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Smooth structures on nonorientable 4-manifolds via twisting operations

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Abstract

Five observations compose the main results of this note. The first records the existence of a smoothly embedded 2-sphere S inside $\mathbb{R}P^2 \times S^2$ such that performing a Gluck twist on S produces a manifold Y that is homeomorphic but not diffeomorphic to the total space of the nontrivial 2-sphere bundle over the real projective plane $S(2\gamma \oplus \mathbb{R})$. The second observation is that there is a 5-dimensional cobordism with a single 2-handle between the 4-manifold Y and a mapping torus that was used by Cappell–Shaneson to construct an exotic $\mathbb{R}P^4$. This construction of Y is similar to the one of the Cappell–Shaneson homotopy 4-spheres. The third observation is that twisting an embedded real projective plane inside Y produces a manifold that is homeomorphic but not diffeomorphic to the circle sum of two copies of $\mathbb{R}P^4$. The fourth observation records new examples of pairs of homeomorphic but not diffeomorphic closed 4-manifolds with Euler characteristic one. These include the total space of the nontrivial $\mathbb{R}P^2$-bundle over $\mathbb{R}P^2$. Knotting phenomena of 2-spheres in nonorientable 4-manifolds that stands in glaring contrast with known phenomena in the orientable domain is pointed out in the fifth observation.

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1 | INTRODUCTION

Akbulut (see [4, 5] and [8, section 9.5]) showed the existence of 2-spheres

$$S \subset (S^3 \widetilde{\times} S^1)\#(S^2 \times S^2) \text{ and } S' \subset (S^3 \widetilde{\times} S^1)\#\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$$

such that performing a Gluck twist to either of them yields a 4-manifold that is homeomorphic but not diffeomorphic to the connected sum of the nontrivial 3-sphere bundle over the circle $S^3\widetilde{\times}S^1$ with a copy of $S^2 \times S^2$. The first main result of this note records such examples in 4-manifolds that are known to be irreducible. Recall that a smooth 4-manifold X is said to be irreducible if for every smooth connected sum decomposition $X = X_1\#X_2$, either X_1 or X_2 is homeomorphic to the 4-sphere [22, Definition 10.1.17].

Theorem A. *There is a smoothly embedded 2-sphere $S \subset \mathbb{R}P^2 \times S^2$ with trivial normal bundle $\nu(S) = D^2 \times S^2$ such that the complements*

$$(\mathbb{R}P^2 \times S^2) \setminus \nu(\{pt\} \times S^2) \text{ and } (\mathbb{R}P^2 \times S^2) \setminus \nu(S)$$

are not homotopy equivalent, and doing a Gluck twist on S yields a 4-manifold Y that is homeomorphic but not diffeomorphic to the nonorientable total space $S(2\gamma \oplus \mathbb{R})$ of the nontrivial 2-sphere bundle over the real projective plane.

Performing a Gluck twist on the 2-sphere $\{pt\} \times S^2 \subset \mathbb{R}P^2 \times S^2$ yields $S(2\gamma \oplus \mathbb{R})$ as explained in Subsection 2.2. The notation $S(2\gamma \oplus \mathbb{R})$ is explained in Subsection 2.3. A key ingredient in our proof of Theorem A is a result due to Akbulut [3, section 4], which we state in Theorem 16. The inequivalent smooth structure Y of Theorem A was constructed in [39, Theorem 1]; cf. Proposition 15. Our next result provides a different perspective on the production of the 4-manifold Y of Theorem A.

Theorem B. *Let M_A be the Cappell–Shaneson's nonorientable 3-torus bundle over S^1 with monodromy A as in (2.21), and let $\alpha_0 \subset M_A$ be a cross-section. Let $\alpha_0^2 \subset M_A$ be an orientation-preserving simple loop that represents the element $[\alpha_0]^2 \in \pi_1(M_A)$.*

- *Surgery on α_0^2 with the canonical framing yields $\mathbb{R}P^2 \times S^2$.*
- *Surgery on $\alpha_0^2 \subset M_A$ with the noncanonical framing yields the 4-manifold Y of Theorem A, which is homeomorphic but not diffeomorphic to $S(2\gamma \oplus \mathbb{R})$.*

  In particular, Y is a boundary component of a Pin^+-cobordism that is obtained by adding a 5-dimensional 2-handle to the product cobordism $M_A \times I$

The reader is directed toward [33] and [38, section 2] for preliminaries and background on $\mathrm{Pin}^{\pm}$-structures. To distinguish the smooth structures of Theorems A and B, we use the η-invariant and results of Stolz [38]. The main property of this spectral invariant that we exploit is that it is a complete Pin^+-cobordism invariant $\mod 2\mathbb{Z}$ (see Theorem 11).

The choice of framing of the loops in Theorem B is taken following Gompf [21, section 2]; see Remark 21. The monodromy of the mapping torus M_A is described in Subsection 2.6. The first clause of Theorem B is essentially due to Akbulut [3, section 4]; see Theorem 16. The simple construction of the 4-manifold Y in terms of adding one single 2-handle resembles the construction of the Cappell–Shaneson homotopy 4-spheres; see Remark 21. Theorem B is proven in Subsection 3.2.

The third main result exemplifies constructions of inequivalent smooth structures obtained by twisting an embedded projective plane in the 4-manifold Y of Theorem A; see Subsection 2.5 for a precise definition of this cut-and-paste operation.

