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We describe a procedure to construct infinite sets of pairwise smoothly inequivalent 2–spheres in simply connected 4–manifolds, which are topologically isotopic and whose complement has a prescribed fundamental group that satisfies some conditions. This class of groups include cyclic groups and the binary icosahedral group. These are the first known examples of such exotic embeddings of 2–spheres in 4–manifolds. Examples of locally flat embedded 2–spheres in a nonsmoothable 4–manifold whose complements are homotopy equivalent to smoothly embedded ones are also given.

57K45, 57R55; 57R40, 57R52

1 Main results

The first main result of this note is the following theorem.

Theorem A *Fix $p \geq 2$. There is an infinite set*

$$\{S_{n,p} : n \in \mathbb{Z}\}$$

of smoothly embedded 2*–spheres in* $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ *that satisfies the following properties*:

- *Any two elements are topologically isotopic.*
- *There is a diffeomorphism of pairs*

$$(2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2, S_{n_1,p}) \to (2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2, S_{n_2,p})$$

  if and only if $n_1 = n_2$.
- *The fundamental group of the complement is*

$$\pi_1(2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2 \setminus \nu(S_{n,p})) = \mathbb{Z}/p$$

  for every $n \in \mathbb{Z}$.
- $[S_{n,p}] \neq 0 \in H_2(2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2; \mathbb{Z})$ *for every* $n \in \mathbb{Z}$.
- *Surgery along each of these* 2*–spheres yields an infinite set of pairwise homeomorphic and pairwise nondiffeomorphic closed smooth* 4*–manifolds with fundamental group* $\mathbb{Z}/p$.

Theorem A provides the first known example of an infinite set of 2–spheres smoothly embedded in a simply connected 4–manifold that are pairwise topologically isotopic, pairwise smoothly inequivalent and having a complement with finite cyclic fundamental group. Schwartz [2019, Theorem 2] pointed out the existence of closed simply connected 4–manifolds containing pairs of smoothly embedded 2–spheres that are both smoothly equivalent and topologically isotopic, but not smoothly isotopic. Examples of these exotic embeddings of 2–spheres in closed 4–manifolds with simply connected complement have been constructed by Akbulut [2015] and Auckly, Kim, Melvin and Ruberman [Auckly et al. 2015]. Exotic embeddings of surfaces with positive genus in simply connected 4–manifolds and complement having nontrivial fundamental group were found by Kim [2006] and Kim and Ruberman [2008]. An ingredient in the proof of Theorem A is of independent interest: we point out in Theorem 1 that constructions of inequivalent smooth structures on simply connected 4–manifolds of Fintushel and Stern [2011; 2012] can be extended to produce such structures on 4–manifolds with nontrivial fundamental group too.

The second main result provides a construction procedure for topologically equivalent yet smoothly inequivalent homologically essential 2–spheres whose complement can be chosen to have the same fundamental group as a wide range of $\mathbb{Q}$–homology 4–spheres. We work with the modified Seiberg–Witten SW'_X invariant of a closed 4–manifold X as defined, for example, in [Fintushel et al. 2007, Section 2], and denote by $\mathcal{B}_X$ the set of basic classes.

Theorem B *Let $\{Z_n : n \in \mathbb{Z}\}$ be an infinite set of closed smooth simply connected 4–manifolds with pairwise different integer invariants*

$$S_n = \max\{|\mathrm{SW}'_{Z_n}(k_{Z_n})| : k_{Z_n} \in \mathcal{B}_{Z_n}\}, \tag{1-1}$$

which are pairwise homeomorphic to a given closed 4–manifold Z and such that the connected sum $Z_n \# S^2 \times S^2$ is diffeomorphic to $Z \# S^2 \times S^2$ for every $n \in \mathbb{Z}$. Let M be a closed smooth 4–manifold with $H_*(M;\mathbb{Q}) \cong H_*(S^4;\mathbb{Q})$ and set $\pi := \pi_1 M$. Suppose that there is a loop $\alpha \subset M$ and a choice of framing such that

$$S^2 \times S^2 = M \setminus \nu(\alpha) \cup D^2 \times S^2. \tag{1-2}$$

There is an infinite set

$$\{S_{n,\pi} : n \in \mathbb{Z}\}$$

of smoothly embedded 2–spheres in $Z \# S^2 \times S^2$ that satisfies the following properties.

- *There is a homeomorphism of pairs*
$$(Z \# S^2 \times S^2, S_{n_1,\pi}) \to (Z \# S^2 \times S^2, S_{n_2,\pi})$$
for every $n_i \in \mathbb{Z}$.
- *There is a diffeomorphism of pairs*
$$(Z \# S^2 \times S^2, S_{n_1,\pi}) \to (Z \# S^2 \times S^2, S_{n_2,\pi})$$
if and only if $n_1 = n_2$.

- *The fundamental group of the complement is*

$$\pi_1(Z \# S^2 \times S^2 \setminus \nu(S_{n,\pi})) = \pi$$

  and its homology class satisfies

$$[S_{n,\pi}] \neq 0 \in H_2(Z \# S^2 \times S^2; \mathbb{Z})$$

  for every $n \in \mathbb{Z}$.

