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5 Spectral geometry of the Steklov problem

Note from the authors

This chapter is a reprint of the article *Spectral geometry of the Steklov problem* by the same authors, published in the Journal of Spectral Theory. We would like to thank the Journal of Spectral Theory and the European Mathematical Society for granting permission to reproduce the paper in this book. Spectral geometry of the Steklov problem is a rapidly developing subject, and there have been a number of important advances since the original version of this article has appeared. In the present text, we have added references to some of these new results in the footnotes. In order to make this chapter coherent with the rest of the book, the dimension is denoted by d, and the trivial Steklov eigenvalue is now denoted by $\sigma_1 = 0$, as opposed to $\sigma_0 = 0$ in the journal version of this article. The numeration has been also changed.

5.1 Introduction

5.1.1 The Steklov problem

Let Ω be a compact Riemannian manifold of dimension $d \geq 2$ with (possibly nonsmooth) boundary $M = \partial\Omega$. The *Steklov problem* on Ω is

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \sigma u & \text{on } M. \end{cases}$$

where Δ is the Laplace-Beltrami operator acting on functions on Ω, and $\frac{\partial u}{\partial n}$ is the outward normal derivative along the boundary M. This problem was introduced by the Russian mathematician V.A. Steklov at the turn of the 20th century (see [613] for a historical discussion). It is well known that the spectrum of the Steklov problem is discrete as long as the trace operator $H^1(\Omega) \to L^2(\partial\Omega)$ is compact (see [50]). In this case, the eigenvalues form a sequence $0 = \sigma_1 \leq \sigma_2 \leq \sigma_3 \leq \cdots \nearrow \infty$. This is true under some mild regularity assumptions, for instance if Ω has Lipschitz boundary (see [725, Theorem 6.2]).

The present paper focuses on the geometric properties of Steklov eigenvalues and eigenfunctions. A lot of progress in this area has been made in the last few years, and

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some fascinating open problems have emerged. We will start by explaining the motivation for studying the Steklov spectrum. In particular, we will emphasize the differences between this eigenvalue problem and its Dirichlet and Neumann counterparts.

5.1.2 Motivation

The Steklov eigenvalues can be interpreted as the eigenvalues of the *Dirichlet-to-Neumann operator* $\mathcal{D} : H^{1/2}(M) \to H^{-1/2}(M)$ which maps a function $f \in H^{1/2}(M)$ to $\mathcal{D}f = \frac{\partial Hf}{\partial n}$, where Hf is the harmonic extension of f to Ω. The study of the Dirichlet-to-Neumann operator (also known as the voltage-to-current map) is essential for applications to electrical impedance tomography, which is used in medical and geophysical imaging (see [858] for a recent survey). The Steklov spectrum also plays a fundamental role in the mathematical analysis of photonic crystals (see [606] for a survey).

A rather striking feature of the asymptotic distribution of Steklov eigenvalues is its unusually (compared to the Dirichlet and Neumann cases) high sensitivity to the regularity of the boundary. On one hand, if the boundary of a domain is smooth, the corresponding Dirichlet-to-Neumann operator is pseudodifferential and elliptic of order one (see [850, pp. 37-38]). As a result, one can show, for instance, that a surprisingly sharp asymptotic formula for Steklov eigenvalues (5.3) holds for smooth surfaces. However, this estimate already fails for polygons (see section 5.3). It is in fact likely that for domains which are not C^∞-smooth but only of class C^k for some $k \geq 1$ that the rate of decay of the remainder in eigenvalue asymptotics depends on k. To our knowledge, for domains with Lipschitz boundaries, even one-term spectral asymptotics have not yet been proved. A summary of the available results is presented in [11] (see also [12]).

One of the oldest topics in spectral geometry is shape optimization. Here again, the Steklov spectrum holds some surprises. For instance, the classical result of Faber–Krahn for the first Dirichlet eigenvalue $\lambda_1(\Omega)$ states that among Euclidean domains with fixed measure, λ_1 is minimized by a ball. Similarly, the Szegő–Weinberger inequality states that the first nonzero Neumann eigenvalue $\mu_2(\Omega)$ is maximized by a ball. In both cases, no topological assumptions are made. The analogous result for Steklov eigenvalues is Weinstock's inequality, which states that among planar domains with fixed perimeter, σ_2 is maximized by a disk provided that Ω is simply–connected. In contrast with the Dirichlet and Neumann case, this assumption cannot be removed. Indeed the result fails for appropriate annuli (see section 5.4.2). Moreover, maximization of the first nonzero Steklov eigenvalue among all planar domains of given perimeter is an open problem. At the same time, it is known that for simply–connected planar domains, the k-th normalized Steklov eigenvalue is maximized in the limit by a disjoint union of $k-1$ identical disks for any $k \geq 2$ [892]. Once again, for the Dirichlet and Neumann eigenvalues the situation is quite different: the extremal domains for $k \geq 3$ are known only at the level of experimental numerics, and, with a few exceptions, are expected to have rather complicated geometries, see the pictures in Chapter 11.

Probably the most well–known question in spectral geometry is "Can one hear the shape of a drum?", or whether there exist domains or manifolds that are isospectral but not isometric. Apart from some easy examples discussed in section 5.5, no examples of Steklov isospectral non-isometric manifolds are presently known. Their construction appears to be even trickier than for the Dirichlet or Neumann problems. In particular, it is not known whether there exist Steklov isospectral Euclidean domains which are not isometric. Note that the standard transplantation techniques (see [127, 232, 233]) are not applicable for the Steklov problem, as it is not clear how to reflect Steklov eigenfunctions across the boundary.

New challenges also arise in the study of the nodal domains and the nodal sets of Steklov eigenfunctions. One of the problems is to understand whether the nodal lines of Steklov eigenfunctions are dense at the "wave-length scale", which is a basic property of the zeros of Laplace eigenfunctions. Another interesting question is the nodal count for the Dirichlet-to-Neumann eigenfunctions. We touch upon these topics in section 5.6.

Let us conclude this discussion by mentioning that the Steklov problem is often considered in the more general form

$$\frac{\partial u}{\partial n} = \sigma \rho u, \tag{5.1}$$

where $\rho \in L^\infty(\partial\Omega)$ is a non-negative weight function on the boundary. If Ω is two-dimensional, the Steklov eigenvalues can be thought of as the squares of the natural frequencies of a vibrating free membrane with its mass concentrated along its boundary with density ρ (see [619]). A special case of the Steklov problem with the boundary condition (5.1) is the sloshing problem, which describes the oscillations of fluid in a container. In this case, $\rho \equiv 1$ on the free surface of the fluid and $\rho \equiv 0$ on the walls of the container. There is an extensive literature on the properties of sloshing eigenvalues and eigenfunctions, see [105, 388, 602] and references therein.

Since the present survey is directed towards geometric questions, in order to simplify the analysis and presentation we focus on the pure Steklov problem with $\rho \equiv 1$.

5.1.3 Computational examples

The Steklov spectrum can be explicitly computed in a few cases. Below we discuss the Steklov eigenvalues and eigenfunctions of cylinders and balls using separation of variables.

Example 5.1. *The Steklov eigenvalues of a unit disk are*

$$0, 1, 1, 2, 2, \dots, k, k, \dots .$$

The corresponding eigenfunctions in polar coordinates (r, ϕ) *are given by*

$$1, r\sin\phi, r\cos\phi, \dots, r^k \sin k\phi, r^k \cos k\phi, \dots.$$

Example 5.2. *The Steklov eigenspaces on the ball* $B(0,R) \subset \mathbb{R}^d$ *are the restrictions of the spaces* H_k^d *of homogeneous harmonic polynomials of degree* $k \in \mathbb{N}$ *on* $\mathbb{R}^d$*. The corresponding eigenvalue is* $\sigma = k/R$ *with multiplicity*

$$\dim H_k^d = \binom{d+k-1}{d-1} - \binom{d+k-3}{d-1}.$$

This is of course a generalization of the previous example.

Example 5.3. *This example is taken from [279]. Let* Σ *be a compact Riemannian manifold without boundary. Let*

$$0 = \lambda_1 < \lambda_2 \le \lambda_3 \cdots \nearrow \infty$$

be the spectrum of the Laplace-Beltrami operator Δ_Σ *on* Σ*, and let* (u_k) *be an orthonormal basis of* $L^2(\Sigma)$ *such that*

$$\Delta_\Sigma u_k = \lambda_k u_k.$$

Given any $L > 0$*, consider the cylinder* $\Omega = [-L, L] \times \Sigma \subset \mathbb{R} \times \Sigma$*. Its Steklov spectrum is given by*

$$0, 1/L, \ \sqrt{\lambda_k}\tanh(\sqrt{\lambda_k}L), \ \sqrt{\lambda_k}\coth(\sqrt{\lambda_k}L).$$

and the corresponding eigenfunctions are

$$1, \ t, \ \cosh(\sqrt{\lambda_k}t)u_k(x), \ \sinh(\sqrt{\lambda_k}t)u_k(x).$$

In sections 5.3.1 and 5.4.2 we will discuss two more computational examples: the Steklov eigenvalues of a square and of annuli.