Theorem C. *There is a smoothly embedded real projective plane $\mathbb{R}P^2 \hookrightarrow Y$ such that twisting it produces a 4-manifold Z that is homeomorphic but not diffeomorphic to the circle sum $\mathbb{R}P^4\#_{S^1}\mathbb{R}P^4$ of two copies of the real projective 4-space. There is a smoothly embedded real projective plane $\mathbb{R}P^2 \hookrightarrow Z$ such that twisting it produces the 4-manifold Y that is homeomorphic but not diffeomorphic to $S(2\gamma \oplus \mathbb{R})$.*

To the best of our knowledge, twisting an $\mathbb{R}P^2$ had not been previously used to construct inequivalent smooth structures. In the orientable realm, the Dolgachev surface can be obtained from an elliptic surface $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ in a similar way [6]; cf. [29]. We discuss in Subsection 3.3 how the examples of Theorem C are made of the same two compact pieces, and the isotopy class of the diffeomorphism used to identify their boundaries determines the homeomorphism type [13, 31]. Handlebodies of the 4-manifolds of Theorem C are drawn in Figure 4.

We enlarge the number of homeomorphism types of 4-manifolds that are known to support inequivalent smooth structures in our next main result.

Theorem D. *Let M be a closed 4-manifold with $\pi_1(M) = \mathbb{Z}/2 \times \mathbb{Z}/2$, Euler characteristic $\chi(M) = 1$ and such that there is an $x \in H^1(M;\mathbb{F}_2)$ with $x^4 \neq 0$. There are closed smooth homeomorphic but not diffeomorphic 4-manifolds X_1 and X_2 with the same homotopy type of M.*

Hambleton–Hillman [23] showed that the 4-manifold M of Theorem D is homotopy equivalent to either the nontrivial $\mathbb{R}P^2$-bundle over $\mathbb{R}P^2$ or to the quotient of the circle sum $\mathbb{R}P^4\#_{S^1}\mathbb{R}P^4$ by a fixed-point free involution. These 4-manifolds are also obtained by twisting an embedded $\mathbb{R}P^2$ (see Remark 22).

Next, we consider smooth 2-knots in nonorientable 4-manifolds. While embeddings of 2-spheres in orientable 4-manifolds have been widely studied in recent years, not much is known in the nonorientable case. Our next result points out further intricacies of smooth embeddings of 2-spheres inside nonorientable 4-manifolds.

Theorem E. *There are five $\{S_i : i = 0, \ldots, 4\}$ smoothly embedded 2-spheres in $(\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2$, which satisfy the following properties.*

(1) *The tubular neighborhood $\nu(S_i)$ is diffeomorphic to the D^2-bundle over S^2 with Euler number one, whose boundary is $\partial(D^2\widetilde{\times}S^2) = S^3$, for $i = 0, \ldots, 4$.*

(2)

$$S_0 = \mathbb{CP}^1 \subset (\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2 \tag{1.1}$$

and

$$[S_i] = [S_0] + [\{pt\} \times S^2] \in H_2((\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2; \mathbb{Z}) \tag{1.2}$$

for $i = 1, \ldots, 4$.

(3) *For* $i = 0, \ldots, 4$*, the complement of* S_i *has cyclic fundamental group*

$$\pi_1((\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2 \setminus \nu(S_i)) = \mathbb{Z}/2. \tag{1.3}$$

(4) *The exteriors of* $\{S_0, S_1, S_2\}$ *are pairwise nonhomotopy equivalent.*

(5) *There are homeomorphisms of pairs*

$$((\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2, S_1) \to ((\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2, S_3) \tag{1.4}$$

and

$$((\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2, S_2) \to ((\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2, S_4), \tag{1.5}$$

but no such diffeomorphisms exist.

Theorem E puts on display phenomena that do not occur in the simply connected realm. Hambleton–Kreck [24] and Lee–Wilczyński [35] showed that for simply connected 4-manifolds any pair of locally flat 2-spheres that belong to the same homology class and whose exteriors have cyclic fundamental group are topologically isotopic; cf. [28, section 5]. Example 19 describes a pair of smoothly inequivalent 2- and 3-spheres in $\mathbb{R}P^4\#\mathbb{CP}^2$ with homeomorphic complements whose existence is an immediate consequence of work of Cappell–Shaneson [12] and Akbulut [3].

The paper is organized as follows. Subsection 2.2 recalls the definition of Gluck twists, which are used in Subsection 2.3 to construct 2-sphere bundles over the real projective plane. In Subsection 2.4, we explain the circle sum operation, while Subsection 2.5 contains a description of the operation of twisting an $\mathbb{R}P^2$ and it includes a procedure to draw the handle diagrams of 4-manifolds that are obtained in this way. The raw materials employed in the construction of the smooth structures and the invariant used to distinguish them are described in Subsections 2.6 and 2.7, and a quick survey of some construction is given in Subsection 2.9. Key ingredients in the proofs of our theorems are collected in Subsections 2.8 and 2.10. The former contains a description of the 4-manifold Y, while in the latter we recall a theorem of Akbulut. Stabilizations with a copy of $\mathbb{CP}^2$ are discussed in Subsection 2.11. The proofs of our main results are found in Section 3.