- *Surgery along each of these* 2*–spheres yields an infinite set* $\{Z_n \# M : n \in \mathbb{Z}\}$ *of pairwise nondiffeomorphic closed smooth* 4*–manifolds with fundamental group* π *that are pairwise homeomorphic to the connected sum* $Z \# M$.

See [Fintushel et al. 2007, Proof of Theorem 1] for details on the definition of the invariant (1-1). Fintushel and Stern [2011; 2012] constructed infinite sets as in the hypothesis of Theorem B for $Z = \mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ for $2 \leq k \leq 7$. Baykur and Sunukian [2013] showed that Fintushel and Stern's examples become diffeomorphic after a connected sum with a single copy of $S^2 \times S^2$. Examples of $\mathbb{Q}$–homology 4–spheres M that satisfy the hypothesis are spun 4–manifolds with the fundamental group of any lens space and the Poincaré homology 3–sphere. A similar result holds if (1-2) is substituted for the nontrivial bundle $S^2 \tilde{\times} S^2$. It is possible to strengthen the conclusion of Theorem B to topologically isotopic 2–spheres, although we do not pursue this endeavor here; see Sunukjian [2015].

A contribution of this note is to point out the simplicity of the proofs of Theorems A and B. The reader will notice that the 4–manifolds in the last clause of Theorem B are smoothly reducible (see [Gompf and Stipsicz 1999, Definition 10.1.17]), while those in the last clause of Theorem A are not. We explain in Remark 10 how an instance of Theorem B implies the claims on the existence of the homeomorphism of pairs and the nonexistence of the diffeomorphism of pairs of Theorem A. An independent proof of Theorem A is given in Section 2.7 as well. The following consequence of Theorem B is another contribution.

Corollary C *Let* G *be a finite cyclic group or the icosahedral group*

$$G = \langle g_1, g_2 : {g_1}^5 = (g_1 g_2)^2 = {g_2}^3 \rangle.$$

There is an infinite set of smoothly embedded 2*–spheres in* $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ *that are pairwise topologically equivalent, yet pairwise smoothly inequivalent, and the fundamental group of the complement is* G.

These are the first examples of exotic embeddings of 2–spheres in simply connected 4–manifolds whose complement has a fundamental group isomorphic to the binary icosahedral group among several other choices of groups. We exhibit interesting smooth embeddings of nullhomotopic 2–spheres in the fourth main result of this note.

Theorem D *There is an infinite set*

$$\{S_n : n \in \mathbb{Z}\} \tag{1-3}$$

of 2*–spheres smoothly embedded in* $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ *that satisfies the following properties.*

- *The fundamental group of the complement of an element in* (1-3) *is*
$$\pi_1(2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2 \setminus \nu(S_n)) = \mathbb{Z}$$
and $[S_n] = 0 \in \pi_2(2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2)$ *for every* $n \in \mathbb{Z}$.
- *There is a diffeomorphism of pairs*
$$(2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2, S_{n_1}) \to (2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2, S_{n_2})$$
if and only if $n_1 = n_2$.

Notice that elements in (1-3) do not bound a smoothly embedded 3–ball in $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$. The smoothly inequivalent embeddings of homotopically trivial 2–spheres of Theorem D are related to a construction of an infinite set of closed smooth 4–manifolds with infinite cyclic fundamental group and the homology of the connected sum $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2 \# S^1 \times S^3$, which is given in Theorem 2.

While any 2–sphere in a closed simply connected 4–manifold can be assumed to be regularly immersed, Hambleton and Kreck [1993b] and Lee and Wilczyński [1990; 1997] completely characterized when a homology class of nonzero divisibility can be represented by a locally flat embedded 2–sphere. The fifth and last result to be mentioned in this introduction records the existence of a myriad of explicit examples of locally flat embedded 2–spheres in closed simply connected 4–manifolds whose exteriors are homotopy equivalent but not homeomorphic.

Theorem E *For every* $p \geq 2$, *there is a locally flat embedded* 2*–sphere*

$$S_p \subset *\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \tag{1-4}$$

whose complement has finite cyclic group $\mathbb{Z}/p$, *and it is homotopy equivalent to the complement of a smoothly embedded* 2*–sphere*

$$S'_p \subset \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2. \tag{1-5}$$

Theorem E is essentially derived from an existence result of nonsmoothable $\mathbb{Q}$–homology 4–spheres due to Hambleton and Kreck [1993a]. Other interesting examples were found by Kasprowski, Lambert-Cole, Land and Lecuona [Kasprowski et al. 2021].

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2 Proofs

2.1 Infinitely many inequivalent smooth structures

Fintushel and Stern [2012, Theorem 1] showed that there is a nullhomologous 2–torus T smoothly embedded in $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ such that performing surgeries on T results in infinitely many inequivalent smooth structures on $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$. We point out that changing the coefficients of the torus surgery on T introduces homotopically nontrivial loops to the resulting 4–manifold, and their procedure also yields infinitely many smooth structures on 4–manifolds with prescribed cyclic fundamental group. The latter will serve as raw material to construct the knotted 2–spheres.

