero somewhere on B then $c_2(E)$ is semipositive. In particular, if the base Λ is a compact curve and the representation ρ_λ varies modulo conjugation on Λ then $c_2(E)$ has a representative which is a semipositive $(2,2)$- form*

This fact yields a possibility for a constructive approach to the proof of Chen-Donaldson-Sun theorem for a projective surface X. The theorem claims that $c_2(E) > 0$ for a stable vector bundle with $c_1(E) = 0$ unless it is induced from a unitary representation of $\pi_1(B)$. Our explicit local calculation implies this result for a compact surface $f : B \to C$ fibered with all smooth fibres $f^{-1}(c), c \in C$ over a smooth curve and a bundle E which has stable restrictions on the fibers $f^{-1}(c)$.

Corollary 3. *We should note that the theorem [1.1] holds not only for stable bundles $E \to B$ on our class of surfaces. In particular, if the bundle E is not stable, but restricts to a stable one to a curve $C \subset B$, this result [1.1] holds as* *well.*

We also discuss other possible applications and generalizations of the above approach.

One of the key ingridients of the proof of the formula (53) is a special geometric cycle which corresponds to every vector bundle on the families of curves. We construct a method of its' interpretation in the algebraic way via unitary representations and demonstrate how to analyze the notion of stability of a vector bundle on the complex projective algebraic surface. The scheme of the proof is the following:

FIND A DESCRIPTION OF VECTOR BUNDLES ON THE FAMILIES OF CURVES. In [8.1] we introduce an object, which corresponds to a certain class of vector bundles on the families of curves (or any varieties/structures).

Theorem 3 (40)**.** *Every vector bundle E on B, which is stable on the restriction to every non-singular curve and a curve with relatively small number of singularities $E|_{C_\lambda}$ (from our family $B = \{C_\lambda\}$) is a smooth moduli section $\mathscr{S} \in \Gamma(M_\lambda)_{\lambda\in\Lambda}$ of the family of Moduli spaces M_{C_λ}, $\lambda \in \Lambda$.*

Corollary 4. *If $C_\lambda = C$ is a constant curve, then all moduli spaces are the same $M_{C_\lambda} = M_C = M$, thus a vector bundle on B is decoded by a moduli section $\mathscr{S} \subset M$ in the moduli space M:*

$$\mathscr{S} = \{m_\lambda \in M, \lambda \in \Lambda\}$$

Corollary 5. *Moduli space M_B of all vector bundles on B, whose restriction to any curve of the family is stable, consist of all moduli sections $\mathscr{S}$ of family of the* *corresponding moduli spaces M_{C_λ}.*

$$M_B = \{\mathscr{S} | \mathscr{S} \in \Gamma((M_{C_\lambda})_{\lambda\in\Lambda})\}$$

Corollary 6. *If $C_\lambda = C_1$ and $\Lambda = C_2$, thus we have a vector bundle on the $C_1 \times C_2$. Then every subset $\mathscr{S} \subset M$ will form a cycle in the moduli space of vector bundles $\mathscr{S} = \mathscr{C}_2 \subset M_{C_1}$ or $\mathscr{C}_1 \subset M_{C_2}$.*

Actually more general theorem holds true:

Theorem 4. *Every vector bundle $\mathcal{V}$ on the family of algebraic varieties $\{X_\lambda\}_{\lambda\in\Lambda}$, which is stable on the restriction to every $|_{X_\lambda}$ is a smooth moduli section $\mathscr{S}$ of the family of Moduli spaces $\{M_{X_\lambda}\}$, $\lambda \in \Lambda$.*

Or

Theorem 5. ***(General principle)*** *Let $\{Q_\lambda\}$ be a family of sets, which parametrizes objects on family of spaces $\mathcal{B} = X_\lambda$. Then a set, which parametrizes objects on $\mathcal{B}$ is $\Gamma(\{Q_\lambda\}_{\lambda\in\Lambda})$.*

Find an interpretation of $c_2(\mathcal{E})$ on the families of curves

The next goal is to find a suitable interpretation of the second chern class as a particular *moduli section* of families of moduli spaces $\{M_{C_\lambda}\}$. We perform an operation of elementary transformation on a trivial bundle E [8.2]. We have to choose a curve $C' \subset B$ in a way which allows to interpret $c_2(E)$ as a moduli section [8.2.1]. Afterwards we obtain a formula for the second chern class of the bundle W which is the result of an elementary transformation procedure [42, 8.7]:

$$\boxed{c_2(W) = c_1(F_C) - \frac{r(k-r)}{2k}C^2} \tag{1.1}$$

where F_C is a stable bundle on a curve $C \subset B$.

Then the following theorem holds true:

Theorem 6. *The second chern class $c_2(W)$ is represented by a moduli section in M_{C_λ} for some $\lambda \in \Lambda$, depending on the choice of a curve C; since $\Delta(W) = -c_2(W)$, if a bundle W is semistable, then*

$$c_2(W) > 0.$$

CRITERIA FOR STABILITY. THE EXPRESSION FOR $\Delta(E)$.

The next goal is to find a criteria for stability for a vector bundle $\mathcal{E}$ over smooth projective surface X [8.3]. The idea is to expose all possible embedded curves to a surface passing through a fixed point $x \in X$ and to reveal the restrictions of vector bundles to these curves. The major problem which arises in the course of procedure of restriction of vector bundle is a presence of the singular points X^{sing} in X and hence of the corresponding singular fibers of vector bundles $\mathcal{E}^{sing}$. To resolve it, we have to suppose that we are in conditions of Bogomolov's effective restriction theorem.

Theorem 7. *(Bogomolov's effective restriction) Let W be a locally free sheaf of rank $r \geqslant 2$ on a family of curves with $c_1(W) = 0$. Assume that W is μ-stable with respect to an ample class $H \in K^+ \cap Num$ and $C \subset X$ be a smooth curve with $[C] = nH$. Let $2n \geqslant \frac{R}{r}\Delta(F) + 1$. Then $F|_C$ is a stable sheaf.*

Let X be a smooth projective surface. Consider very ample line bundle H on a surface X^2 and the space $H^0(X^2, H)$. Then $\mathbb{P}(H^0(X^2, H)) = \mathbb{P}_H$ is a projective space parametrizing curves in class H. Assume that X satisfies a stability inequality of the Bogomolov's effective restriction theorem so there is an open subset U in the class $P(H^0(X^2, H)) = \mathbb{P}_H$ such that we obtain a section s_E over

U. Notice that the codimension of $P(H) - U_s \subset P(H)$ grows if the class H becomes larger and therefore for a generic pencil $\mathbb{P}^1_t \subset P(H)$ of such curves we can define a one-dimensional algebraic class which is given by the intersection of a section $s(E)$ with a *moduli section* $\mathscr{S}_{\mathbb{P}^1_s}$ which corresponds to a vector bundle over a pencil $\mathbb{P}^1_s$.

Theorem 8. *(44) Intersection of a section $s(E)$ and a moduli section $\mathscr{S}_{P^1_s}$ is given by one-dimesional algebraic cycle [8.16],*

$$\boxed{\varkappa = s(E) \cap \mathscr{S}_{P^1_s}} \tag{1.2}$$

and

$$\boxed{\varkappa = (\mathscr{L}.\Delta_P)} \tag{1.3}$$

where $\mathscr{L}$ is a line bundle which corresponds to $s(E)$ on the families of moduli spaces of vector bundles and Δ_P coincides with $\mathscr{S}_{\mathbb{P}^1_s}$ in this model. It holds if a moduli section $\mathscr{S}_{P^1_s}$ has a natural polarizatoion $\mathscr{L}'$. In particular, since $c_2(\mathcal{E})$ is represented by some moduli section $\mathscr{S}_{c_2(\mathcal{E})}$, then locally on the open subset U the intersection is equal to our one-dimesional cycle:

$$\boxed{(2rc_2(\mathcal{E}) \cap [B])|_U = \varkappa}$$

for a polarization $\mathscr{L}$ on the families of moduli spaces of vector bundles.

By (Moriwaki, 1995) for a map $f : B \to X$, where B is a union of all curves passing through all points in X,

$$dis_{X/Y}(E)|_U = f_*(\varkappa|_U)$$

and in (46) we obtain

$$dis_{X/Y}(E)|_U > 0$$

Corollary 7. *If $\varkappa = 0$, then $dis_{X/Y}(E) = 0$ and therefore we obtain a flat bundle E on a surface X. So $\varkappa = 0$ sections correspond to a bundles which are given by the same representation - invariant under the monodromy of the family of curves.*

Corollary 8. *If we work over a field k, then $\varkappa = h_L(P)$. Note that $Gal(\bar{k}/k)$ acts as a monodromy operator on $h_L(P)$. As the geometric height is invariant under the action of Galois group, we have*

$$\sigma(\varkappa) = h_L(\sigma(P)) = h_L(P) = \varkappa, \sigma \in Gal(\bar{k}/k).$$

NUMBER OF SINGULAR POINTS AND REPRESENTATIONS In 8.6 we give an expression of the number of singular points of X.

Lemma 9. *(50) Let N_P be the total number of singular points X^{Sing}. Then it is given by the Euler charactrestics formula for a surface X blown up at the H^2 intersection points of curves parametrized by $\mathbb{P}^1_{C_t}$.*

$$\boxed{N_p = \chi(X) + H^2 + 2H(K + H)}$$

Lemma 10 (51)**.** *Assume that we have a family of representations $\rho_{E,t} : \pi_1(C_t) \to U(n)$ over C_t defined by a stable bundle E. If this representation extends to a representation of at least one singular curve $C_s, s \in X^{sing}$ then it extends to X and hence the bundle E is defined by the representation $\rho_E : \pi_1(X) \to U(n)$.*

MONODROMY

The action of the monodromy m on the unitary representation can be described

[8.17] group-theoretically as:

$$\boxed{m_\eta^\gamma : A_1 \to A_1, A_1 \to A_1A_2, A_i \to A_1A_iA_1{}^{-1}}$$

For each unitary representation of a fundamental group of a given surface X_t we have a normal subgroup N_t generated by g_1^t which is a kernel of projection of $\pi_1(X_t)$ to $\pi_1(X_0)$. If we consider all unitary representations of groups N_t denoted by N_t^U then the monodromy operator m_η^γ acts on each of the corresponding groups N_t^U. For a loop Γ of nonsingular curves X_t we can take $\Pi_{\gamma\in\Gamma}N_\gamma^U$ and the corresponding action of m on it.

Galois group acting on singular points

Assume [8.6.2] that a surface X is defined over a number field $[K : \mathbb{Q}] < \infty$ as well as the pencil $f : X \to \mathbb{P}^1$ and the bundle E. The union of singular fibers in this case is also defined over K. Monodromy group T of the fibration $X^b \to \mathbb{P}^1$ permutes vanishing cycles c_s of different fibers transitively in the fundamental group of the complementary of X to all singular fibers $\pi_1(X - X^{sing})$.
The action of the Galois group $Gal(\bar{K}/K)$ approximates the action of monodromy T on the vanishing cycles $\{c_s\}$.
In particular, if there is only one orbit of the $Gal(\bar{K}/K)$ on the vanishing cycles in a completion of $\hat{\pi}_1(X_t)$ then either the family of representations is obtained from the representation $\pi_1(X)$ or any fiber X_t with a singular point has the latter as singular point of the family of partial connection.

Stability of vector bundles with singularities

In [9.1] we find an expression for $c_2(E)$ for a bundle which has some number of singular points.

Theorem 11. *(55) For a vector bundle $E \to X$, where X^{sing} is the set of singular* *points of X, as $c_1(E) = 0$, the second chern class is given by (9.1)*

$$\boxed{c_2(E) = c_2(\mathbb{P}(T_E)) = \Delta = \sum_p Res_p \begin{pmatrix} \phi(A)dz_k..dz_n \\ a_k, ..., a_n \end{pmatrix} - F^2 = N - F^2}$$

Since $c_1(F) > 0$, it implies that $\boxed{c_2(E) = N - F^2}$.

Corollary 9. *If the number of points N grows sufficiently fast and F is small we get that $c_2 > 0$, i.e. $N > F^2$.*

Bounds for a curves on the surfaces X of general type

It is known by [10.1,61] and (Bogomolov, 1977) that if the canonical class of X is very ample, for except a finite number of curves, there is a linear estimate from below for a genus of the curve inside each cone $K^C_{D,\varepsilon}$ (see 10.1). By the result of [62] (see also (Brunebarbe, Klingler, & Totaro, 2013)) there are nontrivial symmetric tensors on a surface X if it has a finite-dimensional representation with an infinite image. We prove that

Theorem 12. *(10.2) Let X be a compact Kähler manifold. Suppose that there is a finite-dimensional representation of $\pi_1(X)$ over some field with infinite image. Then*

$$h_L(P) \leq A \cdot d(P) + O(1)$$

Deformations of the surfaces of general type

Denote by $T(M^n) = T$ the tangent bundle to M^n, a projective manifold. Then, following [11.1.2] and (Bogomolov, 1978), define $\mathcal{F}^k \subset \mathcal{O}(T)$ be a k-dimensional coherent subsheaf of $\mathcal{O}(T)$. An embedding $r : \mathcal{F}^k \hookrightarrow \mathcal{O}(T) \rightsquigarrow$ a map of one-dimensional bundles $r_{(k)} : det\mathcal{F}^k \to \bigwedge^k T$. If one uses the isomorphism $\bigwedge^{n-1} \Omega^1 \otimes -K \simeq T \rightsquigarrow$ a map $r^{\odot}_{(k)} : det\mathcal{F}^k \otimes -kK \to \Omega^{n-k}$. This way, one can study subbundles in $T(M^n)$ just considering one-dimensional subbundles in

$\Omega^i(M^n)$. The following condition holds:

Theorem 13. *(Bogomolov, 1978) Let M^n be a projective variety, $\Omega^i = \Omega^i(M^n)$, the i^{th} wedge power of the cotangent bundle $\Omega^1 = \Omega^1(M^n)$, $L \to M^n$ be the one-dimensional bundle on M^n and $h : L \to \Omega^i$- a non-trivial homomorphism. Then $\exists$ constants c_M^L, β such that*

$$dimH^0(M^n, sL) < c_M^L s^i + \beta, \forall s > 0$$

Another useful theorem which will be used is

Theorem 14. *(70) Let $E \to X$ be a vector bundle of $dimE = n$ and suppose that $c_2 - \frac{n-1}{2n}{c_1}^2 > 0 \Rightarrow \exists$ a subbundle $F \subset E, dimF = k$ and a homomorphism $h : F \to E :$*

1. *$h : \mathcal{O}(F) \to \mathcal{O}(E)$ is a monomorphism;*
2. *$(c_1(F) - \frac{k}{n}c_1(E))^2 > -c_2(E_0), E_0 = E \otimes \frac{detE}{n}$;*
3. *some multiplicity of the bundle $(ndetF - kdetE)$ has a section; i.e. there exists $\exists s \in H^0(X, l(ndetF - kdetE))$ for some l*

In[11.1.2, 11.2] we prove our main result:

Theorem 15. *(72) Let X^2 be a complex projective surface with an ample canonical class K which generates $PicX^2 = Z$ and assume that $H^1(X,T) \neq 0$ where T is the tangent bundle and $H^1(X,T) \neq 0$. Then we have Severi inequality $c_1^2(X) < 3c_2(X)$.*

1.2 Future directions

1. One of our future projects (joint with Fedor Bogomolov) concerns the questions about the metric expressions on the vector bundles over surfaces. Let M^n be a compact smooth projective manifold with trivial canonical class. Then there is a Calabi-Yau Kähler metric with a corresponding $(1, 1)$-form in each Kähler class - it is a well known Yau theorem. It is essentially existence theorem and no good formula for Kähler metrics exist even for $K3$ surfaces.
We plan to find and write down an explicit formula for the Calabi-Yau metric on a vector bundle $E \to X$ over the complex surface X.
2. A big variety of a more concrete applications of the results obtained in the thesis (some particular cases of algebraic surfaces, vector bundles, etc.)

1.3 Thesis Outline

The thesis is organised as follows:

Chapter 2 — Introduces the concept of vector bundle and discusses its' properties and interpretations;

Chapter 3 — Studies the holomorphic tensors on vector bundles and the notions of T- and H-stability;

Chapter 4 — Gives an overview on Atiyah classes;

Chapter 5 — Provides a useful dictionary of terms in the theory of vector bundles and algebraic curves from the algebraic and analytic perspectives;

Chapter 6 — Discusses results on the expression of $c_2(E)$, stability of vector bundles without singular fibers and monodromy along singular fibers from both algebraic and differential-geometric points of view;

Chapters 7 — Suggests an expression for $c_2(E)$ when the bundle E has certain singular fibers;

Chapter 9 — Introduces a bound for a curves on a surface with symmetric tensors;

Chapter 10 — proves the Severi inequality $c_1^2(X) < 3c_2(X)$ for a X^2 be a complex projective surface with an ample canonical class K which generates $PicX^2 = \mathbb{Z}$.

Chapter 10 — Offers a possible description of vector bundles on particular families of curves.

2 | Preliminaries: Vector bundles and its' interpretations

We first start with the concept of Vector bundles. There are different styles to work with these objects, and here we are collecting some preliminaries of different points of view. The main references therein are (A. N. Tyurin, 1992; A. Tyurin, 1997, 2008). The Basic theory is provided by (Shafarevich, 1988a, 1988b; Iskovskikh & Shafarevich, 1989; Shokurov, 1998; Shokurov, 1988).

One can associate to each vector bundle $\pi : E \to X$ of dimension n on a manifold X an element (called cocycle) $\rightsquigarrow h \in H^1(X, \mathbf{GL(n)})$ in the first cohomology group with values in the sheaf of germs of mappings of $f : X \to GL(n)$ to a full linear group $GL(n)$.

For a $h \in H^1(X, \mathbf{GL(n)}) \rightsquigarrow$ an affine covering $X \subset \{U_i\}_{i \in I}$. A set $\{h_{ij}\}$ of matrix functions such that h_{ij} is regular and regularly invertible on $U_i \cap U_j$ and $h_{ij}h_{jk}h_{ki} = 1$ on triple intersections $U_{ijk} := U_i \cap U_j \cap U_k$ allows us to glue a vector bundle.

More concretely, if V_0 is an n-dimensional vector space, then the matrix functions h_{ij} can help to glue the sets $V_0 \times U_i$ in such a way that $\pi^{-1}(U_i) = V_0 \times U_i$. The triple (V, π, X) is called a *vector bundle*. In algebraic geometry vector bundle has four different interpretations.

2.0.1 The sheaf interpretation of a vector bundle

Vector bundle $E \to X$can be seen as a locally free sheaf on X.

$\Rightarrow$: Let us associate to $E \rightsquigarrow \mathbf{\Gamma}(\mathbf{E})$- its' sheaf of germs of sections.

$\Leftarrow$: Each locally free sheaf is in fact a sheaf of germs $\mathbf{\Gamma}(\mathbf{E})$ of a unique bundle E.

The rank of the locally free sheaf is the dimension of the bundle.

2.0.2 The geometric interpretation

A vector bundle $E \to X$, $\pi^{-1}(x) = V_0$ as a variety itself is neither affine nor complete. To turn it to a complete one, we have to projectivize it. Let $\mathbb{P}(V_0)$ be a projectivization of a fiber V_0, then, just gluing the spaces $\mathbb{P}(V_0) \times U_i$ all together with the same matrices $\{h_{ij}\}_{i,j\in I}$ we will get a projective variety $\pi : \mathbb{P}(E) \to X$ with the same projection map π. It is now a complete variety, a geometrical object. If we tensor E by any line bundle $E \otimes L$ the projectivizations will coincide $\mathbb{P}(E) = \mathbb{P}(E \otimes L)$.

The bundle $\pi^*(E) \to \mathbb{P}(E)$ has a one-dimensional subbundle $L \subset \pi^*(E)$, which is called *tautological* and has a property, that its' fiber $L_p = \ell_p$ defines the same one-dimensional subspace which defines the point of the projectivization $\mathbb{P}(E)$.

A bundle $\tau = L^*$ is called *antitautological* on $\mathbb{P}(E)$. The pair $(\mathbb{P}(E), \tau)$ uniquely determines the bundle V on X (in fact, constructively, by $E = (R^0\pi(\tau))^*$).

2.0.3 The arithmetic interpretation

A vector bundle E as a class of matrix divisors. If X is a curve, then a matrix bundle on X is the following data: for a $x \in X \rightsquigarrow M_x$, where M_x is a functional matrix, in such a way, that there are only finitely many points $x \in X$ at which M_x is not regular and regularly invertible at x.

Two assignments $M \sim M'$ are called equivalent if the expression ${M_x}^{-1}M'_x$ is regular and regularly invertible at x, $\forall x \in X$. A class of matrix assignments $\mathbf{M}_x$ is called a *matrix divisor*.

The matrix divisors $\mathbf{M}_x \sim \mathbf{M'}_x$ are equivalent if $M_x\dot{M'}_x^{-1} = G$ is a matrix of rational functions on X not depending on the point $x \in X$. The concept of a matrix divisor is analogous to that of a divisor, and the connection between matrix divisors and vector bundles is the same as that between divisors and one-dimensional (linear) bundles.

2.0.4 The analytic interpretation.

There is a class of bundles on X which is reasonable to distinguish, the so-called *flat bundles*, which one can get from the representation $\rho : \pi_1(X) \to GL(n)$.

Holonomy of a connection and representation of a fundamental group $\pi_1(X)$
In fact, we can say that an element of a fundamental group corresponds to a certain loop, and the representation of a certain loop to a linear group $GL(n)$ obviously gives an action of a holonomy on a vector bundle: while we are moving around the loop on the manifold X, at the same time we are moving along the fibers of a vector bundle E, starting from the initial vector v_0, using the certain connection ∇_E. Thus when we return to the same fiber,(i.e. the same point of the beginning of the loop x_0 on X), the vector changes to another one v'_o. Thus it defines the action of the holonomy connection ∇_E. It is clear, that the vector will move to some $v'_0 = A \cdot v_o, A \in GL(n)$. Thus, this loop will give us an element of a group $GL(n)$ through the action of the holonomy of connection. Hence, it is clear why every representation $\rho : \pi_1(X) \to GL(n)$ is equivalent to set an action of a certain connection and thus defines a bundle.

The precise construction is the following: let $\rho : \pi_1(X) \to GL(n)$ be a representation of the fundamental group, $\widetilde{X}$ the universal covering manifold of X, and V an n-dimensional vector space. Then $\pi_1(X)$ acts diagonally on $\widetilde{X} \times V$, that is, $g(u, v) = (g(u), \rho g(v))$ and $\widetilde{X} \times V/\pi_1(X) = E$ is a bundle on X. The condition for a bundle to be flat is purely algebraic and for curves is very simple.

We can illustrate those four interpretations of a vector bundle in the one-dimensional case. The equivalence of a divisor class $Div(X)$ with a locally free sheaf of rank one $\mathcal{F} \to X$. Recall that the equivalence of a divisor class with a locally free sheaf of rank 1 provides the divisors with higher cohomology; in fact, cohomology first entered into arithmetic through this equivalence. it is known that one-dimensional bundle can be obtained from a representation of the fundamental group $\Leftrightarrow$ its Chern class is 0. Hence we get the analytic construction of the Picard variety: $PicX = \{$set of unitary characters of $\pi_1(X)\}$.

2.1 Properties of vector bundles on curves

Let X be an algebraic curve. For a line bundle $L \to X$ one can associate a degree of a corresponding divisor D_L, which is called $degL$. For a higher-dimensional bundle $E \to X$, the degree is defined as $degE = degdetE$.

This invariant allows us to split vector bundles in classes and define the notion of stability of vector bundles, i.e. to distiguish the components of the highest dimension.

Definition 1. *A bundle $E \to X$ on a curve X is called stable, if for any proper subbundle $V \subset E$, $\frac{degV}{dimV} < \frac{degE}{dimE}$*

Note that $EndV$ is the direct sum of the trivial bundle I and adV. Consequently, for sections we have: $H^0(X, EndV) = I \oplus H^0(X, adV)$. The sections of the sheaf adV are those endomorphisms of V for which the image is a proper subbundle in V; such endomorphisms are also called non-trivial. Notice that we can represent every vector bundle as an extension for $e \in H^0(X, adV)$,

$$e \leftrightsquigarrow 0 \to kere \to E \to Ime \to 0,$$

here $degE = deg(kere)+deg(Ime)$and $dimE = dim(ker(e))+dim(Im(e))$.Then the fraction $\frac{degE}{dimE}$ lies between $\frac{degkere}{dimkere}$ and $\frac{degIme}{dimIme}$ and is less than one of them, violating the inequality above. Hence stable bundles have no non-trivial endomorphisms.

2.1.1 Parabolic bundles on the curves with nodes

Definition 2. *Let X be a compact Riemann surface with a finite set of points $P_1, .., P_n$ and W a vector bundle on X. By (Bhosle, 1996), a parabolic structure on W is giving at each P_i,*
a) a flag $W_p = F_1W_p \supset F_2W_p \supset ... \supset F_rW_p$,
b)The system of weights $\alpha_1, .., \alpha_r$, attached to $F_1W_p, .., F_rW_P$ such that $0 \leq \alpha_1 < \alpha_2 < .. < \alpha_r < 1$
We call $k_1 = dimF_1W_p - dimF_2W_P, k_r = dimF_rW_P$ the multiplicities of $\alpha_1, .., \alpha_r$.

2.1.2 Serre duality for vector bundles

Following [wikipedia] and (Serre, 1955, 1956), let X be a smooth variety of dimension n over a field k. Define the canonical line bundle K_X to be the bundle of on X, the top exterior power of the cotangent bundle: :$K_X = \Omega^n_X = \bigwedge^n(T^*X)$. Suppose in addition that X is proper morphism over k. Then Serre duality says: for an algebraic vector bundle E on X and an integer i, there is a natural isomorphism :$H^i(X, E) \cong H^{n-i}(X, K_X \otimes E^*)^*$ of finite-dimensional k-vector spaces. Here $\otimes$ denotes the tensor product of vector bundles. It follows that the dimensions of the two cohomology groups are equal: :$h^i(X, E) = h^{n-i}(X, K_X \otimes E^*)$. As in Poincaré duality, the isomorphism in Serre duality comes from the cup product in sheaf cohomology. Namely, the composition of the cup product with a natural trace map on $H^n(X, K_X)$ is a perfect pairing: :$H^i(X, E) \times H^{n-i}(X, K_X \otimes E^*) \to H^n(X, K_X) \to k$. The trace map is the analogue for coherent sheaf cohomology of integration in de Rham cohomology.