5.1.4 Plan of the chapter

The chapter is organized as follows. In section 5.2 we survey results on the asymptotics and invariants of the Steklov spectrum on smooth Riemannian manifolds. In section 5.3 we discuss asymptotics of Steklov eigenvalues on polygons, which turns out to be quite different from the case of smooth planar domains. Section 5.4 is concerned with geometric inequalities. In section 5.5 we discuss Steklov isospectrality and spectral rigidity. Finally, section 5.6 deals with the nodal geometry of Steklov eigenfunctions and the multiplicity bounds for Steklov eigenvalues.

5.2 Asymptotics and invariants of the Steklov spectrum

5.2.1 Eigenvalue asymptotics

As above, let $d \geq 2$ be the dimension of the manifold Ω, so that the dimension of the boundary $M = \partial\Omega$ is $d-1$. As was mentioned in the introduction, the Steklov eigenvalues of a compact manifold Ω with boundary $M = \partial\Omega$ are the eigenvalues of the Dirichlet-to-Neumann map. It is a first order elliptic pseudodifferential operator which has the same principal symbol as the square root of the Laplace-Beltrami operator on M. Therefore, applying the results of Hörmander [540, 541][5.1] we obtain the following Weyl's law for Steklov eigenvalues:

$$\#(\sigma_j < \sigma) = \frac{\mathrm{Vol}(\mathbb{B}^{d-1})\,\mathrm{Vol}(M)}{(2\pi)^{d-1}}\sigma^{d-1} + \mathcal{O}(\sigma^{d-2}),$$

where $\mathbb{B}^{d-1}$ is a unit ball in $\mathbb{R}^{d-1}$. This formula can be rewritten

$$\sigma_j = 2\pi\left(\frac{j}{\mathrm{Vol}(\mathbb{B}^{d-1})\,\mathrm{Vol}(M)}\right)^{\frac{1}{d-1}} + \mathcal{O}(1). \tag{5.2}$$

In two dimensions, a much more precise asymptotic formula was proved in [429]. Given a finite sequence $C = \{\alpha_1, \cdots, \alpha_k\}$ of positive numbers, consider the following union of multisets (i.e. sets with multiplicities): $\{0, \ldots\ldots, 0\} \cup \alpha_1\mathbb{N} \cup \alpha_1\mathbb{N} \cup \alpha_2\mathbb{N} \cup \alpha_2\mathbb{N} \cup \cdots \cup \alpha_k\mathbb{N} \cup \alpha_k\mathbb{N}$, where the first multiset contains k zeros and $\alpha\mathbb{N} = \{\alpha, 2\alpha, 3\alpha, \ldots, n\alpha, \ldots\}$. We rearrange the elements of this multiset into a monotone increasing sequence[5.2] $S(C)$. For example, $S(\{1\}) = \{0, 1, 1, 2, 2, 3, 3, \cdots\}$ and $S(\{1, \pi\}) = \{0, 0, 1, 1, 2, 2, 3, 3, \pi, \pi, 4, 4, 5, 5, 6, 6, 2\pi, 2\pi, 7, 7, \cdots\}$. The following sharp spectral estimate was proved in [429].

Theorem 5.4. *Let Ω be a smooth compact Riemannian surface with boundary M. Let $M_1, \cdots, M_k$ be the connected components of the boundary $M = \partial\Omega$, with lengths $\ell(M_i)$, $1 \leq i \leq k$. Set $R = \left\{\frac{2\pi}{\ell(M_1)}, \cdots, \frac{2\pi}{\ell(M_k)}\right\}$. Then*

$$\sigma_j = S(R)_j + \mathcal{O}(j^{-\infty}), \tag{5.3}$$

where $\mathcal{O}(j^{-\infty})$ means that the error term decays faster than any negative power of j.

In particular, for simply–connected surfaces we recover the following result proved earlier by Rozenblyum and Guillemin–Melrose (see [354, 794]):

$$\sigma_{2j} = \sigma_{2j+1} + \mathcal{O}(j^{-\infty}) = \frac{2\pi}{\ell(M)}j + \mathcal{O}(j^{-\infty}). \tag{5.4}$$

5.1 The authors thank Y. Kannai for providing them a copy of L. Hörmander's unpublished manuscript [540].

5.2 In this chapter, the sequence starts with $S(C)_1 = 0$, as opposed to $S(C)_0 = 0$ in the original paper.

The idea of the proof of Theorem 5.4 is as follows. For each boundary component M_i, $i = 1, \dots, k$, we cut off a "collar" neighbourhood of the boundary and smoothly glue a cap onto it. In this way, one obtains k simply–connected surfaces, whose boundaries are precisely $M_1, \dots, M_k$, and the Riemannian metric in the neighbourhood of each M_i, $i = 1, \dots k$, coincides with the metric on Ω. Denote by $\Omega^\star$ the union of these simply–connected surfaces. Using an explicit formula for the full symbol of the Dirichlet-to-Neumann operator [646] we notice that the Dirichlet-to-Neumann operators associated with Ω and $\Omega^\star$ differ by a smoothing operator, that is, by a pseudodifferential operator with a smooth integral kernel; such operators are bounded as maps between any two Sobolev spaces $H^s(M)$ and $H^t(M)$, $s, t \in \mathbb{R}$. Applying pseudodifferential techniques, we deduce that the corresponding Steklov eigenvalues of $\sigma_j(\Omega)$ and $\sigma_j(\Omega^\star)$ differ by $\mathcal{O}(j^{-\infty})$. Note that a similar idea was used in [529]. Now, in order to study the asymptotics of the Steklov spectrum of $\Omega^\star$, we map each of its connected components to a disk by a conformal transformation and apply the approach of Rozenblyum-Guillemin-Melrose which is also based on pseudodifferential calculus.

5.2.2 Spectral invariants

The following result is an immediate corollary of Weyl's law (5.2).

Corollary 5.5. *The Steklov spectrum determines the dimension of the manifold and the volume of its boundary.*

More refined information can be extracted from the Steklov spectrum of surfaces.

Theorem 5.6. *The Steklov spectrum determines the number k and the lengths $\ell_1 \geq \ell_2 \geq \dots \geq \ell_k$ of the boundary components of a smooth compact Riemannian surface. Moreover, if $\{\sigma_j\}$ is the monotone increasing sequence of Steklov eigenvalues, then:*

$$\ell_1 = \frac{2\pi}{\limsup_{j\to\infty}(\sigma_{j+1} - \sigma_j)}.$$

This result is proved in [429] by a combination of Theorem 5.4 and certain number-theoretic arguments involving the Dirichlet theorem on simultaneous approximation of irrational numbers.

As was shown in [429], a direct generalization of Theorem 5.6 to higher dimensions is false. Indeed, consider four flat rectangular tori: $T_{1,1} = \mathbb{R}^2/\mathbb{Z}^2$, $T_{2,1} = \mathbb{R}/2\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, $T_{2,2} = \mathbb{R}^2/(2\mathbb{Z})^2$ and $T_{\sqrt{2},\sqrt{2}} = \mathbb{R}^2/(\sqrt{2}\mathbb{Z})^2$. It was shown in [344, 741] that the disjoint union $\mathcal{T} = T_{1,1} \sqcup T_{1,1} \sqcup T_{2,2}$ is Laplace–Beltrami isospectral to the disjoint union $\mathcal{T}' = T_{2,1} \sqcup T_{2,1} \sqcup T_{\sqrt{2},\sqrt{2}}$. It follows from Example 5.3 that for any $L > 0$, the two disjoint unions of cylinders $\Omega_1 = [0, L] \times \mathcal{T}$ and $\Omega_2 = [0, L] \times \mathcal{T}'$ are Steklov isospectral. At the same time, Ω_1 has four boundary components of area 1 and two boundary components of area 4, while Ω_2 has six boundary components of area 2. Therefore, the collection of areas of boundary components cannot be determined from the Steklov spectrum. Still, the following question can be asked:

Open problem 5.7. *Is the number of boundary components of a manifold of dimension* ≥ 3 *a Steklov spectral invariant?*

Further spectral invariants can be deduced using the heat trace of the Dirichlet-to-Neumann operator $\mathcal{D}$. By the results of [10, 348, 440], the heat trace admits an asymptotic expansion

$$\sum_{i=1}^{\infty} e^{-t\sigma_i} = \operatorname{Tr} e^{-t\mathcal{D}} = \int_M e^{-t\mathcal{D}}(x,x)\,dx \sim \sum_{k=0}^{\infty} a_k t^{-d+1+k} + \sum_{l=1}^{\infty} b_l t^l \log t. \tag{5.5}$$

The coefficients a_k and b_l are called the *Steklov heat invariants*, and it follows from (5.5) that they are determined by the Steklov spectrum. The invariants $a_0, \ldots, a_{d-1}$, as well as b_l for all l, are local, in the sense that they are integrals over M of corresponding functions $a_k(x)$ and $b_l(x)$ which may be computed directly from the symbol of the Dirichlet-to-Neumann operator $\mathcal{D}$. The coefficients a_k are not local for $k \geq d$ [422, 423] and hence are significantly more difficult to study.

