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Quantum Error Correction and Holography, Krylov Complexity, and continuous Tensor Networks

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Abstract

In this thesis, we study a couple of different but related questions: holography and quantum error correction, the gravitationally dressed operators and locality in holographic theory, introducing generalized versions of the class of continuous matrix product state as the most famous class of tensor network, and the notion of Krylov complexity of matrix quantum mechanics. In the first part, we use the notion of the Petz map and apply it to reconstruct operators in the entanglement wedge. Moreover, for geometries that contain black holes, we generalize the notion of the Petz map by itself and reconstruct black hole interior modes. In particular, we show that the Petz map reconstruction of the black hole interior is equivalent to the Papadodimas-Raju Proposal. In the second part, within the AdS/CFT correspondence, we identify a class of CFT operators which represent diff-invariant and approximately local observables in the gravitational dual. The interpretation of these observables is that they are not gravitationally dressed with respect to the boundary, but instead to features of the state. We also provide evidence that there are bulk observables whose commutator vanishes to all orders in $1/N$ with the entire algebra of single-trace operators defined in a space-like separated time-band. In the third part, we defined two new classes of continuous tensor networks by generalizing the class of continuous matrix product states. One is appropriate to describe the ground state of the relativistic field theory at strong coupling and the other is proper for the theories on compact spacetime. We conjecture that the possible bulk dual of it in the AdS/CFT can be an evaporating black hole microstates with the end of the world branes. In the last project, we study the Krylov complexity for 1-matrix quantum mechanics. In the ground state, the Lancsoz coefficients have linear behavior. The b_n coefficients have a positive slope while a_n coefficients have a negative slope. In the thermal states, the a_n coefficients are zero while b_n have two branches of linear growth. Although matrix quantum mechanics is a solvable theory rather than chaotic, we find the linear behavior of the Lancsoz coefficients of this theory.

Declaration of authorship. I, Niloofar Vardian, declare that this thesis and the work presented in it are my own. Wherever contributions of others are involved, every effort is made to indicate this clearly with due reference to the literature and acknowledgment of collaborative research. This thesis contains no material that has been submitted previously, in whole or in part, for the award of any other academic degree or diploma. This thesis is based on the following papers:

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Introduction

In this thesis, I will go through some different topics. I will start with the reconstruction of the bulk in the holographic setup from the Petz map, a universal recovery channel that comes from the quantum information theory. The chapter is divided into two parts. First, we study the case of entanglement wedge reconstruction and in the second part, we consider the interior reconstruction for the black hole microstates. In the next chapter, we study the concept of Locality in AdS/CFT and in particular provide an idea to gravitationally dress the bulk operators. Next, we define two new classes of continuous tensor networks using the entanglement renormalization idea and boundary states. And in the last chapter, we will study the notion of Krylov complexity for Matrix quantum mechanics

0.0.1 Petz map and holography

To describe our universe, we need to seek a theory of quantum gravity. Quantum gravity refers to the combination of theories, including string theory. Seeking to unify the macroscopic world of gravity, governed by general relativity, with the microscopic world of quantum physics. Based on the classical Einstein-Hilbert action, gravity is perturbatively non-renormalizable and hence can not be expanded about a vanishing gravitational coupling, the Newton coupling. The study of black hole physics can be the simplest way toward this big aim. The black hole information paradox [1, 2] is one of the important questions of physics in the last few decades [3–10]. It is related to the smoothness of the horizon of the black hole. Among all the work related to it, I will focus on a proposal by Papadodimas-Raju, who proposed a construction of black hole interior operator on the boundary side [11–14], called the *mirror operators*¹. For other relevant discussion one can look at [15]

To simplify our study, firstly, we consider the AdS geometry or in other word empty AdS. In this setup, one of the main important questions in AdS/CFT is the subregion duality: finding the bulk region associated or dual to the given region of the boundary.

In the beginning, it has been conjectured that the subregion dual to the given region A on the boundary is the causal wedge of that region shown as $\mathcal{C}(A)$. Later, by using the JLMS statement (the relative entropy of two quantum states is the same as the relative entropy of quantum states in the entanglement wedge of that region), it has been proved that the maximum subregion dual to the given region on the boundary A is the entanglement wedge $\mathcal{E}(A)$. The entanglement wedge $\mathcal{E}(A)$ is the domain of dependence of the bulk region between the Ryu-Takayanagi surface and region A itself on the one Cauchy slice.

In general, the entanglement wedge $\mathcal{E}(A)$ is bigger than the causal wedge $\mathcal{C}(A)$ or in other words, the entanglement wedge contains the causal wedge. Entanglement wedge

¹The mirror operator in the Papadodimas-Raju proposal is different from the mirror operators that are defined in the appendix.

reconstruction (EWR) means that bulk operators acting inside the entanglement wedge can be expressed in terms of CFT operators in region A .

At a large N limit in AdS/CFT, the logic for causal wedge reconstruction is well known. By using the bulk equation of motion for fields in the curved spacetime background and the boundary condition or extrapolate dictionary using the mode-sum approach, one can write the bulk degree of freedom in terms of their dual operators on the boundary. The most familiar example is the AdS-Rindler wedge reconstruction.

In the following chapter, we show how to explicitly reconstruct the entanglement wedge from a quantum information technique called Petz map.

Shortly, the Petz map is a dual of the universal recovery channel for the quantum channel that maps the relevant region on the bulk to the dual CFT region. In the case of the EWR in the empty AdS, one can use the global HKLL reconstruction and take the partial trace to find the appropriate quantum channel, and then by using the Petz theorem, finding the corresponding Petz recovery channel, one can reconstruct the operators in the entanglement wedge in terms of the boundary operators. This map has been known as Petz map.

In other words, in the case of subregion duality, in the large N limit, following the JLMS argument [16], there must be a recovery channel that maps the operator in the entanglement wedge $a = \mathcal{E}_A$ to the given region of the boundary A . In [17], the authors found an expression of the recovery channel using the global HKLL map as a global isometry that embeds the entire bulk to the entire boundary which is known as the Petz map. However, the resulting formula was somewhat abstract. In this chapter it will be shown how this formula can be used explicitly to reconstruct a bulk operator in the entanglement wedge.

After that, we come back to geometries containing black holes. On this side, in parallel, we have the island conjecture related to the information paradox which postulates abstractly that we can construct the modes in the island from the Hawking radiation by using the Petz map [17]. The Petz map has its origin in quantum information theory [18, 19]. Recently, it has been found that the best way to understand the semi-classical limit of AdS/CFT is in the language of quantum error correction codes [20]. The error-correcting codes then are the isometries from the bulk Hilbert space to the dual boundary theory. When the geometry contains a black hole, because of the lack of such an isometry we can not follow the discussion in [17] and write the explicit form of the mapping.

In the second part of the first chapter, instead of following that work to write the quantum channel by taking a trace over the complementary region, we use the definition of the Petz map in modular theory. It is useful to note that the Petz recovery channel has its origin in modular theory, roughly speaking at the age when the theory of quantum computation and information was born. We find the Petz reconstruction of the interior modes and we reach the same result as the Papadodimas-Raju proposal.

0.0.2 Locality and gravitationally dressed operator

Locality is one of the most important concepts in quantum gravity. Locality in non-gravitational QFT is well understood. It can be expressed by the axiom of microcausality [21] or more broadly by the structure of “net of algebras” [22, 23]. While it is not straightforward to factorize the Hilbert space into subsystems corresponding to spacelike separated regions, due to UV-divergent entanglement and the type III₁ nature of local algebras [24], there is a way of thinking about subsystems in terms of the split property [25]. In particular, on any given time-slice quantum information can in principle be strictly localized in

finite spatial regions.

Classical general relativity also respects the principle of locality. While the constraints in the Hamiltonian formulation impose non-trivial conditions on the initial value problem allowing some properties of the state, like the total mass, to be read-off from infinity, it is still possible to localize information in subregions of space, see for example [26] for a recent discussion.

On the other hand, there are indications that in quantum gravity locality is an approximate, emergent notion: the absence of fundamental local degrees of freedom is at the foundations of holography [27–29] and various proposals for resolving the black hole information paradox [30–41] rely on the existence of non-local quantum effects.

Understanding the fate of locality in quantum gravity is thus of primordial importance. More precisely, it remains to be understood if locality breaks down at the level of perturbation theory, or whether the aforementioned non-local effects are always exponentially suppressed in $1/G_N$.

It is not straightforward to answer the question since in order to even define what we mean by locality we first need to identify candidate local observables. In a theory of gravity, these must be diffeomorphism invariant. Defining local, diffeomorphism invariant observables in quantum gravity has proven to be challenging. This question has a rich history, see [42–51] and references therein.

If the spacetime has a well-defined boundary, one approach is to define diffeomorphism invariant observables *relationally*, by gravitationally dressing them with respect to the boundary, but then they are not really local. Moreover in a closed universe with no boundary, this approach is not available.

An alternative would be to define observables relationally with respect to some feature of the geometry without making use of a boundary. This has been discussed in various earlier works and related ideas have been useful in the context of cosmology [42–48, 52–54]. However, it is not clear how to give a precise mathematical definition of such observables at the quantum level, ensuring that they are exactly diffeomorphism invariant.

In this chapter, we revisit the question in the framework of the AdS/CFT correspondence and we attempt to define observables dressed with respect to features of the state directly in the dual CFT. An advantage of this approach is that on the boundary diffeomorphism invariance is automatically built in. A price that we pay in the construction is that the observables are defined only for a class of states.

In order to investigate locality in AdS/CFT, we need to know how subregions in the bulk are encoded in the CFT. For bulk regions corresponding to the entanglement wedge of boundary subregions, this is generally understood [55–57]. However, for the purposes of this work we want to find the CFT dual of a bulk subregion corresponding to a bounded causal diamond containing the candidate approximately local, diffeomorphism invariant observable. Such regions are generally not the entanglement wedge of any boundary subregion so the mapping is of different nature. Previous attempts to understand the CFT mapping of such regions include [58–60]. Here we will follow a different approach by focusing on the algebra of single-trace operators.

In a large N holographic CFT, it is natural to define the algebra \mathcal{A} generated by single-trace operators in a time-band \mathcal{D}_{t_1, t_2} . This was first discussed explicitly in [61], inspired by earlier work [12, 13, 34]. In [61] it was proposed that the algebra \mathcal{A} is dual to the causal wedge of the region \mathcal{D}_{t_1, t_2} in the bulk and the commutant of \mathcal{A} dual to the spacelike-separated causal diamond in the interior. Algebras of this type have received attention

recently [62–64].

The discussion of [61] focused on perturbations around empty AdS. In this case, the bulk geometry is homogeneous and “featureless” which, as we will see, introduces additional challenges in defining local diff-invariant observables. In this paper, we revisit the algebra in a time band, in cases where the bulk state is highly excited and time-dependent.

At infinite N the problem can be understood in terms of QFT on a time-dependent bulk geometry, where gravitational backreaction of quantum fields can be ignored and the existence of the commutant is obvious². When considering $1/N$ corrections, the existence of the commutant is less obvious due to the gravitational Gauss law. Usually, in AdS/CFT bulk operators are gravitationally dressed with respect to the boundary, hence at order $1/N$ they do not commute with the Hamiltonian, which is an element of the algebra \mathcal{A} . This raises the question of whether the algebra \mathcal{A} still has a commutant at subleading orders in $1/N$.

In our works, we provide evidence for the existence of a commutant by identifying a class of operators that are gravitationally dressed with respect to *features of the state*. As they are not dressed with respect to the boundary, these operators have vanishing commutators with the Hamiltonian, to all orders in $1/N$, thus bypassing the previous problems with the gravitational Gauss law. Here, we focus only on ensuring that bulk operators have vanishing commutators with the Hamiltonian, but an extension to all single-trace operators in \mathcal{D}_{t_1, t_2} is necessary. We emphasize that it is really the asymptotic charges that one should be concerned with, since in the absence of gravity, bulk QFT in AdS is manifestly local. Understanding the algebra \mathcal{A} in the $1/N$ expansion around empty AdS, and other static states also requires further attention³.

The existence of a commutant for \mathcal{A} in $1/N$ perturbation theory would imply that information can be localized in regions of the bulk and is not visible from the boundary at the level of perturbative quantum gravity⁴.

0.0.3 Generalization of the continuous matrix product states

The idea that spacetime might emerge from more fundamental degrees of freedom has long fascinated physicists. The holographic principle suggests that a $(d + 1)$ -dimensional spacetime might emerge from degrees of freedom in a d -dimensional theory without gravity [75, 76]

While a completely general implementation of this idea is still lacking, the AdS/CFT correspondence provides a specific example in which to probe the holographic emergence of spacetime.

In other words, given a CFT state, one may think of bulk distance and geometry (at least near the boundary) as being charted out by the entanglement properties of the CFT state. A central question in this picture of spacetime emerging from entanglement is:

²If the Hamiltonian, which is an element of the time band algebra, is normalized appropriately, its commutator with bulk fields is suppressed by $1/N$.

³We strongly believe that our construction can be suitably generalized to include vanishing commutation with all single-trace operators, and we give a prescription to do so in [65]. Concerning the acceptable class of states, the fact that the AdS vacuum is not included in this class is a feature not a bug. Indeed, even in classical gravity, one cannot define local and diff-invariant observables around a maximally symmetric state like the AdS vacuum.

⁴Other discussions of localization of information in perturbative quantum gravity, with varying conclusions, include [66–74].

What is the precise relationship between bulk degrees of freedom and boundary degrees of freedom?

From a very different perspective, tensor networks have arisen as a useful way to calculate quantum states in strongly interacting many-body systems [77]. One significant example is the Multi-scale Entanglement Renormalization Ansatz (MERA) [78] which is relevant for critical (gapless) systems, i.e., CFTs. Starting from a simple state in a low-dimensional Hilbert space, acting repeatedly with fixed tensors living on a network lattice produces an entangled wave function for the quantum system of interest; varying with respect to the tensor parameters efficiently computes the system's ground state. Working "backward" in the MERA, starting with the ground state and gradually removing entanglement, produces a set of consecutively renormalized quantum states. This process reveals a renormalization direction along the graph, which may be thought of as an emergent radial direction of space. As pointed out by Swingle [79], the MERA graph can serve as a lattice discretization of spatial slices of AdS. Furthermore, one can use the MERA to calculate the entanglement entropy of regions of the original (boundary) critical system; this calculation amounts to tracing over bonds in the tensor network that cross the causal cone of the boundary region. The causal cone is a sort of extremal surface for the MERA, motivating comparison to the Ryu–Takayanagi formula.

It is therefore natural to conjecture that the MERA provides a concrete implementation of the emergence of spacetime, in the form of a correspondence between boundary and bulk regions reminiscent of AdS/CFT [80].

Application of the tensor network in high energy physics is not limited to the AdS/MERA connection. One can introduce a continuous version of the tensor networks to describe the low-energy spectrum of the relativistic field theory.

A continuous tensor network gives a variational ansatz for the ground state of the quantum field theories (QFTs). The notable examples are the continuous matrix product state (cMPS) [81] and the continuous multiscale entanglement renormalization ansatz (cMERA). While cMPS is just adapted to the non-relativistic QFTs, only the Gaussian cMERA is well-understood which we can not use to approximate the ground state of the interacting relativistic QFTs. But instead, cMERA also corresponds to a real-space renormalization group flow in the context of the wave functions. In this Chapter, we investigate the backward Gaussian cMERA renormalization group flow of the class of cMPS by putting the standard cMPS at the IR scale. At the UV scale, for the bosonic systems in the thermodynamic limit, we achieve the variational class of states that has been proposed recently as the relativistic cMPS (RCMPS) is adapted to the relativistic QFTs without requiring to introduce of any additional IR or UV cut-off. We also extend the RCMPS to fermionic systems and theories on a finite circle.

In this chapter we also present work on finding the appropriate version of the cMPS states for the theory on compact spacetime or in other words, boundary cMPS. We present a proposal for the holographic dual of this class of states in the AdS/CFT setup which as will be discussed can be the Black hole microstates with the end of the world brane which is coupled to a bath system that absorbs the Hawking radiation of the black hole.

We start with the possible connection with a class of tensor networks with the geometry of empty AdS and reach the point that conjectures a new connection with a class of tensor networks and the geometry dual of that.

0.0.4 Krylov complexity in 1-matrix quantum mechanics

After defining the time-band algebra, the question that emerges is whether the operators in the time-band are simple or complex.

Ideas from quantum information theory have become increasingly relevant in high-energy physics. One such concept is complexity, a quantity that is also important in experimental realizations of quantum computers, describing how difficult it is to perform an operation or task. In quantum field theories, there have been two different proposals on how complexity can be defined. These are known as circuit complexity, based on an operatorial approach, and path integral optimization, originating from ideas in tensor networks - methods that allow efficient simulations of quantum many-body systems. There are two motivations behind these studies. First, Einstein-Rosen bridge behind the horizon keeps evolving even after thermalization. The dual quantity of this growth on QFT on the boundary is conjectured to be complex. Second, the entanglement entropy approach as an emergent of space cannot provide a time part of the metric in the bulk. Complexity can do this.

Operator complexity describes the phenomenon that a simple operator O becomes complex under Heisenberg evolution $O(t)$ in chaotic local quantum systems. Krylov Complexity as a novel measure of operator complexity exhibits many interesting universal behaviors and also bounds many other complexity measures.

In our work, we will study the notion of the Krylov complexity for a model of 1-matrix quantum mechanics. Over the thermal state, we find that the Lanczos coefficients contain the even and odd linear branches. Moreover, we will find the radius of convergence when we are using the correlator to find the Krylov complexity. We see only growth of the Krylov complexity till that point. It is almost the same point as the first peak of the correlation function.

Chapter 1

Bulk Reconstruction in AdS/CFT and Petz Map

In this chapter, our goal is to use a technique from quantum error correction, called Petz map, to construct the bulk side in holographic theories from the boundary side. First, we review the construction of the bulk theory as it is known in the literature which is the HKLL approach. Then, in two parts we will apply the Petz map to the entanglement wedge reconstruction and the construction of the black hole interior modes.

1.1 Bulk reconstruction in ordinary AdS/CFT

According to the AdS/CFT correspondence, a holographic CFT on $\mathbb{R} \times \mathbb{S}^{d-1}$ can be interpreted as a theory of quantum gravity in an asymptotically $AdS_{d+1} \times M$ spacetime, where M is some compact manifold. Usually, this involves taking a large N limit in the CFT, and bulk interactions are suppressed by powers of $1/N$. Thus, to lead order at large N , the bulk quantum theory consists of free fields.

The correspondence also involves an identification between fields in the bulk and operators in the boundary CFT. For example, the CFT operator dual to a bulk scalar field ϕ is a scalar primary O with conformal dimension Δ related to the mass of the field ϕ by $\Delta = d/2 + \sqrt{m^2 + d^2/4}$ and the extrapolate dictionary defines O as the dual of ϕ at infinity. For simplicity, in the following, we will just focus on scalar fields and discuss the identification with the dual CFT operator O at large N .

First, on the bulk side of the duality, we start with AdS_{d+1} in global coordinates (t, ρ, Ω) which is described with the metric below

$$ds^2 = \frac{1}{\cos^2(\rho)} (-dt^2 + d\rho^2 + \sin^2(\rho)d\Omega_{d-1}^2). \quad (1.1)$$

Consider a scalar field on the AdS_{d+1} background with the action

$$S = \int d^{d+1}x \sqrt{-g} \frac{1}{2} (g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - m^2 \phi^2) \quad (1.2)$$

and corresponding equation of motion

$$(\square - m^2)\phi = 0. \quad (1.3)$$

This equation has to be supplemented with normalizable boundary conditions at infinity, which implies that near the AdS boundary $\rho = \frac{\pi}{2}$, the field has to decay as $\phi \sim (\cos \rho)^\Delta$. With these boundary conditions at infinity and demanding regularity in the interior, we find a basis of solutions for (1.3) denoted as $f_{nlm}(t, \rho, \Omega)$ which is labeled by the quantum numbers n, l and m , where $n \in \{0, 1, 2, \dots\}$, l is the total angular momentum of the corresponding mode and m is related to the other angular quantum numbers needed to specify a mode. These modes are proportional to

$$f_{nlm}(t, \rho, \Omega) \propto e^{-iE_{nl}t} Y_{lm}(\Omega) \sin^l(\rho) \cos^\Delta(\rho) P_n^{(l+d/2-1, \Delta-d/2)}(\cos 2\rho) \quad (1.4)$$

while

$$E_{nl} = \Delta + 2n + l, \quad (1.5)$$

and $\Delta = d/2 + \sqrt{m^2 + d^2/4}$ is the conformal dimension of the dual CFT $_d$ operator O .

To quantize the scalar field, we associate an annihilation operator a_{nlm} to each mode f_{nlm} with normalized commutation relation

$$[a_{nlm}, a_{n'l'm'}^\dagger] = \delta_{nn'} \delta_{ll'} \delta_{mm'}. \quad (1.6)$$

The quantized free scalar field in AdS $_{d+1}$ is given by

$$\phi(t, \rho, \Omega) = \sum_{nlm} f_{nlm}(t, \rho, \Omega) a_{nlm} + f_{nlm}^*(t, \rho, \Omega) a_{nlm}^\dagger. \quad (1.7)$$

The modes $f_{nlm}(t, \rho, \Omega)$ should be normalized in such a way that the field ϕ obeys the canonical commutation relation. To find the correct normalization factor we consider the Klein-Gordon inner product defined on a Cauchy surface Σ . If we assume that t direction is orthogonal to Σ , for every two functions ϕ_1 and ϕ_2 , it is defined as

$$\langle \phi_1, \phi_2 \rangle_{KG} \equiv i \int_{\Sigma} d^d x \sqrt{-g} g^{tt} (\phi_1^* \nabla_t \phi_2 - \phi_2 \nabla_t \phi_1^*). \quad (1.8)$$

If both ϕ_1 and ϕ_2 obey the equation of motion, the integral above defines a conserved inner product in t . In particular, it says that if we normalize the modes f_{nlm} at some time such that $\langle f_{nlm}, f_{n'l'm'} \rangle = \delta_{nn'} \delta_{ll'} \delta_{mm'}$ and $\langle f_{nlm}, f_{n'l'm'}^\dagger \rangle = 0$, they will remain normalized also at later times [82]. Following these steps, in the end, one can write the modes explicitly as

$$f_{nlm}(t, \rho, \Omega) = \frac{1}{N_{nlm}} e^{-i(\Delta+2n+l)t} Y_{lm}(\Omega) \sin^l(\rho) \cos^\Delta(\rho) P_n^{(l+d/2-1, \Delta-d/2)}(\cos 2\rho) \quad (1.9)$$

where

$$N_{nlm} = \sqrt{\frac{\Gamma(n+l+d/2)\Gamma(n+\Delta-d/2+1)}{n!\Gamma(n+l+\Delta)}}. \quad (1.10)$$

The conformal boundary of AdS $_{d+1}$ is the cylinder $\mathbb{R} \times \mathbb{S}^{d-1}$ which in terms of the global coordinates we obtain by taking $\rho \rightarrow \pi/2$ limit. We can use the coordinate t and Ω to parametrize the boundary theory with metric

$$ds^2 = -dt^2 + d\Omega_{d-1}^2. \quad (1.11)$$

In the boundary, using the state-operator correspondence in the CFT, the formula (1.5) has a nice interpretation. The state created by the $n = l = 0$ creation operator is identified

with the state in the CFT that is produced by inserting the single-trace primary operator O dual to ϕ into the center of the Euclidean path integral and other excited states come from inserting its descendants.

More generally, to leading order at large N , the Fourier modes O_{nlm} of the single trace primary operator and a_{nlm} for the mode f_{nlm} are the same up to a priori arbitrary constant M_{nlm} . The extrapolate dictionary in the global coordinates is given by

$$O(t, \Omega) = \lim_{\rho \rightarrow \pi/2} \frac{1}{\cos^\Delta \rho} \phi(t, \rho, \Omega). \quad (1.12)$$

As a result, we can define the CFT operator $\hat{O}_{nlm} = \frac{1}{M_{nlm}} O_{nlm}$ which is identified with the bulk operator

$$\hat{O}_{nlm} = a_{nlm}. \quad (1.13)$$

As we will see later, this allows us to write a CFT expression for a local bulk field at any point in the bulk.

The single trace primary operator O has a mode expansion on $\mathbb{R} \times \mathbb{S}^{d-1}$ as

$$O(t, \Omega) = \sum_{nlm} g_{nlm}(t, \Omega) O_{nlm} + g_{nlm}^*(t, \Omega) O_{nlm}^\dagger. \quad (1.14)$$

Following 1.7, we have $g_{nlm} = \frac{1}{M_{nlm}} \lim_{\rho \rightarrow \pi/2} \frac{1}{\cos^\Delta \rho} f_{nlm}(t, \rho, \Omega)$. Thus M_{nlm} can be chosen so that mode functions g_{nlm} are orthonormal.

At the large N limit, since we have a free theory in the bulk, all correlators can be reduced to products of 2-point functions by Wick contractions. Therefore, on the boundary side, we already know all the n -point functions of the operator O by taking the spacetime points to the boundary in the expression we found for the bulk and using the extrapolate dictionary (1.12), we have

$$\langle O(x_1) O(x_2) \rangle \propto \frac{1}{(x_1 - x_2)^{2\Delta}} \quad (1.15)$$

$$\langle O(x_1) O(x_2) \dots O(x_{2n}) \rangle = \langle O(x_1) O(x_2) \rangle \dots \langle O(x_{2n-1}) O(x_{2n}) \rangle + \text{permutations}. \quad (1.16)$$

Although the correlation functions of O factorize to the product of 2-point functions, the scalar primary operator is not really a free scalar field. In a CFT in d spacetime dimensions, the condition that a scalar operator is free, i.e. $\nabla^2 O = 0$, is equivalent to the fact that its conformal dimension is $\Delta = d/2 - 1$. Therefore, as the conformal dimension for the scalar primary operator O in a holographic CFT is $\Delta = d/2 + \sqrt{m^2 + d^2/4}$, it is actually not a free scalar theory on the boundary. For the free scalar primaries, the factorization is a consequence of the equation of motion. More generally, the scalar fields with $\Delta \geq d/2 - 1$ are called *generalized free fields* (GFFs) [83–85] if their correlators take the form of Eqs. (1.15) and (1.16). However, because they do not obey the linear equation of motion, we can not describe them in terms of a local free lagrangian in the spacetime background in which the CFT lives. The reason that such fields can be called free is that their Hilbert space has a Fock space structure that is the Hilbert space of a free theory. Nevertheless, such a CFT, extrapolated to high temperatures, has the wrong thermodynamic properties, and therefore it is inconsistent by itself. For the operators with large conformal dimensions, the spectrum can not have the structure of a freely generated Fock space. One way to solve the problem is that think about the GFF as the low-conformal dimension sector of a

much larger CFT with a large central charge while all the additional states correspond to the black hole microstates in the bulk [85].

As a result, we observe that at the large N limit, a free scalar field in pure AdS can be identified as GFF of the boundary.

1.1.1 HKLL reconstruction method: mode sum approach

The study of bulk reconstruction, usually called HKLL, was developed by Hamilton, Kabat, Lifschytz and Lowe in a series of papers [86–91] building on previous works. They attempt to reconstruct the operators of the bulk gravitational theory in the non-interacting regime from the operators of the boundary. Bulk operators are expressed in terms of smeared single trace operators in the CFT as

$$\phi(X) \longleftrightarrow \int dx K(X|x)O(x), \quad (1.17)$$

where the kernel $K(X|x)$ is called *smearing function*. At large N limit, finding the smearing function can be implemented through Fourier expansion. Consider $f_n(X)$ as the set of orthogonal solutions to the Klein-Gordon equation. For simplicity here we denote the set of labels (nlm) discussed in the previous subsection collectively by n . One can quantize the bulk field in terms of creation and annihilation operators as

$$\phi(X) = \sum_n f_n(X)a_n + h.c. \quad (1.18)$$

Taking the points to the boundary and using the extrapolate dictionary, we get the mode expansion of the boundary operators as

$$O(x) = \sum_n \tilde{g}_n(x)a_n + h.c. \quad (1.19)$$

where $\tilde{g}_n(x) = \lim_{r \rightarrow \infty} r^\Delta f_n(r, x)$. If one defines orthonormal boundary mode functions $g_n(x) = \frac{1}{M_n} \tilde{g}_n(x)$, one can invert (1.19) and obtain

$$a_n = \frac{1}{M_n} \int dx O(x)g_n^*(x). \quad (1.20)$$

By plugging it back to (1.18), we reach

$$\phi(X) = \sum_n \frac{1}{M_n} f_n(X) \int dx O(x)\tilde{g}_n^*(x) + h.c.. \quad (1.21)$$

In case we are able to exchange the summation and integration in (1.21), we will get

$$\phi(X) = \int dx K(X|x)O(x), \quad (1.22)$$

where $K(X|x) = \sum_n M_n^{-1} f_n(X) \tilde{g}_n^*(x) + h.c.$ is the smearing function [92]. We review the HKLL construction for a free scalar field in pure AdS in global and AdS-Rindler coordinates here.

1.1.2 HKLL reconstruction in global and Rindler coordinates

We review here the HKLL reconstruction in global and AdS-Rindler coordinates [89] where authors constructed the smearing functions based on the mode sum approach.

HKLL reconstruction in global coordinates

Before going through it, one point that one might be interested in is if there is any possibility to find a smearing function that has compact support on the boundary of AdS. In particular, we are interested the smearing function has support only on the points that are spacelike separated from $\phi(X)$. The HKLL method provides us with a way of reconstruction in the large N limit where the field ϕ satisfies the free equation of motion. Therefore, the smearing function can be constructed from a suitable Green's function that by definition satisfies

$$(\square - m^2)G(X|X') = \frac{1}{\sqrt{-g}} \delta^{d+1}(X - X'). \quad (1.23)$$

Using the third Green identity, the field ϕ can be written in global coordinates as

$$\phi(X') = \int dx \sqrt{-g} \left(\phi(X) \partial^\rho G(X|X') - G(X|X') \partial^\rho \phi(X) \right) \Big|_{\rho=\rho_0} \quad (1.24)$$

where $X = (\rho, x)$, and by sending $\rho_0 \rightarrow \pi/2$, one can find the smearing function in (1.24) in terms of the Green's function. For this purpose, let us take the ansatz of Green's function that is non-zero only at spacelike separation

$$G(X|X') = f(\sigma(X|X')) \theta(\sigma(X|X') - 1), \quad (1.25)$$

where σ is an AdS-invariant distant function which in global coordinates is

$$\sigma(X|X') = \frac{\cos(t - t') - \sin(\rho) \sin(\rho') \cos(\Omega - \Omega')}{\cos(\rho) \cos(\rho')} \quad (1.26)$$

and $\Omega - \Omega'$ is the angular separation on the sphere. The points that can be connected by a geodesic necessarily lie in the unit cell $-\pi < t - t' < \pi$. Spacelike separated points are the ones with $\sigma > 1$ that connected by a geodesic proper distance. By plugging back the ansatz (1.25) to (1.23), we can see that $f(\sigma)$ satisfies the AdS wave equation. Then, if we start from the beginning by the ansatz (1.25) [88], we can find the smearing function with compact support only at spacelike separated region. We note here that this result has been found in global coordinates and in general, it could not be the case. For example, for odd-dimensional AdS in Poincare coordinates, the smearing function can have support only on the entire boundary.

Now, let us find the smearing function in global coordinates. The exact form of the smearing function depends on the dimension. The scalar field solution can be expanded as a linear combination of independent modes that in global coordinates it is given by (Eq. 1.9). We can split the field into positive and negative frequency components

$$\phi(X) = \phi(X)_+ + \phi(X)_- \quad (1.27)$$

where

$$\phi(X)_+ = \phi(X)_-^\dagger = \sum_{nlm} f_{nlm} a_{nlm} \quad (1.28)$$

and for the boundary operator $O(x)$ as well. Since we can use the AdS isometries to bring one point to another one, it is enough to find the smearing function just at one point. In the center ($\rho = 0$), f_{nlm} vanishes for all $l \neq 0$, therefore only the s-waves contributes to the field in the center of Ads which will simplify the calculation drastically. Let us take $a_n = a_{n00} = \hat{O}_{n00}$, we can read a_n in terms of $O(x)$ as

$$a_n = \frac{1}{\pi \text{vol}(S^{d-1}) P_n^{(\Delta-d/2, d/2-1)}(1)} \int_{-\pi/2}^{\pi/2} dt \int d\Omega \sqrt{g_\Omega} e^{i(2n+\Delta)t} O_+(t, \Omega). \quad (1.29)$$

Plugging it back into the bulk mode expansion, one can find the bulk field at the origin as

$$\phi(t', \rho' = 0, \Omega') = \int_{-\pi/2}^{\pi/2} dt \int d\Omega \sqrt{g_\Omega} K_+(t', \rho' = 0, \Omega'|t, \Omega) O_+(t, \Omega) + h.c. \quad (1.30)$$

where

$$K_+(t', \rho' = 0, \Omega'|t, \Omega) = \frac{1}{\pi \text{vol}(S^{d-1})} e^{i\Delta t} F(1, d/2, \Delta - d/2 + 1, e^{-i2t}). \quad (1.31)$$

It is important to know that smearing functions are not necessarily unique. It can be shifted by terms which vanish when integrated against the boundary operators. It can happen in cases that the boundary fields do not involve a complete set of Foureier modes. This freedom enable us to find K_+ which is real and then we can find the kernel such that $K = K_+ = K_-$.

Finally, for an arbitrary bulk point by action of an isometry map, we have

$$\phi(X) = \int_{x \in \text{bdy}} dx K(X|x) O(x) \quad (1.32)$$

which for the AdS_{d+1} in even dimension, the smearing function is

$$K_G(X|x) = \frac{\Gamma(\Delta - d/2 + 1) \Gamma(1 - d/2)}{\pi \text{vol}(S^{d-1}) \Gamma(\Delta - d + 1)} \lim_{\rho \rightarrow \pi/2} (\sigma(X|x) \cos \rho)^{\Delta-d} \theta(\sigma(xX|x) - 1) \quad (1.33)$$

and in odd dimension it is given by

$$K_G(X|x) = \frac{2(-1)^{d/2-1} \Gamma(\Delta - d/2 + 1)}{\pi \text{vol}(S^{d-1}) \Gamma(\Delta - d + 1) \Gamma(d/2)} \lim_{\rho \rightarrow \pi/2} (\sigma(X|x) \cos \rho)^{\Delta-d} \log(\sigma(X|x) \cos \rho) \theta(\sigma(X|x) - 1). \quad (1.34)$$

HKLL reconstruction in AdS-Rindler coordinates

Consider a CFT Cauchy surface Σ and divide it into two regions A and its complementary part \bar{A} . The domain of dependence $\mathcal{D}(A)$ of A which is the set of points on the boundary that every inextendible causal curve that passes through it must also insert A . The causal wedge of a CFT subregion A is defined as

$$\mathcal{C}_A = \mathcal{J}^+[\mathcal{D}(A)] \cap \mathcal{J}^-[\mathcal{D}(A)] \quad (1.35)$$

where $\mathcal{J}^\pm[R]$ is the bulk causal future/past of region R in the boundary.

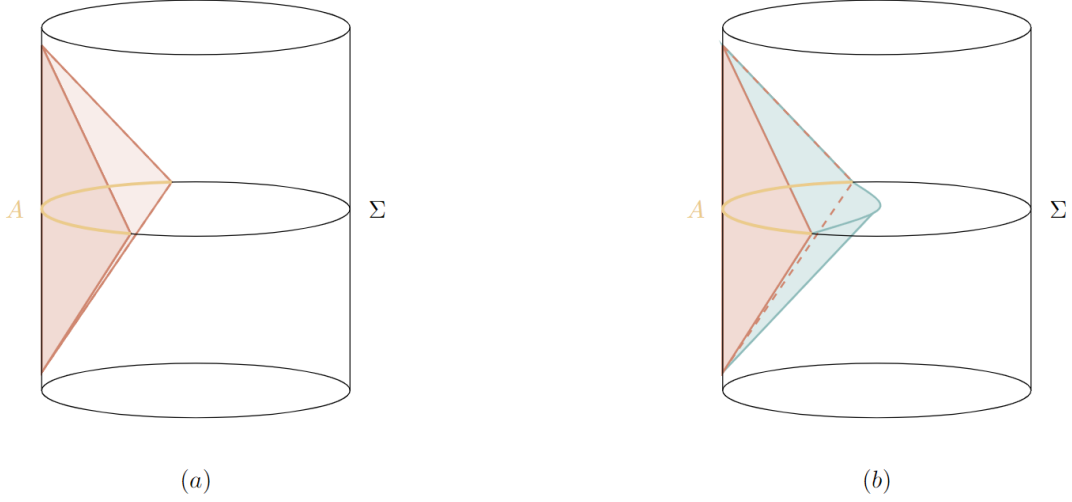


Figure 1.1: (a) Domain of dependence of the spherical region of the boundary. (b) The entanglement wedge of the region A in the bulk which is called AdS-Rindler wedge.

Consider the pure AdS_{d+1} in the bulk. If we take the $t = 0$ slice as the Cauchy surface and A to be one hemisphere of Σ , the causal wedge of A is the region of bulk that referred to as the AdS-Rindler wedge. Although it is naturally associated to a boundary region that covers half of the spatial surface, the patch can be mapped by an isometry to a patch that ends on arbitrary spatial region on the boundary. The coordinate system that covers the AdS-Rindler patch is (r, τ, x) with the metric

$$ds^2 = -(r^2 - 1)d\tau^2 + \frac{dr^2}{r^2 - 1} + r^2 dx^2 \quad (1.36)$$

where x is the set of coordinates on the $(d - 1)$ dimensional hyperbolic ball H_{d-1} . We can find the mode expansion of free scalar field in the AdS-Rindler wedge by solving the Klein-Gordon equation on this background

$$\phi(r, \tau, x) = \int \frac{d\omega}{2\pi} \frac{d\lambda}{2\pi} \left(f_{\omega\lambda}(r, \tau, x) b_{\omega\lambda} + f_{\omega\lambda}^*(r, \tau, x) b_{\omega\lambda}^\dagger \right) \quad (1.37)$$

where the modes $b_{\omega\lambda}$ satisfy the usual commutation relation and the wave function is in the form of

$$f_{\omega\lambda}(r, \tau, x) = e^{-i\omega\tau} Y_\lambda(x) \psi_{\omega\lambda}(r). \quad (1.38)$$

The exact expression for the $\psi_{\omega\lambda}(r)$ in terms of hypergeometric function is [93]

$$\begin{aligned} \psi_{\omega\lambda}(r) = M_{\omega\lambda} r^{-\Delta} \left(1 - \frac{1}{r^2}\right)^{-i\omega/2} F\left(-\frac{d-2}{4} + \frac{\Delta}{2} - \frac{i\omega}{2} + \frac{1}{2}\sqrt{\frac{(d-2)^2}{4} - \lambda}, \right. \\ \left. -\frac{d-2}{4} + \frac{\Delta}{2} - \frac{i\omega}{2} - \frac{1}{2}\sqrt{\frac{(d-2)^2}{4} - \lambda}, \Delta - \frac{d-2}{2}, \frac{1}{r^2}\right) \end{aligned} \quad (1.39)$$

that

$$M_{\omega\lambda} = \frac{1}{\sqrt{2|\omega|}} \frac{\Gamma(-\frac{d-2}{4} + \frac{\Delta}{2} + \frac{i\omega}{2} + \frac{1}{2}\sqrt{\frac{(d-2)^2}{4} - \lambda}) \Gamma(-\frac{d-2}{4} + \frac{\Delta}{2} + \frac{i\omega}{2} - \frac{1}{2}\sqrt{\frac{(d-2)^2}{4} - \lambda})}{\Gamma(\Delta - \frac{d-2}{2}) \Gamma(i\omega)}. \quad (1.40)$$

By taking Fourier transformation of the boundary operator $O(\tau, x) = \lim_{r \rightarrow \infty} r^\Delta \phi(r, \tau, x)$, we have

$$O_{\omega\lambda} = \int d\tau dx e^{i\omega\tau} Y_\lambda^*(x) O(\tau, x) \quad (1.41)$$

that is in the form of $O_{\omega\lambda} = M_{\omega\lambda} b_{\omega\lambda}$. Therefore, $\hat{O}_{\omega\lambda} = \frac{1}{M_{\omega\lambda}} O_{\omega\lambda}$ is the boundary operator identified with Rindler mode functions

$$\hat{O}_{\omega\lambda} = b_{\omega\lambda}. \quad (1.42)$$

By substituting (1.41) into (1.37) and exchange the order of integration we get

$$\phi(r, \tau, x) = \int d\tau' dx' K(r, \tau, x | \tau', x') O(\tau', x') \quad (1.43)$$

where the smearing function is

$$K_R(r, \tau, x | \tau', x') = \int \frac{d\omega d\lambda}{2\pi 2\pi} \frac{1}{M_{\omega\lambda}} f_{\omega\lambda}(r, \tau, x) e^{i\omega\tau'} Y_\lambda^*(x'). \quad (1.44)$$

The issue here is that if we substitute the exact expression of $f_{\omega\lambda}(r, \tau, x)$ in (1.44), we find out that the integral does not converge for any choice of bulk and boundary points [92, 94, 95]. In the original paper [89], authors argued that they can make the integral convergent by analytically continuation of x coordinates. However, there is still this question that if it is actually well-defined in the physically correct Lorentz signature. The issue was illuminated in [94] when they gave an interpretation of the divergent smearing function in the context of distribution theory.

1.1.3 Bulk reconstruction and subregion duality

As we had in the previous sections, a bulk operator $\phi(X)$ can be represented as a smearing integral of boundary operators

$$\phi(X) = \int_{bdy} d^{d-1}x dt K(X|t, x) O(t, x) + O(1/N). \quad (1.45)$$

We can use the CFT Hamiltonian to re-express all operators $O(t, x)$ in terms of fields on a Cauchy surface Σ of the boundary by explicitly evolving the operators with the boundary Hamiltonian. Let us consider the pure AdS case and Σ to be the $t = 0$ slice

$$\phi(X) = \int_{bdy} d^{d-1}x dt K(X|t, x) e^{iH_{CFT}t} O(x) e^{-iH_{CFT}t} \quad (1.46)$$

where $O(t = 0, x) = O(x)$. In general, operators of the form $e^{iH_{CFT}t} O(x) e^{-iH_{CFT}t}$ have support on a large part of the slice Σ , $t = 0$. An interesting question in AdS/CFT is whether the CFT representation of $\phi(X)$ can be localized to a subregion of Σ . Intuitively, it is expected that the boundary support of $\phi(X)$ shrinks as the operator approaches the boundary. However, one can see from (1.46) that even if we take X to be very close to the boundary, the CFT reconstruction still has support on the entire Σ .

In fact, it is possible to reconstruct bulk operators so that they are supported on smaller regions on the boundary. Consider a spherical subregion R on Σ and the corresponding

causal wedge in the bulk. Consider a local field $\phi(X)$ localized inside this causal wedge. Then it is possible to represent the bulk field as

$$\phi(X) = \int_{\mathcal{D}(R)} d\tau d^{d-1}x K_R(X|\tau, x) O(\tau, x). \quad (1.47)$$

where the integral is over the domain of dependence $\mathcal{D}(R)$ of R and $K_R(X|\tau, x)$ is a new smearing function called the AdS-Rindler smearing function.

Again we can write it in terms of non-local operators in the Heisenberg picture which evolves with Rindler Hamiltonian H_τ

$$\phi(X) = \int_{-\infty}^{\infty} d\tau \int_R d^{d-1}x K_R(X|x, \tau) e^{iH_\tau\tau} O(x) e^{-iH_\tau\tau}. \quad (1.48)$$

The operators $e^{iH_\tau\tau} O(x) e^{-iH_\tau\tau}$ are again some non-local operators but this time they have support only on region R instead of entire Σ . Therefore, the AdS-Rindler reconstruction provides us a way to localize the representation of $\phi(X)$ in the boundary. More generally, it suggests the proposal that a given region R on a Cauchy slice of the boundary encodes the bulk data inside the causal wedge of its boundary domain of dependence.

Nevertheless, one can go ahead and look at the Rindler Hamiltonian in (1.48) as the *modular Hamiltonian* of the region R that generates the modular flow of operators on R . For the case of AdS-Rindler, it is just translation in the τ direction. In [16], authors showed that the boundary modular flow is dual to the bulk modular flow in the entanglement wedge \mathcal{E}_R and conjectured that operators in the entanglement wedge of the region R are the ones can be constructed on the boundary region R by replacing τ in (1.48) by the modular parameter s as

$$\phi(X) = \int_R d^{d-1}x \int_{-\infty}^{\infty} ds K'_R(X|x, s) O(x, s) \quad (1.49)$$

for every $X \in \mathcal{E}_R$ where $O(x, s) = e^{iK_R s} O(x, s=0) e^{-iK_R s}$.

1.2 Black holes in AdS

In this section, We briefly review the geometry of the eternal two-sided AdS black holes, quantizing the free field theory on this background which is needed for studying the bulk reconstruction in AdS/CFT at strict large N limit. We also discuss the operator algebra of observable in the case that an eternal black hole in the bulk is dual to the two CFTs in the thermo-field-double (TFD) state, and in the end, talk about the typical one-sided black holes in AdS.

1.2.1 AdS eternal black holes

There is a unique spherically symmetric solution of the Einstein equation with a negative cosmological constant known as AdS-Schwarzschild geometry. Its metric in $(d+1)$ -dimension ($d \geq 3$) is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2. \quad (1.50)$$

In the AdS unit, we have

$$f(r) = 1 + r^2 - \frac{\alpha}{r^{d-2}} \quad (1.51)$$

where α is a parameter proportional to the ADM mass M . Like in flat space, one can maximally extend the solution by introducing AdS-Kruskal coordinates

$$\begin{aligned} U &\equiv -e^{2\pi(r_*-t)/\beta} \\ V &\equiv e^{2\pi(r_*+t)/\beta} \end{aligned} \quad (1.52)$$

where here, r_* is the tortoise coordinate defined as $\frac{dr_*}{dr} = f^{-1}(r)$. The metric in the new coordinates can be written as

$$ds^2 = \left(\frac{\beta}{2\pi}\right)^2 \frac{f(r)}{UV} dU dV + r^2 d\Omega_{d-1}^2. \quad (1.53)$$

The metric is originally defined in the region $U < 0, V > 0$ corresponding to outside the horizon. By extending the geometry in a maximal way when we assume that there is no matter anywhere, one can describe eternal black holes in AdS spacetime (Fig. 1.2). In all regions, one can introduce Schwarzschild coordinates. Their relation with Kruskal coordinates is given as [14] :

regions	Kruskal coordinates	relationship with the AdS-Schwarzschild coordinates
I	$U < 0, V > 0$	$U = -e^{\frac{2\pi}{\beta}(r_*-t)}, V = e^{\frac{2\pi}{\beta}(r_*+t)}$
II	$U > 0, V > 0$	$U = e^{\frac{2\pi}{\beta}(r_*-t)}, V = e^{\frac{2\pi}{\beta}(r_*+t)}$
III	$U > 0, V < 0$	$U = e^{\frac{2\pi}{\beta}(r_*-t)}, V = -e^{\frac{2\pi}{\beta}(r_*+t)}$
IV	$U < 0, V < 0$	$U = -e^{\frac{2\pi}{\beta}(r_*-t)}, V = -e^{\frac{2\pi}{\beta}(r_*+t)}$

Regions I and III are two asymptotically AdS regions corresponding to black hole exteriors that for each of them, another one is behind the horizon. In the U-V plane, surfaces of constant r_* are hyperboloids that always stay within a single region. On the other hand, the surfaces of constant t are simply straight lines running through the origin which means we can think of time translations as rotations of the Kruskal diagram about the bifurcation point. Although, we should keep in mind that a line can not be rotated past the horizon by a finite rotation. Moreover, since there is no global timelike isometry, the entire geometry is time-dependent.

It is good to note that the natural choice for the vacuum of the bulk effective theory on the AdS eternal black hole is the Hartle-Hawking (HH) state $|HH\rangle$. It has been conjectured by Maldacena [96] that the AdS eternal black hole has a holographic description in terms of two copies of an identical CFT in the TFD state

$$|\Psi_{TFD}\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_i e^{-\beta E_i/2} |i^*\rangle_L |i\rangle_R. \quad (1.54)$$

where β^{-1} is the Hawking temperature of the black hole. Therefore, one can describe each holographic CFT dual to the eternal black hole with the thermal density matrix $\rho_{th} = \frac{1}{Z_\beta} e^{-\beta H}$, where Z_β is the partition function of the CFT at the temperature β^{-1} . Here, the states $|i\rangle$ are the energy eigenstates of one single CFT and $|i^*\rangle_{L(R)} = \Theta |i\rangle_{R(L)}$. The Θ is an anti-unitary operator that reverses the time direction after exchanging the CFTs. Having the states $|i^*\rangle$ for the left CFT comes from the point that the left CFT is glued to the region III with a flip in the AdS-Schwarzschild time coordinate. In other

words, the time in left and right CFTs are identified as $t_R = t$ and $t_L = -t$, respectively. The isometry of the entire bulk generated by $t \rightarrow t + T$ also corresponds to the identity

$$e^{i\hat{H}}|\Psi_{TFD}\rangle = |\Psi_{TFD}\rangle, \quad \hat{H} = H_R - H_L. \quad (1.55)$$

The time translation on the entire geometry is generated by a Killing vector field denoted by V . It is future-directed timelike on the right exterior and past-directed timelike on the left exterior. The conserved charge associated to the global time translation is

$$\hat{h} = \int_{\Sigma} d\Sigma^{\mu} V^{\nu} T_{\mu\nu} \quad (1.56)$$

where Σ is a Cauchy hypersurface and $T_{\mu\nu}$ is the energy-momentum tensor of the bulk fields. The boundary dual of the operator \hat{h} is

$$\hat{h} = \beta\hat{H}. \quad (1.57)$$

It comes from the fact that the vector field V on the right boundary reduces to $\beta\partial_t$ and to $-\beta\partial_t$ on the left one. We will discuss these quantity's interpretation in the algebraic context later.

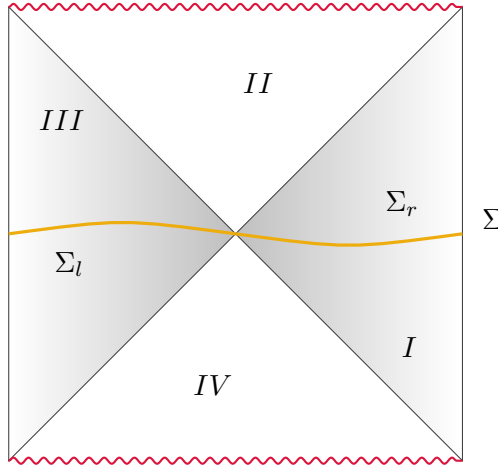


Figure 1.2: The Penrose diagram of the eternal two-sided black hole.

1.2.2 Scalar field quantization in AdS eternal black hole background

Consider a free scalar field propagating on a curved spacetime background. The equation of motion is the Klein-Gordon (KG) equation, $(\square - m^2)\phi = 0$. The field then has a Heisenberg picture expression as

$$\phi(x) = \sum_n f_n(x) a_n + f_n^*(x) a_n^\dagger, \quad (1.58)$$

where f_n are the classical solutions of the KG equation in the given background that should be normalized with respect to the KG norm. We should also impose normalizable boundary conditions at infinity. To each mode f_n , we associate the annihilation and creation operators a_n, a_n^\dagger with normalized commutation relation. The Hilbert space of

QFT at every Cauchy slice of the entire background of interest can be constructed as a Fock space using these ladder operators.

On the other hand, we can always decompose a Cauchy slice Σ into the smaller slices Σ_r such that their union covers the entire Cauchy surface, $\Sigma = \cup_r \Sigma_r$. We can find an alternative expression for the field in the domain of dependence of each subregion denoted by $\mathcal{D}(\Sigma_r)$, by solving the equation of motion on the coordinate system that covers only $\mathcal{D}(\Sigma_r)$

$$\phi(x_r) = \sum_{\alpha} f_{r,\alpha}(x_r) a_{r,\alpha} + f_{r,\alpha}^*(x_r) a_{r,\alpha}^{\dagger}. \quad (1.59)$$

The new operators $a_{r,\alpha}$ has support only on Σ_r . Here, we can use the Bogoliubov transformation and write the mode functions on the entire Cauchy slice as linear combinations of $a_{r,\alpha}$. Hence, we can in principle expand the field ϕ at each point in terms of $a_{r,\alpha}$.

Now, let us quantize a scalar field on the eternal AdS black hole background. In principle, we can do it by solving the equation of motion in Kruskal coordinates. However, there is another possibility when we take the Cauchy slice Σ such that it passes through the bifurcation point. We can consider Σ as the union of two smaller slices as $\Sigma = \Sigma_r \cup \Sigma_l$ (Fig. 1.2). Then, if we consider just region I(III), $\Sigma_r(\Sigma_l)$ itself is a complete Cauchy slice on that. We also know coordinate systems that cover regions I and III which are nothing but two copies of AdS-Schwarzschild coordinates. Therefore, we can first start by region I and solve the KG equation outside the horizon in AdS-Schwarzschild coordinates (1.50). One can find its solutions as $f_{\omega,m}$ which are the modes labeled by quantum numbers ω, m . To each of them, we associate a couple of creation and annihilation operators with normalized commutation relation, denoted by $a_{\omega,m}$. Therefore, one can express the fields lies in region I as

$$\phi_I(x) = \sum_m \int_0^{\infty} \frac{d\omega}{2\pi} \frac{1}{\sqrt{2\omega}} (f_{\omega,m}(x) a_{\omega,m} + f_{\omega,m}^*(x) a_{\omega,m}^{\dagger}). \quad (1.60)$$

We can follow the same analysis in region III and find another set of operators $\tilde{a}_{\omega,m}$ with the same algebra as $a_{\omega,m}$ while they commute with all $a_{\omega,m}$. One can write the fields in region III like region I as

$$\phi_{III}(x) = \sum_m \int_0^{\infty} \frac{d\omega}{2\pi} \frac{1}{\sqrt{2\omega}} (\tilde{f}_{\omega,m}(x) \tilde{a}_{\omega,m} + \tilde{f}_{\omega,m}^*(x) \tilde{a}_{\omega,m}^{\dagger}). \quad (1.61)$$

Therefore, we have the expansion of the field in the entire Cauchy slice Σ and so, it is straightforward to find the expression for fields in region II and IV by evolving them with respect to the total Hamiltonian.

As it is mentioned, the vacuum of the quantum field in the eternal black hole background is an analog of the HH state corresponding to the black hole temperature $T = 1/\beta$ which is defined to satisfy

$$(a_{\omega,m} - e^{-\beta\omega} a_{\omega,m}^{\dagger}) |HH\rangle = 0, \quad (1.62)$$

and characterized by thermal occupation levels for both modes $a_{\omega,k}$ and $\tilde{a}_{\omega,k}$

$$\begin{aligned} \langle a_{\omega,m} a_{\omega',m'}^{\dagger} \rangle_{HH} &= \frac{e^{\beta\omega}}{e^{\beta\omega} - 1} \delta(\omega - \omega') \delta_{m,m'} \\ \langle a_{\omega,m}^{\dagger} a_{\omega',m'} \rangle_{HH} &= \frac{1}{e^{\beta\omega} - 1} \delta(\omega - \omega') \delta_{m,m'} \end{aligned} \quad (1.63)$$

and the same for the modes $\tilde{a}_{\omega,m}$ [34]. Using this standard formalism of quantization of the field theory on the curved spacetime background, one can also describe the Hilbert space of the theory as the Fock space built on the HH vacuum, denoted as $\mathcal{H}_{BH}^{(Fock)}$.

1.2.3 Operator algebra of observables

Consider two copies of the boundary CFT in the TFD state dual to a two-sided eternal black hole in AdS. As explained in the previous section, small perturbations around the black hole background can be described by quantizing the QFT in curved spacetime background. The algebra of low-energy effective field theory in the left and right exteriors is denoted by $\mathcal{A}_{l,0}$ and $\mathcal{A}_{r,0}$ respectively. In the large N limit, the algebras of observables outside the horizon of the black hole, i.e. $\mathcal{A}_{l,0}$ and $\mathcal{A}_{r,0}$, are of Type III₁ von Neumann algebra since the algebra of observables of a local region in QFT must be of this Type. As these two spacetime regions are spacelike separated, the corresponding operator algebras are each other's commutants.

One can split \hat{h} in (1.56) as a difference between right and left operators as

$$\hat{h} = h_r - h_l \quad (1.64)$$

while

$$h_r = \int_{\Sigma_r} d\Sigma^\mu V^\nu T_{\mu\nu}, \quad h_l = \int_{\Sigma_l} d\Sigma^\mu V^\nu T_{\mu\nu}. \quad (1.65)$$

To obtain this splitting we should choose the Cauchy hypersurface Σ to pass through the bifurcate horizon, i.e. $\Sigma = \Sigma_r \cup \Sigma_l$. The question that can arise at this point is whether the operators h_l and h_r belong to the operator algebras $\mathcal{A}_{l,0}$ and $\mathcal{A}_{r,0}$ or not?

As explained in [97] besides this formal splitting, due to the ultraviolet divergences near the horizon, i.e.

$$|h_{r(l)}|HH\rangle|^2 = \langle HH|h_{r(l)}^2|HH\rangle = \infty, \quad (1.66)$$

the operators h_r , h_l do not make sense as an operator on the bulk Hilbert space $\mathcal{H}_{BH}^{(Fock)}$. There is another way of answering this question: in the Tomita-Takesaki theory, the operator \hat{h} is related to the modular Hamiltonian of the HH state for the algebra $\mathcal{A}_{r,0}$

$$\Delta = e^{-\hat{h}} \quad (1.67)$$

A modular Hamiltonian of a Type III₁ algebra never has a splitting as in (1.64) and so the operators h_r , h_l are not well-defined and so they do not belong to the operator algebras $\mathcal{A}_{l,0}$ and $\mathcal{A}_{r,0}$ at strict large N limit.

The operators dual to the low energy effective field theory are the subtracted single trace operators of the boundary theory which have Gaussian correlation functions in the large N limit and we denote them here as $\mathcal{A}_{L,0}$ and $\mathcal{A}_{R,0}$ for the left and right CFTs. Therefore, the commutator of the single trace operators in the large N limit is c -number. Since the operators h_l and h_r are not part of the bulk operator algebras, the gauge theory Hamiltonian of the boundary theories, H_L and H_R must not be part of the algebras $\mathcal{A}_{L,0}$ and $\mathcal{A}_{R,0}$ as well.

Above the Hawking-Page temperature that the two CFTs in the TFD state are dual to the two-sided eternal black hole in the bulk, H_R and H_L have the thermal expectation value and connected two-point function of order N^2 . Although the operators H_R and H_L do not have a large N limit, their difference does have as its bulk dual \hat{h} (1.57). The modular

operator of the TFD state of the boundary for the algebra $\mathcal{A}_{R,0}$ is $\Delta = e^{-\beta\hat{H}}$. To obtain the modular operator, one can also start with finite N where the algebra of observables on both sides are of Type I von Neumann algebras. The left and right Hamiltonians can be written in terms of the usual Hamiltonian of a single copy of the system H as $H_L = H \otimes I$ and $H_R = I \otimes H$. In the case of Type I algebra, each system individually can be described by a reduced density matrix which here for the TFD state, they are obtained to be the thermal density matrices $\rho_L = \rho_R = e^{-\beta H}/Z$. Then, one can find the modular operator of the TFD state by using the relation (A.20) as

$$\Delta = \rho_L^{-1} \otimes \rho_R = e^{-\beta H_R + \beta H_L} = e^{-\beta\hat{H}} \quad (1.68)$$

which is also valid at large N limit. By subtracting the expectation value of the Hamiltonian, one can define the operator $H'_R = H_R - \langle H_R \rangle$ and the same for the left side. We have $\langle H_R^2 \rangle \sim N^2$ and thus H'_R does not have a large N limit. By dividing it by N , we can introduce an operator

$$U = \frac{1}{N} H'_R \quad (1.69)$$

with Gaussian correlation function at large N limit, the same as any other single trace operators. Therefore, U has a large N limit, but at this limit, it is central as

$$[U, O] = \frac{1}{N} [H_R, O] = -\frac{i}{N} \frac{\partial O}{\partial t} \quad (1.70)$$

and at $N = \infty$, it commutes with all the rest of single trace operators.

As explained, H_R and thus U is not part of the algebra $\mathcal{A}_{R,0}$. Therefore, we can also define $\mathcal{A}_{R,0}$ to consist of only the single trace operators that have non-zero commutators. In other words, the operator algebra of low-energy excitations around the black hole background is dual to the single trace operators of the boundary with a nontrivial commutator, they describe a *generalized free field* (GFF) over the thermal state of the CFT. Using the AdS/CFT argument, we can identify the operator algebra of the bulk and boundary as

$$\mathcal{A}_{r,0} = \mathcal{A}_{R,0}, \quad \mathcal{A}_{l,0} = \mathcal{A}_{L,0} \quad (1.71)$$

which by itself requires that $\mathcal{A}_{L,0}$ and $\mathcal{A}_{R,0}$ be of Type III₁ as well [97].

In addition to the argument above about the nature of the algebra $\mathcal{A}_{R,0}$, it has been also studied recently by Leutheusser and Liu in [98, 99] purely in the boundary theory without requiring the duality with the bulk theory. Using the half-sided modular inclusion, they argued that above the Hawking-Page temperature, there is an emergent operator algebra which is a von Neumann algebra of Type III₁. Take B to be a time band in the right boundary and denote the algebra generated by subtracted single-trace operators in B as $\mathcal{A}_{R,0}^B$ which is dual to the bulk algebra of operators in the causal wedge of the time band B . Since the generator of the boundary time translation is not part of the algebra $\mathcal{A}_{R,0}$, the algebra $\mathcal{A}_{R,0}^B$ is not coincide with the algebra of subtracted single-trace operators on the entire boundary and rather $\mathcal{A}_{R,0}^B$ is just a subalgebra of $\mathcal{A}_{R,0}$. As \hat{H} is the generator of the boundary time translation, the modular flow of the algebra $\mathcal{A}_{R,0}$ shifts the boundary time $t \rightarrow t + \beta u$. Therefore, the operator algebra in the time band $B = (t_0, \infty)$ maps to itself under conjugation by Δ^{iu} for $u > 0$

$$\Delta^{iu} \mathcal{A}_{R,0}^B \Delta^{-iu} = \mathcal{A}_{R,0}^B \quad u > 0. \quad (1.72)$$

This structure is called half-sided modular inclusion and exists only if $\mathcal{A}_{R,0}$ is a Type III₁ von Neumann algebra.

The $1/N$ corrections to this picture have been discussed by Witten in [97]. In particular, he showed that it modifies the emergent Type III₁ algebra to an algebra of Type II_∞. At large N limit, one can define algebra \mathcal{A}_R as an extension of the algebra $\mathcal{A}_{R,0}$ by adding an additional generator U as

$$\mathcal{A}_R = \mathcal{A}_{R,0} \otimes \mathcal{A}_U. \quad (1.73)$$

The algebra \mathcal{A}_R is no longer a factor since U is central. Similarly, one can define the algebra \mathcal{A}_L on the left CFT by defining the operator $U' = H'_L/N$. Note that the operators U and U' are not completely independent since $U - U' = \tilde{H}/N$. At strict large N limit, $U - U'$ vanishes, and therefore, the algebra \mathcal{A}_L can also be defined in terms of U as

$$\mathcal{A}_L = \mathcal{A}_{L,0} \otimes \mathcal{A}_U. \quad (1.74)$$

In the large N limit, the algebras \mathcal{A}_L and \mathcal{A}_R are of Type III₁ von Neumann algebra [98, 99]. Once we go beyond the large N limit and consider $1/N$ corrections, the algebras \mathcal{A}_L and \mathcal{A}_R become of Type II_∞. Mathematically, the Type II_∞ algebra is the crossed product of the Type III₁ algebra of the strict large N limit by its modular automorphism group. By duality, these boundary algebras are dual to the bulk algebras of observables on each side of the black hole denoted by \mathcal{A}_l and \mathcal{A}_r

$$\mathcal{A}_r = \mathcal{A}_R, \quad \mathcal{A}_l = \mathcal{A}_L \quad (1.75)$$

which incorporates the algebra $\mathcal{A}_{r(l),0}$, the observable U that is central at large N limit and $1/N$ corrections. Beyond $N = \infty$, U is no longer central and the $1/N$ corrections modify the algebra in such a way that its center becomes trivial. More precisely perturbatively in $1/N$, the algebra of observables deforms to the factor of Type II_∞.

1.2.4 One-sided black holes in AdS/CFT

The full AdS-Schwarzschild geometry in (1.53) describes an additional asymptotically AdS region that is connected to our universe by a wormhole. Besides them, like in flat space, black holes can also be created by some sort of collapsing shell. In such a case, there is no wormhole connecting to another universe since the geometry at the earlier time looks nothing like the full AdS-Schwarzschild geometry as the interior is non-vacuum. Nevertheless, the one-sided geometry may share some features such as singularity and future horizon with maximally extended solutions.

Here, there is a new important feature in comparison with flat space. The Hawking radiation in AdS black hole background will reach the boundary in a finite time and then, it is reflected back by the boundary. If the black hole is small enough, the entire black hole evaporates before the radiation gets to the boundary. By increasing the size of the black hole, while the Schwarzschild radius of the black hole reaches the AdS radius, the radiation will be reflected back into the black hole very fast. So, the black hole will quickly reach equilibrium with the Hawking radiation and remain constant in size up to small fluctuation. As a result, with the usual reflecting boundary condition, big black holes in AdS never evaporate and they are eternal. Thus, for a big black hole in AdS formed from collapse, at a late enough time when all the matter has fallen into the black hole and the fluctuations of the horizon have decayed away, the quantum fields start behaving like ones

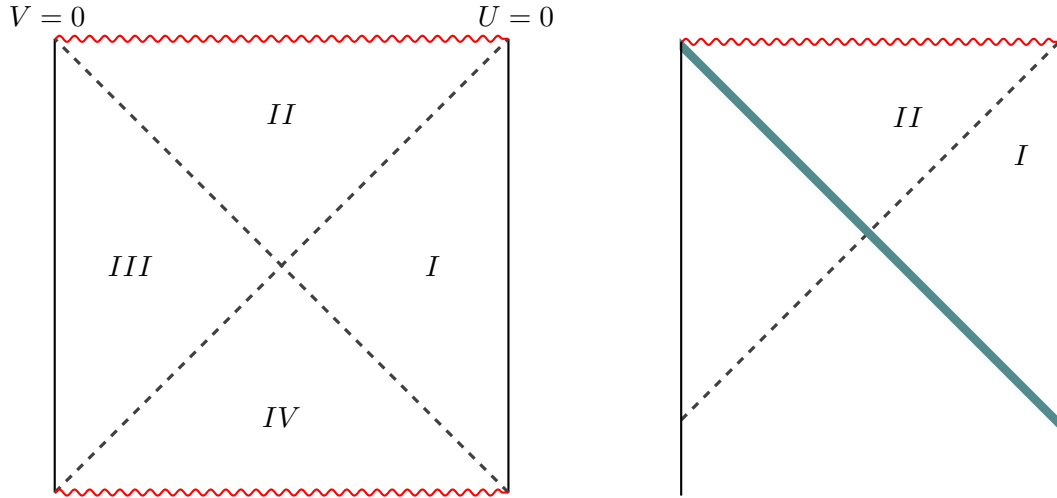


Figure 1.3: The two sided eternal Black hole and the one-sided geometry for a stable AdS black hole created by some sort of collapsing shell.

in the eternal black hole background. It is known that the small black holes in AdS are not stable while the big ones are.

Black hole formation by collapse has a holographic interpretation as the thermalization of the CFT pure state on the boundary of the AdS space. In the dual CFT, we start in a pure state and then allow it to settle down and thermalize after a while. It will evolve to a state that is indistinguishable from a thermal state for the set of interesting observables. Therefore, the late time CFT correlation functions on a massive pure state can be approximated by correlation functions on the thermal density matrix.

As we said, a big black hole in a pure state is dual to a high-energy pure state in the CFT. For a black hole at fixed energy, as Bekenstein proposed, the number of black hole microstates is counted by black hole entropy S . At sufficiently large energy, the black hole microstates dominate the microcanonical ensemble of the CFT. In other words, almost all high-energy states in the CFT have a bulk description as a single black hole. In general, we can think of an equilibrium pure state as a typical state. A typical black hole microstate of energy E_0 in the CFT is defined as a pure state which is a random superposition of energy eigenstates in a narrow energy band

$$|\Psi_0\rangle = \sum_{E_i \in (E_0 - \delta E, E_0 + \delta E)} c_i |E_i\rangle, \quad (1.76)$$

where c_i are random numbers selected with the uniform Haar measure. These typical states represent the majority of black hole microstates of a given energy. They are approximately in equilibrium and so, it is expected that correlators in these states will be the same as thermal correlators at large N limit

$$\langle \Psi_0 | O(x_1) O(x_2) \dots O(x_n) | \Psi_0 \rangle = \frac{1}{Z_\beta} \text{tr}(e^{-\beta H} O(x_1) O(x_2) \dots O(x_n)) \quad (1.77)$$

where β^{-1} is the temperature corresponding to the energy E_0 . We note that these typical states are not exactly the same as the late-time configuration of a black hole forming by collapse, as they have a narrower energy band.

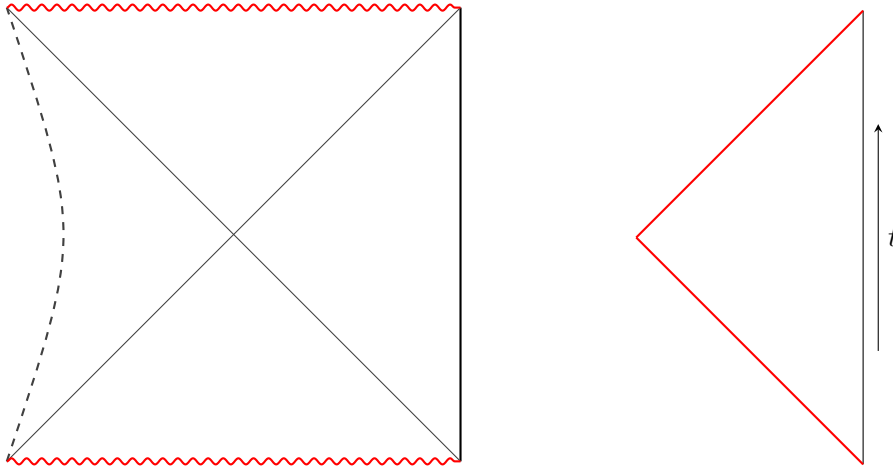


Figure 1.4: Two proposals for the geometry dual to the typical black hole microstate.

