

Physics Area - PhD course in
Theoretical Particle Physics

Generalized Symmetries and Entanglement in Quantum Field Theory

Candidate:
Marco Venuti

Advisor:
Prof. Francesco Benini

Academic Year 2025-26



Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. The discussion is based on the following works:

- [1] Francesco Benini, Pasquale Calabrese, Michele Fossati, Amartya Harsh Singh, and Marco Venuti. *Entanglement asymmetry for higher and noninvertible symmetries*. Sept. 2025. arXiv: [2509.16311 \[hep-th\]](#).
- [2] Francesco Benini, Michele Tortora, and Marco Venuti. *Symmetry-Resolved Entanglement with Anomalous and Noninvertible Symmetries*. To appear.

Abstract

This thesis studies two quantum-information quantities in theories with generalized symmetries: entanglement asymmetry, which measures symmetry breaking, and symmetry-resolved entanglement entropy, which decomposes entanglement into symmetry sectors. Generalized symmetries are described by topological defects and include ordinary group, higher-form, and noninvertible symmetries. In this setting, both quantities must be defined with respect to the algebra of symmetry operators acting on the Hilbert space under consideration. In two spacetime dimensions, this may be the fusion or tube algebra on a circle, depending on whether twisted sectors are included, or the strip algebra on an interval. The interval case also depends on the boundary conditions introduced at the entangling surface; since anomalies constrain the available boundary conditions, they can affect the subsystem symmetry algebra and its representations. The first work presented here [1] develops entanglement asymmetry for generalized symmetries by constructing a unique universal symmetrizer for strongly separable algebras and applying it to fusion, tube, and strip algebras. The second [2] constructs symmetry-resolved entanglement entropy in the presence of anomalous and noninvertible symmetries, showing through several examples how the allowed entangling boundaries affect sector probabilities and entanglement equipartition. Several technical appendices provide some categorical and algebraic background, as well as detailed constructions and proofs.

Contents

1	Introduction	7
1.1	From group symmetries to topological defects	7
1.2	Entanglement entropy in quantum field theory	9
1.3	Symmetry-resolved entanglement entropy	10
1.4	Entanglement asymmetry	11
1.5	Anomalies, boundary conditions, and subsystem algebras	12
1.6	Summary of results	13
1.6.1	Asymmetry for generalized symmetries	13
1.6.2	Noninvertible symmetry-resolved entanglement entropy	14
1.7	Outlook and organization of the thesis	15
2	Symmetry categories and symmetry algebras	17
2.1	Fusion algebra	20
2.2	Tube algebra	21
2.3	Strip algebra	24
3	The asymmetry	27
3.1	Symmetrizer for strongly separable algebras	27
3.1.1	Abelian algebras	31
3.2	Asymmetry for the circle Hilbert space	32
3.2.1	Excited states in CFTs	32
3.2.2	Spontaneous symmetry breaking	34
3.3	Asymmetry for the interval Hilbert space	35
3.3.1	Strongly-symmetric boundary conditions	37
3.3.2	Weakly-symmetric boundary conditions	39
3.3.3	Non-symmetric boundary conditions	39
3.3.4	Symmetrizer from (weak) Hopf algebra structure	42
4	Noninvertible symmetry-resolved entanglement entropy	45
4.1	Strip-algebra representations	46
4.2	Symmetry-resolved entanglement in CFTs	50
5	Examples of asymmetry calculations	54
5.1	Group-like symmetries	54
5.1.1	Tube algebra	55
5.1.2	Strip algebras	55
5.2	Ising symmetry – circle Hilbert space	60
5.2.1	Fusion algebra	60
5.2.2	Tube algebra	63
5.3	$\mathbb{Z}_2 \times \mathbb{Z}_2$ Tambara-Yamagami fusion category	65

5.3.1	Fusion algebra	66
5.3.2	Strip algebra	66
5.3.3	Example: the $(\text{Ising})^2$ CFT	68
5.4	Ising symmetry – interval Hilbert space	73
5.5	Fibonacci fusion category	76
5.6	Haagerup fusion ring	81
5.7	The $\frac{1}{2}E_6$ fusion ring	82
6	Examples of SREE calculations	84
6.1	Group-like symmetries	85
6.1.1	$G = (\mathbb{Z}_2)^3$ with cubic anomaly	86
6.1.2	$G = S_3$	88
6.2	Fibonacci fusion category	91
6.3	Ising CFT	91
6.4	$\mathfrak{su}(2)_{10}$ WZW model with E_6 modular invariant	91
A	Some facts in category theory	101
A.1	Summary of definitions	101
A.2	Direct sums, tensor products and bases of morphisms	102
A.3	Rigid and pivotal structures	105
A.4	Self-dual objects and Frobenius-Schur indicators	107
A.5	Summary of conventions	109
B	Complements on the tube algebra	111
C	Strongly separable and C^*-algebras	114
C.1	Strongly separable algebras	114
C.2	General traces and projectors	115
C.3	C^* -algebras	117
C.4	Quantum trace on the strip algebra	119
C.5	Explicit form of ρ_S	121
C.6	Some properties of (Rényi) asymmetry	122
D	Hopf algebras and their modules	124
D.1	Basic definitions	124
D.2	Representations of WHAs and their characters	126
D.3	Symmetrizer in Hopf and weak Kac algebras	127
E	Strip algebras as weak Hopf algebras	129
F	Module categories over Vec_G^ω	133

G	Anomaly cocycles	135
G.1	$G = \mathbb{Z}_N$	135
G.2	$G = S_3$	136

1 Introduction

Symmetry and entanglement provide complementary tools to study quantum systems. A symmetry organizes states and operators into sectors and leads to conservation laws, selection rules, and constraints on phases and renormalization-group flows. Entanglement instead characterizes the correlations between a region and its complement and provides information that is not contained in individual local observables. Combining the two raises two questions: how is entanglement distributed among symmetry sectors, and to what extent does the state of a given region break the symmetry? This thesis studies how these questions should be formulated for generalized symmetries. In quantum field theory, symmetries can be described by topological defects, a framework that includes ordinary groups, higher-form symmetries, and noninvertible symmetries organized by fusion categories. For generalized symmetries, however, the fusion rules alone do not determine the symmetry action on a Hilbert space. The relevant algebra of symmetry operators depends on the spatial geometry, the sectors included, and, for an interval, the boundary conditions imposed at the entangling surface. In two spacetime dimensions, this leads to the fusion and tube algebras on a circle and to the strip algebra on an interval. Since anomalies constrain the available boundary conditions, they can also affect the symmetry properties of the reduced density matrix and the sectors into which its entanglement can be resolved. Motivated by these issues, the first work presented here [1] develops entanglement asymmetry as a measure of symmetry breaking for generalized symmetries, while the second [2] constructs symmetry-resolved entanglement entropy in the presence of anomalous and noninvertible symmetries.

1.1 From group symmetries to topological defects

The importance of symmetry in QFT is difficult to overstate. Since a global symmetry commutes with the Hamiltonian, its representations provide labels for states and particles. Conservation laws and selection rules constrain processes without requiring a perturbative analysis. Spontaneous symmetry breaking underlies the Landau description of phases and phase transitions [3, 4] and is closely tied to important questions such as confinement [5]. Most powerfully, an 't Hooft anomaly should be matched along the renormalization-group flow. That is, an anomaly present in the ultraviolet must be appropriately reproduced in the infrared, thereby constraining the possible degrees of freedom, vacuum structure, topological order, or interactions. Global symmetries are physical properties of the theory itself in this sense, whereas a gauge “symmetry” is a redundancy of a particular description.

Traditionally, an internal symmetry was specified by a group G acting on local fields and states. Said in the modern language, for an ordinary *zero-form symmetry* element g , one can insert an operator $U_g(\Sigma_{d-1})$ supported on a codimension-one manifold Σ_{d-1} , in Euclidean space-time. Ward identities imply that this operator may be deformed without changing correlation functions, provided that it does not cross a charged insertion. Its action is detected when the defect surrounds a local operator. Topological operator exists for a discrete symmetry, even though no Noether current generates it. The conceptual step of generalized symmetries is to

regard the topologicity of such operators as the fundamental property defining a symmetry, without requiring that they arise from a group action on local fields [5–7].

Two extensions follow immediately. A p -form symmetry in d spacetime dimensions is generated by topological defects of codimension $p + 1$ and acts naturally on p -dimensional operators through linking. For example, in gauge theories Wilson and 't Hooft lines can be charged under one-form symmetries [5, 8]. The linking picture, however, does not exhaust the possible action of generalized symmetries. For instance, condensation defects obtained by gauging a higher-form symmetry along a submanifold may act trivially on local operators while acting nontrivially on extended ones [9].

The second extension relaxes invertibility. Topological defects can always be fused, but their fusion need not be a group product. In two dimensions, a finite collection of simple topological lines satisfies

$$\mathcal{L}_a \otimes \mathcal{L}_b = \bigoplus_c N_{ab}^c \mathcal{L}_c, \quad N_{ab}^c \in \mathbb{Z}_{\geq 0}. \quad (1.1)$$

If for a given line $\mathcal{L}_a$ there is no other line such that their fusion gives only the identity, $\mathcal{L}_a$ is called *noninvertible*. Such lines were present long before the modern terminology: the Verlinde lines of rational CFT, e.g. the Kramers-Wannier duality line of the Ising model, are canonical examples [10–13]. What changed was the realization that they should be treated as symmetries and can be used as broadly as ordinary ones [14–16].

The fusion rules (1.1), however, do not determine the full symmetry structure: they record how lines can join together, but not how different fusion channels are related. In a unitary two-dimensional QFT with a finite internal symmetry, the topological lines, their topological junction operators, and the transformations between fusion channels assemble into a *unitary fusion category* $\mathcal{C}$. Its objects are topological lines, its morphisms are junction operators, its tensor product is fusion, and the transformations between channels are encoded by the F -symbols. In higher dimensions, defects of different codimensions and their junctions organize into higher fusion categories [17–20]. Categories also make contact with the classification of topological phases and their gapped boundaries [19, 21–28].

Generalized symmetries have produced concrete results both in high-energy and condensed-matter physics. Higher-form symmetries provide order parameters for confinement, while their anomalies restrict the possible infrared phases of gauge theories [5, 29–33]. Noninvertible defects constrain RG flows, impose selection rules on correlation functions, and organize boundary conditions [15, 34–36]; in massive theories, they enforce particle-soliton degeneracies and constrain scattering amplitudes [37–39]. Generalized Landau paradigms classify gapped phases and organize phase transitions even when the symmetry is categorical [4, 25, 26, 40–42]. In four dimensions, duality and condensation defects furnish noninvertible symmetries of Maxwell and Yang-Mills theories [43–45], while massless QED and QCD possess noninvertible axial symmetries [46]. In the latter setting, symmetry matching accounts for familiar selection rules and requires the infrared coupling responsible for pion decay [16, 46]. These examples make clear that generalized symmetries are not limited to toy models.

The categorical technology used in this thesis assumes a finite symmetry, so that there are finitely many simple defects and the resulting operator algebras are finite dimensional. Infinite

and continuous noninvertible symmetries are still an active frontier. Examples and proposals include [47–50], but the appropriate (higher-)categorical framework is still being developed.

1.2 Entanglement entropy in quantum field theory

Quantum information provides tools for studying a quantum state without choosing a set of local observables. Instead of looking only at expectation values, one studies density matrices and quantum channels between them, relative entropies, and the correlations produced by restricting a state to a subsystem. This viewpoint has been particularly fruitful in many-body physics and QFT. The scaling of entanglement entropy with subsystem size diagnoses critical points and extracts central charges in one dimension [51, 52]; area laws explain why ground states of many local gapped Hamiltonians admit efficient tensor-network descriptions [53]; in $2 + 1d$, the entanglement entropy gives information that can be used to identify topologically ordered phases [54–56]; entanglement also characterizes nonequilibrium phenomena [57–59], and in holography it connects quantum information to spacetime geometry [60–63].

Let Σ be a spatial manifold and $\mathcal{H}_\Sigma$ the Hilbert space assigned to it. For the moment, assume that a bipartition $\Sigma = A \cup B$ induces a tensor-product factorization

$$\mathcal{H}_\Sigma = \mathcal{H}_A \otimes \mathcal{H}_B. \quad (1.2)$$

Given a state described by a density matrix ρ_{full} on $\mathcal{H}_\Sigma$, its restriction to A is the partial trace

$$\rho_A = \text{tr}_{\mathcal{H}_B}[\rho_{\text{full}}]. \quad (1.3)$$

The von Neumann entropy and the Rényi entropies are defined as

$$S_A \equiv -\text{tr}[\rho_A \log \rho_A], \quad S_A^{(n)} \equiv \frac{1}{1-n} \log \text{tr}[\rho_A^n], \quad n > 0, \quad (1.4)$$

with $S_A = \lim_{n \rightarrow 1} S_A^{(n)}$. When ρ_{full} is pure, it is $S_A = S_B$ and this entropy measures the entanglement between the two regions.

At a critical point, the infrared theory is often a two-dimensional CFT. Conformal symmetry then fixes the leading dependence of the ground-state entanglement entropy of an interval of length L in the infinite line [52, 64]:

$$S_A^{(n)} \sim \frac{c}{6} \left(1 + \frac{1}{n}\right) \log \frac{L}{\varepsilon} + \dots, \quad S_A \sim \frac{c}{3} \log \frac{L}{\varepsilon} + \dots, \quad (1.5)$$

where c is the central charge and ε is a UV regulator. The formula on the left can be obtained from partition functions on an n -sheeted Riemann surface, or equivalently from correlation functions of twist fields.

Eqn. (1.2), however, is not literally true for a continuum QFT. The small-distance correlations near the entangling surface are responsible for the divergence in (1.5) as ε goes to zero. A useful prescription is to replace the sharp cut with a finite-size entangling surface [65]. In

this construction, one removes a small neighborhood of size ε around each component of ∂A , imposes a boundary condition $\mathcal{B}$ on the resulting physical boundary, and defines a cutting map

$$\psi_{\mathcal{B}}: \mathcal{H}_{\Sigma} \rightarrow \mathcal{H}_A \otimes \mathcal{H}_B. \quad (1.6)$$

via a path-integral construction (as will be discussed in more detail in Sec. 3.3). The path integral for $\psi_{\mathcal{B}}$ and its adjoint, followed by a trace over B , gives a map from operators on the original Hilbert space to operators on the regulated interval Hilbert space. In two dimensions the entangling surfaces can be chosen to be two small circles or semicircles of radius ε ; the two endpoints can carry different boundary conditions. After a conformal transformation, the reduced density matrix is described by an annulus or strip partition function.

The coefficient of the leading logarithm in a unitary CFT is independent of the boundary condition, but the subleading constant term is given by the Affleck-Ludwig boundary entropies [66]. Schematically, for conformal boundary conditions $\mathcal{B}_r$ and $\mathcal{B}_m$ one has

$$S_A = \frac{c}{3} \log \frac{L}{\varepsilon} + \log(g_m g_r) + \dots \quad (1.7)$$

where g_r and g_m are the g -factors of the corresponding boundary states, and the dots denote terms that go to zero in ε . Keeping track of these finite contributions is important for symmetry resolution (introduced in Sec. 1.3), since the boundary conditions at the entangling surface determine how the symmetry acts on the regulated subsystem Hilbert space and hence affect its decomposition into symmetry sectors. If the chosen boundary conditions are not symmetry invariant, they may also change the symmetry properties of the reduced density matrix and must therefore be treated carefully when defining the asymmetry (introduced in Sec. 1.4).

1.3 Symmetry-resolved entanglement entropy

Suppose for now that a compact group G acts internally and locally, so that $U(g) = U_A(g) \otimes U_B(g)$ in the factorized setting. If the full state is invariant, then

$$[\rho_A, U_A(g)] = 0, \quad \forall g \in G. \quad (1.8)$$

The subsystem Hilbert space decomposes into irreps of G (with multiplicities) as

$$\mathcal{H}_A = \bigoplus_{\mathfrak{R}} n_{\mathfrak{R}} V_{\mathfrak{R}}, \quad n_{\mathfrak{R}} \in \mathbb{Z}_{\geq 0}, \quad (1.9)$$

and Schur's lemma forces ρ_A to be proportional to the identity in every block that connects equivalent irreps, and zero elsewhere. One can thus break ρ_A into the different sectors as

$$\rho_A = \sum_{\mathfrak{R}} \tilde{\rho}_{\mathfrak{R}}, \quad \tilde{\rho}_{\mathfrak{R}} \equiv \Pi_{\mathfrak{R}} \rho_A \Pi_{\mathfrak{R}}, \quad \rho_{\mathfrak{R}} \equiv \tilde{\rho}_{\mathfrak{R}} / q_{\mathfrak{R}}, \quad q_{\mathfrak{R}} \equiv \text{tr}[\tilde{\rho}_{\mathfrak{R}}], \quad (1.10)$$

where $\Pi_{\mathfrak{R}}$ are the projectors onto the various representations. The entanglement entropy consequently decomposes as

$$S[\rho_A] = - \sum_{\mathfrak{R}} q_{\mathfrak{R}} \log q_{\mathfrak{R}} + \sum_{\mathfrak{R}} q_{\mathfrak{R}} S[\rho_{\mathfrak{R}}]. \quad (1.11)$$

The first term is the number or fluctuation entropy. The term $S[\rho_{\mathfrak{R}}]$ is known as *symmetry-resolved entanglement entropy* (SREE) [67]. As for the EE, it is often easier to compute the Rényi versions of $S[\rho_{\mathfrak{R}}]$ instead.

Symmetry resolution contains information that is lost in the total entropy: states with the same $S[\rho_A]$ can have different sector probabilities and resolved entropies. It also organizes the entanglement spectrum according to charge or representation [67, 68]. Such quantities are also experimentally accessible in many-body systems [67, 69].

A recurring phenomenon is the equipartition of SREE, namely the independence of the resolved entropy from the symmetry sector. In a U(1)-symmetric critical system, sectors in the central region of the charge distribution have the same leading-order SREE (in the vacuum). This does not imply that their probabilities $q_{\mathfrak{R}}$ are equal: these approach a Gaussian distribution for large subsystem size [67, 70]. For compact nonabelian Lie groups, equipartition is violated at subleading order, as the SREE contains a finite term $2\log(\dim(\mathfrak{R}))$ [71]. For a finite (non-anomalous) group, the sector probabilities instead take a Plancherel distribution, $q_{\mathfrak{R}} = \dim(\mathfrak{R})^2/|G|$, and the entropies also display a finite $\log q_{\mathfrak{R}}$ correction [72]. Finite abelian groups therefore exhibit equipartition at this order, whereas finite nonabelian groups have a universal representation-dependent correction. Lattice corrections, or considering excited states can introduce further sector dependence thereby breaking equipartition [73–75]. One of the results of this thesis is that an anomaly can break equipartition even for an abelian group, by constraining the boundary conditions available at the entangling surface.

1.4 Entanglement asymmetry

Symmetry-resolved entanglement presupposes a symmetric reduced density matrix. A complementary question is therefore how to quantify the failure of (1.8). For an ordinary compact group, define the symmetrizing channel

$$\mathbf{S}(\rho_A) \equiv \frac{1}{\text{vol}(G)} \int_G dg U_A(g) \rho_A U_A(g)^\dagger, \quad (1.12)$$

where the integral is over the Haar measure. One can then define the *entanglement asymmetry* [76] as the relative entropy

$$\Delta S[\rho_A] \equiv S[\rho_A || \mathbf{S}(\rho_A)] = S[\mathbf{S}(\rho_A)] - S[\rho_A]. \quad (1.13)$$

Positivity of relative entropy makes $\Delta S[\rho_A] \geq 0$, and the quantity is faithful: it vanishes exactly when ρ_A is invariant. For practical calculations, Rényi versions of the asymmetry can be useful:

$$\Delta S^{(n)}[\rho_A] \equiv \frac{1}{1-n} \log \frac{\text{tr}[\mathbf{S}(\rho_A)^n]}{\text{tr}[\rho_A^n]}, \quad \lim_{n \rightarrow 1} \Delta S^{(n)}[\rho_A] = \Delta S[\rho_A]. \quad (1.14)$$

Closely related quantities appeared earlier in [77–83].

Despite its name, entanglement asymmetry quantifies symmetry breaking rather than entanglement: a product state can have zero bipartite entanglement but nonzero asymmetry. Unlike the Landau approach [4], it requires no prior choice of order parameter and can therefore detect

symmetry breaking even when the relevant order parameter is unknown or difficult to evaluate. It also can measure symmetry breaking as a function of the subregion. The asymmetry is monotonic under restriction:¹ for $A \subseteq B$, $\Delta S_A \leq \Delta S_B$. Thus, symmetry breaking may disappear locally but cannot emerge upon tracing out degrees of freedom.

The dependence of the asymmetry on the spatial region makes it especially useful out of equilibrium. After preparing a symmetry-breaking initial state, evolution under a symmetry-preserving Hamiltonian may restore the symmetry locally: the reduced state of a subsystem can become symmetric even though the global state remains non-symmetric. The time-dependent asymmetry measures this restoration directly, and reveals a *quantum Mpemba effect*: among the various initial states, a state with more initial symmetry breaking can restore the symmetry faster than one that starts closer to being symmetric [76, 84–87]. The effect has also been observed in quantum simulations [88]; a review is given in [89].

1.5 Anomalies, boundary conditions, and subsystem algebras

An ’t Hooft anomaly is commonly described as an obstruction to gauging a global symmetry. Equivalently, the partition function fails to be invariant under background gauge transformations unless the theory is regarded as the boundary of a higher-dimensional invertible topological theory, coupled to the same background. Because the anomaly is unchanged along symmetry-preserving RG flows, it is a nonperturbative datum of a QFT. For an ordinary finite group-like symmetry in two dimensions, a 3-cocycle ω can appear as the associator of the symmetry lines, leading to the category Vec_G^ω . For a general fusion category, the anomaly is more intrinsically tied to the category itself and cannot in general be disentangled from the fusion rules. Generalized symmetries thus lead to a broader range of anomaly-matching conditions [7, 90–95].

A nontrivial anomaly also constrains the existence of boundary conditions with certain properties under fusion with the symmetry lines. For example, it obstructs boundary conditions preserving an ordinary invertible symmetry [96, 97]. For noninvertible symmetries, a boundary can be symmetric in two senses [36]. Strong symmetry requires each boundary to be invariant under fusion (up to a factor), whereas weak symmetry only requires each line to be able to end topologically on it. These notions coincide for group symmetries, but an anomalous fusion category may admit weakly, though not strongly, symmetric boundaries.

These facts have consequences both for the asymmetry and the SREE. The boundary conditions in (1.6) define a subsystem Hilbert space, and one might hope to be able to choose them without affecting the symmetry properties of the reduced state. This is possible when a strongly-symmetric boundary exists. For a weakly-symmetric boundary, it is possible to preserve the boundary condition only by an appropriate restriction of the symmetry algebra. Thus, even when the global state is invariant, the regulated ρ_A can have nonzero asymmetry. In this case, ρ_A is not block diagonal, and a naive symmetry resolution discards off-diagonal information and fails to reproduce the full entropy.

¹Strictly speaking, this is true only if the symmetrization is constructed in such a way that it commutes with taking the partial trace.

The relevant symmetry operators depend on the Hilbert space under consideration. In a two-dimensional theory with fusion-category symmetry $\mathcal{C}$, three finite-dimensional C^* -algebras play recurring roles: the fusion, tube, and strip algebras. The fusion algebra is obtained by laying topological lines along a circle and acting on the ordinary, untwisted Hilbert space. If one includes Hilbert spaces twisted by defects running in the time direction, the symmetry can mix sectors; the tube algebra captures this enlarged action. On an interval, lines have topological endpoints on the boundary conditions. Fusing these lines and their endpoints defines the strip algebra $\text{Strip}_{\mathcal{C}}(\mathcal{M})$, which depends on the chosen $\mathcal{C}$ -module category $\mathcal{M}$, that encodes the multiplet of boundaries [98–100]. The strip algebra is a weak Hopf algebra, and is generally different from the fusion algebra of $\mathcal{C}$. Thus, on an interval, charge sectors are irreducible representations of the strip rather than the fusion algebra. Even for an anomalous group symmetry, the symmetry can permute boundary conditions, so the strip algebra is different from the group algebra, and its irreps therefore do not coincide with ordinary group representations. For noninvertible symmetries, boundary fusion multiplicities provide additional structure. When no symmetric boundary condition is available, one can restore symmetry at the cut by appropriately averaging over the whole multiplet of boundary conditions. We introduce in Sec. 3.3.3 an averaging protocol that produces a reduced density matrix with vanishing asymmetry. Only then does a decomposition into irreducible strip-algebra blocks define a meaningful symmetry-resolved entropy. Moreover, for a non-symmetric full-system state, this protocol allows one to compute the intrinsic asymmetry of the state, without introducing spurious boundary symmetry breaking.

1.6 Summary of results

As we said, a common line of the two works presented in this thesis is that both the asymmetry and the SREE must be defined with respect to the algebra of symmetry operators acting on the relevant Hilbert space. The fusion ring alone is generally insufficient: depending on the spatial geometry, the sectors included, and the boundary conditions, the appropriate algebra may be the fusion, tube, or strip algebra.

1.6.1 Asymmetry for generalized symmetries

In [1], we define asymmetry with respect to a finite-dimensional algebra $\mathcal{A}$ of symmetry operators. This is possible as long as the algebra is *strongly separable*, a condition that, as we discuss, is equivalent to semisimplicity. For such an algebra, the inverse of the regular trace pairing

$$K_{ab} \equiv \text{tr}[\mathcal{X}_a \mathcal{X}_b] \quad (1.15)$$

determines the unique universal symmetrizer, namely the unique symmetrizer constructed solely with the action of $\mathcal{A}$:

$$\rho_S = \sum_{ab} (K^{-1})^{ab} \mathcal{X}_a \rho \mathcal{X}_b. \quad (1.16)$$

For the C^* -algebras relevant to unitary theories, this is a trace-preserving completely positive projector onto the operators that commute with $\mathcal{A}$. It removes matrix elements between inequivalent irreps and averages over each representation space while preserving its trace.

We define the corresponding von Neumann and Rényi asymmetries as

$$\Delta S[\rho] \equiv S(\rho||\rho_S) = S(\rho_S) - S(\rho) , \quad \Delta S^{(n)}[\rho] \equiv \frac{1}{1-n} \log \frac{\text{tr}[\rho_S^n]}{\text{tr}[\rho^n]} . \quad (1.17)$$

The construction applies whenever a Hilbert space and a finite-dimensional algebra of symmetry operators have been identified, including lattice systems and finite higher-form symmetries.

In two-dimensional QFTs, different choices of Hilbert space lead to different symmetry algebras. On a circle, the fusion algebra acts on the untwisted Hilbert space, while the tube algebra acts on the direct sum of untwisted and twisted sectors. Since the latter also contains transformations between twisted sectors, it can detect symmetry breaking that is invisible to the fusion algebra. We illustrate this distinction using excited CFT states created by local operator insertions.

For two-dimensional gapped phases, an infrared TQFT description allows us to compute the asymmetry of degenerate vacua. Vacua related by an ordinary group symmetry have equal asymmetry, whereas a noninvertible line can relate vacua with different asymmetries. This reflects the fact that such vacua need not have identical physical properties [101, 102].

On an interval, the symmetry acts through the strip algebra $\text{Strip}_{\mathcal{C}}(\mathcal{M})$ determined by the boundary conditions. We treat strongly symmetric, weakly symmetric, and non-symmetric boundaries separately. In the last case, we introduce an averaging protocol over a multiplet of boundary conditions related by the symmetry action; this removes the boundary contribution to the asymmetry while leaving only the bulk one. Calculations for excited CFT states show how the interval asymmetry can approach the full-system result as the interval grows. They also show that Rényi asymmetries need not be monotone in the subsystem size, a phenomenon subsequently studied further in [103].

For group-like symmetries the dependence on the boundary condition eventually drops out of the asymmetry, explaining the success of earlier calculations that left the boundary implicit. For noninvertible symmetries, by contrast, the boundary data generally remain essential.

We test the construction for group symmetries and for the Ising, Fibonacci, $\text{Rep}(H_8)$, Haagerup, and $\frac{1}{2}E_6$ fusion rings. When the symmetry algebra is additionally a Hopf or weak Kac algebra, we also derive equivalent expressions for the symmetrizer using the Hopf structure.

1.6.2 Noninvertible symmetry-resolved entanglement entropy

For an interval, the boundary conditions at the entangling surfaces determine both the reduced density matrix and the strip algebra whose representations label its symmetry sectors. Previous BCFT constructions for group symmetries imposed symmetry-preserving entangling boundaries [72], and the first extensions to categorical symmetries similarly required strongly or weakly symmetric boundaries [104]. Subsequent work formulated symmetry resolution for more general boundary choices using the strip algebra [100, 105]. We propose a construction that applies even when no invariant simple boundary exists: we employ the previously introduced averaging protocol to produce a symmetric reduced density matrix, and then resolve this into representations of the associated strip algebra. This procedure allows for a meaningful definition of SREE even if no symmetric boundaries are available.

Using this procedure, we study the vacuum reduced density matrix of a single interval in a unitary two-dimensional CFT with a finite, possibly anomalous or noninvertible, internal symmetry. The examples provided show that anomalies can modify symmetry resolution by restricting the available boundary conditions and hence changing the strip algebra. Possibly anomalous cyclic groups and non-anomalous finite abelian groups exhibit equipartition for the boundaries considered, whereas an anomalous $(\mathbb{Z}_2)^3$ example violates it. This provides an abelian example of representation-dependent SREE. For S_3 , the resolved entropies likewise depend strongly on the boundary multiplet allowed by the anomaly.

For the Fibonacci and Ising symmetries, the corresponding CFTs admit a unique conformal boundary multiplet, and the associated strip-algebra resolution violates equipartition. The $\mathfrak{su}(2)_{10}$ WZW model with non-diagonal E_6 modular invariant further shows that an abstract module category need not be realized by physical conformal boundary conditions. In particular, the regular module category of the full symmetry cannot be used to define the interval Hilbert space in this theory.

Similar ideas were proposed in [106]. Their construction, however, uses a cloaking boundary state rather than averaging over a multiplet of physical conformal boundaries, and consequently yields different symmetry-resolved entropies.

1.7 Outlook and organization of the thesis

The results of this thesis suggest several extensions. First, it would be useful to move beyond finite symmetries. Although continuous groups are well understood in conventional symmetry-resolved entanglement and entanglement asymmetry, continuous or otherwise infinite noninvertible symmetries require extending the present construction beyond finite-dimensional symmetry algebras.

A second direction concerns higher dimensions, where entangling surfaces have less trivial geometry and can intersect topological defects in more varied ways. A basic problem is to identify the algebras acting on the regulated subsystem Hilbert space and to understand their dependence on the boundary conditions imposed at the entangling surface. Natural candidates are higher-dimensional tube and strip algebras [107–110]. Recent work on higher-form asymmetry provides a first step [111, 112].

Finally, nonequilibrium dynamics provides another natural direction. It would be nice to extend Quantum Mpemba effects to generic categorical symmetry, and to determine when a noninvertible symmetry is dynamically restored after a quench.

The thesis is organized as follows:

In Sec. 2 we introduce fusion categories and explain why they describe finite generalized symmetries in two spacetime dimensions. We then discuss the fusion, tube, and strip algebras in detail in Secs. 2.1 to 2.3.

In Sec. 3 we turn to the construction of asymmetry for generalized symmetries. We first present the general construction for an arbitrary strongly separable algebra in Sec. 3.1. We then specialize it to the circle Hilbert space in Sec. 3.2 and to the interval Hilbert space in Sec. 3.3,

treating the different types of boundary conditions in turn. In particular, the averaging protocol for non-symmetric boundaries is introduced in Sec. 3.3.3.

In Sec. 4 we develop symmetry-resolved entanglement entropy for categorical symmetries using this averaging protocol. We study the representations of the strip algebra and their relation to the dual symmetry in Sec. 4.1, before deriving in Sec. 4.2 the general formula for the SREE associated with an arbitrary module category.

The main body concludes with explicit computations of the asymmetry in Sec. 5 and of the SREE in Sec. 6.

Several technical appendices complete the thesis. In App. A we give a more formal account of the relevant categorical structures and summarize the conventions and formulas used in the main text. In App. B we provide additional computations and complementary proofs concerning tube algebras. In App. C we establish the existence and uniqueness of the symmetrizer and derive several useful formulas. In App. D we review Hopf algebras and express the symmetrizer in terms of their structure, while in App. E we explain how strip algebras fit into this framework. Finally, in App. F we review the classification of module categories for group-like symmetries, and in App. G the anomalies of the finite groups used in the examples.

2 Symmetry categories and symmetry algebras

In this section we aim to give a more technical introduction to the concept of finite *generalized symmetries*, focusing on unitary QFTs in two spacetime dimensions. As we already mentioned, the structure of such symmetries is precisely captured by a mathematical concept known as (unitary) *fusion categories* [14]. We begin by describing these and by explaining why they are suitable for the purpose. We then proceed to describe different ways in which such categories can be used to construct algebras of symmetry operators in different physical settings.

Here we limit ourselves to a practical and simplified exposition, and we refer to App. A and to [113] for more formal discussions.

For our purposes, a finite, semisimple fusion category $\mathcal{C}$ is given by the following data:

1. A collection of *objects* $a, b, c, \dots \in \text{Ob}(\mathcal{C})$ ².
2. For each pair of objects a and b , a collection $\text{Hom}_{\mathcal{C}}(a, b)$ of *morphisms* (or “arrows”) between them.³ When compatible, morphisms can be composed: $\text{Hom}_{\mathcal{C}}(a, b) \times \text{Hom}_{\mathcal{C}}(b, c) \xrightarrow{\circ} \text{Hom}_{\mathcal{C}}(a, c)$. Furthermore, there are identity morphisms $\text{id}_a \in \text{Hom}_{\mathcal{C}}(a, a)$ such that $\text{id}_a \circ f = f$ and $g \circ \text{id}_a = g$ (whenever composition is possible), and composition is associative. This structure is supposed to mimic that of functions between sets – but the setting here is more generic than that.

Moreover, one requires $\text{Hom}_{\mathcal{C}}(a, b)$ to have the structure of a finite-dimensional $\mathbb{C}$ -vector space. If $f \in \text{Hom}_{\mathcal{C}}(a, b)$, we can write $a \xrightarrow{f} b$ or $f: a \rightarrow b$.

3. A *direct sum* $\oplus$ of objects. Namely, for every pair of objects $a, b \in \text{Ob}(\mathcal{C})$, also $a \oplus b \in \text{Ob}(\mathcal{C})$. The adjectives “finite” and “semisimple” mean that every object can be decomposed as a finite direct sum of objects taken from a finite collection of so-called *simple* objects. These are in turn objects that cannot be written as a direct sum of other objects. We denote the set of simple objects (modulo isomorphism) as $\text{Irr}(\mathcal{C})$. For simple a, b , it further holds that $\text{Hom}(a, b) \cong \delta_{ab} \mathbb{C}$.

4. A *tensor product* $\otimes$ of objects. For every pair $a, b \in \text{Ob}(\mathcal{C})$, also $a \otimes b \in \text{Ob}(\mathcal{C})$. Furthermore, as a consequence of the previous point, one can decompose the tensor product into a finite direct sum

$$a \otimes b = \bigoplus_{c \in \text{Irr}(\mathcal{C})} N_{ab}^c c, \quad N_{ab}^c \in \mathbb{Z}_{\geq 0}. \quad (2.1)$$

The numbers N_{ab}^c are known as *fusion coefficients*. One requires there to exist isomorphisms $(a \otimes b) \otimes c \cong a \otimes (b \otimes c)$ subject to certain coherence conditions. This implies the relation

$$\sum_e N_{ab}^e N_{ec}^d = \sum_f N_{af}^d N_{bc}^f. \quad (2.2)$$

²We will usually denote simply $a, b, c, \dots \in \mathcal{C}$.

³We will drop the subscript $\mathcal{C}$ when there is no ambiguity.

Furthermore, there is a unit $1 \in \text{Irr}(\mathcal{C})$,⁴ such that $a \otimes 1 \cong a$ and $1 \otimes a \cong a$. In general, on the RHS of (2.1) more than one object will appear, and for a given a there might not exist a b such that $a \otimes b = 1$. In this case a is called “non-invertible”.

5. A *dual structure* (also called *rigid*). Namely, for every object $a \in \text{Ob}(\mathcal{C})$, there is a dual object $\bar{a} \in \text{Ob}(\mathcal{C})$ and certain morphisms $\bar{a} \otimes a \rightarrow 1$ and $1 \rightarrow a \otimes \bar{a}$, subject to coherence conditions. For fixed simple a , the dual $\bar{a}$ is the only simple object such that the products $\bar{a} \otimes a$ and $a \otimes \bar{a}$ contain 1 in the decomposition (2.1).

Each of the above points has a direct physical counterpart [14, 15]:

1. Each object a corresponds to a topological line $\mathcal{L}_a$.
2. A morphism $a \rightarrow b$ corresponds to a topological junction operator between lines $\mathcal{L}_a$ and $\mathcal{L}_b$. Namely, this is a point-like operator that interpolates between the two line defects, and the overall configuration can be deformed arbitrarily as long as no other operator insertions are crossed.
3. Computing a partition function with the insertion of a direct sum $\mathcal{L}_a \oplus \mathcal{L}_b$ simply amounts to computing the partition function in the presence of $\mathcal{L}_a$ alone, plus that in the presence of $\mathcal{L}_b$ alone.

When the lines are inserted on a fixed time slice, the interpretation of the corresponding partition functions is clear: each line defines a conserved operator $\mathcal{O}_{a,b}$ on the (untwisted) Hilbert space $\mathcal{H}$, and inserting a sum of lines amounts to acting on $\mathcal{H}$ with the sum of the corresponding operators.

However, in a Euclidean QFT, one has to treat all spacetime directions on the same footing. In particular, one can instead insert a line $\mathcal{L}$ *along* the time direction, and the corresponding partition function should still be equal to a sum over states in some Hilbert space. In this case, however, the presence of the line defines a *new* Hilbert space, that of $\mathcal{L}$ -twisted states.⁵ Moreover, one has $\mathcal{H}_{\mathcal{L}_a \oplus \mathcal{L}_b} = \mathcal{H}_{\mathcal{L}_a} \oplus \mathcal{H}_{\mathcal{L}_b}$.⁶ In particular, we see that only non-negative integer combinations of topological lines make sense in partition functions. If this were not the case, some combinations of lines could not be interpreted as a twisted partition function. This constraint is referred to in [16] as the “operator/defect principle”.

This rule holds for defects of any dimension, except for point-like operators. Indeed, these cannot be used to create a new twisted Hilbert space, and thus the operator/defect principle

⁴More generally, the unit could be non-simple, leading to a *multi-fusion* category. This however physically implies the existence of a nontrivial topological local (genuine) operator, a scenario which is typically excluded.

⁵The defect in such a configuration can be thought of as imposing specific boundary conditions as fields approach the defect from the two sides.

⁶One intuitive way to argue this is that a partition function (interpreted as a wave functional) that creates a state in $\mathcal{H}_{\mathcal{L}_a \oplus \mathcal{L}_b}$ splits as the sum of two partition functions, one for each defect: $Z_{ab}[\phi] = Z_a[\phi] + Z_b[\phi]$. The test function ϕ then belongs to the direct sum of two configurations spaces, one for each kind of boundary conditions across the defect (see Fn. 5). See [14–16, 36, 114] for related discussions.

$$\begin{array}{c} d \\ | \\ e \text{---} \nu \\ / \quad \backslash \\ a \quad b \quad c \end{array} \quad = \quad \sum_{f, \rho, \sigma} [F^d_{abc}]_{(e; \mu \nu)(f; \rho \sigma)} \quad \begin{array}{c} d \\ | \\ \sigma \text{---} f \\ / \quad \backslash \\ a \quad b \quad c \end{array}$$

Figure 1: Definition of the F -symbols with respect to chosen bases of junction spaces. They have to satisfy the pentagon identity (A.17).

does not apply. This justifies why it is a sensible requirement to ask that topological junctions form a vector space.⁷

The additive structure of lines thus closely mirrors that of vector spaces, rather than that of numbers. This justifies the choice of notation and is a crucial reason why categories (and semisimple⁸ abelian categories in particular, see App. A) are the right language to describe line defects. Also see [16, 114] for further discussions.

4. Given two topological lines $\mathcal{L}_a$ and $\mathcal{L}_b$, these can be brought arbitrarily close without incurring singularities. This limiting procedure, inside partition functions, defines their product or “fusion”. The result is still a topological line, which – by the previous point – must either be simple or decomposable as a sum of simples.

When nearby parallel defects a, b, c are brought together, their fusion is associative. However, one can first fuse a with b forming a junction μ , and then fuse the resulting line with c forming the junction ν , as in Fig. 1 left. Alternatively, one can first fuse b with c , and then the result with a , as in Fig. 1 right. The two results do not agree exactly, as there is a change of basis (or isomorphism, depending on the point of view) relating the two situations. One way to think of this is that the two configurations of lines create two – generically different – states in the same twisted Hilbert space, with Fig. 1 describing their precise relation. The coefficients in the sum are known as “ F -symbols”, and they depend on a choice of bases for all the junction

⁷More precisely, for $\mathcal{L}_a$ simple, we can think of a topological junction $\mathcal{L}_a \rightarrow \mathcal{L}_a$ as coming from the stacking of the trivial junction $\text{id}_{\mathcal{L}_a}$ with a genuine point-like operator. Because of the argument above, any complex multiple of such a point-like operator is another valid operator of the theory, and can be stacked onto the line a . This operation produces all possible junctions $\mathcal{L}_a \rightarrow \mathcal{L}_a$.

⁸It is interesting to argue why simple lines should exist. A way to do this is to study the algebra $A_{\mathcal{L}}$ of topological operators supported on a line $\mathcal{L}$. The product is given by fusing the operators together, and the unit is the identity operator. Assuming this algebra is finite-dimensional and semisimple, one can split the identity as a sum of primitive orthogonal idempotents, $\text{id}_{\mathcal{L}} = \sum_i e_i$, where $e_i e_j = \delta_{ij} e_i$. These are precisely the projectors onto the various simple components of $\mathcal{L}$. Given a loop of $\mathcal{L}$, one can thus insert for free $\sum_i e_i e_i$, and slide one of the two operators along the loop to finally annihilate it from the other side (note that each of them behaves as the identity on the simple components of $\mathcal{L}$). This procedure allows one to write a loop of $\mathcal{L}$ as a sum of simple lines.

Note that if the algebra were infinite-dimensional – since these operators, being topological, have zero energy – there would be infinitely many ground states in the $\mathcal{L}$ -twisted Hilbert space, a scenario which we physically exclude. In [114] it is furthermore argued that, in unitary/reflection-positive theories, $A_{\mathcal{L}}$ should be semisimple.

vector spaces.⁹ For fixed external lines, F_{abc}^d is a square invertible matrix, and, in unitary theories, can be chosen to be unitary by a change of basis.

Another way to interpret Fig. 1 is in the context of partition functions with a background for the symmetry in question. A background for a symmetry can be seen as the insertion of a network of defects on a triangulation of the spacetime manifold (see e.g. [5, 115]). If one changes the triangulation by a Pachner move, the partition function will not in general remain the same. From a dual point of view, a change in triangulation corresponds to a gauge transformation of the background field.¹⁰ The F -symbols capture precisely the change of the partition function when such a transformation is performed. This is one of the possible manifestations of an 't Hooft anomaly.

5. Physically, taking the dual of a line amounts to reflecting it, or changing its orientation, $\mathcal{L}_{\bar{a}} = \overline{\mathcal{L}_a}$.

The origin of the morphisms mentioned in point 5 above is explained as follows. Consider a line $\mathcal{L}_a$ stretched along the time direction and bend it in half so that it forms a “cap” shape. Fusing together the two lines forming the cap, we obtain the line $\overline{\mathcal{L}_a} \otimes \mathcal{L}_a$ terminating in a point-like operator. We can interpret such an operator precisely as a junction $\overline{\mathcal{L}_a} \otimes \mathcal{L}_a \rightarrow \mathcal{L}_1$. Through a similar procedure one can construct a junction $\mathcal{L}_1 \rightarrow \mathcal{L}_a \otimes \overline{\mathcal{L}_a}$.

As already mentioned, when laid on a fixed time slice, topological lines define conserved operators on the untwisted Hilbert space. The algebra spanned by such operators is known as *fusion algebra* and will be described in more detail in Sec. 2.1. Moreover, symmetries can also act on twisted sectors. However, to study such an action, one has to specify the precise configuration of defects, and in particular how a line laid along the time direction intersects a line laid at fixed time. In Sec. 2.2 we describe the so-called *tube algebra*, which is a way of describing such defect configurations. Finally, one can study the theory in the presence of boundary conditions, and how the symmetry interacts with them. In particular, a symmetry can act by mapping states with certain boundary conditions to ones with different boundary conditions. This gives rise to the concept of *strip algebra*, described in Sec. 2.3.

As a side note, while in this thesis we focus particularly on two-dimensional conformal field theories and on conformal boundary conditions, the mathematical constructions of all such objects are totally generic.

2.1 Fusion algebra

Associated to every fusion category $\mathcal{C}$ is a fusion algebra $\mathcal{A}$ spanned by the objects $\mathcal{L}_a$. Such an algebra is unital and associative. Furthermore, in the basis given by the simple lines $\mathcal{L}_a$, the multiplication coefficients are constrained to be nonnegative integer numbers $N_{ab}^c \in \mathbb{Z}_{\geq 0}$. We thus write the fusion algebra as

$$\mathcal{L}_a \times \mathcal{L}_b = \sum_c N_{ab}^c \mathcal{L}_c . \quad (2.3)$$

⁹The F -symbols are nothing but a component-wise expression of the isomorphism $(a \otimes b) \otimes c \cong a \otimes (b \otimes c)$.

¹⁰Strictly speaking, this statement is true only for invertible symmetries.

Besides, associated to the rigid structure of $\mathcal{C}$ is a conjugation operation on $\mathcal{A}$: there exists an element of the chosen basis, that we call $\mathcal{L}_1$, such that the matrix $C_{ab} \equiv N_{ab}^1$ is involutive, namely $C^2 = 1$. Since the coefficients are natural numbers, this implies that for every a there exists a unique $\bar{a}$ such that $N_{a\bar{a}}^1 = \delta_{\bar{a}a}$ and $\bar{\bar{a}} = a$. Conjugation is then the map

$$\mathcal{L}_a \mapsto \mathcal{L}_a^\dagger \equiv \mathcal{L}_{\bar{a}} = \sum_b C_{ab} \mathcal{L}_b = \sum_b N_{ab}^1 \mathcal{L}_b . \quad (2.4)$$

This is also called duality. Conjugation is required to be an algebra anti-automorphism: $N_{\bar{a}\bar{b}}^{\bar{c}} = N_{ba}^c$. This is then extended to the whole fusion algebra by anti-linearity. Conjugation and associativity allow one to derive that $N_{ab}^c = N_{b\bar{c}}^{\bar{a}}$, hence one obtains the pivotal relations

$$N_{ab}^c = N_{\bar{a}\bar{c}}^b = N_{c\bar{b}}^a = N_{b\bar{c}}^{\bar{a}} = N_{\bar{c}a}^{\bar{b}} = N_{\bar{b}\bar{a}}^{\bar{c}} . \quad (2.5)$$

When applied to the conjugation matrix, they imply that $N_{1a}^b = N_{a1}^b = \delta_a^b$ so that $\mathcal{L}_1$ is in fact the unit element for the fusion algebra, it is unique and self-dual, and indeed it corresponds to the identity line of $\mathcal{C}$.

Finally, there exists a one-dimensional representation of (2.3) given by the *quantum dimensions* (or Frobenius-Perron dimensions, also see the discussion around (A.26)):

$$d_a d_b = \sum_c N_{ab}^c d_c . \quad (2.6)$$

For each simple line $\mathcal{L}_a$, the number d_a is defined as the largest positive eigenvalue of the non-negative matrix N_a , where $(N_a)_{bc} = N_{ab}^c$, and it exists thanks to the Frobenius-Perron theorem (see e.g. [113]). It turns out that $d_a \geq 1$, with the equality holding if and only if the element is invertible.

The fusion algebra describes the action of the symmetry on the untwisted Hilbert space $\mathcal{H}_1$ on S^1 , as depicted in Fig. 2 on the left. It also describes the action of the symmetry on untwisted local operators, as depicted in Fig. 2 on the right.¹¹ One can prove (see App. C.3) that finite-dimensional fusion algebras are automatically C^* -algebras and thus semisimple and strongly separable (we refer to Sec. 3.1 for the definition). As anticipated, the asymmetry can therefore be defined with respect to the fusion algebra using the general construction developed in Sec. 3.1. In Sec. 3.2, we provide some general computations for the full-system asymmetry, while in Sec. 5 we provide a wide range of examples.

2.2 Tube algebra

On a spatial S^1 , a noninvertible symmetry can map (un)twisted sectors to other (un)twisted sectors. One can thus construct a more sophisticated and complete action of the symmetry on the total Hilbert space

$$\mathcal{H}_{\text{Tube}} = \bigoplus_{a \in \text{Irr}(\mathcal{C})} \mathcal{H}_a , \quad (2.7)$$

¹¹In CFTs there is a one-to-one correspondence between untwisted states on S^1 and untwisted local operators. In generic QFTs there is no such correspondence, but it is still true that both states and local operators transform in representations of the fusion algebra.



Figure 2: Left: action of the fusion algebra (2.3) of the symmetry category $\mathcal{C}$ on the untwisted Hilbert space on S^1 . The product of two elements of the algebra is realized by stacking the corresponding lines on the cylinder and fusing them. Right: the corresponding action of the fusion algebra on local operators.



Figure 3: Left: action of the tube algebra $\text{Tube}(\mathcal{C})$ on the total Hilbert space on S^1 (left) and on (un)twisted point operators (right). The product of two elements of the algebra is realized by stacking the corresponding elements on the cylinder and then resolving the configuration.

where $\mathcal{H}_a$ are the Hilbert spaces on S^1 twisted by the lines $\mathcal{L}_a$ (and thus $\mathcal{H}_1$ is the untwisted sector), and $\text{Irr}(\mathcal{C})$ is the set of simple objects in $\mathcal{C}$. The finite-dimensional C^* -algebra $\mathcal{A}$ that is obtained in this way is called the *tube algebra* $\text{Tube}(\mathcal{C})$ [116]. This algebra does not only depend on the fusion rules of $\mathcal{C}$, but rather also on its F -symbols.

In order to describe the algebra, we resort to the standard notation and conventions (that follow [117]) and that is summarized in App. A.5. We work in a gauge in which all F -symbols are unitary matrices¹² and $[F_{abc}^d] = \mathbb{1}$ whenever a, b , or c is 1 (and fusion is possible).

The tube algebra is spanned by operators defined by a setup in which a simple line $\mathcal{L}_a$ – stretched horizontally – interpolates between two (un)twisted sectors $\mathcal{H}_g$ and $\mathcal{H}_h$, as in Fig. 3 left. Equivalently, the generators map (un)twisted local operators $\mathcal{O}_g$ (that live at the end of a line $\mathcal{L}_g$) to (un)twisted operators $\mathcal{O}_h$ via the so-called *lasso action*, as in Fig. 3 right. The configuration is described by a “cross”, where the horizontal line $\mathcal{L}_a$ is closed in a circle, while the vertical line $\mathcal{L}_g$ lies below and $\mathcal{L}_h$ lies above, as in (2.8). The junction Δ is an element of $\text{Hom}(\mathcal{L}_a \otimes \mathcal{L}_g, \mathcal{L}_h \otimes \mathcal{L}_a)$, which has dimension $\sum_d N_{ag}^d N_{ha}^d$ (note that $N_{ha}^d = N_{hd}^a$). It is convenient to decompose Δ into trivalent junctions:

$$\begin{array}{c}
 \begin{array}{c}
 \text{Diagram 1: A cylinder with a horizontal blue line } \mathcal{L}_a \text{ and a vertical blue line } \mathcal{L}_g. \text{ A red dot at the intersection is labeled } \Delta. \text{ The cylinder is labeled } a \text{ on both sides.} \\
 \end{array}
 \equiv
 \begin{array}{c}
 \text{Diagram 2: A trivalent junction with three lines labeled } a, d, \text{ and } h. \text{ The junction is labeled } \mu. \\
 \end{array}
 .
 \end{array}
 \quad (2.8)$$

¹²The existence of such bases is guaranteed by the fact that the fusion category is unitary. Without such an assumption, our formulas below would still be correct but in terms of inverse matrices, as opposed to Hermitian conjugate matrices.

We will use the graphical notation on the RHS, and use the indicated basis for the tube algebra. All lines have an implicit arrow from bottom to top, and the tick at the beginning or end of a line indicates the “trace” described in [117]. The dimension of the tube algebra is

$$\dim(\text{Tube}(\mathcal{C})) = \sum_{g,a,d,h} N_{ag}^d N_{ha}^d. \quad (2.9)$$

In the tube algebra we multiply elements by placing them one on top of the other (the product is such that bottom acts first, top acts second). The product is zero if there is no possible junction between adjacent vertical lines. In the case of the basis elements (2.8), because only simple lines are involved, this forces the vertical lines to be the same. We define a product matrix:

$$\begin{array}{c} \begin{array}{c} \text{Diagram 1: A junction with four lines. Top line is labeled k, bottom line is labeled g. Left line is labeled b, right line is labeled a. Internal labels include $e, \sigma, \rho, h, d, \nu, \mu$.$$
 \end{array} \equiv \sum_{c,s,\alpha,\beta} M_{g,a,(d;\mu\nu),h,b,(e;\rho\sigma),k}^{c,(s;\alpha\beta)} \begin{array}{c} \text{Diagram 2: A junction with four lines. Top line is labeled \$k\$, bottom line is labeled \$g\$. Left line is labeled \$c\$, right line is labeled \$c\$. Internal labels include \$s, \beta, \alpha\$. \end{array} . \quad (2.10)

The computation of M , described in App. B, yields:

$$M_{g,a,(d;\mu\nu),h,b,(e;\rho\sigma),k}^{c,(s;\alpha\beta)} = \sqrt{\frac{d_a d_b d_h d_s}{d_d d_e d_c}} \sum_{\gamma,\eta,\lambda} [F_{kba}^s]^*_{(e;\sigma\eta)(c;\gamma\beta)} [F_{bha}^s]_{(e;\rho\eta)(d;\nu\lambda)} [F_{bag}^s]^*_{(c;\gamma\alpha)(d;\mu\lambda)}. \quad (2.11)$$

The sums are over bases of junctions $\mathcal{L}_b \otimes \mathcal{L}_a \xrightarrow{\gamma} \mathcal{L}_c$, $\mathcal{L}_e \otimes \mathcal{L}_a \xrightarrow{\eta} \mathcal{L}_s$, and $\mathcal{L}_b \otimes \mathcal{L}_d \xrightarrow{\lambda} \mathcal{L}_s$. In the multiplicity-free case, namely when $N_{ab}^c \in \{0, 1\}$, this simplifies to:

$$M_{g,a,d,h,b,e,k}^{c,s} = \sqrt{\frac{d_a d_b d_h d_s}{d_d d_e d_c}} [F_{kba}^s]^*_{ec} [F_{bha}^s]_{ed} [F_{bag}^s]^*_{cd}. \quad (2.12)$$

In App. B we explicitly verify that the product is associative. One can also check that the basis elements of the tube algebra that map the untwisted sector to itself, i.e. the elements with $g = h = 1$ in (2.8), form a subalgebra¹³ that coincides with the fusion algebra.

Tube algebras are finite-dimensional C^* -algebras (see e.g. [108]) and so in particular they are semisimple and strongly-separable (see App. C.1). Therefore, the asymmetry with respect to it can be defined, following the general procedure that we will describe in Sec. 3.1. Since the fusion algebra embeds into the tube algebra, we expect the asymmetry with respect to the tube algebra to be bigger or equal to that computed with respect to the fusion algebra.¹⁴ In other words, we expect $\text{Tube}(\mathcal{C})$ to be a more accurate detector of symmetry breaking. We discuss explicit examples of this fact in Sec. 5.

¹³In this thesis, by subalgebra we simply mean a subspace of the algebra that is closed under product. We do not require that the identity of the subalgebra be equal to that of the original algebra, which indeed does not happen in this case.

¹⁴More explicitly, there is an inclusion $\text{End}(\mathcal{H}_1) \xrightarrow{\iota} \text{End}(\mathcal{H}_{\text{Tube}})$ given by extending a given untwisted-sector density matrix as zero in all other sectors. We then conjecture that $\Delta S_{\text{Tube}}(\iota(\rho)) \geq \Delta S_{\text{Fusion}}(\rho)$. It would be interesting to prove this expectation, or to find a counterexample.

2.3 Strip algebra

The action of the symmetry on the interval Hilbert space is described by the strip algebra $\text{Strip}_{\mathcal{C}}(\mathcal{M})$ [21, 98, 99, 118, 119]. Let us first explain how the symmetry properties of boundary conditions are described in terms of *module categories*.

A (left-)module category $\mathcal{M}$ over a fusion category $\mathcal{C}$ is a (abelian) category that carries an action of the elements of $\mathcal{C}$ (see [113] for detailed explanations). In our context, simple objects $\mathcal{B}_m$ correspond to right-hand simple boundary conditions. The action of simple bulk lines $\mathcal{L}_a$ with simple boundaries $\mathcal{B}_m$ is described by:

$$\mathcal{L}_a \otimes \mathcal{B}_m = \bigoplus_n \tilde{N}_{am}^n \mathcal{B}_n, \quad \text{depicted as} \quad \begin{array}{c} \uparrow n \\ | \\ \textcolor{blue}{\nearrow} a \\ | \\ \textcolor{red}{\nearrow} \rho \\ | \\ \downarrow m \end{array} \quad (2.13)$$

and where $\tilde{N}_{am}^n \in \mathbb{Z}_{\geq 0}$. The coefficients $\tilde{N}_{am}^n$ are equal to the dimension of the vector space of morphisms $\text{Hom}(\mathcal{L}_a \otimes \mathcal{B}_m, \mathcal{B}_n)$, and the index ρ labels a basis of this vector space. As a consequence of the pivotal relations, we have that

$$\tilde{N}_m^n = \delta_m^n, \quad \tilde{N}_{am}^n = \tilde{N}_{\bar{a}n}^m. \quad (2.14)$$

Similarly to the case of fusion categories, one requires that

$$(\mathcal{L}_a \otimes \mathcal{L}_b) \otimes \mathcal{B}_m \cong \mathcal{L}_a \otimes (\mathcal{L}_b \otimes \mathcal{B}_m) \quad (2.15)$$

. Therefore, similarly to (2.2), we obtain

$$\sum_p \tilde{N}_{ap}^n \tilde{N}_{bm}^p = \sum_c N_{ab}^c \tilde{N}_{cm}^n \equiv \tilde{N}_{abm}^n . \quad (2.16)$$

This means that the matrices $(\tilde{N}_a)^n_m$ furnish a nonnegative-integer-valued matrix representation (NIM-rep) of the fusion ring of $\mathcal{C}$. Expressing the isomorphisms (2.15) in terms of a basis of junctions, one can define the $\tilde{F}$ -symbols:

$$\begin{array}{c} \text{Diagram 1: A blue line from } a \text{ to } b \text{ (labeled } e \text{)}, \text{ then a vertical blue line from } b \text{ to } m \text{ (labeled } \mu \text{)}. \end{array} = \sum_{p, \nu, \rho} [\tilde{F}_{abm}^n]_{(e; \alpha \mu)(p; \nu \rho)} \begin{array}{c} \text{Diagram 2: A blue line from } a \text{ to } b \text{ (labeled } e \text{)}, \text{ then a vertical blue line from } b \text{ to } m \text{ (labeled } \mu \text{)}. \end{array} \quad (2.17)$$

We indicate right-hand boundaries with upward arrows, while all bulk lines run from bottom to top. The $\tilde{F}$ -symbols are unitary square matrices of size $\tilde{N}_{abm}^n$ and hence relations similar to (2.17) can be obtained using unitarity or the associator diagram. We work in a basis in which $\tilde{F}_{abm}^n = \mathbb{1}$ whenever a or b are equal to 1. The $\tilde{F}$ -symbols satisfy the left-module pentagon equation, that also involves the F -symbols, reported in (A.43). One can define the quantum dimension $\tilde{d}_m$ as the disc partition function of a $\mathcal{C}$ -symmetric TQFT with that boundary condition. They satisfy:

$$d_a \tilde{d}_m = \sum_n \tilde{N}_{am}^n \tilde{d}_n, \quad (2.18)$$

where $\{d_a\}$ are the bulk quantum dimensions introduced in (2.6). Thus the boundary quantum dimensions are the components of a common eigenvector of the matrices $\tilde{N}_a$ (with eigenvalues

d_a). In a unitary theory it is $\tilde{d}_m > 0$. Notice that the quantum dimensions $\tilde{d}_m$ are ambiguous by a common rescaling: this is because the disc partition functions can be rescaled by adding an Euler counterterm to the action. In Sec. 4 we will fix this ambiguity by imposing an extra relation involving the fusion of two topological boundaries.

A right-module category is defined in a similar way, but the action of $\mathcal{C}$ is from the right, and the objects correspond to left-hand boundary conditions. In analogy with (2.17), we have another set of $\tilde{F}$ -symbols:

$$\begin{array}{c} \begin{array}{|c|} \hline n \\ \hline \end{array} \begin{array}{|c|} \hline a \\ \hline \end{array} \begin{array}{|c|} \hline b \\ \hline \end{array} \\ \begin{array}{|c|} \hline \mu \\ \hline \end{array} \begin{array}{|c|} \hline e \\ \hline \end{array} \begin{array}{|c|} \hline \alpha \\ \hline \end{array} \\ \begin{array}{|c|} \hline m \\ \hline \end{array} \end{array} = \sum_{p,\nu,\rho} [{}^R\tilde{F}_{\bar{n}ab}^{\tilde{m}}]_{(\bar{p};\rho\nu)(e;\alpha\mu)} \begin{array}{c} \begin{array}{|c|} \hline n \\ \hline \end{array} \begin{array}{|c|} \hline a \\ \hline \end{array} \begin{array}{|c|} \hline b \\ \hline \end{array} \\ \begin{array}{|c|} \hline \rho \\ \hline \end{array} \begin{array}{|c|} \hline p \\ \hline \end{array} \begin{array}{|c|} \hline \nu \\ \hline \end{array} \\ \begin{array}{|c|} \hline m \\ \hline \end{array} \end{array} . \quad (2.19)$$

We use downward arrows to indicate left-hand boundary conditions.¹⁵ The symbols ${}^R\tilde{F}$ satisfy their own right-module pentagon equation, that we report in (A.44). For simplicity, we will restrict to the case that the same module is used on the right and on the left. If $\tilde{F}$ solves the left-module pentagon equation (A.43), then ${}^R\tilde{F}$ defined as

$$[{}^R\tilde{F}_{\bar{n}ab}^{\tilde{m}}]_{(\bar{p};\rho\nu)(e;\alpha\mu)} \equiv [\tilde{F}_{abm}^n]^*_{(e;\alpha\mu)(p;\nu\rho)} \quad \text{or simply} \quad [{}^R\tilde{F}_{\bar{n}ab}^{\tilde{m}}]_{\bar{p}e} \equiv [\tilde{F}_{abm}^n]^*_{ep} \quad (2.20)$$

solves the right-module pentagon equation (A.44). Besides, this identification guarantees that if right-hand boundary conditions are bent into left-hand boundary conditions, their junctions with bulk lines can be moved from right to left.

A module category that always exists is the *regular module*, in which we identify the boundary conditions with the bulk lines, $\mathcal{B}_m = \mathcal{L}_m$, and thus also the boundary and bulk fusion rings are equal: $\tilde{N} = N$. This is the module in which the fusion category $\mathcal{C}$ acts on itself. The $\tilde{F}$ -symbols are simply equal to the F -symbols:

$$\text{regular module:} \quad \tilde{F} = F . \quad (2.21)$$

For the right module, one could either use the gauge (2.20) as we will do, or use a different gauge in which $\tilde{F} = {}^R\tilde{F} = F$. Both solve the right-module pentagon equation (A.44).

The strip algebra is spanned by the following basis elements:

$$\mathbf{H}_{r,m,(a;\rho\mu)}^{s,n} \equiv \begin{array}{c} \begin{array}{|c|} \hline s \\ \hline \end{array} \begin{array}{|c|} \hline a \\ \hline \end{array} \begin{array}{|c|} \hline n \\ \hline \end{array} \\ \begin{array}{|c|} \hline \rho \\ \hline \end{array} \begin{array}{|c|} \hline \mu \\ \hline \end{array} \\ \begin{array}{|c|} \hline r \\ \hline \end{array} \begin{array}{|c|} \hline m \\ \hline \end{array} \end{array} \quad (2.22)$$

where the line $\mathcal{L}_a$ acts both on the bulk as well as on the boundaries by changing the boundary conditions. We will always make the same choice of module category for the left and right boundary condition. It is however easy to extend all formulas to the general case. The dimension of the algebra is

$$\dim(\text{Strip}_{\mathcal{C}}(\mathcal{M})) = \sum_{m,r,a,n,s} \tilde{N}_{am}^n \tilde{N}_{ar}^s . \quad (2.23)$$

¹⁵Also see Fn. 48.

The product in the algebra is easily computed using (2.17), (2.19) and (A.40):

$$\begin{array}{|c|} \hline t \\ \hline \downarrow \\ \hline \sigma \\ \hline \downarrow \\ \hline s \\ \hline \end{array} \begin{array}{|c|} \hline b \\ \hline \nearrow \\ \hline \nu \\ \hline \nearrow \\ \hline n \\ \hline \end{array} \times \begin{array}{|c|} \hline s \\ \hline \downarrow \\ \hline \rho \\ \hline \downarrow \\ \hline r \\ \hline \end{array} \begin{array}{|c|} \hline a \\ \hline \nearrow \\ \hline \mu \\ \hline \nearrow \\ \hline m \\ \hline \end{array} = \sum_{c,\tau,\pi} Q_{m,r,(a;\rho\mu),n,s,(b;\sigma\nu),p,t}^{(c;\tau\pi)} \begin{array}{|c|} \hline t \\ \hline \downarrow \\ \hline \tau \\ \hline \downarrow \\ \hline r \\ \hline \end{array} \begin{array}{|c|} \hline c \\ \hline \nearrow \\ \hline \pi \\ \hline \nearrow \\ \hline m \\ \hline \end{array}, \quad (2.24)$$

where

$$Q_{m,r,(a;\rho\mu),n,s,(b;\sigma\nu),p,t}^{(c;\tau\pi)} = \sum_{\alpha} \sqrt{\frac{d_a d_b}{d_c}} [\tilde{F}_{bar}^t]_{(c;\alpha\tau)(s;\rho\sigma)} [\tilde{F}_{bam}^p]^*_{(c;\alpha\pi)(n;\mu\nu)}. \quad (2.25)$$

The sum is over a basis of morphisms $\mathcal{L}_b \otimes \mathcal{L}_a \xrightarrow{\alpha} \mathcal{L}_c$. With no multiplicities this simplifies to

$$Q_{m,r,a,n,s,b,p,t}^c = \sqrt{\frac{d_a d_b}{d_c}} [\tilde{F}_{bar}^t]_{cs} [\tilde{F}_{bam}^p]^*_{cn}. \quad (2.26)$$

We verify in App. E that this product is associative.

3 The asymmetry

As discussed in Sec. 1.4, the asymmetry is a quantum-information-based measure of symmetry breaking in pure or mixed states. Given a density matrix ρ , in order to quantify how much this breaks the symmetry, we compare it with a symmetrized version of itself, that we call ρ_S . The (von Neumann) asymmetry is then defined as the relative entropy

$$\Delta S[\rho] \equiv S[\rho||\rho_S] \equiv \text{tr}[\rho(\log \rho - \log \rho_S)] . \quad (3.1)$$

Because of the properties of ρ_S , the asymmetry can also be written, as we show below around (3.11), as a difference

$$\Delta S[\rho] = S[\rho_S] - S[\rho] , \quad (3.2)$$

where $S[\rho] \equiv -\text{tr}[\rho \log \rho]$ is the von Neumann entropy. From standard properties of relative entropy, it follows that the asymmetry is nonnegative, $\Delta S[\rho] \geq 0$, and that it vanishes if and only if $\rho = \rho_S$. One can also define Rényi asymmetries, characterized by an index n :¹⁶

$$\Delta S^{(n)}[\rho] = \frac{1}{1-n} \log \frac{\text{tr}(\rho_S^n)}{\text{tr}(\rho^n)} . \quad (3.3)$$

The asymmetry is reproduced in the limit $n \rightarrow 1$.