In [764], explicit expressions for the Steklov heat invariants a_0, a_1 and a_2 for manifolds of dimensions three or higher were given in terms of the scalar curvatures of M and Ω, as well as the mean curvature and the second order mean curvature of M (for further results in this direction, see [675]). For example, the formula for a_1 yields the following corollary:

Corollary 5.8. *Let* $\dim \Omega \geq 3$. *Then the integral of the mean curvature over* $\partial\Omega = M$ *(i.e. the total mean curvature of* M*) is an invariant of the Steklov spectrum.*

The Steklov heat invariants will be revisited in section 5.5.

Remark 5.9. *Other spectral invariants have also been studied. For smooth simply connected planar domains it was shown in [353] that the regularized determinant* $\det(\mathcal{D})$ *of the Dirichlet–to–Neumann map is equal to the perimeter of the domain. In fact, on an arbitrary smooth compact Riemannian surface with boundary,* $\det(\mathcal{D})/L(\partial\Omega)$ *is a conformal invariant. This was proved in [441], where an explicit formula for the determinant was given in terms of particular values of Selberg and Ruelle zeta functions and of the Euler characteristic of* Ω.

One should also mention the recent paper [690] where special values of the zeta function are computed for smooth simply connected planar domains, providing a seemingly large number of new spectral invariants which are expressed in terms of the Fourier coefficients of a bihilomorphism from the disk (see also [352]).

Table 5.1. Eigenfunctions obtained by separation of variables on the square $(-1,1) \times (-1,1)$.

Eigenspace basis	Conditions on α	Eigenvalues	Asymptotic behaviour
$\cos(\alpha x)\cosh(\alpha y)$ $\cos(\alpha y)\cosh(\alpha x)$	$\tan(\alpha) = -\tanh(\alpha)$	$\alpha\tanh(\alpha)$	$\frac{3\pi}{4} + \pi j + \mathcal{O}(j^{-\infty})$
$\sin(\alpha x)\cosh(\alpha y)$ $\sin(\alpha y)\cosh(\alpha x)$	$\tan(\alpha) = \coth(\alpha)$	$\alpha\tanh(\alpha)$	$\frac{\pi}{4} + \pi j + \mathcal{O}(j^{-\infty})$
$\cos(\alpha x)\sinh(\alpha y)$ $\cos(\alpha y)\sinh(\alpha x)$	$\tan(\alpha) = -\coth(\alpha)$	$\alpha\coth(\alpha)$	$\frac{3\pi}{4} + \pi j + \mathcal{O}(j^{-\infty})$
$\sin(\alpha x)\sinh(\alpha y)$ $\sin(\alpha y)\sinh(\alpha x)$	$\tan(\alpha) = \tanh(\alpha)$	$\alpha\coth(\alpha)$	$\frac{\pi}{4} + \pi j + \mathcal{O}(j^{-\infty})$
xy		1	

5.3 Spectral asymptotics on polygons

The spectral asymptotics given by formula (5.2) and Theorem 5.4 are obtained using pseudodifferential techniques which are valid only for manifolds with smooth boundaries. In the presence of singularities, the study of the asymptotic distribution of Steklov eigenvalues is more difficult, and the known remainder estimates are significantly weaker (see [11] and references therein). Moreover, Theorem 5.4 fails even for planar domains with corners. This can be seen from the explicit computation of the spectrum for the simplest nonsmooth domain: the square.

5.3.1 Spectral asymptotics on the square

The Steklov spectrum of the square $\Omega = (-1,1) \times (-1,1)$ is described as follows. For each positive root α of the following equations:

$$\tan(\alpha) + \tanh(\alpha) = 0, \quad \tan(\alpha) - \coth(\alpha) = 0,$$
$$\tan(\alpha) + \coth(\alpha) = 0, \quad \tan(\alpha) - \tanh(\alpha) = 0$$

the number $\alpha\tanh(\alpha)$ or $\alpha\coth(\alpha)$ is a Steklov eigenvalue of multiplicity two (see Table 5.1 and Figure 5.1). The function $f(x,y) = xy$ is also an eigenfunction, with a simple eigenvalue $\sigma_4 = 1$. Starting from σ_5, the normalized eigenvalues are clustered in groups of 4 around the odd multiples of 2π:

$$\sigma_{4j+l}L = (2j+1)2\pi + \mathcal{O}(j^{-\infty}), \qquad \text{for } l = 1,2,3,4.$$

This is compatible with Weyl's law since for $k = 4j + l$ it follows that

$$\sigma_k L = \left(\frac{k-l}{2} + 1\right) 2\pi + \mathcal{O}(j^{-\infty}) = \pi k + \mathcal{O}(1).$$

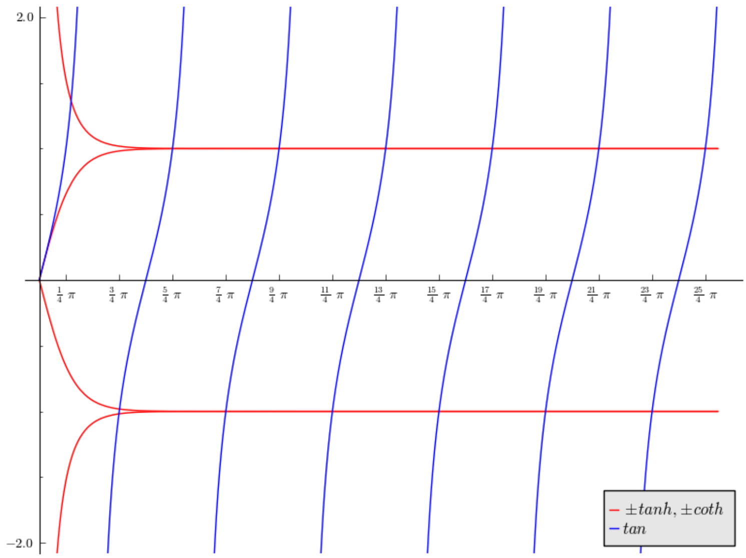

Fig. 5.1. The Steklov eigenvalues of a square. Each intersection corresponds to a double eigenvalue.

Nevertheless, the refined asymptotics (5.4) does not hold.

Let us discuss the spectrum of a square in more detail. Separation of variables quickly leads to the 8 families of Steklov eigenfunctions presented in Table 5.1 plus an "exceptional" eigenfunction $f(x, y) = xy$. One now needs to prove the completeness of this system of orthogonal functions in $L^2(\partial\Omega)$. Using the diagonal symmetries of the square (see Figure 5.2), we obtain symmetrized functions spanning the same eigenspaces. Splitting the eigenfunctions into odd and even parts with respect to the diagonal symmetries, we represent the spectrum as the union of the spectra of four mixed Steklov problems on a right isosceles triangle. In each of these problems the Steklov condition is imposed on the hypotenuse, and on each of the sides the condition is either Dirichlet or Neumann, depending on whether the corresponding eigenfunctions are odd or even when reflected across this side. In order to prove the completeness of this system of Steklov eigenfunctions, it is sufficient to show that the corresponding symmetrized eigenfunctions form a complete set of solutions for each of the four mixed problems.

Let us show this property for the problem corresponding to even symmetries across the diagonal. In this way, one gets a sloshing (mixed Steklov–Neumann) problem on a right isosceles triangle. Solutions of this problem were known since 1840s (see [618]). The restrictions of the solutions to the hypotenuse (i.e. to the side of the

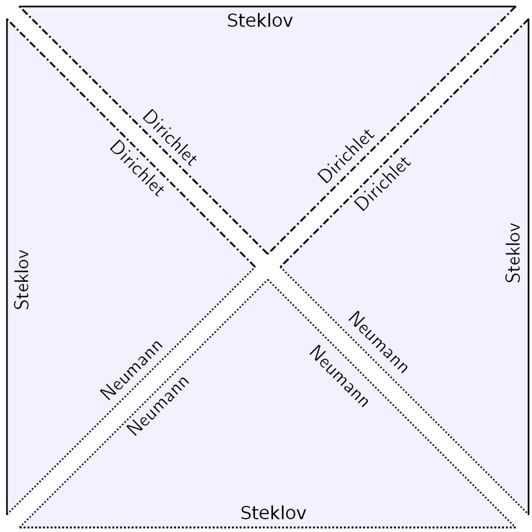

Fig. 5.2. Decomposition of the Steklov problem on a square into four mixed problems on a triangle.

original square) turn out to be the eigenfunctions of the *free beam equation*:

$$\frac{d^4}{dx^4}f = \omega^4 f \qquad \text{on } (-1,1)$$
$$\frac{d^3}{dx^3}f = \frac{d^2}{dx^2}f = 0 \quad \text{at } x = -1, 1.$$

This is a fourth order self-adjoint Sturm-Liouvillle equation. It is known that its solutions form a complete set of functions on the interval $(-1, 1)$.

The remaining three mixed problems are dealt with similarly: one reduces the problem to the study of solutions of the vibrating beam equation with either the Dirichlet condition on both ends, or the Dirichlet condition on one end and the Neumann on the other.

Remark 5.10. *The idea to replace the Dirichlet–to–Neumann map on the boundary of a non-smooth domain by a higher order differential problem has been also used in the mathematical analysis of photonic crystals (see [606, section 7.5.3]).*

5.3.2 Numerical experiments

Understanding fine spectral asymptotics for the Steklov problem on arbitrary polygonal domains is a difficult question. We have used software from the FEniCS Project (see http://fenicsproject.org/ and [677]) to investigate the behaviour of the Steklov eigenvalues for some specific examples. This was done using an implementation due to B.