Since these typical states look time-independent for the simple observables, their dual geometry should be characterized by an approximate killing isometry which is timelike near the horizon. It is mostly accepted that the geometry contains one exterior region described by the AdS-Schwarzschild metric. It was proposed in [3, 6, 8] that the entire dual geometry is just the exterior region terminates on the horizon by a firewall which is consistent with the time translation symmetry we have. However, since the curvature near the horizon of a big black hole is low, their proposal demands a modification of general relativity at low curvature. In addition to this solution, it has been conjectured in [100, 101] that if we have a smooth horizon, the dual geometry to a typical pure state contains the black and white hole interiors and part of the left region as well (Fig. 1.4).

Finally, we mentioned that to study the evaporation of stable black holes in AdS, one can impose the absorbing boundary condition for the big black holes instead of reflecting ones. Here instead, the Hawking radiation never returns to the black hole since the outgoing modes are absorbed by the boundary, and so, the black hole evaporates. In the dual theory then, the CFT is not a closed system and it does not evolve unitarily. However, one can as usual add an auxiliary system which here stores the outgoing Hawking radiation when it reaches the boundary.

1.3 Quantum Error Correction and Petz map

1.3.1 Quantum channels

Real systems suffer from unwanted interactions with the outside world that show up as noise in quantum information processing systems. In order to describe such systems, it is useful to introduce the notion of a quantum channel, i.e. a linear map $\mathcal{E} : L(\mathcal{H}) \rightarrow L(\mathcal{H})$ which is completely positive and trace-preserving. Here, $L(\mathcal{H})$ denotes the set of linear operators acting on the Hilbert space \mathcal{H} .¹ Every quantum channel \mathcal{E} has an operator sum representation in terms of a non-unique set of operators $\{A_i\}$ known as *Kraus operators*

¹More generally a quantum channel can be a map between $L(\mathcal{H}_1), L(\mathcal{H}_2)$ for two different Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$.

[102–104] such that,

$$\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger \quad \sum_i A_i^\dagger A_i = I. \quad (1.78)$$

They are the most general transformation of a quantum state in an open quantum system.

A natural way to describe the dynamics of an open quantum system is to regard it as arising from an interaction between the system and an environment which together transform under a unitary. After the evolution we perform a partial trace over the environment to obtain the state of the system. For every quantum channel, there exists a model environment starting in an initial state σ_{en} and model dynamics specified by a unitary operator U such that

$$\mathcal{E}(\cdot) = \text{tr}_{en}(U(\cdot \otimes \sigma_{en})U^\dagger), \quad (1.79)$$

which is a version of the *Stinespring dilation* theorem. If $\sigma_{en} = \sum_j \lambda_j |j\rangle \langle j|$, one can find the Kraus representation of (3.57) as

$$\mathcal{E}(\rho) = \sum_{j,k} A_{j,k} \rho A_{j,k}^\dagger \quad (1.80)$$

which $A_{j,k} = \sqrt{\lambda_j} \langle k|U|j\rangle$ are the Kraus operators. Therefore, we can describe the dynamics of the system by using the operator-sum representation without having to explicitly consider the properties of the environment. One advantage of this Kraus representation is that it is well adapted to describe discrete state change without explicit reference to the passage of time.

1.3.2 Quantum error correction

In order to do quantum information and communication in presence of the noise, we need *quantum error correction* (QEC) codes. The basic ideas of the theory are inspired by the classical information theory, but all the limitations of quantum mechanics have been considered in its formulation.

The quantum error correcting code is just selecting an appropriate subspace of some larger Hilbert space called *code subspace* (\mathcal{C} or \mathcal{H}_{code}) that has the same dimension as the system. Quantum states are encoded by a unitary operation into a quantum error-correcting code. The encoding is simply as follows:

$$|\psi\rangle = \sum_i \lambda_i |i\rangle \in \mathcal{H}_{system} \longrightarrow |\psi\rangle_{\mathcal{C}} = \sum_i \lambda_i |i\rangle_{\mathcal{C}} \in \mathcal{H}_{\mathcal{C}} \quad (1.81)$$

which $\{|i\rangle_{\mathcal{C}}\}$ forms a basis for \mathcal{H}_{code} .

Similar to the classical case, each QEC code is able to correct only a specific set of errors

$$\mathcal{E} = \{E_1, E_2, \dots, E_n\}. \quad (1.82)$$

A necessary and sufficient condition for the quantum error correcting code to be able to correct the set of errors \mathcal{E} is

$$\langle \psi | E_a^\dagger E_b | \psi \rangle = \Lambda_{ab} \quad \forall |\psi\rangle \in \mathcal{C}, \forall E_a, E_b \in \mathcal{E}. \quad (1.83)$$

One can always choose a new combination of errors $\{F_a\}$, $F_a = U_{ab}E_b$ which U is a unitary matrix, such that $\langle \psi | F_a^\dagger F_b | \psi \rangle = \omega_a \delta_{ab}$. It says that the sub-spaces we will get after the action of the errors are all orthogonal

$$F_a \mathcal{H}_C \perp F_b \mathcal{H}_C \quad a \neq b. \quad (1.84)$$

Thus, different errors correspond to orthogonal sub-spaces of the larger Hilbert space and therefore, there is a possibility to distinguish them by some measurements called error measurements. After recognizing which error happened, one can find the initial state via the appropriate procedure. Hence, to recover the correct quantum state, the error-correction procedure can be done in two steps:

1. *Error detection*: We perform a measurement, that tells us which error occurred on the quantum state.
2. *Recovery*: We use the result of the measurement to know which procedure to do to return the quantum state to the initial one.

We can see one simple case of error correction procedure in the example below.

Example : Consider the one to three qubit encoding. Suppose the only possible error that may happen is the action of X operator -the first one of Pauli matrices- on one qubit. Then, the set of errors is

$$\mathcal{E} = \{X_1, X_2, X_3\} \quad (1.85)$$

where $X_1 = X \otimes I \otimes I$, etc.

To detect and recover the errors, we can encode the $|0\rangle, |1\rangle$ as following:

$$|0\rangle \longrightarrow |0\rangle_C = |000\rangle \quad |1\rangle \longrightarrow |1\rangle_C = |111\rangle. \quad (1.86)$$

Therefore, the state of one qubit can be encoded as

$$|\psi\rangle = a|0\rangle + b|1\rangle \quad \longrightarrow \quad |\psi\rangle_C = a|000\rangle + b|111\rangle \quad (1.87)$$

for every values of a and b . In other words, the quantum error correcting code in our example is $\mathcal{H}_{code} = \text{span}\{|000\rangle, |111\rangle\}$.

In the end, we either have the encoding state or one of the noises of (1.85) occurred on it. Therefore, the quantum state is one of the states below

$$\{|\psi\rangle_C, X_1 |\psi\rangle_C, X_2 |\psi\rangle_C, X_3 |\psi\rangle_C\} \quad (1.88)$$

for any $|\psi\rangle_C \in \mathcal{H}_{code}$. It means the code subspace maps to orthogonal sub-spaces corresponding to different errors (Fig. 1.5).

We should search for some appropriate operators to detect which error happened by measuring them on the output state of the channel and be notified that it should be done in such a way that the state we have in hand does not change. In general, these operators are known as *error syndromes*. In our example, all of the states of the set (F.4) are the eigenvectors of the two operators $Z_1 Z_3$ and $Z_1 Z_2$ with different couples of eigenvalues (Tab.1.1). Hence, just by looking at the result of their measurements, we can find out in which subspace we are and by acting appropriate X_i undo the effects of error.

The example above is a simple case of a wide range of quantum error-correcting codes known as *stabilizer codes*. The theory of *stabilizer formalism* is used for the correction of

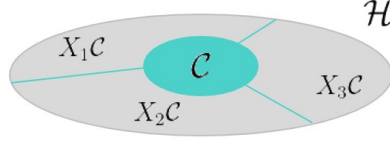


Figure 1.5: The decomposition of the Hilbert space to subspaces orthogonal to the code subspace.

	$ \psi\rangle_E$	$X_1 \psi\rangle_E$	$X_2 \psi\rangle_E$	$X_3 \psi\rangle_E$
Z_1Z_2	1	-1	-1	1
Z_1Z_3	1	-1	1	-1

Table 1.1: The results for measuring the error syndromes on different output states.

this useful class of codes in quantum information processing. Despite this, we should take into account that for some real-world quantum systems, it is very difficult to perform the quantum measurement we want for QEC. Thus, it is required to develop a general theory of QEC that is not necessarily done via a two-step detection-recovery method.

In the general theory of QEC, the noise model is described by a quantum channel \mathcal{E} , that the Kraus operators $\{A_i\}$ of the channel \mathcal{E} are corresponding to the set of the errors in (F.6). Here, the complete error correction procedure is done by another quantum channel \mathcal{R} called *Recovery Channel*. The code subspace can be corrected if we require for every state ρ whose support lies in the \mathcal{H}_{code}

$$\mathcal{R} \circ \mathcal{E}(\rho) = \rho \quad \forall \rho = P\rho P, \quad (1.89)$$

that P is the projection into the code subspace.

In the example above, we can model the errors with the quantum channel

$$\mathcal{E}(\rho) = q_0\rho + \sum_i q_i X_i \rho X_i \quad \sum_i q_i = 1, \quad (1.90)$$

which error X_i happens with probability q_i . The Kraus operators here are $A_i = X_i$. One can write the recovery channel for this model as

$$\mathcal{R}(\rho) = P_0 \rho P_0 + \sum_i X_i P_i \rho P_i X_i, \quad (1.91)$$

that P_i is projection on the subspace corresponding to error X_i .

One might be interested to consider the physical system instead of code subspace. In such a case, if we take $V : \mathcal{H}_{system} \rightarrow \mathcal{H}$ as the isometry that embeds the \mathcal{H}_{system} into \mathcal{H} , we can rewrite (E.11) as the following

$$\mathcal{R} \circ \mathcal{E}(V\rho V^\dagger) = V\rho V^\dagger \quad \forall \rho \in S(\mathcal{H}_{system}) \quad (1.92)$$

that is equivalent to having \mathcal{E}' and \mathcal{R}' such that $\mathcal{R}' \circ \mathcal{E}'(\rho) = \rho$ where $\mathcal{E}'(\cdot) = \mathcal{E}(V^\dagger(\cdot)V)$ and $\mathcal{R}'(\cdot) = V^\dagger \mathcal{R}(\cdot)V$ [105].

Here, it is good to note that in information theory, on top of error correction, another important problem is storing the information which is highly sensitive and important. This problem is referred to as the *secret sharing* problem. There is a strong connection between QEC and secret sharing so it is even suggested in [106] that secret sharing is a better classical analog to QEC. In a quantum secret sharing scheme which is shown as (n, k) , a secret quantum state is divided into n pieces such that any k or more pieces can perfectly reconstruct the secret quantum state and no $k - 1$ pieces reveal any information about the secret.

As an example, consider the case that three qutrits (the 3-level quantum system) are used to store a single-qutrit that is explained in [93, 107]. The quantum message $|\psi\rangle = \sum_{i=0}^2 \lambda_i |i\rangle$ is shared between three people as $|\tilde{\psi}\rangle = \sum_{i=0}^2 \lambda_i |\tilde{i}\rangle$ where

$$\begin{aligned} |0\rangle &\longrightarrow |\tilde{0}\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle) \\ |1\rangle &\longrightarrow |\tilde{1}\rangle = \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle) \\ |2\rangle &\longrightarrow |\tilde{2}\rangle = \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle). \end{aligned}$$

No matter what the encoded state is, the reduced density matrix of each qutrit is maximally mixed. Therefore, one person alone has no information about the message. On the other hand, any two people can reconstruct the secret as explained in [93]. Such a scheme can be interpreted as an error-correcting code that can protect against up to $(n - k)$ erasures, in other words, the set of errors is equivalent to erasures on any d single sites for $1 \leq d \leq n - k$ [107].

1.3.3 Generalized approach to QEC

The formulation we had in the previous section is called the *standard model for QEC*. It contains a triplet $(\mathcal{R}, \mathcal{E}, \mathcal{C})$ where \mathcal{C} is a subspace of a Hilbert space $\mathcal{H} = \mathcal{C} \oplus \mathcal{C}^\perp$ and \mathcal{R}, \mathcal{E} are quantum channels on $\mathcal{L}(\mathcal{H})$. The subspace \mathcal{C} is said to be correctable for \mathcal{E} and conserved by $\mathcal{R} \circ \mathcal{E}$.

The standard QEC has been formulated in the Schrodinger picture. It is known that any quantum mechanical system can be equivalently written in the Heisenberg picture. Although, in the quantum field theory is not always the case.

In order to write the theory of QEC in the Heisenberg picture, consider the case that the evolution of the system is described by the channel \mathcal{E} . After the evolution, the result of a measurement of an observable O will be in the form of $Tr(\mathcal{E}(\rho)O)$, where ρ describes the state of the system. As always, one can alternatively formulate the evolution of the system in the Heisenberg picture by requiring to get the same measurement results. For this purpose, we describe the evolution of the observables by the channel \mathcal{E}^* that is called Hilbert-Schmidt dual map or just *dual channel* which defined as

$$Tr(\rho \mathcal{E}^*(O)) = Tr(\mathcal{E}(\rho)O) \quad \forall \rho, O. \quad (1.93)$$

The set of Kraus operators for \mathcal{E}^* is given easily by cyclicity property of trace as $\{A_a^\dagger\}$ instead of $\{A_a\}$, and trace preservation of \mathcal{E} is equivalent to the requirement that \mathcal{E}^* is unital, $\mathcal{E}^*(I) = I$.

The conservation of a state by $\mathcal{R} \circ \mathcal{E}$ implies that in the Heisenberg picture for all the operators $O \in \mathcal{L}(\mathcal{H})$ we have

$$P(\mathcal{R} \circ \mathcal{E})^*(O)P = P\mathcal{E}^* \circ \mathcal{R}^*(O)P = POP. \quad (1.94)$$

Consider that observables can be characterized by a family of operators $\{X_a\}$. If for every X_a there exists Y_a such that $X_a = \mathcal{E}^*(Y_a)$, to correct for the errors induced by \mathcal{E} , we need the channel \mathcal{R} maps each X_a to one of the operators Y_a through $\mathcal{R}^*(X_a) = Y_a$, so that $(\mathcal{R} \circ \mathcal{E})^*(X_a) = (\mathcal{E}^* \circ \mathcal{R}^*)(X_a) = X_a$. In such a case, we will say that X_a is correctable for \mathcal{E} and conserved by $\mathcal{R} \circ \mathcal{E}$. Therefore, for a given noise model, the conservation of a state by a given noise model implies the conservation of all of its observables. One can think about a natural generalization of the theory of QEC by deducing this strong requirement. Instead of entire observables, we can assume the conservation of just a selecting set of them. Thus, for a noise model \mathcal{E} , we say that a set \mathcal{S} of operators on \mathcal{H} is correctable on states in the code subspace if there exists a channel \mathcal{R} such that

$$P(\mathcal{R} \circ \mathcal{E})^*(O)P = P\mathcal{E}^* \circ \mathcal{R}^*(O)P = POP, \quad \forall O \in \mathcal{S}. \quad (1.95)$$

Here, we shall continue to focus on the conservation of sets of operators that have the structure of an algebra where the result is a new theory referred to as *operator algebra quantum error correction* (OAQEC) [108, 109].

When we have the conservation of just an algebra \mathcal{A} in the code subspace, the information is not encoded into the entire subspace. It is just encoded into some parts of code induced from the algebra structure of \mathcal{A} . In finite-dimensional case, our algebra is the set of linear operators that are closed under addition, multiplication, and hermitian conjugation, that is finite-dimensional von-Neumann algebra. Given the algebra \mathcal{A} , the commutant of \mathcal{A} is another algebra on \mathcal{H} that is defined as

$$\mathcal{A}' = \{y \in \mathcal{L}(\mathcal{H}) | xy = yx, \forall x \in \mathcal{A}\}. \quad (1.96)$$

In order to study the error correction of an algebra, it is better to have a classification of them. To do this, first, one can focus on the simple algebras called factor. An algebra \mathcal{A} on \mathcal{H} is a factor if its center is trivial $Z_{\mathcal{A}} = \mathcal{A} \cap \mathcal{A}' = \lambda I$. For a given factor, there always exists a tensor factorization of the Hilbert space as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ such that $\mathcal{A} = \mathcal{L}(\mathcal{H}_A) \otimes I_{\bar{A}}$, which means each factor is just the set of all linear operators on a subsystem A of the system.

Consider the decomposition of the Hilbert space as $\mathcal{H} = \mathcal{C} \oplus \mathcal{C}^\perp = (\mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}) \oplus \mathcal{C}^\perp$. When we just require the correction of a factor, the generalization of the theory of QEC is known as *Operator Quantum Error Correction* (OQEC) which is a simple case of OAQEC. The standard QEC also corresponds to the special case when \bar{A} is one-dimensional. In the Schrodinger picture, the correctability of the observables on subsystem A is equivalent to require that information is encoded in a subsystem A of the code space and errors need only to be corrected up to a transformation on \bar{A} . It means there exists $\tau' \in \mathcal{S}(\mathcal{H}_{\bar{A}})$ such that $\mathcal{R} \circ \mathcal{E}(\rho \otimes \tau) = \rho \otimes \tau'$ for all $\rho \otimes \tau \in \mathcal{S}(\mathcal{C})$. ($\mathcal{S}(\mathcal{H})$ denote the set of density operators on \mathcal{H}) These are also known as *subsystem codes* in comparison with the standard quantum error correcting codes that are known as *subspace codes*. that encoded in the algebra $\mathcal{L}(\mathcal{H}_A) \otimes I_{\bar{A}}$ for the general A and \bar{A} .

Finally, let us consider the general case, where \mathcal{A} is not necessarily a factor. For an arbitrary algebra, there is always a block decomposition of the Hilbert space as $\mathcal{H} =$

$\mathcal{C} \oplus \mathcal{C}^\perp = \oplus_\alpha (\mathcal{H}_{A_\alpha} \otimes \mathcal{H}_{\bar{A}_\alpha}) \oplus \mathcal{C}^\perp$ which the algebra is block-diagonal with decomposition as $\mathcal{A} = \oplus_\alpha (\mathcal{L}_{\mathcal{H}_{A_\alpha}} \otimes I_{\bar{A}_\alpha}) \oplus 0_{\mathcal{C}^\perp}$. For the noise model \mathcal{E} , \mathcal{A} is correctable on the code subspace if there exists a channel \mathcal{R} such that for all $\rho_\alpha \in \mathcal{S}(\mathcal{H}_{A_\alpha})$, $\tau_\alpha \in \mathcal{S}(\mathcal{H}_{\bar{A}_\alpha})$ there exist some states $\tau'_\alpha \in \mathcal{S}(\mathcal{H}_{\bar{A}_\alpha})$ such that

$$\mathcal{R} \circ \mathcal{E}(\rho) = \mathcal{R} \circ \mathcal{E}\left(\sum_\alpha \lambda_\alpha \rho_\alpha \otimes \tau_\alpha\right) = \sum_\alpha \lambda_\alpha (\rho_\alpha \otimes \tau'_\alpha). \quad (1.97)$$

Here, each of the subsystems encodes the information individually and they all can be corrected simultaneously via the same recovery channel [110].

1.3.4 Universal recovery channel and the Petz map

A quantum channel \mathcal{E} is called reversible if one can find a *recovery channel* $\mathcal{R} : L(\mathcal{H}) \rightarrow L(\mathcal{H})$ such that

$$\mathcal{R} \circ \mathcal{E}(\rho) = \rho \quad \forall \rho \in \mathcal{S}(\mathcal{H}). \quad (1.98)$$

Most quantum channels, which correspond to open or noisy systems, are not reversible. We will return to the question of reversibility later in this subsection.

A quantum error correcting code corresponds to selecting an appropriate subspace, called *code subspace* (\mathcal{C} or \mathcal{H}_{code}) that has the same dimension as the system, of some larger Hilbert space. In the general theory of QEC, the noise model is described by a quantum channel \mathcal{E} . The code subspace can be corrected if we can find a recovery channel \mathcal{R} , such that for every state ρ whose support lies within \mathcal{H}_{code} , the channel can be reversed, i.e.

$$\mathcal{R} \circ \mathcal{E}(\rho) = \rho \quad \forall \rho = P\rho P \quad (1.99)$$

where P is the projection into the code subspace. One might be interested to consider a physical system instead of a code subspace. In such a case, if we take $V : \mathcal{H}_{system} \rightarrow \mathcal{H}$ as the isometry that embeds the \mathcal{H}_{system} into \mathcal{H} , we can rewrite (E.11) as the following

$$\mathcal{R} \circ \mathcal{E}(V\rho V^\dagger) = V\rho V^\dagger \quad \forall \rho \in \mathcal{S}(\mathcal{H}_{system}) \quad (1.100)$$

that is equivalent to having \mathcal{E}' and \mathcal{R}' such that $\mathcal{R}' \circ \mathcal{E}'(\rho) = \rho$ where $\mathcal{E}'(\cdot) = \mathcal{E}(V(\cdot)V^\dagger)$ and $\mathcal{R}'(\cdot) = V^\dagger \mathcal{R}(\cdot)V$ [105].

Given a quantum channel \mathcal{E} , it is useful to consider the Hilbert-Schmidt dual channel which defines a mapping of *observables* rather than of *states*. This is also sometimes referred to as the Heisenberg picture of the channel. The idea is to think of \mathcal{E} as a (discrete) evolution of a state. After the evolution, the result of a measurement of an observable O will be in the form of $\text{tr}(\mathcal{E}(\rho)O)$, where ρ describes the state of the system. As we usually do when going to the Heisenberg picture, we can alternatively formulate the evolution of the system by transforming the operators, requiring to get the same measurement results. For this purpose, we describe the evolution of the observables by the channel \mathcal{E}^* that is called *Hilbert-Schmidt dual map* defined as

$$\text{tr}(\rho \mathcal{E}^*(O)) = \text{tr}(\mathcal{E}(\rho)O) \quad \forall \rho, O. \quad (1.101)$$

The set of Kraus operators for \mathcal{E}^* is given easily by cyclicity property of trace as $\{A_a^\dagger\}$ instead of $\{A_a\}$, and trace preservation of \mathcal{E} is equivalent to the requirement that \mathcal{E}^* is

unital, $\mathcal{E}^*(I) = I$. In the case of QEC, the conservation of a state by $\mathcal{R} \circ \mathcal{E}$ (E.11) implies that in the Heisenberg picture for all the operators $O \in \mathcal{L}(\mathcal{H})$ we have

$$P(\mathcal{R} \circ \mathcal{E})^*(O)P = P\mathcal{E}^* \circ \mathcal{R}^*(O)P = POP. \quad (1.102)$$

We now return to the general question of the reversibility of a quantum channel. This has been studied widely in [18, 19, 111, 112]. The reversibility of \mathcal{E} is related to the quantum relative entropy of states under the action of \mathcal{E} . The relative entropy between two states ρ and σ is defined as $S(\rho||\sigma) = \text{tr}(\rho \log \rho - \rho \log \sigma)$ and it is a measure of distinguishability between two quantum states. The most important theorem related to this quantity known as *monotonicity of relative entropy* or the *data processing inequality*, whose proof is discussed in appendix ?? for finite dimensions, states that $S(\rho||\sigma)$ is non-increasing under the action of any quantum channel \mathcal{E} [113, 114], i.e.,

$$S(\rho||\sigma) \geq S(\mathcal{E}(\rho)||\mathcal{E}(\sigma)). \quad (1.103)$$

It has been shown in [115, 116] that there exists a quantum channel \mathcal{R} such that for all states $\rho \in S(\mathcal{H})$, $\mathcal{R} \circ \mathcal{E}(\rho) = \rho$ if and only if $S(\rho||\sigma) = S(\mathcal{E}(\rho)||\mathcal{E}(\sigma))$ for all $\rho, \sigma \in S(\mathcal{H})$. Moreover, the explicit form of the quantum channel \mathcal{R} for the set of states $\{\mathcal{E}(\rho)|\forall \rho \in S(\mathcal{H})\}$ has been found in [116]. It is given as a function of a reference quantum state $\sigma \in S(\mathcal{H}_A)$ and the channel \mathcal{E} itself as

$$\mathcal{R}(\cdot) = \mathcal{P}_{\sigma, \mathcal{E}}(\cdot) = \sigma^{1/2} \mathcal{E}^*(\mathcal{E}(\sigma)^{-1/2}(\cdot)\mathcal{E}(\sigma)^{-1/2})\sigma^{1/2} \quad (1.104)$$

where \mathcal{E}^* is the dual channel of \mathcal{E} . $\mathcal{P}_{\sigma, \mathcal{E}}$ is known as *Petz recovery channel*. This result has been also independently obtained by Barnum and Knill in [117].

1.3.5 Petz map and modular theory

In order to study the recovery of information in quantum field theories, it would be really helpful to have an alternative description for the Petz recovery channel in the context of the Tomita-Takesaki theory (A brief review can be found in the Appendix A). Luckily, the Petz recovery channel indeed has its origin in the study of operator algebras [18, 19, 118]. It has also been studied in some recent works [119–121]. We will now review the definition of the Petz map in the algebraic approach. All the discussion below is in the Heisenberg picture and it is almost based on [122].

Consider two Type I von Neumann algebras in their standard forms: $(\mathcal{A}, \mathcal{H}_A, J_A, \mathcal{P}_A)$ and $(\mathcal{B}, \mathcal{H}_B, J_B, \mathcal{P}_B)$. Assume two faithful states ρ_A and ρ_B respectively on \mathcal{A} and \mathcal{B} that we will refer to their unique vector representations by $|\rho_A^{1/2}\rangle \in \mathcal{P}_A$ and $|\rho_B^{1/2}\rangle \in \mathcal{P}_B$. The corresponding GNS Hilbert spaces of the algebras over the states $|\rho_A^{1/2}\rangle$ and $|\rho_B^{1/2}\rangle$ are denoted by \mathcal{H}_A and \mathcal{H}_B .

Let us consider a linear superoperator $\mathcal{T} : \mathcal{A} \rightarrow \mathcal{B}$ and denote its corresponding operator between the corresponding GNS Hilbert spaces by $T : \mathcal{H}_A \rightarrow \mathcal{H}_B$. One can define a dual of it $\mathcal{T}_\rho^* : \mathcal{B} \rightarrow \mathcal{A}$ as a solution to

$$\langle b|\mathcal{T}(a)\rangle_{\rho_B} = \langle b|Ta\rangle_{\rho_B} = \langle T^\dagger b|a\rangle_{\rho_A} = \langle \mathcal{T}_\rho^*(b)|a\rangle_{\rho_A} \quad (1.105)$$

in the GNS Hilbert space (A.9) for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. In the case of matrix algebra, the definition (1.105) can be rewritten as

$$\text{tr}(\rho_B b^\dagger \mathcal{T}(a)) = \text{tr}(\rho_A \mathcal{T}_\rho^*(b^\dagger) a). \quad (1.106)$$

We note it here that if we replace both ρ_A and ρ_B with the identity operators (unnormalized maximally mixed states), the dual map \mathcal{T}^* we will get is the usual dual map in the quantum information theory defined in (??). One can find \mathcal{T}_ρ^* in (1.106) in terms of \mathcal{T}^* as

$$\mathcal{T}_\rho^*(b) = \rho_A^{-1} \mathcal{T}^*(\rho_B b). \quad (1.107)$$

On the other hand, the GNS Hilbert space \mathcal{H}_A can be created by acting with the commutant of the algebra $\mathcal{H}_A = \{\mathcal{A}' | \rho_A^{1/2}\}$ and the same for another algebra $\mathcal{H}_B = \{\mathcal{B}' | \rho_B^{1/2}\}$. Therefore, given T between the GNS Hilbert spaces, we can in principle associate to it one superoperator between the commutants $\mathcal{T}'_\rho : \mathcal{B}' \rightarrow \mathcal{A}'$ which defined as

$$\langle b' | \mathcal{T}(a) \rangle_{\rho_B} = \langle \mathcal{T}'_\rho(b') | a \rangle_{\rho_A} \quad (1.108)$$

for all $a \in \mathcal{A}$ and $b' \in \mathcal{B}'$ that is called ρ -dual of the superoperator \mathcal{T} . Then, one can use the modular conjugations in both Hilbert spaces to define a superoperator between the original algebras $\mathcal{T}_\rho^P : \mathcal{B} \rightarrow \mathcal{A}$ as

$$\mathcal{T}_\rho^P(\cdot) = \mathcal{J}_A \circ \mathcal{T}'_\rho \circ \mathcal{J}_B(\cdot) = J_A \mathcal{T}'_\rho (J_B(\cdot) J_B) J_A \quad (1.109)$$

which is exactly the Petz dual map we are interested in that for the Type I von Neumann algebra, one can explicitly find its form as

$$\mathcal{T}_\rho^P(\cdot) = \rho_A^{-1/2} \mathcal{T}(\rho_B^{1/2}(\cdot) \rho_B^{1/2}) \rho_A^{-1/2}. \quad (1.110)$$

The Petz dual map can also be realized as the solution to the relation (1.105) with respect to the KMS inner product. If the state $|\Psi\rangle$ is cyclic and separating for a von Neumann algebra \mathcal{A} , the KMS inner product on \mathcal{A} is defined as

$$\langle a_1 | a_2 \rangle_{\psi, \text{KMS}} = \langle \mathcal{J}_\psi(a_1^\dagger) | a_2 \rangle_\psi = \langle \Psi | a_1^\dagger \Delta_\psi^{1/2} a_2 | \Psi \rangle. \quad (1.111)$$

while the last expression can be found using (A.21). In the case of matrix algebra, it is reduced to $\langle a_1 | a_2 \rangle_\rho = \text{tr}(\rho^{1/2} a_1^\dagger \rho^{1/2} a_2)$.

While the definition (1.110) is only for Type I von Neumann algebras, the one in (1.109) can be generalized to the mapping between general von Neumann algebras. In particular, it is helpful for high-energy physics applications where the von Neumann algebras under consideration are of Type III₁.

To summarize, consider $(\mathcal{A}, \mathcal{H}_A, J_A, \mathcal{P}_A)$, $(\mathcal{B}, \mathcal{H}_B, J_B, \mathcal{P}_B)$ and let $\mathcal{T} : \mathcal{A} \rightarrow \mathcal{B}$ be a unital completely positive map between the algebras. One can choose an arbitrary state $\rho_B \in \mathcal{S}(\mathcal{B})$ and if both ρ_B and $\rho_A = \mathcal{T}^*(\rho_B)$ are faithful states on the corresponding von Neumann algebras, construct the Hilbert space representation of the algebras over them. Then, if for all $\rho, \sigma \in \mathcal{S}(\mathcal{B})$ we have

$$S(\rho | \sigma) = S(\mathcal{T}^*(\rho) | \mathcal{T}^*(\sigma)), \quad (1.112)$$

there exists a unital completely positive map $\tilde{\mathcal{T}} : \mathcal{B} \rightarrow \mathcal{A}$ that $\mathcal{T} \circ \tilde{\mathcal{T}}$ acts as an identity operator on \mathcal{H}_B . The $\tilde{\mathcal{T}}$ is nothing but the Petz dual map given in (1.109) and can be also shown that $(\tilde{\mathcal{T}})^* = \mathcal{P}_{\rho_B, \mathcal{T}^*}$ in (1.104). In [120, 121], one can find the discussion for generalization of the Petz dual map in cases where the states are not faithful.

1.3.6 Approximate recoverability

The equality of relative entropy is a necessary and sufficient condition for exact recoverability. In the case of approximate quantum error correction, the quality of recovery is controlled by the behavior of the relative entropy under the action of the quantum channel. At the heart of this result is a strengthened version of the monotonicity of the relative entropy which undergoes a slight change in relative entropy through the channel provides the approximate recoverability of the states.

For a given channel \mathcal{E} , Junge et al [123] found an expression for the recovery channel $\mathcal{R}_{\rho,\mathcal{E}}$, which is closely related to Petz recovery channel and it is also universal. In terms of \mathcal{E} and an arbitrary full rank density matrix ρ , it is given by

$$\mathcal{R}_{\rho,\mathcal{E}}(\cdot) = \int_{-\infty}^{\infty} dt p(t) \rho^{-it} \mathcal{P}_{\rho,\mathcal{E}}(\mathcal{E}(\rho)^{it} (\cdot) \mathcal{E}(\rho)^{-it}) \rho^{it} \quad (1.113)$$

while $p(t) = \pi/(\cosh(2\pi t) + 1)$ and $\mathcal{P}_{\rho,\mathcal{E}}$ is the Petz recovery channel given in (1.104). Moreover, they gave a lower bound on the difference between the relative entropy in terms of the fidelity between the original state and the recovered one as

$$S(\rho|\sigma) - S(\mathcal{E}(\rho)|\mathcal{E}(\sigma)) \geq -2 \log F(\rho, \mathcal{R}_{\sigma,\mathcal{E}} \circ \mathcal{E}(\rho)). \quad (1.114)$$

The result above has been found in the context of the quantum information theory *i.e.*, Type I von Neumann algebra. While one should go beyond it in the case of QFTs and gravity. In [120, 121], authors generalized the previous result to quantum channels between general von Neumann algebras in the context of modular theory. In the Heisenberg picture, consider $\mathcal{T} : \mathcal{A} \rightarrow \mathcal{B}$ be a unital, normal and two-positive map between von Neumann algebras.

One can associate a dual Petz map $\mathcal{T}_{\psi}^P : \mathcal{B} \rightarrow \mathcal{A}$ (1.109). They found the recovery map in the form of

$$\alpha(\cdot) = \int_{-\infty}^{\infty} dt p(t) \alpha_{\psi,\mathcal{T}}^t(\cdot) \quad (1.115)$$

while

$$\alpha_{\psi,\mathcal{T}}^t(\cdot) = \zeta_{\psi,\mathcal{A}}^t \circ \mathcal{T}_{\psi}^P \circ \zeta_{\psi,\mathcal{B}}^{-t}(\cdot) \quad (1.116)$$

is called the *rotated Petz map*, and $\zeta_{\psi,\mathcal{A}}^t$ is the modular flow for $|\Psi_{\mathcal{A}}\rangle$, \mathcal{A} is define as

$$\zeta_{\psi,\mathcal{A}}^t(a) = Ad \Delta_{\psi,\mathcal{A}}^{it}(a) = \Delta_{\psi,\mathcal{A}}^{it}(a) \Delta_{\psi,\mathcal{A}}^{-it} \quad \forall a \in \mathcal{A}. \quad (1.117)$$

In the case of finite-dimensional Type I factor, the recovery map in (1.115) reduced to the dual of the recovery channel in (1.113).

1.4 Entanglement wedge reconstruction via universal recovery channels

In this section, we review the arguments of [17], on how the Petz map can be used to reconstruct bulk operators in the *entanglement wedge* of a boundary subregion.

1.4.1 Background

Before we proceed we introduce an ingredient that will be useful in what follows. This is the idea of a *code subspace* around a given state. For example, starting with the global AdS vacuum state $|\Omega\rangle$ we define

$$\mathcal{H}_C = \text{span}\{|\Omega\rangle, \phi_i(x)|\Omega\rangle, \dots, \phi_i(x_1)\phi_j(x_2)|\Omega\rangle, \dots\}, \quad (1.118)$$

where the range of i and the number of ϕ insertions are finite. More generally we can define the code subspace around any semi-classical state. This subspace is the one where low-energy experiments in the bulk can be described and we will study bulk reconstruction within a given code subspace.

The entanglement wedge of a boundary region A is defined as the bulk domain of dependence of any bulk spacelike surface whose boundary is the union of A and the codimension two extremal area surface of minimal area (more precisely, quantum extremal surface) whose boundary is ∂A . It is generally believed that bulk operators inside the entanglement wedge can be reconstructed by operators in the region A on the boundary.

An important ingredient supporting this is the observation of JLMS [16] that the relative entropy of two states in the boundary region A is equal to the relative entropy of the two corresponding bulk states in \mathcal{E}_A up to subleading correction.

$$S(\rho_A||\sigma_A) = S(\rho_a||\sigma_a) + O(1/N) \quad (1.119)$$

which already suggests that information in the entanglement wedge is contained in region A on the boundary. Using (1.119) arguments in favor of entanglement wedge reconstruction were given in [20].

Assume that the bulk Hilbert space has a decomposition as $\mathcal{H}_{bulk} = \mathcal{H}_a \otimes \mathcal{H}_{\bar{a}}$, while $a = \mathcal{E}_A$. For the cases where the setup is symmetric, like the vacuum sector of the system, the complement region of a in the bulk is also the entanglement wedge of the region \bar{A} in the boundary, $\bar{a} = \mathcal{E}_{\bar{A}}$, so the same argument applies for \bar{A} and \bar{a} . In general, the entanglement wedge of a given boundary region A can be bigger than its causal wedge. Finally, the entanglement wedge reconstruction is a statement that says any bulk operator ϕ_a acting within \mathcal{H}_a can always be represented in the CFT as an operator O_A has support only on \mathcal{H}_A .

1.4.2 Entanglement wedge reconstruction with a universal recovery channel

We now discuss entanglement wedge reconstruction in terms of the universal recovery channels described in Sec. 1.3.4, based on [17].

First, consider the entanglement wedge reconstruction and for simplicity assume both bulk and CFT Hilbert spaces have a tensor decomposition as $\mathcal{H}_{bulk} = \mathcal{H}_a \otimes \mathcal{H}_{\bar{a}}$ and $\mathcal{H}_{CFT} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$. At large N when the equality between the relative entropy of the states in the entanglement wedge and the boundary region A is exact, i.e.,

$$S(\rho_A||\sigma_A) = S(\rho_a||\sigma_a) \quad (1.120)$$

from the discussion in Sec. 1.3.4, one can say that there exists a quantum channel \mathcal{R} which recovers the information in the entanglement wedge from the boundary region A . Using the dual channel \mathcal{R}^* we can map operators on \mathcal{H}_a to operators on \mathcal{H}_A as $O_A = \mathcal{R}^*(\phi_a)$.

If we assume that there is no black hole in the bulk, the global HKLL reconstruction reviewed in section 2 provides us a map from states of the entire bulk to states of the entire boundary. We can therefore define an isometry of embedding V_{HKLL} that embeds the bulk effective field theory Hilbert space to the CFT Hilbert space $V_{HKLL} : \mathcal{H}_{bulk} \hookrightarrow \mathcal{H}_{CFT}$, which $\mathcal{H}_{code} = V_{HKLL} \mathcal{H}_{bulk} V_{HKLL}^\dagger$.

We now define a quantum channel $\mathcal{E} : S(\mathcal{H}_a) \rightarrow S(\mathcal{H}_A)$. Here $S(\mathcal{H}_a)$ denotes the set of possible density matrices in the bulk region a while $S(\mathcal{H}_A)$ is the set of density matrices in the boundary region A . As the entire AdS space is a closed system, the noise model $\mathcal{E} : S(\mathcal{H}_a) \rightarrow S(\mathcal{H}_A)$ can be written in terms of a model environment (3.57) using the global HKLL map. We take the complementary bulk region \bar{a} in a fixed reference state $\sigma_{\bar{a}}$ and then, we can write the quantum channel \mathcal{E} as

$$\mathcal{E}(\cdot) = \text{tr}_{\bar{A}}(V_{HKLL}(\cdot \otimes \sigma_{\bar{a}})V_{HKLL}^\dagger). \quad (1.121)$$

To map the operators, as we had in Sec. 1.3.4, one can go to the Heisenberg picture and write the dual of Petz recovery channel of \mathcal{E} by taking a fixed reduced density matrix on the entanglement wedge σ_a , using expression (1.104), we reach to

$$O_A = \mathcal{R}^*(\phi_a) = \mathcal{E}(\sigma_a)^{-1/2} \mathcal{E}(\sigma_a^{1/2} \phi_a \sigma_a^{1/2}) \mathcal{E}(\sigma_a)^{-1/2}, \quad (1.122)$$

which for the quantum channel (1.121), it will give us

$$O_A = \mathcal{E}(\sigma_a)^{-1/2} \text{tr}_{\bar{A}}(V_{HKLL}(\sigma_a^{1/2} \otimes \sigma_{\bar{a}}^{1/2})(\phi_a \otimes I_{\bar{a}})(\sigma_a^{1/2} \otimes \sigma_{\bar{a}}^{1/2})V_{HKLL}^\dagger) \mathcal{E}(\sigma_a)^{-1/2}, \quad (1.123)$$

where $\mathcal{E}(\sigma_a) = \text{tr}_{\bar{A}}(V_{HKLL}(\sigma_a \otimes \sigma_{\bar{a}})V_{HKLL}^\dagger)$. If we take both σ_a and $\sigma_{\bar{a}}$ two maximally mixed states or equivalently putting the bulk in the maximally mixed state τ , the map (1.123) will be simplified as

$$O_A = \frac{1}{d_{code}} \tau_A^{-1/2} \text{tr}_{\bar{A}}(V_{HKLL}(\phi_a)V_{HKLL}^\dagger) \tau_A^{-1/2}, \quad (1.124)$$

where $\tau_A = \frac{1}{d_{code}} \text{tr}_{\bar{A}} P_{code}$. It is good to note here that the condition

$$\langle \phi_a \rangle_{\rho_{bulk}} = \langle \Phi_{a,HKLL} \rangle_{\rho_{CFT}} \quad (1.125)$$

implies that $V_{HKLL}(\phi_a)V_{HKLL}^\dagger = P_{code} \Phi_{a,HKLL} P_{code}$, and so the bulk operator ϕ in the entanglement wedge can map to a boundary operator that has support only in the region A as

$$O_A = \frac{1}{d_{code}} \tau_A^{-1/2} \text{tr}_{\bar{A}}(P_{code} \Phi_{HKLL} P_{code}) \tau_A^{-1/2}. \quad (1.126)$$

This is the main result of the section, that we will use in the rest of the paper.

When including $1/N$ corrections, (3.66) will no longer be exact and we do not expect to have an exact recoverability. In that case, we can try to reconstruct the entanglement wedge using the twirled Petz map (F.1). For the maximally mixed state in the code subspace, the mapping is as below

$$O_A = \frac{1}{d_{code}} \int_{\mathbb{R}} dt \beta_0(t) \tau_A^{-1/2(1-it)} \text{tr}_{\bar{A}}(V_{HKLL}(\phi_a)V_{HKLL}^\dagger) \tau_A^{-1/2(1+it)} \quad (1.127)$$

which at large N limit gives us the same formula as (1.124) [17]. It has been argued that for the reconstruction of the entanglement wedge for any finite-dimensional code subspace

as well as code subspaces with dimensions that do not grow exponentially fast in N , while the error is non perturbatively small, the ordinary Petz map works well enough [124].

In the large N limit it is possible to take the size of the code subspace to infinity. In that case, the maximally mixed state on the code subspace does not really exist and we would need to use some regulated version of it² that we denote by ρ .

One should be careful at this point that the quantum channel in (1.121), which takes as input the reduced density matrix of the entanglement wedge $\rho_a = \text{tr}_{\bar{a}}(\rho)$ and gives as output a state on A , will not generally provide us exactly the same state on A as $\rho_A = \text{tr}_{\bar{A}}(V_{HKLL}(\rho)V_{HKLL}^\dagger)$ which depends on the state ρ defined on the entire bulk. Only in the case that the bulk reference state itself is a tensor factor of two states in a and \bar{a} , like the maximally mixed state, they will give us the same result. However, their difference is controlled by $1/N$: if we say that $|S(\rho_A||\sigma_A) - S(\rho_a||\sigma_a)| \leq \epsilon$, then

$$\|\mathcal{E}(\rho_a) - \rho_A\|^2 \leq 2 \ln 2 S(\mathcal{E}(\rho_a)||\rho_A) \leq (2 \ln 2)\epsilon. \quad (1.128)$$

Hence, at large N limit that ϵ goes to zero and we have the exact reconstruction of the entanglement wedge, one can exchange the $\mathcal{E}(\rho_a)$ and ρ_A . Then, we can introduce a general version of the Petz map in terms of an arbitrary fixed state ρ as [125]

$$O_A^{(\rho)} = \rho_A^{-1/2} \text{tr}_{\bar{A}}(V_{HKLL}(\rho^{1/2}\phi_a\rho^{1/2})V_{HKLL}^\dagger)\rho_A^{-1/2}. \quad (1.129)$$

We note here that, for this reconstruction, the only source of the error is not the $1/N$ correction, but rather the entanglement in the state ρ between the inside and outside of the entanglement wedge causes to not recover the original state.

1.5 AdS-Rindler reconstruction and Petz map

As we saw in the previous chapters, a free scalar field in pure AdS is dual to a GFF of the boundary that can be thought of as a sector of a much larger CFT with a large central charge. In addition, Petz map is a tool that comes from the quantum information theory which provides us the CFT representation of the bulk field $\phi(X)$ that is localized in any region A when the field lies in the entanglement wedge of A . It is given by

$$\Phi_A(X) = \tau_A^{-1/2} \text{tr}_{\bar{A}}(P_{code}\Phi_{HKLL}(X)P_{code})\tau_A^{-1/2} \quad (1.130)$$

where we redefine τ_A to the unnormalized maximally mixed state $\tau_A = \text{tr}_{\bar{A}}P_{code}$ and $\Phi_{HKLL}(X)$ is the boundary reconstruction of the field in global coordinates

$$\Phi_{HKLL}(X) = \int_{bdy} dt d\Omega K^g(X|t, \Omega) O(t, \Omega) \quad (1.131)$$

that $K^g(X|t, \Omega)$ is the smearing function for the AdS space which in even and odd dimension given by (1.33) and (1.34) respectively. By plugging (1.131) back into (1.130) and considering the linearity of the trace we will get

$$\Phi_A(X) = \int_{bdy} dt d\Omega K^g(X|t, \Omega) \tau_A^{-1/2} \text{tr}_{\bar{A}}(P_{code}O(t, \Omega)P_{code})\tau_A^{-1/2}. \quad (1.132)$$

²For example, this could be a thermal state, which approximates the maximally mixed state as $T \rightarrow \infty$.

Therefore, to find $\Phi_A(X)$ we need to deal with terms

$$\text{tr}_{\bar{A}}(P_{code}O(t, \Omega)P_{code}) \quad (1.133)$$

for every $O(t, \Omega)$. In order to take trace over \bar{A} , we need to re-express them in terms of the operators that act just on the Cauchy surface Σ . In other words, we should use the Heisenberg picture and rewrite all $O(t, \Omega)$ in terms of the scalar primaries on Σ by evolving them with boundary Hamiltonian. Let us consider Σ to be $t = 0$ slice. Then, (1.133) can be read off as

$$\text{tr}_{\bar{A}}(P_{code}e^{iH_{CFT}t}O(\Omega)e^{-iH_{CFT}t}P_{code}) = \text{tr}_{\bar{A}}(P_{code}e^{iH_{GFF}t}O(\Omega)e^{-iH_{GFF}t}P_{code}). \quad (1.134)$$

Since we project the Heisenberg picture operators on the code subspace, which should be a subspace of the GFF sector of the CFT, the CFT Hamiltonian can be replaced by the Hamiltonian of generalized free theory, which is

$$H_{GFF} = \sum_{nlm} \omega_{nlm} O_{nlm}^\dagger O_{nlm}. \quad (1.135)$$

It is important to note that all the operators in (1.134) have support on entire Σ , even when $t = 0$ and $O(x_A)$ is localized in region A , $P_{code}O(x_A)P_{code}$ still can have support on \bar{A} .

To do the calculation, it can be more convenient to go to Fourier space. By substituting (1.14) into (1.134) and plugging it back to (1.132) we arrive to

$$\begin{aligned} \Phi_A(X) = \sum_{nlm} G_{nlm}(X) \tau_A^{-1/2} \text{tr}_{\bar{A}}(P_{code}O_{nlm}P_{code}) \tau_A^{-1/2} \\ + G_{nlm}^*(X) \tau_A^{-1/2} \text{tr}_{\bar{A}}(P_{code}O_{nlm}^\dagger P_{code}) \tau_A^{-1/2} \end{aligned} \quad (1.136)$$

where

$$G_{nlm}(X) = \int_{bdy} dt d\Omega K^g(X|t, \Omega) g_{nlm}(t, \Omega). \quad (1.137)$$

To go ahead, we need to determine more precisely the setup we want to study and in particular, specify the region A on the boundary. Let us start with a simple case. Take A to be just one hemisphere of Σ , then as a result, $a = \mathcal{E}_A$ is an AdS-Rindler wedge in the bulk which its entanglement wedge coincides with its causal wedge. In the rest of this chapter, we will focus on finding the boundary representation for the operators that lie in the AdS-Rindler wedge. First, we work on rewriting the operator O_{nlm} in terms of the operators that act just on A or \bar{A} . We will then define the code subspace in this case and in particular, we will try to find a suitable choice of basis for code subspace to be able to do the calculation. Finally, we will compare our result with the boundary representation of the field one can find from the HKLL procedure in the AdS-Rindler coordinates.

1.5.1 Bogoliubov coefficients from Rindler mode expansion of bulk field

Now we proceed with the second way of arriving at (??). At every Cauchy surface, the Hilbert space of a QFT is constructed as the Fock space obtained from creation and annihilation operators a_k and a_k^\dagger , corresponding to the global modes of the field operator which is

$$\phi(t, x) = \sum_k f_k(t, x) a_k + f_k^*(t, x) a_k^\dagger. \quad (1.138)$$

k is a collection of indices we need to describe the mode. We can use the same approach to find the mode expansion of the field that lies in the region r by directly solving the equation of motion just in this region to find the appropriate wave functions which have support only on r . Let us take the time slice Σ and decompose it into the subregions Σ_r such that $\Sigma_r \cap \Sigma_{r'} = \emptyset$. For all Σ_r , we should first find a coordinate system U_r which cover $\mathcal{D}(\Sigma_r)$. Then, solve the equation of motion on U_r to find the mode expansion of fields on $\mathcal{D}(\Sigma_r)$

$$\phi(t_r, x_r) = \sum_k f_k^r(t_r, x_r) a_k^r + f_k^{r*}(t_r, x_r) a_k^{r\dagger}. \quad (1.139)$$

The Hilbert space of the QFT restricted to Σ_r is denoted by \mathcal{H}_r and the Hilbert space of the total theory on Σ is naively a tensor product of the subregion Hilbert spaces $\mathcal{H} = \otimes_r \mathcal{H}_r$.

One can expand the field $\phi(X)$ in global coordinates in terms of subregion mode functions as

$$\phi(t, x) = \sum_r \sum_k f_k^r(t_r, x_r) a_k^r + f_k^{r*}(t_r, x_r) a_k^{r\dagger}. \quad (1.140)$$

The point X is labeled in global coordinates and the coordinate system U_r by (t, x) and (t_r, x_r) , respectively. As a result, the creation and annihilation operators of the full Hilbert space can be written as a linear combination of subregions mode functions and vice versa, by comparing (1.138) and (1.140) which is a generalized version of Bogoliubov transformation [126].

Let us come back to our problem. To proceed in the Petz map calculation, it can help us if we could find an expression for O_{nlm} in terms of the mode function corresponding to the subregions A and \bar{A} . The subtlety here is the point that GFF on the boundary do not obey the equation of motion and hence, the discussion above is not applied to the boundary QFT. However, in AdS/CFT correspondence, the extrapolate dictionary leads us to the identification between some bulk and boundary operators. As a result, we expect that bulk Bogoliubov transformation can help us to find one expression for O_{nlm} as a linear combination of the operators has support only on one subregion.

The boundary Cauchy slice Σ is divided into two hemispheres A and \bar{A} . As a result, their entanglement wedges are AdS-Rindler patches in the bulk which both together cover the entire AdS space. To quantize the free fields in the entire AdS space in Rindler coordinates, we need two copies of the creation and annihilation operators that obey the commutation relation

$$[b_{\omega\lambda, I}, b_{\omega'\lambda', I'}^\dagger] = (2\pi)^2 \delta(\omega - \omega') \delta(\lambda - \lambda') \delta_{II'} \quad (1.141)$$

where the mode functions $b_{\omega\lambda, a}$ and $b_{\omega\lambda, \bar{a}}$ have support only in a and \bar{a} patches respectively.

One can globally expand the bulk field $\phi(X)$ in terms of these mode functions as

$$\phi(X) = \sum_{I \in \{a, \bar{a}\}} \int \frac{d\omega}{2\pi} \frac{d\lambda}{2\pi} \left(f_{\omega\lambda, I}(X) b_{\omega\lambda, I} + f_{\omega\lambda, I}^*(X) b_{\omega\lambda, I}^\dagger \right) \quad (1.142)$$

where $f_{\omega\lambda, I}(X)$ is given by (1.38) if the point X belongs to the patch I , otherwise it vanishes. The global mode a_{nlm} in AdS are related to these mode functions by Bogoliubov coefficients α and β as

$$a_{nlm} = \sum_{I \in \{a, \bar{a}\}} \int d\omega d\lambda \left(\alpha_{nlm; \omega\lambda}^I b_{\omega\lambda, I} + \beta_{nlm; \omega\lambda}^{*I} b_{\omega\lambda, I}^\dagger \right). \quad (1.143)$$

The commutation relations (1.141) lead to the following constrain on the Bogoliubov coefficients

$$\sum_{I \in \{a, \bar{a}\}} \int d\omega d\lambda \left(\alpha_{nlm; \omega\lambda}^I \alpha_{n'l'm'; \omega'\lambda'}^{*I} - \beta_{nlm; \omega\lambda}^{*I} \beta_{n'l'm'; \omega'\lambda'}^I \right) = \delta_{nn'} \delta_{ll'} \delta_{mm'}. \quad (1.144)$$

We can substitute (1.143) in the bulk global mode expansion (1.7) which lead us to the relations

$$\begin{aligned} \sum_{nlm} f_{nlm}(X) \alpha_{nlm; \omega\lambda}^a + f_{nlm}^*(X) \beta_{nlm; \omega\lambda}^a &= 0 \quad \forall X \in \bar{a} \\ \sum_{nlm} f_{nlm}(X) \alpha_{nlm; \omega\lambda}^{\bar{a}} + f_{nlm}^*(X) \beta_{nlm; \omega\lambda}^{\bar{a}} &= 0 \quad \forall X \in a. \end{aligned} \quad (1.145)$$

We will use them in what follows.

For the case that we are studying, where on the boundary of pure AdS the GFF lives, the mode functions a_{nlm} and $b_{\omega\lambda}$ are identified with the boundary operators given by (1.13) and (1.42) respectively. By plugging them back into (1.143), one can find

$$O_{nlm} = \sum_{I \in \{A, \bar{A}\}} \int d\omega d\lambda \left(\frac{M_{nlm}}{M_{\omega\lambda}} \alpha_{nlm; \omega\lambda}^I O_{\omega\lambda, I} + \frac{M_{nlm}}{M_{\omega\lambda}} \beta_{nlm; \omega\lambda}^{*I} O_{\omega\lambda, I}^\dagger \right). \quad (1.146)$$

while $\alpha_{nlm; \omega\lambda}^A = \alpha_{nlm; \omega\lambda}^a$, etc .

1.5.2 Appropriate basis for the code subspace

The code subspace has a Fock space structure $\mathcal{H}_{code} = span\{\prod_{nlm} (O_{nlm}^\dagger)^{i_{nlm}} |\Omega\rangle\}$, where $|\Omega\rangle$ is the global vacuum defined as $O_{nlm} |\Omega\rangle = 0$ for all n, l and m . The powers i_{nlm} are some non-negative integers and we can also put a cut-off on them. In order to compute the Petz map reconstruction of the bulk field $\phi(X)$ that lies in the AdS-Rindler patch, we need to compute the terms $\text{tr}_{\bar{A}} P_{code}$ and $\text{tr}_{\bar{A}} (P_{code} O_{nlm} P_{code})$.

Before going through the calculation, we need to choose a basis for code subspace. The natural choice one can take is

$$|\{i_{nlm}\}\rangle \propto \prod_{nlm} (O_{nlm}^\dagger)^{i_{nlm}} |\Omega\rangle. \quad (1.147)$$

In this basis, we should calculate the terms of the form

$$\text{tr}_{\bar{A}} \left((O_{nlm}^\dagger)^i |\Omega\rangle \langle \Omega| (O_{n'l'm'})^{i'} \right) \quad (1.148)$$

for every arbitrary integers i and i' . One way to deal with trace can be using (1.146). As we know, the action of Rindler modes on $|\Omega\rangle$ in two wedges are related to each other by

$$\begin{aligned} O_{\omega, \lambda; \bar{A}} |\Omega\rangle &= e^{\pi\omega} O_{\omega, -\lambda; A}^\dagger |\Omega\rangle \\ O_{\omega, \lambda; \bar{A}}^\dagger |\Omega\rangle &= e^{-\pi\omega} O_{\omega, -\lambda; A} |\Omega\rangle \end{aligned} \quad (1.149)$$

while $|\Omega\rangle = \otimes_{\omega, \lambda} |\Omega_{\omega, \lambda}\rangle = \otimes_{\omega, \lambda} \sqrt{1 - e^{-2\pi\omega}} \sum_n e^{-\pi\omega n} |n\rangle_{\omega, \lambda}^A |\bar{n}\rangle_{\omega, -\lambda}^{\bar{A}}$. As a result, for each choice of i , one can in principle find an operator A_i that has support only on region A such that

$$(O_{nlm}^\dagger)^i |\Omega\rangle = A_i |\Omega\rangle. \quad (1.150)$$

Therefore, (1.148) can be simplified as

$$A_i \text{tr}_{\bar{A}}(|\Omega\rangle\langle\Omega|) A_i^\dagger = A_i \rho_A^{(0)} A_i^\dagger \quad (1.151)$$

that is an operator has support only on A , while $\rho_A^{(0)}$ is a thermal density matrix in the region A . Nevertheless, the equation (1.150) is somewhat abstract, and indeed finding an expression for A_i can be difficult.

To find a more convenient basis for the code subspace we can use the *Reeh-Schlieder* theorem for relativistic QFT. Consider a QFT in Minkowski spacetime \mathcal{M} with a Hilbert space \mathcal{H} and the vacuum state denoted by $|\Omega\rangle \in \mathcal{H}$. For a small open set $\mathcal{U} \subset \mathcal{M}$, there is a bounded algebra of local operators $\mathcal{A}_{\mathcal{U}}$ supported in \mathcal{U} . The Reeh-Schlieder theorem says that every arbitrary state in \mathcal{H} can be approximated by $\mathcal{A}_{\mathcal{U}}|\Omega\rangle$ that means states created by applying elements of the local algebra to the vacuum are not localized to the region \mathcal{U} . In other words, the vacuum is a cyclic and separating vector for the field algebra corresponding to any open set \mathcal{U} in Minkowski spacetime. This is the key point in our work that causes the manageability of the Petz map calculation.

We can construct the code subspace using the Reeh-Schlieder theorem to the boundary QFT by acting on the global vacuum with the operator algebra on region A , $\mathcal{H}_{GFF} = \{\mathcal{L}(\mathcal{H}_A)|\Omega\rangle\}$. Since one choice of basis for the operator algebra on A is the set of Rindler modes $O_{\omega\lambda;A}$ and $O_{\omega\lambda;A}^\dagger$, one can take a basis for code subspace at large N as

$$| \{j_{\omega,\lambda}, \Delta_{\omega,\lambda}\} \rangle = \prod_{\omega,\lambda} (O_{\omega\lambda;A})^{j_{\omega,\lambda}} (O_{\omega\lambda;A}^\dagger)^{j_{\omega,\lambda} + \Delta_{\omega,\lambda}} |\Omega\rangle \quad (1.152)$$

where $j \in \mathbb{N}$ and $\Delta \in \mathbb{Z}$.³ As the theory is free, different modes are decoupled and we can rewrite the code subspace basis as

$$| \{j_{\omega,\lambda}, \Delta_{\omega,\lambda}\} \rangle = \otimes_{\omega,\lambda} |j_{\omega,\lambda}, \Delta_{\omega,\lambda}\rangle = \otimes_{\omega,\lambda} (O_{\omega\lambda;A})^{j_{\omega,\lambda}} (O_{\omega\lambda;A}^\dagger)^{j_{\omega,\lambda} + \Delta_{\omega,\lambda}} |\Omega_{\omega,\lambda}\rangle \quad (1.153)$$

In the following, for simplicity we will just focus on a single mode which the corresponding Hilbert space is $\text{span}\{|j, \Delta\rangle = (O_A)^j (O_A^\dagger)^{j+\Delta} |\Omega\rangle\}$. In the new basis, instead of (1.148), we should calculate the terms $\text{tr}_{\bar{A}} |j, \Delta\rangle \langle j', \Delta'|$ which one can simply find as

$$\text{tr}_{\bar{A}} |j, \Delta\rangle \langle j', \Delta'| = (O_A)^j (O_A^\dagger)^{j+\Delta} \rho_A^{(0)} (O_A)^{j'+\Delta'} (O_A^\dagger)^{j'} \quad (1.154)$$

We should be careful here that although this set of vectors spans the GFF sector of the boundary, they are not orthonormal as we have

$$\langle j, \Delta | j', \Delta' \rangle = \delta_{\Delta, \Delta'} (1 - e^{-2\pi\omega}) \sum_{n=\max\{0, -\Delta\}} e^{-2\omega n} \sqrt{\frac{(n+j+\Delta)!}{(n+\Delta)!}} \sqrt{\frac{(n+j'+\Delta')!}{(n+\Delta')!}} \quad (1.155)$$

which is proportional to $\delta_{\Delta, \Delta'}$ not $\delta_{j, j'} \delta_{\Delta, \Delta'}$. Nevertheless, we can still use this set of vectors as a basis for the code subspace by considering the correct form of the projection on a non-orthonormal basis.

³More precise statement here is that, since the representation of the vacuum state in terms of the Rindler modes is cyclic and separating with respect to the operator algebra of the Rindler wedge, the vacuum sector of the Hilbert space is isomorphic to the GNS Hilbert space of the operator algebra of the Rindler wedge over the vacuum.

Consider a vector space $V = \text{span}\{|v_i\rangle\}$. One can construct the metric tensor for this basis $G = [g_{ij}]$ that by definition $g_{ij} = \langle v_i | v_j \rangle$. The inverse metric $G^{-1} = [g^{ij}]$ is defined to be the inverse of the matrix G , so the relations

$$\sum_j g^{ij} g_{jk} = \delta_k^i, \quad \sum_j g_{ij} g^{jk} = \delta_i^k \quad (1.156)$$

should satisfy and the projection on the subspace $V_I = \text{span}\{|v_i\rangle, i \in I\}$ is given by

$$P_I = \sum_{i,j \in I} g^{ij} |v_i\rangle \langle v_j|. \quad (1.157)$$

1.5.3 AdS-Rindler wedge reconstruction via Petz map

Now we have all relations we need to find the Petz reconstruction for the fields in the AdS-Rindler patch in the bulk. By plugging (1.146) back into (1.136), we arrive to

$$\begin{aligned} \Phi_A(X) = \sum_{I \in \{A, \bar{A}\}} \int d\omega d\lambda \mathcal{F}_{\omega, \lambda; I}(X) \tau_A^{-1/2} \text{tr}_{\bar{A}}(P_{code} O_{\omega, \lambda, I} P_{code}) \tau_A^{-1/2} \\ + \mathcal{F}_{\omega, \lambda; I}^*(X) \tau_A^{-1/2} \text{tr}_{\bar{A}}(P_{code} O_{\omega, \lambda, I}^\dagger P_{code}) \tau_A^{-1/2} \end{aligned} \quad (1.158)$$

while

$$\mathcal{F}_{\omega, \lambda; I}(X) = \sum_{nlm} \frac{M_{nlm}}{M_{\omega\lambda}} (G_{nlm}(X) \alpha_{nlm; \omega\lambda}^I + G_{nlm}^*(X) \beta_{nlm; \omega\lambda}^I). \quad (1.159)$$

By comparing the global mode expansion of $\Phi_{HKLL}(X)$ with $\phi(X)$, one can find that $G_{nlm}(X) = \frac{1}{M_{nlm}} f_{nlm}(X)$. If we substitute it in (1.159), we can find that

$$\mathcal{F}_{\omega, \lambda; I}(X) = \frac{1}{M_{\omega\lambda}} \sum_{nlm} (f_{nlm}(X) \alpha_{nlm; \omega\lambda}^I + f_{nlm}^*(X) \beta_{nlm; \omega\lambda}^I) \quad (1.160)$$

As $\phi(X)$ lies in the AdS-Rindler wedge homologous to the region A , by using the relations (1.145), we find that $\mathcal{F}_{\omega, \lambda; \bar{A}}(X) = 0$ for all $X \in \mathcal{E}_A$. Therefore, the Petz reconstruction of $\phi(X)$ gets simplified as

$$\begin{aligned} \Phi_A(X) = \int d\omega d\lambda \mathcal{F}_{\omega, \lambda; A}(X) \tau_A^{-1/2} \text{tr}_{\bar{A}}(P_{code} O_{\omega, \lambda, A} P_{code}) \tau_A^{-1/2} \\ + \mathcal{F}_{\omega, \lambda; A}^*(X) \tau_A^{-1/2} \text{tr}_{\bar{A}}(P_{code} O_{\omega, \lambda, A}^\dagger P_{code}) \tau_A^{-1/2}. \end{aligned} \quad (1.161)$$

In our basis the projection to the code subspace is

$$P_{code} = \sum_{j, j'} \sum_{\Delta, \Delta'} g^{(j, \Delta); (j', \Delta')} |j, \Delta\rangle \langle j', \Delta'|. \quad (1.162)$$

From the inner product between $\{|j, \Delta\rangle\}$, we see that the metric tensor here is block-diagonal while each block labeled by Δ

$$G = \oplus_{\Delta} G_{\Delta} = \oplus_{\Delta} [g_{j, j'; \Delta}] \quad (1.163)$$

where $g_{j, j'; \Delta} = \langle j, \Delta | j', \Delta \rangle$. As a result, the inverse metric should have the form of

$$G^{-1} = \oplus_{\Delta} G_{\Delta}^{-1} = \oplus_{\Delta} [A_{j, j'}^{\Delta}] \quad (1.164)$$

for some unknown elements $A_{j,j'}^\Delta$ which should satisfy the relations below

$$\begin{aligned} \sum_{j'} A_{j,j'}^\Delta \langle j', \Delta | j'', \Delta \rangle &= \delta_{j,j''} \\ \sum_{j'} \langle j, \Delta | j', \Delta \rangle A_{j',j''}^\Delta &= \delta_{j,j''}. \end{aligned} \quad (1.165)$$

Since $g^{(j,\Delta);(j',\Delta')} = A_{j,j'}^\Delta \delta_{\Delta,\Delta'}$, we can write the projection on the code subspace in terms of $A_{j,j'}^\Delta$ as

$$P_{code} = \sum_{\Delta} \sum_{j,j'} A_{j,j'}^\Delta |j, \Delta\rangle \langle j', \Delta|. \quad (1.166)$$

Now, we can use the form of the code subspace projection to find the three terms we need to find the Petz reconstruction of $\phi(X)$. First, we start with τ_A which is

$$\begin{aligned} \tau_A &= \text{tr}_{\bar{A}} P_{code} = \sum_{\Delta} \sum_{j,j'} A_{j,j'}^\Delta \text{tr}_{\bar{A}} |j, \Delta\rangle \langle j', \Delta| \\ &= \sum_{\Delta} \sum_{j,j'} A_{j,j'}^\Delta (O_A)^j (O_A^\dagger)^{j+\Delta} \rho_A^{(0)} (O_A)^{j'+\Delta'} (O_A^\dagger)^{j'}. \end{aligned} \quad (1.167)$$

We also need to calculate the terms in the form of

$$\text{tr}_{\bar{A}} (P_{code} O P_{code}) = \sum_{\Delta,\Delta'} \sum_{j,j'} \sum_{k,k'} A_{j,k}^\Delta A_{k',j'}^{\Delta'} \langle k, \Delta | O | k', \Delta' \rangle \text{tr}_{\bar{A}} |j, \Delta\rangle \langle j', \Delta'|, \quad (1.168)$$

For O that is O_A or O_A^\dagger , we get

$$\begin{aligned} \langle k, \Delta | O_A | k', \Delta' \rangle &= \langle k, \Delta | k' + 1, \Delta' - 1 \rangle \\ \langle k, \Delta | O_A^\dagger | k', \Delta' \rangle &= \langle k + 1, \Delta - 1 | k', \Delta' \rangle. \end{aligned} \quad (1.169)$$

By using the relations (1.165) and (1.169), one can find that

$$\begin{aligned} \text{tr}_{\bar{A}} (P_{code} O_{\omega,\lambda;A} P_{code}) &= O_{\omega,\lambda;A} \tau_A \\ \text{tr}_{\bar{A}} (P_{code} O_{\omega,\lambda;A}^\dagger P_{code}) &= \tau_A O_{\omega,\lambda;A}^\dagger. \end{aligned} \quad (1.170)$$

The operators $O_{\omega,\lambda;A}$ and $O_{\omega,\lambda;A}^\dagger$ have support only on region A and commute with every operator $X_{\bar{A}}$. We can show that here, it is equivalent to say that τ_A^{-1} commute with the $\text{tr}_{\bar{A}} (P_{code} O P_{code})$ for O being $O_{\omega,\lambda;A}$ and $O_{\omega,\lambda;A}^\dagger$. One can conclude that if they commute with τ_A^{-1} , they commute with $\tau_A^{-1/2}$ as well. Therefore, we reach

$$\begin{aligned} \tau_A^{-1/2} \text{tr}_{\bar{A}} (P_{code} O_{\omega,\lambda;A} P_{code}) \tau_A^{-1/2} &= \text{tr}_{\bar{A}} (P_{code} O_{\omega,\lambda;A} P_{code}) \tau_A^{-1} = O_{\omega,\lambda;A} \\ \tau_A^{-1/2} \text{tr}_{\bar{A}} (P_{code} O_{\omega,\lambda;A}^\dagger P_{code}) \tau_A^{-1/2} &= \tau_A^{-1} \text{tr}_{\bar{A}} (P_{code} O_{\omega,\lambda;A}^\dagger P_{code}) = O_{\omega,\lambda;A}^\dagger. \end{aligned} \quad (1.171)$$

Finally, we find the Petz reconstruction of the bulk field $\phi(X)$ in the AdS-Rindler wedge as

$$\Phi_A(X) = \int d\omega d\lambda \left(\mathcal{F}_{\omega,\lambda;A}(X) O_{\omega,\lambda;A} + \mathcal{F}_{\omega,\lambda;A}^*(X) O_{\omega,\lambda;A}^\dagger \right). \quad (1.172)$$

By substituting (1.41) in (1.172), we will arrive at

$$\Phi_A(X) = \int d\tau dx K_{Petz,A}(X|\tau, x) O(\tau, x) \quad (1.173)$$

where the smearing function is

$$\begin{aligned} K_{Petz,A}(X|\tau, x) &= \int d\omega d\lambda \mathcal{F}_{\omega,\lambda;A}(X) e^{i\omega\tau Y_\lambda^*(x)} \\ &= \int d\omega d\lambda e^{i\omega\tau Y_\lambda^*(x)} \frac{1}{M_{\omega,\lambda}} \sum_{nlm} \left(f_{nlm}(X) \alpha_{nlm;\omega\lambda}^a + f_{nlm}^*(X) \beta_{nlm;\omega\lambda}^a \right) \\ &= \int \frac{d\omega}{2\pi} \frac{d\lambda}{2\pi} \frac{1}{M_{\omega,\lambda}} f_{\omega\lambda,A}(X) e^{i\omega\tau Y_\lambda^*(x)}. \end{aligned} \quad (1.174)$$

By comparing with (1.44), we see that the result one can find by applying the Petz map in an AdS-Rindler patch is exactly the same as the result of the HKLL procedure in the AdS-Rindler coordinate.