We introduce terminology to state the result and follow the notation in [Fintushel and Stern 2012, Section 3]. Let $T \subset X$ be a smoothly embedded 2–torus with trivial tubular neighborhood $\nu(T) = T^2 \times D^2$. Let $\{a, b\}$ be loops in T that form a symplectic basis of $\pi_1 T = \mathbb{Z}^2$, and let $\{S^1_a, S^1_b\}$ be loops in $\partial\nu(T) = T^2 \times \partial D^2 = T^2 \times S^1$ that are homologous to a and b, respectively. The meridian of T is denoted by μ_T and it is any curve in the same isotopy class of the curve $\{x\} \times \partial D^2 \subset \partial\nu(T)$. The smooth 4–manifold

$$X_{T,S^1_b}(p/n) := (X \setminus \nu(T)) \cup_\varphi (T^2 \times D^2), \tag{2-1}$$

where the gluing diffeomorphism satisfies $\varphi_*([\partial D^2]) = n[S^1_b] + p[\mu_T]$, is said to be obtained by performing a p/n–torus surgery to X on T along the curve b.

We first consider the case of finite cyclic fundamental group and postpone the infinite cyclic case to the end of the section.

Theorem 1 *Fix $p \geq 2$. There is a smoothly embedded nullhomologous 2–torus $T \subset \mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ and a nullhomologous curve in its complement $S^1_b \subset \mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \setminus \nu(T)$ such that performing a p/n–torus surgery to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ on T along S^1_b yields an infinite set*

$$\{X_{T,S^1_b}(p/n) : n \in \mathbb{Z}\} \tag{2-2}$$

of pairwise nondiffeomorphic 4–manifolds such that every element is homeomorphic to the connected sum

$$\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \# \Sigma_p, \tag{2-3}$$

where Σ_p is a $\mathbb{Q}$–homology 4–sphere with fundamental group $\pi_1 \Sigma_p = \mathbb{Z}/p$.

Proof The only contribution in this note to the work of Fintushel and Stern [2011; 2012] that provides a proof of Theorem 1 is the change in a coefficient of the torus surgery. We then employ a homeomorphism criteria of Hambleton and Kreck to pin down the homeomorphism class of the closed 4–manifolds that are constructed this way. Set $X := \mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ so to not overload the notation. Fintushel and Stern [2012, Theorem 1.1] showed the existence of a nullhomologous torus $T \subset X$ and the curve $b \subset T$

with framing $S_b^1 \subset X \setminus \nu(T)$ as in the statement of Theorem 1. Build $X_{T,S^1}(p/n)$ as in (2-1). Since $[T] = 0 \in H_2(X;\mathbb{Z})$, we have that $H_1(X_{T,S_b^1}(p/n);\mathbb{Z}) = \mathbb{Z}/p$ for every $n \in \mathbb{Z}$; in this notation, $p = 0$ corresponds to $\mathbb{Z}$. We now fix $p \geq 2$.

To see that the fundamental group of $X_{T,S_b^1}(p/n)$ is $\mathbb{Z}/p$, we take a closer look at the constructions of Fintushel and Stern [2011; 2012], where six torus surgeries along six nullhomologous 2–tori $\{T_{1,i}, T_{2,i} : i = 1,2,3\}$ are performed to X to produce a symplectic 4–manifold Q with fundamental group $\pi_1(Q) = \mathbb{Z}^6$ and that contains six Lagrangian 2–tori $\{L_{1,i}, L_{2,i} : i = 1,2,3\}$ [Fintushel and Stern 2011, Proposition 7; 2012, page 77]. The complements of these 2–tori are the same, ie

$$X \setminus \bigcup_{i=1}^{3} (\nu(T_{1,i}) \sqcup \nu(T_{2,i})) = Q \setminus \bigcup_{i=1}^{3} (\nu(L_{1,i}) \sqcup \nu(L_{2,i})). \tag{2-4}$$

By applying six surgeries to the symplectic 4–manifold Q along the Lagrangian 2–tori with a given choice of surgery curves [Fintushel and Stern 2011, Theorem 2], one obtains an infinite set of inequivalent smooth structures on X. The first five surgeries are $|1/1|$–torus surgeries, while the last one is a $1/n$–torus surgery [Fintushel and Stern 2011, page 1685]. In particular, this infinite set can be obtained by performing torus surgeries to X on six nullhomologous 2–tori. For our purposes, we perform the first five surgeries verbatim as in the proof of [Fintushel and Stern 2011, Theorem 2], but change the surgery coefficients of the sixth surgery to perform a p/n–torus surgery in order to obtain an infinite set

$$\{X_{T,S_b^1}(p/n) : n \in \mathbb{Z}\} \tag{2-5}$$

for a fixed $p \geq 2$. It follows from the Seifert–van Kampen theorem that the fundamental group is $\pi_1(X_{T,S_b^1}(p/n)) = \mathbb{Z}/p$ [Baldridge and Kirk 2009, page 321] for every $n \in \mathbb{Z}$; a detailed account on the computation of the fundamental group of the 4–manifolds obtained with such a change in the surgery coefficient can be found in several places in the literature, for example [Akhmedov and Park 2010, page 595; Baldridge and Kirk 2009, Section 5]. We have explained so far that six surgeries on six nullhomologous 2–tori in X as in [Fintushel and Stern 2011, Theorem 2] produce an infinite set (2-5) of 4–manifolds with fundamental group $\mathbb{Z}/p$.