3.1 Symmetrizer for strongly separable algebras

Given a Hilbert space $\mathcal{H}$ and a density matrix $\rho \in \text{End}(\mathcal{H})$, in order to obtain the symmetrized version of ρ we need to construct a *symmetrizer* map S :

$$\begin{aligned} S: \text{End}(\mathcal{H}) &\rightarrow \text{End}(\mathcal{H}) , \\ \rho &\mapsto \rho_S . \end{aligned} \quad (3.4)$$

We demand that the symmetrizer satisfies the following properties:

1. It is a linear projector, in particular $(\rho_S)_S = \rho_S$.
2. Its image is symmetry-invariant, namely $\mathcal{X}_a \rho_S = \rho_S \mathcal{X}_a$ for all $\mathcal{X}_a \in \mathcal{A}$.¹⁷
3. If ρ is already symmetry-invariant, then $\rho_S = \rho$.
4. It is trace-preserving, namely $\text{tr}(\rho_S) = \text{tr}(\rho)$.
5. It maps density matrices to density matrices: $\rho_S^\dagger = \rho_S$ and $\rho_S \geq 0$.

¹⁶If ρ is a pure state, this reduces to the Rényi entropy of ρ_S , which is nonnegative (independently of the details of the symmetrization procedure). For a generic state, it has been proven in [120, 121] that the Rényi asymmetries are nonnegative for every n for the case of a $U(1)$ symmetry. The proof therein generalizes to any abelian algebra, because the symmetrizer still takes the form (3.14). We show in App. C.6, in complete generality, that Rényi asymmetries are nonnegative.

¹⁷It is understood that $\mathcal{A}$ acts on the Hilbert space via some (in general reducible) representation which is fixed by the physical theory.

$$\rho = \left(\begin{array}{ccc|ccc|ccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array} \right) \Rightarrow \rho_S = \left(\begin{array}{cc|ccc|c} \boxed{\blacksquare} & \boxed{\blacksquare} & 0 & 0 & 0 & 0 \\ \boxed{\blacksquare} & \boxed{\blacksquare} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \boxed{\blacksquare} & \boxed{\blacksquare} & \boxed{\blacksquare} & 0 \\ 0 & 0 & \boxed{\blacksquare} & \boxed{\blacksquare} & \boxed{\blacksquare} & 0 \\ 0 & 0 & \boxed{\blacksquare} & \boxed{\blacksquare} & \boxed{\blacksquare} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \boxed{\blacksquare} \end{array} \right) \quad \text{where} \quad \boxed{\blacksquare} = \frac{\text{tr}(\blacksquare)}{\dim(V_r)} \mathbb{1}$$

Figure 4: The Hilbert space splits as $\mathcal{H} = \bigoplus_r n_r V_r$ where V_r is the r -th irreducible representation of $\mathcal{A}$, while n_r is its degeneracy. A similar block decomposition applies to density matrices. The symmetrizer acts by setting to zero the off-diagonal blocks, and keeping sub-blocks proportional to the identity on the diagonal.

Notice that while properties 1. – 4. only depend on the product structure on $\mathcal{A}$, property 5. also depends on its $*$ -structure. Properties 1. – 3. can be more compactly stated as follows: S is a linear projector to the vector space of operators that commute with the algebra $\mathcal{A}$. Indeed, property 2. states that the image of S is contained in the space of commuting operators, while property 3. states that commuting operators are contained in the image of S . Properties 4. and 5., on the other hand, guarantee that ρ_S is a normalized density matrix (as long as ρ is) so that it makes sense to compute its von Neumann or Rényi entropies.

It turns out that such a symmetrizer has a simple and universal construction¹⁸ when the symmetry action on the Hilbert space gives rise to a *strongly-separable algebra*. Consider a unital associative algebra $\mathcal{A}$ over $\mathbb{C}$ defined by the product

$$\mathcal{X}_a \times \mathcal{X}_b = \sum_c T_{ab}^c \mathcal{X}_c, \quad (3.5)$$

where $\{\mathcal{X}_a\}_a$ is a basis and $T_{ab}^c \in \mathbb{C}$ are the multiplication coefficients. We will often omit the product sign $\times$. Associativity corresponds to the identity $\sum_m T_{ab}^m T_{mc}^d = \sum_m T_{am}^d T_{bc}^m$. We denote the unit element, which in general is a nontrivial linear combination of the chosen basis elements, as 1. It satisfies $1\mathcal{X}_a = \mathcal{X}_a 1 = \mathcal{X}_a$ for all a .

The *regular representation* is defined by the left action of the algebra on itself, and is explicitly given by the matrices $(T_a)^c_b = T_{ab}^c$. We can then consider the trace of an algebra element $X = \sum_a x^a \mathcal{X}_a$ over the regular representation:

$$\text{tr}(X) = \sum_a x^a \text{tr}(T_a) = \sum_{am} x^a T_{am}^m. \quad (3.6)$$

Using this trace we define a symmetric bilinear form on the algebra. In the chosen basis, this has components:

$$K_{ab} = \text{tr}(\mathcal{X}_a \mathcal{X}_b) = \text{tr}(T_a T_b) = \sum_{mn} T_{an}^m T_{bm}^n. \quad (3.7)$$

As explained above, in this thesis we only deal with finite-dimensional unital semisimple algebras over $\mathbb{C}$. For such algebras, using the Wedderburn-Artin theorem (see App. C.1), one can verify that the bilinear form (3.7) is non-degenerate. The inverse matrix $\tilde{K} \equiv K^{-1}$ allows one to construct a special element $e \in \mathcal{A} \otimes \mathcal{A}$ called a *symmetric separability idempotent* (SSI):

$$e = \sum_{ab} \tilde{K}^{ab} \mathcal{X}_a \otimes \mathcal{X}_b. \quad (3.8)$$

¹⁸What we mean by “universal” is that the symmetrizer can be constructed just by using elements of the symmetry algebra under consideration, without other specific knowledge of the particular theory.

An algebra for which such an object exists is called *strongly separable*. A symmetric separability idempotent is defined by the following properties:

- (i) It is symmetric under the exchange of the two factors in the tensor product.
- (ii) It satisfies $\mu(e) = 1$, where $\mu: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ denotes the algebra product.
- (iii) It “commutes” with every element of the algebra: $(\mathcal{X}_a \otimes 1)e = e(1 \otimes \mathcal{X}_a)$ for all a .
- (iv) Seen as element of $\mathcal{A} \otimes \mathcal{A}^{\text{op}}$, it is idempotent, $e^2 = e$.¹⁹

It turns out that, under our hypotheses, the one in (3.8) is the unique element with these properties. Indeed, uniqueness of the SSI is stated in Theorem 3.1 of [122] and Propositions 2.30 and 2.46 of [123]. Besides, it is easy to verify it by direct computation for a sum of matrix algebras. Thanks to the WA theorem, this result is general.

The algebras we are interested in are also C^* -algebras and in particular have a $*$ -structure, namely an anti-linear involution $*$: $\mathcal{A} \rightarrow \mathcal{A}$ such that $(XY)^* = Y^*X^*$. By a version of the Wedderburn-Artin theorem [124], one can show that the SSI is Hermitian, $e^* = e$, and that there exists a special basis $\{\mathcal{X}_a\}$ in which it takes the simple form $e = \sum_a \mathcal{X}_a^* \otimes \mathcal{X}_a$. See App. C.1 for more details.

The SSI provides then the general formula for the symmetrizer $\mathbf{S}$. It is enough to represent e on $\text{End}(\mathcal{H})$ by letting the first factor act from the left and the second act from the right:

$$\boxed{\rho_{\mathbf{S}} = \sum_{ab} \tilde{K}^{ab} \mathcal{X}_a \rho \mathcal{X}_b}. \quad (3.9)$$

One can easily verify that the defining properties of the SSI guarantee the properties 1. – 4. of the symmetrizer. Moreover, assuming that $\rho^\dagger = \rho$ and $\rho \geq 0$, one verifies that $\rho_{\mathbf{S}}^\dagger = \rho_{\mathbf{S}}$ (using that $e^* = e$) and that $\rho_{\mathbf{S}} \geq 0$. In fact, it turns out that $\rho_{\mathbf{S}}$ is a trace-preserving unital quantum channel. As we will show, one can also find equivalent alternative expressions for the SSI when the algebra $\mathcal{A}$ has extra properties or extra structure. For instance, for abelian algebras one can rewrite (3.9) as a sum over projectors on one-dimensional representations, while for Hopf and weak Kac algebras one can rewrite (3.9) in terms of the coproduct and antipode operations.

We now turn to some final comments:

- One can ask to what extent the symmetrizer $\mathbf{S}$ is unique. With the additional assumption of “universality” (see Fn. 18), we show in App. C.5 that $\mathbf{S}$ is in fact uniquely fixed.

We report graphically in Fig. 4 the action of the symmetrizer on a density matrix. To see explicitly what this action is, decompose the Hilbert space $\mathcal{H}$ into a direct sum of irreducible representations of $\mathcal{A}$ (with multiplicities). Then, decompose both ρ and $\rho_{\mathbf{S}}$ into blocks that map irreps to irreps. Because $\rho_{\mathbf{S}}$ commutes with the symmetry, by Schur’s lemma (see e.g. [125]) it must be that every block of $\rho_{\mathbf{S}}$ that connects two equivalent irreps is proportional to

¹⁹Here, $\mathcal{A}^{\text{op}}$ denotes the “opposite algebra”, which shares the same underlying vector space as $\mathcal{A}$, but the product is performed in the opposite order.

the identity matrix, while all other blocks are zero. In App. C.5 we furthermore show that the proportionality constant in each nonzero block of ρ_S is the trace of the corresponding block of ρ , divided by its dimension.

On the other hand, the SSI is unique for a finite-dimensional semisimple algebra. As we commented above, once represented, the SSI defines an operator on $\text{End}(\mathcal{H})$ that satisfies the axioms. Thus, by uniqueness, this operator must be the same as S .

Alternatively, one could give up on the axioms 1. – 5. and instead define S to be *the* orthogonal (or self-adjoint) projector²⁰ onto the space of operators that commute with the algebra, where orthogonality is defined with respect to the Hilbert-Schmidt scalar product. This definition gives exactly the same symmetrizer as (3.9), since one can verify that this is indeed a self-adjoint projector.

Finally, one could relax the requirement of universality and look for more general symmetrizers, which would however require the knowledge of the specific theory.

- Our symmetrizer (3.9) is a generalization of the symmetrizer for invertible symmetries previously constructed in the literature for a finite group G . To see this, consider the group algebra $\mathcal{A} = \mathbb{C}[G]$, which is the fusion algebra associated to this symmetry. Its basis elements $\mathcal{X}_g$ correspond to the group elements $g \in G$ and to the simple objects $\mathcal{L}_g$ of the fusion category Vec_G . The multiplication coefficients are $T_{gh}^k = \delta_{(gh)}^k$, where (gh) is the group multiplication. The bilinear form is then $K_{gh} = |G| \delta_{g,h^{-1}}$ and its inverse is $\tilde{K}^{gh} = |G|^{-1} \delta^{g,h^{-1}}$. The symmetrizer then reads:

$$\rho_S = \frac{1}{|G|} \sum_{g \in G} \mathcal{L}_g \rho \mathcal{L}_{g^{-1}} . \quad (3.10)$$

This expression agrees with what has already been used in the literature (e.g. [76, 77, 126, 127]).

- For the symmetrizer (3.9) it is easy to prove that the two definitions (3.1) and (3.2) of asymmetry coincide. It is enough to notice that

$$\text{tr}(\rho_S \log \rho_S) = \sum_{ab} \tilde{K}_{ab} \text{tr}(\mathcal{X}_a \rho \mathcal{X}_b \log \rho_S) = \sum_{ab} \tilde{K}_{ab} \text{tr}(\rho \mathcal{X}_b \log(\rho_S) \mathcal{X}_a) = \text{tr}(\rho (\log \rho_S)_S) , \quad (3.11)$$

but $(\log \rho_S)_S = \log \rho_S$ because $\log \rho_S$ commutes with the algebra, as does ρ_S . It turns out that all quantities appearing in (3.11), as well as the asymmetry itself, are well-defined and finite. We comment on this in App. C.6.

- For future reference, using the non-degenerate bilinear form (3.7), one can prove (we give some details in App. C.1) the “pivotal” relation

$$T_{ca}^b = \sum_{b'c'} \tilde{K}^{bb'} K_{cc'} T_{ab'}^{c'} . \quad (3.12)$$

Moreover, the unit can be written as

$$1 = \sum_{abc} T_{ab}^b \tilde{K}^{ac} \mathcal{X}_c = \sum_{abc} \tilde{K}^{ab} T_{ab}^c \mathcal{X}_c . \quad (3.13)$$

²⁰Recall that a projector is not uniquely defined by its image, however it is if one demands that it is self-adjoint with respect to a given scalar product.

3.1.1 Abelian algebras

In the case of an abelian algebra, the symmetrizer S can be described in an alternative but equivalent way, that we now present. The irreducible representations of an abelian algebra are one-dimensional. Therefore, the blocks shown in Fig. 4 are one-dimensional and the symmetrizer simply acts by setting to zero all blocks that connect different irreducible representations. Let $P_r: \mathcal{H} \rightarrow \mathcal{H}$ denote the projector onto the r -th irreducible representation. The symmetrizer then reads²¹

$$\rho_S = \sum_r P_r \rho P_r. \quad (3.14)$$

If we further assume that the algebra is an abelian *fusion algebra*, an explicit formula for P_r in terms of simple lines is available. This was discussed in [34] in the context of fusion algebras of modular fusion categories, but in fact it holds more generally for arbitrary fusion algebras. In the following we review the construction, partially based on [128].

The matrices $(N_a)^b{}_c \equiv N_{ac}^b$ give the regular representation of the algebra. They satisfy $N_{\bar{a}} = N_a^\top = N_a^\dagger$ and in the abelian case they all commute. It follows that they are normal matrices and can be simultaneously diagonalized with a unitary transformation. Let us call $w^{(r)}$ the common eigenvectors, that are orthogonal with respect to the standard Hermitian product, and $\nu_{a(r)}$ the eigenvalues:

$$N_a w^{(r)} = \nu_{a(r)} w^{(r)}. \quad (3.15)$$

Note that $r = 1, \dots, \dim \mathcal{A}$. For each r , the eigenvalues provide a one-dimensional representation of the fusion algebra, namely

$$\nu_{a(r)} \nu_{b(r)} = \sum_c N_{ab}^c \nu_{c(r)}. \quad (3.16)$$

This also shows that the eigenvectors can be taken to be equal to the eigenvalues: $w_a^{(r)} = \nu_{\bar{a}(r)}$, as one can check. Orthogonality of the vectors implies that $\sum_a \nu_{a(r)}^* \nu_{a(m)} = |s_r|^{-2} \delta_{rm}$ for some numbers s_r that we can take to be in $\mathbb{R}_{>0}$. One defines the matrix

$$S_{ar} = \nu_{a(r)} s_r. \quad (3.17)$$

Note that $\nu_{1(r)} = 1$ in all representations r since $\mathcal{L}_1$ is the identity, thus $s_r = S_{1r}$. The action of the fusion algebra on its representations is usually written as

$$N_a w^{(r)} = \frac{S_{ar}}{S_{1r}} w^{(r)}, \quad (3.18)$$

and the matrix S_{ar} is called the S -matrix. One checks that $S^\dagger S = \mathbb{1}$, and therefore also $SS^\dagger = \mathbb{1}$. Such identities correspond to the orthogonality and completeness of characters. The relations (3.16) can then be written as

$$N_{ab}^c = \sum_r \frac{S_{ar} S_{br} S_{cr}^*}{S_{1r}}, \quad (3.19)$$

which is the Verlinde formula [10]. It also shows that $S^\top$ diagonalizes the matrices N_a , and that $\nu_{\bar{a}(r)} = \nu_{a(r)}^*$. Notice that when $\mathcal{C}$ is a modular fusion category, the S -matrix is also symmetric, but in general it is not.

²¹The expression (3.14) agrees with the formula originally proposed in [76] for invertible abelian symmetries.

The projector onto the r -th irreducible representation is then given by

$$P_r = S_{1r} \sum_a S_{ar}^* \mathcal{L}_a . \quad (3.20)$$

One can verify, using the Verlinde formula and unitarity of the S -matrix, that the set $\{P_r\}$ forms a family of orthogonal projectors, i.e. they satisfy $P_r P_s = \delta_{rs} P_r$, and that if $|\psi\rangle$ belongs to the r -th irreducible representation then $P_{r'} |\psi\rangle = \delta_{rr'} |\psi\rangle$. Finally, one can verify that the expression (3.14) with P_r given by (3.20) agrees with (3.9).²²

Note that for a nonabelian algebra the formula (3.14) would fail. From Fig. 4 it is clear that this would only put the density matrix in a block-diagonal form, without further forcing each sub-block to be proportional to the identity matrix. In Sec. 5.6 we study the simplest noninvertible and nonabelian example: the Haagerup fusion algebra $H_{(3)}$.

3.2 Asymmetry for the circle Hilbert space

The construction of Sec. 3.1 is general and allows one to define the asymmetry for any type of finite internal symmetry, including higher-form [5] and noninvertible [14] symmetries. In the following, however, we will present concrete constructions focusing on the case of two-dimensional unitary theories. In two dimensions, one can reduce to the case that only 0-form symmetries are present [129, 130]. Hence, the internal symmetries of a 2d unitary theory are described by a unitary fusion category $\mathcal{C}$ whose objects are the topological lines that implement the symmetry in the theory. We call $\{\mathcal{L}_a\}_a$ the simple lines, which are finite in number.

3.2.1 Excited states in CFTs

An interesting class of theories where certain asymmetries can be computed analytically is given by two-dimensional CFTs. The vacuum $|0\rangle$ of a 2d unitary CFT does not break any internal symmetry, therefore its asymmetries vanish. What is more interesting is to compute the asymmetry of *excited states*. An accessible class of states is obtained by performing the Euclidean path-integral on a half-space with the insertion of local operators:

$$|\psi\rangle = \left(\sum_i \mathcal{O}_i(z_i) \right) |0\rangle . \quad (3.21)$$

Here $\mathcal{O}_i$ are local operators, while $z_i \in \mathbb{C}$ denote points in the Euclidean lower half-plane.

In the case of invertible symmetries, the asymmetries of such states have been computed e.g. in [131–133]. Here we compare the asymmetries (of the whole system on $\mathbb{R}$ or S^1) with respect to the fusion algebras of invertible and noninvertible symmetries, and with respect to the tube algebra. We show, in a specific example, that the larger the algebra is, the more powerful the asymmetry is in detecting symmetry breaking.

Consider the example of the Ising CFT, which has three untwisted primary local operators: the identity $\mathbb{1}$, the energy operator ε , and the spin operator σ . We study this example in

²²The expression in (3.14) is $\rho_S = \sum_{ab} \left(\sum_r S_{1r}^2 S_{ar}^* S_{br}^* \right) \mathcal{L}_a \rho \mathcal{L}_b$. We need to show that $\sum_r S_{1r}^2 S_{ar}^* S_{br}^* = \tilde{K}^{ab}$. This is easily verified by multiplying both sides by the matrix K_{bc} , using its definition (3.7) and the Verlinde formula (3.19) twice.

Sec. 5.2. First, consider the $\mathbb{Z}_2$ spin-flip symmetry that maps $\sigma \mapsto -\sigma$. Since $\mathbb{1}$ and ε are neutral, any state constructed using them has vanishing asymmetry. The spin operator is odd, but the states $\sigma(z)|0\rangle$ have vanishing asymmetry because, as rays in the Hilbert space, they are mapped to themselves (equivalently, the density matrices $\rho \propto \sigma(z)|0\rangle\langle 0|\sigma(\bar{z})$ commute with the fusion algebra). Therefore, in order to detect any asymmetry, one should consider states such as $(\mathbb{1} + \sigma(z))|0\rangle$.

Next, consider the full noninvertible symmetry, namely the fusion category $\mathcal{C} = \text{Ising}$ (see Sec. 5.2), which includes Kramers-Wannier duality defect. This symmetry is more powerful because ε is not invariant. We can thus detect symmetry breaking in states such as $(\mathbb{1} + \sigma(z))|0\rangle$ and also $(\mathbb{1} + \varepsilon(z))|0\rangle$. We compute the asymmetries of the latter on $\mathbb{R}$ with respect to the fusion algebra of Ising in (5.37), and plot the second Rényi asymmetry in Fig. 11.

Finally, consider the tube algebra of Ising . Since the Kramers-Wannier duality maps the spin operator σ to the disorder (or twist) operator μ , under the action of the tube algebra the operators σ and μ should be able to mix. Thus, this time, even the state $\sigma(z)|0\rangle$ exhibits nonvanishing asymmetry, because they are mapped to different states in $\mathcal{H}_{\text{Tube}}$ (even when considered as rays). We compute the corresponding asymmetry around (5.50).

Abelian algebras. For CFTs, whenever the symmetry algebra $\mathcal{A}$ is abelian, the asymmetries of the full system in the excited states (3.21) can be easily computed in terms of two-point functions.

Consider a pure state prepared by a Euclidean path-integral with the insertion of a sum of local operators in the Euclidean past. The Hilbert space could be the one on the circle or on the real line. The operators transform in irreducible representations of the symmetry algebra $\mathcal{A}$, which in the abelian case are all one-dimensional. The state is

$$|\psi\rangle = \sum_a |\psi_a\rangle = Z^{-\frac{1}{2}} \sum_a \boxed{\bullet \phi_a} . \quad (3.22)$$

The sum is over irreducible representations of $\mathcal{A}$, and each component is the Euclidean path-integral on a half-plane or a half-cylinder (in order to prepare a state on $\mathbb{R}$ or S^1 , respectively) with the insertion of $\phi_a(z)$, as depicted, while Z is the path-integral on the plane or the cylinder with no insertions. The normalization of the states is provided by the 2-point functions:

$$\langle \psi_a | \psi_a \rangle = \langle \phi_a^\dagger(\bar{z}) \phi_a(z) \rangle , \quad (3.23)$$

where $\bar{z}$ is the point specular to z with respect to the spatial cut. The density matrix for $|\psi\rangle$ is

$$\rho = \frac{1}{R} |\psi\rangle\langle\psi| = \frac{1}{R} \sum_{ab} |\psi_a\rangle\langle\psi_b| \quad \text{with} \quad R = \sum_a \langle \psi_a | \psi_a \rangle . \quad (3.24)$$

When the algebra is abelian we can use (3.14), hence the symmetrized matrix is

$$\rho_S = \frac{1}{R} \sum_a |\psi_a\rangle\langle\psi_a| , \quad (3.25)$$

and the asymmetries are

$$\Delta S^{(n)} = \frac{1}{1-n} \log \left[\sum_a \frac{\langle \psi_a | \psi_a \rangle^n}{R^n} \right] , \quad \Delta S = - \sum_a \frac{\langle \psi_a | \psi_a \rangle}{R} \log \frac{\langle \psi_a | \psi_a \rangle}{R} , \quad (3.26)$$

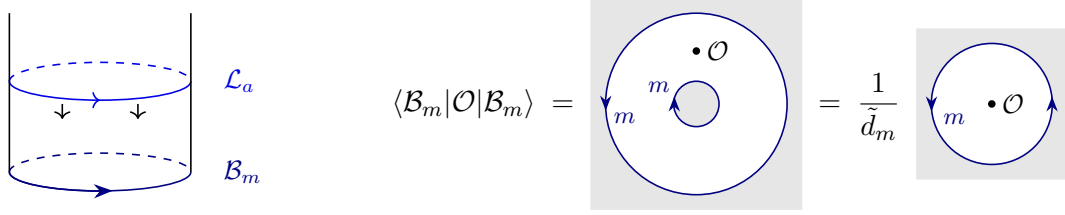


Figure 5: Left: the clustering states, or vacua, $|\mathcal{B}_m\rangle$ on the circle are created by the simple boundaries. Right: Correlation functions in those states are computed by disc partition functions.

where we used orthogonality between the states $|\psi_a\rangle$. Because of (3.23), the asymmetries are expressed in terms of two-point functions.

3.2.2 Spontaneous symmetry breaking

A natural application of the asymmetry is to detect spontaneous symmetry breaking in gapped phases. In the far IR, a gapped phase is described by a topological field theory on which the symmetry acts. Therefore, for a given fusion category $\mathcal{C}$, the possible indecomposable and inequivalent (from the symmetry point of view) two-dimensional gapped phases are in one-to-one correspondence with indecomposable two-dimensional $\mathcal{C}$ -symmetric TQFTs. In turn, the latter are in one-to-one correspondence with indecomposable module categories $\mathcal{M}$ for $\mathcal{C}$ [93].²³ We can then try to use the asymmetry to distinguish the various phases.

Consider a 2d gapped phase described by a certain 2d $\mathcal{C}$ -symmetric TQFT in the IR, and let $\mathcal{M}$ be the corresponding module category. The circle Hilbert space of the TQFT is finite-dimensional. If the symmetry is unbroken, it is one-dimensional and the asymmetry with respect to the fusion algebra vanishes.²⁴ On the other hand, when a finite symmetry is spontaneously broken we expect a higher-dimensional circle Hilbert space on which the symmetry acts. We are not interested in computing the asymmetry of generic vectors in that Hilbert space, but only of very special vectors that we call “vacua”: these are the states that satisfy the cluster decomposition property. Such states form a preferred basis of the Hilbert space. It turns out that they are in correspondence with the simple objects $\mathcal{B}_m$ of the corresponding module category $\mathcal{M}$, and can be constructed using open/closed duality [114]: one simply considers the cylinder with the boundary condition $\mathcal{B}_m$ in the past, as in Fig. 5 left.

To prove that the states created by simple boundaries satisfy cluster decomposition one uses a boundary crossing relation proven in [114]:

$$\begin{array}{c} \text{Diagram 1: A cylinder with two vertical boundaries. The left boundary is labeled } n \text{ and the right boundary is labeled } m. \end{array} = \frac{\delta_{m,n}}{\tilde{d}_m} \begin{array}{c} \text{Diagram 2: A cylinder with two horizontal boundaries. The top boundary is labeled } m \text{ and the bottom boundary is labeled } m. \end{array} . \quad (3.27)$$

²³See Sec. 2.3 for a brief introduction to module categories and an explanation of how these describe boundary conditions.

²⁴Here we only consider indecomposable gapped phases.

A correlator $\langle \mathcal{B}_m | \mathcal{O} | \mathcal{B}_m \rangle$ is the annulus partition function with boundaries $\mathcal{B}_m$ and with the insertion of a local operator $\mathcal{O}$, as in Fig. 5 right. Using the crossing relation, such a correlator is equal to $1/\tilde{d}_m$ times the disc one-point function of $\mathcal{O}$ with boundary $\mathcal{B}_m$, that we call $Z_{D_2}[\mathcal{O}]$. If $\mathcal{O}$ is the identity operator, the latter is equal to $\tilde{d}_m$ and thus the correlator is normalized. The crossing relation can then be used to break a disc into two discs, hence one obtains:

$$\langle \mathcal{B}_m | \mathcal{O}_1 \mathcal{O}_2 | \mathcal{B}_m \rangle = \frac{1}{\tilde{d}_m} Z_{D_2}[\mathcal{O}_1 \mathcal{O}_2] = \frac{1}{\tilde{d}_m^2} Z_{D_2}[\mathcal{O}_1] Z_{D_2}[\mathcal{O}_2] = \langle \mathcal{B}_m | \mathcal{O}_1 | \mathcal{B}_m \rangle \langle \mathcal{B}_m | \mathcal{O}_2 | \mathcal{B}_m \rangle . \quad (3.28)$$

This shows that in the states $|\mathcal{B}_m\rangle$ all correlation functions are equal to products of one-point functions, which is cluster decomposition for topological theories.

The fusion category $\mathcal{C}$ acts on the untwisted circle Hilbert space through its fusion algebra, as described in Sec. 2.1. As is clear from Fig. 5 left, the action of the lines on the vacua is given by the fusion coefficients $\tilde{N}_{am}^n$ (2.13) of the corresponding module category $\mathcal{M}$:

$$\mathcal{L}_a |\mathcal{B}_m\rangle = \sum_n \tilde{N}_{am}^n |\mathcal{B}_n\rangle . \quad (3.29)$$

Thus the clustering states of a gapped phase furnish a NIM-rep of the fusion algebra. This allows one to easily compute the asymmetry of the various vacua.

In Sec. 5 we analyze various examples of spontaneous symmetry breaking and compute the asymmetry of the different vacua. In Sec. 5.1 we consider an finite group-like symmetry G spontaneously broken to a subgroup H , and reproduce the result of [134] that the von Neumann and Rényi asymmetries are equal to $\Delta S = \log(N_{\text{vac}})$, where $N_{\text{vac}} = |G|/|H|$ is the number of vacua. In the rest of Sec. 5 we study broken noninvertible symmetries. One of the outcomes is that the asymmetries can take more general values, and are not directly related to the number of vacua. Another outcome is that, as opposed to the invertible case, different vacua can exhibit inequivalent physical properties (as already anticipated in [101, 102]), and in particular can have different asymmetries. For instance, in a theory that spontaneously breaks the Ising symmetry, one vacuum has $\Delta S = \log 2$ while two other vacua (which are related by the $\mathbb{Z}_2$ subsymmetry) have $\Delta S = \frac{3}{2} \log 2$ (see Sec. 5.2.1).

3.3 Asymmetry for the interval Hilbert space

The entanglement entropy – and likewise the asymmetry – of a subsystem is normally defined assuming that the Hilbert space factorizes: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Here A is the spatial subregion of interest, while B is its complement. This allows one to define the reduced density matrix $\rho_A = \text{tr}_B(\rho)$ from the complete density matrix ρ (that could be pure or mixed), and then compute the entropy or the asymmetry of ρ_A . In quantum field theory one usually takes A to be a spatial subregion, and here we will consider the case that A is an interval. However, in quantum field theory the Hilbert space does not actually factorize. A consequence is that entanglement entropy is divergent and it requires a regularization.

A convenient regularization was proposed in [65]. One constructs a linear map

$$\psi_B: \mathcal{H} \rightarrow \mathcal{H}_A \otimes \mathcal{H}_B \quad (3.30)$$

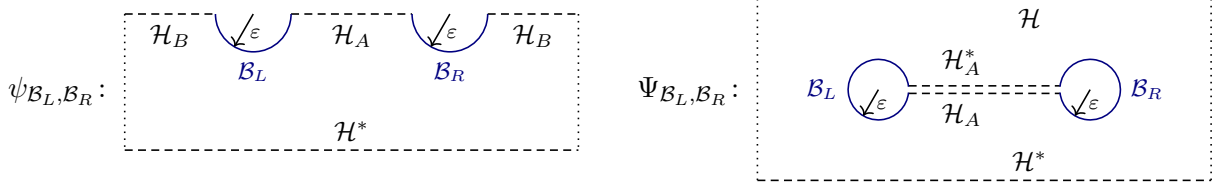


Figure 6: Left: the map $\psi_{\mathcal{B}_L, \mathcal{B}_R}: \mathcal{H} \rightarrow \mathcal{H}_A \otimes \mathcal{H}_B$ constructed with the path-integral on a strip, where $\mathcal{B}_{L,R}$ are boundary conditions at two removed half-discs of radius ε . Right: the map $\Psi_{\mathcal{B}_L, \mathcal{B}_R}: \text{End}(\mathcal{H}) \rightarrow \text{End}(\mathcal{H}_A)$ constructed via the path-integral on a strip with a finite-size cut, with the same boundary conditions imposed.

and computes the entanglement entropy in the image of $\psi_{\mathcal{B}}$. Such a map can be constructed as the path-integral on a strip, in which the lower (or incoming) boundary has the geometry underlying $\mathcal{H}$, while the upper (or outgoing) boundary is the union of two disjoint parts with geometries underlying $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. In the case that $\mathcal{H}_A$ is an interval, one removes two small half-discs of radius ε at the two entangling points that bound A , and specifies some boundary conditions $\mathcal{B}_{L,R}$ there.²⁵ Eventually, the limit $\varepsilon \rightarrow 0$ is taken. We depict the setup in Fig. 6 left. Given the map $\psi_{\mathcal{B}}$ between Hilbert spaces, one also constructs the linear map

$$\Psi_{\mathcal{B}}: \text{End}(\mathcal{H}) \rightarrow \text{End}(\mathcal{H}_A) \quad (3.31)$$

that acts on density matrices and already includes the partial trace on $\mathcal{H}_B$. It can be constructed as the path-integral on a strip obtained from a copy of $\psi_{\mathcal{B}}$ and a copy of its reflected image, glued along B . We depict the setup in Fig. 6 right. The path-integral constructions of those maps are particularly useful when also the states we want to analyze have a path-integral preparation, as we will see.

Once we have a (generically mixed) state in $\mathcal{H}_A$, we can define the asymmetry in the general way described in Sec. 3.1. To this end, we need to identify how the symmetry acts on $\mathcal{H}_A$, which is the Hilbert space on an interval with boundary conditions $\mathcal{B}_{L,R}$. This action is given by the strip algebra, which we described in Sec. 2.3. Note that $\mathcal{H}_A$ could also describe the asymmetry of the full system on an interval with boundary conditions; in this case one would not take the limit $\varepsilon \rightarrow 0$. For invertible symmetries, asymmetry in the presence of boundaries has been studied in [135, 136].

Since, in our context, the boundary conditions are introduced as a tool to regularize the entanglement entropy, it is desirable to use symmetry-invariant boundary conditions, in order not to contaminate the measurement of symmetry breaking coming from the state with an effect due to the boundaries.²⁶ Moreover, in order to compute the symmetry-resolved entanglement entropy, one starts from a symmetric full-system density matrix, and one would like it to remain symmetric even after applying $\psi_{\mathcal{B}}$. The type of symmetry-invariant boundary conditions available in a given theory depends on the structure of the symmetry, and in particular on the presence of 't Hooft anomalies [93, 96, 97]. For noninvertible symmetries, one can distinguish three situations [36]: theories with strongly-symmetric simple boundaries, theories with only

²⁵In (3.30) we used the compact notation $\mathcal{B}$ to indicate the collection of boundary conditions used.

²⁶On the other hand, as previously mentioned, it is an interesting question to study the asymmetry of boundary conditions. This has been analyzed in [135, 136].

weakly-symmetric simple boundaries, and theories with no symmetric simple boundaries. We now discuss these three cases in turn.

3.3.1 Strongly-symmetric boundary conditions

The most favorable case is when $\mathcal{C}$ admits a module category with single simple object (that we interpret as a boundary condition, as usual). Such a boundary condition is then automatically symmetry-invariant, meaning that it is invariant (up to rescaling) under parallel fusion with the lines $\mathcal{L}_a \in \mathcal{C}$. Yet, there are in general multiple junctions between a given bulk line $\mathcal{L}_a$ and the boundary.²⁷ Such a boundary condition was called *strongly-symmetric* in [36]. Choosing it makes the definition and computation of the asymmetry particularly simple, because the density matrix obtained via the map $\Psi_{\mathcal{B}}$ (see Fig. 6) can be used directly. Such a boundary condition exists if and only if the symmetry is free from 't Hooft anomalies.²⁸

Let $\mathcal{B}$ denote the single simple object of $\mathcal{M}$. Since the boundary index takes a single value, we henceforth indicate it with a dot. The fusion coefficients and the $\tilde{F}$ -symbols simplify:

$$\tilde{N}_{a\cdot} = \tilde{N}_{\cdot a} = d_a \geq 1, \quad [\tilde{F}_{ab\cdot}]_{(c;\delta\rho)(\cdot;\nu\mu)}^* \equiv m_{a\mu,b\nu}^{c\rho;\delta}. \quad (3.32)$$

The fact that the boundary fusion coefficients are equal to the quantum dimensions immediately follows from (2.18). Thus a necessary condition for the existence of strongly-symmetric boundary conditions in $\mathcal{C}$ is that all quantum dimensions are integer. Notice that when $\tilde{N}_{a\cdot} > 1$ there are multiple junctions between the line $\mathcal{L}_a$ and the boundary. It turns out that there is a one-to-one correspondence between the following objects:

- left-module categories over $\mathcal{C}$ with only one simple object;
- fiber functors of the fusion category $\mathcal{C}$;
- maximal haploid algebra objects $\mathfrak{A}$ in $\mathcal{C}$.²⁹

The first entry corresponds to a strongly-symmetric boundary condition. The second entry corresponds to a trivially-gapped phase (a.k.a. SPT phase, or invertible TQFT) for the symmetry, and thus the symmetry is non-anomalous. We see that strongly-symmetric boundary conditions exist if and only if the symmetry is non-anomalous. The third entry corresponds to a maximal gauging of the whole symmetry (possible only in the absence of anomalies).

²⁷This is always the case for a noninvertible line, as its quantum dimension is greater than 1.

²⁸More precisely, when there is no 't Hooft anomaly, there always exists a module category with a single simple object, that can describe a strongly-symmetric boundary condition. Whether such a boundary condition actually exists in a given theory depends on the theory itself and on extra properties we might require to the boundary condition, for instance conformality. A similar comment applies to the regular module category that always exists, irrespective of 't Hooft anomalies.

²⁹More precisely, one is typically interested in equivalence classes of the objects listed above. The relevant equivalence relations are, respectively: equivalence of $\mathcal{C}$ -module categories; equivalence of tensor functors; Morita equivalence of algebras. In particular, the last one turns out to be the correct equivalence that corresponds to equivalent module categories, as opposed to the stronger notion of isomorphism of algebras [113, 137]. See also Fn. 31.

The redefinition in (3.32) right makes the correspondence with the maximal algebra object $\mathfrak{A}$ explicit. At the level of objects, this is given by

$$\mathfrak{A} = \bigoplus_a d_a \mathcal{L}_a, \quad (3.33)$$

where $d_a \in \mathbb{Z}_{\geq 1}$ is equal to the dimension of $\text{Hom}(\mathcal{L}_a, \mathfrak{A})$.³⁰ The symbols m are the components of the multiplication morphism $m: \mathfrak{A} \otimes \mathfrak{A} \rightarrow \mathfrak{A}$:

$$\begin{array}{c} c \\ | \\ \rho \\ \textcircled{\mathfrak{A}} \\ / \quad \backslash \\ \mu \quad \nu \\ | \quad | \\ a \quad b \end{array} = \sum_{\delta} m_{a\mu, b\nu}^{c\rho; \delta} \begin{array}{c} c \\ | \\ \delta \\ / \quad \backslash \\ a \quad b \end{array}. \quad (3.34)$$

The requirement that m be associative, namely that $m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m)$, when written in components, boils down to the constraint [138]:

$$\sum_{e, \epsilon, \alpha, \beta} m_{a\mu, b\nu}^{e\epsilon; \alpha} m_{e\epsilon, c\rho}^{d\sigma; \beta} [F_{abc}^d]_{(e; \alpha\beta)(f; \gamma\delta)} = \sum_{\phi} m_{a\mu, f\phi}^{d\sigma; \delta} m_{b\nu, c\rho}^{f\phi; \gamma}. \quad (3.35)$$

Using the correspondence in (3.32) and unitarity, this equation is equivalent to the left-module pentagon equation (A.43) (for module categories with one simple object). Notice that a fusion category $\mathcal{C}$ can admit multiple inequivalent maximal algebra objects $\mathfrak{A}$: they will have the same underlying object (3.33), but would differ by the product m .³¹

The corresponding strip algebra is generated by the basis elements

$$H_{(a; \rho\mu)} \equiv \begin{array}{c} \text{strip diagram with } a \text{ and } \mu \text{ labels} \end{array} \quad (3.36)$$

and the product is given by

$$\begin{aligned} Q_{(a; \rho\mu)(b; \sigma\nu)}^{(c; \tau\pi)} &= \sum_{\alpha} \sqrt{\frac{d_a d_b}{d_c}} [\tilde{F}_{ba}^{\cdot}]_{(c; \alpha\tau)(\cdot; \rho\sigma)} [\tilde{F}_{ba}^{\cdot}]_{(c; \alpha\pi)(\cdot; \mu\nu)}^* \\ &= \sum_{\alpha} \sqrt{\frac{d_a d_b}{d_c}} (m_{b\sigma, a\rho}^{c\tau; \alpha})^* (m_{b\nu, a\mu}^{c\pi; \alpha}). \end{aligned} \quad (3.37)$$

The sum is over a basis of morphisms $\mathcal{L}_b \otimes \mathcal{L}_a \xrightarrow{\alpha} \mathcal{L}_c$. This defines a finite-dimensional semisimple algebra $\mathcal{A}$, whose symmetrizer can be constructed using the universal formula (3.9).

³⁰Haploid means that $\dim(\text{Hom}(\mathcal{L}_1, \mathfrak{A})) = 1$.

³¹The construction is more general, see [36, 139] as well as [14, 113, 114]. Let $\mathcal{M}$ be indecomposable semisimple, and let $m \in \mathcal{M}$ be simple. Then, the so-called internal Hom construction produces a (in general not maximal) haploid algebra object $\mathfrak{A}_m \in \mathcal{C}$ as $\mathfrak{A}_m \equiv \underline{\text{Hom}}(m, m) = \bigoplus_a \tilde{N}_{am}^m a$ (with a certain product structure). This gives $\mathcal{M}$ back as the category of right $\mathfrak{A}$ -modules in $\mathcal{C}$, namely $\mathcal{C}_{\mathfrak{A}} \cong \mathcal{M}$ as $\mathcal{C}$ -module categories. Different choices of m can produce nonisomorphic algebra objects, but these algebras are Morita equivalent, meaning they determine equivalent $\mathcal{C}$ -module categories. All haploid algebra objects are obtained in this way, by considering all possible indecomposable module categories. Conversely, all indecomposable module categories can be obtained by listing (Morita-equivalence classes of) haploid algebras in $\mathcal{C}$.

3.3.2 Weakly-symmetric boundary conditions

When strongly-symmetric boundary conditions do not exist, we are forced to consider modules with multiple simple objects. This means that we have to deal with a set of boundary conditions that form a “multiplet” under fusion with the bulk symmetry lines. It might happen, however, that the module contains one particular simple boundary $\mathcal{B}_{\underline{m}} \in \text{Irr}(\mathcal{M})$ such that the fusion of any bulk line $\mathcal{L}_a$ with it contains $\mathcal{B}_{\underline{m}}$ itself, $\mathcal{B}_{\underline{m}} \subseteq \mathcal{L}_a \otimes \mathcal{B}_{\underline{m}}$. This happens precisely when

$$\tilde{N}_{a\underline{m}}^{\underline{m}} \geq 1 \quad \text{for all } a. \quad (3.38)$$

This means that every line $\mathcal{L}_a$ has at least one topological junction with the boundary $\mathcal{B}_{\underline{m}}$. Such a boundary condition was called “weakly-symmetric” in [36]. It allows us to restrict to a subspace of the strip algebra in which we have all bulk lines but only the boundary $\mathcal{B}_{\underline{m}}$, and that is closed under strip-algebra product. When multiple junctions exist, $N_{a\underline{m}}^{\underline{m}} > 1$, in general one has to keep all of them in order to obtain a closed algebra.³² We call the resulting algebra *reduced strip algebra*.

Note that the reduced strip algebra always admits a symmetrizer. Indeed, its C^* -structure is simply the restriction of the C^* -structure of the full strip algebra. In particular, the C^* condition (C.28) is still satisfied and the reduced strip algebra is thus semisimple and strongly separable. The symmetrizer can be constructed using the universal formula (3.9).

It turns out [36] that a fusion category $\mathcal{C}$ admits a weakly-symmetric boundary condition if and only if there exists a (not necessarily maximal) haploid algebra object

$$\mathfrak{A} = \bigoplus_a N_a^{(\mathfrak{A})} \mathcal{L}_a \quad \text{such that } N_a^{(\mathfrak{A})} \geq 1 \text{ for all } a. \quad (3.39)$$

We introduced the nonnegative integer numbers $N_a^{(\mathfrak{A})} \equiv \dim(\text{Hom}(\mathcal{L}_a, \mathfrak{A}))$, that for haploid algebra objects always satisfy $N_a^{(\mathfrak{A})} \leq d_a$ [140]. The connection between boundaries and algebra objects goes through the internal Hom construction (see Fn. 31). Haploid algebra objects are the noninvertible generalization of anomaly-free subgroups of an invertible symmetry group, along with a choice of discrete torsion for it. Thus, a weakly-symmetric boundary condition exists if and only if there exists a gauging of $\mathcal{C}$ that involves all simple lines in $\mathcal{C}$. When the gauging is maximal, $N_a^{(\mathfrak{A})} = d_a$ (which is possible only in the absence of ’t Hooft anomalies) the boundary condition is actually strongly symmetric.

3.3.3 Non-symmetric boundary conditions

It might happen that a given theory admits neither strongly- nor weakly-symmetric boundary conditions, either because the symmetry has a severe enough ’t Hooft anomaly to prevent them, or because we insist on some extra property, for instance conformality, of the boundary conditions. In such cases, the boundary condition at the entangling surface necessarily breaks the symmetry, implying a nonzero asymmetry on subsystems even when the original state is invariant. To overcome this, and to detect symmetry breaking of states independently of the

³²In some cases it might be possible to restrict to a subspace of the junction space.

boundary conditions at the entangling surface, we consider a map $\Psi_{\mathcal{B}}$ (see (3.31)) that produces a statistical mixture of density matrices with different boundary conditions. We show that there exists a choice of coefficients such that the invariant state prepared by the empty path-integral exhibits indeed vanishing asymmetry. The procedure can be carried out for any choice of module category $\mathcal{M}$, as we show here. However, in the examples in Sec. 5 we specialize to the regular module category.

At a formal level, we could define a symmetric “composite” boundary condition $|\mathcal{B}_S\rangle$ if we allow for arbitrary linear combinations of boundaries (which we denote here as kets because we regard them as elements of a vector space):

$$|\mathcal{B}_S\rangle = \sum_m \tilde{d}_m |\mathcal{B}_m\rangle \quad \text{so that} \quad \mathcal{L}_a |\mathcal{B}_S\rangle = d_a |\mathcal{B}_S\rangle . \quad (3.40)$$

Here d_a and $\tilde{d}_m$ are the bulk and boundary quantum dimensions, and we used (2.18) to prove the second equality.³³

We exploit this idea – in the case of unitary and normalizable CFTs and insisting on conformal boundary conditions – to construct a symmetric density matrix ρ on the interval Hilbert space starting from an invariant state, even in the absence of invariant simple boundary conditions. Consider a module category $\mathcal{M}$, and let ρ_{mn} be the reduced density matrix of the vacuum (or of another state invariant under the symmetry) constructed with simple boundary conditions $\mathcal{B}_m, \mathcal{B}_n \in \mathcal{M}$. We construct an “average” over boundary conditions as follows:

$$\rho = \frac{1}{D^2} \sum_{m,n \in \text{Irr}(\mathcal{M})} \tilde{d}_m \tilde{d}_n \rho_{mn} = \frac{1}{D^2} \sum_{m,n \in \text{Irr}(\mathcal{M})} \frac{\tilde{d}_m \tilde{d}_n}{Z_{mn}} \tilde{\rho}_{mn} , \quad D = \sum_{m \in \text{Irr}(\mathcal{M})} \tilde{d}_m \quad (3.41)$$

Here $\tilde{\rho}_{mn}$ is the unnormalized density matrix produced by the path-integral realization of $\Psi_{\mathcal{B}}$ in (3.31) with boundary conditions $\mathcal{B}_m$ and $\mathcal{B}_n$, as depicted in Fig. 15, while $Z_{mn} = \text{tr}(\tilde{\rho}_{mn})$. We show that in the limit $\varepsilon \rightarrow 0$, ρ commutes with the strip algebra and thus it has vanishing asymmetry. The interval Hilbert space $\mathcal{H}$ decomposes in the various sectors with simple boundary conditions as $\mathcal{H} = \bigoplus_{m,n} \mathcal{H}_{m,n}$. We want to show that, for every $O \in \text{End}(\mathcal{H})$:

$$\lim_{\varepsilon \rightarrow 0} \text{tr}(\rho H_{m,n}^{r,s} O) = \lim_{\varepsilon \rightarrow 0} \text{tr}(H_{m,n}^{r,s} \rho O) . \quad (3.42)$$

Here $H_{m,n}^{r,s}$ is a basis element of the strip algebra in $\text{Hom}(\mathcal{H}_{m,n}, \mathcal{H}_{r,s})$. By matching boundary conditions and using (3.41), the previous equation is equivalent to

$$\lim_{\varepsilon \rightarrow 0} \frac{\tilde{d}_r \tilde{d}_s}{Z_{rs}} \text{tr}(\tilde{\rho}_{rs} H_{m,n}^{r,s} O_{r,s}^{m,n}) = \lim_{\varepsilon \rightarrow 0} \frac{\tilde{d}_m \tilde{d}_n}{Z_{mn}} \text{tr}(H_{m,n}^{r,s} \tilde{\rho}_{mn} O_{r,s}^{m,n}) , \quad (3.43)$$

where $O_{r,s}^{m,n}$ is the component of O in $\text{Hom}(\mathcal{H}_{r,s}, \mathcal{H}_{m,n})$. Notice that $\tilde{\rho}_{rs} H_{m,n}^{r,s} = H_{m,n}^{r,s} \tilde{\rho}_{mn}$ since, by assumption, lines can be slid past the matrices $\tilde{\rho}_{mn}$ (even for finite ε). Thus, because the operators $O_{r,s}^{m,n}$ are completely arbitrary, we only need to check that

$$\lim_{\varepsilon \rightarrow 0} \frac{\tilde{d}_r \tilde{d}_s}{Z_{rs}} = \lim_{\varepsilon \rightarrow 0} \frac{\tilde{d}_m \tilde{d}_n}{Z_{mn}} . \quad (3.44)$$

³³One also uses the pivotal relation: $\mathcal{L}_a |\mathcal{B}_S\rangle = \sum_{mn} \tilde{d}_m \tilde{N}_{am}^n |\mathcal{B}_n\rangle = \sum_{mn} \tilde{N}_{an}^m \tilde{d}_m |\mathcal{B}_n\rangle$, as well as $d_a = d_{\bar{a}}$.

This equation is indeed satisfied because

$$\lim_{\varepsilon \rightarrow 0} Z_{rs} = \tilde{d}_r \tilde{d}_s Z(\mathbb{C}) , \quad (3.45)$$

where $Z(\mathbb{C})$ is the Euclidean partition function on the plane. This can be argued as follows. Recall that Z_{rs} is the partition function on the plane with two discs of radius ε removed (see Fig. 15). The discs are centered, say, at $z = 0$ and $z = \ell$. Following [65], this geometry can be mapped to a cylinder of length $2\log(\ell/\varepsilon)$ with the transformation $w = \log(z/(\ell - z))$. The partition function can then be rewritten as a closed-channel amplitude:

$$Z_{rs} = W \langle s | (\varepsilon/\ell)^{L_0 + \bar{L}_0 - c/12} | r \rangle , \quad (3.46)$$

where W is a Weyl factor, while $|r\rangle, |s\rangle$ are Cardy states. Inserting a resolution of the identity with eigenstates of the cylinder Hamiltonian, in the limit $\varepsilon \rightarrow 0$ (which corresponds to a cylinder of infinite length) only the ground state $|0\rangle$ remains.³⁴ Thus one gets

$$\lim_{\varepsilon \rightarrow 0} Z_{rs} = \lim_{\varepsilon \rightarrow 0} g_r g_s W \langle 0 | (\varepsilon/\ell)^{L_0 + \bar{L}_0 - c/12} | 0 \rangle , \quad (3.47)$$

where $g_r \equiv \langle r | 0 \rangle$ is the g -factor [66]. Because of the operator/state correspondence, the term $\lim_{\varepsilon \rightarrow 0} \langle 0 | (\varepsilon/\ell)^{L_0 + \bar{L}_0 - c/12} | 0 \rangle$ becomes the partition function of an infinite cylinder, with no insertions at infinity. We can then undo the Weyl transformation, reabsorbing the factor W , to obtain $\lim_{\varepsilon \rightarrow 0} Z_{rs} = g_r g_s Z(\mathbb{C})$. The g -factors can be interpreted as disc partition functions [141] for a specific choice of Euler counterterm, and thus they satisfy the equation of boundary quantum dimensions (2.18) (see also [142]). Hence we can choose a normalization such that $\tilde{d}_r = g_r$, and obtain (3.45).³⁵

Large subsystem limits. Given the asymmetry of some state, computed on the interval subsystem using one of the types of boundary conditions discussed above, what is the relation between its large subsystem limit and the asymmetry of the full system, computed using the fusion or tube algebras?

In Sec. 5.1 we analyze invertible group-like symmetries. In particular, we consider excited CFT states created by the insertion of local operators in Euclidean time. We find that the von Neumann and Rényi asymmetries on an interval, as long as our averaging protocol is used, are independent of the choice of boundary conditions. Moreover, for untwisted states, the large subsystem limits of the asymmetries coincide with the asymmetries of the full system, computed with respect to either the fusion or the tube algebra (since they coincide for untwisted states).

In Sec. 5.3 we study certain excited states in two copies of the Ising CFT, utilizing the $\text{Rep}(H_8)$ (a.k.a. $\mathbb{Z}_2 \times \mathbb{Z}_2$ Tambara-Yamagami) noninvertible symmetry that admits a strongly-symmetric boundary condition. The result is similar: the second Rényi entropy on the interval

³⁴For unitary CFTs with discrete spectrum, $|0\rangle$ is the ground state. More generally, $|0\rangle$ is the lowest-dimension state that overlaps with the Cardy states $|r\rangle, |s\rangle$. See the discussion in [65].

³⁵With a generic normalization (i.e., choice of Euler counterterm), the g -factors g_m are proportional to the boundary quantum dimensions $\tilde{d}_m$ and thus they satisfy (2.18). This implies that $\lim_{\varepsilon \rightarrow 0} \tilde{d}_m \tilde{d}_n / Z_{mn}$ is independent from m, n . This is enough to prove (3.44), even with generic normalization.

subsystem asymptotes in the large subsystem limit to the second Rényi entropy of the full system, computed with respect to the tube algebra (not the fusion algebra, which in general is different).

On the other hand, in Secs. 5.2 and 5.4 we study certain excited states in the Ising CFT, utilizing its `Ising` fusion category (a.k.a. $\mathbb{Z}_2$ Tambara-Yamagami) symmetry, which does not admit symmetric boundary conditions and which then forces us to resort to the averaging protocol. This time we find that the large subsystem limit of the second Rényi asymmetry on an interval is qualitatively similar to, but slightly smaller than, the asymmetry of the full system with respect to the tube algebra. A similar behavior is found in Sec. 5.5 when studying the Fibonacci fusion category with its weakly-symmetric boundary condition. It would be interesting to understand these results more systematically.

3.3.4 Symmetrizer from (weak) Hopf algebra structure

We can find an alternative, but equivalent, formula for the symmetrizer by leveraging the following reasoning that holds in the invertible case for a symmetry group G (or equivalently, for the group algebra $\mathbb{C}[G]$). Consider a Hilbert space $\mathcal{H}$ that carries a representation of G . A density matrix $\rho \in \text{End}(\mathcal{H})$ is an element of $\mathcal{H} \otimes \mathcal{H}^\vee$ (the second factor being the dual space). This tensor product itself carries a representation of G , which decomposes into a sum of irreducible representations. Then, the symmetrization operation is reproduced by applying to this space the projector onto the trivial representation. The obstacle in generalizing this reasoning to the noninvertible case is that for a generic algebra $\mathcal{A}$ the concepts of tensor product of representations, of dual (or conjugate) representation, and of trivial representation, are not defined. In the special case that the algebra is *Hopf*, however, these concepts are available and the symmetrizer can be written as the projector onto the trivial representation. Moreover, we will see that if the algebra is only weak Hopf but its antipode map is involutive (such an algebra is called *weak Kac* [143]), a formula similar to that of the Hopf case applies, even though it lacks that interpretation. It was shown in [21, 98] that the strip algebras are always weak Hopf, therefore when they satisfy the weak Kac condition we can apply the strategy above to the case of the interval Hilbert space.³⁶

Let us first consider the case that a strip algebra $\mathcal{A}$ is constructed using a strongly-symmetric boundary condition. One can show that such a strip algebra is a Hopf algebra. We provide definitions and basic properties in App. D and a proof of this fact around (E.9).

Concisely, a Hopf algebra comes equipped with (besides its unit and its associative product) the following algebra homomorphisms: a coassociative coproduct $\Delta: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ such that $\Delta(1) = 1 \otimes 1$ and a counit $\epsilon: \mathcal{A} \rightarrow \mathbb{C}$. In addition, it comes with an antipode $S: \mathcal{A} \rightarrow \mathcal{A}$, which is a linear algebra anti-homomorphism, i.e. $S(xy) = S(y)S(x)$. In finite-dimensional semisimple Hopf algebras one proves that $S^2 = \text{id}$. The one-dimensional “trivial” representation $\mathfrak{R}_\epsilon: \mathcal{A} \rightarrow \text{End}(\mathbb{C}) \cong \mathbb{C}$, which behaves as the identity under the tensor product of representations, is

³⁶On the contrary, the tube algebra is in general not a weak Hopf algebra. One can still define a tensor product of representations using the so-called Day convolution product [144] (also see [145]). We do not explore this possibility here.

provided by the counit:

$$\mathfrak{R}_\epsilon(x) \equiv \epsilon(x) \quad \forall x \in \mathcal{A}. \quad (3.48)$$

The tensor product of representations is constructed using the coproduct Δ . Given two representations $\mathfrak{R}_i: \mathcal{A} \rightarrow \text{End}(V_i)$ with $i = 1, 2$, their tensor product representation $\mathfrak{R}_{12}: \mathcal{A} \rightarrow \text{End}(V_1 \otimes V_2)$ is defined as

$$\mathfrak{R}_{12} = (\mathfrak{R}_1 \otimes \mathfrak{R}_2) \circ \Delta \quad \text{i.e.} \quad \mathfrak{R}_{12}(x) v_1 \otimes v_2 = \sum_i \left[\mathfrak{R}_1(x_{(1)}^i) v_1 \right] \otimes \left[\mathfrak{R}_2(x_{(2)}^i) v_2 \right], \quad (3.49)$$

where $v_i \in V_i$ and we used Sweedler's notation to write $\Delta(x) = \sum_i x_{(1)}^i \otimes x_{(2)}^i$. Using (D.2), one sees that the representation $\mathfrak{R}_\epsilon$ multiplies trivially with any other representation. Finally, given a representation $\mathfrak{R}: \mathcal{A} \rightarrow \text{End}(V)$, its conjugate representation $\mathfrak{R}^*: \mathcal{A} \rightarrow \text{End}(V^\vee)$ is constructed using the antipode:

$$\langle \mathfrak{R}^*(x)f, v \rangle \equiv \langle f, \mathfrak{R}(S(x))v \rangle \quad \forall x \in \mathcal{A}, v \in V, f \in V^\vee, \quad (3.50)$$

where the angular brackets denote the evaluation of a functional on a vector. More compactly, this can be written as $\mathfrak{R}^* = (\mathfrak{R} \circ S)^\top$. Using (D.6) one sees that $\mathfrak{R}_\epsilon$ is self-conjugate.

Associated to $\mathfrak{R}_\epsilon$ there is a minimal central idempotent (MCI) $P_\epsilon \in \mathcal{A}$, i.e. the projector onto the trivial representation (also see Fn. 103). It satisfies³⁷

$$P_\epsilon x = x P_\epsilon = \epsilon(x) P_\epsilon \quad \forall x \in \mathcal{A}. \quad (3.51)$$

Taking x equal to one of the MCIs P_i (i.e. a projector to an irreducible representation of $\mathcal{A}$), one obtains $\epsilon(P_i) = \delta_{i\epsilon}$, confirming that P_ϵ projects onto the trivial representation.

We claim that the symmetrizer can be constructed using the object

$$\mathbb{P} = (\text{id} \otimes S) \circ \Delta(P_\epsilon), \quad (3.52)$$

which was proven to be a separability idempotent for a Hopf algebra in [146]. When acting on the space $\mathcal{H} \otimes \mathcal{H}^\vee \cong \text{End}(\mathcal{H})$, this operator is precisely the action of the projector to the trivial representation P_ϵ on density matrices. More explicitly:

$$\rho \mathbb{S} = \sum_i (P_\epsilon)_{(1)}^i \rho S((P_\epsilon)_{(2)}^i) \quad (3.53)$$

where $\Delta(P_\epsilon) = \sum_i (P_\epsilon)_{(1)}^i \otimes (P_\epsilon)_{(2)}^i$. The disadvantage of this formula is that one needs to explicitly construct the projector P_ϵ to the trivial representation. In App. D.3 we prove that the symmetrizer defined in (3.53) satisfies properties 1. – 4. listed in Sec. 3.1. The expression in (3.53) is also universal, in the sense that it is written solely in terms of the elements of $\mathcal{A}$. As we commented at the end of Sec. 3.1, this completely fixes the symmetrizer and thus (3.53) agrees with the symmetrizer defined by (3.9). Notice that the universal formula (3.9) only depends

³⁷The first equality is the centrality of MCIs. For the second equality, since $P_\epsilon^2 = P_\epsilon$ and the trivial representation is one-dimensional, then P_ϵ is a basis for the image of the projector in the regular representation (recall that a 1d representation occurs only once inside the regular representation), and thus $xP_\epsilon = \alpha(x)P_\epsilon$ for some $\alpha \in \mathbb{C}$. Applying ϵ to both sides we find $\alpha(x) = \epsilon(x)$.

on the product structure on $\mathcal{A}$, therefore different Hopf algebras that share the same product structure will give rise, through (3.53), to the same symmetrizer.

For weak Hopf algebras a similar recipe exists to compute the symmetrizer, provided they satisfy the additional condition $S^2 = \text{id}$ (such algebras are called weak Kac).³⁸ Referring to [148], one introduces the left and right projectors $\Pi^L, \Pi^R: \mathcal{A} \rightarrow \mathcal{A}$ as

$$\Pi^L(x) = \sum_i \epsilon(1_{(1)}^i x) 1_{(2)}^i, \quad \Pi^R(x) = \sum_i 1_{(1)}^i \epsilon(x 1_{(2)}^i), \quad (3.54)$$

acting on $x \in \mathcal{A}$. Then one defines the left and right normalized integrals as those elements $\ell, r \in \mathcal{A}$ such that

$$x \ell = \Pi^L(x) \ell, \quad \Pi^L(\ell) = 1, \quad \text{or} \quad r x = r \Pi^R(x), \quad \Pi^R(r) = 1, \quad (3.55)$$

for all $x \in \mathcal{A}$. In general, being a left integral does not imply being a right integral. When this happens, one talks about *Haar integrals*. That is, a Haar integral h is defined as a simultaneous left and right normalized integral. When it exists, it is unique. The necessary and sufficient conditions for its existence are spelled out in Theorem 3.27 of [148]. Such conditions are indeed satisfied under our working hypotheses.³⁹ Then, the analogue of (3.52) is given by

$$\mathbb{P} = (\text{id} \otimes S) \circ \Delta(h), \quad (3.56)$$

where $h \in \mathcal{A}$ is the Haar integral. In App. D.3 we verify that this expression satisfies the properties 1. – 4., once we let it act on density matrices. Thus, being also universal, it must define the same symmetrizer as (3.9).⁴⁰

Note that in the case of a Hopf algebra, both the left and right projectors reduce to

$$\Pi^L(x) = \Pi^R(x) = \epsilon(x) 1 \quad (3.57)$$

because $\Delta(1) = 1 \otimes 1$. Then the Haar integral h is equal to the projector P_ϵ .

³⁸This is not the case for a generic strip algebra (see [147], eq. (D.14)).

³⁹In the notation of Th. 3.27 in [148], since we are assuming $S^2 = \text{id}$, we have $g = 1$. Then, the second condition therein is satisfied because the trace of 1 in any representation is the dimension of that representation.

⁴⁰Note that, had we not required the property $S^2 = \text{id}$, the object (3.56) could still be defined. However, the resulting symmetrized density matrix will fail to satisfy property 4. In this case, the existence of the Haar integral is guaranteed provided the weak Hopf algebra also possesses a C^* -structure compatible with the former, see Theorem 4.5 of [148].

4 Noninvertible symmetry-resolved entanglement entropy

The goal of this section is to define a protocol to compute the entanglement entropy of a region A with its complement, resolved with respect to a generic categorical symmetry $\mathcal{C}$. Here, we only consider unitary CFTs in two spacetime dimensions. We are interested in (possibly mixed) states ρ_{full} of the full system that are invariant under the symmetry. Indeed, as discussed in Sec. 3.1, the density matrices of those (and only those) states take a block-diagonal form in which each block corresponds to an irrep of the symmetry algebra. Moreover, each block further decomposes into sub-blocks connecting the various copies of that irrep, each of which is proportional to the identity (see Fig. 4). In other words, the asymmetry of the state must be zero $\Delta S[\rho_{\text{full}}] = 0$.⁴¹

We will take the spatial region A to be an interval. Therefore, given a state ρ_{full} of the full system, we need to construct the reduced density matrix ρ on A . As discussed in Sec. 3.3, in QFT this is a subtle operation because the Hilbert space does not factorize, and one has to resort to a regularization procedure. We follow here the same protocol described in Sec. 3.3 (first proposed in [65]) and using the maps $\Psi_{\mathcal{B}}$ depicted in Fig. 6. As explained there, a proper definition of ρ requires to choose boundary conditions transforming in some $\mathcal{C}$ -module category $\mathcal{M}$, to be used in the map $\Psi_{\mathcal{B}}$. When symmetric boundary conditions are available,⁴² the standard construction of symmetry-resolved entanglement entropy of [67, 68, 70] goes through. However, as commented in Sec. 3.3, there are situations in which symmetric boundary conditions do not exist. For instance, in the presence of 't Hooft anomalies there are no simple strongly-symmetric boundary conditions, and most noninvertible symmetries do not admit symmetric boundary conditions. In those situations, one is forced to consider multiplets of boundary conditions that transform among themselves under the symmetry, rather than being invariant. This raises two issues that one should take into account and that we want to stress:

1. The symmetry acts on the interval Hilbert space via an action of the strip algebra $\text{Strip}_{\mathcal{C}}(\mathcal{M})$. Therefore, symmetry resolution can only be meaningfully defined with respect to representations of the strip algebra, which in general differ from the representations of the symmetry group G or of the fusion algebra of the symmetry category $\mathcal{C}$.⁴³
2. The strip algebra does not have, in general, a consistent action on an interval Hilbert space $\mathcal{H}_{rm}$ with fixed boundary conditions $\mathcal{B}_r, \mathcal{B}_m$; rather, it acts on the total interval Hilbert space

$$\mathcal{H} = \bigoplus_{r,m} \mathcal{H}_{rm} \quad (4.1)$$

obtained using all boundary conditions in the chosen module category $\mathcal{M}$, because the symmetry transforms the boundary conditions among themselves.

⁴¹In practice, we will focus on the CFT vacuum, $\rho_{\text{full}} = |0\rangle\langle 0|$.

⁴²Here we always require more, namely that the boundary conditions be also conformal.

⁴³For a non-anomalous symmetry group G , the strip algebra $\mathcal{A} = \text{Strip}_{\text{Vec}_G}(\text{Vec})$ constructed using the symmetric simple boundary condition is isomorphic to the group algebra $\mathbb{C}[G]$ and therefore the strip-algebra representations are the same as the representations of G . However for noninvertible symmetries, even when a strongly-symmetric boundary condition exists, the strip algebra constructed out of it differs from the fusion algebra of $\mathcal{C}$ because some of the junctions acquire multiplicities. We elaborate on this point below.

For each choice of $\mathcal{H}_{rm}$ with fixed boundary conditions, one can construct a normalized reduced density matrix ρ_{rm} (using the map $\Psi_{\mathcal{B}}$, as explained above). In general each ρ_{rm} has nonvanishing asymmetry, $\Delta S[\rho_{rm}] > 0$, and therefore does not have a clean symmetry resolution. This problem can be easily solved using the same “averaging protocol” that has been introduced in Sec. 3.3.3 to study the asymmetry when neither strongly nor weakly-symmetric boundary conditions exist. Indeed, this protocol produces a normalized reduced density matrix $\rho \in \text{End}(\mathcal{H})$ with vanishing asymmetry:

$$\rho = \sum_{r,m} \frac{\tilde{d}_r \tilde{d}_m}{D^2} \rho_{rm} , \quad D = \sum_m \tilde{d}_m . \quad (4.2)$$

In Sec. 3.3.3 we explicitly showed that whenever ρ_{full} has zero asymmetry, the reduced density matrix (4.2) has zero asymmetry.

Since the reduced density matrix ρ has vanishing asymmetry with respect to the strip algebra $\mathcal{A}$, it can be resolved in terms of representations of $\mathcal{A}$, which we now describe.