1.6 Entanglement wedge reconstruction and Petz map

In the previous chapter, we used the Petz map to find the CFT reconstruction of a bulk field in the AdS-Rindler wedge. In principle, this approach can be used to reconstruct the entanglement wedge of any region on the boundary explicitly. Let us consider CFT_d in a semi-classical state $|\Psi\rangle$ which is dual to a smooth asymptotically AdS spacetime \mathcal{M} . We also assume that there is no black hole in the bulk. Consider a Cauchy surface Σ of the boundary and divide it into an arbitrary region A and its complementary part \bar{A} . In the rest, we focus on finding the reconstruction of the entanglement wedge of A via the Petz map.

In the bulk, one can find the global mode expansion of the field ϕ as

$$\phi(X) = \sum_n \left(f_n(X) a_n + f_n^*(X) a_n^\dagger \right) \quad (1.175)$$

where $f_n(X)$ is the solution of the Klein-Gordon equation on \mathcal{M} and a_n is the mode corresponding to it that obeys the usual canonical commutation relations. All the labels needed to define the modes are shown collectively by n . By applying the HKLL method to an appropriate coordinate system that covers the entire bulk, labeled here by (r, t, x) , one can find that

$$\Phi_{HKLL}(X) = \int_{bdy} dt dx K_{\partial\mathcal{M}}^g(X|t, x) O(t, x) \quad (1.176)$$

where $K_{\partial\mathcal{M}}^g(X|t, x)$ is the global smearing function. As in the AdS-Rindler case, it is convenient to go to the Fourier modes where the single trace primaries have a mode expansion as

$$O(t, x) = \sum_n \left(\tilde{g}_n(t, x) \hat{O}_n + \tilde{g}_n^*(t, x) \hat{O}_n^\dagger \right) \quad (1.177)$$

where

$$\hat{O}_n = \frac{1}{M_n} \int dt dx O(t, x) g_n^*(t, x). \quad (1.178)$$

If we choose \hat{O}_n with standard commutation relation, i.e. identified it with a_n , from extrapolate dictionary, we have $\tilde{g}_n(t, x) = \lim_{r \rightarrow \infty} r^\Delta f_n(r, x)$, where M_n and g_n are defined as in Sec. 1.1.1. Therefore

$$\Phi_{HKLL}(X) = \sum_n \left(G_{n, \mathcal{M}}(X) \hat{O}_n + G_{n, \mathcal{M}}^*(X) \hat{O}_n^\dagger \right) \quad (1.179)$$

while

$$G_{n, \mathcal{M}}(X) = \int_{bdy} dt dx K_{\partial \mathcal{M}}^g(X|t, x) \tilde{g}_n(t, x). \quad (1.180)$$

By comparing (1.175) and (1.179), one can find that $G_{n, \mathcal{M}}(X) = f_n(X)$.

Let us choose a basis for the operator algebra of the regions A and \bar{A} which we denote them by $\{A_\nu\}$ and $\{\bar{A}_\nu\}$ respectively. In order to find the Petz reconstruction of $\phi(X)$, we need to write the mode functions \hat{O}_n as a linear combination of $\{A_\nu\}$ and $\{\bar{A}_\nu\}$. If it is in the form of

$$\hat{O}_n = \sum_\nu \alpha_{n, \nu}^A A_\nu + \alpha_{n, \nu}^{\bar{A}} \bar{A}_\nu. \quad (1.181)$$

The Petz reconstruction of $\phi(X)$ arrives to

$$\Phi_A(X) = \sum_\nu \mathcal{F}_\nu^A(X) \tau_A^{-1/2} \text{tr}_{\bar{A}}(P_{code} A_\nu P_{code}) \tau_A^{-1/2} + \mathcal{F}_\nu^{\bar{A}*}(X) \tau_A^{-1/2} \text{tr}_{\bar{A}}(P_{code} \bar{A}_\nu P_{code}) \tau_A^{-1/2} \quad (1.182)$$

where

$$\mathcal{F}_\nu^I(X) = \sum_n f_n(X) \alpha_{n, \nu}^I + f_n^*(X) \alpha_{n, -\nu}^{I*} \quad (1.183)$$

for $I \in \{A, \bar{A}\}$.

For a generic choice of basis, the coefficients behind $\tau_A^{-1/2} \text{tr}_{\bar{A}}(P_{code} \bar{A}_\nu P_{code}) \tau_A^{-1/2}$ does not vanish like the case of AdS-Rindler in the previous chapter. Hence, we also need to calculate these terms here. Moreover, for a generic case of the basis of the operator algebra, the sets of $\{A_\nu\}$ and $\{\bar{A}_\nu\}$ do not have in general a simple bulk dual, and therefore, we can not find the Bogoliubov coefficients in (1.181) from the bulk theory.

For now, let us assume that we can somehow find the Bogoliubov coefficients in (1.181). Then in order to proceed, similar to the AdS-Rindler case, we can use the Reeh-Schlieder theorem and write the code subspace as

$$\mathcal{H}_{code} = \mathcal{L}(\mathcal{H}_A) |\Psi\rangle. \quad (1.184)$$

In principle, to find the Petz reconstruction in (1.182), it is needed to know the commutation relation between the operator algebras of the regions A , and rewrite the action of the operator \bar{A}_ν on the state $|\Psi\rangle$ in terms of the operators in region A on the state, *i.e.*, finding the operator $O_{A, \nu}$ as a function of $\{A_\nu\}$ such that

$$\bar{A}_\nu |\Psi\rangle = O_{A, \nu} |\Psi\rangle. \quad (1.185)$$

But practically, whether or not we can explicitly compute all the terms in (1.182) depends on the basis we take, and for an appropriate choice of it, we will be able to find an explicit expression for the operator $\Phi_A(X)$.

In the following, we will describe an appropriate choice of the sets $\{A_\nu\}$ and $\{\bar{A}_\nu\}$ that by using them, the calculation becomes attainable. We will see that there is a clever choice of basis, the eigenfunctions of the modular Hamiltonian, that the Petz calculation will get drastically simplified.

1.6.1 Petz map and modular flow

In this section, we will focus on a special choice of basis for operator algebra in the region A and \bar{A} that is the eigenfunctions of modular Hamiltonian of the regions.

The modular Hamiltonian of a given region R is defined as $K_R = -\log \rho_R$ where ρ_R is the reduced density matrix of the region R . K_R generates an automorphism for the operator algebra \mathcal{A}_R associated to ρ_R [127] as

$$A \in \mathcal{A}_R \quad \longrightarrow \quad A_s = e^{iK_R s} A e^{-iK_R s} \in \mathcal{A}_R. \quad (1.186)$$

called *modular flow*. The modular flow originally introduced in the context of the algebraic QFT [127–131] which recently played a key role in using the concepts of quantum information theory in QFT and gravity [132–155].

In modular Fourier space, the Fourier transformation of A_s is

$$A_\omega = \int_{-\infty}^{\infty} ds e^{is\omega} e^{iK_R s} A e^{-iK_R s}. \quad (1.187)$$

The operators A_ω are the eigenfunctions of modular Hamiltonian $[K_R, A_\omega] = \omega A_\omega$. They also form a basis for operator algebra on region R . Therefore, we can take the eigenfunctions of the modular Hamiltonian of the both regions A and \bar{A} as the basis for the corresponding operator algebras on these regions.

Moreover, as we assume that there is no black hole in the bulk, the entanglement wedge of the complementary part of A in the boundary is the same region as the complementary part of the entanglement wedge of the region A and hence, the union of a and \bar{a} covers the entire Cauchy surface. As a result, we can expand both the bulk and boundary global modes as a linear combination of the modular eigenbasis as

$$\begin{aligned} a_n &= \sum_{\omega} \alpha_{n,\omega}^a A_\omega^a + \alpha_{n,\omega}^{\bar{a}} A_\omega^{\bar{a}} \\ \hat{O}_n &= \sum_{\omega} \alpha_{n,\omega}^A A_\omega^A + \alpha_{n,\omega}^{\bar{A}} A_\omega^{\bar{A}}. \end{aligned} \quad (1.188)$$

In such a case, we can use the JLMS statement that relates the modular Hamiltonian of a given boundary region A to the modular Hamiltonian of its entanglement wedge a as

$$K_A = K_a + \frac{Area}{4G} + O(1/N). \quad (1.189)$$

Since the area term in the right hand side of (1.189) is proportional to the identity, both $K_{A(\bar{A})}$ and $K_{a(\bar{a})}$ have the same spectrum and we can identify their eigenfunctions as

$$\begin{aligned} A_\omega^A &= A_\omega^a \equiv A_\omega \\ A_\omega^{\bar{A}} &= A_\omega^{\bar{a}} \equiv \bar{A}_\omega. \end{aligned} \quad (1.190)$$

Therefore as $\hat{O}_n = a_n$, by comparing (1.188) and (1.190), we see that both \hat{O}_n and a_n have the same Bogoliubov coefficients $\alpha_{n,\omega}^{A(\bar{A})} = \alpha_{n,\omega}^{a(\bar{a})}$. One can replace it into (1.183) and find that when we take the eigenfunctions of modular Hamiltonian as the basis of operator algebra of subregions, by definition

$$\mathcal{F}_\omega^{\bar{A}}(X) = \sum_n f_n(X) \alpha_{n,\omega}^{\bar{a}} + f_n^*(X) \alpha_{n,-\omega}^{\bar{a}*} = 0, \quad \forall X \in a. \quad (1.191)$$

Therefore, the Petz reconstruction of $\phi(X)$ in the entanglement wedge of the region A can be read off as

$$\Phi_A(X) = \sum_{\omega} \mathcal{F}_{\omega}^A(X) \tau_A^{-1/2} \text{tr}_{\bar{A}}(P_{code} A_{\omega} P_{code}) \tau_A^{-1/2} = \sum_{\omega} \mathcal{F}_{\omega}^A(X) A_{\omega} \quad (1.192)$$

where A_{ω} is the eigenfunction of K_A and $\mathcal{F}_{\omega}^A(X)$ is given by

$$\mathcal{F}_{\omega}^A(X) = \sum_n \left(f_n(X) \alpha_{n,\omega}^a + f_n^*(X) \alpha_n^{a*} \right) \quad (1.193)$$

which by using (1.180), it can be rewritten in terms of the global smearing function as

$$\mathcal{F}_{\omega}^A(X) = \int_{bdy} dt dx K_{\partial\mathcal{M}}^g(X|t, x) \sum_n \left(\tilde{g}_n(t, x) \alpha_{n,\omega}^A + \tilde{g}_n^*(t, x) \alpha_{n,-\omega}^{A*} \right). \quad (1.194)$$

At this point to write the operator Φ_A more precisely, we should know more about the A_{ω} themselves. To leading order in AdS/CFT, the bulk field consists of free fields. For free scalar fields on any region R that all correlators are fixed by the two-point function, the density matrix is Gaussian and the modular Hamiltonian is bilinear. Its eigenfunctions can be labeled by ω and X_S where the coordinates X_S corresponds to a codimension 2 surface $S \in R$ on one Cauchy slice [156]. Therefore, we have

$$[K_R, \Phi_{\omega}(X_S)] = \omega \Phi_{\omega}(X_S) \quad (1.195)$$

where

$$\Phi_{\omega}(X_S) = \int ds e^{is\omega} e^{iK_R s} \phi(X_S) e^{-iK_R s} \quad \forall X_S \in S. \quad (1.196)$$

Now, let us consider the free scalar field in the entanglement wedge of the region A and the Cauchy surface as the slice of bulk that intersects with A . One clever choice for S can be A itself. By using (1.189) and the identification on the boundary $\phi_0(x_A) = O(x_A)$, we can find the modular eigenfunction of K_A as

$$O_{\omega}(x_A) = \int ds e^{is\omega} e^{iK_A s} O(x_A) e^{-iK_A s} \quad \forall x_A \in A. \quad (1.197)$$

By substituting it in (1.192), we find the Petz reconstruction of $\phi(X)$ lies in the entanglement wedge of A as

$$\Phi_A(X) = \int_{-\infty}^{\infty} ds \int_A dx_A K_{Petz,A}(X|x_A, s) e^{iK_A s} O(x_A) e^{-iK_A s} \quad (1.198)$$

where the smearing function is given by

$$K_{Petz,A}(X|x_A, s) = \sum_n \int d\omega e^{is\omega} \left(f_n(X) \alpha_n^A(\omega, x_A) + f_n^*(X) \alpha_n^{A*}(-\omega, x_A) \right) \quad (1.199)$$

while $\alpha_n^A(\omega, x_A)$ is the Bogoliubov coefficient between \hat{O}_n and $O_{\omega}(x_A)$. As we mentioned in (1.1.3), the equation (1.198) has been conjectured in [16], and also derived in [156] through acting with the modular flow on the extrapolate dictionary. Here, we could again obtain it by using the Petz map formula which is a more generic approach.

As a consistency check, let us calculate (1.199) for the AdS-Rindler wedge. In this patch, the modular parameter is just the Rindler time τ and the modular Hamiltonian is the Rindler Hamiltonian H_τ . To find the smearing function in (1.199), we need to find the Bogoliubov coefficients between \hat{O}_{nlm} and $O_\omega(x_A) = \int d\tau e^{i\omega\tau} O(\tau, x_A)$ which is

$$\begin{aligned}\alpha_{nlm}^A(\omega, x_A) &= \int d\lambda \frac{1}{M_{\omega,\lambda}} Y_\lambda^*(x_A) \alpha_{nlm;\omega,\lambda} & \forall \omega \geq 0 \\ \alpha_{nlm}^A(\omega, x_A) &= \int d\lambda \frac{1}{M_{\omega,\lambda}} Y_\lambda^*(x_A) \beta_{nlm;\omega,\lambda}^* & \forall \omega < 0\end{aligned}\tag{1.200}$$

By plugging it into (1.199), we get

$$\begin{aligned}K_{Petz,A}(X|x_A, \tau) &= \sum_{nlm} \int d\omega e^{i\omega\tau} \int d\lambda \frac{1}{M_{\omega,\lambda}} Y_\lambda^*(x_A) \left(f_{nlm}(X) \alpha_{nlm;\omega,\lambda}^A + f_{nlm}^*(X) \beta_{nlm;\omega,\lambda}^A \right) \\ &= \int \frac{d\omega}{2\pi} \frac{d\lambda}{2\pi} \frac{1}{M_{\omega,\lambda}} f_{\omega\lambda,A}(X) e^{i\omega\tau} Y_\lambda^*(x)\end{aligned}\tag{1.201}$$

which is exactly the smearing function that we know from AdS-Rindler wedge reconstruction.

As illustrated, to reconstruct the operator in the interior of the entanglement wedge, we need to learn more about the modular Hamiltonian of general regions in QFTs. It might be simpler to study the modular flow in the bulk theory since we treat with real free theory which we know that its modular Hamiltonian is bi-local in the fields, rather than the boundary theory that it contains the GFF.

1.7 Black hole exterior reconstruction and Petz map

In this section, we will discuss the reconstruction of the modes on the two-sided black hole background. First, we review the HKLL procedure and after that, we will explain how we can use the Petz map definition in modular theory to reconstruct the modes on the left exterior from the reconstruction of the modes on the right exterior of the black hole.

1.7.1 Reconstruction of the black hole exterior using HKLL map

At large N we can treat the bulk theory as a quantum field theory on a curved spacetime background. One can then represent the black hole exterior in terms of the CFT operators using the HKLL reconstruction procedure [88–91]. It is known that the free scalar field ϕ in the bulk is dual to the scalar conformal primary of the boundary with conformal dimension $\Delta = d/2 + \sqrt{m^2 + d^2/4}$, which is related to the boundary limit of the field ϕ via extrapolate dictionary as

$$\lim_{r \rightarrow \infty} r^\Delta \phi(t, r, \Omega) = O(t, \Omega).\tag{1.202}$$

In case we have a gauge theory in the boundary, these primary operators are usually some single trace operators.

Consider scalar conformal primary operator O . The same as for the vacuum, large N factorization holds for the thermal correlation functions, i.e.

$$\begin{aligned} \text{tr}(\rho_{th} O(x_1) \dots O(x_{2n})) = \\ \frac{1}{2^n} \sum_{\pi} \text{tr}(\rho_{th} O(x_{\pi_1}) O(x_{\pi_2})) \dots \text{tr}(\rho_{th} O(x_{\pi_{2n-1}}) O(x_{\pi_{2n}})) + O(1/N), \end{aligned} \quad (1.203)$$

where π runs over the set of permutations. From (1.77), one can find out that the large N factorization holds for the typical pure states as well, thus in all cases, each Schwarzschild mode in the bulk is dual to a GFF on the boundary.

We can expand the boundary GFF in terms of its Fourier modes $O_{\omega,m}$ as

$$O(t, \Omega) = \sum_m \int_0^\infty \frac{d\omega}{2\pi} (g_{\omega,m}(t, \Omega) O_{\omega,m} + g_{\omega,m}^*(t, \Omega) O_{\omega,m}^\dagger). \quad (1.204)$$

The thermal expectation values of the Fourier operators also imply that they behave like the unnormalized creation and annihilation operators. One can use the extrapolate dictionary to find the rescaled operators $\hat{O}_{\omega,m} = M_{\omega,m}^{-1} O_{\omega,m}$, which are identified with the bulk modes $a_{\omega,m}$. These CFT operators $\hat{O}_{\omega,m}$ are the ones thermally populated at the Hawking temperature of the black hole β^{-1}

$$\begin{aligned} \frac{1}{Z_\beta} \text{tr}(e^{-\beta H} \hat{O}_{\omega,m} \hat{O}_{\omega',m'}^\dagger) &= \frac{e^{\beta\omega}}{e^{\beta\omega} - 1} \delta(\omega - \omega') \delta_{m,m'} \\ \frac{1}{Z_\beta} \text{tr}(e^{-\beta H} \hat{O}_{\omega,m}^\dagger \hat{O}_{\omega',m'}) &= \frac{1}{e^{\beta\omega} - 1} \delta(\omega - \omega') \delta_{m,m'}. \end{aligned} \quad (1.205)$$

Having the identification between bulk and boundary modes, we can follow the mode sum approach in [90] and find the CFT expression for every bulk field outside the horizon as

$$\Phi_{HKLL}(t, r, \Omega) = \int dt' d\Omega' K(t, r, \Omega | t', \Omega') O(t', \Omega') \quad (1.206)$$

for an appropriate choice of smearing function K . We can also find the field expression in terms of Fourier modes by plugging (1.204) into (1.206) as

$$\Phi_{HKLL}(t, r, \Omega) = \sum_m \int_0^\infty \frac{d\omega}{2\pi} (\mathcal{F}_{\omega,m}(t, r, \Omega) O_{\omega,m} + \mathcal{F}_{\omega,m}^*(t, r, \Omega) O_{\omega,m}^\dagger) \quad (1.207)$$

where

$$\mathcal{F}_{\omega,m}(t, r, \Omega) = \int dt' d\Omega' K(t, r, \Omega | t', \Omega') g_{\omega,m}(t', \Omega'). \quad (1.208)$$

By comparing with (1.60), we can find that $\mathcal{F}_{\omega,m}(t, r, \Omega) = M_{\omega,m}^{-1} f_{\omega,m}(t, r, \Omega)$.

1.7.2 Coarse-grained vs fine-grained observables

Consider a big AdS black hole in equilibrium. An observable outside the horizon of the black hole has access just to the information in the exterior of the black hole, referred to as region I. This bulk observable can not distinguish the microstate of the black hole and more generally, the fact that it is a one-sided black hole or connected to another universe through a wormhole. However, in all cases, if we are just interested to the low

energy observables outside the horizon, we find from (1.63) that it is enough to describe the system by a thermal density matrix.

On the other hand, having a big black hole in AdS is dual to the thermalization of the boundary theory. In general, the thermalization of a closed quantum system leads to the division of the observables of the theory into two parts: the coarse-grained or macroscopic observables and the fine-grained or microscopic ones, denoted by \mathcal{A}_c and \mathcal{A}_f , respectively. The coarse-grained observables are the ones that can be easily measured by the low-energy observer. More precisely, the thermalization of the system means that if we are just interested to measure the macroscopic observable, the system can be approximately described by a thermal density matrix, *i.e.*

$$\rho|_{\mathcal{A}_c} = \rho_{th,c} = \frac{1}{Z_\beta^c} e^{-\beta H_c}, \quad (1.209)$$

where H_c is the coarse-grained Hamiltonian and Z_β^c is the coarse-grained partition function [11].

In AdS/CFT where we have the duality between the bulk and boundary theories, the bulk Hilbert space is isomorphic to the boundary one. In the bulk, we have a fundamental theory of quantum gravity that in low energy described by a local quantum field theory on a curved spacetime background. These are usually the macroscopic degrees of freedom in the bulk while the stringy and trans-Planckian observables are the non-perturbative microscopic degrees of freedom. When we have a black hole in the bulk, the coarse-grained observables are just the operators lie outside the horizon while the fine-grained one contains the degrees of freedom of the black hole interior as well as the non-perturbative ones on the entire bulk. In the rest, we are interested to study the bulk gravity in the low energy and we denote the algebra of operator in this regime in the exterior and interior of the black hole by \mathcal{A}_{ext} and \mathcal{A}_{int} , respectively.

As it is mentioned above, for the low-energy observables outside the horizon we can describe the bulk theory by the thermal density matrix for the bulk effective field theory lives in the AdS-Schwarzschild coordinates, in other words

$$\rho_{bulk}|_{\mathcal{A}_{ext}} = \rho_{th,I}. \quad (1.210)$$

1.7.3 Reconstruction of the black hole exterior using the Petz map

As previously described, one can map the local bulk field in the exterior of a black hole into the non-local CFT operators using the HKLL procedure

$$\phi(t, r, \Omega) \longrightarrow \Phi_{HKLL}(t, r, \Omega) = \int_{bdy} dt' d\Omega' K(t, r, \Omega|t', \Omega') O(t', \Omega'). \quad (1.211)$$

In other words, the HKLL map provides us an isometry of embedding $V_{HKLL} : \mathcal{H}_{ext} \rightarrow \mathcal{H}_{CFT}$ which in the Heisenberg picture maps the operators as

$$\phi(t, r, \Omega) \longrightarrow \Phi_{HKLL}(t, r, \Omega) = V_{HKLL} \phi(t, r, \Omega) V_{HKLL}^\dagger. \quad (1.212)$$

It is equivalent to consider the quantum channel which maps the density matrices in the exterior region to the boundary density matrices $\mathcal{E} : \mathcal{S}(\mathcal{A}_{ext}) \rightarrow \mathcal{S}(\mathcal{A}_{CFT})$ as

$$\mathcal{E}(\cdot) = V_{HKLL} (\cdot) V_{HKLL}^\dagger. \quad (1.213)$$

However, we should be careful that the low-energy observers can only measure the coarse-grained operators, for any GFF $O(t, \Omega)$, they can measure just

$$O_c(t, \Omega) = P_{coarse} O(t, \Omega) P_{coarse} \quad (1.214)$$

where P_{coarse} is the projection onto the coarse-grained Hilbert space which traces out the fine-grained degrees of freedom. Thus, the actual map we have is

$$\begin{aligned} \phi(t, r, \Omega) &\longrightarrow P_{coarse} \Phi_{HKLL}(t, r, \Omega) P_{coarse} \\ &= P_{coarse} V_{HKLL} \phi(t, r, \Omega) V_{HKLL}^\dagger P_{coarse} = \int_{bdy} dt' d\Omega' K(t, r, \Omega|t', \Omega') O_c(t', \Omega'). \end{aligned} \quad (1.215)$$

The Hilbert space of these coarse-grained GFF has the Fock space structure which is isomorphic to the Hilbert space of the free fields on the AdS-Schwarzschild background. Therefore, we can introduce the quantum channel $\mathcal{E}_c : \mathcal{S}(\mathcal{A}_{ext}) \rightarrow \mathcal{S}(\mathcal{A}_{coarse})$ as

$$\mathcal{E}_c(\cdot) = V_{HKLL}(\cdot) V_{HKLL}^\dagger \Big|_{\mathcal{A}_{coarse}} = V_c(\cdot) V_c^\dagger \quad (1.216)$$

where $V_c = P_{coarse} V_{HKLL}$. Unlike \mathcal{E} , the quantum channel \mathcal{E}_c is invertible as the evolution is done via a unitary evolution. In this case, the recovery channel is simply known as

$$\mathcal{R}_c(\cdot) = V_c^\dagger(\cdot) V_c = \mathcal{E}_c^*(\cdot), \quad (1.217)$$

which one can also find using the Petz recovery channel formula (1.206). Therefore, one can use the dual of the recovery channel to map the operators in the Heisenberg picture, $\mathcal{R}_c^* : \mathcal{A}_{ext} \rightarrow \mathcal{A}_{coarse}$

$$\mathcal{R}_c^*(\cdot) = V_c(\cdot) V_c^\dagger = \mathcal{E}_c(\cdot). \quad (1.218)$$

As a result, we have

$$\begin{array}{ccc} \mathcal{S}(\mathcal{A}_{ext}) & & \mathcal{A}_{ext} \\ \downarrow \mathcal{E}_c & \searrow \mathcal{E} & \downarrow \mathcal{R}_c^* \\ \mathcal{S}(\mathcal{A}_{coarse}) & \xleftarrow{res} \mathcal{S}(\mathcal{A}_{CFT}) & \mathcal{A}_{coarse} \end{array}$$

$$\mathcal{R}_c^*(\phi(t, r, \Omega)) = \int_{bdy} dt' d\Omega' K(t, r, \Omega|t', \Omega') O_c(t', \Omega') \quad (1.219)$$

From now on, we drop the subscript c , but we mean by O the coarse-grained GFF. Since $\mathcal{A}_{ext} = \text{span}\{a_{\omega, m}, a_{\omega, m}^\dagger\}$, it is enough to find the action of the recovery map \mathcal{R}_c^* on these set of operators. One can easily use (1.60), (1.207) and reach to

$$\mathcal{R}_c^*(a_{\omega, m}) = \hat{O}_{\omega, m}, \quad \mathcal{R}_c^*(a_{\omega, m}^\dagger) = \hat{O}_{\omega, m}^\dagger. \quad (1.220)$$

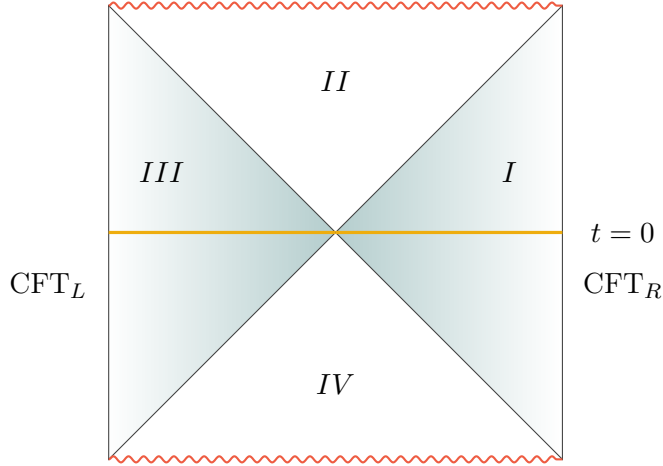


Figure 1.6: The two-sided eternal black hole in holography.

For now, let us consider a two-sided geometry that is dual to the TFD state of the two identical non-interacting CFTs in the boundary which is given by (1.4), and take the $t = 0$ Cauchy slice in the bulk, Fig. 1.6. As we had in Sec. 1.2, the bulk Hilbert space corresponding to the quantizing the small fluctuations around the black hole geometry is denoted as $\mathcal{H}_{BH}^{(Fock)}$ and the algebra of low-energy observables on two sides of the black hole as $\mathcal{A}_{l,0}$ and $\mathcal{A}_{r,0}$. The bulk Hilbert space can be constructed through the action of the operator algebra of only the right exterior on the HH state (one can obtain it equivalently from the algebra of the left exterior), i.e.

$$\mathcal{H}_{BH}^{(Fock)} = \mathcal{A}_{r,0} |HH\rangle = \mathcal{A}_{l,0} |HH\rangle. \quad (1.221)$$

The Hilbert space of the full boundary theory is

$$\mathcal{H} = \mathcal{H}_{CFT_L} \otimes \mathcal{H}_{CFT_R}. \quad (1.222)$$

The bulk states in the Hilbert space $\mathcal{H}_{BH}^{(Fock)}$ are dual to the set of states called *code subspace* in the boundary theory corresponding to the excitations around the TFD state. They can be obtained by acting with the dual single-trace boundary operators on the TFD state denoted by \mathcal{H}_{TFD} . The TFD code subspace spanned by the states $a |\Psi_{TFD}\rangle$ with $a \in \mathcal{A}_{R,0}$. The corresponding code subspace has a structure of a Hilbert space that can be made via GNS construction by using only $\mathcal{A}_{R,0}$ or $\mathcal{A}_{L,0}$ over TFD state (We will discuss it more precisely later). At large N limit where the algebras are of Type III₁, the TFD Hilbert space does not have a tensor product structure, in other words, there are no candidates for \mathcal{H}_L and \mathcal{H}_R such that $\mathcal{H}_{TFD} = \mathcal{H}_L \otimes \mathcal{H}_R$. The GFFs are then the representations of the single trace operators on the TFD Hilbert space and thus the algebras $\mathcal{A}_{L,0}$ and $\mathcal{A}_{R,0}$ are the von Neumann algebras on \mathcal{H}_{TFD} . We note that these representations are not exactly the same as the original operators since they only define on \mathcal{H}_{TFD} while the single trace operators act on the full CFT Hilbert space.

We can follow the discussion above and map the algebra of operators in each region to the coarse-grained operators on the boundary via two copies of the Petz recovery channel as

$$\mathcal{R}_{c,R}^* : \mathcal{A}_{r,0} \longrightarrow \mathcal{A}_{R,0}, \quad \mathcal{R}_{c,L}^* : \mathcal{A}_{l,0} \longrightarrow \mathcal{A}_{L,0} \quad (1.223)$$

such that

$$\begin{aligned}\mathcal{R}_{c,R}^*(a_{\omega,m}) &= \hat{O}_{\omega,m;R}, & \mathcal{R}_{c,R}^*(a_{\omega,m}^\dagger) &= \hat{O}_{\omega,m;R}^\dagger \\ \mathcal{R}_{c,L}^*(\tilde{a}_{\omega,m}) &= \hat{O}_{\omega,m;L}, & \mathcal{R}_{c,L}^*(\tilde{a}_{\omega,m}^\dagger) &= \hat{O}_{\omega,m;L}^\dagger.\end{aligned}\quad (1.224)$$

Knowing the boundary reconstruction of the Schwarzschild modes, we can find the boundary representation of the bulk field in every bulk point. Although we can simply map the operator algebra in region III to the left CFT through $\mathcal{R}_{c,L}^*$, there is an alternative way to write this mapping by using the Petz map formula in the GNS Hilbert space (1.109). This will be helpful later when we are interested to find the CFT representation of the one-sided black hole interior modes.

As we mentioned, the effective field theory on the eternal black hole background is described by the HH state, thus

$$\rho_{bulk} \Big|_{\mathcal{A}_{r,0}} = \rho_{th,I} \qquad \rho_{bulk} \Big|_{\mathcal{A}_{l,0}} = \rho_{th,III}, \quad (1.225)$$

and it is cyclic and separating for the operator algebra both in regions I and III. Moreover, since these two regions are spacelike separated, we have $[\mathcal{A}_{l,0}, \mathcal{A}_{r,0}] = 0$ and so they are each others commutants

$$(\mathcal{A}_{l,0})' = \mathcal{A}_{r,0}. \quad (1.226)$$

On the other hand, the boundary theory is in the TFD state, therefore each CFT itself is described by the CFT thermal state and as we mentioned above, its restriction to the coarse-grained algebra is also a thermal state but this time with respect to the coarse-grained Hamiltonian. They are also the commutants of each other on the \mathcal{H}_{TFD}

$$(\mathcal{A}_{L,0})' = \mathcal{A}_{R,0}. \quad (1.227)$$

Therefore, one can use the modular theory expression (1.109) and write the Petz map $\mathcal{R}_{c,L}^*$ as

$$\mathcal{R}_{c,L}^*(\cdot) = \mathcal{J}_{TFD} \circ \mathcal{R}'_{c,R-\rho_{th}} \circ \mathcal{J}_{HH}(\cdot) = \mathcal{J}_{TFD} \mathcal{R}'_{c,R-\rho_{th}} (J_{HH}(\cdot) J_{HH}) \mathcal{J}_{TFD} \quad (1.228)$$

while \mathcal{J}_{HH} and \mathcal{J}_{CFT} are respectively the modular conjugations of the bulk and boundary theories with respect to the low-energy observables, and, $\mathcal{R}'_{c,R-\rho_{th}}$ is defined based on the relation in (1.108).

$$\begin{array}{ccc} \mathcal{A}_{l,0} & \xrightarrow{\mathcal{J}_{HH}} & \mathcal{A}_{r,0} \\ \mathcal{R}_{c,L}^* \downarrow & & \downarrow \mathcal{R}'_{c,R-\rho_{th}} \\ \mathcal{A}_{L,0} & \xleftarrow{\mathcal{J}_{TFD}} & \mathcal{A}_{R,0} \end{array}$$

In the bulk where the theory is described by HH state, the modular conjugation operator is the anti-unitary CPT operator-more precisely, CRT transformation- which in the AdS-Schwarzschild coordinates acts as

$$\mathcal{J}_{HH}(\phi_I(t, r, \Omega)) = J_{HH} \phi_I(t, r, \Omega) J_{HH} = \phi_{III}(-t, r, \Omega). \quad (1.229)$$

One then can find that

$$\mathcal{J}_{HH} : a_{\omega,m} \longleftrightarrow \tilde{a}_{\omega,m}. \quad (1.230)$$

On the boundary side where the theory is described by the TFD state, the modular conjugation acts in the TFD Hilbert space as

$$\mathcal{J}_{TFD} : O_{\omega,m;L} \longleftrightarrow O_{\omega,m;R}. \quad (1.231)$$

Therefore, one can simply check that

$$\begin{aligned} \mathcal{R}_{c,L}^*(\tilde{a}_{\omega,m}) &= \mathcal{J}_{TFD} \circ \mathcal{R}'_{c,R-\rho_{th}} \circ \mathcal{J}_{HH}(\tilde{a}_{\omega,m}) = \hat{O}_{\omega,m;L} \\ \mathcal{R}_{c,L}^*(\tilde{a}_{\omega,m}^\dagger) &= \mathcal{J}_{TFD} \circ \mathcal{R}'_{c,R-\rho_{th}} \circ \mathcal{J}_{HH}(\tilde{a}_{\omega,m}^\dagger) = \hat{O}_{\omega,m;L}^\dagger. \end{aligned} \quad (1.232)$$

Before going ahead we note here that after considering the $1/N$ corrections, the picture needs some modifications. The algebra of observables on each side of the eternal black hole is \mathcal{A}_l and \mathcal{A}_r which are of Type II_∞ . They are dual to the crossed product of $\mathcal{A}_{L,0}$ and $\mathcal{A}_{R,0}$ and their group of modular automorphism. The mapping now is as below. It

$$\begin{array}{ccc} \mathcal{A}_l & & \mathcal{A}_r \\ \downarrow \mathcal{R}_{c,L}^* & & \downarrow \mathcal{R}'_{c,R-\rho_{th}} \\ \mathcal{A}_L & & \mathcal{A}_R \\ \mathcal{J}_{TFD} & & \mathcal{J}_{HH} \end{array}$$

should be considered that since we are working at $1/N$ correction, it is better even to use the approximate version of the recovery channel introduced in Sec. 1.3.6.

1.7.4 Vacuum of the GNS Hilbert space

As we had in the previous section, the Hilbert space of the effective field theory in bulk is dual to the TFD Hilbert space \mathcal{H}_{TFD} . In this section, we will discuss more precisely the structure of dual code subspace on the boundary.

The boundary theory is the tensor factor of two identical CFTs, $\mathcal{H} = \mathcal{H}_{CFT_L} \otimes \mathcal{H}_{CFT_R}$. The algebra of bounded operators on \mathcal{H} is

$$\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}_{CFT_L}) \otimes \mathcal{L}(\mathcal{H}_{CFT_R}) \quad (1.233)$$

and the algebra of low-energy observables is a subalgebra of the full algebra

$$\mathcal{A} = \mathcal{A}_{L,0} \otimes \mathcal{A}_{R,0}. \quad (1.234)$$

In order to define the TFD Hilbert space as explained in [99] we should associate a state $|a\rangle$ to each operator $a \in \mathcal{A}$ with the inner product among them which is defined as

$$\langle a|b\rangle = \langle \Psi_{TFD} | a^\dagger b | \Psi_{TFD} \rangle \quad (1.235)$$

for all $a, b \in \mathcal{A}$ and in particular if both a, b belong to $\mathcal{A}_{R,0}$ or $\mathcal{A}_{L,0}$, it is reduced to

$$\langle a | b \rangle = \text{tr}(\rho_{th} a^\dagger b). \quad (1.236)$$

The set of vectors $|a\rangle$ does not have a Hilbert space structure since there exists non-zero operators $y \neq 0$ in the algebra \mathcal{A} such that $\langle y | y \rangle = 0$ [99]. In other words, the TFD state is not separating for the algebra \mathcal{A} . It is just cyclic and separating for the full operator algebra on each CFT. In such a case to construct a Hilbert space from this set of vectors, we can use the GNS construction to set such a vectors to zero by introducing the set of equivalence classes. The equivalence relation between them is defined as

$$a \sim a + y \quad a \in \mathcal{A}, y \in \mathcal{Y} \quad (1.237)$$

while \mathcal{Y} is the set of operators such that $\langle y | y \rangle = 0$. Moreover, since the action of the single trace operators on both sides on TFD state are related to each other, only the algebra $\mathcal{A}_{R,0}$ or $\mathcal{A}_{L,0}$ is enough to generate the full TFD Hilbert space. In other words, all the vectors in \mathcal{H}_{TFD} can be written as $|a\rangle$ with $a \in \mathcal{A}_{R,0}$.

Now let us consider just the algebra of single trace operators $\mathcal{A}_{R,0}$ on the right boundary theory and build the GNS Hilbert space of the algebra with respect to the thermal density matrix which is denoted as $\mathcal{H}_{\rho_{th}}^{GNS}$. In the GNS Hilbert space, the thermal state is represented by a pure state denoted by $|\Omega_0\rangle$ which is called the GNS vacuum. It is also the state in $\mathcal{H}_{\rho_{th}}^{GNS}$ corresponds to the identity operator of the algebra $\mathcal{A}_{R,0}$. The GNS construction provides a representation for the algebra $\mathcal{A}_{R,0}$ on $\mathcal{H}_{\rho_{th}}^{GNS}$ which we denote here as \mathcal{M}_R . The representation of any operator $a \in \mathcal{A}_{R,0}$ is $\pi(a) \in \mathcal{M}_R$ that acts only on $\mathcal{H}_{\rho_{th}}^{GNS}$ and thus, it is *state-dependent* since it depends on the state that the GNS Hilbert space is built over that. On the other hand, the original operator acts on the full CFT Hilbert space and is *state-independent*. The inner product among the states in the GNS Hilbert space is written as

$$\langle a | b \rangle = \langle \Omega_0 | \pi(a)^\dagger \pi(b) | \Omega_0 \rangle. \quad (1.238)$$

Here the algebra consists of the single trace operators of the CFT. Their representations on the GNS Hilbert space are the GFFs acting only on $\mathcal{H}_{\rho_{th}}^{GNS}$ and we also have

$$\langle \Omega_0 | \pi(O_R(x_1))^\dagger \pi(O_R(x_2)) | \Omega_0 \rangle = \text{tr}(\rho_{th} O(x_1)^\dagger O(x_2)). \quad (1.239)$$

As we had, the TFD Hilbert space can be obtained using just $\mathcal{A}_{R,0}$ or $\mathcal{A}_{L,0}$ alone over the TFD state. While the boundary theory is in the TFD state, the CFT_R is described by thermal state and so there should be some relation between the TFD Hilbert space and the GNS Hilbert space corresponding to the thermal matrix over $\mathcal{A}_{R,0}$. Indeed, it can be shown that they are isomorphic

$$\mathcal{H}_{TFD} \cong \mathcal{H}_{\rho_{th}}^{GNS}. \quad (1.240)$$

We defined the algebra \mathcal{M}_R to be the representation of the $\mathcal{A}_{R,0}$ on the GNS Hilbert space. Therefore, its commutant which we denote as \mathcal{M}_L can be interpreted as the representation of the $\mathcal{A}_{L,0}$ on $\mathcal{H}_{\rho_{th}}^{GNS}$.

The GNS vacuum $|\Omega_0\rangle$ is cyclic and separating for both \mathcal{M}_R and \mathcal{M}_L . Therefore, there is a modular operator for the algebras which generate automorphisms of them and leaves $|\Omega_0\rangle$ invariant. If we denote the modular operator for \mathcal{M}_R as Δ_0 , then Δ_0^{-1} is the modular

operator for the algebra \mathcal{M}_L . It can be seen as the representation of the Δ , the modular operator for $\mathcal{A}_{R,0}$, in the GNS Hilbert space and in particular, it should satisfy

$$\pi(\Delta^{-iu} a \Delta^{iu}) = \Delta_0^{-iu} \pi(a) \Delta_0^{iu} \quad (1.241)$$

for all $a \in \mathcal{A}_{R,0}$.

Now consider bulk field ϕ which is dual to the boundary operator O in the AdS/CFT dictionary. To be more precise from the algebraic point of view, we should say that the bulk field restricted in the regions I and III are dual to the representations of the O_R and O_L in the GNS Hilbert space that here they are nothing but GFFs. The extrapolate dictionary (1.202) should also be written more carefully as

$$\begin{aligned} \pi(O_R(t, \Omega)) &= \lim_{r \rightarrow \infty} r^\Delta \phi_R(t, r, \Omega) \\ \pi(O_L(t, \Omega)) &= \lim_{r \rightarrow \infty} r^\Delta \phi_L(t, r, \Omega). \end{aligned} \quad (1.242)$$

Under the duality, at strict large N limit we reach to the identifications:

$$\mathcal{H}_{\rho_{th}}^{GNS} = \mathcal{H}_{BH}^{(Fock)}, \quad |\Omega_0\rangle = |HH\rangle, \quad \mathcal{M}_R = \mathcal{A}_{r,0}, \quad \mathcal{M}_L = \mathcal{A}_{l,0}. \quad (1.243)$$

Moreover, the state-dependence of the GNS representations of the boundary dual operators is indeed a reflection of the fact that when we treat with the gravity at weak coupling in bulk, its mode expansion that identifies the bulk operator algebra for us depends on the bulk semi-classical geometry.

In order to create a GNS Hilbert space to describe the code subspace in the boundary, one can alternatively start with another cyclic and separating vector $|\Omega\rangle$ for the algebra \mathcal{M}_R as the vacuum. Therefore, by duality, we have

$$|\Omega\rangle = |HH\rangle. \quad (1.244)$$

In general, the new vacuum can be related to $|\Omega_0\rangle$ by a unitary as

$$|\Omega\rangle = U |\Omega_0\rangle. \quad (1.245)$$

In particular, the simple cases are in the form of

$$|\Omega\rangle = v_L w_R |\Omega_0\rangle \quad (1.246)$$

while $v_L \in \mathcal{M}_L$ and $w_R \in \mathcal{M}_R$. The vacuum $|\Omega_0\rangle$ chose to be related to building the GNS Hilbert space around the TFD state of the boundary theory. Then the GNS vacuum $|\Omega\rangle$ interpretation depends on whether it belongs to $\mathcal{H}_{\Omega_0}^{GNS}$ or not. If it does so, *i.e.*

$$\mathcal{H}_{\Omega}^{GNS} = \mathcal{H}_{\Omega_0}^{GNS} \quad (1.247)$$

the new GNS vacuum corresponds to having some small excitations around the eternal black hole background and the unitaries w_R and v_L are related to the excitations which lie just in region I and III respectively. On the other hand, if the state $|\Omega\rangle$ does not lie in the GNS Hilbert space, bulk geometry is no longer described by the eternal black hole. The new vacuum can be related to excitations due to the unitary that changes the energy of the system by an amount that scales with N and thus its backreaction changes the geometry

of the spacetime. Another possible example could be the time-shifted TFD state, defined as

$$|\Psi_T\rangle = e^{i(H_L+H_R)T/2} |\Psi_{TFD}\rangle = e^{iH_L T} |\Psi_{TFD}\rangle = e^{iH_R T} |\Psi_{TFD}\rangle \quad (1.248)$$

or more generally evolving the TFD state with some other global charges. They correspond to large diffeomorphisms in the bulk which for some large value of T , it changes the bulk geometry (for more detail see [14]).

More generally we can extend the discussion in Sec. 1.7.1 and write the mapping from the bulk algebra of observables to the representation of the corresponding algebra on the boundary as

$$\mathcal{R}_{c,R}^* : \mathcal{A}_{r,0} \longrightarrow \mathcal{M}_R, \quad \mathcal{R}_{c,L}^* : \mathcal{A}_{l,0} \longrightarrow \mathcal{M}_L \quad (1.249)$$

while again to map the operators lies in the region III to the left CFT, we can use the

$$\begin{array}{ccc} \mathcal{A}_{l,0} & \xrightarrow{\mathcal{J}_{HH}} & \mathcal{A}_{r,0} \\ & & \downarrow \mathcal{R}'_{c,R-\Omega} \\ \mathcal{M}_L & \xleftarrow{\mathcal{J}_{GNS,\Omega}} & \mathcal{M}_R \end{array}$$

Petz map definition in the modular theory

$$\mathcal{R}_{c,L}^*(\cdot) = \mathcal{J}_{GNS,\Omega} \circ \mathcal{R}'_{c,R-\Omega} \circ \mathcal{J}_{HH}(\cdot) \quad (1.250)$$

where the bulk mode maps as

$$\begin{aligned} \tilde{a}_{\omega,m} &\longrightarrow \mathcal{J}_{GNS,\Omega} \circ \mathcal{R}'_{c,R-\Omega} \circ \mathcal{J}_{HH}(\tilde{a}_{\omega,m}) = \pi(\hat{O}_{\omega,m;L}) \\ \tilde{a}_{\omega,m}^\dagger &\longrightarrow \mathcal{J}_{GNS,\Omega} \circ \mathcal{R}'_{c,R-\Omega} \circ \mathcal{J}_{HH}(\tilde{a}_{\omega,m}^\dagger) = \pi(\hat{O}_{\omega,m;L}^\dagger) \end{aligned} \quad (1.251)$$

and π provides the representation of the single trace operators in \mathcal{H}_Ω^{GNS} .

Let us consider two vacua, $|\Omega\rangle$ and $|\Omega_0\rangle$, lie in the same GNS Hilbert space. In such a case since the Petz reconstruction of the modes act on the same Hilbert space, it is interesting to compare them and find the relation between them. Take the Petz map corresponding to two different GNS Hilbert space as \mathcal{R}_Ω^* and $\mathcal{R}_{\Omega_0}^*$. From (1.134), we get

$$\begin{aligned} \langle \tilde{a}_1 | \Delta_{bulk}^{1/2} | \tilde{a}_2 \rangle &= \langle \mathcal{R}_{\Omega_0}^*(\tilde{a}_1) | \Delta_0^{1/2} | \mathcal{R}_{\Omega_0}^*(\tilde{a}_2) \rangle_{\Omega_0} \\ \langle \tilde{a}_1 | \Delta_{bulk}^{1/2} | \tilde{a}_2 \rangle &= \langle \mathcal{R}_\Omega^*(\tilde{a}_1) | \Delta^{1/2} | \mathcal{R}_\Omega^*(\tilde{a}_2) \rangle_\Omega. \end{aligned} \quad (1.252)$$

Since the two are in the same GNS Hilbert space, their dual bulk geometry is the same and the left-hand sides of the equalities (1.252) coincide. By comparing the left-hand sides

$$\langle \Omega_0 | \mathcal{R}_{\Omega_0}^*(\tilde{a}_1^\dagger) \Delta_0^{1/2} \mathcal{R}_{\Omega_0}^*(\tilde{a}_2) | \Omega_0 \rangle = \langle \Omega | \mathcal{R}_\Omega^*(\tilde{a}_1^\dagger) \Delta^{1/2} \mathcal{R}_\Omega^*(\tilde{a}_2) | \Omega \rangle \quad (1.253)$$

and using (1.245), one reach to

$$\mathcal{R}_\Omega^*(\tilde{a}_\omega) = U \mathcal{R}_{\Omega_0}^*(\tilde{a}_\omega) U^\dagger. \quad (1.254)$$

Consider that we have the eternal black hole in the bulk and the vacuums $|\Omega_0\rangle$ and $|\Omega\rangle$ respectively correspond to the eternal black hole in equilibrium and have some excitations on the eternal black hole background created by U . Then, we get

$$\mathcal{R}_\Omega^*(\tilde{a}_{\omega,m}^\dagger) = \pi(\hat{O}_{\omega,m;L}) \quad \mathcal{R}_{\Omega_0}^*(\tilde{a}_{\omega,m}^\dagger) = \pi_0(\hat{O}_{\omega,m;L}) \quad (1.255)$$

where π_0 and π are representations of the single trace operators on the GNS Hilbert spaces. From (1.254), we reach to

$$\pi(\hat{O}_{\omega,m;L}) = U \pi_0(\hat{O}_{\omega,m;L}) U^\dagger. \quad (1.256)$$

1.8 Interior Petz reconstruction and Papadodimas-Raju proposal

The idea of reconstructing the bulk modes in the left exterior via the Petz map (1.250) is helpful even in the cases in which we have a one-sided black hole in the bulk. In this section, we attempt to construct the interior modes of a typical black hole microstate and we will see that we arrive to the same result as the Papadodimas-Raju proposal.

1.8.1 Papadodimas-Raju proposal

Consider a big one-sided black hole in AdS. In this case, only the bulk modes outside the horizon can be described in the boundary theory using the HKLL procedure, while describing the interior modes is much more challenging. For this purpose, a remarkable proposal has been introduced by Papadodimas and Raju in a series of papers [11–14] to find a CFT description of the black hole interior when the system is in a pure state. Here we first shortly review the PR proposal. The main idea of the PR proposal is to focus on a code subspace of the CFT theory, which is created by acting with a *small algebra* on the corresponding pure state and then find the CFT description of the interior operator in a state-dependent manner just in the chosen code subspace.

If the CFT pure state describes the stable black hole in AdS, it should be close to the thermal state. More precisely, we consider a typical pure state in the high-temperature phase of the gauge theory denoted by $|\Psi_0\rangle$ (see Sec. 1.2.4). The small algebra \mathcal{A} corresponds to simple experiments in the effective field theory in the bulk, i.e. the observables outside the horizon of the black hole. At large N limit, \mathcal{A} can be thought of as the set of products of simple trace operators of low conformal dimensions up to K number of these operators

$$\mathcal{A} = \text{span} \{O_{\omega_1}, O_{\omega_1} O_{\omega_2}, \dots, O_{\omega_1} O_{\omega_2} \dots O_{\omega_K}\} \quad (1.257)$$

such that O_{ω_i} are the Fourier modes of the single trace operators and $\sum_i \omega_i \ll \mathcal{N}$, while \mathcal{N} is the CFT's central charge. Therefore, we do not have too many insertions and so $K \ll \mathcal{N}$. Taking \mathcal{A} as the linear span of the products of the operators is equivalent to considering it as the set of all polynomials in the modes of the operators

$$A_\alpha = \sum_I \alpha(I) (O_{\omega_i})^{I(\omega_i)} \quad (1.258)$$

where $\alpha(I)$ are arbitrary coefficients and the sum runs over all functions I . This set of polynomials forms a linear space. The size of the set all such polynomials scales like \mathcal{N}^K

and limit the dimension of this space K should satisfy the constraint

$$\dim(\mathcal{A}) = \mathcal{N}^K \ll e^{\mathcal{N}}. \quad (1.259)$$

Given a typical black hole microstate $|\Psi_0\rangle$, one can define the code subspace, also called the small Hilbert space as

$$\mathcal{H}_{\psi_0} = \text{span} \{ \mathcal{A} |\Psi_0\rangle \}. \quad (1.260)$$

A typical pure state for the observables in $O \in \mathcal{A}$ can be approximated by the thermal state, i.e.

$$\langle \Psi_0 | O^\dagger O | \Psi_0 \rangle = \frac{1}{Z} \text{tr}(e^{-\beta H} O^\dagger O) + O(1/\mathcal{N}). \quad (1.261)$$

To describe interior modes in the dual CFT theory, the PR proposal requires doubling the set of operators \tilde{O}_ω corresponding to the operator in the small algebra which they call *mirror operator*. These mirror operators commute with the original operators and moreover, they should be entangled with them in the pure state $|\Psi_0\rangle$ in an appropriate way to ensure they have the right properties in a given state of the CFT. More concretely, the mirror operators defined as

$$\begin{aligned} \tilde{O}_\omega |\Psi_0\rangle &= e^{-\beta\omega/2} O_\omega^\dagger |\Psi_0\rangle \\ \tilde{O}_\omega O_{\omega_1} \dots O_{\omega_n} |\Psi_0\rangle &= O_{\omega_1} \dots O_{\omega_n} \tilde{O}_\omega |\Psi_0\rangle \end{aligned} \quad (1.262)$$

Thus, demanding that the mirror operator has the correct behavior within low-point correlators in a given pure state leads to the set of linear equations for the mirror operators. As far as we do not have too many operator insertions, this set of operators can be solved in the full Hilbert space of the CFT.

1.8.2 Black hole interior reconstruction using Petz map

As we had in Sec. 1.7.4, we can start with just one black hole exterior. Having only access to the black hole exterior is equivalent to doing simple experiments on the boundary theory. In other words, the algebra of low-energy operators which we denote here as \mathcal{A}_{ex} is identified with the algebra of coarse-grained operators of the CFT, denoted by \mathcal{A}_c (see Sec. 1.7.2)

$$\mathcal{A}_{ex} = \mathcal{A}_c. \quad (1.263)$$

From (1.209) and (1.210), we know that for the black hole in equilibrium both $\rho_{bulk}|_{\mathcal{A}_{ex}}$ and $\rho_{CFT}|_{\mathcal{A}_c}$ are thermal and under the duality, we can identify them.

One can follow the discussion in appendix A and build the GNS Hilbert spaces corresponding to these thermal states in the bulk and boundary over the algebras \mathcal{A}_{ex} and \mathcal{A}_c , we denote them as \mathcal{H}_{ex}^{GNS} and \mathcal{H}_c^{GNS} respectively. We take the vectors

$$|\Omega_{ex}\rangle \in \mathcal{H}_{ex}^{GNS} \quad |\Omega_c\rangle \in \mathcal{H}_c^{GNS} \quad (1.264)$$

as the cyclic vectors corresponding to $\rho_{bulk}|_{\mathcal{A}_{ex}}$ and $\rho_{CFT}|_{\mathcal{A}_c}$ which satisfy (1.204). Then, we have

$$\begin{aligned} \mathcal{H}_{ex}^{GNS} &= \text{span}\{ \mathcal{A}_{ex} |\Omega_{ex}\rangle \} \\ \mathcal{H}_c^{GNS} &= \text{span}\{ \mathcal{A}_c |\Omega_c\rangle \}. \end{aligned} \quad (1.265)$$

The same as in Sec. 1.7.4, we will refer to the cyclic vectors in (1.264) as the GNS vacuums of the GNS Hilbert spaces (1.265). We also identify the algebras with their representations on the GNS Hilbert spaces.

By accessing to the information outside the black hole, the observable can not distinguish between all possible geometries of the entire bulk. The bulk can be described as an eternal two-sided black hole or a one-sided black hole. If we know that in the bulk we have an eternal black hole that is dual to the TFD state on the boundary, we will reach exactly the setup that we discussed in Sec. 1.7.4. In this case, we have

$$|\Omega_{ex}\rangle = |HH\rangle, \quad |\Omega_c\rangle = |\Omega_0\rangle, \quad \mathcal{H}_{ex}^{GNS} = \mathcal{H}_{\rho_{th}}^{(Fock)}, \quad \mathcal{H}_c^{GNS} = \mathcal{H}_{TFD} \quad (1.266)$$

and

$$\mathcal{A}_{ex} = \mathcal{A}_{r,0}, \quad (\mathcal{A}_{ex})' = \mathcal{A}_{l,0}, \quad \mathcal{A}_c = \mathcal{M}_R, \quad (\mathcal{A}_c)' = \mathcal{M}_L. \quad (1.267)$$

There is another possibility that we have a black hole microstate in equilibrium. The CFT dual of such a geometry is a typical state defined in (1.76). We assume that the geometry corresponds to a typical state in the bulk has a smooth horizon and contains an interior region. Here, there does not exist a second copy of the CFT as the left system and entire bulk is dual to just one CFT.

Consider a Cauchy slice Σ in the bulk. We can divide it as $\Sigma = \Sigma_{ex} \cup \Sigma_{in}$. \mathcal{A}_{ex} is the operator algebra of observable on Σ_{ex} and the same we can denote the algebra of operators on Σ_{in} as \mathcal{A}_{in} . Since the two regions are spacelike separated, they commute with each other, and as they cover the entire Cauchy slice

$$(\mathcal{A}_{ex})' = \mathcal{A}_{in}. \quad (1.268)$$

Therefore, the commutant of the algebra \mathcal{A}_{ex} in the GNS Hilbert space \mathcal{H}_{ex}^{GNS} can be interpreted as the representation of the \mathcal{A}_{in} in the GNS Hilbert space. We identify them and denote the representation of the operator algebra inside the black hole in the GNS Hilbert space as \mathcal{A}_{in} too. We can build \mathcal{A}_{in} in the GNS Hilbert space by conjugating with the modular conjugation as

$$\mathcal{A}_{in} = J_{bulk} \mathcal{A}_{ex} J_{bulk}. \quad (1.269)$$

Hence, to each element of the algebra we associate an operator in the commutant as

$$a_\omega \in \mathcal{A}_{ex} \longrightarrow \tilde{a}_\omega \in \mathcal{A}_{in} \quad (1.270)$$

where corresponds to the modes in the black hole interior.

On the other hand on the boundary, the degree of freedoms inside the black hole are encoded in the fine-grained observables of the CFT. In other words, the image of \mathcal{A}_{in} on the boundary, which we denote as \mathcal{A}_{in-bdy} is a subalgebra of fine-grained algebra of the CFT

$$\mathcal{A}_{in-bdy} \subset \mathcal{A}_f \subset \mathcal{L}(\mathcal{H}_{CFT}). \quad (1.271)$$

Since \mathcal{A}_{in} is the commutant of \mathcal{A}_{ex} , under the duality it should map to the commutant of \mathcal{A}_c on the \mathcal{H}_c^{GNS} . Therefore, we can identify the commutant of the algebra \mathcal{A}_c on the \mathcal{H}_c^{GNS} with the representation of the \mathcal{A}_{in-bdy} on the GNS Hilbert space

$$(\mathcal{A}_c)' = J_{bdy} \mathcal{A}_c J_{bdy} = \mathcal{A}_{in-bdy} \quad (1.272)$$

where J_{bdy} is the modular conjugation for the vacuum $|\Omega_c\rangle$ corresponding to the algebra \mathcal{A}_c . We also note that if the black hole is in the microstate $|\Psi_0\rangle$, the GNS Hilbert space \mathcal{H}_c^{GNS} is isomorphic to the GNS Hilbert space obtained by acting the elements of \mathcal{A}_c on the state $|\Psi_0\rangle$.

Following the discussion in Sec. 1.7.3, we can map the algebra of the operator outside the horizon \mathcal{A}_{ext} to the coarse-grained algebra \mathcal{A}_{cg} of the boundary theory through the Petz map in (1.217) while the Schwarzschild modes mapped as (1.220). But here instead we do not know the isometry that map the interior modes to the boundary theory unlike in the case of the eternal black holes where the left exterior can also be mapped to the left CFT via the second copy of the HKLL map. Moreover, we do not even know any global mapping like the global HKLL map in the pure AdS spacetime that maps the entire bulk to the entire boundary for us to use the same logic as the one has been done to find the Petz map in order to reconstruct the entanglement wedge of a given boundary region.

The discussion we had in the previous section suggest the idea that we can use the definition of the Petz map in modular theory and map the interior modes to the boundary via (1.250). In addition to that from the bulk perspective for a black hole in a typical microstates the geometry locally is the same as eternal black holes, and even for the late time bulk correlation functions between the operators inside and outside the horizon of the collapsing star geometry, it is known that they can be approximated very well by the correlators of the operators in regions I and II of an eternal black hole [11]. Thus indeed here the interior modes play the role of the modes coming from the left side on the eternal geometry. But the important difference is that the commutant of the image of the operator algebra in the region I is no longer in the second CFT but rather it represents a subalgebra of fine-grained operators in the original CFT.

As a consequence of all the discussions above, we introduce the Petz map that encodes the interior modes of a black hole microstate in the dual CFT $\mathcal{R}_{c,in}^* : \mathcal{A}_{in} \rightarrow \mathcal{A}_{in-bdr}$ as

$$\mathcal{R}_{c,in}^*(\cdot) = \mathcal{J}_{bdy} \circ \mathcal{R}'_{c,\Omega} \circ \mathcal{J}_{bulk}(\cdot) \quad (1.273)$$

which leads to

$$\begin{aligned} \tilde{a}_{\omega,m} &\longrightarrow \mathcal{R}_{c,in}^*(\tilde{a}_{\omega,m}) = J_{bdy} \hat{O}_{\omega,m;c} J_{bdy} \equiv \tilde{O}_{\omega,m} \in \mathcal{L}(\mathcal{H}_c^{GNS}) \\ \tilde{a}_{\omega,m}^\dagger &\longrightarrow \mathcal{R}_{c,in}^*(\tilde{a}_{\omega,m}^\dagger) = J_{bdy} \hat{O}_{\omega,m;c}^\dagger J_{bdy} \equiv \tilde{O}_{\omega,m}^\dagger \in \mathcal{L}(\mathcal{H}_c^{GNS}). \end{aligned} \quad (1.274)$$

$$\begin{array}{ccc} \mathcal{A}_{in} & \xrightarrow{\mathcal{J}_{bulk}} & \mathcal{A}_{ex} \\ & & \vdots \mathcal{R}'_{c-\Omega} \\ & & \mathcal{A}_c \\ \mathcal{A}_{in-bdy} & \xleftarrow{\mathcal{J}_{bdy}} & \mathcal{A}_c \end{array}$$

We can also find the dual operator through its insertion between the vectors in the GNS Hilbert space. From (??), it is clear that every vector $|a\rangle \in \mathcal{H}_c^{GNS}$ can be obtained

by the action of an element of the algebra $a \in \mathcal{A}_c$ on the GNS vacuum

$$|a\rangle \equiv \pi(a) |\Omega_c\rangle \quad (1.275)$$

where $\pi(a)$ is the representation of a in the GNS Hilbert space. Then, we have

$$\begin{aligned} \langle a | \tilde{O}_{\omega,m} | b \rangle &= \langle \Omega_c | \pi(a)^\dagger \tilde{O}_{\omega,m} \pi(b) | \Omega_c \rangle = \langle \Omega_c | \pi(a)^\dagger \pi(b) \tilde{O}_{\omega,m} | \Omega_c \rangle \\ \langle a | \tilde{O}_{\omega,m}^\dagger | b \rangle &= \langle \Omega_c | \pi(a)^\dagger \tilde{O}_{\omega,m}^\dagger \pi(b) | \Omega_c \rangle = \langle \Omega_c | \pi(a)^\dagger \pi(b) \tilde{O}_{\omega,m}^\dagger | \Omega_c \rangle. \end{aligned} \quad (1.276)$$

To go ahead, we remind that when the black hole is in equilibrium from (??) we have

$$|\Omega_c\rangle \Big|_{\mathcal{A}_c} = \rho_{th,c} = \frac{1}{Z_\beta^c} e^{-\beta H_c}, \quad (1.277)$$

One can obtain that for every $a \in \mathcal{A}_{cg}$

$$J_{bdy} a J_{bdy} |\Omega_{cg}\rangle = \rho_{th,c}^{1/2} a^\dagger \rho_{th,c}^{-1/2} |\Omega_c\rangle = e^{-\beta H_c/2} a^\dagger e^{\beta H_c/2} |\Omega_c\rangle. \quad (1.278)$$

Considering (1.277) roughly speaking, we can also interpret the vacuum $|\Omega_c\rangle$ as a TFD state with respect to H_c in the GNS Hilbert space as

$$|\Omega_c\rangle = \sum_i e^{-\beta E_i/2} |E_i\rangle_c |E_i\rangle_f \quad (1.279)$$

where E_i s are the energy eigenvalues of the coarse-grained Hamiltonian. As it is mentioned, the coarse-grained observables are the GFFs around the thermal state of the CFT. Thus, the coarse-grained Hamiltonian should be in the form of

$$H_c = \sum_{\omega,m} \omega O_{\omega,m;c}^\dagger O_{\omega,m;c} \quad (1.280)$$

where the operators $O_{\omega,m;c}$, the projection of the Fourier modes of the single trace operators onto the coarse-grained part of the system, can be identified with the representation of the single trace operators in the GNS Hilbert space \mathcal{H}_c^{GNS} . They satisfy

$$[H_c, O_{\omega,m;c}] = -\omega O_{\omega,m;c} \quad [H_c, O_{\omega,m;c}^\dagger] = \omega O_{\omega,m;c}^\dagger \quad (1.281)$$

and therefore, one can obtain that

$$\begin{aligned} e^{-\beta H_c/2} O_{\omega,m;c} e^{\beta H_c/2} &= e^{\beta\omega/2} O_{\omega,m;c} \\ e^{-\beta H_c/2} O_{\omega,m;c}^\dagger e^{\beta H_c/2} &= e^{-\beta\omega/2} O_{\omega,m;c}^\dagger. \end{aligned} \quad (1.282)$$

In the end, using the definition of the operator $\tilde{O}_{\omega,m}$ (1.274) and the relations (1.282), we reach to

$$\begin{aligned} \langle a | \tilde{O}_{\omega,m} | b \rangle &= e^{-\beta\omega/2} \langle \Omega_c | \pi(a)^\dagger \pi(b) O_{\omega,m;c}^\dagger | \Omega_c \rangle = e^{-\beta\omega/2} \text{tr}(\rho_{th,c} a^\dagger b O_{\omega,m}^\dagger) + O(1/\mathcal{N}) \\ \langle a | \tilde{O}_{\omega,m}^\dagger | b \rangle &= e^{\beta\omega/2} \langle \Omega_c | \pi(a)^\dagger \pi(b) O_{\omega,m;c} | \Omega_c \rangle = e^{\beta\omega/2} \text{tr}(\rho_{th,c} a^\dagger b O_{\omega,m}) + O(1/\mathcal{N}). \end{aligned} \quad (1.283)$$

Moreover, we also obtain

$$\begin{aligned} \tilde{O}_{\omega,m} |\Omega_c\rangle &= e^{-\beta\omega/2} O_{\omega,m;c}^\dagger |\Omega_c\rangle \\ \tilde{O}_{\omega,m} O_{\omega_1,m_1;c} \dots O_{\omega_n,m_n;c} |\Omega_c\rangle &= O_{\omega_1,m_1;c} \dots O_{\omega_n,m_n;c} \tilde{O}_{\omega,m} |\Omega_c\rangle \end{aligned} \quad (1.284)$$

which is equivalent to the operators result in the PR proposal but it is obtained in a more concrete way. The black hole microstate is replaced by the GNS vacuum state $|\Omega_c\rangle$ and considering the code subspace in the PR proposal is equivalent to work in the GNS Hilbert space.

The GNS Hilbert space \mathcal{H}_c^{GNS} can be also constructed through the action of \mathcal{A}_c on other cyclic and separating vectors $|\Omega'_c\rangle$ belong to \mathcal{H}_c^{GNS} . They can be related to $|\Omega_c\rangle$ via a unitary operator $u \in \mathcal{L}(\mathcal{H}_c^{GNS})$ as

$$|\Omega'_c\rangle = U |\Omega_c\rangle \quad (1.285)$$

and the vectors in the GNS Hilbert space can be identified as

$$\mathcal{H}_c^{GNS} = \text{span}\{|a'\rangle \equiv \pi(a)|\Omega'_c\rangle \mid \forall a \in \mathcal{A}_c\}. \quad (1.286)$$

If we denote the representation of the operator dual to the interior mode $\tilde{a}_{\omega,m}$ in the GNS Hilbert space is built over the vector $|\Omega'_c\rangle$ by $\tilde{O}'_{\omega,m}$, from (??) we reach to

$$\tilde{O}'_{\omega,m} = U \tilde{O}_{\omega,m} U^\dagger \quad (1.287)$$

while $\tilde{O}_{\omega,m}$ is the representation of the dual operator we defined in the GNS Hilbert space over the thermal state (1.274). In particular, for the matrix elements of the operator we find that

$$\begin{aligned} \langle a' | \tilde{O}'_{\omega,m} | b' \rangle &= \langle \Omega'_c | \pi(a)^\dagger \tilde{O}'_{\omega,m} \pi(b) | \Omega'_c \rangle \\ &= \langle \Omega_c | U^\dagger \pi(a)^\dagger U \tilde{O}_{\omega,m} U^\dagger \pi(b) U | \Omega_c \rangle. \end{aligned} \quad (1.288)$$

and to be compatible with PR proposal, we can also find that

$$\begin{aligned} \tilde{O}'_{\omega,m} | \Omega'_c \rangle &= U \tilde{O}_{\omega,m} | \Omega_c \rangle \\ &= U \rho_{th,c}^{1/2} O_{\omega,m;c}^\dagger \rho_{th,c}^{-1/2} U^\dagger | \Omega'_c \rangle \\ &= U e^{-\beta\omega/2} O_{\omega,m;c}^\dagger U^\dagger | \Omega'_c \rangle. \end{aligned} \quad (1.289)$$

The vectors in (??) correspond to an equilibrium black hole background which is excited by some sources. It can be provided by turning on a source for some CFT operators. The unitary operator in (??) is indeed the representation of composite operator which creates that excitation. The simplest cases for the unitary operator U that result in a cyclic and separating vector $|\Omega'_c\rangle$ correspond to the local unitaries as

$$V_c \in \mathcal{A}_c \quad (1.290)$$

related to the excitation only on the region I, or

$$W_f \in \mathcal{A}'_c = \mathcal{A}_{in-bdy}. \quad (1.291)$$

correspond to the excitation behind the black hole horizon. If we have the unitaries as in (1.290), we get

$$\langle a' | \tilde{O}'_{\omega,m} | b' \rangle = \langle V_c^\dagger a V_c | \tilde{O}_{\omega,m} | V_c^\dagger b V_c \rangle \quad (1.292)$$

since $V_c^\dagger a V_c \in \mathcal{A}_c$, and in the case that we have some excitation inside the black hole, i.e. act with some unitary in the form of (1.291), we get

$$\langle a' | \tilde{O}'_{\omega,m} | b' \rangle = \langle a | \tilde{O}_{\omega,m} | b \rangle \quad (1.293)$$

as W_f and W_f^\dagger commute with every $a, b \in \mathcal{A}_c$. As a result, as it is expected we see that if we access only on the coarse-grained observables, we can not detect what happening behind the black hole horizon.