We now appeal to the main result of Fintushel and Stern [2012, Section 8], which is that the first five surgeries on X do not change the diffeomorphism type of X and, thus, there is a single nullhomologous 2–torus $T \subset X$ along with a nullhomologous curve $S_b^1 \subset X \setminus \nu(T)$ such that a $1/n$–torus surgery produces an infinite set of smooth structures on X, as we had mentioned before [Fintushel and Stern 2012, Theorem 1.1]. Thus, we conclude that each element in the set (2-5) is obtained by performing a p/n–torus surgery on $T \subset X$ along S_b^1.

We now argue that these 4–manifolds are homeomorphic to (2-3). An inclusion-exclusion argument indicates that the Euler characteristic is unchanged under torus surgeries, ie

$$\chi(X_{T,S_b^1}(p/n)) = \chi(X) = 6. \tag{2-6}$$

Novikov additivity [Gompf and Stipsicz 1999, Remark 9.1.7] implies

$$\sigma(X_{T,S^1_b}(p/n)) = \sigma(X) = -2, \tag{2-7}$$

and we conclude that the second Stiefel–Whitney class of $X_{T,S^1_b}(p/n)$ does not vanish employing a result of Rohklin; see [Gompf and Stipsicz 1999, Theorem 1.2.29]. A classification result of Hambleton and Kreck [1993a, Theorem C] allows us to conclude that the 4–manifold $X_{T,S^1_b}(p/n)$ is homeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \# \Sigma_p$, where Σ_p is a closed smooth 4–manifold with Euler characteristic two and signature zero for every $n \in \mathbb{Z}$ and $p \geq 2$.

To argue that we have constructed infinitely many 4–manifolds that are pairwise nondiffeomorphic, we compute their Seiberg–Witten invariants using an argument well documented in the literature [Akhmedov et al. 2008; Baldridge and Kirk 2009; Fintushel et al. 2007; Fintushel and Stern 2011; 2012]. We reproduce the argument here for the sake of completeness, which requires us to describe the relation between the Seiberg–Witten invariants of the 4–manifolds $X_{T,S^1_b}(p/n)$, X and $X_{T,S^1_b}(0/1)$. Given a characteristic element $k_0 \in H_2(X_{T,S^1_b}(0/1);\mathbb{Z})$, there are unique characteristic elements $k_X \in H_2(X_{T,S^1_b}(1/0);\mathbb{Z}) = H_2(X;\mathbb{Z})$ and $k_{p/n} \in H_2(X_{T,S^1_b}(p/n);\mathbb{Z})$ [Akhmedov et al. 2008, Remark 4; Fintushel and Stern 2011, page 64]. The 4–manifolds $X_{T,S^1_b}(1/0) = X$ and $X_{T,S^1_b}(0/1)$ will have at most one basic class up to sign in our setting; cf [Akhmedov et al. 2008; Fintushel and Stern 2011; 2012]. As described in [Fintushel and Stern 2012, Section 3], the 4–manifold $X_{T,S^1_b}(0/1)$ has infinite cyclic fundamental group and it admits a symplectic structure [Fintushel and Stern 2012, Section 4]; cf [Fintushel et al. 2007, Section 3]. A result of Taubes [1994] says that the canonical class $k_0 = -c_1(X_{T,S^1_b}(0/1))$ is a basic class of $X_{T,S^1_b}(0/1)$ and $\mathrm{SW}_{X_{T,S^1_b}(0/1)}(\pm k_0) = \pm 1$. Moreover, the adjunction inequality — see [Akhmedov et al. 2008, Section 2.1] — implies that $k_0 \in \mathcal{B}$ is the only basic class up to sign.

It follows that there is a unique $k_{p/n} \in \mathcal{B}_{X_{T,S^1_b}(p/n)}$ up to sign for every $n \geq 1$, and the product formula of Morgan, Mrowka and Szabó [Morgan et al. 1997, Theorem 1.1] yields

$$\mathrm{SW}_{X_{T,S^1_b}(p/n)}(k_{p/n}) = p \cdot \mathrm{SW}_X(k_X) + n \cdot \sum_i \mathrm{SW}_{X_{T,S^1_b}(0/1)}(k_0 + i[T_0]). \tag{2-8}$$

There is a 2–torus $T_d \subset X_{T,S^1_b}(0/1)$ that is geometrically dual to the core 2–torus $T_0 \subset X_{T,S^1_b}(0/1)$ of the surgery. Along with this fact, an adjunction inequality argument implies that the sum on the right-hand side of (2-8) contains at most one nonvanishing term; see [Akhmedov et al. 2008, Section 4.1] for the argument. We have the equality

$$\mathrm{SW}_{X_{T,S^1_b}(p/n)}(k_{p/n}) = p \cdot \mathrm{SW}_X(k_X) + n \cdot \mathrm{SW}_{X_{T,S^1_b}(0/1)}(k_0) \tag{2-9}$$

and we conclude that there is an infinite set of pairwise nondiffeomorphic closed 4–manifolds (2-2). □

What is obtained when we set $p = 0$ in the statement of Theorem 1 and the previous proof, is an infinite set $\{X_{T,S^1_b}(0/n) : n \in \mathbb{Z} - \{0\}\}$ of pairwise nondiffeomorphic closed 4–manifolds with infinite cyclic fundamental group and the same homology of the connected sum $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2 \# S^1 \times S^3$; see Fintushel and Stern [2012, Theorem 1.1].