Siudeja [814] which was already applied in [613]. For the sake of completeness, we discuss two of these experiments here.

Example 5.11. *(Equilateral triangle) We have computed the first 60 normalized eigenvalues $\sigma_j L$ of an equilateral triangle. The results lead to a conjecture that*

$$\sigma_{2j}L = \sigma_{2j-1}L + o(1) = \pi(2j-1) + o(1).$$

Example 5.12. *(Right isosceles triangle) For the right isosceles triangle with sides of lengths $1, 1, \sqrt{2}$, we have also computed the first 60 normalized eigenvalues. The numerics indicate that the spectrum is composed of two sequences of eigenvalues, one of which is behaving as a sequence of double eigenvalues*

$$\pi j + o(1)$$

and the other one as a sequence of simple eigenvalues

$$\frac{\pi}{\sqrt{2}}(j + 1/2) + o(1).$$

In the context of the sloshing problem, some related conjectures have been proposed in [388].

5.4 Geometric inequalities for Steklov eigenvalues

5.4.1 Preliminaries

Let us start with the following simple observation. If a Euclidean domain $\Omega \subset \mathbb{R}^d$ is scaled by a factor $c > 0$, then

$$\sigma_k(c\,\Omega) = c^{-1}\sigma_k(\Omega). \tag{5.6}$$

Because of this scaling property, maximizing $\sigma_k(\Omega)$ among domains with fixed perimeter is equivalent to maximizing the normalized eigenvalues $\sigma_k(\Omega)|\partial\Omega|^{1/(d-1)}$ on arbitrary domains. Here and further on we use the notation $|\cdot|$ to denote the volume of a manifold.

All the results concerning geometric bounds are proved using a variational characterization of the eigenvalues. Let $\mathcal{E}(k)$ be the set of all k dimensional subspaces of the Sobolev space $H^1(\Omega)$ which are orthogonal to constants on the boundary $\partial\Omega$, then for each $k \geq 1$,

$$\sigma_{k+1}(\Omega) = \min_{E \in \mathcal{E}(k)} \sup_{0 \neq u \in E} R(u), \tag{5.7}$$

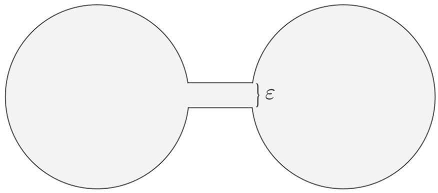

Fig. 5.3. A domain with a thin passage.

where the *Rayleigh quotient* is

$$R(u) = \frac{\int_\Omega |\nabla u|^2\, dA}{\int_{\partial\Omega} u^2\, dS}.$$

In particular, the first nonzero eigenvalue is given by

$$\sigma_2(\Omega) = \min\left\{R(u)\ :\ u \in H^1(\Omega),\ \int_{\partial\Omega} u\, dS = 0\right\}.$$

These variational characterizations are similar to those of Neumann eigenvalues on Ω, where the integral in the denominator of $R(u)$ would be on the domain Ω rather than on its boundary.

One last observation is in order before we discuss isoperimetric bounds. Let $\Omega_\epsilon := (-1, 1) \times (-\epsilon, \epsilon)$ be a thin rectangle ($0 < \epsilon << 1$). It is easy to see using using (5.7) that

$$\lim_{\epsilon\to 0} \sigma_k(\Omega_\epsilon) = 0, \qquad \text{for each } k \in \mathbb{N}. \tag{5.8}$$

In fact, it suffices for a family Ω_ϵ of domains to have a *thin collapsing passage* (see Figure 5.3) to guarantee that $\sigma_k(\Omega_\epsilon)$ becomes arbitrarily small as $\epsilon \searrow 0$ (see [892, section 2.2].) This follows from the variational characterization: the idea is to construct a sequence of k pairwise orthogonal test functions that oscillate inside the thin passage and vanish outside. Then the Dirichlet energy of such functions will be very small, while the denominator in the Rayleigh quotient remains bounded away from zero, due to the integration over the side of the passage. Hence, the Rayleigh quotient will tend to zero, yielding (5.8). When considering an isoperimetric constraint, it is therefore more interesting to maximize eigenvalues.

5.4.2 Isoperimetric upper bounds for Steklov eigenvalues on surfaces

On a compact surface with boundary, the following theorem gives a general upper bound in terms of the genus and the number of boundary components.

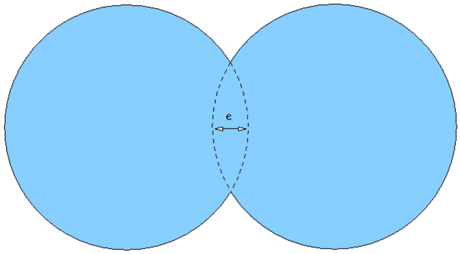

Fig. 5.4. A family of domains Ω_ϵ maximizing $\sigma_3 L$ in the limit as $\epsilon \to 0$.

Theorem 5.13 ([431]). [5.3] *Let Ω be a smooth orientable compact Riemannian surface with boundary $M = \partial\Omega$ of length L. Let γ be the genus of Ω and let l be the number of its boundary components. Then the following holds:*

$$\sigma_p \sigma_q L^2 \leq \begin{cases} \pi^2(\gamma + l)^2(p + q - 2)^2 & \text{if } p + q \text{ is even}, \\ \pi^2(\gamma + l)^2(p + q - 3)^2 & \text{if } p + q \text{ is odd}, \end{cases} \tag{5.9}$$

for any pair of integers $p, q \geq 2$. In particular by setting $p = q = k$ one obtains the following bound:

$$\sigma_k(\Omega)L(M) \leq 2\pi(\gamma + l)(k - 1). \tag{5.10}$$

The proof of Theorem 5.13 is based on the existence of a proper holomorphic covering map $\phi : \Omega \to \mathbb{D}$ of degree $\gamma + l$ (the Ahlfors map), which was proved in [414], and on an ingenious complex analytic argument due to J. Hersch, L. Payne and M. Schiffer [513], who used it to prove inequality (5.9) for planar domains. In this particular case, inequality (5.10) is known to be sharp. Indeed, it was proved in [892] that equality is attained in the limit by a family Ω_ϵ of domains degenerating to a disjoint union of $k - 1$ identical disks (see Figure 5.4). For $k = 2$, inequality (5.10) was proved in [392].

The earliest isoperimetric inequality for Steklov eigenvalues is that of R. Weinstock [873]. For simply–connected planar domains ($\gamma = 0$, $l = 1$), he proved that

$$\sigma_2(\Omega)L(\partial\Omega) \leq 2\pi \tag{5.11}$$

with equality if and only if Ω is a disk. Weinstock used an argument similar to that of G. Szegő [838], who obtained an isoperimetric inequality for the first nonzero Neumann eigenvalue of a simply–connected domain Ω normalized by the measure $|\Omega|$ rather than its perimeter. In fact, Weinstock's proof is the simplest application of the center of mass renormalization (also known as Hersch's lemma, see [428, 430, 519, 802]).

5.3 This result has been recently improved by M. Karpukhin [573]. Karpukhin's bound involves an explicit linear comibination of the genus, the number of boundary components and the index of the eigenvalue. It should also be compared to Theorem 5.18 below, where the constants are implicit, but the estimate is independent on the number of boundary components.

While Szegő's inequality can be generalized to an arbitrary Euclidean domain (see [871]), this is not true for Weinstock's inequality. In particular, as follows from the example below, Weinstock's inequality fails for non-simply–connected planar domains.

Example 5.14. *The Steklov eigenvalues and eigenfunctions of an annulus have been computed in [338]. On the annulus $A_\epsilon = \mathbb{D}\setminus B(0,\epsilon)$, there is a radially symmetric Steklov eigenfunction*

$$f(r) = -\left(\frac{1+\epsilon}{\epsilon\log(\epsilon)}\right)\log(r) + 1,$$

with the corresponding eigenvalue $\sigma = \frac{1+\epsilon}{\epsilon\log(1/\epsilon)}$. All other eigenfunctions are of the form

$$f_k(r,\theta) = (A_k r^k + A_{-k} r^{-k})T(k\theta) \qquad (\textit{with } k \in \mathbb{N})$$

where $T(k\theta) = \cos(k\theta)$ or $T(k\theta) = \sin(k\theta)$. In order for $f_k(r,\theta)$ to be a Steklov eigenfunction it is required that

$$\frac{\partial}{\partial_r} f_k(1,\theta) = \sigma f_k(1,\theta) \qquad \textit{and} \qquad -\frac{\partial}{\partial_r} f_k(\epsilon,\theta) = \sigma f_k(\epsilon,\theta),$$

which leads to the following system:

$$\begin{pmatrix} \sigma\epsilon^k + k\epsilon^{k-1} & \sigma\epsilon^{-k} - k\epsilon^{-k-1} \\ \sigma - k & \sigma + k \end{pmatrix}\begin{pmatrix} A_k \\ A_{-k}\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This system has a nonzero solution if and only if its determinant vanishes. After some simplifications, the Steklov eigenvalues of the annulus $A_\epsilon = \mathbb{D}\setminus B(0,\epsilon)$ are seen to be the roots of the quadratic polynomials

$$p_k(\sigma) = \sigma^2 - \sigma k\left(\frac{\epsilon+1}{\epsilon}\right)\left(\frac{1+\epsilon^{2k}}{1-\epsilon^{2k}}\right) + \frac{1}{\epsilon}k^2 \qquad (k \in \mathbb{N}).$$