Chapter 2

Localization of Information in Quantum Gravity

: Within the AdS/CFT correspondence, we identify a class of CFT operators which represent diff-invariant and approximately local observables in the gravitational dual. Provided that the bulk state breaks all asymptotic symmetries, we show that these operators commute to all orders in $1/N$ with asymptotic charges, thus resolving an apparent tension between locality in perturbative quantum gravity and the gravitational Gauss law. The interpretation of these observables is that they are not gravitationally dressed with respect to the boundary, but instead to features of the state. We also provide evidence that there are bulk observables whose commutator vanishes to all orders in $1/N$ with the entire algebra of single-trace operators defined in a space-like separated time-band. This implies that in a large N holographic CFT, the algebra generated by single-trace operators in a short-enough timeband has a non-trivial commutant when acting on states which break the symmetries. It also implies that information deep in the interior of the bulk is invisible to single-trace correlators in the time-band and hence that it is possible to localize information in perturbative quantum gravity

2.1 Aspects of locality in field theory and gravity

In this section, mostly addressed to non-experts, we review some background necessary to explore the question of localizing information in different regions of space. A closely related question is the association of algebras of observables to subregions and the factorization of the Hilbert space. We start with non-gravitational field theories, where a non-dynamical background space-time can be used in order to define sub-regions and their causal relations, and then we consider the additional complications when gravity is taken into account.

In relativistic theories we expect that signals and information cannot travel faster than light. We then want to address the following question: consider an initial space-like slice Σ and divide it into a compact subregion D and its complement D' . We denote by $J(D')$ the domain of dependence of D' . The question is the following: is it possible to modify the state¹ in region D without affecting the state in $J(D')$. If the answer is positive then an observer initially in D' , and confined to move in $J(D')$, cannot reconstruct information about the interior of D . Then we say that information can be localized.

¹Either classical state, or quantum density matrix.

2.1.1 Classical field theories

At the classical level this question can be addressed by studying the initial value problem: we specify initial data \mathcal{C} on a spacelike slice Σ and then look for a solution in the entire space-time, or at least a neighborhood of the slice Σ , compatible with the initial data. The initial data will typically include the values and time-derivatives of various fields of the theory. The theories we will be considering have gauge invariance. One of the implications is that the existence of a solution is guaranteed only if the initial data satisfy certain *constraints*. In relativistic field theories the dynamical equations are hyperbolic, which ensures that signals propagate forward from Σ at most at the speed of light. On the other hand the constraint equations for initial data are of elliptic nature. This makes the question of being able to specify the initial data independently in region D and its complement D' non-trivial. It is thus convenient to divide the question formulated above in two steps:

- **A. Localized preparation of states:** for given initial data \mathcal{C}_1 on Σ satisfying the constraints, to what extent can we deform to other initial data \mathcal{C}_2 , also satisfying the constraints, such that $\mathcal{C}_1, \mathcal{C}_2$ agree on D' , possibly up to a gauge transformation, but differ essentially² on D ?
- B. No super-luminal propagation:** suppose we are given two initial data $\mathcal{C}_1, \mathcal{C}_2$ which satisfy the constraints, which agree on D' and differ on D . We then want to show that the two corresponding solutions agree on $J(D')$, possibly up to a gauge transformation.

We will return to the classical problem in theories with gauge invariance in the following subsections. For now we briefly consider the simplest example of a free Klein-Gordon field in flat space obeying $\square\phi = m^2\phi$. We consider initial data on the slice Σ corresponding to $t = 0$. The initial data on this slice are parametrized by $\mathcal{C} = \{\phi(t = 0, x), \partial_0\phi(t = 0, x)\}$. In this case condition *A* mentioned earlier is clearly satisfied: the initial data do not need to obey any constraint, so we can simply select the functions $\phi, \partial_0\phi$ to have any smooth profile with features strictly localized in D . Notice that this requires the use of non-analytic initial data. Condition *B* is also satisfied, see [157] for a basic review.³

2.1.2 Localization of information in QFT

In non-gravitational QFT we can associate algebras of observables to space-time regions [21–23]. Locality is exact, and is expressed by the condition that algebras corresponding to space-like separated regions commute. An analogue of the initial value problem in QFT is expressed by the condition of *primitive causality* or relatedly the *time-slice axiom* which postulates that the only operators commuting with the algebra generated by operators in a time-band are proportional to the identity. Moreover a local version of these statements postulates that the algebra of operators in a subregion coincides with the algebra of operators in the causal domain of dependence of the subregion [158].

An intuitive way to see that that information can be localized in QFT is as follows: suppose $|\Psi_0\rangle$ is a state in the Hilbert space of the QFT. Consider a unitary operator U_D

²i.e. cannot be matched by a gauge transformation on D .

³In the case of non-relativistic theories, for example the heat equation, which is first order in time and hence not hyperbolic, we are able to specify the initial data in subregions independently but the speed of propagation is unbounded. Hence the heat equation obeys condition *A* but not *B*.

constructed out of observables localized in D and the new state $|\Psi\rangle = U_D|\Psi_0\rangle$. The unitary U_D modifies the state by creating an excitation in region D which encodes the desired information in that region. For any observation $\mathcal{O}_{D'}$ in region D' , and more generally in $J(D')$, we have

$$\langle\Psi|\mathcal{O}_{D'}|\Psi\rangle = \langle\Psi_0|U_D^\dagger\mathcal{O}_{D'}U_D|\Psi_0\rangle = \langle\Psi_0|\mathcal{O}_{D'}|\Psi_0\rangle, \quad (2.1)$$

where we used $[U_D, \mathcal{O}_{D'}] = 0$. Hence states $|\Psi\rangle, |\Psi_0\rangle$ are indistinguishable by measurements in $J(D')$ and the excitation created by U_D in D is invisible in $J(D')$.

Comments on the split property

More generally we would like to know whether it is possible to independently specify the quantum state in space-like separated regions. The question is non-trivial since in most quantum states these regions will be entangled. It is believed that, as long as the regions in question are separated by a finite buffer region, then the answer should be positive. This is related to the *split property* of quantum field theory [23, 25, 159, 160].

The split property can be defined as follows: consider the causal diamond whose base is a ball D_1 and the corresponding operator algebra \mathcal{A}_{D_1} . Consider a slightly larger ball D_2 , containing D_1 , with corresponding operator algebra \mathcal{A}_{D_2} in its causal diamond. The split property is satisfied if we can find a type I von Neumann algebra of operators \mathcal{N} such that $\mathcal{A}_{D_1} \subset \mathcal{N} \subset \mathcal{A}_{D_2}$. It has been shown that quantum field theories with a reasonable thermodynamic behavior, as expressed in terms of nuclearity conditions (see [23] for an introduction), satisfy the split property. Using the algebra \mathcal{N} we can have strict localization of quantum information which is completely inaccessible from $J(D'_2)$.

Equivalently, the split property can be defined by the existence of a state $|\phi\rangle$ which is cyclic and separating for the algebra $\mathcal{A}_{D_1 \cup D'_2}$ and such that

$$\langle\phi|a b|\phi\rangle = \langle 0|a|0\rangle\langle 0|b|0\rangle \quad \forall a \in \mathcal{A}_{D_1}, b \in \mathcal{A}_{D'_2}, \quad (2.2)$$

where $|0\rangle$ is the Minkowski vacuum and D'_2 denotes the complement of D_2 . In the state $|\phi\rangle$, the mutual information between regions D_1 and D'_2 is vanishing. Such a state is not uniquely defined, since for any unitary $U \in \mathcal{A}_{(D'_1 \cap D_2)}$ a state of the form $U|\phi\rangle$ will also satisfy (2.2).

Starting with a split state $|\phi\rangle$ we can construct more general states by exciting the two regions D_1 and D'_2 acting with localized operators in the corresponding algebras. Since there is no entanglement between D_1 and D'_2 in the split state $|\phi\rangle$ the two algebras act independently and we can arbitrarily approximate an excited state in D_1 and another state in D'_2 .

An interesting question is to estimate the energy of a split state⁴. We do not expect a split state to be an energy eigenstate, so in general it will have non-vanishing energy variance. Here we provide some very heuristic arguments about the expectation value of the energy. As a starting point, let us consider a CFT on $\mathbb{R}^{1,d-1}$ with coordinates x^0, x^1, \dots, x^{d-1} . We define two regions to be the causal domains of two slightly displaced Rindler wedges with bases $x^0 = 0, x^1 < -\epsilon$ and $x^0 = 0, x^1 > \epsilon$ respectively. The two wedges are separated by the buffer region $-\epsilon < x^1 < \epsilon$. In this case the total energy of the split state will be infinite due to the infinite planar extension of the regions in the

⁴Since the split state is not unique, a reasonable question might be finding the lowest possible expectation value for the energy of a split state.

transverse directions. However, we expect to have a finite energy per unit area \mathcal{E} . Since we are dealing with a CFT then the only scale in the problem is the size ϵ of the buffer region. Hence by dimensional analysis the energy per unit area will scale like $\mathcal{E} = \frac{s}{\epsilon^{d-1}}$ where s is a constant depending on the CFT. If we now consider a more general compact region D_1 of typical size R , which is separated by a small buffer region of typical size ϵ from D'_2 then we would expect that a split state with respect to D_1, D'_2 will have energy which in the $\epsilon \rightarrow 0$ limit will scale like

$$E = s \frac{A_{(\partial D_1)}}{\epsilon^{d-1}} + O\left(\frac{\epsilon}{R}\right), \quad (2.3)$$

where $A_{(\partial D_1)}$ is the area of the boundary of D_1 . This is a heuristic estimate and it would be interesting to investigate it more carefully.

As mentioned above, this is the expectation value of the energy and it would be interesting to understand the spectral decomposition of a split state in the energy basis. Notice that a split state does not respect the Reeh-Schlieder property with respect to the algebra \mathcal{A}_{D_1} ⁵. This implies in particular that the split state should have non-compact support in energy, since otherwise the Reeh-Schlieder property would have to hold for D_1 , see for example [161].

Subtleties with gauge invariance

Consider $U(1)$ gauge theory minimally coupled to a charged scalar with Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\phi)^*D^\mu\phi$, $D_\mu\phi = \partial_\mu\phi - igA_\mu\phi$. The system has $U(1)$ gauge invariance $A_\mu \rightarrow A_\mu + \partial_\mu\Lambda$, $\phi \rightarrow e^{ig\Lambda}\phi$. The dynamical equations are

$$\begin{aligned} \partial^\nu F_{\mu\nu} &= ig(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) - 2g^2A_\mu\phi^*\phi \\ \square\phi &= ig(\partial_\mu A^\mu)\phi + 2igA^\mu\partial_\mu\phi + g^2A_\mu A^\mu\phi. \end{aligned} \quad (2.4)$$

In this case the initial data are $\mathcal{C} = \{A_\mu(t=0, x), \partial_0 A_\mu(t=0, x), \phi(t=0, x), \partial_0\phi(t=0, x)\}$. Here we encounter the subtleties mentioned for gauge systems: initial data related by a gauge transformation are physically equivalent and initial data are admissible (i.e. lead to a solution) only if they obey a constraint, the Gauss law, which is the $\mu=0$ component of the first equation in (2.4)

$$\partial^i(\partial_0 A_i - \partial_i A_0) = ig(\phi\partial_0\phi^* - \phi^*\partial_0\phi) - 2g^2A_0\phi^*\phi. \quad (2.5)$$

We now revisit the two properties mentioned in subsection 2.1.1. The fact that the dynamical part of (2.4) obey condition *B* follows from general properties of hyperbolic equations of this type. Let us now examine question *A* in this theory. From (2.5) we see that if we try to deform the initial data in region D , then we may be forced to change them in D' too. For example if we turn on a profile for the scalar in region D with total non-zero charge, then the gauge field has to be turned on in region D' . The Gauss law constraint (2.5) is of the familiar form $\nabla \cdot \vec{E} = \rho$. This imposes the constraint that $\oint_{\partial D} \vec{E} \cdot d\vec{S} = Q_D$.

However it is clear that once we make sure that the initial data in \mathcal{D}' are compatible with the Gauss constraint from the total charge Q_D enclosed in D , there are many ways of rearranging the initial data in region D keeping those in D' fixed. In other words there are

⁵Since there is no entanglement between D_1 and D'_2 we cannot create excitations in region D'_2 by acting with operators in D_1 .

deformations of the constraint equation (2.5), which are not gauge-equivalent, and which have compact support localized in D . This means that theory under consideration obeys condition A.

Moving on to the quantum theory, we can consider $U(1)$ gauge theory weakly coupled to matter. As in the classical theory the total charge Q enclosed in a region can be measured on its boundary and the total charge of the entire state can be measured at space-like infinity. At the quantum level, we can get information not only about the expectation value of the charge but all the higher moments

$$\langle \Psi | Q^n | \Psi \rangle \quad , \quad n = 1, 2, \dots \quad (2.6)$$

To proceed it is useful to consider observables in this theory. Physical observables must be gauge invariant. In a $U(1)$ gauge theory there are several examples of such observables which are also local, for example, local operators constructed out of $F_{\mu\nu}(x)$ or $\phi^*(x)\phi(x)$. Other interesting gauge invariant operators which are not completely local, but can be contained in compact regions are closed Wilson loops $e^{ig \oint_C A_\mu dx^\mu}$ or bilocals of the form $\phi^*(x)e^{ig \int_{C,x}^y A_\mu dx^\mu} \phi(y)$. All these operators are neutral and do not change the electric charge of the region D , if they are entirely contained in D . We can use such operators localized in region D to construct unitaries U_D which can be used to modify the state inside D leaving all correlators outside invariant, as in (2.1). So information can be localized in this theory if we work with neutral operators.

But what if we want to create an excitation in region D which has non-zero charge? We already know from the classical problem that it will not be possible to add a charge in D without affecting the exterior due to Gauss law (2.5). The same is true at the quantum level. A charged operator ϕ in D is not gauge invariant. It can be made gauge invariant by dressing it with a Wilson line extending all the way to infinity. We can think of the Wilson line as a localized tube of electric flux ensuring that Gauss law is satisfied. It may be energetically better to smear the Wilson line in a spherically symmetric configuration. The important point is that the dressed operator $\Phi(x) = e^{ig \int_\infty^x A_\mu dx^\mu} \phi(x)$ is no longer a local operator, though it is gauge invariant. If we act with a unitary made out of this operator, we will modify correlators outside D and (2.1) will fail. This means that the addition of the charge in D can be detected immediately outside. This is not surprising, as the same thing is already visible at the classical level.

However, looking a bit more carefully, we run into certain somewhat surprising features of the quantum theory. Suppose we have several charged fields ϕ_i , labeled by a flavor index i , with the same electric charge. We construct the corresponding dressed operators $\Phi_i(x) = e^{ig \int_\infty^x A_\mu dx^\mu} \phi_i(x)$, using some particular prescription for the Wilson line. These obey

$$[Q, \Phi_i(x)] = g \Phi_i(x) \quad , \quad (2.7)$$

where $Q = \int_{\mathbb{S}_\infty^2} *F$ is the charge operator which can be measured at space-like infinity. Suppose the point $x = 0$ is inside D . We create a charged excitation of type i in region D by acting on $|0\rangle$ with a unitary $U_i = e^{i\epsilon\Phi_i(0)}$. Then we study correlators in region D' in the state $U_i|0\rangle$ in perturbation theory. Consider a correlator of Q and $\Phi_j(x)$ in region D' .

$$\langle 0 | U_i^\dagger \Phi_j(x) Q U_i | 0 \rangle = \langle 0 | \Phi_j(x) | 0 \rangle + i\epsilon \langle 0 | [\Phi_j(x) Q, \Phi_i(0)] | 0 \rangle + \mathcal{O}(\epsilon^2) \quad , \quad (2.8)$$

where to leading order in the perturbative expansion the second term is

$$\langle 0 | [\Phi_j(x) Q, \Phi_i(0)] | 0 \rangle = g \langle 0 | \Phi_j(x) \Phi_i(0) | 0 \rangle \propto \delta_{ij} \quad . \quad (2.9)$$

Hence by measuring correlators of all $\phi_j(x)$ and Q in \overline{D} it seems that we can detect not only the presence of a charge in D , which is expected by Gauss's law, but we can even identify the flavor of the charged particle, i.e. the value of the index i in the interior of D . A similar argument in the gravitational case was discussed in [12, 162] for black hole states and in [72] around empty AdS.

The reason we were able to get information beyond the total charge in D is that in the vacuum the fields have non-trivial entanglement, on which the non-vanishing 2-point function (2.9) depends. When we act with the unitary containing the Wilson line, the Wilson line disturbs the pattern of entanglement in such a way that it breaks the symmetry between the fields ϕ_i and we can detect from D' the flavor of the excitation in D .

This suggests a way to avoid the issue and succeed in hiding the flavor of charge in D . We start with the analogue of a split state in the U(1) gauge theory, see the discussion in [67], and then create the charged excitation in D by acting with the same unitary. In that case there is no entanglement between D and D' and hence (2.9) will vanish making it impossible to tell from measurements in D' what is the type of charged excitation in D .⁶ This requires creating the charged excitation on top of the split state, with typical energy scaling like (2.3), rather than the ground state.

2.1.3 Classical and Quantum Gravity

First we notice that in non-perturbative quantum gravity we do not expect to be able to localize information in space: holography and AdS/CFT suggest that the fundamental degrees of freedom in quantum gravity are not local, but rather lie at the boundary. Moreover there is strong evidence that an ingredient towards the resolution of the black hole information paradox is that the naive factorization of the Hilbert space in space-like separated subregions may not be true in the underlying theory of quantum gravity.

On the other hand at the classical level in General Relativity we do have an exact notion of locality and information can be localized, as we will discuss below. An interesting question, which is the main focus of this paper, is to understand the fate of locality at the level of perturbative quantum gravity.

On the initial value problem of general relativity

In General Relativity the initial value problem is formulated by starting with a spacelike slice Σ and specifying the data $\mathcal{C} = (h_{ab}, K_{ab})$ where h_{ab} is the intrinsic metric and K_{ab} the extrinsic curvature of Σ . If we have matter then the values of the fields and their normal derivatives need to be specified. Initial data related by spatial diffeomorphisms on the slice Σ are gauge-equivalent and have to be physically identified. In general relativity there is one more subtlety: even if we have two initial data on the slice Σ which are not related by a spatial diffeomorphism, they may still correspond to the same physical solution in space-time. This is related to the freedom of choosing the initial slice Σ in space-time and diffeomorphism invariance in full space-time.

Admissible initial data, which can be extended into a solution of the Einstein equations must obey the following constraints

$$R + (K_a^a)^2 - K_{ab}K^{ab} = 16\pi G\rho \quad (2.10)$$

⁶A more mundane way to hide the charge is to add "screening charges" in the buffer region, but here we want to discuss how information can be localized even though a Wilson line extends all the way to infinity.

$$\nabla^a K_{ab} - \nabla_b K_c^c = -8\pi G J_b, \quad (2.11)$$

where R is the Ricci scalar of h_{ab} on Σ , the covariant derivatives are with respect to h_{ab} on Σ , n^a is the unit normal to Σ and $\rho = T_{ab}n^an^b$ and $J_b = -h_b^c T_{ca}n^a$.

We now want to address the question of localization of information in classical general relativity, as formulated in subsection 2.1.1. A theorem, see for example [157, 163], settles question *B* for pure general relativity: if we have two admissible initial data which agree, up to spatial diffeomorphism, on a part D' of Σ , then the corresponding solutions will agree, up to a space-time diff, on the development of D' . This continues to be true in the presence of matter provided certain reasonable conditions are satisfied. This shows that in general relativity signals propagate at most at the speed of light: if we modify the initial data only in the region D , then the signals will propagate in the causal future of D .

Then we come to question *A*, that of localizing information on compact regions on Σ : to what extent is it possible to find two initial data satisfying the constraints (2.10), (2.11), which agree on D' but differ on D ?⁷ The equations (2.10) and (2.11) are non-linear and of elliptic nature, though underdetermined. Understanding the space of solutions of the constraint equations is an interesting problem which has been studied extensively in the literature. Here we summarize some relevant points:

1. **Gravitational Gauss law:** in asymptotically flat or AdS space-times, the energy and other conserved charges are defined at space-like infinity. The constraints of general relativity relate these asymptotic charges to contributions from excitations in the interior of space-time. For example, in the Newtonian limit the constraint equations reduce to the gravitational analogue of Gauss's law

$$\square\phi = 4\pi G\rho.$$

As in electromagnetism this implies that the initial data in region D' know about the total mass enclosed in D .

2. **Existence of localized deformations:** it is possible to find many solutions of the constraint equations which look the same in the domain D' but differ on D . For example, if we restrict our attention to spherically symmetric solutions, Birkhoff's theorem implies that there is a large number of solutions of (2.10) and (2.11) which all look like the Schwarzschild metric of mass M in D' but differ in D . Examples include static, interior, star-like geometries supported by matter or more generally spherically symmetric, time-dependent collapsing geometries of mass M . More generally, it has been shown [164] that under reasonable conditions a compact patch D of a solution of the constraints (2.10) and (2.11) can be glued to a boosted, Kerr solution in D' of appropriate mass, angular momentum, momentum and center of mass position. The existence of a large number of solutions, which all look exactly the same in D' demonstrates that it is possible to localize information in classical general relativity.
3. **Comments on the vacuum:** For asymptotically AdS geometries, if a solution looks like empty AdS in D' ⁸ then it is guaranteed to be empty AdS in D as well. In other words, starting with the vacuum it is not possible to modify the initial data in D into a new solution, without at the same time modifying the solution in D' .

⁷Here we need to keep in mind that even if the initial data differ on D they may correspond to the same solution in space-time, as they may correspond to two different choices of the slice Σ in the same space-time solution.

⁸Here we assume that D is compact so D' includes the region near space-like infinity.

Diff-invariant observables in classical GR

We now consider the question of defining local diff-invariant observables in gravity. This is a long-standing problem which is subtle even at the classical level. Let us consider general relativity, possibly coupled to other fields, defined with certain asymptotic boundary conditions at infinity (for example asymptotically flat or AdS) or on a closed manifold of fixed topology. We denote by $\overline{\mathcal{X}}$ the space of solutions of the equations of motion, in any possible coordinate system. On this space we have the action of the group Diff of diffeomorphisms⁹. Solutions related by a diffeomorphism are physically identified and we introduce

$$\mathcal{X} = \overline{\mathcal{X}}/\text{Diff}. \quad (2.12)$$

We can think of a diff-invariant observable as a function which has definite values on points of \mathcal{X} . However, we do not demand an observable to be necessarily defined on the entire space of solutions \mathcal{X} . Instead we will allow observables to possibly have a limited domain of definition. Hence a diff-invariant observable is a map

$$A : U \subset \mathcal{X} \rightarrow \mathbb{R}, \quad (2.13)$$

where U is an open subset of \mathcal{X} . Such observables can also be expressed as functions on $\overline{\mathcal{X}}$ which must obey $\overline{A}(s) = \overline{A}(f_*s)$, where s denotes a solution in some coordinate system and f_* the action of a diffeomorphism.

In order for a diff-invariant observable to be local we need to impose additional conditions. To formulate these conditions it is useful to introduce the Peierls bracket $\{A, B\}$ between two diff-invariant observables [165], which is a covariant generalization of the Poisson bracket. To compute the value of $\{A, B\}$ we consider a modification of the action as $S \rightarrow S + \epsilon A$ and compute the difference of the first order change of observable B on the perturbed solutions with advanced (+) and retarded (-) boundary conditions. The Peierls bracket is defined as¹⁰

$$\{A, B\} = \delta_A^- B - \delta_A^+ B. \quad (2.14)$$

It can be shown that the Peierls bracket has similar properties as the Poisson bracket, for example linearity, antisymmetry and the Jacobi identity, and in fact coincides with the Poisson bracket if a Hamiltonian formalism is introduced. One of the advantages of the Peierls bracket is that we do not need to pass to the Hamiltonian formalism which is somewhat complicated due to the constraints. Notice that to define the Peierls bracket of two observables A, B they must have a common domain of definition on \mathcal{X} and the bracket will be generally a non-trivial function on this overlap.

We would like to define diff-invariant observables which can be associated to *points* in space-time with the property that if two such observables are associated to space-like separated points the corresponding Peierls bracket must vanish. The difficulty in doing this is that in order to define an observable we need to define it at least in an open neighborhood around a state as in (2.13), so we need some prescription for following "the same point", on which the candidate diff-invariant observable will be localized, as we move on the space of solutions \mathcal{X} . General covariance implies that there is no canonical way to keep track of the point as we change the state.

⁹If the space-time is non-compact along space we only consider *small* diffeomorphism, i.e. those which become trivial fast enough at infinity.

¹⁰The first order solutions are not unique due to diffeomorphism invariance, however the ambiguity drops out when computing the change of the diff-invariant observable B .

If the space-time has a well-defined boundary we can find prescriptions which select a point in space-time for each solution in \mathcal{X} *relationally* with respect to the boundary. For example in AdS we can define a diff-invariant observable which *seems* to be localized at a point by considering a radial geodesic at right angle from a specific point on the boundary, moving a fixed regularized distance along it and measuring the value of a scalar quantity, for example a scalar field or a scalar combination of the curvature, at the resulting point. This gives a map from the space of solutions \mathcal{X} to \mathbb{R} , so it is a diff-invariant observable which could potentially be local. Notice however that the location of the resulting point depends on the entire geometry along the geodesic, all the way from the boundary. Changing the metric anywhere along this geodesic will move the resulting point. Hence the value of the observable will not strictly depend on local data near the point. Similarly, if we act with one of the asymptotic symmetries the boundary starting point will move and also the resulting bulk point will move. This implies that the Peierls brackets of this candidate observable with the boundary charges, or other observables along the geodesic will be non-zero, even though these regions are space-like separated. Hence this relational observable is not really local.

Another way to to define candidate local diff-invariant observables is to consider a complete gauge fixing scheme. Then observables in the particular gauge labeled by a space-time coordinates are automatically diff-invariant. However they will generally have non-local Peierls brackets, since the assignment of a coordinate value to a point in space-time in the particular gauge, will generally depend on the solution everywhere.

Additional difficulties arise in space-times without boundaries, for example in de Sitter space. A boundary is an (asymptotic) part of the spacetime where gravity is not dynamical anymore. This is why we can for example anchor geodesics to the boundary, and define relational diff-invariant observables. Without a boundary, there is no part of the space-time where gravity is turned off, and consequently no place to anchor geodesics.

State-dressed observables

If we consider a solution that is sufficiently complicated it is possible to specify points, and hence define local diff-invariant observables, by using *features of the state*. We emphasize that these observables will not have all the desired properties over the entire space of solutions \mathcal{X} , so these observables have certain aspects of state-dependence as discussed around (2.13). One approach based on this idea was studied by DeWitt [166], building on [42, 43]. For a D -dimensional space-time we start by identifying D scalar quantities $Z^a, a = 1, \dots, D$. These can be combinations of curvature invariants and other scalars formed by the fields of the theory. We could try to fix a coordinate system by using these D -scalars as coordinates. We can use this intuition to introduce candidate local diff-invariant observables of the form

$$\phi(Z_0^a) = \int d^D x \phi(x) \delta^D(Z^a - Z_0^a) \det \frac{\partial Z}{\partial x}. \quad (2.15)$$

Here Z^a are the D scalar quantities introduced above and ϕ is any other scalar combination of the fundamental fields of the theory. Similar constructions can be done for fields with tensor indices.

Some comments are in order:

1. For a general space-time which is in-homogenous, and for certain choices of the values Z_0^a , the delta function in (2.15) will click on a finite number of points, so the quantity

above is well-defined and finite. In symmetric space-times it will either not click at all, hence the observable will be zero, or an infinite number of times so the observable will be ill-defined. This shows that (2.15) is a quantity which is defined only on part of the phase space. This is in accordance with our expectation that state-dressed observables have to be state-dependent (2.13).

2. Suppose that the observable (2.15) is well defined on a state s and a neighborhood U of the space of solutions \mathcal{X} around it. It is clear that, at least at the classical level, this observable is diff-invariant, i.e. a well defined map $\phi(Z_0^a) : U \subset \mathcal{X} \rightarrow \mathbb{R}$ and hence a good observable according to the definition (2.13).
3. One can show that under certain conditions, observables (2.15) are also *local*. If we have a state s on which two such observables $\phi(Z_A^a), \phi(Z_B^b)$ are well defined, with the property that the delta functions click at single points A, B and that these points are space-like separated with respect to the metric of s , then the corresponding observables have vanishing Peierls brackets $\{\phi(Z_A^a), \phi(Z_B^b)\} = 0$, see [167] for a review. This follows from the causality properties of linearized Green's functions appearing in (2.14) around the solution s . Notice that if two points A, B are spacelike separated on a solution s , then there is a small enough neighborhood of s in which they remain space-like separated. Hence their Peierls bracket will vanish in this entire neighborhood.
4. This shows that, as long as we accept that observables may be defined only locally on the phase space of solutions, it is possible to find local, diff-invariant observables in classical general relativity around states which are complicated enough. These are also the interesting states, i.e. those containing bulk observers who want to study physics in their environment.
5. Similar ideas are useful in cosmology, where the value of a scalar field can be used as as clock [52–54].

The next question is whether it is possible to define similar observables at the quantum level. Aspects of this question were discussed in [47] and [48], where it was argued that there is a quantum version of these observables which retain their locality properties to all orders in the \hbar expansion, even though they are not expected to be local at the non-perturbative level. Various difficulties are encountered at the quantum level including the question of the renormalization of the composite operators (2.15), establishing diffeomorphism invariance at the quantum level and the role of Poincare recurrences which will generally introduce infinite copies where the delta function will have support [48]. In this paper we provide support in favor of this conjecture by finding observables with certain similarities in spirit to (2.15) directly in CFT language. This has the advantage that any object built directly in the CFT is by construction diff-invariant.

A time-band in AdS

We now specialize to a setup that will allow us to make contact with AdS/CFT. We consider geometries that are asymptotically AdS $_{d+1}$ and we consider a short time-band $\mathcal{T}_{-\epsilon, \epsilon}$ on the boundary in global coordinates, defined as the set of points $(-\epsilon, +\epsilon) \times \mathbb{S}^{d-1}$, $\epsilon > 0$, where the first interval refers to the time coordinate t . Near the boundary we can select a

Fefferman-Graham coordinate system where the fields, for example the metric and a scalar of mass m^2 , have the behavior

$$ds^2 = \frac{dr^2}{r^2} + r^2(-dt^2 + d\Omega_{d-1}^2) + r^{2-d} g_{\mu\nu}(r, x) dx^\mu dx^\nu \quad g_{\mu\nu}(r, x) = g_{\mu\nu}^{(0)}(x) + g_{\mu\nu}^{(2)}(x)r^{-2} + \dots$$

$$\phi = r^{-\Delta}(\phi^{(0)}(x) + \phi^{(2)}(x)r^{-2} + \dots), \quad (2.16)$$

where $x = (t, \Omega_{d-1})$ and $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}$. Here we consider normalizable states so the growing modes, which would be dual to sources in the CFT, are set to zero¹¹. The Fefferman-Graham coefficients $g_{\mu\nu}^{(0)}(x), \phi^{(0)}(x)$ are diff-invariant observables and are labelled by boundary coordinates¹². This set of observables includes the asymptotic charges, for example the ADM Hamiltonian can be computed as

$$H = \frac{1}{\text{const}} \int_{\mathbb{S}^{d-1}} d\Omega^{d-1} g_{00}^{(0)}(x).$$

We focus on these Fefferman-Graham observables restricted in the time band $\mathcal{T}_{-\epsilon, \epsilon}$. This set of observables is closed under Peierls brackets and form a Poisson algebra \mathcal{A} . Notice that in this algebra we do not include observables which would be finite distance under Poisson flow, otherwise flowing by finite distance with H would take us out of the time-band, see also the discussion in [66].

Starting with the classical theory, we ask whether we can find observables localized deep in the interior of AdS which are space-like with respect to the time-band and which have vanishing Peierls brackets with observables in the time-band algebra \mathcal{A} . These candidate observables are to be defined as in (2.13), in particular they need to be defined on a neighborhood $U \subset \mathcal{X}$ of a solution $s \in U$ and not necessarily on the entire space of solutions \mathcal{X} .

It is clear that observables defined relationally with respect to the boundary, or with a gauge fixing condition which makes use of the boundary, do not satisfy these conditions. Due to their gravitational Wilson lines they will have non-vanishing Peierls brackets with the Hamiltonian and other charges on the boundary [49, 50]. Such observables generally change the energy of the state, which due to the gravitational Gauss law can be measured in the time band $\mathcal{T}_{-\epsilon, \epsilon}$ by (??). Another point of view is that such observables identify a point in the bulk, and in particular *a moment in time*, relationally with respect to the boundary. Thus an infinitesimal motion in time of the starting point on the boundary is translated via the relational prescription into an infinitesimal time motion of the corresponding bulk point. Then the Peierls bracket of the candidate bulk observable with H generates time-derivatives of the point in the bulk and is non-vanishing.

The discussion of the previous subsection implies that if we start with an asymptotically AdS_{d+1} solution s of the bulk equations which is complicated enough, then we can define diff-invariant observables of the form (2.15) in a neighborhood of s so that they have vanishing Peierls bracket with all elements of the time-band algebra \mathcal{A} including charges like the Hamiltonian (??). Such observables do not change the total energy of the state but instead they rearrange the energy, "absorbing" from the background solution the amount of energy they themselves create. These observables select a point in the bulk, and a moment in time, by using features of the state.

¹¹We only assume that the sources are zero in the time band \mathcal{T} , they could be turned on in the far past in order to prepare a state.

¹²The subleading coefficients are fixed by the equations of motion in terms of the leading ones.

In what follows we will provide evidence that the same conclusions are true in perturbative quantum gravity. We will proceed by translating the question in CFT language and using the AdS/CFT correspondence.

2.2 Holographic setup

In this section, we will study the question of locality in quantum gravity in the context of the AdS/CFT correspondence. A question we would like to understand is how certain bulk subregions are encoded in the boundary CFT. There are cases where this is well understood. For example, the bulk dual of a boundary subregion is known as the entanglement wedge, which is the bulk region extending between the boundary subregions and the relevant Ryu-Takayanagi surface extending in the bulk [168]. This correspondence between parts of the boundary and bulk is known as subregion-subregion duality [55–57], and it is worthwhile to mention that in general, the entanglement wedge of a boundary subregion is much larger than its causal wedge (the part of the bulk contained by lightrays shot from the causal developments of the boundary subregion).

Subregion-subregion duality and entanglement wedge reconstruction utilizes the organization and entanglement of CFT degrees of freedom organized spatially. We will be interested in rather different bulk subregions, which lie deep down in the bulk and never extend to the boundary CFT. What is the CFT dual of a causal diamond located deep near the center of AdS? The answer to this question remains elusive, and in particular it is understood that in general, these bulk regions do not correspond to the entanglement wedge of any boundary subregion. There have been previous attempts to understand the CFT mapping of such regions, see for example [58–60, 169] which attempt to assign a meaning to the entropy of a general closed codimension-2 spatial curve in AdS. Here we will follow a different approach by focusing on the algebra of single-trace operators [61].

We will start by reviewing some basic but relevant features of AdS/CFT, before turning to a discussion of the class of states that we will be considering throughout this paper and their salient properties.

2.2.1 Gravitational states in AdS, large diffeomorphisms and asymptotic symmetries

We will be interested in gravitational solutions which are asymptotically AdS_{d+1} . We have in mind an embedding in a top-down setup with a holographic dual CFT, like $\mathcal{N} = 4$ SYM at strong coupling, on $\mathbb{S}^3 \times \mathbb{R}$ and the N -scaling we indicate in most of the paper refers to this theory. However for most of the discussion the details of the embedding in string theory, the extra fields, as well as the presence of a compact internal manifold are not important unless explicitly stated.

Solutions to the bulk equations of motion can be thought of as *states* in the dual CFT. If we think of a bulk geometry described by a Penrose diagram, the diagram really represents the entire time-history of the state. We can take the state to live at $t = 0$ on a boundary Cauchy slice, and the portion of the geometry relevant to describing the state is an initial data surface given by a bulk Cauchy slice (or the Wheeler-de Witt patch associated to the boundary Cauchy slice). To view these geometries as states of the dual CFT, it is important that the bulk fields have a fall-off corresponding to normalizable modes with

vanishing CFT sources.¹³

We want to consider semi-classical solutions with non-trivial bulk geometries, i.e. where backreaction is strong. The corresponding CFT states $|\Psi_0\rangle$, which we take to be pure, have large energies which scale as

$$\langle\Psi_0|H|\Psi_0\rangle\sim\mathcal{O}(N^2),\quad(2.18)$$

and as we will see, they will generally also have an energy variance of the same order. We will also consider perturbative excitations of the quantum fields on top of the background geometry. These excitations add/subtract quantum particles which change the energy by an $\mathcal{O}(N^0)$ amount, and whose backreaction on the geometry is thus generally small.

Geometries of this type will often be macroscopically time-dependent, such that the initial data on a bulk Cauchy slice changes as we perform time-evolution of the state. This has consequences for the variance of the energy, as we will now see. Any state $|\Psi_0\rangle$ can be expanded in the basis of CFT energy eigenstates as

$$|\Psi_0\rangle=\sum_i c_i|E_i\rangle.\quad(2.19)$$

The time-dependence of the bulk geometry implies that such states will have energy variance

$$(\Delta H)^2\equiv\langle\Psi_0|H^2|\Psi_0\rangle-\langle\Psi_0|H|\Psi_0\rangle^2\sim\mathcal{O}(N^2).\quad(2.20)$$

To see this, consider the inequality

$$\frac{1}{2}|\langle[H,A]\rangle|=\frac{1}{2}\langle\partial_t A\rangle\leq\Delta H\cdot\Delta A,\quad(2.21)$$

where in the first equality we assumed that the operator A is not explicitly time-dependent. Then we have

$$\Delta H\geq\frac{1}{2}\frac{\langle\partial_t A\rangle}{\Delta A}\sim\mathcal{O}(N),\quad(2.22)$$

where we have used large N factorization for the operator A . This shows that provided there is macroscopic time-dependence (the classical vev of A changes at leading order), the variance of the energy scales at least as N^2 .¹⁴ Some bulk geometries we will consider are macroscopically time-dependent, but only inside the horizon. In this case, we cannot use the argument above, but we still expect the variance to be of order N^2 . It is interesting to ask whether the variance is a quantity that can be extracted from the semi-classical geometry alone. In general, we expect that the quantum state of the fields in the bulk is important as well. We discuss this further in Appendix B.

There are various types of explicit constructions of states of this kind. There are states prepared by Euclidean path integral with sources for single-trace operators [170–173]. These states should be interpreted as coherent states of the quantum gravitational dual, which are labelled by phase-space points corresponding to initial data¹⁵. There are

¹³If these states are prepared by a Euclidean path-integral [170–173], sources can be turned on in the Euclidean past which prepares the state, but it is important that they vanish as $t_E\rightarrow 0$ for the geometries to be interpreted as states in the undeformed CFT.

¹⁴Note that if the variance is parametrically larger than $\mathcal{O}(N^2)$, the state may no longer have a good semi-classical interpretation. An example would be a superposition of black holes of different masses.

¹⁵It appears that one may not construct arbitrary initial data this way, see [174]. This will not affect our construction and for states prepared by a Euclidean path integral, we should simply keep in mind that we have access to a restricted class of initial data.

also states prepared by a boundary state of the CFT, further evolved by some amount of Euclidean time [175–178]. The bulk interpretation of these states is that they correspond to black hole geometries with End-of-the-World branes sitting behind the horizon. This is an example where the bulk geometry is macroscopically time-dependent, but only behind the horizon. Similarly, for two-dimensional CFTs, we can construct pure states by performing the path integral over a surface of higher topology, for example half a genus-2 surface, see [179]. These geometries are also macroscopically time-dependent behind the horizon, but instead of having a brane behind the horizon, they have topology. Finally, it is worth noting that there are semi-classical geometries that also preserve supersymmetry, the most famous of which are the LLM geometries [180]. In these cases, one can obtain a better understanding of the dual CFT states. We will come back to these geometries in section 2.5.

As usual in gravity, we should identify solutions which are related by *small diffeomorphisms*, i.e. diffeomorphisms that vanish near the AdS boundary. There is also a class of large diffeomorphisms, which are compatible with the boundary conditions imposed in the definition of our theory of AdS gravity. This set of diffeomorphisms forms what is called the *asymptotic symmetry group*. In the case of AdS_{d+1} , $d \geq 3$ this is the conformal group $SO(2, d)$, while for $d = 2$ it gets enhanced to the Virasoro group [181]. When acting on a given bulk solution these large diffeomorphisms will generally transform the geometry into a new state, which is physically distinguished from the previous one, unless of course the original state happens to be invariant under the symmetry. We will later also discuss solutions with two asymptotic boundaries, such as the eternal black hole in AdS, in which case the asymptotic symmetry group is larger. Let us now discuss the various elements of the asymptotic group/conformal group:

- **Time translations:** One particular class of states we will discuss are those with semiclassical time-dependence in the bulk, for example a state corresponding to the gravitational collapse of a star. In this case large diffeomorphisms corresponding to asymptotic time translations transform the state as $|\Psi_0\rangle \rightarrow e^{-iHt}|\Psi_0\rangle$. The initial data corresponding to $|\Psi_0\rangle$ is not the same as that of $e^{-iHt}|\Psi_0\rangle$. Our end goal will be to provide local operators whose gravitational dressing is done towards a feature of the state. If the state is time-dependent then we can select a moment in time by using the features of the state, as opposed to the boundary time coordinate. On the other hand if the state is static, then the only way to identify a moment in time is by dressing to the boundary. This is why it will be important for us to consider time-dependent states.
- **SO(d) rotations:** If the state breaks $SO(d)$, then asymptotic rotations transform it to a new state. In this case we can use the features of the state to identify the angular location of a point. On the other hand, if the state is $SO(d)$ invariant it will generally not be possible and at best we can obtain an operator smeared over the bulk angular coordinates, or alternatively we can fix the angular location by dressing to the boundary.
- **AdS boosts:** The Lorentzian conformal group acting on $S^{d-1} \times \mathbb{R}$ has another $2d$ generators which correspond to boosts in various directions. These can be realized as d non-independent copies of an $SL(2, \mathbb{R})$ algebra, see for example [182]. Any state with finite energy cannot be annihilated by Hermitian combinations of these

generators, which we show in Appendix C. The only state which is annihilated by these generators is the global vacuum and any other state will necessarily transform under the action of these boosts¹⁶. Therefore, in any non-trivial state, we can fix the radial position of an operator without referring to the boundary.

In a top-down setup, the gravity dual may have an internal manifold, like the S^5 in the context of $\mathcal{N} = 4$ SYM. In such cases, we would need to break the R-symmetry to localize a bulk operator in the internal space. In this paper, we will mostly restrict to a bottom up construction without an internal manifold but it would be an interesting generalization.

2.2.2 Locality in AdS

We are now ready to discuss locality in quantum gravity with asymptotically AdS boundary conditions. We would like to understand whether one can define local observables and whether we can localize information deep in the center of the AdS.

The presence of the AdS boundary allows us to define one natural class of diff-invariant observables: The fields in AdS can be expanded in a Fefferman-Graham expansion. The coefficients of this expansion are themselves diff-invariant observables, which are dressed to the boundary since the Fefferman-Graham gauge is chosen with respect to the boundary. Let us call these observables FG-observables. For example, the AdM Hamiltonian is one particular observable in this class. In perturbative quantum gravity, we can also consider the expectation values of these observables as well as their higher-point correlation functions. As we will discuss below, if we want to stay within the regime which can be described by semi-classical gravity we may need to restrict the complexity of the correlators (for example the number of operator insertions in the correlation function). We emphasize again that all these observables are dressed with respect to the boundary. In particular, they will generally not commute with the Hamiltonian or the other charges described in the previous section.

The question we would like to address is the following. If we start with a state with a semi-classical geometric description, is there a way to modify the state in the interior of AdS, without modifying any of the correlators of FG-observables localized in a short time-band of the boundary? If the answer is yes, this means we can localize information since an observer living near the boundary will have no way to know whether or not we modified the state. Rather than trying to come up with bulk objects that achieve this goal, we will address this question directly in the dual CFT. This has the following advantage: any object built out of CFT degrees of freedom is necessarily diff-invariant and non-perturbatively well defined. Provided the object acts in the right away, we can be assured that the construction is fully consistent.

2.2.3 The CFT description and the time band algebra

Consider a large N holographic CFT which is dual to semi-classical general relativity coupled to matter fields. In the large N limit, we can define the algebra \mathcal{A} generated by single-trace operators in a time-band \mathcal{D}_{t_1, t_2} , where we allow products of single-trace

¹⁶States with infinite energy like the AdS-Rindler vacuum could also potentially be annihilated by some boost generators.

operators where the number of factors is arbitrary but scales like $O(N^0)$.¹⁷ This was originally discussed in [61], inspired by the earlier work [12, 13, 34]. In [61] it was proposed that the algebra \mathcal{A} can be thought of as being dual to the causal wedge of the region \mathcal{D}_{t_1, t_2} in the bulk (see Fig. 4.3). This picture also suggests that the algebra \mathcal{A} has a commutant which can be identified with a spacelike-separated causal diamond in the interior. Algebras of this type have received attention recently [62–64, 183, 184].

The work [61] studied this setup for states which are small perturbations around the AdS vacuum. The geometry of AdS is homogeneous and featureless since it is a maximally symmetric space. As already discussed in the previous section, this makes the definition of local diff-invariant observables challenging. We would like to revisit the time-band algebra, this time in cases where the bulk state has features, which in particular are time-dependent. This means the state must be highly excited as can be seen for example from its energy (2.18).

At infinite N the problem can be understood in terms of QFT on a curved and in general time-dependent background. In particular, gravitational backreaction of the quantum fields can be ignored and one does not need to talk about gravitational dressing, which is a form of backreaction. In this case, the existence of the commutant is obvious because we are in a QFT situation. Note that if the Hamiltonian (which is always an element of the time band algebra) is normalized appropriately¹⁸, its commutator with the other single-trace operators is suppressed by $1/N$ and thus vanishes when N is infinite.

At the level of $1/N$ corrections, the existence of the commutant is less obvious. Backreaction must now be taken into account and the gravitational Gauss law can spoil the commutator between H and the other operators of the time-band algebra. For example, the standard way to write bulk fields in terms of CFT operators is the HKLL construction [86, 87, 185–189]

$$\Phi(t, r, \Omega) = \int_{\text{bdry}} dt' d\Omega'_{d-1} K(t, r, \Omega; t', \Omega') \mathcal{O}(t', \Omega'), \quad (2.23)$$

where K is related to a Green's function of the Klein-Gordon operator on the appropriate bulk geometry. This operator is defined purely within the CFT so it is manifestly diff-invariant. To leading order at large N , it acts as a bulk field and commutes with other bulk fields at spacelike separation. Notice however that in order to define the kernel K we have to choose a coordinate system in the bulk, which often is taken using Fefferman-Graham gauge. As we already mentioned, this gauge choice is defined by making use of the asymptotic boundary, and an HKLL operator is thus dressed to the boundary. Because of this, the commutator between an HKLL operator and the Hamiltonian will not vanish at subleading orders in the $1/N$ expansion.

The physical origin of this effect is the gravitational Gauss law: acting with (2.23) will generally create or destroy a particle in the bulk, thus changing the energy of the state, which can be immediately measured at spacelike infinity by H . One can try to correct the HKLL operators at higher orders in $1/N$ by mixing it with other single- and multi-trace operators, see [189–191], but the commutator with the Hamiltonian is universal and generally cannot be removed in this way. It is also possible to think about the dressing in terms of

¹⁷Notice that at finite N the algebra in a time-band would be the same as the full algebra. In the large N limit, a natural hierarchy emerges between "small products" of single-trace operators and the rest of the algebra, which allows us to consider the notion of a time-band algebra.

¹⁸A useful normalization is $h = \frac{1}{N}(H - \langle \Psi_0 | H | \Psi_0 \rangle)$, which ensures that $\langle \Psi_0 | h^2 | \Psi_0 \rangle \sim O(N^0)$.

(smeared) gravitational Wilson lines connecting the bulk operator to the boundary, which make it diff-invariant at the price of making it non-local [26, 192–194]. The commutator with H is nonzero because H picks up the contribution of the Wilson line.

This raises the question of whether the algebra \mathcal{A} still has a commutant at subleading orders in $1/N$. The main goal of this paper is to provide evidence for the existence of such a commutant. We will do so by identifying a class of operators that are gravitationally dressed with respect to *features of the state*, rather than dressed to the boundary. In particular, these operators will have vanishing commutators with the Hamiltonian, to all orders in $1/N$. In this paper, we will focus mostly on ensuring that bulk operators have a vanishing commutator with the Hamiltonian (and the other charges), but it would be important to extend our construction to all single-trace operators in \mathcal{D}_{t_1, t_2} . We give an alternative argument for the existence of a commutant to all orders in $1/N$ in section 2.4.

The existence of a commutant for \mathcal{A} in $1/N$ perturbation theory would imply that information can be localized in regions of the bulk and is not visible from the boundary at the level of perturbative quantum gravity¹⁹. We are now ready to formulate the concrete goal that we will achieve in this paper.

2.2.4 Formulating the main goal

Our goal is to improve the locality properties of (2.23) by moving the gravitational dressing from the boundary to the state. From a technical point of view, we will find CFT operators $\widehat{\Phi}$ which obey two properties:

1. $[Q_i, \widehat{\Phi}] = 0$ to all orders in $1/N$, for all asymptotic charges $Q_i \in SO(2, d)$.
2. The correlators of $\widehat{\Phi}$ agree with those of Φ_{HKLL} to leading order in the large N expansion, on the code subspace of $|\Psi_0\rangle$.

In taking the large N limit it is important to track how various effects scale with N . As we will see, our new operators $\widehat{\Phi}$ have vanishing commutator with Q_i to all orders in the $1/N$ expansion, but have a non-vanishing commutator at the level of e^{-N^2} corrections.

In what follows we will first focus on ensuring a vanishing commutator of $\widehat{\Phi}$ with the Hamiltonian H to all orders in $1/N$ and then discuss the generalization to the other charges in $SO(2, d)$.

As we will see, our construction will not work for $|\Psi_0\rangle = |0\rangle$. Technically, this is because the vacuum does not comply with the properties (2.18) and (2.20). Physically, it is because the AdS vacuum has no feature that we can use to attach the dressing of our local operator. Note that this is in line with the results of [73], where a protocol to reconstruct the bulk state from correlators in the time-band was discussed.

2.2.5 Time-shifted states and return probability

We will now present the main technical tool that will enable us to define state-dressed operators: the return probability. Let us start with a state $|\Psi_0\rangle$ satisfying the properties (2.18) and (2.20). We define the following one-parameter family of states

$$|\Psi_T\rangle = e^{-iTH}|\Psi_0\rangle \quad T \in \mathbb{R}. \quad (2.24)$$

¹⁹See [66–74] for other discussions of localization of information in perturbative quantum gravity, with varying conclusions.

In the bulk, the states $|\Psi_T\rangle$ are related to $|\Psi_0\rangle$ by a large diffeomorphism, i.e. one that does not vanish near the boundary and induces a boundary time-translation. It is important to emphasize that they are *different* quantum states, even though they are related by a symmetry. If we think about the phase space of gravity in AdS, the family of states correspond to different phase space points, just like a particle moves on phase space as a function of time in classical mechanics. From the bulk perspective, if $|\Psi_0\rangle$ was a coherent state, we can also think of $|\Psi_T\rangle$ as coherent states.

We would now like to consider the overlap of such states. In particular, we would like to study the overlap

$$\langle\Psi_0|\Psi_T\rangle. \quad (2.25)$$

Thinking of these states as coherent states is useful to gain intuition about such overlaps. For the simple harmonic oscillator, the overlap of two coherent states is $\beta\langle\alpha|\beta\rangle = e^{-\frac{1}{\hbar}f(\alpha,\beta)}$, for a very simple quadratic function f . For states on the gravitational phase space, recalling that $\hbar \sim G_N \sim 1/N^2$, we thus expect

$$\langle\Psi_0|\Psi_T\rangle = e^{-N^2 f_0(T)}, \quad (2.26)$$

for a function f_0 whose real part is positive. In the gravitational setting, it is not straightforward to directly compute $f_0(T)$ from the phase space information, see [14] for a discussion on nearby states. There is a general way to compute $f_0(T)$ based on a Euclidean preparation of the states [173], but it requires some effort (in particular solving the non-linear Einstein equations). The computation of $f_0(T)$ directly from the information on an initial data slice, which specifies the point on phase-space, is an interesting problem.²⁰

It is also instructive to think about the overlap from a microscopic point of view. In the CFT, the overlap is given by

$$\langle\Psi_0|\Psi_T\rangle = \sum_i |c_i|^2 e^{-iTE_i}. \quad (2.27)$$

Note that there are $e^{S(E)}$ terms here, each of size $e^{-S(E)}$. The suppression (2.26) must therefore come from the summation over a large number of phases.

If the bulk state has no periodicities in time, we expect the real part of $f_0(T)$ to increase as we increase T . However, this increase will not continue forever. We will shortly give an estimate of the time-average of (2.27), and argue that the decay will saturate at some point. Physically, the non-trivial overlaps (2.27) imply that it is not correct to think that all the states $|\Psi_T\rangle$ are independent, see also [14, 195, 196] for related discussions. In particular, even if the bulk state is not macroscopically periodic, there will still be a microscopic periodicity of the state due to Poincare recurrences, that will happen at very large $T \sim \mathcal{O}(e^{e^{N^2}})$. Throughout this paper, we will be interested in much earlier time scales so it will be sufficient for us to treat the states $|\Psi_T\rangle$ as quasi-orthogonal since all overlaps will be exponentially small.

We will also need to define the notion of code subspace. Starting with the state $|\Psi_0\rangle$ we define the code subspace as

$$\mathcal{H}_0 = \text{span}\{|\Psi_0\rangle, \mathcal{O}(t, \Omega)|\Psi_0\rangle, \dots, \mathcal{O}_1(t_1, \Omega_1)\dots\mathcal{O}_n(t_n, \Omega_n)|\Psi_0\rangle\}, \quad (2.28)$$

²⁰Similarly, we do not know of a gravitational argument that guarantees that the real part of $f_0(T)$ is positive, which must be the case if the geometries have a state interpretation in the dual CFT. We comment on this further in the discussion.

generated by acting on $|\Psi_0\rangle$ with a small number ($n \ll N$) of single-trace operators²¹. It will also be useful to define the projector P_0 on this subspace. Similarly, a code subspace can be defined for each of the time-shifted states

$$\mathcal{H}_T = \text{span}\{|\Psi\rangle_T, \mathcal{O}(t, \Omega)|\Psi_T\rangle, \dots, \mathcal{O}_1(t_1, \Omega_1)\dots\mathcal{O}_n(t_n, \Omega_n)|\Psi_T\rangle\}, \quad (2.29)$$

with the corresponding projector P_T . The projectors P_0 and P_T are simply related by time-evolution, i.e. we have

$$P_T = e^{-iTH} P_0 e^{iTH}, \quad (2.30)$$

and in particular, we emphasize again that $P_T \neq P_0$. In what follows, it will be convenient to work with real quantities rather than the overlap (2.25), and we are now ready to define the return probability.

2.2.6 The return probability

We now ready to examine the T -dependence of the overlap (2.27) in more detail. As explained above, it is more convenient to work with a real quantity so let us define *the return probability*

$$R(T) := |\langle\Psi_0|e^{-iTH}|\Psi_0\rangle|^2. \quad (2.31)$$

It is similar to the spectral form factor (the two coincide when $|\Psi_0\rangle = |\text{TFD}\rangle$ and $H = H_L + H_R$). Recently, the spectral form factor has been extensively discussed in connection to the black hole information paradox and quantum chaos, see for example [197]. The time-scales of interest in that context are again late times such as $t \sim e^{N^2}$ (note this is much shorter than the Poincare recurrence time which is doubly exponential). Here again, we will be interested in much earlier time-scales.

In general, it is difficult to compute (2.31). As we mentioned above, the overlaps can be computed from time-shifted coherent states in gravity but the best known technology to do so uses the Euclidean path integral and involves solving the non-linear Einstein's equations. Nevertheless, we can compute the very early time dependence using large N factorization. We present this calculation in Appendix D. At early times, we have

$$R(T) = e^{-(\Delta H)^2 T^2}, \quad (2.32)$$

which is generally valid for times up to $T \sim O(N^{-1})$. For the purposes of this paper, we want to understand how the return probability behaves at time-scales $T \sim O(1)$. Here, the decay does not follow from large N factorization and it is in general not an easy task to compute it.

In Appendix D, we review that for the TFD state, the return probability (which is the spectral form factor) decays as

$$R_{\text{TFD}}(T) = e^{-N^2 f_{\text{TFD}}(T)}, \quad (2.33)$$

where $f_{\text{TFD}}(T)$ is $O(N^0)$ and for early times $T \sim O(N^0) \ll \beta$ behaves like $f_{\text{TFD}}(T) \approx \alpha T^2$, where α is an $O(N^0)$ constant which depends on the temperature. This is an extremely fast decay, much faster than thermalization where the prefactor in the exponent is of order

²¹To be precise, we should also give a small smearing to the single-trace operators in order to avoid UV divergences of operator insertions at coincident points. We will leave it as implicit in what follows.

N^0 , and shows that thermofield double states at different times orthogonalize exponentially fast.

We expect similar behaviour for many other semi-classically time-dependent states, that is for timescales of $T \sim \mathcal{O}(1)$, we expect

$$R(T) \sim e^{-N^2 \tilde{f}_0(T)}, \quad (2.34)$$

for a positive and $\mathcal{O}(N^0)$ function $\tilde{f}_0(T)$ which depends on the state $|\Psi_0\rangle$. We expect that for small T the function $\tilde{f}_0(T)$ starts quadratically, as in (2.32). Note that this fast decay is not even a consequence of quantum chaos, as it can occur at weak coupling or even in free theories, provided they have a large number of degrees of freedom (see [198] for a study of this question in weakly coupled $\mathcal{N} = 4$ SYM). The difference between a free theory and a holographic one will manifest itself in the time-scale during which the exponentially small overlap remains valid. For free $\mathcal{N} = 4$ SYM, the spectrum is integer spaced and so the return probability will be periodic with period 2π , while in a chaotic theory it will take doubly exponentially long for the signal to return to unity.

The average late-time value of the signal is also highly dependent on whether the theory is chaotic or not. For a system with no degeneracies,²²

$$\bar{R} = \lim_{t_* \rightarrow \infty} \frac{1}{2t_*} \int_{-t_*}^{t_*} dT R(T) = \sum_i |c_i|^4. \quad (2.35)$$

For the type of states we are considering, i.e. those with a large energy variance, this is exponentially small, and scales as $e^{-\alpha' N^2}$, where α' is an $\mathcal{O}(1)$ constant which depends on the particular $|\Psi_0\rangle$ we have picked. This value is often referred to as the plateau, especially in the context of the spectral form factor.

Between the initial decay (2.33) and the plateau (2.35), there can be other regimes, which are particularly interesting in connection to quantum chaos [199, 200]. For example, in the spectral form factor, the plateau is preceded by a ramp where the signal grows linearly. These effects will not be important for the present work, as we will only consider $\mathcal{O}(1)$ timescales. The crucial point we will exploit throughout the paper is that the signal is already exponentially small in N^2 at those timescales.

The overlap (2.25) obeys the property

$$\langle \Psi_{t_0} | \Psi_{t_0+T} \rangle = \langle \Psi_0 | \Psi_T \rangle. \quad (2.36)$$

This may appear trivial, but it means that even if the bulk geometry appears to be static at the semi-classical level, the return probability may still decay following (2.33) if the state had a period of manifest bulk time-dependence in the far past. Said differently, the variance in energy which determines the decay is unchanged under time-evolution, so even if the 1-point functions have stabilized, the variance remains large. This observation is particularly relevant in the case of a black hole formed by gravitational collapse.

The exponential decay (2.33) can be extended to more general correlators of the form $\langle \Psi_0 | \mathcal{O}(t_1) \dots \mathcal{O}(t_n) | \Psi_T \rangle$, where \mathcal{O} are single-trace operators. We expect

$$\langle \Psi_0 | \mathcal{O}(t_1) \dots \mathcal{O}(t_n) | \Psi_T \rangle = F(T) \langle \Psi_0 | \Psi_T \rangle, \quad (2.37)$$

²²Systems like $\mathcal{N} = 4$ SYM will have degeneracies due to superconformal symmetry. For example, for every primary, there are towers of descendants with degenerate energy levels. Nevertheless, the number of degenerate states is exponentially smaller than the number of all states, at least in the high-energy sector of the theory, so the degeneracy only contributes a subleading effect.

where $F(T)$ is finite in the large N limit and satisfies

$$F(0) = \langle \Psi_0 | \mathcal{O}(t_1) \dots \mathcal{O}(t_n) | \Psi_0 \rangle \quad , \quad \frac{d^k F(T)}{dT^k} \Big|_{T=0} = O(N^0). \quad (2.38)$$

To see the exponential decay we write (2.37) as

$$\langle \Psi_0 | \mathcal{O}(t_1) \dots \mathcal{O}(t_n) | \Psi_T \rangle = \frac{\langle \Psi_0 | \mathcal{O}(t_1) \dots \mathcal{O}(t_n) | \Psi_T \rangle}{\langle \Psi_0 | \Psi_T \rangle} \langle \Psi_0 | \Psi_T \rangle. \quad (2.39)$$

The second term in this product is really responsible for the decay of the correlator. The first term is hard to evaluate from first principles, but in holography its meaning is clearer. In the bulk theory, it is computed by computing a correlation function on a background dictated by the Euclidean path integral with different sources on the northern and southern hemisphere (corresponding to $|\Psi_0\rangle$ and $|\Psi_T\rangle$, respectively). This correlator is $\mathcal{O}(1)$ and a smooth function of the background, which will generally change slowly with T , so we expect its time derivatives not to scale with N as indicated in (2.39). We check this statement in a few examples in section 2.5.

To sum up, any state in the code subspace (2.28) has an exponentially small overlap with any state in the code subspace (2.29). This can be summarized by the relation

$$R_{code}(T) = \frac{1}{d_{code}} \text{Tr}[P_T P_0] = \mathcal{O}(e^{-N^2 \tilde{f}(T)}) \quad (2.40)$$

where d_{code} is the dimensionality of the code subspace, and for the time-scales we have discussed. The decay (2.40) can be used in combination with other useful inequalities. For example, for a Hermitian operator \mathcal{O} with eigenvalues λ_i , and if $[P_0, \mathcal{O}] = 0$, we have $|\langle \Psi_0 | \mathcal{O} | \Psi_T \rangle|^2 \leq \sqrt{\text{Tr}[\mathcal{O}^4]} \sqrt{\text{Tr}[P_T P_0]}$ and $|\langle \Psi_0 | \mathcal{O} | \Psi_T \rangle|^2 \leq \max(\lambda_i^2) \text{Tr}[P_T P_0]$.

2.2.7 Other asymptotic charges

More generally we can consider the change of the state by large diffeomorphisms corresponding to the other asymptotic symmetries of the theory, in the case of AdS_{d+1} the conformal group $SO(2, d)$ with the generators we discussed in section 2.2.1. This leads us to define a natural generalization of the return probability

$$R(g) = |\langle \Psi_0 | U(g) | \Psi_0 \rangle|^2 \quad , \quad g \in SO(2, d), \quad (2.41)$$

where $U(g)$ is the unitary realizing the conformal transformation of the CFT on $S^{d-1} \times \text{time}$.

What can we expect for these overlaps? To start, let us suppose the state $|\Psi_0\rangle$ breaks rotational $SO(d)$ symmetry at the classical level. By this, we mean that bulk dual geometry breaks the symmetry, which would be the case for some spherically asymmetric lump of matter. Take J to be the angular momentum generator, then we expect that the variance of J will be of $O(N^2)$ for such a state. Hence we expect that for small values of a rotation angle ϕ dual to J we will have

$$R(\phi) = e^{-(\Delta J)^2 \phi^2} = e^{-\kappa N^2 \phi^2}, \quad (2.42)$$

for $\kappa \sim \mathcal{O}(1)$. For more general angles, we expect

$$R(\phi) = e^{-N^2 f_{rot}(\phi)}. \quad (2.43)$$

However, because angular momentum is quantized, we have

$$R(\phi + 2\pi) = R(\phi), \quad (2.44)$$

hence the function $f_{\text{rot}}(\phi)$ has period 2π . In this direction of the conformal group the return probability has a very short Poincare recurrence equal to 2π .

All in all we find that as we increase ϕ away from 0 the return probability $R(\phi)$ very quickly dips down to exponentially small values and stays there until the Poincare recurrence at $\phi = 2\pi$. As we see from (2.43), for any fixed ϕ which is in the range $(0, 2\pi)$, we have $R(\phi)$ being exponentially small in the large N limit.

Of course if the state respects spherical symmetry then the return probability will not decay in the corresponding $SO(d)$ directions. It is worthwhile to discuss several distinct scenarios. In the simplest case, the state preserves the symmetry and is thus annihilated by the generators of rotations. The second simplest situation is the case where the symmetry is manifestly broken at the classical level (for example an asymmetric lump of matter). In this case, the breaking of the symmetry is manifest, and would be visible in the 1-point function of single-trace operators. There are also more subtle situations where the state breaks the symmetry classically in the bulk, but this may be invisible in the 1-point functions. An example of this are states by prepared by the path integral on higher genus surfaces in $d = 2$, and have topology behind the horizon [179].²³

Finally as discussed in section 2.2.1, we expect that semi-classical states also break the other conformal symmetries. We can get some intuition by considering a state dual to a conformal primary of dimension Δ . In this case the return probability along one of the conformal boost directions is determined by a group theoretic computation

$$R(s) = |\langle \Delta | e^{-isK} | \Delta \rangle|^2 = \left(\frac{1}{\cosh^2 s} \right)^{2\Delta}. \quad (2.45)$$

For primary states with $\Delta \sim O(N^2)$, we get exponential decay of the form $e^{-N^2 f(s)}$ for any non-zero s . Notice that for the conformal boosts we do not expect any Poincare recurrence for large s , which in the case of primaries is obvious from the formula above, since such a transformation monotonically increases the energy of the state.

In the case of AdS_3 the asymptotic symmetry group is enhanced to Virasoro and similar statements hold for the flow of the state under more general large diffeomorphisms generated by L_n, \bar{L}_n .

To summarize, if we start with a state $|\Psi_0\rangle$ which breaks all conformal symmetries at the level of the semi-classical geometry we expect that $R(g)$ defined in (2.41) will decay exponentially fast in all directions away from the identity element on the conformal group manifold.

2.3 State-dressed operators

We are now in a position to introduce operators $\hat{\Phi}$ which satisfy the two properties described in section 2.2.4, namely their commutator with the Hamiltonian and other asymptotic charges is zero to all orders in the $1/N$ expansion and they act like HKLL operators to

²³The thermofield double also has this property. It breaks rotational symmetry of each CFT individually, but the breaking is invisible in 1-point functions. It would be interesting to understand if this type of breaking always requires a horizon.

leading order at large N on the code subspaces $\{\mathcal{H}_T, T \in (-t_*, t_*)\}$. Here t_* is an order one (i.e. N^0) time of our choice. We define the HKLL operator Φ , (2.23), in the $N \rightarrow \infty$ limit. In this limit the bulk is described by a quantum field theory on a curved spacetime and code subspaces for different T will be strictly orthogonal to one another. In addition, Φ is a local bulk operator which commutes with all the boundary single-trace operators in the time band algebra, including the appropriately normalized Hamiltonian [186, 190]. But it will no longer be commuting once $1/N$ corrections are included. In particular, we will have

$$[\Phi, \frac{H - \langle H \rangle}{N}] = O(1/N) \neq 0. \quad (2.46)$$

Again, the physical reason behind this is that (2.23) is a diff-invariant operator that is dressed to the boundary. Note that for the naive HKLL operator (2.23), the commutator with other single-trace operators will also be non-zero at order $O(1/N)$. For almost all single-trace operators, this can be removed order by order in $1/N$ by adding the appropriate corrections to Φ [190]. However, these modifications will not be able to remove the non-vanishing commutator with the Hamiltonian (2.46). Thus, to remove the gravitational dressing to the boundary CFT, a more sophisticated procedure is required.

We start by focusing on setting the commutator with the Hamiltonian to zero and discuss the extension to other asymptotic charges later. To this end, we introduce the following operator²⁴

$$\widehat{\Phi} = c \int_{-t_*}^{t_*} dT e^{-iTH} P_0 \Phi P_0 e^{iTH}, \quad (2.47)$$

where t_* is an $O(N^0)$ timescale of our choice, and c is an overall normalization constant

$$c^{-1} = \int_{-t_*}^{t_*} dT \langle \Psi_0 | P_T | \Psi_0 \rangle. \quad (2.48)$$

As we will see, the projector P_0 will be key and will make $\widehat{\Phi}$ act appropriately on the code subspaces. The range $(-t_*, t_*)$ determines the set of code subspaces on which $\widehat{\Phi}$ acts in the desired fashion, and ultimately cannot be taken to be bigger than the time range where the exponential decay of the return probability (2.33) is valid. To make the operator (2.47) have the desired properties on as many states as possible, we can take this range to be the time range where the return probability decays exponentially, though this is not strictly necessary and a t_* of $O(N^0)$ is sufficient. We also provide an alternative presentation of the operators in subsection 2.3.4. In the following subsections, we will study the action of these operators in the relevant code subspaces, and will be particularly interested in their commutator with the Hamiltonian.