3 | Preliminaries: Holomorphic tensors and Vector bundles

In this chapter we are fully following a fundamental work (Bogomolov, 1978), where all the concepts have appeared at first time.
The idea of Holomorphic tensors is the following: For a $(E, \{\varphi_{ij}\} \in \Gamma_G) \to X$ vector bundle over X, the cocycles $\Gamma_G \to GL(V)$ define a vector bundle. Thus, one studies the group G and its' representations and associated important subgroups and charaters. In more details,

$(E, \Gamma_G) \rightsquigarrow$ parabolic subgroup $P \subset G$, torus $T \subset G$, $Z(T)$ center of T and a Weyl group $W(G) \rightsquigarrow$ a root system $R \subset \chi(T) : T \to k^*$, Weyl chambers $K_S \subset \chi(T)$ and metric on $\chi(T) \Rightarrow$ it allows to define special one-dimensional bundles $E(-\chi) = G \otimes k(-\chi)$, and the set of models $A_\chi := Im(\phi : G \otimes k(-\chi) \to H^0(G/P_\pi, E(-\chi)))$, which classify unstable points of X. All of it is achieved using the theory of representations of reductive groups and, in particular, the $\chi(T)$. Applying all this theory to groups $SL(n), GL(n)$ one gets the stability results for vector bundles which are familiar to most of us.

3.0.1 Cocycles associated to vector bundles.

Every vector bundle E_γ corresponds to a cocycle $\gamma \in H^1(X, \Gamma_G)$. A group homomorphism $G \to H$ in turn induces a homomorphism of the sheaves $\Gamma_G \to \Gamma_H \rightsquigarrow$ and hence a map $\rho : H^1(X, \Gamma_G) \to H^1(X, \Gamma_H)$.

It is the same as a fiberwise map of vector bundles $\hat{\rho}_* : E_\gamma \to E_{\hat{c}\gamma} \rightsquigarrow \hat{\rho}_*(gx) = \rho(g)\hat{\rho}_*(x)$. A bundle E_γ is said to be isomorphic to a direct product $X \times G$ iff there exists a Γ_G-section s: $s|_{U^i} \in \Gamma_G(U^i), U^i \in X, E_\gamma|_{U^i} \simeq U^i \times G$, where

$G \subset GL(r;k)$ is an algebraic subgroup. Principal bundle, associated with E^r is built via cocyle $\gamma \in H^1(X, \mathcal{O}(G))$.

3.0.2 Group of characters of G.

Consider the group of characters of a group G, $\chi(G) : \{\rho : G \to k^*\}$. One can represent $\chi(G) = \bigoplus\{$infinite cyclic groups$\}$, if G is connected.
Let k^n be an affine G- space and f be a semiinvariant $(f(gx) = \chi(g)f(x))$ regular function on k^n for a certain character $\chi(G)$. Then, one can associate to $f \rightsquigarrow$ a holomorphic map $E^N_\gamma \to k_\gamma(\chi)$, where $k_\gamma(\chi)$ is a one-dimensional bundle. In this way, we get a non-linear map $f : H^\rho(X, E^N_\gamma) \to H^0(X, k_\gamma(\chi))$.
A subgroup $P \subset G$ is called parabolic, if G/P is complete. $RadG$ is a connected component of $\bigcap_{P \subset G} P$ of all parabolic subgroups in G. In particular, $RadG$ is a normal subgroup in G. The unipotent part of $RadG$, $U(G) \ltimes G$. G is called semisimple (or reductive) if $RadG = 0$. Commutative reductive group is called torus. Any reductive group is isogenic to a $T \times Q$, where T is a torus and Q is a semisimple group. Let $T \subset G$ be a maximal torus in G. Any parabolic subgroup H contains a maximal torus T in G. $Z(T)$ is a center of a torus T, $N(T)$ is a normalizator of a torus T. The group $W(G) := N(T)/Z(T)$ is finite and called a Weyl group of G. $W(G) \curvearrowright T$ by conjugation and hence acts on the lattice of characters $\chi(T)$. $W(G) = \{$reflections $s_\alpha, s^2_\alpha = 1\}$.

A reductive group G can be described in terms of a finite set $R \subset \chi(T)$, which is called a root system. Consider an action of $G \curvearrowright G$ on itself by conjugation $\rightsquigarrow$ we get an action $G \curvearrowright \mathfrak{g}|_T$, restricted to a torus T. One can write $G \curvearrowright \mathfrak{g}|_T = \bigoplus\{$ representations, corresponding to $\chi(T)\} = \bigoplus \mathfrak{g}_x$. *A root* is $\alpha \neq 0, \alpha \in \chi(T)$ such that $\mathfrak{g}_\alpha \neq 0$. The space $\mathfrak{g}_\alpha$ is one-dimesional. $\mathfrak{g}_0 \simeq$ Lie algebra of a torus T. There exist a connected T-invariant unipotent subgroup $U_\alpha \subset G$ for each root $\alpha \in R$.

A subset S is a basis or the system of simple roots if

- elements of S are linearly independent;
- $R \subset NS \cap (-NS)$, where $NS = \sum n_i s_i, s_i \in S, n_i \geq 0$

$W \curvearrowright S$ acts transitively on the sets of bases R.

Definition 3. *A Weyl chamber K_S w.r.t. basis S is a cone in $\chi(T)_R = \chi(T)_Z \otimes R$. $K_S \subset \chi(T)_R, y \in K_S \Leftrightarrow (y, \pi) \geqslant 0\ \forall \pi \in S$.*

K_S is a fundamental domain in $\chi(T)_R$ w.r.t. action of W.

Definition 4. *A dual basis Ω to S is described as: $\omega_\alpha \in \Omega, 2\frac{(\omega_\alpha, \beta)}{(\beta,\beta)} = \delta_{\alpha,\beta}$*

Among invariant metrics on $\chi(T)_R$ for a reductive G, there exists a unique one, such that $\chi(T) = \{\xi \in \chi(T)_R, 2\frac{(\xi,\alpha)}{(\alpha,\alpha)} \in Z(T)\ \forall \alpha \in R\}$

DESCRIPTION OF PARABOLIC SUBGROUPS Let $\pi \subset R$. Then:

- π contains a certain basis
- π is closed under addition, i.e. $(\pi + \pi) \cap R = \pi$

Consider $P_\pi =< T, U_\alpha, \alpha \in \pi >$. Then P_π is a parabolic subgroup. Using this procedure, one can get any parabolic subgroup P, which contains a torus T. In case π is a basis, P_π is a Borel subgroup. Bruhat decomposition is defined as follows. Let P be a parabolic with an unipotent radical $U(P)$, $T \subset P$. Then $G = \bigcup_{w\in W} U(P)wP = \bigcup PwU(P)$. If $P = P_\pi$, where $\pi \subset R$, then w can be extracted from W/W_π.

3.0.3 Representations and invariant theory

Let $\rho : G \to GL(V)$ be a linear representation of a group G. Restricting it to a maximal torus, we get a decomposition w.r.t. weights: $V = \bigoplus_{\chi\in\chi(T)} V_\chi$.

DESCRIPTION OF REPRESENTATIONS OF THE REDUCTIVE GROUPS As is well-known, any finite-dimensional representation splits into a sum of irreducible representations. Any irreducible representation is uniquely defined by its' character trg.

Correspondence between the characters of a maximal torus T and irreducible representations of a group G

Let $\chi \in \chi(T); \pi \subset R, \pi = \{\alpha \in R, (\alpha, \chi) \geqslant 0\}$. Let P_π be a parabolic subgroup, defined via π. As $\chi \perp$ the root system G_π hence χ is a character for P_π.

One can get a one-dimensional vector bundle $E(-\chi)$ with a base G/P_π by twisting a representation $k(-\chi)$ of P_π via principal P_π-bundle $G \to G/P_\pi$, where $E(-\chi) = G \otimes_P k(-\chi)$. As $G \curvearrowright E(-\chi) \rightsquigarrow$ it acts on a space of sections of $E(-\chi) : G \curvearrowright H^0(G/P_\pi, E(-\chi))$. Some properties of $E(-\chi)$ are:

- If $i > 0$, then $H^i(G/P_\pi, E(-\chi)) = 0$
- $H^0(G/P_\pi, E(-\chi))|_{c \in G/P_\pi} \rightsquigarrow$ a surjective homomorphism of P- representations $\varphi : H^0(G/P_\pi, E(-\chi)) \to k(-\chi)$
- Denote $V_\chi := H^0(G/P_\pi, E(-\chi))^*$, i.e. linear functionals on sections of $E(-\chi)$. Dualizing φ , we get a map $k(\chi) \to H^0(G/P_\pi, E(-\chi))^*$. Thus we get a representation $V_\chi \leftrightarrow \chi$
- $V_\chi \simeq V_{\chi'} \Leftrightarrow \chi' = \omega\chi$, where $\omega \in W$.

THE CLASS OF MODELS $\mathbf{A}_\chi$. Let us associate to a character $\chi \leftrightarrow$ a manifold A_χ which is a G-orbit of $k(\chi)$ in the space V_χ, χ is a character of P. A model is defined as $A\chi := Im(\phi : G \otimes_P k(\chi) \to V(\chi))$. This map is an embedding for $x \notin s^{-1}(0)$ for points not lying on the zero section of $E(\chi)$. It also contracts to 0 the zero section of $E(\chi)$. A_χ is closed.

$\mathbf{M_H}$. Consider a reductive subgroup $H \subset G$ and a manifold $G/H := M_H$, which we shall call an M_H-model.

There are 3 classes of models: $\mathbf{A}_\chi, \mathbf{M_H}$ and $\mathbf{C}$.

Lemma 16. *Let V be a representation of G, $V_0 \subset V$ be a subspace, invariant under $P \subset G$ a parabolic subgroup. Then the set GV_0, which consists of shifts of V_o, is closed in V.*

Let us classify all points under the action of a reductive group G over $\mathbb{C}$ and its' representation $\rho : G \to GL(N, \mathbb{C})$.

For a $x \in \mathbb{C}^N \rightsquigarrow$ a manifold of a Gx-orbit. The points of Gx can be classified as:

- Unstable: $0 \in \bar{Gx}$
- Stable: $Gx = \bar{Gx}$ and dimGx is maximal, i.e. for any $y \in \mathbb{C}^N, dimGy \leq dimGx$
- Polystable:
  - Points x, which have a closed orbit Gx of a non-maximal dimension, which does not coincide with 0;
  - Points x, which have a non-closed orbit, i.e. $\bar{Gx} \neq Gx$, but $0 \notin \bar{Gx}$.

G-invariant polynomials help us to differ the closed orbits of G in $\mathbb{C}^N$. If X, Y are the closed orbits of G, then there exists a polynomial h on $\mathbb{C}^N : h(gx) = h(x), x \in \mathbb{C}^N$ and $h(X) = 1, h(Y) = 0$.

The representation theory of reductive algebraic groups allows us to give a geometric description of the manifold of unstable points W_0^G. Namely, it strongly depends on the isomophism $W_0^G \simeq G(W_0^T)$ (T is a maximal torus) and relies on the description of W_0^T. A representation $\rho : T \to \mathbb{C}^N$ splits into direct sum of weighted representations C_χ; $\mathbb{C}^N = \sum C\chi$, where $C\chi := \{v \in \mathbb{C}^N : T \curvearrowright \mathbb{C}^N$ via $\chi(T)\}$.

Definition 5. *A support of a vector* $v \in \mathbb{C}^N, Supp(v) \subset \chi(T)$ *is a subset in the lattice of characters of* T. $\alpha \in Supp(v) \Leftrightarrow$ *for a projection* $P : V \to C\chi$, $P(v) = v_\alpha \neq 0$.

One can consider a convex set of $L(Supp(v))$ in the space of $\chi(T)_\mathbb{Q} = \chi(T)_\mathbb{Z} \otimes \mathbb{Q}$. The support of a set $X \subset \mathbb{C}^N = \{\bigcup Supp(x), x \in X\}$. The following two Lemmas are in particular interest for us.

Lemma 17. $v \in \mathbb{C}^N$ *is unstable w.r.t.* $T \Leftrightarrow 0 \notin L(Supp(v))$.

Denote W_0^G a manifold of unstable points.

Theorem 18. *For any closed G-submanifolds of* $X \subset W_0^G$ *there exists a filtration of G-invariant submanifolds* $X = X_0 \supset X_1 \supset X_2 .. \supset X_N = 0$ *which is dual to a system of regular mappings* $f_i : X_i \to A\chi_i, X_{i+1} = f_i^{-1}(0), 0 \in A_{\chi_i}$.

Thus, the **set of unstable points is determined by the models** $\mathbf{A}_\chi$.

3.0.4 Parabolic and character groups for $\mathbf{G = GL(n), SL(n)}$.

Definition 6. *For* E *a vector space,* $dimE = n$ *and a sequence of integer numbers* $n_1, .., n_r \leq n$, *a flag of type* $\pi = (n_1, .., n_r)$ *in* E *is a set* $F_1 \subset F_2 \subset ... \subset F_r \subset E$, $dimE_i = n_i$. *The set of all flags* $\Phi_\pi(E)$ *of type* π *forms an algebraic variety.*

$GL(E) \curvearrowright \Phi_\pi$ transitively. A Stabilizer $F_n = Stab(GL(E)) := P_\pi$ is a parabolic subgroup in $GL(E)$. A group P_π uniquelly defines a flag, whose stabilizer represents itself. P_π consists of block-diagonal matrices of a known form.

Consider a maximal torus $T \subset GL(E), T = \{diag(\lambda_1, .., .\lambda_n), \lambda_i \in k^*\}$, $\chi(T) \simeq \mathbb{Z}^n$. For a $(m_1, .., m_n) \in \mathbb{Z}^n$, $\chi(diag(\lambda_1, .., .\lambda_n)) = \lambda_1^{m_1} .. \lambda_n^{m_n}$. Note that $\{$Characters of group $GL(E)\} \leftrightarrow_{1:1} \{$Characters of torus T of type $(r, .., r)\}$.

Let $s \in H^0(X, \hat{E}^\rho)$ be a non-zero section. Suppose there exists a non-zero GL-

map $f_\alpha : E^\rho \to C, f_\alpha(s_x) \neq 0, f_\alpha(0) = 0$; $f_\alpha(s)$ is a section of trivial bundle $\Rightarrow f_\alpha(s) = const \neq 0$ Then for any $x \in X s_x \neq 0$. Moreover, s_x is not an unstable point of a fiber E^ρ_x w.r.t. $GL(E)$.

Consider a full system of a GL-invariant regular functions on E^ρ. Note that for any $x, y \in X, f_\alpha(s_x) = f_\alpha(s_y)$ and it implies that $\bar{GL}s_x$ and $\bar{GL}s_y$ are adjoint to the same closed GL-orbit in the space E^ρ.

There are two cases to keep in mind:

- The point s_x is stable, but then s_y is stable as well $\Rightarrow GLs_x = GLs_y$, i.e. $s \in$ subbundle $GLs_x \otimes E_\gamma \subset \hat{E}^\rho$. The fiber of this bundle is a homogeneous space GL/H, where $H \subset GL(E)$ is a reductive group.
  - $H = SL(n), \rho = detE$
  - $H = Sp(n), \rho = \bigwedge^2 E$
  - $H = SO(n), \rho = S^2 E$
  - H is a finite group, ρ is a mult.
- If s_x is polystable, let $s_x \in W_s \subset E^\rho$, for any x, W_s are the points adjoint to the closed orbit S

Consider a bundle $E_\gamma, G = GL(k)$. Let ρ be a representation of $GL(k)$ and $\rightsquigarrow E^\rho$. Note that $\rho_1 + \rho_2 \rightsquigarrow E^{\rho_1} \oplus E^{\rho_2}$ and $\rho_1 \otimes \rho_2 \rightsquigarrow E^{\rho_1} \otimes E^{\rho_2}$. Thus, we get a monoid of representations $R(GL(k))$ to which one can associate $\rightsquigarrow A(E) = \sum E^\rho, \rho \in R(GL(k))$.

Definition 7. *An associative algebra of sections* $A(M, E) = \sum\limits_{\rho \subset R(GL(k))} H^0(M, E^\rho)$ *is called a tensor algebra of fibration.*

Irreducible representations of $GL(k, \mathbb{C})$ are represented by its' highest weight.

Let us fix the following subalgebras in $A(E)$ and $A(M, E)$.

Among all $\rho \in R(GL(k))$ let us take the ones which are trivial on the center $C^* \subset GL(k, \mathbb{C})$. They form a subalgebra $R(PGL) \subset R(GL)$. Note that $\{E^\rho, \rho \in R(PGL) : detE^\rho = 0\}$. $detE$ is associated with a principal bundle C^*_γ with a fiber C^* over M via a character $\chi(C^*) = det\gamma$.

$detE^\rho \rightsquigarrow$ via the character $det\rho|_{C^*}$, but it is trivial if $\rho \in R(PGL)$. Let us define a few natural subalgebras:

- Consider subalgebras defined via $R(PGL) : A_0(M, E) = \sum H^0(M, E^\rho), \rho \in R(PGL)$ and $A_0^H(M, E) = \sum\limits_{\rho \subset R(PGL)} H^0(M, E^\rho \otimes H)$
- Consider a subalgebra $R^+(GL) : \rho|_{C^*} \in Z^+ \subset Z \simeq \chi(C^*)$- is a character of a standard representation to which a bundle E_γ is associated.
- R_p are strictly positive representations: they are the components of tensor powers of representation $\gamma^{\otimes k}, k \in Z^+$. The corresponding subalgebra is denoted by $A_R(M, E) \subset \sum_{\rho \in R_\rho} H^0(M, E^\rho)$.
- One can also associate to γ a subalgebra $\sum \bigoplus S^m\gamma$ and an algebra of symmetric tensors $A_S(M, E) = \sum_{m>0} H^0(M, S^m E)$. $A_S(M, E)$ is commutative.
- For an algebra $S^{mn}\gamma \otimes -mdet\gamma \rightsquigarrow$ an algebra of tensors $A_{0s}(M, E) = \sum H^0(M, S^{mn}E \otimes -mdetE)$

Definition 8. *The cone of the positive characters K_T of a maximal torus $T \subset GL(K, \mathbb{C}), K_T := K_{T(PGL)} \times Z$, where Z is a representation of the center C^*.*

Recall that if ω is a representation, then $supp\omega = \cup\chi, R(\omega) = \bigoplus R(\chi), \chi \in K_T$ for the irreducible components of $R(\omega)$.

Lemma 19. *These assertions hold true:*

1. $SuppA^+ = \{(x, y), y > 0\}$, $x \in K_{T(PGL)}, y \in Z$
2. $SuppA_0 = \{(x, 0)\}$
3. $SuppA_s = n\chi(\gamma)$
4. $SuppA_{0,s} = \{mk\chi(\gamma) - mdet\gamma\}$

To a Flag bundle $\mathcal{F}_1 \subset \mathcal{F}_2 \subset .. \subset \mathcal{F}_k = \mathcal{F}$ one can associate a one-dimensional bundle$F(n_1, .., n_k) = \sum_i n_i(dim\mathcal{F}det\mathcal{F}_i - dim\mathcal{F}_i det\mathcal{F})$ on X.
Set of flag tensors, which one can build using a fixed tensor $s \in C^N$: consider $\{s_\alpha\}$ such that

1. $s_\alpha, s_\beta \in \{s\} \Leftrightarrow s_\alpha \otimes s_\beta \in \{s\}$
2. $\lambda_1 s_\alpha + \lambda_2 s_\beta \in \{s\}$
3. For $\forall G$-map of $f : \bar{G}s \to C^i$, C^i is a G-module , $f(0) = 0, f(s) \in \{s\}$

Remark 1. *1. If $\lim_{n\to\infty} g_n s = 0, g_n \in G$, then $\lim_{n\to\infty} g_n s_\alpha = 0, \forall s_\alpha \in \{s\}$*

2. If s is stable and has a trivial stabilizer in G, then $\forall s_\beta \in \{s\}$. Indeed, in this case , $\bar{G}s \simeq G$ and $\forall s_\beta$ there exists a G-map $f : G \to Gs_\beta$ such that $f(s) = s_\beta$.

Let us reiterate it using language of supports. Fix a maximal torus T. Then, if s is polystable and has a closed orbit G/H, a support $Supphs_\alpha$ for some $h \in G$ also lies in the subspace L_χ, $\{\chi(T \cap H) = 1\}$ and any s' with support in a lattice with a finite index in L_χ, $s' \in \{hs\}$. For an algebra, generated via unstable tensor s the following statement holds.

Theorem 20. *There exists a cone $R_s \subset \chi_R(T)$, for a certain maximal torus T: A_s- algebra, generated by elements s, has R_s as a support. $\{$set of flag tensors$f_{n\alpha}\} \in A_s$, where $\alpha \notin \partial R_s$, ∂R_s is a boundary , $n \in Z, n >> 0$.*

STABILITY OF BUNDLES E $\to$ M

Let us fix a number of conditions:

1. $\forall \mathcal{F} \subset \mathcal{O}(E)$ coherent subsheaves

2. Fix a direction of polarization of M, i.e. a line of curves $\{nH\} \subset PicM \otimes R, n > 0$.

Modifying a bit Takemoto's stability definition, we get

H-STABILITY

We call a coherent sheaf $\mathcal{O}(E)$ H-stable if for any $\mathcal{F} \subset \mathcal{O}(E)$, $dim\mathcal{F} > 0$ the inequality holds $dimE \wedge H^{n-1} < dim\mathcal{F}c_1(E) \wedge H^{n-1}$. $c_1(\mathcal{F})$ is a divisor of 1-dimensional bundle $det\mathcal{F}$.

If $\exists \mathcal{F} \subset \mathcal{O}(E)$, for which the other inequality holds: $dimEc_1(\mathcal{F}) \wedge H^{n-1} > dim\mathcal{F}c_1(\mathcal{E}) \wedge H^{n-1}$, then $\mathcal{O}(E)$ is H-non-stable.

Let us introduce the notion of T-stability, which is equivalent to H- stability, but uses the language of symmetric tensors.

3.0.5 T-stability

A vector bundle E (corr. $\mathcal{O}(E)$) is called T-*stable* if $\forall$ flag of subsheaves $\mathcal{F}_1 \subset \mathcal{F}_2 \subset .. \subset \mathcal{F}_k \simeq \mathcal{O}(E)$ and $\forall$ positive vector $(n_1, .., n_k)$, $H^0(M^n, F_\chi(n_1, .., n_k)) = 0, \forall \chi, F_\chi(n_1, .., n_k) = F(n_1, .., n_k) \otimes L_\chi, L_\chi \in Pic^0(M)$.

A vector bundle E is called T-polystable if $H^0(M^n, F_\chi(n_1, .., n_k)) > 0 \Rightarrow$

$F_\chi(n_1, .., n_k) = 0$. The others are unstable.

Definition 9. *A vector bundle E is called T-unstable if $\exists$ a flag of coherent submodules $\mathcal{F}_1 \subset^{n_1} \mathcal{F}_1 \subset .. \subset \mathcal{F}_k \simeq \mathcal{O}(E)$ such that for a certain positive $n = (n_1, .., n_k)$ the divisor $F_\chi(n_1, .., n_k)$ and $L_\chi \in Pic^0(M)$ is effective.*

Definition 10. *A vector bundle E on M is α-unstable if $\exists$ a free submodule $\mathcal{F} \subset \mathcal{O}(E)$: the divisor $(dimEdet\mathcal{F} - dim\mathcal{F}detE)$ has a square > 0 and lies in the positive component of the cone $\{x^2 > 0\}$ in a group $Pic_R V = PicV \otimes R$.*

The notion of H-stability is equivalent to T-stability. The main point is that we can extract the information about stability of E from the algebra $A_0^H(E)$ as one can do it using the description of E^ρ via unitary representation.

Lemma 21. *A bundle E is called polystable, if for $\forall H \subset Pic^0 X$ holds that:*

1. *$\exists s \in A_0^H(E) : s_x$ is unstable at least at one point x w.r.t. group $GL(E)_x$*
2. *$\forall s \in A_0^H(E) : s \neq 0 \Rightarrow s_x \neq 0$ for any point $x \in X$*

Knowing the existence of a non-trivial unstable section $s \in A_0^H$ and using the family of G-maps to the cones A_χ, one can build a section $s_{\chi,\gamma}$ of the bundle $A_{\chi,\gamma} \otimes nH \rightsquigarrow$ we have a flag of bundles $E_1 \subset .. \subset E_k \simeq E$ and non-zero section $\hat{s} \in H^0(F(n_1,..,n_k) \otimes nH), \hat{s} \neq 0$, thence $F(n_1,..,n_k) \otimes nH \simeq 0$, i.e. $\forall E_i \subset E, deg(dimEdetE_i - dimE_idetE) = 0$, otherwise E is unstable bundle.

Lemma 22. *If E is stable, then $A_0(E)$ is isomorphic to th algebra of invariants for a reductive group $H_E \subset PGL$, which is an iso to the image of the structure group E in the bundle $E_\gamma/C^* \simeq PGL$.*

Lemma 23. *The following are equivalent:*

1. *$\exists s \in A_0^H(E)s \neq 0, s_x = 0$ for some $x \in X, H \in Pic^0(X)$*
2. *Vector bundle E is T-unstable on the blow-up $\hat{X}$*

Lemma 24. *If E is T-unstable on $X \Rightarrow$ it is unstable w.r.t. $\forall H$ polarization.*

$s \in K_F \rightsquigarrow$ an effective divisor in $Pic_R V$. Then there exists a non-zero tensor $\hat{s} = L_\chi \sum_i n_i detF_H \oplus detE(-\frac{\sum n_i dimF_H}{dimE}) \in \bigotimes_i S^{n_i}E_0 \otimes L_\chi$, $L_\chi \in Pic^0 V, \hat{s} \in$

$A_0^{L_\chi}(E), \hat{s}|_D = 0, D$ is an effective divisor.

Consider $K_D \subset Pic_R V, K_D$ is closed and generated by effective divisors.

Lemma 25. *Let $\bar{K}_D = K_D$ and vector bundle E is H-unstable for $\forall$ polarizations H. Then E is T-unstable.*

Proof. Consider a cone K_F generated by the elements $f_H = \{\frac{detF_H}{dimF_H} - \frac{detE}{dimE}\} \in Pic_R V, F \subset E$ is a subbundle of E. $\bar{K}_D \cap \bar{K}_F = K_D \cap K_F \neq \emptyset$. Let us choose a H-polarization: C is a curve on $M, [C] \in H$. $E|_X \rightsquigarrow$ a number $m_e(X) = max\{\frac{detF}{dimF} - \frac{detE}{dimE}\}$. If $F \subset E$ is a subbundle, then $\forall f_H$, it is true that $\{f_H, H\} \leq m_e(X)$. If we consider a cone K_R^H with an axis $\{H\}$ in $Pic_R V, K_R^H \subset K_D$, then $\#\{f_H \subset K_R^H\} < \infty$, because $f_{H'} \in Z[\frac{1}{n!}]$-lattice in $Pic_R V, n = dimE$ and $\{f_{H'}, H\} \leq m_e(X)$. It implies that $\bar{K}_F \cap K_D = K_F \cap K_D$. On the contrary, if $K_F \cap K_D \neq 0 \Rightarrow \exists$ an integers $n_i > 0 : \sum\limits_i n_i f_H \in K_D$ and $\sum\limits_i n_i f_H \otimes L_\chi$ is an effective for some $L_\chi \in Pic^0 V \Rightarrow T$-unstable bundle E.