2 | BACKGROUND RESULTS, TOOLS AND CONSTRUCTIONS

2.1 | Handlebody diagrams of nonorientable 4-manifolds

Foundational results concerning handlebodies of nonorientable 4-manifolds have been obtained by Akbulut [3, 8], César de Sá [13], and Miller–Naylor [36]. We refer the reader to these sources for background material, and proceed to briefly recall the conventions of Akbulut (see [3] and [8, section 1.5]) that we follow to draw handlebodies in this paper. The pair of round balls

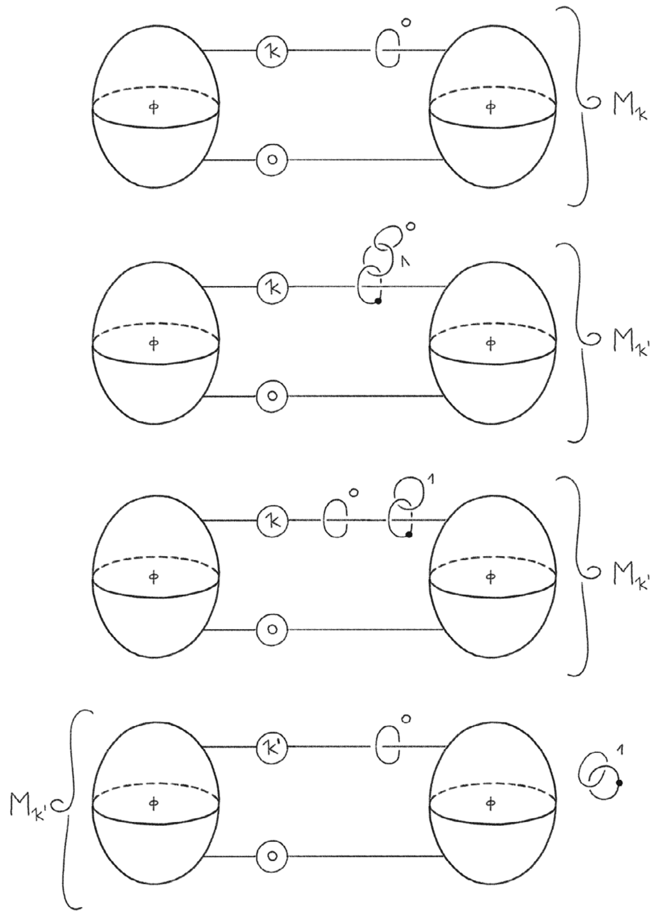

FIGURE 1 Gluck twists of Lemma 1.

representing the attaching region of the orientation-reversing 1-handles are drawn with small arcs at their centers to denote that the standard oriented 1-handle identification [22, fig. 4.15] is composed with a reflection across the plane orthogonal to the arcs [8, fig. 1.25]. In the handlebody depicted in Figure 2, one of the round balls is at ∞. In this case, the identification is the composition of such reflection with the radial map that identifies a point on the sphere centered at the origin with a point on the sphere centered at ∞ [8, fig. 1.25]. The dotted circle notation will be used for any orientable 1-handle [2, 13].

The canonical framing of a 2-handle is the blackboard framing [22, section 4.5], which depends on the chosen projection. This is the framing coming from the normal vector field to the surface of the page of the paper, and the framing of any 2-handle is taken with respect to this framing [22, p. 117]. The notation used to record the framing of the attaching circle of a 2-handle in a nonorientable handlebody depends on whether the 2-handle runs or does not run over a nonorientable 1-handle as follows [3, section 0]. When the 2-handle does not run over any nonorientable 1-handle, the framing is specified with a single integer number. For example, this is the case of the 0-framed circle in any of the handlebodies of Figure 1. If the 2-handle runs over a nonorientable 1-handle, an integer number is assigned to each arc of the attaching circle minus the nonorientable 1-handles. Following [8, section 1.5], we record the framing of such a 2-handle as a pair of circled integer numbers $(m, n) \in \mathbb{Z} \times \mathbb{Z}$. For example, in the handlebody on top of Figure 1, the framing of the 2-handle that runs over the nonorientable 1-handle is $(k, 0)$; it indicates that one adds k twists to the blackboard framing. Moreover, the sign of the framing of a 2-handle flips when going across an orientation-reversing 1-handle [3, pp. 78–79]. The 3- and 4-handles do not need to be drawn to determine the diffeomorphism type of a nonorientable 4-manifold due to a foundational result of César de Sá [13] and Miller–Naylor [36].

2.2 | Gluck twists

Let X be a smooth 4-manifold and $S \hookrightarrow X$ be a smoothly embedded 2-sphere with trivial normal bundle $\nu(S) = D^2 \times S^2$; there is a unique framing for S [22, section 4.1]. The result of performing a Gluck twist to X on S is the 4-manifold

$$X_S := (X \backslash \nu(S)) \cup_\varphi (D^2 \times S^2) \tag{2.1}$$

obtained by using a diffeomorphism $\varphi : \partial(X \backslash \nu(S)) \to \partial(D^2 \times S^2) = S^1 \times S^2$ to identify the common boundaries given by

$$\varphi(\theta, x) = (\theta, \rho_\theta(x)) \tag{2.2}$$

for $(\theta, x) \in S^1 \times S^2$. The diffeomorphism $\rho_\theta : S^2 \to S^2$ is a rotation of the 2-sphere through an angle $\theta \in \partial D^2 = S^1$ [18, section 6].

2.3 | 2-Sphere bundles over the real projective plane

There are two nonorientable total spaces of S^2-bundles over $\mathbb{R}P^2$ [26, chapter 12]. The nontrivial bundle is given as the quotient

$$S(2\gamma \oplus \mathbb{R}) = \frac{S^2 \times S^2}{(x, y) \sim (-x, \mathrm{rot}_\pi(y))}, \tag{2.3}$$

where $\mathrm{rot}_\pi : S^2 \to S^2$ is a rotation of π radians [26]. The bundle (2.3) can also be regarded as the unit sphere bundle of the rank 3 vector bundle over $\mathbb{R}P^2$ given by the Whitney sum of two copies of the canonical bundle γ with one copy of the trivial line bundle. The latter description explains the notation (2.3).