Theorem 2 *There is a smoothly embedded nullhomologous* 2*–torus* $T \subset \mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ *and a nullhomologous curve in its complement* $S_b^1 \subset \mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \setminus \nu(T)$ *such that performing a* $0/n$*–torus surgery to* $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ *on* T *along* S_b^1 *yields an infinite set*

$$\{X_{T,S_b^1}(0/n) : n \in \mathbb{Z} - \{0\}\} \tag{2-10}$$

of pairwise nondiffeomorphic 4*–manifolds with infinite cyclic fundamental group and such that every element has the homology of the connected sum*

$$2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2 \# S^1 \times S^3. \tag{2-11}$$

Similar statements to Theorems 1 and 2 for further choices of homeomorphism types of 4–manifolds with cyclic fundamental group are produced by employing other results of Fintushel and Stern [2012].

2.2 2–spheres whose complement has a prescribed fundamental group

Let X be a closed smooth 4–manifold whose fundamental group has a presentation

$$\pi_1 X = \langle g_1, \dots, g_j : r_1, \dots, r_k \rangle \tag{2-12}$$

such that adding the relation $g_1 = 1$ to it for a given generator g_1, one obtains the trivial group. Cyclic groups and the group $\langle g_1, g_2 : {g_1}^5 = (g_1 g_2)^2 = {g_2}^3 \rangle$ are examples of such groups.

Let $\alpha_1 \subset X$ be a based loop whose homotopy class is $[\alpha_1] = g_1 \in \pi_1 X$. Build the closed smooth simply connected 4–manifold

$$Z := X \setminus \nu(\alpha_1) \cup (D^2 \times S^2) \tag{2-13}$$

and consider the belt 2–sphere

$$S := \{0\} \times S^2 \subset D^2 \times S^2 \subset Z. \tag{2-14}$$

Further information is needed on the framing of the loop $\alpha_1 \subset X$ to pin down the diffeomorphism or homeomorphism type of Z. Once this is taken care of, this process provides a 2–sphere (2-14) smoothly embedded in Z and whose complement has fundamental group G. A topological construction of locally flat 2–surfaces in topological 4–manifolds is obtained by using locally flat embedded submanifolds in the surgery (2-13); see [Freedman and Quinn 1990, Section 9.3] for existence and uniqueness results on tubular neighborhoods of locally flat embedded submanifolds.

We set some notation and construct the 2–spheres of Theorem A using this procedure in the following example. It includes the choice of framing on the loop whose homotopy class generates the fundamental group.

Example 3 Fix $p \geq 2$ and an integer $n \in \mathbb{Z}$. Consider the 4–manifold $X_{T,S_b^1}(p/n)$ in the set (2-2) and let $\widehat{T} \subset X_{T,S_b^1}(p/n)$ be the core 2–torus of the surgery. Let $\alpha \subset X_{T,S_b^1}(p/n)$ be a loop such that the 4–manifold

$$Z_{n,p} := (X_{T,S_b^1}(p/n) \setminus \nu(\alpha)) \cup (D^2 \times S^2) \tag{2-15}$$

is simply connected and consider the belt 2–sphere

$$S_{n,p} := \{0\} \times S^2 \subset D^2 \times S^2 \subset Z_{n,p}. \tag{2-16}$$

Notice that the loop α lies on the boundary of $\partial\nu(\widehat{T})$. The framing on the loop α is induced by the product framing of core torus of the p/n–torus surgery. The complement of the 2–sphere (2-16) has fundamental group

$$\pi_1(Z_{n,p} \setminus \nu(S_n)) = \mathbb{Z}/p, \tag{2-17}$$

and the homology class of (2-16) satisfies $[S_{n,p}] \neq 0 \in H_2(Z_{n,p}; \mathbb{Z})$. Moreover, the 4–manifold $X_{n,p}$ is recovered by applying surgery to $Z_{n,p}$ along $S_{n,p}$.

2.3 The ambient 4–manifold of Theorems A and D

We prove in this section that the 2–spheres (2-16) of Example 3 are all smoothly embedded in $2\mathbb{CP}^2\#4\overline{\mathbb{CP}}^2$, and postpone to Section 2.7 the proof that they are pairwise smoothly inequivalent.

Proposition 4 *The 4–manifold $Z_{n,p}$ from (2-15) is diffeomorphic to the connected sum $2\mathbb{CP}^2\#4\overline{\mathbb{CP}}^2$ for every $n \in \mathbb{Z}$ and a fixed $p \geq 2$. In particular, there is an infinite set*

$$\{S_{n,p} : n \in \mathbb{Z}\} \tag{2-18}$$

of 2–spheres smoothly embedded in $Z = 2\mathbb{CP}^2\#4\overline{\mathbb{CP}}^2$ such that the complement $Z \setminus \nu(S_{n,p})$ has fundamental group $\mathbb{Z}/p$ for every $n \in \mathbb{Z}$.