4.1 Strip-algebra representations

The category of representations of the algebra $\text{Strip}_{\mathcal{C}}(\mathcal{M})$ is monoidally equivalent to the category dual to $\mathcal{C}$ with respect to $\mathcal{M}$ [21]:⁴⁴

$$\text{Rep}(\text{Strip}_{\mathcal{C}}(\mathcal{M})) \cong \mathcal{C}_{\mathcal{M}}^* \equiv \text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})^{\text{op}} . \quad (4.3)$$

Note that, although the representation category of a generic algebra need not be monoidal, the representation categories of strip algebras are, owing to their weak Hopf structure. See App. E for more details. Eqn. (4.3) defines $\mathcal{C}_{\mathcal{M}}^*$ as the category of $\mathcal{C}$ -module functors from $\mathcal{M}$ to $\mathcal{M}$, with monoidal structure given by (opposite) composition. We indicate its simple objects as $\mathcal{R}_{\hat{a}}$. The left $\mathcal{C}$ -module category $\mathcal{M}$ also possesses a right module structure over $\mathcal{C}_{\mathcal{M}}^*$, given by functor evaluation, namely $\mathcal{B}_m \otimes \mathcal{R}_{\hat{a}} \equiv \mathcal{R}_{\hat{a}}(\mathcal{B}_m)$ (see Fig. 7 center right). For simple objects $\mathcal{L}_a \in \mathcal{C}$, $\mathcal{B}_m \in \mathcal{M}$, $\mathcal{R}_{\hat{a}} \in \mathcal{C}_{\mathcal{M}}^*$, the condition of being a module-functor means that there are isomorphisms $(\mathcal{L}_a \otimes \mathcal{B}_m) \otimes \mathcal{R}_{\hat{a}} \cong \mathcal{L}_a \otimes (\mathcal{B}_m \otimes \mathcal{R}_{\hat{a}})$ described in a basis by suitable F -symbols subject to certain consistency conditions.

The relation (4.3) means that the strip-algebra representations $\mathfrak{R}_{\hat{a}}$ are in correspondence with the simple objects $\mathcal{R}_{\hat{a}}$ of $\mathcal{C}_{\mathcal{M}}^*$, and the decomposition of tensor products of strip-algebra representations corresponds to the fusion ring of $\mathcal{C}_{\mathcal{M}}^*$. Recall from Sec. 3.3.4 that strip algebras are weak Hopf algebras and thus the tensor product of its representations is defined. Note that such a tensor product is not necessarily commutative (when the coproduct is not co-commutative), and the dimension of the tensor product is not necessarily the product of the individual dimensions.⁴⁵ It follows that to each representation are associated both a classical dimension $\dim(\mathfrak{R}_{\hat{a}})$ (i.e. the dimension of the vector space that carries that representation) and a quantum dimension $d_{\hat{a}}$ (that satisfies the fusion algebra of $\mathcal{C}_{\mathcal{M}}^*$).

⁴⁴See [149] for a mathematical proof. Note that we are using a different convention with respect to [113] in the definition of $\mathcal{C}_{\mathcal{M}}^*$. This is motivated by the fact that $\text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})^{\text{op}}$ (“op” meaning that multiplication is in the opposite order), rather than $\text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$, has a right module action on $\mathcal{M}$ given by functor evaluation.

⁴⁵This is instead true for (strong) Hopf algebras.

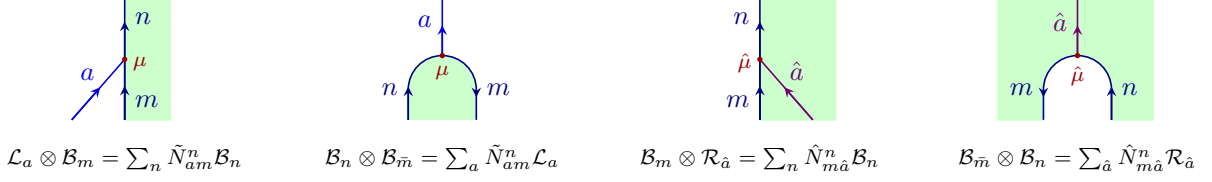


Figure 7: Fusions involving boundaries, represented here as interfaces between $\mathcal{C}$ (in white) and $\mathcal{C}_{\mathcal{M}}^*$ (in green). The definition and role played by $\mathcal{C}_{\mathcal{M}}^*$ are described around (4.3) and (4.4). Left: $\mathcal{C}$ -action on a left $\mathcal{C}$ -module category. Center left: fusion of boundaries into an object of $\mathcal{C}$. Center right: $\mathcal{C}_{\mathcal{M}}^*$ -action on the $\mathcal{C}$ -module category from the right. Right: fusion of boundaries into an object of $\mathcal{C}_{\mathcal{M}}^*$. This diagram, once turned upside down, also represents a basis vector $V_{(\hat{a})\hat{\mu}}^{mn}$ of the strip-algebra representation $\mathfrak{R}_{\hat{a}}$.

It is convenient to think of $\mathcal{C}_{\mathcal{M}}^*$ as the quantum symmetry emerging after gauging an algebra object in $\mathcal{C}$ associated to $\mathcal{M}$,⁴⁶ and think of the module category $\mathcal{M}$ as a $(\mathcal{C}, \mathcal{C}_{\mathcal{M}}^*)$ -bimodule category, namely $\mathcal{M}$ is acted upon by $\mathcal{C}$ from the left and by $\mathcal{C}_{\mathcal{M}}^*$ from the right (see Fig. 7), in a compatible way. In other words, we can think of $\mathcal{M}$ as describing interfaces between $\mathcal{C}$ and $\mathcal{C}_{\mathcal{M}}^*$, instead of boundaries of $\mathcal{C}$. In this new picture, $\mathcal{R}_{\hat{a}}$ are the simple objects of $\mathcal{C}_{\mathcal{M}}^*$ giving rise to the following two fusion rings:

$$\mathcal{B}_m \otimes \mathcal{R}_{\hat{a}} = \bigoplus_n \hat{N}_{m\hat{a}}^n \mathcal{B}_n \quad \text{and} \quad \mathcal{R}_{\hat{a}} \otimes \mathcal{R}_{\hat{b}} = \bigoplus_{\hat{c}} \bar{N}_{\hat{a}\hat{b}}^{\hat{c}} \mathcal{R}_{\hat{c}}. \quad (4.4)$$

The first one is the right action of $\mathcal{C}_{\mathcal{M}}^*$ on $\mathcal{M}$. The integers $\hat{N}_{m\hat{a}}^n \in \mathbb{Z}_{\geq 0}$ turn out to coincide with the dimensions of the restrictions of the representation $\mathfrak{R}_{\hat{a}}$ to the sectors $\mathcal{H}_{mn}$ of the interval Hilbert space in (4.1). Indeed, the representation space of $\mathfrak{R}_{\hat{a}}$ restricted to $\mathcal{H}_{mn}$ is precisely the space of junctions with basis vectors labeled by $\hat{\mu}$ depicted in Fig. 7 right, whose dimension is $\hat{N}_{m\hat{a}}^{\hat{\mu}} = \hat{N}_{m\hat{a}}^n$ [99, 100]. Hence, the classical dimension of $\mathfrak{R}_{\hat{a}}$ is

$$\dim(\mathfrak{R}_{\hat{a}}) = \sum_{m,n} \hat{N}_{m\hat{a}}^n. \quad (4.5)$$

Taking the sum of the squares of the irrep dimensions, one obtains⁴⁷

$$\dim(\text{Strip}_{\mathcal{C}}(\mathcal{M})) = \sum_{\hat{a}, m, n, r, s} \hat{N}_{m\hat{a}}^n \hat{N}_{r\hat{a}}^s. \quad (4.6)$$

This should be compared with (2.23). The second equation (4.4) is the fusion ring of $\mathcal{C}_{\mathcal{M}}^*$. The quantum dimensions $d_{\hat{a}}$ furnish a one-dimensional representation of both fusion algebras.

Finally, it is useful to also describe boundary-boundary fusions. The fusion of left and right boundaries (belonging to the same module category⁴⁸) gives the following fusion ring (see

⁴⁶Calling $\mathfrak{A}$ the algebra object in $\mathcal{C}$ that is gauged, the direct mathematical counterpart of the quantum symmetry is the category of $\mathfrak{A}$ -bimodules in $\mathcal{C}$ [14], denoted ${}_{\mathfrak{A}}\mathcal{C}_{\mathfrak{A}}$. Indeed it turns out that, in our conventions, ${}_{\mathfrak{A}}\mathcal{C}_{\mathfrak{A}} \cong \mathcal{C}_{\mathcal{M}}^*$ [113].

⁴⁷Notice that $\mathcal{M}$ is a right module category over $\mathcal{C}_{\mathcal{M}}^*$, and thus $\mathcal{M}^{\vee}$ is a left module category over $\mathcal{C}_{\mathcal{M}}^*$ (see Fn. 48). Since $(\mathcal{C}_{\mathcal{M}}^*)^*_{\mathcal{M}^{\vee}} \cong \mathcal{C}$ (see Theorem 7.12.11 of [113], adapted to our conventions), it follows that the strip algebra $\text{Strip}_{\mathcal{C}_{\mathcal{M}}^*}(\mathcal{M}^{\vee})$ has a representation category equivalent to $\mathcal{C}$. Moreover, the two strip algebras $\text{Strip}_{\mathcal{C}}(\mathcal{M})$ and $\text{Strip}_{\mathcal{C}_{\mathcal{M}}^*}(\mathcal{M}^{\vee})$ have the same dimension.

⁴⁸More precisely, if the boundaries with $\mathcal{C}$ on their left are in a left module category $\mathcal{M}$, the boundaries with $\mathcal{C}$ on their right are in the right module category $\mathcal{M}^{\vee}$ defined as in Remark 7.1.5 of [113]. Our notation here is

e.g. [138, 139], where it is defined in terms of the internal Hom construction):

$$\mathcal{B}_n \otimes \mathcal{B}_{\bar{m}} = \bigoplus_a \tilde{N}_{n\bar{m}}^a \mathcal{L}_a, \quad (4.7)$$

where the boundary $\mathcal{B}_n$ has $\mathcal{C}$ on its left, $\mathcal{B}_{\bar{m}}$ on its right, and the coefficients satisfy the pivotal relation $\tilde{N}_{n\bar{m}}^a = \tilde{N}_{\bar{m}n}^a$. This is represented in Fig. 7 center left. Instead, the fusion of left and right boundaries, but in the opposite order with respect to (4.7), gives the fusion rules:

$$\mathcal{B}_{\bar{m}} \otimes \mathcal{B}_n = \bigoplus_{\hat{a}} \hat{N}_{\bar{m}n}^{\hat{a}} \mathcal{R}_{\hat{a}}, \quad (4.8)$$

with coefficients $\hat{N}_{\bar{m}n}^{\hat{a}} = \hat{N}_{m\hat{a}}^n$, as represented in Fig. 7 right.

To each boundary $\mathcal{B}_m$ one also associates a positive real number $\tilde{d}_m = \tilde{d}_{\bar{m}}$ which is its quantum dimension. As in (2.18), the boundary quantum dimensions provide a representation of the bulk-boundary fusion ring. However, for the purposes of studying the SREE we fix a specific normalization, by demanding that they form a representation of (4.7):

$$\tilde{d}_m \tilde{d}_n = \sum_a \tilde{N}_{am}^n d_a. \quad (4.9)$$

This normalization is such that the total quantum dimension of any $\mathcal{M}$ is equal to that of $\mathcal{C}$, $\mathcal{D}_{\mathcal{M}}^2 \equiv \sum_m \tilde{d}_m^2 = \sum_a d_a^2 \equiv \mathcal{D}_{\mathcal{C}}^2$, as can be checked. Furthermore, it turns out that also $\mathcal{C}_{\mathcal{M}}^*$ has the same total dimension as $\mathcal{C}$ (see Corollary 7.16.7 of [113]): $\mathcal{D}_{\mathcal{C}_{\mathcal{M}}^*}^2 \equiv \sum_{\hat{a}} d_{\hat{a}}^2 = \mathcal{D}_{\mathcal{C}}^2$. Using this fact, one can show that, with this normalization, the boundary quantum dimension also provide a representation of (4.8):

$$\tilde{d}_m \tilde{d}_n = \sum_{\hat{a}} \hat{N}_{m\hat{a}}^n d_{\hat{a}}. \quad (4.10)$$

Regular module category. A given fusion category $\mathcal{C}$ admits one or more (in fact, finitely many⁴⁹) indecomposable module categories $\mathcal{M}$. A module that always exists is the *regular module* $\mathcal{M} = \mathcal{C}$. This case is particularly simple: the simple boundaries $\{\mathcal{B}_a\}$ have the same label as the bulk lines, and the fusion of lines with boundaries is identical to the fusion between bulk lines. In particular $\tilde{N} = N$ and $\tilde{F} = F$, therefore

$$\dim(\text{Strip}_{\mathcal{C}}(\mathcal{C})) = \sum_{a,b,c,d,e} N_{ab}^c N_{ad}^e. \quad (4.11)$$

Moreover, the dual category is equivalent to the symmetry category itself (see Example 7.12.3 of [113]). In particular, the fusion of lines of the dual category with the boundaries is identical to the fusion between bulk lines. In other words, the category of strip-algebra representations is equivalent to the category $\mathcal{C}$ itself, and the product of representations is identical to the fusion algebra of $\mathcal{C}$:

$$\text{Rep}(\text{Strip}_{\mathcal{C}}(\mathcal{C})) \cong \mathcal{C}_{\mathcal{C}}^* \cong \mathcal{C}. \quad (4.12)$$

that boundaries with $\mathcal{C}$ on their left are labeled by unbarred letters $m, n, \dots$ while, once rotated by 180° so as to become boundaries with $\mathcal{C}$ on their right, they are labeled by barred letters $\bar{m}, \bar{n}, \dots$. The orientation is always such that, looking in the direction of the boundary, $\mathcal{C}$ is on the left.

⁴⁹Indeed, there is a bound on the number of simple objects that an indecomposable module category can have: $|\text{Irr}(\mathcal{M})| \leq \mathcal{D}_{\mathcal{M}}^2 = \mathcal{D}_{\mathcal{C}}^2$. Moreover, at fixed number of simple objects, the number of module categories is finite, see Corollary 2.35 of [150].

The irreducible representations $\mathfrak{R}_a$ are in correspondence with the bulk lines $\mathcal{L}_a$, their classical dimensions are given by $\dim(\mathfrak{R}_a) = \sum_{b,c} N_{ba}^c = \sum_{b,c} N_{ab}^c$, whereas their quantum dimensions are simply d_a .

In fact, for the regular module, the strip-algebra representations can be explicitly constructed in terms of the F -symbols of $\mathcal{C}$. As suggested by the SymTFT construction [100], for a general module $\mathcal{M}$ the basis vectors of the representation space of $\mathfrak{R}_{\hat{a}}$ are

$$V_{(\hat{a})\hat{\mu}}^{rm} \sim \text{Diagram} \quad , \quad (4.13)$$

as in Fig. 7 right but reflected upside down, and $\sim$ means that we have not fixed the normalization. This reproduces the formula (4.5) for the representation dimensions. In the case of the regular module $\mathcal{M} = \mathcal{C}$, since $\mathcal{C}_{\mathcal{C}}^* \cong \mathcal{C}$, it is convenient to use instead the following strip-algebra elements as basis vectors of $\mathfrak{R}_a$:

$$V_{(a)\mu}^{rm} = \frac{1}{\sqrt{d_r}} \text{Diagram} \quad . \quad (4.14)$$

The action of the strip algebra on them simply follows from the strip-algebra product, and its associativity guarantees that this is a representation. From the expression for the product in (2.24) we obtain the matrix elements of the representation $\mathfrak{R}_a$:

$$\left[\text{Diagram} \right]_{tp\pi}^{uq\rho} = \delta^{su} \delta^{nq} \delta_{rt} \delta_{mp} \sqrt{d_b} [F_{bra}^n]_{(s;\sigma\rho)(m;\pi\nu)}^* . \quad (4.15)$$

The expression simplifies in cases with no multiplicities.

Anomaly-free symmetries. For an anomaly-free symmetry group G , the symmetry category is $\mathcal{C} = \text{Vec}_G$ with trivial F -symbols. Such a symmetry admits a simple symmetric boundary condition, described by the module category $\mathcal{M} = \text{Vec}$, which has a single simple object $\mathcal{B}$. The strip algebra constructed out of it is isomorphic to the group algebra of G , $\text{Strip}_{\text{Vec}_G}(\text{Vec}) \cong \mathbb{C}[G]$. Its representations form the category $\mathcal{C}_{\mathcal{M}}^* \cong \text{Rep}(G)$. This means that the strip-algebra representations are identical to the representations of G , and our procedure reproduces the standard treatment of symmetry-resolved entanglement entropy in the literature [67, 68, 70].

The case of anomaly-free noninvertible symmetries is, however, more complicated: the strip-algebra representations furnish a “refinement” of the fusion-algebra representations. So, in a sense, our procedure produces a refinement of symmetry-resolved entanglement entropy with respect to what one could have expected from the action of the symmetry on the infinite line. More in detail, an anomaly-free noninvertible fusion category $\mathcal{C}$ admits a fiber functor $F: \mathcal{C} \rightarrow \text{Vec}$ [93], from which one constructs a strongly-symmetric simple boundary condition $\mathcal{B}$ [36] described by a module category $\mathcal{M}$ with a single simple object.⁵⁰ The representations of the

⁵⁰The category $\mathcal{C}$ could admit multiple inequivalent fiber functors, from which one constructs multiple inequivalent strongly-symmetric boundary conditions.

corresponding strip algebra restrict to representations of the fusion algebra of $\mathcal{C}$, but the map is not injective.⁵¹ In this case, our procedure is identical to what was already proposed in [99, 100, 105, 151].

The biggest surprises come from anomalous symmetries, which do not admit simple symmetric boundary conditions. As we will see in Sec. 6, anomalous symmetry groups G can lead to strip-algebra representations that are very different from the representations of G .

4.2 Symmetry-resolved entanglement in CFTs

Let us first quickly review how one can compute the entanglement entropy of the vacuum $|0\rangle$ on an interval of length L in a 2d unitary, compact⁵² CFT of central charge c using the regularization scheme of [65]. To construct the reduced density matrix ρ_{rm} on the interval, one excises from the plane two small discs of radius ε around the entangling points and chooses conformal boundary conditions $\mathcal{B}_r, \mathcal{B}_m$ there, as in Fig. 6 right. By a conformal transformation, the setup can be mapped to a cylinder of modular parameter q where

$$q = e^{-\pi\beta/\ell}, \quad L/\varepsilon = e^{\pi\ell/\beta}, \quad (4.16)$$

and the length and circumference of the cylinder are ℓ and β , respectively.⁵³ Then traces of powers of the reduced density matrix are computed by the partition functions $Z_{\mathcal{B}_r\mathcal{B}_m}(q^n)$ of the theory on cylinders of circumference $n\beta$ with boundary conditions $\mathcal{B}_r, \mathcal{B}_m$ on the two sides:

$$\text{tr}(\rho_{rm}^n) = \frac{Z_{\mathcal{B}_r\mathcal{B}_m}(q^n)}{Z_{\mathcal{B}_r\mathcal{B}_m}(q)^n}, \quad Z_{\mathcal{B}_r\mathcal{B}_m}(q^n) \equiv \text{tr}_{\mathcal{H}_{rm}}[e^{-n\beta H}], \quad (4.18)$$

for any integer $n \geq 1$. Here H is the Hamiltonian on $\mathcal{H}_{rm}$. We are interested in the limit $\varepsilon \ll L$ in which the regulator is removed, which corresponds to the high-temperature limit $\beta \ll \ell$ on the cylinder. It is convenient to go to the so-called closed-string channel:

$$Z_{\mathcal{B}_r\mathcal{B}_m}(q^n) = \langle \mathcal{B}_m | \tilde{q}^{\frac{1}{2n}(L_0 + \bar{L}_0 - \frac{c}{12})} | \mathcal{B}_r \rangle, \quad \tilde{q} = e^{-4\pi\ell/\beta}, \quad (4.19)$$

where $|\mathcal{B}_m\rangle$ is the Cardy state created by the conformal boundary condition $\mathcal{B}_m$ (and similarly for $\mathcal{B}_r$) [152]. For $\ell \gg \beta$, corresponding to $\tilde{q} \rightarrow 0$, the amplitude projects to the vacuum state:⁵⁴

$$Z_{\mathcal{B}_r\mathcal{B}_m}(q^n) \simeq \langle \mathcal{B}_m | 0 \rangle \tilde{q}^{-\frac{c}{24n}} \langle 0 | \mathcal{B}_r \rangle = g_m g_r e^{\frac{c\pi}{6} \frac{\ell}{n\beta}}. \quad (4.20)$$

⁵¹As an example, consider the case $\mathcal{C} = \text{Rep}(G)$ (where G is a finite group). This symmetry admits a strongly-symmetric boundary condition $\mathcal{M} = \text{Vec}$, and the corresponding dual category is $\mathcal{C}_{\mathcal{M}}^* \cong \text{Vec}_G$. Thus, the strip-algebra representations are labeled by elements g of G . On the other hand, the representations of the fusion algebra of $\mathcal{C}$ are class functions on G , labeled by the conjugacy classes $[g]$ of G . There is a restriction of representations that maps $g \mapsto [g]$, but this is not injective.

⁵²By “compact” we mean that the CFT has a unique normalizable vacuum and a discrete spectrum.

⁵³In detail, let w be a coordinate on the plane and $z \bmod i\beta$ a coordinate on the cylinder, related by

$$z = \frac{\beta}{\pi} \text{arctanh} \frac{2w}{L} = \frac{\beta}{2\pi} \log \frac{L/2 + w}{L/2 - w}, \quad w = \frac{L}{2} \tanh \frac{\pi z}{\beta} = \frac{L}{2} \frac{e^{\frac{2\pi z}{\beta}} - 1}{e^{\frac{2\pi z}{\beta}} + 1}. \quad (4.17)$$

The exact cylinder $|\text{Re}(z)| \leq \frac{\beta}{2\pi} \log \frac{L}{\varepsilon}$, which has length $\ell = \frac{\beta}{\pi} \log \frac{L}{\varepsilon}$ and circumference (or height) β , is mapped, for $\varepsilon \ll L$, to the w -plane with two small approximate discs of radius ε removed around the two points $w = \pm \frac{L}{2}$.

⁵⁴As explained in [65], the computation generalizes to CFTs that are non-unitary or with continuous spectrum by substituting $|0\rangle$ with the lowest-energy state that has nonvanishing superposition with both $|\mathcal{B}_m\rangle$ and $|\mathcal{B}_r\rangle$.

Here $g_m \equiv \langle \mathcal{B}_m | 0 \rangle \in \mathbb{R}_{>0}$ is the exponential of the Affleck-Ludwig boundary entropy [66] of $\mathcal{B}_m$, also known as g -function. We can thus compute the Rényi entropy in the limit $q \rightarrow 1$:

$$S^{(n)} = \frac{1}{1-n} \log \frac{Z_{\mathcal{B}_r \mathcal{B}_m}(q^n)}{Z_{\mathcal{B}_r \mathcal{B}_m}(q)^n} \simeq \frac{c}{3} \frac{n+1}{2n} \log \frac{L}{\varepsilon} + \log(g_m g_r), \quad (4.21)$$

and from here the entanglement entropy:

$$S = \lim_{n \rightarrow 1} S^{(n)} = \frac{c}{3} \log \frac{L}{\varepsilon} + \log(g_m g_r). \quad (4.22)$$

In both formulas the leading term is logarithmically divergent with the cutoff ε , while the subleading term depends on the specific chosen regularization.

When a state ρ has vanishing asymmetry with respect to a (semisimple) symmetry algebra $\mathcal{A}$, it commutes with all elements of $\mathcal{A}$ and thus, by Schur's lemma, it is block diagonal in the symmetry indices, each block corresponding to a certain irrep $\mathfrak{R}$ of $\mathcal{A}$. Hence one can define density matrices restricted to a single irrep:

$$\tilde{\rho}_{\mathfrak{R}} \equiv \Pi_{\mathfrak{R}} \rho \Pi_{\mathfrak{R}}, \quad q_{\mathfrak{R}} \equiv \text{tr}(\tilde{\rho}_{\mathfrak{R}}), \quad \rho_{\mathfrak{R}} = \frac{1}{q_{\mathfrak{R}}} \tilde{\rho}_{\mathfrak{R}}. \quad (4.23)$$

Here $\Pi_{\mathfrak{R}}$ is minimal central idempotent (MCI), see Fn. 103, associated to the irrep $\mathfrak{R}$ (which we analyze below), and this definition guarantees that $\rho_{\mathfrak{R}}$ are normalized density matrices. Moreover, $\sum_{\mathfrak{R}} q_{\mathfrak{R}} = 1$ since ρ is normalized and indeed $\{q_{\mathfrak{R}}\}$ are probabilities. The fact that ρ is a block-diagonal matrix guarantees that $\tilde{\rho}_{\mathfrak{R}}^n = \Pi_{\mathfrak{R}} \rho^n = \rho^n \Pi_{\mathfrak{R}}$ for all $n \geq 1$. This allows one to decompose

$$\rho^n = \sum_{\mathfrak{R}} q_{\mathfrak{R}}^n \rho_{\mathfrak{R}}^n \quad \text{for all } n \geq 1. \quad (4.24)$$

It implies the following decomposition of the entanglement entropy [70]:⁵⁵

$$S[\rho] \equiv -\text{tr}(\rho \log \rho) = S(\{q_{\mathfrak{R}}\}) + \sum_{\mathfrak{R}} q_{\mathfrak{R}} S[\rho_{\mathfrak{R}}]. \quad (4.25)$$

The first contribution $S(\{q_{\mathfrak{R}}\}) \equiv -\sum_{\mathfrak{R}} q_{\mathfrak{R}} \log q_{\mathfrak{R}}$ is called fluctuation or number entropy [153], the second contribution is called configurational entropy [153] (see also [154]), while $S[\rho_{\mathfrak{R}}]$ are the *symmetry-resolved entanglement entropies* (SREE) [67, 70].

As we discussed, the strip-algebra irreps are in correspondence with the simple lines $\mathcal{R}_{\hat{a}}$ of $\mathcal{C}_{\mathcal{M}}^*$. The MCI $\Pi_{\hat{a}}$ is equal to the identity matrix in the representation $\mathfrak{R}_{\hat{a}}$ and to the zero matrix in all other representations. Let us expand the MCI $\Pi_{\hat{a}}$ in a basis. Centrality implies that only those elements with the same boundary condition above and below each junction can appear.⁵⁶ We thus have:

$$\Pi_{\hat{a}} = \sum_{(r,m) \in \mathcal{I}_{\hat{a}}} \sum_{(b;\rho\mu)} \alpha_{r,m,(b;\rho\mu)}^{(\hat{a})} \quad \begin{array}{c} \text{Diagram: A vertical line with a blue arrow pointing down labeled } r \text{ and a red arrow pointing up labeled } \rho. A blue arrow points from the junction to the right labeled } b. \text{ Another vertical line has a blue arrow pointing up labeled } m \text{ and a red arrow pointing down labeled } \mu. \end{array} \quad (4.26)$$

⁵⁵Sometimes a different definition of the Rényi entropies is used: $\tilde{S}^{(n)}[\rho] \equiv \text{tr}(\rho^n)$ so that the entanglement entropy is $S[\rho] = -\lim_{n \rightarrow 1} \partial_n \tilde{S}^{(n)}[\rho]$. Such Rényi entropies satisfy $\tilde{S}[\rho] = \sum_{\mathfrak{R}} q_{\mathfrak{R}}^n \tilde{S}^{(n)}[\rho_{\mathfrak{R}}]$.

⁵⁶Indeed one has $e_{sn} \Pi_{\hat{a}} e_{rm} = \Pi_{\hat{a}} e_{sn} e_{rm} \propto \delta_{rs} \delta_{mn}$, where e_{rm} is the identity element (E.6) restricted to the sector (r, m) .

with certain coefficients $\alpha_{r,m,(b;\rho\mu)}^{(\hat{a})}$. The first sum is over the set $\mathcal{I}_{\hat{a}}$ of pairs (r, m) with $\hat{N}_{r\hat{a}}^m > 0$, whereas the second sum is over all junctions $\mathcal{L}_b \otimes \mathcal{B}_m \xrightarrow{\mu} \mathcal{B}_m$, $\mathcal{L}_b \otimes \mathcal{B}_r \xrightarrow{\rho} \mathcal{B}_r$, and all b such that those junctions exist, namely such that $N_{bm}^m N_{br}^r > 0$. One can write a formal expression for the coefficients α (see (C.22)), however in what follows we will only need the coefficients for the strip-algebra basis elements with a suspended identity line:

$$\Pi_{\hat{a}} \supset \frac{d_{\hat{a}} \hat{N}_{r\hat{a}}^m}{\tilde{d}_r \tilde{d}_m} \begin{array}{c} \text{Diagram: A vertical line with a blue segment labeled '1' connecting two points. The top point is labeled 'r' and the bottom point is labeled 'r'. To the right, a vertical line has a blue segment labeled 'm' connecting two points. The top point is labeled 'm' and the bottom point is labeled 'm'. A blue line connects the two points on the left to the two points on the right, forming a parallelogram-like shape. The left vertical line is labeled 'r' at both ends, and the right vertical line is labeled 'm' at both ends. The blue segment on the left is labeled '1'. $\Leftrightarrow \alpha_{r,m,1}^{(\hat{a})} = \frac{d_{\hat{a}} \hat{N}_{r\hat{a}}^m}{\tilde{d}_r \tilde{d}_m} . \quad (4.27)$$$

This result was found in Sec. 5.2 of [99] (see also [100]) using the so-called generalized half-linking numbers. We provide a more direct derivation in App. C.4.

The partition functions restricted to a given representation $\mathfrak{R}$ are computed similarly to above [100]:

$$\begin{aligned} Z_{\mathcal{B}_r \mathcal{B}_m, \mathfrak{R}}(q^n) &\equiv \text{tr}_{\mathcal{H}_{rm}} \left[\Pi_{\mathfrak{R}} e^{-n\beta H} \right] = \sum_{(b;\rho\mu)} \alpha_{r,m,(b;\rho\mu)}^{(\mathfrak{R})} \text{tr}_{\mathcal{H}_{rm}} \left[\begin{array}{c} \text{Diagram: Similar to the previous one, but with a blue segment labeled 'b' connecting the two points. The left vertical line is labeled 'r' and the right vertical line is labeled 'm'. The blue segment on the left is labeled 'b'. $e^{-n\beta H} \right] \\ &= \sum_{(b;\rho\mu)} \alpha_{r,m,(b;\rho\mu)}^{(\mathfrak{R})} \langle \mathcal{B}_m; b\mu | \tilde{q}^{\frac{1}{2n}(L_0 + \bar{L}_0 - \frac{c}{12})} | \mathcal{B}_r; b\rho \rangle . \end{aligned} \quad (4.28)$$$

In the last line we went to the closed-string channel, and $|\mathcal{B}_m; b\mu\rangle$ is the state in the b -twisted sector produced by the boundary condition $\mathcal{B}_m$ with junction to the topological line $\mathcal{L}_b$ labeled by μ (whenever $\tilde{N}_{bm}^m > 0$). In the limit $\tilde{q} \rightarrow 0$, the amplitude is dominated by the lightest state in the twisted sector, of energy proportional to $\Delta_b - \frac{c}{12}$. In a unitary CFT with faithfully acting symmetry, $\Delta_b > 0$ for all $b \neq 1$ [34]. Therefore the sum in the last line is dominated by the strip-algebra elements with suspended identity, namely by the untwisted sector:

$$\begin{aligned} Z_{\mathcal{B}_r \mathcal{B}_m, \mathfrak{R}}(q^n) &\simeq \sum_{(b;\rho\mu)} \alpha_{r,m,(b;\rho\mu)}^{(\mathfrak{R})} \langle \mathcal{B}_m; b\mu | 0, b \rangle \tilde{q}^{\frac{1}{2n}(\Delta_b - \frac{c}{12})} \langle 0, b | \mathcal{B}_r; b\rho \rangle \\ &\simeq \alpha_{r,m,1}^{(\mathfrak{R})} g_m g_r e^{\frac{c\pi}{6} \frac{\ell}{n\beta}} \end{aligned} \quad (4.29)$$

in the limit $\tilde{q} \rightarrow 0$. When $\mathcal{B}_m$ are simple conformal boundaries belonging to a module category $\mathcal{M}$, their respective g -functions are proportional to the quantum dimensions:

$$g_m = g \tilde{d}_m, \quad (4.30)$$

where g is the overall g -function, constant over the module, to be determined dynamically.

We now have all ingredients to compute the symmetry-resolved entanglement entropy of the CFT vacuum. The reduced density matrix ρ on the interval is computed using the averaging protocol (4.2) for a choice of module category $\mathcal{M}$. Using orthogonality between the sectors $\mathcal{H}_{rm}$ of the interval Hilbert space (4.1), we compute the trace:

$$\text{tr}(\rho^n) = \sum_{rm} \left(\frac{\tilde{d}_r \tilde{d}_m}{D^2} \right)^n \text{tr}(\rho_{rm}^n) = (gD)^{2-2n} e^{\frac{c\pi}{6} \frac{\ell}{\beta} (\frac{1}{n} - n)} . \quad (4.31)$$

From here we can already compute the Rényi and entanglement entropies in the averaging-protocol regularization scheme of the reduced density matrix:

$$S^{(n)}[\rho] = \frac{c}{3} \frac{n+1}{2n} \log \frac{L}{\varepsilon} + 2 \log(gD) , \quad S[\rho] = \frac{c}{3} \log \frac{L}{\varepsilon} + 2 \log(gD) . \quad (4.32)$$

Notice that $gD = \sum_m g_m$ using (4.2). Terms that vanish in the limit $\varepsilon \rightarrow 0$ have been dropped.

Labeling with $\hat{a}$ the strip-algebra irreps $\mathfrak{R}_{\hat{a}}$, the probabilities are

$$q_{\hat{a}} \equiv \text{tr}(\Pi_{\hat{a}} \rho) = \sum_{rm} \frac{\tilde{d}_r \tilde{d}_m}{D^2} \lim_{q \rightarrow 1} \frac{Z_{\mathcal{B}_r \mathcal{B}_m, \hat{a}}(q)}{Z_{\mathcal{B}_r \mathcal{B}_m}(q)} = \sum_{rm} \frac{\tilde{d}_r \tilde{d}_m}{D^2} \alpha_{r,m,1}^{(\hat{a})} = \frac{\dim(\mathfrak{R}_{\hat{a}}) d_{\hat{a}}}{D^2}, \quad (4.33)$$

where we used (4.27) and (4.5). In this formula, both the classical and the quantum dimensions of the representation appear. Note that, using (4.5) and (4.10), we have $\sum_{\hat{a}} q_{\hat{a}} = 1$.

The last trace to compute is

$$\begin{aligned} \text{tr}(\rho_{\hat{a}}^n) &= \frac{1}{q_{\hat{a}}^n} \sum_{rm} \left(\frac{\tilde{d}_r \tilde{d}_m}{D^2} \right)^n \text{tr}(\Pi_{\hat{a}} \rho_{rm}^n) = \sum_{rm} \left(\frac{\tilde{d}_r \tilde{d}_m}{q_{\hat{a}} D^2} \right)^n \lim_{q \rightarrow 1} \frac{Z_{\mathcal{B}_r \mathcal{B}_m, \hat{a}}(q^n)}{Z_{\mathcal{B}_r \mathcal{B}_m}(q)^n} \\ &= \left(g^2 \dim(\mathfrak{R}_{\hat{a}}) d_{\hat{a}} \right)^{1-n} e^{\frac{c\pi}{6} \frac{\ell}{\beta} \left(\frac{1}{n} - n \right)}. \end{aligned} \quad (4.34)$$

From there we compute the symmetry-resolved Rényi and entanglement entropies:

$$\begin{aligned} S^{(n)}[\rho_{\hat{a}}] &= \frac{c}{3} \frac{n+1}{2n} \log \frac{L}{\varepsilon} + 2 \log(g) + \log \left(\dim(\mathfrak{R}_{\hat{a}}) d_{\hat{a}} \right), \\ S[\rho_{\hat{a}}] &= \frac{c}{3} \log \frac{L}{\varepsilon} + 2 \log(g) + \log \left(\dim(\mathfrak{R}_{\hat{a}}) d_{\hat{a}} \right). \end{aligned} \quad (4.35)$$

They can also be written as:

$$\boxed{\begin{aligned} S^{(n)}[\rho_{\hat{a}}] &= \frac{c}{3} \frac{n+1}{2n} \log \frac{L}{\varepsilon} + 2 \log \left(\sum_m g_m \right) + \log q_{\hat{a}} \\ S[\rho_{\hat{a}}] &= \frac{c}{3} \log \frac{L}{\varepsilon} + 2 \log \left(\sum_m g_m \right) + \log q_{\hat{a}} \end{aligned} \quad q_{\hat{a}} = \frac{\dim(\mathfrak{R}_{\hat{a}}) d_{\hat{a}}}{D^2}} \quad (4.36)$$

The leading term, as well as the dependence on the g -functions, are the same as for the entropies in the averaging-protocol regularization scheme (4.32). The symmetry-resolved entropies, however, contain an additional finite contribution that depends on the strip-algebra representation and can be expressed in terms of the classical and quantum dimensions of the representation and of the boundaries. Taking the difference between the symmetry-resolved and the ordinary entanglement entropy within the same regularization scheme, one can remove the dependence on the cutoff ε :

$$\lim_{\varepsilon \rightarrow 0} \left(S[\rho_{\hat{a}}] - S[\rho] \right) = \log q_{\hat{a}}. \quad (4.37)$$

Moreover, the entanglement entropies satisfy the decomposition law (4.25), as one can verify.

5 Examples of asymmetry calculations

In this section we collect a number of examples in two-dimensional theories with details about the computation of von Neumann and Rényi asymmetries, in order to exhibit general features.

5.1 Group-like symmetries

Consider an invertible symmetry given by a finite group G . The corresponding fusion category is Vec_G^ω , whose simple lines $\mathcal{L}_g$ are in correspondence with the elements $g \in G$ and they all have dimension $d_g = 1$. The fusion coefficients are $N_{gh}^k = \delta_{(gh)}^k$ where (gh) denotes the group multiplication of g with h . In particular, there are no multiplicities at the junctions. The symmetrizer with respect to the fusion algebra is

$$\rho_S = \frac{1}{|G|} \sum_{g \in G} \mathcal{L}_g \rho \mathcal{L}_{g^{-1}} . \quad (5.1)$$

A possible 't Hooft anomaly is described by a class $\omega \in H^3(G, \text{U}(1))$, which fixes the nonvanishing F -symbols to be

$$[F_{g,h,k}^{ghk}]_{gh,hk} = \omega(g, h, k) . \quad (5.2)$$

We use normalized representatives ω , meaning that $\omega(g, h, k) = 1$ whenever one or more of g, h, k are equal to the identity $1 \in G$. Different representatives of the same class produce the F -symbols in different gauges.

Spontaneous symmetry breaking. The possible indecomposable G -symmetric TQFTs, in correspondence with the possible indecomposable $\mathcal{C}$ -module categories, are characterized by the conjugation class of a subgroup $H \subseteq G$ (besides a cochain ψ , which however will not play a role here), as reviewed in App. F. They describe the SSB pattern $G \rightarrow H$ at very low energies. The clustering vacua $|m\rangle$, in correspondence with the simple boundaries $\mathcal{B}_m$, are labeled by the cosets $m \in G/H$. The vacua furnish a simple NIM-rep of the fusion algebra:

$$\mathcal{L}_g |m\rangle = |gm\rangle . \quad (5.3)$$

Here gm is the coset with representative ga , if $a \in G$ is a representative of m .

Consider the pure state $\rho = |m\rangle\langle m|$ of a vacuum on the circle. Using (5.1), the corresponding symmetrized density matrix with respect to the fusion algebra is

$$\rho_S = \frac{1}{|G|} \sum_{g \in G} \mathcal{L}_g \rho \mathcal{L}_{g^{-1}} = \frac{1}{|G|} \sum_{g \in G} |gm\rangle\langle gm| = \frac{1}{|G/H|} \sum_{p \in G/H} |p\rangle\langle p| . \quad (5.4)$$

In the second equality we used that $\mathcal{L}_{g^{-1}} = \mathcal{L}_g^\dagger$. Because of the crossing relation (3.27), the vacua are orthonormal and thus $\text{tr}(\rho_S^n) = |G/H|^{1-n}$. The asymmetries easily follow:

$$\Delta S[\rho] = \Delta S^{(n)}[\rho] = \log |G/H| . \quad (5.5)$$

This computation reproduces the result of [134], obtained using matrix product states. Notice that the derivation here applies verbatim to spacetime dimensions higher than two as well.

5.1.1 Tube algebra

The tube algebra of Vec_G^ω (which is given by the twisted quantum double of G) is generated by the basis elements:

$$\tau_{g,a} \equiv \begin{array}{c} \text{aga}^{-1} \\ | \\ \text{ag} \text{---} \text{a} \\ / \quad \backslash \\ \text{a} \quad \text{g} \end{array} . \quad (5.6)$$

The product is given by:

$$\tau_{h,b} \times \tau_{g,a} = \delta_{h,aga^{-1}} \frac{\omega(b, aga^{-1}, a)}{\omega(baga^{-1}b^{-1}, b, a) \omega(b, a, g)} \tau_{g,ba} . \quad (5.7)$$

From here the multiplication coefficients $T_{h,b;g,a}^{k,c}$ are immediately read off. The phase factor appearing in the product can be interpreted as a twisted 2-cocycle. Indeed, define

$$\beta_g(x, y) = \frac{\omega(g, x, y) \omega(x, y, y^{-1}x^{-1}gxy)}{\omega(x, x^{-1}gx, y)} . \quad (5.8)$$

One verifies that this is a normalized twisted 2-cocycle, where x has an adjoint action on $g \in G$ (this also appeared in [155] Sec. 4.B).⁵⁷ We can write the tube algebra product as

$$\tau_{h,b} \times \tau_{g,a} = \frac{\delta_{h,aga^{-1}}}{\beta_{baga^{-1}b^{-1}}(b, a)} \tau_{g,ba} . \quad (5.9)$$

The identity element is $1 = \sum_g \tau_{g,1}$. Associativity of the product coincides with the twisted cocycle condition for β . The non-degenerate bilinear form can be written as

$$K_{h,b;g,a} = \frac{|G| \delta_{h,aga^{-1}} \delta_{b,a^{-1}}}{\beta_g(a^{-1}, a)} . \quad (5.10)$$

From its inverse matrix $\tilde{K}^{g,a;h,b}$ one obtains the symmetrizer:

$$\rho_S = \frac{1}{|G|} \sum_{g,a} \beta_g(a^{-1}, a) \tau_{g,a} \rho \tau_{aga^{-1}, a^{-1}} . \quad (5.11)$$

Using the twisted cocycle condition one proves the relation $\beta_{aga^{-1}}(a, a^{-1}) = \beta_g(a^{-1}, a)$ which is useful to check directly some properties of the symmetrizer.

5.1.2 Strip algebras

Let us now study the case of the interval Hilbert space using the strip algebras. The possible boundary conditions are given in terms of module categories, whose classification in the case of invertible finite symmetries is reviewed in App. F.

In the non-anomalous case, one can always choose strongly-symmetric boundary conditions (but see Fn. 28). From App. F, those are classified by $\psi \in H^2(G, \text{U}(1))$. Explicitly, one has

$$\tilde{N}_a^\cdot = 1 , \quad [\tilde{F}_{a,b}^\cdot]_{c,\cdot} = (m_{a,b}^c)^* = \delta_{ab}^c \psi(a, b) . \quad (5.12)$$

⁵⁷The twisted cocycle condition is $d\beta_g(x, y, z) = \beta_{x^{-1}gx}(y, z) \beta_g(x, yz) / \beta_g(xy, z) \beta_g(x, y) = 1$. See also App. A of [156] for a review on twisted cocycles.

Note that there are no multiplicities at the boundary junctions. In terms of the basis elements H_g defined in (3.36), we see that the cocycle cancels out from the product (3.37) so that $H_g \times H_h = H_{gh}$. The strip algebra $\text{Strip}_{\text{Vec}_G}(\text{Vec})$ is thus isomorphic to the group algebra $\mathbb{C}[G]$ (which is also the fusion algebra) regardless of the class of ψ .⁵⁸ One finds the symmetrizer

$$\rho_S = \frac{1}{|G|} \sum_{g \in G} H_g \rho H_{g^{-1}} . \quad (5.13)$$

On the other hand, a boundary condition that can always be chosen regardless of the value of the anomaly cocycle $\omega \in H^3(G, \text{U}(1))$ is the regular module (but see Fn. 28). In this case we obtain the algebra $\text{Strip}_{\text{Vec}_G^\omega}(\text{Vec}_G^\omega)$, generated by the basis elements:

$$H_{r,m}^g \equiv \begin{array}{c} \text{Diagram: A vertical line with a red dot. To the left, a blue arrow labeled 'gr' points down. To the right, a blue arrow labeled 'gm' points up. A blue arrow labeled 'g' points from the red dot to the right. Below the red dot, a blue arrow labeled 'r' points down. To the right of the red dot, a blue arrow labeled 'm' points up. The entire diagram is enclosed in a light gray box. \end{array} \quad (5.14)$$

with $g, m, r \in G$ and $\tilde{N}_{gm}^n = \delta_{(gm)}^n$. The product is given by:

$$H_{s,n}^b \times H_{r,m}^a = \delta_{s,ar} \delta_{n,am} \frac{\omega(b, a, r)}{\omega(b, a, m)} H_{r,m}^{ba} . \quad (5.15)$$

Also in this case the phase can be interpreted as a twisted cocycle (in which the action on r, m is in the regular representation) and associativity as its closure. In this case, however, there exists an isomorphism with $\text{Strip}_{\text{Vec}_G}(\text{Vec}_G)$, namely, with the strip algebra of the symmetry without anomaly [98]. The isomorphism is given by⁵⁹

$$\begin{aligned} \varphi: \text{Strip}_{\text{Vec}_G}(\text{Vec}_G) &\rightarrow \text{Strip}_{\text{Vec}_G^\omega}(\text{Vec}_G^\omega) , \\ \varphi(H_{r,m}^a) &= \omega(a, r, r^{-1}m) H_{r,m}^a , \end{aligned} \quad (5.16)$$

then extended by linearity. Indeed one verifies that

$$\varphi(H_{ar,am}^b) \times_\omega \varphi(H_{r,m}^a) = \varphi(H_{ar,am}^b \times_{\omega=1} H_{r,m}^a) \quad (5.17)$$

using that $d\omega(b, a, r, r^{-1}m) = 1$. Here we made explicit where we used the anomalous product (5.15) or the non-anomalous one. The non-degenerate bilinear form can be written as

$$K_{s,n,b;r,m,a} = |G| \frac{\omega(a^{-1}, a, r)}{\omega(a^{-1}, a, m)} \delta_{s,ar} \delta_{n,am} \delta_{b,a^{-1}} . \quad (5.18)$$

From its inverse one obtains the symmetrizer

$$\rho_S = \frac{1}{|G|} \sum_{r,m,a} \frac{\omega(a^{-1}, a, m)}{\omega(a^{-1}, a, r)} H_{r,m}^a \rho H_{ar,am}^{a^{-1}} = \frac{1}{|G|} \sum_{r,m,a} \varphi(H_{r,m}^a) \rho \varphi(H_{ar,am}^{a^{-1}}) . \quad (5.19)$$

The rewriting in terms of φ makes it manifest that the symmetrizers for the two strip algebras are the same up to a phase redefinition of the basis elements.

⁵⁸Alternatively, the cocycle can be reabsorbed with a change of basis of the junction vector spaces [98].

⁵⁹The two algebras share the same underlying vector space, hence we denote a basis of both with the same notation $H_{r,m}^a$.

Figure 8: Left: path-integral representation of the state on the real line. The operator $\mathcal{O}$ is inserted at $z = x - i\tau$, while the operator $\mathcal{O}^\dagger$ is inserted at the specular image $\bar{z} = x + i\tau$. The normalization by the partition function Z ensures that $\text{tr}(\rho^{\text{full}}) = 1$. Right: path-integral representation of $\mathcal{L}_g \rho^{\text{full}} \mathcal{L}_{g^{-1}}$.

Figure 9: Left: path-integral construction of the reduced density matrix $\rho_{a,b}$. The dashed lines are the two open boundaries (cuts). The constant $Z_{a,b}$ is the partition function obtained by gluing the two cuts, so as to ensure that $\text{tr}(\rho_{a,b}) = 1$. Right: path-integral construction of the rotated density matrix $\rho_{ga,gb}^{(g)}$ where $Z_{ga,gb}^{(g)} \equiv Z_{a,b} \omega(g^{-1}, g, a) / \omega(g^{-1}, g, b)$.

Asymmetries in CFTs. Consider a CFT and a state on the real line $\mathbb{R}$ created by the Euclidean path-integral on a half space with the insertion of local operators. The corresponding density matrix ρ^{full} is prepared by a Euclidean path-integral in which the future half-space is the specular image of the past half-space, and there is a cut along the real line. This is depicted in Fig. 8 left. There, Z is the partition function on the plane obtained by closing the cut, so as to ensure that $\text{tr}(\rho^{\text{full}}) = 1$.

To construct the reduced density matrix for a single interval along the real line, we apply the procedure of Sec. 3.3. Consider first the case that we use boundary conditions a, b chosen from the regular module. We depict the path-integral construction of $\rho_{a,b}$ in Fig. 9 left, where $Z_{a,b}$ is the partition function obtained by closing the cut so as to ensure that $\text{tr}(\rho_{a,b}) = 1$. For simplicity, in the figures we indicate only one operator insertion $\mathcal{O}$ and its specular image $\mathcal{O}^\dagger$, but the argument applies to any number of insertions. To obtain the symmetrized density matrix we apply (5.19). Since the symmetry elements $H_{r,m}^g$ project to zero when their boundary conditions do not match those of the state, we remain with a single sum:

$$\rho_S = \frac{1}{|G|} \sum_g \frac{\omega(g^{-1}, g, b)}{\omega(g^{-1}, g, a)} H_{a,b}^g \rho_{a,b} H_{ga,gb}^{g^{-1}} \equiv \frac{1}{|G|} \sum_g \rho_{ga,gb}^{(g)}. \quad (5.20)$$

In the last equality we introduced the rotated density matrices $\rho_{ga,gb}^{(g)}$ whose path-integral construction is depicted in Fig. 9 right. They belong to $\text{End}(\mathcal{H}_{ga,gb})$. The constant prefactor is $Z_{ga,gb}^{(g)} = Z_{a,b} \omega(g^{-1}, g, a) / \omega(g^{-1}, g, b)$. To obtain the second presentation one moves the line of $H_{a,b}^g$ through infinity and then annihilates it with $H_{ga,gb}^{g^{-1}}$. When the line crosses an operator $\mathcal{O}$, the latter gets transformed to ${}^g\mathcal{O}$. Since Hilbert spaces constructed with different simple boundary conditions are orthogonal, the moments of ρ_S simplify:

$$\text{tr}(\rho_S^n) = \frac{1}{|G|^n} \sum_g \text{tr}[(\rho_{ga,gb}^{(g)})^n] = \frac{1}{|G|^{n-1}} \text{tr}(\rho_{a,b}^n). \quad (5.21)$$

The Rényi and von Neumann asymmetries are then

$$\Delta S^{(n)}[\rho_{a,b}] = \Delta S[\rho_{a,b}] = \log |G| \quad \text{for all } n. \quad (5.22)$$

This result does not depend on the state, and in particular it applies to the vacuum as well. Its interpretation is that the boundary conditions completely break the symmetry, therefore $\Delta S^{(n)}$ and ΔS reach their maximal value irrespective of the state. This observable is clearly not very interesting. To make it interesting, we should rather use a symmetric combination of boundary conditions, as described in Sec. 3.3.3.

Consider then the “averaged” reduced density matrix

$$\rho = \frac{1}{|G|^2} \sum_{a,b} \rho_{a,b} . \quad (5.23)$$

The symmetrized matrix this time can be written as

$$\rho_S = \frac{1}{|G|^3} \sum_{a,b,g} \rho_{a,b}^{(g)} , \quad (5.24)$$

where $\rho_{a,b}^{(g)}$ is defined in Fig. 9 right. One computes

$$\text{tr}(\rho^n) = \frac{1}{|G|^{2n}} \sum_{a,b} \text{tr}(\rho_{a,b}^n) , \quad \text{tr}(\rho_S^n) = \frac{1}{|G|^{3n}} \sum_{a,b} \sum_{g_1, \dots, g_n} \text{tr}(\rho_{a,b}^{(g_1)} \dots \rho_{a,b}^{(g_n)}) . \quad (5.25)$$

From the path-integral representation of $\rho_{a,b}$ (see Fig. 9), $\text{tr}(\rho_{a,b}^n)$ is equal to $Z_{a,b}^{-n}$ times the partition function on an n -sheeted Riemann surface with operators $\mathcal{O}$ and $\mathcal{O}^\dagger$ inserted in each replica. In this manifold, the n copies of the boundary with boundary condition a are glued together to form a single boundary that spans all replicas, and the same holds for the boundaries with boundary condition b . Similarly, $\text{tr}(\rho_{a,b}^{(g_1)} \dots \rho_{a,b}^{(g_n)})$ is equal to $\prod_i 1/Z_{g_i^{-1}a, g_i^{-1}b}$ times the partition function on an n -sheeted Riemann surface with operators $^{g_j}\mathcal{O}$ and $^{g_j}\mathcal{O}^\dagger$ inserted in the j -th replica and with one a and one b boundary.

In the limit $\varepsilon \rightarrow 0$, the dependence of the traces on the boundary conditions becomes very simple. As described in (3.45), in the limit each partition function produces a factor of the boundary quantum dimension from each shrinking disc. Thus, the partition function on the n -sheeted Riemann surface becomes $\tilde{d}_a \tilde{d}_b$ times the partition function on a similar Riemann surface, where the boundary discs are shrunk to a branch point and all insertions of fields remain at the same position. For a group-like symmetry all boundary quantum dimensions $\tilde{d}$ are equal. We thus obtain the same factor $\tilde{d}^{2(1-n)}$ from both $\lim_{\varepsilon \rightarrow 0} \text{tr}(\rho^n)$ and $\lim_{\varepsilon \rightarrow 0} \text{tr}(\rho_S^n)$, which simplifies in the ratio. Besides, once the $\varepsilon \rightarrow 0$ limit is taken, there is no dependence on a and b left and we can easily sum over them. The result can be expressed in terms of correlation functions on a Riemann surface with branch points:

$$\lim_{\varepsilon \rightarrow 0} \Delta S^{(n)}[\rho] = \frac{1}{1-n} \log \left[\frac{1}{|G|^n} \sum_{g_1, \dots, g_n} \frac{\langle ^{g_1}\mathcal{O}^\dagger ^{g_1}\mathcal{O} \dots ^{g_n}\mathcal{O}^\dagger ^{g_n}\mathcal{O} \rangle_{\mathcal{M}_n}}{\langle \mathcal{O}^\dagger \mathcal{O} \dots \mathcal{O}^\dagger \mathcal{O} \rangle_{\mathcal{M}_n}} \right] , \quad (5.26)$$

where the notation $\langle \cdot \rangle_{\mathcal{M}_n}$ indicates a correlation function on the branched Riemann surface, as detailed in Fig. 10.

We make three observations. First, if there are no insertions, i.e., if we compute the asymmetry of the vacuum, then all correlators in (5.26) are equal to 1 and thus $\Delta S^{(n)} = \Delta S = 0$, as expected from the general discussion in Sec. 3.3.3. Second, the asymmetries have no explicit

$$\langle g_1 \mathcal{O}^\dagger g_1 \mathcal{O} \dots g_n \mathcal{O}^\dagger g_n \mathcal{O} \rangle_{\mathcal{M}_n} \equiv \left\langle \begin{array}{c} \text{Sheet 1: } g_1 \mathcal{O}^\dagger, g_1 \mathcal{O} \\ \vdots \\ \text{Sheet } n: g_n \mathcal{O}^\dagger, g_n \mathcal{O} \end{array} \right\rangle$$

Figure 10: Correlator on the n -sheeted Riemann surface $\mathcal{M}_n$. It is understood that the lower rim of each cut (denoted by a dashed segment) is glued to the upper rim of the cut of the sheet above it (and the uppermost sheet is connected to the lowermost sheet). Two branch points sit at the ends of the branch cut. In the j -th replica the operator $g_j \mathcal{O}$ is inserted at $z = x - i\tau$ while the operator $g_j \mathcal{O}^\dagger$ is inserted at $\bar{z} = x + i\tau$.

dependence on the anomaly, as already anticipated in [132]. Third, in the absence of an anomaly, one could compute the asymmetries using the strongly-symmetric boundary condition, and this would lead to exactly the same formula. In particular, earlier works on asymmetry for groups ignored the entangling surface and used the group algebra instead of the strip algebra; we see here that those less rigorous approaches are in fact validated by our more rigorous approach of using the strip algebra.

Large subsystem limit. In the limit of a large subsystem of size ℓ , the branch cut extends along the whole real axis. Then effectively each sheet gets split into two half-planes, and each lower half-plane is glued to the upper half-plane of the next sheet (*cfr.* with Fig. 10). One can perform a conformal block expansion of the $2n$ -point functions appearing in (5.26), in the channel in which each operator is first contracted with the operator found across the cut after the gluing. When that channel is dominated by exchanges of the identity operator, the $2n$ -point function factorizes into the product of n two-point functions. This is always the case for the $2n$ -point function at denominator in (5.26). When exchanges of the identity operator are not possible in the numerator, the contribution to the sum in (5.26) is subdominant – and on the other hand some of the two-point functions vanish. Therefore, in the large subsystem limit we find the formula:

$$\lim_{\ell \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \Delta S^{(n)}[\rho] = \frac{1}{1-n} \log \left[\frac{1}{|G|^n} \sum_{g_1, \dots, g_n} \frac{\langle g_1 \mathcal{O}^\dagger g_2 \mathcal{O} \rangle_{\mathbb{C}} \dots \langle g_n \mathcal{O}^\dagger g_1 \mathcal{O} \rangle_{\mathbb{C}}}{\langle \mathcal{O}^\dagger \mathcal{O} \rangle_{\mathbb{C}}^n} \right]. \quad (5.27)$$

We claim that this is the same as the Rényi asymmetry of the full system, i.e., of the state on the real line symmetrized with respect to the fusion algebra $\mathbb{C}[G]$. Indeed, let ρ^{full} be the state on the real line, depicted in Fig. 8 left. The symmetrized density matrix for the full system is

$$\rho_S^{\text{full}} = \frac{1}{|G|} \sum_{g \in G} \mathcal{L}_g \rho^{\text{full}} \mathcal{L}_{g^{-1}}. \quad (5.28)$$

The summand $\mathcal{L}_g \rho^{\text{full}} \mathcal{L}_{g^{-1}}$ is depicted in Fig. 8 right. The Rényi asymmetry is then

$$\Delta S^{(n)}[\rho^{\text{full}}] = \frac{1}{1-n} \log \left\{ \frac{1}{|G|^n} \text{tr} \left[\frac{1}{Z^n} \left(\sum_{g \in G} \begin{array}{c} \bullet g \mathcal{O}^\dagger \\ \text{---} \\ \bullet g \mathcal{O} \end{array} \right)^n \right] \right\}. \quad (5.29)$$

We used that ρ^{full} is pure and thus $\text{tr}[(\rho^{\text{full}})^n] = 1$. The n -th power of the operator appearing inside the trace is obtained by taking n copies of the cut plane and then connecting the lower rim of each cut (which is the “output” of the operator) to the upper rim of the next sheet (which is the “input”). The trace connects them cyclically, in a fashion similar to Fig. 10. The partition functions can then be replaced by expectation values, since both at numerator and denominator the required normalization is given by the same partition function with no operator insertions. We conclude that (5.29) can be written as in (5.27) and therefore

$$\lim_{\ell \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \Delta S^{(n)}[\rho] = \Delta S^{(n)}[\rho^{\text{full}}] . \quad (5.30)$$

5.2 Ising symmetry – circle Hilbert space

The internal symmetry of the two-dimensional Ising CFT is described by a fusion category $\mathcal{C}$ (that we call the Ising fusion category) with 3 simple lines, that we indicate as $\mathcal{L}_1$, $\mathcal{L}_\eta$, $\mathcal{L}_\mathcal{N}$, satisfying the commutative fusion ring:

$$\mathcal{L}_\eta \times \mathcal{L}_\eta = \mathcal{L}_1 , \quad \mathcal{L}_\eta \times \mathcal{L}_\mathcal{N} = \mathcal{L}_\mathcal{N} , \quad \mathcal{L}_\mathcal{N} \times \mathcal{L}_\mathcal{N} = \mathcal{L}_1 + \mathcal{L}_\eta . \quad (5.31)$$

Here $\mathcal{L}_1$ is the identity, $\mathcal{L}_\eta$ generates the $\mathbb{Z}_2$ spin-flip symmetry, and $\mathcal{L}_\mathcal{N}$ is the line of Kramers-Wannier duality. The junctions have no multiplicities. The nontrivial F -symbols are:

$$[F_{\eta\mathcal{N}\eta}^\mathcal{N}]_{\mathcal{N}\mathcal{N}} = [F_{\mathcal{N}\eta\mathcal{N}}^\eta]_{\mathcal{N}\mathcal{N}} = -1 , \quad [F_{\mathcal{N}\mathcal{N}\mathcal{N}}^\mathcal{N}]_{ab} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} , \quad (5.32)$$

where $a, b \in \{1, \eta\}$, while all other F -symbols allowed by fusion are equal to 1. All lines are self-dual. One reads off the Frobenius-Schur (FS) indicators $\varkappa_\eta = \varkappa_\mathcal{N} = 1$ and the quantum dimensions $d_\eta = 1$, $d_\mathcal{N} = \sqrt{2}$. The category $\mathcal{C}$ is the Tambara-Yamagami fusion category $\text{TY}(\mathbb{Z}_2, \chi, +)$ [157] where $\chi(a, b) = e^{i\pi ab}$ is a symmetric nondegenerate $\mathbb{Z}_2$ bicharacter that equals -1 only if a, b are both the nontrivial element η , otherwise it equals 1.⁶⁰

5.2.1 Fusion algebra

The fusion algebra has three 1-dimensional representations that we indicate as $\mathbb{1}$, ε , σ . In the Ising CFT these are realized in the Verma module of the identity, the energy, and the spin operator, respectively. The fusion algebra acts on them through (3.18), in terms of the unitary and symmetric modular S -matrix:

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} . \quad (5.33)$$

⁶⁰The $\text{TY}(\mathbb{Z}_2, \chi, +)$ symmetry appears, for instance, in a system of $\nu = 1, 7, 9, 15 \pmod{16}$ Dirac fermions. A slight generalization is $\text{TY}(\mathbb{Z}_2, \chi, -)$, whose only difference is that $[F_{\mathcal{N}\mathcal{N}\mathcal{N}}^\mathcal{N}]_{ab} = -\chi(a, b)/\sqrt{2}$ and thus $\varkappa_\mathcal{N} = -1$, which appears, for instance, in a system of $\nu = 3, 5, 11, 13 \pmod{16}$ Dirac fermions.

The symmetrizer for the fusion algebra is given by the inverse bilinear form:

$$\tilde{K} = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} & 0 \\ -\frac{1}{8} & \frac{3}{8} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, \quad (5.34)$$

and in particular it takes the form:

$$\rho_S = \frac{1}{8} \left[3\rho + 3\mathcal{L}_\eta \rho \mathcal{L}_\eta - \mathcal{L}_\eta \rho - \rho \mathcal{L}_\eta + 2\mathcal{L}_\mathcal{N} \rho \mathcal{L}_\mathcal{N} \right]. \quad (5.35)$$

Asymmetry of excited states in CFTs. We consider a CFT whose symmetry includes the Ising fusion category. This could be the Ising CFT itself, or a more complicated one (for instance, the tricritical Ising model). The local operators fall into representations of the fusion algebra, that we indicate as $\mathbb{1}$, ε , σ as in (5.33). We consider excited states on the line $\mathbb{R}$ produced using a scalar operator $\varepsilon(z)$ of dimension Δ_ε in the Euclidean past, which by definition transforms in the representation ε :

$$|\psi\rangle = \boxed{\bullet 1 + \lambda \varepsilon}. \quad (5.36)$$

Here λ is a constant of mass dimension $-\Delta_\varepsilon$. Notice that this state is neutral under the $\mathbb{Z}_2$ spin-flip symmetry. To compute its asymmetry with respect to the fusion algebra we can use (3.26). Setting the operator insertion at $z = -i\tau$ with $\tau > 0$ (thus $\bar{z} = i\tau$), the relevant two-point function is $\langle \varepsilon(\bar{z}) \varepsilon(z) \rangle \equiv \langle \varepsilon^2 \rangle = 1/(2\tau)^{2\Delta_\varepsilon}$. The second Rényi and von Neumann asymmetries are then

$$\Delta S^{(2)}[|\psi\rangle\langle\psi|] = -\log \left[\frac{1 + \lambda^4 \langle \varepsilon^2 \rangle^2}{(1 + \lambda^2 \langle \varepsilon^2 \rangle)^2} \right], \quad \Delta S[|\psi\rangle\langle\psi|] = f \left[\frac{1}{1 + \lambda^2 \langle \varepsilon^2 \rangle} \right] + f \left[\frac{\lambda^2 \langle \varepsilon^2 \rangle}{1 + \lambda^2 \langle \varepsilon^2 \rangle} \right], \quad (5.37)$$

where we used the function $f(x) = -x \log x$.

In the case of the Ising CFT we can take $\varepsilon(x)$ to be the energy conformal primary operator of dimension $\Delta_\varepsilon = 1$. In Fig. 11 we plot the asymmetry as a function of τ/λ .

Spontaneous symmetry breaking. The only indecomposable $\mathcal{C}$ -module category of the Ising fusion category is the regular one.⁶¹ This means that in a gapped phase the Ising symmetry is always completely spontaneously broken, and this is due to the presence of an 't Hooft anomaly. In the regular module one could indicate both lines and boundary conditions using the same set of labels. However, in order to adhere to a more commonly used notation, we relabel the boundary conditions as:⁶²

$$1 \rightarrow +, \quad \eta \rightarrow -, \quad \mathcal{N} \rightarrow f. \quad (5.38)$$

⁶¹By Th. 6 of [137] (where the Ising category is called $\mathcal{C}_2$ and the module is of type A_3), the Ising fusion category admits only one indecomposable module category. Hence, this must be the regular module, which always exists.

⁶²These boundary conditions have a counterpart in a lattice spin model: $\pm$ correspond to up/down fixed boundary conditions for the spins, while f corresponds to the free boundary condition.

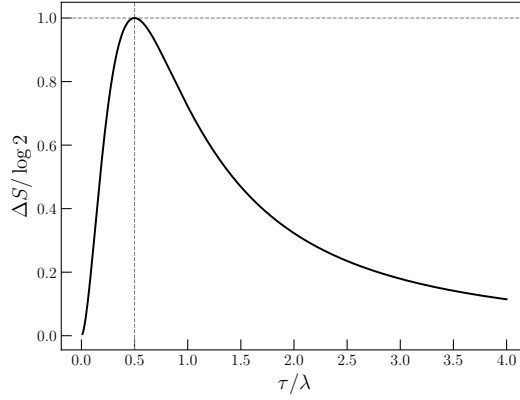


Figure 11: Asymmetry ΔS with respect to the Ising fusion algebra of a state on the real line prepared by inserting $(1 + \lambda \varepsilon)$ with $\Delta_\varepsilon = 1$ in the path-integral. The asymmetry is plotted as a function of τ/λ , where $-\tau$ is the Euclidean time of the insertion. In the limit $\tau/\lambda \rightarrow \infty$ only the identity contributes, so the state becomes symmetric and the asymmetry vanishes. Similarly, in the limit $\tau/\lambda \rightarrow 0$ only ε contributes, leading again to a symmetric state with vanishing asymmetry. The asymmetry reaches its maximum at $\tau/\lambda = 1/2$.

Correspondingly, we indicate the three normalized vacua (i.e. the three clustering states) as $\{|+\rangle, |-\rangle, |f\rangle\}$. They furnish a NIM-rep of the fusion algebra:

$$\begin{aligned} \mathcal{L}_\eta |+\rangle &= |-\rangle, & \mathcal{L}_\eta |-\rangle &= |+\rangle, & \mathcal{L}_\eta |f\rangle &= |f\rangle, \\ \mathcal{L}_\mathcal{N} |+\rangle &= |f\rangle, & \mathcal{L}_\mathcal{N} |-\rangle &= |f\rangle, & \mathcal{L}_\mathcal{N} |f\rangle &= |+\rangle + |-\rangle. \end{aligned} \quad (5.39)$$

The gapped phase (some curious properties thereof were already noticed in [101]) can be obtained, for instance, from the tricritical Ising model deforming it by the relevant operator ε' , as explained in Sec. 7.2.3 of [15].