Each of these roots contributes double eigenvalues, corresponding to the choice of a cos *or* sin *function for the angular part $T(k\theta)$ of the corresponding eigenfunction. For $\epsilon > 0$ small enough, this leads in particular to*

$$\sigma_2(A_\epsilon) = \frac{1}{2\epsilon}\frac{1+\epsilon^2}{1-\epsilon}\left(1 - \sqrt{1 - 4\epsilon\left(\frac{1-\epsilon}{1+\epsilon^2}\right)^2}\right). \tag{5.12}$$

It follows from formula (5.12) that for the annulus $A_\epsilon = B(0,1)\setminus B(0,\epsilon)$ one has

$$\sigma_2(A_\epsilon)L(\partial A_\epsilon) = 2\pi\sigma_2(\mathbb{D}) + 2\pi\epsilon + o(\epsilon) \quad \text{as } \epsilon \searrow 0. \tag{5.13}$$

Therefore, $\sigma_2(A_\epsilon)L(\partial A_\epsilon) > 2\pi\sigma_2(\mathbb{D})$ for $\epsilon > 0$ small enough (see Figure 5.5), and hence Weinstock's inequality (5.11) fails.

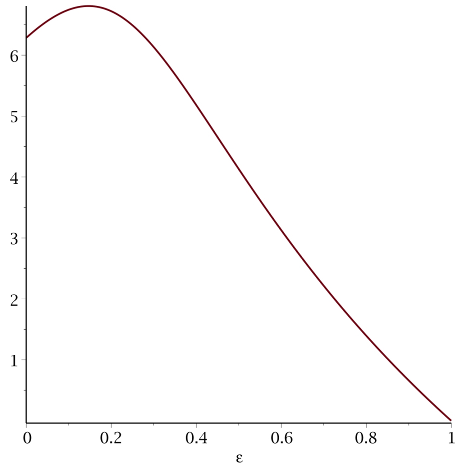

Fig. 5.5. The normalized eigenvalue $\sigma_2(A_\epsilon)L(\partial A_\epsilon)$

Remark 5.15. *One can also compute the Steklov eigenvalues of the spherical shell $\Omega_\epsilon := B(0,1) \setminus B(0,\epsilon) \subset \mathbb{R}^d$ for $d \geq 3$. The eigenvalues are the roots of certain quadratic polynomials which can be computed explicitly. Here again, it is true that for $\epsilon > 0$ small enough, $\sigma_2(\Omega_\epsilon)|\partial\Omega_\epsilon|^{\frac{1}{d-1}} > \sigma_2(\mathbb{B})|\partial\mathbb{B}|^{\frac{1}{d-1}}$. This computation was part of an unpublished undergraduate research project of E. Martel at Université Laval.*

Given that Weinstock's inequality is no longer true for non-simply–connected planar domains, one may ask whether the supremum of $\sigma_2 L$ among all planar domains of fixed perimeter is finite. This is indeed the case, as follows from the following theorem for $k = 2$ and $\gamma = 0$.

Theorem 5.16 ([279]). *There exists a universal constant $C > 0$ such that*

$$\sigma_k(\Omega)L(\partial\Omega) \leq C(\gamma + 1)k. \tag{5.14}$$

Theorem 5.16 leads to the following question:

Open problem 5.17. *What is the maximal value of $\sigma_2(\Omega)$ among Euclidean domains $\Omega \subset \mathbb{R}^d$ of fixed perimeter? On which domain (or in the limit of which sequence of domains) is it realized?*

Some related results will be discussed in subsection 5.4.3. In particular, in view of Theorem 5.26 [392], it is tempting to suggest that the maximum is realized in the limit by a sequence of domains with the number of boundary components tending to infinity.

The proof of Theorem 5.16 is based on N. Korevaar's metric geometry approach [598] as described in [438]. For $k = 2$, inequality (5.14) holds with $C = 4\pi$

(see [594]). For $k = 2$ and $\gamma = 0$, it holds with $C = 2\pi$ [392] (see Theorem 5.26 below). It is also possible to "decouple" the genus γ and the index k. The following theorem was proved by A. Hassannezhad [467], using a generalization of the Korevaar method in combination with concentration results from [281].

Theorem 5.18. *There exists two constants $A, B > 0$ such that*

$$\sigma_k(\Omega)L(\partial\Omega) \leq A\gamma + Bk.$$

At this point, we have considered maximization of the Steklov eigenvalues under the constraint of fixed perimeter. This is natural, since they are the eigenvalues of to the Dirichlet-to-Neumann operator, which acts on the boundary. Nevertheless, it is also possible to normalize the eigenvalues by fixing the measure of Ω. The following theorem was proved by F. Brock [192].

Theorem 5.19. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then*

$$\sigma_2(\Omega)|\Omega|^{1/d} \leq \omega_d^{1/d}, \tag{5.15}$$

with equality if and only if Ω is a ball. Here ω_d is the volume of the unit ball $\mathbb{B}^d \subset \mathbb{R}^d$.

Observe that no connectedness assumption is required this time.[5.4] The proof of Theorem 5.19 is based on a weighted isoperimetric inequality for moments of inertia of the boundary $\partial\Omega$. A quantitative improvement of Brock's theorem was obtained in [180] in terms of the *Fraenkel asymmetry* of a bounded domain $\Omega \subset \mathbb{R}^d$,:

$$\mathcal{A}(\Omega) := \inf\left\{\frac{\|1_\Omega - 1_B\|_{L^1}}{|\Omega|} \ :\ B \text{ is a ball with } |B| = |\Omega|\right\}.$$

Theorem 5.20. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then*

$$\sigma_2(\Omega)|\Omega|^{1/d} \leq \omega_d^{1/d}(1 - \alpha_d\mathcal{A}(\Omega)^2), \tag{5.16}$$

where $\alpha_d > 0$ depends only on the dimension.

The proof of Theorem 5.20 is based on a quantitative refinement of the isoperimetric inequality, see also [178] for related results on stability of the Dirichlet and Neumann eigenvalues and Chapter 7 for a complete overview on that topic. It would be interesting to prove a similar stability result for Weinstock's inequality:

Open problem 5.21. *Let Ω be a planar simply–connected domain such that the difference $2\pi - \sigma_2(\Omega)L(\partial\Omega)$ is small. Show that Ω must be close to a disk (in the sense of Fraenkel asymmetry or some other measure of proximity).*

5.4 Recent numerical results [16, 153] show that in two dimensions, the maximizer of σ_k under the area constraint is connected for each $k \geq 2$. Moreover, the maximizing domains appear to be smooth and have exactly k axes of symmetry.

5.4.3 Existence of maximizers and free boundary minimal surfaces

A *free boundary submanifold* is a proper minimal submanifold of some unit ball $\mathbb{B}^d$ with its boundary meeting the sphere $\mathbb{S}^{d-1}$ orthogonally. These are characterized by their Steklov eigenfunctions.

Lemma 5.22 ([393]). *A properly immersed submanifold Ω of the ball $\mathbb{B}^d$ is a free boundary submanifold if and only if the restriction to Ω of the coordinate functions $x_1, \cdots, x_d$ satisfy*

$$\begin{cases} \Delta x_i = 0 & \text{in } \Omega, \\ \frac{\partial x_i}{\partial n} = x_i & \text{on } \partial\Omega. \end{cases}$$

This link was exploited by A. Fraser and R. Schoen who developed the theory of extremal metrics for Steklov eigenvalues. See [392, 393] and especially [394] where an overview is presented.

Let $\sigma^\star(\gamma, l)$ be the supremum of $\sigma_2 L$ taken over all Riemannian metrics on a compact surface of genus γ with l boundary components. In [392], a geometric characterization of maximizers was proved.

Proposition 5.23. *Let Ω be a compact surface of genus γ with l boundary components and let g_0 be a smooth metric on Ω such that*

$$\sigma_2(\Omega, g_0)L(\partial\Omega, g_0) = \sigma^\star(\gamma, l).$$

Then there exist eigenfunctions $u_1, \cdots, u_d$ corresponding to $\sigma_2(\Omega)$ such that the map

$$u = (u_1, \cdots, u_d) : \Omega \to \mathbb{B}^d$$

is a conformal minimal immersion such that $u(\Omega) \subset \mathbb{B}^d$ is a free boundary submanifold, and u is an isometry on $\partial\Omega$ up to a rescaling by a constant factor.

This result was extended to higher eigenvalues σ_k in [394]. This characterization is similar to that of extremizers of the eigenvalues of the Laplace operator on surfaces (see [361, 362, 720]).

For surfaces of genus zero, Fraser and Schoen could also obtain an existence and regularity result for maximizers, which is the main result of their paper [392].