2.3.1 Vanishing commutator with H to all orders in $1/N$

We now show that the operator (2.47) has vanishing commutator with H to all orders in $1/N$. We start by rewriting the commutator as

$$[H, \widehat{\Phi}] = -i \frac{d}{ds} (e^{isH} \widehat{\Phi} e^{-isH}) \Big|_{s=0}, \quad (2.49)$$

²⁴Recall that P_0 is the projector on the code subspace of $|\Psi_0\rangle$, and thus $[\Phi, P_0] = 0$ in that code subspace. Therefore, we could have defined operators with the same action on the code subspace as (2.47), using a single projector on the left (or right) of Φ . Even though the resulting operators would act in the same way on the relevant code subspace, the operators would not be exactly identical: they would have additional non-zero matrix elements associated to subspaces orthogonal to \mathcal{H}_0 .

and performing a change of variables, we find

$$\begin{aligned} [H, \widehat{\Phi}] &= -i \frac{d}{ds} \left(c \int_{-t_*-s}^{t_*-s} dT e^{-iTH} P_0 \Phi P_0 e^{iTH} \right) \Big|_{s=0} \\ &= ic(P_{t_*} \Phi_{t_*} P_{t_*} - P_{-t_*} \Phi_{-t_*} P_{-t_*}), \end{aligned} \quad (2.50)$$

where we defined $\Phi_{t_*} = e^{-iHt_*} \Phi e^{iHt_*}$. Using the decay of the return probability through (2.40), we see that the commutator inserted inside a correlator of a small number of single-trace operators and evaluated on the state $|\Psi_T\rangle$ will give an exponentially small answer, since each of the two terms in (2.50) give exponentially small numbers. This is valid for any T as long as $|T| < t_*$ and $|T| - t_* \sim O(N^0)$. Thus,

$$[H, \widehat{\Phi}] = O(e^{-\gamma N^2}), \quad (2.51)$$

where γ is positive and $O(N^0)$, proving property 1, defined in subsection 2.2.4, for these operators. Note (2.51) is true for our set of code subspaces with T constrained as above, but not for all states. For example, the commutator is not exponentially suppressed in the state $|\Psi_{t_*}\rangle$.

2.3.2 Similar action as HKLL operators

A vanishing commutator with the Hamiltonian is necessary but not sufficient. There are many CFT operators that commute with the Hamiltonian up to exponentially small corrections in N^2 , but they will not have the same effect as acting with a local bulk operator. Therefore, we also need to show that the operator $\widehat{\Phi}$ behaves in the same way as the HKLL operator (2.23) to leading order at large N inside correlation functions of single-trace operators. For that we consider

$$\begin{aligned} \langle \Psi_0 | \mathcal{O} \dots \widehat{\Phi} \dots \mathcal{O} | \Psi_0 \rangle &= \\ &= c \int_{-t_*}^{t_*} dT \langle \Psi_0 | \mathcal{O} \dots e^{-iTH} P_0 \Phi P_0 e^{iTH} \dots \mathcal{O} | \Psi_0 \rangle \\ &= c \int_{-t_*}^{t_*} dT \langle \Psi_0 | \mathcal{O} \dots P_0 P_T (e^{-iTH} \Phi e^{iTH}) P_T P_0 \dots \mathcal{O} | \Psi_0 \rangle. \end{aligned} \quad (2.52)$$

In the last line, we have inserted two projectors P_0 , which we are free to do since the correlators is evaluated in the state $|\Psi_0\rangle$. The integrand above corresponds to $\text{Tr} P_T P_0$, up to some operator insertions that do not affect its general structure. From (2.40) we see that the integrand will be exponentially suppressed as $|T|$ increases (and is not $O(1/N)$) because of the exponentially small overlap of the code subspaces. We can thus evaluate the integral by a saddle-point method controlled by the large N limit. The dominant contribution comes from $T = 0^{25}$. Using (2.37) and (2.48) we have

$$\langle \Psi_0 | \mathcal{O} \dots \widehat{\Phi} \dots \mathcal{O} | \Psi_0 \rangle = \langle \Psi_0 | \mathcal{O} \dots \Phi \dots \mathcal{O} | \Psi_0 \rangle + O(1/N), \quad (2.53)$$

as desired. The $1/N$ corrections can be thought of coming from corrections to the leading saddle-point, and would be sensitive to the more detailed form of $F(T)$ in (2.37).

²⁵One might worry about the possibility of rapidly oscillating phases, such as the one in $\langle \Psi_0 | \Psi_T \rangle$ displacing the location of the saddle point. Notice however that from (2.37),(2.38) it follows that such rapidly oscillating phases cancel between the bra and ket contribution.

Notice that if we apply the operator $\hat{\Phi}$ to one of the time-shifted states, then as long as $|T| < t_*$, we find

$$\langle \Psi_T | \mathcal{O} \dots \hat{\Phi} \dots \mathcal{O} | \Psi_T \rangle = \langle \Psi_T | \mathcal{O} \dots (e^{-iT_H} \Phi e^{iT_H}) \dots \mathcal{O} | \Psi_T \rangle + O(1/N) \quad (2.54)$$

Thus in the code subspace \mathcal{H}_T , $\hat{\Phi}$ acts as $e^{-iT_H} \Phi e^{iT_H}$ to leading order at large N . To make this more manifest, we can also write (2.47) as

$$\hat{\Phi} = c \int_{-t_*}^{t_*} dT P_T (e^{-iT_H} \Phi e^{iT_H}) P_T. \quad (2.55)$$

Since we have shown that, to leading order at large N , $\hat{\Phi}$ and Φ have the same matrix elements on the entire code subspace it follows that higher point functions of $\hat{\Phi}$ will also agree at large N with those of Φ . Consider for instance,

$$\hat{\Phi}_i = c \int_{-t_*}^{t_*} dT e^{-iT_H} P_0 \Phi_i P_0 e^{iT_H} \quad (2.56)$$

where $\Phi_i \equiv \Phi(x_i)$ is an HKLL operator located at a certain spacetime point x_i , then in the large N limit

$$\begin{aligned} \langle \Psi_0 | \mathcal{O} \dots \hat{\Phi}_1 \hat{\Phi}_2 \dots \hat{\Phi}_n \dots \mathcal{O} | \Psi_0 \rangle &= c^n \int_{-t_*}^{t_*} dT_1 \dots dT_n \langle \Psi_0 | \mathcal{O} \dots P_{T_1} (e^{-iT_1 H} \Phi_1 e^{iT_1 H}) P_{T_1} P_{T_2} \\ &\quad (e^{-iT_2 H} \Phi_2 e^{iT_2 H}) P_{T_2} \dots P_{T_n} (e^{-iT_n H} \Phi_n e^{iT_n H}) P_{T_n} \dots \mathcal{O} | \Psi_0 \rangle \\ &\approx \langle \Psi_0 | \mathcal{O} \dots \Phi_1 \Phi_2 \dots \Phi_n \dots \mathcal{O} | \Psi_0 \rangle. \end{aligned} \quad (2.57)$$

In addition, this implies that the commutator of $\hat{\Phi}_i$'s is the same as that of HKLL operators in the large N limit. Two operators, $\hat{\Phi}(x_i)$ and $\hat{\Phi}(x_j)$, will have zero commutator at space-like separated points whereas they have $O(1)$ commutator if they are timelike-separated. This is true even though these operators do not translate under commutation with the boundary Hamiltonian, up to exponentially small corrections in N . Nevertheless, they still have bulk space-time labels and preserve the causal properties of HKLL operators in the large N limit.

2.3.3 Interpretation and comments

We have just seen that to leading order in the large N limit, the operator (2.47) acts like the HKLL operator (2.23) in the appropriate code subspace. However, it commutes with H to all orders in $1/N$. The existence of these operators provides strong evidence that the algebra of single-trace operators in a short time band can have a non-trivial commutant when acting on time-dependent states of high energy.

The vanishing of the commutator with H should be interpreted as (2.47) being gravitationally dressed not with respect to the boundary, but instead with respect to features of the bulk state, in particular its time-dependence. This can be seen by the fact that $\hat{\Phi}$ acts differently on different states. On the time-shifted states $|\Psi_T\rangle$ and their code subspaces, it acts as $e^{-iT_H} \Phi e^{iT_H}$. For example, imagine that in the state $|\Psi_0\rangle$ we have a supernova explosion taking place at $t = 0$ and we chose the operator (2.23) so that it acts right next to the explosion. In the state $|\Psi_T\rangle$ the explosion obviously takes place at $t = -T$. From

equation (2.54), we can see that the operator $\hat{\Phi}$ will act again right next to the supernova explosion, even though the supernova is now at $t = -T$. Therefore, one and the same operator $\hat{\Phi}$ knows how to always act at the correct moment (right next to the explosion) for the entire family of states $|\Psi_T\rangle$, as long as $|T| < t_*$. The finiteness of t_* indicates that there is still some residual boundary dressing, which however is not visible in perturbation theory²⁶.

The property of being dressed with respect to features of the state is also present in the local observables one defines in general relativity, discussed in section 2.1.3. These state dressed observables are defined at points where a set of D scalars, like the Ricci scalar or $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$ where $\mathcal{R}_{\mu\nu\rho\sigma}$ is the Riemann tensor, 'click' with a certain set of numbers. The observables are labeled by these values and they are evaluated precisely where the scalars take those values in each state. Locality of these observables requires them to be defined only in some neighbourhood of a classical solution. In the same spirit, the operators discussed in this section are also local for a certain family of code subspaces, see section 2.3.1.

As mentioned earlier, if the spacetime is so symmetric that the scalars take the same values throughout the spacetime, then these classical observables are not well defined. Since every point in the spacetime is physically equivalent, it is reasonable that local observables are ill defined for these solutions. For this reason, the observables are state dependent. Similarly, it is not possible to apply the same logic discussed in the previous subsections to empty AdS, or other static states, as there are no time-dependent features in the bulk that can be used as a 'clock' to define a moment in time where the operator acts. Technically, the return probability for such states does not exhibit the rapid decay (2.26). We thus see a nice parallel between the classical and quantum situations.

The definition of our operator gives a bulk operator which is dressed with respect to features of the state, but in an implicit manner. Our construction does not permit us to extract the details of the dressing. Going back to our example of a supernova explosion, one might guess that the dressing is with respect to the supernova and that one could in principle define a gravitational Wilson line between the operator and the supernova. But what if the state described instead two supernovas exploding at the same or different times? To which explosion would our operator be dressed to? The construction does not give a definite answer, and the way to address this question would be to enlarge the set of code subspaces on which our operator correctly acts. For example, if our operator did not move under the time-translation of one of two supernovae, we would say that it is dressed to the other one. We hope to return to this question in the future, but see subsection 2.6.3 for some related remarks.

2.3.4 A similarity transformation

We briefly mention a variant of operators with properties similar to those of (2.47). We first define the shifted Hamiltonian²⁷

$$\hat{H} = H - \langle \Psi_0 | H | \Psi_0 \rangle \mathbb{1}. \quad (2.58)$$

Then we introduce

$$V = \frac{c}{\sqrt{2}} \int_{-t_*}^{t_*} dT e^{-i\hat{H}T} P_0, \quad (2.59)$$

²⁶Similar remarks were made in [48] for the DeWitt observables in AdS.

²⁷This shift is useful in order to avoid rapidly oscillating phases in the discussion below.

with c given in (2.48). We have

$$VV^\dagger = \frac{c^2}{2} \int_{-t_*}^{t_*} dT \int_{-t_*}^{t_*} dT' e^{-i\hat{H}T} P_0 e^{i\hat{H}T'}, \quad (2.60)$$

where we used $P_0^2 = P_0$. Following arguments similar to those of the previous subsection, we find that to leading order at large N , and when computing the matrix elements of (2.60) within the code subspace, the two integrals in (2.60) can be computed by a saddle point method, where the dominant saddle is $T = T' = 0$. We then find that in this class of states and at large N

$$VV^\dagger \simeq \mathbb{1}, \quad \text{and} \quad V^\dagger V \simeq \mathbb{1}. \quad (2.61)$$

in the sense that, within the code subspace V behaves like a unitary, up to $1/N$ corrections.

Then we start with a boundary-dressed operator Φ and define

$$\hat{\Phi} = V\Phi V^\dagger. \quad (2.62)$$

Following similar arguments as before we can show that the operator (2.62) satisfies properties 1 and 2 of subsection 2.2.4. To check the commutator of $\hat{\Phi}$ with H . We write

$$\begin{aligned} [H, \hat{\Phi}] &= -i \frac{d}{ds} \left(e^{i\hat{H}s} V\Phi V^\dagger e^{-i\hat{H}s} \right) \Big|_{s=0} \\ &= -i \frac{d}{ds} \frac{c^2}{2} \left(\int_{-t_*-s}^{t_*-s} dT e^{-i\hat{H}T} \right) P_0 \Phi P_0 \left(\int_{-t_*-s}^{t_*-s} dT' e^{i\hat{H}T'} \right) \Big|_{s=0}, \end{aligned} \quad (2.63)$$

which again localizes on boundary terms and is thus exponentially suppressed.

Second, to show that the leading large N correlators of $\hat{\Phi}$ are the same as those of Φ we follow exactly the same reasoning as in the previous subsection, but now we will have two time-integrals. Each one of these time integrals will lead to a sharply suppressed Gaussian around $T = T' = 0$ and can be evaluated by saddle-point at large N , reproducing the desired result.

2.3.5 Other asymptotic charges

More generally we need to make (2.23) commute with all boundary symmetry generators corresponding to asymptotic symmetries. For asymptotically AdS_{d+1} space-times this is the conformal group $SO(2, d)$ and we consider a generalization of the form

$$\hat{\Phi} = c \int_B d\mu(g) U(g) P_0 \Phi P_0 U(g)^{-1}, \quad (2.64)$$

where now

$$c^{-1} = \int_B d\mu(g) \langle \Psi_0 | U(g) P_0 U(g)^{-1} | \Psi_0 \rangle. \quad (2.65)$$

Above, $d\mu(g)$ is the Haar measure on $SO(2, d)$ and B is a reasonably sized connected submanifold of $SO(2, d)$ containing the identity. The commutator with conformal generators will then be given by operators in the code subspace of states $U(g_*)|\Psi_0\rangle$, where g_* lies on the boundary ∂B . For the construction to work in this generalization we must make sure that the overlaps

$$R(g) = |\langle \Psi_0 | U(g) | \Psi_0 \rangle|^2, \quad (2.66)$$

decay exponentially in the geodesic distance of g from the identity. As discussed in subsection 2.2.7 we expect this to be true for states which break all symmetries at the semiclassical level²⁸. The quantity $R(g)$ is an interesting generalization of the return probability (2.31) that would be interesting to study further.

2.4 A more general argument for the commutant

The operators (2.64) constructed in the previous section commute with the asymptotic charges to all orders in $1/N$, however they commute with the other single-trace operators in the time-band generally only to leading order in $1/N$. To identify a commutant for the time-band algebra \mathcal{A} , the operators (2.64) have to be improved. In this short section we outline a somewhat different argument suggesting that it is indeed possible to find a commutant to all orders in $1/N$. We caution the reader that the argument that follows is based on certain assumptions which seem physically plausible, but for which a rigorous proof is still lacking. A more careful treatment for the existence of a commutant (as well as a mathematically precise definition of the time-band algebra in the first place) would be desirable.

Let us start with a standard HKLL operator Φ . We also introduce the notation $q_i = \frac{Q_i - \langle Q_i \rangle}{N}$ for where Q_i denotes any of the asymptotic $SO(2, d)$ charges and \mathcal{O}_j a general single-trace operator in the time-band. Our goal is to find an operator $\hat{\Phi}$ which has the following properties:

1. $[\hat{\Phi}, q_i] = 0$ and $[\hat{\Phi}, \mathcal{O}_j] = 0$ for all $q_i \in SO(2, d)$ and $\mathcal{O}_j \in \mathcal{A}$, to all orders in $1/N$.
2. To *leading order* at large N the correlators of $\hat{\Phi}$ with q_j, \mathcal{O}_i must be the same as those of Φ . In particular this means that for single-trace operators \mathcal{O}_i outside the time-band we generally expect $[\mathcal{O}_i, \hat{\Phi}] = O(N^0)$.

The first condition is obvious. The second condition is necessary in order to ensure that the operator $\hat{\Phi}$ acts in the expected way, at least to leading order at large N , and creates particles that can be detected with an $O(1)$ effect by operators outside the time-band when light rays from the diamond hit the boundary.

Here we remark that in order for the two conditions to be mutually consistent, it is important that we impose the second condition only to leading order at large N . The point is that $[q_i, \Phi] = O(1/N)$ hence when looking at leading order correlators it is indeed consistent to demand simultaneously that i) $\hat{\Phi}$ commutes with q_i and that ii) $\hat{\Phi}$ acts like Φ . However, when moving on to subleading corrections we have a non-vanishing commutator $[q_i, \Phi]$ hence we cannot impose both conditions at the same time. We choose to impose that our operators $\hat{\Phi}$ continue to commute with q_i to all orders in $1/N$, but we allow their correlators to depart from those of Φ_i at subleading orders in $1/N$.

We now define the desired operators $\hat{\Phi}$ by specifying how they act on the code subspace \mathcal{H}_0 . Earlier we defined the code subspace as the space generated by acting on $|\Psi_0\rangle$ with single-trace operators, which are not necessarily restricted in the time-band. However, by an analogue of the Reeh-Schlieder theorem²⁹ we expect that for reasonable bulk states $|\Psi_0\rangle$

²⁸For compact symmetries, such as rotations, $R(g)$ will have recurrences every 2π . Hence along the compact directions we take $g_* \sim O(1) < 2\pi$.

²⁹This was discussed in [61] for the case of empty AdS and at large N . We believe that a similar result should hold for more general heavy states and even when taking $1/N$ corrections into account, but it would be interesting to develop a more careful proof.

the code subspace \mathcal{H}_0 can also be generated by acting on $|\Psi_0\rangle$ with only elements of the time-band algebra \mathcal{A}

$$\mathcal{H}_0 = \text{span}\{\mathcal{A}|\Psi_0\rangle\}. \quad (2.67)$$

We now define the action of the operator $\hat{\Phi}$ on the code subspace by the following conditions

$$\hat{\Phi}A|\Psi_0\rangle = A\hat{\Phi}|\Psi_0\rangle, \quad \forall A \in \mathcal{A}. \quad (2.68)$$

This set of linear equations, one for every element of the small algebra \mathcal{A} , defines the action of $\hat{\Phi}$ on the code subspace, in a way which satisfies the desired properties as we will see below.

Notice that these equations can also be represented as follows: we first select a basis of linearly independent elements A_i of the algebra \mathcal{A} . then we define the matrix of 2-point functions

$$g_{ij} = \langle \Psi_0 | A_i^\dagger A_j | \Psi_0 \rangle. \quad (2.69)$$

From (2.67), it follows that the set of states $|i\rangle = A_i|\Psi_0\rangle$ form a (possibly over-complete) basis of the code subspace. Since $\hat{\Phi}$ is an operator on the code subspace it can be written as

$$\hat{\Phi} = K^{ij} |i\rangle \langle j| = K^{ij} A_i |\Psi_0\rangle \langle \Psi_0 | A_j^\dagger. \quad (2.70)$$

for an appropriate choice of K^{ij} . To find the matrix K , we start with the desired relation (2.68) written as

$$\hat{\Phi}A_l|\Psi_0\rangle = A_l\hat{\Phi}|\Psi_0\rangle, \quad (2.71)$$

then we replace $\hat{\Phi}$ with (2.70) and multiply from the left with $\langle \Psi_0 | A_k^\dagger$ to get

$$g_{jl} g_{ki} K^{ij} = \langle \Psi_0 | A_k^\dagger A_l \hat{\Phi} | \Psi_0 \rangle. \quad (2.72)$$

If the set of states $|i\rangle = A_i|\Psi_0\rangle$ are linearly independent then the matrix g_{ij} is positive definite and invertible. In that case we can solve for K as

$$K_{ij} = g^{ik} g^{jl} \langle \Psi_0 | A_k^\dagger A_l \hat{\Phi} | \Psi_0 \rangle, \quad (2.73)$$

where $g^{ij} g_{jk} = \delta_k^i$. When (2.73) is replaced in expression (2.70), we find an explicit solution of the desired equation (2.68).

We emphasize that the necessary ingredient to arrive at (2.73) was the linear independence of the states $A_i|\Psi_0\rangle$, which is equivalent to the statement that there is no non-vanishing operator in \mathcal{A} which annihilates the state $|\Psi_0\rangle$. We discuss this condition in the following subsection.

2.4.1 On the consistency of the defining equations

Before checking that the operators $\hat{\Phi}$ defined by (2.68), or equivalently via (2.70),(2.73), have the desired properties, we need to check that equations (2.68) are self-consistent linear equations. The only possible source of inconsistency is the following: if there was an element $A \neq 0$ of the time-band algebra \mathcal{A} such that $A|\Psi_0\rangle = 0$, this could potentially be a problem since we would then have $A|\Psi_0\rangle = 0$, while in general $A\hat{\Phi}|\Psi_0\rangle \neq 0$. Then the equation (2.68) would imply $0 = A|\Psi_0\rangle = A\hat{\Phi}|\Psi_0\rangle \neq 0$ which is a contradiction. Relatedly, g_{ij} defined in (2.69) would not be invertible and we would not be able to get to (2.73).

We will now show that this situation does not arise, that is

$$A|\Psi_0\rangle \neq 0 \quad \forall A \in \mathcal{A}, A \neq 0. \quad (2.74)$$

We will prove this by first proving that at large N (2.74) is true and then we will argue that $1/N$ corrections cannot change the conclusion.

We have been working under the assumption that the time-band is short enough, which means that in the bulk there will be a region which is space-like relative to the time band. In the large N limit, where gravitational backreaction is turned off, operators inside that region (for example usual HKLL operators) commute with all elements of the algebra \mathcal{A} , including the appropriately normalized asymptotic charges q_i . Hence, in the large N limit the algebra \mathcal{A} has a non-trivial commutant \mathcal{A}' . We want to argue that this commutant continues to exist when $1/N$ corrections are taken into account, provided that the state $|\Psi_0\rangle$ has non-vanishing variance of $O(N^2)$ under the asymptotic charges.

Assuming that at large N the theory in the bulk behaves like usual QFT on a curved background, we expect that an analogue of the Reeh-Schlieder theorem will hold for the commutant \mathcal{A}' , which means that we can generate the code subspace \mathcal{H}_0 by acting on $|\Psi_0\rangle$ with elements of \mathcal{A}' .

Suppose now that there was an element A of the time-band algebra \mathcal{A} which annihilated the state $|\Psi_0\rangle$. Then for any element $a' \in \mathcal{A}'$ we have

$$Aa'|\Psi_0\rangle = a'A|\Psi_0\rangle = 0. \quad (2.75)$$

Since states of the form $a'|\Psi_0\rangle$ generate \mathcal{H}_0 we conclude that the operator A has vanishing matrix elements in \mathcal{H}_0 at large N . From this we can not immediately conclude that $A = 0$ as an operator when $1/N$ corrections are included. For example, for $|\Psi_0\rangle = |0\rangle$ the normalized $SO(2, d)$ generators $q_i = \frac{Q_i}{N}$ have vanishing matrix elements at large N , since they annihilate $|0\rangle$ and commute with all other operators. However they are non-vanishing operators at order $1/N$. If A is a non-vanishing operator which has vanishing matrix elements at large N on \mathcal{H}_0 then it means that it acts as a central element at large N . Here we make an additional assumption, that the only central elements are the $SO(2, d)$ generators q_i and their functions³⁰. Since, by assumption, the state $|\Psi_0\rangle$ has non-trivial variance under these generators, we conclude that it cannot be annihilated by a non-trivial A .

Let us assume now that we have a state of the form $A|\Psi_0\rangle$ which has finite (i.e. $O(N^0)$) positive norm at large N . Including $1/N$ corrections will generally modify the norm of this state, but it will do so by corrections suppressed by powers of $1/N$. Since the previous argument established that the leading large N norm of the state $A|\Psi_0\rangle$ is a finite positive number, perturbative $1/N$ corrections cannot make it vanish. Hence we expect property (2.74) to be true to all orders in $1/N$ perturbation theory.

We emphasize that the fact that we cannot annihilate the state by the time-band algebra \mathcal{A} relies on the fact that we have restricted our attention to small products of single-trace operators. As discussed in a related context [12, 61], if we consider the *full* algebra of operators in the time-band we can find sufficiently complicated combinations which can annihilate the state³¹.

³⁰We believe this assumption to be quite weak, but it would be interesting to prove it more thoroughly.

³¹For example, consider a state $|\Psi\rangle$ with $\langle\Psi_0|\Psi\rangle = 0$. Then the (complicated) operator $|\Psi\rangle\langle\Psi|$ annihilates $|\Psi_0\rangle$.

Finally, as should be clear from the above, if the state $|\Psi_0\rangle$ has very small or vanishing variance in the asymptotic charges then (2.74) fails and it is not possible to define operators obeying (2.68).

2.4.2 Proof that $\hat{\Phi}$ has the desired properties

Having established that equations (2.68) are consistent, we argue that the operator $\hat{\Phi}$ has the desired properties.

First it is obvious by (2.68) that the operator $\hat{\Phi}$ has vanishing commutators with elements of \mathcal{A} . To see that consider $A_1 \in \mathcal{A}$ and a general state in the code subspace which can be written as $A_2|\Psi_0\rangle$, with $A_2 \in \mathcal{A}$. Then we have

$$[\hat{\Phi}, A_1]A_2|\Psi_0\rangle = \hat{\Phi}(A_1A_2)|\Psi_0\rangle - A_1(\hat{\Phi}A_2|\Psi_0\rangle) = A_1A_2\Phi|\Psi_0\rangle - A_1A_2\Phi|\Psi_0\rangle = 0, \quad (2.76)$$

where in the second equality we used (2.68). Since this is true for all A_2 , we find

$$[\hat{\Phi}, A_1] = 0 \quad \forall A_1 \in \mathcal{A}, \quad (2.77)$$

where it should be understood that this equation holds on the relevant code subspace.

Second, we will show that *to leading order at large N* , the operator $\hat{\Phi}$ acts like the HKLL operator Φ . To see this, consider an arbitrary matrix element on the code subspace. Two general states of the code subspace can be written as $A_1|\Psi_0\rangle, A_2|\Psi_0\rangle$. Then we have

$$\langle\Psi_0|A_1^\dagger\hat{\Phi}A_2|\Psi_0\rangle = \langle\Psi_0|A_1^\dagger A_2\Phi|\Psi_0\rangle = \langle\Psi_0|A_1^\dagger\Phi A_2|\Psi_0\rangle + \langle\Psi_0|A_1^\dagger[\Phi, A_2]|\Psi_0\rangle. \quad (2.78)$$

In the first equality we used (2.68). Now, the operator A_2 is some combination of single-trace operators in the time band, as well as the normalized $SO(2, d)$ generators q_i . All of these operators have commutators with Φ which are suppressed by powers of $1/N$. Hence the last term in the equation above is suppressed. All in all, we find

$$\langle\Psi_0|A_1^\dagger\hat{\Phi}A_2|\Psi_0\rangle = \langle\Psi_0|A_1^\dagger\Phi A_2|\Psi_0\rangle + O(1/N), \quad (2.79)$$

which establishes the desired result. This ensures that large N correlators of $\hat{\Phi}$ are the same as Φ .

We emphasize that the operators defined in this section are not exactly the same as the operators (2.47) discussed earlier. For example, unlike (2.47) the operators (2.68) were defined to act only on the code subspace \mathcal{H}_0 of $|\Psi_0\rangle$ and not on the code subspace \mathcal{H}_T for $T = O(N^0)$. Also, the commutator of (2.47) with the Hamiltonian is of order e^{-N^2} while it is exactly zero, within the code subspace, for the operators (2.68).

2.5 Examples

In this section, we consider various examples. Our primary focus will be on examining the validity of equations (2.34), (2.37), (2.38), on which the construction of our operators relies.

2.5.1 Coherent states

In general, we are interested in time-dependent semi-classical geometries. Many of these states can be thought of as bulk coherent states. We will discuss the overlap of these states closely following [173]. In the CFT, these states are prepared by a Euclidean path integral

$$|\Psi\rangle = T e^{-\int_{t_E < 0} dt_E d^{d-1}x \phi_b(t_E, x) \mathcal{O}(t_E, x)} |0\rangle, \quad (2.80)$$

where \mathcal{O} is a single-trace operator dual to a supergravity field, and the source is scaled appropriately so that it leads to states with non-trivial gravitational backreaction, i.e. the expectation value of the energy and variance of this state will scale like (2.18) and (2.20).

In the large N limit the overlap of two such states can be computed by a Euclidean gravitational path integration which in the semi-classical limit can be approximated by a saddle point computation. For example, the norm of the state is

$$\langle \Psi | \Psi \rangle \approx e^{-I_{grav}(\lambda_b)}, \quad (2.81)$$

where λ_b is the following boundary condition for the bulk field

$$\lambda_b = \begin{cases} \phi_b(t_E, x), & t_E < 0 \\ \phi_b^*(-t_E, x), & t_E > 0, \end{cases} \quad (2.82)$$

and $I_{grav}(\lambda_b)$ is the on-shell gravitational action in the presence of the sources specified above.

Generalizing to two states $|\Psi_1\rangle$ and $|\Psi_2\rangle$, the normalized inner product between them is

$$\mathcal{R} = \frac{|\langle \Psi_1 | \Psi_2 \rangle|^2}{\langle \Psi_1 | \Psi_1 \rangle \langle \Psi_2 | \Psi_2 \rangle}, \quad (2.83)$$

which at large N can be computed by a supergravity saddle-point computation

$$\mathcal{R} \approx \exp \left[-2 \operatorname{Re}(I_{grav}(\tilde{\lambda})) + I_{grav}(\lambda_1) + I_{grav}(\lambda_2) \right], \quad (2.84)$$

where the supergravity solutions have the boundary sources $\tilde{\lambda}$, λ_1 and λ_2 which take the following form

$$\tilde{\lambda} = \begin{cases} \phi_2(t_E, x), & t_E < 0 \\ \phi_1^*(-t_E, x), & t_E > 0, \end{cases} \quad \lambda_i = \begin{cases} \phi_i(t_E, x), & t_E < 0 \\ \phi_i^*(-t_E, x), & t_E > 0, \end{cases} \quad (2.85)$$

where $i = 1, 2$ ³².

Notice that in each of the terms of (2.84), the gravitational on-shell action is proportional to $\frac{1}{G_N} \sim N^2$. Since quantum mechanically we need $\mathcal{R} \leq 1$, we find that the following inequality has to be satisfied

$$2 \operatorname{Re}(I_{grav}(\tilde{\lambda})) \geq I_{grav}(\lambda_1) + I_{grav}(\lambda_2), \quad (2.86)$$

for the on-shell value of solutions of the Einstein plus matter equations, for any choice of sources of the form (2.85). If the two sources are different, we expect a strict inequality.

³²The sources $\phi_2(t_E, x)$ and $\phi_1^*(-t_E, x)$ should decay sufficiently fast at the $t = 0$ surface such that the states are normalizable. This also implies that the bra and ket preparations of different states can be smoothly glued to each other.

It would be interesting to explore this inequality directly from the gravitational point of view. We discuss this further in the discussion.

We now move on to the computation of the return probability for states of the form (2.80) after a small (not N -dependent) time evolution. That is, we take the time-evolved state, $|\Psi(T)\rangle = e^{-iHT} |\Psi\rangle$, and consider the following quantity

$$R(T) = \frac{|\langle \Psi(0) | \Psi(T) \rangle|^2}{\langle \Psi(0) | \Psi(0) \rangle \langle \Psi(T) | \Psi(T) \rangle}. \quad (2.87)$$

To apply the general formalism described above, we need to analyze how the Euclidean sources ϕ_0 preparing the state $|\Psi(0)\rangle$ need to be modified to ϕ_T , in order to prepare $|\Psi(T)\rangle$. From a technical point of view computing ϕ_T in terms of ϕ_0 is not straightforward, as it requires a solution of the Einstein equations. Nevertheless, we can in principle compute the return probability using (2.83) and (2.84) with a modified source

$$\tilde{\lambda} = \begin{cases} \phi_T(t_E, x), & t_E < 0 \\ \phi_0^*(-t_E, x), & t_E > 0, \end{cases} \quad \lambda_T = \begin{cases} \phi_T(t_E, x), & t_E < 0 \\ \phi_T^*(-t_E, x), & t_E > 0. \end{cases} \quad (2.88)$$

Thus we get

$$R(T) = \exp \left[-2 \operatorname{Re}(I_{grav}(\tilde{\lambda})) + I_{grav}(\lambda_0) + I_{grav}(\lambda_t) \right], \quad (2.89)$$

and this is exponentially suppressed in the semi-classical limit because of the $1/G_N \sim N^2$ coefficient in the gravitational action and the condition (2.86).

2.5.2 Thermofield double state

We now consider the thermofield double state

$$|\text{TFD}\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta E_n}{2}} |E_n\rangle_L \otimes |E_n\rangle_R, \quad (2.90)$$

where the $|E_n\rangle$'s are the energy eigenstates and $Z(\beta)$ is the partition function at inverse temperature β . In the strong coupling limit, for temperatures below the Hawking-Page temperature, the state is dual to two entangled thermal AdS geometries, while for temperatures higher than the Hawking-Page temperature, it is expected to be dual to the eternal black hole in AdS [201]. This geometry has two asymptotically AdS boundaries, on the "left" and the "right", hence the asymptotic symmetry group is $SO(2, d)_L \times SO(2, d)_R$. The state (2.90) is invariant under certain combinations of the asymptotic charges, for example, we have

$$(H_R - H_L) |\text{TFD}\rangle = 0 \quad \text{but} \quad (H_R + H_L) |\text{TFD}\rangle \neq 0 \quad (2.91)$$

and similarly for the other charges. In this case, we can generalize the return probability to include all possible large diffeomorphisms on the two sides

$$R(g_L, g_R) = |\langle \text{TFD} | U_L(g_L) U_R(g_R) | \text{TFD} \rangle|^2, \quad g_{L/R} \in SO(2, d)_{L/R}. \quad (2.92)$$

In this case, we expect $R(g_L, g_R)$ to rapidly decay along certain directions but remain constant along others due to the symmetries of the state (2.90).

In what follows we focus on a particular class of deformations, corresponding to evolving with $H_L + H_R$. This gives what is usually called the spectral form factor (SFF) defined as

$$R(t) = |\langle \text{TFD} | e^{-i\frac{T}{2}(H_L + H_R)} | \text{TFD} \rangle|^2 = \left| \frac{Z(\beta + iT)}{Z(\beta)} \right|^2, \quad (2.93)$$

which was introduced in the context of the eternal AdS black hole in [195] and studied in detail in [197].

We are interested in studying (2.93) above the Hawking-Page temperature for small times, i.e., $T \sim O(1)$. One way to proceed is by computing $Z(\beta)$ and then analytically continuing $\beta \rightarrow \beta + iT$. If we are above the Hawking-Page temperature $Z(\beta)$ can be estimated by the Euclidean AdS-Schwarzschild black hole saddle point

$$Z(\beta) \approx e^{-I_{BH}(\beta)}, \quad (2.94)$$

where $I_{BH}(\beta)$ is the on-shell action on the Euclidean black hole background. For example, we find

$$I_{BH}(\beta) = -\frac{\pi^2}{2G_N\beta} \quad (\text{for } AdS_3) \quad I_{BH}(\beta) = \frac{\beta}{G_N} g(r_H) \quad (\text{for } AdS_5), \quad (2.95)$$

where we have set the AdS radius $\ell_{\text{AdS}} = 1$ and r_H is the horizon radius, while

$$g(r_H) = \frac{V_3}{8\pi} (-r_H^4 + r_H^2) \quad (2.96)$$

where V_3 is the dimensionless volume associated with the metric on a unit sphere. For the AdS_5 case, $r_H \approx \pi/\beta$ for small real β . A detailed discussion of the action can be found in [202]. The central charge of the CFT_2 is $c = 3/2G_N$ and the rank for the gauge group of the dual four dimensional $SU(N)$ $\mathcal{N} = 4$ super Yang Mills theory is given by $N^2 = \pi/2G_N$.

For small T the complexified partition function $Z(\beta + iT)$ will be given in terms of the analytic continuation of the above actions. Thus for $T \ll \beta$, one gets the following for AdS_3 ,

$$R(T) \approx e^{-\frac{2\pi^2}{\beta^3} c T^2}, \quad (2.97)$$

which is exponentially small in the large central charge limit³³. Similarly for AdS_5 , we find that $Z(\beta) \sim e^{\frac{\pi N^2}{\beta^3}}$ in the high temperature limit. Again for $T \ll \beta$, we have

$$R(T) \approx e^{-\frac{12\pi}{\beta^5} N^2 T^2}, \quad (2.98)$$

As T becomes larger and approaches $T \sim \beta$, the dominant saddle point will no longer be the black hole, as the analytically continued action can start to compete with thermal AdS. In addition the analytically continued black hole saddle point corresponds to a geometry with a complex metric, and as $T \sim O(\beta)$ this metric becomes 'unallowable' according to the criteria of [203], see also [204]. Interestingly, thermal AdS becomes the dominant saddle point before the metric becomes not allowable [198].

An exponential decay of $R(T)$ in N is to be expected even when $T \sim \beta$, since in this case the thermal AdS saddle dominates and, $|Z(\beta + iT)|^2 \sim e^{\tilde{g}(T)/\beta^3}$ where \tilde{g} is $O(N^0)$ periodic function of time. Thus, the numerator of (2.93) $|Z(\beta + iT)|^2$ is N^0 while the denominator is $O(e^{N^2})$ leading to an exponentially suppressed $R(T)$.

³³There will be additional terms suppressed in T^2/β^2 which will not affect the exponential decay in the large c limit as long as t is smaller than β .

2.5.3 Weakly coupled, large N gauge theories

It is interesting to consider the behavior of the SFF at small, or even vanishing 't Hooft coupling λ . In this case, the bulk dual is stringy and moreover, at $\lambda = 0$, the spectrum of the dual CFT is (half)-integer-spaced and thus not chaotic at all. Nevertheless, the decay (2.34) is still valid for a certain timescale, even in the free theory. This was discussed in detail in [198]. For concreteness, we consider the partition function of free $\mathcal{N} = 4$ SYM on $\mathbb{S}^3 \times \mathbb{R}$, where the sphere has a unit radius. It has the form [205, 206]

$$Z(\beta) = \int \mathcal{D}U e^{\sum_R \sum_m^\infty \frac{1}{m} z_m^R(\beta) \chi_R(U^m)}, \quad (2.99)$$

where $\mathcal{D}U$ is the invariant Haar measure on the gauge group normalized to one, χ_R is character in the representation R and

$$z_m^R(\beta) = \sum_{R_i, B=R} e^{-m\beta E_i} + (-1)^{m+1} \sum_{R_i, F=R} e^{-m\beta E_i}, \quad (2.100)$$

where the first sum is over bosonic states and the sum in the second term is over fermionic states.

The behavior of the SFF $\left| \frac{Z(\beta+iT)}{Z(\beta)} \right|^2$, as well as of the microcanonical analogue $Y_{E, \Delta E}(T)$, based on the analytic continuation of (2.99) was discussed in [198].

Even at $\lambda = 0$ the SFF obeys (2.34), though in this case the Poincare recurrence time is very short, i.e. 4π .³⁴ While in this limit the bulk theory does not admit a semiclassical gravitational description, we could still apply the procedure (2.47) to identify operators with vanishing commutators with the Hamiltonian to all orders in $1/N$, though now they do not have a nice bulk interpretation.³⁵ In doing so, we would need to be careful to take t_* to be a short $O(1)$ time-scale which is less than 4π .

Here we notice that similar results have been derived for the analytically continued superconformal index [207], which can be thought of as the SFF for 2-sided eternal super-symmetric AdS black holes.

2.5.4 Perturbative states around empty AdS

We now briefly discuss the return probability for perturbative states around empty AdS. We want to consider states that have a large number of particles, but are still small enough so that we can ignore gravitational backreaction. We can get some useful estimates by considering a thermal gas of particles in AdS_{d+1} . These are dual to a gas generated by single-trace operators in the CFT. Suppose we have low-lying single-trace operators with conformal dimension Δ_i . For simplicity, we consider only scalars and we take the radius of AdS_{d+1} to be 1. Then the partition function of single-particle states $z(\beta)$ and the multi-trace Fock-space partition function are respectively

$$z(\beta) = \sum_i \frac{e^{-\beta \Delta_i}}{(1 - e^{-\beta})^d}, \quad Z(\beta) = \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} z(n\beta) \right]. \quad (2.101)$$

³⁴In our conventions conformal dimensions in the free theory are half-integers.

³⁵To start with, the HKLL procedure cannot be implemented at subleading orders in $1/N$ due to the many stringy fields present in the bulk. Therefore, the issue of non-commutativity with the Hamiltonian does not stand out like it does in the case of Einstein gravity.

It is now straightforward to do the analytic continuation

$$Z(\beta + iT) = \exp \left\{ \sum_{n=1}^{\infty} \sum_i \frac{e^{-(n\beta + iT)\Delta_i}}{(1 - e^{-n\beta + iT})^d} \right\}. \quad (2.102)$$

For scalar BPS operators dual to SUGRA modes, Δ_i is integer. Then it is obvious that the SFF $R(T) = \left| \frac{Z(\beta + iT)}{Z(\beta)} \right|^2$ has periodicity $T = T + 2\pi$, as expected. What we want to estimate is the decay rate of the SFF at early times, and how close to 0 the SFF drops between the recurrences.

First, we notice that the partition function factorizes to a product over Δ_i . Hence we can study the behavior of a given Δ_i and we drop the sum over i . If we first take the small β limit, before analytically continuing, we find

$$Z(\beta) \sim \exp \left[\zeta(d+1) \frac{1}{\beta^d} \right]. \quad (2.103)$$

Using this approximation we find that for early times

$$R(T) \sim e^{-\frac{d(d+1)\zeta(d+1)}{\beta^{d+2}} T^2}. \quad (2.104)$$

As expected the decay is controlled by the variance of H . Of course, if we use the high-temperature approximation (2.103) to perform the analytic continuation, then we do not see the recurrences. At high temperatures, the SFF starts decaying quite rapidly, stays close to zero for a while, and then goes back to 1 every $T = 2\pi \times \text{integer}$. To find an estimate of how closely it approaches zero it is convenient to evaluate it at $T = \pi$. Suppose that the conformal dimension is an even integer. Then we find

$$R(\pi) = \frac{\exp \left[2 \sum_{n=1}^{\infty} \frac{1}{n} \frac{e^{-n\beta\Delta}}{(1 - (-1)^n e^{-n\beta})^d} \right]}{\exp \left[2 \sum_{n=1}^{\infty} \frac{1}{n} \frac{e^{-n\beta\Delta}}{(1 - e^{-n\beta})^d} \right]} \sim e^{-(2-2^{-d})\zeta(d+1)\frac{1}{\beta^d} - \frac{1}{2^d} \log \frac{\beta\Delta}{2}} \quad (2.105)$$

So we see significant suppression at small β , though, of course, the suppression does not scale like e^{-N^2} .

We expect a similar qualitative behavior for $R(T)$ for generic pure states of similar energy as the states studied above (namely high energy states whose energy scales as $O(N^0)$): they will have recurrences every 2π , but the return probability will quickly decay to small values for $0 < t < 2\pi$. If we use (2.47) for such states, with $t_* \sim O(1) < \pi$, then the commutator with H will be suppressed by a factor of the order of (2.105) rather than e^{-N^2} . Note that this is not good enough, since the commutator we are trying to cancel is $\mathcal{O}(1/N)$, which in the large N limit is much smaller than the suppression controlled by (2.105).

2.5.5 Kourkoulou-Maldacena states in SYK model

The SYK model is a quantum mechanical model of N Majorana fermions interacting with random interactions which is given by the Hamiltonian

$$H = \sum_{iklm} j_{iklm} \psi_i \psi_k \psi_l \psi_m, \quad (2.106)$$

where ψ_i are the Majorana fermions $\{\psi_i, \psi_j\} = \delta_{ij}$, and the coupling j_{iklm} has drawn from the distribution

$$P(j_{iklm}) \sim \exp(-N^3 j_{iklm}^2 / 12J^2), \quad (2.107)$$

leading to disorder average of

$$\overline{j_{iklm}} = 0, \quad \overline{j_{iklm}^2} = \frac{3!J^2}{N^3}. \quad (2.108)$$

In a particular realization of the couplings, we consider pure states which are obtained by using the Jordan-Wigner transformation and combining pairs of Majorana fermions into qubit like operators and choosing states with definite eigenvalues for the σ_3 components of all qubits. These states are denoted by $|B_s\rangle$, where $s = (s_1, s_2, \dots, s_{N/2})$ with $s_k = \pm 1$, and they satisfy the relations below

$$S_k |B_s\rangle = s_k |B_s\rangle, \quad (2.109)$$

where $S_k = \sigma_3^k / 2 \equiv 2i \psi^{2k-1} \psi^{2k}$ is the spin operator. By choosing all possible combinations of the $\{s_k\}$'s we get a basis of the Hilbert space whose dimension is $2^{N/2}$ (N is an even integer number). We further evolve these states over some distance l in Euclidean time in order to get low energy states $|B_{s,l}\rangle = e^{-lH} |B_s\rangle$ which we will refer to as Kourkoulou-Maldacena (KM) states. To stay in the low-energy regime where the SYK model exhibits conformal invariance we take $1 \ll lJ \ll N$ [175].

As discussed in [175] the KM states can be thought of as a toy model of pure black hole microstates which are out of equilibrium and which contain excitations behind the horizon. Hence they are states which exhibit time-dependence and our general formalism should be applicable. We start by discussing the behavior of the return probability for these states.

Analytical computation of the return probability at large N

We start with the normalization of the KM states. In the large N limit, due to the approximate $O(N)$ symmetry of the theory it can be shown [175] that

$$\langle B_{s,l} | B_{s,l} \rangle = \langle B_s | e^{-2lH} | B_s \rangle = 2^{-N/2} Z(\beta), \quad (2.110)$$

where $\beta = 2l$ [175]. The return probability then in the large N limit is given by

$$R(T) = \left| \frac{\langle B_{s,l} | e^{-iHT} | B_{s,l} \rangle}{\langle B_{s,l} | B_{s,l} \rangle} \right|^2 = \left| \frac{Z(\beta + iT)}{Z(\beta)} \right|^2. \quad (2.111)$$

In a low-temperature expansion, the partition function can be estimated [208] using the Schwarzian approximation to be

$$Z(\beta) \propto \frac{e^{2\sqrt{2}\pi^2 \alpha_S \frac{N}{\beta J}}}{(\beta J)^{3/2}}. \quad (2.112)$$

Using (3.65) we find for the return probability

$$R(T) = \frac{1}{\left(1 + \frac{T^2}{\beta^2}\right)^{3/2}} e^{-(4\sqrt{2}\pi^2 \alpha_S \frac{N}{J\beta^3})T^2}, \quad (2.113)$$

which is compatible with (2.34), after we take into account the different N -dependence in the SYK model vs $\mathcal{N} = 4$ SYM.

We can now try to test the more general decay of the inner product between states in time-shifted code subspaces (2.37). Let us denote the unit-normalized KM states as

(2.114)

$|s,l\rangle = \frac{|B_{s,l}\rangle}{\sqrt{\langle B_{s,l}|B_{s,l}\rangle}}$, and denote their time-dependence as $|\hat{B}_{s,l}(T)\rangle = e^{-iHT}|\widehat{B}_{s,l}\rangle$. We consider an operator $A(t)$ which is a simple combination of the fermions, so that the state $A(t)|\hat{B}_{s,l}\rangle$ is in the code subspace. Then we write

$$\langle \widehat{B}_{s,l}(0)|A(t)|\widehat{B}_{s,l}(T)\rangle = \langle \widehat{B}_{s,l}(0)|\widehat{B}_{s,l}(T)\rangle \times \frac{\langle B_{s,l}(0)|A(t)|B_{s,l}(T)\rangle}{\langle B_{s,l}(0)|B_{s,l}(T)\rangle}. \quad (2.115)$$

Let us focus on the last ratio. We can rewrite it as

$$\frac{\langle B_{s,l}(0)|A(t)|B_{s,l}(T)\rangle}{\langle B_{s,l}(0)|B_{s,l}(T)\rangle} = \frac{\langle B_s|e^{-(l+i\frac{T}{2})H}A(t-\frac{T}{2})e^{-(l+i\frac{T}{2})H}|B_s\rangle}{\langle B_s|e^{-(l+i\frac{T}{2})H}e^{-(l+i\frac{T}{2})H}|B_s\rangle}, \quad (2.116)$$

which depends holomorphically on $l + i\frac{T}{2}$, so we can evaluate it by analytic continuation. All in all we find

$$\langle \widehat{B}_{s,l}(0)|A(t)|\widehat{B}_{s,l}(T)\rangle = \langle \widehat{B}_{s,l}(0)|\widehat{B}_{s,l}(T)\rangle \times \left[\langle \hat{B}_{s,l}(0)|A(t-\frac{T}{2})|\hat{B}_{s,l}(0)\rangle \right]_{l \rightarrow l+i\frac{T}{2}}. \quad (2.117)$$

At large N and for flip-invariant operators [175] we can also write this as

$$\langle \widehat{B}_{s,l}(0)|A(t)|\widehat{B}_{s,l}(T)\rangle = \langle \widehat{B}_{s,l}(0)|\widehat{B}_{s,l}(T)\rangle \times \langle A(t-\frac{T}{2})\rangle_{\beta|\beta \rightarrow \beta+iT}, \quad (2.118)$$

where in the last term we first compute the thermal 1-point function $\langle A(t-\frac{T}{2})\rangle_{\beta}$ as a function of β and then analytically continue β .

As an example, we consider the case where $A = \psi^k(t)\psi^k(t')$ (no summation over k implied). Following [175] we have for real time and large N

$$\langle \widehat{B}_{s,l}(0)|\psi^k(t)\psi^k(t')|\widehat{B}_{s,l}(0)\rangle = G_{\beta}(t-t'), \quad (2.119)$$

where, for $t > t'$, we have

$$G_{\beta}(t-t') = \frac{\pi^{1/4}}{\sqrt{2\beta J}} \frac{e^{-i\pi/4}}{\sqrt{\sinh[\pi(t-i\epsilon)/\beta]}}, \quad (2.120)$$

Therefore, using (2.117) we get

$$\langle \widehat{B}_{s,l}(0)|\psi^k(t)\psi^k(t')|\widehat{B}_{s,l}(T)\rangle = \langle \widehat{B}_{s,l}(0)|\widehat{B}_{s,l}(T)\rangle G_{\beta+iT}(t-t'), \quad (2.121)$$

where the last term can be computed as the analytic continuation of (2.120).

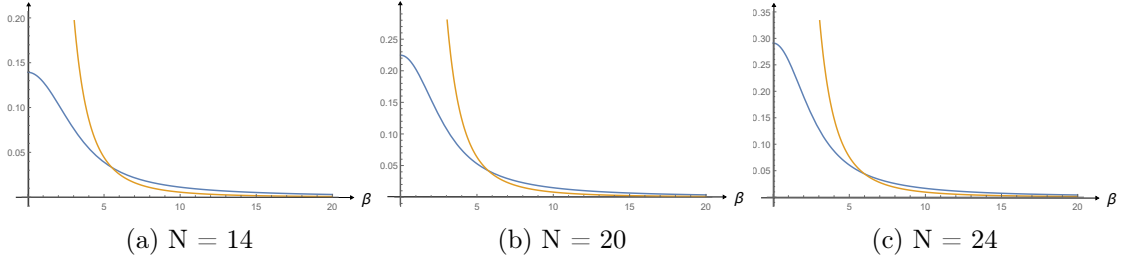


Figure 2.1: The blue lines are the numerical results for the variance of Hamiltonian as a function of β while the yellow ones are the Schwarzian approximation $\Delta H^2 = 0.396N/\beta^3$.

Similarly for $A = \psi^{2k-1}(t)\psi^{2k}(t')S_k$ we have [175]

$$\langle \widehat{B}_{s,l}(0) | \psi^{2k-1}(t)\psi^{2k}(t')S_k | \widehat{B}_{s,l}(0) \rangle = -2is_k G_\beta(t)G_\beta(t') + O(1/N), \quad (2.122)$$

hence

$$\langle \widehat{B}_{s,l}(0) | \psi^{2k-1}(t)\psi^{2k}(t')S_k | \widehat{B}_{s,l}(T) \rangle = \langle \widehat{B}_{s,l}(0) | \widehat{B}_{s,l}(T) \rangle \times \quad (2.123)$$

$$\times \left[-2is_k G_{\beta+iT}(t - \frac{T}{2})G_{\beta+iT}(t' - \frac{T}{2}) + O(1/N) \right]. \quad (2.124)$$

The examples (2.121) and (2.123) are consistent with our general expectations, see (2.37) and (2.38).

Some numerical checks

In this subsection we perform some simple numerical checks of (2.37) and (2.40), as well as the behavior of the operators (2.47) for KM states in the SYK model. The first step is to select an appropriate value for the inverse temperature $\beta = 2l$. The early time decay of the return probability is

$$R(T) = e^{-\Delta H^2 T^2}. \quad (2.125)$$

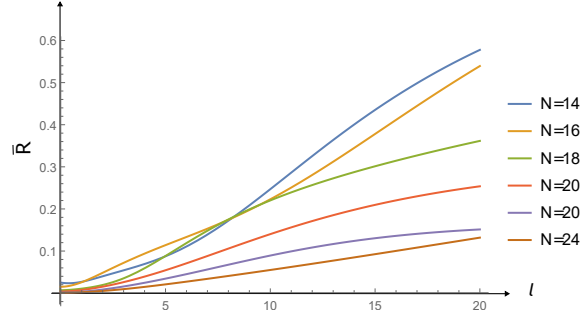
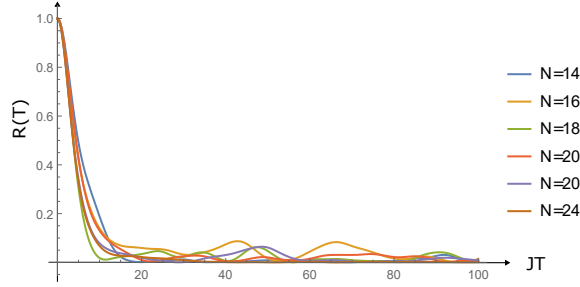
Earlier we used the Schwarzian approximation to compute the partition function (3.65) from which we can also get the variance

$$\Delta H^2 = 4\sqrt{2}\pi^2\alpha_S \frac{N}{\beta^3} = 0.396 \frac{N}{\beta^3}. \quad (2.126)$$

We compare this result with a numerical computation of the variance ΔH^2 for a KM state constructed from $|B_s\rangle = |+-\dots\rangle$. This is shown in Figure 4.4. In Figure 4.6, we show the value of the plateau for the KM state, as defined in (2.35) for various values of N and β . For the range of values of N we are interested in, we can take the inverse temperature to be $\beta = 5$, which is the value we will use in what follows.

In Figure 4.2 we can see the return probability as a function of t for different values of N for the corresponding KM state. As discussed in subsection 2.2.6, we expect that the overlap between any state in the code subspace at $t = 0$ will and the one at $t = T$ will also decay exponentially fast. We can encode the overlap between all such pairs of states by

$$R_{code}(T) = \frac{1}{d_{code}} \text{Tr}[P_T P_0]. \quad (2.127)$$


 Figure 2.2: The plateau height \bar{R} as a function of $l = \beta/2$.

 Figure 2.3: Return probability as a function of T for different values of N

For the numerical computation we need to make some choice about the code subspace. One condition is that the dimension d_{code} of the code subspace should satisfy $d_{code} \ll 2^{N/2}$. As an example, and for the purpose of the numerical computation, we can define the code subspace as

$$\mathcal{H}_{code} = \text{span}\{\mathcal{O}_1^{i_1} \dots \mathcal{O}_k^{i_k} |B_s\rangle; i_j = 0, 1\}, \quad (2.128)$$

for some choice of the operators \mathcal{O}_i . Here $D_{code} = 2^k$ the value of k should be such that $D \ll 2^{N/2}$. Note that the states in (2.128) are generally not orthonormal but it is easy to write a projector on the code subspace in terms of elements of this basis, see [209] for a related discussion.

In Fig. 4.7, we see plots of the behavior of $R_{code}(T)$ as a function of time for some specific choices of such a code subspace:

- **a** : the dimension of the code subspace is $D = 8$ and the operators are chosen to be

$$\mathcal{O}_1 = \psi_1(t = 0), \quad \mathcal{O}_2 = \psi_1(t = 0.1), \quad \mathcal{O}_3 = \psi_1(t = 0.5).$$

- **b** : the dimension of the code subspace is $D = 8$ and the operators are chosen to be

$$\mathcal{O}_1 = \psi_1(t = 0), \quad \mathcal{O}_2 = \psi_1(t = 0.1), \quad \mathcal{O}_3 = h.$$

- **c** : the dimension of the code subspace is $D = 16$ and the operators are chosen to be

$$\mathcal{O}_1 = \psi_1(t = 0), \quad \mathcal{O}_2 = \psi_1(t = 0.1), \quad \mathcal{O}_3 = \psi_1(t = 0.5), \quad \mathcal{O}_4 = \psi_1(t = 1).$$

where in case (b) the operator h is the normalized Hamiltonian

$$h = \frac{1}{\sqrt{N}}(H - \langle H \rangle). \quad (2.129)$$

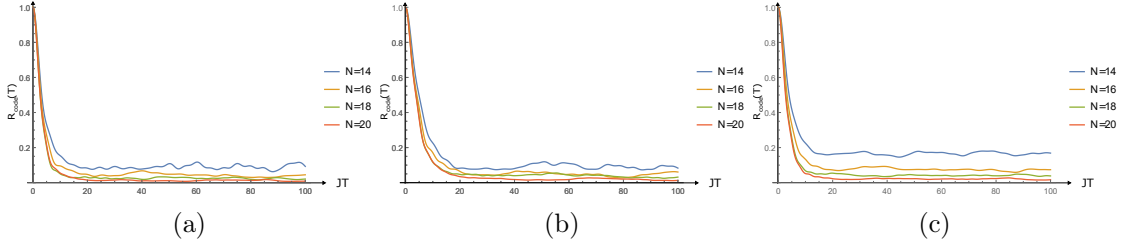


Figure 2.4: $R_{code}(T)$ as a function of T for three different examples of codesubspaces in the form of (2.128).

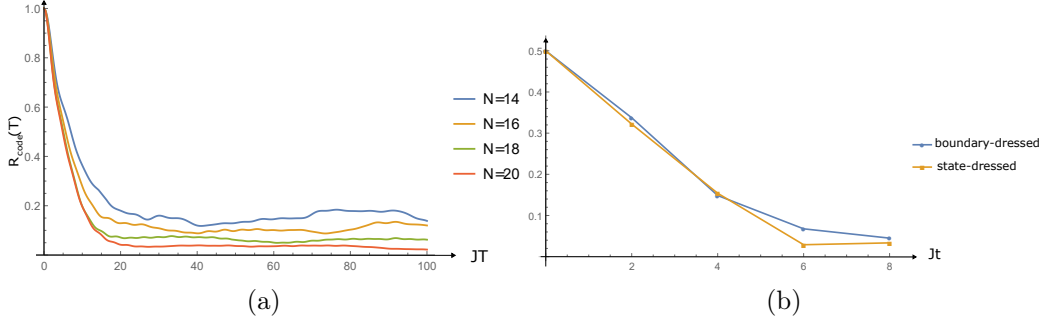


Figure 2.5: Results for the code subspace (2.130). (a) $R_{code}(T)$ as a function of T . (b) The blue line is $\langle \psi_3(0)\psi_3(t) \rangle$ as a function of t , while in the case of the yellow line, $\psi_3(0)$ is replaced by the dressed operator obtained from our proposal. Here $N=20$.

We finally check that the operator (2.47) has similar correlators as the boundary-dressed operator. We take the code subspace as

$$\mathcal{H}_{code} = \text{span}\{|B_s\rangle, \mathcal{O}_1|B_s\rangle, \dots, \mathcal{O}_k|B_s\rangle, h|B_s\rangle, h\mathcal{O}_1|B_s\rangle, \dots, h\mathcal{O}_k|B_s\rangle\}, \quad (2.130)$$

where the dimension of the code subspace is $d_{code} = 2(k+1) \ll 2^{N/2}$. In Fig. 4.8, we plot the result for the case of $k=5$ and where the operators chosen to be

$$\mathcal{O}_1 = \psi_1(t=0), \quad \mathcal{O}_2 = \psi_1(t=2), \quad \mathcal{O}_3 = \psi_1(t=4) \quad \mathcal{O}_4 = \psi_1(t=6), \quad \mathcal{O}_5 = \psi_1(t=8)$$

for $N=20$ ($d_{code} = 12 \ll 2^{10}$) are plotted. One can see from Fig.2.5b that the state-dressed operator for ψ_3 has approximately the same correlation function as the original one.

2.5.6 Holographic boundary states

The KM states discussed in the previous section can be thought of as certain atypical black hole microstates in the context of SYK/AdS₂. Interesting analogs in higher dimensional examples of AdS/CFT can be found by considering boundary states in CFTs [210–212]. A boundary state characterizes boundary conditions which can be imposed on a boundary of space-time on which the CFT lives. For each allowed boundary condition, we can evolve the state along the Euclidean time to suppress the high-energy contributions and obtain a state of finite energy which is called a regularized boundary state of the CFT.

For holographic theories, the CFT path integral maps onto the gravity path integral. Therefore, we will be able to make use of the AdS/CFT correspondence to deduce the

corresponding geometries if we can choose a state for which we can understand a gravity prescription for dealing with the boundary condition at the initial Euclidean time. As discussed in [213], we can describe boundary states by starting with the TFD state of two CFTs labeled by L and R

$$|\text{TFD}(\beta/2)\rangle = \frac{1}{Z} \sum_i e^{-\beta E_i/4} |E_i\rangle_L \otimes |E_i\rangle_R, \quad (2.131)$$

and then project the TFD state onto some particular pure state $|B\rangle$ of the left CFT. As a result, we obtain a pure state of the right CFT given by

$$|\Psi_{B,\beta}\rangle = \frac{1}{Z} \sum_i e^{-\beta E_i/4} \langle B | E_i \rangle |E_i\rangle. \quad (2.132)$$

If the temperature is high enough, the TFD state is dual to the maximally extended AdS-Schwarzschild black hole in the bulk. The geometry which is dual to these regularized boundary states is expected to contain a significant portion of the left asymptotic region. Therefore, in a holographic CFT, this class of regularized boundary states can be regarded as microstates of a single-sided black hole. These black hole microstates can be thought of as black holes with end of the world (EOW) branes on the left side.³⁶ Generally the EOW brane configuration is time-dependent at the macroscopic level. Hence these are states with energy and energy variance compatible with (2.18) and (2.20), so we expect to be able to apply our construction and define operators (2.47). As we will discuss in the next section, one way to think of them is that the gravitational dressing has been moved over to the EOW brane.

Computation of the return probability and correlators

First, we define unit-normalized boundary states

$$|\widehat{B}_a(0)\rangle = \frac{e^{-\frac{\beta H}{4}} |B_a\rangle}{\sqrt{\langle B_a | e^{-\frac{\beta H}{2}} |B_a\rangle}}. \quad (2.133)$$

Then we want to show that return probability of a boundary state

$$R(T) = |\langle \widehat{B}_a(0) | \widehat{B}_a(T) \rangle|^2, \quad (2.134)$$

decays exponentially fast at early time. For boundary states in holographic 2d CFTs we have (??)

$$G(\beta) = \langle B_a | e^{-\frac{\beta H}{2}} |B_a\rangle \simeq e^{\frac{\pi^2 c}{6\beta}}. \quad (2.135)$$

where we have taken the CFT to be defined on a spatial circle of length 2π . For small T we have

$$R(T) = \frac{|G(\beta + 2iT)|^2}{|G(\beta)|^2} \simeq e^{-\frac{4\pi^2 c}{3\beta^3} T^2}. \quad (2.136)$$

³⁶Proving from first principles that boundary states dual to EOW branes exist is far from trivial. It has been investigated from a bootstrap perspective in [214], where it was suggested that such boundary states must be extremely fine-tuned. In [215], the full classification of boundary states in large N symmetric orbifolds was carried out, and typical boundary states are not of this form.

The energy variance of the boundary state can be easily computed from (2.135) and we find

$$\Delta H^2 = \langle H^2 \rangle - \langle H \rangle^2 = \frac{4\pi^2 c}{3\beta^3}, \quad (2.137)$$

so the initial decay (2.136) is, not surprisingly, consistent with (2.32), (2.34) and (2.137).

In higher dimensional cases we can read from (E.13)

$$G(\beta) = e^{\frac{\alpha_d}{\beta^{d-1}}}, \quad (2.138)$$

thus

$$R(T) = \frac{|G(\beta + 2iT)|^2}{|G(\beta)|^2} \simeq \exp \left[-\frac{\alpha_d}{\beta^{d+1}} 4d(d-1)T^2 \right]. \quad (2.139)$$

We can again check that

$$\Delta H^2 = \langle H^2 \rangle - \langle H \rangle^2 = \frac{\alpha_d}{\beta^{d+1}} 4d(d-1), \quad (2.140)$$

which is compatible with (2.139).

We now proceed with checking that the other states in the code subspace around a boundary state are orthogonal to the time evolved code subspace. Consider for example the state $\mathcal{O}(t, x)|\widehat{B}_a\rangle$. Following similar reasoning as in subsection 2.5.5 we can show that

$$|\langle \widehat{B}_a(0)|\mathcal{O}(t, x)|\widehat{B}_a(T)\rangle|^2 = \langle \widehat{B}_a(0)|\mathcal{O}(t, x)|\widehat{B}_a(T)\rangle \langle \mathcal{O}(t - \frac{T}{2}, x) \rangle_{\beta \rightarrow \beta + 2iT}. \quad (2.141)$$

where $\langle \widehat{B}_a(0)|\mathcal{O}(t, x)|\widehat{B}_a(T)\rangle = \frac{G_a(I, \beta + 2iT)}{G_a(I, \beta)}$. More generally

$$\begin{aligned} & \langle \widehat{B}_a(0)|\mathcal{O}(t_1, x_1)\mathcal{O}(t_2, x_2)\dots\mathcal{O}(t_n, x_n)|\widehat{B}_a(T)\rangle = \\ & \langle \widehat{B}_a(0)|\mathcal{O}(t, x)|\widehat{B}_a(T)\rangle \langle \mathcal{O}(t_1 - \frac{T}{2}, x_1)\mathcal{O}(t_2 - \frac{T}{2}, x_2)\dots\mathcal{O}(t_n - \frac{T}{2}, x_n) \rangle_{\beta \rightarrow \beta + 2iT}. \end{aligned} \quad (2.142)$$

Thus, as long as the analytical continuation of the correlation function in β does not introduce any surprising N -dependent factors we will get the expected behavior (2.38). We now check this condition for low-point functions in 2d boundary states.

Here we assume that for a holographic CFT, and if we are working in the large N limit, the 1-point function of light conformal primaries can be computed by a method of images. Then for a 1-point function of a scalar primary \mathcal{O} with dimension Δ on a boundary state we have

$$\langle \widehat{B}_a(0)|\mathcal{O}(t, x)|\widehat{B}_a(0)\rangle = \frac{A_{\mathcal{O}}}{(\frac{\beta}{\pi} \cosh[\frac{2\pi}{\beta}t])^\Delta}. \quad (2.143)$$

for some constant $A_{\mathcal{O}}$ which depends on the boundary state a and the operator \mathcal{O} . After the analytic continuation necessary for (2.141) we find

$$\langle \mathcal{O}(t - \frac{T}{2}, x) \rangle_{\beta \rightarrow \beta + 2iT} = \frac{A_{\mathcal{O}}}{(\frac{\beta + 2iT}{\pi} \cosh[\frac{2\pi}{\beta + 2iT}(t - \frac{T}{2})])^\Delta}. \quad (2.144)$$

Hence we notice that the results (2.141), (2.144) are consistent with our general expectations (2.37), (2.38).

We can also check 2-point functions, which we can compute in the large N limit. First we compute the 2-point function on the boundary state, using the method of images

$$\langle \widehat{B}_a(0) | \mathcal{O}(t_1, x_1) \mathcal{O}(t_2, x_2) | \widehat{B}_a(0) \rangle = \sum_{n=-\infty}^{+\infty} \frac{1}{\left| \frac{\beta}{\pi} \sinh\left(\frac{\pi}{\beta}[(x_1 - x_2 + 2\pi n) - (t_1 - t_2)]\right) \right|^{2\Delta}} \pm \frac{1}{\left| \frac{\beta}{\pi} \cosh\left(\frac{\pi}{\beta}[(x_1 - x_2 + 2\pi n) - (t_1 + t_2)]\right) \right|^{2\Delta}}, \quad (2.145)$$

After the analytic continuation necessary for (2.142) we find from (2.145) that we do not notice any unexpected behavior of this part of the correlator as T increases, so the result (2.142) is dominated by the decay of the return probability, and is consistent with our expectations (2.37),(2.38).

2.6 Black Hole microstates

One question that is particularly interesting is whether we can apply our construction to black hole microstates. We have already mentioned in section 2.2.1 that there are various classes of black hole microstates, some of which have macroscopic time dependence and some of which do not. We will now discuss these various cases in more detail and interpret our operators for these types of states.

2.6.1 States with macroscopic time-dependence

We will start with the simplest situation: states with macroscopic time dependence. This can be visible outside the horizon, for example, black holes in the presence of infalling matter. Alternatively, it can be that the geometry appears to be static outside the horizon but there is no corresponding Killing isometry in the interior. As the first case is more straightforward, we focus on the second case. Two examples of such states are boundary states of the CFT, corresponding to end-of-the-world branes inside the horizon, which have already been discussed in the previous section. A second example is states prepared by the Euclidean path integral on some surfaces of higher topology. The dual geometries have topology behind the horizon and are often referred to as geons [179, 216, 217]. It is worth re-emphasizing that both of these states are usually prepared by the Euclidean path integral and are in fact very atypical states, even if the CFT 1-point functions are very close to those in a thermal state (or said differently, even if the classical geometry is exactly that of a black hole outside the horizon).