Consider the pure state $\rho = |f\rangle\langle f|$ for the vacuum $|f\rangle$. Using the symmetrizer (5.35) with respect to the fusion algebra, one computes ρ_S . Its diagonalization is $\rho_S \sim \text{diag}(\frac{1}{2}, \frac{1}{2}, 0)$ and therefore $\text{tr}(\rho_S^n) = 2^{1-n}$. The Rényi and von Neumann asymmetries are then

$$\Delta S^{(n)} = \Delta S = \log 2. \quad (5.40)$$

Notice that this is the same asymmetry as for a spontaneously broken $\mathbb{Z}_2$ symmetry [127].⁶³ On the other hand, consider the pure state $\rho = |+\rangle\langle +|$ for the vacuum $|+\rangle$ (the vacuum $|-\rangle$ would give the same result). The symmetrized matrix ρ_S has diagonalization $\rho_S \sim \text{diag}(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and hence $\text{tr}(\rho_S^n) = (2^{n-1} + 1)/2^{2n-1}$, from which the Rényi and von Neumann asymmetries are

$$\Delta S^{(n)} = \frac{1}{1-n} \log \frac{2^{n-1} + 1}{2^{2n-1}}, \quad \Delta S = \frac{3}{2} \log 2. \quad (5.41)$$

This time the asymmetry is larger than for a broken $\mathbb{Z}_2$, but smaller than for a broken $\mathbb{Z}_3$. We conclude that, while for a completely broken invertible symmetry G the asymmetry of a vacuum is $\Delta S = \log |G|$ [134], for noninvertible symmetries the asymmetry may take more general values.

⁶³This is explained by the fact that the “orbit” of $|f\rangle$ under the noninvertible symmetry is given by two rays, the ones of $|f\rangle$ and $(|+\rangle + |-\rangle)$, as it happens for a $\mathbb{Z}_2$ symmetry.

We also observe that different vacua in the same gapped phase can have inequivalent physical properties (as already stressed in [101, 102]) and in particular different asymmetries.

5.2.2 Tube algebra

The tube algebra is generated by the following 12 basis elements (see also [107]):

$$\begin{aligned} & \begin{array}{c} 1 \\ | \\ 1 \end{array}, \quad \begin{array}{c} \eta \\ | \\ 1 \end{array}, \quad \begin{array}{c} \mathcal{N} \\ | \\ 1 \end{array}, \quad \begin{array}{c} \mathcal{N} \\ | \\ \eta \end{array}, \quad \begin{array}{c} 1 \\ | \\ \eta \end{array}, \quad \begin{array}{c} \eta \\ | \\ \eta \end{array}, \quad \begin{array}{c} \mathcal{N} \\ | \\ \eta \end{array}, \quad \begin{array}{c} \mathcal{N} \\ | \\ \mathcal{N} \end{array}, \\ & \begin{array}{c} 1 \\ | \\ \mathcal{N} \end{array}, \quad \begin{array}{c} \eta \\ | \\ \mathcal{N} \end{array}, \quad \begin{array}{c} \mathcal{N} \\ | \\ \mathcal{N} \end{array}, \quad \begin{array}{c} \mathcal{N} \\ | \\ \mathcal{N} \end{array}. \end{aligned} \quad (5.42)$$

By applying the formula (2.12) we compute the algebra product. We use the bicharacter $\chi(a, b)$ defined on $\mathbb{Z}_2 = \{1, \eta\}$ above. As in the general discussion, we adopt the convention that the element on the left is stacked on top of the one on the right. We find:

$$\begin{aligned} & \begin{array}{c} c \\ | \\ a \end{array} \times \begin{array}{c} b \\ | \\ a \end{array} = \begin{array}{c} cb \\ | \\ a \end{array}, & \begin{array}{c} \mathcal{N} \\ | \\ \mathcal{N} \end{array} \times \begin{array}{c} b \\ | \\ \mathcal{N} \end{array} = \chi(b, c) \begin{array}{c} cb \\ | \\ \mathcal{N} \end{array}, \\ & \begin{array}{c} c \\ | \\ b \end{array} \times \begin{array}{c} \mathcal{N} \\ | \\ a \end{array} = \chi(a, c) \begin{array}{c} \mathcal{N} \\ | \\ b \end{array}, & \begin{array}{c} \mathcal{N} \\ | \\ a \end{array} \times \begin{array}{c} c \\ | \\ a \end{array} = \chi(b, c) \begin{array}{c} \mathcal{N} \\ | \\ a \end{array}, \end{aligned} \quad (5.43)$$

as well as

$$\begin{aligned} & \begin{array}{c} b \\ | \\ \mathcal{N} \end{array} \times \begin{array}{c} \mathcal{N} \\ | \\ \mathcal{N} \end{array} = \begin{array}{c} \mathcal{N} \\ | \\ \mathcal{N} \end{array} \times \begin{array}{c} b \\ | \\ \mathcal{N} \end{array} = \chi(b, ab) \begin{array}{c} \mathcal{N} \\ | \\ \mathcal{N} \end{array}, \\ & \begin{array}{c} \mathcal{N} \\ | \\ b \end{array} \times \begin{array}{c} \mathcal{N} \\ | \\ a \end{array} = \delta_{ac} \sum_d \chi(b, da) \begin{array}{c} d \\ | \\ a \end{array}, \\ & \begin{array}{c} \mathcal{N} \\ | \\ b \end{array} \times \begin{array}{c} \mathcal{N} \\ | \\ \mathcal{N} \end{array} = \sum_d \frac{\chi(a, b)\chi(ab, d)}{\sqrt{2}} \begin{array}{c} d \\ | \\ \mathcal{N} \end{array}. \end{aligned} \quad (5.44)$$

The unit of the algebra is

$$1 = \begin{array}{c} 1 \\ | \\ 1 \end{array} + \begin{array}{c} \eta \\ | \\ \eta \end{array} + \begin{array}{c} \mathcal{N} \\ | \\ \mathcal{N} \end{array}, \quad (5.45)$$

which agrees with (3.13).

Untwisted and twisted point operators fall into representations of the tube algebra. Such representations are in one-to-one correspondence with simple objects in the Drinfeld center of $\mathcal{C}$. Because $\mathcal{C}$ is modular, its Drinfeld center is isomorphic to the product of two copies of the TY fusion category, therefore it has 9 simple lines, listed in [107].

The total circle Hilbert space (2.7) is decomposed as a sum of twisted sectors, each graded by a simple element $a \in \text{Irr}(\mathcal{C})$ of the symmetry category. Similarly, one can show that (see e.g. [145]) any representation of the tube algebra has a grading by elements in $\text{Irr}(\mathcal{C})$: $V = \bigoplus_a V_a$.

This grading is such that elements of the tube algebra that have g as the lower vertical line act as zero on those $V_{g'}$ with $g' \neq g$. In other words, the twist of the tube algebra must be compatible with that of the state it is acting on. When listing the irreducible representations of the tube algebra, we thus explicitly report how each vector space decomposes in graded components. In this case one finds:

- Two 1-dimensional irreducible representations $\mathcal{F}_1^\pm$ with space $V = \mathbb{C}_1$, and

$$\mathcal{F}_1^\pm \left(\begin{array}{c|c} a & 1 \\ \hline & 1 \end{array} \right) = 1, \quad \mathcal{F}_1^\pm \left(\begin{array}{c|c} \mathcal{N} & 1 \\ \hline & 1 \end{array} \right) = \pm\sqrt{2}, \quad (5.46)$$

while all other ones are mapped to zero. They represent genuine local operators, not charged under $\mathbb{Z}_2$, which could be charged under $\mathcal{N}$ but that remain genuine local operators under its action.

- Two 1-dimensional irreducible representations $\mathcal{F}_\eta$ and $\mathcal{F}_\eta^*$ with space $V = \mathbb{C}_\eta$, and

$$\mathcal{F}_\eta \left(\begin{array}{c|c} a & \eta \\ \hline & \eta \end{array} \right) = \chi(a, \eta), \quad \mathcal{F}_\eta \left(\begin{array}{c|c} \mathcal{N} & \eta \\ \hline & \eta \end{array} \right) = i\sqrt{2}, \quad (5.47)$$

while all other ones are mapped to zero. They represent η -twisted operators that are charged under $\mathbb{Z}_2$, and that remain twisted-sector operators under the action of $\mathcal{N}$.

- Four 1-dimensional irreducible representations $\mathcal{F}_\mathcal{N}^\pm$ and $(\mathcal{F}_\mathcal{N}^\pm)^*$ with space $V = \mathbb{C}_\mathcal{N}$, and

$$\begin{aligned} \mathcal{F}_\mathcal{N}^\pm \left(\begin{array}{c|c} 1 & \mathcal{N} \\ \hline & \mathcal{N} \end{array} \right) &= 1, & \mathcal{F}_\mathcal{N}^\pm \left(\begin{array}{c|c} \eta & \mathcal{N} \\ \hline & \mathcal{N} \end{array} \right) &= i, \\ \mathcal{F}_\mathcal{N}^\pm \left(\begin{array}{c|c} \mathcal{N} & 1 \\ \hline & \mathcal{N} \end{array} \right) &= \pm e^{\frac{i\pi}{8}}, & \mathcal{F}_\mathcal{N}^\pm \left(\begin{array}{c|c} \mathcal{N} & \eta \\ \hline & \mathcal{N} \end{array} \right) &= \pm e^{-\frac{3i\pi}{8}}, \end{aligned} \quad (5.48)$$

while all other ones are mapped to zero. They represent $\mathcal{N}$ -twisted operators, charged under η and $\mathcal{N}$, that remain $\mathcal{N}$ -twisted operators under the action of $\mathcal{N}$.

- One 2-dimensional irreducible representation $\mathcal{F}_2$ with space $V = \mathbb{C}_1 \oplus \mathbb{C}_\eta$, and

$$\begin{aligned} \mathcal{F}_2 \left(\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array} \right) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & \mathcal{F}_2 \left(\begin{array}{c|c} \eta & 1 \\ \hline & 1 \end{array} \right) &= \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, & \mathcal{F}_2 \left(\begin{array}{c|c} \mathcal{N} & \eta \\ \hline & 1 \end{array} \right) &= \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} \\ \mathcal{F}_2 \left(\begin{array}{c|c} 1 & \eta \\ \hline & \eta \end{array} \right) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & \mathcal{F}_2 \left(\begin{array}{c|c} \eta & \eta \\ \hline & \eta \end{array} \right) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & \mathcal{F}_2 \left(\begin{array}{c|c} \mathcal{N} & 1 \\ \hline & \eta \end{array} \right) &= \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \end{aligned} \quad (5.49)$$

while all other ones are mapped to zero. It represents a doublet of a genuine local operator $\mathcal{O}_1$ and an η -twisted operator $\mathcal{O}_\eta$ that are exchanged by $\mathcal{N}$. Note that $\mathcal{O}_1$ is charged under $\mathbb{Z}_2$ while $\mathcal{O}_\eta$ is neutral.

Asymmetry of the tube algebra. The inverse bilinear form, in the ordered basis (5.42), reads: $\tilde{K} = \frac{1}{4} \text{diag}(1, 1, 1, 0, 1, 1, 0, -1, 1, -1, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) + \frac{1}{4} \text{offdiag}_{[4,7]} + \frac{1}{4\sqrt{2}} \text{offdiag}_{[11,12]}$, where the first term is a diagonal matrix while $\text{offdiag}_{[i,j]}$ is an off-diagonal matrix with a 1 at positions (i, j) and (j, i) . The symmetrizer can be computed from $\tilde{K}$.

Let us specialize to a state belonging to the 2d irreducible representation (5.49) of the tube algebra. For example, we can choose a pure state $\mathcal{O}_1(z)|0\rangle$ created by the untwisted local operator $\mathcal{O}_1$, which in the Ising model is $\sigma(z)$. In this case the only terms that survive in the symmetrized density matrix are

$$\rho_S = \frac{1}{4} \left(\rho + \begin{array}{c|c} \eta & 1 \\ \hline 1 & \rho \end{array} + \begin{array}{c|c} \eta & 1 \\ \hline 1 & \rho \end{array} + \begin{array}{c|c} \mathcal{N} & \eta \\ \hline 1 & \rho \end{array} + \begin{array}{c|c} \mathcal{N} & \eta \\ \hline 1 & \rho \end{array} \right). \quad (5.50)$$

One immediately obtains $\Delta S = \log(2)$. More generally, the asymmetry of a pure state that belongs to a single d -dimensional irreducible representation of an algebra is $\Delta S = \log(d)$. Indeed, following the discussion in Fig. 4, the symmetrized density matrix is supported on a single block, where it takes the form $\frac{1}{d} \mathbb{1}_{d \times d}$. Because the entropy of the initial state is zero, one obtains that both the von Neumann and Rényi asymmetries are equal to $\log(d)$.

5.3 $\mathbb{Z}_2 \times \mathbb{Z}_2$ Tambara-Yamagami fusion category

The Ising (or $\mathbb{Z}_2$ TY) fusion category does not admit a fiber functor. This can be seen, for example, from the fact that the noninvertible line has non-integer quantum dimension. Therefore, the symmetry carries an 't Hooft anomaly and does not admit strongly-symmetric boundary conditions (in the language of [36]). Furthermore, it does not admit weakly-symmetric boundary conditions either, because the only indecomposable module is the regular one, which is not weakly symmetric.

However, one can take the stacking of two decoupled copies of the Ising symmetry and then look at a subsymmetry therein [36, 139, 158]. The full set of simple lines of the stacking is made of objects of the form $a_1 \boxtimes a_2$ with $a_j \in \{1, \eta, \mathcal{N}\}$. We consider the subcategory generated by

$$\mathcal{L}_1 = 1 \boxtimes 1, \quad \mathcal{L}_\eta = \eta \boxtimes 1, \quad \mathcal{L}_{\eta'} = 1 \boxtimes \eta, \quad \mathcal{L}_{\eta\eta'} = \eta \boxtimes \eta, \quad \mathcal{L}_\mathcal{V} = \mathcal{N} \boxtimes \mathcal{N}. \quad (5.51)$$

Their fusion ring is abelian and given by:

$$\begin{aligned} \mathcal{L}_\eta^2 &= \mathcal{L}_{\eta'}^2 = \mathcal{L}_{\eta\eta'}^2 = \mathcal{L}_1, & \mathcal{L}_\eta \times \mathcal{L}_\mathcal{V} &= \mathcal{L}_{\eta'} \times \mathcal{L}_\mathcal{V} = \mathcal{L}_\mathcal{V}, \\ \mathcal{L}_\eta \times \mathcal{L}_{\eta'} &= \mathcal{L}_{\eta\eta'}, & \mathcal{L}_\mathcal{V} \times \mathcal{L}_\mathcal{V} &= \mathcal{L}_1 + \mathcal{L}_\eta + \mathcal{L}_{\eta'} + \mathcal{L}_{\eta\eta'}. \end{aligned} \quad (5.52)$$

This is in fact the fusion ring of $\mathbb{Z}_2 \times \mathbb{Z}_2$ Tambara-Yamagami fusion categories [157]. In this case, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric bilinear character is $\gamma((s_1, s_2), (s_3, s_4)) = e^{i\pi(s_1 s_3 + s_2 s_4)}$ where each $s_j \in \{0, 1\}$, the Frobenius-Schur indicator is $\kappa_\mathcal{V} = 1$, therefore the nontrivial F -symbols are

$$[F_{i\mathcal{V}j}^\mathcal{V}]_{\mathcal{V}\mathcal{V}} = [F_{\mathcal{V}i\mathcal{V}}^j]_{\mathcal{V}\mathcal{V}} = \gamma(i, j), \quad [F_{\mathcal{V}\mathcal{V}\mathcal{V}}^\mathcal{V}]_{ij} = \frac{1}{2} \gamma(i, j), \quad (5.53)$$

where $i, j \in \mathbb{Z}_2^2$. The four invertible lines have quantum dimension 1, while $\mathcal{L}_\mathcal{V}$ has quantum dimension 2. This fusion category is also equivalent to $\text{Rep}(H_8)$, where H_8 is the Kac-Paljutkin

Hopf algebra [159]. As all categories of representations of a Hopf algebra, it admits at least one fiber functor given by the forgetful functor that maps a representation to its underlying vector space [113]. In this case one can show that this is the only fiber functor.⁶⁴ We conclude that the $\text{Rep}(H_8)$ symmetry admits one (and only one) strongly-symmetric boundary condition. This is our motivation to study this example.

5.3.1 Fusion algebra

The fusion algebra has five 1-dimensional representations that we call $\mathbb{1}$, ε , σ , σ' , σ'' .⁶⁵ The unitary (but not symmetric) S -matrix is

$$S = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & 1 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 1 & 1 & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 1 & 1 & -\sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 2 & -2 & 0 & 0 & 0 \end{pmatrix}. \quad (5.54)$$

The symmetrizer for the fusion algebra is

$$\rho_S = \frac{1}{32} \left(8 \sum_{a \in \mathbb{Z}_2 \times \mathbb{Z}_2} \mathcal{L}_a \rho \mathcal{L}_a - B \rho B + 4 \mathcal{L}_V \rho \mathcal{L}_V \right) \quad \text{with} \quad B = \sum_{a \in \mathbb{Z}_2 \times \mathbb{Z}_2} \mathcal{L}_a. \quad (5.55)$$

Asymmetry of an excited state in CFT. Let us compute the asymmetry of excited states on the line $\mathbb{R}$ produced by Euclidean operator insertions in CFTs. For concreteness, consider the states

$$|\psi\rangle = \left(1 + \lambda \varepsilon(-i\tau)\right) |0\rangle, \quad (5.56)$$

that are neutral under the invertible part of the symmetry. Using (3.26), we obtain exactly the same formula (5.37) as before. In the case of two decoupled copies of the Ising CFT, we can take $\varepsilon(z)$ to be the energy operator of one of the two copies of Ising, so that $\Delta_\varepsilon = 1$. We obtain exactly the same function as in Fig. 11.

5.3.2 Strip algebra

Strongly-symmetric boundary condition. The left-module category with a single object is described by the fusion coefficients $\tilde{N}_a^\cdot = \{1, 1, 1, 1, 2\}$. This means that the $\mathbb{Z}_2^2$ lines $\mathcal{L}_{\eta^m \eta'^n}$ have a single junction to the boundary, while the noninvertible line $\mathcal{L}_V$ has two. Writing the $\tilde{F}$ -symbols in terms of m -symbols,

$$[\tilde{F}_{ab}^\cdot]_{(c; -\rho)(\cdot; \nu\mu)} = (m_{a\mu, b\nu}^{c\rho})^*, \quad (5.57)$$

⁶⁴Indeed, there is a unique maximal haploid algebra object in $\text{Rep}(H_8)$ as shown in [139].

⁶⁵In the case of two decoupled copies of the Ising CFT, the nine conformal primaries split as $1 \supset 1 \boxtimes 1, \varepsilon \boxtimes \varepsilon; \sigma \supset \varepsilon \boxtimes 1, 1 \boxtimes \varepsilon; \sigma \supset \sigma \boxtimes 1, \sigma \boxtimes \varepsilon; \sigma' \supset 1 \boxtimes \sigma, \varepsilon \boxtimes \sigma; \sigma'' \supset \sigma \boxtimes \sigma$.

the nontrivial ones are given by [139, 158]:⁶⁶

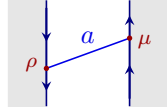
$$\begin{aligned} m_{i,j}^{(ij)} &= K_{ij} , & m_{\eta, \mathcal{V}\mu}^{\mathcal{V}\rho} &= m_{\mathcal{V}\rho, \eta}^{\mathcal{V}\mu} = \sqrt{2} m_{\mathcal{V}\mu, \mathcal{V}\rho}^{\eta} = m_{\eta', \mathcal{V}\rho}^{\mathcal{V}\mu} = m_{\mathcal{V}\mu, \eta'}^{\mathcal{V}\rho} = \sqrt{2} m_{\mathcal{V}\rho, \mathcal{V}\mu}^{\eta'} = M_{\mu\rho} , \\ m_{1, \mathcal{V}\mu}^{\mathcal{V}\rho} &= m_{\mathcal{V}\mu, 1}^{\mathcal{V}\rho} = \sqrt{2} m_{\mathcal{V}\mu, \mathcal{V}\rho}^1 = \delta_{\mu\rho} , & m_{\eta\eta', \mathcal{V}\mu}^{\mathcal{V}\rho} &= m_{\mathcal{V}\mu, \eta\eta'}^{\mathcal{V}\rho} = \sqrt{2} m_{\mathcal{V}\mu, \mathcal{V}\rho}^{\eta\eta'} = (\sigma_z)_{\mu\rho} , \end{aligned} \quad (5.58)$$

in terms of matrices

$$K_{ij} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -i & i \\ 1 & i & 1 & -i \\ 1 & -i & i & 1 \end{pmatrix}, \quad M_{\mu\rho} = \begin{pmatrix} 0 & \xi^{-1} \\ \xi & 0 \end{pmatrix}, \quad \xi = e^{\frac{3\pi i}{4}}, \quad (\sigma_z)_{\mu\rho} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5.59)$$

The m -symbols satisfy the associativity condition (3.35), while the $\tilde{F}$ -symbols satisfy the left-module pentagon equation (A.43). We chose a gauge in which the $\tilde{F}$ -symbols are unitary.

Strip algebra. For the unique strongly-symmetric boundary condition described above, the strip algebra $\text{Strip}_{\text{Rep}(H_8)}(\text{Vec})$ has dimension 8. The fusion coefficients are $\tilde{N}_a = d_a$ and the basis elements are


(5.60)

where $a \in \{1, \eta, \eta', \eta\eta', \mathcal{V}\}$, and for $a = \mathcal{V}$ the junctions are labeled by $\rho, \mu \in \{0, 1\}$. We will use the notation H_i with $i \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and $H_{\mathcal{V}}^{\rho, \mu}$ for the basis elements. Applying (3.37) we find the following algebra product (see also [100]):

$$\begin{aligned} H_{\mathcal{V}}^{s, s'} \times H_{\eta^m \eta'^n} &= e^{-\frac{i\pi}{2}(s-s')(m-n+2mn)} H_{\mathcal{V}}^{s+m+n, s'+m+n}, & H_i \times H_j &= H_{(ij)}, \\ H_{\eta^m \eta'^n} \times H_{\mathcal{V}}^{s, s'} &= e^{\frac{i\pi}{2}(s-s')(m-n+2mn)} H_{\mathcal{V}}^{s+m+n, s'+m+n}, \\ H_{\mathcal{V}}^{s_1, s_2} \times H_{\mathcal{V}}^{s_3, s_4} &= \delta_{s_1 s_3} \delta_{s_2 s_4} \left(H_1 + (-1)^{s_1+s_2} H_{\eta\eta'} \right) \\ &\quad + \delta_{s_1, 1-s_3} \delta_{s_2, 1-s_4} \left(e^{\frac{i\pi}{2}(s_1-s_2)} H_{\eta} + e^{\frac{i\pi}{2}(s_2-s_1)} H_{\eta'} \right). \end{aligned} \quad (5.61)$$

The symmetrizer, computed with (3.9), reads:

$$\rho_S = \frac{1}{8} \left[\sum_{j \in \mathbb{Z}_2 \times \mathbb{Z}_2} H_j \rho H_j + \sum_{s, s' \in \{0, 1\}} H_{\mathcal{V}}^{s, s'} \rho H_{\mathcal{V}}^{s, s'} \right]. \quad (5.62)$$

Strip algebras constructed using strongly-symmetric boundary conditions are Hopf algebras. We review the definition and some properties of Hopf algebras in App. D. In this case one obtains an eight-dimensional Hopf algebra which is neither commutative nor cocommutative. It is known that there is only one such Hopf algebra, the Kac-Paljutkin H_8 Hopf algebra [159]. This algebra is not isomorphic to the group algebra of some group, nor to the dual of a group algebra, and it is the smallest-dimensional algebra with such properties [160, 161]. The coproduct is given by

$$\Delta(H_i) = H_i \otimes H_i, \quad \Delta(H_{\mathcal{V}}^{s, s'}) = \frac{1}{\sqrt{2}} \sum_{t \in \{0, 1\}} H_{\mathcal{V}}^{s, t} \otimes H_{\mathcal{V}}^{t, s'}. \quad (5.63)$$

⁶⁶Which we have adapted to our choice of unitary gauge.

The counit and the antipode map are

$$\epsilon(H_i) = 1, \quad \epsilon(H_{\mathcal{V}}^{s,s'}) = \sqrt{2} \delta^{s,s'}, \quad S(H_i) = H_i, \quad S(H_{\mathcal{V}}^{s,s'}) = H_{\mathcal{V}}^{s',s}. \quad (5.64)$$

In the mathematical literature, the algebra H_8 is usually presented (see, e.g., Sec. 4.2 of [161]) as generated by the elements $1, x, y, z$ where 1 is the unit

$$x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy), \quad xy = yx, \quad xz = zy, \quad yz = zx. \quad (5.65)$$

The basis elements are thus $\{1, x, y, z, xy, xz, yz, xyz\}$. The Hopf algebra structure is

$$\begin{aligned} \epsilon(x) &= 1, & S(x) &= x, & \Delta(x) &= x \otimes x, \\ \epsilon(y) &= 1, & S(y) &= y, & \Delta(y) &= y \otimes y, \\ \epsilon(z) &= 1, & S(z) &= z, & \Delta(z) &= \frac{1}{2}(z \otimes z + z \otimes xz + yz \otimes z - yz \otimes xz). \end{aligned} \quad (5.66)$$

An isomorphism is given by

$$1 = H_1, \quad x = H_\eta, \quad y = H_{\eta'}, \quad z = \frac{1-i}{\sqrt{8}} H_{\mathcal{V}}^{0,0} + \frac{1}{2}(H_{\mathcal{V}}^{1,0} + H_{\mathcal{V}}^{0,1}) + \frac{1+i}{\sqrt{8}} H_{\mathcal{V}}^{1,1}. \quad (5.67)$$

The algebra has four 1-dimensional and one 2-dimensional irreducible representations. The four 1d representations $\mathcal{F}_1^\pm$ and $\mathcal{F}_\eta^\pm$ are

$$\begin{aligned} \mathcal{F}_1^\pm: & \quad H_i \mapsto 1, & H_{\mathcal{V}}^{s,s'} & \mapsto \pm \sqrt{2} \delta^{s,s'}, \\ \mathcal{F}_\eta^\pm: & \quad H_1, H_{\eta\eta'} \mapsto 1, \quad H_\eta, H_{\eta'} \mapsto -1, & H_{\mathcal{V}}^{s,s'} & \mapsto \pm \sqrt{2} \delta^{s,s'} e^{i\pi s s'}. \end{aligned} \quad (5.68)$$

The 2d representation is given by

$$\mathcal{F}_2: \quad H_{\eta^m \eta'^n} \mapsto \begin{pmatrix} (-1)^m & 0 \\ 0 & (-1)^n \end{pmatrix}, \quad H_{\mathcal{V}}^{s,s'} \mapsto \delta^{s,1-s'} \sqrt{2} \begin{pmatrix} 0 & (i)^{s'} \\ (-i)^{s'} & 0 \end{pmatrix}. \quad (5.69)$$

The projector, or MCI (see Fn. 103), to the trivial representation is

$$P_\epsilon = \frac{1}{8} (H_1 + H_\eta + H_{\eta'} + H_{\eta\eta'} + \sqrt{2} H_{\mathcal{V}}^{0,0} + \sqrt{2} H_{\mathcal{V}}^{1,1}). \quad (5.70)$$

This is the only MCI on which the counit ϵ takes value 1. Applying the formula (3.53) for the symmetrizer with respect to a Hopf algebra, one reproduces the result (5.62) obtained using the general formula.

5.3.3 Example: the (Ising)² CFT

Consider the theory given by two decoupled copies of the Ising CFT (which has central charge $c = 1$ and is equivalent to an orbifold of the free compact boson at radius $\sqrt{2}$ times the self-dual radius). We study a subsymmetry of this theory, given by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Tambara-Yamagami fusion category for the specific choice of bicharacter and Frobenius-Schur indicator already made after (5.52), and equivalent to the $\text{Rep}(H_8)$ fusion category. This category admits a strongly-symmetric boundary condition, with which we construct the strip algebra H_8 .

Case 1 $\boxtimes 1 + \lambda \varepsilon \boxtimes 1$. We study the state on the real line prepared by inserting the operator $\mathcal{O}(z) = 1 \boxtimes 1 + \lambda \varepsilon(z) \boxtimes 1$ at position $z = x - i\tau$. Since ε has scaling dimension $\Delta_\varepsilon = 1$, the coefficient λ has dimensions of a length. The path-integral that prepares the state is done on a half-plane. We study the asymmetry of the reduced density matrix on an interval $A = [-\frac{\ell}{2}, \frac{\ell}{2}]$, using the setup of Sec. 3.3.1: to prepare the state we cut out two discs of size ε around the entangling surface centered at $(\pm\frac{\ell}{2}, 0)$ and impose the strongly-symmetric boundary condition $|\mathcal{B}\rangle$ there. We compute the second Rényi asymmetry using the symmetrization formula derived above. This produces

$$\text{tr}(\rho_S^2) = \frac{1}{8} \text{tr} \left[\sum_{j \in \mathbb{Z}_2 \times \mathbb{Z}_2} \rho H_j \rho H_j + \sum_{s, s'} \rho H_{\mathcal{V}}^{s, s'} \rho H_{\mathcal{V}}^{s, s'} \right] = \frac{1}{2} \text{tr}(\rho^2 + \rho \rho') . \quad (5.71)$$

Here ρ' is obtained by swiping a line $\mathcal{L}_{\mathcal{V}}$ across the whole density matrix, transforming the operator $\mathcal{O} = 1 \boxtimes 1 + \lambda \varepsilon \boxtimes 1$ to $\mathcal{O}' = 1 \boxtimes 1 - \lambda \varepsilon \boxtimes 1$. We thus get:

$$\Delta S^{(2)}[\rho] = -\log \frac{\text{tr}(\rho_S^2)}{\text{tr}(\rho^2)} = \log 2 - \log \left[1 + \frac{\text{tr}(\rho \rho')}{\text{tr}(\rho^2)} \right] . \quad (5.72)$$

We can express $\text{tr}(\rho \rho') / \text{tr}(\rho^2)$ as a ratio of partition functions on a Riemann surface obtained by gluing two copies of the complex plane with the entangling discs cut out. In the $\varepsilon \rightarrow 0$ limit, each entangling disc is replaced by a branch point and the partition functions are multiplied by the g -factor $g_{\mathcal{B}} = \langle \mathcal{B}|0 \rangle$ (we already discussed such a procedure in Sec. 3.3.3). The g -factors at numerator and denominator are equal and cancel out. The same procedure is discussed in the paragraph above (5.26). In turn, the ratio of partition functions is equal to a ratio of correlation functions on a Riemann surface $\mathcal{M}_2$ with two branch points. We denote a point on $\mathcal{M}_2$ by z_j where $z \in \mathbb{C}$ denotes a point on the complex plane while $j \in \{1, 2\}$ labels the sheet. We get:

$$\frac{\text{tr}(\rho \rho')}{\text{tr}(\rho^2)} = \frac{\left\langle 1 + 2\lambda^2 \left(\varepsilon(z_1) \varepsilon(\bar{z}_1) - \varepsilon(z_1) \varepsilon(z_2) - \varepsilon(z_1) \varepsilon(\bar{z}_2) \right) + \lambda^4 \varepsilon(z_1) \varepsilon(\bar{z}_1) \varepsilon(z_2) \varepsilon(\bar{z}_2) \right\rangle_{\mathcal{M}_2}}{\left\langle 1 + 2\lambda^2 \left(\varepsilon(z_1) \varepsilon(\bar{z}_1) + \varepsilon(z_1) \varepsilon(z_2) + \varepsilon(z_1) \varepsilon(\bar{z}_2) \right) + \lambda^4 \varepsilon(z_1) \varepsilon(\bar{z}_1) \varepsilon(z_2) \varepsilon(\bar{z}_2) \right\rangle_{\mathcal{M}_2}} . \quad (5.73)$$

We used the symmetry that cyclically exchanges the sheets, and that the 3-point function of ε vanishes. We can also write

$$\frac{\text{tr}(\rho_S^2)}{\text{tr}(\rho^2)} = 1 - \frac{2\lambda^2 \left\langle \varepsilon(z_1) \varepsilon(z_2) + \varepsilon(z_1) \varepsilon(\bar{z}_2) \right\rangle_{\mathcal{M}_2}}{\left\langle 1 + 2\lambda^2 \left(\varepsilon(z_1) \varepsilon(\bar{z}_1) + \varepsilon(z_1) \varepsilon(z_2) + \varepsilon(z_1) \varepsilon(\bar{z}_2) \right) + \lambda^4 \varepsilon(z_1) \varepsilon(\bar{z}_1) \varepsilon(z_2) \varepsilon(\bar{z}_2) \right\rangle_{\mathcal{M}_2}} . \quad (5.74)$$

Correlation functions on $\mathcal{M}_2$ are obtained from the ones on the plane by a conformal transformation:

$$z_1 \mapsto \sqrt{\frac{z - \ell/2}{z + \ell/2}} \equiv g(z) , \quad z_2 \mapsto -g(z) , \quad (5.75)$$

where the square root is taken with its branch cut along the negative real axis. The 2-point functions are then:

$$\langle \varepsilon(z_1) \varepsilon(\bar{z}_{1,2}) \rangle_{\mathcal{M}_2} = \frac{|g'(z)|^{2\Delta_\varepsilon}}{|g(z) \mp g(\bar{z})|^{2\Delta_\varepsilon}} , \quad \langle \varepsilon(z_1) \varepsilon(z_2) \rangle_{\mathcal{M}_2} = \frac{|g'(z)|^{2\Delta_\varepsilon}}{|2g(z)|^{2\Delta_\varepsilon}} . \quad (5.76)$$

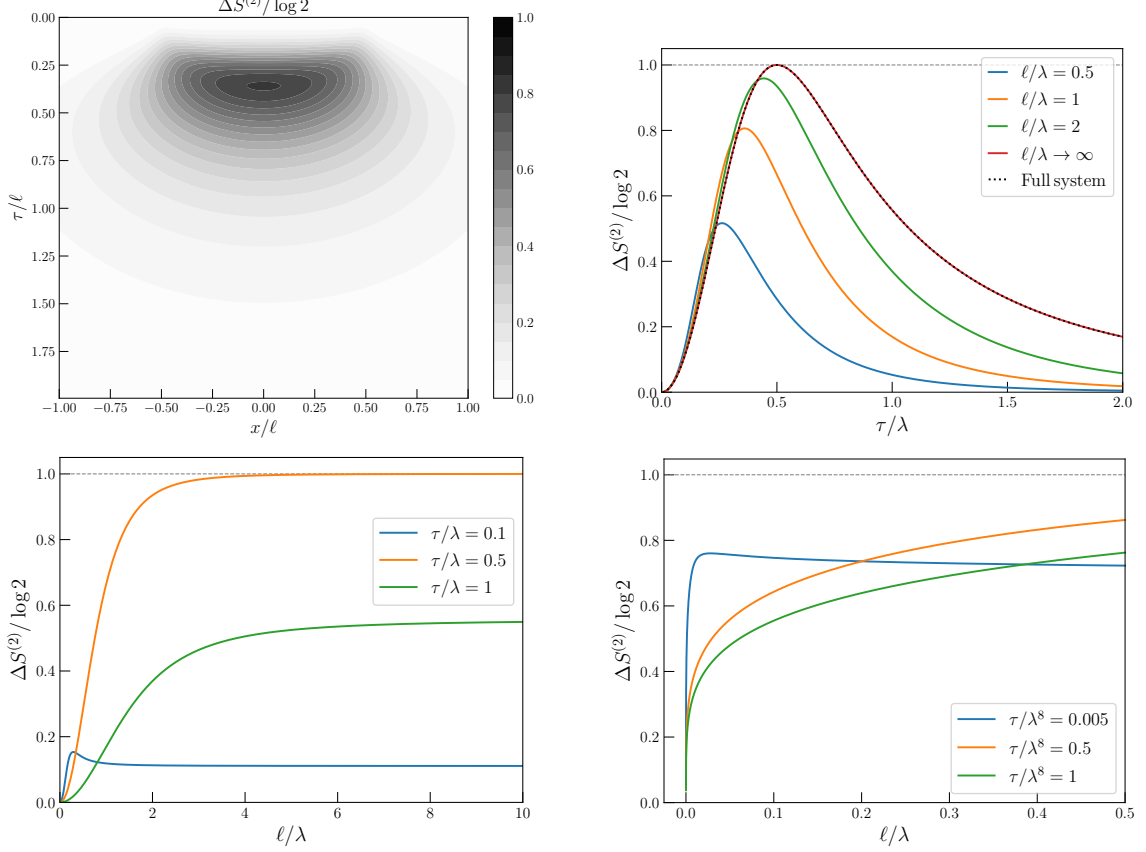


Figure 12: Top left: density plot of the second Rényi asymmetry of the state $(1 \boxtimes 1 + \lambda \varepsilon(z) \boxtimes 1)|0\rangle$ for $\lambda = \ell = 1$ and with $z = x - i\tau$. Top right: second Rényi asymmetry when the field is inserted at $z = -i\tau$ as a function of τ , for different values of ℓ (in units of λ). Bottom left: second Rényi asymmetry as a function of ℓ , for different positions of the field insertion at $z = -i\tau$. Bottom right: second Rényi asymmetry of the state $(1 + \lambda \sigma(-i\tau))|0\rangle$ in the Ising CFT with respect to the ordinary $\mathbb{Z}_2$ spin-flip symmetry (that acts as the strip algebra of the $\text{Vec}_{\mathbb{Z}_2}$ fusion category on the interval Hilbert space with symmetric boundary conditions).

The 4-point function on $\mathcal{M}_2$ is related to the one on the complex plane as

$$\langle \varepsilon(z_1) \varepsilon(\bar{z}_1) \varepsilon(z_2) \varepsilon(\bar{z}_2) \rangle_{\mathcal{M}_2} = |g'(z)|^{4\Delta_\varepsilon} \langle \varepsilon(g(z)) \varepsilon(g(\bar{z})) \varepsilon(-g(z)) \varepsilon(-g(\bar{z})) \rangle_{\mathbb{C}}, \quad (5.77)$$

and the latter is given by [162]:

$$\langle \varepsilon(w_1) \varepsilon(w_2) \varepsilon(w_3) \varepsilon(w_4) \rangle_{\mathbb{C}} = \left| \frac{1 - \chi + \chi^2}{w_{14} w_{23} \chi} \right|^2, \quad (5.78)$$

where $w_{ij} = w_i - w_j$ and $\chi = (w_{12} w_{34}) / (w_{13} w_{24})$ is the conformal cross-ratio.

We report a few plots of the resulting second Rényi asymmetry in Fig. 12. Notice that the asymmetry takes values in the interval $[0, \log 2]$. The asymmetry vanishes in both limits $\tau \rightarrow \infty$ and $\tau \rightarrow 0$: indeed the state is dominated by $1 \boxtimes 1 |0\rangle$ or by $\varepsilon \boxtimes 1 |0\rangle$, respectively.

Large subsystem limit. When $\ell \rightarrow \infty$, the branch cut extends along the real line and correlation functions factorize. The 2-point functions behave as

$$\langle \varepsilon(z_1) \varepsilon(\bar{z}_1) \rangle_{\mathcal{M}_2} \sim \langle \varepsilon(z_1) \varepsilon(z_2) \rangle_{\mathcal{M}_2} \sim \frac{1}{\ell^2}, \quad \langle \varepsilon(z_1) \varepsilon(\bar{z}_2) \rangle_{\mathcal{M}_2} \sim \frac{1}{(2\tau)^2}, \quad (5.79)$$

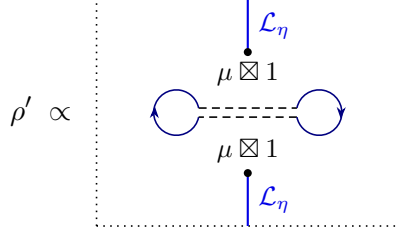


Figure 13: Path-integral representation of ρ' , described above (5.82).

while the 4-point function factorizes through the identity channel:

$$\langle \varepsilon(z_1) \varepsilon(\bar{z}_1) \varepsilon(z_2) \varepsilon(\bar{z}_2) \rangle_{\mathcal{M}_2} \sim \frac{1}{(2\tau)^4} . \quad (5.80)$$

This gives the asymptotic behavior of the second Rényi asymmetry:

$$\Delta S^{(2)} = \log \frac{[1 + (\frac{\lambda}{2\tau})^2]^2}{1 + (\frac{\lambda}{2\tau})^4} + \mathcal{O}(\lambda^2/\ell^2) . \quad (5.81)$$

This expression matches with the Rényi asymmetry (5.37) with respect to the fusion algebra of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ TY symmetry acting on the circle Hilbert space of the full system.

Non-monotonicity of Rényi asymmetry. It is interesting to notice that Rényi asymmetries are not always monotonic in the size of the subsystem. Indeed, when the field is inserted at $z = -i\tau$ and $\tau < \lambda$, the second Rényi asymmetry is not monotonic in ℓ , as shown in the bottom left panel of Fig. 12. This raises the question of whether such a counterintuitive behavior is a distinctive feature of asymmetries associated with noninvertible symmetries. The answer is negative: a simpler example involving a group-like symmetry also exhibits non-monotonic behavior. Specifically, we consider the state $(1 + \lambda\sigma(z))|0\rangle$ in the Ising CFT (a single copy) and compute its second Rényi asymmetry with respect to the $\mathbb{Z}_2$ spin-flip symmetry over a subsystem, imposing symmetric conformal boundary conditions at the entangling surface – thus realizing a strip algebra isomorphic to $\mathbb{C}[\mathbb{Z}_2]$. The computation closely parallels the one described above: the second Rényi asymmetry is given by (5.72) and (5.73), with ε replaced by σ and λ assigned length dimension $\Delta_\sigma = \frac{1}{8}$. The relevant 4-point function of σ is provided in (5.83). In the bottom right panel of Fig. 12 we plot the second Rényi asymmetry as a function of the subsystem size ℓ , for a field insertion at $z = -i\tau$, showing a non-monotonic behavior for small values of τ .

Case $\sigma \boxtimes 1$. Consider now the state created by inserting the operator $\mathcal{O}(z) = \sigma(z) \boxtimes 1$ with dimension $\Delta_\sigma = \frac{1}{8}$ at $z = x - i\tau$. As before, let ρ be the reduced density matrix on $A = [-\frac{\ell}{2}, \frac{\ell}{2}]$ obtained using the strongly-symmetric boundary condition. The symmetrized matrix ρ_S is obtained applying (5.62). The invertible lines can freely pass through ρ_S , while the noninvertible one changes the untwisted operator σ to the twisted (or disorder) operator μ (with a line $\mathcal{L}_\eta$ attached). Therefore $\rho_S = \frac{1}{2}(\rho + \rho')$, where ρ' is the reduced density matrix of

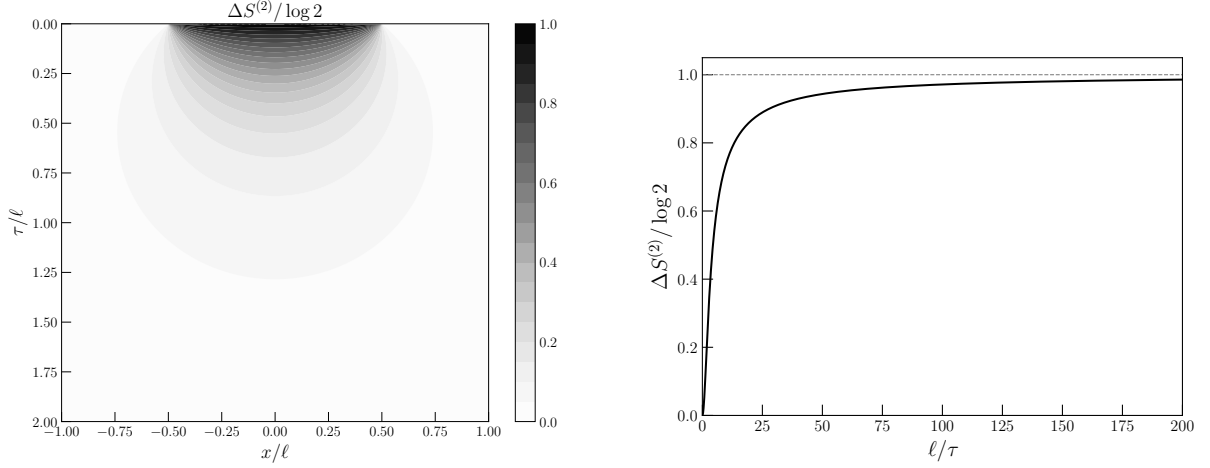


Figure 14: Second Rényi asymmetry of the state $(\sigma(z) \boxtimes 1)|0\rangle$ in two copies of the Ising CFT. Left: density plot of the Rényi asymmetry, as a function of the Euclidean insertion point $z = x - i\tau$. Right: Rényi asymmetry as a function of the subsystem size ℓ (measured in units of τ) for a field insertion at $z = -i\tau$.

the state created by $\mathcal{O}' = \mu \boxtimes 1$. Such a density matrix contains a line $\mathcal{L}_\eta$ that goes through infinity, as depicted in Fig. 13.

The second Rényi asymmetry takes the form

$$\Delta S^{(2)}[\rho] = \log 2 - \log \left[1 + \frac{\langle \sigma(z_1) \sigma(\bar{z}_1) \mu(z_2) \mu(\bar{z}_2) \rangle_{\mathcal{M}_2}}{\langle \sigma(z_1) \sigma(\bar{z}_1) \sigma(z_2) \sigma(\bar{z}_2) \rangle_{\mathcal{M}_2}} \right], \quad (5.82)$$

where we have used the equality of 4-point functions $\langle \sigma^4 \rangle = \langle \mu^4 \rangle$ as a consequence of Kramers-Wannier duality. On the plane one finds the following 4-point functions [163]:

$$\begin{aligned} \langle \sigma(w_1) \sigma(w_2) \sigma(w_3) \sigma(w_4) \rangle_{\mathbb{C}} &= \frac{(1 + |\chi| + |1 - \chi|)^{1/2}}{\sqrt{2} |w_{14} w_{23} \chi|^{1/4}} = \frac{|1 + \sqrt{\chi}| + |1 - \sqrt{\chi}|}{2 |w_{14} w_{23} \chi|^{1/4}}, \\ \langle \sigma(w_1) \sigma(w_2) \mu(w_3) \mu(w_4) \rangle_{\mathbb{C}} &= \frac{(1 - |\chi| + |1 - \chi|)^{1/2}}{\sqrt{2} |w_{14} w_{23} \chi|^{1/4}}, \end{aligned} \quad (5.83)$$

where χ is the cross-ratio. The fields σ and μ , because of duality, have the same conformal dimension, therefore the conformal factors in the map between $\mathcal{M}_2$ and $\mathbb{C}$ simplify. This leads to the expression:

$$\Delta S^{(2)} = \log 2 - \log \left[1 + \left(\frac{1 - |\chi| + |1 - \chi|}{1 + |\chi| + |1 - \chi|} \right)^{1/2} \right], \quad (5.84)$$

where $w_1 = g(z)$, $w_2 = g(\bar{z})$, $w_3 = -g(z)$, $w_4 = -g(\bar{z})$. At finite subsystem size ℓ , the asymmetry vanishes in the limit $\tau \rightarrow \infty$ in which the field is inserted in the far Euclidean past. This is because symmetry breaking spreads over $\mathbb{R}$ and thus the subsystem only catches an infinitesimal fraction thereof. On the other hand, in the limit $\tau \rightarrow 0$ in which the field is inserted close to the real axis, the asymmetry is equal to $\log(2)$ if $x \in A$ and it vanishes if $x \notin A$. Finally, in the limit $\ell \rightarrow \infty$ that the subsystem size goes to infinity, the asymmetry is equal to $\log(2)$ irrespective of the insertion point. Plots of the Rényi asymmetry are in Fig. 14.

5.4 Ising symmetry – interval Hilbert space

As already mentioned in Sec. 5.2.1, the Ising fusion category admits only one indecomposable module category (that can describe boundary conditions): the regular one. This is physically realized in the Ising CFT, which is the setting of our computations here. In order to study the asymmetry on a subsystem, then, one has to employ the general procedure of Sec. 3.3.3. In the regular module the $\tilde{F}$ -symbols are equal to the F -symbols, and we indicate the boundary conditions as $\{+, -, f\}$. Recall that the Ising symmetry has a multiplicity-free fusion ring, so no junction labels are necessary. Hence, we use the notation

$$H_{r,m}^{s,n}(a) \equiv \begin{array}{c} \begin{array}{|c|c|} \hline s & n \\ \hline \end{array} \\ \begin{array}{c} \downarrow \quad \quad \downarrow \\ \text{---} a \text{---} \\ \uparrow \quad \quad \uparrow \\ \begin{array}{|c|c|} \hline r & m \\ \hline \end{array} \end{array} \end{array} \quad (5.85)$$

for basis elements of the strip algebra, which has dimension 34. The formula (2.24) for the strip-algebra product simplifies considerably, given the fact that every F -symbol is a symmetric and involutive matrix, and that every line is self-dual. We will not report it here, though.

The symmetrizer is given by a long expression. Let us write it sector by sector, in terms of the boundary conditions (r, m) for the reduced density matrix ρ .

- For a density matrix in the $(+, +)$ sector we have:

$$\rho_S = \frac{1}{3} \left(H_{++}^{++}(1) \rho H_{++}^{++}(1) + H_{++}^{--}(\eta) \rho H_{++}^{--}(\eta) + \frac{1}{\sqrt{2}} H_{++}^{ff}(\mathcal{N}) \rho H_{ff}^{++}(\mathcal{N}) \right). \quad (5.86)$$

- In the $(-, -)$ sector:

$$\rho_S = \frac{1}{3} \left(H_{--}^{--}(1) \rho H_{--}^{--}(1) + H_{--}^{++}(\eta) \rho H_{--}^{++}(\eta) + \frac{1}{\sqrt{2}} H_{--}^{ff}(\mathcal{N}) \rho H_{ff}^{--}(\mathcal{N}) \right). \quad (5.87)$$

- In the $(+, -)$ sector:

$$\rho_S = \frac{1}{3} \left(H_{+-}^{+-}(1) \rho H_{+-}^{+-}(1) + H_{+-}^{+}(\eta) \rho H_{+-}^{+}(\eta) + \frac{1}{\sqrt{2}} H_{+-}^{ff}(\mathcal{N}) \rho H_{ff}^{+-}(\mathcal{N}) \right). \quad (5.88)$$

- In the $(f, +)$ sector:

$$\rho_S = \frac{1}{4} \left(H_{f+}^{f+}(1) \rho H_{f+}^{f+}(1) + H_{f+}^{f-}(\eta) \rho H_{f+}^{f-}(\eta) + H_{f+}^{+f}(\mathcal{N}) \rho H_{f+}^{+f}(\mathcal{N}) + H_{f+}^{-f}(\mathcal{N}) \rho H_{f+}^{-f}(\mathcal{N}) \right). \quad (5.89)$$

- In the $(f, -)$ sector:

$$\rho_S = \frac{1}{4} \left(H_{f-}^{f-}(1) \rho H_{f-}^{f-}(1) + H_{f-}^{f+}(\eta) \rho H_{f-}^{f+}(\eta) + H_{f-}^{-f}(\mathcal{N}) \rho H_{f-}^{-f}(\mathcal{N}) + H_{f-}^{+f}(\mathcal{N}) \rho H_{f-}^{+f}(\mathcal{N}) \right). \quad (5.90)$$

- In the (f, f) sector:

$$\rho_S = \frac{1}{6} \left(H_{ff}^{ff}(1) \rho H_{ff}^{ff}(1) + H_{ff}^{ff}(\eta) \rho H_{ff}^{ff}(\eta) \right) + \frac{1}{3\sqrt{2}} \sum_{s, s' \in \{+, -\}} H_{ff}^{ss'}(\mathcal{N}) \rho H_{ss'}^{ff}(\mathcal{N}). \quad (5.91)$$

The sectors not reported are easily obtained by exchanging left and right boundary conditions.

We now proceed to compute the von Neumann and Rényi asymmetries of the vacuum (i.e. without operator insertions). The procedure is analogous to the group-like case in Sec. 5.1.2.

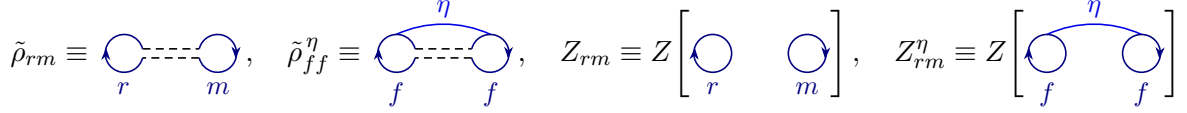


Figure 15: Path-integral definitions of various density matrices that enter in the symmetrization with respect to the strip algebra of the Ising fusion category with regular boundary conditions.

Consider first a generic density matrix ρ in the $(+, +)$ sector, that we indicate as ρ_{++} (as in Fig. 9 left), to which we apply the symmetrizer (5.86). Using the strip algebra products $[H_{++}^{++}(1)]^2 = H_{++}^{++}(\eta)H_{++}^{--}(\eta) = \frac{1}{\sqrt{2}}H_{ff}^{++}(\mathcal{N})H_{++}^{ff}(\mathcal{N}) = H_{++}^{++}(1)$ and orthogonality between different boundary conditions, we find $\text{tr}[(\rho_{++})_{\mathbb{S}}^n] = 3^{1-n} \text{tr}(\rho_{++}^n)$. Therefore the asymmetry of a state belonging to this sector is always

$$\Delta S^{(n)}[\rho_{++}] = \Delta S[\rho_{++}] = \log(3) . \quad (5.92)$$

Exactly the same result is found for ρ_{--} and ρ_{+-} . This result is valid for any state, and in particular also for the vacuum. By the same argument one finds $\text{tr}[(\rho_{f+})_{\mathbb{S}}^n] = 4^{1-n} \text{tr}(\rho_{f+}^n)$ and thus, for a generic density matrix ρ_{f+} :

$$\Delta S^{(n)}[\rho_{f+}] = \Delta S[\rho_{f+}] = \log(4) . \quad (5.93)$$

The same result is found for ρ_{f-} . Consider then the reduced density matrix of the vacuum in the (f, f) sector, that we indicate as ρ_{ff} , and whose symmetrizer is (5.91). When computing $\text{tr}[(\rho_{ff})_{\mathbb{S}}^n]$, to treat the first term in (5.91) one uses the products $H_{ff}^{ff}(a)H_{ff}^{ff}(b) = H_{ff}^{ff}(ab)$ for $a, b \in \{1, \eta\} \cong \mathbb{Z}_2$ and the fact that $H_{ff}^{ff}(a)$ commutes with ρ_{ff} . Since the number of lines $\mathcal{L}_\eta$ is always even, they annihilate among themselves. For the second term one uses $H_{ss'}^{ff}(\mathcal{N})H_{ff}^{ss'}(\mathcal{N}) = \frac{1}{\sqrt{2}}(H_{ff}^{ff}(1) + (ss')H_{ff}^{ff}(\eta))$ with $s, s' = \pm 1$. When expanding $\text{tr}[(\rho_{ff})_{\mathbb{S}}^n]$, those terms with an odd number of lines $\mathcal{L}_\eta$ cancel out. Eventually $\text{tr}[(\rho_{ff})_{\mathbb{S}}^n] = 3^{1-n} \text{tr}(\rho_{ff}^n)$ and hence

$$\Delta S^{(n)}[\rho_{ff}] = \Delta S[\rho_{ff}] = \log(3) . \quad (5.94)$$

In all cases, the asymmetries of the vacuum in a subsystem with fixed simple boundary conditions attain a nonzero value. This is expected since each simple boundary condition breaks the symmetry.

On the other hand, let us apply the averaging procedure of Sec. 3.3.3 to the vacuum state and consider the matrix (recall that $\tilde{d}_m = d_m$ in the regular module):

$$\rho = \frac{1}{D^2} \sum_{rm} d_r d_m \rho_{rm} . \quad (5.95)$$

Here ρ_{rm} is the reduced density matrix of the vacuum with boundary conditions (r, m) , as in Fig. 9 left but without insertions, the sum is over $r, m \in \{+, -, f\}$, and $D = \sum_m d_m$. It is convenient to write $\rho_{rm} = Z_{rm}^{-1} \tilde{\rho}_{rm}$ where, as in Sec. 3.3.3, $\tilde{\rho}_{rm}$ is the unnormalized density matrix while Z_{rm} is its trace. They are depicted in Fig. 15. Since $\tilde{\rho}_{rm}$ come from the vacuum,

the lines can move through them and the effect of the symmetrizer can be written as⁶⁷

$$\begin{aligned}
(\tilde{\rho}_{++})_{\mathbf{S}} &= (\tilde{\rho}_{--})_{\mathbf{S}} = \frac{1}{3} \left(\tilde{\rho}_{++} + \tilde{\rho}_{--} + \frac{1}{2} \tilde{\rho}_{ff} + \frac{1}{2} \tilde{\rho}_{ff}^{\eta} \right), \\
(\tilde{\rho}_{+-})_{\mathbf{S}} &= (\tilde{\rho}_{-+})_{\mathbf{S}} = \frac{1}{3} \left(\tilde{\rho}_{+-} + \tilde{\rho}_{-+} + \frac{1}{2} \tilde{\rho}_{ff} - \frac{1}{2} \tilde{\rho}_{ff}^{\eta} \right), \\
(\tilde{\rho}_{f+})_{\mathbf{S}} &= (\tilde{\rho}_{f-})_{\mathbf{S}} = (\tilde{\rho}_{+f})_{\mathbf{S}} = (\tilde{\rho}_{-f})_{\mathbf{S}} = \frac{1}{4} \left(\tilde{\rho}_{f+} + \tilde{\rho}_{f-} + \tilde{\rho}_{+f} + \tilde{\rho}_{-f} \right), \\
(\tilde{\rho}_{ff})_{\mathbf{S}} &= \frac{1}{3} \left(\tilde{\rho}_{++} + \tilde{\rho}_{--} + \tilde{\rho}_{+-} + \tilde{\rho}_{-+} + \tilde{\rho}_{ff} \right).
\end{aligned} \tag{5.96}$$

Here $\tilde{\rho}_{ff}^{\eta}$ is a density matrix that contains a line $\mathcal{L}_{\eta}$ connecting the two boundaries of type f , as described in Fig. 15. To proceed, we use that $\lim_{\varepsilon \rightarrow 0} Z_{rm} \propto d_r d_m$, proven in Sec. 3.3.3. In the $\varepsilon \rightarrow 0$ limit, then, one immediately obtains the weak identity $\rho_{\mathbf{S}} = \rho$, valid when inserted into traces. We thus explicitly confirm that, in the limit, the asymmetries of the reduced vacuum density matrix vanish: $\Delta S^{(n)}[\rho] = \Delta S[\rho] = 0$.

Case 1 + $\lambda \varepsilon$. We now wish to compute the asymmetry of an excited state $\mathcal{O}(z)|0\rangle$ obtained by the operator insertion $\mathcal{O}(z) = 1 + \lambda \varepsilon(z)$ at $z = -i\tau$, along with its mirror image $\mathcal{O}^{\dagger}(\bar{z})$. The reduced density matrix is as in (5.95), but now ρ_{rm} contain the insertions. When we swipe a line $\mathcal{L}_{\mathcal{N}}$ through them, this time $\mathcal{O} \mapsto \mathcal{O}' = 1 - \lambda \varepsilon$ and we call ρ'_{rm} the reduced density matrices with that insertion. The action of the symmetrizer can be written as

$$\begin{aligned}
(\tilde{\rho}_{++})_{\mathbf{S}} &= (\tilde{\rho}_{--})_{\mathbf{S}} = \frac{\tilde{\rho}_{++} + \tilde{\rho}_{--}}{3} + \frac{\tilde{\rho}'_{ff} + \tilde{\rho}^{\eta'}_{ff}}{6}, & (\tilde{\rho}_{f+})_{\mathbf{S}} &= (\tilde{\rho}_{f-})_{\mathbf{S}} = \frac{\tilde{\rho}_{f+} + \tilde{\rho}_{f-} + \tilde{\rho}'_{+f} + \tilde{\rho}'_{-f}}{4}, \\
(\tilde{\rho}_{+-})_{\mathbf{S}} &= (\tilde{\rho}_{-+})_{\mathbf{S}} = \frac{\tilde{\rho}_{+-} + \tilde{\rho}_{-+}}{3} + \frac{\tilde{\rho}'_{ff} - \tilde{\rho}^{\eta'}_{ff}}{6}, & (\tilde{\rho}_{+f})_{\mathbf{S}} &= (\tilde{\rho}_{-f})_{\mathbf{S}} = \frac{\tilde{\rho}'_{f+} + \tilde{\rho}'_{f-} + \tilde{\rho}_{+f} + \tilde{\rho}_{-f}}{4}, \\
(\tilde{\rho}_{ff})_{\mathbf{S}} &= \frac{\tilde{\rho}'_{++} + \tilde{\rho}'_{--} + \tilde{\rho}'_{+-} + \tilde{\rho}'_{-+} + \tilde{\rho}_{ff}}{3}.
\end{aligned} \tag{5.97}$$

This allows one to compute traces of powers of $\rho_{\mathbf{S}}$. For example, in the $\varepsilon \rightarrow 0$ limit, one can write the compact expression

$$\lim_{\varepsilon \rightarrow 0} \frac{\text{tr}(\rho_{\mathbf{S}}^2)}{\text{tr}(\rho^2)} = \frac{1}{3} \frac{\text{tr}[(3 - \sqrt{2})\tilde{\rho}^2 + \sqrt{2}\tilde{\rho}\tilde{\rho}']}{\text{tr}(\tilde{\rho}^2)}. \tag{5.98}$$

The notation on the RHS should merely be interpreted as an instruction to construct a branched Riemann surface, obtained from the gluing of unnormalized density matrices followed by the $\varepsilon \rightarrow 0$ limit. The expression can be rewritten in terms of correlation functions on a two-sheeted Riemann surface $\mathcal{M}_2$:

$$\frac{\text{tr}(\rho_{\mathbf{S}}^2)}{\text{tr}(\rho^2)} = 1 - \frac{\frac{4\sqrt{2}}{3} \lambda^2 \left\langle \varepsilon(z_1)\varepsilon(z_2) + \varepsilon(z_1)\varepsilon(\bar{z}_2) \right\rangle_{\mathcal{M}_2}}{\left\langle 1 + 2\lambda^2 \left(\varepsilon(z_1)\varepsilon(\bar{z}_1) + \varepsilon(z_1)\varepsilon(z_2) + \varepsilon(z_1)\varepsilon(\bar{z}_2) \right) + \lambda^4 \varepsilon(z_1)\varepsilon(\bar{z}_1)\varepsilon(z_2)\varepsilon(\bar{z}_2) \right\rangle_{\mathcal{M}_2}} \tag{5.99}$$

⁶⁷Using the tube algebra action (B.10) on boundary conditions, one can prove various relations among the vacuum traces, for $n \in \mathbb{Z}_{>0}$: $\text{tr}(\tilde{\rho}_{++}^n) = \text{tr}(\tilde{\rho}_{--}^n)$, $\text{tr}(\tilde{\rho}_{+-}^n) = \text{tr}(\tilde{\rho}_{-+}^n)$, $\text{tr}(\tilde{\rho}_{ff}^n) = \text{tr}(\tilde{\rho}_{++}^n + \tilde{\rho}_{--}^n)$, $\text{tr}(\tilde{\rho}_{ff}^{n-1} \tilde{\rho}_{ff}^{\eta}) = \text{tr}(\tilde{\rho}_{++}^n - \tilde{\rho}_{--}^n)$, as well as $\text{tr}(\tilde{\rho}_{f+}^n) = \text{tr}(\tilde{\rho}_{f-}^n) = \text{tr}(\tilde{\rho}_{+f}^n) = \text{tr}(\tilde{\rho}_{-f}^n)$. Besides, one finds the operator relations $\tilde{\rho}_{ff}^{\eta} \tilde{\rho}_{ff}^{\eta} = \tilde{\rho}_{ff} \tilde{\rho}_{ff}$ and $[\tilde{\rho}_{ff}, \tilde{\rho}_{ff}^{\eta}] = 0$. One can then verify that $\mathbf{S}$ is trace-preserving.

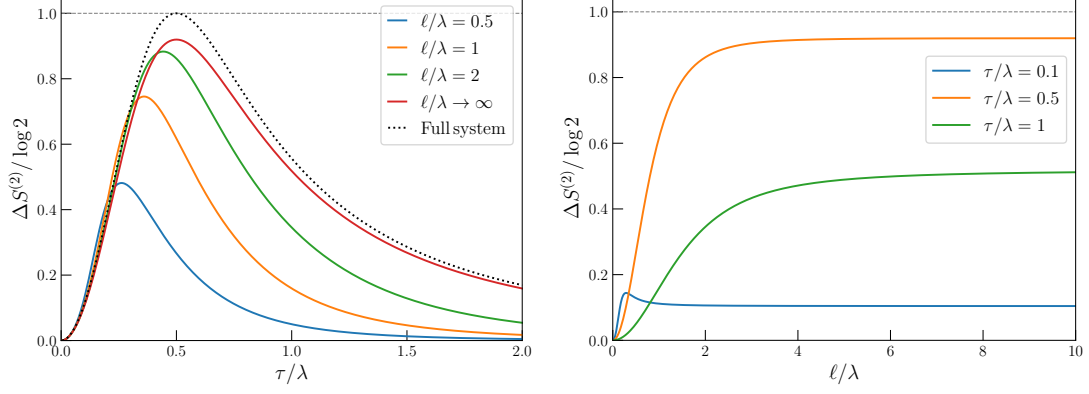


Figure 16: Second Rényi asymmetry of the state $(1 + \lambda \varepsilon(z))|0\rangle$ in the Ising CFT, with “averaged” boundary conditions at the entangling surface. Left: asymmetry when the field is inserted at $z = -i\tau$, as a function of τ , for different values of ℓ (in units of λ). We compare it with the asymmetry of the full system (dotted line) computed with respect to the fusion algebra. Right: asymmetry as a function of ℓ , for different positions of the field insertion at $z = -i\tau$.

where we used the same notation as in Sec. 5.3.3 and cyclic permutation symmetry of the sheets. This can be used to obtain the second Rényi asymmetry, of which we present some plots in Fig. 16 (more generally, the n -th Rényi asymmetry can be written in terms of correlation functions up to $2n$ points). Comparing (5.99) with (5.74), the two expressions are very similar. Indeed, the second Rényi asymmetry $\Delta S^{(2)}[\rho] = -\log[\text{tr}(\rho_S^2)/\text{tr}(\rho^2)]$ of $(1 + \lambda \varepsilon)|0\rangle$ in the Ising CFT is very similar to (in fact, just slightly smaller than) the one of $(1 + \lambda \varepsilon \boxtimes 1)|0\rangle$ in two copies of the Ising CFT, plotted in Fig. 12.

In the large subsystem limit $\ell \rightarrow \infty$, correlation functions factorize as discussed around (5.81). For the Ising CFT, where $\Delta_\varepsilon = 1$, we obtain:

$$\lim_{\ell \rightarrow \infty} \Delta S^{(2)} = -\log \left[1 - \frac{\frac{4\sqrt{2}}{3} \left(\frac{\lambda}{2\tau}\right)^2}{\left(1 + \left(\frac{\lambda}{2\tau}\right)^2\right)^2} \right]. \quad (5.100)$$

This, as a function of τ/λ , yields a very similar curve to the second Rényi asymmetry of the full system (5.37) computed with respect to the fusion algebra, just slightly smaller (the maximum of (5.100) is $\log(\frac{3}{3-\sqrt{2}})$, smaller than $\log(2)$). We compare them in Fig. 16. So in this case the large subsystem limit of the asymmetry computed with the averaging procedure is not identical to the asymmetry of the full system.