Both of these bundles are obtained by adding a copy of $D^2 \times S^2$ to

$$(\mathbb{R}P^2 \times S^2) \setminus \nu(\Sigma') = Mb \times S^2 = S(2\gamma \oplus \mathbb{R}) \setminus \nu(\Sigma), \tag{2.4}$$

where Mb is the Möbius band, $\Sigma' := \{pt\} \times S^2 \subset \mathbb{R}P^2 \times S^2$ and $\Sigma \subset S(2\gamma \oplus \mathbb{R})$ is a fiber. The diffeomorphism class of the resulting 4-manifold depends on the isotopy class of the diffeomorphism $\phi : \partial Mb \times S^2 \to \partial D^2 \times S^2 = S^1 \times S^2$ that is used to glue in $D^2 \times S^2$ to (2.4). Using $\phi = \mathrm{id}$ yields $\mathbb{R}P^2 \times S^2$, while using the diffeomorphism (2.2) yields $S(2\gamma \oplus \mathbb{R})$. The next lemma is a summary of this discussion.

Lemma 1.

- *The nontrivial bundle* $S(2\gamma \oplus \mathbb{R})$ *is obtained by performing a Gluck twist on* $\Sigma' \subset \mathbb{R}P^2 \times S^2$.
- *The trivial bundle* $\mathbb{R}P^2 \times S^2$ *is obtained by performing a Gluck twist on* $\Sigma \subset S(2\gamma \oplus \mathbb{R})$.

Proof. Let M_k be the 4-manifold given by the handlebody on the top level of Figure 1 along with a 3- and a 4-handle. We have $M_k = \mathbb{R}P^2 \times S^2$ if $k = 0 \mod 2$, and $M_k = S(2\gamma \oplus \mathbb{R})$ if $k = 1 \mod 2$. This can be seen as follows. There are two nonorientable 2-sphere bundles over the real projective plane [26, section 12.3] and both of them arise as the double of a nonorientable 2-disk bundle over $\mathbb{R}P^2$ [22, Example 4.6.3]. Handlebodies of these 2-disk bundles are obtained by removing the 0-framed 2-handle, and the 3- and 4-handles from the handlebody of M_K [3, p. 79] (cf. [36]). The 0-framed 2-handle (coming from the double construction) in the handlebody of M_K of Figure 1 represents the tubular neighborhood $\nu(S) = D^2 \times S^2$ of an embedded 2-sphere $S \subset M_k$ [22, Exercise 5.4.3(d)]. Let $M_{k'}$ be the 4-manifold obtained by doing a Gluck twist on this 2-sphere. Its handlebody is drawn on the second stage from top to bottom in Figure 1 following Akbulut–Yasui's method [11, fig. 1]. Take the 0-framed 2-handle that links the 1-framed 2-handle, and slide it under the 1-handle represented by the dotted circle to obtain the third stage of Figure 1. Sliding the k-framed 2-handle over the 1-framed 2-handle and then under the dotted circle results in the handlebody drawn at the bottom of Figure 1: the framing k' is an odd integer if and only if k is an even integer. The bottom level of Figure 1 is the handlebody of M'_k, where the Hopf link of the dotted circle and the 1-framed unknot is a cancelling pair of 1- and 2-handles. $\square$

2.4 | Circle sums

Let $\{X_i : i = 1, 2\}$ be closed 4-manifolds with $H^1(X_i; \mathbb{Z}/2) \cong \mathbb{Z}/2$ and that admit a Pin^+-structure, and let $D^3\widetilde{\times}S^1$ be the nontrivial 3-disk bundle over S^1 endowed with a fixed Pin^+-structure. There is a diffeomorphism $\nu(\gamma_i) \to D^3\widetilde{\times}S^1$, where $\nu(\gamma_i)$ is the tubular neighborhood of an orientation-reversing simple loop $\gamma_i \subset X_i$. The circle sum of X_1 and X_2 along γ_1 and γ_2 (cf. [25]) is defined as the smooth Pin^+ 4-manifold

$$X_1 \#_{S^1} X_2 := (X_1 \setminus \nu(i_1(D^3\widetilde{\times}S^1)) \cup_\varphi (X_2 \setminus \nu(i_2(D^3\widetilde{\times}S^1)) \tag{2.5}$$

where the gluing diffeomorphism is

$$\varphi := i_2|_{\partial D^3\widetilde{\times}S^1} \cdot (i_1|_{\partial D^3\widetilde{\times}S^1})^{-1} : \partial(X_1 \setminus \nu(i_1(D^3\widetilde{\times}S^1)) \to \partial(X_2 \setminus \nu(i_2(D^3\widetilde{\times}S^1)) \tag{2.6}$$

and $i_i : D^3\widetilde{\times}S^1 \to X_i$ are embeddings representing the loop generating $H^1(X_i; \mathbb{Z}/2)$ via the universal coefficient theorem for $i = 1, 2$. Moreover, we ask i_1 to preserve the Pin^+-structure and i_2 to reverse it.