Proof We use an argument due to Moishezon [1977, Lemma 13] (see also Gompf [1991, Lemma 3]) and work of Baykur and Sunukjian [2013] to establish the diffeomorphism type of our 4–manifolds. We follow the notation in [Gompf 1991, Lemma 3], fix an $n \in \mathbb{Z}$ and a $p \geq 2$, and consider the 4–manifold $X_{T,S^1_b}(p/n)$ in (2-2) that is constructed from $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ using torus surgeries and the 4–manifold $Z_{n,p}$ built in (2-15). Perform a torus surgery to $X_{T,b}(p/n)$ which identifies the loop that generates its fundamental group with the normal disk to the 2–torus to obtain a simply connected 4–manifold $\widehat{N}$; this gluing map is described on [Gompf 1991, page 101]. The latter 4–manifold can also be obtained by applying a torus surgery to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$. Moishezon's argument implies that $Z_{n,p}$ is obtained from $\widehat{N}$ by doing surgery along a loop [Gompf and Stipsicz 1999, Section 5.2], ie $Z_{n,p} = N^* = \widehat{N}\#S^2\times S^2$ [Gompf and Stipsicz 1999, Propositions 5.2.3 and 5.2.4]. Results of Baykur and Sunukjian [2013, Section 3] imply that $\widehat{N}\#S^2\times S^2$ is diffeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2\#S^2\times S^2 = 2\mathbb{CP}^2\#4\overline{\mathbb{CP}}^2$. Since the choice of n and p was arbitrary, we conclude that $Z_{n,p}$ is diffeomorphic to $2\mathbb{CP}^2\#4\overline{\mathbb{CP}}^2$ for every $n \in \mathbb{Z}$ and $p \geq 2$. □

A tweak to the proof of Proposition 4 pins down the diffeomorphism type of the 4–manifolds constructed in the proof of Theorem D.

2.4 Topological isotopy

Locally flat embeddings of 2–spheres in 4–manifolds whose complement has finite cyclic fundamental group have been studied by Lee and Wilczyński [1990] and Hambleton and Kreck [1993b, Theorem 4.5]. The next result from their work is of particular importance for our purposes.

Theorem 5 (Lee–Wilczyński, Hambleton–Kreck) *Let X be a closed simply connected topological 4–manifold such that $b_2(X) > |\sigma(X)| + 2$ and let $h \in H_2(X;\mathbb{Z})$ be a homology class of nonzero divisibility $p \neq 0$. Let $S_1, S_2 \subset X$ be locally flat embedded 2–spheres with homology classes*

$$[S_1] = [S_2] = h \in H_2(X;\mathbb{Z}),$$

and whose complement has fundamental group $\pi_1(X \setminus \nu(S_1)) = \mathbb{Z}/p = \pi_1(X \setminus \nu(S_2))$ for $p \geq 2$. If

$$b_2(X) > \max_{0 \leq j < p} |\sigma(X) - 2j(p-j)(1/p^2)h \cdot h|, \tag{2-19}$$

then there is a topological isotopy between S_1 and S_2.

Notice that our ambient 4–manifold $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ is within the range of the hypothesis of Theorem 5. Moreover, the homology class of the belt 2–sphere (2-16) of Example 3 has nonzero divisibility and self-intersection zero by construction. We conclude that the 2–spheres that were constructed in the previous sections are all topologically isotopic to each other by Theorem 5.

Corollary 6 *The infinite set $\{S_{n,p} : n \in \mathbb{Z}\}$ of Proposition 4 is made of smoothly embedded 2–spheres in $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ that are pairwise topologically isotopic.*

2.5 Some examples of $\mathbb{F}$–homology 4–spheres

Constructions of 4–manifolds that have the same $\mathbb{F}$–homology as S^4 are not scarce in the literature. For example, a surgery theory construction of $\mathbb{Q}$–homology 4–spheres with finite cyclic fundamental group is given by Hambleton and Kreck [1993a, Proposition 4.1]. Their examples include 4–manifolds with nonvanishing Kirby–Siebermann invariant and they admit no smooth structure. We describe two constructions of such objects in this section.

The first involves doing surgery on the product of a 3–manifold with a circle. Spun closed smooth 4–manifolds form a classical set of examples of 4–manifolds that share the homology of S^4 with $\mathbb{F}$–coefficients and whose fundamental group is a 3–manifold group. We briefly recall their construction and suppose that N is a closed orientable 3–manifold. A homology 4–sphere Σ_N with fundamental group $\pi_1\Sigma_N = \pi_1 N$ is constructed as

$$\Sigma_N := (N \times S^1) \setminus \nu(\{\mathrm{pt}\} \times S^1) \cup_{\mathrm{id}} (D^2 \times S^2), \tag{2-20}$$

where we use the identity map to identify the common boundary. There is another choice of framing, yet results of Plotnick [1986] state that there is a unique diffeomorphism class of (2-20) for the 3–manifolds employed in this paper. If N is an $\mathbb{F}$–homology 3–sphere, then Σ_N is an $\mathbb{F}$–homology 4–sphere. There are two principal choices of 3–manifold used in the proofs of our results:

- For $N = L(p, 1)$, we obtain a $\mathbb{Q}$–homology 4–sphere $\Sigma_{L(p,1)}$ with fundamental group
$$\pi_1 \Sigma_{L(p,1)} = \mathbb{Z}/p.$$
- For $N = \Sigma(2, 3, 5)$, we obtain a $\mathbb{Z}$–homology 4–sphere $\Sigma_{\Sigma(2,3,5)}$ with fundamental group
$$\pi_1 \Sigma_{\Sigma(2,3,5)} = \langle a, b : a^5 = (ab)^2 = b^3 \rangle.$$