5.5 Fibonacci fusion category

The Fibonacci fusion category has 2 simple objects: $\mathcal{L}_1, \mathcal{L}_W$. Their fusion rule is⁶⁸

$$\mathcal{L}_W \times \mathcal{L}_W = \mathcal{L}_1 + \mathcal{L}_W \quad (5.101)$$

and it is commutative. The quantum dimension of $\mathcal{L}_W$ is

$$d_W = \frac{1 + \sqrt{5}}{2} \equiv \varphi \quad (5.102)$$

⁶⁸The Fibonacci fusion category is also a specific instance ($N = 1$) of the Haagerup-Izumi categories, which we study in Sec. 5.6.

which is the golden ratio (i.e., the positive solution of the equation $\varphi^2 = \varphi + 1$). Both lines are self-dual and $\varkappa_W = 1$. The only nontrivial F -symbol, in terms of $a, b \in \{1, W\}$, is⁶⁹

$$[F_{WWW}^W]_{ab} = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix}, \quad (5.103)$$

while all other F -symbols allowed by fusion are equal to 1. The symmetrizer for the fusion algebra is determined by the inverse bilinear form $\tilde{K}$ and takes the form:

$$\rho_S = \frac{1}{5} \left(3\rho - \mathcal{L}_W \rho - \rho \mathcal{L}_W + 2\mathcal{L}_W \rho \mathcal{L}_W \right). \quad (5.104)$$

The fusion algebra has two one-dimensional representations $R_\pm$, where $R_+(\mathcal{L}_W) = \varphi$ while $R_-(\mathcal{L}_W) = -\varphi^{-1}$. The corresponding projectors are given by

$$P_+ = 5^{-\frac{1}{2}} (\varphi^{-1} \mathcal{L}_1 + \mathcal{L}_W), \quad P_- = 5^{-\frac{1}{2}} (\varphi \mathcal{L}_1 - \mathcal{L}_W). \quad (5.105)$$

Spontaneous symmetry breaking. Like in the case of Ising, the only indecomposable module category over the Fibonacci fusion category is the regular one [98]. This means that in a Fibonacci-symmetric gapped phase, the symmetry is always completely spontaneously broken. The clustering vacua form a 2-dimensional NIM-rep spanned by $|1\rangle$ and $|W\rangle$, and

$$\mathcal{L}_W |1\rangle = |W\rangle, \quad \mathcal{L}_W |W\rangle = |1\rangle + |W\rangle, \quad (5.106)$$

where $\langle 1|1\rangle = \langle W|W\rangle = 1$. Consider the pure state $\rho = |1\rangle\langle 1|$. Using the symmetrizer (5.104), in the basis above we find

$$\rho = |1\rangle\langle 1| \quad \Rightarrow \quad \rho_S = \frac{1}{5} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}. \quad (5.107)$$

Defining

$$\phi_+ = \frac{\sqrt{5} + 1}{2\sqrt{5}} = \frac{\varphi}{\sqrt{5}}, \quad \phi_- = \frac{\sqrt{5} - 1}{2\sqrt{5}} = \frac{\varphi^{-1}}{\sqrt{5}}, \quad (5.108)$$

the eigenvalues of ρ_S are $\{\phi_+, \phi_-\}$. Then the Rényi asymmetries are $\Delta S^{(n)} = \frac{1}{1-n} \log(\phi_+^n + \phi_-^n)$ while the asymmetry is

$$\Delta S = -\phi_+ \log \phi_+ - \phi_- \log \phi_- = \frac{1}{2} \log 5 - \frac{1}{\sqrt{5}} \operatorname{arccoth}(\sqrt{5}) \simeq 0.5895. \quad (5.109)$$

This value is smaller than $\log(2)$, the asymmetry for a spontaneously broken group of order 2. For the other vacuum:

$$\rho = |W\rangle\langle W| \quad \Rightarrow \quad \rho_S = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \quad (5.110)$$

which gives the same asymmetries (because ρ_S has the same eigenvalues).

⁶⁹There is a unique fusion category whose fusion ring is (5.101) and that has unitary F -symbols, see e.g. [108]. This category occurs for example as a subsymmetry of the tricritical Ising model [15].

$$\begin{aligned}
\begin{array}{c} W \\ \hline 1 \end{array} \times \begin{array}{c} 1 \\ \hline 1 \end{array} &= \begin{array}{c} 1 \\ \hline 1 \end{array} + \begin{array}{c} W \\ \hline 1 \end{array}, \quad \begin{array}{c} W \\ \hline W \end{array} \times \begin{array}{c} 1 \\ \hline 1 \end{array} = \varphi^{\frac{1}{2}} \begin{array}{c} 1 \\ \hline 1 \end{array} - \varphi^{-\frac{1}{2}} \begin{array}{c} W \\ \hline 1 \end{array}, \\
\begin{array}{c} W \\ \hline 1 \end{array} \times \begin{array}{c} 1 \\ \hline W \end{array} &= -\varphi^{-1} \begin{array}{c} W \\ \hline W \end{array}, \quad \begin{array}{c} W \\ \hline 1 \end{array} \times \begin{array}{c} W \\ \hline 1 \end{array} = -\varphi^{-1} \begin{array}{c} W \\ \hline W \end{array}, \\
\begin{array}{c} W \\ \hline 1 \end{array} \times \begin{array}{c} W \\ \hline W \end{array} &= \varphi^{-\frac{1}{2}} \begin{array}{c} 1 \\ \hline W \end{array} + \varphi^{-\frac{1}{2}} \begin{array}{c} W \\ \hline 1 \end{array} + \varphi^{-2} \begin{array}{c} W \\ \hline W \end{array}, \\
\begin{array}{c} W \\ \hline W \end{array} \times \begin{array}{c} 1 \\ \hline W \end{array} &= (1, \varphi^{-\frac{3}{2}})_a \begin{array}{c} 1 \\ \hline W \end{array}, \quad \begin{array}{c} W \\ \hline W \end{array} \times \begin{array}{c} W \\ \hline 1 \end{array} = (1, \varphi^{-\frac{3}{2}})_a \begin{array}{c} W \\ \hline 1 \end{array}, \\
\begin{array}{c} W \\ \hline W \end{array} \times \begin{array}{c} W \\ \hline W \end{array} &= \begin{pmatrix} \varphi^{-1} & \varphi^{-\frac{1}{2}} \\ \varphi^{-\frac{1}{2}} & -\varphi^{-1} \end{pmatrix}_{ab} \begin{array}{c} 1 \\ \hline W \end{array} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{ab} \begin{array}{c} W \\ \hline 1 \end{array} + \begin{pmatrix} \varphi^{-\frac{1}{2}} & -\varphi^{-1} \\ -\varphi^{-1} & -\varphi^{-\frac{5}{2}} \end{pmatrix}_{ab} \begin{array}{c} W \\ \hline W \end{array}.
\end{aligned}$$

Figure 17: Nontrivial products in the tube algebra of the Fibonacci fusion category.

Tube algebra. This has been discussed, e.g., in [108]. It has dimension 7. The elements

$$\begin{array}{c} 1 \\ \hline 1 \end{array} \quad \text{and} \quad \begin{array}{c} W \\ \hline W \end{array} \tag{5.111}$$

behave as the identity in the untwisted and twisted sector, respectively. The nonvanishing products are in Fig. 17. The algebra has three 1-dimensional and one 2-dimensional irreducible representations, that we now describe.

- A 1-dimensional representation $\mathcal{F}_1$ with space $V = \mathbb{C}_1$:

$$\mathcal{F}_1 \left(\begin{array}{c} 1 \\ \hline 1 \end{array} \right) = 1, \quad \mathcal{F}_1 \left(\begin{array}{c} W \\ \hline 1 \end{array} \right) = \varphi, \tag{5.112}$$

while all other ones are mapped to zero. It represents genuine local operators invariant under the symmetry (up to multiplication by the quantum dimension of $\mathcal{L}_W$).

- Two 1-dimensional representations $\mathcal{F}_W$ and $\mathcal{F}_W^*$ with space $V = \mathbb{C}_W$, and

$$\mathcal{F}_W \left(\begin{array}{c} W \\ \hline W \end{array} \right) = 1, \quad \mathcal{F}_W \left(\begin{array}{c} W \\ \hline 1 \end{array} \right) = e^{\frac{4\pi i}{5}}, \quad \mathcal{F}_W \left(\begin{array}{c} W \\ \hline W \end{array} \right) = e^{-\frac{3\pi i}{5}} \varphi^{\frac{1}{2}}, \tag{5.113}$$

while all other ones are zero. It represents W -twisted operators, that remain W -twisted under the action of $\mathcal{L}_W$. Notice that there are two actions of $\mathcal{L}_W$ on these operators.

- One 2-dimensional representation $\mathcal{F}_2$ with space $V = \mathbb{C}_1 \oplus \mathbb{C}_W$ and

$$\begin{aligned} \mathcal{F}_2\left(\begin{array}{c|c} 1 & 1 \\ \hline 1 & 1 \end{array}\right) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & \mathcal{F}_2\left(\begin{array}{c|c} W & 1 \\ \hline 1 & 1 \end{array}\right) &= \begin{pmatrix} -\varphi^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \\ \mathcal{F}_2\left(\begin{array}{c|c} 1 & W \\ \hline 1 & W \end{array}\right) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & \mathcal{F}_2\left(\begin{array}{c|c} W & 1 \\ \hline W & W \end{array}\right) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & \mathcal{F}_2\left(\begin{array}{c|c} W & W \\ \hline W & W \end{array}\right) &= \begin{pmatrix} 0 & 0 \\ 0 & \varphi^{-\frac{3}{2}} \end{pmatrix}, \\ \mathcal{F}_2\left(\begin{array}{c|c} W & W \\ \hline 1 & 1 \end{array}\right) &= \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}, & \mathcal{F}_2\left(\begin{array}{c|c} W & 1 \\ \hline W & W \end{array}\right) &= \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, & x &= 5^{\frac{1}{4}}\varphi^{-\frac{1}{4}}. \end{aligned} \quad (5.114)$$

It represents a doublet of an untwisted local operator $\mathcal{O}_1$ and a W -twisted operator $\mathcal{O}_W$ that get mixed under the action of $\mathcal{L}_W$.

Full and reduced strip algebras. The only indecomposable module category is the regular one. The corresponding strip algebra has dimension 13, and a basis is given by the elements:

where dashed lines are $\mathcal{L}_1$ while solid lines are $\mathcal{L}_W$. One easily verifies that the boundary $\hat{\mathcal{B}} = \mathcal{L}_W$ provides a weakly-symmetric boundary condition [36], and the corresponding reduced strip algebra has dimension 2. In terms of the generators

$$\mathcal{S}_1 = \begin{array}{|c|} \hline \text{dashed line} \\ \hline \end{array}, \quad \mathcal{S}_W = \begin{array}{|c|} \hline \text{solid line} \\ \hline \end{array}, \quad (5.115)$$

the reduced strip algebra is commutative, $\mathcal{S}_1$ is its identity, and the nontrivial product is

$$\mathcal{S}_W \times \mathcal{S}_W = \mathcal{S}_1 + \varphi^{-\frac{3}{2}} \mathcal{S}_W. \quad (5.116)$$

This algebra is very similar to the original fusion algebra (5.101), and it is semisimple. The symmetrizer with respect to the reduced strip algebra, computed using (3.9), takes the form

$$\rho_S = \frac{1}{2} \left(\rho + \frac{1}{4\varphi^3 + 1} (1 - 2\varphi^{3/2} \mathcal{S}_W) \rho (1 - 2\varphi^{3/2} \mathcal{S}_W) \right). \quad (5.117)$$

For the asymmetry of the vacuum, it is clear that the density matrix $\rho = \tilde{\rho}_{WW}/Z_{WW}$ (using the notation of Fig. 15) commutes with the action of the reduced strip algebra (5.115) (because there is a single boundary condition involved). Hence its asymmetry is zero.

We then compute the asymmetry of an excited state created by the insertion of an untwisted operator. We choose an operator transforming as $\mathcal{O}_1$ in the two-dimensional tube-algebra representation $\mathcal{F}_2$ in (5.114) (because the one-dimensional representations are either trivial or contain twisted-sector operators only). As above, we denote by $\mathcal{O}_1$ and $\mathcal{O}_W$ the two operators spanning the 2d irreducible representation, and we let ρ be the reduced density matrix for the state $\mathcal{O}_1(z)|0\rangle$. In order to compute the second Rényi asymmetry, we need to understand the swiping action of the line $\mathcal{L}_W$ across the operator $\mathcal{O}_1$. By looking at (5.114), it is clear that this will be a

$$\begin{aligned}
\text{Circle with } \mathcal{O}_1 &= -\varphi^{-1} \cdot \mathcal{O}_1 \\
\text{Circle with } \mathcal{O}_W &= 5^{1/4} \varphi^{-1/4} \cdot \mathcal{O}_1 \\
\overline{\cdot \mathcal{O}_1} &= -\varphi^{-2} \overline{\cdot \mathcal{O}_1} + 5^{1/4} \varphi^{-3/4} \overline{\cdot \mathcal{O}_W}
\end{aligned}$$

Figure 18: Left: lasso action of the Fibonacci tube algebra on the doublet of operators $(\mathcal{O}_1, \mathcal{O}_W)$, extracted from (5.114). Right: swiping action of the line $\mathcal{L}_W$ across $\mathcal{O}_1$. The coefficients have been fixed by separately completing the diagram from above and from below to form a circle, and by using the lasso actions on the left.

linear combination of $\mathcal{O}_1$ and $\mathcal{O}_W$. The result is reported in Fig. 18. When swiping $\mathcal{L}_W$ through ρ , which contains two instances of $\mathcal{O}_1$, it is also useful to perform an F -move described by the F -symbols (5.103). Consider now the limit $\varepsilon \rightarrow 0$ of small radius for the discs at the entangling surfaces. The traces of products of density matrices can be written in terms of partition functions on a replicated manifold, with the weakly-symmetric boundary condition at the entangling surfaces, and possibly with a line $\mathcal{L}_W$ connecting the two boundaries or connecting $\mathcal{O}_W$ to one of the boundaries. As discussed around (3.47), those partition functions can be analyzed by mapping the geometry to a cylinder and expressing them as amplitudes of Cardy states evolved in the closed channel. If the Cardy states are untwisted, the limit $\varepsilon \rightarrow 0$ simply produces two multiplicative g -factors, as in (3.45). If the Cardy states are twisted, instead, the ground state in the closed channel necessarily has higher energy than the untwisted vacuum, and the amplitude is suppressed by a positive power of ε . It follows that, in the $\varepsilon \rightarrow 0$ limit, partition functions involving a residual line $\mathcal{L}_W$ connected to one or both boundaries do not contribute.

For instance, in the computation of the second moment $\text{tr}(\rho_S^2)$, we can write

$$\rho(1 - 2\varphi^{3/2}\mathcal{S}_W)\rho(1 - 2\varphi^{3/2}\mathcal{S}_W) = (1 + 4\varphi^{-1})\rho + 4 \cdot 5^{1/2}\varphi\rho' + \dots, \quad (5.118)$$

where ρ' is as ρ but with $\mathcal{O}_W$ in place of $\mathcal{O}_1$ and with a line $\mathcal{L}_W$ connecting the two operators, similarly to Fig. 13, while the dots represent terms that are suppressed in the $\varepsilon \rightarrow 0$ limit once the trace is taken. In the end, up to an overall normalization which cancels in the ratio, we find:

$$\begin{aligned}
\text{tr}(\rho_S^2) &\propto (11\sqrt{5} - 24) \langle \mathcal{O}_1^\dagger \mathcal{O}_1 \mathcal{O}_1^\dagger \mathcal{O}_1 \rangle_{\mathcal{M}_2} + (25 - 11\sqrt{5}) \langle \mathcal{O}_1^\dagger \mathcal{O}_1 \mathcal{O}_W^\dagger \mathcal{O}_W \rangle_{\mathcal{M}_2}, \\
\text{tr}(\rho^2) &\propto \langle \mathcal{O}_1^\dagger \mathcal{O}_1 \mathcal{O}_1^\dagger \mathcal{O}_1 \rangle_{\mathcal{M}_2},
\end{aligned} \quad (5.119)$$

where the correlators are computed on the 2-sheeted Riemann surface $\mathcal{M}_2$. Hence:

$$\Delta S^{(2)} = -\log \left[(11\sqrt{5} - 24) + (25 - 11\sqrt{5}) \frac{\langle \mathcal{O}_1^\dagger \mathcal{O}_1 \mathcal{O}_W^\dagger \mathcal{O}_W \rangle_{\mathcal{M}_2}}{\langle \mathcal{O}_1^\dagger \mathcal{O}_1 \mathcal{O}_1^\dagger \mathcal{O}_1 \rangle_{\mathcal{M}_2}} \right]. \quad (5.120)$$

In the limit of a large subsystem, the correlation four-point functions factorize into two-point functions, see also the discussion around (5.27). While the correlator at denominator $\langle \mathcal{O}_1^\dagger \mathcal{O}_1 \mathcal{O}_1^\dagger \mathcal{O}_1 \rangle_{\mathcal{M}_2}$ has a conformal block decomposition dominated by the identity channel and thus $\langle \mathcal{O}_1^\dagger \mathcal{O}_1 \mathcal{O}_1^\dagger \mathcal{O}_1 \rangle_{\mathcal{M}_2} \simeq \langle \mathcal{O}_1^\dagger \mathcal{O}_1 \rangle_{\mathbb{C}}^2$, in the numerator the OPE $\mathcal{O}_1^\dagger \mathcal{O}_W$ does not contain the

identity channel and therefore, keeping into account that $\mathcal{O}_1$ and $\mathcal{O}_W$ have the same conformal dimension, the correlator $\langle \mathcal{O}_1^\dagger \mathcal{O}_1 \mathcal{O}_W^\dagger \mathcal{O}_W \rangle_{\mathcal{M}_2}$ is further suppressed. We conclude that, in the large subsystem limit, the second Rényi asymmetry becomes

$$\Delta S^{(2)} = -\log(11\sqrt{5} - 24) \simeq 0.516. \quad (5.121)$$

It is interesting to notice that this value is smaller than the second Rényi asymmetry of the full system computed with respect to the tube algebra. Indeed, by symmetrizing with respect to the tube algebra, one immediately gets $\Delta S^{(n)} = \Delta S = \log(2)$ because $\mathcal{O}_1$ belongs to an irreducible representation of dimension 2, see the discussion after (5.50).⁷⁰

5.6 Haagerup fusion ring

The simplest fusion category whose product rules are both noninvertible and nonabelian is the Haagerup fusion category [164–166], that we indicate as $H_{(3)}$. There exists a generalization to Haagerup-Izumi fusion categories [166–169] that we indicate as $H_{(N)}$, with $N > 3$. In this section we will only analyze the fusion algebra, hence we do not report the F -symbols of the category.⁷¹ The fusion ring of $H_{(N)}$ is generated by the objects $\{1, a, \varrho\}$ subject to the conditions

$$a^N = 1, \quad a \times \varrho = \varrho \times a^{-1}, \quad \varrho^2 = 1 + \sum_{j=0}^{N-1} a^j \times \varrho. \quad (5.122)$$

We see that there is a $\mathbb{Z}_N$ subsymmetry and that, for $N \geq 3$, the element ϱ does not commute with a and thus the multiplication is nonabelian. Conjugation is given by

$$\bar{a} = a^{-1}, \quad \bar{\varrho} = \varrho, \quad \overline{a^j \times \varrho} = a^j \times \varrho. \quad (5.123)$$

Notice that $H_{(1)}$ coincides with the Fibonacci (commutative) fusion ring that we already discussed in Sec. 5.5, while $H_{(2)}$ is the fusion ring of the $SO(3)_6$ WZW model.

The fusion algebra is $2N$ -dimensional and a basis is given by the elements

$$\{1, a, \dots, a^{N-1}, \varrho, a\varrho, \dots, a^{N-1}\varrho\}. \quad (5.124)$$

The symmetrizer with respect to it can be written as:

$$\rho_S = \frac{1}{2N} \sum_{\mathcal{L}} \mathcal{L} \rho \mathcal{L}^\dagger + \frac{1}{2(N^2 + 4)} (A \rho A - B \rho B) - \frac{1}{N(N^2 + 4)} (A \rho B + B \rho A) \quad (5.125)$$

where in the first term we summed over all simple lines, while

$$A = 1 + a + \dots + a^{N-1} = A^\dagger, \quad B = \varrho + a\varrho + \dots + a^{N-1}\varrho = B^\dagger. \quad (5.126)$$

⁷⁰On the other hand, the asymmetry with respect to the fusion algebra vanishes: $\Delta S^{(n)} = \Delta S = 0$. To see this, we should restrict the 2d representation $\mathcal{F}_2$ of the tube algebra to its untwisted sector, corresponding to the fusion algebra, and determine how $\mathcal{O}_1$ decomposes in irreducible fusion-algebra representations. From (5.114), $\mathcal{O}_1$ lands in the single representation R_- . Since this is one-dimensional, the asymmetry vanishes.

⁷¹The F -symbols of $H_{(3)}$ can be found in [169, 170]. The F -symbols in [169] are in a particularly convenient gauge, which makes the expressions extremely compact. Numerical evidence for the existence of a CFT with Haagerup symmetry was given in [171]. The F -symbols of $H_{(N)}$ with $3 \leq N \leq 15$ odd can be found in [169].

Let us list the irreducible representations of the fusion algebra. For N odd, there are two 1-dimensional representations R_+ , R_- and $\frac{N-1}{2}$ 2-dimensional representations R_j . For N even, there are four 1-dimensional representations R_+ , R_- , R_+ , R_- and $\frac{N-2}{2}$ 2-dimensional representations R_j . The 1-dimensional representations are:

$$\begin{aligned} R_{\pm}: \quad a &= 1, & \varrho &= \frac{1}{2}(N \pm \sqrt{N^2 + 4}), \\ R_{\pm}: \quad a &= -1, & \varrho &= \pm 1. \end{aligned} \quad (5.127)$$

The 2-dimensional representations R_j with $j = 1, \dots, \frac{N-1}{2}$ (for N odd) or $j = 1, \dots, \frac{N-2}{2}$ (for N even) are:

$$R_j: \quad a = \begin{pmatrix} \zeta^j & 0 \\ 0 & \zeta^{-j} \end{pmatrix}, \quad \varrho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \zeta = e^{\frac{2\pi i}{N}}. \quad (5.128)$$

Spontaneous symmetry breaking. For the Haagerup fusion ring with $N = 3$, let us consider a TQFT describing the full SSB of the symmetry, thus giving 6 vacua. These transform in the regular representation. The inverse bilinear form in the fusion algebra is given by

$$\tilde{K} = \frac{1}{39} \begin{pmatrix} 8 & \frac{3}{2} & \frac{3}{2} & -1 & -1 & -1 \\ \frac{3}{2} & \frac{3}{2} & 8 & -1 & -1 & -1 \\ \frac{3}{2} & 8 & \frac{3}{2} & -1 & -1 & -1 \\ -1 & -1 & -1 & 5 & -\frac{3}{2} & -\frac{3}{2} \\ -1 & -1 & -1 & -\frac{3}{2} & 5 & -\frac{3}{2} \\ -1 & -1 & -1 & -\frac{3}{2} & -\frac{3}{2} & 5 \end{pmatrix}. \quad (5.129)$$

We take a pure density matrix $\rho = |b\rangle\langle b|$. The symmetrized density matrix ρ_S has eigenvalues

$$\lambda_i = \left\{ \frac{\sqrt{13}+3}{6\sqrt{13}}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{\sqrt{13}-3}{6\sqrt{13}} \right\} \quad (5.130)$$

for any of the six vacua. This leads to the asymmetry $\Delta S \simeq 1.6568$. This value is smaller than $\log(6)$, the asymmetry for a spontaneously broken group of order 6.

5.7 The $\frac{1}{2}E_6$ fusion ring

Finally, we briefly discuss the example of the $\frac{1}{2}E_6$ fusion ring, where a multiplicity greater than 1 appears. For the purposes of this example we limit ourselves to the study of the fusion ring, and we will not be concerned with its categorifications, which do in fact exist (see [172] and Sec. 6.4 of [15]). One of the corresponding fusion categories appears as a subsymmetry of the non-diagonal $SU(2)_{10}$ WZW model of E_6 type (which we will study in greater detail in Sec. 6.4), or of the (A_{10}, E_6) minimal model.

The ring has three elements $\{\mathcal{L}_1, \mathcal{L}_X, \mathcal{L}_Y\}$ with abelian product rules

$$\mathcal{L}_X \times \mathcal{L}_X = \mathcal{L}_1, \quad \mathcal{L}_X \times \mathcal{L}_Y = \mathcal{L}_Y \times \mathcal{L}_X = \mathcal{L}_Y, \quad \mathcal{L}_Y \times \mathcal{L}_Y = \mathcal{L}_1 + \mathcal{L}_X + 2\mathcal{L}_Y. \quad (5.131)$$

The non-degenerate inverse bilinear form, that provides the symmetrizer, is

$$\tilde{K} = \frac{1}{12} \begin{pmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \quad (5.132)$$

With this, we can compute the asymmetry in the case of fully broken symmetry. We get, for the three vacua, the following eigenvalues of ρ_S and the corresponding asymmetries:

$$\begin{aligned}
|1\rangle, |X\rangle : \quad \lambda_i &= \left\{ \frac{1}{2}, \frac{3+\sqrt{3}}{12}, \frac{3-\sqrt{3}}{12} \right\} & \Delta S &\simeq 0.951, \\
|Y\rangle : \quad \lambda_i &= \left\{ \frac{3+\sqrt{3}}{6}, \frac{3-\sqrt{3}}{6}, 0 \right\} & \Delta S &\simeq 0.516.
\end{aligned} \tag{5.133}$$

As in the case of the Ising symmetry, the first two vacua are related by a $\mathbb{Z}_2$ action and thus share the same asymmetry. The third vacuum, instead, has a different asymmetry.

6 Examples of SREE calculations

In this section we present various computations of entanglement entropy resolved with respect to a categorical symmetry. We start by analyzing different cases of group-like symmetries. We observe that many cases realize equipartition of symmetry-resolved entanglement entropy. These include (possibly anomalous) $\mathbb{Z}_n$ symmetries as well as non-anomalous abelian groups. However, in Sec. 6.1.1 we provide the example of an abelian group for which – with an appropriate choice of boundary conditions – entanglement equipartition is violated due to the presence of an anomaly. This should be compared, for instance, with [71], where violations of entanglement equipartition at next-to-leading order in the interval size were observed only for nonabelian groups. In Sec. 6.1.2, the example of a nonabelian group is presented; to keep the discussion simple, we consider the smallest possible nonabelian group, namely the group of permutations of three objects, S_3 . We find that the result of symmetry resolution is highly sensitive to the choice of boundary conditions, which is in turn constrained by the anomaly. We then proceed to analyze the simplest examples of noninvertible symmetries with no simple (strongly) symmetric boundary conditions, namely the Fibonacci and Ising symmetries. In these two cases, there turns out to be only a single choice of a multiplet of conformal boundary conditions,⁷² with respect to which equipartition is also violated. Finally, we turn to the richer example of the Wess-Zumino-Witten model with $\mathfrak{su}(2)_{10}$ chiral algebra and non-diagonal modular invariant of type E_6 . Besides exhibiting lines which multiply according to a fusion ring with multiplicities, this example highlights the fact that the existence of a module category does not always imply that there exists an associated multiplet of conformal boundary conditions for the physical theory. In this case indeed the regular module category for the full symmetry is not realized in the theory, and thus cannot be used to compute the symmetry-resolved entanglement entropy.

In practice, once we have fixed a symmetry category $\mathcal{C}$ and a module category $\mathcal{M}$ over it, in order to compute the symmetry-resolved entanglement (4.36) we need the following pieces of data:

- the list of simple objects $\{\mathcal{R}_{\hat{a}}\}_{\hat{a}}$ in the dual category $\mathcal{C}_{\mathcal{M}}^*$;
- their quantum dimensions $d_{\hat{a}}$;
- the dimensions of the representations $\mathfrak{R}_{\hat{a}}$ of the strip algebra $\text{Strip}_{\mathcal{C}}(\mathcal{M})$ associated to $\mathcal{R}_{\hat{a}}$;
- the quantum dimensions $\{\tilde{d}_m\}_m$ of the simple boundary conditions in $\mathcal{M}$, normalized so that (4.9) holds.

The easiest way to obtain these data depends on the specific example under consideration. A generic algorithmic procedure to extract them is discussed at the end of App. E.

⁷²Mathematically, the symmetry fusion category $\mathcal{C}$ admits a unique indecomposable module category $\mathcal{M}$: the regular one.

6.1 Group-like symmetries

Here we study examples in which the symmetry category is $\mathcal{C} = \text{Vec}_G^\omega$ where G is a finite group and $\omega \in H^3(G, \text{U}(1))$ is the anomaly 3-cocycle. Let us first state some general facts:

- The indecomposable module categories over $\mathcal{C}$ are labeled by pairs (H, ψ) where $H \subseteq G$ is a subgroup (considered up to conjugation) and $\psi \in C^2(H, \text{U}(1))$ trivializes the restriction of the anomaly,⁷³ namely $\omega|_H = -\delta\psi$ (see App. F for a review). In other words, H must be anomaly free. We denote such a module category by $\mathcal{M}(H, \psi)$. An algebra object associated with such a module category is the group algebra of H whose multiplication is twisted by the 2-cochain ψ , $\mathbb{C}[H; \psi]$.⁷⁴
- The normalized quantum dimensions of the boundaries, $\{\tilde{d}_m\}_{m \in \mathcal{M}}$, can be computed using equation (4.7) describing the fusion of boundaries.

In the case at hand, we also have that all the boundary quantum dimensions are equal. Indeed, consider $\mathcal{M}(H, \psi)$. Its elements are labeled by right-cosets G/H , on which G acts from the left, sending a coset to a (generally different) coset. This means that we must have

$$d_g \tilde{d}_m = \tilde{d}_{gm}, \quad \forall g \in G. \quad (6.1)$$

Because all $d_g = 1$, and because all cosets can be connected by left-multiplication by some $g \in G$, we have that all $\tilde{d}_m$ are equal to each other. Now choosing $m = [1]_H$ we have $[1]_H \otimes [1]_H = \mathbb{C}[H; \psi]$ (see Fn. 74), and thus $\tilde{d}_{[1]}^2 = d_{\mathbb{C}[H; \psi]} = |H|$. Finally, we obtain

$$\tilde{d}_m = \sqrt{|H|}, \quad \forall m \in \text{Irr}(\mathcal{M}). \quad (6.2)$$

Let us now consider some possible situations.

Group-like dual category. Consider the module category $\mathcal{M} = \mathcal{M}(H, \psi)$. Suppose we know that the dual symmetry $\mathcal{C}_{\mathcal{M}}^*$ is an invertible symmetry with group $\tilde{G}$. Theorem 3.4 of [173] gives necessary and sufficient conditions for the dual category to be group-like. In particular, H must be an abelian normal subgroup of G . As noted in Example 3.5 therein, these conditions are always satisfied when $G = \mathbb{Z}_n$.

Since the total quantum dimension of a fusion category is invariant under gauging, it must be $|G| = |\tilde{G}|$ (see Corollary 7.16.7 of [113]). Furthermore, $\mathcal{M}$ also has a module-category structure over the dual symmetry category, with action $\tilde{N}_{\tilde{g}m}^n = \delta_{n, \tilde{g} \cdot m}$, where $m, n \in \tilde{G}/\tilde{H}$ for some $\tilde{H} \subset \tilde{G}$ (with $|\tilde{H}| = |H|$). This implies that the dimension of every irreducible representation of

⁷³Such cochains form a torsor over $H^2(H, \text{U}(1))$.

⁷⁴Such an algebra object can be obtained via the internal Hom construction, see Fn. 31, which in our setting takes the form (4.7) with $m = n$. In the group-like case, there is a privileged element $[1]_H$ in $\mathcal{M}$, the H -coset of the identity of G . This gives precisely $\mathbb{C}[H; \psi]$ as the associated algebra object. Choosing other elements of the form $[g]_H$ in $\mathcal{M}$ corresponds to g -conjugation of H ; however, this does not modify the quantum dimension of the algebra object.

$\text{Strip}_{\mathcal{C}}(\mathcal{M})$ is, according to (4.5),⁷⁵

$$\dim(\mathfrak{R}_{\tilde{g}}) = \sum_{m,n \in \tilde{G}/\tilde{H}} N_{\tilde{g}m}^n = |\mathcal{M}| = |G|/|H|. \quad (6.4)$$

Putting everything together and using (6.2) we obtain

$$q_{\tilde{g}} = \frac{1}{|G|}, \quad \forall \tilde{g} \in \tilde{G}, \quad (6.5)$$

and finally

$$\begin{aligned} S[\rho_{\tilde{g}}] &= \frac{c}{3} \log \frac{L}{\epsilon} + 2 \log \sum_m g \tilde{d}_m + \log \frac{1}{|G|} \\ &= \frac{c}{3} \log \frac{L}{\epsilon} + 2 \log(g) + \log \frac{|G|}{|H|}. \end{aligned} \quad (6.6)$$

We see that, when the dual symmetry is group-like, the symmetry-resolved entanglement entropy is always equipartite among representations.

For example, consider $G = \mathbb{Z}_n$, with anomaly labeled by $p \in \mathbb{Z}_n$. As explained in App. G.1, the maximal non-anomalous subgroup is given by $H = \mathbb{Z}_k$ with $k = \gcd(n, p)$. Therefore, with respect to the module category $\mathcal{M}(H; 1)$, one obtains

$$q_{\tilde{g}} = \frac{1}{n}, \quad S[\rho_{\tilde{g}}] = \frac{c}{3} \log \frac{L}{\epsilon} + 2 \log(g) + \log \frac{n}{k}. \quad (6.7)$$

Regular module category. It is always possible to choose as module category the regular one which, for a group-like symmetry associated to a finite group G , is given by $\mathcal{M}(\{1\}, 1) \equiv \mathcal{C} \cong \text{Vec}_G^\omega$. In this case we have that the dual symmetry is equivalent to the original one, see (4.12). Therefore, we are under the hypotheses of the paragraph above, namely the dual symmetry is still group-like. This is the only possible choice of indecomposable module category if there exists no non-trivial anomaly-free subgroup of G . With this choice, the representations of the strip algebra are also labeled by G , and using (6.6) we immediately get

$$q_h = \frac{1}{|G|}, \quad S[\rho_h] = \frac{c}{3} \log \frac{L}{\epsilon} + 2 \log(g) + \log |G|, \quad \forall h \in G. \quad (6.8)$$

This result is independent of the anomaly $[\omega] \in H^3(G, \text{U}(1))$ and it is the same for every representation of the strip algebra, therefore realizing entanglement equipartition.

6.1.1 $G = (\mathbb{Z}_2)^3$ with cubic anomaly

Let us now consider the interesting case of $(\mathbb{Z}_2)^3$ group with a type III anomaly.⁷⁶ One cocycle representative of this anomaly is

$$\omega((a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3)) = e^{i\pi a_1 b_2 c_3}, \quad (6.9)$$

⁷⁵This is in agreement with the fact that $\dim(\text{Strip}_{\mathcal{C}}(\mathcal{M})) = |G|^3/|H|^2$. Indeed

$$\sum_{\tilde{g} \in \tilde{G}} \dim(\mathfrak{R}_{\tilde{g}})^2 = |\tilde{G}| \cdot \left(\frac{|G|}{|H|} \right)^2 = |G|^3/|H|^2. \quad (6.3)$$

⁷⁶This is a specific choice of mixed anomaly between the three $\mathbb{Z}_2$ factors (see e.g. [174]). More generally, the anomaly for $(\mathbb{Z}_2)^3$ is given by an element of $H^3(\mathbb{Z}_2^3, \text{U}(1)) \cong \mathbb{Z}_2^7$.

where $(a_i, b_i, c_i) \in (\mathbb{Z}_2)^3$. The relevant characteristic of this example is that there exists a particular choice of (discrete) gauging that yields a noninvertible dual symmetry, despite the group being abelian [139, 173, 174]. Let us review this fact from the perspective of boundary conditions and strip-algebra representations. Let us denote the generators by

$$A = (1, 0, 0) , \quad B = (0, 1, 0) , \quad C = (0, 0, 1) . \quad (6.10)$$

From the form of the cocycle in (6.9), it is easy to show that there are three maximal subgroups where the anomaly is trivial, which are the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroups obtained by deleting one generator, namely

$$H_1 = \{1, B, C, BC\} , \quad H_2 = \{1, A, C, AC\} , \quad H_3 = \{1, A, B, AB\} . \quad (6.11)$$

Let us choose H_3 , without loss of generality. The restriction of the anomaly cocycle is $\omega|_{H_3} = 1$, so we can also choose the trivialization $\psi = 1$. There are two cosets $\mu_0, \mu_1 \in G/H_3$ given by

$$\mu_0 = \{1, A, B, AB\} , \quad \mu_1 = \{C, AC, BC, ABC\} . \quad (6.12)$$

The left action of G on the cosets is given by

$$(a, b, c) \cdot \mu_i = \mu_{i+c} , \quad (6.13)$$

and, choosing the lift $\sigma(\mu_i) = C^i$, the h -function (defined in (F.3)) is $h((a, b, c), \mu_i) = (a, b, 0)$. These data allow us to compute the $\tilde{F}$ -symbols and thus the whole weak Hopf algebra structure of the strip algebra corresponding to the $\text{Vec}_{\mathbb{Z}_2^3}^\omega$ -module category $\mathcal{M}(H_3, 1)$. This strip algebra is 32-dimensional, and turns out to have 5 irreducible representations of dimensions 2, 2, 2, 2 and 4. These fuse according to the abelian product of $\text{Rep}(D_8)$, which is indeed known to be (tensor equivalent to) the dual category $\mathcal{C}_{\mathcal{M}}^*$ of $\text{Vec}_{\mathbb{Z}_2^3}^\omega$ with respect to $\mathcal{M}(H, 1)$ (see e.g. [139]). Let us denote these representations by $1, \eta, \eta', \eta\eta'$ and $\mathcal{V}$ respectively. The lines $\mathcal{R}_\eta$ and $\mathcal{R}_{\eta'}$ in the dual category generate a $\mathbb{Z}_2 \times \mathbb{Z}_2$ invertible part, whereas $\mathcal{R}_\mathcal{V}$ is a Kramers-Wannier-type defect that satisfies

$$\mathcal{R}_\mathcal{V} \times \mathcal{R}_{\hat{a}} = \mathcal{R}_\mathcal{V} \quad \forall \hat{a} \in \{1, \eta, \eta', \eta\eta'\} , \quad \mathcal{R}_\mathcal{V} \times \mathcal{R}_\mathcal{V} = \mathcal{R}_1 + \mathcal{R}_\eta + \mathcal{R}_{\eta'} + \mathcal{R}_{\eta\eta'} , \quad (6.14)$$

from which we see that the noninvertible object has quantum dimension $d_\mathcal{V} = 2$. To summarize, we obtain

$$\begin{aligned} d_{\hat{a}} &= 1 , \quad \dim(\mathfrak{R}_{\hat{a}}) = 2 , \quad \forall \hat{a} \in \{1, \eta, \eta', \eta\eta'\} ; \\ d_\mathcal{V} &= 2 , \quad \dim(\mathfrak{R}_\mathcal{V}) = 4 ; \\ \tilde{d}_{\mu_i} &= 2 , \quad \forall i \in \{0, 1\} . \end{aligned} \quad (6.15)$$

The probabilities are therefore $q_{\hat{a}} = 1/8$ for the representations corresponding to invertible objects in $\text{Rep}(D_8)$, and $q_\mathcal{V} = 1/2$ for the one corresponding to the noninvertible, while $D = 4$. Finally, the symmetry-resolved entanglement entropies are

$$\begin{aligned} S[\rho_{\hat{a}}] &= \frac{c}{3} \log \frac{L}{\epsilon} + 2 \log(g) + \log 2 , \quad \forall \hat{a} \in \{1, \eta, \eta', \eta\eta'\} ; \\ S[\rho_\mathcal{V}] &= \frac{c}{3} \log \frac{L}{\epsilon} + 2 \log(g) + 3 \log 2 . \end{aligned} \quad (6.16)$$

This is an example of an abelian group that, due to the anomaly, does not admit a (simple) symmetric boundary condition. Instead, it admits a nontrivial multiplet of boundary conditions which, as we have shown, does not realize entanglement equipartition. However, notice that, as for any symmetry category, one can always choose the regular module for $\mathbf{Vec}_{\mathbb{Z}_3}^\omega$, if the corresponding boundary conditions are realized in the physical theory. In this case one would indeed obtain entanglement equipartition as explained in Sec. 6.1.

6.1.2 $G = S_3$

We will now discuss an example of group-like symmetry with nonabelian fusion algebra, the easiest example being the permutation group S_3 . Following [174], we present this group as

$$\begin{aligned} S_3 &\equiv \langle R, X : R^3 = 1, X^2 = 1, XR = R^{-1}X \rangle \\ &= \{1, R, R^2, X, XR, XR^2\}. \end{aligned} \quad (6.17)$$

Writing the elements, as above, in the form

$$(A, a) \equiv X^A R^a \quad \text{with} \quad A \in \{0, 1\}, \quad a \in \{0, 1, 2\}, \quad (6.18)$$

the product reads:

$$(A, a) \cdot (B, b) = \left([A + B]_0^1, [(-1)^B a + b]_0^2 \right). \quad (6.19)$$

Here $[n]_r^s$ is the mod function in the range $[r, s]$, namely $[n]_r^s = n \bmod (s - r + 1)$ and $r \leq n \leq s$. Furthermore, it turns out (see e.g. [174]) that

$$H^3(S_3, \mathbf{U}(1)) \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6, \quad (6.20)$$

classifies the possible anomalies for S_3 -symmetric (oriented, bosonic) theories. We refer to App. G.2 for more details.

We now wish to list all possible module categories depending on the value of the anomaly, which we label with $p \in \mathbb{Z}_6$. First notice that neither S_3 nor any of its subgroups (which are cyclic groups of order 2 or 3) can have discrete torsion in two dimensions.⁷⁷ Hence, module categories are just labeled by an anomaly-free subgroup (see App. G.2 for the classification of such subgroups).

The possible inequivalent module categories/gaugings are summarized in Table 1, together with the fusion ring of corresponding dual categories. The first line therein corresponds to the trivial module category/full gauging. The last line corresponds to the regular module category/trivial gauging. These two cases have already been treated for a generic group, so we concentrate on the two intermediate cases, which we examine in turn.

$H = \mathbb{Z}_3$. This is the normal subgroup generated by R . It is anomaly free when $p = 0, 3$. We have that the (right) cosets are $S_3/Z_3 \cong \mathbb{Z}_2$. Explicitly, these are given by

$$\mu_0 = \{1, R, R^2\}, \quad \mu_1 = \{X, XR, XR^2\}. \quad (6.21)$$

⁷⁷This follows from $H^2(S_3, \mathbf{U}(1)) \cong H^2(\mathbb{Z}_N, \mathbf{U}(1)) \cong 0$, see e.g. Table 3.1 and Eqs. (2.A.14), (2.A.15) of [174].

H	G/H	Dual fusion ring					
		$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$
S_3	$\{1\}$	$\text{Rep}(S_3)$	–	–	–	–	–
$\mathbb{Z}_3$	$\mathbb{Z}_2$	S_3	–	–	S_3	–	–
$\mathbb{Z}_2$	$\{\mu_0, \mu_1, \mu_{-1}\}$	$\text{Rep}(S_3)$	–	$\text{Rep}(S_3)$	–	$\text{Rep}(S_3)$	–
$\{0\}$	S_3	S_3	S_3	S_3	S_3	S_3	S_3

Table 1: Dual symmetry upon gauging a subgroup H starting from S_3 with anomaly p . Note that there is no discrete torsion in 2d neither for S_3 nor for cyclic groups. For the sake of this thesis we juts care about the fusion ring of the dual category, so we do not specify its F -symbols.

The left S_3 -action on the cosets can be written as

$$(A, a) \cdot \mu_k = \mu_{k+A} , \quad k \in \mathbb{Z}_2 , \quad (6.22)$$

and the lift $\sigma: G/H \rightarrow G$ can be taken to be

$$\sigma(\mu_k) = X^k = (k, 0) . \quad (6.23)$$

The h function can be computed to be

$$h((A, a), \mu_k) = \sigma((A, a) \mu_k)^{-1} (A, a) \sigma(\mu_k) = (0, (-1)^k a) \in H . \quad (6.24)$$

It can be checked that this satisfies the product rule

$$h((A, a)(B, b), \mu_k) = h((A, a), (B, b) \mu_k) h((B, b), \mu_k) . \quad (6.25)$$

When $p = 0$, the F and $\tilde{F}$ -symbols are all trivial:

$$[F_{g,h,k}^{ghk}]_{gh,hk} = \omega_0(g, h, k) = 1 , \quad [\tilde{F}_{g,h,\mu_k}^{gh\mu_k}]_{gh,h\mu_k} = 1 . \quad (6.26)$$

The resulting strip algebra is 24-dimensional. Following the procedure described at the end of App. E, it turns out to have six 2-dimensional irreducible representations, which fuse according to the product rule of S_3 . For $p = 3$, the bulk and boundary F -symbols are nontrivial but one still gets the same results.

To summarize, we get

$$d_{\mathfrak{R}} = 1, \quad \dim(\mathfrak{R}) = 2, \quad \tilde{d}_{\mu_k} = \sqrt{3}, \quad \forall \mathfrak{R} \in S_3, \quad \forall k \in \mathbb{Z}_2, \quad (\mathbf{p} = \mathbf{0}, \mathbf{3}) . \quad (6.27)$$

The probabilities are therefore all $q_{\mathfrak{R}} = 1/6$ and we have

$$S[\rho_{\mathfrak{R}}] = \frac{c}{3} \log \frac{L}{\epsilon} + 2 \log(g) + \log 2 \quad \forall \mathfrak{R} \in S_3 , \quad (6.28)$$

realizing entanglement equipartition.

$H = \mathbb{Z}_2$. Since $\mathbb{Z}_2$ is not a normal subgroup of S_3 , there are three different choices, pairwise related by conjugation. We choose here $H = \mathbb{Z}_2^{(a)} = \{1, X\}$, as defined in (G.13). The right-cosets are $S_3/\mathbb{Z}_2^{(a)} = \{\mu_0, \mu_1, \mu_{-1}\}$, which consist of

$$\mu_0 = \{1, X\}, \quad \mu_1 = \{R, RX\}, \quad \mu_{-1} = \{R^{-1}, R^{-1}X\}. \quad (6.29)$$

The left S_3 -action on the cosets can be written as

$$(A, a) \cdot \mu_k = \mu_{(-1)^A(k+a)}, \quad (6.30)$$

where the subscript is considered modulo 3 in the interval $\{-1, 0, 1\}$. The lift $\sigma: G/H \rightarrow G$ can be taken to be

$$\sigma(\mu_k) = (0, k), \quad (6.31)$$

and the h function can be computed to be

$$h((A, a), \mu_k) = (A, 0) \in H. \quad (6.32)$$

For the strip algebra and its ring of representations, we get the same results regardless of the value of $p \in \{0, 2, 4\}$. In particular, the strip algebra has dimension 54, and three representations of dimensions 3, 3 and 6, which have the fusion ring of $\text{Rep}(S_3)$. We thus denote them by 1, ϵ and s respectively, referring to the “trivial”, “sign”, and “standard” representations of S_3 . Explicitly, the fusion ring is then

$$\mathcal{R}_\epsilon \times \mathcal{R}_\epsilon = \mathcal{R}_1, \quad \mathcal{R}_\epsilon \times \mathcal{R}_s = \mathcal{R}_s \times \mathcal{R}_\epsilon = \mathcal{R}_s, \quad \mathcal{R}_s \times \mathcal{R}_s = \mathcal{R}_1 + \mathcal{R}_\epsilon + \mathcal{R}_s. \quad (6.33)$$

From this one can see that the quantum dimensions are $d_1 = d_\epsilon = 1$ and $d_s = 2$.

To summarize, we obtain

$$\begin{aligned} d_1 &= 1 & \dim(\mathfrak{R}_1) &= 3 \\ d_\epsilon &= 1 & \dim(\mathfrak{R}_\epsilon) &= 3 \\ d_s &= 2 & \dim(\mathfrak{R}_s) &= 6 & (\mathbf{p} = \mathbf{0}, \mathbf{2}, \mathbf{4}) \\ \tilde{d}_{\mu_k} &= \sqrt{2} & k &\in \{-1, 0, 1\} \end{aligned} \quad (6.34)$$

The probabilities are therefore $q_1 = q_\epsilon = 1/6$ and $q_s = 2/3$ and the symmetry-resolved entanglement entropies are

$$\begin{aligned} S[\rho_1] &= S[\rho_\epsilon] = \frac{c}{3} \log \frac{L}{\epsilon} + 2 \log(g) + \log 3, \\ S[\rho_s] &= \frac{c}{3} \log \frac{L}{\epsilon} + 2 \log(g) + \log 12. \end{aligned} \quad (6.35)$$

Equipartition is manifestly not realized in this case.

If more than one module categories are available (and realized in the theory), there is no canonical way of choosing one. In general, one can try to take the one(s) with the smallest number of simple objects, but we consider all choices to be valid schemes for the computation of the SREE.

6.2 Fibonacci fusion category

Consider the unitary Fibonacci fusion category, introduced in Sec. 5.5. As already mentioned, the only indecomposable module category is the regular. The corresponding strip algebra is 13-dimensional and has two irreps of dimensions [98]

$$\dim(\mathfrak{R}_1) = 2, \quad \dim(\mathfrak{R}_W) = 3. \quad (6.36)$$

From this data, the symmetry-resolved entanglement entropy for the reduced density matrix ρ constructed using the protocol (4.2), can be easily computed. In particular one has $D = 1 + \varphi = \varphi^2$ and

$$q_1 = \frac{2}{\varphi^4} = 7 - 3\sqrt{5}, \quad q_W = \frac{3}{\varphi^3} = -6 + 3\sqrt{5}. \quad (6.37)$$

For the entanglement we get

$$\begin{aligned} S[\rho_1] &= \frac{c}{3} \log \frac{L}{\epsilon} + 2 \log(g) + \log 2, \\ S[\rho_W] &= \frac{c}{3} \log \frac{L}{\epsilon} + 2 \log(g) + \log \frac{3}{2}(1 + \sqrt{5}). \end{aligned} \quad (6.38)$$

6.3 Ising CFT

Another simple and interesting example is the **Ising** fusion category. For concreteness, we focus Ising CFT, whose symmetry and operator content is reviewed in Sec. 5.2. The theory has 3 simple conformal boundary conditions $\mathcal{B}_+, \mathcal{B}_-, \mathcal{B}_f$, with g -functions⁷⁸ given by $g_+ = g_- = \frac{1}{\sqrt{2}}$ and $g_f = 1$. These boundary conditions generate the only indecomposable module category, the regular one. The corresponding strip algebra (see Sec. 5.4) is 34-dimensional and has 3 irreps (corresponding to the simple objects of the dual category $\mathcal{C}_{\mathcal{C}}^* \cong \text{Ising}$) of dimensions $\dim(\mathfrak{R}_1) = \dim(\mathfrak{R}_\eta) = 3$ and $\dim(\mathfrak{R}_\mathcal{N}) = 4$. Using this data, it is easy to compute the symmetry-resolved entanglement entropy for the reduced density matrix ρ constructed using the protocol (4.2). In particular, we have $D = 2 + \sqrt{2}$ and

$$q_1 = q_\eta = \frac{3}{2}(3 - 2\sqrt{2}), \quad q_\mathcal{N} = 2\sqrt{2}(3 - 2\sqrt{2}). \quad (6.39)$$

Therefore, using (4.36) we obtain

$$\begin{aligned} S[\rho_1] &= S[\rho_\eta] = \frac{c}{3} \log \frac{L}{\epsilon} + 2 \log(1 + \sqrt{2}) + \log \left(\frac{3}{2}(3 - 2\sqrt{2}) \right), \\ S[\rho_\mathcal{N}] &= \frac{c}{3} \log \frac{L}{\epsilon} + 2 \log(1 + \sqrt{2}) + \log (2\sqrt{2}(3 - 2\sqrt{2})). \end{aligned} \quad (6.40)$$

6.4 $\mathfrak{su}(2)_{10}$ WZW model with E_6 modular invariant

Let us first review some facts about the $\mathfrak{su}(2)_\ell$ current algebra, where $\ell \in \mathbb{Z}_{\geq 0}$ is the level. This has $\ell + 1$ integrable representations labeled by the $\mathfrak{su}(2)$ Dynkin label $i \in \mathcal{I} \equiv \{0, \dots, \ell\}$. The

⁷⁸The theory-dependent factor g in equation (4.30) is therefore $g_{\text{Ising}} = 1/\sqrt{2}$.

OPE of such representations is

$$i \otimes j = \bigoplus_{\substack{k=|i-j| \\ k=i-j \bmod 2}}^{\min(i+j, 2\ell-i-j)} k, \quad (6.41)$$

from which the fusion coefficients $\mathcal{N}_{ij}^k \in \{0, 1\}$ follow. We can also see that every line is self-dual, $\bar{i} = i$. Such representations form a richer structure called a Modular Tensor Category (MTC), whose full data can be found e.g. in [138, 175]. Among the various pieces of data that it contains we will only need the so-called modular S -matrix:

$$S_{ij} = \sqrt{\frac{2}{\ell+2}} \sin\left(\pi \frac{(i+1)(j+1)}{\ell+2}\right). \quad (6.42)$$

This is a real, symmetric and unitary matrix, and thus squares to the identity. In rational CFTs that enjoy such an $\mathfrak{su}(2)_\ell$ chiral algebra, this is precisely the matrix that appears in the modular S -transformation of characters

$$\chi_i(\tilde{q}) = \sum_{j \in \mathcal{I}} S_{ij} \chi_j(q), \quad (6.43)$$

where $q = e^{2\pi i \tau}$, $\tilde{q} = e^{-2\pi i/\tau}$, and τ is the modular parameter of the torus, belonging to the complex upper half-plane.

Consider now the Wess-Zumino-Witten model with $\mathfrak{su}(2)_{10}$ extended chiral algebra, and modular invariant of type E_6 (see e.g. [163]). The partition function on the torus is

$$Z_{E_6} = |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2, \quad (6.44)$$

where χ_i are the irreducible characters of the chiral algebra. Notice that in this model only a subset of 6 out of the 11 possible irreps appears (at least in the untwisted sector). We denote this subset as $\mathcal{E} \equiv \{0, 3, 4, 6, 7, 10\}$.

Let us start by discussing the conformal boundary conditions⁷⁹ of the model, following [176]. We write the partition function of the BCFT on an annulus of modular parameter q and boundary conditions A and B in the open string channel as

$$Z_{AB}(q) = \sum_{i \in \mathcal{I}} n_{AB}^i \chi_i(q), \quad n_{AB}^i \in \mathbb{Z}_{\geq 0}, \quad (6.45)$$

where n_{AB}^i is the multiplicity of the representation i in the interval Hilbert space. This can be rewritten in the closed string channel as

$$Z_{AB}(q) = \langle A | \tilde{q}^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{12})} | B \rangle, \quad (6.46)$$

where $|A\rangle$ and $|B\rangle$ are Cardy boundary states. These states are of the form

$$|A\rangle = \sum_{i \in \mathcal{E}} \frac{\psi_A^i}{\sqrt{S_{0i}}} |i\rangle, \quad A, B \in \mathcal{M}, \quad (6.47)$$

where $|i\rangle = \sum_n |i, i; n, n\rangle$ are known as “Ishibashi states” and are the sum over all descendants n of a given primary $|i, i\rangle$ that appears diagonally in the partition function (6.44). Moreover,

⁷⁹More precisely, we mean the boundary conditions invariant under the full $\mathfrak{su}(2)_{10}$ algebra.

the coefficients in (6.47) are such that the so-called ‘‘Cardy condition’’ is satisfied. This is the requirement that the right-hand sides of (6.46) and (6.45) are the same, after a modular S -transformation. In terms of the coefficients ψ_A^i , the Cardy condition becomes

$$n_{AB}^i = \sum_{j \in \mathcal{E}} \frac{S_{ij}}{S_{0j}} \psi_A^j (\psi_B^j)^* . \quad (6.48)$$

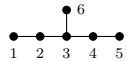
This equation has the interpretation of the spectral decomposition of the matrices n^i in terms of a common set of eigenvectors $\{|\psi^j\rangle\}_{j \in \mathcal{E}}$, provided that these form an orthonormal and complete set. In this model, a solution to (6.48) is given by the following procedure. Consider the adjacency matrix G of the E_6 graph.⁸⁰ The indices of G correspond to the nodes of the graph, and the entries are defined to be $G_{ab} = 1$ if a and b are connected, and zero otherwise. Then, the matrices n^i defined in (6.45) that solve the Cardy condition are recursively given by letting

$$n^0 = \mathbb{1}_6 , \quad n^1 = G , \quad n^i = n^1 n^{i-1} - n^{i-2} . \quad (6.49)$$

All such matrices turn out to have non-negative integer entries. Explicitly, we obtain

$$\begin{aligned} n^0 &= \mathbb{1}_6 & n^4 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} & n^8 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \\ n^1 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} & n^5 &= \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 & 2 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix} & n^9 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\ n^2 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} & n^6 &= \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} & n^{10} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ n^3 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} & n^7 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{aligned} \quad (6.50)$$

One verifies that these matrices provide a 6-dimensional non-negative-integer representation (NIM rep) of the fusion ring of $\mathfrak{su}(2)_{10}$.⁸¹ In particular, all these matrices commute pairwise and thus admit a common set of eigenvectors. Since the spectrum of n^1 is non-degenerate, one

⁸⁰We label the nodes of the E_6 diagram as , which is the convention used in [13] and [177].

⁸¹More explicitly, this means that $n^i n^j = \sum_{k=0}^{10} \mathcal{N}_{ij}^k n^k$, where the entries of n^i are in $\mathbb{Z}_{\geq 0}$.

can choose its (normalized) eigenvectors as the common set for all matrices. It is then

$$\begin{aligned}
|\psi^1\rangle &= \frac{1}{2\sqrt{3+\sqrt{3}}} \left(1, \sqrt{2+\sqrt{3}}, 1+\sqrt{3}, \sqrt{2+\sqrt{3}}, 1, \sqrt{2} \right), \\
|\psi^2\rangle &= \frac{1}{2} (1, 1, 0, -1, -1, 0), \\
|\psi^3\rangle &= \frac{1}{2\sqrt{3+\sqrt{3}}} \left(\sqrt{2+\sqrt{3}}, 1, -\sqrt{2}, 1, \sqrt{2+\sqrt{3}}, -1-\sqrt{3} \right), \\
|\psi^4\rangle &= \frac{1}{2\sqrt{3+\sqrt{3}}} \left(\sqrt{2+\sqrt{3}}, -1, -\sqrt{2}, -1, \sqrt{2+\sqrt{3}}, 1+\sqrt{3} \right), \\
|\psi^5\rangle &= \frac{1}{2} (1, -1, 0, 1, -1, 0), \\
|\psi^6\rangle &= \frac{1}{2\sqrt{3+\sqrt{3}}} \left(1, -\sqrt{2+\sqrt{3}}, 1+\sqrt{3}, -\sqrt{2+\sqrt{3}}, 1, -\sqrt{2} \right).
\end{aligned} \tag{6.51}$$

From these, using (6.47) we obtain the Cardy states in terms of the Ishibashi states:

$$\begin{aligned}
|1\rangle &= \frac{|1\rangle\rangle + \sqrt[4]{2}|2\rangle\rangle + |3\rangle\rangle + |4\rangle\rangle + \sqrt[4]{2}|5\rangle\rangle + |6\rangle\rangle}{\sqrt{2}}, \\
|2\rangle &= \frac{\sqrt[4]{3} \left((2+\sqrt{3})|1\rangle\rangle + |3\rangle\rangle - |4\rangle\rangle + \sqrt[4]{14+8\sqrt{3}}(|2\rangle\rangle - |5\rangle\rangle) - (2+\sqrt{3})|6\rangle\rangle \right)}{\sqrt{6+4\sqrt{3}}}, \\
|3\rangle &= \sqrt{2+\sqrt{3}}(|1\rangle\rangle + |6\rangle\rangle) - \sqrt{2-\sqrt{3}}(|3\rangle\rangle + |4\rangle\rangle), \\
|4\rangle &= \frac{\sqrt[4]{3} \left((2+\sqrt{3})|1\rangle\rangle + |3\rangle\rangle - |4\rangle\rangle + \sqrt[4]{14+8\sqrt{3}}(|5\rangle\rangle - |2\rangle\rangle) - (2+\sqrt{3})|6\rangle\rangle \right)}{\sqrt{6+4\sqrt{3}}}, \\
|5\rangle &= \frac{|1\rangle\rangle - \sqrt[4]{2}|2\rangle\rangle + |3\rangle\rangle + |4\rangle\rangle - \sqrt[4]{2}|5\rangle\rangle + |6\rangle\rangle}{\sqrt{2}}, \\
|6\rangle &= |1\rangle\rangle - |3\rangle\rangle + |4\rangle\rangle - |6\rangle\rangle.
\end{aligned} \tag{6.52}$$

Let us now discuss the topological lines⁸² of the theory, following [13].⁸³ One can define the torus partition function twisted by the insertion of a line $\mathcal{L}_x$ along the time direction as

$$Z_{\mathcal{L}_x}(q) = \sum_{i,j \in \mathcal{I}} n_x^{ij} \chi_i(q) \chi_j(\bar{q}). \tag{6.53}$$

Similarly to above, one requires that this partition function be equal, after an S -transformation, to the partition function with the line $\mathcal{L}_x$ inserted along the space direction. This imposes a constraint on what lines can exist and on the coefficients n_x^{ij} .

In this case, there turn out to be 12 lines, labeled by a pair $x = (a, b)$, where a and b label the nodes of the E_6 graphs (following the conventions of Fn. 80) but, for this particular modular invariant, b only takes values 1 or 2. Furthermore, the coefficients appearing in (6.53) are given by

$$n_{(a,b)}^{ij} = \sum_{c \in \{1,5,6\}} n_{ca}^i n_{cb}^j, \tag{6.54}$$

⁸²More specifically, we mean the lines that commute with the whole $\mathfrak{su}(2)_{10}$ chiral algebra, and not merely with the Virasoro algebra.

⁸³The corresponding twisted partition functions have also been classified in [178].

where on the right-hand side the matrices (6.50) appear. Using this, with an S -transformation we can determine how the line $\mathcal{L}_{(a,b)}$ acts on each module of the chiral algebra:

$$\mathrm{tr}[\mathcal{L}_{(a,b)} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}] = \sum_{i,j \in \mathcal{I}} \underbrace{(S n_{(a,b)} S^{-1})^{ij}}_{\equiv \lambda_{(a,b)}^{ij}} \chi_i(q) \chi_j(\bar{q}) . \quad (6.55)$$

For $(a,b) = (1,1)$ this reproduces the modular invariant partition function (6.44). More generally, one verifies that the partition function (6.55) has the same block structure as (6.44) and each line acts – as it should be – by multiplication by a scalar in each module of $\mathfrak{su}(2)_{10} \oplus \mathfrak{su}(2)_{10}$. From here, we can extract the fusion ring of the lines as

$$\lambda_x^{ij} \lambda_y^{ij} = \sum_C N_{xy}^z \lambda_z^{ij} , \quad (6.56)$$

where the x, y, z indices denote pairs of lowercase indices, i.e. line operators. The resulting fusion rules are reported in Table 2, along with a relabeling of the line operators. This fusion ring is known in the literature to span the so-called Ocneanu’s algebra of E_6 [178, 179]:

$$\mathrm{Oc}(E_6) \equiv E_6 \otimes_{A_3} E_6 . \quad (6.57)$$

Here, the algebra E_6 is the so-called graph algebra associated to the corresponding Dynkin diagram (see e.g. [163]). As a vector space, it is spanned by the nodes of the E_6 graph, which we denote (the ordering of the lines reflects that of the nodes of the diagram in Footnote Fn. 80) as

$$E_6 = \langle 1, D_2, D_3, D_4, \eta, \mathcal{N} \rangle . \quad (6.58)$$

The adjacency matrix G of the diagram gives the product of D_2 with all other basis vectors. All other product rules are fixed by imposing commutativity and associativity.⁸⁴ It is a nontrivial fact that the resulting algebra turns out to have – in this preferred basis – non-negative integer structure coefficients.⁸⁵ The Ocneanu algebra (6.57) is then defined by tensoring two copies of the graph algebra of E_6 while identifying a common A_3 subalgebra, the graph algebra of the A_3 Dynkin diagram. More explicitly, the symbol $\otimes_{A_3}$ means that $a u \otimes_{A_3} b = a \otimes_{A_3} u b$ for any $u \in A_3$. This subalgebra is explicitly given by

$$A_3 = \langle 1, \mathcal{N}, \eta \rangle , \quad (6.59)$$

which multiply according to the fusion ring of the $\mathbb{Z}_2$ Tambara-Yamagami categories (cf. with Table 2). Denoting the basis vectors in the second copy of E_6 with primed symbols (apart from those belonging to A_3) we have that the full algebra is spanned by the 12 elements:

$$\mathrm{Oc}(E_6) = \langle 1, D_2, D_3, D_4, \eta, \mathcal{N}, D'_2, D'_3, D'_4, X, Y, Z \rangle . \quad (6.60)$$

⁸⁴For instance, consider the A_3 Dynkin diagram: $\bullet \xrightarrow{\mathcal{N}} \bullet \xrightarrow{\eta} \bullet$. Here 1 is the identity and $\mathcal{N}$ is the generator of algebra. We see from the graph that $\mathcal{N}1 = \mathcal{N}$, $\mathcal{N}^2 = 1 + \eta$, $\mathcal{N}\eta = \mathcal{N}$. Then associativity gives $\eta\mathcal{N}\mathcal{N} = 1 + \eta = \eta + \eta^2$. This implies $\eta^2 = 1$. The result is the fusion algebra of the Ising, or $\mathbb{Z}_2$ Tambara-Yamagami fusion category.

⁸⁵The resulting fusion ring appears as a sub-ring of Table 2.

In the notation of Table 2 top, the two copies of E_6 are associated with the two entries of the pair (a, b) . The list therein already takes into account the identification by the common A_3 , thus there is no double counting.

We highlight the fact that the construction of all such algebras yields a preferred basis, with respect to which it is the fusion algebra of a fusion category.

(a, b)	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)	(1, 2)	(6, 2)	(5, 2)	(2, 2)	(4, 2)	(3, 2)
line	1	D_2	D_3	D_4	η	$\mathcal{N}$	D'_2	D'_3	D'_4	X	Y	Z

$\times$	1	D_2	D_3	D_4	η	$\mathcal{N}$	D'_2	D'_3	D'_4
1	1								
D_2	D_2	$1+D_3$							
D_3	D_3	$D_2+D_4+\mathcal{N}$	$1+\eta+2D_3$						
D_4	D_4	$\eta+D_3$	$D_2+D_4+\mathcal{N}$	$1+D_3$					
η	η	D_4	D_3	D_2	1				
$\mathcal{N}$	$\mathcal{N}$	D_3	D_2+D_4	D_3	$\mathcal{N}$	$1+\eta$			
D'_2	D'_2	X	Z	Y	D'_4	D'_3	$1+D'_3$		
D'_3	D'_3	Z	$X+Y$	Z	D'_3	$D'_2+D'_4$	$D'_2+D'_4+\mathcal{N}$	$1+\eta+2D'_3$	
D'_4	D'_4	Y	Z	X	D'_2	D'_3	$\eta+D'_3$	$D'_2+D'_4+\mathcal{N}$	$1+D'_3$
X	X	D'_2+Z	D'_3+X+Y	D'_4+Z	Y	Z	D_2+Z	D_3+X+Y	D_4+Z
Y	Y	D'_4+Z	D'_3+X+Y	D'_2+Z	X	Z	D_4+Z	D_3+X+Y	D_2+Z
Z	Z	D'_3+X+Y	$D'_2+D'_4+2Z$	D'_3+X+Y	Z	$X+Y$	D_3+X+Y	D_2+D_4+2Z	D_3+X+Y

$\times$	X	Y	Z
X	$1+D_3+D'_3+X+Y$		
Y	$\eta+D_3+D'_3+X+Y$	$1+D_3+D'_3+X+Y$	
Z	$\mathcal{N}+D_2+D_4+D'_2+D'_4+2Z$	$\mathcal{N}+D_2+D_4+D'_2+D'_4+2Z$	$1+\eta+2D_3+2D'_3+2X+2Y$

Table 2: Top: list of topological line operators of the $\mathfrak{su}(2)_{10}$ WZW model with E_6 modular invariant, both in the notation of [13] and in shorthand notation. Center and bottom: fusion ring of the $\text{Oc}(E_6) \equiv E_6 \otimes_{A_3} E_6$ fusion algebra. Since multiplication is commutative, only one ordering of the products is indicated.

As is clear from the E_6 untwisted partition function (6.44), the spectrum of local primary operators of the model, written in terms of representations of the $\mathfrak{su}(2)_{10} \oplus \mathfrak{su}(2)_{10}$ extended chiral algebra, is given by the 12 operators

$$\{\Phi_{i,\bar{j}}\}_{i,j=0,6} \cup \{\Phi_{i,\bar{j}}\}_{i,j=3,7} \cup \{\Phi_{i,\bar{j}}\}_{i,j=4,10} . \quad (6.61)$$

Recall from (6.55) that the action of a line $\mathcal{L}_{(a,b)}$ on one of these operators (and on their descendants) is given by

$$\mathcal{L}_{(a,b)} \Phi_{i,\bar{j}} = \lambda_{(a,b)}^{ij} \Phi_{i,\bar{j}} . \quad (6.62)$$

Using this action and the operator-state map, we can determine how the Cardy states (6.52) – and thus the boundary conditions – transform under the action of the lines. We find in particular that

$$\mathcal{L}_{(a,1)} |c\rangle = \sum_{d=1}^6 M_{ac}^d |d\rangle , \quad a, c \in \{1, \dots, 6\} , \quad (6.63)$$

$$\mathcal{L}_{(1,b)} |c\rangle = \sum_{d=1}^6 M_{bc}^d |d\rangle , \quad b \in \{1, 2\}, \quad c \in \{1, \dots, 6\} , \quad (6.64)$$

where M_{ab}^c are the fusion coefficients of the E_6 graph algebra. In other words, the boundary conditions transform in the regular representation of the fusion ring of the (anti)chiral lines. Using that $\mathcal{L}_{(a,b)} = \mathcal{L}_{(a,1)} \mathcal{L}_{(1,b)}$ we recover the full action of the lines on the boundaries:

$$\mathcal{L}_{(a,b)} |c\rangle = \sum_d M_{abc}^d |d\rangle , \quad (6.65)$$

where $M_{abc}^d \equiv \sum_e M_{ab}^e M_{ec}^d = \sum_f M_{af}^d M_{bc}^f$.