Remark 2. *Takemoto's stability is equivalent to T-stability if X is compact, so that we can cover it with a locally finite covering (locally) and thus one gets a finite linear combinations of "lines of curves $[H]$" inside the cones.*

For a coherent sheaf F on X, denote $\Delta(F) = 2rc_2(F) - (r-1){c_1}^2$. $\Delta(F)$ is called *discriminant*.

The following theorem holds true (Huybrechts & Lehn, 2010), [page 72]:

Theorem 26. *(Bogomolov's theorem) Let X be a smooth projective variety of dimension n and H an ample divisor on X. If F is a semistable torsion free sheaf, then*

$$\Delta(F).H^{n-1} \geq 0$$

3.0.6 Bogomolov's effective restriction theorem

It was asked, what are the particular requirements for the restriction of a bundle $E \to X$ on a surface X to a curve $C \subset X$ to remain $E|_C$ stay stable. The answer to this question is provided by the Bogomolov's effective restriction theorem (Bogomolov, 1978; Bogomolov, 1994; Huybrechts & Lehn, 2010). Here we are mostly following the exposition presented in the book (Huybrechts & Lehn, 2010).

An important notion in algebraic geometry plays the concept of rearrangement of a vector bundle along a curve or elementary transformations. In the case of smooth projective surfaces we have a non-degenerate hyperbolic scalar product on the group $Pic_R V$ and duality between the cone of polarizations and the cone of effective divisors. Let Num denote the free Z-module $Pic(X)/\equiv$, where $\equiv$ means numerical equivalence. Its rank ρ is called the Picard number of X. The intersection product defines an integral quadratic form on Num. The Hodge Index Theorem says, that, over $\mathbb{R}$, the positive definite part is 1-dimensional. In other words, $Num_{\mathbb{R}}$ carries the Minkowski metric. For any class $u \in Num_R$ let $|u| = |u^2|^{1/2}$. Actually it is not a norm.

Denote as A a subcone consisting of all ample divisors. A polarization of X is a ray $R_{>0}.H$, where $H \in A$. Let $\mathcal{H}$ denote the set of rays in K_+ (Barth, Hulek, Peters, & Van de Ven, 2004).

This set can be identified with the hyperbolic manifold $\{H \in K_+ || H| = 1\}$.

The hyperbolic metric β is defined as follows:

for points $[H], [H'] \in \mathcal{H}$ let

$$\beta([H], [H']) = arcosh(\frac{H.H'}{|H|.|H'|})$$

Denote the open cone $K^+ = \{D \in Num_R | D^2 > 0, D.H > 0$ for all ample divisors$H\}$. Note that the second condition is added only to pick one of the two connected components of the set of all D with $D^2 > 0$. This cone contains the cone of ample divisors and in turn is contained in the cone of effective divisors. K^+ satisfies the property: $D \in K^+ \iff D.L > 0$ for all $L \in \overline{K^+} \setminus \{0\}$.
For any pair of sheaves G, G' with nonzero rank let

$$\xi_{G',G} := \left(\frac{c_1(G')}{rk(G')} - \frac{c_1(G)}{rk(G)} \right) \in Num_R$$

This invariant in the case of destabilizing sheaf plays role as "destabilizing" point in the corresponding cone and defines kind of "class of curve" which we cannot take as one for elementary transformation.

Suppose W_1 is a bundle on X such that $c_1(W_1) = 0$.

- Every vector bundle E of rank r can be obtained by an elementary transformation of $\mathcal{O}_X^{\oplus r}(nH)$ with $n >> 0$ along a line bundle on a smooth curve $C \subset X$.

- For a given $\alpha \in Pic(X)$, $r \geq 2$ ample divisor H and integer $c_0 \in Z$ there exists a μ-stable vector bundle E with $detE \cong \alpha$, $rkE = r$ and $c_2(E) \geq c_0$.

Let us analyze a bundle which we get on a curve, restricting initial one to C. We want it to stay stable under restriction $|_C, C \subset X$. This condition is given by the Bogomolov's effective restriction theorem.

Theorem 27. *(Bogomolov's effective restriction) Let F be a locally free sheaf of rank $r \geqslant 2$. Assume that F is μ-stable with respect to an ample class $H \in K^+ \cap Num$. Let $C \subset X$ be a smooth curve with $[C] = nH$. If $2n \geqslant \frac{R}{r}\Delta(F) + 1$, then $F|_C$ is a stable sheaf.*

Suppose that F_C has a destabilizing quotient N of rank s. Following (Huybrechts

& Lehn, 2010) we can change a condition $2n \geqslant \frac{R}{r}\Delta(F) + 1$ to the following two conditions in assumption that $F_C \to E$ is a destabilizing quotient of rank one:

$$2n \geq \Delta(\wedge^s F) + 1$$

$$n^2 H^2 = C^2 > \Delta(\wedge^s F)$$

We have a following theorem:

Theorem 28. *Let F be a μ-semistable vector bundle and $\wedge^s F \to M$ be a rank one torsion free quotient with $\mu(\wedge^s F) = \mu(M)$. If the restriction $\wedge^s F|_C \to M_C$ to a curve C is the s-th power of a locally free quotient $F|_C \to E$ of rank s, then $\wedge^s F \to M$ is induced by a torsion free quotient $F \to E$‘ of rank s. In particular, if F is μ-stable, then $s = rkF$.*

4 | Bogomolov's effective restriction and Mehta-Ramanathan theorems

Generally, there are several restrictions theorems in algebraic geometry which deal with vector bundles. It is reasonable to understand what each of them asserts and which one is better to use in our situation.

4.0.1 Result of Mehta-Ramanathan

By (Mehta & Ramanathan, 1984), let X be a projective non-singular algebraic variety of dimension $n \geq 2$ over an algebraically closed field k. Let H be a given very ample line bundle on X corresponding to a projectively normal embedding $X \subset \mathbb{P}^N$. Consider the projective set of lines in the vector space $H^0(X, H^m)$, $m = (m_1, ..., m_t)$ is a multivector and let

$$H^0(X, H^{m_1}) \times H^0(X, H^{m_2}) .. \times H^0(X, H^{m_t}) \times X \supset Z_m \to X,$$

a projection, where the correspondence variety $Z_m = \{(x, s_1, .., s_t) \in X \times S_m | s_i(x) = 0, 1 \leq i \leq t\}$. The projection is a fibration with the fibre over $x \in X$ being identified with the product of hyperplanes $H_1 \times ... \times H_t$, $H_i = \{s \in H^0(X, H^{m_i}) | s(x) = 0\}$.

Consider the other projection: $q_m : Z_m \to H^0(X, H^{m_1}) \times H^0(X, H^{m_2}) .. \times H^0(X, H^{m_t}))$. Let Y_m be a generic fiber of q_m. Note that Y_m are irreducible and nonsingular. Denote $\varphi_m : Y_m \to Z_m$.

Theorem 29. *(Mehta & Ramanathan, 1984) Let V be a semistable torsion free sheaf on X (with respect to the polarisation H). Let Y_m be the generic curve of type (m). Then there is an m_0 such that for $m \geq m_0$ the restriction of V to Y_m (i.e.*

$\phi^*_{(m)}p^*_{(m)}V$) is semistable, or equivalently for $m \geq m_0$ and for s in a nonempty open subset of $S_{(m)}$, $V|_{q^{-1}_{(m)}(s)}$ is semistable.

4.0.2 Mehta-Ramanathan theorem as corollary of Bogomolov's effective restriction theorem

The following theorem holds true:

Theorem 30. *Mehta-Ramanathan theorem follows from Bogomolov's effective restriction theorem.*

Proof. We want to show that Mehta-Ramanathan theorem follows from Bogomolov's restriction theorem. If X^n is a projective manifold then for any algebraic family of bundle F_s we have $H^0(X, F_s) = H^0(X_H, F_s)$ for a sufficiently ample line bundle H and X_H a hyperplane section of H if $dimX_H \geq 1$ since $H^1(X, F_s \times O(-H)) = 0$ for sufficiently big H.

If F_{X_H} is destabilizing subsheaf E on X_H then we can extend this sheaf to X if $dimX_H \geq 2$ (isomorphism for H^1). Note that if $F^r \subset E$ is a destabilizing subsheaf then $detF^r \to \Lambda^r E$ is a destabilzing line bundle and vice versa. Since the line bundles on X and X_H are the same we obtain the result. $\square$

5 | Overview of the concepts of stability

There are many notions of stability that have been introduced. Here we list a few of them. Sometimes we follow the relevant formulations from [wikipedia].

1. **TAKEMOTO, MARUYAMA T-STABILITY (TAKEMOTO, 1972; MARUYAMA, 1977; MARUYAMA, 1976; MARUYAMA, 1975)**

   Let E be a coherent sheaf on a scheme X and $H^i(X, E)$ be its' sheaf cohomology. Holomorphic Euler characteristic is $\chi(X, E) = \sum_{i=0}^{n}(-1)^i dim H^i(X, E)$. Let H be an ample line bundle on X. A Hilbert polynomial in n is $p_H(E, n) = \chi(x, E \otimes H^{\otimes n})$.

   Definition 11. *(Maruyama, 1976) A coherent sheaf E on X is called semistable if for all subsheaves*

   $$\frac{p_H(F, n)}{rk(F)} \leq \frac{p_H(E, n)}{rk(E)}, n >> 0$$

   Theorem 31. *(Maruyama, 1976) There exists a moduli space $M_S^H(r, c_1, c_2)$ of H-semistable coherent sheaves of rkr with chern classes c_1, c_2. There is an open subset $M_S^H(r, c_1, c_2)^{st}$ parametrizing stable bundles.*

2. **GIESEKER STABILITY (GIESEKER, 1977)**

   A *slope* of a holomorphic vector bundle W over a nonsingular algebraic curve (or over a Riemann surface) is a rational number $\mu(W) = \frac{deg(W)}{rank(W)}$. A bundle W is stable if and only if $\mu(V) < \mu(W)$ for all proper non-zero subbundles $V \subset W$ and is semistable if $\mu(V) \leq \mu(W)$ for all proper non-zero subbundles $V \subset W$. Informally this says that a bundle is stable if it is "more ample" than any proper subbundle, and is unstable if it contains a "more ample" subbundle.

Definition 12. *([Gieseker, 1977]) If W and V are semistable vector bundles and $\mu(W) > \mu(V)$, then there are no nonzero maps $W \to V$. If X is a smooth projective variety of dimension m and H is a hyperplane section, then a vector bundle W is called stable (or sometimes Gieseker- stable ([Gieseker, 1977])) if:*

$$\frac{\chi(V(nH))}{rank(V)} < \frac{\chi(W(nH))}{rank(W)} \text{ for } n \text{ large}$$

for all proper non-zero subbundles (or subsheaves) $V \subset W$, where χ denotes the Euler characteristic of an algebraic vector bundle and the vector bundle $V(nH)$ means the n-th Serre twist of V by H, i.e. $V \otimes H^{\otimes n}$. W is called semistable if the above holds with $<$ replaced by $\leq$.

3. **T-STABILITY VIA SYMMETRIC TENSORS AND REPRESENTATIONS OF REDUCTIVE GROUPS (BOGOMOLOV, 1978) AND [3.0.4] , [3.0.5].** Let us classify all points under the action of a reductive group G over $\mathbb{C}$ and its' representation $\rho : G \to GL(N, \mathbb{C})$.

For a $x \in \mathbb{C}^N \rightsquigarrow$ a manifold of a Gx-orbit. The points of Gx can be classified as:

- Unstable: $0 \in \bar{Gx}$
- Stable: $Gx = \bar{Gx}$ and dimGx is maximal, i.e. for any $y \in \mathbb{C}^N, dimGy \leq dimGx$
- Polystable:
  - Points x, which have a closed orbit Gx of a non-maximal dimension, which does not coincide with 0;
  - Points x, which have a non-closed orbit, i.e. $\bar{Gx} \neq Gx$, but $0 \notin \bar{Gx}$.

G-invariant polynomials help us to differ the closed orbits of G in $\mathbb{C}^N$. If X, Y are the closed orbits of G, then there exists a polynomial h on $\mathbb{C}^N$: $h(gx) = h(x), x \in \mathbb{C}^N$ and $h(X) = 1, h(Y) = 0$.

The representation theory of reductive algebraic groups allows us to give a geometric description of the manifold of unstable points W_0^G. Namely, it strongly depends on the isomophism $W_0^G \simeq G(W_0^T)$ (T is a maximal torus) and relies on the description of W_0^T. A representation $\rho : T \to \mathbb{C}^N$ splits into direct sum of weighted representations C_χ; $\mathbb{C}^N = \sum C\chi$, where $C\chi := \{v \in \mathbb{C}^N : T \curvearrowright \mathbb{C}^N \text{ via } \chi(T)\}$.

Definition 13. *A support of a vector* $v \in \mathbb{C}^N, Supp(v) \subset \chi(T)$ *is a subset in the lattice of characters of* T. $\alpha \in Supp(v) \Leftrightarrow$ *for a projection* $P : V \to C\chi$, $P(v) = v_\alpha \neq 0$.

One can consider a convex set of $L(Supp(v))$ in the space of $\chi(T)_{\mathbb{Q}} = \chi(T)_{\mathbb{Z}} \otimes \mathbb{Q}$. The support of a set $X \subset \mathbb{C}^N = \{\bigcup Supp(x), x \in X\}$. The following two Lemmas are in particular interest to us.

Lemma 32. $v \in \mathbb{C}^N$ *is unstable w.r.t.* $T \Leftrightarrow 0 \notin L(Supp(v))$.

Denote W_0^G a manifold of unstable points.

Theorem 33. *For any closed G-submanifolds of* $X \subset W_0^G$ *there exists a filtration of G-invariant submanifolds* $X = X_0 \supset X_1 \supset X_2 .. \supset X_N = 0$ *which is dual to a system of regular mappings* $f_i : X_i \to A\chi_i, X_{i+1} = f_i^{-1}(0), 0 \in A_{\chi_i}$.

T-STABILITY

A vector bundle E (corr. $\mathcal{O}(E)$) is called T-*stable* if $\forall$ flag of subsheaves $\mathcal{F}_1 \subset \mathcal{F}_2 \subset .. \subset \mathcal{F}_k \simeq \mathcal{O}(E)$ and $\forall$ positive vector $(n_1, .., n_k)$, $H^0(M^n, F_\chi(n_1, .., n_k)) = 0, \forall \chi, F_\chi(n_1, .., n_k) = F(n_1, .., n_k) \otimes L_\chi, L_\chi \in Pic^0(M)$.

A vector bundle E is called T-polystable if $H^0(M^n, F_\chi(n_1,..,n_k)) > 0 \Rightarrow F_\chi(n_1,..,n_k) = 0$. The others are unstable.

Definition 14. *A vector bundle E is called T-unstable if $\exists$ a flag of coherent submodules $\mathcal{F}_1 \subset^{n_1} \mathcal{F}_1 \subset .. \subset \mathcal{F}_k \simeq \mathcal{O}(E)$ such that for a certain positive $n = (n_1,..,n_k)$ the divisor $F_\chi(n_1,..,n_k)$ and $L_\chi \in Pic^0(M)$ is effective.*

Definition 15. *A vector bundle E on M is α-unstable if $\exists$ a free submodule $\mathcal{F} \subset \mathcal{O}(E)$: the divisor $(\mathbf{dimEdet}\mathcal{F} - \mathbf{dim}\mathcal{F}\mathbf{detE})^\mathbf{2} > \mathbf{0}$ and lies in the positive component of the cone $\{x^2 > 0\}$ in a group $Pic_R V = PicV \otimes R$.*

The notion of H-stability is equivalent to T-stability. The main point is that we can extract the information about stability of E from the algebra $A_0^H(E)$ as one can do it using the description of E^ρ using unitary representation.

Theorem 34. *(Bogomolov's effective restriction) (Huybrechts & Lehn, 2010) Let W be a locally free sheaf of rank $r \geqslant 2$ on a family of curves with $c_1(W) = 0$. Assume that W is μ-stable with respect to an ample class $H \in K^+ \cap Num$ and $C \subset X$ be a smooth curve with $[C] = nH$. Let $2n \geqslant \frac{R}{r}\Delta(F) + 1$. Then $F|_C$ is a stable sheaf.*

4. **Kempf-Ness theorem, GIT and slope-stability (Kempf, , & Ness, 1979), (Mumford et al., 1994), (Gieseker, 1979)** The Kempf-Ness theorem (Kempf et al., 1979), gives a criterion for the stability of a vector v in a representation ρ of a complex reductive group G. If the complex vector space is given a norm that is invariant under a maximal compact subgroup of the reductive group, then the Kempf-Ness theorem states that a vector is stable if and only if the norm attains a minimum value on the orbit of the vector.

Theorem 35. *(Kempf et al., 1979) Let G be a reductive algebraic group* *acting linearly on a projective variety X. A point $x \in X$ is polystable if and only if the orbit Gx contains a zero of the moment map. If x is polystable, the intersection $\mu^{-1}(0) \cap Gx$ is a unique K-orbit. The inclusion $\mu_K^{-1}(0) \subset M^{ss}$ induces the homeomorphism $\mu_K(0) \to M//G$*

For bundles on curves the stability defined by slopes and by growth of Hilbert polynomial coincide. In higher dimensions, these two notions are different and have different advantages. Gieseker stability has an interpretation in terms of geometric invariant theory.

Let X be a smooth projective variety of dimension n, H its hyperplane section. A slope of a vector bundle E with respect to H is a rational number defined as:

$$\mu(E) := \frac{c_1(E) \cdot H^{n-1}}{\mathrm{rk}(E)}$$

where c_1 is the first Chern class. A torsion-free coherent sheaf E is μ-semistable if for any nonzero subsheaf $F \subseteq E$ the slopes satisfy the inequality $\mu(F) \leq \mu(E)$. It's μ-stable if, in addition, for any nonzero subsheaf $F \subseteq E$ of smaller rank the strict inequality $\mu(F) < \mu(E)$ holds. This notion of stability may be called slope stability.

For a vector bundle E the following chain of implications holds: E is μ-stable $\Rightarrow$ E is stable $\Rightarrow$ E is semistable $\Rightarrow$ E is μ-semistable.

5. K-STABILITY,-POLYSTABILITY

The Hilbert-Mumford criterion shows that to test the stability of a point x in a projective algebraic variety $X \subset \mathbb{CP}^N$ under the action of a reductive algebraic group $G \subset \mathrm{GL}(N+1, \mathbb{C})$, it is enough to consider the one parameter subgroups 1-PS of G. To proceed, one takes a 1-PS of G, say $\lambda : \mathbb{C}^* \hookrightarrow G$, and looks at the limiting point: $x_0 = \lim_{t\to 0} \lambda(t) \cdot x$.

This is a fixed point of the action of the 1-PS λ, and so the line over x in the affine space $\mathbb{C}^{N+1}$ is preserved by the action of λ. An action of the multiplicative group $\mathbb{C}^*$ on a one dimensional vector space comes with a weight, an integer we label $\mu(x, \lambda)$, with the property that:$\lambda(t) \cdot \tilde{x} = t^{\mu(x,\lambda)}\tilde{x}$ for any $\tilde{x}$ in the fibre over x_0. The Hilbert-Mumford criterion says:

- The point x is "'semistable"' if $\mu(x, \lambda) \leq 0$ for all 1-PS $\lambda < G$.
- The point x is "'stable"' if $\mu(x, \lambda) < 0$ for all 1-PS $\lambda < G$.
- The point x is "'unstable"' if $\mu(x, \lambda) > 0$ for any 1-PS $\lambda < G$.

For a definition of *test-configuration* we refer reader to a Algebro-Analytic dictionary [7].

We follow (Stoppa, 2011; Ross & Dervan, 2017; Székelyhidi, 2014).

Definition 16. *(Slope of a polarized variety, projective case) Define a slope of (X, L) of a polarized variety to be*

$$\mu(X, L) := \frac{-K_X.L^{n-1}}{L^n} = \frac{-\int_X c_1(K_X).c_1(L)^{n-1}}{\int_X c_1(L)^n}$$

Definition 17. *(Donaldson-Futaki invariant) D-F invariant of a test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) of the exponent r is*

$$DF(\mathcal{X}, \mathcal{L}) := \frac{n}{n+1}\mu(X, L^{\otimes r})\mathcal{L}^{n+1} + \mathcal{L}^n.K_{\mathcal{X}/P}$$

Definition 18. *(K-stability for projective varieties) We say a polarized variety (X, L) is*

- *K-semistable if $DF(\mathcal{X}, \mathcal{L}) \geqslant 0$ for t.c. $(\mathcal{X}, \mathcal{L})$*
- *K-stable if $DF(\mathcal{X}, \mathcal{L}) > 0$ for t.c. $(\mathcal{X}, \mathcal{L})$ with $||(\mathcal{X}, \mathcal{L})||_m > 0$*

- *uniformly K-stable if* $\exists$ *an* $\varepsilon > 0$ *s.t.* $DF(\mathcal{X}, \mathcal{L}) \geqslant \varepsilon ||(\mathcal{X}, \mathcal{L})||_m \forall$ *test-configurations* $(\mathcal{X}, \mathcal{L})$ *for* (X, L)

6 | Preliminaries:
Atiyah class of a vector bundle

6.0.1 Rank 2 Higgs bundles

Higgs bundle for the basic representation of $\mathbb{C}^2$ is the rank 2 holomorphic vector bundle with a symplectic form and trace zero Higgs field $\phi \in H^0(\Sigma, EndV \otimes K)$. Characteristic equation

$$det(x - \phi) = x^2 - q = 0$$

defines a curve in the total space of canonical bundle $\pi : K \to \Sigma$.

Let $L \subset E$ be a ϕ-invariant line bundle. Then near a zero of $det\phi$, which has only simple zeroes we can find a holomorphic trivialization of E w.r.t. which

$$\phi = \begin{bmatrix} a(z) & b(z) \\ 0 & -a(z) \end{bmatrix} dz$$

ϕ takes values at trace-free $EndE$ for holomorphic functions a, b.
$det\phi$ is a holomorphic quadratic differential in this case. We obtain a *Hitchin fibration*

$$\mathcal{M}_d^s \to QD(\Sigma)$$

$$(A, \phi) \to det\phi$$

This map is surjective with fibres $det^{-1}(q)$ diffeomorphic to a compact tori.

6.0.2 The Atiyah class of a vector bundle

We mostly follow (Kapranov, 1999). Let X be a complex analytic manifold. Let E be a holomorphic vector bundle on X, and $J^1(E)$ be the bundle of first jets of sections of E. By (Atiyah, 1957a) it fits into an exact sequence

$$0 \to \Omega_1 \otimes E \to J^1(E) \to E \to 0, \tag{6.1}$$

which therefore gives rise to the extension class $\alpha_E \in Ext^1X(E, \Omega_1 \otimes E) = H^1(X, \Omega_1 \otimes End(E))$ known as the Atiyah class of E. An equivalent way of getting α_E is as follows. Let $Conn(E)$ be the sheaf on X whose sections over $U \subset X$ are holomorphic connections in $E|_U$. As well known, the space of such connections is an affine space over $\Gamma(U, \Omega_1 \otimes End(E))$, so $Conn(E)$ is a sheaf of $\Omega_1 \otimes End(E)$ torsors. Sheaves of torsors over any sheaf A of Abelian groups are classified by elements of $H^1(X, A)$, and α_E is the element classifying $Conn(E)$. So α_E is an obstruction to the existence of a global holomorphic connection

6.0.3 Atiyah class and curvature

The class α_E can be easily calculated both in Cech and Dolbeault models for cohomology.

6.0.4 Čech model

In the Čech model, we take an open covering $X = \cup U_i$ and pick connections ∇_i in $E|_{U_i}$. Then the differences $\varphi_{ij} = \nabla_i - \nabla_j \in \Gamma(U_i \cap U_j, \Omega_1 \otimes End(E))$ form a Cech cocycle representing α_E.

6.0.5 Dolbeault model

In the Dolbeault model, we pick a C^∞-connection in E of type $(1,0)$, i.e., a differential operator $\nabla : E \to \Omega^{1,0} \otimes E, \nabla(fs) = \partial(f)s + f(\nabla s)$. Let $\bar{\nabla} = \nabla + \bar{\partial}$ where $\bar{\nabla}$ is the $(0,1)$-connection deï¬$ningtheholomorphicstructure.Thecurvature$F$_{\bar{\nabla}}$ splits into the sum $F_{\bar{\nabla}} = F_{\bar{\nabla}}^{2,0} + F_{\bar{\nabla}}^{1,1}$ according to the number of antiholomorphic differentials. Then

Theorem 36. *If ∇ is any smooth connection in E of type $(1,0)$, then $F_{\bar{\nabla}}$ is a Dolbeault representative of α_E.*

Remark 3. *Holomorphic connections in E can be identified with holomorphic sections of a natural holomorphic fiber bundle $C(E)$, which is an affine bundle over $\Omega^1 \otimes End(E)$. The fiber $C(E)_x$ of $C(E)$ at $x \in X$ is the space of first jets of fiberwise linear isomorphisms $E_x \times X \to E$ defined near and identical on $E_x \times x$. Clearly, this is an affine space over $T_x X \otimes End(E_x)$. Now, $(1,0)$-connections ∇ in E are in natural bijection with arbitary C^∞ sections σ of $C(E)$. Since $C(E)$ is a holomorphic affine bundle, every such σ has a well defined antiholomorphic derivative $\bar{\partial}\sigma$ which is a $(0,1)$-form with values in the corresponding vector bundle, i.e., $\bar{\partial}\sigma \in \Omega^{0,1} \otimes \Omega^{1,0} \otimes End(E) = \Omega^{1,1} \otimes End(E)$. If σ corresponds to ∇, then $\bar{\partial}\sigma = F_{\bar{\nabla}}^{1,1}$.*

Recall that a Hermitian metric in a holomorphic vector bundle E gives rise to a unique connection $\bar{\nabla} = \nabla + \bar{\partial}$ of the above type which preserves the metric. This connection is called the canonical connection of the hermitian holomorphic bundle. It is known that $F_{\bar{\nabla}}$ in this case is of type $(1,1)$. It implies at once the following

Proposition If E is equipped with a Hermitian metric and $\bar{\nabla}$ is its canonical connection, then $F_{\bar{\nabla}}$ is a Dolbeault representative of α_E.