Theorem 5.24. *For each $l > 0$, there exists a smooth metric g on the surface of genus zero with l boundary components such that*

$$\sigma_2(\Omega, g)L_g(\partial\Omega) = \sigma^\star(0, l).$$

Similar existence results have been proved for the first nonzero eigenvalue of the Laplace–Beltrami operator in a fixed conformal class of a closed surface of arbitrary genus, in which case conical singularities have to be allowed (see [556, 756]).

Proposition 5.23 and Theorem 5.24 can be used to study optimal upper bounds for σ_2 on surfaces of genus zero. Observe that inequality (5.10) can be restated as

$$\sigma^\star(\gamma, l) \le 2\pi(\gamma + l).$$

This bound is not sharp in general. For instance, Fraser and Schoen [392] proved that on annuli ($\gamma = 0, l = 2$), the maximal value of $\sigma_2(\Omega)L(\partial\Omega)$ is attained by the *critical catenoid* ($\sigma_2 L \sim 4\pi/1.2$), which is the minimal surface $\Omega \subset B^3$ parametrized by

$$\phi(t, \theta) = c(\cosh(t)\cos(\theta), \cosh(t)\sin(\theta), t),$$

where the scaling factor $c > 0$ is chosen so that the boundary of the surface Ω meets the sphere $\mathbb{S}^2$ orthogonally.

Theorem 5.25 ([392]). *The supremum of $\sigma_2(\Omega)L(\partial\Omega)$ among surfaces of genus 0 with two boundary components is attained by the* critical catenoid. *The maximizer is unique up to conformal changes of the metric which are constant on the boundary.*

The uniqueness can be proved using Proposition 5.23 by showing that the critical catenoid is the unique free boundary annulus in a Euclidean ball. The maximization of $\sigma_2 L$ for the Möbius bands has also been considered in [392].

For surfaces of genus zero with arbitrary number of boundary components, the maximizers are not known explicitly, but the asymptotic behaviour for large number of boundary components is understood [392].

Theorem 5.26. *The sequence $\sigma^\star(0, l)$ is strictly increasing and converges to 4π. For each $l \in \mathbb{N}$ a maximizing metric is achieved by a free boundary minimal surface Ω_l of area less than 2π. The limit of these minimal surfaces as $l \nearrow +\infty$ is a double disk.*

The results discussed above lead to the following question:

Open problem 5.27. [5.5] *Let Ω be a surface of genus γ with l boundary components. Does there exist a smooth Riemannian metric g_0 such that*

$$\sigma_2(\Omega, g_0)L(\partial\Omega, g_0) \ge \sigma_2(\Omega, g)L(\partial\Omega, g)$$

for each Riemannian metric g?

Free boundary minimal surfaces were used as a tool in the study of maximizers for σ_2, but this interplay can be reversed and used to obtain interesting geometric results.

5.5 Under certain assumptions, the existence of a regular maximizer of σ_k, $k \ge 2$, on an arbitrary Riemannian surface has recently been established in [755]. However, this result does not provide an answer to the open problem.

Corollary 5.28. *For each $l \geq 1$, there exists an embedded minimal surface of genus zero in $\mathbb{B}^3$ with l boundary components satisfying the free boundary condition.*

5.4.4 Geometric bounds in higher dimensions

In dimensions $d = \dim(\Omega) \geq 3$, isoperimetric inequalities for Steklov eigenvalues are more complicated, as they involve other geometric quantities, such as the isoperimetric ratio:

$$I(\Omega) = \frac{|M|}{|\Omega|^{\frac{d-1}{d}}}.$$

For the first nonzero eigenvalue σ_2, it is possible to obtain upper bounds for general compact manifolds with boundary in terms of $I(\Omega)$ and of the relative conformal volume, which is defined below. Let Ω be a compact manifold of dimension d with smooth boundary M. Let $m \in \mathbb{N}$ be a positive integer. The *relative m-conformal volume* of Ω is

$$V_{rc}(\Omega, m) = \inf_{\phi:\Omega \hookrightarrow B^m} \sup_{\gamma \in M(m)} \mathrm{Vol}(\gamma \circ \phi(\Omega)),$$

where the infimum is over all conformal immersions $\phi : \Omega \hookrightarrow \mathbb{B}^m$ such that $\phi(M) \subset \partial\mathbb{B}^m$, and $M(m)$ is the group of conformal diffeomorphisms of the ball. This conformal invariant was introduced in [393]. It is similar to the celebrated conformal volume of P. Li and S.-T. Yau [659].

Theorem 5.29. *[393] Let Ω be a compact Riemannian manifold of dimension d with smooth boundary M. For each positive integer m, the following holds:*

$$\sigma_2(\Omega)|M|^{\frac{1}{d-1}} \leq \frac{d V_{rc}(\Omega, m)^{2/d}}{I(\Omega)^{\frac{d-2}{d-1}}}. \tag{5.17}$$

In case of equality, there exists a conformal harmonic map $\phi : \Omega \to \mathbb{B}^m$ which is a homothety on $M = \partial\Omega$ and such that $\phi(\Omega)$ meets ∂B^m orthogonally. If $d \geq 3$, then ϕ is an isometric minimal immersion of Ω and it is given by a subspace of the first eigenspace.

The proof uses coordinate functions as test functions and is based on the Hersch center of mass renormalization procedure. It is similar to the proof of the Li-Yau inequality [659].

For higher eigenvalues, the following upper bound for bounded domains was proved by B. Colbois, A. El Soufi and the first author in [279].

Theorem 5.30. *Let N be a Riemannian manifold of dimension d. If N is conformally equivalent to a complete Riemannian manifold with non-negative Ricci curvature, then for each domain $\Omega \subset N$, the following holds for each $k \geq 1$,*

$$\sigma_k(\Omega)|M|^{\frac{1}{d-1}} \leq \frac{\alpha(d)}{I(\Omega)^{\frac{d-2}{d-1}}} k^{2/d}. \tag{5.18}$$

where $\alpha(d)$ is a constant depending only on d.

The proof of Theorem 5.30 is based on the methods of metric geometry initiated in [598] and further developed in [438]. In combination with the classical isoperimetric inequality, Theorem 5.30 leads to the following corollary.

Corollary 5.31. *There exists a constant C_d such that for any Euclidean domain $\Omega \subset \mathbb{R}^d$*

$$\sigma_k(\Omega)|\partial\Omega|^{\frac{1}{d-1}} \leq C_d k^{2/d}.$$

Similar results also hold for domains in the hyperbolic space $\mathbb{H}^d$ and in the hemisphere of $\mathbb{S}^d$. An interesting question raised in [279] is whether or not one can replace the exponent $2/d$ in Corollary 5.31 by $1/(d-1)$, which should be optimal in view of Weyl's law (5.2):

Open problem 5.32. *Does there exist a constant C_d such that any bounded Euclidean domain $\Omega \subset \mathbb{R}^d$ satisfies*

$$\sigma_k(\Omega)|\partial\Omega|^{\frac{1}{d-1}} \leq C_d k^{\frac{1}{d-1}}?$$

While it might be tempting to think that inequality (5.18) should also hold with the exponent $1/(d-1)$, this is necessarily false since it would imply a universal upper bound on the isoperimetric ratio $I(\Omega)$ for Euclidean domains.

5.4.5 Lower bounds

In [365], J. Escobar proved the following lower bound.

Theorem 5.33. *Let Ω be a smooth compact Riemannian manifold of dimension ≥ 3 with boundary $M = \partial\Omega$. Suppose that the Ricci curvature of Ω is non-negative and that the second fundamental form of M is bounded below by $k_0 > 0$, then $\sigma_2 > k_0/2$.*

The proof is a simple application of Reilly's formula. In [366], Escobar conjectured the stronger bound $\sigma_2 \geq k_0$, which he proved for surfaces. For convex planar domains, this had already been proved by Payne [745]. Earlier lower bounds for convex and star-shaped planar domains are due to Kuttler and Sigillito [608, 609].

In more general situations (e.g. no convexity assumption), it is still possible to bound the first eigenvalue from below, in a way similar to the classical Cheeger inequality. The classical Cheeger constant associated to a compact Riemannian manifold Ω with boundary $M = \partial\Omega$ is defined by

$$h_c(\Omega) := \inf_{|A| \leq \frac{|\Omega|}{2}} \frac{|\partial A \cap \operatorname{int} \Omega|}{|A|}.$$

where the infimum is over all Borel subsets of Ω such that $|A| \le |\Omega|/2$. In [559] P. Jammes introduced the following Cheeger type constant for the Steklov problem:

$$h_{\mathrm{j}}(\Omega) := \inf_{|A| \le \frac{|\Omega|}{2}} \frac{|\partial A \cap \operatorname{int} \Omega|}{|A \cap \partial\Omega|}.$$

He proved the following lower bound.

Theorem 5.34. *Let Ω be a smooth compact Riemannian manifold with boundary $M = \partial\Omega$. Then*

$$\sigma_2(\Omega) \ge \frac{1}{4} h_c(\Omega) h_j(\Omega) \tag{5.19}$$

The proof of this theorem uses the coarea formula and follows the proof of the classical Cheeger inequality quite closely. Previous lower bounds were also obtained in [365] in terms of a related Cheeger type constant and of the first eigenvalue of a Robin problem on Ω.