From now on, let us focus on the (anti)chiral part only. Such lines form one of the unitary fusion categories with fusion ring E_6 . For our purposes it will not be important to identify precisely which choice of F -symbols appears, so we will denote one such category also as E_6 . We can argue that the boundary conditions belong to the regular module category $\mathcal{M} = \mathcal{C}$ as this is the only possible categorification of the regular action.⁸⁶ In particular, notice that the regular representation of the category of all 12 lines is not realized in this model as conformal boundary conditions. Therefore, while symmetries can be studied just from the “kinematical” point of view – and computations can be done with respect to any module category – it is important to check which modules are indeed physically realized. Only these constitute valid choices for computing the SREE in a given theory.

⁸⁶Indeed we know that, through the internal Hom construction (see Fn. 31), it is possible to recover an algebra object associated to the $\mathcal{C}$ -module category $\mathcal{M}$ as $\mathfrak{A}_m = \bigoplus_c \tilde{N}_{am}^c a$, with different m yielding Morita-equivalent algebras. For the regular module category we can choose $m = 1$ which gives $\mathfrak{A}_1 = 1$. It can be checked that this object admits (up to equivalence) only one algebra structure, and thus there is only one module category with the fusion ring of the regular. Physically, it can be thought of as the only possible way of gauging the trivial symmetry.

We proceed here with the computation of the SREE for the “chiral” part of the symmetry category, with respect to the conformal boundary conditions described above. The quantum dimensions of the lines are

$$d_1 = 1, \quad d_{D_2} = \sqrt{2 + \sqrt{3}}, \quad d_{D_3} = 1 + \sqrt{3}, \quad d_{D_4} = \sqrt{2 + \sqrt{3}}, \quad d_\eta = 1, \quad d_{\mathcal{N}} = \sqrt{2}, \quad (6.66)$$

and the dimensions of strip representations are, from (4.11),

$$\begin{aligned} \dim(\mathfrak{R}_1) &= 6, & \dim(\mathfrak{R}_{D_2}) &= 10, & \dim(\mathfrak{R}_{D_3}) &= 14, \\ \dim(\mathfrak{R}_{D_4}) &= 10, & \dim(\mathfrak{R}_\eta) &= 6, & \dim(\mathfrak{R}_{\mathcal{N}}) &= 8. \end{aligned} \quad (6.67)$$

We obtain the probabilities

$$\begin{aligned} q_1 &= \frac{3}{2} \left(5 - 3\sqrt{2} + \sqrt{3} - \sqrt{6} \right), \\ q_{D_2} &= \frac{5}{2} \left(-6 + 4\sqrt{2} - 4\sqrt{3} + 3\sqrt{6} \right), \\ q_{D_3} &= -7 \left(-4 - 3\sqrt{3} + \sqrt{6(7 + 4\sqrt{3})} \right), \\ q_{D_4} &= \frac{5}{2} \left(-6 + 4\sqrt{2} - 4\sqrt{3} + 3\sqrt{6} \right), \\ q_\eta &= \frac{3}{2} \left(5 - 3\sqrt{2} + \sqrt{3} - \sqrt{6} \right), \\ q_{\mathcal{N}} &= -2 \left(6 - 5\sqrt{2} + \sqrt{6(3 - 2\sqrt{2})} \right), \end{aligned} \quad (6.68)$$

from which the computation of the symmetry-resolved entanglement entropies follows via (4.36). We do not report it for brevity.

We can also decide to restrict to the $\frac{1}{2}E_6$ subsymmetry,⁸⁷ generated by

$$\frac{1}{2}E_6 = \langle 1, D_3, \eta \rangle. \quad (6.69)$$

One can check from Table 2 that this is closed under product. Moreover, one can see that E_6 carries a $\mathbb{Z}_2$ -grading compatible with fusion, with $E_6^{\text{even}} = \frac{1}{2}E_6$ and $E_6^{\text{odd}} = \langle D_2, D_4, \mathcal{N} \rangle$. This implies that the regular module category of E_6 is reducible when seen as a module category over $\frac{1}{2}E_6$, namely $E_6 = E_6^{\text{even}} \oplus E_6^{\text{odd}}$.⁸⁸ The regular module category over $\frac{1}{2}E_6$ is clearly E_6^{even} . In this case we obtain

$$\dim(\mathfrak{R}_1) = 3, \quad \dim(\mathfrak{R}_{D_3}) = 6, \quad \dim(\mathfrak{R}_\eta) = 3, \quad (6.70)$$

and thus

$$q_1 = 1 - \frac{\sqrt{3}}{2}, \quad q_{D_3} = \sqrt{3} - 1, \quad q_\eta = 1 - \frac{\sqrt{3}}{2}. \quad (6.71)$$

⁸⁷This was briefly introduced in Sec. 5.7, though with a different notation. The correspondence is $\mathcal{L}_X \leftrightarrow \eta$, $\mathcal{L}_Y \leftrightarrow D_3$.

⁸⁸Also see [180]. Note that, in the notation therein, $\frac{1}{2}E_6$ is indeed the adjoint subcategory of E_6 , as one can check it is $\otimes$ -generated by $D_2 \otimes \bar{D}_2 = 1 \oplus D_3$.

For the module category E_6^{odd} , we need to understand what is the dual module category with respect to it. We can reason as follows. Notice that E_6 is an (E_6, E_6) -bimodule category in an obvious way. We think of the objects in this bimodule category as the topological interfaces between a theory with symmetry E_6 and itself (the theory after the trivial gauging). We can restrict the lines of both sides to those in $\frac{1}{2}E_6$, and – since they are closed under the action – the interfaces to those of E_6^{odd} . This is telling us that E_6^{odd} is a $(\frac{1}{2}E_6, \frac{1}{2}E_6)$ -bimodule category. In general, this is not enough to conclude that the first category in the pair is dual (with respect to the module category considered) to the second one. However, this is true when we are dealing with an *invertible* bimodule category, see Proposition 4.2 of [181]. In our example, it is indeed true that E_6^{odd} is an invertible bimodule category. It is in fact the generator of the Brauer-Picard group of $\frac{1}{2}E_6$, see [180]. The NIM reps associated to this module category can be extracted from Table 2 and are

$$\tilde{N}_1 = \mathbb{1}_{3 \times 3}, \quad \tilde{N}_{D_3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \tilde{N}_\eta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6.72)$$

The boundary quantum dimensions of objects in E_6^{odd} coincide with the corresponding ones in E_6 , as can be checked. The representations of the strip algebra have dimensions

$$\dim(\mathfrak{R}_1) = 3, \quad \dim(\mathfrak{R}_{D_3}) = 8, \quad \dim(\mathfrak{R}_\eta) = 3. \quad (6.73)$$

We finally obtain

$$q_1 = \frac{21}{2} - 6\sqrt{3}, \quad q_{D_3} = 4(3\sqrt{3} - 5), \quad q_\eta = \frac{21}{2} - 6\sqrt{3}, \quad (6.74)$$

from which the entropies immediately follow.

A Some facts in category theory

Here, we aim to provide a more detailed – though necessarily incomplete – introduction to some categorical concepts used throughout the thesis. We first give, in App. A.1, piecemeal definitions that build up to that of fusion categories, which is the concept needed in physics – at least in the context of this thesis.⁸⁹ Then, in Apps. A.2 to A.4, we elaborate on some of the ingredients that enter in those definitions. We refer to [113] for more detailed explanations. Other useful references include [138, 182–184]. Finally, in App. A.5 we summarize the conventions and results used throughout the thesis.

A.1 Summary of definitions

Let us sketch here the meaning of various structures and adjectives associated to categories. These are necessary to build up to the definition of fusion categories, which are the ones of physical interest, at least in this thesis. We do not enter deeply in all definitions, as it is enough to have an intuition of which category has which structures:

- *Additive category.* A category where: all hom-sets are abelian groups and composition is bilinear; there exists a *zero* object 0 such that $\text{Hom}(0, 0) = 0$;⁹⁰ there exist all direct sums (which will be defined in App. A.2).
- *$\mathbb{K}$ -linear category.* An additive category where hom-sets are $\mathbb{K}$ -vector spaces, and composition is linear.
- *Abelian category.* An additive category that has all kernels, cokernels, and images.⁹¹
- *Semisimple category.* An abelian category where every object is a direct sum of simple objects.⁹²
- *Finite $\mathbb{K}$ -linear abelian category.* An abelian, $\mathbb{K}$ -linear category where all hom-sets are finite-dimensional and where there are finitely many simple objects (modulo isomorphism).⁹³
- *Monoidal category.* A category with a tensor product, associator and unit, subject to certain coherence conditions. We will explain this more in detail in App. A.2.
- *Rigid monoidal category.* A monoidal category with duals (see App. A.3).
- *Finite multi-tensor category.* A finite, $\mathbb{K}$ -linear, abelian, rigid monoidal category where the functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is bilinear on morphisms. It is called *tensor* if $\text{Hom}(1, 1) \cong \mathbb{K}$.

⁸⁹Other types of categories, including braided and modular categories, also play important roles in physics, see e.g. [138].

⁹⁰The zero on the RHS denotes the trivial group.

⁹¹We will not need the definitions of such objects here, but these are generalizations of the familiar concepts defined for vector spaces. Such structures allow one to define monomorphisms (a generalization of injective functions), epimorphisms (a generalization of surjective functions) and subobjects.

⁹²A nonzero object X is simple if it has only 0 and X as subobjects.

⁹³Actually, there are two more technical conditions that we do not report for simplicity, see Def. 1.8.6 of [113].

- *Finite multi-fusion category.* A finite semisimple multi-tensor category. It is called *fusion* if $\text{Hom}(1, 1) \cong \mathbb{K}$.

An important result for abelian categories is Schur's lemma [113]: any morphism between simple objects, $x \rightarrow y$, is either zero or an isomorphism. For finite $\mathbb{K}$ -linear abelian categories a stronger result holds if $\mathbb{K}$ is algebraically closed:⁹⁴ $\text{Hom}(x, y)$ is $\mathbb{K}$ if $x \cong y$ or 0 otherwise.

Note that in the case $\mathcal{C} = \text{Rep}(G)$ this reduces to the usual Schur lemma for group representations.

A.2 Direct sums, tensor products and bases of morphisms

Let $\mathcal{C}$ be an abelian $\mathbb{C}$ -linear category. As anticipated in App. A.1 this is in particular additive and thus has all direct sums, which we now turn to define. Given two objects $\{X_i\}_{i=1,2}$, their *direct sum* is an object Y with morphisms $\iota_i: X_i \rightarrow Y$ and $\pi_i: Y \rightarrow X_i$ such that

$$\pi_i \iota_i = \text{id}_{X_i}, \quad \sum_i \iota_i \pi_i = \text{id}_Y. \quad (\text{A.1})$$

We denote $Y = X_1 \oplus X_2$. Note that this implies $\pi_2 \iota_1 = 0$ and $\pi_1 \iota_2 = 0$ (even when X_1 and X_2 are isomorphic). To see this, precompose the equation on the right of (A.1) by ι_1 :

$$\iota_1 \pi_1 \iota_1 + \iota_2 \pi_2 \iota_1 = \iota_1 \Rightarrow \iota_1 + \iota_2 \pi_2 \iota_1 = \iota_1 \Rightarrow \iota_2 \pi_2 \iota_1 = 0 \Rightarrow \pi_2 \iota_2 \pi_2 \iota_1 = 0 \Rightarrow \pi_2 \iota_1 = 0. \quad (\text{A.2})$$

We have made repeated use of (A.1), and in the second-to-last step we have composed by π_2 . Analogously, one can show that $\pi_1 \iota_2 = 0$. Using this, one can show that, for fixed i , the linear maps

$$\begin{aligned} \text{Hom}(X_i, Y) &\rightarrow \text{Hom}(X_i, X_1) \oplus \text{Hom}(X_i, X_2) \\ f &\mapsto (\pi_1 f, \pi_2 f) \\ \text{Hom}(X_i, Y) &\leftarrow \text{Hom}(X_i, X_1) \oplus \text{Hom}(X_i, X_2) \\ \iota_1 g_1 + \iota_2 g_2 &\leftarrow (g_1, g_2) \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} \text{Hom}(Y, X_i) &\rightarrow \text{Hom}(X_1, X_i) \oplus \text{Hom}(X_2, X_i) \\ f &\mapsto (f \iota_1, f \iota_2) \\ \text{Hom}(Y, X_i) &\leftarrow \text{Hom}(X_1, X_i) \oplus \text{Hom}(X_2, X_i) \\ g_1 \pi_1 + g_2 \pi_2 &\leftarrow (g_1, g_2) \end{aligned} \quad (\text{A.4})$$

are pairwise inverses.

Let now $\mathcal{C}$ be in addition finite semisimple. Then, every object is isomorphic to a direct sum of the form

$$Y = \bigoplus_i n_i x_i, \quad n_i \in \mathbb{Z}_{\geq 0}, \quad (\text{A.5})$$

⁹⁴This includes the case of finite fusion categories over $\mathbb{K} = \mathbb{C}$.

for simple objects x_i (which are pairwise non-isomorphic). In this case, we have inclusions $\iota_{x_i}^{(k)} : x_i \rightarrow Y$ and projections $\pi_{x_i}^{(k)} : Y \rightarrow x_i$, that satisfy a generalization of (A.1):

$$\pi_{x_i}^{(k)} \iota_{x_i}^{(k)} = \text{id}_{x_i}, \quad \sum_{i,k} \iota_{x_i}^{(k)} \pi_{x_i}^{(k)} = \text{id}_Y, \quad (\text{A.6})$$

as well as the “orthogonality” relation $\pi_{x_i}^{(k)} \iota_{x_i}^{(l)} = 0$ for $k \neq l$. All the arguments above can be generalized by iteration until all direct summands are simple objects. Using a generalized version of (A.3) and Schur’s Lemma, we obtain

$$\text{Hom}(x_i, Y) \cong \bigoplus_j \text{Hom}(x_i, x_j)^{\oplus n_j} \cong \text{Hom}(x_i, x_i)^{\oplus n_i}. \quad (\text{A.7})$$

Let $\text{id}_{x_i}^{(k)}$ be morphism $0 \oplus \dots \oplus \text{id}_{x_i} \oplus \dots \oplus 0$ with the identity in position k . This is an element of the RHS of (A.7). The preimage of $\text{id}_{x_i}^{(k)}$ through the isomorphism is simply $\iota_{x_i}^{(k)}$. Since $\text{Hom}(x_i, x_i) \cong \mathbb{C}$, the set $\{\text{id}_{x_i}^{(k)}\}_k$ is a basis of the RHS, and therefore $\{\iota_{x_i}^{(k)}\}_k$ is a basis of $\text{Hom}(x_i, Y)$. Similarly, $\{\pi_{x_i}^{(k)}\}_k$ is a basis of $\text{Hom}(Y, x_i)$.

Note that composition of morphisms gives a bilinear pairing

$$\begin{aligned} \text{Hom}(Y, x_i) \times \text{Hom}(x_i, Y) &\rightarrow \mathbb{C}, \\ (f, g) &\mapsto fg. \end{aligned} \quad (\text{A.8})$$

Here, we are regarding the composition fg as a number via the canonical isomorphism $\text{Hom}(x_i, x_i) \cong \mathbb{C}$, which maps identity morphism to 1. The already mentioned statement that

$$\pi_{x_i}^{(k)} \iota_{x_i}^{(l)} = \delta_{kl} \text{id}_{x_i} \quad (\text{A.9})$$

implies that this pairing is nondegenerate and that the two bases $\{\iota_{x_i}^{(k)}\}_k$ and $\{\pi_{x_i}^{(k)}\}_k$ are, in fact, dual to each other.

Let us now introduce a *monoidal structure* on $\mathcal{C}$. This is given by the following data: a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$; a collection of isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ which are the components of a natural isomorphism of functors (called associator); a unit $1 \in \mathcal{C}$; isomorphisms $l_X : 1 \otimes X \xrightarrow{\sim} X$ and $r_X : X \otimes 1 \xrightarrow{\sim} X$ which are also components of natural isomorphisms (called left and right unitors). These are required to make the pentagon diagram

$$\begin{array}{ccccc} & & ((X \otimes Y) \otimes Z) \otimes W & & \\ & \swarrow a_{X,Y,Z} \otimes \text{id}_W & & \searrow a_{X \otimes Y, Z, W} & \\ (X \otimes (Y \otimes Z)) \otimes W & & & & (X \otimes Y) \otimes (Z \otimes W) \\ & \searrow a_{X, Y \otimes Z, W} & & \swarrow a_{X, Y, Z \otimes W} & \\ X \otimes ((Y \otimes Z) \otimes W) & \xrightarrow{\text{id}_X \otimes a_{Y, Z, W}} & & & X \otimes (Y \otimes (Z \otimes W)) \end{array} \quad (\text{A.10})$$

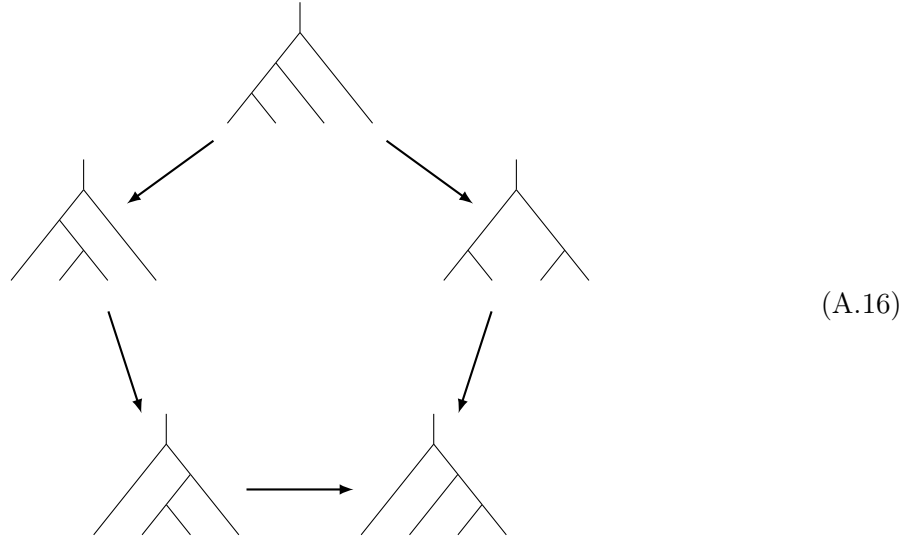
and the triangle diagram

$$(\text{id}_X \otimes l_Y) a_{X,1,Y} = r_X \otimes \text{id}_Y \quad (\text{A.11})$$

where the sum is over simple objects that appear in a decomposition of $x \otimes y$, and over a basis of the corresponding hom-spaces (A.12).

In practice, one does not use a totally arbitrary basis, as partial “gauge-fixing” can be done. In particular, when for example $x = 1$, one has $\text{Hom}(1 \otimes y, y)$, and thus one can choose $v_{1y}^{y,1} = l_y$, and similarly for r_y .

Once a choice of basis for all junction vector spaces has been made, the associator a can be expressed in components.⁹⁸ This leads to the notion of F -symbols, see Fig. 1. Then, the commutativity of the pentagon diagram (A.10) can be physically understood as follows. In the following figure, there are two possible ways to connect the top configuration of line defects to the bottom-right, by performing sequences of F -moves:⁹⁹



Since the path-integral should only be a function of the network of inserted lines, the results of the two sequences of F -moves should agree. This imposes a constraint on the F -symbols themselves. With the definition in Fig. 1, the pentagon identity then reads:

$$\sum_{\delta} [F_{fcd}^e]_{(g;\beta\gamma)(l;\nu\delta)} [F_{abl}^e]_{(f;\alpha\delta)(k;\mu\lambda)} = \sum_{h,\psi,\sigma,\rho} [F_{abc}^g]_{(f;\alpha\beta)(h;\psi\sigma)} [F_{ahd}^e]_{(g;\sigma\gamma)(k;\rho\lambda)} [F_{bcd}^k]_{(h;\psi\rho)(l;\nu\mu)} \cdot \quad (\text{A.17})$$

A.3 Rigid and pivotal structures

Here we explain the concept of *rigid* (or *dual*) structure on a monoidal category $\mathcal{C}$. For concreteness, we assume that $\mathcal{C}$ is also semisimple, finite, $\mathbb{C}$ -linear, abelian, and that 1 is simple. Such a category is required to have, for each object X , a dual $\bar{X}$ and *evaluations* and *coevaluations*

⁹⁸More precisely, the F -symbols are matrices describing the isomorphisms:

$$\bigoplus_e \text{Hom}(a \otimes b, e) \otimes \text{Hom}(e \otimes c, d) \xrightarrow{\sim} \bigoplus_f \text{Hom}(a \otimes f, d) \otimes \text{Hom}(b \otimes c, f) \quad (\text{A.15})$$

induced by the associator.

⁹⁹I would like to thank Niccolò Porciani for letting me use his amazingly parametric code for this picture.

morphisms:

$$\begin{aligned} \text{ev}_X: \bar{X} \otimes X &\rightarrow 1, \\ \text{coev}_X: 1 &\rightarrow X \otimes \bar{X}, \end{aligned} \quad (\text{A.18})$$

which respect the zigzag identities¹⁰⁰

$$\begin{aligned} \text{X} \text{ (vertical line)} \text{ with cap } \text{ev}_X \text{ and cup } \text{coev}_X &= \text{X} \text{ (vertical line)}, \\ \text{X-bar} \text{ (vertical line)} \text{ with cap } \text{ev}_X \text{ and cup } \text{coev}_X &= \text{X-bar} \text{ (vertical line)}. \end{aligned} \quad (\text{A.19})$$

One can also define dual (or transpose) of a morphism $f: X \rightarrow Y$ as the morphism $\bar{f}: \bar{Y} \rightarrow \bar{X}$ given by

$$\bar{X} \text{ (vertical line)} \text{ with dot } \bar{f} \equiv \bar{Y} \text{ (vertical line)} \text{ with cap } \text{ev}_Y \text{ and cup } \text{coev}_X \text{ and } f: X \rightarrow Y. \quad (\text{A.20})$$

One shows that $\overline{\bar{f}g} = \bar{g}\bar{f}$ and that dual and inverse commute, $\overline{f^{-1}} = \bar{f}^{-1}$.

Once a rigid structure has been chosen, one can choose the morphisms (A.18) as basis vectors of $\text{Hom}(\bar{X} \otimes X, 1)$ and $\text{Hom}(1, X \otimes \bar{X})$ respectively.

A *pivotal structure* on a rigid monoidal category is defined as a natural isomorphism of monoidal functors

$$\eta: \text{id}_{\mathcal{C}} \xrightarrow{\sim} \overline{(-)}. \quad (\text{A.21})$$

Explicitly, this means we have a collection of isomorphisms $\{\eta_X\}_{X \in \mathcal{C}}$ such that for every morphism $f: X \rightarrow Y$ the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \bar{X} \\ f \downarrow & & \downarrow \bar{f} \\ Y & \xrightarrow{\eta_Y} & \bar{Y} \end{array} \quad (\text{A.22})$$

commutes. We can write this condition as $\bar{f} = \eta_Y \circ f \circ \eta_X^{-1}$. Moreover, we require $\eta_{X \otimes Y} = \eta_X \otimes \eta_Y$. One can show that the relations

$$\overline{\eta_X} = \eta_{\bar{X}}^{-1}, \quad \eta_{\bar{X}} = \overline{\eta_X}. \quad (\text{A.23})$$

hold for every X . Furthermore, defining

$$\begin{aligned} \text{X} \text{ (vertical line)} \text{ with cap } \text{ev}_X &\equiv \text{X-bar} \text{ (vertical line)} \text{ with cap } \text{ev}_{\bar{X}}, \\ \text{X-bar} \text{ (vertical line)} \text{ with cap } \text{ev}_{\bar{X}} &\equiv \text{X} \text{ (vertical line)} \text{ with cap } \text{ev}_X, \\ \text{X-bar} \text{ (vertical line)} \text{ with cup } \text{coev}_X &\equiv \text{X} \text{ (vertical line)} \text{ with cup } \text{coev}_{\bar{X}}, \\ \text{X} \text{ (vertical line)} \text{ with cup } \text{coev}_{\bar{X}} &\equiv \text{X-bar} \text{ (vertical line)} \text{ with cup } \text{coev}_X. \end{aligned} \quad (\text{A.24})$$

¹⁰⁰In this whole appendix, in order to be as clear as possible, all lines are oriented upwards. Moreover, near evaluations and coevaluations, we explicitly denote the orientation with arrows, in order to avoid any possible confusion.

the following zigzag identities hold:

$$\begin{array}{c} \bar{X} \\ \downarrow \\ \text{coev}_X \\ \uparrow \\ X \end{array} \begin{array}{c} \text{ev}_X \\ \downarrow \\ X \end{array} \begin{array}{c} \downarrow \\ \bar{X} \end{array} = \begin{array}{c} \downarrow \\ \bar{X} \end{array}, \quad \begin{array}{c} \bar{X} \\ \downarrow \\ \text{coev}_X \\ \uparrow \\ X \end{array} \begin{array}{c} \text{ev}_X \\ \downarrow \\ X \end{array} \begin{array}{c} \downarrow \\ X \end{array} = \begin{array}{c} \downarrow \\ X \end{array}. \quad (\text{A.25})$$

With the pivotal structure one can define the following “quantum dimension”:

$$\dim_\eta(X) \equiv \eta_X \begin{array}{c} \text{ev}_{\bar{X}} \\ \downarrow \\ \text{coev}_X \end{array} = \begin{array}{c} \text{ev}_X \\ \downarrow \\ \text{coev}_X \end{array}. \quad (\text{A.26})$$

This is a priori a different concept than that of Frobenius-Perron (FP) dimension of an object x , $\text{FPdim}(x)$. This is defined as the largest positive eigenvalue of the non-negative real matrix N_x , where $(N_x)_{yz} = N_{xy}^z$, and is always greater or equal to 1. It exists thanks to the FP theorem. However, one can show that if $\dim_\eta(x) > 0$ for all simple x , then this must be equal to the Frobenius-Perron dimension.¹⁰¹ In fact, such a pivotal structure always exists for unitary/reflection-positive theories. We thus assume that topological line defects are described by such a category. Furthermore, in this case the category is automatically *spherical*, namely $\dim_\eta(x) = \dim_\eta(\bar{x})$.

A.4 Self-dual objects and Frobenius-Schur indicators

Let $\mathcal{C}$ be a pivotal monoidal (abelian $\mathbb{C}$ -linear) category, with pivotal structure η . Suppose a simple object x is isomorphic to its dual, and let

$$f: x \xrightarrow{\sim} \bar{x} \quad (\text{A.27})$$

be such an isomorphism. Then, consider the composition

$$x \xrightarrow{\eta_x} \bar{\bar{x}} \xrightarrow{\bar{f}} \bar{x}, \quad (\text{A.28})$$

which is also an isomorphism. Because x is simple, these two isomorphisms must be proportional to each other. We define the proportionality to be the *Frobenius-Schur (FS) indicator* of x :

$$\bar{f} \circ \eta_x = \kappa_x f, \quad (\text{A.29})$$

with $\kappa_x \in \mathbb{C}^\times$. Note that any other isomorphism f differs from it by a nonzero multiplicative constant, which cancels in the definition (A.29). Using string diagrams, we can express (A.29) as:

$$\begin{array}{c} \bar{x} \\ \downarrow \\ \bar{f} \end{array} = \begin{array}{c} \text{ev}_{\bar{x}} \\ \downarrow \\ f \\ \uparrow \\ \text{coev}_x \end{array} \begin{array}{c} \downarrow \\ \bar{x} \end{array} = \kappa_x \begin{array}{c} \bar{x} \\ \downarrow \\ f \end{array}. \quad (\text{A.30})$$

¹⁰¹Using that $\eta_{X \otimes Y} = \eta_X \otimes \eta_Y$, one can in fact show that $d_x \equiv \dim_\eta(x)$ respects the tensor product structure, namely $d_x d_y = \sum_z N_{xy}^z d_z$. Equivalently, we can write $N_x \mathbf{d} = d_x \mathbf{d}$, where $\mathbf{d}$ is the positive vector with entries $\{d_y\}_y$. By the FP theorem, the eigenvalue of $\mathbf{d}$ must be the FP one, namely it is $d_x = \text{FPdim}(x)$.

Alternatively, we can write:

$$\begin{array}{c} \text{ev}_{\bar{x}} \\ \curvearrowright \\ \eta_x \quad x \quad f \quad \bar{x} \quad f^{-1} \\ \curvearrowleft \\ \text{coev}_x \end{array} \Bigg|_x = \kappa_x \Bigg|_x . \quad (\text{A.31})$$

Note that, unlike in the zigzag identities (A.19), the evaluation and coevaluation morphisms appearing in (A.31) are associated with $\bar{x}$ and x respectively, rather than both with x .

From (A.29) we can immediately deduce that $\kappa_x \in \{1, -1\}$. Indeed, plugging that equation into itself we obtain

$$f = \frac{1}{\kappa_x} \overline{\left(\frac{1}{\kappa_x} \bar{f} \circ \eta_x \right)} \circ \eta_x = \frac{1}{\kappa_x^2} \bar{\eta}_x \circ \bar{f} \circ \eta_x = \frac{1}{\kappa_x^2} f, \quad (\text{A.32})$$

where we have used (A.22) and (A.23). We conclude that $\kappa_x^2 = 1$.

There are slightly different – but equivalent – definitions of the FS indicator. For example, in [138] it is defined as

$$\begin{array}{c} \text{ev}_x \\ \curvearrowright \\ f^{-1} \quad \bar{x} \quad f \\ \curvearrowleft \\ \text{coev}_{\bar{x}} \end{array} \Bigg|_{\bar{x}} = \kappa'_x \Bigg|_{\bar{x}} . \quad (\text{A.33})$$

As an equation, this reads $\eta_x^{-1} \circ \bar{f}^{-1} = \kappa'_x f^{-1}$. By proving that $\kappa'_x \in \{1, -1\}$ and by taking an inverse one can readily see that $\kappa'_x = \kappa_x$.

We now show that $\kappa_x = \kappa_{\bar{x}}$. Given an isomorphism $g: \bar{x} \rightarrow \bar{\bar{x}}$, the FS indicator is defined as $\bar{g} \circ \eta_{\bar{x}} = \kappa_{\bar{x}} g$. Choosing $g = (\bar{f})^{-1}$, we obtain

$$\overline{\bar{f}^{-1}} \circ \eta_{\bar{x}} = \kappa_{\bar{x}} (\bar{f})^{-1}. \quad (\text{A.34})$$

Composing with η_x^{-1} on the left and using (A.22) we obtain

$$f^{-1} = \kappa_{\bar{x}} \eta_x^{-1} \circ (\bar{f})^{-1}. \quad (\text{A.35})$$

Finally, taking the inverse and comparing with the definition of κ_x , we see that it must be $\kappa_x = \kappa_{\bar{x}}$.

In practice, one does not want to distinguish between isomorphic, but different, objects. In particular, if $x \cong \bar{x}$, we want to avoid dealing with $\bar{x}$ and morphisms involving it. In the notation of (A.12), we want to avoid using $v_{x\bar{x}}^1$ or $v_1^{x\bar{x}}$, and we instead define new morphisms

$$\begin{array}{c} \text{ev}_{\bar{x}} \\ \curvearrowright \\ \eta_x \quad x \quad f \quad \bar{x} \\ \curvearrowleft \\ \text{coev}_x \end{array} \equiv \begin{array}{c} \text{ev}_{\bar{x}} \\ \curvearrowright \\ \eta_x \quad x \quad f \quad \bar{x} \\ \curvearrowleft \\ \text{coev}_x \end{array} \quad \text{and} \quad \begin{array}{c} x \\ \curvearrowright \\ \text{coev}_x \end{array} \equiv \begin{array}{c} x \\ \curvearrowright \\ \text{coev}_x \end{array} . \quad (\text{A.36})$$

These are basis vectors of $\text{Hom}(x \otimes x, 1)$ and $\text{Hom}(1, x \otimes x)$ respectively (note that these are one-dimensional vector spaces). With this definition, (A.31) becomes

$$\begin{array}{c} \text{---} x \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \text{---} x \end{array} = \kappa_x \begin{array}{c} | \\ | \\ | \\ | \\ | \end{array} \quad . \quad (\text{A.37})$$

Furthermore, we have

$$\begin{array}{c} \text{---} x \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \text{---} x \end{array} = \begin{array}{c} \text{ev}_{\bar{x}} \\ \bullet \\ \circlearrowleft \\ \bullet \\ \text{coev}_x \end{array} = \dim_{\eta}(x) \equiv d_x. \quad (\text{A.38})$$

Choosing such a basis, we can perform an F -move in (A.37) and rewrite it as (see Fig. 1 for the notation)

$$[F_{xxx}^x]_{11} d_x \begin{array}{c} | \\ | \\ | \\ | \\ | \end{array} = \kappa_x \begin{array}{c} | \\ | \\ | \\ | \\ | \end{array} \quad (\text{A.39})$$

from which we deduce that $[F_{xxx}^x]_{11} = \kappa_x / d_x$.

A.5 Summary of conventions

It is common to rescale the basis vectors so that a slightly modified version of (A.13) and (A.14) holds (this is the normalization used e.g. in [117], see also [188] and [182] for a more detailed discussion):

$$\begin{array}{c} z' \\ | \\ \beta \\ \diamond \\ \alpha \\ | \\ z \end{array} \begin{array}{c} x \\ y \end{array} = \delta_{zz'} \delta_{\alpha\beta} \sqrt{\frac{d_x d_y}{d_z}} \begin{array}{c} | \\ | \\ | \\ | \\ | \end{array} \quad , \quad \sum_{z, \alpha} \sqrt{\frac{d_z}{d_x d_y}} \begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ \alpha \\ | \\ \alpha \\ \diagup \quad \diagdown \\ x \quad y \end{array} = \begin{array}{c} | \\ | \\ | \\ | \\ | \end{array} \begin{array}{c} x \\ y \end{array} \quad (\text{A.40})$$

where $\{d_x\}_x$ are the quantum dimensions (also see discussion after (A.26)), and we have adapted the same coloring scheme as the rest of the thesis. Here, we denoted the new basis vector with square vertices, rather than round. We make this choice of normalization everywhere outside of this section (regardless of the shape that vertices have in the various drawings).

The definition of F -symbols (which respect the pentagon identities (A.17)) was given in Fig. 1. We report it here along with the inverse F -move, and the corresponding dual moves. Here, as everywhere in the rest of the thesis, it is assumed that $\mathcal{C}$ is unitary, namely that it is

possible to choose the F -symbols to be unitary matrices. We then have:

$$\begin{aligned}
\begin{array}{c} d \\ \text{---} \nu \\ \text{---} \mu \\ \text{---} e \\ \text{---} a \quad b \quad c \end{array} &= \sum_{f,\rho,\sigma} [F_{abc}^d]_{(e;\mu\nu)(f;\rho\sigma)} \begin{array}{c} d \\ \text{---} \sigma \\ \text{---} f \\ \text{---} \rho \\ \text{---} a \quad b \quad c \end{array}, & \begin{array}{c} d \\ \text{---} \sigma \\ \text{---} f \\ \text{---} \rho \\ \text{---} a \quad b \quad c \end{array} &= \sum_{e,\mu,\nu} [F_{abc}^d]^*_{(e;\mu\nu)(f;\rho\sigma)} \begin{array}{c} d \\ \text{---} \nu \\ \text{---} e \\ \text{---} \mu \\ \text{---} a \quad b \quad c \end{array}, \\
\begin{array}{c} a \quad b \quad c \\ \text{---} \mu \\ \text{---} e \\ \text{---} \nu \\ \text{---} d \end{array} &= \sum_{f,\rho,\sigma} [F_{abc}^d]^*_{(e;\mu\nu)(f;\rho\sigma)} \begin{array}{c} a \quad b \quad c \\ \text{---} \rho \\ \text{---} f \\ \text{---} \sigma \\ \text{---} d \end{array}, & \begin{array}{c} a \quad b \quad c \\ \text{---} \rho \\ \text{---} f \\ \text{---} \sigma \\ \text{---} d \end{array} &= \sum_{f,\rho,\sigma} [F_{abc}^d]_{(e;\mu\nu)(f;\rho\sigma)} \begin{array}{c} a \quad b \quad c \\ \text{---} \nu \\ \text{---} e \\ \text{---} \mu \\ \text{---} d \end{array}.
\end{aligned} \tag{A.41}$$

Finally, we have the following alternative moves, derived from the ones in (A.41) by manipulating the associator diagrams:

$$\begin{aligned}
\begin{array}{c} c \quad d \\ \text{---} \nu \quad e \quad \mu \\ \text{---} a \quad b \end{array} &= \sum_{f,\rho,\sigma} \sqrt{\frac{d_e d_f}{d_b d_c}} [F_{aed}^f]_{(c;\nu\sigma)(b;\mu\rho)} \begin{array}{c} c \quad d \\ \text{---} \sigma \quad f \\ \text{---} \rho \quad a \quad b \end{array}, \\
\begin{array}{c} c \quad d \\ \text{---} \mu \quad e \quad \nu \\ \text{---} a \quad b \end{array} &= \sum_{f,\rho,\sigma} \sqrt{\frac{d_e d_f}{d_a d_d}} [F_{ceb}^f]^*_{(a;\mu\rho)(d;\nu\sigma)} \begin{array}{c} c \quad d \\ \text{---} \sigma \quad f \\ \text{---} \rho \quad a \quad b \end{array}.
\end{aligned} \tag{A.42}$$

Conventions for module categories. We refer to Sec. 2.3 for explanations on module categories. The $\tilde{F}$ -symbols of a left $\mathcal{C}$ -module category $\mathcal{M}$, as defined in (2.17), satisfy a pentagon equation that also involves the F -symbols of $\mathcal{C}$:

$$\sum_{\rho} [\tilde{F}_{dcm}^n]_{(e;\beta\mu)(p;\nu\rho)} [\tilde{F}_{abp}^n]_{(d;\alpha\rho)(q;\sigma\tau)} = \sum_{f,\gamma,\delta,\omega} [F_{abc}^e]_{(d;\alpha\beta)(f;\gamma\delta)} [\tilde{F}_{afm}^n]_{(e;\delta\mu)(q;\omega\tau)} [\tilde{F}_{bcm}^q]_{(f;\gamma\omega)(p;\nu\sigma)}. \tag{A.43}$$

Similarly, right-module $\tilde{F}$ -symbols are defined in (2.19) and satisfy

$$\sum_{\omega} [{}^R\tilde{F}_{\bar{q}bc}^{\bar{m}}]_{(\bar{p};\sigma\nu)(f;\gamma\omega)} [{}^R\tilde{F}_{\bar{n}af}^{\bar{m}}]_{(\bar{q};\tau\omega)(e;\delta\mu)} = \sum_{d\alpha\beta\rho} [{}^R\tilde{F}_{\bar{n}ab}^{\bar{p}}]_{(\bar{q};\tau\sigma)(d;\alpha\rho)} [{}^R\tilde{F}_{\bar{n}dc}^{\bar{m}}]_{(\bar{p};\rho\nu)(e;\beta\mu)} [F_{abc}^e]_{(d;\alpha\beta)(f;\gamma\delta)}. \tag{A.44}$$

If $\tilde{F}$ solves the left module pentagon identity, then

$$[{}^R\tilde{F}_{\bar{n}ab}^{\bar{m}}]_{(\bar{p};\rho\nu)(e;\alpha\mu)} \equiv [\tilde{F}_{abm}^n]^*_{(e;\alpha\mu)(p;\nu\rho)} \tag{A.45}$$

solves the right module pentagon identity. In this thesis we always make this choice of right module $\tilde{F}$ -symbols.

Finally, analogously to (A.40), we normalize bulk-boundary junctions in such a way that

$$\begin{array}{c} a \\ \text{---} \\ \text{---} m \end{array} = \sum_{n,\mu} \sqrt{\frac{d_n}{d_a d_m}} \begin{array}{c} a \quad m \\ \text{---} \mu \\ \text{---} n \\ \text{---} m \end{array}, \quad \begin{array}{c} n \\ \text{---} \nu \\ \text{---} p \\ \text{---} \mu \\ \text{---} m \end{array} = \delta_{mn} \delta_{\mu\nu} \sqrt{\frac{d_a d_p}{d_m}} \begin{array}{c} a \quad m \\ \text{---} \mu \\ \text{---} p \\ \text{---} m \end{array}, \tag{A.46}$$

for simple objects $a \in \mathcal{C}$ and $m, n, p \in \mathcal{M}$. Similar equations hold for a right module category.

B Complements on the tube algebra

In this appendix we present some further computations regarding the product structure of tube algebras, and prove that it is associative. Finally, we provide a useful formula for the action of the tube algebra on twisted boundary states.

The basis elements of the tube algebra are defined in (2.8), and the multiplication coefficients M of the algebra are introduced in (2.10). To compute them we proceed as follows. First we fuse b with a into c using (A.40), then we section the diagram horizontally into three parts:

$$(2.10) = \sum_{c,\gamma} \sqrt{\frac{d_c}{d_a d_b}} \quad \begin{array}{c} \begin{array}{c} k \\ \sigma \\ e \\ \rho \\ b \\ c \end{array} \begin{array}{c} b \\ \gamma \\ h \\ a \\ \mu \\ g \end{array} \begin{array}{c} c \\ \nu \\ d \\ \end{array} \end{array} \quad (B.1)$$

and finally we apply the F -moves in (A.42). The resulting expression for M is in (2.11).

Case of an invertible symmetry. In the special case of an invertible symmetry group G , possibly with 't Hooft anomaly ω , the fusion category $\mathcal{C}$ is $\mathbf{Vec}_G^\omega$. The anomaly is parametrized by $\omega \in H^3(G, \mathbb{U}(1))$. The only nonvanishing F -symbols are

$$[F_{abc}^{(abc)}]_{(ab)(bc)} \equiv F_{abc} = \omega(a, b, c) . \quad (B.2)$$

On the left we used parenthesis (gh) to indicate the group multiplication, while in the middle we introduced a lighter notation for the F -symbols since they only depend on three group elements. The pentagon equation (A.17) is nonvanishing only if $e = (abcd)$, $f = (ab)$, $g = (abc)$, $h = (bc)$, $k = (bcd)$, $l = (cd)$, and it reads $F_{(ab)cd} F_{ab(cd)} = F_{abc} F_{a(bc)d} F_{bcd}$. This is precisely the cocycle condition $d\omega = 1$.

The elements of the tube algebra are specified by two labels: the vertical line incoming from below, and the horizontal line. Referring to (2.8), we specify g and a , while $d = ag$ and $h = aga^{-1}$. The nonvanishing multiplication coefficients of the tube algebra are then specified by three labels:

$$M_{g,a,ag,aga^{-1},b,baga^{-1},baga^{-1}b^{-1}}^{ba,bag} \equiv M_{g,a,b} . \quad (B.3)$$

Formula (2.12) gives:

$$M_{g,a,b} = [F_{baga^{-1}b^{-1},b,a}]^* [F_{b,aga^{-1},a}] [F_{b,a,g}]^* = \frac{\omega(b, aga^{-1}, a)}{\omega(baga^{-1}b^{-1}, b, a) \omega(b, a, g)} . \quad (B.4)$$

Associativity of the tube algebra is guaranteed by the following identity:

$$M_{g,a,b} M_{g,ba,c} = M_{g,a,cb} M_{aga^{-1},b,c} . \quad (B.5)$$

When expanded in terms of ω , this identity becomes equivalent to

$$\frac{d\omega(c, b, a, g) d\omega(c, baga^{-1}b^{-1}, b, a)}{d\omega(c, b, aga^{-1}, a) d\omega(cbaga^{-1}b^{-1}c^{-1}, c, b, a)} = 1 , \quad (B.6)$$

which is indeed a true equation because ω is a cocycle.

Associativity in the noninvertible case. In the general case, the tube algebra product is given by (2.11). Using an argument similar to the previous one, but with lengthier algebra, and in particular applying the pentagon equation (A.17) four times, one can show that the product is associative. Let us sketch the proof in the multiplicity-free case. Associativity is the equation

$$\sum_{ds} M_{g,a,m,h,b,n,k}^{d,s} M_{g,d,s,k,c,p,l}^{f,t} = \sum_{er} M_{g,a,m,h,e,r,l}^{f,t} M_{h,b,n,k,c,p,l}^{e,r} \quad \text{for} \quad \begin{array}{c|c} c & p \\ \hline b & n \\ \hline a & m \\ \hline & g \end{array} \begin{array}{c|c} l & \\ \hline k & \\ \hline h & \\ \hline g & \end{array}. \quad (\text{B.7})$$

Substituting the expression for M in (2.12), the quantum dimensions are the same on the two sides and can be dropped. One is left with the relation

$$\begin{aligned} \sum_{ds} [F_{kba}^s]_{nd}^* [F_{bha}^s]_{nm} [F_{bag}^s]_{dm}^* \cdot [F_{lcd}^t]_{pf}^* [F_{ckd}^t]_{ps} [F_{cdg}^t]_{fs}^* \\ = \sum_{er} [F_{lea}^t]_{rf}^* [F_{eha}^t]_{rm} [F_{eag}^t]_{fm}^* \cdot [F_{lcb}^r]_{pe}^* [F_{ckb}^r]_{pn} [F_{cbh}^r]_{en}^*. \end{aligned} \quad (\text{B.8})$$

This can be proven using the following set of four pentagon equations, obtained from (A.17) with various indices and possibly applying unitarity once or twice:

$$\begin{aligned} [F_{ckd}^t]_{ps} [F_{kba}^s]_{nd}^* &= \sum_r [F_{pba}^t]_{rd}^* [F_{ckb}^r]_{pn} [F_{cna}^t]_{rs}, \\ \sum_e [F_{cba}^f]_{ed}^* [F_{eag}^t]_{fm} [F_{cbm}^t]_{es} &= [F_{cdg}^t]_{fs} [F_{bag}^s]_{dm}, \\ [F_{cbh}^r]_{en}^* [F_{eha}^t]_{rm} &= \sum_s [F_{cna}^t]_{rs} [F_{bha}^s]_{nm} [F_{cbm}^t]_{es}, \\ \sum_d [F_{pba}^t]_{rd} [F_{lcd}^t]_{pf} [F_{cba}^f]_{ed}^* &= [F_{lcb}^r]_{pe} [F_{lea}^t]_{rf}. \end{aligned} \quad (\text{B.9})$$

The first one and the complex conjugate of the second one are used on the LHS of (B.8), while the third one and the complex conjugate of the fourth one are used on the RHS. On each side one ends up with a quadruple sum over d, s, e, r of the same string, proving associativity.

Action on boundaries. Let us write a useful formula for the action of the tube algebra onto a twisted boundary state, namely on a boundary condition where a bulk line ends topologically. This computation applies for example to $\mathcal{C}$ -symmetric TQFTs (see e.g. [114]). By open/closed duality, this is computed from the data of the corresponding module category $\mathcal{M}$.

Using the various moves, in particular those in (A.40) and in (A.41) and (A.42), adapted to module categories, we find the following relation:

$$\begin{array}{c} h \\ \swarrow \\ a \quad \beta \quad d \quad \alpha \quad g \\ \downarrow \\ a \quad \mu \quad m \end{array} = \sum_{n,\nu,\rho,\sigma} \sqrt{\frac{d_g d_a}{d_d}} [\tilde{F}_{agm}^n]_{(d;\alpha\rho)(m;\mu\sigma)} [\tilde{F}_{ham}^n]_{(d;\beta\rho)(n;\sigma\nu)}^* \begin{array}{c} h \\ \swarrow \\ \nu \\ \downarrow \\ n \end{array}. \quad (\text{B.10})$$

The sum is over a basis of junctions $\mathcal{L}_h \otimes \mathcal{B}_n \xrightarrow{\nu} \mathcal{B}_n$, $\mathcal{L}_d \otimes \mathcal{B}_m \xrightarrow{\rho} \mathcal{B}_n$, $\mathcal{L}_a \otimes \mathcal{B}_m \xrightarrow{\sigma} \mathcal{B}_n$, while the ticks indicate that the lines or the boundaries are glued together. The configuration on the RHS

represents, via open/closed duality, a state $|\mathcal{B}_{n,\nu}^{(h)}\rangle$ belonging to the h -twisted Hilbert space. The configuration on the LHS of (B.10), on the other hand, represents the action of the tube algebra basis element (2.8) on a twisted state $|\mathcal{B}_{m,\mu}^{(g)}\rangle$. In the special case where $g = h = 1$ and $d = a$, Eqn. (B.10) reproduces the fusion algebra action on the untwisted circle Hilbert space.

C Strongly separable and C^* -algebras

C.1 Strongly separable algebras

Consider a finite-dimensional unital semisimple associative algebra $\mathcal{A}$ ¹⁰²

$$\mathcal{X}_a \times \mathcal{X}_b = \sum_c T_{ab}^c \mathcal{X}_c \quad (\text{C.1})$$

over $\mathbb{C}$ as in (3.5). Here $\{\mathcal{X}_a\}_a$ is a basis of the algebra, and one defines a trace as $\text{tr}(\mathcal{X}_a) = \sum_m T_{am}^m$ then extended by linearity. This is the trace over the regular representation. In the finite-dimensional case, such algebras are automatically strongly separable and the symmetric bilinear form K_{ab} defined in (3.7) is non-degenerate. We call $\tilde{K} \equiv K^{-1}$ its inverse. Indeed, the Wedderburn-Artin theorem states that every finite-dimensional unital semisimple associative algebra over $\mathbb{C}$ is isomorphic to a sum of matrix algebras,

$$\mathcal{A} \cong \bigoplus_{\alpha=1}^r \text{Mat}_{N_\alpha \times N_\alpha}(\mathbb{C}), \quad (\text{C.2})$$

where r is the number of blocks, N_α are the sizes of the matrices, and those numbers are unique up to permutation. Note that r is also the number of irreducible representations, and N_α are their dimensions. Using the isomorphism (C.2), all algebraic properties can be explicitly checked.

Following [122], we construct the dual basis $\{\mathcal{X}^a\}_a$ with respect to the bilinear form:

$$\mathcal{X}^a = \sum_b \tilde{K}^{ab} \mathcal{X}_b. \quad (\text{C.3})$$

It satisfies $\text{tr}(\mathcal{X}_a \mathcal{X}^b) = \delta_a^b$. Going to a basis, one proves that for every $x \in \mathcal{A}$:

$$x = \sum_a \text{tr}(x \mathcal{X}_a) \mathcal{X}^a = \sum_a \text{tr}(x \mathcal{X}^a) \mathcal{X}_a. \quad (\text{C.4})$$

The symmetric separability idempotent (SSI) defined in Sec. 3.1 can be written as

$$e = \sum_{ab} \tilde{K}^{ab} \mathcal{X}_a \otimes \mathcal{X}_b = \sum_a \mathcal{X}_a \otimes \mathcal{X}^a = \sum_a \mathcal{X}^a \otimes \mathcal{X}_a. \quad (\text{C.5})$$

It is clearly symmetric under the exchange of the two factors. We can prove that e commutes with every element of $\mathcal{A}$. Using the shorthand notations $\mathcal{X}_b e \equiv (\mathcal{X}_b \otimes 1)e$ and $e \mathcal{X}_b \equiv e(1 \otimes \mathcal{X}_b)$:

$$\begin{aligned} \mathcal{X}_b e &= \sum_a \mathcal{X}_b \mathcal{X}_a \otimes \mathcal{X}^a = \sum_{ac} \text{tr}(\mathcal{X}_b \mathcal{X}_a \mathcal{X}^c) \mathcal{X}_c \otimes \mathcal{X}^a = \\ &= \sum_{ac} \mathcal{X}_c \otimes \text{tr}(\mathcal{X}^c \mathcal{X}_b \mathcal{X}_a) \mathcal{X}^a = \sum_c \mathcal{X}_c \otimes \mathcal{X}^c \mathcal{X}_b = e \mathcal{X}_b. \end{aligned} \quad (\text{C.6})$$

To prove the remaining properties, we first use K to obtain a pivotal relation. Consider the following element of $\mathcal{A} \otimes \mathcal{A}$:

$$\sum_b \mathcal{X}_b \otimes \mathcal{X}_a \mathcal{X}^b = \sum_b \mathcal{X}_b \mathcal{X}_a \otimes \mathcal{X}^b = \sum_{bc} T_{ba}^c \mathcal{X}_c \otimes \mathcal{X}^b = \sum_{bc} \mathcal{X}_b \otimes T_{ca}^b \mathcal{X}^c. \quad (\text{C.7})$$

In the first equality we used that e commutes with $\mathcal{A}$ also “from the inside”, which follows from the fact that e is symmetric. In the last equality we moved the scalars and renamed the indices. Since $\{\mathcal{X}_b\}_b$ is a basis of the first factor of $\mathcal{A} \otimes \mathcal{A}$, we conclude that

$$\mathcal{X}_a \times \mathcal{X}^b = \sum_c T_{ca}^b \mathcal{X}^c. \quad (\text{C.8})$$

¹⁰²Often we will omit the product sign $\times$.

This can be rewritten as the pivotal relation (3.12). One can also obtain

$$\sum_{ab} T_{ab}^c \mathcal{X}^b \mathcal{X}^a = \mathcal{X}^c. \quad (\text{C.9})$$

Using the trace relation (C.4), the unit (3.13) can be written as

$$1 = \sum_a \text{tr}(1\mathcal{X}_a) \mathcal{X}^a = \sum_{ab} T_{ab}^b \mathcal{X}^a = \sum_b \mathcal{X}_b \mathcal{X}^b = \sum_b \mathcal{X}^b \mathcal{X}_b. \quad (\text{C.10})$$

In the third equality we used (C.8). Using the notation $xy \equiv \mu(x \otimes y)$ for the product, we can regard the previous relation as the contraction $\mu(e) = 1$. Finally, let us show that e is idempotent as an element of $\mathcal{A} \otimes \mathcal{A}^{\text{op}}$ (where $\mathcal{A}^{\text{op}}$ is the same as $\mathcal{A}$ but elements are multiplied in the opposite order). In other words, $e^2 = e$ where e acts on itself from the outside:

$$e^2 = \sum_{ab} \mathcal{X}_a \mathcal{X}_b \otimes \mathcal{X}^b \mathcal{X}^a = \sum_{abc} T_{ab}^c \mathcal{X}_c \otimes \mathcal{X}^b \mathcal{X}^a = \sum_c \mathcal{X}_c \otimes \mathcal{X}^c = e. \quad (\text{C.11})$$

In the third equality we moved the scalars to the other side and used (C.9). We have thus shown properties (i) – (iv) of Sec. 3.1, and thus that e is a *symmetric separability idempotent*. It is in fact the unique element with these properties, as commented near the end of Sec. 3.1.

Relation with special Frobenius algebras. The strong separability condition allows one to make $\mathcal{A}$ into a special Frobenius algebra $(\mathcal{A}, \mu, u, \Delta, \epsilon)$. The product $\mu: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is the one already defined. The unit $u: \mathbb{C} \rightarrow \mathcal{A}$ is given by $u(1) = 1$ and linearity. They satisfy $\mu(x \otimes u(1)) = \mu(u(1) \otimes x) = x$. The coproduct $\Delta: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ is defined as

$$\Delta(x) = x e = e x. \quad (\text{C.12})$$

It satisfies $\mu \circ \Delta = \text{id}_{\mathcal{A}}$ because $\mu(\Delta(x)) = \mu(xe) = \mu(\sum_a x \mathcal{X}_a \otimes \mathcal{X}^a) = \sum_a x \mathcal{X}_a \mathcal{X}^a = x1$. Note that $\Delta \circ \mu \neq \text{id}_{\mathcal{A} \otimes \mathcal{A}}$. The Frobenius relation

$$(\text{id}_{\mathcal{A}} \otimes \mu) \circ (\Delta \otimes \text{id}_{\mathcal{A}}) = (\mu \otimes \text{id}_{\mathcal{A}}) \circ (\text{id}_{\mathcal{A}} \otimes \Delta) = \Delta \circ \mu \quad (\text{C.13})$$

simply follows from $xey = xye$. One can also define the counit $\epsilon: \mathcal{A} \rightarrow \mathbb{C}$ using the trace, $\epsilon(x) = \text{tr}(x)$. It satisfies $(\text{id}_{\mathcal{A}} \otimes \epsilon) \circ \Delta = \text{id}_{\mathcal{A}} = (\epsilon \otimes \text{id}_{\mathcal{A}}) \circ \Delta$. The first equation is

$$(\text{id}_{\mathcal{A}} \otimes \epsilon) \circ \Delta(x) = (\text{id}_{\mathcal{A}} \otimes \epsilon) \left(\sum_a x \mathcal{X}_a \otimes \mathcal{X}^a \right) = \sum_a x \mathcal{X}_a \text{tr}(\mathcal{X}^a) = x1 = x. \quad (\text{C.14})$$

The second equation is similar. One finds $\Delta(u(1)) = e$ and $\epsilon(\mu(e)) = \dim \mathcal{A}$.

C.2 General traces and projectors

Under the same hypotheses of the previous section, we now define a generic *trace* Θ on $\mathcal{A}$ as a cyclic linear functional $\Theta: \mathcal{A} \rightarrow \mathbb{C}$, namely

$$\Theta(\mathcal{X}_a \mathcal{X}_b) = \Theta(\mathcal{X}_b \mathcal{X}_a) \equiv \mathcal{K}_{ab}. \quad (\text{C.15})$$

Then $\mathcal{K}_{ab}$ is a symmetric pairing, and we call the trace nondegenerate if the pairing is nondegenerate: $\det(\mathcal{K}_{ab}) \neq 0$. We can use the WA theorem to characterize such functionals.

It is convenient to introduce the canonical basis for a sum of matrix algebras: we let $a \equiv (\alpha ij)$ be a multi-index, where α indicates the matrix block while $i, j = 1, \dots, N_\alpha$. Then $\mathcal{X}_a \equiv \mathcal{X}_{\alpha ij}$ is the matrix with a single 1 at position (i, j) of the α -th block, and zero elsewhere. In this basis we have

$$\mathcal{X}_a \times \mathcal{X}_b \equiv \mathcal{X}_{\alpha ij} \times \mathcal{X}_{\beta kl} = \delta_{\alpha\beta} \delta_{jk} \mathcal{X}_{\alpha il} , \quad (\text{C.16})$$

therefore the multiplication coefficients and trace over the regular representation are

$$T_{ab}^c \equiv T_{\alpha ij, \beta kl}^{\gamma mn} = \delta_\alpha^\gamma \delta_\beta^\gamma \delta_i^m \delta_j^n \delta_{kl} , \quad \text{tr}(\mathcal{X}_a) \equiv \text{tr}(\mathcal{X}_{\alpha ij}) = N_\alpha \delta_{ij} . \quad (\text{C.17})$$

Coming back to a generic trace Θ , using that $\mathcal{X}_{\alpha ij} \mathcal{X}_{\alpha jk} = \mathcal{X}_{\alpha ik}$ and $\mathcal{X}_{\alpha jk} \mathcal{X}_{\alpha ij} = \delta_{ik} \mathcal{X}_{\alpha jj}$ we deduce that, for any j :

$$\Theta(\mathcal{X}_{\alpha ik}) = \delta_{ik} \Theta(\mathcal{X}_{\alpha jj}) = \delta_{ik} \Theta(\mathcal{X}_{\alpha 11}) \equiv \theta_\alpha \delta_{ik} = \theta_\alpha \chi_\alpha(\mathcal{X}_{\alpha ik}) . \quad (\text{C.18})$$

where

$$\chi_\alpha(x) \equiv \text{tr}[\mathfrak{R}_\alpha(x)] \quad (\text{C.19})$$

is the character in the representation $\mathfrak{R}_\alpha$. Thus the traces Θ form a vector space, and they are parametrized by the numbers θ_α :

$$\Theta = \sum_\alpha \theta_\alpha \chi_\alpha . \quad (\text{C.20})$$

In other words, Θ is an arbitrary linear combination of the irreducible characters. A trace is nondegenerate if and only if $\theta_\alpha \neq 0$ for all α 's.

In a WA basis the minimal central idempotent¹⁰³ (MCI) corresponding to the irrep $\mathfrak{R}_\alpha$ is

$$\Pi_\alpha = \sum_i \mathcal{X}_{\alpha ii} = \sum_{ij} \chi_\alpha(\mathcal{X}_{\alpha ij}) \mathcal{X}_{\alpha ij} . \quad (\text{C.21})$$

This can be written in terms of Θ in a generic basis as

$$\Pi_\alpha = \theta_\alpha \sum_{ab} \tilde{\mathcal{K}}^{ab} \chi_\alpha(\mathcal{X}_a) \mathcal{X}_b . \quad (\text{C.22})$$

This formula is valid in any basis and for any choice of nondegenerate trace. Taking the character of (C.22) in the irrep $\mathfrak{R}_\beta$ one obtains the orthogonality relation

$$\sum_{ab} \tilde{\mathcal{K}}^{ab} \chi_\alpha(\mathcal{X}_a) \chi_\beta(\mathcal{X}_b) = \delta_{\alpha\beta} \frac{N_\alpha}{\theta_\alpha} , \quad (\text{C.23})$$

as can be easily verified using the WA theorem.

For future reference, note that the characters can be computed in terms of the trace Θ and the projectors as

$$\chi_\alpha(X) = \frac{1}{\theta_\alpha} \Theta(X \Pi_\alpha) . \quad (\text{C.24})$$

This can be derived using that $\chi_\beta(X \Pi_\alpha) = \delta_{\alpha\beta} \chi_\alpha(X)$.

¹⁰³These are those elements of the algebra that belong to the center, square to themselves, and cannot be written as a sum of other such elements. They are in 1-to-1 correspondences with the irreps, and implement a projection onto each of them.

Finally, notice that the trace over the regular representation corresponds to choosing $\theta_\alpha = N_\alpha = \dim(\mathfrak{R}_\alpha)$. In this case, the general formula (C.22) becomes

$$\Pi_\alpha = \dim(\mathfrak{R}_\alpha) \sum_{ab} \tilde{K}^{ab} \chi_\alpha(\mathcal{X}_a) \mathcal{X}_b. \quad (\text{C.25})$$

In the case of the group algebra $\mathbb{C}[G]$ of a finite group G we have

$$K_{gh} = \sum_k T_{(gh)k}^k = \sum_k \delta_{g,h^{-1}} = |G| \delta_{g,h^{-1}} \quad \text{and} \quad K^{gh} = \frac{1}{|G|} \delta_{g,h^{-1}}, \quad (\text{C.26})$$

so that

$$\Pi_\alpha = \dim(\mathfrak{R}_\alpha) \sum_{gh} \frac{1}{|G|} \delta_{g,h^{-1}} \chi_\alpha(g) h = \frac{\dim(\mathfrak{R}_\alpha)}{|G|} \sum_g \chi_\alpha(g^{-1}) g, \quad (\text{C.27})$$

which is the known formula for the projector onto a representation $\mathfrak{R}_\alpha$ of a finite group.

C.3 C^* -algebras

A $*$ -structure on an algebra $\mathcal{A}$ over $\mathbb{C}$ is an anti-linear involution $*$: $\mathcal{A} \rightarrow \mathcal{A}$ (i.e. such that $X^{**} = X$ and $(\alpha X + \beta Y)^* = \alpha^* X^* + \beta^* Y^*$ for $\alpha, \beta \in \mathbb{C}$) which is also an anti-homomorphism: $(XY)^* = Y^* X^*$.

A C^* -algebra $\mathcal{A}$ is a Banach algebra¹⁰⁴ over $\mathbb{C}$, with a $*$ -structure and whose norm is such that

$$\|x^* x\| = \|x\|^2 \quad \text{for all } x \in \mathcal{A}. \quad (\text{C.28})$$

This is called C^* condition and it is very restrictive. Indeed, if it exists, such a norm is uniquely fixed by the algebraic structure as

$$\|x\|^2 = \|x^* x\| = \sup\{|\lambda| : (x^* x - \lambda 1) \text{ is not invertible}\}. \quad (\text{C.29})$$

One can prove that finite-dimensional C^* -algebras are semisimple, and thus the WA theorem applies. Moreover, a stronger version of this theorem (see Th. 6.3.8 of [124]) states that the isomorphism to a sum of matrix algebras can be chosen such that the star operation is mapped to the standard matrix adjoint operation (that we also denote here as $*$). Consider for simplicity the case of a single block of size N : $\mathcal{A} \cong \text{Mat}_{N \times N}(\mathbb{C})$, which means that $\mathcal{A}$ is simple (the following arguments extend to the general case of multiple blocks with minimal modifications). Then we can represent the index a as $a \equiv ij$ with $i, j = 1, \dots, N$.

Working in the notation (C.16) (but suppressing the block indices¹⁰⁵), the regular-trace form is

$$K_{ab} \equiv K_{ij,kl} = \sum_{mnr} T_{ij,mn}^{rs} T_{kl,rs}^{mn} = N \delta_{il} \delta_{jk}. \quad (\text{C.30})$$

This is indeed invertible, with inverse

$$\tilde{K}^{ab} \equiv \tilde{K}^{ij,kl} = N^{-1} \delta^{il} \delta^{jk}. \quad (\text{C.31})$$

¹⁰⁴Meaning that it is associative, it has a norm such that $\|xy\| \leq \|x\| \cdot \|y\|$, and it is complete with respect to the metric induced by that norm.

¹⁰⁵For a sum of matrix algebras, it is $K_{\alpha ij, \beta kl} = N_\alpha \delta_{\alpha\beta} \delta_{il} \delta_{jk}$. All the following formulas apply block by block.

In this basis the symmetric separability idempotent takes the simple form

$$e = \sum_{ijkl} \tilde{K}^{ij,kl} \mathcal{X}_{ij} \otimes \mathcal{X}_{kl} = N^{-1} \sum_{ij} \mathcal{X}_{ij} \otimes \mathcal{X}_{ji} . \quad (\text{C.32})$$

The star operation takes a simple form as well:

$$\mathcal{X}_a^* \equiv \mathcal{X}_{ij}^* = \mathcal{X}_{ji} . \quad (\text{C.33})$$

We see that e is Hermitian: $e^* = e$. Since Hermiticity does not depend on the basis (it is invariant under isomorphism), we conclude that e is Hermitian in general:

$$e^* = \sum_{ab} (\tilde{K}^{ab})^* \mathcal{X}_a^* \otimes \mathcal{X}_b^* = e . \quad (\text{C.34})$$

In the WA basis the SSI can also be written as follows:

$$e = N^{-1} \sum_{ij} \mathcal{X}_{ij}^* \otimes \mathcal{X}_{ij} . \quad (\text{C.35})$$

By a rescaling of the basis, $\mathcal{X}_{\underline{a}} \equiv N^{-\frac{1}{2}} \mathcal{X}_{ij}$, it reads $e = \sum_{\underline{a}} \mathcal{X}_{\underline{a}}^* \otimes \mathcal{X}_{\underline{a}}$. This very simple form (which is reached also in the general case of multiple blocks) is not invariant under change of basis, however it is useful that such an expression exists.

Consequences for the symmetrizer. The symmetrizer constructed in Sec. 3.1 can be defined in terms of the SSI e as the linear map $\rho_S = e(\rho) = \sum_{ab} \tilde{K}^{ab} \mathcal{X}_a \rho \mathcal{X}_b$. Idempotency of e implies that $(\rho_S)_S = \rho_S$ and thus the symmetrizer is a projector. The property $\mu(e) = 1$ and symmetry of e imply that $\text{tr}(\rho_S) = \text{tr}(\rho)$ and that, if ρ commutes with $\mathcal{A}$, then $\rho_S = \rho$. Since e commutes with $\mathcal{A}$ then ρ_S commutes with $\mathcal{A}$. In other words, the image of the projector is precisely given by the matrices that commute with the action of the algebra, and not a subspace thereof.

Since $e = e^*$, when $\rho = \rho^\dagger$ one finds that also ρ_S is Hermitian. Finally, assume that the density matrix ρ is nonnegative, namely that $\langle \psi | \rho | \psi \rangle \geq 0$ for every vector $|\psi\rangle$ in the Hilbert space. Then

$$\langle \psi | \rho_S | \psi \rangle = \sum_{ab} \tilde{K}^{ab} \langle \psi | \mathcal{X}_a \rho \mathcal{X}_b | \psi \rangle \stackrel{\text{special basis}}{=} \sum_{\underline{a}} \langle \psi | \mathcal{X}_{\underline{a}}^* \rho \mathcal{X}_{\underline{a}} | \psi \rangle \geq 0 . \quad (\text{C.36})$$

In the second equality we used the special basis adapted to the $*$ -structure. In that basis we find a sum of nonnegative numbers, which is nonnegative. This proves that also ρ_S is nonnegative.

Alternatively, note that the expression below (C.35) provides a Kraus representation of the symmetrizer. This implies that the symmetrizer is in fact a trace preserving (unital) quantum channel, and in particular is completely positive [189].