6.0.6 Atiyah class and Chern classes.

If X is Kähler, then $c_m(E) \in H^{2m}(X, C)$, the m-th Chern class of E, can be seen as lying in $H^m(X, \Omega^m)$, and it follows that it is recovered from the Atiyah class as follows:

$$c_m(E) = Alt(tr(\alpha_E^m))$$

Here α_E^m is an element of $H^m(E, (\Omega^1)^{\otimes m} \otimes EndE)$ obtained using the tensor product in the tensor algebra and the associative algebra structure in $End(E)$, while Alt is the antisymmetrization $(\Omega^1)^{\otimes m} \to \Omega^m$. Note that the antisymmetrization constitutes in fact an extra step which disregards a part of information: without it, we get an element

$$\bar{c_m} = tr(\alpha_E^m) \in H^m(x, (\Omega^1)^{\otimes m})$$

. For a vector space V let us denote by $Cyc^m(V)$ the cyclic antisymmetric tensor power of V , i.e.,

$$Cyc^m(V) = \{a \in V^{\otimes m} : ta = (1)^{m+1}a\}, t = (12...m),$$

where t is the cyclic permutation. Then, the cyclic invariance of the trace implies that

$$c_m\bar{(}E) \in H^m(X, Cyc^m(\Omega^1)),$$

but it is not, in general, totally antisymmetric. We will call $\bar{c_m}(E)$ the big Chern class of E; the standard Chern class is obtained from it by total antisymmetrization.

6.0.7 The Atiyah class of a principal bundle.

Let G be a complex Lie group with Lie algebra g and $P \to G$ be a principal G-bundle on X. Let $ad(P)$ be the vector bundle on X associated with the adjoint representation of G. By considering connections in P, we obtain, similarly to the above, its Atiyah class $\alpha_P \in H^1(X, \Omega^1 \otimes ad(P))$. All the above properties of Atiyah classes are obviously generalized to this case.

6.0.8 Atiyah class via curvature tensor

As we have observed before, $\alpha_E := \bar{\partial}\sigma$, where σ is a C^∞-section of $C(E)$, where $C(E)_x := J^1(E_x \times X \to E)$, where $E_x \times X \to E$ are fiberwise linear isomorphisms defined near and identical on $E_x \times \{x\}$.

$$\bar{\partial}\sigma \in \Omega^{1,1} \otimes EndE,$$

at the same time,

$$\bar{\partial}\sigma = F^{1,1}_{\bar{\nabla}},$$

where $F^{1,1}_{\bar{\nabla}}$ is a $(1,1)$-part $F_{\bar{\nabla}} = F^{2,0}_{\bar{\nabla}} + F^{1,1}_{\bar{\nabla}}$ of the curvature of connection $\bar{\nabla} = \nabla + \bar{\partial}$, where ∇ is $(1,0)$-type connection

6.0.9 Weak Lie algebra in Kähler geometry

Suppose that X is equipped with a Kaehler metric h. Let ∇ be the canonical $(1,0)$-connection in TX associated with h, so that

$$[\nabla, \nabla] = 0 \in \Omega^{2,0}(End(T)).$$

Set $\bar{\nabla} = \nabla + \bar{\partial}$, where $\bar{\partial}$ is the $(0,1)$-connection defining the complex structure.

The curvature of $\bar{\nabla}$ is just

$$R = [\bar{\partial}, \nabla] \in \Omega^{1,1}(End(T)) = \Omega^{0,1}(Hom(T \otimes T, T))$$

This is a Dolbeault representative of the Atiyah class $\alpha_T X$, in particular, $\bar{\partial} R = 0$ in $\Omega^{0,2}(Hom(T \otimes T, T))$ (Bianchi identity). Further, the condition for h to be Kaehler is equivalent, as it is well known, to torsion-freeness of ∇, so actually $R \in \Omega^{0,1}(Hom(S^2 T, T))$.

Let us now define tensor fields R_n, $n \leq 2$, as higher covariant derivatives of the curvature: $R_n \in \Omega^{0,1}(Hom(S^2 T) \otimes T^{\otimes(n-2)}, T))$,$R_2 := R, R_{i+1} = \nabla R_i$.

Lemma 37. *Each R_n is totally symmetric, i.e., $R_n \in \Omega^{0,1}(Hom(S^n T, T))$.*

Proof. Except for $R_2 = R$ the forms R_n are not, in general, $\bar{\partial}$-closed. Let $\Omega^{0,\bullet}(T)$ be the Dolbeault complex of global smooth $(0, i)$-forms with values in T, and $\Omega^{0,\bullet-1}(T)$ be the shifted complex. □

Theorem 38. *The maps*

$$R_n : \Omega^{0,j_1}(T) \otimes .. \otimes \Omega^{0,j_n}(T) \to \Omega^{0,j_1+\ldots+j_n+1}(T), n \leq 2,$$

given by composing the wedge product (with values in ${}^{0,\bullet}(T^{\otimes n})$) with $R_n \in \Omega^{0,1}(Hom(T^{\otimes n}, T))$, make the shifted Dolbeault complex $\Omega^{0,\bullet 1}(T)$ into a weak Lie algebra.

If X is a Hermitian symmetric space, then R makes $\Omega^{0,\bullet 1(T)}$ into a genuine Lie dg-algebra.

6.0.10 Companion theorem for vector bundles

Let now (E, h_E) be a Hermitian holomorphic vector bundle on a Kähler manifold X, and let ∇_E be its canonical $(0, 1)$ -connection, so that

$$[\nabla_E, \nabla_E] = 0 \in \Omega^{2,0}(End(E)).$$

Let $F = [\bar{\partial}, \nabla_E] \in \Omega^{1,1}(End(E)) = \Omega^{0,1}(Hom(T \otimes E))$ be the total curvature of ∇_E. Then $\bar{\partial}F = 0$ in $\Omega^{2,0}(Hom(T \otimes E, E))$, and F is the Dolbeault representative of the Atiyah class α_E. Define the tensor fields $F_n \in \Omega^{0,1}(Hom(S^{n-1}T \otimes E, E))$ by setting $F_2 = F, F_n = \nabla F_{n1}$, $n \leq 3$. As before, the required symmetry of F follows from.

Theorem 39. *The maps*

$$c_n : (\Omega^{0,\bullet-1}(T)^{\otimes(n-1)}) \otimes \Omega^{0,\bullet-1}(E) \to \Omega^{0,\bullet-1}$$

given by composing the wedge product with F_n, make the Dolbeault complex $\Omega^{0,\bullet-1}(E)$ into a weak module over the weak Lie algebra $\Omega^{0,\bullet-1}(T)$.

Corollary 10. *If (E, h_E) is a homogeneous Hermitian bundle over a Hermitian symmetric space X, then F makes $\Omega^{0,\bullet-1}(E)$ into a dg-module over the dg-Lie algebra $\Omega^{0,\bullet-1}(T)$.*

7 | Algebro-Analytic dictionary: Vector bundles

Algebraic side

Unitary representation ρ

ρ: $\pi_1(X) \to U(n) \rightsquigarrow$ stable bundle E_ρ

1. (Narasimhan & Seshadri, 1965): A vector bundle $E \to X$, X is a Riemann surface is stable if and only if it comes from the unitary representation $\rho : \pi_1(X) \to U(n)$.

ε-subcones in the cones of effective divisors see [10.1] and (Bogomolov, 1977): Let us define a subcone $K^C_{D,\varepsilon}$ in the cone of effective divisors $K_D \subset Pic(V^2_R)$ as

$K^C_{D,\varepsilon}$:= will be a set of such $\{x \in K_D, x^2 \geq \frac{\alpha-\varepsilon}{K^2}(K.\Delta_P)^2\}$ where $\varepsilon \geq 0, \alpha = \frac{3K^2}{4\chi-K^2}$ and α measures the "inclination" of the cone $K^C_{D,\varepsilon}$ inside K_D.

[a]

[a]We should note that in the case of algebraic surface X the elements of K_D exactly correspond to the elements of $H^{1,1}(X,R)$ and the positivity condition of the intersection a certain elements reads as an integration over a certain cycle in differential-geometrical language, so the correspondence becomes evident.

Analytic side

Unitary connection ∇_E on X

having constant central curvature $*F = -2\pi\mu(E)$, whose holonomy corresponds to the representation ρ.

1. (S. K. Donaldson, 1983): An indecomposable holomorphic bundle $E \to X$ over X is stable if and only if there is a unitary connection ∇_E on X having constant central curvature $*F = -2\pi\mu(E)$. Such a connection is unique up to isomorphism.

Kähler cone (Demailly & Paun, 2004)

X is a compact Kähler manifold

$H^{1,1}(X,R)$ the space of real $(1,1)$-cohomology classes. Then the Kähler cone K of X, $K \subset H^{1,1}(X,R)$ is one of the connected components of the set P of real $(1,1)$-cohomology classes α which are numerically positive on analytic cycles Y, i.e. such that $\int_Y \alpha_P \geq 0$ for every irreducible analytic set Y in X, $p = dimY$.

Algebraic side

Model of the smooth projective variety According to (Moriwaki, 1995), also see [10.1, 10.1.2] and (Bogomolov, 1977),

Let X be a geometrically irreducible smooth projective variety over F. Let X and C be smooth projective varieties over k, and $f : \mathcal{X} \to C$ a k-morphism such that the function field of C is F and the generic fiber of f is X, i.e. $\mathcal{X} = X \otimes F$.

X is said to be non-isotrivial if there is a non-empty open set C_0 of C such that, for all $t \in C_0$, the Kodaira-Spencer map is not zero. Let $\bar{F}$ be the algebraic closure of F. For a point $P \in X(F)$, let us denote by Δ_P the corresponding integral curve on X . We fix a line bundle L on X. Let $\mathscr{L}$ be a line bundle on $\mathcal{X}$ with $\mathscr{L} \otimes F = L$.

Definition 19. *The pair $(f : \mathcal{X} \to C, \mathscr{L})$ is called a model of (X, L). A geometric height $h_L(P)$ of P with respect to L is defined by*

$$h_L(P) = \frac{(\mathscr{L} \cdot \Delta_P)}{[F(P) : F]}$$

Analytic side

Test-configuration (S. Donaldson, 2002), [pages 5-6]

A t.c. for a polarised variety (X, L) is a pair $(\mathcal{X}, \mathcal{L})$ where $\mathcal{X}$ is a Scheme with a flat morphism $\pi : \mathcal{X} \to \mathbb{C}$ and $\mathcal{L}$ is a relatively ample line bundle for the morphism π, such that: For every $t \in \mathbb{C}$, the Hilbert polynomial of the fibre $(\mathcal{X}_t, \mathcal{L}_t)$ is equal to the Hilbert polynomial $\mathcal{P}(k)$of (X, L). This is a consequence of the flatness of π. There is an action of $\mathbb{C}^*$on the family $(\mathcal{X}, \mathcal{L})$ covering the standard action of $\mathbb{C}^*$ on $\mathbb{C}$. For any (and hence every) $t \in \mathbb{C}^*$, $(\mathcal{X}_t, \mathcal{L}_t) \cong (X, L)$ as polarised varieties. In particular away from $0 \in \mathbb{C}$, the family is trivial: $(\mathcal{X}_{t\neq 0}, \mathcal{L}_{t\neq 0}) \cong (X \times \mathbb{C}^*, \mathrm{pr}_1^* L)$ where $\mathrm{pr}_1 : X \times \mathbb{C}^* \to X$is projection onto the first factor.

We say that a test configuration $(\mathcal{X}, \mathcal{L})$is a product configuration if $\mathcal{X} \cong X \times \mathbb{C}$, and a 'trivial configuration' if the $\mathbb{C}^*$-action on $\mathcal{X} \cong X \times \mathbb{C}$ is trivial on the first factor.

Algebraic side

Moriwaki constant (Moriwaki, 1995), also see [10.1.2] and (Bogomolov, 1977)

The pair $(f : \mathcal{X} \to C, \mathscr{L})$ is called a model of (X, L). A geometric height $h_L(P)$ of P with respect to L is defined by

$$h_L(P) = \frac{(\mathscr{L} \cdot \Delta_P)}{[F(P) : F]}$$

Define a geometric logarithmic discriminant as

$$d(P) = \frac{2g(\Delta_P^{\sim}) - 2}{[F(P) : F]}$$

where $\Delta_P^{\sim}$ is a normalization of Δ_P.

Theorem (Moriwaki, 1995)

If the cotangent bundle to X is very ample, the following holds:

$$h_L(P) \leq A \cdot d(P) + O(1)$$

Analytic side

Seshadri constant (Demailly, 1992)

Let X be a smooth projective variety, L an ample line bundle on it, x a point of X, $\mathcal{C}_x$ = all irreducible curves passing through x .

$$\epsilon(L, x) := \inf_{C \in \mathcal{C}_x} \frac{L \cdot C}{mult_x(C)}$$

Here, $L \cdot C$ denotes the intersection number of L and C, $mult_x(C)$ measures how many times C passing through x.

Definition: One says that $\epsilon(L, x)$ is the Seshadri constant of L at the point x, a real number. When X is an abelian variety, it can be shown that $\epsilon(L, x)$ is independent of the point chosen, and it is written simply $\epsilon(L)$.

8 | Results

8.1 Vector bundle as a section of family of Moduli spaces

Let us introduce a new object, arising from every vector bundle on particular families. Suppose we have B a family of curves and $\mathcal{E}$ a vector bundle on it. We can say that $\mathcal{E}|_{C_\lambda}$ restricted to a point of the family, i.e. a curve C_λ, is a bundle on a Riemann surface with the corresponding unitary connection A. The information of stability of $\mathcal{E}$ is contained in its second chern class $c_2(\mathcal{E})$.

Theorem 40. *Every vector bundle $\mathcal{E}$ on B, which is stable on the restriction to any curve of the family B, is a smooth section of the family of moduli spaces of vector bundles M_{C_λ}, $\lambda \in \Lambda$.*

$$\mathcal{E} \in \Gamma(M_{C_\lambda})_{\lambda\in\Lambda}$$

Proof. Vector bundle $\mathcal{E}$ on the family of curves B on the restriction to the point $\mathcal{E}|_{C_{\lambda_0}}$, i.e. a particular curve C_{λ_0}, is a vector bundle , arising from the representation of the fundamental group of $\pi_1(C_{\lambda_0})$ (by the Narasimhan-Seshadri theorem). Therefore we can simply consider the following fibration on B: the fiber on every curve C_λ corresponds to its‘ moduli space M_{C_λ}. Consequently, to "fix a vector bundle on the family B " simply means that on each fiber, i.e. on the corresponding moduli space, we have to choose one element. It precisely repeats the statement of the proposition. □

Remark 4. *To supplement our definition we should note that stable bundles come from*

- *Stable bundles on the normalizations of singular curves: $\mathcal{E} \to \hat{C}$, where $\nu : \hat{C} \to C$ is a normalizarion. These bundles one can get from representations of a fundametal group of the normalization $\pi_1(\hat{C})$ and thus destabilizing sheaf is not induced on the initial curve C*

- *Bundles which are unstable on the normalization $\hat{C}$, but are stable on the singular curve C (parabolic semistable vector bundles). They appear in the following way: we can take an unstable bundle $\hat{\mathcal{E}}$ (i.e. with destabilizing subsheaf $\mathscr{F}$) on the normalization $\hat{C}$. When we arrive to a base C it transforms to a point $\mathcal{E}^{sing}$. In such way we obtain the identification of the parameters of the destabilizing sheaf $\mathscr{F}$ with the structure of the parabolic bundle on C. It means that on $\hat{\mathcal{E}} \to \hat{X}$, where $\hat{\mathcal{E}}$ is an unstable bundle at the point on the normalization $\hat{X}$ ($\hat{X}$ denotes X blown-up in a a node of a singular curve C, $\nu : \hat{C} \to C$). Then it has a normal destabilizing subsheaf $\mathscr{F} \subset \hat{\mathcal{E}}$ such that $c_1(\mathscr{F}) - (\frac{rk\mathscr{F}}{k})c_1(\mathcal{E}) \in K^+$.*
  A blow-down map $\hat{\nu}$ induces a morphism:

  $$\hat{\nu} : \mathscr{F} \to \mathcal{E}_p,$$

  where $\mathcal{E}_p$ is a flag variety of type determined by a fixed quasiparabolic structure i.e. a flag $\mathcal{E}_p = F_1\mathcal{E}_p \supset F_2\mathcal{E}_p \supset ... \supset F_r\mathcal{E}_p$ and weights $\alpha_1, .., \alpha_r$ attached to $F_1\mathcal{E}_p, .., F_r\mathcal{E}_P$ such that $0 \leq \alpha_1 < \alpha_2 < ... < \alpha_r < 1$, $k_1 = dimF_1\mathcal{E}_p - dimF_2\mathcal{E}_p, k_r = dimF_r\mathcal{E}_P$ the multiplicities of $\alpha_1, .., \alpha_r$.

These are parabolic bundles. Thus we get a strata, which is glued to our families of moduli spaces of vector bundles. It is known that the set of all parabolic semistable bundles is an open subset of a suitable Hilbert Scheme, which has the usual properties, i.e. non-singular, irreducible and of a given dimension. This open set could be mapped to a product of Grassmannians and Flag varieties.

Corollary 11. *If $C_\lambda = C$ is a constant curve, then all the moduli spaces are the same $M_{C_\lambda} = M_C = M$, thus every vector bundle on B is decoded by the subset $\mathscr{S} \subset M$ in the moduli space M: $\mathscr{S} = \{m_\lambda \in M, \lambda \in \Lambda\}$*

Corollary 12. *Moduli space M_B of all the vector bundles on B consist of all the sections S of family of the corresponding moduli spaces M_{C_λ}.*

Corollary 13. *If $C_\lambda = C_1$ and $\Lambda = C_2$, thus we have a vector bundle on the $C_1 \times C_2$. Then every subset $\mathscr{C}_1 \subset M_{C_2}$ or $\mathscr{C}_2 \subset M_{C_1}$ will form a cycle in the corresponding Moduli space of vector bundles.*

8.2 Second chern class as a cycle in the moduli space

Suppose, as above $\mathcal{E}$ be a vector bundle on B and $C \subset B$ is some other smooth projective curve. Let $F_C \to C$ be an arbitrary stable vector bundle on C. $\mathcal{O}(\mathcal{E})$ will denote the space of smooth sections of vector bundle $\mathcal{E}$ and $\mathcal{O}(\mathcal{E})|_C$ its restriction to a curve C. Therefore we obtain a diagram which describes the operation of elementary transformation $T_{C,F}$:

$$T_{C,F} : \mathcal{O}(\mathcal{E}) \longrightarrow \mathcal{O}(\mathcal{E})|_C \xrightarrow{\mathfrak{S}} F_C \longrightarrow 0 \tag{8.1}$$

This way we get a bundle $W =: T_{C,F}(\mathcal{E}) = ker(\mathfrak{S})$. We can take the inverse of the map $\mathfrak{S}$.

Following work (Bogomolov, 1994), we obtain the formulae for the change of Chern classes under the operation of elementary transformation:

$$c_1(W) = c_1(\mathcal{E}) - rC \tag{8.2}$$

$$c_2(W) = c_2(\mathcal{E}) + \frac{(r^2 - r)}{2} C^2 - rCc_1(\mathcal{E}) + detF_C, \tag{8.3}$$

where $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$ are the chern classes of the initial bundle. One can start performing elementary transformation procedure assuming that the initial bundle $\mathcal{E}$ is trivial.

Thus we can rewrite a map $\mathfrak{S}$ on the level of sections of moduli spaces.

By Theorem (40) any vector bundle on family of curves is a smooth section of families of moduli spaces of vector bundles. Rewrite a diagram as:

$$\Gamma(\{M_{C_\lambda}\}_{\lambda \in \Lambda}) \longrightarrow \Gamma(\{M_{C_\lambda}\}^{|C}_{\lambda \in \Lambda}) \xrightarrow{\mathfrak{S}} \Gamma(M_C) \longrightarrow 0 \tag{8.4}$$

where $\Gamma(-)$ denotes taking the corresponding sections. Let us mention that the kernel of the first map is a locally free sheaf and is isomorphic to $\mathcal{O}(\mathcal{E} \otimes -C)$. It is natural to take the inverse of the map $\mathfrak{S}$. It will act just as a tensor product to $\otimes \mathcal{O}(C)$=$\mathfrak{S}^{-1}$.

$\mathfrak{S}^{-1}$ induces a morphism from the curve lying in the base space $C \subset B$ to the space of unitary bundles.

$$\begin{tikzcd}
\Gamma(\{M_{C_\lambda}\}^{|C}_{\lambda \in \Lambda}) & \Gamma(M_C) \arrow[l, bend left, "\mathfrak{S}^{-1}"'] \arrow[d, "\pi"] \\
& C \arrow[u, bend right, "s"']
\end{tikzcd}$$

Now we should understand how to choose a curve $C \subset B$ and a bundle F_C on it to remain a bundle $W = ker\mathfrak{S}$ stay stable.

Let us count the second chern class of a bundle $W' = W \otimes \frac{1}{r}\mathcal{O}(C)$.

Before we proceed let us count the determinant bundle for L_1.

$$detW' = detW \otimes \frac{1}{r}\mathcal{O}(C) = det\mathcal{E} \otimes \mathcal{O}_X(-rC) \otimes \frac{1}{r}\mathcal{O}(C) = det\mathcal{E} \tag{8.5}$$

Thus the determinant bundle does not change in our case. Simple calculation also shows that $c_1(W') = 0$.

Now we can start counting the second Chern class of the bundle W'. Recall that the discriminant of a bundle does not change under multiplication to any line bundle.

Thus,

$$\Delta(W') = \Delta(W)$$

$$\Delta(W) = \Delta(\mathcal{E}) + \frac{r}{k}c_1(\mathcal{E})C - c_1(F_C) + \frac{r(k-r)}{2k}C^2,$$

since $c_1(W') = 0$,

it implies that

$$-c_2(W) = \Delta(\mathcal{E}) + \frac{r}{k}c_1(\mathcal{E})C - c_1(F_C) + \frac{r(k-r)}{2k}C^2$$

and since initial $\mathcal{E}$ was a trivial bundle, we get:

$$-c_2(W') = -c_1(F_C) + \frac{r(k-r)}{2k}C^2 \tag{8.6}$$

The question of stability of a vector bundle on a family of curves reduces to a much simplier notion of stability for particular curve and bundle on it. Our observations result to a theorem which states that:

Theorem 41. *Let W be a locally free sheaf of rank $r \geqslant 2$ on family of curves*

with $c_1(W) = 0$. Assume that W is μ-stable with respect to an ample class $H \in K^+ \cap Num$ and $C \subset X$ be a smooth curve with $[C] = nH$.
Let $2n \geqslant \frac{R}{r}\delta(F) + 1$. Then if a bundle W is stable it implies that

$$c_2(W) = c_1(F_C) - \frac{r(k-r)}{2k}C^2, \tag{8.7}$$

where F_C is a bundle on a curve C;
since $\delta(W) = -c_2(W)$, it follows that

$$c_2(W) > 0$$

Below we will modify our theorem according to the interpretation of second chern class as a cycle in the moduli space of vector bundles on a curve C_λ for some $\lambda \in \Lambda$.

8.2.1 Optimal choice of the curve C and the bundle F_C

The purpose of this section is to interpret $c_2(\mathcal{E})$ as a cycle in a moduli space M_{C_λ} for some $\lambda \in \Lambda$ and prove

Theorem 42. *Operation of the elementary transformation works on any algebraic surface. In particular, for a family of curves B. For example, we can perform an elementary transformation along a curve C_λ. Therefore the second chern class is indeed represented by some moduli section $\mathscr{S}_{c_2(\mathcal{E})}$ for B in some of moduli spaces M_{C_λ}. Moreover, it is given by a formula:*

$$\mathscr{S}_{c_2(\mathscr{E})} = c_2(W) = c_1(F_C) - \frac{r(k-r)}{2k}C^2, \tag{8.8}$$

Proof. Let us consider a map in a more detailed way:

$$\mathfrak{S}^{-1} : M_C \to \{M_{C_\lambda}\}^{|C}_{\lambda \in \Lambda} \tag{8.9}$$

Let us analyze in detail the sense of elementary transformation.

$$\Gamma(\{M_{C_\lambda}\}_{\lambda\in\Lambda}) \xrightarrow{\phi} \Gamma(\{M_{C_\lambda}\}^{|C}_{\lambda\in\Lambda}) \xrightarrow{\mathfrak{S}} \Gamma(M_C) \longrightarrow 0 \tag{8.10}$$

or corresponding diagram:

$$\mathrm{E} \xrightarrow{\phi} \mathrm{E}|_C \xrightarrow{\mathfrak{S}} \mathrm{F}_C \longrightarrow 0 \tag{8.11}$$

Take $F_C \in \Gamma(M_C)$, $E \in \Gamma(\{M^t_{C^1}\}_{t\in C_2})$, where E is the initial trivial bundle. The new bundle W which we get is a kernel of a composite of all the maps on the diagram. When one uses an operation of elementary transformation, a curve $C \subset B$ is embedded to B. We can think of a curve C as a sum of divisors along each curve C_λ for a family B. Assume that C intersects each curve C_λ only at one point. Put $C \cap C_\lambda = D_\lambda$. Clearly, the restrictions $E|_{D_\lambda} = V|_{D_\lambda}$, where $V \in M_{C_\lambda}$, coincide.