Lemma 2. *Let* X_1 *and* X_2 *be two closed smooth nonorientable 4-manifolds with* $H^1(X_i; \mathbb{Z}/2) = \mathbb{Z}/2$ *that admit a* Pin^+*-structure and let* $\gamma_i \subset X_i$ *be an orientation-reversing simple loop for* $i = 1, 2$. *There is a* Pin^+*-cobordism between the disjoint union* $X_1 \sqcup X_2$ *and their circle sum (2.5).*

Proof (See [38, p. 160]). A cobordism W is obtained from $(X_1 \sqcup X_2) \times [0, 1]$ by identifying its upper lid with the boundary of $(S^2\widetilde{\times}S^1) \times [0, 1]$ as

$$(X_1 \setminus \nu(i_1(D^3\widetilde{\times}S^1)) \times \{1\}) \cup ((S^2\widetilde{\times}S^1) \times [0, 1]) \cup (X_2 \setminus \nu(i_2(D^3\widetilde{\times}S^1)) \times \{1\}) \tag{2.7}$$

using the map (2.6). A Pin^+-structure on W is obtained by gluing together the matching Pin^+-structures on the building blocks in (2.7) and it induces a Pin^+-structure on $X_1\#_{S^1}X_2$ [32, 33]. □

We point out that the circle sum of two irreducible 4-manifolds need not result in an irreducible 4-manifold.

Lemma 3. *The circle sum $S(2\gamma\oplus\mathbb{R})\#_{S^1}\mathbb{R}P^4$ is diffeomorphic to $\mathbb{R}P^4\#(S^2\times S^2)$.*

Proof. A handlebody of $\mathbb{R}P^4\#(S^2\times S^2)$ is obtained by modifying the handlebody on the third stage from top to bottom of Figure 1 as follows: erase the 0-framed circle, substitute the dotted circle with a 2-framed circle, change the framing of the linking circle from 1 to 0, and set $k = 1$. The 3-handle and the 4-handle do not need to be drawn [13, 36]. A handlebody for $S(2\gamma\oplus\mathbb{R})\#_{S^1}\mathbb{R}P^4$ is obtained by adding a $(1,0)$-framed 2-handle to the first stage from bottom to top of Figure 3. Slide the $(-1,0)$-framed 2-handle over the $(1,0)$-handle and then slide the latter over the other $(1,0)$-handle to obtain the handlebody of $\mathbb{R}P^4\#(S^2\times S^2)$. □

2.5 | Twisting an embedded $\mathbb{R}P^2$

Let X be a smooth 4-manifold and let $R\hookrightarrow X$ be a smoothly embedded real projective plane with tubular neighborhood $\nu(R)$ diffeomorphic to the nontrivial and nonorientable 2-disk bundle

$$D^2\widetilde{\times}\mathbb{R}P^2 = \frac{D^2\times S^2}{(d,y)\sim(-d,-y)} \tag{2.8}$$

for every $(d,y)\in D^2\times S^2$ [23, section 5]. The result of twisting $\nu(R)\subset X$ is the 4-manifold

$$X_R := (X\setminus\nu(R))\cup_{\varphi'}(D^2\widetilde{\times}\mathbb{R}P^2) \tag{2.9}$$

obtained by using a diffeomorphism $\varphi' : \partial(X\setminus\nu(R))\to\partial(D^2\widetilde{\times}\mathbb{R}P^2) = S^2\widetilde{\times}S^1$ to identify the common boundaries given by

$$\varphi'(x,\theta) = (\rho_\theta(x),\theta) \tag{2.10}$$

for θ in the circle base and x in the 2-sphere fiber of the 2-sphere bundle over the circle $S^2\widetilde{\times}S^1$ (see [13, section 2] and [31]).

We describe a method to draw a handlebody of the construction (2.9) by building heavily on the operation of blowing down an $\mathbb{R}P^2$ that was introduced by Akbulut in [3, pp. 79–80]. The method is described in Figure 2, where one of the round balls of the attaching region of the 1-handle is at the point of infinity and a single round ball is drawn. The handlebody of X is depicted by the diagram at the top level of Figure 2, where the union of the 1-handle and the 2-handle h with strands drawn horizontally with framing $(1,0)$ is the tubular neighborhood $\nu(R) = D^2\widetilde{\times}\mathbb{R}P^2$. We first argue that this is the only case that needs to be considered. If the 2-handle in the handle decomposition of $\nu(R)$ has framing $(-1,0)$, first rotate the round ball drawn 360° around the axis indicated by the arc [3, p. 79] to change its framing to $(0,-1)$, and then transfer the framing through the orientation-reversing 1-handle to obtain a 2-handle with framing $(1,0)$ as in the top level of Figure 2 (see [8, fig. 1.26]). Denote the 2-handles that link h by $\{K_1,\dots,K_i\}$ with their respective framings given