Both of these examples involve pure states $|\Psi_0\rangle$ that have a large energy variance, of order N^2 , such that the return probability will decay as (2.33). We can thus apply our construction to build local operators that are not dressed to the boundary CFT. The interpretation is that the operators are dressed with respect to the time-dependence of the interior. Consider for example the genus-2 geon in $d = 2$, which is prepared by the Euclidean path integral on half of a genus-2 surface [179, 218]. Microscopically, the state can be described by

$$|\Psi_0\rangle \sim \sum_{i,j} C_{ij} e^{-E_i \beta_i / 2 - E_j \beta_j} |E_j\rangle, \quad (2.146)$$

where \sim indicates that we have not been careful about the parametrization of the genus-2 surface, but $\beta_{i,j}$ are related to the moduli of the surface. The un-normalized overlap of

this state corresponds to a genus-2 partition function in the dumbbell channel, where β_j parametrizes the length of the two handles, and β_i parametrizes the length of the neck between them.

It is not straightforward to write down a metric that covers the entire space-time of such states. Outside the horizon whose size is controlled by β_j , they look exactly like the BTZ geometry. Inside the horizon, they have macroscopic time-dependence. A nice coordinate patch that covers the Wheeler-de Witt patch of the $t = 0$ slice of the geometry can be written down in a very simple form

$$ds^2 = -dt^2 + \cos^2 t d\Sigma_2^2, \quad (2.147)$$

where $d\Sigma_2^2$ is the constant negative curvature metric on half of a genus-2 surface. This coordinate patch covers the entire $t = 0$ slice of the geometry, which is precisely half of a genus-2 surface. The neck corresponds to the horizon, and there is topology (one handle) behind the horizon. From this metric, we explicitly see the time dependence of the geometry, even if a metric for the full spacetime is hard to write down. The interpretation of our operator is that the dressing is to the time-dependence of the geometry that sits inside the horizon. For end-of-the-world brane geometries, the situation is similar and the operator is dressed to the end-of-the-world brane.

2.6.2 Typical states

The question we would now like to ask is whether our prescription works in typical black hole microstates. Contrary to states with end-of-the-world branes or topology behind the horizon, it seems reasonable to expect that typical states should also look like the thermal state a finite distance inside the black hole (see for example [100, 101]).

Whether or not our prescription works depends on the definition of a typical black hole microstate, and in particular on the energy spread we are choosing. One possibility is to define typical states using an ensemble of energy eigenstates with spread $\mathcal{O}(N^0)$ in energy (recall that there are still e^S with $S \sim \mathcal{O}(N^2)$ states in this energy band). In that situation, our prescription does not work, as the variance of energy is $\mathcal{O}(N^0)$ and the return probability will not decay fast enough. Another possibility is to consider typical states with an energy spread similar to that of the canonical ensemble, that is

$$(\Delta E)^2 \sim \mathcal{O}(N^2). \quad (2.148)$$

For such states, the return probability will decay following the behavior (2.33). Therefore, we can follow our prescription and define the operators in the same way and they will satisfy the two properties of commuting with the Hamiltonian to all orders in $1/N$ and acting like HKLL operators to leading order at large N .

While these operators are certainly diff-invariant, since they are operators defined in the CFT, the bulk interpretation of their gravitational dressing on typical black hole microstates is not entirely clear. When the gravitational configurations are macroscopically time-dependent, our operators are dressed with respect to the features of the geometry. The typical states are still time-dependent, but only microscopically, as it seems plausible to assume that macroscopically they are featureless. In some sense our operators are dressed to the microscopic time-dependence of the state (the phases of the c_i in (2.19)), but it is unclear exactly what that means in the bulk.

Notice however, that if we start with a particular typical pure state $|\Psi_0\rangle$ and act with a unitary made out of the operator (2.47), associated to that state, then the predictions for what an infalling observer jumping into the black hole will see are unambiguous. For example, the operators (2.47) will generally create an excitation in the bulk and the location in time relative to that of the infalling observer who jumps from the boundary at a particular boundary time, can be unambiguously computed for each state $|\Psi_0\rangle$ and corresponding operators (2.47). We emphasize that for this interpretation it is important to remember that the operators (2.47) are state-dependent and cannot generally be promoted to a single operator which acts in a specific way globally on most typical states.

We briefly comment on black hole interior reconstruction. Suppose we start with a typical black hole microstate with energy spread of order (2.20). If we assume that the interior geometry contains part of the left asymptotic region, then the possibility of removing the dressing of the operators implies that we can deform the state behind the horizon by creating some particles there, in such a way that these excitations cannot be detected from the boundary CFT by the measurement of single-trace correlators, including the Hamiltonian, in the $1/N$ expansion. This was also discussed in [219, 220]. We emphasize that this does not contradict the statements made in [12, 100, 101] that for typical states with microcanonical energy spread, it is impossible to add excitations without affecting single-trace correlators.

2.6.3 Two entangled CFTs

Similar considerations apply to geometries with two asymptotically AdS regions. Consider two non-interacting CFTs with total Hamiltonian $H = H_L + H_R$. We take the full system to be in a pure state $|\Psi_0\rangle$ which may be entangled, but we will assume the pattern of entanglement is generic. In particular, we *do not* consider states like the thermofield-double which have a very fine-tuned structure of entanglement. We can imagine the state $|\Psi_0\rangle$ to be, for example, $U_L |\text{TFD}\rangle$, where U_L is a complicated random unitary acting on the left CFT. In this case we can consider the following generalization of our construction. Let us consider the 2-parameter family of time-shifted states

$$e^{-i(T_L H_L + T_R H_R)} |\Psi_0\rangle.$$

We start with an HKLL operator Φ dressed with respect to the left system, which commutes with H_R but not H_L . We now consider the following generalization of the operators (2.47)

$$\hat{\Phi} = c \int dT_L dT_R e^{-i(T_L H_L + T_R H_R)} P_0 \Phi P_0 e^{i(T_L H_L + T_R H_R)} \quad (2.149)$$

using $P_0 = P_0^L \otimes P_0^R$ and $[\Phi, P_0^R] = 0$ then

$$\hat{\Phi} = c \int dT_L e^{-i T_L H_L} P_0^L \Phi P_0^L e^{i T_L H_L} \otimes \int dT_R P_{T_R}^R \quad (2.150)$$

The resulting operator commutes with both H_L and H_R on the relevant code subspaces. In this case, the operator is not dressed with respect to the overall time-dependence of the full system, but rather to the time dependence of the “left” subsystem.

There are states with special entanglement pattern such as the TFD state, which was already discussed in section 2.5.2. The generalized return amplitude $\langle \Psi_0 | e^{-i(H_L T_L + H_R T_R)} | \Psi_0 \rangle$ which is a function of T_L and T_R does not decay in all directions for these special states.

For example, in the TFD state it is constant along the line $T_L = -T_R$. In those cases we cannot set both commutators with H_L, H_R to zero. So we can move the dressing from one side to another if we wish to, but there it is always dressed to one of the boundaries. This happens because the TFD state has a symmetry, it is annihilated by $H_L - H_R$.

2.6.4 Island discussion

Our prescription is also useful to resolve some paradoxes in the context of black hole evaporation and islands. Consider a setup where a holographic CFT is coupled to a bath such that the bulk description is given by an evaporating black hole. After the Page time, a non-trivial quantum extremal surface appears in the bulk delimiting an island, i.e. a part of the interior of the black hole that is encoded in the bath degrees of freedom rather than in those of the CFT [221, 222].

There is an apparent tension in this context related to gravitational dressing [223]. If we create an excitation in the island by acting with a local operator ϕ_{island} , where does the gravitational dressing go? It appears that the only place for the dressing to go is the boundary CFT. But this implies that the local operator will have the property

$$[\phi_{\text{island}}, H_{\text{CFT}}] \neq 0. \quad (2.151)$$

But this seems to be inconsistent because since the operator is in the island, it should be reconstructable from the bath degrees of freedom, and commute with the CFT degrees of freedom.

Our operators provide a way out of this paradox. We can apply our prescription above in terms of two entangled systems with a generic pattern of entanglement (there is a subtlety here since the bath and CFT are actually coupled rather than non-interacting, but we can treat this interaction as weak). In that case, even if we did start with an operator that had a non-trivial commutator (2.151), we would engineer a new operator that commutes with H_{CFT} up to exponentially small corrections. This new operator is now dressed with respect to the radiation, rather than the boundary CFT.

The interpretation of the dressing is similar to that of the typical states. While it would be tempting to imagine dressing the operator to the quantum extremal surface, the bulk geometry only has extremely slow time-dependence so it is unclear if time-dependent features of the geometry are sharp enough to dress with respect to them. It appears that the dressing is towards the microscopic time-dependence of the radiation. The story becomes less subtle if we consider a doubly holographic model (see for example [38, 224]). In that case, the dressing to the bath can be directly geometrized in the higher-dimensional geometry. Our operators can perhaps be thought as a counterpart of the operators in the doubly-holographic setup, but in cases where the dressing cannot be so easily geometrized.

Finally, we would like to clarify the distinction between reconstruction and dressing. To make things simple, let us consider the TFD state and consider an HKLL operator on the left ϕ_L . This operator is dressed to the left CFT. Now we run our protocol, and as explained above, we can move the dressing to the right. The operator $\hat{\phi}_L$ now commutes with H_L but no longer with H_R [195]. This *does not* mean that it can be reconstructed from the right degrees of freedom, but that it can be detected from the right CFT via the Gauss law tail. It is still mostly built from the left CFT degrees of freedom, only its dressing has been pushed to the right.

Chapter 3

Generalized continuous Matrix Product States

In this chapter, we introduce two new classes of continuous matrix product states. First, we start by briefly reviewing the tensor network in physics.

3.1 Brief review of the tensor network in physics

Tensor Network states are the entanglement-based ansatz that has arisen in recent years based on the renormalization group (RG) ideas and later on developed using tools and concepts from quantum information theory. The main examples include *matrix product states* (MPS) [225], *projected entangled-pair states* (PEPS) [226], and *multiscale entanglement renormalization ansatz* (MERA) [227]. By construction, they obey the entropy/area law [228–231] and are able to encode both global and local symmetries [232–236]. Therefore, they provide an efficient class of symmetric variational ansatz to approximate the ground state of the local Hamiltonian. In general, the understanding of the low-energy behavior of many-body quantum systems is one of the major challenges of modern physics, both in high-energy and condensed matter physics. There are plenty of methods based on RG introduced to tackle this problem. To study the weakly coupled system, one can use the momentum space RG [237–241]. But instead, in the case of the strongly interacting systems where the perturbation theory fails, this question is usually addressed by real-space RG methods.

In the case of the many-body system on the lattice, Kadanoff’s spin-blocking idea [242] was replaced by Wilson’s real space RG [241] which is improved later by White’s *density matrix renormalization group* (DMRG) [243, 244]. This technique is extraordinarily powerful in the study of quantum systems on the 1-D lattice. It has been generalized as *tensor renormalization group* (TRG) by Levin and Nave [245] to study the Euclidean path integral of 1-D quantum systems or the 2-D classical lattice models. Although, both the DMRG and TRG are very successful, they provide a coarse-grained system that still contains irrelevant microscopic information which implies the breakdown of both methods at criticality [245], and the resulting RG flow has the wrong structure of noncritical fixed points [246]. In the context of wave functions, this problem was resolved with the introduction of *entanglement renormalization* (ER) by Vidal [227]. A key aspect of ER is the ability to remove the short-range entanglement at each coarse-graining step by introducing a disentangler operator. This leads to the restoration of scale-invariant at criticality and

results in a proper RG flow with the correct structure of fixed points both at criticality and off criticality. More recently, this technique has been adapted to tackle the same problem in TRG in the context of the Euclidean path integral of quantum many-body systems and the partition function of a classical statistical system by removing short-range correlations this time from the partition function, known as *tensor network renormalization* (TNR) [247]. ER and TNR represent a powerful alternative to Wilsonian real-space RG methods in the context of the wave function and partition function respectively.

Beginning with the DMRG, it has been shown that this technique can be understood as a variational method within the class of MPS [248]. In addition, it justifies the point that DMRG is powerful just in one spatial dimension because of the area law. More generally, any variational class corresponds to an RG scheme. As another important example, the ER is naturally associated with the class of MERA [227].

Tensor Network formalism can be also applied to study the low-energy limit of quantum field theories (QFTs) after an appropriate discretization of the theory on the lattice [249–254]. However, the symmetries of spacetime in this way will be destroyed. Thus, it would be desirable to work directly in the continuum which can provide a powerful non-perturbative approach for studying the strongly interacting QFTs. In the last decade, the generalization from lattice to continuum has been done for some classes of tensor network states [81, 255–258]. In particular, the continuous version of MPS and MERA, known as cMPS [81] and cMERA [255]. To date, only the Gaussian cMERA is well-understood, which limits the interest of cMERA to use as a variational ansatz to study the strongly coupled QFTs. Instead, cMERA has already attracted considerable attention in the context of holography [79, 259–271]. On the other hand, the cMPS provides a variational class of non-Gaussian wave functional which is just adapted to the non-relativistic interacting QFTs in $1 + 1$ dimensions. In the case of relativistic QFTs, the cMPS construction suffers from regularization ambiguity. One can still use cMPS to study the low energy limit of the theory in practice by introducing a UV cut-off [272, 273]. But still by construction, using the cMPS approach, one can not capture the short-distance behavior of the system. Moreover, defining a UV cut-off by itself is in contrast with the purpose of working directly in the continuum.

In this chapter, we will propose the class of boundary cMPS by following the same logic as the one that has been defined to create the class of standard cMPS. But, this time we replace the non-relativistic vacuum with one of the boundary states in 2 dimension CFT. Moreover, motivated by [274], we study the one-parameter family of cMPS generated by ER which maps a free non-relativistic theory at the IR scale to a free relativistic theory at UV. We will find that at the UV scale, the resulting wave functional is exactly the variational ansatz known as *relativistic* cMPS (RCMPS) introduced in [274] which is adapted to relativistic QFTs in $1 + 1$ dimensions. In the following, one can find the brief review of the ER and the class of cMPS that we need in the main discussion of the paper and introduce the notation there.

3.2 Continuous Matrix Product States

The family of MPS [275–277] is probably the most famous example of Tensor Network states. This is because it is behind some very powerful methods to simulate the one-dimensional quantum many-body systems.

MPS are a special class of tensors that can be written as products over many rank-3

tensors, See Fig. 4.3. Each square have represent a rank-3 tensor (rank-2 for the left and right boundaries) $A_{\alpha_j, \alpha_{j+1}}^{s_j}$. The vertical lines represent the physical indices and the

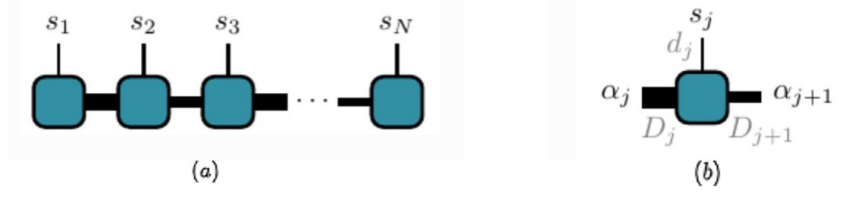


Figure 3.1: (a) MPS class of Tensor Network. (b) A rank-3 tensor.

horizontal lines are called ancillary indices. The MPS diagram in Fig. 4.3 is a rigorous representation of the mathematical expression

$$C^{s_1 s_2 \dots s_N} = \sum_{\{\alpha\}} A_{\alpha_1}^{s_1}[1] A_{\alpha_1 \alpha_2}^{s_2}[2] \dots A_{\alpha_{N-2} \alpha_{N-1}}^{s_{N-1}}[N-1] A_{\alpha_{N-1}}^{s_N}[N]. \quad (3.1)$$

where $\alpha_i \in \{1, \dots, D_i\}$. MPS can represent any quantum state of the many-body Hilbert space just by increasing sufficiently the value of D_i . To see that, consider a quantum many-body system of N particles. Let us take

$$|\psi\rangle = \sum_{s_1, \dots, s_N=0}^{d-1} C^{s_1, \dots, s_N} |s_1\rangle \otimes \dots \otimes |s_N\rangle \quad (3.2)$$

be the state of the N qudit (d-dimensional quantum systems). The state is completely specified by knowledge of the rank- N tensor C . One can obtain the MPS representation by breaking the wave function into small pieces. By starting from the first index and split it out from the rest and perform a singular value decomposition, we can get the Schmidt decomposition. We can now perform successive singular value decomposition along the indices and obtain that

$$|\psi\rangle = \sum_{s_1, \dots, s_N=0}^{d-1} A^{s_1}[1] A^{s_2}[2] \dots A^{s_{N-1}}[N-1] A^{s_N}[N] |s_1\rangle \otimes \dots \otimes |s_N\rangle. \quad (3.3)$$

We can redefine the first and last tensors as

$$\begin{aligned} A^{s_1}[1] &\longrightarrow \langle v_L | A^{s_1}[1] \\ A^{s_N}[N] &\longrightarrow A^{s_N}[N] |v_R\rangle. \end{aligned} \quad (3.4)$$

Thus, the tensors $A^{s_1}[1]$ and $A^{s_N}[N]$ are the rank-2 tensors as well. Therefore, we have

$$C^{s_1, \dots, s_N} = \langle v_L | A^{s_1}[1] A^{s_2}[2] \dots A^{s_{N-1}}[N-1] A^{s_N}[N] |v_R\rangle. \quad (3.5)$$

Moreover, one can take the periodic boundary condition by putting $N+1 \equiv 1$. As a result

$$C^{s_1, \dots, s_N} = \text{Tr} [A^{s_1}[1] A^{s_2}[2] \dots A^{s_{N-1}}[N-1] A^{s_N}[N]]. \quad (3.6)$$

For states that are translationally symmetric, we can choose

$$A^s[1] = A^s[2] = \dots = A^s[N] \equiv A^s \quad (3.7)$$

and take all D_i equal to single D . In the end, the MPS representation can be obtained as

$$|\psi\rangle = \sum_{s_1, \dots, s_N=0}^{d-1} \text{Tr}[BA^{s_1}A^{s_2}\dots A^{s_{N-1}}A^{s_N}] |s_1\rangle \otimes \dots \otimes |s_N\rangle \quad (3.8)$$

where the information about the boundary conditions is encoded in the matrix B . We have $B = I$ in the case of periodic boundary conditions and $B = |v_R\rangle\langle v_L|$ in the case of open boundary conditions. As it is mentioned, every state can be generally written in the MPS form with D growing exponentially with the particle number N . However, MPS is practical when D is small. It is particularly useful for dealing with the ground state of a one-dimensional quantum spin model.

The continuum limit of MPS known as continuous MPS (cMPS) was proposed in [81] by Verstraete and Cirac. It is originally introduced as a variational ansatz for the ground state of non-relativistic QFT Hamiltonians in $1 + 1$ dimensions.

To find a generalization of MPS in the continuum limit, one can approximate the QFT on a line of length L by a lattice with lattice spacing ϵ and $N = L/\epsilon$ sites. At each site of the lattice, there is a bosonic (or fermionic) mode a_i obeys the commutation relation $[a_i, a_j^\dagger]_\pm = \delta_{ij}$. Therefore, the Hilbert space spanned by $\{|n_i\rangle\}$ while $|n_i\rangle$ corresponding to having n_i particles on that site. For the many-body state we have

$$|i_1, i_2, \dots, i_N\rangle = a_1^{\dagger i_1} a_2^{\dagger i_2} \dots a_N^{\dagger i_N} |\mathbf{0}\rangle, \quad (3.9)$$

where $|\mathbf{0}\rangle = \otimes_{n=1}^N |0\rangle_n$ is the vacuum that

$$a_j |\mathbf{0}\rangle = 0 \quad \forall j. \quad (3.10)$$

On the lattice, we can define a certain family of MPS as

$$\begin{aligned} A_i^0 &= I + \epsilon Q(i\epsilon) \\ A_i^n &= \frac{1}{n!} (\sqrt{\epsilon} R(i\epsilon))^n \quad n \geq 1. \end{aligned} \quad (3.11)$$

For higher n , the matrices A^n have been determined by the requirement that a doubly occupied site gives the same physics as 2 bosons on 2 neighboring sites in the limit $\epsilon \rightarrow 0$. By taking the $\epsilon \rightarrow 0$ limit of this specific class of MPS, we can find the class of cMPS as

$$|\psi[Q, R]\rangle = \text{Tr}_{aux} \{ B \mathcal{P} \exp \int_{-L/2}^{L/2} dx (Q(x) \otimes I + R(x) \otimes \psi^\dagger(x)) \} |\Omega\rangle \quad (3.12)$$

where Tr_{aux} denotes a partial trace over the auxiliary system where the matrices Q and R act. For the translational invariant cMPS the matrices Q, R are position independent. The field $\psi(x)$ is the continuum limit of the rescale modes $\psi(i\epsilon) = a_i/\sqrt{\epsilon}$, that satisfying $[\psi(x), \psi^\dagger(y)]_\pm = \delta(x - y)$, and $|\Omega\rangle$, the empty vacuum is the continuum limit of $|\mathbf{0}\rangle$ that defined as

$$\psi(x) |\Omega\rangle = 0 \quad \forall x. \quad (3.13)$$

One can express the expectation value of local operators and in particular, the Hamiltonian on the cMPS representation of the ground state in terms of the matrices Q and R .

Specifically, all normal ordered correlation functions of local field operators can be deduced from a generating functional as

$$\langle : F[\psi^\dagger(x), \psi(y)] : \rangle = F\left[\frac{\delta}{\delta \bar{j}(x)}, \frac{\delta}{\delta j(y)}\right] \mathcal{Z}_{\bar{j}, j} \Big|_{\bar{j}, j=0}, \quad (3.14)$$

while its explicit form can be given in terms of the cMPS matrices Q and R as

$$\mathcal{Z}_{\bar{j}, j} = \text{Tr} \left\{ B \otimes \bar{B} \mathcal{P} \exp \left[\int dx T + j(x) R \otimes I + \bar{j}(x) I \otimes \bar{R} \right] \right\} \quad (3.15)$$

where

$$T = Q \otimes I + I \otimes \bar{Q} + R \otimes \bar{R} \quad (3.16)$$

is the cMPS transfer matrix [278].

In order to find the cMPS approximation of the ground state, it is just needed to minimize the expectation value of the Hamiltonian over the cMPS matrices Q and R . After that, correlation functions can be straightforwardly computed. The cMPS representation has gauge freedom

$$\begin{aligned} Q(x) &\longrightarrow g(x)Q(x)g^{-1}(x) - \frac{dg(x)}{dx}g^{-1}(x) \\ R(x) &\longrightarrow g(x)R(x)g^{-1}(x) \end{aligned} \quad (3.17)$$

that one can use to impose certain conditions on the cMPS matrices, including symmetry conditions. Moreover, for the continuum version, the left orthogonality condition of MPS can be read as

$$Q(x) + Q^\dagger(x) + R^\dagger(x)R(x) = 0 \quad (3.18)$$

for all x . A better approximation of the ground state can be found by increasing D . In the last decade, several optimization algorithms have been developed to study a number of theories, both bosonic and fermionic [272, 279–290]. The cMPS provides an efficient variational ansatz for non-relativistic QFTs. It is not adapted to relativistic theories because of a lack of sensitivity to short-distance behavior.

3.3 Regularized Boundary States

Regularized boundary states play a crucial role in understanding the behavior of CFTs in the presence of boundaries. These boundary states capture the impact of boundary conditions on the CFT living on the boundary, allowing us to study various physical phenomena related to open quantum systems or the presence of interfaces.

The regularization process involves evolving the boundary state along Euclidean time, effectively smearing out high-energy contributions, and ensuring that the state possesses finite energy. This regularization is essential to render the theory well-defined and to make physical predictions that are consistent and meaningful.

Regularized boundary states are valuable tools in exploring the physics of CFTs in various contexts, such as boundary critical phenomena, quantum entanglement at boundaries, and interface dynamics in condensed matter systems. They provide a natural framework to understand the interplay between bulk and boundary degrees of freedom and the emergence of universal features near the boundary.

Moreover, regularized boundary states facilitate the study of entanglement entropy and entanglement spectra at the interface between different phases of matter, helping to reveal the underlying quantum phase transitions and topological properties of the system. They also find applications in holography, where they correspond to boundary states of the corresponding AdS in the AdS/CFT correspondence, connecting insights from gravity and quantum field theory.

3.3.1 Boundary states in 2D CFT

In a 2D CFT, boundary states are required to fulfill the condition stated in [291]

$$(L_n - \tilde{L}_n) |B\rangle = 0. \quad (3.19)$$

Here, L_n and \tilde{L}_n represent the Virasoro generators associated with the left and right-moving sectors, respectively, and $|B\rangle$ denotes the boundary state. Within any Verma module, a straightforward solution to these conditions can be found as follows

$$|I_h\rangle = \sum_{\vec{k}} |\vec{k}, h\rangle_L \otimes |\vec{k}, h\rangle_R, \quad (3.20)$$

Here, $|\vec{k}, h\rangle_L$ is a linear combination of Virasoro descendants of the primary state $|h\rangle$, which is characterized by an infinite-dimensional vector $\vec{k} = (k_1, k_2, \dots)$ with non-negative integer components. We recognize these states by considering descendants of the following form:

$$\dots L_{-n}^{K_n} \dots L_{-1}^{K_1} |h\rangle_L, \quad (3.21)$$

where we construct an orthonormal basis, ensuring that $L(\vec{k}, h | \vec{k}', h)_L = \delta_{\vec{k}, \vec{k}'}$.

The state $|I_h\rangle$ is referred to as the Ishibashi state associated with the primary state $|h\rangle_L$, where the states $|\vec{k}, h\rangle$ represent descendants built upon the primary state labeled by h . It is readily apparent that the following relation holds:

$$L_n |I_h\rangle = \tilde{L}_n |I_h\rangle. \quad (3.22)$$

The Ishibashi states exhibit maximum entanglement between the left-moving and right-moving sectors. Furthermore, linear combinations of Ishibashi states also satisfy the constraint (F.5).

Physical boundary states are expressed as specific linear combinations of Ishibashi states, referred to as Cardy states:

$$|B_a\rangle = \sum_h C_{a,h} |I_h\rangle, \quad (3.23)$$

To be considered as physically valid, these boundary states must fulfill a consistency condition related to open-closed duality, which emerges from the partition function on a finite cylinder, as described in [291].

The Cardy states become singular due to the divergent norm of the Ishibashi states. To address this, one can introduce regularized boundary states by evolving them in Euclidean time:

$$|B_{a,\beta}\rangle = e^{-\frac{\beta}{4} H_c} |B_a\rangle, \quad (3.24)$$

where β is a positive constant and $H_c = L_0 + \tilde{L}_0 - \frac{c}{12}$. This regularization ensures that the state (F.4) remains space-translationally invariant on the circle but becomes time-dependent.

3.3.2 Entanglement entropy of boundary states

Let us start with the massless Dirac fermion theory in 2d. The system is in the boundary state $e^{-\epsilon H} |B\rangle$ in either the Dirichlet or Neumann boundary condition. We need to calculate the entanglement entropy S_A when A is an interval. In [292] based on the work in [293], the entanglement entropy S_A for the corresponding set of states has been calculated. It has been shown that

$$S_A \sim O(1). \quad (3.25)$$

Since we need to introduce the cut-off ϵ used as the damping factor, we have the ambiguity of shifting ϵ . This means that $O(1)$ entropy can be changed by the choice of the UV cut-off and this is enough to argue that boundary states essentially have no real-space entanglement.

3.3.3 Holographic boundary states

In holographic theories, the gravity path integral can be related to the CFT path integral through the AdS/CFT correspondence. Consequently, by selecting an appropriate state with a well-understood gravity prescription for handling the boundary condition at the initial Euclidean time, we can derive the corresponding geometries. Cooper et al. [213] discussed the method of describing boundary states by initiating with the Thermofield Double (TFD) state of two CFTs denoted as L and R

$$|\text{TFD}(\beta/2)\rangle = \frac{1}{Z} \sum_i e^{-\beta E_i/4} |E_i\rangle_L \otimes |E_i\rangle_R. \quad (3.26)$$

Subsequently, we perform a projection of the TFD state onto a specific pure state $|B\rangle$ belonging to the left CFT. Hence, the outcome is a pure state in the right Conformal Field Theory (CFT) represented as

$$|B_{a,\beta}\rangle = \frac{1}{Z} \sum_i e^{-\beta E_i/4} \langle B_a | E_i \rangle |E_i\rangle. \quad (3.27)$$

In case of a sufficiently high temperature, the TFD state corresponds to the maximally extended AdS-Schwarzschild black hole in the bulk according to the duality. The geometry associated with these regularized boundary states is anticipated to encompass a substantial portion of the left asymptotic region. Thus, the state $\rho \propto e^{-\epsilon H} |B\rangle \langle B| e^{-\epsilon H}$, which corresponds to a 2d CFT on a strip of width 2ϵ , has a gravity dual described by a section of the Euclidean BTZ black hole. The metric of the Euclidean black hole is given as

$$ds^2 = R^2 \left(\frac{h(z) dt^2}{z^2} + \frac{dz^2}{h(z) z^2} + \frac{dx^2}{z^2} \right), \quad h(z) = 1 - \frac{\pi^2 z^2}{4\epsilon^2} \quad (3.28)$$

where R is the AdS radius and the ranges of the coordinates (t, z, x) are $-2\epsilon \leq t \leq 2\epsilon$ with periodicity of 4ϵ , $0 < z \leq 2\epsilon/\pi$ and $-\infty < x < \infty$. Consequently, in the context of a holographic CFT, this set of regularized boundary states can be considered as microstates of a single-sided black hole. These black hole microstates can be conceptualized as black holes accompanied by end-of-the-world (EOW) branes positioned on the left side. Typically, the EOW brane setup manifests as a time-dependent configuration on a macroscopic scale.

3.4 Bulding a class of cMPS over the boundary states

As it has been discussed in Sec. 3.3.2, the entanglement entropy of spatial regions in boundary states vanishes. Therefore, there exists a set of operators denoted as $O(x)$ such that

$$O(x) |B_{a,\beta}\rangle = 0 \quad \forall x. \quad (3.29)$$

Let us consider a boundary state $|B_{a,\beta}\rangle$ in 2 dimensions. One can approximate the QFT on a special line in 2 dimensions by a lattice spacing ϵ and $N = L/\epsilon$ sites. One set of basis can be given as

$$O^{\dagger i_1}(\epsilon) O^{\dagger i_2}(2\epsilon) \dots O^{\dagger i_N}(N\epsilon) |B_{a,\beta}\rangle \quad (3.30)$$

and the MPS representation of one such class of states is given as

$$|\Psi\rangle = \sum_{i_1, \dots, i_N=0}^{\infty} \text{Tr}[BA^{i_1} A^{i_2} \dots A^{i_N}] O^{\dagger i_1}(\epsilon) O^{\dagger i_2}(2\epsilon) \dots O^{\dagger i_N}(N\epsilon) |B_{a,\beta}\rangle. \quad (3.31)$$

Define a specific class of MPS as

$$\begin{aligned} A^0(n\epsilon) &= I + \epsilon Q(n\epsilon) \\ A^1(n\epsilon) &= \sqrt{\epsilon} R(n\epsilon) \\ A^k(n\epsilon) &= A^1(n\epsilon)^k / k!. \end{aligned} \quad (3.32)$$

To find an explicit form of the MPS representation in the continuum limit one can introduce

$$|\Psi\rangle = \sum_{n=0}^{\infty} |\Psi_n\rangle \quad (3.33)$$

while

$$|\Psi_n\rangle = \sum_{i_1 + \dots + i_N = n} \text{Tr}[BA^{i_1} A^{i_2} \dots A^{i_N}] O^{\dagger i_1}(\epsilon) O^{\dagger i_2}(2\epsilon) \dots O^{\dagger i_N}(N\epsilon) |B_{a,\beta}\rangle. \quad (3.34)$$

For $n = 0$ and small value of ϵ we have

$$\begin{aligned} |\Psi_0\rangle &= \text{Tr}[BA^0(\epsilon) \dots A^0(N\epsilon)] |B_{a,\beta}\rangle \\ &= \text{Tr}[B(I + \epsilon Q(\epsilon)) \dots (I + \epsilon Q(N\epsilon))] |B_{a,\beta}\rangle \\ &= \text{Tr}[B\mathcal{P} \exp\left(\sum_{m=1}^N \epsilon Q(m\epsilon)\right)] |B_{a,\beta}\rangle \end{aligned} \quad (3.35)$$

where \mathcal{P} is the path order. In the $\epsilon \rightarrow 0$ limit, we reach

$$|\Psi_0\rangle = \text{Tr}[B\mathcal{P} \exp\left(\int_0^L dx Q(x)\right)] |B_{a,\beta}\rangle. \quad (3.36)$$

Then we consider $n = 1$ term

$$\begin{aligned} |\Psi_1\rangle &= \sum_{j=1}^{\infty} \text{Tr}[BA^0(\epsilon) \dots A^0((j-1)\epsilon) A^1(j\epsilon) A^0((j+1)\epsilon) \dots A^0(N\epsilon)] O^{\dagger}(j\epsilon) |B_{a,\beta}\rangle \\ &= \sum_{j=1}^N \epsilon \text{Tr}[B\mathcal{P} e^{\sum_{m=1}^{j-1} \epsilon Q(m\epsilon)} R(j\epsilon) \mathcal{P} e^{\sum_{m=j+1}^N \epsilon Q(m\epsilon)}] O^{\dagger}(j\epsilon) |B_{a,\beta}\rangle \end{aligned} \quad (3.37)$$

and in the $\epsilon \rightarrow 0$ limit, we get

$$|\Psi_1\rangle = \int_0^L dx \operatorname{Tr} \left[B\mathcal{P} \left\{ e^{\int_0^L ds Q(s)} R(x) \right\} \right] O^\dagger(x) |B_{a,\beta}\rangle. \quad (3.38)$$

For a generic n , one can find that

$$|\Psi_n\rangle = \frac{1}{n!} \int_0^L dx_1 dx_2 \dots dx_n \operatorname{Tr} \left[B\mathcal{P} \left\{ e^{\int_0^L ds Q(s)} R(x_1) \dots R(x_n) \right\} \right] O^\dagger(x_1) \dots O^\dagger(x_n) |B_{a,\beta}\rangle. \quad (3.39)$$

Therefore, we find

$$|\Psi\rangle = \sum_{n=0}^{\infty} \int_{0 \leq x_1 \leq \dots \leq x_n \leq L} dx_1 \dots dx_n \Phi_n(x_1, \dots, x_n) O^\dagger(x_1) \dots O^\dagger(x_n) |B_{a,\beta}\rangle \quad (3.40)$$

while

$$\Phi_n(x_1, \dots, x_n) = \operatorname{Tr} \left[B\mathcal{P} \left\{ e^{\int_0^L Q(s) ds} R(x_1) \dots R(x_n) \right\} \right]. \quad (3.41)$$

We can rewrite (3.40) as

$$\begin{aligned} |\Psi\rangle &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^L dx_1 \dots dx_n \operatorname{Tr}_{aux} \left[B\mathcal{P} \left\{ (e^{\int_0^L dx Q(x)} \otimes I) (R(x_1) \dots R(x_n) \otimes O^\dagger(x_1) \dots O^\dagger(x_n)) \right\} \right] |B_{a,\beta}\rangle \\ &= \operatorname{Tr}_{aux} \left[B\mathcal{P} \left\{ e^{\int_0^L dx Q(x) \otimes I} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^L dx_1 \dots dx_n R(x_1) \dots R(x_n) \otimes O^\dagger(x_1) \dots O^\dagger(x_n) \right\} \right] |B_{a,\beta}\rangle \\ &= \operatorname{Tr}_{aux} \left[B\mathcal{P} \left\{ e^{\int_0^L dx Q(x) \otimes I} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_0^L dx R(x) \otimes O^\dagger(x) \right)^n \right\} \right] |B_{a,\beta}\rangle \end{aligned} \quad (3.42)$$

and finally one can find the representation of the boundary cMPS (BCMPS) as

$$|\Psi_{a,\beta}\rangle = \operatorname{Tr}_{aux} \left[B\mathcal{P} \exp \int_0^L dx (Q(x) \otimes I + R(x) \otimes O^\dagger(x)) \right] |B_{a,\beta}\rangle. \quad (3.43)$$

3.5 Holographic interpretation of the boundary cMPS

3.5.1 Continuous measurement

In this section, based on [294], we provide a natural physical interpretation of this variational class.

In the context of cavity electrodynamics, we can directly understand the bulk and boundary fields as follows[295–298]: think of the cavity modes as the auxiliary system and the quantum field as describing the photons escaping from the cavity.

One can start by describing how we measure something, like a physical observable "M", on a quantum system with D levels. This approach is called "von Neumann's prescription" [299]. We attach a quantum system with a continuous degree of freedom, known as meter, in a fiducial state vector $|0\rangle$ and couple it with the system for some time t according to the interaction $H_I = M \otimes p$. If initially, the system is in the state $|\phi\rangle$, then after the interaction the state is

$$e^{-itH_I} |\phi\rangle |0\rangle = \sum_{j=0}^D \phi_j |m_j\rangle |x = m_j t\rangle \quad (3.44)$$

while $M|m_j\rangle = m_j|m_j\rangle$ and the initial state in the basis of the eigenstate of M is written as $|\phi\rangle = \sum_{j=1}^D \phi_j|m_j\rangle$.

The main idea in [294] is to reverse von Neumann's measurement approach. Instead of focusing on the system as the primary element, we treat the meter as the central system A, and the original system becomes an extra part B. This approach allows us to view it as a state creation tool: we can create various quantum states for meter A by using the measurement approach and then either remove or measure system B. This way, we can generate quantum states for a system with a continuously changing characteristic.

To proceed, let us consider a family of $D \times D$ complex matrices $R(x)$, $x \in [0, L]$ which we measure at time $t = x$ on B. B additionally evolve with a Hamiltonian $K(x)$. The total Hamiltonian is given by

$$H(t) = K(t) \otimes I + H_I \quad (3.45)$$

where $H_I = iR(x) \otimes O^\dagger(x) + h.c.$. Integrating the Schrodinger equation for (3.45) we get

$$U(L) = \mathcal{P} \exp -i \int_0^L ds \left(K(s) \otimes I + iR(s) \otimes O^\dagger(s) - iR^\dagger(s) \otimes O(s) \right) \quad (3.46)$$

The evolution (3.46) prepare the class of BCMPS. If we initialize the meter A in the specific boundary state $|B_{a,\beta}\rangle_A$ and system B in the initial state $|v_i\rangle$ we have

$$U(L) |v_i\rangle \otimes |B_{a,\beta}\rangle_A \quad (3.47)$$

Using the Baker-Hausdroff formula, we have

$$\begin{aligned} & \exp \left(ds(K(s) \otimes I + R(s) \otimes O^\dagger(s) - iR^\dagger(s) \otimes O(s)) \right) \\ &= \exp \left(ds(K(s) \otimes I + R(s) \otimes O^\dagger(s)) \right) \\ & \quad \times \exp \left(ds - iR^\dagger(s) \otimes O(s) \right) \\ & \quad \times \exp \left(\frac{1}{2} ds ds' [iR^\dagger(s) \otimes O(s), K(s') \otimes I + R(s') \otimes O^\dagger(s')] \right) \\ &= \exp \left(ds(K(s) \otimes I + R(s) \otimes O^\dagger(s)) \right) \\ & \quad \times \exp \left(ds - iR^\dagger(s) \otimes O(s) \right) \\ & \quad \times \exp \left(-\frac{1}{2} ds R^\dagger(s) R(s) \otimes I + \frac{1}{2} ds ds' [iR^\dagger(s), K(s')] \otimes O(s) \right. \\ & \quad \left. + \frac{1}{2} ds ds' [iR^\dagger(s), iR(s')] \otimes O^\dagger(s') O(s) + \dots \right). \end{aligned} \quad (3.48)$$

Consider the fact that

$$e^{O(x)} |B_{a,\beta}\rangle = |B_{a,\beta}\rangle, \quad (3.49)$$

we reach to

$$U(L, 0) |v_i\rangle \otimes |B_{a,\beta}\rangle = \mathcal{P} \exp -i \int_0^L ds \left(Q(s) \otimes I + R(s) \otimes O^\dagger(s) \right) |v_i\rangle \otimes |B_{a,\beta}\rangle \quad (3.50)$$

while

$$Q(x) = -iK(x) - \frac{1}{2} R^\dagger(x) R(x). \quad (3.51)$$

After projecting the system B on the final state $|v_f\rangle$, we will reach to the class of BCMPS

$$|\Psi_{a,\beta}\rangle = \langle v_f|U(L,0)|v_i\rangle|B_{a,\beta}\rangle = \text{Tr}_B\left[\mathcal{BP}\exp -i\int_0^L ds\left(Q(s)\otimes I + R(s)\otimes O^\dagger(s)\right)\right]|B_{a,\beta}\rangle \quad (3.52)$$

while the matrix B here is

$$B = |v_i\rangle\langle v_f|. \quad (3.53)$$

Thus, a BCMPS like the class of cMPS can be found from a continuous measurement and the dynamic of system B described by a Lindblad equation.

3.5.2 A toy model for evaporating black hole

In Sec. 3.3.3 we saw that a given regularized boundary state of a CFT can be written as a TFD state of two CFTs, let us refer to them as left and right CFTs while right CFT stands for the original one, projecting on the corresponding boundary state.

In Sec. 3.3.3, we discussed that at high temperature a regularized boundary state is dual to a microstate of a single-sided black hole. Therefore, the class of BCMPS can be dual to the microstate of the black hole coupled to an ancilla that can represent a bath that absorbs Hawking radiation.

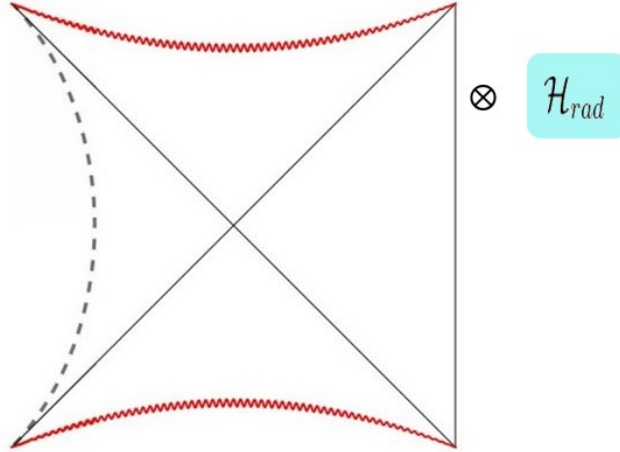


Figure 3.2: BCMPS as a toy model for evaporating black hole

$$\begin{aligned} \langle v_f|U(L,0)|v_i\rangle|B_{a,\beta}\rangle &= \langle v_f|\langle B_a|U(L,0)\otimes I|v_i\rangle|TFD(\beta)\rangle \\ &= \langle v_f|\langle B_a|\mathcal{P}\exp\left(\int_0^L dt Q(t)\otimes I_{LR} + R(t)\otimes O_R^\dagger(t)\otimes I_L\right)|v_i\rangle|TFD(\beta)\rangle \end{aligned} \quad (3.54)$$

while $Q(t) = -iK(t) - \frac{1}{2}R^\dagger(t)R(t)$. The interpretation is that at the time $t = 0$ we put the ancilla in the state $|v_i\rangle$ and two CFTs in the TFD state. They coupled together via the evolution

$$H_{tot} = K(t)\otimes I_{LR} + iR(t)\otimes O_R^\dagger(t)\otimes I_L - iR^\dagger(t)\otimes O_R(t)\otimes I_L \quad (3.55)$$

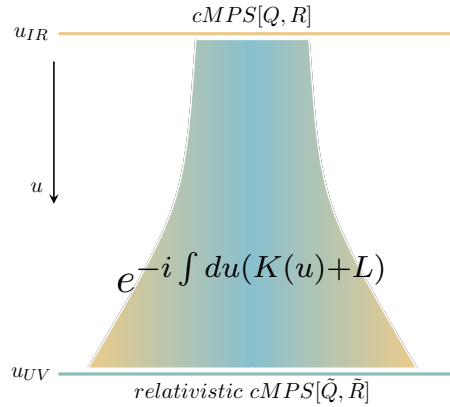


Figure 3.3: Entanglement renormalization group flow of the class of cMPS

In the end, we project the ancilla and left CFT on the states $|v_f\rangle$ and $|B_a\rangle$ respectively. The two CFTs are dual to an eternal black hole if the temperature is high enough. Thus, the ancilla can be interpreted as the bath that interacts with the bulk and absorbs the Hawking radiation.

3.6 cMERA RG flow of the class of cMPS

The cMPS representation is a mathematical framework used in the context of QFT. It's a way of describing the ground state of a non-relativistic QFT as a special kind of state generated by transforming the ground state of the free part of the non-relativistic QFT's Hamiltonian, i.e. $|\Omega\rangle$ in (??).

Now, when we're dealing with relativistic QFT, things get more complicated due to the relativistic nature of the theory. However, this interpretation of cMPS suggests a way to adapt it to represent the ground state of a relativistic QFT. This adaptation involves transforming the ground state of the free relativistic QFT in a certain manner. Interestingly, there exists a concept known as cMERA RG flow. This concept establishes a connection or flow between the ground states of the non-relativistic and relativistic free theories. In simpler terms, it provides a way to relate the ground states of these two different types of quantum field theories.

In the context of this discussion, the focus is on studying a particular family of cMPS that evolves using the corresponding cMERA evolution. This means we're investigating how a specific set of cMPS changes or transforms as we apply the principles of cMERA, particularly in the context of the ground states of both non-relativistic and relativistic free quantum field theories. In summary, this discussion revolves around using the cMPS framework to represent ground states in QFT, extending it to relativistic cases, and exploring the connections and transformations between these states through the concept of cMERA RG flow.

There is a cMERA RG flow that relates the ground states of the non-relativistic and relativistic free theories to each other. To do this, we start by placing the vacuum of the cMPS representation at the "IR level" in the cMERA framework. In simpler terms, we're setting up our system with the non-relativistic vacuum state as a starting point. Then, we follow this process as we move up to the "UV level" within the cMERA framework.

At this UV level, the goal is to reach the vacuum state of the free relativistic quantum field theory. In the following, we study the one-parameter family of the cMPS evolves with the corresponding cMERA evolution. The cMERA formalism was originally formulated for infinite systems [255]. However, its generalization to systems with open boundary conditions [300] and on a finite circle [301] has been introduced more recently. First, we work in the thermodynamic limit, *i.e.*, $L \rightarrow \infty$, and after that, we will discuss the extension for the theories defined on a finite circle. To proceed, we should find the generator of the RG flow in the case that we are interested in, *i.e.* the cMERA generator of mapping the IR state $|\Omega\rangle$ to the ground state of the relativistic field theory.

Consider the free scalar field in the 1 + 1 dimension. The Hamiltonian is given by

$$H_{f.b.} = \frac{1}{2} \int dx [\pi^2(x) + (\partial_x \phi(x))^2 + m^2 \phi(x)^2] \quad (3.56)$$

where the field operator and its conjugate momentum satisfy $[\phi(x), \pi(y)] = i\delta(x - y)$. One can expand them in terms of creation and annihilation operators a_k and a_k^\dagger satisfying $[a_k, a_{k'}^\dagger] = 2\pi\delta(k - k')$. The ground state of the theory is known to be the Fock space vacuum denoted by $|\mathbf{0}\rangle_a$ while $a_k |\mathbf{0}\rangle_a = 0$ for all k .

In order to specify the cMERA representation of the ground state, we need to first define the unentangled reference state $|\Omega\rangle$ which is the same as the vacuum of the cMPS state in terms of the fundamental fields of the given theory. In general, one can define a Gaussian factorized state with width Δ^{-1} as $\psi(x) |\Omega\rangle = 0$ for all x , while [255]

$$\psi(x) = \sqrt{\frac{\Delta}{2}} \phi(x) + i\sqrt{\frac{1}{2\Delta}} \pi(x). \quad (3.57)$$

Notice that the operator (3.57) here is equivalent to the cMPS operators $\psi(x)$. By substituting (3.57) into (F.4) and (F.5), we have the form of the cMERA Hamiltonian in terms of ϕ and π . The function $g(k, u)$ in (F.5) is assumed to be real-valued in the form

$$g(k, u) = \chi(u)\Theta(1 - |k|/\Lambda), \quad (3.58)$$

where $\Theta(x)$ is the step function. By considering

$$|\psi(u = 0)\rangle = |\mathbf{0}\rangle_a \quad (3.59)$$

in (F.1), we find an ansatz to represent the exact ground state of the theory as a circuit cMERA. As the last step, one should apply the variational principle and minimize the energy

$$E = \langle \psi(u = 0) | H_{f.b.} | \psi(u = 0) \rangle \quad (3.60)$$

to exactly find Δ and $\chi(u)$.

In order to do the calculation, it is useful to go to the interaction picture where L can be understood as the free part of the cMERA Hamiltonian while $K(u)$ is the interacting part. One can rewrite the unitary evolution in scale in the interaction picture as

$$\begin{aligned} U(u_1, u_2) &= e^{-iu_1 L} \hat{U}(u_1, u_2) e^{iu_2 L} \\ &= e^{-iu_1 L} \mathcal{P} e^{-i \int_{u_2}^{u_1} \hat{K}(u) du} e^{iu_2 L} \end{aligned} \quad (3.61)$$

where $\hat{K}(u) = e^{iuL}K(u)e^{-iuL}$ can be read off as

$$\hat{K}(u) = \frac{i}{2} \int dk g(ke^{-u}, u) (a_k^\dagger a_{-k}^\dagger - a_{-k} a_k). \quad (3.62)$$

Finally, by requiring $\delta E/\delta\chi(u) = 0$ for every u , we find that

$$\Delta = \sqrt{\Lambda^2 + m^2} \quad (3.63)$$

and

$$\chi(u) = \Lambda^2 e^{2u} / 2(\Lambda^2 e^{2u} + m^2). \quad (3.64)$$

Before going ahead to find the RG flow of the class of cMPS, in order to find the renormalized operators via the evolution in scale, it is good to know that

$$e^{-iuL}\psi(k)e^{iuL} = e^{-u/2}\psi(ke^{-u}) \quad (3.65)$$

and under the action of $\hat{U}(u, u_{IR})$

$$\begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} \longrightarrow \begin{pmatrix} \cosh \theta(u) & -\sinh \theta(u) \\ -\sinh \theta(u) & \cosh \theta(u) \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} \quad (3.66)$$

where $\theta(u) = \int_{u_{IR}}^u ds g(ke^{-s}, s)$ and $\theta(u=0) = \ln \sqrt{\frac{\Delta}{\omega_k}}$, while $\omega_k = \sqrt{k^2 + m^2}$ [263].

Now, we are ready to define a one-parameter family of states by relating them to the IR state through the entangling evolution in scale as

$$|\Psi(u)\rangle = U(u, u_{IR}) |\psi[Q, R]\rangle. \quad (3.67)$$

Here, $|\Psi(u_{IR})\rangle = |\psi[Q, R]\rangle$ is the standard class of cMPS which is suitable for the ground state of the non-relativistic QFT and $U(u, u_{IR})$ is the cMERA RG flow that maps the state $|\Omega\rangle$ to the ground state of the free relativistic theory, *i.e.* $|0\rangle_a = U(0, u_{IR}) |\Omega\rangle$.

In the standard cMPS, one can explicitly expand the path order in (E.11) and obtain

$$\begin{aligned} |\psi[Q, R]\rangle &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} dx_1 dx_2 \dots dx_n \\ &\Phi_n(x_1, x_2, \dots, x_n) \psi^\dagger(x_1) \psi^\dagger(x_2) \dots \psi^\dagger(x_n) |\Omega\rangle \end{aligned} \quad (3.68)$$

while

$$\Phi_n(x_1, \dots, x_n) = \text{Tr}[\mathcal{P}\{e^{\int_{-\infty}^{\infty} Q(y)dy} R(x_1) \dots R(x_n)\}]. \quad (3.69)$$

Therefore, the one-parameter family of states in (3.67) can be read as

$$\begin{aligned} |\Psi(u)\rangle &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} dx_1 dx_2 \dots dx_n \\ &\Phi_n(x_1, x_2, \dots, x_n) U(u, u_{IR}) \psi^\dagger(x_1) \psi^\dagger(x_2) \dots \psi^\dagger(x_n) |\Omega\rangle \end{aligned} \quad (3.70)$$

Therefore, to find the explicit form of $|\Psi(u)\rangle$, it is enough to determine the transformation of the $\psi^\dagger(x_1) \dots \psi^\dagger(x_n) |\Omega\rangle$ under the action of unitary evolution which is

$$\begin{aligned} U(u_{IR}, u) \psi^\dagger(x_1) \dots \psi^\dagger(x_n) |\Omega\rangle &= \\ &\psi^\dagger(x_1, u) \dots \psi^\dagger(x_n, u) |\psi(u)\rangle, \end{aligned} \quad (3.71)$$

where we define

$$\psi^\dagger(x, u) = U(u, u_{IR})\psi^\dagger(x)U^{-1}(u, u_{IR}) \quad (3.72)$$

and

$$|\psi(u)\rangle = U(u, u_{IR})|\Omega\rangle. \quad (3.73)$$

In particular by using (3.65) and (3.66), one can obtain that at the UV scale

$$\psi(x, u = 0) = e^{u_{IR}/2}a(xe^{u_{IR}}) \quad (3.74)$$

where

$$a(x) = 1/\sqrt{2\pi} \int dk e^{ikx} a_k \quad (3.75)$$

is defined to be the Fourier transform of the annihilation operator a_k . By construction, we also have

$$|\psi(0)\rangle = |\mathbf{0}\rangle_a. \quad (3.76)$$

In the end, we obtain the UV state as

$$\begin{aligned} |\Psi(u = 0)\rangle = |\Psi[\tilde{Q}, \tilde{R}]\rangle = Tr_{aux} \left\{ \mathcal{P} \exp \int_{-\infty}^{\infty} dx \right. \\ \left. (\tilde{Q}(x) \otimes I + \tilde{R}(x) \otimes a^\dagger(x)) \right\} |\mathbf{0}\rangle_a \end{aligned} \quad (3.77)$$

while $\tilde{Q}(x)$ and $\tilde{R}(x)$ in terms of the $Q(x)$ and $R(x)$ can be given as

$$\tilde{Q}(x) = e^{-u_{IR}}Q(xe^{-u_{IR}}) \quad \tilde{R}(x) = e^{-u_{IR}/2}R(xe^{-u_{IR}}). \quad (3.78)$$

It is nothing but the class of RCMPS introduced in [274] as an ansatz to approximate the ground state of a relativistic QFT without requiring any additional UV cut-off, and thus, the result is valid even at high momenta. As the operator $a(x)$ has the same algebra as $\psi(x)$, RCMPS inherits the properties of the class of cMPS by replacing $\psi(x)$ with $a(x)$. Specifically, the correlation function of the $a(x)$, $a^\dagger(x)$ can be obtained via the same generation functional (3.15). The only important point is that since $a(x)$ is not local in terms of ϕ and π , the computation of the expectation value of the Hamiltonian is more difficult than in the non-relativistic cases. Moreover, the naive optimization, which works well for the standard cMPS, fails for RCMPS and one should use some more precise methods like the tangent space approach [302]. In [274], RCMPS was used to study the self-interacting ϕ^4 theory and provided some remarkable results.

Finally, one can also check the cMERA RG flow of the Hamiltonian. We define the Hamiltonian as $H(u = 0) = H_{f.b.}$. Here, $H_{f.b.}$ represents the Hamiltonian of the free boson system in a relativistic context given in (E.20). At the IR scale, we will get

$$H(u_{IR}) = U^\dagger(u = 0, u_{IR})H(u = 0)U(u = 0, u_{IR}) \quad (3.79)$$

One can explicitly find that at the IR scale, we reach exactly the Hamiltonian of the non-relativistic free boson as

$$H(u_{IR}) = \frac{1}{2\tilde{m}} \int dx \partial_x \psi^\dagger(x) \partial_x \psi(x) + \mu \int dx \psi^\dagger(x) \psi(x), \quad (3.80)$$

while $\tilde{m} = me^{2u_{IR}}$ and $\mu = m$ is the so-called chemical potential. Thus, $|\Omega\rangle$ is really represent the ground state of the free non-relativistic field theory.

3.7 RCMPS for Fermionic Theories

The free relativistic fermions in the 1 + 1 dimensions given by Dirac Hamiltonian

$$H_{Dirac} = \int dx [\bar{\psi}(x)\sigma_2\partial_x\psi(x) + m\bar{\psi}\psi] \quad (3.81)$$

where $\psi = (\psi_1, \psi_2)^T$ is the two-component complex fermions and $\bar{\psi} = \psi^\dagger\sigma_3$. Here, one can choose the unentangled state as

$$\psi_1(x) |\Omega\rangle = 0 = \psi_2^\dagger(x) |\Omega\rangle. \quad (3.82)$$

The standard class of cMPS is defined as

$$|\psi[Q, R_1, R_2]\rangle = Tr_{aux} \left\{ \mathcal{P} \exp \int dx (Q(x) \otimes I + R_1(x) \otimes \psi_1^\dagger(x) + R_2(x) \otimes \psi_2(x)) \right\} |\Omega\rangle. \quad (3.83)$$

To find the related class of states appropriate for relativistic theories, we need to find the exact form of the RG flow such that $|\mathbf{0}\rangle = U(u=0, u_{IR}) |\Omega\rangle$, where $|\mathbf{0}\rangle$ is the exact ground state of the Dirac Hamiltonian. The entangler is given as

$$K(u) = i \int dk g(k, u) (\psi_1^\dagger\psi_2(k) + \psi_1(k)\psi_2(k)^\dagger). \quad (3.84)$$

In this case, the Bogoliubov angle is antisymmetric and we can suppose its form as

$$g(k, u) = k\chi(u)\theta(1 - |k|/\Lambda). \quad (3.85)$$

The same as free bosons, one can find $\chi(u)$ by minimizing the expectation value of the Hamiltonian [255]. Moreover, one can derive that

$$e^{-iuL}\psi_{1,2}(k)e^{iuL} = e^{-u/2}\psi_{1,2}(ke^{-u}) \quad (3.86)$$

while $\psi_i(k)$ is the Fourier transform of $\psi_i(x)$, and under the action of the unitary evolution in the interaction picture

$$\begin{pmatrix} \psi_1(k) \\ \psi_2(k) \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta_f(u) & -\sin\theta_f(u) \\ \sin\theta_f(u) & \cos\theta_f(u) \end{pmatrix} \begin{pmatrix} \psi_1(k) \\ \psi_2(k) \end{pmatrix} \quad (3.87)$$

where

$$\theta_f(u) = \int_{u_{IR}}^u ds g(ke^{-s}, s) \quad (3.88)$$

and $\theta_f(u=0) = \frac{1}{2} \arcsin(-k/\omega_k)$. By considering (3.83) as the IR state, we can find the fermionic RCMPS at UV scale, *i.e.*, $u=0$ as

$$|\Psi[\tilde{Q}, \tilde{R}_1, \tilde{R}_2]\rangle = Tr_{aux} \left\{ \mathcal{P} \exp \int dx (\tilde{Q}(x) \otimes I + \tilde{R}_1(x) \otimes b_1^\dagger(x) + \tilde{R}_2(x) \otimes b_2(x)) \right\} |\mathbf{0}\rangle. \quad (3.89)$$

while \tilde{Q} and \tilde{R} defined by (E.13), and $b_{1,2}(x)$ are the Fourier transform of the $b_{1,2}(k)$ which can be found in terms of $\psi_{1,2}(k)$ as

$$\begin{aligned} b_1(k) &= \alpha_k\psi_1(k) + \beta_k\psi_2(k) \\ b_2(k) &= -\beta_k\psi_1(k) + \alpha_k\psi_2(k) \end{aligned} \quad (3.90)$$

while

$$\begin{aligned}\alpha_k &= -k/\sqrt{k^2 + (\omega_k - m)^2} \\ \beta_k &= (m - \omega_k)/\sqrt{k^2 + (\omega_k - m)^2}.\end{aligned}\tag{3.91}$$

One can check that $[H, b_1^\dagger(k)] = \omega_k b_1^\dagger(k)$ and $[H, b_2(k)] = \omega_k b_2(k)$ or in other words, the set of operators $b_{1,2}(k)$ are the modes diagonalizing the Dirac Hamiltonian.

3.8 RCMPS on a Circle

Finding the RCMPS on a circle requires having the cMERA RG flow for relativistic free fields on the circle. In [301], it has been shown that if a Gaussian cMERA describes the ground state of a theory on a line, the ground state of the same theory on a circle has a cMERA representation as well. Furthermore, the cMERA entangler can be obtained using the method of images. The unentangled reference state is defined as

$$\psi(x) |\Omega^c\rangle = 0\tag{3.92}$$

for $x \in [0, l_c)$ when $\psi(x)$ is again given by (3.57). The entangler has the form of

$$K^c(u) = \frac{i}{2} \sum_{n \in \mathbb{N}} \tilde{g}^c(n, u) [\psi_n^\dagger \psi_{-n}^\dagger - \psi_n \psi_{-n}]\tag{3.93}$$

where

$$\psi_n = 1/\sqrt{l_c} \int_0^{l_c} dx e^{-ik_n x} \psi(x)\tag{3.94}$$

for $n \in \mathbb{Z}$ and $k_n = 2\pi n/l_c$. The entangling profile on the circle is defined as

$$\tilde{g}^c(x, u) = 1/\sqrt{l_c} \sum_n e^{ik_n x} \tilde{g}^c(n, u)\tag{3.95}$$

can be obtained from the one on the line $g(x, u)$ through the method of images as

$$\tilde{g}^c(x, u) = \sum_{n \in \mathbb{Z}} g(x + nl_c, u).\tag{3.96}$$

It implies that $\tilde{g}^c(n, u) = g(k, u)|_{k=k_n}$. Following the procedure described above, one can generalize RCMPS to an ansatz as a variational class to approximate the ground state of the relativistic theory on a finite circle as

$$|\Psi[Q, R]\rangle^c = Tr_{aux} \left\{ \mathcal{P} e^{\int_0^{l_c} dx (\tilde{Q}(x) \otimes I + \tilde{R}(x) \otimes a^{c\dagger}(x))} \right\} |\mathbf{0}^c\rangle_a\tag{3.97}$$

where $a^c(x) |\mathbf{0}^c\rangle_a = 0$ for all $x \in [0, l_c)$ and $a^c(x)$ is defined as the Fourier transform of the modes which diagonalize the free theory on the circle [301].

Chapter 4

Krylov complexity of the Matrix Quantum Mechanics

In this chapter, we report the Krylov complexity study of the matrix quantum mechanics. After reviewing the concepts related to the study of Krylov complexity we first, study the simple example of the collection of the harmonic oscillators and then, we go through the matrix model.

4.1 Krylov Complexity

We start with the definition of the notion of the Krylov complexity. It is defined as the recursion method in [303] and recently has been used in [304].

4.1.1 Krylov state complexity

Consider a quantum system with a time-independent Hamiltonian H . A state $|\psi(t)\rangle$ is time evolved under the Schrodinger equation $i\partial_t |\psi(t)\rangle = H |\psi(t)\rangle$. Its solution $|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle$ has a formal power series expansion

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} |\psi_n\rangle \quad (4.1)$$

while $|\psi_n\rangle = H^n |\psi(0)\rangle$. The time-evolved state is a linear combination of

$$|\psi(0)\rangle, \quad |\psi_1\rangle = H |\psi(0)\rangle, \quad |\psi_2\rangle = H^2 |\psi(0)\rangle, \quad \dots \quad (4.2)$$

The subspace \mathcal{H}_ψ which is spanned by (4.2) is called *Krylov subspace*. Notice that in general, this basis is not orthogonal. The Gram-Schmidt procedure applied to $|\psi_n\rangle$ generate an orthogonal basis $\mathcal{K} = \{|K_n\rangle : n = 0, 1, 2, \dots, K_\psi\}$ when we define $K_\psi = \dim \mathcal{H}_\psi$ for one subspace of the full Hilbert space explored by the evolution of $|\psi(0)\rangle = |K_0\rangle$. In general, this code subspace can be infinite dimension.

Using the ordinary inner product, one can orthogonalize the basis (4.2) through the *Lanczos algorithm*:

1. $b_0 \equiv 0, \quad |K_{-1}\rangle = 0$
2. $|K_0\rangle \equiv |\psi(0)\rangle, \quad a_0 = \langle K_0 | H | K_0 \rangle$

3. For $n \geq 1$, $|A_n\rangle = (H - a_0)|K_{n-1}\rangle - b_{n-1}|K_{n-2}\rangle$
4. Set $b_n = \sqrt{\langle A_n|A_n\rangle}$
5. If $b_n = 0$ stop, otherwise set $|K_n\rangle = \frac{1}{b_n}|A_n\rangle$, $a_n = \langle K_n|H|K_n\rangle$ and go to step 3 [305].

In the case that K_ψ is finite, the Lanczos algorithm will end at some point that $b_{K_\psi} = 0$. The result of the Lanczos algorithm is two sets of Lanczos coefficients $\{a_n\}$ and $\{b_n\}$.

We can expand the time-evolved state in terms of the Krylov basis as

$$|\psi(t)\rangle = \sum_{n=0}^{K_\psi-1} \phi_n(t) |K_n\rangle \quad (4.3)$$

by substituting it into the Schrodinger equation, one gets

$$\dot{\phi}_n(t) = a_n\phi_n(t) + b_{n+1}\phi_{n+1}(t) + b_n\phi_{n-1}(t) \quad (4.4)$$

and the initial condition is $\phi_n(0) = \delta_{n,0}$.

The Krylov state complexity of the state $|\psi(t)\rangle$ is defined as

$$C_\psi(t) \equiv \sum_{n=0}^{K_\psi-1} n |\phi_n(t)|^2. \quad (4.5)$$

4.1.2 Krylov operator complexity

Similar to the Krylov state complexity, we can define Krylov complexity for quantum operators. Motivated by the time evolution of the operators, one can create the Krylov basis for a given operator in terms of the nested commutators with the Hamiltonian as they determine the time Taylor expansion of the Heisenberg operator.

Consider a time-independent Hamiltonian of a quantum system H and a given Hermitian operator O . The operator undergoing a Heisenberg evolution

$$O(t) = e^{itH}O(0)e^{-itH}. \quad (4.6)$$

Just as states evolved under the Hamiltonian operator, operators evolved under the Liouvillian operator $\mathcal{L} \equiv [H, \cdot]$

$$\begin{aligned} O(t) &= e^{itH}O(0)e^{-itH} = O(0) + it[H, O(0)] + \dots \\ &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \mathcal{L}^n O(0) \equiv e^{i\mathcal{L}t} O(0) \end{aligned} \quad (4.7)$$

This is a linear combination of the sequence of operators

$$O, \quad \mathcal{L}O = [H, O], \quad \mathcal{L}^2O = [H, [H, O]], \quad \dots \quad (4.8)$$

where O stands for $O(0)$ [306]. The linear span of operators forms an invariant subspace \mathcal{H}_O . A convenient way to study the growth of a simple operator is to realize them as states, $O \equiv |O\rangle$, and to introduce a notion of an inner product. It can be any non-degenerate

inner product in the operator algebra such as the trace inner product for finite-dimensional Hilbert space (also known as infinite temperature inner product or Frobenius norm)

$$\langle O|O' \rangle = \frac{\text{Tr}[O^\dagger O']}{\text{Tr}[I]} \quad (4.9)$$

and we write $\|O\| = \langle O|O \rangle^{1/2}$ for the norm [307]. Thereby any operator within this subspace can be thought of as a vector in the linear vector space. Such a vector space endowed with a valid inner product is called the *Krylov subspace*.

The set of operators (4.8) are not orthogonal. The idea is to apply the Gram-Schmidt to orthogonalize it. As in the case of the state complexity, it is called Lanczos algorithm. It is as follows

1. $b_0 \equiv 0, O_{-1} \equiv 0$
2. $O_0 = O/\|O\|$
3. For $n \geq 1$: $A_n = \mathcal{L}O_{n-1} - b_{n-1}O_{n-2}$
4. Set $b_n = \|A_n\|$
5. If $b_n = 0$ stop; otherwise set $O_n = A_n/b_n$ and go to step3 [305].

The output of the algorithm is a sequence of positive numbers, $\{b_n\}$, called the Lanczos coefficients and an orthogonal set of operators $\{O_n\}_{n=0}^{K_O-1}$ called the Krylov basis.

The time-evolved operator can now be expanded on the Krylov basis

$$O(t) = e^{iHt}O_0e^{-iHt} = \sum_{n=0}^{K_O-1} i^n \phi_n(t)O_n \quad (4.10)$$

where $\phi_n(t)$ can be thought of as the wavefunction over the Krylov basis. From the orthogonality, we obtain

$$\phi_n(t) = i^{-n} \langle O_n|O(t) \rangle. \quad (4.11)$$

The time evolution of the operator follows

$$\begin{aligned} \frac{dO(t)}{dt} &= \sum_n i^n \frac{d\phi_n(t)}{dt} O_n \\ &= i[H, O(t)] = i\mathcal{L}O(t) = \sum_n i^{n+1} \phi_n(t) \mathcal{L}O_n \end{aligned} \quad (4.12)$$

thus via the Heisenberg equation $\phi_n(t)$ satisfies the equation

$$\partial_t \phi_n(t) = b_n \phi_{n-1} - b_{n+1} \phi_{n+1} \quad (4.13)$$

with boundary condition $\phi_{-1}(t) = 0$ and $\phi_n(t=0) = \delta_{0,n}$. From unitarity, since the initial operator is normalized at the first step of the Lanczos algorithm, the wavefunction $\phi_n(t)$ is normalized at all times $\sum_{n=0}^{K_O-1} |\phi_n(t)|^2 = 1$.

Krylov complexity or K-complexity is defined as the time-dependent average position over the Krylov chain

$$C_K(t) = \langle O(t)|n|O(t) \rangle = \sum_n n |\phi_n(t)|^2 \quad (4.14)$$

which can be viewed as the expectation value of the Krylov operator

$$K_O = \sum_n n |O_n\rangle \langle O_n|. \quad (4.15)$$

Intuitively, $C_K(t)$ describes the mean width of a wavepacket in the Krylov space and hence quantitatively measures how the size of the operator increases as time goes by [308].

4.1.3 Krylov operator complexity over pure and mixed states

Given an normalized operator O , by acting the operator on a pure state $|\psi\rangle$, one can construct a state

$$|O\rangle := O|\psi\rangle \quad \mathcal{L}^n |O\rangle := [H, [H, \dots [H, O]]] |\psi\rangle. \quad (4.16)$$

The choice of pure state $|\psi\rangle$ depends on the two-point function of the operator that we have in hand. For example, when we have the zero-temperature two-point function of O , one can take the state $|\psi\rangle$ to be the ground state of the theory.