Consider a map $\mathfrak{S} : E|_C \to F_C$ and take its kernel: $W := Ker(E|_C \to F_C) = ker\mathfrak{S}$ we obtain a bundle which shows how do not coincide these two bundles (the set where they coincide maps surjectively). Composing with ϕ we obtain a new bundle on a surface X. Note that ϕ is just a restriction map. From definition of $T_{F,C}$ follows that $Im\phi \subset W$.

$$\mathrm{E} \xrightarrow{\phi} \mathrm{W} \longrightarrow 0 \tag{8.12}$$

To get a fiberwise non-trivial bundle W, elementary transformation should be performed on each fiber, as we have started from the trivial one E. Therefore a curve C should be chosen in a way that it intersects each fiber C_λ. Therefore our initial choice of a curve C was a right one. To conclude, we have to analyze a restriction map $\phi : E \to W$. Note that the bundle W depends on a bundle F_C. And we should choose a bundle F_C in a special way. Let us note that at the end of this procedure our goal is to get a new bundle on $\{C_\lambda\}_{\lambda\in\Lambda}$ which is an element $Q \in \Gamma(\{M_{C_\lambda}\}_{\lambda\in\Lambda})$. W should be such bundle along the curve, that at each fiber D_λ, the kernel of a morphism $\phi : E|_{D_\lambda} \to W|_{D_\lambda}$ is exactly a bundle, which we want to get after elementary transformation procedure, i.e. Q. Let us consider the case when the elementary transformation is being performed along a curve C_λ. We can choose a bundle F_C as:

$$F_C|_{D_\lambda} = E|_{C_\lambda}/Q|_{C_\lambda} \tag{8.13}$$

Note that $Q|_{C_\lambda} \in M_{C_\lambda}$ is just an element of the moduli space of vector bundles over curve C_λ. Thus we get a map

$$\mathfrak{S}^{-1} : M_C \to \{M_{C_\lambda}\}_{\lambda\in\Lambda}|_C, \text{ for a } F_C \in M_C, \text{ the map is defined as}$$

$$\mathfrak{S}^{-1}|_{D_\lambda} : F_C|_{D_\lambda} \to E|_{C_\lambda}/\{\text{ some } \alpha \in M_{C_\lambda}\}$$

The rank of the bundle $rkF_C = r$ depends on the rank of the chosen bundle $\alpha \in M_{C_\lambda}$.

It means that F_C maps a curve $C = C_\lambda$ to a moduli space M_{C_λ}. Thus we obtain a *moduli cycle* in the moduli space M_{C_λ}, corresponding to a bundle F_C.
If a curve C is arbitrary then λ will vary and F_C will map a curve C to a moduli section $\mathscr{S}$.

Consequently, chern class $c_2(W)$ will be itself a cycle (or section) in the families of moduli spaces of vector bundles $\{M_{C_\lambda}\}_{\lambda\in\Lambda}$. Denote a map [8.13] as F_C : $C \to \{M_{C_\lambda}\}_{\lambda\in\Lambda}$. If we look carefully, we will deduce that the maps $\mathfrak{S}^{-1}$ and F_C coincide by definition. Now we had above:

$$c_2(W) = c_1(F_C) - \frac{r(k-r)}{2k}C^2, \tag{8.14}$$

We have to understand what is c_1 of a moduli cycle F_C. Since F_C is a stable bundle on a curve C it follows that $c_1(F_C) > 0$. This way a question of positivity of $c_2(W)$ reduces to an apropriate choice of a curve C.But if we choose a curve C as C_λ we will get a cycle in the Moduli space M_{C_λ}. □

8.3 Stability of bundles on smooth projective surfaces

By the effective stability restricton result (Bogomolov, 1994; Huybrechts & Lehn, 2010) any stable vector bundle on a smooth projective surface restricts to a stable bundle on any ample curve belonging to sufficiently big class. More precisely:

Theorem 43. *Let F be a locally free sheaf of rank $r \geqslant 2$. Assume that F is μ-stable with respect to an ample class $H \in K^+ \cap Num$. Let $C \subset X$ be a smooth curve with $[C] = nH$. If $2n \geqslant \frac{R}{r}\delta(F) + 1$, then $F|_C$ is a stable sheaf.*

In particular the restriction is stable on the singular fibers with relatively small

number of singularities and on normalizations of such curves. Hence we obtain a section s_E over an open subset of $U_s \subset P(H)$. Note that the codimension of $P(H) - U_s \subset P(H)$ grows if the class H becomes larger and therefore for a generic pencil $P_t^1 \subset P(H)$ of such curves we can define a one-dimensional algebraic class which is given by the intersection of a section $s(E)$ with a *moduli section* $\mathscr{S}_{P_s^1}$ which corresponds to a vector bundle over a pencil $\mathbb{P}_s^1$.
We want to adress first the relation of this class with an invariant δ defined for vector bundles on surfaces.

Lemma 44. *Intersection of a section $s(E)$ and a moduli section $\mathscr{S}_{\mathbb{P}_s^1}$ is indeed given by one-dimensional algebraic class, which is given by $(\mathscr{L}.\Delta_P)$, where $\mathscr{L}$ is a line bundle, corresponding to a divisor $s(E)$ and Δ_P is an integral curve, corresponding to a moduli section $\mathscr{S}_{\mathbb{P}_s^1}$, if a moduli section $\mathscr{S}_{P_s^1}$ has a natural polarizatoion $\mathscr{L}'$.*

$$\varkappa := s(E) \cap \mathscr{S}_{\mathbb{P}_s^1} = (\mathscr{L}.\Delta_P) \tag{8.15}$$

Proof. Take as a model $\mathcal{X}$ a family of moduli spaces of vector bundles $\{M_t\}, t \in \mathbb{P}_s^1$ for the curves in the pencil $\mathbb{P}_s^1$. Then the section $s(E)$ corresponds to a divisor on $\{M_t\}$ and we can find a line bundle $\mathscr{L}$ which corresponds to it. Integral curve Δ_P is the same as a moduli section $\mathscr{S}_{\mathbb{P}_s^1}$ by definition. $\square$

Note that as we have already observed $c_2(\mathcal{E})$ is represented by a particular moduli section $\mathscr{S}_{c_2(\mathcal{E})}$. In this case a class $\varkappa$ up to a constant is equal to a discriminant Δ.

$$\varkappa = A \cdot \Delta$$

Locally, the section $s(E)$ represents a family of curves in the class $[H]$.

8.4 Stability in the case we have no singular fibers

We are going to find a description of stability of a vector bundle on Y through stability of vector bundle $f_*(\mathcal{E})$ induced from vector bundle on families of curves B.

A good candidate for a family of curves B is a family, obtained in the following way: take a point $x \in X$ and consider all possible embedded curves. They are parametrized by a set which can be described in the following way. Consider very ample line bundle H on a surface X^2 and the space $H^0(X^2, H)$. Then $\mathbb{P}(H^0(X^2, H)) = \mathbb{P}_H$ is a projective space parametrizing the curves in the class H. For any $x \in X^2$ there is a subspace $\mathbb{P}_x$ parametrzing all such x. Therefore a family of all such curves will be parametrized by a bigger projective space P^N. What we have discussed so far is essentially the same families of curves in the class H. The following theorem (?, ?) gives us a positive answer whether the bundle obtained in the way written in the previous section is stable:

Theorem 45. *(Moriwaki, 1997)*

Let $f : X \to Y$ be a surjective and projective morphism of quasi-projective varieties over k with $dim f = 1$. Let E be a vector bundle of rank r on X. Then, we define the discriminant divisor of E with respect to $f : X \to Y$ to be

$$dis_{X/Y}(E) = f_*(2rc_2(E) - (r-1)c_1(E)^2) \cap [X])$$

Let E be a locally free sheaf on X and y a point of Y.
If f is flat over y, the geometric fiber X_y over y is reduced and Gorenstein, and E is strongly semistable on each connected component of the normalization of X_y, then $dis_{X/Y}(E)$ is weakly positive at y.

In our case, $f : B \to X$, $c_1(\mathcal{E}) = 0$, thus $dis_{X/Y}(\mathcal{E}) = f_*(2rc_2(\mathcal{E}) \cap [B])$. Notice that $2rc_2(\mathcal{E}) \cap [B]$ is an intersection of 2 cycles in the families of moduli spaces of vector bundles $\{M_{C_\lambda}\}$. Locally, we can assume that a section $s(E)$ represents a class $[B]$. It implies that, since $c_2(\mathcal{E})$ is represented by some moduli section $\mathscr{S}_{c_2(\mathcal{E})}$, then locally on the open subset U the intersection is equal to a kappa-class:

$$(2rc_2(\mathcal{E}) \cap [B])|_U = \varkappa_{\mathscr{L}} \tag{8.16}$$

The only thing is that we should find a polarization $\mathscr{L}$ for a moduli section $\mathscr{S}$. It is given by a polarization $\mathscr{L}$ on the families of moduli spaces. Let us recall some facts from (Beauville, 2006):

Suppose J^k is a Jacobian variety parametrizing line bundles of degree k on a curve C. Let us fix a line bundle $L \in J^k$ and consider a set

$$\Delta_L := \{E \in M_r | H^0(C, E \otimes L) \neq 0\}$$

Δ_L is a Cartier divisor on M_r. The line bundle $\mathscr{L} = \mathcal{O}_{M_r}(\Delta_L)$ is called a determinant line bundle. In our observations, we can take $\mathscr{L} = \mathcal{O}_{M_r}(\Delta_L)$ as a polarization we need above.

So, we have a theorem

Theorem 46. *In the assumptions of Theorem(45), it is true that for a map* $f : B \to X$

$$dis_{X/Y}(E)|_U = f_*(h_{\mathscr{L}}|_U)$$

and

$$dis_{X/Y}(E)|_U > 0,$$

where the polarization $\mathscr{L} = \mathcal{O}_{M_r}(\Delta_L)$.

8.5 Monodromy along singular fibers

8.5.1 Gauss-Manin connection and quadratic Hitchin map

We are recalling a number of facts of non-abelian Hodge theory, following (T. Chen, 2012) and (Donagi & Pantev, 2009).

Denote $\mathfrak{Higgs}_X$ the moduli space of principal G-bundles over X together with a Higgs field. Bun_X is denoted a coarse moduli space of regular stable G-bundles on X.

$\mathfrak{Higgs}_X$ has a symplectic structure because it is equal to T^*Bun_X.

Let $\mathfrak{g}$ be a Lie algebra of a group G and $<,>$ is a corresponding Killing form. Then the quadratic Hitchin map is defined as

$$qh : \mathfrak{Higgs}_X \to H^0(X, \Omega^{\otimes 2})$$

$$(P, \theta) \to < \theta, \theta >$$

where $\theta \in H^0(X, adP \otimes \Omega^1_X)$ is a Higgs field.

We can define a lifting of tangent vectors associated with this map:

$$L_{qh} : T_X \mathscr{M}_g \to T_{(P,\theta)} \mathfrak{Higgs}_X$$

$$f \mapsto H_{qh^* f}|_{(P,\theta)}$$

where $f \in T_X \mathscr{M}_g \cong H^1(X, TX)$ is viewed as a linear function on $H^0(X, \Omega^{\otimes 2})$ by Serre duality, and $H_{qh^* f}$ is a hamiltonian vector field of $qh^* f$ on $\mathfrak{Higgs}_X$.

The following theorem holds true:

Theorem 47. *The limit lifting of tangent vectors L_0 associated to the isomon-*

odromy lifting L is equal to $\frac{1}{2}L_{qh}$; or,equivalently,
the lifting of tangent vectors on $\mathfrak{Higgs} \to \mathscr{M}_g$ is representing the associated map of the non-abelian Gauss-Manin connection is equal up to a constant multiple to the lifting of tangent vectors induced from the quadratic Hitchin map.

Let E be a vector bundle on X of degree d and rank n. Recall that in our case the restriction of a bundle $E|_{C_t}$ to any curve $C_t \in X^b$ is stable, then, as it is well-known, we have a flat unitary connection $\nabla_{E|_{C_t}}$ along the fibers C_t in X^b. Therefore we have a family of such "fiberwise" connections $\{\nabla_{E|_{C_t}}\} := T_E$. Also, we can find the projective connections along the singular fibers in a natural way: we can assume the existence of connections on the smooth normalized curves X'_s with a singularity for a corresponding connection ∇_{X_s} over the points $x_s \in X^{sing}$. Those are called parabolic connections and are constructed in the following way: if we have a non-singular curve with points x_i corresponding to the singularities one can define λ-connections(logarithmic connections), which have the poles along a divisor $D = \sum x_i$. We can take the residues along these singular points and put an order on it. In particular, they should satisfy the Fuchs relation. Then one can define Parabolic connections:

1) taking logarithmic connections singular over X^{sing}, i.e. a $\mathbb{C}$-linear maps

$$D : E \to E \otimes \Omega^1_X(logX^{sing}) = E \otimes \Omega^1_X \otimes \mathcal{O}_X(X^{sing}),$$

which satisfy the Leibniz identity.

The fiber $\Omega^1_X \otimes \mathcal{O}_X(X^{sing})(x_\beta)$ is canonically identified with $\mathbb{C}$ by sending a meromorphic form to its residue at x_β. In particular, $ResD(x_\beta) \in End(E)(x_\beta)$

2) forming a full flag of subspaces on each fiber of $E|_{x_i}$,

3)Let D be a logarithmic connection on X. Let $Res(D, x_i) = \lambda_i Id_{E(x_i)}, x_i \in$

X^{sing}, $i = 1, ..., m$. Then it follows that

$$d + n \sum_{j=1}^{m} \lambda_i = 0$$

Lemma 48. *Smooth deformation between the Higgs bundles and connections is given by the deformation of the λ-connections.*

8.5.2 Monodromy from differential-geometric perspective

Stable vector bundles of degree 0 on curves have a special flat hermitian metrics. We can reduce our study to this case for now.

Notice that if we have a moduli space of vector bundles M_C , coming from unitary representation from one curve C it is topologically isomorphic to a moduli space coming from unitary representations from any the other curve $M_{C'}$. Thus we have a locally topologically constant family of the moduli spaces $M_H := \{M_t\}^H_{t \in P^1_t - X^{sing}}$ over each pencil of curves $P^1_t - X^{sing}$ in the class H with the same degenerate fibers corresponding to a finite number X^{sing} of points in P^1_t where the curves have singular points. Denote $T := \{C_t, t \in \mathbb{P}^1_t - X^{sing}\} \subset [H]$.

$$M_H \to T \subset \mathscr{M}_g,$$

We can assume that all degenerate curves have exactly one singular point. Consequently, we obtain a family with a monodromy. Regarded as a family of smooth manifolds over T this is a locally trivial fibration by a theorem of Ehresmann. In particular, the fibres are diffeomorphic, but the complex structure may vary. However, the symplectic structure does not vary: it is a symplectic fibre bundle. By taking the annihilator (with respect to the symplectic form) of $T_{M_H/T}$ we obtain a connection (horizontal subbundle of TM_H) on M_H over T. The parallel transport map associated to a path in T is a symplectomorphism, and for loops we in particular obtain the monodromy representation

$$Mon : \pi_1(T, t) \to Aut(M_t)$$

Collecting it for all $t \in T$, we conclude that we can take a loop around a singular curve, which consists of a non-singular curves and hence obtain a monodromy which will correspond to some automorphism of a moduli section $\phi \in Aut(\mathscr{S}_{P^1_t})$.

Recall that the space of sections $\Gamma(M_H)$ is exactly the space of vector bundles which are stable on the restriction to the curve of the family. We can consider a $\mathscr{C}onn \to M_g$, where $\mathscr{C}onn$ is a nonabelian cohomology space for X. Then there exists a Gauss-Manin ∇_{GM} connection on this bundle.
We can differentiate any element of $\Gamma(M_H)$ using Gauss-Manin connection ∇_{GM}. Thus a monodromy of a moduli section will correspond to a $\nabla_{GM}(\gamma(t))$, where $\gamma(t)$ is a loop , consisting of non-singular curves, taken around a curve with singularity $\gamma(t) \subset T$.
The tangent space to a moduli space at a regular point is identified with the infinitesimal deformations of the object corresponding to that point. In fact, tangent space to a Bun_X at a point P is naturally isomorphic to $H^1(X, adP)$.

In our case, it holds that

Theorem 49. *The operator Mon acts as*

$$Mon : f \mapsto H_{qh^*f}|_{(P,\theta)},$$

where $f \in T_X\mathscr{M}_g \cong H^1(X, TX)$ is viewed as a linear function on $H^0(X, \Omega^{\otimes 2})$ by Serre duality, and H_{qh^*f} is a hamiltonian vector field of qh^*f on $\mathfrak{Higgs}_X$.

8.6 Relation between the fundamental groups

Consider a pencil $\mathbb{P}^1_{C_t}$ of curves $\{C_t\}$ on a surface X corresponding to a polarization $\mathscr{L} = H$ on X. We can assume that:

1. The corresponding curves are smooth over $\mathbb{P}^1_{C_t} - X^{sing}$ and have exactly one quadratic singular point p_s over the set X^{sing}
2. The kernel of the map $\pi_1(C_t) \to \pi_1(X)$ is normally generated by a vanishing cycle c_s at any point p_s
3. The fundamental group $\pi_1(X)$ is a quotient of $\pi_1(C_t)$ by a normal subgroup containing a subgroup normally generated by an arbitary vanishing cycle c_s

Both are standard assumptions on the generic pencil of curves.

Lemma 50. *Let N_P be the total number of singular points X^{Sing}. Then it is given by the Euler charactrestics formula for a surface X blown up at the H^2 intersection points of curves parametrized by $\mathbb{P}^1_{C_t}$.*

Proof. Indeed, after blowing up X^b at all intersection points, the pencil defines a projection map $X^b \to \mathbb{P}^1_{C_t}$ with curves C_t as fibers. Thus, by the changing Euler characteristic (Hirzebruch, 1962) under blow up, we have

$$\chi(X^b) = \chi(X) + H^2$$

on the other hand,

$$\chi(X^b) = (2 - 2g(C_t))(2 - N_P) + N_P((2 - 2g(C_t) + 1)$$

and

$$2g(C_t) - 2 = H(K + H)$$

and thus we have the formula of number N_P through $KH, \chi(X), H^2$

$$N_p = \chi(X) + H^2 + 2H(K + H)$$

$\square$

Lemma 51. *Assume that we have a family of representations $\rho_{E,t} : \pi_1(C_t) \to U(n)$ over C_t defined by a stable bundle E. If this representation extends to a representation of at least one singular curve $C_s, s \in X^{sing}$ then it extends to X and hence the bundle E is defined via representation $\rho_E : \pi_1(X) \to U(n)$.*

Proof. Indeed the representaion $\rho_{E,s}$ is defined if $\pi_1(X_t)(c_s) = 0$ for all vanishing cycles c_s. Since by our assumption any c_s normally generates the kernel of $\pi_1(X_t) \to \pi_1(X)$ any representation $\rho_{E,t}$ comes from a representation $\rho_E : \pi_1(X) \to U(n)$ hence the result. $\square$

Corollary 14. *The bundle E is not induced from a representation only if we have a singularity for fiberwise unitary connection on E over any singular point p_s.*

8.6.1 Monodromy from group-theoretic perspective

Recall that in the situation above the monodromy operator m acts via Dehn twist on each bond around each vanishing cycle c_s (for details see (Bogomolov & Katzarkov, 1998) and (Bogomolov & Katzarkov, 1999)). The composition of all monodromies which correspond to a loop Γ consisting of non-singular curves X_s around a singular curve X_0 is given by composition of all corresponding Dehn twists T_{c_s} of bonds around each vanishing cycle c_s corresponding to a singular point in each fiber X_s.

$$m = \Pi_{\gamma \in \Gamma} m_{c_{s\gamma}},$$

Assume that X_0 is a fiber with a singular point and generic fiber has fundamental group generated by g_i with standard commutator relation $< \Pi_i[g_i, g_{i+1}] = 1 >$. Then we have the monodromy automorphism (standart action on the fundamental group $\pi_1(X_t)$ induced by the action of Dehn twist around g_1)

$$m : g_1 \to g_1, g_2 \to g_1 g_2$$

and their commutator $[g_1, g_2]$ maps to $g_1^2 g_2 g_1^{-1} g_2^{-1} g_1^{-1}$, i.e. $g_1[g_1, g_2]g_1^{-1}$.
Thus if m maps $g_i \to g_1 g_i g_1^{-1}, i > 1$ then the product of commutators $m(\Pi_i[g_{2i-1}, g_{2i}]) = 1$.
The kernel of projection $\pi_1(X_t)$ to $\pi_1(X_0)$ is generated by g_1 as a normal subgroup of $\pi_1(X_t)$.
Now we have to understand how does monodromy operator for each vanishing cycle g_1 act on the unitary representation of its' fundametal group, which corresponds to a concrete vector bundle E.
Recall the basic construction of unitary representations of fundametal groups of Riemann surfaces.
Consider a set $\Omega = U(n) \times .. \times U(n)$ of $2g$ copies of Unitary group and canonical projections: $p_i : \Omega \to X_i$ and $q_i : \Omega \to Y_i$. Consider the condition $\Pi_i A_i B_i A_i^{-1} B_i^{-1} = I$. There exists a unique homomorphism $\eta : \pi_1 \to U(n)$ such that $\eta(g_i) \to A_i$ and $\eta(g_{i+1}) \to B_i$.
Recall that if we have a map $f : \Omega \to SU(n)$ such that

$$f : (A_1, B_1, ..., A_g, B_g) \to A_1 B_1 A_1^{-1} B_1^{-1} .. A_g B_g A_g^{-1} B_g^{-1}$$

it is known that

$$Hom(\pi_1(X_t), U(n)) \simeq f^{-1}(I).$$

Therefore the action of the monodromy m on the unitary representation is group-theoretically the same:

$$m_{\eta}^{\gamma} : A_1 \to A_1, A_1 \to A_1 A_2, A_i \to A_1 A_i {A_1}^{-1} \tag{8.17}$$

For each unitary representation of a fundamental group of a given surface X_t we have a normal subgroup N_t generated by g_1^t which is a kernel of projection of $\pi_1(X_t)$ to $\pi_1(X_0)$. If we consider all unitary representations of groups N_t denoted by N_t^U then the monodromy operator m_{η}^{γ} acts on each of the corresponding groups N_t^U. For a loop Γ of nonsingular curves X_t we can take $\Pi_{\gamma \in \Gamma} N_{\gamma}^U$ and the corresponding action of m on it.

8.6.2 Action of the Galois group on the singular points

To understand our problem better, we will treat it extending our arguments to the case when we have our constructions over a field K.

Assume that a surface X is defined over a number field $[K : \mathbb{Q}] < \infty$ as well as the pencil $f : X \to \mathbb{P}^1$ and the bundle E. The union of singular fibers in this case is also defined over K. Note that monodromy group T of the fibration $X^b \to \mathbb{P}^1$ permutes vanishing cycles c_s of different fibers transitively in the fundamental group of the complementary of X to all singular fibers $\pi_1(X - X^{sing})$.
Additionally, the action of Galois group $Gal(\bar{K}/K)$ approximates the action of monodromy T on the vanishing cycles $\{c_s\}$.
In particular, if there is only one orbit of the $Gal(\bar{K}/K)$ on the vanishing cycles in a completion of $\hat{\pi}_1(X_t)$ then either the family of representations is obtained from the representation $\pi_1(X)$ or any fiber X_t with a singular point has the latter as singular point of the family of partial connection.

8.7 Geometry of the Second Chern class

Recall, as we have already observed, for a vector bundle $\mathcal{E} \to X$ on a surface we can consider its' restriction to a pencil of curves in the class $[H]$, which is $\mathcal{E} \to \mathbb{P}^1_{C_t}$.
For every such restriction, we have an associated second chern class $c_2(\mathcal{E})$.
As we already know by Theorem (42) , $c_2(\mathcal{E})$ is represented by a cycle $\mathscr{C}_{c_2(\mathcal{E})}$ in the moduli space of vector bundles M_{c_λ} for some $\lambda \in \mathbb{P}^1$.
Notice that when the pencil of curves $\mathbb{P}^1_{C_t}$ is moving on the surface, the associated cycle $\mathscr{C}_{c_2(\mathcal{E})}$ is also changing.

8.7.1 Geometry of the cycle $\mathscr{C}_{c_2(\mathcal{E})}$ and associated invariants

We want to find an invariant associated to a cycle $\mathscr{C}_{c_2(\mathcal{E})}$ ($\mathscr{C}$ for short), which does not change in course of moving of pencil of curves $\mathbb{P}^1_{C_t}$ on a surface. As $c_2(\mathcal{E})$ is represented by a cycle $\mathscr{C}$ in some moduli space M_{C_λ} we can regard this cycle as a vector bundle on a product of curves $C_\lambda \times C$, where C is a curve along those we have performed an operation of elementary transformation. As we have noticed before we can use Gauss-Manin ∇_{GM} connection to deal with moduli sections and cycles.

8.7.2 Vector bundles on the product of curves and the curvature

Here we are following (Bogomolov & Lukzen, 2022). Let $\mathcal{E} \to C_1 \times C_2$ be a vector bundle such that $\mathcal{E}|_{C_1}$ and $\mathcal{E}|_{C_2}$ stable. Coordinates on $\mathcal{E} \to C_1 \times C_2$ could be described as a pair of representations (τ, ρ), where τ corresponds to a horizontal bundle and ρ corresponds to a vertical bundle. The fiber is equal to

$\mathcal{E}|_{(x,y)} = E_\rho \cap E_\tau = E|_{(\rho,\tau)}$

As is well-known along E_ρ, E_τ exist an Ehresmann connections which can be viewed as horizontal subbundles H_ρ,H_τ:

$$TE_\rho = H_\rho \oplus V_\rho, TE_\tau = H_\tau \oplus V_\tau.$$

Thus we have two families of Ehresmann connections: $\{H_\rho, \rho \in \mathscr{C}_1\}$, $\{H_\tau, \tau \in \mathscr{C}_2\}$, where $\mathscr{C}_1, \mathscr{C}_2$ are the moduli cycles.
We are seeking for the connection H on the bundle $\mathcal{E}$ such that $T\mathcal{E} = H \oplus V$, where V is a tangent space to a fiber. Note that $V = T(E_\rho \cap E_\tau) = (H_\rho \oplus V_\rho) \cap (H_\tau \oplus V_\tau)$.
The tangent to $T\mathcal{E}$ is a tangent bundle of a moduli cycle $\mathscr{C}_1$.
Locally $T\mathcal{E} = TE_\rho \oplus TE_\tau$, therefore set-theoretically a possible connection is

$$H = (H_\rho \oplus V_\rho) \oplus (H_\tau \oplus V_\tau)/\{(H_\rho \oplus V_\rho) \cap (H_\tau \oplus V_\tau)\}$$

Theorem 52. *Denote Θ_E a curvature of connection ∇_E. The obstruction to Jordan property - failure for the tangent fields V, W to connection to generate a Lie algebra- is exactly the curvature $\Theta_{\mathcal{E}}$.*

Proof. Our goal, in general, is to find an expression for the curvature tensor Θ of the connection on the vector bundle $\mathcal{E}$. Therefore we would be able to express the second Chern class $c_2(\mathcal{E})$ explicitly, depending on a moduli cycle $\mathscr{S}$:

$$c_2(\mathcal{E}) = \frac{tr(\Theta(\mathscr{S})^2) - tr^2(\Theta(\mathscr{S}))}{8\pi^2} \tag{8.18}$$

Notice that there exist only two "directions" on the base space B: a direction along a curve C_λ and a direction along a curve C, which parameterizes the family.