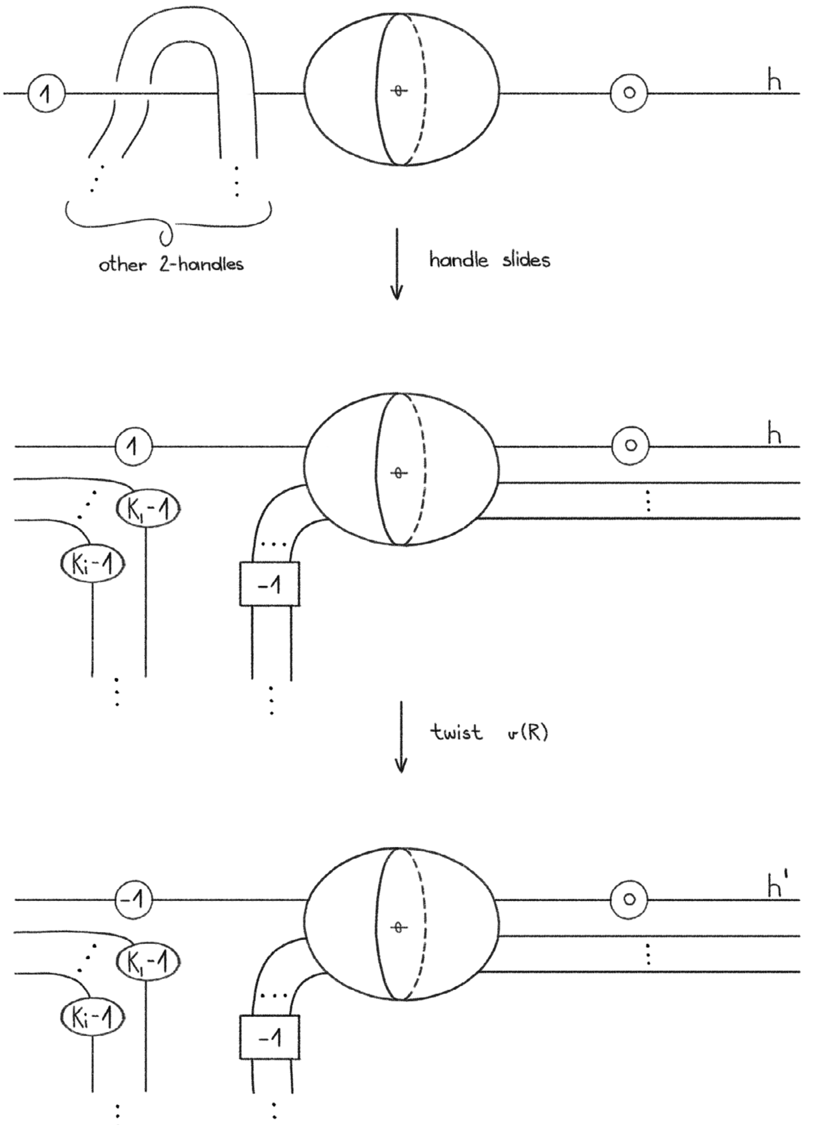

FIGURE 2 Twisting an embedded real projective plane.

by $\{k_1, \ldots, k_i\}$. Slide the attaching circles of all the 2-handles $\{K_r : r = 1, \ldots, i\}$ over h until they do not link it anymore and obtain the second level of Figure 2 (cf. [3, p. 80]). The framings are now $\{k_1 - 1, \ldots, k_i - 1\}$ and the strands have a -1 full twist as indicated in the handlebody of X drawn in the second level of Figure 2. At this point we perform the twist (2.9) on $\nu(R)$ and draw the handlebody of X_R at the bottom level of Figure 2. This is done by first blowing down the real projective plane R by erasing the 2-handle h from the handlebody of X drawn in the second level of Figure 2, and then adding a 2-handle h' with framing $(-1, 0)$ that replaces h.

So far we have described a method that produces a 4-manifold X', which is obtained by carving $\nu(R)$ out of X and gluing it back in using a self-diffeomorphism of the boundary $\partial(X \setminus \partial\nu(R)) = S^2\widetilde{\times}S^1$. To conclude that X' is diffeomorphic to (2.9), we argue as follows (cf. [11, Proof of Lemma 2.2]). A result of Kim–Raymond [31] on the diffeotopy group of $S^2\widetilde{\times}S^1$ (cf. [36, Remark

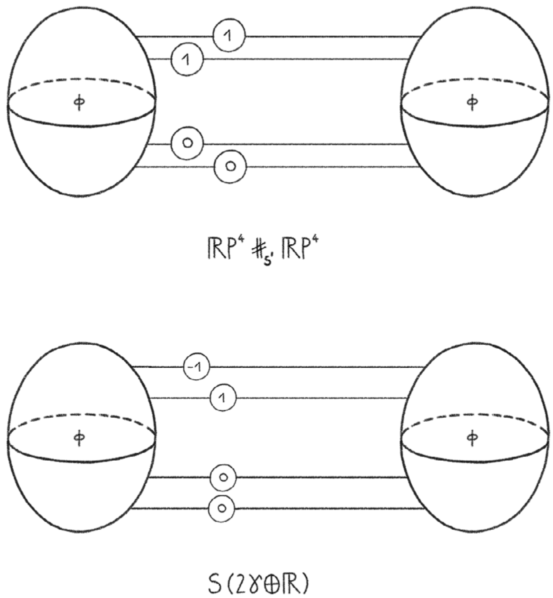

FIGURE 3 Handlebodies in Example 4.

3.12]) implies that we need to check that the gluing diffeomorphism (2.10) does not extend to a self-diffeomorphism of $D^2\widetilde{\times}\mathbb{R}P^2$. This is done in the following example.

Example 4. The homotopy equivalence classes of the double

$$S(2\gamma \oplus \mathbb{R}) = (D^2\widetilde{\times}\mathbb{R}P^2) \cup_{\mathrm{id}} (D^2\widetilde{\times}\mathbb{R}P^2) \tag{2.11}$$

and of the circle sum

$$\mathbb{R}P^4\#_{S^1}\mathbb{R}P^4 = (D^2\widetilde{\times}\mathbb{R}P^2) \cup_{\varphi'} (D^2\widetilde{\times}\mathbb{R}P^2) \tag{2.12}$$

do not coincide [30]. Handlebodies of these two nonorientable 4-manifolds are drawn in Figure 3 with an additional 3- and 4-handle in each, making visible the procedure of twisting an $\mathbb{R}P^2$.