A second construction of smooth $\mathbb{Q}$–homology 4–spheres with finite cyclic fundamental group is through handlebodies. Gompf and Stipsicz's [1999, Figure 5.46] depiction of a pair of orientable S^2–bundles over $\mathbb{RP}^2$ describes a handlebody of a pair of $\mathbb{Q}$–homology 4–spheres with fundamental group of order two whose second Stiefel–Whitney class can be chosen to vanish or not depending on the n–framing of one of the two 2–handles. Handlebodies of pairs of $\mathbb{Q}$–homology 4–spheres $\{\Sigma_{p,n} : n \in \{0, 1\}\}$ with fundamental group
$$\pi_1 \Sigma_{p,n} = \mathbb{Z}/p$$
for every $p \geq 2$ and second Stiefel–Whitney class
$$w_2 \Sigma_{p,n} = n$$
consisting of one 0–handle, one 1–handle, one 0–framed 2–handle, one n–framed 2–handle, one 3–handle, and one 4–handle are drawn as a straight-forward extension of the $p = 2$ case [Gompf and Stipsicz 1999, Figure 5.46].

2.6 2–spheres in simply connected 4–manifolds via $\mathbb{F}$–homology 4–spheres

The 4–manifolds of the previous section and the procedure of Section 2.2 yields knotted 2–spheres smoothly embedded in the total space of an S^2–bundle over S^2. The case of most interest for us is summarized in the following lemma.

Lemma 7 [Sato 1991, Section 3] *There is a smoothly embedded 2–sphere $S_p \hookrightarrow S^2 \times S^2$ whose complement has fundamental group $\mathbb{Z}/p$ for every $p \geq 2$.*

There is a smoothly embedded 2–sphere $S_G \hookrightarrow S^2 \times S^2$ whose complement has fundamental group $G = \langle a, b : a^5 = (ab)^2 = b^3 \rangle$ or $\mathbb{Z}/p$.

A variation of the proof of Proposition 4 yields a proof of Lemma 7 by using Moishezon's argument [1977], a lemma of Gompf [1991, Lemma 1.6] and a result of Akbulut [1999, Theorem]; cf [Tange 2014]. Another proof of Lemma 7 is obtained by using handlebodies [Akbulut 1999; 2016; Gompf and Stipsicz 1999].

2.7 Proof of Theorem A

We collect the results of previous sections into a proof of the following theorem, which is equivalent to Theorem A.

Theorem 8 *Fix $p \geq 2$. There is an infinite set*

$$\{S_{n,p} \subset 2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2 : n \in \mathbb{Z}\} \tag{2-21}$$

made of topologically isotopic 2–spheres whose complement has fundamental group $\mathbb{Z}/p$, and for which doing surgery on each element yields the infinite set (2-2) of pairwise nondiffeomorphic smooth 4–manifolds in the homeomorphism class of $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \# \Sigma_p$.

In particular, there is a diffeomorphism of pairs

$$(2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2, S_{n_1,p}) \to (2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2, S_{n_2,p}) \tag{2-22}$$

if and only if $n_1 = n_2$, and the infinite set (2-21) consists of pairwise smoothly inequivalent 2–spheres.

Proof The infinite set (2-21) was constructed in Section 2.2. The fundamental group of the complement of any 2–sphere is a prescribed finite cyclic group; see Example 3. Corollary 6 says that elements in (2-21) are pairwise topologically isotopic. As indicated in Example 3, the 4–manifold $X_{T,S^1_b}(p/n)$ in the infinite set (2-2) is obtained by carving out a tubular neighborhood $\nu(S_{n,p})$ of a 2–sphere in (2-21) from $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$, and capping off the boundary with $S^1 \times D^3$. Given that the infinite set (2-2) is made of pairwise nondiffeomorphic 4–manifolds, we conclude that the infinite set (2-21) is made of pairwise smoothly inequivalent 2–spheres. □

Remark 9 A minor modification to the previous argument yields a proof of Theorem D.

2.8 Proof of Theorem B

Let $\{Z_n : n \in \mathbb{Z}\}$ be an infinite set of pairwise nondiffeomorphic 4–manifolds in the homeomorphism class of Z. Taking a connected sum with any $\mathbb{Q}$–homology 4–sphere M yields an infinite set

$$\{Z_n \# M : n \in \mathbb{Z}\} \tag{2-23}$$

of reducible pairwise nondiffeomorphic 4–manifolds that are pairwise homeomorphic to $Z \# M$. The smooth structures are distinguished with the Seiberg–Witten invariant of the connected sums using the fact that $b_1(M) = 0 = b_2^+(M)$ and results of Kotschick, Morgan and Taubes [Kotschick et al. 1995]. By hypothesis, there is a $\mathrm{Spin}^{\mathbb{C}}$–structure on Z_n for which the Seiberg–Witten invariant SW_{Z_n} is nonzero. As explained in [Kotschick et al. 1995, Proof of Proposition 2], the $\mathrm{Spin}^{\mathbb{C}}$–structure can be extended to the connected sum $Z_n \# M$ and conclude that there is a $\mathrm{Spin}^{\mathbb{C}}$–structure for which $\mathrm{SW}_{Z_n \# M} = \mathrm{SW}_{Z_n}$. This implies that the infinite set $\{Z_n \# M : n \in \mathbb{Z}\}$ consists of pairwise nondiffeomorphic 4–manifolds that are pairwise homeomorphic to $Z \# M$.