When we lift vector fields from the base B to a vector bundle, we get the vector fields V, W along the "cycle/section" $\mathscr{C}_1$ and along the curve C. V and W induce diffeomorphisms φ_V, ψ_W on a vector bundle $\mathcal{E}$, which act on particular sections $s \in \Gamma(\mathcal{E})$. If we differentiate the result twice $\partial^2 \varphi_V \psi_W^{-1} \varphi_V^{-1} \psi_W(s)$, we will get $\Theta(s)$ a value of curvature on s. If we take a corresponding basis of sections of $\mathcal{E}$ and lift the basis of vector fields V, W from the base to our cycle C in the moduli space of vector bundles and look how do the diffeomorphisms act on the basis of sections, we would be able to count $\Theta_{\mathcal{E}}$ explicitly. As is well-known in this case, the obstruction to Jordan property - failure for the tangent fields V, W to connection to generate a Lie algebra- is the curvature $\Theta_{\mathcal{E}}$. $\square$

Note that we can use Gauss-Manin connection which acts as $\nabla_{GM} : T\mathscr{C} \to V$ (by the reason that ∇_{GM} is flat), where V, as above, is a tangent space to a fiber.

Recall that by non-abelian Hodge theory, the tangent space at the point (C, U) which is equal to $M_{dR}(C_u)$ is cannonically identified with $H^1_{dR}(C_u, EndE)$.

Therefore a Gauss-Manin connection computes the derivative of our section S. $c_2(E) > 0$, because the corresponding curvature matrix has a block-diagonal form and positive determinant as we will show below in more details.

8.8 Curvature matrix on hermitian vector bundles

8.8.1 Connection on a vector bundle E

Let D be a connection on a vector bundle acting on the local basis of sections $s = (s_1, .., s_n)$. Then, following (Kobayashi, 2014),

$$Ds_i = \sum s_j \omega_i^j,$$

where $\Omega = (\omega_i^j)$ is a matrix of connection form. If $\xi = \xi^i s_i$ is an arbitrary section, then

$$D\xi = d\xi + \omega\xi$$

Recall that $\Omega = d\omega + \omega \wedge \omega$.

A bundle is called flat if $Ds = 0$.

For a complex vector bundle E, a connection is decomposed as $D = D' + D''$, $D' : A^{p,q}(E) \to A^{p+1,q}(E)$, $D'' : A^{p,q}(E) \to A^{p,q+1}(E)$

Thus we can write in local coordinates:

$$\omega = \omega^{0,1} + \omega^{1,0}$$

and

$$\Omega = \Omega^{2,0} + \Omega^{1,1} + \Omega^{0,2}$$

8.8.2 Hermitian metric on E

Recall if we have a vector bundle $E \to M$ consider a Hermitian metric h in E which is a C^∞ field of Hermitian inner products in the fibers of E.

$$h(\xi, \eta) = \bar{h}(\eta, \xi), \xi, \eta \in E_x$$

Let ξ and η be a C^∞-section and $s_U = (s_1, .., s_r)$ is a local frame field of E over U, we set

$$h_{ij} = h(s_i, s_j)$$

and denote $H_U = (h_{i\bar{j}})$ is a positive definite Hermitian matrix at every point of U.
Recall that for a curvature form Ω we write

$$\Omega^i_j = \sum R^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$$

so that

$$R_{j\bar{k}\alpha\bar{\beta}} = \sum h_{i\bar{k}} R^i_{j\alpha\bar{\beta}} = -\partial_{\bar{\beta}} \partial_\alpha h_{j\bar{k}} + \sum h^{a\bar{b}} \partial_\alpha h_{j\bar{b}} \partial_{\bar{\beta}} h_{a\bar{k}},$$

where $\partial_\alpha = \partial / \partial z_\alpha$ and $\partial_{\bar{\beta}} = \partial / \partial \bar{z}_\beta$
We say that holomorphic normal frame field is normal at $x_0 \in M$ if

$$h_{i\bar{j}} = \delta_{ij}$$

$$\omega^i_j = \sum h^{i\bar{k}} d' h_{j\bar{k}} = 0$$

at x_0.
In matrix notation we have $dH = {}^t\omega H + H\bar{\omega}$ and subsequently,

$${}^t\Omega H + H\bar{\Omega} = 0$$

It leads that ω and Ω are skew-symmetric.
So the curvature form is

$$\Omega = d''\omega$$

For the line bundle $detE = \wedge^r E$ we have

$$D(s_1 \wedge .. \wedge s_r) = (\sum \omega^j_i) s_1 \wedge .. \wedge s_r,$$

i.e. the connection form for D in $det(E)$ is given by the trace of ω

$$tr\omega = \sum \omega_i^i,$$

similarly, its' curvature is given by

$$tr\Omega = \sum \Omega_i^i$$

We have a Ricci form

$$tr\Omega = \sum R_{\alpha\bar{\beta}} dz^{\alpha} d\bar{z^{\beta}},$$

where

$$R_{\alpha\bar{\beta}} = \sum R^i_{i\alpha\bar{\beta}} = -\partial_{\alpha}\partial_{\bar{\beta}} det(h i\bar{j})$$

8.9 Curvature of $E \to C_1 \times C_2$

Let $E \to C_1 \times C_2$ is a bundle , which is stable on $C_1 \times \{x\}$ and $C_2 \times \{y\}$ for any $x, y \in C_1, C_2$.

8.9.1 $rkE = 2$

In the case $C_1 \times C_2$ and the bundle is of rank 2 we consider a connection defined by unitary flat connections in both directions. (can define at least set-theoretically) We can take a basis of sections which is given by basis of sections on $E|_{C_1}$ which we denote as (s_1, s_2) and $E|_{C_2}$ (s_1', s_2'), thus a local basis of sections is (s_1, s_2, s_1', s_2').

If we consider a curvature matrix R of second derivatives $d^2 h_{i,k}/dz_i dz_k$ (we differentiate also by $\bar{d}z_i$). As curvature vanishes on $E|_{C_1}$ and on $E|_{C_2}$ we have vanishing of the corresponding second derivatives of $h_{12}, h_{11}, h_{21}, h_{22}$ and $h'_{12}, h'_{11}, h'_{21}, h'_{22}$. Then by hermitian condition in our 4×4 matrix 2×2 diagonal squares are zero and antidiagonal matrices are conjugated. We will act on each h_{ij} by an operator matrix which is 4×4 matrix, schematically depicted as :

$$\partial^2 = \begin{pmatrix} \partial z_1 \partial z_1 & \partial z_1 \partial \bar{z_1} & \partial z_1 \partial z_2 & \partial z_1 \partial \bar{z_2} \\ \partial \bar{z_1} \partial z_1 & \partial \bar{z_1} \partial \bar{z_1} & \partial \bar{z_1} \partial z_2 & \partial \bar{z_1} \partial \bar{z_2} \\ \partial z_2 \partial z_1 & \partial z_2 \partial \bar{z_1} & \partial z_2 \partial z_2 & \partial z_2 \partial \bar{z_2} \\ \partial \bar{z_2} \partial z_1 & \partial \bar{z_2} \partial \bar{z_1} & \partial \bar{z_2} \partial z_2 & \partial \bar{z_2} \partial \bar{z_2} \end{pmatrix}$$

$$\partial^2(h_{ij}) = \begin{pmatrix} 0 & 0 & a_{\bar{i},i} & a_{\bar{i},j} \\ 0 & 0 & a_{\bar{j},i} & a_{\bar{j},j} \\ a_{i,i} & a_{i,j} & 0 & 0 \\ a_{i,j} & a_{j,j} & 0 & 0 \end{pmatrix} \tag{8.19}$$

8.9.2 $rkE > 2$

In the higher rank bundles we have the same situation for some 2-minors. The curvature matrix is expressed through the second derivatives and so the above inequality (if true) should also tell about the class c_2.

As the basis of a local frame field is $s_U = (s_1, .., s_r, s'_1, .., s'_l)$ then analogously on $E|_{C_1}$ and $E|_{C_2}$ a connection D is flat therefore the curvature vanishes and second derivatives of $h_{11}, h_{12}..h_{r1}.., h_{rr}$ and $h'_{11}, h'_{12}..h'_{r1}.., h'_{rr}$ as well. Only in the mixed directions (i.e. s_i, s'_j) the mixed derivatives would not vanish, i.e. $\partial_i \partial_j$, $\partial_i \bar{\partial}_j$ etc. I.e. denote $h_{ij} = h(s_i, s_j)$, $h'_{ij} = h(s_i, s'_j)$,$h''_{ij} = h(s'_i, s'_j)$.

8.10 Ehressman-induced connection on $E \to \mathcal{C}$

Let $\varphi : \mathcal{C} \to C$ be a fibration of curves over a curve. Then we can again form a frame of sections lifting vector fields from a curve C and every curve $\varphi^{-1}(c), c \in \mathcal{C}$. Thus when we are differentiating a section of a bundle E using connection D induced from flat connections from both directions. We have a vector fields along $\varphi^{-1}(c)$ and vector fields along C. As D is flat in both directions, we will get only action of vertical connection along horizontal vector fields and horizontal connection along vertical vector fields.

Thus action $D(s) = \nabla_{\varphi^{-1}(c)}(s^{hor}) + \nabla_C(s^{vert})$. We can consider a matrix of the second derivatives of the metric $h_{i\bar{j}}$. Since the basis of a local frame field $s_U = (s_1, .., s_r, s'_1, .., s'_l)$, where $s = (s_1, .., s_r)$ is a frame field along $E|_{\varphi^{-1}(c)}$ and $s' = (s'_1, .., s'_l)$ is a local frame field along $E|_C$. Thus

$$D(s) = \nabla_{\varphi^{-1}(c)}(s) + \nabla_C(s')$$

Therefore we can form a corresponding matrix of an action of connection D on the local frame section. It is if we will write $D(s) = (\nabla_{\varphi^{-1}(c)} + \nabla_C)(s_1, .., s_r, s'_1, .., s'_r)$. The action of the Ehresmann-induced connection is

$$\begin{cases} Ds_1 = 0s_1 + .. + 0s_r + a_{11}s'_1 + .. + a_{1r}s'_r \\ Ds_2 = 0s_1 + .. + 0s_r + a_{21}s'_1 + .. + a_{2r}s'_r \\ \vdots \\ Ds'_1 = a'_{11}s_1 + .. + a'_{1r}s_r + 0s'_1 + .. + 0s'_r \\ Ds'_2 = a'_{21}s_1 + .. + a'_{2r}s_r + 0s'_1 + .. + 0s'_r \\ \vdots \end{cases}$$

Let us consider the coordinates: $(z_1, \bar{z}_1, z_2, \bar{z}_2)$. The first part of coordinates correspond to coordinates on a curve C and the second on the curve $\varphi^{-1}(c)$.
Thus we have $s_U = (s_1, .., s_r, s'_1, .., s'_r)$. Again denote $h_{ij} = h(s_i, s_j)$, $h'_{ij} = h(s_i, s'_j)$,$h''_{ij} = h(s'_i, s'_j)$.
Notice that
$\partial^2 h_{ij}$ vanishes along directions $(z_1, \bar{z}_1)$ and does not vanish along z_2's
$\partial^2 h'_{ij}$ does not vanish
$\partial^2 h''_{ij}$ vanishes along $(z_2, \bar{z}_2)$ and does not vanish along z's
Again we will act on each h_{ij} by an operator matrix which is 4×4 matrix, schematically :

$$\partial^2 = \begin{pmatrix} \partial z_1 \partial z_1 & \partial z_1 \partial \bar{z}_1 & \partial z_1 \partial z_2 & \partial z_1 \partial \bar{z}_2 \\ \partial \bar{z}_1 \partial z_1 & \partial \bar{z}_1 \partial \bar{z}_1 & \partial \bar{z}_1 \partial z_2 & \partial \bar{z}_1 \partial \bar{z}_2 \\ \partial z_2 \partial z_1 & \partial z_2 \partial \bar{z}_1 & \partial z_2 \partial z_2 & \partial z_2 \partial \bar{z}_2 \\ \partial \bar{z}_2 \partial z_1 & \partial \bar{z}_2 \partial \bar{z}_1 & \partial \bar{z}_2 \partial z_2 & \partial \bar{z}_2 \partial \bar{z}_2 \end{pmatrix}$$

Then the matrix of the second derivatives $\mathcal{D}^2$ of a metric is

$$\mathcal{D}^2 = \begin{pmatrix} 0 & 0 & a_{2,n-1} & a_{1,n-1} \\ 0 & 0 & a_{2,n-1} & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{1,n-1} \\ a_{2,1} & a_{2,2} & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,n-1} & a_{n,n-1} \\ a_{n,1} & a_{n,2} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & 0 & 0 \end{pmatrix}$$

8.11 Calculation of the Second Chern class

Recall that the Dolbeaut representative of the Atiyah class is given by $[\bar{\partial}, \Omega]$, where Ω is a curvature tensor on the vector bundle E. In coordinates, $\Omega^i_j = \sum R^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$, therefore, for $C = \Omega \wedge \Omega = \sum_k (\Omega^i_k \wedge \Omega^k_j)$. As every Ω_{ij} has a form $\Omega^i_j = \sum R^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$, one has

$$
\begin{aligned}
C_{ij} = & \sum \Omega^i_k \wedge \Omega^k_j = \sum R^i_{k\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta \wedge \sum R^k_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta = \\
= & \sum_k \sum_{\alpha,\beta,\gamma,\theta} \sum_{\sigma\in S_4} (-1)^\sigma R^i_{k\sigma(\alpha\bar{\beta})} R^k_{j\sigma(\gamma\bar{\theta})} dz^\alpha \wedge dz^\beta \wedge d\bar{z}^\gamma \wedge d\bar{z}^\theta
\end{aligned}
$$

Denote $(\alpha) = (\alpha, \beta, \gamma, \theta)$ a multindex consisting of 4 variables. Then,

$$
\begin{aligned}
(\Omega^4)_{ii} = \sum_j (\Omega^2)_{ij} \wedge (\Omega^2)_{ji} = \sum_j \sum_{(\alpha)} \Big[\sum_k \sum_{\sigma\in S_4} (-1)^\sigma R^i_{k\sigma'\cdot\sigma(\alpha\bar{\beta})} R^k_{j\sigma'\cdot\sigma(\gamma\bar{\theta})} \Big] \cdot \\
\Big[\sum_k \sum_{\sigma\in S_4} (-1)^\sigma R^j_{k\sigma'\cdot\sigma(\alpha'\bar{\beta}')} R^k_{i\sigma'\cdot\sigma(\gamma'\bar{\theta}')} \Big] dz^\alpha \wedge dz^\beta \wedge d\bar{z}^\gamma \wedge d\bar{z}^\theta
\end{aligned}
$$

Recall that

$$
R_{j\bar{k}\alpha\bar{\beta}} = \sum h_{i\bar{k}} R^i_{j\alpha\bar{\beta}} = -\partial_{\bar{\beta}} \partial_\alpha h_{j\bar{k}} + \sum h^{a\bar{b}} \partial_\alpha h_{j\bar{b}} \partial_{\bar{\beta}} h_{a\bar{k}}
$$

,

Recall that $(h^{\beta\gamma}) = h^{-1}$ and

$$
R^\beta_{i\bar{j}\alpha} = -h^{\beta\gamma} \Big(\partial^2 \frac{h_{\alpha\bar{\gamma}}}{\partial z^i \partial \bar{z}^j} - h^{\delta\mu} \frac{\partial h_{\alpha\mu}}{\partial z^i} \frac{\partial h_{\delta\bar{\gamma}}}{\partial \bar{z}^j} \Big)
$$

And the curvature

$$\Omega = \sum R^{\beta}_{i\bar{j}\alpha} dz^i \wedge d\bar{z}^j \otimes e^{*\alpha} \otimes e_\beta$$

For our expression we would need:

$$R^{j}_{\sigma'(k)\cdot\sigma(\alpha\bar{\beta})} = h^{jk}(-\partial_{\sigma'\cdot\sigma(\bar{\beta})}\partial_{\sigma'\cdot\sigma(\alpha)}h_{i\bar{k}} + \sum h^{a\bar{b}}\partial_{\sigma'\cdot\sigma(\alpha)}h_{i\bar{b}}\partial_{\sigma'\cdot\sigma\beta}h_{a\bar{k}})$$

Then we will have:

$$\begin{aligned}(\Omega^4)_{ii} = \sum_j \sum_\alpha \Big[\sum_k \sum_{\sigma\in S_4} (-1)^\sigma h^{ik}h^{kj}h^{jk}h^{ki}(-\partial_{\sigma(\bar{\beta})}\partial_{\sigma(\alpha)}h_{i\bar{k}} + \sum h^{a\bar{b}}\partial_{\sigma(\alpha)}h_{i\bar{b}}\partial_{\sigma\beta}h_{a\bar{k}})\cdot \\ \cdot(-\partial_{\sigma(\bar{\theta})}\partial_{\sigma(\gamma)}h_{k\bar{j}} + \sum h^{a\bar{b}}\partial_{\sigma(\gamma)}h_{k\bar{b}}\partial_{\sigma\theta}h_{a\bar{j}})\Big] \\ \cdot\Big[\sum_k \sum_{\sigma\in S_4}(-1)^\sigma(-\partial_{\sigma(\bar{\beta})}\partial_{\sigma(\alpha)}h_{j\bar{k}} + \sum h^{a\bar{b}}\partial_{\sigma(\alpha)}h_{j\bar{b}}\partial_{\sigma\beta}h_{a\bar{k}})\cdot \\ \cdot(-\partial_{\sigma(\bar{\theta})}\partial_{\sigma(\gamma)}h_{k\bar{i}} + \sum h^{a\bar{b}}\partial_{\sigma(\gamma)}h_{k\bar{b}}\partial_{\sigma\theta}h_{a\bar{i}})\Big] dz^\alpha \wedge dz^\beta \wedge d\bar{z}^\gamma \wedge d\bar{z}^\theta\end{aligned}$$

8.11.1 Curvature of $E \to \mathcal{C}$

Again, return to the case when $\varphi : \mathcal{C} \to C$ be a fibration of curves over a curve and, using the induced connection D bearing it from $E|_C$ and every curve $E|_{\varphi^{-1}(c)}, c \in \mathcal{C}$ we can differentiate a section of a bundle E. Consider a matrix of the second derivatives of the metric $h_{i\bar{j}}$. Notice that $\partial^2 h_{ij}, \partial^2 h'_{ij}, \partial^2 h''_{ij}$ vanish along $(z_1, \bar{z}_1)$ and nowhere else.

Consequently, we will have

$$\partial^2(h_{ij}) = \begin{pmatrix} 0 & 0 & a_{\bar{i},i} & a_{\bar{i},j} \\ 0 & 0 & a_{\bar{j},i} & a_{\bar{j},j} \\ a_{i,i} & a_{i,j} & * & * \\ a_{i,j} & a_{j,j} & * & * \end{pmatrix} \tag{8.20}$$

8.11.2 Second chern class for a $E \to \mathcal{C}$

To analyze a formula for Ω^4 we have to understand how do the first derivatives ∂h_{ij} involved in the expression

$$\begin{aligned}(\Omega^4)_{ii} = &\sum_j \sum_\alpha [\sum_k \sum_{\sigma \in S_4} (-1)^\sigma (-\partial_{\sigma(\bar{\beta})}\partial_{\sigma(\alpha)} h_{i\bar{k}} + \sum h^{a\bar{b}} \partial_{\sigma(\alpha)} h_{i\bar{b}} \partial_{\sigma\beta} h_{a\bar{k}}) \cdot \\ &\cdot (-\partial_{\sigma(\bar{\theta})}\partial_{\sigma(\gamma)} h_{k\bar{j}} + \sum h^{a\bar{b}} \partial_{\sigma(\gamma)} h_{k\bar{b}} \partial_{\sigma\theta} h_{a\bar{j}})] \\ &\cdot [\sum_k \sum_{\sigma \in S_4} (-1)^\sigma (-\partial_{\sigma(\bar{\beta})}\partial_{\sigma(\alpha)} h_{j\bar{k}} + \sum h^{a\bar{b}} \partial_{\sigma(\alpha)} h_{j\bar{b}} \partial_{\sigma\beta} h_{a\bar{k}}) \cdot \\ &\cdot (-\partial_{\sigma(\bar{\theta})}\partial_{\sigma(\gamma)} h_{k\bar{i}} + \sum h^{a\bar{b}} \partial_{\sigma(\gamma)} h_{k\bar{b}} \partial_{\sigma\theta} h_{a\bar{i}})] dz^\alpha \wedge dz^\beta \wedge d\bar{z}^\gamma \wedge d\bar{z}^\theta\end{aligned}$$

behave in this case. It is clear that $\partial_{z_1} h_{ij} = 0, \partial_{\bar{z_1}} h_{ij} = 0$, meanwhile the derivatives from the other direction may not be zero (i.e. $\partial_z h_{ij}$ is not zero, $z \in C$). If we look at the expression above, it could be deduced that the first derivatives are included in the expression $\sum h^{a\bar{b}} \partial_{\partial_{z_i}} h_{k\bar{b}} \partial_{\bar{z_j}} h_{a\bar{i}}$ in pairs (so that at least one of them is always zero as the connection is flat in one direction and hence the first derivative is zero) therefore the terms involving it go to zero and what lasts are the only expressions for the second derivatives. If we permute the columns or rows of the matrix , while working with the expression for Ω^4, the essential form will stay the same. This way we will get an expression

$$\kappa = \sum_i (\Omega^4)_{ii} = \sum_{i,j,\alpha,\sigma \in S_4} \partial_{\sigma(\bar{\beta})}\partial_{\sigma(\alpha)} h_{i\bar{k}} \cdot \partial_{\sigma(\bar{\beta})}\partial_{\sigma(\alpha)} h_{j\bar{k}} \cdot$$

$$\cdot \partial_{\sigma(\bar{\theta})}\partial_{\sigma(\gamma)} h_{k\bar{j}} \cdot \partial_{\sigma(\bar{\beta})}\partial_{\sigma(\alpha)} h_{j\bar{k}} dz^\alpha \wedge dz^\beta \wedge d\bar{z}^\gamma \wedge d\bar{z}^\theta =$$

$$= \sum_{i,j} det(\partial^2 h_{ij}) dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2$$

If all the expressions with the first derivatives go to zero, we will get:

$$\partial^2(h_{ij}) = \begin{pmatrix} 0 & 0 & \bar{a_{i,i}} & \bar{a_{i,j}} \\ 0 & 0 & \bar{a_{j,i}} & \bar{a_{j,j}} \\ a_{i,i} & a_{i,j} & * & * \\ a_{i,j} & a_{j,j} & * & * \end{pmatrix} \tag{8.21}$$

We denote by $*$ the elements of derivatives of metric h_{ij} not in the mixed directions. Denote the matrices corresponding to the mixed directions as

$$D_{ij} = \begin{pmatrix} a_{i,i} & a_{i,j} \\ a_{i,j} & a_{j,j} \end{pmatrix}$$

and $\bar{D}_{ij}$ correspondingly. In general, for $E \to \mathcal{C}$, $det(\partial^2(h_{ij})) = det(D_{ij})det(\bar{D}_{ij})$ (in the second case the additional minor doesn't influence on the expression of determinant by the Linear Algebra rules).

Theorem 53. *The resulting formula for $E \to \mathcal{C}$ is*

$$c_2(E) = \sum_i (\Omega^4)_{ii} = \sum_{i,j} \frac{det(\partial^2 h_{ij})}{det(h_{ij})} dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 = \sum_{i,j} \frac{det(D_{ij})det(\bar{D}_{ij})}{det(h_{ij})} dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2$$

Thus from the last theorem follows the stability result:

Corollary 15. *As we have for* $E \to \mathcal{C}$

$$c_2(E) = \sum_{i,j} \frac{det(D_{ij})det(\bar{D}_{ij})}{det(h_{ij})} dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 \tag{8.22}$$

therefore,

$$c_2(E) > 0 \tag{8.23}$$

Remark 5. *Local positivity of* c_2*-form holds outside of singular points and in fact, the impact of singular point depends on the dilation (distance on symmetric space* $GL(n, C)/R^*U(nC)$*) –two points corresponding to unitary structures from different branches of the singular curve.*

9 | Results: Stability for vector bundles with singularities

9.1 Bott-Baum argument

Recall that if we have a sequence

$$0 \to \underline{L} \to \underline{T} \to Q \to 0$$

where $\underline{L}$ is a space of sections of a line bundle L on X, $\underline{T}$ is a space of sections of a tangent bundle to X and $Q = \underline{T}/\xi$, where $\xi = \underline{T}/\underline{\eta}(\underline{L})$, where $\underline{\eta} : \underline{L} \to \underline{T}$. It implies that $c_i(Q) = c_i(T - L)$, $i = 1, .., n$.
Therefore the following classical formula holds true

$$\phi(Q)[M] = \sum_{p \in Zero(\eta)} \phi(\eta, p) = \sum_{p \in Zero(\eta)} Res_\phi(\xi, p)$$

So we can conclude that the formula above computes the Chern numbers of Q in terms of local information at the singularities of the foliation. Indeed, written in a slightly different way, the following holds:

$$\phi(Q) = \sum_Z \mu_* Res_\phi(\xi, Z),$$

where $\mu_* : H_j(Z; C) \to H^{2n-j}(M, C)$ a map, induced by the inclusion of a singular set $Z \hookrightarrow M$ and the classical Poincare duality map on the homologies, $\phi(Q)$ is a given symmetric polynomial of the Chern classes.

Bott and Baum had also noticed that one can think of

$$Res_\phi(\xi, Z) = \sum_{i=1}^{s} \#(\phi, \xi, Z_i)[Z_i]$$

as a sum of quantities $\#(\phi, \xi, Z_i) = Res_p \begin{pmatrix} \phi(A)dz_k..dz_n \\ a_k, ..., a_n \end{pmatrix}$,

which are not difficult to count in the local coordinates.

Let E be a vector bundle on X of degree d and rank n. Recall that in our case the restriction of a bundle $E|_{C_t}$ to any curve $C_t \in X^b$ is stable, then, as it is well-known, we have a flat unitary connection $\nabla_{E|_{C_t}}$ along the fibers C_t in X^b. Therefore we have a family of such "fiberwise" connections $\{\nabla_{E|_{C_t}}\} := T_E$. Also, we can find projective connections along the singular fibers in this natural way: we can assume the existence of connections on the smooth normalized curves X'_s with a singularity for a corresponding connection ∇_{X_s} over the points $x_s \in X^{sing}$. They are called parabolic connections. They are constructed in the following way: if we have a non-singular curve with points x_i corresponding to the singularities we can define λ-connections(logarithmic connections), which have the poles along a divisor $D = \sum x_i$. We can take the residues along these singular points and put an order on it. They should satisfy in particular Fuchs relation. Then we can define Parabolic connections:

1) taking logarithmic connections singular over X^{sing}, i.e. a $\mathbb{C}$-linear maps

$$D : E \to E \otimes \Omega^1_X(logX^{sing}) = E \otimes \Omega^1_X \otimes \mathcal{O}_X(X^{sing}),$$

which satisfy Leibniz identity.