Hambleton–Hillman [23] studied the homotopy and homeomorphism types of closed 4-manifolds whose universal cover is $S^2 \times S^2$. A representative of a homotopy equivalence class in the quadratic 2-type of PD_4-complexes with Euler characteristic one and fundamental group $\mathbb{Z}/2 \times \mathbb{Z}/2$ is a 4-manifold W, which arises as the quotient space of $\mathbb{R}P^4\#_{S^1}\mathbb{R}P^4$ by a free involution [23, Definition 7.1]. To exemplify the utility of the construction (2.9), we now remark that W can be obtained from the nontrivial bundle $\mathbb{R}P^2\widetilde{\times}\mathbb{R}P^2$ by twisting an $\mathbb{R}P^2$.

Example 5. Use the decomposition $\mathbb{R}P^2 = D^2 \cup Mb$ to deconstruct the bundle as

$$\mathbb{R}P^2\widetilde{\times}\mathbb{R}P^2 = (D^2\widetilde{\times}\mathbb{R}P^2) \cup_{\varphi} (Mb\widetilde{\times}\mathbb{R}P^2), \tag{2.13}$$

where $Mb\widetilde{\times}\mathbb{R}P^2$ is a nontrivial Möbius band bundle over $\mathbb{R}P^2$ with boundary $\partial(Mb\widetilde{\times}\mathbb{R}P^2) = S^2\widetilde{\times}S^1$. In the previous equality, there is a choice of diffeomorphism $f : \partial(Mb\widetilde{\times}\mathbb{R}P^2) \to S^2\widetilde{\times}S^1$

that is made as follows. A Pin$^+$-structure on $\mathbb{R}P^2\widetilde{\times}\mathbb{R}P^2$ induces a Pin$^+$-structure on each of the building blocks in (2.13) as codimension zero submanifolds [33, Construction 1.5]. The induced Pin$^+$-structures restrict to the same Pin$^+$-structure $\phi^{S^2\widetilde{\times}S^1}$ on each of the $S^2\widetilde{\times}S^1$ boundary components $\partial(D^2\widetilde{\times}\mathbb{R}P^2)$ and $\partial(Mb\widetilde{\times}\mathbb{R}P^2)$. The diffeomorphism f is chosen so that it preserves the bundle structure and satisfies $f^*\phi^{S^2\widetilde{\times}S^1} = \phi^{S^2\widetilde{\times}S^1}$. A result of Kim–Raymond [31] says that there are two isotopy classes of self-diffeomorphisms of $S^2\widetilde{\times}S^1$ that respect the bundle structure. Taking into account the induced Pin$^+$-structures, we conclude that the gluing diffeomorphism φ in (2.13) is isotopic to the identity map. From (2.13), we see that $S(2\gamma \oplus \mathbb{R})$ is obtained by taking a double cover (fiberwise) of $\mathbb{R}P^2\widetilde{\times}\mathbb{R}P^2$. Using a self-diffeomorphism φ' of $S^2\widetilde{\times}S^1$ that preserves the bundle structure and that is not isotopic to the identity map, we obtain

$$W = (D^2\widetilde{\times}\mathbb{R}P^2) \cup_{\varphi'} (Mb\widetilde{\times}\mathbb{R}P^2) \tag{2.14}$$

with $\mathbb{R}P^4\#_{S^1}\mathbb{R}P^4$ as a double cover.

The 4-manifolds X and X_R are $\mathbb{CP}^2$-stable in the following sense.

Lemma 6. *There is a diffeomorphism*

$$X\#\mathbb{CP}^2 \to X_R\#\mathbb{CP}^2. \tag{2.15}$$

Proof. Erase the 2-handle h from the handlebody in the middle level of Figure 2 to produce the compact 4-manifold $X_0' = X \setminus \nu(R)$. To draw a handlebody of $X\#\overline{\mathbb{CP}^2}$, add a 1-framed unknot to the middle diagram of Figure 2, and slide the 2-handle h over it. The new 2-handle h now has $(0,0)$-framing and it is linked by a 1-framed circle. This exhibits a decomposition of the connected sum as

$$X\#\overline{\mathbb{CP}^2} = X_0' \cup ((D^2\widetilde{\times}\mathbb{R}P^2)\#\overline{\mathbb{CP}^2}). \tag{2.16}$$

To see that there is a decomposition

$$X_R\#\mathbb{CP}^2 = X_0' \cup ((D^2\widetilde{\times}\mathbb{R}P^2)\#\mathbb{CP}^2), \tag{2.17}$$

draw a handlebody for this connected sum by adding a 1-framed unknot to the bottom diagram of Figure 2 and sliding the $(-1,0)$-framed 2-handle h' over it. These two handlebodies of $X\#\overline{\mathbb{CP}^2}$ and $X_R\#\mathbb{CP}^2$ differ solely on the sign of the circle that links the $(0,0)$-framed 2-handle and, in particular, the gluing maps in (2.16) and (2.17) coincide. As $M\#\mathbb{CP}^2$ is diffeomorphic to $M\#\overline{\mathbb{CP}^2}$ for every nonorientable 4-manifold M, we conclude that the diffeomorphism (2.15) exists. □

There are instances where twisting an $\mathbb{R}P^2$ does not change the diffeomorphism type of X and it is equivalent to twisting a nonorientable 1-handle (cf. [3, p. 82] and [22, fig. 5.42]). In such a scenario, X is diffeomorphic to X_R and the result of this self-diffeomorphism is to relate the Pin$^+$-structures on X as we now exemplify.