A time-dependent state $|O(t)\rangle := O(t)|\psi\rangle$ for $O(t) = e^{\mathcal{L}t}O$ can be expanded by $\mathcal{L}^n |O\rangle$. Although they do not create on an orthonormal basis. We need to apply the Gram-Schmidt procedure to make them orthogonal. By using it, one can obtain the Krylov basis $|O_n\rangle$ such that $\langle O_m | O_n \rangle = \delta_{m,n}$ as follows

$$|O_0\rangle = |O\rangle \quad \mathcal{L} |O_n\rangle = \sum_{i=0}^{n+1} h_{i,n} |O_i\rangle. \quad (4.17)$$

This construction of the basis is called *Arnoldi iteration* for general matrices. If $\langle O_m | \mathcal{L} | O_n \rangle$ is a Hermitian matrix, then (4.17) is simplified as

$$\mathcal{L} |O_n\rangle = a_n |O_n\rangle + b_n |O_{n-1}\rangle + b_{n+1} |O_{n+1}\rangle \quad (4.18)$$

$$\langle O_n | \mathcal{L} | O_m \rangle = \begin{pmatrix} a_0 & b_1 & 0 & 0 & \dots \\ b_1 & a_2 & b_2 & 0 & \dots \\ 0 & b_2 & a_2 & b_3 & \dots \\ 0 & 0 & b_3 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (4.19)$$

while $|O_{-1}\rangle = 0$. As before, this construction is called the Lanczos algorithm.

If $|\psi\rangle$ is an eigenstate of H , let us say $H|\psi\rangle = \lambda|\psi\rangle$, we have

$$\langle O_m | \mathcal{L} | O_n \rangle = \langle \psi | O_m^\dagger (H - \lambda) O_n | \psi \rangle \quad (4.20)$$

which is Hermitian. Assuming that O and H are Hermitian and we have an appropriate inner product by trace and Hermitian conjugation, we find that

$$a_n = 0. \quad (4.21)$$

a_n is the Hamiltonian eigenvalue in the absence of b_n , which would be not directly related to the spreads of operators. On the other hand, b_n , especially at large n , represents how much the operator spreads into an orthogonal direction in the Hilbert space at a later time.

By introducing an inner product between operators at finite temperature one can generalize the above procedure

$$\langle A|B\rangle_\beta := \frac{1}{Z} \text{Tr}(e^{-\beta H} A^\dagger B), \quad Z = \text{Tr}(e^{-\beta H}) \quad (4.22)$$

where β is the inverse temperature. We define

$$\langle A|\mathcal{L}^n|B\rangle_\beta := \langle A|\mathcal{L}^n B\rangle_\beta = \langle \mathcal{L}^n A|B\rangle_\beta. \quad (4.23)$$

Once the inner product is defined, one can construct the Krylov basis as $\langle O_m|O_n\rangle = \delta_{m,n}$. On top of it

$$L_{mn} = \langle O_m|\mathcal{L}|O_n\rangle = \frac{1}{Z} \text{Tr}[e^{-\beta H}(O_m^\dagger H O_n - O_m^\dagger O_n H)] \quad (4.24)$$

which is Hermitian. Hence, one can use the Lanczos algorithm instead of the Arnoldi iteration. For mixed states, it is more convenient that define the Lanczos coefficients in terms of operators as

$$\begin{aligned} O_{-1} &= 0, & O_0 &= O \\ \mathcal{L}O_n &= a_n O_n + b_n O_{n-1} + b_{n+1} O_{n+1}. \end{aligned} \quad (4.25)$$

One can also obtain

$$\langle O_m|\mathcal{L}^n|O_n\rangle = (L^n)_{mn}. \quad (4.26)$$

In the zero temperature limit, this reduces to the Lanczos algorithm for the pure state case when $|\psi\rangle$ is the ground state of the theory and in the infinite temperature limit, it reaches the discussion of the Krylov operator complexity [309].

4.1.4 Recursion Method and Moment Expansion

This part is mostly based on [303].

The dynamical behavior of a quantum system is determined by its Hamiltonian H and the operator A representing the observable we're interested in tracking over time. Our objective is to compute the dynamical correlation function $\langle A(t)A(0)\rangle$, which provides insights into how A evolves with time. Here, we assume that the correlators are even in the time. This evolution is governed by the Heisenberg equation of motion:

$$\frac{dA}{dt} = i[H, A] \quad (4.27)$$

Here, the commutator $[H, \cdot]$, known as the quantum Liouvillian operator L , plays a crucial role. It's a Hermitian superoperator. The formal solution to the equation of motion is expressed as:

$$A(t) = e^{i\mathcal{L}t} A(0). \quad (4.28)$$

To implement the recursion method effectively, besides H and A , we need to define an inner product for operators within the Hilbert space associated with H and A . This choice influences the nature of the resulting dynamic correlation function.

The heart of the Liouvillian representation in the recursion method lies in the orthogonal expansion of the observable under examination:

$$A(t) = \sum_{k=0}^{\infty} \phi_k(t) A_k. \quad (4.29)$$

For classical systems, A_k comprises an orthonormal set of functions in phase space. In contrast, for quantum systems, it constitutes an orthonormal set of operators. Regardless, these sets span a Hilbert space, typically of infinite dimensionality. The Liouvillian operator acts on the vectors A_k within this space. The orthogonal expansion is executed in two successive steps.

- Determine a particular orthogonal basis A_k in the Hilbert space of the dynamical variables by applying the Gram-Schmidt procedure with the Liouvillian \mathcal{L} as the generator of the new direction.
- Insert the expansion (4.29) into the equation of motion to obtain a set of differential equations for the time-dependent coefficients $\phi_k(t)$.

As the first step, we note that the general inner product between the vectors A and $i\mathcal{L}A$ for arbitrary A vanishes

$$\langle A, i\mathcal{L}A \rangle = 0. \quad (4.30)$$

This simplifies the Gram-Schmidt orthogonalization process and results in the subsequent set of recurrence relations for the vectors A_k

$$A_{k+1} = i\mathcal{L}A_k + \Delta_k A_{k-1}, \quad k = 0, 1, 2, \dots \quad (4.31)$$

$$\Delta_k = \frac{\langle A_k, A_k \rangle}{\langle A_{k-1}, A_{k-1} \rangle} \quad k = 1, 2, 3, \dots \quad (4.32)$$

with $A_{-1} = 0$ and $A_0 = A$. The sequence of numbers Δ_k contains all the information for the reconstruction of the fluctuation function $\langle A(t), A(0) \rangle$.

In the second step, we plug in the orthogonal expansion (4.29) into the equation of motion. The differential operator acts on the $\phi_k(t)$ and the Liouvillian acts on the A_k , which yields the following set of coupled linear differential equations for the function $\phi_k(t)$:

$$\frac{d\phi_k(t)}{dt} = \phi_{k-1}(t) - \Delta_{k+1}\phi_{k+1}(t), \quad k = 0, 1, 2, \dots \quad (4.33)$$

with $\phi_{-1} \equiv 0$, $\phi_k(0) = \delta_{k,0}$. Unlike the vectors A_k , the functions $\phi_k(t)$ can not be determined recursively.

If our goal is to determine the fluctuation function of the dynamical variable $A(t)$, then it is sufficient to know just one of the functions $\phi_k(t)$. Follows directly from the orthogonal expansion

$$\phi_0(t) = \frac{\langle A(t), A(0) \rangle}{\langle A(0), A(0) \rangle}. \quad (4.34)$$

There is a way to calculate the Δ_k sequence for specific correlation functions of a given model system. It is called the moment expansion. The normalized fluctuation function can be expanded in a Taylor series

$$\phi_0(t) = \sum_{k=0}^{\infty} \frac{i^{2k} t^{2k}}{(2k)!} M_{2k} \quad (4.35)$$

with $M_0 \equiv 1$. The coefficients M_{2k} are the frequency moments of the normalized spectral density

$$M_{2k} = \langle \omega^{2k} \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega^{2k} f(\omega) = i^{2k} \left[\frac{d^{2k}}{dt^{2k}} \phi_0(t) \right]_{t=0}, \quad k = 1, 2, \dots \quad (4.36)$$

while

$$f(\omega) = \int_{-\infty}^{\infty} d\omega e^{i\omega t} \phi_0(t). \quad (4.37)$$

for a given set of moments M_{2k} , $k = 0, 1, \dots, K$ with $M_0 = 1$ the first K coefficients Δ_n are determined by

$$M_{2k}^{(n)} = \frac{M_{2k}^{(n-1)}}{\Delta_{n-1}} - \frac{M_{2k}^{(n-2)}}{\Delta_{n-2}}, \quad \Delta_n = M_{2n}^{(n)} \quad (4.38)$$

for $k = n, n+1, \dots, K$ and $n = 1, 2, \dots, K$, and with set values $M_{2k}^{(0)} = M_{2k}$, $M_{2k}^{-1} = 0$, $\Delta_{-1} = \Delta_0 = 1$.

The set of coefficients Δ_n is equivalent to the square of the set of b_n as discussed earlier.

Recursion method of the quantum Hamiltonian system in its ground state:

This application of the recursion method is tailored for investigating dynamic correlation functions within the quantum Hamiltonian system's ground state. An essential preliminary step in more practical scenarios involves identifying the ground state wave function of the system.

For a given quantum Hamiltonian H and its ground state wave function $|\phi_0\rangle$, our goal is to determine the normalized correlation function of the dynamical variable represented by the Hermitian operator A

$$C(t) = \frac{\langle \phi_0 | A(t)A(0) | \phi_0 \rangle}{\langle \phi_0 | A(0)A(0) | \phi_0 \rangle} \quad (4.39)$$

In such a case the result of the Lanczos algorithm is two sets of coefficients a_k and b_k . The relation between the moments and these sets of coefficients are most conveniently expressed in terms of two arrays of auxiliary quantities $L_k^{(n)}$ and $M_k^{(n)}$:

Given a set of moments $M_0 \equiv 1$, M_1, \dots, M_{2K+1} the coefficients a_0, \dots, a_K and b_1, \dots, b_K are obtained by initializing

$$M^{(0)} = (-1)_k^k, \quad L_k^{(0)} = (-1)^{k+1} M_{k+1} \quad (4.40)$$

for $k = 0, \dots, 2K$ and then applying the recursion relations [309]

$$\begin{aligned} M_k^{(n)} &= L_k^{(n-1)} - L_{n-1}^{(n-1)} \frac{M_k^{(n-1)}}{M_{n-1}^{(n-1)}} \\ L_k^{(n)} &= \frac{M_{k+1}^{(n)}}{M_n^{(n)}} - \frac{M_k^{(n-1)}}{M_{n-1}^{(n-1)}} \end{aligned} \quad (4.41)$$

for $k = n, \dots, 2K - n + 1$ and $n = 1, \dots, 2K$. The resulting coefficients are

$$b_n = \sqrt{M_n^{(n)}}, \quad a_n = -L_n^{(n)}, \quad n = 0, \dots, K. \quad (4.42)$$

4.2 Simple example: Krylov Complexity of free field theory

For a single harmonic oscillator, the Euclidean two-point function must obey the equation

$$\left(-\frac{d^2}{d\tau^2} + \omega^2\right) \langle X(\tau)X(0) \rangle = \delta(\tau). \quad (4.43)$$

The solution to this equation is

$$C_0(\tau) = \langle X(\tau)X(0) \rangle = \frac{1}{2\omega} e^{-\omega|\tau|} \quad (4.44)$$

which can also be found by using the path integral method. To find the finite-temperature two-point function, one can use the method of images

$$G_\beta(\tau) = \sum_{n=-\infty}^{\infty} C_0(\tau + n\beta). \quad (4.45)$$

For simplicity, consider the case that $0 < \tau < \beta$, then we have

$$\begin{aligned} G_0(\tau) &= \sum_{n=-\infty}^{-1} \frac{1}{2\omega} e^{\omega(\tau+n\beta)} + \sum_{n=0}^{\infty} \frac{1}{2\omega} e^{-\omega(\tau+n\beta)} \\ &= \frac{e^{\beta\omega - \tau\omega}}{2\omega(-1 + \beta\omega)} + \frac{e^{\tau\omega}}{2\omega(-1 + \beta\omega)}. \end{aligned} \quad (4.46)$$

Therefore, the thermal correlator is given as

$$\text{Tr}(e^{-\beta H} X(t)X(0)) = G_\beta(t) = \frac{e^{\beta\omega - it\omega}}{2\omega(-1 + \beta\omega)} + \frac{e^{it\omega}}{2\omega(-1 + \beta\omega)} \quad (4.47)$$

In order to find the complexity we can use the inner product which can be motivated or inspired by a two-sided correlator on the TFD state or KMS inner product as

$$\langle O_1, O_2 \rangle = \text{Tr}(e^{-\beta H/2} O_1^\dagger e^{-\beta H/2} O_2) \quad (4.48)$$

To find the inner product between the single harmonic oscillator and its time-shifted we can use the thermal two-point function and shift the time as $t \rightarrow t - i\beta/2$

$$\text{Tr}(e^{-\beta H/2} X(t)e^{-\beta H/2} X(0)) = G_\beta(t - i\beta/2) = \frac{e^{\beta\omega/2}}{2\omega(-1 + e^{\beta\omega})} e^{it\omega} + \frac{e^{\beta\omega/2}}{2\omega(-1 + e^{\beta\omega})} e^{-it\omega} \quad (4.49)$$

considering a free quantum field on a circle of length L . We can expand it in modes and get a collection of harmonic oscillators with frequency ω_j . In the following, we consider a J number of modes over the ground state and thermal states respectively.

4.2.1 Krylov complexity of the operator X over the ground state

The correlator for J different modes of harmonic oscillator in the ground state is

$$C(t) = \frac{1}{N} \sum_{j=1}^J \frac{1}{2\omega_j} e^{-i\omega_j t} \quad (4.50)$$

while N in the normalization factor such that $C(t=0) = 1$. The moments are

$$M_n = \frac{1}{N} \sum_{j=1}^J \frac{1}{2\omega_j} \frac{(-i\omega_j)^n}{i^n}. \quad (4.51)$$

Here both sets of odd and even moments are nonzero, thus we get the nonzero values for both sets of a_n and b_n . In Fig. 4.1, one can find the non-zero value of a_n and b_n for different value of J .

In Fig. 4.2, one can find the Krylov complexity for the limit $J \rightarrow \infty$.

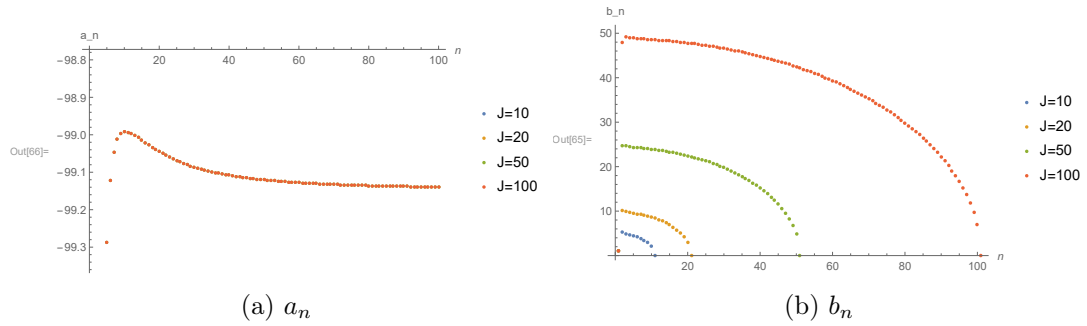


Figure 4.1: The non-zero values of a_n and b_n for different J . The plots of a_n are on top of each other for different values of J , and only the number of nonzero values of a_n will increase as one increases the J .

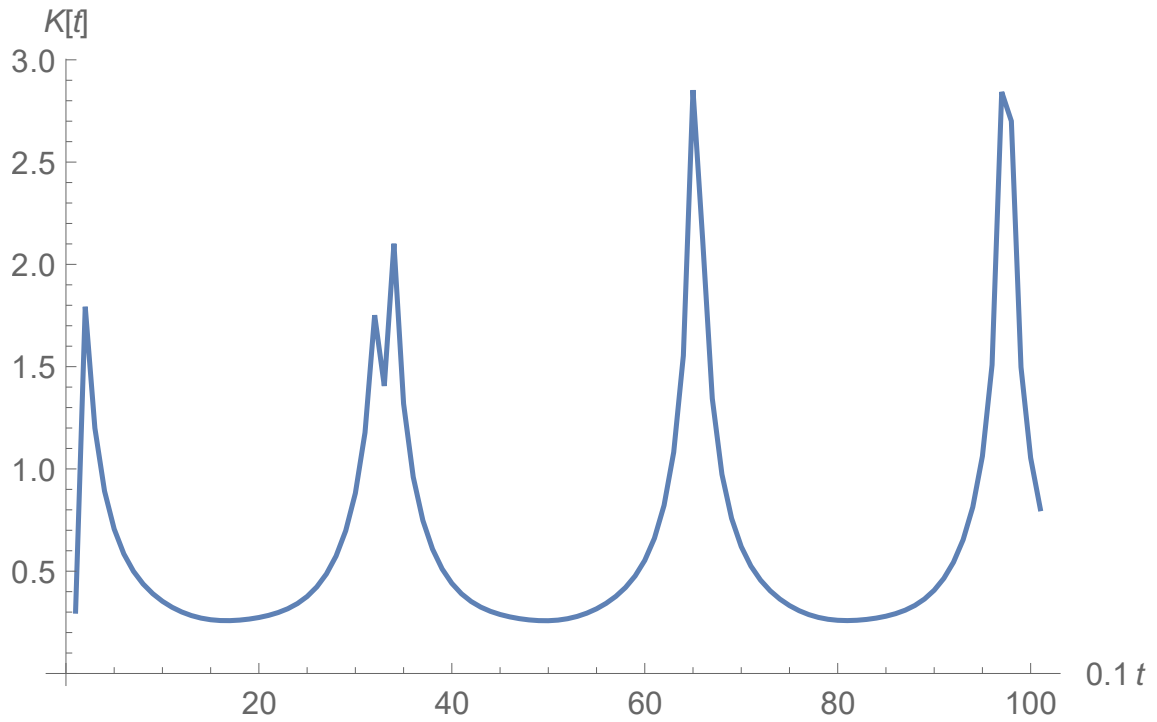
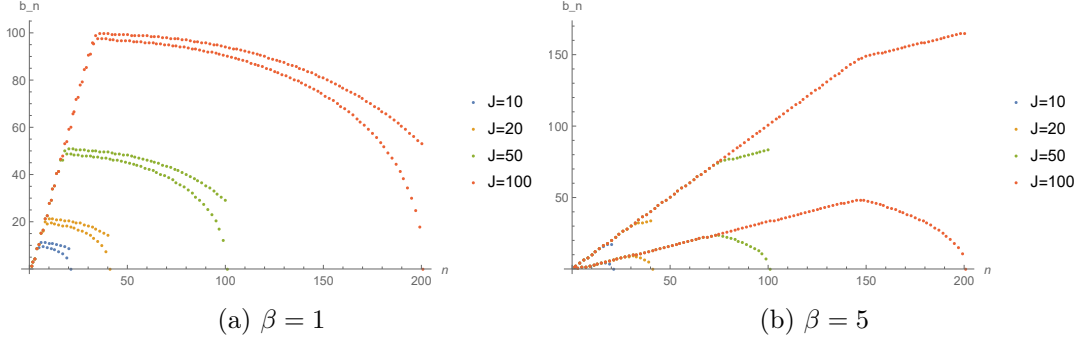


Figure 4.2: Krylov complexity as a function of time for the free field theory over the ground state in the limit $J \rightarrow \infty$

Figure 4.3: The non-zero b_n for different value of J .

4.2.2 Krylov complexity of the operator X over the thermal state

To find the correlator we use the inner product defined in (4.48). The correlator for J different mode of the harmonic oscillator in the thermal state with inverse temperature β is given by

$$C(t, \beta) = \frac{1}{N} \sum_{j=1}^J \frac{e^{\beta\omega_j/2}}{2\omega_j(-1 + e^{\beta\omega_j})} e^{it\omega_j} + \frac{e^{\beta\omega_j/2}}{2\omega_j(-1 + e^{\beta\omega_j})} e^{-it\omega_j} \quad (4.52)$$

while

$$\omega_j = j \frac{\pi}{2L} \quad (4.53)$$

and N the normalization factor such that $C(t=0, \beta) = 1$. The moments are

$$M_n = \frac{1}{N} \sum_{j=1}^J \frac{e^{\beta\omega_j/2}}{2\omega_j(-1 + e^{\beta\omega_j})} \left[\frac{(-i\omega_j)^n}{i^n} + \frac{(i\omega_j)^n}{i^n} \right]. \quad (4.54)$$

One can calculate the Lanczos coefficients using (4.42). As it is clear $M_{2n+1} = 0$ and thus

$$a_n = 0 \quad \forall n. \quad (4.55)$$

In Fig. 4.3 and 4.4, one can see the behavior of the non-zero b_n for different values of J and β . In general, in this case, b_n has two branches. For small n , it increases linearly, and at some point, it starts to decrease and goes to zero. The number of non-zero valued b_n increases as we increase the J and it is almost twice the value of J for this range of β . Considering both positive and negative modes in the thermal case, the number of non-zero b_n is equal to the number of different modes (in this case $2J$). As β increases, the linear behavior of the plots is dominant, and for $\beta = 10$ in Fig.4.4 one can see that we just have two linear branches. Moreover, by increasing the β the branches get more separated, and in the high value of β , it means the small value of T the second branch is getting to vanish and we will reach the one linear branch as in the ground state. However, for a fixed β , the slopes of two branches remain constant. As one can see in the 4.3 that the linear growth part of the plots for different J are on top of each other.

In Fig. 4.5, one can see the behavior of b_n when $J \rightarrow \infty$. It contains two linear branches and the slopes of two branches for different values of β are different. Finally, in Fig. 4.6, one can see the behavior of the Krylov complexity for different values of β . As correlators are periodic in time, the Krylov complexity is also periodic with period of $4L$.

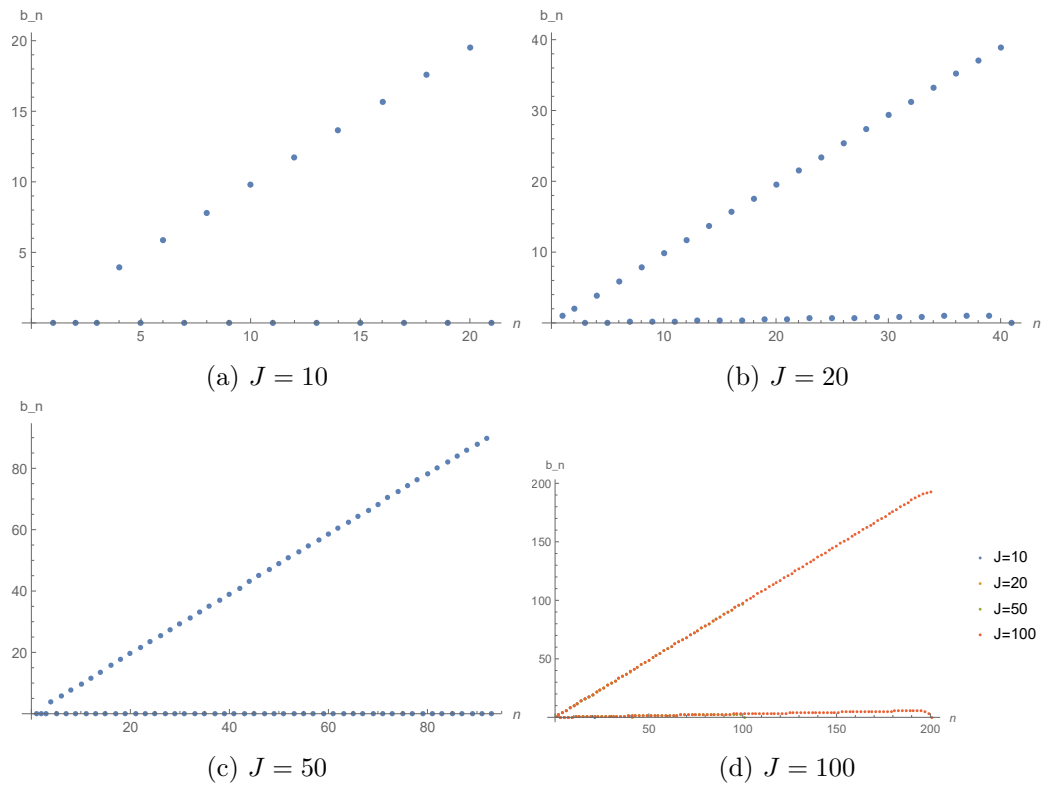


Figure 4.4: The non-zero b_n for $\beta = 10$ and different value of J . They contain two linear branches with the same slopes but the numbers of non-zero value of b_n depend on J and it increases when J increases.

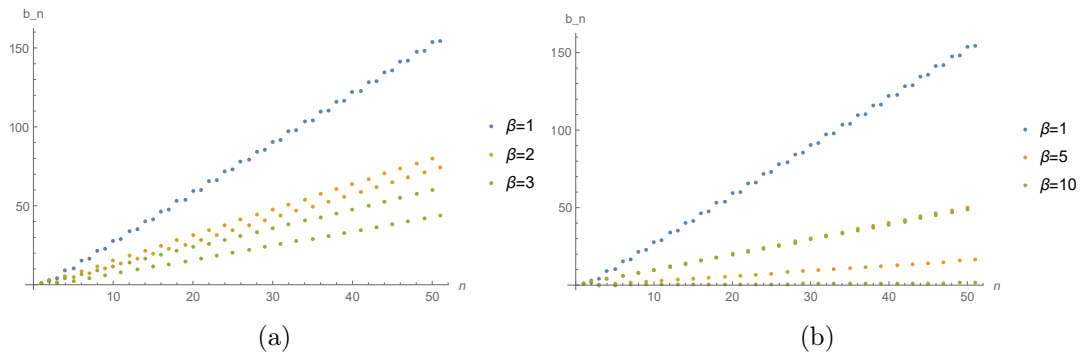


Figure 4.5: The non-zero b_n for the J goes to infinity limit.

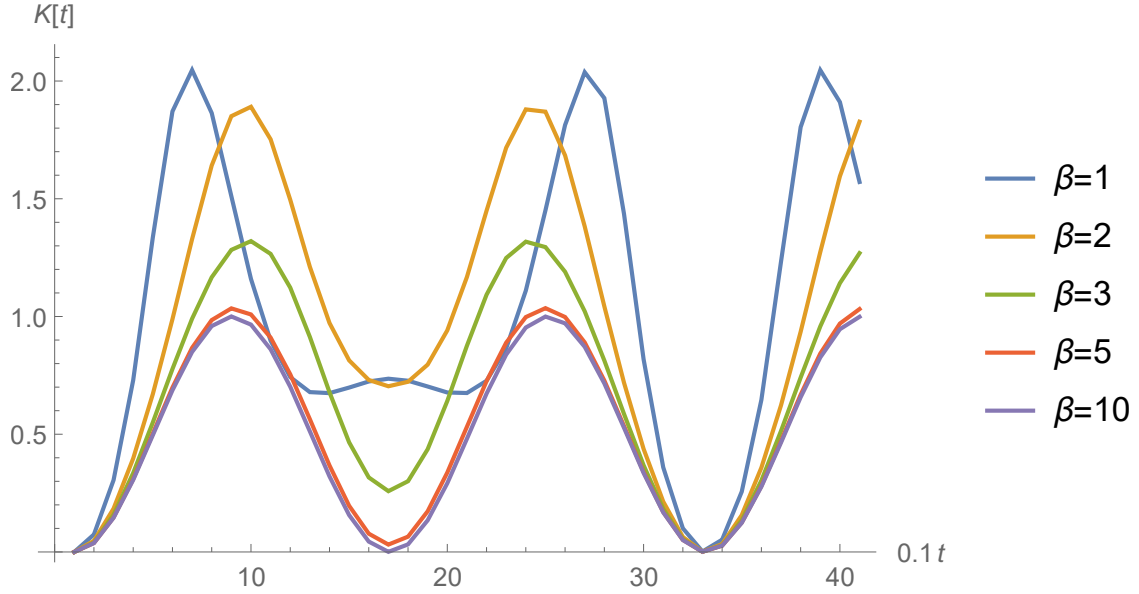


Figure 4.6: Krylov complexity as a function of time for different values of β .

In [310], the authors discussed the simple example of harmonic oscillator analytically. In particular, they find that for a very generic choice for the frequency of the modes, only the first $2J$ Lanczos coefficients are nonzero. For just one harmonic oscillator the theory describes with the Hamiltonian

$$H = \frac{1}{2}(X^2 + P^2) \quad (4.56)$$

while $[X, P] = i\hbar$. The position operator can be written in terms of the creation and annihilation operator $X = \sqrt{\frac{\hbar}{2}}(a + a^\dagger)$. The calculation for the momentum operator is similar to the position operator. From the partition function, one can include the normalization factor

$$\text{tr}(e^{-\beta H}) = \frac{1}{2 \sinh\left(\frac{\beta\hbar}{2}\right)}. \quad (4.57)$$

we find

$$\|X\|^2 = \|P\|^2 = \frac{\hbar}{2 \sinh\left(\frac{\beta\hbar}{2}\right)}. \quad (4.58)$$

One can apply the Lanczos algorithm starting from a normalized operator

$$O_0 = \sqrt{\frac{2 \sinh(\beta\hbar/2)}{\hbar}} X \quad (4.59)$$

The first recursion gives

$$O_1 = -i \sqrt{\frac{2 \sinh(\beta\hbar/2)}{\hbar}} P \quad (4.60)$$

while

$$b_1 = \hbar \quad (4.61)$$

Then the second operator in the recursion actually vanishes $A_2 = 0, b_2 = 0$. So the harmonic oscillator is a rather trivial model with the Lanczos algorithm terminating at the second step.

To generalize this case, consider a quantum system of N decoupled harmonic oscillators of different frequencies

$$H = \sum_{i=0}^J \frac{1}{2} (P_i^2 + \omega_i^2 X_i^2) \quad (4.62)$$

with a properly normalized initial operator

$$O_0 = \sum_{i=1}^J X_i. \quad (4.63)$$

It is easy to compute the moments in the case

$$M_n = \frac{1}{N} (\hbar^n \sum_{i=1}^J \omega_i^n), \quad n : \text{even} \quad (4.64)$$

The determinant in

$$b_1^{2n} b_2^{2(n-1)} \dots b_n^2 = \det(M_{(i+j)})_{0 \leq i, j \leq n} \quad (4.65)$$

vanishes for

$$n \geq 2J, \quad (4.66)$$

so the Lanczos algorithm terminates at the $2J$ steps with $b_{2J+1} = 0$.

For a more general discussion on free theory, one can look at [311]. They consider free massive scalar and Dirac fermion in d spacetime dimension. In the first case, Lanczos coefficients split into even and odd branches, growing linearly with n albeit with different intercepts. b_n grows linearly with the universal slope, but even and odd branches have different finite terms. In the second case of free massless fermions $C(t)$ is not an even function, hence besides b_n , Lanczos coefficients also include a_n . In [311], one can see the numerical results as a function of β .

They also consider a CFT on a sphere and calculate Lanczos coefficients and Krylov complexity associated with the thermal two-point function of the model. They consider $4d$ free massless scalar compacted on a S^3 . The corresponding two-point function has some singularity on the imaginary time axis. The correlator is in terms of a parameter R which is the radius of S^3 measured in the units of β which is the radius of S^1 . Their numerical results are good for $R < 1$. The Lanczos coefficients split into even and odd branches which grow linearly with n but with different slopes. The same as our results in the thermal case. The behavior of Krylov complexity is the same as the β around one of our results (see Fig. 4.6).

4.3 Matrix Quantum Mechanics

This chapter is based on [312, 313]. The one dimension takes to be timelike and the Lagrangian defines the theory

$$\mathcal{L} = \text{Tr} \left(\frac{1}{2} \dot{M}^2(t) - V(M) \right) \quad (4.67)$$

where M is a Hermitian matrix variable. The Lagrangian is invariant under a global $U(N)$ symmetry, $M \rightarrow U^{-1}MU$ with the conserved $U(N)$ angular momentum

$$J = i[M, \dot{M}] \quad \frac{dJ}{dt} = 0. \quad (4.68)$$

The quantum theory then is defined by the Hamiltonian

$$H = \text{Tr} \left(-\frac{1}{2} \frac{\partial^2}{\partial M^2} + V(M) \right) \quad (4.69)$$

and we restrict ourselves to the singlet sector $J | \rangle = 0$.

A set of basic singlet vertex operators is given by

$$\phi_m = \text{Tr}(M^m). \quad (4.70)$$

To work with the collective field theory approach, a natural set of singlet operators is given by the vertex operators

$$\phi_k = \text{Tr}(e^{ikM}) \quad (4.71)$$

and one considers the collective field as its Fourier transform

$$\phi(x) = \int \frac{dk}{2\pi} e^{-ikx} \phi_k = \int \frac{dk}{2\pi} e^{-ikx} \text{Tr}(e^{ikM}). \quad (4.72)$$

In terms of the eigenvalues

$$M = U^{-1} \text{diag}(\lambda_i) U \quad (4.73)$$

one has $\phi_k = \sum_{i=1}^N e^{ik\lambda_i}$ and thus

$$\phi(x) = \sum_{i=1}^N \delta(x - \lambda_i). \quad (4.74)$$

$\phi(x)$ is simply the density of eigenvalues λ_i . The Collective field is constrained by

$$\phi(x) \geq 0, \quad \int \phi(x) dx = N \quad (4.75)$$

and other constrained which disappear as $N \rightarrow \infty$.

To reformulate the theory with ϕ as the coordinate, one not only needs to change variables in the Hamiltonian but also to rescale the wavefunctions by the Jacobian of the transformation from M to ϕ . While the Jacobian is singular for finite N , the $N \rightarrow \infty$ may be found from the hermiticity of the Hamiltonian. One can compute

$$\begin{aligned} \omega(k, \phi) &= -\frac{\partial^2}{\partial M^2} \phi_k = k^2 \int_0^1 d\alpha \phi_{\alpha k} \phi_{k(1-\alpha)} \\ \Omega(k, k'; \phi) &= \frac{\partial \phi_k}{\partial M} \frac{\partial \phi_{k'}}{\partial M} = k k' \phi_{k+k'} \end{aligned} \quad (4.76)$$

One can easily verify the following useful identity

$$\omega(k, \phi) = \int dk' \Omega(k, k', \phi) \frac{1}{|k'|} \phi_{-k'} \quad (4.77)$$

The Fourier transform of $\omega(k, \phi)$ is the singular form

$$\omega(x, \phi) = 2\partial_x \int \frac{\phi(x)\phi(y)}{x-y} dy \quad (4.78)$$

In the end, one can write down the following field theory Hamiltonian

$$H_\phi = \int dx \left(\frac{1}{2} \partial_x \Pi \phi \partial_x \Pi + V(\phi)\phi(x) - \mu_F \left(\phi(x) - \frac{N}{V} \right) + \frac{1}{2} \int dx \phi(x) \left(\int dy \frac{\phi(y)}{x-y} \right)^2 \right) \quad (4.79)$$

where Π is the momentum conjugation to ϕ , $-i\frac{\delta}{\delta\phi(x)}$ and μ_F represent a multiplier for the density constraint and we also have some additional singular terms associated with the derivative terms. The kinetic energy piece is local. The effective potential is given by

$$V_{eff} = \frac{1}{2} \int dx \phi(x) \left(\int dy \frac{\phi(y)}{x-y} \right)^2 - \int (\mu_F - V(x))\phi(x) dx \quad (4.80)$$

One can evaluate the integral and find

$$V_{eff} = \int dx \left(\frac{\pi^2}{6} \phi^3(x) - (\mu_F - V(x))\phi(x) \right) \quad (4.81)$$

We also have two other terms which are of lower order

$$\Delta V = \frac{1}{2} \int_{y=x} dx \phi(x) \partial_x \partial_y \ln(x-y) + \frac{1}{2} \int \frac{\partial \Omega}{\partial \phi} \int \ln|x-y| \phi(y). \quad (4.82)$$

They do not contribute to the planar limit but begin to contribute in the first torus correction.

We should find the classical equation of motion. Since the constraints (4.75) should satisfy, the ground state has $\partial\Pi/\partial x = 0$ and in the leading order minimize

$$V(\phi) - \mu_F \int \phi(x) dx. \quad (4.83)$$

This gives

$$\phi_0(x) = \begin{cases} \frac{1}{\pi} \sqrt{2(\mu_F - v(x))} & |x| < \Lambda \\ 0 & |x| > \Lambda \end{cases} \quad (4.84)$$

where Λ is the point at which the square root vanishes. the planar ground state energy is then given by

$$E_{0,GS} = \mu_F - \frac{1}{3\pi} \int dx (2(\mu_F - v(x)))^{3/2}. \quad (4.85)$$

We now proceed to the computation of the propagator. This corresponds to the study of fluctuations in the collective field method. By shifting the field

$$\phi(x, t) = \phi_0(x) + \xi(x, t) \quad (4.86)$$

the propagator is determined by the quadratic action

$$S = \int dx dt \left(\frac{1}{2} \partial_x^{-1} \frac{1}{\phi_0(x)} \partial_x^{-1} + \frac{1}{2} \pi^2 \phi_0(x) \xi^2 \right). \quad (4.87)$$

It is convenient to introduce a new variable q as

$$q = \frac{1}{\pi} \int^x \frac{dx}{\phi_0(x)}. \quad (4.88)$$

For a classical particle moving in the potential $v(x)$, q is the time taken for the particle to go from the origin to the point x . The range of q is given by $-L < q < L$ where $4L$ is the time period of the classical motion and it is determined by

$$\frac{1}{\pi} \int_0^\Lambda \frac{dx}{\phi_0(x)} = L \quad (4.89)$$

where $\pm\Lambda$ are the turning points of the classical motion. by redefining the field variable

$$\xi = \frac{1}{\phi_0(x)} \eta \quad (4.90)$$

we will give

$$S = \pi^3 \int dt \int_{-L}^L dq \left(\frac{1}{2} \partial_q^{-1} \dot{\eta} \partial_q^{-1} \dot{\eta} - \frac{1}{2} \eta^2 \right). \quad (4.91)$$

Notice that the background field $\phi_0(x)$ has disappeared. The only remnant is the new integration region $(-L, L)$ for the variable q . With the further transformation

$$\eta = \partial_q \psi \quad (4.92)$$

the action is brought into the form

$$S = \pi^3 \int dt \int_{-L}^L dq \left(\frac{1}{2} (\partial_t \psi)^2 - \frac{1}{2} (\partial_q \psi)^2 \right) \quad (4.93)$$

the propagator of the scalar field $\psi(q, t)$ are obtained by implementing the constraint

$$\frac{d}{dt} \int dx \phi(x) = 0 \quad (4.94)$$

which leads to the Dirichlet boundary condition on $\psi : \psi(-L, t) = \psi(L, t) = 0$. The small fluctuation eigenfunctions are found to be

$$\psi_j(q) = \begin{cases} \frac{1}{\sqrt{L}} \sin\left(\frac{j\pi q}{L}\right) & j = 0, 1, 2, \dots \\ \frac{1}{\sqrt{L}} \cos\left(\left(j + \frac{1}{2}\right) \frac{\pi q}{L}\right) & \end{cases} \quad (4.95)$$

with the frequencies

$$\omega_j = \frac{j\pi}{2L} = j\omega_c \quad j = 0, 1, 2, \dots \quad (4.96)$$

The propagator is then

$$D(t - t'; q, q') = \int \frac{dE}{\pi} e^{iE(t-t')} \sum_j \frac{\psi_j(q) \psi_j(q')}{E^2 - \omega_j^2 + i\epsilon}. \quad (4.97)$$

To find the two-point function in the matrix model, we have

$$\text{Tr}(M^n) = (-i)^n \left. \frac{\partial^n \phi_k}{\partial k^n} \right|_{k=0}. \quad (4.98)$$

In terms of the collective field, one can find that

$$\text{Tr}M^n(t) = \int dx x^n \phi(x, t), \quad (4.99)$$

therefore

$$\langle \text{Tr}M^n(t) \text{Tr}M^m(0) \rangle = \int dx dx' x^n x'^m \langle \phi(x, t) \phi(x', 0) \rangle. \quad (4.100)$$

By substituting $\phi(x, t) = \phi_0(x) + \frac{1}{\phi_0(x)} \partial_q \psi(x, t) = \phi_0(x) + \partial_x \psi(x, t)$, we reach to

$$\langle \phi(x, t) \phi(x', 0) \rangle = \phi_0(x) \phi_0(x') + \partial_x \partial_{x'} \langle \psi(x, t) \psi(x', 0) \rangle \quad (4.101)$$

and thus

$$\begin{aligned} \langle \text{Tr}M^n(t) \text{Tr}M^m(0) \rangle &= \int dx dx' x^n x'^m \phi_0(x) \phi_0(x') + \\ &\int dx dx' x^n x'^m \partial_x \partial_{x'} \langle \psi(x, t) \psi(x', 0) \rangle \end{aligned} \quad (4.102)$$

and the connected two-point function in terms of the propagator in (4.97) can be written as

$$\langle \text{Tr}M^n(t) \text{Tr}M^m(0) \rangle_c = \int dq dq' x^n[q] x'^m[q'] \partial_q \partial_{q'} D(t; q, q'). \quad (4.103)$$

To evaluate the integration over E in (4.97) for $t > 0$ we can take the integration over upper half plane and in the case of $t < 0$ over the lower half plane and we will find that

$$D(t; q, q') = \theta(t) \sum_j \frac{\psi_j(q) \psi_j(q') e^{-i\omega_j t}}{i\omega_j} + \theta(-t) \sum_j \frac{\psi_j(q) \psi_j(q') e^{i\omega_j t}}{i\omega_j} \quad (4.104)$$

From now on we assume that $t > 0$ and thus

$$D(t; q, q') = \sum_j \frac{e^{-i\omega_j t}}{i\omega_j L} \left(\sin\left(\frac{j\pi q}{L}\right) \sin\left(\frac{j\pi q'}{L}\right) + \cos\left(\left(j + \frac{1}{2}\right) \frac{\pi q}{L}\right) \cos\left(\left(j + \frac{1}{2}\right) \frac{\pi q'}{L}\right) \right) \quad (4.105)$$

therefore

$$\begin{aligned} &\langle \text{Tr}M^n(t) \text{Tr}M^m(0) \rangle_c \\ &= \int dq dq' x^n[q] x'^m[q'] \left(\sum_j \frac{e^{-i\omega_c j t} j \pi^2}{i\omega_c L^3} \cos\left(\frac{j\pi q}{L}\right) \cos\left(\frac{j\pi q'}{L}\right) \right. \\ &\quad \left. + \sum_j \frac{e^{-i\omega_c (j+1/2)^2 t} j \pi^2}{i\omega_c j L^3} \sin\left(\left(j + \frac{1}{2}\right) \frac{\pi q}{L}\right) \sin\left(\left(j + \frac{1}{2}\right) \frac{\pi q'}{L}\right) \right) \\ &= \sum_j \frac{e^{-i\omega_c j t} j \pi^2}{i\omega_c L^3} \int dq x^n[q] \cos\left(\frac{j\pi q}{L}\right) \int dq' x'^m[q'] \cos\left(\frac{j\pi q'}{L}\right) \\ &\quad + \sum_j \frac{e^{-i\omega_c j t} (j + 1/2)^2 \pi^2}{i\omega_c j L^3} \int dq x^n[q] \sin\left(\left(j + \frac{1}{2}\right) \frac{\pi q}{L}\right) \int dq' x'^m[q'] \cos\left(\left(j + \frac{1}{2}\right) \frac{\pi q'}{L}\right) \end{aligned} \quad (4.106)$$

4.3.1 Collective field theory formalism

Broadly, the collective approach involves a variable transformation. Consider an operator Hamiltonian

$$\hat{H} = \frac{1}{2} \sum_{i=1}^N P_i^2 + V(q_1, \dots, q_M) \quad (4.107)$$

in a manner that allows its representation using an infinite set of new variables

$$\phi(x) = f(x, q_1, \dots, q_M). \quad (4.108)$$

This set would be generally over-complete for finite M . One can make a standard canonical transformation and express the theory using $\phi(x)$. Thus, the wave function of the theory should be written in terms of $\phi(x)$. This can come about as a restriction on invariant singlet subspace of the full Hilbert space. On the wave functional, the kinetic term takes the form

$$K \equiv -\frac{1}{2} \sum \frac{\partial^2}{\partial q_i^2} = \frac{1}{2} \int dx \omega(x, \phi) \frac{\delta}{\delta \phi(x)} - \frac{1}{2} \int dx dy \Omega(x, y, \phi) \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi(y)} \quad (4.109)$$

where

$$\begin{aligned} \omega(x, \phi) &= \sum_i \partial_i^2 f(x, q) \\ \Omega(x, y, \phi) &= \sum_i \partial_i f(x, q) \partial_i f(y, q) \end{aligned} \quad (4.110)$$

The kinetic term in the new collective representation is not Hermitian. It is because of the fact that the new scalar product involves a Jacobian.

Using a similar transformation, one finds the following Hermitian Hamiltonian

$$H = \frac{1}{2} \Pi \Omega \Pi + \frac{1}{8} \left(\omega + \frac{\partial \Omega}{\partial \phi} \right) \Omega^{-1} \left(\omega + \frac{\partial \Omega}{\partial \phi} \right) + V[\phi] - \frac{1}{4} \frac{\delta \omega}{\delta \phi} - \frac{1}{4} \frac{\partial^2 \Omega}{\partial \phi \partial \phi} \quad (4.111)$$

whith $\phi(x)$ and $\Pi(x)$ being a conjugate set of fields variables when $\Pi(x) = -i\delta/\delta\phi(x)$.

4.3.2 Quadratic Potential

We start with the free theory. Taking $v(x) = x^2$, we have

$$\phi_0(x) = \frac{1}{\pi} \sqrt{2\mu_F - 2x^2}, \quad (4.112)$$

thus the x variable in terms of q can be find as

$$x[q] = \sqrt{m\mu_F} \sin(\sqrt{2}q). \quad (4.113)$$

We have $-\frac{\pi}{2} < q < \frac{\pi}{2}$ and so

$$L = \frac{\pi}{2\sqrt{2}}. \quad (4.114)$$

In the end, for free theory, we find that

$$\begin{aligned} \langle \text{Tr} M^n(t) \text{Tr} M^m(0) \rangle_c &= \\ \frac{16}{\pi} (\sqrt{\mu_F})^{m+n} &\int dq \sin^m(\sqrt{2}q) \cos(2\sqrt{2}jq) \int dq' \sin^n(\sqrt{2}q') \cos(2\sqrt{2}jq') \end{aligned} \quad (4.115)$$

For some value of m , n , the result is as below

$$\begin{aligned}\langle \text{Tr}M^2(t)\text{Tr}M^2(0) \rangle_c &= \frac{1}{2}\mu_F^2\pi e^{-i\sqrt{2}t} \\ \langle \text{Tr}M^4(t)\text{Tr}M^4(0) \rangle_c &= \frac{1}{2}\mu_F^4\pi e^{-i\sqrt{2}t} + \frac{1}{16}\mu_F^4\pi e^{-i2\sqrt{2}t} \\ \langle \text{Tr}M^6(t)\text{Tr}M^6(0) \rangle_c &= \frac{225}{512}\mu_F^6\pi e^{-i\sqrt{2}t} + \frac{9}{64}\mu_F^6\pi e^{-i2\sqrt{2}t} + \frac{3}{512}\mu_F^6\pi e^{-i3\sqrt{2}t}\end{aligned}\quad (4.116)$$

4.3.3 Quartic Potential

Now let us consider the interacting theory, the simplest potential is

$$v(x) = x^2 + gx^4 \quad (4.117)$$

and we set $2\mu_F = 1$ here. Hence, we have

$$q = \int \frac{dx}{\sqrt{1-2x^2-2gx^4}} \quad (4.118)$$

by change of variable $t = \sqrt{2gx}/\sqrt{-1+\sqrt{1+2g}}$ we will reach to

$$\begin{aligned}q &= \frac{\sqrt{-1+\sqrt{1+2g}}}{\sqrt{2g}} \int_0^t \frac{dt'}{\sqrt{(1-t'^2)(1+\frac{1+g-\sqrt{1+2g}}{g}t'^2)}} \\ &= \frac{\sqrt{-1+\sqrt{1+2g}}}{\sqrt{2g}} F\left(t, -\frac{1+g-\sqrt{1+2g}}{g}\right) \\ &= \frac{\sqrt{-1+\sqrt{1+2g}}}{\sqrt{2g}} F\left(\frac{\sqrt{2gx}}{\sqrt{-1+\sqrt{1+2g}}}, -\frac{1+g-\sqrt{1+2g}}{g}\right)\end{aligned}\quad (4.119)$$

where

$$F(x, m) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}} \quad (4.120)$$

is the elliptic integral of the first kind. The turning point of the classical particle is at $\Lambda = \frac{1}{\sqrt{2g}\sqrt{-1+\sqrt{1+2g}}}$. Therefore

$$L = \frac{\sqrt{-1+\sqrt{1+2g}}}{\sqrt{2g}} \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1+\frac{1+g-\sqrt{1+2g}}{g}t^2)}} = \frac{\sqrt{-1+\sqrt{1+2g}}}{\sqrt{2g}} K\left(\frac{-(g+1)+\sqrt{1+2g}}{g}\right) \quad (4.121)$$

where

$$K(m) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}} \quad (4.122)$$

is the complete elliptic integral of the first kind. Solving x in terms of q , one can find that

$$x[q] = \frac{1}{\sqrt{2g}} \sqrt{-1+\sqrt{1+2g}} \text{sn}\left(\frac{1}{\sqrt{2g}} \sqrt{-1+\sqrt{1+2g}}(1+\sqrt{1+2g})q \middle| \frac{1}{g}(-1+g)+\sqrt{1+2g}\right) \quad (4.123)$$

where $\text{sn}(z|m)$ is the Jacobi elliptic function.

In order to find the connected two-point function, we need to calculate $\int dq x^m \cos(j\pi q/L)$. To proceed, we can use the series definition of The Jacobi elliptic function

$$\operatorname{sn}(z|m) = \frac{2\pi}{\sqrt{m}K(m)} \sum_{n=0}^{\infty} \frac{q(m)^{n+1/2}}{1-q(m)^{2n+1}} \sin\left((2n+1)\frac{\pi z}{2K(m)}\right) \quad (4.124)$$

In our case $m = \frac{1}{g}(-1+g) + \sqrt{1+2g}$, and $L/K(m) = \frac{1}{\sqrt{2g}}\sqrt{-1+\sqrt{1+2g}}$. Let us first calculate the two-point function for the singlet $\operatorname{Tr}M^2$. Thus, we have

$$\begin{aligned} \int dq x^2[q] \cos(j\pi q/L) &= -\frac{1}{2L} \sum_{n,l=0}^{\infty} \frac{q(m)^{n+l+1}}{(1-q(m)^{2n+1})(1-q(m)^{2l+1})} \\ &\int_{-L}^L dq \sin\left((2n+1)\frac{\pi q}{2L}\right) \sin\left((2l+1)\frac{\pi q}{2L}\right) \cos\left(\frac{j\pi q}{L}\right). \end{aligned} \quad (4.125)$$

We have

$$\begin{aligned} \int_{-L}^L \sin\left((2n+1)\frac{\pi q}{2L}\right) \sin\left((2l+1)\frac{\pi q}{2L}\right) \cos\left(\frac{j\pi q}{L}\right) &= \\ \frac{L}{2\pi} \left\{ \frac{\sin(j+l-n)\pi}{j+l-n} + \frac{\sin(j-l+n)\pi}{j-l+n} - \frac{\sin(1-j+l+n)\pi}{1-j+l+n} - \frac{\sin(1+j+l+n)\pi}{1+j+l+n} \right\} \end{aligned} \quad (4.126)$$

and in the end

$$\int_{-L}^L dq x^2[q] \cos\left(\frac{j\pi q}{L}\right) = -\frac{1}{4}\{2A_j + B_j\} \quad (4.127)$$

while

$$\begin{aligned} A_j &= \sum_{l=0}^{\infty} \frac{q(m)^{2l+1+j}}{(1-q(m)^{2l+2j+1})(1-q(m)^{2l+1})} \\ B_j &= \sum_{l=0}^{j-1} \frac{q(m)^j}{(1-q(m)^{2l+1})(1-q(m)^{2j-2l-1})}. \end{aligned} \quad (4.128)$$

Finally, we reach to

$$\langle \operatorname{Tr}M^2(t)\operatorname{Tr}M^2(0) \rangle_c = \sum_{j=1}^{\infty} \frac{-ij\pi}{8L^2} e^{-\frac{i\pi jt}{2L}} \{2A_j + B_j\}^2 \quad (4.129)$$

4.4 Krylov complexity for MQM via the Lanczos algorithm

Now it is time to attempt to find the notion of Krylov complexity for the MQM.

4.4.1 Over the Ground State

The correlator in the ground state is

$$C(t) = \frac{1}{N} \sum_j \frac{-ij\pi}{8L^2} e^{-ij\pi t/2L} \{2A_j + B_j\}^2 \quad (4.130)$$

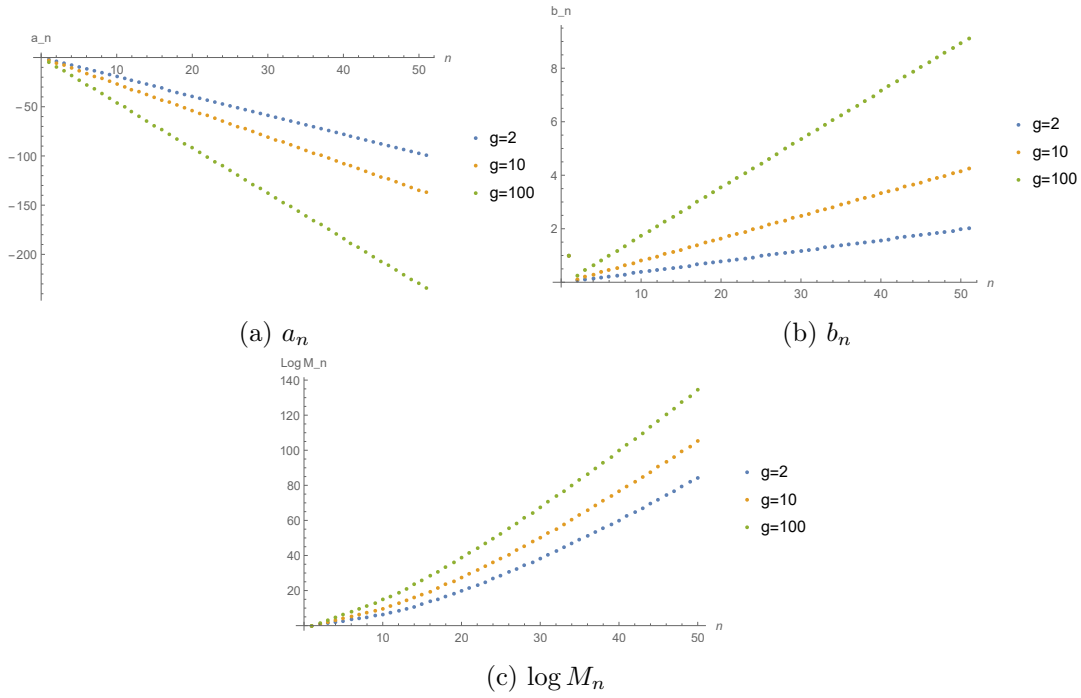


Figure 4.7: The moments and Lanczos coefficients for different values of g .

while N is the normalization factor such that $C(t=0) = 1$. The moments are given by

$$M_n = \frac{1}{N} \sum \frac{-ij\pi}{8L^2} \{2A_j + B_j\}^2 \left(\frac{-ij\pi}{2L}\right)^n \frac{1}{i^n}. \quad (4.131)$$

In Fig. 4.7, one can see the moments and Lanczos coefficients of the matrix quantum mechanics in the ground state for different values of g . The Lanczos coefficients have a linear behavior in that the absolute value of the slope increases for higher values of the g parameter. The slopes of the a_n coefficients are negative while in the case of b_n , they are positive.

Finally, in Fig. 4.8, one can find the Krylov complexity for different values of g . The peak of the complexity grows while g increases. The complexity is periodic as the correlation function is periodic. However, we should consider the behavior of the complexity as a function of time for the time less than the radius of convergence in the time direction. The period of complexity is related to the L and it decreases while g increases and it is expected that it saturates for infinite g .

4.4.2 Over the Thermal State

The correlator for the inner product (4.48) at inverse temperature β is

$$C(t, \beta) = \frac{1}{N'} \sum_j \frac{j\pi}{8L} \frac{e^{\beta\pi j/4L}}{-1 + e^{\beta\pi j/2L}} \{2A_j + B_j\}^2 [e^{-i\pi jt/2L} + e^{i\pi jt/2L}]. \quad (4.132)$$

Thus, the moments are given by

$$M_n = \frac{1}{N'} \sum_j \frac{j\pi}{16L} \frac{e^{\beta\pi j/4L}}{-1 + e^{\beta\pi j/2L}} \{2A_j + B_j\}^2 \left(\frac{(i\pi j/2L)^n}{i^n} + \frac{(-i\pi j/2L)^n}{i^n} \right). \quad (4.133)$$

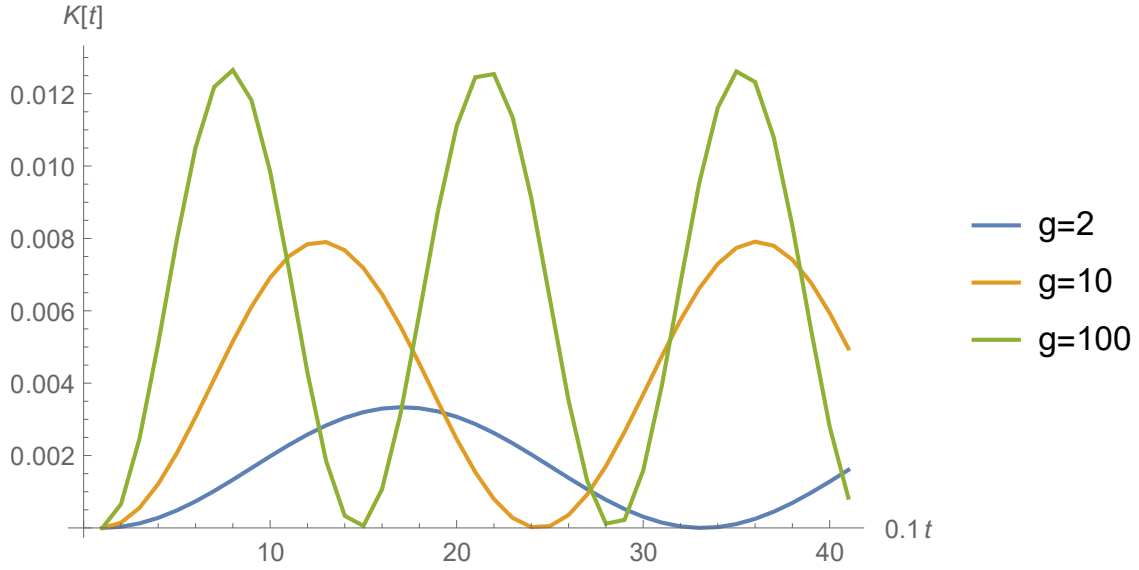


Figure 4.8: Krylov complexity of the matrix quantum mechanics in the ground state as a function of time for different values of g .

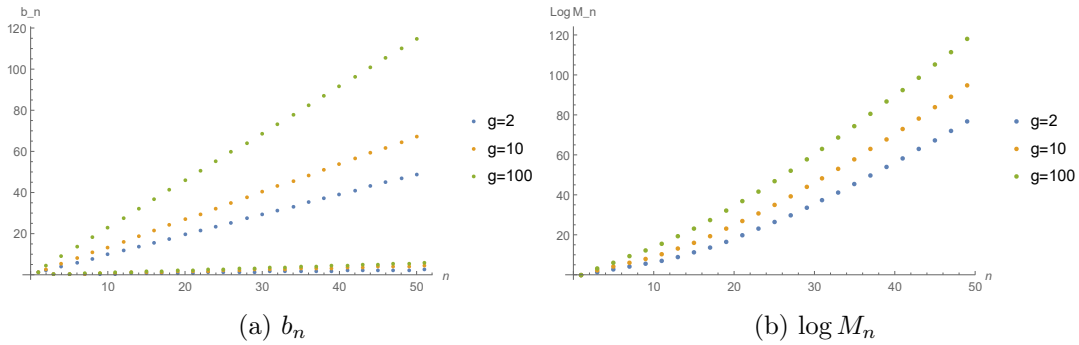


Figure 4.9: The moments and Lanczos coefficients for different values of g .

As it is obvious from the formula odd moments are zero and thus the set of the Lanczos coefficients

$$a_n = 0. \tag{4.134}$$

In Fig. 4.9, one can see the plots for the set of b_n and the even moments in this case. The b_n coefficients have two linear branches with two different positive slopes. One of the slopes is almost the same for different values of g while another slope increases while the g parameter grows. In other words, one slope is a function of g while another one is constant and g -independent. (Look at the example in chapter 4 in [303])

In Fig. 4.10 and 4.11, one can see the Krylov complexity of the matrix quantum mechanics over the thermal states for different values of β and g . For a fixed β , the periodicity of the Krylov complexity decreases as g increases. In this case, unlike over the ground state, the peak of the complexity remains the same for different values of g and β . Moreover, for the fixed g , the periodicity remains the same for different values of β .

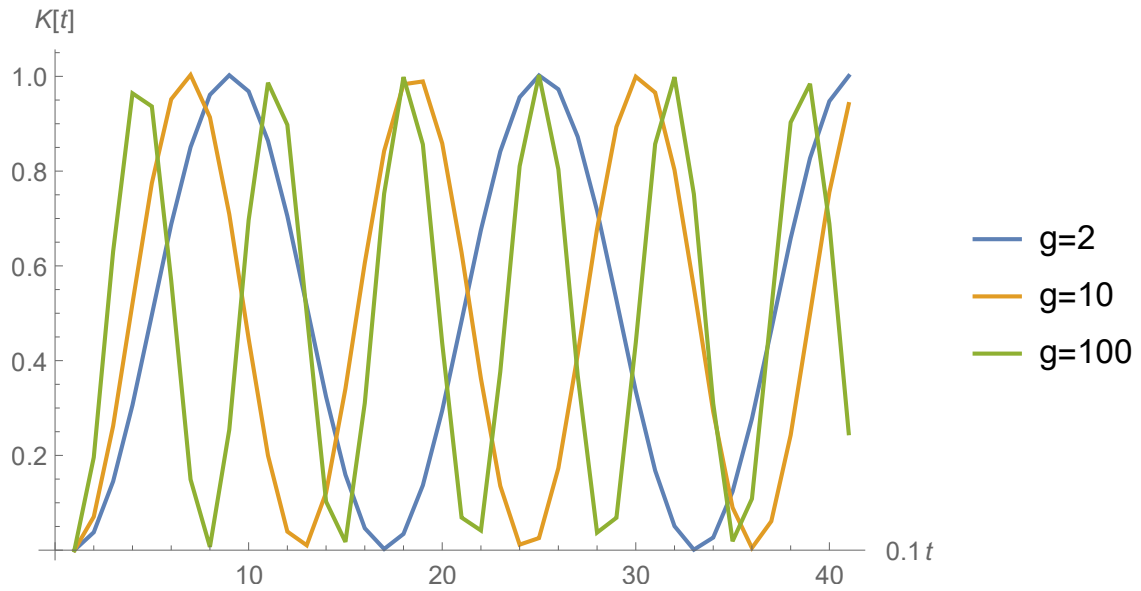


Figure 4.10: Krylov complexity of the matrix quantum mechanics in the thermal state as a function of time for different values of g at $\beta = 1$.

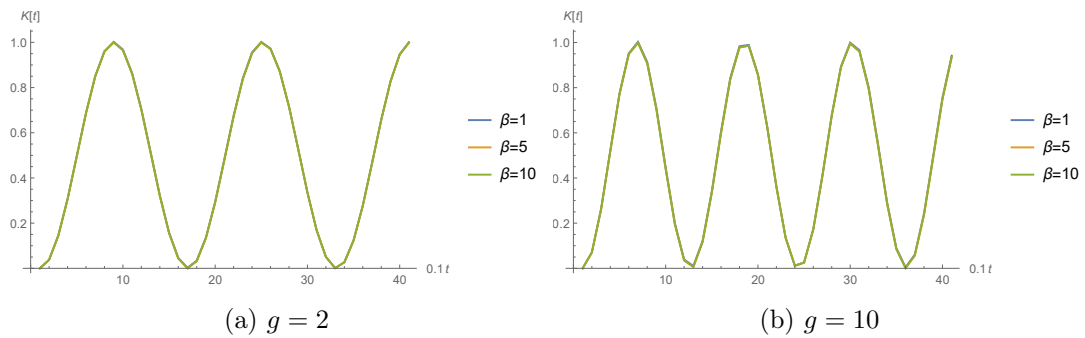


Figure 4.11: Krylov complexity for different values of β .

4.5 Toda chain flow in Krylov space and radius of convergence of Krylov complexity

4.5.1 Toda chain flow in Krylov space

This section is mostly based on [314].

They begin by reviewing the basics of the recursion method. First, we start with the time-correlation function of some operator A ,

$$C(t) = \langle A(t), A \rangle \quad (4.135)$$

it is defined based on the Hermitian form in the space of operators

$$\langle A, B \rangle \equiv \text{tr}(A^\dagger \rho_1 B \rho_2) = \langle B, A^\dagger \rangle^* \quad (4.136)$$

here ρ_1, ρ_2 are some hermitian positive semi-definite operators which commute with the Hamiltonian H .

It is convenient to introduce

$$q_n = \ln \langle A_n, A_m \rangle \quad (4.137)$$

such that

$$G_n m = \langle A_n, A_m \rangle = \delta_{nm} e^n \quad (4.138)$$

In [314], authors focus on the Euclidean time evolution. For a given $O(t)$ where t is Euclidean time, an operator evolved in Minkowski time is $O(-it)$.

The adjoint action of H in the Krylov basis A_n can be represented by Jacobi matrix L ,

$$\begin{aligned} [H, A_n] &= \sum_m L_{nm} A_m, & L &= g M g^{-1} \\ g &= (e^{q_0/2}, e^{q_1/2}, \dots) \\ M &= \begin{pmatrix} a_0 & b_1 & 0 & 0 & \dots \\ b_1 & a_2 & b_2 & 0 & \dots \\ 0 & b_2 & a_2 & b_3 & \dots \\ 0 & 0 & b_3 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \end{aligned} \quad (4.139)$$

As a generalization of (4.138) we define

$$G_n m(t) = \langle A_n(t), A_m \rangle \quad (4.140)$$

and evolution in terms of the Lanczos coefficient

$$G(t) = g e^{Mt} g^T \quad (4.141)$$

the original correlation function is then

$$C(t) = G_{00}(t) = \langle A_0, A_0 \rangle (e^{Mt})_{00} \quad (4.142)$$

Lanczos coefficients a_n, b_n can be promoted to be t -dependent.

Therefore, we can apply the recursion method to define the Krylov basis starting from the same initial A for any given value of t . This defines the orthogonal basis $A_n^t, A_0^t \equiv A$,

$$G_n m^t \equiv \langle A_n^t, A_m^t \rangle_t = \delta_{nm} e^{q_n(t)} \quad (4.143)$$

where $a_n(t), b_n(t)$ and q_n are now t -dependent and thus $M(t)$ and $g(t)$ are time-dependent as well.

An important observation is that $G_n m(t)$ and G_{nm}^t written in terms of two different bases A_n, A_n^t . They are related by a change of coordinates

$$\begin{aligned} G(t) &= z(t)G^t Z(t)^T \\ A_n &= \sum_m z_n m(t) A_m^t. \end{aligned} \quad (4.144)$$

The basis A_n^t has been transformed into basis $A_n = A_n^{t=0}$ by the matrix $z(t)$. One can express G^t in terms of $g(t)$

$$G(t) = g(0)e^{M(0)t}g(0)^T = Z(t)g(t)g(t)^T z(t)^T. \quad (4.145)$$

Explicit time dependence of $G(t)$ provides that

$$\frac{d}{dt}(G^{-1}\dot{G}) = 0. \quad (4.146)$$

It follows that $q_n(t)$ satisfies the Toda equation. The relation between a_n, b_n , and q_n is given by

$$\begin{aligned} a_n(t) &\equiv q^{\dot{n}}, \\ b_n(t) &\equiv e^{(q_{n+1}-q_n)/2}. \end{aligned} \quad (4.147)$$

One can introduce $\tau_n =$

Furthermore, since $e^{-q_n(0)/2A_n}$ and $e^{-q_n(t)/2A_n^t(t/2)}$ are orthonormal bases, they must be related by an orthogonal transformation Q^T (for more detail look at [314])

$$\sum_m Q_{nm}^T(t/2)e^{q_m(0)/2}A_m = e^{-q_n(t)/2}A_n^t(t/2) \quad (4.148)$$

Evolving this equation in time by $-t/2$. We find

$$e^{M(0)t} = Q(t)R(t), \quad R^T(t/2) = g^{-1}(0)Z(t)g(t). \quad (4.149)$$

This QR decomposition of $e^{M(0)t}$ [315].

In [314], the authors apply the relation of Lanczos coefficients to the Toda chain to clarify chaos in quantum many-body systems. An accurate counting of nested commutators appearing in the Taylor series expansion of $C(t)$ will be singular at some finite $t = t^*$.

In general, chaotic behavior is reflected by the linear growth of both a_n and b_n . While the slope of a_n can not exceed twice the slope of b_n .

To study the singular behavior of the time-correlation function, we assume that $C(t) = G_{00}(t)$ together with its derivatives are smooth functions for $0 \leq t < t^*$, and diverges at $t = t^*$. From here follows that $G_{nm}(t)$ are regular for $0 \leq t < t^*$. Using QR decomposition, we have

$$R_{00}(t/2)^2 = C(t)/C(0) \quad (4.150)$$

and conclude that $R_{00}(t)$ is regular for $0 \leq t < t^*/2$ and diverge at $t = t^*/2$. We can decompose $A(t)$ into orthogonal Krylov basis

$$e^{q_0/2}A(t) = \sum_n \phi_n(t)(e^{q_0/2}A_n) \quad (4.151)$$

while

$$\phi_n(t) = R_{00}(t)Q_{n0}(t) \quad (4.152)$$

This is a manifestation of delocalization in Krylov space. At $t = t^*/2$, the operator $A(t)$ spreads across the whole Krylov space.