The fiber $\Omega^1_X \otimes \mathcal{O}_X(X^{sing})(x_\beta)$ is canonically identified with $\mathbb{C}$ by sending a meromorphic form to its residue at x_β. In particular, $ResD(x_\beta) \in End(E)(x_\beta)$

2) forming a full flag of subspaces on each fiber of $E|_{x_i}$,

3)Let D be a logarithmic connection on X. Let $Res(D, x_i) = \lambda_i Id_{E(x_i)}, x_i \in X^{sing}$, $i = 1, ..., m$. Then it follows that

$$d + n \sum_{j=1}^{m} \lambda_i = 0$$

Lemma 54. *Smooth deformation between the Higgs bundles and connections is given by the deformation of the λ-connections.*

Additionally, if we restrict the set of connections to some subspace from a flag, we will get a partial connection.
A consequence of the lemma is that instead of studying a bundle E itself we can restrict our studies to a space of λ-connections. Thus the foliation $P(T_E)$ on the projectivization $\mathbb{P}(E)$ over X acquires a standard singular points.
Put $T = T_E$ and let L be the one-dimensional torsion-free subsheaf, such that the morphism $\eta : L \to T_E$ has zeroes at the singular points X^{sing}.

Although the constructed foliation is not holomorphic but the presence of the standard singular points provides us the following general Bott-Baum formulas (Baum & Bott, 1972) of the Δ-class of E:

Theorem 55. *As $c_1(E) = 0$,*

$$c_2(E) = c_2(\mathbb{P}(T_E)) = \Delta = \sum_p Res_p \begin{pmatrix} \phi(A) dz_k..dz_n \\ a_k, ..., a_n \end{pmatrix} - F^2 = N - F^2$$

Since $c_1(F) > 0$, it implies that $c_2(E) = N - F^2$.

If the number of points N grows fast and F is small we get that $c_2 > 0$, i.e. $N > F^2$.

9.2 Stability of Projective varieties

Let Y be a smooth projective variety with trivial canonical class.

Without loss of generality, we can consider a fibration $B \to Y$, where B, as above, is a family of curves, embedded to Y.

(Bogomolov & de Oliveira, 2005; Bogomolov, Cascini, & de Oliveira, 2006) The line of arguments could be repeated identically as in the case of smooth projective complex surfaces.

10 | Bounds for a curves on the surfaces of general type

10.1 Preliminaries

Consider a families of curves of a fixed genus g, lying on the surface of general type (Bogomolov, 1977). Suppose that the rank of Picard group of a surface $V^2 \geqslant 0, K^2 \geqslant 0$ (i.e. the canonical class is very ample). Let Δ_P be a smooth curve of genus g and $f : \Delta_P \to V^2$ a regular map which induces an epimorphism of function fields: $f^* : C(V^2) \to C(\Delta_P)$. Let us define a subcone $K_D \subset Pic(V^2_R)$ in the cone of effective divisors

$$K^C_{D,\epsilon} = \{x \in K_D, x^2 \geq \frac{\alpha - \epsilon}{K^2}(K.\Delta_P)^2, \alpha = \frac{3K^2}{4\chi - K^2}\}, \epsilon \geq 0$$

The following theorem holds true:

Theorem 56. ***(Bogomolov's finitness theorem, (Bogomolov, 1977))*** *The set of genus g curves such that the image of their fundamental cycle $f(\Delta_P) \notin K^C_{D,\epsilon}$ form an algebraic family. In particular, the number of classes such that $f(\Delta_P) \in PicV^2_R \setminus K^C_{D,\epsilon}$ for the curves of genus g is finite.*

Denote $h_L(P) = \frac{\alpha-\epsilon}{K^2}(K.\Delta_P)^2$ Let us reformulate a theorem in a more convenient form:

Theorem 57. ***(Bogomolov's finitness theorem, (Bogomolov, 1977))*** *For except a finite number of curves, there is a linear estimate from below for a genus of the curve inside each cone:*

$$h_L(P) \leq A \cdot d(P) + O(1)$$

10.1.1 A few details of the proof of Bogomolov's finitness theorem

Recall that the proof of Theorem(10.1.2) is based on the properties of symmetric differential $s \in H^0(V^2, S^i\Omega^1 \otimes iF)$, $F \subset PicV$ is a one-dimensional fibration. Note that $H^0(V^2, S^i\Omega^1 \otimes iF) = H^0(\mathbb{P}(T), iD+iF)$, D is a Grothendieck fibration on $\mathbb{P}(T)$. It means that there exists a sheaf D on $\mathbb{P}(T)$ whose sections coincide with the sections of $S^i\Omega^1$. For a big i the map $\nu_i : \mathbb{P}(T) \to \mathbb{P}^N$ given by linear series $|iD + iF|$ is a birational embedding (Iitaka, 1971). The main idea is to observe a set $\{s(x) = 0\} := \mathcal{S}$ as a surface in $\mathbb{P}(T)$. For a curve X, a map $tf :$ $X \to \mathbb{P}(T)$. One can take a composition $\nu_i tf : X \to \mathbb{P}^N$. Let $F^2 < 0$, $KF > 0$; Let us take such curves that $FX < 0$; They lie in a subcone $\{{K_D}^F \subset K_D$, $FX \leq 0\}$. Let $K^2 - \chi + 3KF + 3F^2 > 0$ and $H^0(V^2, S^i\Omega^1 \otimes iF) \sim ci^3$ (shows how the number of symmetric tensors grows). In this case the base subvariety of ν_i is defined $B_F \subset \mathbb{P}(T)$. We say X is F-regular if $tf(X) \nsubseteq B_F$.

Lemma 58. *F-regular curves of genus g form an algebraic family.*

Its' proof first uses that the divisor of intersection $(iD + iF) \cap tf(X)|_X$ on a curve X coincides with zero divisor of the symmetric tensor $s' \in H^0(S^iK(X) \otimes iF)$ which takes values in F and appears as a restriction of $s \in S^i\Omega^1 \otimes iF$ to a curve X, i.e. $\mathcal{S}$. As $FX \leq 0$, the degree of $s' \leq 2ig(x)$. By this reason, the degree of a curve $\nu_i tf(x) \leq 2ig$ but the curves of the bounded degree form an algebraic family in $\mathbb{P}^N$. Notice that the bound depends on F (which could be not a sheaf apriori). The condition $X \subset \nu_i P(T)$ is given by some number of algebraic conditions. Curves of the small genus satisfy algebraic differential equations. In our case we have one-dimensional families therefore the equations are of the first order. In general case, one gets differential equations of the higher order. It is reasonable to study curves as subvarieties of $\mathbb{P}(T)$. Let us recall some properties of the base manifold for ν_i: $B_F = \cup_i B_i$. Let p denote a projection $p : \mathbb{P}(T) \to V$. Let p denote a projection $p : \mathbb{P}(T) \to V$. As $ptf(X) = f(X)$, the curves whose image under the tangent map lie in one of the components $B_i \subset B_F$, $p(B_i)$ is a proper subvariety of V, constitute a finite set.

$$\begin{tikzcd}
\mathbb{P}^N & & \\
\mathbb{P}(T) \arrow[u, "\nu_i(F)"'] \arrow[rr, "p"] & & V \\
& X \arrow[ul, "tf(X)"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Now let $p(B_i) = V$ and $tf(X) \subset B_j$. The component $B_j \subset \mathbb{P}(T)$ corresponds to a one-dimensional field of directions in $T(B_j)$ if $tf(X) \subset B_j$, then $f(X)$ is the image of the curve on B_j tangent to the field of directions B_j on B_j under the projection $p : B_j \to V$. Thus, F-irregular curves are images of integral curves of a finite set of foliations on branched covers of $B_j \to V$. Further, we will denote $\tau(C) := tf(X)$.

10.1.2 Bogomolov's finitness theorem and result of Moriwaki

Our goal is to generalize result of Moriwaki using Bogomolov's finitness theorem and Brunebarbe- Klingler-Totaro's theorem. Let us recall one more time both theorems:

Theorem 59. *(**Bogomolov's finitness theorem**, (Bogomolov, 1977)) If the canonical class of X is very ample, for except a finite number of curves, there is a linear estimate from below for a genus of the curve inside each cone:*

$$h_L(P) \leq A \cdot d(P) + O(1)$$

Theorem 60. *(Moriwaki, 1995) If the canonical class of X is very ample, the following holds:*

$$h_L(P) \leq A \cdot d(P) + O(1)$$

10.1.3 Moriwaki's model of the projective variety

Here we are recalling a number of facts due to A. Moriwaki (Moriwaki, 1995). Let X be a geometrically irreducible smooth projective variety over F. Let X and C be smooth projective varieties over k, and $f : \mathcal{X} \to C$ a k-morphism such that the function field of C is F and the generic fiber of f is X, i.e. $\mathcal{X} = X \otimes F$. X is said to be non-isotrivial if there is a non-empty open set C_0 of C such that, for all $t \in C_0$, the Kodaira-Spencer map is not zero. Let $\bar{F}$ be the algebraic closure of F. For a point $P \in X(F)$, let us denote by Δ_P the corresponding integral curve on X . We fix a line bundle L on X. Let $\mathscr{L}$ be a line bundle on $\mathcal{X}$ with $\mathscr{L} \otimes F = L$.

Definition 20. *The pair $(f : \mathcal{X} \to C, \mathscr{L})$ is called a model of (X, L). A geometric height $h_L(P)$ of P with respect to L is defined by*

$$h_L(P) = \frac{(\mathscr{L} \cdot \Delta_P)}{[F(P) : F]}$$

Define a geometric logarithmic discriminant as

$$d(P) = \frac{2g(\Delta_P^{\sim}) - 2}{[F(P) : F]}$$

where $\Delta_P^{\sim}$ is a normalization of Δ_P.

Theorem 61. *(Moriwaki, 1995) If the cotangent bundle to X is very ample, the* *following holds:*

$$h_L(P) \leq A \cdot d(P) + O(1)$$

10.1.4 Brunebarbe- Klingler-Totaro theorem

It was reasonable to ask whether a smooth complex projective variety X with infinite fundamental group $\pi_1(X)$ must have a nonzero symmetric differential, meaning that $H^0(X, S^i\Omega^1 X) \neq 0$ for some $i > 0$. The following theorem gives the answer to this question.

Theorem 62. *(Brunebarbe et al., 2013) Let X be a compact Kähler manifold. Suppose that there is a finite-dimensional representation of $\pi_1(X)$ over some field with infinite image. Then X has a nonzero symmetric differential.*

All known varieties with infinite fundamental group have a finite-dimensional complex representation with infinite image, and so the theorem applies to them. Depending on what we know about the representation, the proof gives more precise lower bounds on the ring of symmetric differentials. The theorem is highly non-trivial, because there are many varieties X of general type which have many sections of the line bundles $K_X^{\otimes j}$, so of the bundles $(\Omega_X^1)^{\otimes i}$ which have no symmetric differentials. For example, Schneider showed that a smooth subvariety $X \subset \mathbb{P}^N$ with $dim(X) > N/2$ has no symmetric differentials. Most such varieties are of general type.

10.2 Proof of the bound

By result of Brunebarbe-Klinger-Totaro there are nontrivial symmetric tensors on a surface X. If the zero set of symmetric tensors defines a rational map which is an embedding at general point $\psi : \mathbb{P}^1(T(X)) \to \mathbb{P}^m$ then the curves of degree

$\tau(C) = d$ have geometric genus $\leq 2d - 2$. Thus for exceptional curves $\tau(C)$ are tangent to a zero subset $S \subset \mathbb{P}^1(T)$. Thus the curves with this property constitute either a one-dimensional family (of the same geometric genus) + a ψ-finite number of curves. So we have Moriwaki estimate in this case.

If the map $\psi : \mathbb{P}^1(T(X)) \to \mathbb{P}^m$ has generically one-dimensional fiber then there are two options:

1) ψ is induced from $X \to C$ and so there is a one-dimensional subbundle $F \subset \Omega^1$ with many sections in F^n.(on further slides more details). Thus except from one-dimensional family of curves with trivial restriction of F we have a linear bound on genus with respect to the degree (F, C)

2) If the map ψ is not induced from X then the images of different curves are different under tau (except for a one-dimensional family) (If C_t is one-dimensional family of curves in X then the map ψ is nontrivial on C_t and so curves have bounded genus on the whole family by the bound on generic member.)

If the image of ψ is a curve C then if genus of C is > 1then it is clear. If genus is smaller then there is family of surfaces $S_t \subset P(T)$ which are preimages of points in C. Then for any such surface which surjects on X we obtain a one-dimensional family for each S_t. Each S_t provides at least one-dimensional family. Thus we may have either a two dimensional family of curves of the same genus tangent to S_t or a discrete set of one-dimensional families.

11 | Deformations of surfaces of general type

11.1 Preliminaries: Almost complex structures and symmetric tensors on cotangent bundles

11.1.1 Kuranishi map and almost complex structures

We follow mainly (Hironaka et al., 2013) and (Kodaira & Spencer, 1958b, 1958a; Huybrechts, 2005; Arbarello & Cornalba, 2009).

Definition 21. *A complex structure on a manifold M is a collection of complex-valued charts $\Phi := \{\varphi_\alpha : U_\alpha \to U'_\alpha\}$, where φ_α is a diffeomorphism and $U_\alpha \subset M, U \subset \mathbf{C}^N$ such that*

1. *For $\varphi_\alpha, \varphi_\beta \in \Phi$ their composition $\varphi_\alpha \circ \varphi_\beta$ is complex analytic*
2. *The union of the domains of charts in $\bigcup U_\alpha = M$*
3. *If u is a complex valued chart of M such that $u \circ {\varphi_\alpha}^{-1}$ is complex analytic for all φ_α in Φ, then $u \in \Phi$.*

Recall that $\mathbf{C}TM = TM \oplus iT_xM$. Denoting by $T''M_x$ the set of all $L \in \mathbf{C}T_xM$: $Lf = 0, f \in F_x$ which annihilate all germs of holomorphic functions on neighborhoods of x, we can say that $\mathbf{C}T_xM = T'_xM \oplus T''_xM$, where T' and T'' are complex conjugate to each other.

We define families of analytic charts over the parameter spaces of analytic sets. More concretely,

Definition 22. *Let S be an analytic subset, our parameter set for families of a complex charts. Let $S' \subset S$. Then, for a $U \subset M$, by a family of complex charts* *on M we mean a map*

$$\varkappa : U \times S' \to \mathbf{C}^N,$$

where $\varkappa(s)$ is a complex analytic chart for M for any fixed $s \in S'$.

A collection of families of complex analytic charts, satisfying Properties (21) is called *complex analytic families of complex structures on M.*

An isomorphism of two complex structures of M is a diffeomorphism which sends charts of one structure to those of another. Hence it is natural to define an isomorphism of families as a family of diffeomorphisms satisfying conditions we naturally expect. To be more precise, let $\hat{\theta}$ and $\hat{\varphi}$ be complex analytic families over an analytic set B. Denote by f a family of diffeomorphisms of M parametrized by B $\{f^b : b \in B\}$.

In the following we fix a complex structure M on $\mathbf{M}$ and we are interested in how we can so to speak deform M locally. To be more precise, by "deforming M" we mean that we consider an analytic set S together with a reference point s_0 and a complex analytic family $\mathcal{J}$ of complex structures on $\mathbf{M}$ over S such that the complex structure in $\mathcal{J}$ over s_0 is equal to the given M. In this case we say that $\mathcal{J}$, is a complex analytic family of deformations of M with a parameter in (S, s_0) (or, over (S, s_0)). By "deforming locally M" we mean that we consider the germ of such $\mathcal{J}_s$ over neighborhoods of reference points. Our main aim is to construct a complex analytic family of deformations of M which is complete.

Definition 23. *Let M'' be an almost complex structure on $\mathbf{M}$. We say that it is of finite distance from a complex structure M on $\mathbf{M}$ when $\rho'' : \mathbf{C}TM \to T''M$, such that $ker \rho'' = T'$ induces an isomorphism of T'' to $T''M$.*

Let $M := \{M_t\}_{t \in T}$ be a family of compact complex manifolds, parametrized by the set T. For every $J_t : t \in T \rightsquigarrow M_t$ we have an assignment of a complex analytic structure. For a $\tau : S \to T$ with $\tau(s_0) = t_0$ we associate an assignment $s \to M_{\tau(s)}$ which is a family of deformations of M. So we have germs of deformations of M at the reference points. Note that 1-dimensional cohomology group with coefficients in the sheaf of germs of holomorphic vector fields does not change on a neighborhood of the reference point.

Let us represent the complex analytic structure on M by differential forms

$$\begin{array}{ccc} T'' & \xrightarrow[\omega_{M'}]{} & T' \\ \downarrow & & \downarrow \\ M' & & M \end{array}$$

where $T''(M') = \{-\omega_{M'}(X) + X : X \in T''\}$. Define a set
$A^p = \{$set of C^∞- diff. forms of type $(0,p)$ of complex analytic structures M with values in $T'\}$. We have an operator $\bar\partial : A^p \to A^{p+1}, \phi \in A^p, \psi \in A^q$. Consider a complex-analytic local coordinates $(z^1, .., z^n)$ of M and express $\phi = X_{\alpha_1 \ldots \alpha_p} d\bar z^{\alpha_1} \ldots d\bar z^{\alpha_p}$ and $\psi = Y_{\beta_1 \ldots \beta_p} d\bar z^{\beta_1} \ldots d\bar z^{\beta_p}$. $X_{\alpha_1 \ldots \alpha_p}$ and $Y_{\beta_1 \ldots \beta_p}$ are complex vector fields of type $(0,1)$, antisymmetric w.r.t. indices. Locally, one has $[\varphi, \psi]_{loc} := [X_{\alpha_1 \ldots \alpha_p}, Y_{\beta_1 \ldots \beta_p}] d\bar z^{\alpha_1} \ldots d\bar z^{\alpha_p} d\bar z^{\beta_1} \ldots d\bar z^{\beta_p}$

Let M' be a complex structure such that $T''_x(M')$ is very close to T''_x so the C^∞-fiber mapping $\omega_{M'} : T'' \to T', T''(M') = \{-\omega_{M'}(X) + X : X \in T''\}$. Therefore, M' is completely determined by $\omega_{M'}$. We say that M' *has finite distance to* M when we can find $\omega_{M'}$ as above.

Definition 24. *An almost complex structure M_ω is integrable $\Leftrightarrow$ ω satisfies the PDE: $\bar\partial\omega - [\omega, \omega] = 0$*

Almost complex structure is integrable, when it comes from a complex analytic structure. Set of almost complex structures on **M** having finite distance to M $\leftrightarrow^{1-1}$ with a set of fiber mappings $\varphi : T'' \to T'$.

A^p is a space of C^∞-differential forms of type $(0,p)$ of the complex analytic structure M with values in T'. Hence, A^1 is a space of fiber mappings $T'' \to T'$, thence the set of almost complex structures on $\mathbf{M}$ having finite distance to M is indexed by elements of A^1.

Given any linear map A on each tangent space of M i.e., A is a tensor field of rank $(1,2)$ then the Nijenhuis tensor is a tensor field of rank $(1,2)$ given by

$$N_A(X,Y) = -A^2[X,Y] + A([AX,Y] + [X,AY]) - [AX,AY],$$

or, for the usual case of an almost complex structure $A = J$ such that $J^2 = -Id$,

$$N_J(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY]$$

The *Newlander-Nirenberg theorem* states that an almost complex structure J is integrable if and only if $N_J = 0$. The compatible complex structure is unique. Since the existence of an integrable almost complex structure is equivalent to the existence of a complex structure, this is sometimes taken as the definition of a complex structure.

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Let $f : \mathbf{M} \to \mathbf{M}$ be a diffeomorphism. Take $\omega \in A^1$. Then there exists unique complex structure M' on $\mathbf{M}$ s.t. f induces an isomorphism $f : M' \to M_\omega$. M' has again a finite distance to M. There exists a unique $\theta \in A^1 : M' = M_\theta$. In this case, we set $\theta = \omega \circ f$. For a given ω it is clear that $\omega \circ f$ is defined for f sufficiently close (in the 1-jets topology) to the identity mapping.
Let U be a domain of a local corrdinate (z). Let $f(U_1) \subset U$ for an open $U_1 \subset U$.

Then $\theta = \omega \circ f$ is given by

$$\frac{\partial f^\alpha}{\partial \bar{z^\beta}} + {\omega_\gamma}^\alpha(f(z)\Big(\frac{\partial \bar{f}^\nu}{\partial \bar{z}^\beta}\Big) = \Big(\frac{\partial f^\alpha}{\partial \bar{z^\delta}} + {\omega_\gamma}^\alpha(f(z)\Big)\Big(\frac{\partial \bar{f}^\nu}{\partial \bar{z}^\delta}\Big) \cdot {\theta_\beta}^\delta(z),$$

where $\omega = X_\alpha d\bar{z}^\alpha$, $X_\alpha = {\omega_\alpha}^\beta(z)\frac{\partial}{\partial z^\beta}$ and similarly for θ.
$\omega \circ f$ is defined when the matrix $\frac{\partial f^\alpha}{\partial \bar{z^\beta}} + {\omega_\gamma}^\alpha(f(z))\Big(\frac{\partial \bar{f}^\nu}{\partial \bar{z}^\beta}\Big)$ is non-singular. Let us index f near the identity mapping by elements in A^0, i.e. complex vector fields of type $(1, 0)$. Let X be a vector field on M and $(e'(X))(x)$ be the end point of the geodesic from x moved by the initial velocity X_x after the time interval 1.
$e'(X) : M \to M$ is a C^∞- geodesic mapping. If X is sufficiently small , then $e'(X)$ is a diffeomorphism of $\mathbf{M}$. For a $\xi \in A^0 \rightsquigarrow \xi + \iota(\xi)$ is a real vector field on $\mathbf{M}$.
$e(\xi)(x) = B(\xi_x)$, and say $B : T \to M$. In terms of local coordinates, $\Big(\frac{\partial B^\alpha(\xi)}{\partial \xi^\beta}\Big)|_{\xi=0} = {\delta_\beta}^\alpha$, $\Big(\frac{\partial B^\alpha(\xi)}{\partial \xi^\beta}\Big)|_{\xi=0} = {\delta_\beta}^\alpha \rightsquigarrow \omega \circ e(\xi) = \omega + \bar{\partial}\xi + R(\omega, \xi)$, where $R(t\omega, t\xi) = t^2 R(\omega, \xi, t)$.

Lemma 63. *Given a 1-parameter group of diffeomorphisms* $\{\psi_t := exp(t(\theta + \bar{\theta}))|t \in \mathbb{R}\}$, $(\frac{d}{dt})_{t=0}({\psi_t}^*(J_0))$ *corresponds to the small variation* $\bar{\partial}(\theta)$*.*

$Diff^0(M) \curvearrowright \{M\}$ on the set of complex structures. Locally, at each point J the orbit of diffeomorphisms in $Diff^0(M)$ contains a submanifold having a tangent space, consisting of forms $TS \subset TDiff^0(M)J$, $TS :=$ forms of the Dolbeaut cohomology class of 0, $TS \subset TR$ (TS has a finite codimension in TR), $TR :=$ space of $\bar{\partial}$-closed forms ϕ.
Set $F(\phi) = (\bar{\partial} + \frac{1}{2}[\phi, \phi])$. F is a map of degree 2 between ∞-dimensional spaces: the space of $(0, 1)$-forms with values in the bundle $TM^{1,0}$ and the space of $(0, 2)$-forms with values in $TM^{1,0}$. Note that $F|_{\{\bar{\partial}(\phi)=0\}}$ takes values in the space of $\bar{\partial}$-closed forms. Therefore, we can reduce an equation $\{F = 0\} \rightsquigarrow$ to an equation

$k = 0$, for a map

$$k : H^1(\Theta_Y) \to H^2(\Theta_Y),$$

called the *Kuranishi map*, which is a finite-dimensional problem.

Theorem 64. *(Kuranishi) see (Catanese, 2011) Let X be a compact complex manifold. Then:*

1. *The Kuranishi family $\pi : (\mathbf{X}, X_0) \to (B(X), 0)$ is semiuniversal;*
2. *$(B(X), 0)$ is unique up to isomorphism and as a germ of analytic subspace of the v.s. $H^1(Y, \Theta_Y)$ inverse image of the origin under Kuranishi map k, whose differential vanishes at the origin. A quadratic term in the Taylor development of the Kuranishi map is given by a bilinear map $H^1(Y, \Theta_Y) \times H^1(Y, \Theta_Y) \to H^2(Y, \Theta_Y)$, called the Schouten bracket (Schouten, 1954), which is the composition of the cup product followed by Lie bracket of vector fields.*
3. *The Kuranishi family is a versal deformation of Y_t for $t \in B(X)$.*
4. *The Kuranishi family is a universal if $B(M)$ is reduced and $h^0(X_t, \Theta_t) := dim H^0(X_t, \Theta_{Y_t})$ is constant in a suitable neighborhood of 0.*

11.1.2 Bogomolov subsheaves and Bogomolov-Miyaoka-Yau inequality

Denote $T(M^n) = T$ the tangent bundle to M^n. Let $\mathcal{F}^k \subset \mathcal{O}(T)$ be a k-dimensional coherent subsheaf of $\mathcal{O}(T)$. An embedding $r : \mathcal{F}^k \hookrightarrow \mathcal{O}(T) \rightsquigarrow$ a map of one-dimensional bundles $r_{(k)} : det\mathcal{F}^k \to \bigwedge^k T$. If one uses the isomorphism $\bigwedge^{n-1} \Omega^1 \otimes -K \simeq T \rightsquigarrow$ a map $r_{(k)}^{\odot} : det\mathcal{F}^k \otimes -kK \to \Omega^{n-k}$. This way, one can study subbundles in $T(M^n)$ just considering one-dimensional subbundles in

$\Omega^i(M^n)$.

Theorem 65. *Let M^n be a projective variety, $\Omega^i = \Omega^i(M^n)$, the i^{th} wedge power of the cotangent bundle $\Omega^1 = \Omega^1(M^n)$, $L \to M^n$ be the one-dimensional bundle on M^n and $h : L \to \Omega^i$- a non-trivial homomorphism. Then $\exists$ constants c_M^L, β such that*

$$dimH^0(M^n, sL) < c_M^L s^i + \beta, \forall s > 0$$

Its' proof will be interesting for us. It splits into a few Lemmas.