C^* -structure of fusion algebras. Let us show that finite-dimensional fusion algebras (summarized in Sec. 2.1) are C^* -algebras. We show that

$$x^* x = 0 \quad \Rightarrow \quad x = 0 . \quad (\text{C.37})$$

For finite-dimensional unital $*$ -algebras, this implies that they are C^* (see e.g. Prop. 2.1 of [190] or Th. 2.4.4 of [191]). Write $x = \sum_a x_a \mathcal{L}_a$. Then

$$x^* x = \sum_{abcd} x_a^* x_b N_{ac}^1 N_{cb}^d \mathcal{L}_d . \quad (\text{C.38})$$

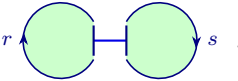
Assume that $x^* x = 0$, then $\sum_{abc} x_a^* x_b N_{ac}^1 N_{cb}^d = 0$ for all d . Taking $d = 1$ we get $\sum_a |x_a|^2 = 0$, therefore $x = 0$. We conclude that finite-dimensional fusion algebras are C^* -algebras, and thus they are also semisimple.

C.4 Quantum trace on the strip algebra

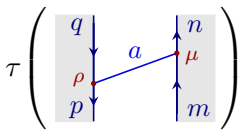
The goal of this section is to compute the coefficients (4.27) for the MCIs (see Fn. 103) of the strip algebra, restricted to those elements where the identity line is suspended. To this end, we construct a particular trace (among those of the form (C.20)) that will make the wanted coefficient stand out in the projector formula (C.22). We call this the *quantum trace* τ on the strip algebra.

It is convenient to think of objects in a $\mathcal{C}$ -module category $\mathcal{M}$ as topological interfaces between $\mathcal{C}$ and $\mathcal{C}_{\mathcal{M}}^*$, as in Fig. 7, rather than as boundary conditions. Moreover, we think of diagrams in this section as belonging to a 2-sphere (or, equivalently, to the plane compactified with the point at infinity) rather than to the plane. One way of interpreting such diagrams is as partition functions of TQFTs with symmetry $\mathcal{C}$ or $\mathcal{C}_{\mathcal{M}}^*$, each defined on subregions of the 2-sphere, with elements of $\mathcal{M}$ acting as interfaces between them.

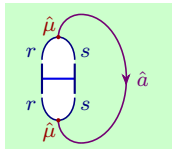
The trace is then defined by closing the two interfaces into an annulus:

$$\tau(\mathbb{H}) \equiv \sum_{rs} \text{Diagram} \quad (\text{C.39})$$


The two ends are glued using the identity of the strip algebra (E.6), in other words they can be directly connected. This immediately shows that

$$\tau \left(\begin{array}{c} q \\ \downarrow \\ \rho \\ \downarrow \\ p \end{array} \begin{array}{c} a \\ \nearrow \\ \mu \\ \downarrow \\ m \end{array} \begin{array}{c} n \\ \downarrow \\ \mu \\ \downarrow \\ m \end{array} \right) \propto \delta_{a1} \delta_p^q \delta_m^n . \quad (\text{C.40})$$


The delta functions of boundary labels follow from the fact that the boundaries are connected. The delta function on a follows from the fact that the two discs of $\mathcal{C}_{\mathcal{M}}^*$ in (C.39) can be shrunk to points producing an open line of $\mathcal{C}$, which vanishes unless the line is the identity. By moving the boundaries through infinity, and fusing them together into a sum of dual lines, we obtain:

$$\tau(\mathbb{H}) = \sum_{r,s,\hat{a},\hat{\mu}} \sqrt{\frac{d_{\hat{a}}}{\tilde{d}_r \tilde{d}_s}} \text{Diagram} \quad (\text{C.41})$$


We claim that, for fixed $\hat{a}$, the configuration on the right-hand side evaluates to a number proportional to the character in representation $\mathfrak{R}_{\hat{a}}$. More precisely, we have:

$$\chi_{\hat{a}}(\text{H}) = \sum_{r,s,\hat{\mu}} \frac{1}{\sqrt{d_{\hat{a}} \tilde{d}_r \tilde{d}_s}} \quad \begin{array}{c} \text{Diagram: A horizontal line with a cup below it and a cap above it. The cup is labeled with } r \text{ and } s \text{ on its sides, and } \hat{\mu} \text{ above it. The cap is labeled with } r \text{ and } s \text{ on its sides, and } \hat{\mu} \text{ below it. The entire diagram is labeled } \hat{a} \text{ on the right.} \end{array} . \quad (\text{C.42})$$

This follows from the following observations. The “cup” below the strip algebra element creates the state $V_{(\hat{a})\hat{\mu}}^{rm}$, as in (4.13), which is a basis vector of the representation space. Similarly, the “cap” above it creates a dual basis vector. Thus, the sum over $\hat{\mu}$ computes a quantity proportional to the trace in $\mathfrak{R}_{\hat{a}}$, provided that the normalizations for each individual vector are correct. To check that this is the case, we consider basis elements of the strip algebra in which the horizontal line is the identity:

$$\chi_{\hat{a}} \left(\begin{array}{c} \text{Diagram: A vertical line with a horizontal identity line crossing it. The vertical line is labeled } p \text{ and } m \text{ on its sides. The horizontal line is labeled } 1. \end{array} \right) = \hat{N}_{p\hat{a}}^m , \quad (\text{C.43})$$

where we used standard relations in graphical calculus (reviewed in App. A.5). This is indeed the correct result, because the elements with horizontal identity are idempotents that project to the sectors $\mathcal{H}_{rm}$, whose character gives the dimension of $\mathfrak{R}_{\hat{a}}$ restricted to $\mathcal{H}_{rm}$. Thus the normalizations in (C.42) are correct. We obtain that the quantum trace has the following decomposition in terms of characters:

$$\tau(\cdot) = \sum_{\hat{a}} d_{\hat{a}} \chi_{\hat{a}}(\cdot) , \quad \text{tr}(\cdot) = \sum_{\hat{a}} \dim(\mathfrak{R}_{\hat{a}}) \chi_{\hat{a}}(\cdot) . \quad (\text{C.44})$$

For comparison, we also wrote the decomposition of the regular trace. We see that the quantum and regular traces have coefficients given by the quantum and classical dimensions of the strip-algebra representations, respectively.

We have thus shown that, in the notation of (C.15), taking $\Theta = \tau$ amounts to choosing $\theta_{\hat{a}} = d_{\hat{a}}$. The general projector (C.22) then becomes:

$$\Pi_{\hat{a}} = d_{\hat{a}} \sum_{CD} \tilde{\mathcal{K}}^{CD} \chi_{\hat{a}}(\mathcal{X}_C) \mathcal{X}_D . \quad (\text{C.45})$$

We are interested in the coefficient in front of the strip-algebra element $\mathbf{H}_{p,m,1}^{p,m}$ with boundaries $\mathcal{B}_p, \mathcal{B}_m$ and horizontal identity line. We indicate this element as $\mathbf{H}$ for short. Using (C.40), $\mathbf{H}$ is orthogonal to all elements other than itself, and thus

$$\mathcal{K}_{\mathbf{H}\mathbf{H}} = \tau(\mathbf{H}\mathbf{H}) = \sum_{\hat{a}} d_{\hat{a}} \chi_{\hat{a}}(\mathbf{H}) = \sum_{\hat{a}} d_{\hat{a}} \hat{N}_{p\hat{a}}^m = \sum_{\hat{a}} d_{\hat{a}} \hat{N}_{\bar{p}\hat{a}}^m = \tilde{d}_p \tilde{d}_m , \quad (\text{C.46})$$

where we used the pivotal relation. Moreover, orthogonality implies $\tilde{\mathcal{K}}^{\mathbf{H}\mathbf{H}} = (\tilde{d}_p \tilde{d}_m)^{-1}$. We obtain

$$\left. \Pi_{\hat{a}} \right|_{\mathbf{H}_{p,m,1}^{p,m}} = \frac{d_{\hat{a}} N_{p\hat{a}}^m}{\tilde{d}_p \tilde{d}_m} . \quad (\text{C.47})$$

Combining the computation of $\mathcal{K}_{\mathbf{H}\mathbf{H}}$ above with (C.40), we also find

$$\tau \left(\begin{array}{c} \text{Diagram: A vertical line with a horizontal identity line crossing it. The vertical line is labeled } q \text{ and } m \text{ on its sides. The horizontal line is labeled } a. \end{array} \right) = \delta_{a1} \delta_p^q \delta_m^n \tilde{d}_p \tilde{d}_m . \quad (\text{C.48})$$

C.5 Explicit form of ρ_S

Let us consider an algebra $\mathcal{A}$, with the same hypotheses as in App. C.1, which we identify with a matrix algebra by virtue of (C.2). We pick a basis of $\mathcal{A}$ given by $\{\mathcal{X}_{\alpha ij}\}_{\alpha, i, j}$ as above (C.16), where

$$\mathcal{X}_{\alpha ij} = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & e_{ij}^\alpha & 0 \\ \hline 0 & 0 & 0 \end{array} \right) \leftarrow \alpha\text{-th block} \quad \begin{array}{l} \alpha = 1, \dots, r \\ i, j = 1, \dots, N_\alpha \end{array} \quad (\text{C.49})$$

and e_{ij}^α is the $N_\alpha \times N_\alpha$ matrix that has a single 1 in position (i, j) and 0 elsewhere. The β -th representation of $\mathcal{A}$ merely selects the β -th diagonal block from the matrix above. The most generic representation $\Re: \mathcal{A} \rightarrow \text{End}(V)$ is a direct sum of irreducible representations:

$$V = \bigoplus_{\alpha=1}^r n_\alpha V_\alpha, \quad (\text{C.50})$$

where $\dim(V_\alpha) = N_\alpha$, and thus the matrix that represents $\mathcal{X}_{\alpha ij}$ is of the form:

$$\Re(\mathcal{X}_{\alpha ij}) = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & e_{ij}^\alpha & 0 \\ \hline 0 & 0 & 0 \end{array} \right) \leftarrow \alpha\text{-th block} \quad (\text{C.51})$$

where e_{ij}^α is repeated n_α times. We now assume that the symmetrizer, which in principle can be any element of $\text{End}(\text{End}(V))$, is written in the following form:¹⁰⁶

$$M_S = \sum_{\alpha, i, j, \alpha', i', j'} c_{(\alpha ij)(\alpha' i' j')} \Re(\mathcal{X}_{\alpha ij}) M \Re(\mathcal{X}_{\alpha' i' j'}), \quad (\text{C.52})$$

where $M \in \text{End}(V)$ is any matrix, that we also decompose in blocks $M^{\alpha\alpha'}$ according to (C.50). Thanks to property 2. and to Schur's lemma,¹⁰⁷ we can conclude that, as a matrix, M_S has vanishing off-diagonal blocks, $(M_S)^{\alpha\alpha'} = 0$ for $\alpha \neq \alpha'$, while each diagonal block is of the form

$$(M_S)^{\alpha\alpha} = \begin{pmatrix} \lambda_{11}^{\alpha\alpha} \mathbb{1}_{N_\alpha} & \cdots & \lambda_{1n_\alpha}^{\alpha\alpha} \mathbb{1}_{N_\alpha} \\ \vdots & \ddots & \vdots \\ \lambda_{n_\alpha 1}^{\alpha\alpha} \mathbb{1}_{N_\alpha} & \cdots & \lambda_{n_\alpha n_\alpha}^{\alpha\alpha} \mathbb{1}_{N_\alpha} \end{pmatrix}. \quad (\text{C.53})$$

In principle each coefficient $\lambda_{ab}^{\alpha\alpha}$ is completely generic. However, using the additional assumption (C.52), we have

$$M_S = \sum_{\alpha, i, j, i', j'} c_{(\alpha ij)(\alpha i' j')} \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & e_{ij}^\alpha m_{11}^{\alpha\alpha} e_{i' j'}^\alpha & \cdots & e_{ij}^\alpha m_{1n_\alpha}^{\alpha\alpha} e_{i' j'}^\alpha \\ \hline 0 & \vdots & \ddots & \vdots \\ \hline 0 & e_{ij}^\alpha m_{n_\alpha 1}^{\alpha\alpha} e_{i' j'}^\alpha & \cdots & e_{ij}^\alpha m_{n_\alpha n_\alpha}^{\alpha\alpha} e_{i' j'}^\alpha \\ \hline 0 & 0 & 0 & 0 \end{array} \right) \leftarrow \alpha\text{-th block} \quad (\text{C.54})$$

¹⁰⁶This assumption is referred to as “universality” in Sec. 3.1.

¹⁰⁷Schur's lemma [125] can be stated (under our working hypotheses) as the fact that the module-endomorphisms of an irreducible $\mathcal{A}$ -module form a division algebra. But there is only one such algebra over $\mathbb{C}$: the complex numbers themselves. Since the identity is clearly a module-endomorphism, we conclude that all such endomorphisms must be proportional to the identity. In more concrete terms, matrices that commute with all elements of an irreducible representation of $\mathcal{A}$ must be proportional to the identity matrix. This result generalizes the well-known analogous result for groups and group algebras.

where $m_{ab}^{\alpha\alpha}$ with $a, b \in \{1, \dots, n_\alpha\}$ are sub-blocks of $M^{\alpha\alpha}$. The symmetrization in each of the sub-blocks $m_{ab}^{\alpha\alpha}$ depends only on the entries of M of that same sub-block. Let us now focus on such a sub-block. Its symmetrized version is

$$\sum_{i,j,i',j'} c_{(\alpha ij)(\alpha i' j')} e_{ij}^\alpha m_{ab}^{\alpha\alpha} e_{i'j'}^\alpha = \sum_{i,j,i',j'} c_{(\alpha ij)(\alpha i' j')} (m_{ab}^{\alpha\alpha})_{ji'} e_{i'j'}^\alpha. \quad (\text{C.55})$$

This must be proportional to the identity, so it must be $c_{(\alpha ij)(\alpha i' j')} = \delta_{ij'} \tilde{c}_{i'j}^\alpha$. Consider now a matrix M that is nonzero only in the block $m_{aa}^{\alpha\alpha}$. Imposing conservation of the trace gives $\sum_j (m_{aa}^{\alpha\alpha})_{jj} = \text{tr}(M) \stackrel{!}{=} \text{tr}(M_S) = N_\alpha \sum_{j,i'} \tilde{c}_{i'j}^\alpha (m_{aa}^{\alpha\alpha})_{ji'}$. We conclude it must be $\tilde{c}_{i'j}^\alpha = \delta_{i'j}/N_\alpha$ and $c_{(\alpha ij)(\alpha i' j')} = \delta_{ij'} \delta_{i'j}/N_\alpha$. This means that the sub-block (a, b) of the block (α, α) is

$$\sum_{i,j,i',j'} c_{(\alpha ij)(\alpha i' j')} e_{ij}^\alpha m_{ab}^{\alpha\alpha} e_{i'j'}^\alpha = \frac{1}{N_\alpha} \text{tr}(m_{ab}^{\alpha\alpha}) \mathbb{1}_{ab}^\alpha. \quad (\text{C.56})$$

Notice that conservation of the trace has also fixed those blocks with $a \neq b$.

Said in words, the matrix M_S is proportional to the identity in blocks that connect isomorphic irreducible representations, and is zero elsewhere. In each block, the proportionality constant is the trace of that block, divided by its dimension¹⁰⁸ (see also Fig. 4).

C.6 Some properties of (Rényi) asymmetry

Here, we first comment on the finiteness of the asymmetry. Recall that the relative entropy $S[\rho||\sigma] \equiv \text{tr}[\rho \log \rho - \rho \log \sigma]$ is finite as long as the condition $\ker(\sigma) \subseteq \ker(\rho)$ holds. This guarantees finiteness of the second term in the relative entropy, while the first term is always finite. We now aim to show that, for the specific choice $\sigma = \rho_S$, such a condition is always satisfied, namely

$$\ker(\rho_S) \subseteq \ker(\rho). \quad (\text{C.57})$$

To show this, we first notice that $S^\dagger = S$ with respect to the Hilbert-Schmidt scalar product on operators:

$$\text{tr}[S(M)^\dagger N] = \text{tr}[M^\dagger S(N)], \quad (\text{C.58})$$

as can be checked by using (3.9), the self-adjointness of the SSI, and cyclicity of the trace. This also implies that the decomposition $\text{End}(\mathcal{H}) = \ker(S) \oplus \text{im}(S)$ is orthogonal with respect to the same scalar product.

Let now P be the orthogonal projector¹⁰⁹ onto $\ker(\rho_S)$ and note that this commutes with the action of the algebra.¹¹⁰ We thus have $S(P) = P$. Then consider the following string of equalities:

$$\text{tr}[P\rho] = \text{tr}[S(P)\rho] = \text{tr}[P\rho_S] = \text{tr}[P\rho_S P] = 0. \quad (\text{C.59})$$

¹⁰⁸Implicit in this statement is a choice of basis such that all copies of a given representation use the same matrices, as was the case in (C.51). This choice implies that also the off-diagonal blocks are proportional to the identity (see Fn. 107). Indeed, such blocks are nonzero module-endomorphisms between isomorphic representations and thus, as matrices, commute with all elements of the corresponding irreducible module of $\mathcal{A}$.

¹⁰⁹Orthogonal with respect to the physical scalar product.

¹¹⁰To see this, write P as a function of ρ_S : $P = \prod_{j=1}^{\text{rk}(\rho_S)} \left(\mathbb{1} - \frac{\rho_S}{\lambda_j} \right)$, where λ_j are the distinct nonzero eigenvalues of ρ_S . One can check that $P|\lambda_k\rangle = 0$ for all k , and that $P|v\rangle = |v\rangle$ if $\rho_S|v\rangle = 0$. It is then obvious that, since ρ_S commutes with the action of the algebra, the same must be true for P .

Then, letting $\rho^\dagger = \rho$ and using the Hilbert-Schmidt norm we have:

$$\|\rho^{1/2}P\|^2 = \text{tr}[P\rho^{1/2}\rho^{1/2}P] = \text{tr}[P\rho] = 0 . \quad (\text{C.60})$$

Therefore, $\rho^{1/2}P = 0$ as an operator, and thus also $\rho P = 0$. To conclude, take $|v\rangle \in \ker(\rho_S)$, so that $P|v\rangle = |v\rangle$. Then $0 = \rho P|v\rangle = \rho|v\rangle$, so also $|v\rangle \in \ker(\rho)$. This establishes the finiteness of the asymmetry, as well as that of all quantities appearing in the computation (3.11).

We finally show here that all Rényi asymmetries (3.3) turn out to be nonnegative. First, we rewrite these as

$$\Delta S^{(n)}[\rho] = H_n(\rho_S) - H_n(\rho) , \quad (\text{C.61})$$

where $H_n[\rho] \equiv \frac{1}{1-n} \log \text{tr}[\rho^n]$ is the n -th Rényi entropy of a density matrix.

Following [192], let ρ be a Hermitian matrix, and let $\lambda(\rho)$ be the vector of its eigenvalues, sorted in decreasing order:

$$\lambda(\rho) = (\lambda_1(\rho), \dots, \lambda_d(\rho)) . \quad (\text{C.62})$$

A unital quantum channel is a linear, completely positive, trace preserving map (between linear operators) Φ such that $\Phi(\mathbb{1}) = \mathbb{1}$. As we have already commented, our symmetrizer S is a trace-preserving quantum channel. It is furthermore unital, as a consequence of property 3. in Sec. 3.1. One can then apply Uhlmann's theorem (see Theorem 4.32 of [192]) to the channel S . The theorem implies, among other things, that

$$\lambda(\rho_S) \prec \lambda(\rho) , \quad (\text{C.63})$$

where $v \prec w$ means that

$$\sum_{i=1}^k v_i \leq \sum_{i=1}^k w_i , \quad (\text{C.64})$$

for every $k = 1, \dots, d$, provided that the entries of the real d -dimensional vectors v and w are ordered in decreasing order. Moreover, the equality holds for $k = d$ (as it should be since the trace is supposed to be preserved). A real function of such vectors is called Schur-convex if $x \prec y \Rightarrow f(x) \leq f(y)$ and Schur-concave when the inequalities are in opposite directions. Rényi entropies H_n are known to be Schur-concave functions [193]. This is precisely the statement of non-negativity of (C.61).

D Hopf algebras and their modules

D.1 Basic definitions

We review here the definition of Hopf algebras, following [98, 146, 148]. An associative algebra $\mathcal{A}$ over $\mathbb{C}$ has an associative product $\mu: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ and a unit $u: \mathbb{C} \rightarrow \mathcal{A}$ such that

$$\mu \circ (u \otimes \text{id}) = \text{id} = \mu \circ (\text{id} \otimes u) . \quad (\text{D.1})$$

We will henceforth denote $1 \equiv u(1)$. Both μ and u are linear, and u is unique when it exists. Dually, a coassociative coalgebra $\mathcal{A}$ over $\mathbb{C}$ has a coassociative coproduct $\Delta: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, such that $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta \equiv \Delta^2$, and a counit $\epsilon: \mathcal{A} \rightarrow \mathbb{C}$ such that

$$(\epsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \epsilon) \circ \Delta . \quad (\text{D.2})$$

Both Δ and ϵ are linear, and ϵ is unique. It is convenient to use Sweedler's notation: it is always possible to write (in a non-unique way) coproducts as sums: $\Delta(x) = \sum_i x_{(1)}^i \otimes x_{(2)}^i$. This is concisely written as $\Delta(x) = x_{(1)} \otimes x_{(2)}$ with the sum kept implicit.

A weak bialgebra $\mathcal{A}$ has both an algebra and a coalgebra structures, which must be compatible in the following sense. The coproduct is an algebra homomorphism: $\Delta(xy) = \Delta(x) \Delta(y)$, where on the RHS pairwise multiplication is used. The counit is weakly multiplicative:

$$\epsilon(xyz) = \epsilon(xy_{(1)}) \epsilon(y_{(2)}z) = \epsilon(xy_{(2)}) \epsilon(y_{(1)}z) , \quad (\text{D.3})$$

where associativity is assumed. The unit is weakly comultiplicative, that can be written as

$$\Delta^2(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1) , \quad (\text{D.4})$$

where coassociativity is assumed in the first expression, while in the second and third expressions pairwise multiplication is used. A *weak Hopf algebra* is a weak bialgebra in which there exists an antipode $S: \mathcal{A} \rightarrow \mathcal{A}$ satisfying the following three axioms:

$$x_{(1)} S(x_{(2)}) = \epsilon(1_{(1)} x) 1_{(2)} , \quad S(x_{(1)}) x_{(2)} = 1_{(1)} \epsilon(x 1_{(2)}) , \quad S(x_{(1)}) x_{(2)} S(x_{(3)}) = S(x) . \quad (\text{D.5})$$

In the last equation we used Sweedler's notation for $\Delta^2(x)$. It follows [148] that the antipode is unique, and that it is an algebra anti-homomorphism satisfying:

$$\begin{aligned} S(xy) &= S(y) S(x) , & S(1) &= 1 , \\ S(x)_{(1)} \otimes S(x)_{(2)} &= S(x_{(2)}) \otimes S(x_{(1)}) , & \epsilon \circ S &= \epsilon . \end{aligned} \quad (\text{D.6})$$

In finite-dimensional weak Hopf algebras, S is always invertible. Weak C^* -Hopf algebras¹¹¹ in which the antipode is involutive, $S^2 = \text{id}$, are called weak Kac algebras [143, 194].

A bialgebra (sometimes called strong bialgebra) is a weak bialgebra in which the counit is multiplicative and the unit is comultiplicative:

$$\epsilon(xy) = \epsilon(x) \epsilon(y) , \quad \Delta(1) = 1 \otimes 1 . \quad (\text{D.7})$$

¹¹¹These are algebras that have both a WHA and a C^* structure, subject to the compatibility condition $\Delta(x^*) = x_{(1)}^* \otimes x_{(2)}^*$. Strip algebras satisfy this condition [98].

These stronger properties imply the corresponding weaker ones. A *Hopf algebra* (sometimes called a strong Hopf algebra) is a weak Hopf algebra in which one requires, equivalently, multiplicativity of the counit, comultiplicativity of the unit, or the “hexagon diagram”:

$$x_{(1)} S(x_{(2)}) = S(x_{(1)}) x_{(2)} = \epsilon(x) 1 . \quad (\text{D.8})$$

If one of these three properties holds for a weak Hopf algebra, then the other two follow automatically. This means that a Hopf algebra is in particular a bialgebra. Also in this case the antipode S is unique, it satisfies (D.6), and it is always invertible in the finite-dimensional case [195]. Moreover,¹¹²

$$S^2 = \text{id} \quad (\text{D.9})$$

in finite-dimensional semisimple Hopf algebras over $\mathbb{C}$ [196, 197].

Group algebra. The group algebra $\mathbb{C}[G]$ is a typical example of Hopf algebra, where

$$\mu(g_1 \otimes g_2) = (g_1 g_2) , \quad u(1) = 1_G , \quad \Delta(g) = g \otimes g , \quad \epsilon(g) = 1 , \quad S(g) = g^{-1} . \quad (\text{D.10})$$

One easily verifies that all properties are satisfied.

Integrals. Following [148], in a weak Hopf algebra one introduces the left and right projectors

$$\begin{aligned} \Pi^L(x) &= \epsilon(1_{(1)} x) 1_{(2)} = x_{(1)} S(x_{(2)}) , \\ \Pi^R(x) &= 1_{(1)} \epsilon(x 1_{(2)}) = S(x_{(1)}) x_{(2)} , \end{aligned} \quad (\text{D.11})$$

acting on $x \in \mathcal{A}$. These are projectors because of (D.3), and the second equality is as in (D.5). They are related by the antipode:¹¹³

$$\Pi^L \circ S = \Pi^L \circ \Pi^R = S \circ \Pi^R , \quad \Pi^R \circ S = \Pi^R \circ \Pi^L = S \circ \Pi^L . \quad (\text{D.12})$$

Then one defines left and right normalized integrals ℓ and r , respectively, such that:

$$x \ell = \Pi^L(x) \ell , \quad \Pi^L(\ell) = 1 , \quad \text{or} \quad r x = r \Pi^R(x) , \quad \Pi^R(r) = 1 , \quad (\text{D.13})$$

for all $x \in \mathcal{A}$. Note that the normalized integrals ℓ, r are themselves projectors. Besides, $S(\ell)$ is a normalized right integral, and $S(r)$ is a normalized left integral (proven by applying S to (D.13) and using that S is invertible). An element that is simultaneously a left and right normalized integral is called a Haar integral h . When it exists, the Haar integral is unique. It follows that $S(h) = h$.

¹¹²In finite-dimensional semisimple weak Hopf algebras over $\mathbb{C}$, S is not necessarily involutive.

¹¹³Let us prove the first relation. First, notice that $\Pi^L(1) = \Pi^R(1) = 1$ and $\epsilon \circ \Pi^L = \epsilon \circ \Pi^R = \epsilon$. Then $\Pi^L \circ \Pi^R(x) = \epsilon(1_{(1)} S(x_{(1)}) x_{(2)}) 1_{(2)} = \epsilon(1_{(1)} S(x_{(1)})_{(2)}) \epsilon(S(x_{(1)})_{(1)} x_{(2)}) 1_{(2)} = \epsilon(1_{(1)} S(x_{(1)})) \epsilon(S(x_{(2)}) x_{(3)}) 1_{(2)}$. Use $\epsilon(S(x_{(2)}) x_{(3)}) = \epsilon \circ \Pi^R(x_{(2)}) = \epsilon(x_{(2)})$, so that $\Pi^L \circ \Pi^R(x) = \epsilon(1_{(1)} S(x)) 1_{(2)} = \Pi^L \circ S(x)$.

D.2 Representations of WHAs and their characters

A representation (or left-module) of $\mathcal{A}$ consists of a vector space V and an algebra homomorphism $\mathfrak{R}: \mathcal{A} \rightarrow \text{End}(V)$. For a general algebra there is no concept of conjugate representation or of tensor product of representations. Those concepts become available in a (weak) Hopf algebra [143].

Let $\mathcal{M} = (V, \mathfrak{R})$ be a left-module for a weak Hopf algebra $\mathcal{A}$ over $\mathbb{C}$. There is a canonical right action on the dual space $V^\vee$, induced by the left action on V simply as $\langle f \mathfrak{R}(x), v \rangle \equiv \langle f, \mathfrak{R}(x) v \rangle$ for all $f \in V^\vee$, $v \in V$ and $x \in \mathcal{A}$. In order to construct a left action, instead, one uses the antipode S :

$$\mathfrak{R}^*(x) f \equiv f \mathfrak{R}(S(x)) . \quad (\text{D.14})$$

The fact that $S(xy) = S(y)S(x)$ guarantees that $\mathfrak{R}^*$ is an algebra homomorphism. Thus $\mathcal{M}^* = (V^\vee, \mathfrak{R}^*)$ is a left-module that we call the conjugate representation to $\mathcal{M}$.

Let $\mathcal{M}_\alpha, \mathcal{M}_\beta$ be two left-modules. The tensor product representation $\mathcal{M}_\alpha \otimes \mathcal{M}_\beta$ on $V_\alpha \otimes V_\beta$ is constructed using the coproduct:

$$\mathfrak{R}_{\alpha\beta} \equiv (\mathfrak{R}_\alpha \otimes \mathfrak{R}_\beta) \circ \Delta \quad \text{i.e.} \quad \mathfrak{R}_{\alpha\beta}(x) v_\alpha \otimes v_\beta \equiv \sum_i \left[\mathfrak{R}_\alpha(x_{(1)}^i) v_\alpha \right] \otimes \left[\mathfrak{R}_\beta(x_{(2)}^i) v_\beta \right] . \quad (\text{D.15})$$

This defines an algebra homomorphism $\mathfrak{R}_{\alpha\beta}: \mathcal{A} \rightarrow \text{End}(V_\alpha \otimes V_\beta)$ in the case of Hopf algebras.

Note however that for a strictly weak Hopf algebra, it is

$$\mathfrak{R}_{\alpha\beta}(1) = \mathfrak{R}_\alpha(1_{(1)}) \otimes \mathfrak{R}_\beta(1_{(2)}) \neq \mathbb{1} \otimes \mathbb{1}, \quad (\text{D.16})$$

and thus the identity would not behave trivially in the tensor product representation. This problem is easily solved by noticing that $\mathfrak{R}_{\alpha\beta}(1)$ is a projector: $\mathfrak{R}_{\alpha\beta}(1)^2 = \mathfrak{R}_{\alpha\beta}(1)$. Therefore, one truncates the support of $\mathfrak{R}_{\alpha\beta}$ from the vector space $V_\alpha \otimes V_\beta$ to the image of this projector.¹¹⁴

In the case of Hopf algebras one can also define the so-called “trivial” one-dimensional representation $\mathfrak{R}_\epsilon: \mathcal{A} \rightarrow \mathbb{C}$ using the counit:

$$\mathfrak{R}_\epsilon(x) \equiv \epsilon(x) . \quad (\text{D.17})$$

This is an algebra homomorphism thanks to multiplicativity of the counit (D.7). The trivial representation multiplies trivially with all other representations under tensor product, since (D.2) implies $(\epsilon \otimes \mathfrak{R}) \circ \Delta = (\mathfrak{R} \otimes \epsilon) \circ \Delta = \mathfrak{R}$, and it is moreover self-conjugate since $\epsilon \circ S = \epsilon$. In strictly weak Hopf algebras, a representation that multiplies trivially with all the other ones exists as well, however it is not one-dimensional.¹¹⁵

¹¹⁴This is a quite mild operation: notice that, because $\mathfrak{R}_{\alpha\beta}(1) \mathfrak{R}_{\alpha\beta}(x) \mathfrak{R}_{\alpha\beta}(1) = \mathfrak{R}_{\alpha\beta}(x)$ for every $x \in \mathcal{A}$, $\mathfrak{R}_{\alpha\beta}(x)$ is in fact nonzero only in the subspace which is the image of $\mathfrak{R}_{\alpha\beta}(1)$. So, by projecting, we are in fact removing a subspace of $V_\alpha \otimes V_\beta$ on which $\mathcal{A}$ acts as the zero map. This operation however has the side effect that taking the dimension of a representation is not a homomorphism of rings. In other words, the dimension is not multiplicative under tensor product.

¹¹⁵Furthermore, it is not always irreducible [148] (while it obviously is in the Hopf case). Thus, the representation category is in general $\text{Rep}(\mathcal{A})$ is multi-fusion, rather than fusion. However, in our case of interest, namely when the WHA is a strip algebra, the representation category is the fusion category $\mathcal{C}_{\mathcal{M}}^*$ on general grounds, see (4.3). This implies that the trivial representation is always irreducible, and can be read off from the fusion ring considered above. It is the only irrep $\mathfrak{R}_1$ that satisfies $N_{1\alpha}^\beta = \delta_{\alpha\beta}$.

Let us now discuss how the character defined in (C.19) behaves in WHAs. While it is clearly additive under the direct sum of representations, under tensor product one has¹¹⁶

$$\chi_{\alpha\beta}(h) = \text{tr}[\mathfrak{R}_\alpha(h_{(1)}) \otimes \mathfrak{R}_\beta(h_{(2)})] = \text{tr}[\mathfrak{R}_\alpha(h_{(1)})] \text{tr}[\mathfrak{R}_\beta(h_{(2)})] = \chi_\alpha(h_{(1)}) \chi_\beta(h_{(2)}) . \quad (\text{D.18})$$

The trace can be used to extract the fusion ring of representations of H . First, define the fusion coefficients $N_{\alpha\beta}^\gamma$ of the irreducible representations as

$$\mathfrak{R}_\alpha \otimes_H \mathfrak{R}_\beta = \bigoplus_\gamma N_{\alpha\beta}^\gamma \mathfrak{R}_\gamma . \quad (\text{D.19})$$

For the corresponding characters we thus have

$$\chi_\alpha(h_{(1)}) \chi_\beta(h_{(2)}) = \sum_\gamma N_{\alpha\beta}^\gamma \chi_\gamma(h) . \quad (\text{D.20})$$

Combining this with (C.23) we obtain

$$N_{\alpha\beta}^\gamma = \frac{\theta_\gamma}{\dim(\mathfrak{R}_\gamma)} \sum_{ab} \tilde{\mathcal{K}}^{ab} \chi_\alpha(h_{a,(1)}) \chi_\beta(h_{a,(2)}) \chi_\gamma(h_b) , \quad (\text{D.21})$$

which provides an operative way of computing the fusion ring of $\text{Rep}(\mathcal{A})$, assuming one knows the characters. These can in turn be computed from the MCI (see Fn. 103) via (C.24).

D.3 Symmetrizer in Hopf and weak Kac algebras

In semisimple Hopf algebras, the symmetrizer is constructed using the object

$$\mathbb{P} = (\text{id} \otimes S) \circ \Delta(P_\epsilon) \in \mathcal{A} \otimes \mathcal{A}^{\text{op}} . \quad (\text{D.22})$$

This acts on density matrices by letting $\mathcal{A}$ act from the left and $\mathcal{A}^{\text{op}}$ from the right. $\mathbb{P}$ is known to provide a separability idempotent with respect to the canonical Frobenius structure of a Hopf algebra, see Remark 3.2.31 of [146]. This means that $\mathbb{P}$ satisfies the properties (ii) – (iv) listed in Sec. 3.1. Idempotency is easy to check:

$$\begin{aligned} \mathbb{P} \times \mathbb{P} &= P_{\epsilon,(1)'} P_{\epsilon,(1)} \otimes S(P_{\epsilon,(2)}) S(P_{\epsilon,(2)'}) = P_{\epsilon,(1)'} P_{\epsilon,(1)} \otimes S(P_{\epsilon,(2)'} P_{\epsilon,(2)}) \\ &= (\text{id} \otimes S)(\Delta(P_\epsilon) \Delta(P_\epsilon)) = (\text{id} \otimes S) \circ \Delta(P_\epsilon^2) = \mathbb{P} , \end{aligned} \quad (\text{D.23})$$

and the property $\mu(\mathbb{P}) = P_{\epsilon,(1)} S(P_{\epsilon,(2)}) = 1$ immediately follows from (D.8) and $\epsilon(P_\epsilon) = 1$. Properties (ii) – (iv) guarantee 1. – 3. Instead, property 4. is guaranteed by the fact that

$$S(P_{\epsilon,(2)}) P_{\epsilon,(1)} = S\left(S(P_{\epsilon,(1)}) P_{\epsilon,(2)}\right) = S(\epsilon(P_\epsilon) 1) = 1 , \quad (\text{D.24})$$

where we used $S^2 = \text{id}$ repeatedly. Hence, $\mathbb{P}$ defines a map on density matrices that satisfies 1. – 4. and is universal. By uniqueness, it must thus define the same operation as $\mathbb{S}$ (see comments in Sec. 3.1).

¹¹⁶Note that one does not have to explicitly restrict the trace to the image of $\mathfrak{R}_{\alpha\beta}(1)$. Indeed, as discussed in Fn. 114, $\mathfrak{R}_{\alpha\beta}$ is zero on the complement of that image and so it gives no contribution to the trace.

In semisimple weak Hopf algebras where $S^2 = \text{id}$, the symmetrizer is constructed using

$$\mathbb{P} = (\text{id} \otimes S) \circ \Delta(h) , \quad (\text{D.25})$$

where h is the Haar integral. By Theorem 3.13 of [148], the properties 1. – 3. are satisfied. In particular, the preservation of symmetry-invariant matrices follows from

$$\mu(\mathbb{P}) = h_{(1)} S(h_{(2)}) = \Pi^L(h) = 1 . \quad (\text{D.26})$$

The property of being trace-preserving instead follows from

$$S(h_{(2)}) h_{(1)} = S\left(S(h_{(1)}) h_{(2)}\right) = S(\Pi^R(h)) = S(1) = 1 , \quad (\text{D.27})$$

where we used $S^2 = \text{id}$.

E Strip algebras as weak Hopf algebras

In this section we explain more in detail how the results of App. D.2 can be used in the context of this thesis, namely when the WHA is a strip algebra. In this case, the WHA structure is known (see e.g. [21, 98, 147]), but we repeat the calculations here in order to match our conventions and keep into account more general moves that may carry nontrivial factors.

In particular, a self-dual line $\mathcal{L}_a \in \mathcal{C}$ might carry a nontrivial *Frobenius-Schur indicator* κ_a (see App. A.4 for a discussion). With an appropriate choice of basis of the junction spaces, this is associated to the following zigzag moves:

$$\begin{array}{c} \text{zigzag} \\ \text{a} \end{array} = \kappa_a \begin{array}{c} \text{vertical} \\ \text{a} \end{array} = \begin{array}{c} \text{zigzag} \\ \text{a} \end{array}, \quad \kappa_a = \pm 1. \quad (\text{E.1})$$

It can be obtained from the relation $[F_{a\bar{a}a}^a]_{11} = \kappa_a/d_a$. For non-self-dual lines, it can always be chosen to be 1 by an appropriate gauge transformation.

Consider now a module category $\mathcal{M}$ over $\mathcal{C}$ and $\mathcal{B}_m, \mathcal{B}_n \in \mathcal{M}$. We henceforth suppress shadings and colors for ease of notation. Then the following relations hold and can be proven with an appropriate boundary F -move and by using (A.46) (also cf. with [99]):

$$\begin{array}{cc} \begin{array}{c} n \\ \diagup \quad \diagdown \\ a \quad m \end{array} = [\tilde{F}_{a\bar{a}n}^n]_{1m} \sqrt{\frac{d_a \tilde{d}_n}{\tilde{d}_m}} \begin{array}{c} n \\ \diagdown \quad \diagup \\ a \quad m \end{array}, & \begin{array}{c} a \quad m \\ \diagdown \quad \diagup \\ n \end{array} = [\tilde{F}_{a\bar{a}n}^n]^*_{1m} \sqrt{\frac{d_a \tilde{d}_n}{\tilde{d}_m}} \begin{array}{c} a \quad m \\ \diagup \quad \diagdown \\ n \end{array}, \\ \begin{array}{c} n \\ \diagdown \quad \diagup \\ m \quad a \end{array} = [R\tilde{F}_{\bar{n}a\bar{a}}^{\bar{n}}]^*_{\bar{m}1} \sqrt{\frac{d_a \tilde{d}_n}{\tilde{d}_m}} \begin{array}{c} n \\ \diagup \quad \diagdown \\ m \quad a \end{array}, & \begin{array}{c} m \quad a \\ \diagup \quad \diagdown \\ n \end{array} = [R\tilde{F}_{\bar{n}a\bar{a}}^{\bar{n}}]_{\bar{m}1} \sqrt{\frac{d_a \tilde{d}_n}{\tilde{d}_m}} \begin{array}{c} m \quad a \\ \diagdown \quad \diagup \\ n \end{array}. \end{array} \quad (\text{E.2})$$

The right module $\tilde{F}$ -symbols are defined as in (2.19). We work in a gauge where (2.20) holds. In this case we thus have $[R\tilde{F}_{\bar{n}a\bar{a}}^{\bar{n}}]_{\bar{m}1} = [\tilde{F}_{a\bar{a}n}^n]^*_{1m}$.

We now turn to the strip algebra and describe its weak Hopf structure.

Product and unit. The algebra product has already been reported in (2.24). Let us verify that it is associative, working for simplicity in the case where there are no multiplicities. We need to show that:

$$\sum_d Q_{m,r,a,n,s,b,p,t}^d Q_{m,r,d,p,t,c,q,u}^f = \sum_e Q_{m,r,a,n,s,e,q,u}^f Q_{n,s,b,p,t,c,q,u}^e. \quad (\text{E.3})$$

Substituting the expression for Q , the quantum dimensions cancel out and we are left with

$$\sum_d [\tilde{F}_{bar}^t]_{ds} [\tilde{F}_{bam}^p]^*_{dn} [\tilde{F}_{cdr}^u]_{ft} [\tilde{F}_{cdm}^q]^*_{fp} = \sum_e [\tilde{F}_{ear}^u]_{fs} [\tilde{F}_{eam}^q]^*_{fn} [\tilde{F}_{cbs}^u]_{et} [\tilde{F}_{cbn}^q]^*_{ep}. \quad (\text{E.4})$$

This identity can be verified using the module pentagon equation (A.43) twice, in particular in the following two forms:

$$\begin{aligned} \sum_e [F_{cba}^f]_{ed}^* [\tilde{F}_{ear}^u]_{fs} [\tilde{F}_{cbs}^u]_{et} &= [\tilde{F}_{cdr}^u]_{ft} [\tilde{F}_{bar}^t]_{ds} , \\ [\tilde{F}_{eam}^q]_{fn} [\tilde{F}_{cbn}^q]_{ep} &= \sum_d [F_{cba}^f]_{ed} [\tilde{F}_{cdm}^q]_{fp} [\tilde{F}_{bam}^p]_{dn} . \end{aligned} \quad (\text{E.5})$$

The second one is used after complex conjugation. On both sides of (E.4) we obtain a double sum over d and e with the same string. This proves the identity.

Finally, it is easy to verify that the unit is

$$1 = \sum_{m,n \in \text{Irr}(\mathcal{M})} \begin{array}{c} m \quad n \\ \downarrow \quad \downarrow \\ \text{---} 1 \text{---} \\ \downarrow \quad \downarrow \\ m \quad n \end{array} . \quad (\text{E.6})$$

Coproduct. To derive the coproduct Δ we perform the following series of moves inside a “pair of pants”:¹¹⁷

$$\begin{aligned} \sum_r \begin{array}{c} p \quad r \quad q \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{---} a \text{---} \\ \downarrow \quad \downarrow \\ m \quad n \end{array} &= \sum_r \kappa_a \begin{array}{c} p \quad r \quad q \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{---} a \text{---} \\ \downarrow \quad \downarrow \\ m \quad n \end{array} = \sum_{r,s} \kappa_a [\tilde{F}_{aar}^r]_{1s} \begin{array}{c} p \quad r \quad r \quad q \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \text{---} s \text{---} \\ \downarrow \quad \downarrow \\ m \quad n \end{array} \\ &= \sum_{r,s} \underbrace{\kappa_a [\tilde{F}_{aar}^r]_{1s} [\tilde{F}_{aas}^s]_{1r}^*}_{G_{ss}^{rr}(a)} \sqrt{\frac{d_a \tilde{d}_s}{\tilde{d}_r}} \begin{array}{c} p \quad r \quad r \quad q \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \text{---} s \text{---} \\ \downarrow \quad \downarrow \\ m \quad n \end{array} , \end{aligned} \quad (\text{E.7})$$

whence

$$\Delta \left(\begin{array}{c} p \quad q \\ \downarrow \quad \downarrow \\ \text{---} a \text{---} \\ \downarrow \quad \downarrow \\ m \quad n \end{array} \right) = \sum_{r,s} \sqrt{\frac{d_a \tilde{d}_s}{\tilde{d}_r}} G_{ss}^{rr}(a) \begin{array}{c} p \quad r \\ \downarrow \quad \downarrow \\ \text{---} a \text{---} \\ \downarrow \quad \downarrow \\ m \quad s \end{array} \otimes \begin{array}{c} r \quad q \\ \downarrow \quad \downarrow \\ \text{---} a \text{---} \\ \downarrow \quad \downarrow \\ s \quad n \end{array} . \quad (\text{E.8})$$

It is interesting to compute $\Delta(1)$ to see when the strip algebra is Hopf. Notice that when $a = 1$ it must be $r = s$ and that $G_{ss}^{ss}(1) = 1$. Then we obtain

$$\Delta(1) = \sum_{m,n,s} \begin{array}{c} m \quad s \\ \downarrow \quad \downarrow \\ \text{---} 1 \text{---} \\ \downarrow \quad \downarrow \\ m \quad s \end{array} \otimes \begin{array}{c} s \quad n \\ \downarrow \quad \downarrow \\ \text{---} 1 \text{---} \\ \downarrow \quad \downarrow \\ s \quad n \end{array} . \quad (\text{E.9})$$

We see that, unless $\mathcal{M}$ has just one simple object, $\Delta(1)$ is different from $1 \otimes 1$ (which would have four independent sums, one for each boundary). Thus, if $\mathcal{M}$ has more than one simple, the corresponding algebra is strictly weak Hopf.

¹¹⁷See (E.14) for the most general definition of the coefficients G .

For group-like symmetries, the expression (E.8) simplifies. Indeed $d_a = 1$ for all $a \in G$ and, moreover, all boundary quantum dimensions are equal, so that they cancel in the ratios. Furthermore, $r = as$ is fixed,¹¹⁸ and the module pentagon identity implies:

$$\frac{\tilde{F}_{\bar{a},a,s} \tilde{F}_{a,\bar{a},s} F_{a,\bar{a},a}}{\tilde{F}_{a\bar{a},a,s} \tilde{F}_{a,\bar{a},as}} = 1 \quad \Rightarrow \quad \tilde{F}_{\bar{a},a,s} \tilde{F}_{a,\bar{a},as}^* \underbrace{F_{a,\bar{a},a}}_{\kappa_a} = 1. \quad (\text{E.10})$$

We used that $\bar{a} = a^{-1}$, the normalization condition, and that F and $\tilde{F}$ -symbols are phases. Finally, the coefficient in the coproduct simplifies to

$$\sqrt{\frac{d_a \tilde{d}_s}{\tilde{d}_r}} G_{s,s}^{as,as}(a) = \kappa_a \tilde{F}_{a,\bar{a},as} \tilde{F}_{\bar{a},a,s}^* = 1. \quad (\text{E.11})$$

Counit. For the counit ϵ we perform the following sequence of moves inside a “cap”:

$$\begin{array}{c} p \quad q \\ \curvearrowright \\ m \quad n \end{array} \quad \begin{array}{c} a \\ \nearrow \\ \searrow \end{array} = \delta_{pq} \begin{array}{c} n \quad a \\ \curvearrowright \\ m \quad n \end{array} = \delta_{pq} \delta_{mn} \sqrt{\frac{\tilde{d}_p d_a}{\tilde{d}_m}} \begin{array}{c} \curvearrowright \\ m \quad m \end{array}, \quad (\text{E.12})$$

whence

$$\epsilon \left(\begin{array}{c} p \quad q \\ \curvearrowright \\ m \quad n \end{array} \right) = \delta_{pq} \delta_{mn} \sqrt{\frac{\tilde{d}_p d_a}{\tilde{d}_m}}. \quad (\text{E.13})$$

For groups, this simplifies to $\delta_{pq} \delta_{mn}$.

Antipode. For the antipode S , combining (E.1) and (E.2), and working in the gauge (2.20), we first obtain the relation:

$$\begin{array}{c} p \quad q \\ \curvearrowright \\ m \quad n \end{array} \quad \begin{array}{c} a \\ \nearrow \\ \searrow \end{array} = \kappa_a \begin{array}{c} p \quad a \quad q \\ \curvearrowright \\ m \quad n \end{array} = \underbrace{\kappa_a d_a \sqrt{\frac{\tilde{d}_m \tilde{d}_q}{\tilde{d}_n \tilde{d}_p}} [\tilde{F}_{\bar{a}am}^m]_{1p}^* [\tilde{F}_{a\bar{a}q}^q]_{1n}}_{G_{mn}^{pq}(a)} \begin{array}{c} p \quad q \\ \curvearrowright \\ m \quad n \end{array}. \quad (\text{E.14})$$

We can then perform the following manipulations inside a “bent strip”:

$$\begin{array}{c} \curvearrowright \\ m \quad n \quad q \quad p \end{array} \quad \begin{array}{c} a \\ \nearrow \\ \searrow \end{array} = G_{mn}^{pq}(a)^{-1} \begin{array}{c} \curvearrowright \\ m \quad n \quad q \quad p \end{array} \quad \begin{array}{c} a \\ \nearrow \\ \searrow \end{array} = G_{mn}^{pq}(a)^{-1} \begin{array}{c} \curvearrowright \\ m \quad n \quad q \quad p \end{array} \quad \begin{array}{c} \bar{a} \\ \nearrow \\ \searrow \end{array}, \quad (\text{E.15})$$

whence

$$S \left(\begin{array}{c} p \quad q \\ \curvearrowright \\ m \quad n \end{array} \right) = G_{mn}^{pq}(a)^{-1} \begin{array}{c} n \quad \bar{a} \quad m \\ \curvearrowright \\ q \quad p \end{array}. \quad (\text{E.16})$$

¹¹⁸Recall that, in this case, objects in $\mathcal{M}$ are right cosets: $r, s \in G/H$ for some subgroup H of G . The classification of module categories over Vec_G^ω is reviewed in App. F.

For group-like symmetries, the coefficient simplifies to

$$G_{m,n}^{am,an}(a)^{-1} = F_{a,\bar{a},a} \tilde{F}_{\bar{a},a,m} \tilde{F}_{a,\bar{a},an}^* = \tilde{F}_{\bar{a},a,m}^* \tilde{F}_{\bar{a},a,n} . \quad (\text{E.17})$$

To conclude, we lay out a systematic procedure to compute the data necessary for the computation of the entanglement entropy resolved with respect to the strip algebra, following the protocol described in Sec. 4, and summarized at the beginning of Sec. 6. To this end, given a fusion category $\mathcal{C}$ and a module category $\mathcal{M}$ over it, we study the associated strip algebra to determine the dimensions of its representations and the quantum dimensions of the dual category.¹¹⁹ Similar ideas have appeared in [198]. The algorithm goes as follows:

- Construct the strip algebra as a vector space:

$$\text{Strip}_{\mathcal{C}}(\mathcal{M}) = \bigoplus_{\substack{m,n,p,q \in \text{Irr}(\mathcal{M}) \\ a \in \text{Irr}(\mathcal{C})}} \text{Hom}_{\mathcal{M}}(a \otimes m, p) \otimes \text{Hom}_{\mathcal{M}}(a \otimes n, q) . \quad (\text{E.18})$$

- The categorical data of $\mathcal{C}$ and $\mathcal{M}$ further allows one to compute the product of this algebra via (2.24), as well as the $\mathcal{K}$ -matrix via (C.15),¹²⁰ and the coproduct using (E.8).
- Compute the central idempotents Π by solving the defining equations:

$$[\Pi, H] = 0 \quad \forall H \in \text{Strip}_{\mathcal{C}}(\mathcal{M}) , \quad \Pi^2 = \Pi . \quad (\text{E.19})$$

These equations have 2^{n_r} solutions, where n_r is the number of irreducible representations.¹²¹ From these, extract the minimal central idempotents (MCIs). One way of doing that is to compute all products $M_{\alpha\beta} \equiv \Pi_{\alpha}\Pi_{\beta}$. The MCIs are labeled by those α 's for which the matrix row M_{α} has exactly 2^{n_r-1} zero entries. Indeed, if $\Pi_{\beta} \supseteq \Pi_{\alpha}$ then $M_{\alpha\beta} = \Pi_{\alpha}$, whereas if $\Pi_{\beta} \not\supseteq \Pi_{\alpha}$ then $M_{\alpha\beta} = 0$, and precisely half of the central idempotents contain (or do not contain) a given representation α .

- The rank of an MCI Π_{α} , seen as a linear operator acting by left multiplication on the algebra, is the square of the dimension of the associated irreducible representation, namely $\dim(\mathfrak{R}_{\alpha}) = \sqrt{\text{rk}(\Pi_{\alpha})}$.
- Eqn. (C.24) allows, with the knowledge of the MCIs, to compute any character in any representation. Then, using (D.21), one can compute the fusion ring of strip-algebra irreps, namely the fusion ring of $\mathcal{C}_{\mathcal{M}}^*$, and finally the quantum dimensions $d_{\bar{a}}$.

¹¹⁹Instead, the boundary quantum dimensions can be simply computed from the bulk ones using (4.9) with $m = n$.

¹²⁰For this purpose it is enough to use $\theta_{\alpha} = \dim(\mathfrak{R}_{\alpha})$ for all α , namely the trace over the regular representation.

¹²¹Indeed, a generic central idempotent is a linear combination of the minimal ones with coefficients either 0 or 1, and there is an MCI for each irrep. Also see Fn. 103.

F Module categories over $\mathbf{Vec}_G^\omega$

Consider a symmetry group G . We first consider the case that there is no anomaly, $\omega = 1$, therefore the fusion category is $\mathbf{Vec}_G$. Module categories are labeled by conjugacy classes of subgroups $H \subseteq G$ and cocycles $\psi \in H^2(H, \mathbf{U}(1))$ (see Example 7.4.10 in [113]). Let us explain how the modules are constructed.

The simple objects $\mathcal{B}_m$ correspond to the right-cosets G/H . We indicate them as $\mu \in G/H$. There is a projection map $[\cdot]: G \rightarrow G/H$ so that $[m] = \mu$ if m is a representative of the class μ . The module action is simply

$$a \times [m] = [am] \quad \text{for } a \in G, \quad [m] \in G/H. \quad (\text{F.1})$$

This product is well defined, in the sense that it does not depend on the representative m chosen, because the coset is a right-coset. We can then write $a \times \mu = a\mu$. Notice that H and kHk^{-1} (where $k \in G$) define equivalent module categories. If we denote $[g]_1 \in G/H$ and $[g]_k \in G/(kHk^{-1})$, an isomorphism is given by $\varphi([g]_1) = [gk^{-1}]_k$.

In order to construct the $\tilde{F}$ -symbols we need a map $h: G \times G/H \rightarrow H$. First, choose a lift $r: G/H \rightarrow G$ of $[\cdot]$, that is a map

$$r(\mu) \quad \text{such that} \quad [r(\mu)] = \mu. \quad (\text{F.2})$$

In general, $a \cdot r(\mu) \neq r(a\mu)$, however $[a \cdot r(\mu)] = a\mu = [r(a\mu)]$, therefore the two terms in the inequality differ by the action of H from the right. This allows us to define

$$h(a, \mu) \quad \text{such that} \quad a \cdot r(\mu) = r(a\mu) \cdot h(a, \mu). \quad (\text{F.3})$$

By considering $ab \cdot r(\mu)$ one obtains the identity

$$h(ab, \mu) = h(a, b\mu) \cdot h(b, \mu). \quad (\text{F.4})$$

Besides, $h(1, \mu) = 1$ and so $h(a^{-1}, \mu) = h(a, a^{-1}\mu)^{-1}$. We then define the $\tilde{F}$ -symbols as

$$\tilde{F}_{ab\mu} = \psi(h(a, b\mu), h(b, \mu)). \quad (\text{F.5})$$

The indices not indicated are fixed. The left-module pentagon equation reads:

$$\tilde{F}_{ab,c,\mu} \tilde{F}_{a,b,c\mu} = F_{a,b,c} \tilde{F}_{a,bc,\mu} \tilde{F}_{b,c,\mu}. \quad (\text{F.6})$$

As we are assuming that there are no anomalies, $\omega = 1$, we have $F_{a,b,c} = 1$. The remaining terms organize as

$$\frac{\psi[h(b, c\mu), h(c, \mu)] \psi[h(a, bc\mu), h(bc, \mu)]}{\psi[h(ab, c\mu), h(c, \mu)] \psi[h(a, bc\mu), h(b, c\mu)]} = d\psi[h(a, bc\mu), h(b, c\mu), h(c, \mu)] = 1. \quad (\text{F.7})$$

This equation is satisfied because ψ is a cocycle. Thus the pentagon equation is satisfied.

There are two extreme cases. When $H = \{1\}$ we obtain the regular module, whose simple objects are all the elements of G . Such a module is completely fixed by G . Indeed $r(\mu) = \mu \in G$,

$h(a, \mu) = 1$, and ψ is trivial so that $\tilde{F} = \omega$ is trivial. When $H = G$, instead, the module has a single simple object (which is then a strongly-symmetric boundary condition) and the inequivalent consistent $\tilde{F}$ -symbols parametrize the different SPT phases of G – or equivalently projective representations of G .

Consider now the general case that ω is not necessarily vanishing. This includes the case that ω vanishes in cohomology, but we have chosen a representative that is not identically 1. We will however use normalized cocycles. The fusion category is $\mathbf{Vec}_G^\omega$. Then module categories are labeled by:

- the conjugacy class of a subgroup $H \subseteq G$ such that $\omega|_H$ is trivial in $H^3(H, \mathbf{U}(1))$;
- a cochain $\psi \in C^2(H, \mathbf{U}(1))$ such that (using multiplicative notation):

$$d\psi = \omega^{-1}|_H . \quad (\text{F.8})$$

Such cochains form a torsor over $H^2(H, \mathbf{U}(1))$.

As before, the simple objects are classes of G/H and fusion is defined in the natural way. Given a lift r of the projection and the map $h(a, \mu) \in H$ defined in (F.3), the $\tilde{F}$ -symbols are

$$\tilde{F}_{ab\mu} = \psi(h(a, b\mu), h(b, \mu)) \frac{\omega(r(ab\mu), h(a, b\mu), h(b, \mu)) \omega(a, b, r(\mu))}{\omega(a, r(b\mu), h(b, \mu))} . \quad (\text{F.9})$$

The only thing to check is that the left-module pentagon equation is satisfied. The condition (F.8) on ψ reads:

$$\frac{\psi[h(b, c\mu), h(c, \mu)] \psi[h(a, bc\mu), h(bc, \mu)]}{\psi[h(ab, c\mu), h(c, \mu)] \psi[h(a, bc\mu), h(b, c\mu)]} = d\psi = \omega[h(a, bc\mu), h(b, c\mu), h(c, \mu)]^{-1} . \quad (\text{F.10})$$

When substituting into the pentagon equation, we obtain

$$\begin{aligned} & \left\{ \omega[a, b, c] \omega[a, bc, r(\mu)] \omega[a, r(bc\mu), h_{b,c\mu}] \omega[b, c, r(\mu)] \omega[ab, r(c\mu), h_{c,\mu}] \times \right. \\ & \left. \omega[r(bc\mu), h_{b,c\mu}, h_{c,\mu}] \omega[r(abc\mu), h_{a,bc\mu}, h_{bc,\mu}] \right\} / \left\{ \omega[a, b, r(c\mu)] \omega[a, r(bc\mu), h_{bc,\mu}] \omega[b, r(c\mu), h_{c,\mu}] \times \right. \\ & \left. \omega[h_{a,bc\mu}, h_{b,c\mu}, h_{c,\mu}] \omega[ab, c, r(\mu)] \omega[r(abc\mu), h_{a,bc\mu}, h_{b,c\mu}] \omega[r(abc\mu), h_{ab,c\mu}, h_{c,\mu}] \right\} \stackrel{?}{=} 1 , \end{aligned} \quad (\text{F.11})$$

where we used a more compact notation for h . The expression on the left is indeed equal to 1, using that $d\omega[a, b, c, r(\mu)]$, $d\omega[a, b, r(c\mu), h_{c,\mu}]$, $d\omega[a, r(bc\mu), h_{b,c\mu}, h_{c,\mu}]$, $d\omega[r(abc\mu), h_{a,bc\mu}, h_{bc,\mu}, h_{c,\mu}]$ are all equal to 1, as well as the multiplication properties of r and h given above.

G Anomaly cocycles

G.1 $G = \mathbb{Z}_N$

The possible anomalies of a (bosonic, oriented) 2-dimensional theory with $\mathbb{Z}_N$ 0-form symmetry are classified by $H^3(\mathbb{Z}_N, \text{U}(1))$. A standard result in group-cohomology [174, 199] is

$$H^3(\mathbb{Z}_N, \text{U}(1)) \cong \mathbb{Z}_N . \quad (\text{G.1})$$

Let us use the additive notation, $\mathbb{Z}_N = \{0, \dots, N-1\}$. A representative of the cohomology class labeled by $p \in \mathbb{Z}_N$ can then be taken to be

$$\omega_p(a, b, c) = \exp\left(\frac{2\pi i}{N} p a \left\lfloor \frac{b+c}{N} \right\rfloor\right) = \exp\left(\frac{2\pi i}{N} p a \frac{b+c - [b+c]_0^{N-1}}{N}\right), \quad (\text{G.2})$$

where $[\cdot]_0^{N-1}$ is the reduction modulo N in the range $0, \dots, N-1$, and a, b, c are taken in the same range (while for p one can take any representative). One can check that this cocycle is already in the normalized form (namely, it is 1 whenever one of the entries is the identity). For $N = 2$ there is a simplified form,

$$\omega_p(a, b, c) = \exp(i\pi p abc), \quad (\text{G.3})$$

which one can check directly to be equal to the general formula.

For N odd, one could instead choose a symmetric range $\mathbb{Z}_N = \{-\frac{N-1}{2}, \dots, 0, \dots, \frac{N-1}{2}\}$ for a, b, c . Let us denote this choice with primed symbols. The two choices are related by

$$a' = a - N\epsilon(a) \quad \text{where} \quad \epsilon(a) \equiv \begin{cases} 0 & \text{if } 0 \leq a \leq \frac{N-1}{2} \\ 1 & \text{if } \frac{N+1}{2} \leq a \leq N-1 \end{cases}. \quad (\text{G.4})$$

In this notation, one can take as representative the cocycle

$$\omega'_p(a', b', c') = \exp\left(\frac{2\pi i}{N} p a' \frac{b' + c' - [b' + c']_{-(N-1)/2}^{(N-1)/2}}{N}\right), \quad (\text{G.5})$$

which is different from (G.2) but cohomologous to it. Indeed, we claim that

$$\omega_p(a, b, c) = \delta\beta_p(a, b, c) \omega'_p(a'(a), b'(b), c'(c)), \quad (\text{G.6})$$

with

$$\beta_p(a, b) \equiv \exp\left(-\frac{2\pi i}{N} p a \epsilon(b)\right). \quad (\text{G.7})$$

To show this, the LHS of (G.6) can easily be reexpressed as

$$\frac{\omega_p}{\omega'_p} = \exp\left[-\frac{2\pi i}{N} p a \left(\frac{[b+c]_0^{N-1} - [b'+c']_{-(N-1)/2}^{(N-1)/2}}{N} - \epsilon(b) - \epsilon(c)\right)\right], \quad (\text{G.8})$$

whereas the RHS becomes

$$\begin{aligned} \delta\beta_p &= \exp\left[-\frac{2\pi i}{N} p \left(b\epsilon(c) - (a+b)\epsilon(c) + a\epsilon([b+c]_0^{N-1}) - a\epsilon(b)\right)\right] \\ &= \exp\left[-\frac{2\pi i}{N} p a \left(\epsilon([b+c]_0^{N-1}) - \epsilon(b) - \epsilon(c)\right)\right]. \end{aligned} \quad (\text{G.9})$$

The two sides are then equal because $([b+c]_0^{N-1})' = [b'+c']_{-(N-1)/2}^{(N-1)/2}$ and using (G.4).

Subgroups. Let us now study the anomaly-free subgroups. The subgroups of $\mathbb{Z}_N$ are isomorphic to $\mathbb{Z}_k$, with $k \mid N$. We want to determine when the restriction of the anomaly is trivializable. Let us use the cocycle (G.2). Then, we have for the restriction

$$\omega_p\left(\frac{N}{k}a, \frac{N}{k}b, \frac{N}{k}c\right) = \exp\left(\frac{2\pi i}{k} p a \left\lfloor \frac{b+c}{k} \right\rfloor\right), \quad (\text{G.10})$$

which takes the same form as the cocycle for a $\mathbb{Z}_k$ anomaly. This implies that the restriction is cohomologically trivial whenever p is an integer multiple of k , equivalently $k \mid p$. It follows that the maximal non-anomalous subgroup is given by $\mathbb{Z}_k$ with $k = \gcd(N, p)$. All other anomaly-free subgroups are then subgroups of $\mathbb{Z}_k$, i.e. $\mathbb{Z}_m$ with $m \mid k$.

G.2 $G = S_3$

We refer to Sec. 6.1.2 for the basic definitions. Recall that the possible anomalies of a (bosonic, oriented) 2-dimensional theory with S_3 0-form symmetry are classified by $H^3(S_3, \text{U}(1)) \cong \mathbb{Z}_6$. An explicit representative for the cocycle labeled by $p \in \mathbb{Z}_6$ is given in Eqn. (3.2.8) of [174]. In the notation of (6.18), it reads:

$$\omega_p((A, a), (B, b), (C, c)) = \exp\left(\frac{2\pi i}{3} p (-1)^{B+C} a \frac{(-1)^C b + c - [(-1)^C b + c]_{-1}^1}{3} + i\pi p ABC\right). \quad (\text{G.11})$$

These cocycles are already in the normalized form.

Subgroups. The group S_3 has one nontrivial normal subgroup, the $\mathbb{Z}_3$ generated by $R = (0, 1)$. The cocycle above restricted to this subgroup is

$$\omega_p((0, a), (0, b), (0, c)) = \exp\left(\frac{2\pi i}{3} p a \frac{b + c - [b + c]_{-1}^1}{3}\right), \quad (\text{G.12})$$

which is of the general form (G.5). Such a cocycle is then trivial if and only if $3 \mid p$, so $p = 0, 3$.

In addition, there are three non-normal $\mathbb{Z}_2$ subgroups, pairwise related by conjugation:

$$\mathbb{Z}_2^{(a)} = \{1, X\}, \quad \mathbb{Z}_2^{(b)} = \{1, XR\}, \quad \mathbb{Z}_2^{(c)} = \{1, XR^2\}. \quad (\text{G.13})$$

The pullback of group cohomologies induced by a conjugation group-automorphism is the identity.¹²² Hence, it suffices to study the restriction of the S_3 cocycle to only one of these $\mathbb{Z}_2$ subgroups. Choosing $\mathbb{Z}_2^{(a)}$ we have

$$\omega_p((A, 0), (B, 0), (C, 0)) = \exp(i\pi p ABC), \quad (\text{G.14})$$

which is a $\mathbb{Z}_2$ cocycle of the form (G.3). Hence, the corresponding cohomology class is trivial if and only if $2 \mid p$, so $p = 0, 2, 4$.

¹²²See for example Proposition 6.2 of [200].