5.4.6 Surfaces with large Steklov eigenvalues

The previous discussion immediately raises the question of whether or not there exist surfaces with an arbitrarily large normalized first Steklov eigenvalue. The question was settled by the first author and B. Colbois in [280].

Theorem 5.35. *There exists a sequence $\{\Omega_N\}_{N \in \mathbb{N}}$ of compact surfaces with boundary and a constant $C > 0$ such that for each $N \in \mathbb{N}$, genus$(\Omega_N) = 1 + N$, and*

$$\sigma_2(\Omega_N) L(\partial\Omega_N) \ge CN.$$

The proof is based on the construction of surfaces which are modelled on a family of expander graphs.

Remark 5.36. *The literature on geometric bounds for Steklov eigenvalues is expanding rather fast. There is some interest in considering the maximization of various functions of the Steklov eigenvalues. See [338, 355, 427, 503]. In the framework of comparison geometry, σ_2 was studied is [367] and more recently in [149]. For submanifolds of $\mathbb{R}^d$, upper bounds involving the mean curvatures of $M = \partial\Omega$ have been obtained in [553]. Higher eigenvalues on annuli have been studied in [378]. Isoperimetric bounds for the first nonzero eigenvalue of the Dirichlet-to-Neumann operator on forms have been recently obtained in [785, 786].*

5.5 Isospectrality and spectral rigidity

5.5.1 Isospectrality and the Steklov problem

Adapting the celebrated question of M. Kac "Can one hear the shape of a drum?" to the Steklov problem, one may ask:

Open problem 5.37. *Do there exist planar domains which are not isometric and have the same Steklov spectrum?*

We believe the answer to this question is negative. Moreover, the problem can be viewed as a special case of a conjecture put forward in [564]: two surfaces have the same Steklov spectrum if and only if there exists a conformal mapping between them such that the conformal factor on the boundary is identically equal to one. Note that the "if" part immediately follows from the variational principle (5.7). Indeed, the numerator of the Rayleigh quotient for Steklov eigenvalues is the usual Dirichlet energy, which is invariant under conformal transformations in two dimensions. The denominator also stays the same if the conformal factor is equal to one on the boundary. Therefore, the Steklov spectra of such conformally equivalent surfaces coincide. For simply connected domains, a closely related question is to find out whether a smooth positive function $a \in C^\infty(S^1)$ is determined by the spectrum of $a\mathcal{D}_{\mathbb{D}}$, up to conformal automorphisms of the disk. A positive answer to this question would imply that smooth simply connected domains are spectrally determined (see [564]). In [352], calculations of the zeta function were used to prove a weaker statement — namely, that a family of smooth simply connected planar domains is pre-compact in the topology of a certain Sobolev space.

In higher dimensions, the Dirichlet energy is not conformally invariant, and therefore the approach described above does not work. However, one can construct Steklov isospectral manifolds of dimension $d \geq 3$ with the help of Example 5.3. Indeed, given two compact manifolds M_1 and M_2 which are Laplace-Beltrami isospectral (there are many known examples of such pairs, see, for instance, [232, 436, 834]), consider two cylinders $\Omega_1 = M_1 \times [0, L]$ and $\Omega_2 = M_2 \times [0, L]$, $L > 0$. It follows from Example 5.3 that Ω_1 and Ω_2 have the same Steklov spectra. Recently, examples of higher-dimesional Steklov isospectral manifolds with connected boundaries were announced in [435].

In all known constructions of Steklov isospectral manifolds, their boundaries are Laplace isospectral. The following question was asked in [429]:

Open problem 5.38. *Do there exist Steklov isospectral manifolds such that their boundaries are not Laplace isospectral?*

5.5.2 Rigidity of the Steklov spectrum: the case of a ball

It is an interesting and challenging question to find examples of manifolds with boundary that are uniquely determined by their Steklov spectrum. In this subsection we discuss the seemingly simple example of Euclidean balls.

Proposition 5.39. *A disk is uniquely determined by its Steklov spectrum among all smooth Euclidean domains.*

Proof. Let Ω be an Euclidean domain which has the same Steklov spectrum as the disk of radius r. Then, by Corollary 5.5 one immediately deduces that Ω is a planar domain of perimeter $2\pi r$. Moreover, it follows from Theorem 5.6 that Ω is simply–connected. Therefore, since the equality in Weinstock's inequality (5.11) is achieved for Ω, the domain Ω is a disk of radius r. □

Remark 5.40. *The smoothness hypothesis in the proposition above seems to be purely technical. We have to make this assumption since we make use of Theorem 5.6.*

The above result motivates the following open problem:

Open problem 5.41. *Let $\Omega \subset \mathbb{R}^d$ be a domain which is isospectral to a ball of radius r. Show that it is a ball of radius r.*

Note that Theorem 5.19 does not yield a solution to this problem because the volume $|\Omega|$ is not a Steklov spectrum invariant. Using the heat invariants of the Dirichlet-to-Neumann operator (see subsection 5.2.2), one can prove the following statement in dimension three.

Proposition 5.42. *Let $\Omega \subset \mathbb{R}^3$ be a domain with connected and smooth boundary M. Suppose its Steklov spectrum is equal to that of a ball of radius r. Then Ω is a ball of radius r.*

This result was obtained in [764], and we sketch its proof below. First, let us show that M is simply–connected. We use an adaptation of a theorem of Zelditch on multiplicities [891] proved using microlocal analysis. Namely, since Ω is Steklov isospectral to a ball, the multiplicities of its Steklov eigenvalues grow as $m_k = Ck + \mathcal{O}(1)$, where $C > 0$ is some constant and m_k is the multiplicity of the k-th *distinct* eigenvalue (cf. Example 5.2). Then one deduces that M is a Zoll surface (that is, all geodesics on M are periodic with a common period), and hence it is simply–connected [141].

Therefore, the following formula holds for the coefficient a_2 in the Steklov heat trace asymptotics (5.5) on Ω:

$$a_2 = \frac{1}{16\pi}\int_M H_1^2 + \frac{1}{12}.$$

Here $H_1(x)$ denotes the mean curvature of M at the point x, and the term $\frac{1}{12}$ is obtained from the Gauss–Bonnet theorem using the fact that M is simply–connected. We have then: $\int_M H_1^2 = \int_{S_r} H_1^2$, where $S_r = \partial B_r$.

On the other hand, it follows from (5.2) and Corollary 5.8 that $\mathrm{Vol}(M)$ and $\int_M H_1$ are Steklov spectral invariants. Therefore,

$$\mathrm{Area}(M) = \mathrm{Area}(S_r), \quad \int_M H_1 = \int_{S_r} H_1.$$

Hence

$$\sqrt{\mathrm{Area}(M)}\left(\int_M H_1^2\right)^{1/2} - \left|\int_M H_1\right| = \sqrt{\mathrm{Area}(S_r)}\left(\int_{S_r} H_1^2\right)^{1/2} - \left|\int_{S_r} H_1\right| = 0.$$

Since the Cauchy-Schwarz inequality becomes an equality only for constant functions, one gets that H_1 must be constant on M. By a theorem of Alexandrov [18], the only compact surfaces of constant mean curvature embedded in $\mathbb{R}^3$ are round spheres. We conclude that M is itself a sphere of radius r and therefore Ω is isometric to B_r. This completes the proof of the proposition.

5.6 Nodal geometry and multiplicity bounds

5.6.1 Nodal domain count

The study of nodal domains and nodal sets of eigenfunctions is probably the oldest topic in geometric spectral theory, going back to the experiments of E. Chladni with vibrating plates. The fundamental result in the subject is Courant's nodal domain theorem which states that the k-th eigenfunction of the Dirichlet boundary value problem has at most k nodal domains. The proof of this statement uses essentially two ingredients: the variational principle (see section 5.4.1) and the unique continuation for solutions of second order elliptic equations. It can therefore be extended essentially verbatim to Steklov eigenfunctions (see [574, 609]).

Theorem 5.43. *Let Ω be a compact Riemannian manifold with boundary and u_k be an eigenfunction corresponding to the Steklov eigenvalue σ_k. Then u_k has at most k nodal domains.*

Apart from the "interior" nodal domains and nodal sets of Steklov eigenfunctions, a natural problem is to study the boundary nodal domains and nodal sets, that is, the nodal domains and nodal sets of the eigenfunctions of the Dirichlet-to-Neumann operator.

The proof of Courant's theorem cannot be generalized to the Dirichlet-to-Neumann operator because it is nonlocal. The following problem therefore arises:

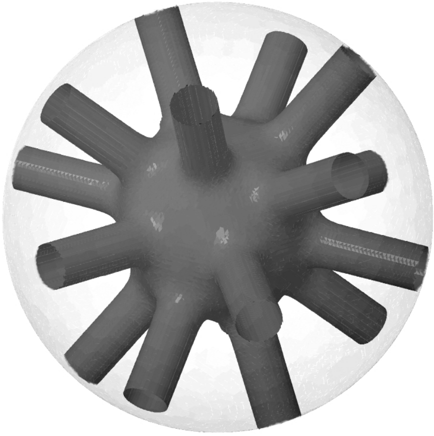

Fig. 5.6. A surface inside a ball creating only two connected components in the interior and a large number of connected components on the boundary sphere.