Lemma 66. *(Induction on dimension of a manifold) Suppose the theorem holds for $\forall M$, s.t. $dimM \leq (i+1)$, then it holds for projective variety X of any dimension.*

Proof. Suppose that for $dimM \leq (i+1)$ the theorem holds. Now consider a projective variety of dimension $n = dimM > i+1$ and that the theorem does not hold for M^n. Let us find a projective projective variety V s.t. $dimV = i+1$, for those the theorem does not hold.

Consider a graded canonical ring associated to M : $\sum\limits_{s>0} H^0(M^n, sE) \rightsquigarrow V^r$ a projective variety. By Iitaka (Iitaka, 1971), $\exists$ a rational epimorphism $h : M^n \to V^r$, whose fiber $h^{-1}(v)$ is a connected variety.

Let us build up a projective subvariety $V^{i+1} \subset M^n$ s.t.

1. V^{i+1} doesn't lie $V^{i+1} \not\subseteq B_h$ in the base subvariety of a map $h : M^n \to V^r$;
2. The dimension of an image $dimh(V^{i+1} \setminus (V^{i+1} \cap B_h)) = i+1$;
3. The composition of the maps $j^* r^*$

$$r^* : L \to \Omega^i(M^n), j^* : \Omega^i(M^n) \to \Omega^i(V^{i+1})$$

is non-trivial.

For this, let us consider an arbitrary point $x \in M^n \setminus B_h \rightsquigarrow$ a map $r_x : L_x \to \Omega^i(M) \neq 0, rkdh_x = r$. Consider a subspace $Q_x \subset T_x(M^n), dimQ_x = (i+1), Q_x \pitchfork kerdh_x$ and $j_* r_* : L_x \to \bigwedge^i Q^*$ is a non-zero map. This condition is satisfied by almost all $(i+1)-$dimensional subspaces of T_x.

Consider an arbitrary submanifold $V^{i+1} \subset_i M^n$ s.t. $x \in V^{i+1}$ is non-special and $i_* T_x(V^{i+1}) = Q_x$. Then the manofold V^{i+1} satisfies the conditions (11.1.2). Note that as $dimh(V^{i+1}) = i+1$, then $dimH^0(V^{i+1}, sE) > \alpha s^{i+1} + \beta$ for some $\alpha > 0$, and, therefore, if Lemma (66) does not hold for $M^n \Rightarrow$ it does not hold for V^{i+1} as well.

$\square$

Let now $dimM = n+1$. Then the subsheaf $\mathcal{O}(L) \subset \Omega^i(M^{i+1})$ induces a one-dimensional foliation $\mathscr{L}_s$ on M^{i+1}. In more details, from the iso $\Omega^i(M^{i+1}) \simeq T(M^{i+1}) \otimes K \rightsquigarrow$ the homomorphism $r : L \to \Omega^i(M^{i+1})$ corresponds to one-dimensional subsheaf $r(L \otimes -K) \subset T(M^{i+1})$. It is an integrable subsheaf if M^{i+1} is a complex projective manifold (in this case vector fields are always integrable). Thus L corresponds to the one-dimensional foliation $\mathscr{L}_s$. It has special points on the submanofold $V_s, codimV_s \geqslant 2$.

In a neighborhood of a point x, where $\mathscr{L}_s$ is non-special, one can write a foliation as $(z_1 = c_1, .., z_i = c_i)$, where $z_1, .., z_{i+1}$ are the local holomorphic functions. Then $(z_1, .., z_i)$ are called the normal coordinates of the foliation. Sections $rH^0(V, sL)$ could be locally expressed as $\hat{s} = f(z_1, .., z_{i+1})(dz_1 \wedge .. \wedge dz_i)^{\otimes s}$.

Lemma 67. *$\hat{s} \in H^0(V, sL)$ is locally given by $\hat{s} = f(z_1, .., z_{i+1})(dz_1 \wedge .. \wedge dz_i)^{\otimes s}$. Then $\frac{\partial f}{\partial z_{i+1}} = 0$.*

Proof. Lemma(67) is equivalent to the statement of f being constant along the fibers $\mathscr{L}_s$ and does not depend on the chosen coordinate system.

- Consider a case when $s \in r(H^0(M^n, L)) \subset H^0(M^n, \Omega^i)$. Then $\hat{s}$ is a holomorphic form on M^{i+1}. Holomorphic form of codimension 1 on the arbitrary compact complex manifold is closed, i.e. $d\hat{s} = 0$. Consider a condition of being closed in a neighborhood of a special point x of a foliation induced via form s. The form could be written as $\hat{s} = f(z_1, .., z_{i+1})dz_1 \wedge .. \wedge dz_i, d\hat{s} = \frac{\partial f}{\partial z_{i+1}} dz_1 \wedge .. \wedge dz_{i+1}$ and $d\hat{s} = 0$ is equivalent to the condition $\frac{\partial f}{\partial z_{i+1}} = 0$.

- For a general case, one wishes to find a holomorphic form on some variety $\widetilde{V^{i+1}}$, which corresponds to a tensor $\hat{s} \in r(H^0(sE))$. Let $\hat{s} \in H^0(M, nE)$. Consider a non-linear map of the bundles $n : E \to nE$ which is given by fiberwise raising to the power $n, E_x \to nE_x$. The preimage of a section $n^{-1}(s)$ is a subvariety $n^{-1}(s) = W^{i+1} \subset L$, which is an n-fold covering of $V^{i+1}, n^{-1}(s) = W^{i+1} \to_{n:1} V^{i+1}$. For a section $s \in H^0(V^{i+1}, nE) \rightsquigarrow$ a section $\tilde{s} \in H^0(W^{i+1}, p^*E)$: to every point s_x in the fiber of the bundle $(nE)_x$ and the chosen point $(\sqrt[n]{s})_{x,i} \subset E_x \rightsquigarrow$ a point $(\widetilde{\sqrt[n]{s}})_{x,i} \in$ fiber p^*E_x via the isomorphism $p^*E_x \simeq E_x$. This way we get a section $s \in H^0(W^{i+1}, p^*E)$ such that $\widetilde{s}^n \in H^0(W^{i+1}, p^*nE)$ and $\widetilde{s}^n = p^*s$. Consider now one of the irreducible components W_θ^{i+1} and let us now build a resolution of singularities $h : \widetilde{W}^{i+1} \to W_0^{i+1}$ and the projection p; The composition $ph : \widetilde{W}^{i+1} \to V^{i+1} \rightsquigarrow$ a homomorphism of the bundles $p^*h^* : \Omega^i(V^{i+1}) \to^{\pounds} \Omega^i(\widetilde{W}^{i+1}) \rightsquigarrow$ a map $\pounds\widetilde{r}p^* : h^*p^*E \to p^*h^*\Omega^i(V^{i+1}) \to^{\widetilde{\pounds}} \Omega^i(\widetilde{W}^{i+1})$. Note that $\pounds\widetilde{r}p^* = \pounds\widetilde{r}\widetilde{s}^n$. Thus, one can extract the n-th root from the tensor form $\hat{s}^n \in r(H^0(\widetilde{W}^{i+1}, nE))$. Note that in the neighborhood of the non-special point of the foliation $\mathscr{L}_s$ on $\widetilde{W}^{i+1}$, one can rewrite $\widetilde{s}^n$ as $\widetilde{s}^n|_{x \in U, x \in E_s} = f(z_1, .., z_{i+1})(dz_1 \wedge .. \wedge dz_i)^{\otimes n}$, therefore $\widetilde{s} = \sqrt[n]{f} dz_1 \wedge .. \wedge dz_i$, where $\sqrt[n]{f}$ denotes some fixed branch of the multivalued function, s is a holomorphic form and so $\frac{\partial \sqrt[n]{f}}{\partial z_{i+1}} = 0$. It means that the statement holds for non-special points x of the foliation $\mathscr{L}_s$ on V^{i+1}, where the map $p^*h : \widetilde{W}^{i+1} \to V^{i+1}$ has rank $i+1$ and $rkp^*h = (i+1)$. Hence, everywhere outside the branch divisor $ph\frac{\partial f}{\partial z_{i+1}} = 0$. But the condition $\frac{\partial f}{\partial z_{i+1}} = 0$ is closed and it implies that it it holds everywhere.

□

Now, to prove a theorem, consider an arbitrary submanifold $V_r^i \subset V^{i+1}$ which is transversal to our foliation $V_r^i \pitchfork \mathscr{L}_s$ in a general point x.

Lemma 68. *A restriction homomorphism $r : H^0(V^{i+1}, nE) \to H^0(V^i, nE)$ is a monomorphism.*

Proof. Take $s_1 \in kerr$. Let $x \in V^i, T_x(V^i)$ and L_x^s are transversal and the foliation $\mathscr{L}_s$ is non-special at the point x. Then the normal coordinates of the foliation $\mathscr{L}_s : z_1, .., z_i$ in the neighborhood of x give the coordinates on V^i and $z_{i+1}(V^i) = 0$. Locally, in $U(x)$, s_1 can be written as $s_1 = f(z_1, .., z_i, z_{i+1})(dz_1 \wedge .. \wedge dz_i)^{\otimes n}$. The equality $s_1|_{V_i} = 0$ means that $f(z_1, .., z_i, 0) = 0$ but from the property $\frac{\partial f}{\partial z_{i+1}} = 0$ follows that then $f \equiv 0$ and, hence, $s_1 = 0$ in $U(x) \subset V^{i+1}$. So, any section $s_1 \in kerr$ is identically equal to zero ($\equiv 0$), so r is a monomorphism. □

Corollary 16. *$dimH^0(V^{i+1}, nE) \leq dimH^0(V^i, nE) \leq c_M n^i + \beta$ as it holds true for an arbitrary one-dimensional bundle on V^i. Thus the theorem holds in case $dimV = i + 1$ and $\Rightarrow$ it holds true for a variety of an arbitrary dimension.*

Remark 6. *The statement of the theorem (65) can be specified in the case when $dimH^0(M^n, kE) > c_1 k^\alpha + \beta_1, \alpha > 0, c_1 > 0, L \hookrightarrow^h \Omega^i(M^n)$. Then $\exists$ a projective manifold V^α and a rational epimorphism $f_\alpha : M^n \to V^\alpha$ such that the image $h\mathcal{O}(E)$ belongs to the ideal $\mathcal{I} \subset$ in the sheaf of exterior algebras $\sum \mathcal{O}(\Omega^i(M^n))$,* *spanned on the integer closure* $f_\alpha^*\mathcal{O}(K(V^\alpha)) \subset \mathcal{O}(\Omega^\alpha(M^n)$*, i.e.* $h(\mathcal{O}(E)) \subset f_\alpha^*\mathcal{O}(\widehat{K(V^\alpha)}) \wedge \Omega^{i-\alpha}(M^n)$

Remark 7. *By* $f_\alpha^*\mathcal{O}(\widehat{K(V^\alpha)})$ *we mean a maximal one-dimensional submodule in* $\mathcal{O}(\Omega^\alpha(M^n))$*, whose sections are* $f_\alpha^*(\mathcal{O}(K(V^\alpha)))$ *up to mult.to a function.*

Proof. Let us build up a manifold V^α and a rational epimorphic map $f : M^n \to V^\alpha$ with a connected general fiber. Similarly, the restriction of L to the general $(i+1)$-submanifold defines a one-dimensional foliation L_s^V. The fraction $\frac{s}{s'}$ of the sections $s, s' \in H^0(M, nL)$ is a rational function on M^n, which when restricted to V, has a form $\frac{f(z_1,..,z_i)}{g(z_1,..,z_i)} = \frac{s}{s'}$, where $z_1, .., z_i$ are the normal coordinates of a foliation L_s^V. This function is constant on the fibers of the foliation L_s^V on V^{i+1}, as f and φ are both constant on L_s^V. On the other hand, taking n sufficiently big, we will get a homogeneous map via the linear series $H^0(M, nL)$: $h_\alpha : M^n \to V^\alpha; (s_1, .., s_m) \in H^0(M, nL)$, where $s_i|_V = f_i(dz_1 \wedge .. \wedge dz_i)^{\otimes n}$. Thus if we consider a fiber of the foliation $L_s^V \subset V^{i+1}$, it fully belongs to the submanifold $Q_x = h_\alpha^{-1}(h(x))$, where $x \in L_{s,x}^V$. Thus the fibers of a map $h_\alpha|_{V^{i+1}} \to V^\alpha$ contain the fibers of a foliation L_s^V. It means that in the neighborhood of a point $x \in V^{i+1}$, where $rankdh = \alpha$ and L_s^V is α-non-special, the form $\sqrt[n]{f} dz_1 \wedge .. \wedge dz_i$, which locally gives L_s^V, belongs to the ideal of the form $det(dh)$. $(z_1, .., z_\alpha)$ coordinates on $V^\alpha \rightsquigarrow$ adding $(i+1-\alpha)$ local derivatives of the coordinates from the set $\{v_1, .., v_n\}$ on M^n, we get that our form $s \in H^0(U(x), r\mathcal{O}(L)), s = \sum\limits_{\omega=(1,..,\gamma,1,..,\beta)} f\omega dz_1 \wedge .. \wedge dz_\gamma \wedge dv_1 \wedge .. \wedge dv_\beta$, can be written as $f\omega = 0$, if the polynomial ω does not contain $(1, .., \alpha)$. Consider the case when $\alpha = i, \mathcal{O}(E) \simeq f_\alpha^*(K(V^\alpha))$. Interchanging varieties M^n and V^α to birationally equivalent ones, we can consider that $h : M^n \to V^\alpha$ is a holomorphic map, whose general fiber is connected. At points $x \in M^n$, where the $rankdf_\alpha = dimV^\alpha = i$, we can define a trace of any tensor $\forall s \in H^0(M^n, nE)$.

Indeed, the restriction $E|_{h^{-1}(x)}$ is trivial and one can build $trs_{f(x)}$ and if $s_x \neq 0$, then $trs_{f(x)} \neq 0$. The tensor $trs_{f(x)}$ will be a section of a bundle $nK(V_\alpha)$ holomorphic at all points $x \in V^\alpha$ for those $\exists y \in h^{-1}(x), rkdh_y = dimV^\alpha = i$. If $\forall y \in h^{-1}(x), dh_y < dimV_\alpha$, then the point y corresponds to a multiple fiber of a map h. Such points always form a divisor $D \subset V^\alpha$, or an empty set $\emptyset$. The section $trs_{f(x)}$ can have singularities of the pole type along $h(D)$ on V^α, if $h(D)$ is a divisor on V^α.

Example. Let V^n be an arbitrary projective manifold of dimension n and $E \hookrightarrow \Omega^1$ is a one-dimensional subbundle such that $H^0(V^n, nE) > cn + \beta, c > 0$. Then $\exists$ a rational map $h_1 : V^n \to X$, where X is a non-special curve. If h_1 doesn't have multiple fibers, then the genus of the curve $g(X^1) > 1$ as $tr_h s \in H^0(X^1, nK(X^1))$ and, hence, $dimH^0(X^1, nK) > cn + \beta$. Thus h_1 authomatically becomes a regular map. Suppose now that h_1 has multiple fibers.

Lemma 69. *$\exists$ a canonical non-ramified covering $\widetilde{X^1} \to X^1$ such that the induced map $\widetilde{h_1} : \widetilde{M^n} \to \widetilde{X^1}$, where $\widetilde{M^n} \simeq M \times \widetilde{X^1}$ doesn't have multiple fibers, i.e. $\forall x \in \widetilde{X^1}$, among components $\widetilde{h_1}^{-1}(x), \exists$ one $\mathcal{U} \subset \widetilde{h_1}^{-1}(x)$, such that $\mathcal{U} \in h^{-1}(x)$ with multiplicity* 1.

Proof. Indeed, the divisor on X^1 is a set of points p_i with multiplicities n_i, where n_i is a minimal multiplicity among the fiber components. Then consider a covering $\widetilde{X^1} \to X^1$ with multiplicities n_i at the points p_i. We will call it $\widetilde{X^1}$. It $\exists!$, if:

1. $g(X^1) \geqslant 0$ and n_i are arbitrary
2. $g(X^1) = 0$, number of ramification points $\geqslant 3$, n_i are arbitrary.

$\widetilde{X^1}$ satisfies Lemma(68) conditions. We have to show that if $g(X) = 0 \Rightarrow$

$\#\{p_i$ which correspond to the multiple fibers $\} \geqslant 3$.

For that, consider a holomorphic tensor s on M^n and a form $tr_h s$. $tr_h s \in nK(\mathbb{P}^1)$ and locally is given by $f(z)dz^n$. Note that at the point p_i the order of the pole $f(z), ordf(x) < n$, as $s = f(z)dz^n$ is resolved, i.e. via a substitution $q^p = z$ it becomes holomorphic and $\Rightarrow$ if $trs = \frac{dz^n}{z^r} + ..$, then $\frac{z^{n(p-1)}}{z^{pr}}$ is a holomorphic function, $(p-1)n > r$ and $r < n$. Hence as the sum over all points $\sum ord_p f(z) > 2n, 2n = -degnK(P^1)$, then $ord_p f < n$ for $\forall$ point p_i and we get that $\#$ of multiple points $\geqslant 3$. $\square$

Remark 8. *If the local covering, induced via $\sqrt[n_i]{z}$ in the neighborhood of p_i is not ramified on $h^{-1}(U(p_i))$ in M, then $f : \widetilde{M}^n \to M^n$ is non-ramified as well.*

Theorem 70. *(Bogomolov, 1978) Let $E \to X$ be a vector bundle of $dimE = n$ and suppose that $c_2 - \frac{n-1}{2n}{c_1}^2 > 0 \Rightarrow \exists$ a subbundle $F \subset E, dimF = k$ and a homomorphism $h : F \to E$:*

1. *$h : \mathcal{O}(F) \to \mathcal{O}(E)$ is a monomorphism;*
2. *$(c_1(F) - \frac{k}{n}c_1(E))^2 > -c_2(E_0), E_0 = E \otimes \frac{detE}{n}$;*
3. *some multiplicity of the bundle $(ndetF - kdetE)$ has a section; i.e. there exists $\exists s \in H^0(X, l(ndetF - kdetE))$ for some l*

Theorem 71. **(Bogomolov-Miyaoka-Yau inequality with const. 4)** *For an arbitrary projective surface of general type $/\mathbb{C}$ the following inequality holds $\mathbf{{c_1}^2 \leq 4c_2}$.*

Proof. Let V be a projective surface of general type. Then $dimH^0(V, nK) < cn^2 + \beta, c > 0$. By Theorem[70] the inequality ${c_1}^2 > 4c_2$ is equivalent to $\exists$ of one-dimensional subbundle $h : \mathcal{F} \to \Omega^1$ such that $H^0(V, n(2\mathcal{F} - K)) > 0$ for some $n > 0$. Then $H^0(V, m\mathcal{F}) > \frac{c}{4}m^2 + \beta$ and it contradicts Theorem (65). $\square$

11.2 Inequalities and deformation theory

It is interesting to point out that a presence of nontrivial infinitesimal deformations provides another way to obtain strong inequalities obtained by Yau for some classes of manifolds of general type.

Here we proof our main result:

Theorem 72. *Let X^2 be a complex projective surface which ample canonical class K which generates $PicX^2 = Z$ and assume that $H^1(X, T) \neq 0$ where T is the tangent bundle. Then we have Severi inequality $c_1^2(X) < 3c_2(X)$.*

Proof. Let X be an algebraic surface. We follow notations in (Déev, 2021). Denote $C_X(X) := \{$the set of tensors $I \in C^\infty(T^*X \otimes TX) : I^2 = -Id_{TX}\}$ and $\mathcal{T}(X) = C_X(X)/Diff^0(X)$ a Teichmüller space. The tangent space to Teichmüller space is $T_I\mathcal{T} \simeq H^1(X, TX)$: take a first order deformation $\mathfrak{X} \to \Delta$ with $\mathfrak{X}_0 \simeq X$. We have a short exact sequence $TX \to T\mathfrak{X}|_X \to \mathcal{O}_X$. The short exact sequences of the isomorphism classes of such extensions are classified by $H^1(X, TX)$. Further, denote $T' := T\mathfrak{X}$. Indeed, consider the extension $\tilde{T}$ obtained from exact sequence $0 \to T \to \tilde{T} \to O \to 0$. Then if $\tilde{T}$ we obtain Severi inequality $c_1^2 < 3c_2$ since $\tilde{T}$ has rank 3. If $\tilde{T}$ is unstable then we have a destabilizing subbundle $F^l \subset \tilde{T}, l = 1, 2$, with $detF^l \geq (-l)/3K$ If $l = 1$ and $F^1 \subset T$ then $F \times K \subset \Omega^1$ is ample contracting the absence of ample subsheafs in Ω^1

If F^1 is not contained in T then $F^1 \leq -K$ since it maps to O and hence can not be destabilizing.

If we have destabilizing F^2 then we have that determinant $detF^2 = detFT + detF^2/FT$ where FT is the intersection of F^2 and $T \subset \tilde{T}$. Thus $FT \geq -2/3K$ if F^2/FT is trivial and we get contradiction as above. Otherwise $F^2/FT \leq -K$ and $FT \times K \subset \Omega^1$ is also ample. This result can be extended to other classes of surfaces and higher dimensional manifolds.

12 | A particular examples of vector bundles on families

Here we observe some descriptions of vector bundles on certain families of curves following the survey (Beauville, 1994).

12.0.1 Case of ruled surfaces $B = \mathbb{P}^1 \times C$

Consider a vector bundle $\mathcal{E}$ on a family B, where a curve C is of arbitrary genus. Then by the Grothendieck theorem, $\mathcal{E}|_x \simeq \mathcal{O}(r_1(x)) \oplus .. \oplus \mathcal{O}(r_k(x))$, where $x \in C$ and $\sum_i r_i(x) = n$, $n = rank(\mathcal{E})$. But the only stable ones are those which are represented as a sum of a two line bundles.

We can say that a moduli space of vector bundles over $\mathbb{P}^1$ is

$$M_{\mathbb{P}^1} = \bigoplus_l \mathcal{O}(l_1) \oplus \mathcal{O}(l_2),$$

Then $\mathcal{E} \simeq C \subset M_{\mathbb{P}^1}$ and $M_B \simeq \{C \subset M_{\mathbb{P}^1}\}$

From the other point of view $\{\mathcal{E} \simeq \mathbb{P}^1 \subset M_C\}$.

Additionally, there is a correspondence between bundles. Namely, they coincide (or are isomorphic) on the restrictions.

Let us treat it in a few well-studied cases, by (Beauville, 1994):

Let J be a jacobian variety of a curve C. Denote by J^k its' Jacobian variety of degree k line bundles. Let us fix some $L \in J^{g-1}$. Consider a reduced subvariety

$$\Delta_L = \{E \in \mathcal{M}_r | H^0(C, E \otimes L) \neq 0\}$$

Then Δ_L is a Cartier divisor in $\mathcal{M}_r$. The line bundle $\mathcal{L} = \mathcal{O}_{\mathcal{M}_r}(\Delta_L)$ is called a determinant line bundle and is independent of the choice of L and generates $Pic(\mathcal{M}_r)$.

For a vector bundle $E \in \mathcal{M}_r$ consider the locus

$$\Theta_E := \{L \in J^{g-1} | H^0(C, E \otimes L) \neq \emptyset\}$$

If it is nonempty we can say that E admits a theta-divisor. This divisor belongs to a linear system $|r\Theta|$, where Θ is a canonical Theta-divisor in J^{g-1}. Thus we got a map:

$$\theta : \mathcal{M}_2 \to |r\Theta|$$

Let us observe more carefully a few cases:

- rank $\mathcal{E} = 2$, genus $g = 2$:

  A map θ is a finite morphism in this case and $\mathcal{M}_2$ is isomorphic to $\mathbb{P}^3$,

  $$i : \mathcal{M}_2 \to \mathbb{P}^3$$

  Thus vector bundles on $\mathbb{P}^1 \times C$ are isomorphic to $i^{-1}(\mathbb{P}^1)$, $\mathbb{P}^1 \subset \mathbb{P}^3$

- genus $g \geq 3$, C is hyperelliptic in this case, a map

  $$i : \mathcal{M}_2 / {<}i^*{>} \hookrightarrow |2\Theta|$$

  is embedding, where i^* is an envolution. There is an explicit geometric description (will write it later), thus vector bundles on $\mathbb{P}^1 \times C$ are isomorphic to $i^{-1}(\mathbb{P}^1)$, $\mathbb{P}^1 \subset |2\Theta|$

- If a curve C is not hyperelliptic, the map θ is again an embedding $i : \mathcal{M}_2 \to$

$|2\Theta|$. Recall that $|2\Theta|$ is isomorphic to $\mathbb{P}^{2^g-1}$. Thus $i(\mathcal{M}_2)$ to a subvariety of $\mathbb{P}^{2^g-1}$ of dimension $3g-3$. $Sing(\mathcal{M}_2) = K$, where K is a Kummer variety.

Thus every vector bundle is $\mathbb{P}^1 \subset i(\mathcal{M}_2)$

- In genus $g = 3$, i identifies $\mathcal{M}_2$ with a Coble quartic hypersurface. This every preimage of $\mathbb{P}^1 \subset$ Coble quartic hypersurface describes our vector bundle. [write Coble quartic hypersurface in coordinates...]

- In genus 4, $\mathcal{M}_2$ is a variety of dimension 9 and degree 96 in $\mathbb{P}^{15}$. Oxbury and Pauly have observed that there exists a unique $J[2]$-invariant quartic hypersurface singular along $\mathcal{M}_2$. A geometric interpretation of this quartic is not known. In arbitrary genus, the quartic hypersurfaces in $|2\Theta|$ containing $\mathcal{M}_2$ have been studied in [vG] and [vG-P]. Here is one sample of their results: Assume that C has no vanishing thetanull. A $J[2]$ invariant quartic form F on $|2\Theta|$ vanishes on $\mathcal{M}_2$ if and only if the hypersurface $F = 0$ is singular along K.

12.0.2 Example: $B = C_1 \times C_2$

In this case, some amount of bundles are coming from $\{C_1 \subset M_{C_2}\} \simeq \{C_2 \subset M_{C_1}\}$. And similarly they coincide on the restrictions. Also, $C_1 \otimes C_2$. To get a more general picture recall that

$$Pic(C_1 \times C_2) = Pic(C_1) \oplus Pic(C_2) \oplus Hom(J_{C_1}, J_{C_2})$$

And for some curve C we can get any vector bundle on $C_1 \times C_2$ using the procedure of elementary transformation along C.

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