References

- [1] Francesco Benini, Pasquale Calabrese, Michele Fossati, Amartya Harsh Singh, and Marco Venuti. *Entanglement asymmetry for higher and noninvertible symmetries*. Sept. 2025. arXiv: [2509.16311 \[hep-th\]](#).
- [2] Francesco Benini, Michele Tortora, and Marco Venuti. *Symmetry-Resolved Entanglement with Anomalous and Noninvertible Symmetries*. To appear.
- [3] John McGreevy. “Generalized Symmetries in Condensed Matter”. In: *Ann. Rev. Condensed Matter Phys.* 14 (2023), pp. 57–82. DOI: [10.1146/annurev-conmatphys-040721-021029](#). arXiv: [2204.03045 \[cond-mat.str-el\]](#).
- [4] Xie Chen. “Essay: Generalized Landau Paradigm for Quantum Phases and Phase Transitions”. In: *Phys. Rev. Lett.* 135.25 (2025), p. 250001. DOI: [10.1103/tmvy-vsqd](#). arXiv: [2511.19793 \[hep-th\]](#).
- [5] Davide Gaiotto, Anton Kapustin, Nathan Seiberg, and Brian Willett. “Generalized global symmetries”. In: *JHEP* 02 (2015), p. 172. DOI: [10.1007/JHEP02\(2015\)172](#). arXiv: [1412.5148 \[hep-th\]](#).
- [6] Lakshya Bhardwaj, Lea E. Bottini, Ludovic Fraser-Taliente, Liam Gladden, Dewi S. W. Gould, Arthur Platschorre, and Hannah Tillim. “Lectures on generalized symmetries”. In: *Phys. Rept.* 1051 (2024), pp. 1–87. DOI: [10.1016/j.physrep.2023.11.002](#). arXiv: [2307.07547 \[hep-th\]](#).
- [7] Clay Cordova, Thomas T. Dumitrescu, Kenneth Intriligator, and Shu-Heng Shao. “Snowmass White Paper: Generalized Symmetries in Quantum Field Theory and Beyond”. In: *Snowmass 2021*. May 2022. arXiv: [2205.09545 \[hep-th\]](#).
- [8] Anton Kapustin and Nathan Seiberg. “Coupling a QFT to a TQFT and Duality”. In: *JHEP* 04 (2014), p. 001. DOI: [10.1007/JHEP04\(2014\)001](#). arXiv: [1401.0740 \[hep-th\]](#).
- [9] Konstantinos Roumpedakis, Sahand Seifnashri, and Shu-Heng Shao. “Higher Gauging and Non-invertible Condensation Defects”. In: *Commun. Math. Phys.* 401.3 (2023), pp. 3043–3107. DOI: [10.1007/s00220-023-04706-9](#). arXiv: [2204.02407 \[hep-th\]](#).
- [10] Erik P. Verlinde. “Fusion Rules and Modular Transformations in 2D Conformal Field Theory”. In: *Nucl. Phys. B* 300 (1988), pp. 360–376. DOI: [10.1016/0550-3213\(88\)90603-7](#).
- [11] Gregory W. Moore and Nathan Seiberg. “Classical and Quantum Conformal Field Theory”. In: *Commun. Math. Phys.* 123 (1989), p. 177. DOI: [10.1007/BF01238857](#).
- [12] Gregory W. Moore and Nathan Seiberg. “Taming the Conformal Zoo”. In: *Phys. Lett. B* 220 (1989), pp. 422–430. DOI: [10.1016/0370-2693\(89\)90897-6](#).
- [13] V. B. Petkova and J. B. Zuber. “Generalized twisted partition functions”. In: *Phys. Lett. B* 504 (2001), pp. 157–164. DOI: [10.1016/S0370-2693\(01\)00276-3](#). arXiv: [hep-th/0011021](#).

- [14] Lakshya Bhardwaj and Yuji Tachikawa. “On finite symmetries and their gauging in two dimensions”. In: *JHEP* 03 (2018), p. 189. DOI: [10.1007/JHEP03\(2018\)189](https://doi.org/10.1007/JHEP03(2018)189). arXiv: [1704.02330](https://arxiv.org/abs/1704.02330) [[hep-th](#)].
- [15] Chi-Ming Chang, Ying-Hsuan Lin, Shu-Heng Shao, Yifan Wang, and Xi Yin. “Topological defect lines and renormalization group flows in two dimensions”. In: *JHEP* 01 (2019), p. 026. DOI: [10.1007/JHEP01\(2019\)026](https://doi.org/10.1007/JHEP01(2019)026). arXiv: [1802.04445](https://arxiv.org/abs/1802.04445) [[hep-th](#)].
- [16] Shu-Heng Shao. “What’s Done Cannot Be Undone: TASI Lectures on Non-Invertible Symmetries”. In: *Theoretical Advanced Study Institute in Elementary Particle Physics 2023: Aspects of Symmetry*. Aug. 2023. arXiv: [2308.00747](https://arxiv.org/abs/2308.00747) [[hep-th](#)].
- [17] Christopher L. Douglas and David J. Reutter. “Fusion 2-categories and a state-sum invariant for 4-manifolds”. In: (2018). arXiv: [1812.11933](https://arxiv.org/abs/1812.11933) [[math.QA](#)].
- [18] Davide Gaiotto and Theo Johnson-Freyd. “Condensations in higher categories”. In: (2019). arXiv: [1905.09566](https://arxiv.org/abs/1905.09566) [[math.CT](#)].
- [19] Theo Johnson-Freyd. “On the classification of topological orders”. In: *Commun. Math. Phys.* 393 (2022), pp. 989–1033. DOI: [10.1007/s00220-022-04380-3](https://doi.org/10.1007/s00220-022-04380-3). arXiv: [2003.06663](https://arxiv.org/abs/2003.06663) [[math.CT](#)].
- [20] Lakshya Bhardwaj, Lea E. Bottini, Sakura Schafer-Nameki, and Apoorv Tiwari. “Non-invertible higher-categorical symmetries”. In: *SciPost Phys.* 14 (2023), p. 007. DOI: [10.21468/SciPostPhys.14.1.007](https://doi.org/10.21468/SciPostPhys.14.1.007). arXiv: [2204.06564](https://arxiv.org/abs/2204.06564) [[hep-th](#)].
- [21] Alexei Kitaev and Liang Kong. “Models for gapped boundaries and domain walls”. In: *Commun. Math. Phys.* 313 (2012), pp. 351–373. DOI: [10.1007/s00220-012-1500-5](https://doi.org/10.1007/s00220-012-1500-5). arXiv: [1104.5047](https://arxiv.org/abs/1104.5047) [[cond-mat.str-el](#)].
- [22] Michael A. Levin and Xiao-Gang Wen. “String net condensation: A Physical mechanism for topological phases”. In: *Phys. Rev. B* 71 (2005), p. 045110. DOI: [10.1103/PhysRevB.71.045110](https://doi.org/10.1103/PhysRevB.71.045110). arXiv: [cond-mat/0404617](https://arxiv.org/abs/cond-mat/0404617).
- [23] Liang Kong. “Anyon condensation and tensor categories”. In: *Nucl. Phys. B* 886 (2014), pp. 436–482. DOI: [10.1016/j.nuclphysb.2014.07.003](https://doi.org/10.1016/j.nuclphysb.2014.07.003). arXiv: [1307.8244](https://arxiv.org/abs/1307.8244) [[cond-mat.str-el](#)].
- [24] Tian Lan, Juven C. Wang, and Xiao-Gang Wen. “Gapped Domain Walls, Gapped Boundaries and Topological Degeneracy”. In: *Phys. Rev. Lett.* 114.7 (2015), p. 076402. DOI: [10.1103/PhysRevLett.114.076402](https://doi.org/10.1103/PhysRevLett.114.076402). arXiv: [1408.6514](https://arxiv.org/abs/1408.6514) [[cond-mat.str-el](#)].
- [25] Lakshya Bhardwaj, Lea E. Bottini, Daniel Pajer, and Sakura Schäfer-Nameki. “Gapped phases with non-invertible symmetries: (1+1)d”. In: *SciPost Phys.* 18 (2025), p. 032. DOI: [10.21468/SciPostPhys.18.1.032](https://doi.org/10.21468/SciPostPhys.18.1.032). arXiv: [2310.03784](https://arxiv.org/abs/2310.03784) [[hep-th](#)].
- [26] Lakshya Bhardwaj, Lea E. Bottini, Daniel Pajer, and Sakura Schafer-Nameki. “Categorical Landau paradigm for gapped phases”. In: *Phys. Rev. Lett.* 133 (2024), p. 161601. DOI: [10.1103/PhysRevLett.133.161601](https://doi.org/10.1103/PhysRevLett.133.161601). arXiv: [2310.03786](https://arxiv.org/abs/2310.03786) [[cond-mat.str-el](#)].

- [27] Lakshya Bhardwaj, Daniel Pajer, Sakura Schafer-Nameki, Apoorv Tiwari, Alison Warman, and Jingxiang Wu. “Gapped phases in (2+1)d with non-invertible symmetries: Part I”. In: *SciPost Phys.* 19.2 (2025), p. 056. DOI: [10.21468/SciPostPhys.19.2.056](https://doi.org/10.21468/SciPostPhys.19.2.056). arXiv: [2408.05266](https://arxiv.org/abs/2408.05266) [hep-th].
- [28] Lakshya Bhardwaj, Sakura Schafer-Nameki, Apoorv Tiwari, and Alison Warman. “Gapped Phases in (2+1)d with Non-Invertible Symmetries: Part II”. In: (Feb. 2025). arXiv: [2502.20440](https://arxiv.org/abs/2502.20440) [hep-th].
- [29] Ofer Aharony, Nathan Seiberg, and Yuji Tachikawa. “Reading between the lines of four-dimensional gauge theories”. In: *JHEP* 08 (2013), p. 115. DOI: [10.1007/JHEP08\(2013\)115](https://doi.org/10.1007/JHEP08(2013)115). arXiv: [1305.0318](https://arxiv.org/abs/1305.0318) [hep-th].
- [30] Davide Gaiotto, Anton Kapustin, Zohar Komargodski, and Nathan Seiberg. “Theta, Time Reversal, and Temperature”. In: *JHEP* 05 (2017), p. 091. DOI: [10.1007/JHEP05\(2017\)091](https://doi.org/10.1007/JHEP05(2017)091). arXiv: [1703.00501](https://arxiv.org/abs/1703.00501) [hep-th].
- [31] Hiroyuki Shimizu and Kazuya Yonekura. “Anomaly constraints on deconfinement and chiral phase transition”. In: *Phys. Rev. D* 97.10 (2018), p. 105011. DOI: [10.1103/PhysRevD.97.105011](https://doi.org/10.1103/PhysRevD.97.105011). arXiv: [1706.06104](https://arxiv.org/abs/1706.06104) [hep-th].
- [32] Po-Shen Hsin, Ho Tat Lam, and Nathan Seiberg. “Comments on One-Form Global Symmetries and Their Gauging in 3d and 4d”. In: *SciPost Phys.* 6.3 (2019), p. 039. DOI: [10.21468/SciPostPhys.6.3.039](https://doi.org/10.21468/SciPostPhys.6.3.039). arXiv: [1812.04716](https://arxiv.org/abs/1812.04716) [hep-th].
- [33] Clay Córdova and Kantaro Ohmori. “Anomaly Obstructions to Symmetry Preserving Gapped Phases”. In: (Oct. 2019). arXiv: [1910.04962](https://arxiv.org/abs/1910.04962) [hep-th].
- [34] Ying-Hsuan Lin, Masaki Okada, Sahand Seifnashri, and Yuji Tachikawa. “Asymptotic density of states in 2d CFTs with non-invertible symmetries”. In: *JHEP* 03 (2023), p. 094. DOI: [10.1007/JHEP03\(2023\)094](https://doi.org/10.1007/JHEP03(2023)094). arXiv: [2208.05495](https://arxiv.org/abs/2208.05495) [hep-th].
- [35] Ying-Hsuan Lin and Shu-Heng Shao. “Bootstrapping noninvertible symmetries”. In: *Phys. Rev. D* 107.12 (2023), p. 125025. DOI: [10.1103/PhysRevD.107.125025](https://doi.org/10.1103/PhysRevD.107.125025). arXiv: [2302.13900](https://arxiv.org/abs/2302.13900) [hep-th].
- [36] Yichul Choi, Brandon C. Rayhaun, Yaman Sanghavi, and Shu-Heng Shao. “Remarks on boundaries, anomalies, and noninvertible symmetries”. In: *Phys. Rev. D* 108 (2023), p. 125005. DOI: [10.1103/PhysRevD.108.125005](https://doi.org/10.1103/PhysRevD.108.125005). arXiv: [2305.09713](https://arxiv.org/abs/2305.09713) [hep-th].
- [37] Clay Córdova, Diego García-Sepúlveda, and Nicholas Holfester. “Particle-soliton degeneracies from spontaneously broken non-invertible symmetry”. In: *JHEP* 07 (2024), p. 154. DOI: [10.1007/JHEP07\(2024\)154](https://doi.org/10.1007/JHEP07(2024)154). arXiv: [2403.08883](https://arxiv.org/abs/2403.08883) [hep-th].
- [38] Christian Copetti, Lucia Cordova, and Shota Komatsu. “S-matrix bootstrap and non-invertible symmetries”. In: *JHEP* 03 (2025), p. 204. DOI: [10.1007/JHEP03\(2025\)204](https://doi.org/10.1007/JHEP03(2025)204). arXiv: [2408.13132](https://arxiv.org/abs/2408.13132) [hep-th].
- [39] Andrea Antinucci, Christian Copetti, Giovanni Galati, and Giovanni Rizi. “A Twist on Scattering from Defect Anomalies”. In: (May 2026). arXiv: [2605.13961](https://arxiv.org/abs/2605.13961) [hep-th].

- [40] Anuj Apte, Clay Cordova, and Ho Tat Lam. “Obstructions to gapped phases from noninvertible symmetries”. In: *Phys. Rev. B* 108.4 (2023), p. 045134. DOI: [10.1103/PhysRevB.108.045134](https://doi.org/10.1103/PhysRevB.108.045134). arXiv: [2212.14605](https://arxiv.org/abs/2212.14605) [hep-th].
- [41] Lakshya Bhardwaj, Lea E. Bottini, Daniel Pajer, and Sakura Schafer-Nameki. “The club sandwich: Gapless phases and phase transitions with non-invertible symmetries”. In: *SciPost Phys.* 18.5 (2025), p. 156. DOI: [10.21468/SciPostPhys.18.5.156](https://doi.org/10.21468/SciPostPhys.18.5.156). arXiv: [2312.17322](https://arxiv.org/abs/2312.17322) [hep-th].
- [42] Lakshya Bhardwaj, Daniel Pajer, Sakura Schafer-Nameki, and Alison Warman. “Hasse diagrams for gapless SPT and SSB phases with non-invertible symmetries”. In: *SciPost Phys.* 19.4 (2025), p. 113. DOI: [10.21468/SciPostPhys.19.4.113](https://doi.org/10.21468/SciPostPhys.19.4.113). arXiv: [2403.00905](https://arxiv.org/abs/2403.00905) [cond-mat.str-el].
- [43] Yichul Choi, Clay Cordova, Po-Shen Hsin, Ho Tat Lam, and Shu-Heng Shao. “Noninvertible duality defects in 3+1 dimensions”. In: *Phys. Rev. D* 105.12 (2022), p. 125016. DOI: [10.1103/PhysRevD.105.125016](https://doi.org/10.1103/PhysRevD.105.125016). arXiv: [2111.01139](https://arxiv.org/abs/2111.01139) [hep-th].
- [44] Justin Kaidi, Kantaro Ohmori, and Yunqin Zheng. “Kramers-Wannier-like Duality Defects in (3+1)D Gauge Theories”. In: *Phys. Rev. Lett.* 128.11 (2022), p. 111601. DOI: [10.1103/PhysRevLett.128.111601](https://doi.org/10.1103/PhysRevLett.128.111601). arXiv: [2111.01141](https://arxiv.org/abs/2111.01141) [hep-th].
- [45] Yichul Choi, Clay Cordova, Po-Shen Hsin, Ho Tat Lam, and Shu-Heng Shao. “Noninvertible Condensation, Duality, and Triality Defects in 3+1 Dimensions”. In: *Commun. Math. Phys.* 402.1 (2023), pp. 489–542. DOI: [10.1007/s00220-023-04727-4](https://doi.org/10.1007/s00220-023-04727-4). arXiv: [2204.09025](https://arxiv.org/abs/2204.09025) [hep-th].
- [46] Yichul Choi, Ho Tat Lam, and Shu-Heng Shao. “Noninvertible Global Symmetries in the Standard Model”. In: *Phys. Rev. Lett.* 129.16 (2022), p. 161601. DOI: [10.1103/PhysRevLett.129.161601](https://doi.org/10.1103/PhysRevLett.129.161601). arXiv: [2205.05086](https://arxiv.org/abs/2205.05086) [hep-th].
- [47] Andrea Antinucci, Giovanni Galati, and Giovanni Rizi. “On continuous 2-category symmetries and Yang-Mills theory”. In: *JHEP* 12 (2022), p. 061. DOI: [10.1007/JHEP12\(2022\)061](https://doi.org/10.1007/JHEP12(2022)061). arXiv: [2206.05646](https://arxiv.org/abs/2206.05646) [hep-th].
- [48] Jeremias Aguilera Damia, Riccardo Argurio, and Soumyadeep Chaudhuri. “When the moduli space is an orbifold: spontaneous breaking of continuous non-invertible symmetries”. In: *JHEP* 03 (2024), p. 042. DOI: [10.1007/JHEP03\(2024\)042](https://doi.org/10.1007/JHEP03(2024)042). arXiv: [2309.06491](https://arxiv.org/abs/2309.06491) [hep-th].
- [49] Azeem Hasan, Shani Meynet, and Daniele Migliorati. “ $SL_2(\mathbb{R})$ symmetries of SymTFT and non-invertible $U(1)$ symmetries of Maxwell theory”. In: *JHEP* 12 (2024), p. 131. DOI: [10.1007/JHEP12\(2024\)131](https://doi.org/10.1007/JHEP12(2024)131). arXiv: [2405.19218](https://arxiv.org/abs/2405.19218) [hep-th].
- [50] Diego Delmastro, Adar Sharon, and Yunqin Zheng. “Non-local conserved currents and continuous non-invertible symmetries”. In: *JHEP* 11 (2025), p. 072. DOI: [10.1007/JHEP11\(2025\)072](https://doi.org/10.1007/JHEP11(2025)072). arXiv: [2507.22976](https://arxiv.org/abs/2507.22976) [hep-th].

- [51] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev. “Entanglement in quantum critical phenomena”. In: *Phys. Rev. Lett.* 90 (2003), p. 227902. DOI: [10.1103/PhysRevLett.90.227902](https://doi.org/10.1103/PhysRevLett.90.227902). arXiv: [quant-ph/0211074](https://arxiv.org/abs/quant-ph/0211074).
- [52] Pasquale Calabrese and John L. Cardy. “Entanglement entropy and quantum field theory”. In: *J. Stat. Mech.* (2004), P06002. DOI: [10.1088/1742-5468/2004/06/P06002](https://doi.org/10.1088/1742-5468/2004/06/P06002). arXiv: [hep-th/0405152](https://arxiv.org/abs/hep-th/0405152).
- [53] J. Eisert, M. Cramer, and M. B. Plenio. “Area laws for the entanglement entropy - a review”. In: *Rev. Mod. Phys.* 82 (2010), pp. 277–306. DOI: [10.1103/RevModPhys.82.277](https://doi.org/10.1103/RevModPhys.82.277). arXiv: [0808.3773 \[quant-ph\]](https://arxiv.org/abs/0808.3773).
- [54] Alexei Kitaev and John Preskill. “Topological entanglement entropy”. In: *Phys. Rev. Lett.* 96 (2006), p. 110404. DOI: [10.1103/PhysRevLett.96.110404](https://doi.org/10.1103/PhysRevLett.96.110404). arXiv: [hep-th/0510092](https://arxiv.org/abs/hep-th/0510092).
- [55] Michael Levin and Xiao-Gang Wen. “Detecting Topological Order in a Ground State Wave Function”. In: *Phys. Rev. Lett.* 96 (2006), p. 110405. DOI: [10.1103/PhysRevLett.96.110405](https://doi.org/10.1103/PhysRevLett.96.110405). arXiv: [cond-mat/0510613](https://arxiv.org/abs/cond-mat/0510613).
- [56] Hui Li and F. Haldane. “Entanglement Spectrum as a Generalization of Entanglement Entropy: Identification of Topological Order in Non-Abelian Fractional Quantum Hall Effect States”. In: *Phys. Rev. Lett.* 101.1 (2008), p. 010504. DOI: [10.1103/PhysRevLett.101.010504](https://doi.org/10.1103/PhysRevLett.101.010504). arXiv: [0805.0332 \[cond-mat.mes-hall\]](https://arxiv.org/abs/0805.0332).
- [57] Pasquale Calabrese and John L. Cardy. “Evolution of entanglement entropy in one-dimensional systems”. In: *J. Stat. Mech.* 0504 (2005), P04010. DOI: [10.1088/1742-5468/2005/04/P04010](https://doi.org/10.1088/1742-5468/2005/04/P04010). arXiv: [cond-mat/0503393](https://arxiv.org/abs/cond-mat/0503393).
- [58] Pasquale Calabrese and John Cardy. “Quantum Quenches in Extended Systems”. In: *J. Stat. Mech.* 0706 (2007), P06008. DOI: [10.1088/1742-5468/2007/06/P06008](https://doi.org/10.1088/1742-5468/2007/06/P06008). arXiv: [0704.1880 \[cond-mat.stat-mech\]](https://arxiv.org/abs/0704.1880).
- [59] Adam M. Kaufman, M. Eric Tai, Alexander Lukin, Matthew Rispoli, Robert Schittko, Philipp M. Preiss, and Markus Greiner. “Quantum thermalization through entanglement in an isolated many-body system”. In: *Science* 353.6301 (2016), aaf6725. DOI: [10.1126/science.aaf6725](https://doi.org/10.1126/science.aaf6725). arXiv: [1603.04409 \[quant-ph\]](https://arxiv.org/abs/1603.04409).
- [60] Shinsei Ryu and Tadashi Takayanagi. “Holographic derivation of entanglement entropy from AdS/CFT”. In: *Phys. Rev. Lett.* 96 (2006), p. 181602. DOI: [10.1103/PhysRevLett.96.181602](https://doi.org/10.1103/PhysRevLett.96.181602). arXiv: [hep-th/0603001](https://arxiv.org/abs/hep-th/0603001).
- [61] Veronika E. Hubeny, Mukund Rangamani, and Tadashi Takayanagi. “A covariant holographic entanglement entropy proposal”. In: *JHEP* 07 (2007), p. 062. DOI: [10.1088/1126-6708/2007/07/062](https://doi.org/10.1088/1126-6708/2007/07/062). arXiv: [0705.0016 \[hep-th\]](https://arxiv.org/abs/0705.0016).
- [62] Mark Van Raamsdonk. “Building up spacetime with quantum entanglement”. In: *Gen. Rel. Grav.* 42 (2010), pp. 2323–2329. DOI: [10.1142/S0218271810018529](https://doi.org/10.1142/S0218271810018529). arXiv: [1005.3035 \[hep-th\]](https://arxiv.org/abs/1005.3035).

- [63] Thomas Faulkner, Monica Guica, Thomas Hartman, Robert C. Myers, and Mark Van Raamsdonk. “Gravitation from Entanglement in Holographic CFTs”. In: *JHEP* 03 (2014), p. 051. DOI: [10.1007/JHEP03\(2014\)051](https://doi.org/10.1007/JHEP03(2014)051). arXiv: [1312.7856](https://arxiv.org/abs/1312.7856) [[hep-th](#)].
- [64] Pasquale Calabrese and John Cardy. “Entanglement entropy and conformal field theory”. In: *J. Phys. A* 42 (2009), p. 504005. DOI: [10.1088/1751-8113/42/50/504005](https://doi.org/10.1088/1751-8113/42/50/504005). arXiv: [0905.4013](https://arxiv.org/abs/0905.4013) [[cond-mat.stat-mech](#)].
- [65] Kantaro Ohmori and Yuji Tachikawa. “Physics at the entangling surface”. In: *J. Stat. Mech.* (2015), P04010. DOI: [10.1088/1742-5468/2015/04/P04010](https://doi.org/10.1088/1742-5468/2015/04/P04010). arXiv: [1406.4167](https://arxiv.org/abs/1406.4167) [[hep-th](#)].
- [66] Ian Affleck and Andreas W. W. Ludwig. “Universal noninteger ‘ground state degeneracy’ in critical quantum systems”. In: *Phys. Rev. Lett.* 67 (1991), pp. 161–164. DOI: [10.1103/PhysRevLett.67.161](https://doi.org/10.1103/PhysRevLett.67.161).
- [67] Moshe Goldstein and Eran Sela. “Symmetry-resolved entanglement in many-body systems”. In: *Phys. Rev. Lett.* 120 (2018), p. 200602. DOI: [10.1103/PhysRevLett.120.200602](https://doi.org/10.1103/PhysRevLett.120.200602). arXiv: [1711.09418](https://arxiv.org/abs/1711.09418) [[cond-mat.stat-mech](#)].
- [68] Nicolas Laflorencie and Stephan Rachel. “Spin-resolved entanglement spectroscopy of critical spin chains and Luttinger liquids”. In: *J. Stat. Mech.* (2014), P11013. DOI: [10.1088/1742-5468/2014/11/P11013](https://doi.org/10.1088/1742-5468/2014/11/P11013). arXiv: [1407.3779](https://arxiv.org/abs/1407.3779) [[cond-mat.str-el](#)].
- [69] Vittorio Vitale, Andreas Elben, Richard Kueng, Antoine Neven, Jose Carrasco, Barbara Kraus, Peter Zoller, Pasquale Calabrese, Benoit Vermersch, and Marcello Dalmonte. “Symmetry-resolved dynamical purification in synthetic quantum matter”. In: *SciPost Phys.* 12.3 (2022), p. 106. DOI: [10.21468/SciPostPhys.12.3.106](https://doi.org/10.21468/SciPostPhys.12.3.106). arXiv: [2101.07814](https://arxiv.org/abs/2101.07814) [[cond-mat.stat-mech](#)].
- [70] J. C. Xavier, F. C. Alcaraz, and G. Sierra. “Equipartition of the entanglement entropy”. In: *Phys. Rev. B* 98 (2018), p. 041106. DOI: [10.1103/PhysRevB.98.041106](https://doi.org/10.1103/PhysRevB.98.041106). arXiv: [1804.06357](https://arxiv.org/abs/1804.06357) [[cond-mat.stat-mech](#)].
- [71] Pasquale Calabrese, Jérôme Dubail, and Sara Murciano. “Symmetry-resolved entanglement entropy in Wess–Zumino–Witten models”. In: *JHEP* 10 (2021), p. 067. DOI: [10.1007/JHEP10\(2021\)067](https://doi.org/10.1007/JHEP10(2021)067). arXiv: [2106.15946](https://arxiv.org/abs/2106.15946) [[hep-th](#)].
- [72] Yuya Kusuki, Sara Murciano, Hiroshi Ooguri, and Sridip Pal. “Symmetry-resolved entanglement entropy, spectra & boundary conformal field theory”. In: *JHEP* 11 (2023), p. 216. DOI: [10.1007/JHEP11\(2023\)216](https://doi.org/10.1007/JHEP11(2023)216). arXiv: [2309.03287](https://arxiv.org/abs/2309.03287) [[hep-th](#)].
- [73] Riccarda Bonsignori, Paola Ruggiero, and Pasquale Calabrese. “Symmetry resolved entanglement in free fermionic systems”. In: *J. Phys. A* 52 (2019), p. 475302. DOI: [10.1088/1751-8121/ab4b77](https://doi.org/10.1088/1751-8121/ab4b77). arXiv: [1907.02084](https://arxiv.org/abs/1907.02084) [[cond-mat.stat-mech](#)].
- [74] Sara Murciano, Giuseppe Di Giulio, and Pasquale Calabrese. “Symmetry resolved entanglement in gapped integrable systems: a corner transfer matrix approach”. In: *SciPost Phys.* 8 (2020), p. 046. DOI: [10.21468/SciPostPhys.8.3.046](https://doi.org/10.21468/SciPostPhys.8.3.046). arXiv: [1911.09588](https://arxiv.org/abs/1911.09588) [[cond-mat.stat-mech](#)].

- [75] Luca Capizzi, Paola Ruggiero, and Pasquale Calabrese. “Symmetry resolved entanglement entropy of excited states in a CFT”. In: *J. Stat. Mech.* 2007 (2020), p. 073101. DOI: [10.1088/1742-5468/ab96b6](https://doi.org/10.1088/1742-5468/ab96b6). arXiv: [2003.04670](https://arxiv.org/abs/2003.04670) [[cond-mat.stat-mech](#)].
- [76] Filiberto Ares, Sara Murciano, and Pasquale Calabrese. “Entanglement asymmetry as a probe of symmetry breaking”. In: *Nature Commun.* 14 (2023), p. 2036. DOI: [10.1038/s41467-023-37747-8](https://doi.org/10.1038/s41467-023-37747-8). arXiv: [2207.14693](https://arxiv.org/abs/2207.14693) [[cond-mat.stat-mech](#)].
- [77] J. A. Vaccaro, F. Anselmi, H. M. Wiseman, and K. Jacobs. “Tradeoff between extractable mechanical work, accessible entanglement, and ability to act as a reference system, under arbitrary superselection rules”. In: *Phys. Rev. A* 77 (2008), p. 032114. DOI: [10.1103/PhysRevA.77.032114](https://doi.org/10.1103/PhysRevA.77.032114). arXiv: [quant-ph/0501121](https://arxiv.org/abs/quant-ph/0501121).
- [78] Gilad Gour, Iman Marvian, and Robert W. Spekkens. “Measuring the quality of a quantum reference frame: The relative entropy of frameness”. In: *Phys. Rev. A* 80 (2009), p. 012307. DOI: [10.1103/PhysRevA.80.012307](https://doi.org/10.1103/PhysRevA.80.012307). arXiv: [0901.0943](https://arxiv.org/abs/0901.0943) [[quant-ph](#)].
- [79] Iman Marvian and Robert W. Spekkens. “Extending Noether’s theorem by quantifying the asymmetry of quantum states”. In: *Nature Commun.* 5 (2014), p. 3821. DOI: [10.1038/ncomms4821](https://doi.org/10.1038/ncomms4821). arXiv: [1404.3236](https://arxiv.org/abs/1404.3236) [[quant-ph](#)].
- [80] Horacio Casini, Marina Huerta, Javier M. Magán, and Diego Pontello. “Entanglement entropy and superselection sectors. Part I. Global symmetries”. In: *JHEP* 02 (2020), p. 014. DOI: [10.1007/JHEP02\(2020\)014](https://doi.org/10.1007/JHEP02(2020)014). arXiv: [1905.10487](https://arxiv.org/abs/1905.10487) [[hep-th](#)].
- [81] Javier M. Magán and Diego Pontello. “Quantum complementarity through entropic certainty principles”. In: *Phys. Rev. A* 103 (2021), p. 012211. DOI: [10.1103/PhysRevA.103.012211](https://doi.org/10.1103/PhysRevA.103.012211). arXiv: [2005.01760](https://arxiv.org/abs/2005.01760) [[hep-th](#)].
- [82] Horacio Casini, Marina Huerta, Javier M. Magán, and Diego Pontello. “Entropic order parameters for the phases of QFT”. In: *JHEP* 04 (2021), p. 277. DOI: [10.1007/JHEP04\(2021\)277](https://doi.org/10.1007/JHEP04(2021)277). arXiv: [2008.11748](https://arxiv.org/abs/2008.11748) [[hep-th](#)].
- [83] Javier M. Magán. “Proof of the universal density of charged states in QFT”. In: *JHEP* 12 (2021), p. 100. DOI: [10.1007/JHEP12\(2021\)100](https://doi.org/10.1007/JHEP12(2021)100). arXiv: [2111.02418](https://arxiv.org/abs/2111.02418) [[hep-th](#)].
- [84] Filiberto Ares, Sara Murciano, Eric Vernier, and Pasquale Calabrese. “Lack of symmetry restoration after a quantum quench: An entanglement asymmetry study”. In: *SciPost Phys.* 15 (2023), p. 089. DOI: [10.21468/SciPostPhys.15.3.089](https://doi.org/10.21468/SciPostPhys.15.3.089). arXiv: [2302.03330](https://arxiv.org/abs/2302.03330) [[cond-mat.stat-mech](#)].
- [85] Colin Rylands, Katja Klobas, Filiberto Ares, Pasquale Calabrese, Sara Murciano, and Bruno Bertini. “Microscopic origin of the quantum Mpemba effect in integrable systems”. In: *Phys. Rev. Lett.* 133 (2024), p. 010401. DOI: [10.1103/PhysRevLett.133.010401](https://doi.org/10.1103/PhysRevLett.133.010401). arXiv: [2310.04419](https://arxiv.org/abs/2310.04419) [[cond-mat.stat-mech](#)].
- [86] Xhek Turkeschi, Pasquale Calabrese, and Andrea De Luca. “Quantum Mpemba effect in random circuits”. In: *Phys. Rev. Lett.* 135 (2025), p. 040403. DOI: [10.1103/5d6p-8d1b](https://doi.org/10.1103/5d6p-8d1b). arXiv: [2405.14514](https://arxiv.org/abs/2405.14514) [[quant-ph](#)].

- [87] Shuo Liu, Hao-Kai Zhang, Shuai Yin, and Shi-Xin Zhang. “Symmetry restoration and quantum Mpemba effect in symmetric random circuits”. In: *Phys. Rev. Lett.* 133 (2024), p. 140405. DOI: [10.1103/PhysRevLett.133.140405](https://doi.org/10.1103/PhysRevLett.133.140405). arXiv: [2403.08459](https://arxiv.org/abs/2403.08459) [quant-ph].
- [88] Lata Kh. Joshi et al. “Observing the quantum Mpemba effect in quantum simulations”. In: *Phys. Rev. Lett.* 133 (2024), p. 010402. DOI: [10.1103/PhysRevLett.133.010402](https://doi.org/10.1103/PhysRevLett.133.010402). arXiv: [2401.04270](https://arxiv.org/abs/2401.04270) [quant-ph].
- [89] Filiberto Ares, Pasquale Calabrese, and Sara Murciano. “The quantum Mpemba effects”. In: *Nature Rev. Phys.* 7 (2025), pp. 451–460. DOI: [10.1038/s42254-025-00838-0](https://doi.org/10.1038/s42254-025-00838-0). arXiv: [2502.08087](https://arxiv.org/abs/2502.08087) [cond-mat.stat-mech].
- [90] Clay Cordova, Po-Shen Hsin, and Carolyn Zhang. “Anomalies of non-invertible symmetries in $(3+1)d$ ”. In: *SciPost Phys.* 17.5 (2024), p. 131. DOI: [10.21468/SciPostPhys.17.5.131](https://doi.org/10.21468/SciPostPhys.17.5.131). arXiv: [2308.11706](https://arxiv.org/abs/2308.11706) [hep-th].
- [91] Andrea Antinucci, Francesco Benini, Christian Copetti, Giovanni Galati, and Giovanni Rizi. “Anomalies of non-invertible self-duality symmetries: fractionalization and gauging”. In: (Aug. 2023). arXiv: [2308.11707](https://arxiv.org/abs/2308.11707) [hep-th].
- [92] Andrea Antinucci, Christian Copetti, Yuhan Gai, and Sakura Schafer-Nameki. “Categorical Anomaly Matching”. In: (Aug. 2025). arXiv: [2508.00982](https://arxiv.org/abs/2508.00982) [hep-th].
- [93] Ryan Thorngren and Yifan Wang. “Fusion category symmetry. Part I. Anomaly in-flow and gapped phases”. In: *JHEP* 04 (2024), p. 132. DOI: [10.1007/JHEP04\(2024\)132](https://doi.org/10.1007/JHEP04(2024)132). arXiv: [1912.02817](https://arxiv.org/abs/1912.02817) [hep-th].
- [94] Justin Kaidi, Emily Nardoni, Gabi Zafrir, and Yunqin Zheng. “Symmetry TFTs and anomalies of non-invertible symmetries”. In: *JHEP* 10 (2023), p. 053. DOI: [10.1007/JHEP10\(2023\)053](https://doi.org/10.1007/JHEP10(2023)053). arXiv: [2301.07112](https://arxiv.org/abs/2301.07112) [hep-th].
- [95] Carolyn Zhang and Clay Córdova. “Anomalies of $(1+1)$ -dimensional categorical symmetries”. In: *Phys. Rev. B* 110.3 (2024), p. 035155. DOI: [10.1103/PhysRevB.110.035155](https://doi.org/10.1103/PhysRevB.110.035155). arXiv: [2304.01262](https://arxiv.org/abs/2304.01262) [cond-mat.str-el].
- [96] Kristan Jensen, Evgeny Shaverin, and Amos Yarom. “’t Hooft anomalies and boundaries”. In: *JHEP* 01 (2018), p. 085. DOI: [10.1007/JHEP01\(2018\)085](https://doi.org/10.1007/JHEP01(2018)085). arXiv: [1710.07299](https://arxiv.org/abs/1710.07299) [hep-th].
- [97] Ryan Thorngren and Yifan Wang. “Anomalous symmetries end at the boundary”. In: *JHEP* 09 (2021), p. 017. DOI: [10.1007/JHEP09\(2021\)017](https://doi.org/10.1007/JHEP09(2021)017). arXiv: [2012.15861](https://arxiv.org/abs/2012.15861) [hep-th].
- [98] Clay Córdova, Nicholas Holfester, and Kantaro Ohmori. “Representation theory of solitons”. In: *JHEP* 06 (2025), p. 001. DOI: [10.1007/JHEP06\(2025\)001](https://doi.org/10.1007/JHEP06(2025)001). arXiv: [2408.11045](https://arxiv.org/abs/2408.11045) [hep-th].
- [99] Yichul Choi, Brandon C. Rayhaun, and Yunqin Zheng. “Generalized tube algebras, symmetry-resolved partition functions, and twisted boundary states”. In: (2024). arXiv: [2409.02159](https://arxiv.org/abs/2409.02159) [hep-th].

- [100] Yichul Choi, Brandon C. Rayhaun, and Yunqin Zheng. “Noninvertible symmetry-resolved Affleck–Ludwig–Cardy formula and entanglement entropy from the boundary tube algebra”. In: *Phys. Rev. Lett.* 133 (2024), p. 251602. DOI: [10.1103/PhysRevLett.133.251602](https://doi.org/10.1103/PhysRevLett.133.251602). arXiv: [2409.02806](https://arxiv.org/abs/2409.02806) [hep-th].
- [101] David A. Huse. “Exact exponents for infinitely many new multicritical points”. In: *Phys. Rev. B* 30 (1984), pp. 3908–3915. DOI: [10.1103/PhysRevB.30.3908](https://doi.org/10.1103/PhysRevB.30.3908).
- [102] Jeremias Aguilera-Damia, Riccardo Argurio, Francesco Benini, Sergio Benvenuti, Christian Copetti, and Luigi Tizzano. “Non-invertible symmetries along 4d RG flows”. In: *JHEP* 02 (2024), p. 084. DOI: [10.1007/JHEP02\(2024\)084](https://doi.org/10.1007/JHEP02(2024)084). arXiv: [2305.17084](https://arxiv.org/abs/2305.17084) [hep-th].
- [103] Yuya Kusuki, Sridip Pal, and Hiroyasu Tajima. “Resource-Theoretic Quantifiers of Weak and Strong Symmetry Breaking: Strong Entanglement Asymmetry and Beyond”. In: (Jan. 2026). arXiv: [2601.20924](https://arxiv.org/abs/2601.20924) [hep-th].
- [104] P. Saura-Bastida, A. Das, G. Sierra, and J. Molina-Vilaplana. “Categorical-symmetry resolved entanglement in conformal field theory”. In: *Phys. Rev. D* 109.10 (2024), p. 105026. DOI: [10.1103/PhysRevD.109.105026](https://doi.org/10.1103/PhysRevD.109.105026). arXiv: [2402.06322](https://arxiv.org/abs/2402.06322) [hep-th].
- [105] Jared Heymann and Thomas Quella. “Revisiting the symmetry-resolved entanglement for noninvertible symmetries in $1+1d$ conformal field theories”. In: *Phys. Rev. D* 112 (2025), p. 025004. DOI: [10.1103/lr47-yv3j](https://doi.org/10.1103/lr47-yv3j). arXiv: [2409.02315](https://arxiv.org/abs/2409.02315) [hep-th].
- [106] Arpan Bhattacharyya, Saptaswa Ghosh, Sounak Pal, and Jagannath Santara. “On the resolution of categorical symmetries in (Non-) Unitary Rational CFTs”. In: (Nov. 2025). arXiv: [2511.16363](https://arxiv.org/abs/2511.16363) [hep-th].
- [107] Thomas Bartsch, Mathew Bullimore, and Andrea Grigoletto. “Representation theory for categorical symmetries”. In: (2023). arXiv: [2305.17165](https://arxiv.org/abs/2305.17165) [hep-th].
- [108] Thomas Bartsch. “Unitary Categorical Symmetries”. In: (2025). arXiv: [2502.04440](https://arxiv.org/abs/2502.04440) [hep-th].
- [109] Alex Bullivant and Clement Delcamp. “Tube algebras, excitations statistics and compactification in gauge models of topological phases”. In: *JHEP* 10 (2019), p. 216. DOI: [10.1007/JHEP10\(2019\)216](https://doi.org/10.1007/JHEP10(2019)216). arXiv: [1905.08673](https://arxiv.org/abs/1905.08673) [cond-mat.str-el].
- [110] Finn Gagliano, Andrea Grigoletto, and Kantaro Ohmori. “Higher Representations and Quark Confinement”. In: (Jan. 2025). arXiv: [2501.09069](https://arxiv.org/abs/2501.09069) [hep-th].
- [111] Amanda Gatto Lamas, Jacopo Gliozzi, and Taylor L. Hughes. “Higher-form entanglement asymmetry and topological order”. In: (Oct. 2025). arXiv: [2510.03967](https://arxiv.org/abs/2510.03967) [cond-mat.str-el].
- [112] Francesco Benini, Eduardo García-Valdecasas, and Stathis Vitouladitis. “Higher-form entanglement asymmetry. Part I. The limits of symmetry breaking”. In: *JHEP* 05 (2026), p. 202. DOI: [10.1007/JHEP05\(2026\)202](https://doi.org/10.1007/JHEP05(2026)202). arXiv: [2512.15898](https://arxiv.org/abs/2512.15898) [hep-th].
- [113] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*. Vol. 205. Mathematical Surveys and Monographs. American Math. Soc., 2015. DOI: [10.1090/surv/205](https://doi.org/10.1090/surv/205). URL: <https://math.mit.edu/~etingof/egnobookfinal.pdf>.

- [114] Tzu-Chen Huang, Ying-Hsuan Lin, and Sahand Seifnashri. “Construction of two-dimensional topological field theories with non-invertible symmetries”. In: *JHEP* 12 (2021), p. 028. DOI: [10.1007/JHEP12\(2021\)028](https://doi.org/10.1007/JHEP12(2021)028). arXiv: [2110.02958](https://arxiv.org/abs/2110.02958) [[hep-th](#)].
- [115] Jurg Frohlich, Jurgen Fuchs, Ingo Runkel, and Christoph Schweigert. “Defect Lines, Dualities and Generalised Orbifolds”. In: *16th International Congress on Mathematical Physics*. 2010, pp. 608–613. DOI: [10.1142/9789814304634_0056](https://doi.org/10.1142/9789814304634_0056). arXiv: [0909.5013](https://arxiv.org/abs/0909.5013) [[math-ph](#)].
- [116] Adrian Ocneanu. “Chirality for operator algebras”. In: *Subfactors: Proceedings of the Taniguchi Symposium on operator algebras*. Ed. by H. Araki et al. Singapore: World Scientific, 1994, pp. 39–63. DOI: [10.1142/2394](https://doi.org/10.1142/2394).
- [117] Maissam Barkeshli, Parsa Bonderson, Meng Cheng, and Zhenghan Wang. “Symmetry Fractionalization, Defects, and Gauging of Topological Phases”. In: *Phys. Rev. B* 100 (2019), p. 115147. DOI: [10.1103/PhysRevB.100.115147](https://doi.org/10.1103/PhysRevB.100.115147). arXiv: [1410.4540](https://arxiv.org/abs/1410.4540) [[cond-mat.str-el](#)].
- [118] Shamindra K. Ghosh and Corey Jones. “Annular representation theory for rigid C^* -tensor categories”. In: *J. Functional Anal.* 270 (2016), pp. 1537–1584. DOI: [10.1016/j.jfa.2015.08.017](https://doi.org/10.1016/j.jfa.2015.08.017). arXiv: [1502.06543](https://arxiv.org/abs/1502.06543) [[math.OA](#)].
- [119] Daniel Barter, Jacob C. Bridgeman, and Corey Jones. “Domain walls in topological phases and the Brauer–Picard ring for $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$ ”. In: *Commun. Math. Phys.* 369 (2019), pp. 1167–1185. DOI: [10.1007/s00220-019-03338-2](https://doi.org/10.1007/s00220-019-03338-2). arXiv: [1806.01279](https://arxiv.org/abs/1806.01279) [[math.QA](#)].
- [120] Zhanyu Ma, Cheolhee Han, Yigal Meir, and Eran Sela. “Symmetric inseparability and number entanglement in charge-conserving mixed states”. In: *Phys. Rev. A* 105 (2022), p. 042416. DOI: [10.1103/PhysRevA.105.042416](https://doi.org/10.1103/PhysRevA.105.042416). arXiv: [2110.09388](https://arxiv.org/abs/2110.09388) [[quant-ph](#)].
- [121] Cheolhee Han, Yigal Meir, and Eran Sela. “Realistic protocol to measure entanglement at finite temperatures”. In: *Phys. Rev. Lett.* 130 (2023), p. 136201. DOI: [10.1103/PhysRevLett.130.136201](https://doi.org/10.1103/PhysRevLett.130.136201). arXiv: [2210.10713](https://arxiv.org/abs/2210.10713) [[cond-mat.mes-hall](#)].
- [122] Marcelo Aguiar. “A note on strongly separable algebras”. In: *Boletín de la Academia Nacional de Ciencias* 65 (2000). [Journal page](#), pp. 51–60.
- [123] Beren Sanders. “A characterization of finite étale morphisms in tensor triangular geometry”. In: *Épjournal de Géométrie Algébrique* Volume 6 (Oct. 2022). ISSN: 2491-6765. DOI: [10.46298/epiga.2022.volume6.7641](https://doi.org/10.46298/epiga.2022.volume6.7641). URL: <http://dx.doi.org/10.46298/epiga.2022.volume6.7641>.
- [124] Gerard J. Murphy. *C^* -algebras and operator theory*. Academic Press, 1990. DOI: [10.1016/C2009-0-22289-6](https://doi.org/10.1016/C2009-0-22289-6).
- [125] Serge Lang. *Algebra*. Vol. 211. Graduate Texts in Mathematics. New York: Springer, 2002. DOI: [10.1007/978-1-4613-0041-0](https://doi.org/10.1007/978-1-4613-0041-0).
- [126] Florent Ferro, Filiberto Ares, and Pasquale Calabrese. “Non-equilibrium entanglement asymmetry for discrete groups: the example of the XY spin chain”. In: *J. Stat. Mech.* (2024), P023101. DOI: [10.1088/1742-5468/ad138f](https://doi.org/10.1088/1742-5468/ad138f). arXiv: [2307.06902](https://arxiv.org/abs/2307.06902) [[cond-mat.stat-mech](#)].

- [127] Luca Capizzi and Michele Mazzoni. “Entanglement asymmetry in the ordered phase of many-body systems: the Ising field theory”. In: *JHEP* 12 (2023), p. 144. DOI: [10.1007/JHEP12\(2023\)144](#). arXiv: [2307.12127 \[cond-mat.stat-mech\]](#).
- [128] Jurgen Fuchs. “Fusion rules in conformal field theory”. In: *Fortsch. Phys.* 42 (1994), pp. 1–48. DOI: [10.1002/prop.2190420102](#). arXiv: [hep-th/9306162](#).
- [129] Simeon Hellerman, Andre Henriques, Tony Pantev, Eric Sharpe, and Matt Ando. “Cluster decomposition, T-duality, and gerby CFT’s”. In: *Adv. Theor. Math. Phys.* 11 (2007), pp. 751–818. DOI: [10.4310/ATMP.2007.v11.n5.a2](#). arXiv: [hep-th/0606034](#).
- [130] Eric Sharpe. “An introduction to decomposition”. In: *2021-2022 MATRIX Annals*. Ed. by D. R. Wood, J. de Gier, and C. E. Praeger. Vol. 5. MATRIX Book Series. Springer, 2024, pp. 145–168. DOI: [10.1007/978-3-031-47417-0_8](#). arXiv: [2204.09117 \[hep-th\]](#).
- [131] Miao Chen and Hui-Huang Chen. “Rényi entanglement asymmetry in (1+1)-dimensional conformal field theories”. In: *Phys. Rev. D* 109 (2024), p. 065009. DOI: [10.1103/PhysRevD.109.065009](#). arXiv: [2310.15480 \[hep-th\]](#).
- [132] Francesco Benini, Victor Godet, and Amartya Harsh Singh. “Entanglement asymmetry in conformal field theory and holography”. In: *PTEP* 2025.6 (2025), 063B05. DOI: [10.1093/ptep/ptaf080](#). arXiv: [2407.07969 \[hep-th\]](#).
- [133] Harunobu Fujimura and Soichiro Shimamori. “Entanglement asymmetry and quantum Mpemba effect for non-Abelian global symmetry”. In: (2025). arXiv: [2509.05597 \[hep-th\]](#).
- [134] Luca Capizzi and Vittorio Vitale. “A universal formula for the entanglement asymmetry of matrix product states”. In: *J. Phys. A: Math. Theor.* 57 (2024), 45LT01. DOI: [10.1088/1751-8121/ad8796](#). arXiv: [2310.01962 \[quant-ph\]](#).
- [135] Michele Fossati, Colin Rylands, and Pasquale Calabrese. “Entanglement asymmetry in CFT with boundary symmetry breaking”. In: *JHEP* 06 (2025), p. 089. DOI: [10.1007/JHEP06\(2025\)089](#). arXiv: [2411.10244 \[hep-th\]](#).
- [136] Yuya Kusuki, Sara Murciano, Hiroshi Ooguri, and Sridip Pal. “Entanglement asymmetry and symmetry defects in boundary conformal field theory”. In: *JHEP* 01 (2025), p. 057. DOI: [10.1007/JHEP01\(2025\)057](#). arXiv: [2411.09792 \[hep-th\]](#).
- [137] Viktor Ostrik. “Module categories, weak Hopf algebras and modular invariants”. In: *Transform. Groups* 8 (2003), pp. 177–206. DOI: [10.1007/s00031-003-0515-6](#). arXiv: [math/0111139](#).
- [138] Jurgen Fuchs, Ingo Runkel, and Christoph Schweigert. “TFT construction of RCFT correlators 1. Partition functions”. In: *Nucl. Phys. B* 646 (2002), pp. 353–497. DOI: [10.1016/S0550-3213\(02\)00744-7](#). arXiv: [hep-th/0204148](#).
- [139] Oleksandr Diatlyk, Conghuan Luo, Yifan Wang, and Quinten Weller. “Gauging non-invertible symmetries: topological interfaces and generalized orbifold groupoid in 2d QFT”. In: *JHEP* 03 (2024), p. 127. DOI: [10.1007/JHEP03\(2024\)127](#). arXiv: [2311.17044 \[hep-th\]](#).

- [140] Jurgen Fuchs, Ingo Runkel, and Christoph Schweigert. “TFT construction of RCFT correlators. 3. Simple currents”. In: *Nucl. Phys. B* 694 (2004), pp. 277–353. DOI: [10.1016/j.nuclphysb.2004.05.014](https://doi.org/10.1016/j.nuclphysb.2004.05.014). arXiv: [hep-th/0403157](https://arxiv.org/abs/hep-th/0403157).
- [141] Andreas Recknagel and Volker Schomerus. *Boundary Conformal Field Theory and the Worldsheet Approach to D-Branes*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2013. DOI: [10.1017/CB09780511806476](https://doi.org/10.1017/CB09780511806476).
- [142] Jurgen Fuchs and Christoph Schweigert. “Category theory for conformal boundary conditions”. In: *Fields Inst. Commun.* 39 (2003), p. 25. DOI: [10.1090/fic/039/03](https://doi.org/10.1090/fic/039/03). arXiv: [math/0106050](https://arxiv.org/abs/math/0106050).
- [143] Gabriella Böhm and Kornel Szlachányi. “Weak Hopf algebras: II. Representation theory, dimensions, and the Markov trace”. In: *J. Algebra* 233 (2000), pp. 156–212. DOI: [10.1006/jabr.2000.8379](https://doi.org/10.1006/jabr.2000.8379). arXiv: [math/9906045](https://arxiv.org/abs/math/9906045) [[math.QA](#)].
- [144] Brian Day. “On closed categories of functors”. In: *Reports of the Midwest Category Seminar IV*. Vol. 137. Lecture Notes in Mathematics. Springer, Berlin, 1970, pp. 1–38. DOI: [10.1007/BFb0060438](https://doi.org/10.1007/BFb0060438).
- [145] Kyle Kawagoe, Corey Jones, Sean Sanford, David Green, and David Penneys. “Levin–Wen is a gauge theory: Entanglement from topology”. In: *Commun. Math. Phys.* 405 (2024), p. 266. DOI: [10.1007/s00220-024-05144-x](https://doi.org/10.1007/s00220-024-05144-x). arXiv: [2401.13838](https://arxiv.org/abs/2401.13838) [[cond-mat.str-el](#)].
- [146] Christoph Schweigert. *Hopf Algebras, Quantum Groups and Topological Field Theory*. 2023. URL: <https://www.math.uni-hamburg.de/home/schweigert/skripten/hskript.pdf>.
- [147] Shadi Ali Ahmad, Marc S. Klinger, and Yifan Wang. “The many faces of non-invertible symmetries”. In: *JHEP* 05 (2026), p. 110. DOI: [10.1007/JHEP05\(2026\)110](https://doi.org/10.1007/JHEP05(2026)110). arXiv: [2509.18072](https://arxiv.org/abs/2509.18072) [[hep-th](#)].
- [148] Gabriella Böhm, Florian Nill, and Kornel Szlachányi. “Weak Hopf algebras: I. Integral theory and C^* -structure”. In: *J. Algebra* 221 (1999), pp. 385–438. DOI: [10.1006/jabr.1999.7984](https://doi.org/10.1006/jabr.1999.7984). arXiv: [math/9805116](https://arxiv.org/abs/math/9805116) [[math.QA](#)].
- [149] Ansi Bai and Zhi-Hao Zhang. “On the representation categories of weak Hopf algebras arising from Levin–Wen models”. In: (2025). arXiv: [2503.06731](https://arxiv.org/abs/2503.06731) [[math.QA](#)].
- [150] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. “On fusion categories”. In: *Annals of Mathematics* 162 (2005), pp. 581–642. arXiv: [math/0203060](https://arxiv.org/abs/math/0203060) [[math.QA](#)]. URL: <http://www.jstor.org/stable/20159926>.
- [151] Arpit Das, Javier Molina-Vilaplana, and Pablo Saura-Bastida. “Generalized symmetry resolution of entanglement in conformal field theory for twisted and anyonic sectors”. In: *Phys. Rev. D* 110 (2024), p. 125005. DOI: [10.1103/PhysRevD.110.125005](https://doi.org/10.1103/PhysRevD.110.125005). arXiv: [2409.02162](https://arxiv.org/abs/2409.02162) [[hep-th](#)].
- [152] John L. Cardy. “Boundary conditions, fusion rules and the Verlinde formula”. In: *Nucl. Phys. B* 324 (1989), pp. 581–596. DOI: [10.1016/0550-3213\(89\)90521-X](https://doi.org/10.1016/0550-3213(89)90521-X).

- [153] Alexander Lukin, Matthew Rispoli, Robert Schittko, M. Eric Tai, Adam M. Kaufman, Soonwon Choi, Vedika Khemani, Julian Léonard, and Markus Greiner. “Probing entanglement in a many-body-localized system”. In: *Science* 364 (2019), pp. 256–260. DOI: [10.1126/science.aau0818](https://doi.org/10.1126/science.aau0818). arXiv: [1805.09819](https://arxiv.org/abs/1805.09819) [cond-mat.quant-gas].
- [154] H. M. Wiseman and John A. Vaccaro. “Entanglement of indistinguishable particles shared between two parties”. In: *Phys. Rev. Lett.* 91 (2003), p. 097902. DOI: [10.1103/PhysRevLett.91.097902](https://doi.org/10.1103/PhysRevLett.91.097902). arXiv: [quant-ph/0210002](https://arxiv.org/abs/quant-ph/0210002).
- [155] Yuting Hu, Yidun Wan, and Yong-Shi Wu. “Twisted quantum double model of topological phases in two dimensions”. In: *Phys. Rev. B* 87 (2013), p. 125114. DOI: [10.1103/PhysRevB.87.125114](https://doi.org/10.1103/PhysRevB.87.125114). arXiv: [1211.3695](https://arxiv.org/abs/1211.3695) [cond-mat.str-el].
- [156] Francesco Benini, Clay Córdova, and Po-Shen Hsin. “On 2-group global symmetries and their anomalies”. In: *JHEP* 03 (2019), p. 118. DOI: [10.1007/JHEP03\(2019\)118](https://doi.org/10.1007/JHEP03(2019)118). arXiv: [1803.09336](https://arxiv.org/abs/1803.09336) [hep-th].
- [157] Daisuke Tambara and Shigeru Yamagami. “Tensor categories with fusion rules of self-duality for finite Abelian groups”. In: *J. Algebra* 209 (1998), pp. 692–707. DOI: [10.1006/jabr.1998.7558](https://doi.org/10.1006/jabr.1998.7558).
- [158] A. Perez-Lona, D. Robbins, E. Sharpe, T. Vandermeulen, and X. Yu. “Notes on gauging noninvertible symmetries. Part I. Multiplicity-free cases”. In: *JHEP* 02 (2024), p. 154. DOI: [10.1007/JHEP02\(2024\)154](https://doi.org/10.1007/JHEP02(2024)154). arXiv: [2311.16230](https://arxiv.org/abs/2311.16230) [hep-th].
- [159] G. I. Kats and V. Palyutkin. “Finite ring groups”. In: *Trudy Moskovskogo Matematicheskogo Obshchestva* 15 (1966), pp. 224–261.
- [160] Arika Masuoka. “Semisimple Hopf algebras of dimension 6, 8”. In: *Israel J. Math.* 92 (1995), pp. 361–373. DOI: [10.1007/BF02762089](https://doi.org/10.1007/BF02762089).
- [161] Daniel S. Sage and Maria D. Vega. “Twisted Frobenius–Schur indicators for Hopf algebras”. In: *J. Algebra* 354 (2012), pp. 136–147. DOI: [10.1016/j.jalgebra.2011.12.026](https://doi.org/10.1016/j.jalgebra.2011.12.026). arXiv: [1107.0742](https://arxiv.org/abs/1107.0742) [math.QA].
- [162] Michael P. Mattis. “Correlations in two-dimensional critical theories”. In: *Nucl. Phys. B* 285 (1987), pp. 671–686. DOI: [10.1016/0550-3213\(87\)90361-0](https://doi.org/10.1016/0550-3213(87)90361-0).
- [163] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. New York: Springer-Verlag, 1997. DOI: [10.1007/978-1-4612-2256-9](https://doi.org/10.1007/978-1-4612-2256-9).
- [164] Uffe Haagerup. “Principal graphs of subfactors in the index range $4 < [M : N] < 3 + \sqrt{2}$ ”. In: *Subfactors: Proceedings of the Taniguchi Symposium on operator algebras*. Ed. by H. Araki et al. Singapore: World Scientific, 1994, pp. 1–38. DOI: [10.1142/2394](https://doi.org/10.1142/2394).
- [165] Masako Asaeda and Uffe Haagerup. “Exotic subfactors of finite depth with Jones indices $(5 + \sqrt{13})/2$ and $(5 + \sqrt{17})/2$ ”. In: *Commun. Math. Phys.* 202 (1999), pp. 1–63. DOI: [10.1007/s002200050574](https://doi.org/10.1007/s002200050574). eprint: [math/9803044](https://arxiv.org/abs/math/9803044) (math.OA).

- [166] Pinhas Grossman and Noah Snyder. “Quantum subgroups of the Haagerup fusion categories”. In: *Commun. Math. Phys.* 311 (2012), pp. 617–643. DOI: [10.1007/s00220-012-1427-x](https://doi.org/10.1007/s00220-012-1427-x). arXiv: [1102.2631](https://arxiv.org/abs/1102.2631) [[math.OA](#)].
- [167] M. Izumi. “The structure of sectors associated with Longo–Rehren inclusions. II: Examples”. In: *Rev. Math. Phys.* 13 (2001), pp. 603–674. DOI: [10.1142/S0129055X01000818](https://doi.org/10.1142/S0129055X01000818).
- [168] Pinhas Grossman and Masaki Izumi. “Drinfeld centers of fusion categories arising from generalized Haagerup subfactors”. In: *Quantum Topol.* 13 (2023), pp. 593–668. DOI: [10.4171/qt/167](https://doi.org/10.4171/qt/167). arXiv: [1501.07679](https://arxiv.org/abs/1501.07679) [[math.OA](#)].
- [169] Tzu-Chen Huang and Ying-Hsuan Lin. “The F -Symbols for Transparent Haagerup–Izumi Categories with $G = \mathbb{Z}_{2n+1}$ ”. In: (2020). arXiv: [2007.00670](https://arxiv.org/abs/2007.00670) [[math.CT](#)].
- [170] Tobias J. Osborne, Deniz E. Stiegemann, and Ramona Wolf. “The F -symbols for the $\mathcal{H}_3$ fusion category”. In: (2019). arXiv: [1906.01322](https://arxiv.org/abs/1906.01322) [[math.CT](#)].
- [171] Tzu-Chen Huang, Ying-Hsuan Lin, Kantaro Ohmori, Yuji Tachikawa, and Masaki Tezuka. “Numerical evidence for a Haagerup conformal field theory”. In: *Phys. Rev. Lett.* 128 (2022), p. 231603. DOI: [10.1103/PhysRevLett.128.231603](https://doi.org/10.1103/PhysRevLett.128.231603). arXiv: [2110.03008](https://arxiv.org/abs/2110.03008) [[cond-mat.stat-mech](#)].
- [172] Tobias J. Hagge and Seung-Moon Hong. “Some non-braided fusion categories of rank 3”. In: *Commun. Contemp. Math.* 11 (2009), pp. 615–637. DOI: [10.1142/S0219199709003521](https://doi.org/10.1142/S0219199709003521).
- [173] Deepak Naidu. “Categorical Morita equivalence for group-theoretical categories”. In: *Commun. Alg.* 35 (2007), pp. 3544–3565. DOI: [10.1080/00927870701511996](https://doi.org/10.1080/00927870701511996). arXiv: [math/0605530](https://arxiv.org/abs/math/0605530) [[math.QA](#)].
- [174] Mark Dirk Frederik de Wild Propitius. “Topological interactions in broken gauge theories”. PhD thesis. Amsterdam U., 1995. arXiv: [hep-th/9511195](https://arxiv.org/abs/hep-th/9511195).
- [175] Lorenz Eberhardt. *Wess–Zumino–Witten models*. Lecture notes. 2019. URL: https://conf.itp.phys.ethz.ch/esi-school/Lecture_notes/WZW%20models.pdf.
- [176] Roger E. Behrend, Paul A. Pearce, Valentina B. Petkova, and Jean-Bernard Zuber. “On the classification of bulk and boundary conformal field theories”. In: *Phys. Lett. B* 444 (1998), pp. 163–166. DOI: [10.1016/S0370-2693\(98\)01374-4](https://doi.org/10.1016/S0370-2693(98)01374-4). arXiv: [hep-th/9809097](https://arxiv.org/abs/hep-th/9809097).
- [177] Roger E. Behrend, Paul A. Pearce, Valentina B. Petkova, and Jean-Bernard Zuber. “Boundary conditions in rational conformal field theories”. In: *Nucl. Phys. B* 570 (2000), pp. 525–589. DOI: [10.1016/S0550-3213\(99\)00592-1](https://doi.org/10.1016/S0550-3213(99)00592-1). arXiv: [hep-th/9908036](https://arxiv.org/abs/hep-th/9908036).
- [178] Robert Coquereaux and Gil Schieber. “Twisted partition functions for ADE boundary conformal field theories and Ocneanu algebras of quantum symmetries”. In: *J. Geom. Phys.* 42 (2002), pp. 216–258. DOI: [10.1016/S0393-0440\(01\)00090-0](https://doi.org/10.1016/S0393-0440(01)00090-0). arXiv: [hep-th/0107001](https://arxiv.org/abs/hep-th/0107001).
- [179] Adrian Ocneanu. “Paths on Coxeter diagrams: From Platonic solids and singularities to minimal models and subfactors”. In: *Lectures on Operator Theory*. Ed. by B. V. Rajarama Bhat et al. Vol. 13. Fields Institute Monographies. AMS, 2000, pp. 243–323. URL: <https://pubs.ams.org/ebooks/fim/013>.

- [180] Cain Edie-Michell. “Classifying fusion categories $\otimes$ -generated by an object of small Frobenius–Perron dimension”. In: *Sel. Math. New Ser.* 26 (2020), p. 24. DOI: [10.1007/s00029-020-00550-3](https://doi.org/10.1007/s00029-020-00550-3). arXiv: [1810.05717](https://arxiv.org/abs/1810.05717) [math.QA].
- [181] Pavel Etingof, Dmitri Nikshych, Victor Ostrik, and Ehud Meir. “Fusion categories and homotopy theory”. In: *Quantum Topol.* 1 (2010), pp. 209–273. DOI: [10.4171/QT/6](https://doi.org/10.4171/QT/6). arXiv: [0909.3140](https://arxiv.org/abs/0909.3140) [math.QA].
- [182] Steven H. Simon. *Topological Quantum*. Oxford University Press, 2023. DOI: [10.1093/oso/9780198886723.001.0001](https://doi.org/10.1093/oso/9780198886723.001.0001).
- [183] David Penneys. *Higher Linear Algebra, Categories*. Lecture notes. 2023. URL: <https://people.math.osu.edu/penneys.2/UQSL/ArchivedNotes/HLA/Cat.pdf>.
- [184] David Penneys. *Higher Linear Algebra, Fusion Categories*. Lecture notes. 2023. URL: <https://people.math.osu.edu/penneys.2/UQSL/ArchivedNotes/HLA/FusionCat.pdf>.
- [185] Peter Schauenburg. “Turning monoidal categories into strict ones”. In: *New York J. Math.* 7.257-265 (2001), p. 102. URL: <https://nyjm.albany.edu/j/2001/7-16.pdf>.
- [186] Saunders Mac Lane. “Natural associativity and commutativity”. In: *Rice University Studies* 49.4 (1963), pp. 28–46. URL: <https://hdl.handle.net/1911/62865>.
- [187] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer New York, 1978. DOI: [10.1007/978-1-4757-4721-8](https://doi.org/10.1007/978-1-4757-4721-8).
- [188] Alexei Kitaev. “Anyons in an exactly solved model and beyond”. In: *Annals Phys.* 321.1 (2006), pp. 2–111. DOI: [10.1016/j.aop.2005.10.005](https://doi.org/10.1016/j.aop.2005.10.005). arXiv: [cond-mat/0506438](https://arxiv.org/abs/cond-mat/0506438).
- [189] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2010. DOI: [10.1017/cbo9780511976667](https://doi.org/10.1017/cbo9780511976667).
- [190] Michael Müger. “Galois theory for braided tensor categories and the modular closure”. In: *Adv. Math.* 150 (2000), pp. 151–201. DOI: [10.1006/aima.1999.1860](https://doi.org/10.1006/aima.1999.1860). arXiv: [math/9812040](https://arxiv.org/abs/math/9812040) [math.CT].
- [191] David Penneys. *Operator algebras and their modules*. Notes on Unitary quantum symmetries (Chapter 2). 2018. URL: <https://people.math.osu.edu/penneys.2/UQSL/PartI/UQSLPartIChapter2.pdf>.
- [192] John Watrous. *The Theory of Quantum Information*. Cambridge University Press, Apr. 2018. DOI: [10.1017/9781316848142](https://doi.org/10.1017/9781316848142).
- [193] Albert W Marshall, Ingram Olkin, and Barry C Arnold. *Inequalities: theory of majorization and its applications*. Springer, 2011. DOI: [10.1007/978-0-387-68276-1](https://doi.org/10.1007/978-0-387-68276-1).
- [194] Dmitri Nikshych and Leonid Vainerman. “Algebraic versions of a finite-dimensional quantum groupoid”. In: *Lecture Notes in Pure and Appl. Math.* (2000), pp. 189–220. arXiv: [math/9808054](https://arxiv.org/abs/math/9808054) [math.QA].

- [195] Richard Gustavus Larson and Moss Eisenberg Sweedler. “An associative orthogonal bilinear form for Hopf algebras”. In: *Amer. J. Math.* 91 (1969), pp. 75–94. DOI: [10.2307/2373270](#).
- [196] Richard G. Larson and David E. Radford. “Semisimple cosemisimple Hopf algebras”. In: *Amer. J. Math.* 110 (1988), pp. 187–195. DOI: [10.2307/2374545](#).
- [197] Richard G. Larson and David E. Radford. “Finite dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple”. In: *J. Alg.* 117 (1988), pp. 267–289. DOI: [10.1016/0021-8693\(88\)90107-X](#).
- [198] Daniel Barter, Jacob C. Bridgeman, and Ramona Wolf. “Computing associators of endomorphism fusion categories”. In: *SciPost Phys.* 13 (2022), p. 029. DOI: [10.21468/SciPostPhys.13.2.029](#). arXiv: [2110.03644 \[math.QA\]](#).
- [199] Edwin H. Spanier. *Algebraic Topology*. New York, NY: Springer, 1966. DOI: [10.1007/978-1-4684-9322-1](#).
- [200] Kenneth S. Brown. *Cohomology of groups*. Vol. 87. Graduate Texts in Mathematics. New York, NY: Springer, 1982. DOI: [10.1007/978-1-4684-9327-6](#).