Open problem 5.44. *Let Ω be a Riemannian manifold with boundary M. Find an upper bound for the number of nodal domains of the k-th eigenfunction of the Dirichlet-to-Neumann operator on M.*

For surfaces, a simple topological argument shows that the bound on the number of interior nodal domains implies an estimate on the number of boundary nodal domains of a Steklov eigenfunction. In particular, the k-th nontrivial Dirichlet-to-Neumann eigenfunction on the boundary of a simply–connected planar domain has at most $2k$ nodal domains [19, Lemma 3.4].

In higher dimensions, the number of interior nodal domains does not control the number of boundary nodal domains (see Figure 5.6), and therefore new ideas are needed to tackle Open Problem 5.44. However, there are indications that a Courant-type (i.e. $\mathcal{O}(k)$) bound should hold in this case as well. For instance, this is the case for cylinders and Euclidean balls (see Examples 5.2 and 5.3).

5.6.2 Geometry of the nodal sets

The nodal sets of Steklov eigenfunctions, both interior and boundary, remain largely unexplored. The basic property of the nodal sets of Laplace–Beltrami eigenfunctions is their density on the scale of $1/\sqrt{\lambda}$, where λ is the eigenvalue (cf. [889], see also Figure 5.7). This means that for any manifold Ω, there exists a constant C such that for any eigenvalue λ large enough, the corresponding eigenfunction ϕ_λ has a zero in any geodesic ball of radius $C/\sqrt{\lambda}$. This motivates the following questions (see also Figure 5.7):

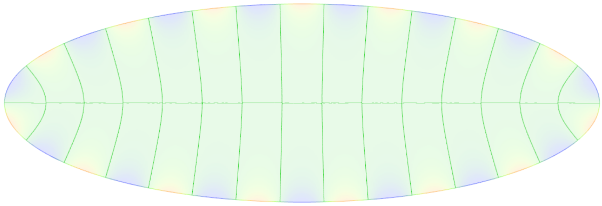

Fig. 5.7. The nodal lines of the 30th eigenfunction on an ellipse.

Open problem 5.45. *(i) Are the nodal sets of Steklov eigenfunctions on a Riemannian manifold Ω dense on the scale $1/\sigma$ in Ω? (ii) Are the nodal sets of the Dirichlet-to-Neumann eigenfunctions dense on the scale $1/\sigma$ in $M = \partial\Omega$?*

For smooth simply–connected planar domains, a positive answer to question (ii) follows from the work of Shamma [807] on the asymptotic behaviour of Steklov eigenfunctions. On the other hand, the explicit representation of eigenfunctions on rectangles implies that there exist eigenfunctions of arbitrary high order which have zeros only on one pair of parallel sides. Therefore, a positive answer to (ii) may hold only under some regularity assumptions on the boundary.

Another fundamental problem in nodal geometry is to estimate the size of the nodal set. It was conjectured by S.-T. Yau that for any Riemannian manifold of dimension d,

$$C_1\sqrt{\lambda} \le \mathcal{H}_{d-1}(\mathcal{N}(\phi_\lambda)) \le C_2\sqrt{\lambda},$$

where $\mathcal{H}_{d-1}(\mathcal{N}(\phi_\lambda))$ denotes the $d-1$-dimensional Hausdorff measure of the nodal set $\mathcal{N}(\phi_\lambda)$ of a Laplace-Beltrami eigenfunction ϕ_λ, and the constants C_1, C_2 depend only on the geometry of the manifold. Similar questions can be asked in the Steklov setting:

Open problem 5.46. *Let Ω be an d-dimensional Riemannian manifold with boundary M. Let u_σ be an eigenfunction of the Steklov problem on Ω corresponding to the eigenvalue σ and let $\phi_\sigma = u_\sigma|_M$ be the corresponding eigenfunction of the Dirichlet-to-Neumann operator on M. Show that*

(i) $C_1\sigma \le \mathcal{H}_{d-1}(\mathcal{N}(u_\sigma)) \le C_2\sigma$,

(ii) $C_1'\sigma \le \mathcal{H}_{d-2}(\mathcal{N}(\phi_\sigma)) \le C_2'\sigma$,

where the constants C_1, C_2, C_1', C_2' depend only on the manifold.

Some partial results on this problem are known. In particular, the upper bound in (ii) was conjectured by [119] and proved in [889] for real analytic manifolds with real analytic boundary. A lower bound on the size of the nodal set $\mathcal{N}(\phi_\sigma)$ for smooth Riemannian manifolds (though weaker than the one conjectured in (ii) in dimensions ≥ 3)

was recently obtained in [868] using an adaptation of the approach of [822] to nonlocal operators.

The upper bound in (i) is related to the question of estimating the size of the zero set of a harmonic function in terms of its frequency (see [445]). In [765], this approach is combined with the methods of potential theory and complex analysis in order to obtain both upper and lower bounds in (i) for real analytic Riemannian surfaces.[5.6] Let us also note that the Steklov eigenfunctions decay rapidly away from the boundary [529], and therefore the problem of understanding the properties of the nodal set in the interior is somewhat analogous to the study of the zero sets of Schrödinger eigenfunctions in the "forbidden regions" (see [446]).

5.6.3 Multiplicity bounds for Steklov eigenvalues

In two dimensions, the estimate on the number of nodal domains allows us to control the eigenvalue multiplicities (see [142, 268]). The argument roughly goes as follows: if the multiplicity of an eigenvalue is high, one can construct a corresponding eigenfunction with a high enough vanishing order at a certain point of a surface. In the neighbourhood of this point the eigenfunction looks like a harmonic polynomial, and therefore the vanishing order together with the topology of a surface yield a lower bound on the number of nodal domains. To avoid a contradiction with Courant's theorem, one deduces a bound on the vanishing order, and hence on the multiplicity.

This general scheme was originally applied to Laplace-Beltrami eigenvalues, but it can be also adapted to prove multiplicity bounds for Steklov eigenvalues. For simply connected surfaces, this idea was used in [19]. For general Riemannian surfaces, interestingly enough, one can obtain estimates of two kinds. Recall that the Euler characteristic χ of an orientable surface of genus γ with l boundary components equals $2 - 2\gamma - l$, and of a non-orientable one is equal to $2 - \gamma - l$. Putting together the results of [392, 558, 560, 574] we get the following bounds:

Theorem 5.47. *Let Σ be a compact surface of Euler characteristic χ with l boundary components. Then the multiplicity $m_k(\Sigma)$ for any $k \geq 2$ satisfies the following inequalities:*

$$m_k(\Sigma) \leq 2k - 2\chi - 2l + 3, \tag{5.20}$$

$$m_k(\Sigma) \leq k - 2\chi + 2. \tag{5.21}$$

Note that the right-hand side of (5.20) depends only on the index of the eigenvalue k and on the genus γ of the surface, while the right-hand side of (5.21) depends also

5.6 See also recent results of J. Zhu [893, 894]. In particular, in [893] the upper bound in (i) was proved for real-analytic Riemannian manifolds of arbitrary dimension.

on the number of boundary components. Inequality (5.21) in this form was proved in [558]. In particular, it is sharp for the first eigenvalue of simply connected surfaces ($\chi = 1$, the maximal multiplicity is two, see also [19]) and for surfaces homeomorphic to a Möbius band ($\chi = 0$, the maximal multiplicity is four). Inequality 5.20 is sharp for surfaces homeomorphic to an annulus ($\chi = 0$, $l = 2$, the maximal multiplicity is three and attained by the critical catenoid, see Theorem 5.25).

While these bounds are sharp in some cases, they are far from optimal for large k. In fact, the following result is an immediate corollary of Theorem 5.4.

Corollary 5.48. *[429] For any smooth compact Riemannian surface Ω with l boundary components, there is a constant N depending on the metric on Ω such that for $j > N$, the multiplicity of σ_j is at most $2l$.*

Remark 5.49. *The multiplicity of the first nonzero eigenvalue σ_2 has been linked to the relative chromatic number of the corresponding surface with boundary in [558].*

Remark 5.50. *It is well-known that the spectrum of the Laplace-Beltrami operator is generically simple [17, 857]. It is likely that the same is true for the Steklov spectrum, however, to our knowledge, such a result has not been established yet.*

For manifolds of dimension $d \geq 3$, no general multiplicity bounds for Steklov eigenvalues are available. Moreover, given a Riemannian manifold Ω of dimension $d \geq 3$ and any non-decreasing sequence of N positive numbers, one can find a Riemannian metric g in a given conformal class, such that this sequence coincides with the first N nonzero Steklov eigenvalues of (M, g) [559].

Theorem 5.51. *Let Ω be a compact manifold of dimension $d \geq 3$ with boundary. Let m be a positive integer and let $0 = s_0 < s_1 \leq \cdots \leq s_m$ be a finite sequence. Then there exists a Riemannian metric g on Ω such that $\sigma_j = s_j$ for $j = 0, \cdots, m$.*

For Laplace-Beltrami eigenvalues, a similar result was obtained in [282]. It is plausible that multiplicity bounds for Steklov eigenvalues in higher dimensions could be obtained under certain geometric assumptions, such as curvature constraints.

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