We do surgery along the loop $\alpha \subset Z_n \# M$ as in the hypothesis of Theorem B verbatim to the procedure described in Example 3 to construct an infinite set

$$\{S_{n,\pi} : n \in \mathbb{Z}, \pi = \pi_1 M\} \tag{2-24}$$

of smoothly embedded 2–spheres in $Z\#S^2\times S^2$ whose complement has fundamental group $\pi=\pi_1 M$. By construction we obtain a homeomorphism of pairs between $(Z\#S^2\times S^2, S_{n_1,\pi})$ and $(Z\#S^2\times S^2, S_{n_2,\pi})$ for every $n_i\in\mathbb{Z}$. Surgery on the belt 2–sphere $S_{n,\pi}\subset Z\# S^2\times S^2$ gives us $Z_n\# M$ back. Since the infinite set (2-23) is made of pairwise nondiffeomorphic 4–manifolds, we conclude that there is no diffeomorphism of pairs

$$(Z\# S^2\times S^2, S_{n_1,\pi})\to(Z\# S^2\times S^2, S_{n_2,\pi}) \tag{2-25}$$

if $n_1\neq n_2$. □

Remark 10 We elaborate on an argument to prove Theorem A by using the construction procedure of Theorem B. The ingredients that satisfy the hypothesis of the latter are the following. Take the infinite set $\{Z_n : n\in\mathbb{Z}\}$ of pairwise nondiffeomorphic 4–manifolds that are homeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ that was constructed by Fintushel and Stern [2012]. These 4–manifolds have different Seiberg–Witten invariant. A result of Baykur and Sunukjian [2013] implies that $Z_n\# S^2\times S^2$ is diffeomorphic to $2\mathbb{CP}^2\#4\overline{\mathbb{CP}}^2$ for every $n\in\mathbb{Z}$. As the 4–manifold M in the statement of Theorem B, use the $\mathbb{Q}$–homology 4–sphere $\Sigma_{p,0}$ that was discussed in Section 2.6 with $\pi_1\Sigma_{p,0}=\mathbb{Z}/p$. Build the infinite set

$$\{Z_n\#\Sigma_{p,0} : n\in\mathbb{Z}\} \tag{2-26}$$

of closed reducible 4–manifolds that are homeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2\#\Sigma_{p,0}$. The set (2-26) consists of pairwise nondiffeomorphic 4–manifolds, where the diffeomorphism classes are distinguished by their Seiberg–Witten invariants [Kotschick et al. 1995, Proposition 2]. Proceed as in the proof of Theorem B and build an infinite set (2-24) of pairwise smoothly inequivalent 2–spheres. These submanifolds have the required properties by construction and they are pairwise topologically isotopic by Theorem 5.

2.9 Proof of Corollary C

We check that the hypothesis of Theorem B are met in these cases. As the infinite set $\{Z_n : n\in\mathbb{Z}\}$ we can take the infinite inequivalent smooth structures on $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ constructed by Fintushel and Stern [2012]. Work of Baykur and Sunukjian [2013, Theorem] implies that $Z_n\# S^2\times S^2=2\mathbb{CP}^2\#4\overline{\mathbb{CP}}^2$ for every $n\in\mathbb{Z}$; this connected sum is the simply connected 4–manifold in the statement of Corollary C. The $\mathbb{Q}$–homology 4–spheres with the desired fundamental group were constructed in Section 2.5; see Lemma 7. □

2.10 Proof of Theorem E

Hambleton and Kreck [1993a, Proposition 4.1] used surgery to prove the existence of a $\mathbb{Q}$–homology 4–sphere M_p with nonzero second Stiefel–Whitney class $w_2(M_p)\neq 0$, nonvanishing Kirby–Siebenmann invariant $\mathrm{KS}(M_p)\neq 0$, and fundamental group $\pi_1 M_p=\mathbb{Z}/p$ for every $p\geq 2$. Carve out the loop in M_p whose homotopy class generates the group $\pi_1 M_p=\mathbb{Z}/p$, and glue back a locally flat copy of

$D^2 \times S^2$ to obtain a simply connected 4–manifold $\widehat{M}$ with Euler characteristic $\chi(\widehat{M}) = \chi(M_p) + 2 = 4$, signature $\sigma(\widehat{M}) = \sigma(M_p)$, second Stiefel–Whitney class $w_2(\widehat{M}) \neq 0$ and Kirby–Siebenmann invariant $\mathrm{KS}(\widehat{M}) \neq 0$. A result of Freedman and Quinn [1990, Section 10.1] states that $\widehat{M}$ is homeomorphic to the connected sum $*\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ of the Chern manifold and the complex projective space with the opposite orientation for every $p \geq 2$; cf [Gompf and Stipsicz 1999, Theorem 1.2.27]. The fundamental group of the complement of the belt 2–sphere S_p of the surgery is isomorphic to $\pi_1 M_p = \mathbb{Z}/p$.

To produce the smoothly embedded 2–sphere $S'_p \subset \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ as in (1-5) and prove the last clause of Theorem E, perform surgery to the smooth $\mathbb{Q}$–homology 4–sphere $\Sigma_{p,1}$ described in Section 2.6. □

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