Just note that this singularity is along the imaginary axis as we consider the Euclidean time.

4.5.2 Radius of convergence of the Krylov complexity

The Krylov complexity is defined in (4.14) can be written in terms of ϕ_n . Thus, in case that in our calculation, the coefficients ϕ_n are regular in the time-band $0 \leq t \leq t^*/2$. Thus the Krylov complexity is also regular in this time band and the radius of convergence for Krylov complexity is $t^*/2$. While t^* is the radius of convergence of the correlation function.

Now in the MQM model we consider in this project, first we should find the radius of convergence of the correlator.

For a given series

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1. \quad (4.153)$$

In the case of MQM over the ground state we have

$$a_n = \frac{-i\pi}{8L^2} n(2A_n + B_n)^2 e^{-i\pi t n/2L} \quad (4.154)$$

therefore

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{n+1}{n} e^{-i\pi t/2L} \left(\frac{2A_{n+1} + B_{n+1}}{2A_n + B_n} \right)^2 < 1. \quad (4.155)$$

After analytically continuation of t , for a complex $z = x + iy$, one can get

$$|e^{\pi y/2L}| = |e^{-i\pi z/2L}| \quad (4.156)$$

Therefore

$$y < \frac{2L}{\pi} \lim_{n \rightarrow \infty} \ln \left(\frac{n}{n+1} \left(\frac{2A_n + B - n}{2A_{n+1} + B_{n+1}} \right)^2 \right) \quad (4.157)$$

Thus the radius of convergence of correlation function $C(t)$ is at most

$$t^* = \frac{2L}{\pi} \lim_{n \rightarrow \infty} \ln \left(\frac{n}{n+1} \left(\frac{2A_n + B - n}{2A_{n+1} + B_{n+1}} \right)^2 \right) \quad (4.158)$$

In Fig. 4.8, one can see that the first peak of the Krylov complexity for all values of g is approximately at $t^*/2$. The radius of convergence of the Krylov complexity is discussed in the previous part.

Chapter 5

Conclusion

- Petz map and holography, EWR: The discussion of EWR in the last section is generic and applies to any desired region on the boundary. In particular, the region can even be disconnected. For example, let us consider the union of two disjoint intervals $A = A_L \cup A_R$, Fig. 5.1, on a Cauchy slice of a 2d holographic CFT dual to AdS_3 in the bulk. If the regions A_L , A_R are sufficiently small, the entanglement wedge of A is union of the entanglement wedges of A_L and A_R , denoted by a_L and a_R , individually, i.e. the union of two AdS-Rindler wedges. It is well known [93, 316] that as we increase the size of the region A , the extremal surface changes discontinuously and in the new configuration the entanglement wedge of A becomes larger and in particular larger than the causal wedge of A .

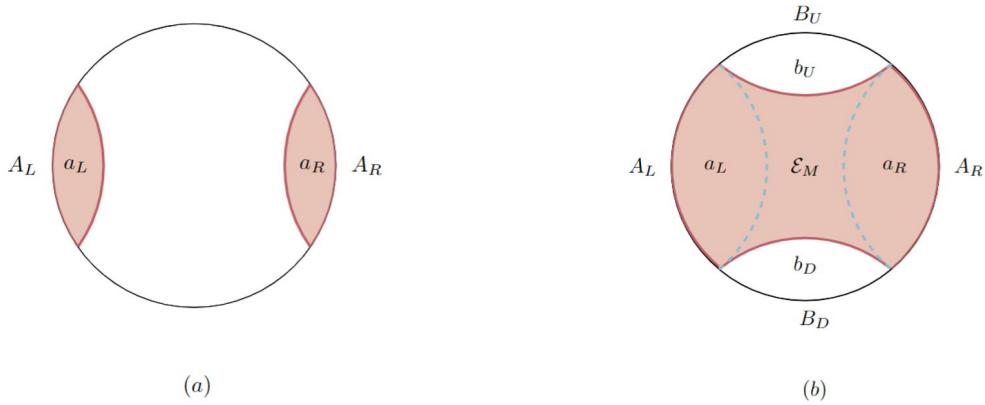


Figure 5.1: The entanglement wedge of a two disjoint intervals $A = A_L \cup A_R$ in $\text{AdS}_3/\text{CFT}_2$. (a) The entanglement is the region bounded by the boundary region A and the minimal area co-dimension 1 surface in the bulk, with the same boundary as A . Thus $\mathcal{E}_A = a_L \cup a_R$. (b) As one increases the sizes of A_L and A_R , the minimal area surface changes and the entanglement wedge is no longer just $a_L \cup a_R$, rather it is all of the shade region, that is $\mathcal{E}_A = a_L \cup a_R \cup \mathcal{E}_M$.

An important question is understanding the nature of observables in the region which is in the entanglement wedge, but not the causal wedge. The Petz formula gives in

principle a CFT representation of these observables, but their microscopic nature is not understood. To make the question more precise, notice that from the point of view of the bulk there is a well defined Bogoliubov transformation between the bulk global modes and the modes in regions a_L , a_R , b_U , b_D , \mathcal{E}_M (see Fig. 5.1). The modes in a_R , a_L , b_U , b_D can be related to modes of single trace operators in the corresponding boundary regions A_L , A_R , B_U , B_D . Entanglement wedge reconstruction and the Petz formula suggests that the modes d , which we take to be localized only in \mathcal{E}_M , should also be representable in region $A = A_L \cup A_R$, but the nature of these observables remains mysterious. Of course the modes d are precisely the modes which are in the entanglement wedge but not the causal wedge of $A = A_L \cup A_R$.

One possibility is that the d modes in region \mathcal{E}_M are combinations of complicated operators in region A_L and A_R

$$d = \sum_{ij} c_{ij} O_i^{A_L} \otimes O_j^{A_R}$$

where $O_i^{A_{L,R}}$ are complicated gauge invariant operators. By *complicated* we mean that they are not single-trace or low-multi-trace operators. In this scenario, while each $O_i^{A_{L,R}}$ by themselves do not behave like GFFs, the particular combination above is expected to behave like a GFF in the large N limit.

Another intriguing possibility is that the modes d are operators which are gauge invariant, but they are made out of constituents in regions A_L , A_R which are not separately gauge invariant. This seems natural from the point of view of, for example, the free $O(N)$ model. In that case we have operators like $\sum_{L,R} \phi^i(x)\phi^i(y)$, with $x \in A_L$ and $y \in A_R$, which are $O(N)$ invariant but the individual constituents are not, see also discussion in [317].

A difficulty with the second possibility is that in a proper gauge theory one would expect that non-gauge invariant operators in regions A_L , A_R have to be connected by Wilson lines which will have to go through the regions B_U or B_D .¹ If the operators in \mathcal{E}_M are actually dual to gauge invariant Wilson line operators with end points in A_L and A_R , this would imply that they cannot strictly commute with all operators in regions B_U and B_D , as generally the Wilson lines can be detected by operators in regions B_U or B_D . This seems to contradict the conventional understanding of EWR, as in the scenario described above the operators on \mathcal{E}_M would not be entirely supported in region A_L, A_R since the Wilson lines are passing through the complementary regions.

It would be interesting to explore whether a particular combination of such Wilson lines connecting the individual non-gauge invariant constituents can be constructed, where commutators of this combination with all simple operators in region B_U, B_D are sufficiently suppressed at large N . This might not directly contradict the arguments supporting EWR. For example, the equality of relative entropies [16] has been established at large N and the arguments are not expected to generalize to imply equality including exponentially suppressed corrections². This might suggest a re-

¹In the $O(N)$ model the symmetry is global hence no Wilson line is necessary.

²For example, at finite N we expect that the bulk geometry is fully quantum and it is not even clear how one can define the entanglement wedge.

finement of EWR where bulk operators are *mostly* supported in $A_R \cup A_L$, allowing some form of Wilson lines connecting the two regions.

In any case, the nature of observables in the entanglement wedge but not the causal wedge, like the operators d in this case, remains somewhat mysterious and further study of their properties is necessary.

- Petz map and holography, Black hole interior: In order to study the evaporating black hole in AdS, one can use absorbing boundary conditions. In [37, 39], it has been shown that exactly at Page time, there is a phase transition in the location of the quantum extremal surface. The new Ryu-Takayanagi surface lies slightly inside the black hole event horizon. Thus after Page time, some parts of the interior are now encoded in the early Hawking radiation, or in other words, it can be reconstructed through the bath.

In order to study the reconstruction of the interior, let us first consider a general entangled system. The CFT can be entangled with another CFT or a collection of qubits. we refer to another system as a bath. Here, we consider the entangled state as

$$|\Psi_{en}\rangle = \sum_i \alpha_i |\psi_i\rangle \otimes |\tilde{i}\rangle \quad (5.1)$$

where α_i are some coefficients, $|\psi_i\rangle$ are orthonormal states in the original CFT, and $|\tilde{i}\rangle$ are states in the bath. The sum can be over a small number of states or an exponentially large number.

We denote the coarse-grained algebra of the original CFT as \mathcal{A}_{cg} and the operator algebra of the bath as \mathcal{B} . We define

$$\mathcal{A}_{bdy} = \mathcal{A}_{cg} \otimes \mathcal{B}. \quad (5.2)$$

The code subspace here is the set of states obtained by acting the algebra \mathcal{A}_{bdy} over the state $|\Psi_{en}\rangle$. The corresponding subspace has the structure of a Hilbert space that can be made via GNS construction

$$\mathcal{H}_{en}^{GNS} \cong \mathcal{A}_{bdy} |\Psi_{en}\rangle. \quad (5.3)$$

This set of states generally is bigger than $\mathcal{A}_{cg} |\Psi_{en}\rangle$ and in some specific cases like when the entangled state is TFD state, these two sets coincide.

In general, as it was discussed in [14], the GNS Hilbert space can be decomposed into the direct sum of $\mathcal{H}_{\Psi_{en}}^j$ while all are closed under the action of the coarse-grained algebra

$$\mathcal{H}_{en}^{GNS} = \bigoplus_j \mathcal{H}_{\psi_{en}}^j. \quad (5.4)$$

For each j , one can identify a unique state $|\psi_{en}^j\rangle \in \mathcal{H}_{\psi_{en}}^j$ which is an equilibrium state with respect to \mathcal{A}_{cg}

$$|\psi_{en}^j\rangle|_{\mathcal{A}_{cg}} = \rho_{th}^c \quad (5.5)$$

and entire $\mathcal{H}_{\psi_{en}}^j$ can be generated by acting with \mathcal{A}_{cg} on $|\psi_{en}^j\rangle$.

For the exterior of the black hole, the same as (1.143), we have the mapping $\mathcal{R}_{ext}^* : \mathcal{A}_{ext} \rightarrow \mathcal{A}_{cg}$ and from its ρ -dual, we can find the Petz map from the operator algebra

of the interior to the commutant of the representation of the coarse-grained algebra on the boundary

$$\mathcal{R}_{in}^* : \mathcal{A}_{in} \rightarrow \mathcal{M}'_{cg}. \quad (5.6)$$

The interior part of a Cauchy slice at late time can be divided into the island and the remaining part of the interior which can be reconstructed from the original CFT. Thus we consider $\mathcal{A}_{in} = \mathcal{A}_{island} \otimes \mathcal{A}_{in-CFT}$. The algebra \mathcal{M}'_{cg} is the representation of the $\mathcal{A}'_{cg} = \mathcal{B} \otimes \mathcal{A}_{fg}$ while \mathcal{A}_{fg} is the fine-grain algebra of the original CFT.

In the GNS Hilbert space \mathcal{R}_{in}^* can be obtained from the direct sum of the mapping in each \mathcal{H}_{en}^j as

$$\mathcal{R}_{in}^* = \oplus_j \mathcal{R}_{in,j}^* \quad (5.7)$$

Each $\mathcal{R}_{in,j}^*$ can be obtained from the same approach as obtaining the Petz dual map in modular theory. In each mapping $\mathcal{R}_{in,j}^*$ depending on the structure of the entanglement in $|\psi_{en}^j\rangle$, the interior can be map to the commutant of the \mathcal{A}_{cg} in the \mathcal{H}_{en}^j that can be the representation of a subalgebra of the fine-grained algebra or the algebra of the bath system.

From the island conjecture, it is expected that

$$\begin{aligned} \mathcal{R}_{in}^*(a) &= \oplus_j \mathcal{R}_{in,j}^*(a) \in \mathcal{A}_{fg} & \forall a \in \mathcal{A}_{in,CFT} \\ \mathcal{R}_{in}^*(a) &= \oplus_j \mathcal{R}_{in,j}^*(a) \in \mathcal{B} & \forall a \in \mathcal{A}_{island} \end{aligned} \quad (5.8)$$

Up to this point, there is not exist any microscopic proof of the island conjecture in the literature. Doing the exact calculation of the Petz map reconstruction of an evaporating black hole can be a good check of the island conjecture.

- **Locality and gravitationally dressed operator:** we have presented a construction of CFT operators that act as local bulk operators in a code subspace, but commute with the Hamiltonian to all orders in the $1/N$ expansion. The gravitational interpretation of such operators is that they are bulk local operators that are gravitationally dressed to features of the state, in particular its time dependence. Because the operators are constructed directly in the CFT, they are manifestly diffeomorphism-invariant. We conclude with some open questions.

It would be interesting to understand if there is a natural way to identify operators whose commutators is zero to all orders in $1/N$ with both H and other single-trace operators in the time band, thus proving the conjecture that the time-band algebra has a commutant in the $1/N$ expansion.

It would also be interesting to understand how to analyze states with very small energy variance, for instance typical black hole microstates in the sharp microcanonical ensemble [100, 101, 162], energy eigenstates, or even empty AdS. In these cases the return amplitude does not decay fast enough and the construction (2.47) cannot be applied. These are also the states where there is no semi-classical feature of the state to dress with respect to, or in other words there is no bulk observer. It may be interesting to clarify the role of the observer, perhaps as in [318], towards identifying a commutant for the time-band algebra in those states.

- **Generalization of the continuous matrix product states:** In this paper, we could obtain the class of RCMPS via an RG flow generated by an appropriate cMERA circuit.

They can be used to approximate the ground states of the relativistic QFTs in $1+1$ dimensions containing both bosonic theories like the sine-Gordon model and fermionic ones such as the Gross-Neveu and Thirring models. Moreover, since the Gaussian cMERA is known in higher dimensions for all bosonic, fermionic, and gauge fields [255, 319], the procedure above can provide a way to find appropriate wave functionals for relativistic theories in higher dimensions, especially, the relativistic version of the continuous PEPS in $2+1$ dimensions. Furthermore, an alternative approach to RCMPS for relativistic theories is the interacting cMERA (icMERA) [320]. It can be found by modifying the entangler and going beyond the Bogoliubov transformation by adding the terms generate n -tuple transformation in fields. Thus, the icMERA evolution is the combination of two Gaussian and non-Gaussian unitaries, exactly the same as RCMPS. However, for icMERA, the important point is the fact that to date, we do not know for a given theory, up to what n -tuple interacting terms are exactly needed to capture the full non-perturbative structure of the theory. But in the case of RCMPS, the form of the ansatz is fixed for all the families of the relevant theories. On the other hand, there is freedom in choosing the entangling profile of the entangler operator of the cMERA. In particular, there is a specific choice that leads to another class of states called magic cMERA [321] which is already shown that has the same UV structure as the standard cMPS. Moreover, its entangler by itself has the continuous matrix product operator representation. Therefore, studying the connection between them might even help us for a better understanding of the interacting disentangler. In the end, we would like to point out that since cMERA is connected to AdS/CFT, it would be desirable to study the possible gravity dual of the states of the form of RCMPS.

- Krylov complexity of the matrix quantum mechanics:

Here we mention some questions remain about the Krylov complexity of the MQM model. First is the reason that we have two linear branches over the thermal case. In both models that have been studied in the case of the thermal case where we use the two-sided inner product, two branches appear. Moreover is there any models that we have more than two branches or not?

Second, how can we calculate the Krylov complexity of the model after $t^*/2$? To see the long-time behavior of the complexity we need to find a way to go beyond this limit. What about the result that in other papers have been found for the Krylov complexity? Are their result in the radius of convergence of the Krylov complexity?

Third, it has been known or conjectured in the literature that for the chaotic system the Lancsoz coefficients b_n grow linearly. Here we can see the linear growth of b_n happen for the MQM model which is not chaotic, it is rather solvable. IS it the case that it can be the case not just for the chaotic system or the conjecture needs some modification.

Appendix A

Tomita-Takesaki theory in a nutshell

In this section, we briefly review the Tomita-Takesaki theory. It is mostly based on [119, 133, 322].

The set of all bounded, linear operators acting on a Hilbert space \mathcal{H} is denoted by $\mathcal{L}(\mathcal{H})$. A subset $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ which is closed under Hermitian conjugation, addition, multiplication, and closed under the weak convergent limit that also contains the unit operator is called a *von Neumann algebra*. For a given \mathcal{A} , the set of all bounded operators which commute with every elements of \mathcal{A} is called the *commutant* of \mathcal{A}

$$\mathcal{A}' = \{b \in \mathcal{L}(\mathcal{H}) | ab = ba, \forall a \in \mathcal{A}\} \quad (\text{A.1})$$

which itself is a von Neumann algebra. For any von Neumann algebra \mathcal{A} on \mathcal{H} , we have $\mathcal{A}'' = (\mathcal{A}')' = \mathcal{A}$. Another von Neumann algebra which is induced by \mathcal{A} is the *center* of the algebra, denoted by $Z_{\mathcal{A}} = \mathcal{A} \cap \mathcal{A}'$.

A representation of the algebra \mathcal{A} in a Hilbert space \mathcal{H} is a map π from the algebra to the bounded operators on \mathcal{H} such that $\pi(ab) = \pi(a)\pi(b)$ and $\pi(a^*) = \pi(a)^\dagger$. The map π is unital if $\pi(I) = I$. A linear form over \mathcal{A} is a function from algebra to the complex numbers $\phi : \mathcal{A} \rightarrow \mathbb{C}$ such that

$$\phi(\alpha a + \beta b) = \alpha\phi(a) + \beta\phi(b) \quad \forall a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}.$$

It is called positive if $\phi(aa^*) \geq 0, \forall a \in \mathcal{A}$, and normalized if $\phi(I) = 1$. A normalized, positive linear form is called a *state* on a von Neumann algebra.

Following the GNS construction, for each positive linear form ϕ over \mathcal{A} , one can build a Hilbert space \mathcal{H}_ϕ and a representation π_ϕ of the algebra \mathcal{A} by linear operators acting on \mathcal{H}_ϕ . The state ϕ defines a Hermitian scalar product on \mathcal{A} as

$$\langle a | b \rangle = \phi(a^*b) \quad \forall a, b \in \mathcal{A}. \quad (\text{A.2})$$

A vector $|\Psi\rangle \in \mathcal{H}$ is called *cyclic* for an algebra \mathcal{A} if the set of $a|\Psi\rangle$ for $a \in \mathcal{A}$ are dense in \mathcal{H} and *separating* if the condition $a|\Psi\rangle = 0$ implies that $a = 0$. If $|\Psi\rangle$ is cyclic and separating for \mathcal{A} , it is also for \mathcal{A}' . And naturally, a representation π is called cyclic if there exists a vector $|\Psi\rangle$ in the representation space \mathcal{H} such that $\pi(\mathcal{A})|\Psi\rangle$ is dense in \mathcal{H} . The GNS construction provides a cyclic representation with the cyclic vector $|\Psi\rangle$ which

$$\psi(a) = \langle \Psi | \pi_\psi(a) | \Psi \rangle \quad (\text{A.3})$$

that is familiar form of the expectation values in quantum mechanics. For the faithful linear form, we will identify \mathcal{A} with $\pi_\psi(\mathcal{A})$.

Moreover, there is a correspondence between superoperators on \mathcal{A} and linear operators acting on \mathcal{H} . A linear map from algebra to itself $\mathcal{T} : \mathcal{A} \rightarrow \mathcal{A}$ is called a *superoperator*. A superoperator is called unital if $\mathcal{T}(I) = I$ and ϕ -preserving if $\phi(\mathcal{T}(a)) = \phi(a)$ for all $a \in \mathcal{A}$. For a generic von Neumann algebra, every normal superoperator has a corresponding operator in the GNS Hilbert space. However, the converse does not always hold, like the local algebra of QFT. Although, in matrix algebra, the correspondence is one-to-one. The GNS Hilbert space operator $T_\psi \in \mathcal{L}(\mathcal{H}_\psi)$ corresponding to the superoperator \mathcal{T} is defined in such a way that

$$T_\psi(a |\Psi\rangle) = \mathcal{T}(a) |\Psi\rangle \quad (\text{A.4})$$

for all $a \in \mathcal{A}$. If \mathcal{T} is unital, T_ψ leaves $|\Psi\rangle$ invariant and if \mathcal{T} is ψ -preserving, T_ψ^\dagger also leaves $|\Psi\rangle$ invariant.

Before proceeding, to have more intuition, let us first consider the Type I von Neumann algebra, *i.e.* the algebra of $d \times d$ complex matrices acting irreducibly on the Hilbert space \mathcal{K} of a d -level system, denoted by $\mathcal{L}(\mathcal{K})$. In such a system, states are described by a positive, semi-definite, Hermitian operator of trace one $\rho \in \mathcal{L}(\mathcal{K})$, which is called *density matrix*. The set of all density matrices on \mathcal{K} denoted by $\mathcal{S}(\mathcal{K})$. Corresponding to the state ρ , one can define a map $\phi_\rho : \mathcal{L}(\mathcal{K}) \rightarrow \mathbb{C}$ given by $\phi_\rho(a) = \text{tr}(\rho a)$, such that for any observable on the system gives us its expectation value on the state ρ .

Given a density matrix

$$\rho = \sum_i \lambda_i^2 |i\rangle \langle i| \quad (\text{A.5})$$

We can always purify the state by coupling it with a second system with the Hilbert space \mathcal{K}' such that $\dim \mathcal{K}' \geq \text{rank } \rho$. The Schmidt decomposition always guarantees that there exists such a system that equality holds. Here, we take $\mathcal{K}' = \text{span}\{|i'\rangle\}$ to be isomorphic to \mathcal{K} . The state ρ on \mathcal{K} can be purified by a vector

$$|\rho^{1/2}\rangle = \sum_i \lambda_i |i\rangle |i'\rangle \in \mathcal{K} \otimes \mathcal{K}' \quad (\text{A.6})$$

such that $\rho = \text{tr}_{\mathcal{K}'} |\rho^{1/2}\rangle \langle \rho^{1/2}|$. Therefore, we can consider $\mathcal{A} = \mathcal{L}(\mathcal{H}) \otimes I_{\mathcal{K}'}$ as a von Neumann algebra on $\mathcal{K} \otimes \mathcal{K}' = \mathcal{H}_\rho$. Here, the commutant is simply $\mathcal{A}' = I_{\mathcal{K}} \otimes \mathcal{L}(\mathcal{K}')$ (Sometimes, for simplicity, we just refer them as $\mathcal{A} = \mathcal{L}(\mathcal{K})$ and $\mathcal{A}' = \mathcal{L}(\mathcal{K}')$). The vector $|\rho^{1/2}\rangle$ is cyclic and separating for the algebra \mathcal{A} if and only if ρ is full-rank. We have also

$$\phi_\rho(a) = \text{tr}(\rho a) = \langle \rho^{1/2} | (a \otimes I) | \rho^{1/2} \rangle \quad (\text{A.7})$$

for all $a \in \mathcal{A}$. Thus, following the GNS construction, we can find a cyclic representation of \mathcal{A} with the cyclic vector $|\rho^{1/2}\rangle$. The map from $\mathcal{A} \rightarrow \mathcal{H}_\rho$ defined as

$$a \longrightarrow |a\rangle_\rho = (a \otimes I) |\rho^{1/2}\rangle = \sum_i \lambda_i (a |i\rangle) |i'\rangle, \quad (\text{A.8})$$

and \mathcal{H}_ρ is nothing but the set of vectors $(a \otimes I) |\rho^{1/2}\rangle$ endowed with the inner product

$$\langle a|b\rangle_\rho = \phi_\rho(a^\dagger b) = \text{tr}(\rho a^\dagger b) = \langle \rho^{1/2} | (a^\dagger b \otimes I) | \rho^{1/2} \rangle. \quad (\text{A.9})$$

In addition, for every $a \in \mathcal{A}$, there exists an operator $a'_m \in \mathcal{A}'$ that creates the same vector in the \mathcal{H}_ρ as a

$$(a \otimes I) |\rho\rangle = (I \otimes a'_m) |\rho\rangle \quad (\text{A.10})$$

which is called the *mirror* operator of a and it is given by

$$a'_m = \rho^{1/2} a^T \rho^{-1/2}, \quad (\text{A.11})$$

where transpose is taken in the ρ eigenbasis.

More generally, one can consider a generic algebra \mathcal{A} on some Hilbert space \mathcal{H} , and for every cyclic and separating state $|\Psi\rangle$ for \mathcal{A} on \mathcal{H} creates a GNS Hilbert space. There is a classification of von Neumann algebras on finite-dimensional Hilbert space corresponding to the center of the algebra. One special case is when the center is trivial $\mathcal{Z}_{\mathcal{A}} = \{\lambda I\}$. In such a case the algebra is called a *factor*. If \mathcal{A} is a factor on \mathcal{H} , there always exists a tensor factorization of Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ such that \mathcal{A} is just the set of all linear operators on one tensor factor $\mathcal{A} = \mathcal{L}(\mathcal{H}_A) \otimes I_{\bar{A}}$. For a generic case that \mathcal{A} is not a factor, there is a decomposition of the Hilbert space as $\mathcal{H} = \oplus_\alpha (\mathcal{H}_{\mathcal{A}_\alpha} \otimes \mathcal{H}_{\bar{\mathcal{A}}_\alpha})$ which \mathcal{A} is block-diagonal $\mathcal{A} = \oplus_\alpha (\mathcal{L}(\mathcal{H}_{\mathcal{A}_\alpha}) \otimes I_{\bar{\mathcal{A}}_\alpha})$.

Here, the set of states on the algebra \mathcal{A} is the intersection of the algebra with the set of states on the Hilbert space $\mathcal{S}(\mathcal{A}) = \mathcal{A} \cap \mathcal{S}(\mathcal{H})$. Any state $\rho \in \mathcal{S}(\mathcal{A})$ is connected with the standard definition of state on von Neumann algebra by linear form $\phi_\rho(a) = \text{tr}(\rho a)$ for all $a \in \mathcal{A}$. Moreover, for any state ρ on \mathcal{H} , there exists a *unique* restriction $\rho|_{\mathcal{A}}$ on $\mathcal{S}(\mathcal{A})$ such that

$$\phi_{\rho|_{\mathcal{A}}}(a) = \phi_\rho(a) \quad \forall a \in \mathcal{A}. \quad (\text{A.12})$$

For now, consider a factor $\mathcal{A} = \mathcal{L}(\mathcal{H}_A) \otimes I_{\bar{A}}$ on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ and a state ρ on \mathcal{H} . The restriction of any $\rho \in \mathcal{S}(\mathcal{H})$ on \mathcal{A} is

$$\rho|_{\mathcal{A}} = \rho_A \otimes \frac{1}{|\bar{A}|} I_{\bar{A}}, \quad (\text{A.13})$$

that $\rho_A \equiv \text{tr}_{\bar{A}} \rho$ is the reduced density matrix of the subsystem A and $|\bar{A}| = \dim(\mathcal{H}_{\bar{A}})$. One can follow the discussion above for ρ_A and create the GNS Hilbert space representation of $\mathcal{A} = \mathcal{L}(\mathcal{H}_A)$ as $\mathcal{H}_{\rho_A} = \mathcal{A} |\rho_A^{1/2}\rangle$. If ρ_A is full-rank and the two tensor factors have the same dimensionality, this GNS Hilbert space is isomorphic to the original \mathcal{H} . Otherwise, it is isomorphic to one subspace of \mathcal{H} .

Let us look at some important superoperators and their corresponding operators in the GNS Hilbert space:

An important anti-linear superoperator which defines complex conjugation is the modular map $\mathcal{S}(a) = a^\dagger$. Its GNS Hilbert space operator correspondence, called *Tomita* operator $S_\rho : \mathcal{H}_\rho \rightarrow \mathcal{H}_\rho$, acts as

$$S_\rho(a \otimes I) |\rho^{1/2}\rangle = (a^\dagger \otimes I) |\rho^{1/2}\rangle. \quad (\text{A.14})$$

It is clear that $S_\rho^2 = I$, thus S_ρ is invertible. We also have $S_\rho |\rho^{1/2}\rangle = |\rho^{1/2}\rangle$. As S_ρ is anti-linear, the S_ρ^\dagger is defined by

$$\langle a | S_\rho^\dagger b \rangle_\rho = \langle S_\rho a | b \rangle_\rho^* = \langle b | S_\rho a \rangle_\rho \quad (\text{A.15})$$

for all $a, b \in \mathcal{A}$. The Tomita operator for the commutant \mathcal{A}' is $S'_\rho = S_\rho^\dagger$.

Another important anti-linear superoperator is the one-to-one corresponding map $\mathcal{J}_\rho : \mathcal{A} \rightarrow \mathcal{A}'$ between operators in \mathcal{A} and \mathcal{A}' , such that

$$\mathcal{J}_\rho(|i\rangle\langle j|) = |i'\rangle\langle j'|. \quad (\text{A.16})$$

The operator corresponding to it is the anti-linear map $J_\rho : \mathcal{H}_\rho \rightarrow \mathcal{H}_\rho$ called *modular conjugation* that acts on the GNS Hilbert space as

$$J_\rho(a \otimes I)|\rho^{1/2}\rangle = (I \otimes (a^\dagger)^T)|\rho^{1/2}\rangle \quad (\text{A.17})$$

where the transpose is in the ρ eigenbasis. In other words in this basis, J_ρ acts as

$$J_\rho c_i |i\rangle |j'\rangle = c_i^* |j\rangle |i'\rangle. \quad (\text{A.18})$$

It also leaves $|\rho^{1/2}\rangle$ invariant.

We also have the *relative modular* operator $\mathcal{D}_{\sigma|\rho} : \mathcal{A} \rightarrow \mathcal{A}$ as an superoperator on \mathcal{A} defined as $\mathcal{D}_{\sigma|\rho}(a) = \sigma a \rho^{-1}$, where ρ and σ are two full-rank density matrices. Its corresponding operator on the GNS Hilbert space is $\Delta_{\sigma|\rho} : \mathcal{H}_\rho \rightarrow \mathcal{H}_\rho$. Since by definition, we have

$$\Delta_{\sigma|\rho}(a \otimes I)|\rho^{1/2}\rangle = (\mathcal{D}_{\sigma|\rho}(a) \otimes I)|\rho^{1/2}\rangle. \quad (\text{A.19})$$

One can find $\Delta_{\sigma|\rho} = \sigma \otimes \rho^{-1}$ using the mirror operator. In case σ is the same as ρ , the operator

$$\Delta_\rho = \rho \otimes \rho^{-1} \quad (\text{A.20})$$

corresponding to $\mathcal{D}_\rho(a) = \rho a \rho^{-1}$ is called *modular* operator. It leaves $|\rho^{1/2}\rangle$ invariant. One can check that

$$\begin{aligned} \Delta_\rho &= S_\rho S_\rho^\dagger \\ J_\rho &= \Delta_\rho^{1/2} S_\rho \\ S_\rho &= J_\rho \Delta_\rho^{1/2} = \Delta_\rho^{-1/2} J_\rho. \end{aligned} \quad (\text{A.21})$$

One can also show that

$$J_\rho \mathcal{A} J_\rho = \mathcal{A}' \quad J_\rho \mathcal{A}' J_\rho = \mathcal{A} \quad (\text{A.22})$$

and

$$\Delta_\rho^z \mathcal{A} \Delta_\rho^{-z} = \mathcal{A} \quad \Delta_\rho^z \mathcal{A}' \Delta_\rho^{-z} = \mathcal{A}', \quad (\text{A.23})$$

for all $z \in \mathbb{C}$. If we write $\Delta_\rho = e^{-K_\rho}$, where K_ρ is called the *modular Hamiltonian*, the later equation can be interpreted as

$$e^{iK_\rho t} \mathcal{A} e^{-iK_\rho t} = \mathcal{A} \quad e^{iK_\rho t} \mathcal{A}' e^{-iK_\rho t} = \mathcal{A}' \quad (\text{A.24})$$

for $z = -it$ which says both \mathcal{A} and \mathcal{A}' are closed under time evolution using the modular Hamiltonian.

It is good to note it now that for every isometry $v' \in \mathcal{A}'$, the vector $v'|\rho^{1/2}\rangle$ or $v'_m|\rho^{1/2}\rangle$ is also a purification of ρ in \mathcal{H}_ρ . These vectors are also cyclic and separating for the algebra \mathcal{A} . Thus, one could start from one of them instead of $|\rho^{1/2}\rangle$ and build \mathcal{H}_ρ by acting the elements of \mathcal{A} on it. Actually, it comes from the point that while the eigenbasis of ρ is the preferred basis for \mathcal{K} , we still have the freedom to choose a basis for \mathcal{K}' . Here, acting with the isometry v' is indeed related to the change of basis in \mathcal{K}' . To have just one unique

vector corresponding to any state ρ , we can use the modular conjugation operator defined in (A.17) or (A.18), fix this operator, and choose the vector which is invariant under J_ρ . The set of all vectors that are invariant under J_ρ is called the *natural cone*. The states on \mathcal{A} are in one-to-one correspondence with the vectors in the natural cone. Take $|e\rangle$ to be the vector corresponding to the maximally mixed state in the natural cone. For every $\sigma \in \mathcal{S}(\mathcal{A})$, the vector $(\sigma^{1/2} \otimes I)|e\rangle$ is also in the natural cone:

$$J_\rho(\sigma^{1/2} \otimes I)|e\rangle = (I \otimes (\sigma^{1/2})^T)|e\rangle = (\sigma^{1/2} \otimes I)|e\rangle. \quad (\text{A.25})$$

The vector $|e\rangle$ itself is given as $|e\rangle = (\rho^{-1/2} \otimes I)|\rho^{1/2}\rangle$ in \mathcal{H}_ρ . Thus, the unique purification of the state σ in the natural cone is

$$|\sigma^{1/2}\rangle = (\sigma^{1/2}\rho^{-1/2} \otimes I)|\rho^{1/2}\rangle = \Delta_{\sigma|\rho}^{1/2}|\rho^{1/2}\rangle. \quad (\text{A.26})$$

Thus, we reach to $J_\rho|\sigma^{1/2}\rangle = \Delta_{\sigma|\rho}^{1/2}|\rho^{1/2}\rangle$, that also holds in infinite dimensional system.

We usually consider the von Neumann algebra in its standard form which is defined as $(\mathcal{A}, \mathcal{H}, J, \mathcal{P}_\mathcal{A})$ where the algebra \mathcal{A} acts on the Hilbert space \mathcal{H} , J is a anti-linear, unitary involution and $\mathcal{P}_\mathcal{A}$ is the natural cone which is invariant by J .

Finally, we note that although in a finite-dimensional system the Hilbert space approach and algebraic approach are equivalent, in an infinite dimension like QFT, it is not the case. In QFT, there is even no tensor factorization of the Hilbert space and indeed the algebraic approach is appropriate to work in. For an open region, \mathcal{O} in the Minkowski spacetime, $\mathcal{A}_\mathcal{O}$ is defined to be the algebra of operators supported only in \mathcal{O} which is called the *local algebra* of the quantum field theory. $\mathcal{A}_\mathcal{O}$ is also a von Neumann algebra that has the properties below:

1. For $\mathcal{O}_1 \subset \mathcal{O}_2$, we have $\mathcal{A}_{\mathcal{O}_1} \subset \mathcal{A}_{\mathcal{O}_2}$.
2. If \mathcal{O}_1 and \mathcal{O}_2 are spacelike separated, we have $[\mathcal{A}_{\mathcal{O}_1}, \mathcal{A}_{\mathcal{O}_2}] = 0$.
3. If \mathcal{O}' denote the causal complement of \mathcal{O} , then $\mathcal{A}'_\mathcal{O} = \mathcal{A}_{\mathcal{O}'}$, that is called *Hagg duality*.
4. If we denote the causal completion of \mathcal{O} as $\tilde{\mathcal{O}}$, then we have $\mathcal{A}_{\tilde{\mathcal{O}}} = \mathcal{A}_\mathcal{O}$.

An important statement for local algebra in QFT is the *Reeh-Schlieder* theorem. It says that the vacuum vector $|\Omega\rangle$ is cyclic and separating for the local algebra in any region \mathcal{O} . It means that to generate the full vacuum sector of the Hilbert space, one needs to act just with the operator restricted to any arbitrary open region. Therefore, although there is not any notion of trace or tensor factorization in QFT, the Tomita-Takesaki theory provides us with a powerful tool to define the quantum information quantities also in QFT.

As we had in Sec. ??, an important quantity to study the recoverability of the quantum channels in the theory of quantum error correction is the relative entropy which is defined in (??). But since, the expression in (??) can be used just for the Type I von Neumann algebra, to use the theory of QEC to study of QFTs and gravity, we need to generalize the definition of relative entropy such that it can be applied for a generic type of von Neumann algebra. One can check that the relative entropy can be also rewritten in terms of the relative modular operator in the GNS Hilbert space as

$$S(\rho|\sigma) = -\langle \rho^{1/2} | \log \Delta_{\sigma|\rho} | \rho^{1/2} \rangle. \quad (\text{A.27})$$

By using the expression (A.27), the relative entropy was generalized to the general v. Neumann algebras by Araki [323, 324] using relative modular hamiltonians. And in the case of the local algebra in QFTs, the suitable definition of the relative entropy between two states $|\Psi\rangle$ and $|\Phi\rangle$ for measurements in the spacetime region \mathcal{O} is define as

$$S_{\mathcal{O}}(\Psi|\Phi) = -\langle\Psi|\log\Delta_{\Phi|\Psi}(\mathcal{O})|\Psi\rangle. \quad (\text{A.28})$$

Appendix B

Changing the variance of H

We would like to understand whether the variance of the energy is accessible within semiclassical gravity, simply from the geometry, or whether it requires more knowledge and in particular, the knowledge of the bulk quantum state for the fields propagating on the background. As we will see, knowledge of the quantum state seems to be required to extract the variance.

The quantity we would like to compute is

$$\langle \Psi_0 | H^2 | \Psi_0 \rangle - \langle \Psi_0 | H | \Psi_0 \rangle^2 \equiv \langle \Psi_0 | H^2 | \Psi_0 \rangle_c . \quad (\text{B.1})$$

This is a connected correlation function in holography, which usually would be computed from the 2-point function of the associated propagating fields on the relevant background. This 2-point function is sensitive both to the geometry and to the bulk quantum state of the propagating fields. However, here the situation is more subtle because we are not studying the local correlation function of an operator, but rather the 2-point function of the spatial integral of a local operator. In this particular case, the situation is a lot more confusing because the dual bulk field would be the s-wave graviton, which is not a propagating degree of freedom in gravity.

So what computes this variance? We will not be able to answer this question, and we believe it to be an interesting open problem which we hope to return to in the future. Nevertheless, we will study some particular states that should be interpreted as adding an s-wave graviton in the bulk. Even though this mode doesn't propagate, we will see that adding it can affect the CFT variance. We will consider two types of deformations of the thermofield double (TFD) state, both of which are related to adding an integrated stress-tensor operator on the cylinder that prepares the TFD state. Let us start with some basics. We consider the TFD state

$$|\text{TFD}\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\beta E_i/2} |E_i\rangle |E_i\rangle . \quad (\text{B.2})$$

We assume that the partition function has the usual large N behavior

$$Z(\beta) = \exp \left[N^2 \left(F_0(\beta) + \frac{1}{N^2} F_1(\beta) + \dots \right) \right] , \quad (\text{B.3})$$

from which we can compute

$$\langle H^n \rangle_\beta = (-1)^n \frac{1}{Z} \frac{d^n}{d\beta^n} Z . \quad (\text{B.4})$$

where H is H_L or H_R . We have

$$\langle \text{TFD} | H | \text{TFD} \rangle = \langle H \rangle_\beta = -N^2 F'_0 - F'_1, \quad (\text{B.5})$$

$$\langle \text{TFD} | H^2 | \text{TFD} \rangle - \langle \text{TFD} | H | \text{TFD} \rangle^2 = \langle H^2 \rangle_{\beta,c} \equiv \langle H^2 \rangle_\beta - \langle H \rangle_\beta^2. \quad (\text{B.6})$$

We have

$$\langle H^2 \rangle_{\beta,c} = N^2 F''_0 + F''_1. \quad (\text{B.7})$$

Now, consider the following state

$$|\psi\rangle = H | \text{TFD} \rangle. \quad (\text{B.8})$$

We now have

$$\langle \psi | \psi \rangle = \langle H^2 \rangle_\beta \quad (\text{B.9})$$

Let us now see how the energy and variance of the state have evolved. We have

$$\frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\langle \text{TFD} | H^3 | \text{TFD} \rangle}{\langle \text{TFD} | H^2 | \text{TFD} \rangle} = \frac{\langle H \rangle_\beta^3 + 3 \langle H^2 \rangle_{\beta,c} \langle H \rangle_\beta + \langle H^3 \rangle_{\beta,c}}{\langle H \rangle_\beta^2 + \langle H^2 \rangle_{\beta,c}}, \quad (\text{B.10})$$

where we defined

$$\langle H^3 \rangle_{\beta,c} \equiv \langle H^3 \rangle_\beta - 3 \langle H^2 \rangle_{\beta,c} \langle H \rangle_\beta - \langle H \rangle_\beta^3. \quad (\text{B.11})$$

Large N factorization implies that we can expand this answer and we find

$$\begin{aligned} \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} &= \langle H \rangle_\beta + 2 \frac{\langle H^2 \rangle_{\beta,c}}{\langle H \rangle_\beta} + \dots \\ &= -N^2 F'_0 - F'_1 - 2 \frac{F''_0}{F'_0} + \dots. \end{aligned} \quad (\text{B.12})$$

We see that we obtain the TFD answer, up to a correction term, which is of size N^0 . This means we have not changed the geometry classically, but only added a quantum particle on top of the TFD state. Similarly, one can compute

$$\begin{aligned} \frac{\langle \psi | H^2 | \psi \rangle}{\langle \psi | \psi \rangle} - \left(\frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} \right)^2 &= \frac{\langle H \rangle_\beta^4 + 6 \langle H^2 \rangle_{\beta,c} \langle H \rangle_\beta^2 + \dots}{\langle H \rangle_\beta^2 + \langle H^2 \rangle_{\beta,c}} - \left(\langle H \rangle_\beta^2 + 4 \langle H^2 \rangle_{\beta,c} + \dots \right) \\ &= \langle H^2 \rangle_{\beta,c} + \dots \\ &= N^2 F''_0 + \dots \end{aligned} \quad (\text{B.13})$$

We see that that the energy has changed at N^0 , but the variance has not changed at order N^2 , only at order N^0 . So this state modifies both the variance and the energy at subleading order compared to the TFD. We will now build a state that modifies the energy at subleading order, but the variance at leading order compared to the TFD.

Consider the state

$$|\phi\rangle = (H - \langle H \rangle_\beta) | \text{TFD} \rangle. \quad (\text{B.14})$$

We now have

$$\langle \phi | \phi \rangle = \langle H^2 \rangle_{\beta,c}, \quad (\text{B.15})$$

and we can now compute the energy in this state:

$$\frac{\langle \phi | H | \phi \rangle}{\langle \phi | \phi \rangle} = \frac{\langle H^3 \rangle_\beta - 2 \langle H^2 \rangle_\beta \langle H \rangle_\beta + \langle H \rangle_\beta^3}{\langle H^2 \rangle_{\beta,c}} = \langle H \rangle_\beta + \frac{\langle H^3 \rangle_{\beta,c}}{\langle H^2 \rangle_{\beta,c}} = -N^2 F'_0 - F'_1 - 2 \frac{F_0'''}{F_0''} + \dots \quad (\text{B.16})$$

We see that this state modifies again the energy only at order N^0 , and in a slightly different way than the previous state. In a similar way, we compute the variance and find

$$\begin{aligned} \frac{\langle \phi | H^2 | \phi \rangle}{\langle \phi | \phi \rangle} - \left(\frac{\langle \phi | H | \phi \rangle}{\langle \phi | \phi \rangle} \right)^2 &= \langle H \rangle_\beta^2 + 3 \langle H \rangle_{\beta,c}^2 + \frac{2 \langle H^3 \rangle_{\beta,c} \langle H \rangle_\beta + \langle H^4 \rangle_{\beta,c}}{\langle H^2 \rangle_{\beta,c}} - \left(\langle H \rangle_\beta + \frac{\langle H^3 \rangle_{\beta,c}}{\langle H^2 \rangle_{\beta,c}} \right)^2 \\ &= 3 \langle H^2 \rangle_{\beta,c} + \frac{\langle H^4 \rangle_{\beta,c}}{\langle H^2 \rangle_{\beta,c}} - \left(\frac{\langle H^3 \rangle_{\beta,c}}{\langle H^2 \rangle_{\beta,c}} \right)^2 \\ &= 3N^2 F_0'' + \frac{3(F_0'')^2 F_1'' - (F_0''')^2 + F_0'' F_0''''}{(F_0'')^2} + \dots \end{aligned} \quad (\text{B.17})$$

One can see that the change in the variance is order N^2 (it is three times the variance of the TFD state), so this is a modification of the variance at the order we were looking for.

From this, we can conclude that the semi-classical geometry is not enough to extract the variance of the energy. The quantum state of the bulk fields is equally important. For the state $|\phi\rangle$, we have the same leading large N properties, but a different quantum state for the graviton. The fact that it is the s-wave of the graviton that enters is still puzzling, and it would be interesting how to properly quantize this non-propagating degree of freedom. We leave this for the future.

Appendix C

Boosts in global AdS

As we have discussed in section 2.2, the conformal generators on the d -dimensional cylinder $\mathbb{R} \times S^{d-1}$ organize themselves as time-translations, rotations, and $2d$ remaining generators which correspond to boosts in the dual AdS geometry. The goal of this section is to discuss whether there exist states that can preserve the boost symmetry. As we have seen throughout the paper, symmetries that are broken by semi-classical states allow us to specify bulk points by dressing the location of a bulk point to the feature of the state that breaks the symmetry. It is important to understand which symmetries are broken, and which symmetries can be preserved by semi-classical states. For time translations and rotations, this is straightforward, but it is somewhat more subtle for boosts, which is the purpose of this section.

The $2d$ boost generators can be realized as d non-independent copies of $SL(2, \mathbb{R})$ [182]. For simplicity, we will study the case of AdS_3 , but the higher dimensional versions follow in a straightforward manner. In $d = 2$, the two copies of $SL(2, \mathbb{R})$ are well-known and correspond to the left and right moving sectors of conformal transformation. The generators are given by L_{-1}, L_0, L_1 and $\bar{L}_{-1}, \bar{L}_0, \bar{L}_1$. Time-translations and rotations are obtained by the combinations

$$H = L_0 + \bar{L}_0, \quad J = L_0 - \bar{L}_0. \quad (C.1)$$

The four residual generators correspond to boosts in AdS_3 . For explicit expressions, see [325]. We would now like to analyze whether non-trivial states can be annihilated by these boosts. As a starting point, notice that there are obviously CFT states which are annihilated by L_{-1} and \bar{L}_{-1} : primary states. However, we would like to consider generators that can be exponentiated to norm-preserving group elements. This means the generators should be Hermitian. The generators L_{-1} and \bar{L}_{-1} do not satisfy this property. However, we can assemble them into the combinations

$$L_+ = L_{-1} + L_1, \quad L_- = i(L_{-1} - L_1) \quad (C.2)$$

Using that $L_{-1}^\dagger = L_1$, we see that L_\pm are hermitian operators and can thus be exponentiated to form unitaries.

The question we would like to ask is whether there are states in the Hilbert space that are eigenstates of L_\pm . We will see that the only finite energy eigenstates of these operators are those where the left-moving part of the CFT is in the vacuum. To see this, we consider the commutator

$$[L_+, L_-] = 4iL_0 \quad (C.3)$$

Suppose now that $|\psi\rangle$ is a normalizable eigenstate of —say— L_+ . Computing the expectation value of this equation we find

$$\langle\psi|L_0|\psi\rangle = 0 \tag{C.4}$$

From the positivity of the energy spectrum this is possible only if $L_0|\psi\rangle = 0$. The only states with this property are states where the left moving sector of the CFT is in the vacuum.

Non-trivial states will thus break boost invariance, which can be used to specify the radial location of an operator. For the construction of operators presented in this paper, this would require considering the states obtained by acting with the unitary operators on semi-classical states $|\psi_0\rangle$ as

$$e^{-i\gamma L_{\pm}} |\psi_0\rangle, \tag{C.5}$$

and studying the generalized return probability

$$R(\gamma) \equiv \left| \langle\psi_0| e^{-i\gamma L_{\pm}} |\psi_0\rangle \right|^2. \tag{C.6}$$

These return probabilities have not been studied but for semi-classical states, it is natural to expect them to be exponentially small for $\gamma \sim \mathcal{O}(1)$.

Appendix D

Early time decay of the return probability

We wish to estimate the early time decay of the return probability (2.31). We will see that at very early times, namely $t \sim \frac{1}{N}$, we can find the decay purely from large N factorization. We will first recall a general property of coherent state overlaps which follows from large N factorization, and then adapt the situation slightly to the return probability.

D.0.1 Overlap of coherent states and large N factorization

Coherent states of quantum gravity in AdS/CFT can be described by states prepared by a Euclidean path integral with sources turned on for single-trace operators. These states are thus given by

$$|\lambda\rangle = e^{\int_{x_0 < 0} dx^d \lambda(x) \mathcal{O}(x)} |0\rangle, \quad (\text{D.1})$$

where we have not written the appropriate time-ordering which is left implicit. We will now show that the overlap is given by

$$\langle \lambda_1 | \lambda_2 \rangle = e^{\int_{\mathbb{R}^d} \lambda_1^*(y) \lambda_2(x) \langle \mathcal{O}(y) \mathcal{O}(x) \rangle} + \mathcal{O}(1/N), \quad (\text{D.2})$$

where it should be understood that y is integrated over the upper half plane while x is integrated over the lower half plane.

We can explicitly expand out the integrals of the bra and the ket states, and use large N factorization: this implies that the operators should be paired up and contracted using Wick's theorem, up to $1/N$ corrections. At a given power in the source, we will have a term of the form

$$\left(\int dx dy \right)^k \frac{1}{(k!)^2} \lambda_1^*(y)^k \lambda_2(x)^k \langle 0 | \mathcal{O}^k(y) \mathcal{O}^k(x) | 0 \rangle. \quad (\text{D.3})$$

We can now apply Wick's theorem and find

$$\left(\int dx dy \right)^k \frac{1}{(k!)^2} \lambda_1^*(y)^k \lambda_2(x)^k \langle 0 | \mathcal{O}^k(y) \mathcal{O}^k(x) | 0 \rangle = \frac{1}{k!} \left(\int dx dy \lambda_1^*(y) \lambda_2(x) \langle 0 | \mathcal{O}(y) \mathcal{O}(x) | 0 \rangle \right)^k,$$

which we can re-exponentiate to find (D.2). Note that we have not written the normalization of the states, which takes care of the Wick contraction between any two operators living both in the lower half plane, or upper half plane. Similarly, terms which have a

different powers of upper and lower operators do not give contributions to leading order at large N because we cannot pair the operators and use Wick's theorem.

For this to work, we have implicitly assumed that $\lambda \sim \mathcal{O}(N^0)$. To see this, note that the connected correlation functions of higher-point operators are suppressed by $1/N$, but also have more sources than lower-point functions. If we scale the sources as $\lambda \sim N^{1/2}$, which is the correct scaling to induce $\mathcal{O}(1)$ back-reaction on the dual spacetime¹, we have to be more careful, as some of the terms we dropped involving connected correlators will be the same size as the Wick contractions. For example, we have

$$\lambda_1^*(y)\lambda_2(x)\langle\mathcal{O}(y)\mathcal{O}(x)\rangle \sim N^2 \quad (\text{D.4})$$

$$(\lambda_1^*(y)\lambda_2(x))^2\langle\mathcal{O}(y)\mathcal{O}(y)\mathcal{O}(x)\mathcal{O}(x)\rangle_c \sim N^2. \quad (\text{D.5})$$

This means that we cannot truncate to the sector of Wick contraction, and we must resum the entire expansion. Note however that the contributions corresponding to loop diagrams in AdS are still suppressed by $1/N$, so we are resumming tree-level diagrams to build the backreacted geometry.

The upshot of this analysis is that we can use large- N factorization to easily compute the overlap of coherent states, but only if the sources are $\mathcal{O}(1)$, in which case the exponent in the exponential is also $\mathcal{O}(1)$. If we try to make the sources scale with N , the exponent will be of order N^2 and then infinitely many contributions must be resummed. We will now apply this logic to the return probability.

D.0.2 The return probability

We can now apply the same logic as above, taking the operator e^{-iHT} to be seen as an imaginary Euclidean source for the Hamiltonian (which is the integral of the stress-tensor). We want to compute

$$R(T) = \langle\Psi_0|e^{-iHT}|\Psi_0\rangle\langle\Psi_0|e^{iHT}|\Psi_0\rangle. \quad (\text{D.6})$$

Applying the logic above, we would find that to leading order we have

$$R(T) = e^{-iT\langle\Psi_0|H_0|\Psi_0\rangle}e^{iT\langle\Psi_0|H_0|\Psi_0\rangle} = 1 + \mathcal{O}(1/N). \quad (\text{D.7})$$

So we see that the candidate leading term vanishes, and we must go to the next order. This is due to the nature of the return probability, which is a square of overlaps. A quick expansion of the exponentials shows that at order T^2 , we have

$$T^2\left(-\langle\Psi_0|H^2|\Psi_0\rangle + (\langle\Psi_0|H|\Psi_0\rangle)^2\right) = -T^2\Delta H^2. \quad (\text{D.8})$$

For reasons similar to those explained above, this term can be exponentiated such that we find

$$R(T) = e^{-T^2\Delta H^2} + \mathcal{O}(1/N). \quad (\text{D.9})$$

As in the previous section, we can only trust this approximation if the exponent is $\mathcal{O}(1)$. Because we are considering states that have $\Delta H \sim N^2$, we see that we can trust this exponential decay of the return probability for time scales up to $t \sim 1/N$.

For larger time scales, it may still hold, but it cannot be justified based solely on large N factorization. It is instructive to consider the case of the thermofield double state and

¹For operators that have unit 2-point function.

the spectral form factor, as we already discussed in section 2.5.2. For simplicity, we set $d = 2$ where we have

$$Z(\beta) = e^{\frac{c}{12} \frac{4\pi^2}{\beta}}. \quad (\text{D.10})$$

The spectral form factor then gives

$$R(T) = e^{\frac{\pi^2 c}{3} \left(\frac{1}{\beta+iT} + \frac{1}{\beta-iT} \right)} = e^{\frac{2\pi^2 c}{3} \frac{\beta}{\beta^2+T^2}}. \quad (\text{D.11})$$

We can expand this expression in T , as long as $T \ll \beta$, to find

$$R(T) \approx Z(\beta)^2 e^{-\frac{2\pi^2 c}{3} \frac{T^2}{\beta^3}}. \quad (\text{D.12})$$

We find the exponential decay that goes like T^2 . What is important is that even though T must be much smaller than β , it is allowed to scale as N^0 . This cannot be justified solely from large N factorization but still holds in this particular context. We expect the return probability to satisfy this property for holographic states more generally.

Appendix E

Notes on boundary states

Some useful references for this section are [178, 291, 326, 327].

E.0.1 Boundary states in 2D CFT

Boundary states in a 2d CFT need to satisfy [291]

$$(L_n - \tilde{L}_n) |B\rangle = 0. \quad (\text{E.1})$$

In any Verma module, one can find a simple solution to these conditions as

$$|I_h\rangle = \sum_{\vec{k}} |\vec{k}, h\rangle_L \otimes |\vec{k}, h\rangle_R, \quad (\text{E.2})$$

where $|\vec{k}, h\rangle_L$ is a linear combination of Virasoro descendants of the primary state $|h\rangle$ characterized by an infinite dimensional vector $\vec{k} = (k_1, k_2, \dots)$ with non-negative integer components. We identify these states by starting with descendants of the form

$$\dots L_{-n}^{K_n} \dots L_{-1}^{K_1} |h\rangle_L. \quad (\text{E.3})$$

and forming an orthonormal basis selected such that ${}_L\langle \vec{k}, h | \vec{k}', h \rangle_L = \delta_{\vec{k}, \vec{k}'}$.

The state $|I_h\rangle$ is called the Ishibashi state for the primary state $|h\rangle_L$, where the states $|\vec{k}, h\rangle$ are the descendant on top of the primary labeled by h . It can be seen easily that

$$L_n |I_h\rangle = \tilde{L}_n |I_h\rangle. \quad (\text{E.4})$$

It is clear that the Ishibashi states have maximal entanglement between the left-moving and right-moving sectors. Linear combinations of the Ishibashi states satisfy the constraint (F.5) as well.

Physical boundary states are given by special linear combinations of Ishibashi states which are called Cardy states

$$|B_a\rangle = \sum_h C_{a,h} |I_h\rangle. \quad (\text{E.5})$$

Physical boundary states should satisfy a consistency condition of the partition function on a finite cylinder related to open-closed duality [291].

The Cardy states are singular because the norm of the Ishibashi states is divergent. One can define regularized boundary states by evolving in Euclidean time as

$$|B_{a,\beta}\rangle = e^{-\frac{\beta}{4} H_c} |B_a\rangle, \quad (\text{E.6})$$

where β is a positive constant and $H_c = L_0 + \tilde{L}_0 - \frac{c}{12}$. Since $[L_0 - \tilde{L}_0, H_c] = 0$, the state (F.4) is still space-translational invariant on the circle, but it is time-dependent.

Ishibashi states are orthogonal to each other. The amplitude of Euclidean time evolution by $\beta/2$ between two such states is computed as

$$\langle I_k | e^{-\beta H_c/2} | I_l \rangle = \delta_{kl} \chi_k(e^{-\beta/2}). \quad (\text{E.7})$$

χ_k is the character for the primary k . On the other hand, the Cardy states are not orthogonal to each other but satisfy the open-closed duality relation as follows

$$\langle B_a | e^{-\frac{\beta}{2} H_c} | B_b \rangle = \sum_k N_{a,b}^{(k)} \text{Tr}_k [e^{-\frac{4\pi^2}{\beta} H_o}] \quad (\text{E.8})$$

where $H_o = L_o - \frac{c}{24}$ denotes the Hamiltonian in the dual channel, characterized by the boundary conditions a, b . On the right hand side, $\text{Tr}_k[\dots]$ denotes a trace in the sector associated to a primary k as well as its descendants. Moreover, $N_{a,b}^{(k)}$ counts the degeneracy of sectors which belong to the primary k with boundary conditions a and b .

In the high temperature limit $\beta \rightarrow 0$, we find that

$$\langle B_a | e^{-\frac{\beta}{2} H_c} | B_b \rangle \simeq N_{a,b}^{(k_m)} e^{-\frac{4\pi^2}{\beta} (h_{a,b}^{(min)} - \frac{c}{24})}, \quad (\text{E.9})$$

where k_m is the lightest primary among those satisfy $N_{a,b}^{(k_m)} \neq 0$, whose conformal dimension is denoted as $h_{a,b}^{(min)}$.

We can estimate the inner products between two normalized boundary states in this limit as

$$\langle \psi_a | e^{-\frac{\beta}{2} H_c} | \psi_b \rangle = \frac{\langle B_a | e^{-\frac{\beta}{2} H_c} | B_b \rangle}{\sqrt{\langle B_a | e^{-\frac{\beta}{2} H_c} | B_a \rangle \langle B_b | e^{-\frac{\beta}{2} H_c} | B_b \rangle}} \simeq \delta_{a,b} + N_{a,b}^{(k_m)} e^{-\frac{4\pi^2}{\beta} h_{a,b}^{(min)}}. \quad (\text{E.10})$$

Note that $N_{a,a}^{(0)} = 1$. In this way, a large gap in the open string channel leads to a large exponential suppression of off-diagonal elements of inner products.

In holographic BCFT, the inner product between two boundary states can be computed by evaluating the gravity action on the dual background. When we consider the gravity dual of a cylinder, there are two candidates of classical gravity solutions depending on whether the end of the world brane is connected or disconnected which are called connected and disconnected solutions. When we consider the overlap for an identical boundary condition a , then both the connected and disconnected solution are allowed. In the limit $\beta \rightarrow 0$, the connected solution is favored and one can find that

$$\langle B_a | e^{-\frac{\beta}{2} H_c} | B_a \rangle \simeq e^{\frac{\pi^2 c}{6\beta}}. \quad (\text{E.11})$$

We will use it later to calculate the return probability for boundary states. In addition to it, one can find the inner product between two boundary states with different boundary conditions. In this case, only the disconnected solutions are allowed and

$$\langle B_a | e^{-\frac{\beta}{2} H_c} | B_b \rangle \simeq e^{\frac{c\beta}{12} + S_{bdy}^{(a)} + S_{bdy}^{(b)}}, \quad (\text{E.12})$$

where $S_{bdy}^{(i)}$, $i = a, b$ are the boundary entropies [178].

E.0.2 Boundary states in higher dimensions

One can generalize to higher dimensions and define a boundary state $|B_a\rangle$ as a state associated to a $(d-1)$ -dimensional boundary in d -dimensional CFT [178, 328]. Taking the boundary to be a torus \mathbb{T}^{d-1} , the inner product between two boundary states in a holographic BCFT can be computed as a partition function on a d -dimensional open manifold $I_{\beta/2} \times \mathbb{T}^{d-1}$ where $I_{\beta/2}$ is a length $\beta/2$ interval. As in the 2d case, there are two bulk solutions, a connected and a disconnected one. In the $\beta \rightarrow 0$ limit the connected solution is dominant and one can find the inner product between two identical boundary states using the gravity solution as

$$\langle B_a | e^{-\frac{\beta}{2} H_c} | B_a \rangle_{con} \simeq e^{\alpha_d / \beta^{d-1}}, \quad (\text{E.13})$$

where

$$\alpha_d = (4\zeta(T))^d \frac{R^{d-1}}{16G_N} L^{d-1}, \quad (\text{E.14})$$

where R is the AdS radius, L is the length of the compactified spatial directions and $\zeta(T)$ is a function of tension which is defined when $T < 0$ as

$$\zeta(T) \equiv \frac{\Gamma(1/d)\Gamma(1/2)}{\Gamma(1/d+1/2)} \frac{R|T|}{d(d-1)} \left(1 - \frac{R^2 T^2}{(d-1)^2}\right)^{1/d-1/2} F\left(1, 1/d, 1/2 + 1/d; 1 - \frac{R^2 T^2}{(d-1)^2}\right), \quad (\text{E.15})$$

and when $T > 0$, $\zeta(T) = \frac{2\pi}{d} - \zeta(-T)$. The tension takes values in the range $|T| < \frac{d-1}{R}$. For $d > 2$, $\zeta(T)$ non-trivially depends on T and there is an upper bound of the tension $T < T_*$ which $T_* > 0$ and $\zeta(T_*) = 0$ [178].

E.0.3 Correlation functions in BCFTs

Let us first start with the simplest case where the CFT is defined on the upper half plane and the boundary state $|B\rangle$ is placed along the real axis. We consider the 1-point function of a local operator placed at z in the upper half plane. In the case of a CFT on the plane, the 1-point function of a primary operator in the vacuum is required to vanish by the symmetries. These are partly broken in a BCFT. The remaining symmetries constraint the 1-point function to have the form

$$\langle \mathcal{O}(z) \rangle_{\text{UHP}} = \frac{A_{\mathcal{O}}}{(2 \text{Im}(z))^{\Delta}}, \quad (\text{E.16})$$

where $A_{\mathcal{O}}$ is determined by the details of the theory and the precise boundary state in question. One could think of this as the boundary providing a source for the operator \mathcal{O} .

The 2-point function of a primary operator in a BCFT is more complicated than the case with no boundaries where it is exactly fixed by the symmetries. Non-trivial information about the operator content and OPE coefficients is necessary to compute the 2-point function exactly in a BCFT. We assume that for large N holographic CFTs the large N 2-point function takes the form

$$\langle \mathcal{O}(z_1) \mathcal{O}(z_2) \rangle_{\text{UHP}} = \langle \mathcal{O}(z_1) \rangle_{\text{UHP}} \langle \mathcal{O}(z_2) \rangle_{\text{UHP}} + \langle \mathcal{O}(z_1) \mathcal{O}(z_2) \rangle \pm \langle \mathcal{O}(z_1) \mathcal{O}(z_2^*) \rangle, \quad (\text{E.17})$$

where

$$\langle \mathcal{O}(z_1) \mathcal{O}(z_2) \rangle = \frac{1}{|z_1 - z_2|^{2\Delta}}, \quad (\text{E.18})$$

where the contribution from an image insertion placed at z_2^* . The sign of the last term is governed by the boundary conditions, being either Dirichlet (-) or Neumann (+).

Mapping the z coordinate to a new coordinate w by

$$w \rightarrow z = \exp(2\pi w/\beta + i2\pi/4), \quad (\text{E.19})$$

we can map the upper half plane to the a strip of width $\beta/2$, where the positive (negative) real axis is mapped to the lower (upper) edge of the strip.

Since primary operators continue to transform in the usual way, the correlation functions now transform to

$$\begin{aligned} \langle \mathcal{O}(w) \rangle_{\text{strip}} &= \frac{A_{\mathcal{O}}}{\left(\frac{\beta}{\pi} \cos\left[\frac{2\pi}{\beta}\tau\right]\right)^{\Delta}} \\ \langle \mathcal{O}(w_1)\mathcal{O}(w_2) \rangle_{\text{strip}}^{\text{connected}} &= \frac{1}{\left|\frac{\beta}{\pi} \sinh\left[\frac{\pi}{\beta}(w_1 - w_2)\right]\right|^{2\Delta}} \pm \frac{1}{\left|\frac{\beta}{\pi} \cosh\left[\frac{\pi}{\beta}(w_1 - \bar{w}_2)\right]\right|^{2\Delta}}, \end{aligned} \quad (\text{E.20})$$

where the second line is only the connected piece of the large N 2-point function [210]. Higher order correlation function can be found through large N factorization.

Correlation functions on a state defined on a circle by

$$|B_{\beta}\rangle = e^{-\beta H/4} |B\rangle, \quad (\text{E.21})$$

can be thought of as correlation function on a cylinder of width $\beta/2$ where the boundary state is placed on both sides. We can instead consider a strip of width $\beta/2$, from $\tau = -\beta/4$ to $\tau = \beta/4$ with periodicity $x \sim x + R$. We choose $R = 2\pi$ for simplicity from now on. In large N holographic CFTs correlation functions on the cylinder can be found from the correlation function on the strip using the method of images

$$\langle O(w_1)O(w_2) \rangle_{\text{cylinder}}^{\text{connected}} = \sum_{n=0}^{\infty} \langle O(w_1 + 2\pi n)O(w_2) \rangle_{\text{strip}}^{\text{connected}}. \quad (\text{E.22})$$

Appendix F

Entanglement Renormalization in Continuum

cMERA [255] was originally introduced as an ansatz wave functional for the ground states of QFT Hamiltonians. The same as the ER that corresponds to MERA tensor network, the continuous version of it implements a real-space RG in the continuum. MERA on a lattice can also be visualized as a quantum circuit [329]. In this representation, the physical state can be obtained by evolving a simple product state with no entanglement that factories with respect to the lattice sites -usually considered as "all sites 0"- by a unitary operator to create entanglement at different scales. The generalization to the continuum is conceptually straightforward. To describe cMERA first assume a QFT and impose a UV cut-off Λ . It is required to start with a finite Λ in order to define the process but, in the end, it can be sent to infinity. One parameter family of scale-dependent states is produced through continuous unitary evolution in scale u as

$$|\Psi(u)\rangle = U(u, u_{IR}) |\Omega\rangle = \mathcal{P} e^{-i \int_{u_{IR}}^u (K(s)+L) ds} |\Omega\rangle \quad (\text{F.1})$$

where the symbol \mathcal{P} is path ordering and $|\Omega\rangle$ is the IR state that is the continuum limit of a product state on the lattice that contains no entanglement between spatial regions, and the UV state is what describes the system we are studying, usually, the ground state of the system. Moreover, it has been shown that any spacetime symmetry of the ground state is also a symmetry of the cMERA representation of it [330]. Only the difference between UV and IR limits is fixed as $u_{UV} - u_{IR} = O(\log \xi \Lambda)$ when ξ is the correlation length of the theory. It is convenient to set $u_{UV} = 0$ and $u_{IR} = -O(\log \xi \Lambda)$. For critical systems $u_{IR} \rightarrow -\infty$.

On the other hand, L is the generator of the scale transformation in spacial directions and $K(u)$ is the so-called entangler (or disentangler, depending on the direction of the RG flow) which contains the variational parameters of the cMERA. The IR state is scale-invariant, thus

$$L |\Omega\rangle = 0. \quad (\text{F.2})$$

Consider a set of field operators of the theory $\psi(x)$, $\psi^\dagger(x)$ satisfying $[\psi(x), \psi^\dagger(y)]_\pm = \delta(x - y)$ with $+(-)$ for fermions (bosons). If the IR state is the vacuum of this set of annihilation and creation operators *i.e.*

$$\psi(x) |\Omega\rangle = 0, \quad \forall x, \quad (\text{F.3})$$

the generator of scale transformation can be read as

$$L = -\frac{i}{2} \int \psi^\dagger(x) x \frac{d\psi(x)}{dx} - x \frac{d\psi^\dagger(x)}{dx} \psi(x) dx. \quad (\text{F.4})$$

Although some steps have been taken towards finding the form of the entangler operator for interacting theories, both at the perturbative level [17, 331, 332] and non-perturbatively [320], it has only been explicitly studied for free theories [255, 319]. The entangler operator for quadratic interactions is the generator of Bogoliubov transformation given by

$$K(u) = \frac{i}{2} \int dk (g(k, u) \psi_k^\dagger \psi_{-k}^\dagger - g^*(k, u) \psi_{-k} \psi_k) \quad (\text{F.5})$$

where $\psi_k = \frac{1}{\sqrt{2\pi}} \int dx e^{-ikx} \psi(x)$ and $g(k/\Lambda, u)$ is even and odd in its first argument for bosons and fermions, respectively. Finally, we mention that the cMERA unitary process provides a RG flow for the operators as

$$\frac{dO(u)}{du} = -i[K(u) + L, O(u)] \quad (\text{F.6})$$

when the physical or bare operators of the theory are defined at the UV scale.

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