Example 7. Deconstruct the real projective 4-space as

$$\mathbb{R}P^4 = (D^2\widetilde{\times}\mathbb{R}P^2) \cup (D^3\widetilde{\times}S^1) \tag{2.18}$$

and twist the obvious $\mathbb{R}P^2$. As any self-diffeomorphism of $S^2\widetilde{\times}S^1$ extends to a self-diffeomorphism of $D^3\widetilde{\times}S^1$, twisting the $\mathbb{R}P^2$ in (2.18) does not change the diffeomorphism type. There are two Pin^+-structures $(\mathbb{R}P^4, \pm\phi_{\mathbb{R}P^4})$ and they are mutual inverses in the fourth Pin^+-cobordism group $\Omega_4^{\mathrm{Pin}^+} = \mathbb{Z}/16$ [33]. The twist of the nonorientable 1-handle gives a self-diffeomorphism of $\mathbb{R}P^4$ that maps $\pm\phi_{\mathbb{R}P^4}$ to $\mp\phi_{\mathbb{R}P^4}$. We use the following convention to distinguish the Pin^+-structures:

$$\eta(\mathbb{R}P^4, \pm\phi_{\mathbb{R}P^4}) = \pm\frac{1}{8} \mod 2\mathbb{Z}. \tag{2.19}$$

Although this cut-and-paste operation is of a local nature in the sense that $X \setminus D^2\widetilde{\times}\mathbb{R}P^2 \subset X_R$ is a codimension zero submanifold, the topology of X_R can well be drastically different to that of X as exemplified in the following lemma.

Lemma 8. *Twisting an $\mathbb{R}P^2$ need not preserve irreducibility.*

Proof. The circle sum of three copies of the real projective 4-space $\#_{S^1} 3 \cdot \mathbb{R}P^4$ contains a copy of $D^2\widetilde{\times}\mathbb{R}P^2$. Twisting this $\mathbb{R}P^2$ yields $S(2\gamma \oplus \mathbb{R})\#_{S^1}\mathbb{R}P^4$ (cf. Figure 3), which is diffeomorphic to $\mathbb{R}P^4\#(S^2 \times S^2)$ by Lemma 3. □

2.6 | Pin$^\pm$-structures on a mapping torus and framings on its cross-section

As it was done by Cappell–Shaneson in [12, section 2], consider the diffeomorphism

$$\varphi_A : T^3 \to T^3 = \mathbb{R}^3/\mathbb{Z}^3 \tag{2.20}$$

whose induced map $(\varphi_A)_* : H_1(T^3;\mathbb{Z}) \to H_1(T^3;\mathbb{Z}) = \mathbb{Z}^3$ is given by the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \in \mathrm{GL}(3;\mathbb{Z}). \tag{2.21}$$

The mapping torus

$$M_A := (T^3 \times [0,1])/\sim \tag{2.22}$$

of the diffeomorphism (2.20) under the identification $(x,0) \sim (\varphi_A(x),1)$ is the total space of a nonorientable T^3-bundle over S^1. Let ν_0 be the tubular neighborhood of the loop

$$\alpha_0 := ((\{(x_0\} \times [0,1])/\sim) \subset M_A, \tag{2.23}$$

where $\varphi_A(x_0) = x_0$ for $x_0 \in T^3$. The loop α_0 inherits a canonical framing from $\{x_0\} \times [0,1]$ (see [1] and [21, section 2]). We will call the other framing the noncanonical framing following Gompf [21, p. 1668]. The compact 4-manifold ν_0 is diffeomorphic to $D^3\widetilde{\times}S^1$ with $\partial\nu_0 = S^2\widetilde{\times}S^1$.

The 4-manifold (2.22) admits a pair of $\mathrm{Pin}^\dagger$-structures for $\dagger = +, -$ and we record some of their properties in the following proposition.

Proposition 9.

- *There is a pair of* $\mathrm{Pin}^\dagger$*-structures* $(M_A, \pm\phi^\dagger)$ *for* $\dagger = +, -$.
- *The* Pin^-*-structure* (M_A, ϕ^-) *induces the* Pin^-*-structure on* $M_A \setminus \nu_0$ *that restricts to the* Pin^-*-structure on* α_0 *that extends to the 2-disk* D^2.
- *The* Pin^+*-structure* (M_A, ϕ^+) *induces the* Pin^+*-structure on* $M_A \setminus \nu_0$ *that restricts to the* Pin^+*-structure on* α_0 *that extends to the Möbius band* Mb.

The proof of Proposition 9 follows from well-known results [33, 38]. We will write $(M_A, \phi^+) = (M_A, \phi^{M_A})$ for the Pin^+-structure on M_A in the sequel.

The circle sum $M\#_{S^1}M_A$ along an orientation-reversing simple loop in a 4-manifold M and the loop (2.23) is a potential exotic smooth structure on M. The homeomorphism type of these 4-manifolds is the same by the following result.

Theorem 10 (See Cappell–Shaneson [12] and Freedman [16]). *Let M be a closed smooth nonorientable 4-manifold and let $\alpha \subset M$ be a simple loop whose homotopy class generates $\pi_1(M) = \mathbb{Z}/2$. Let $\alpha_0 \subset M_A$ be the orientation-reversing simple loop in the mapping torus M_A that was defined in (2.23). The 4-manifolds M and $M\#_{S^1}M_A = (M \setminus \nu(\alpha)) \cup (M \setminus \nu_0)$ are homeomorphic.*

A proof of Theorem 10 is obtained by generalizing an argument due to Ruberman [37, Proof of Theorem 2]. It also follows as a corollary of work of Hambleton–Kreck–Teichner [25, Theorem 1]. We now provide a sketch of the argument for the convenience of the reader (cf. [38, p. 159]). A result of Cappell–Shaneson [12, Theorem 3.1] says that M is simple homotopy equivalent to the circle sum $M\#_{S^1}M_A$ via a simple homotopy equivalence whose normal invariant is the nontrivial element in the kernel of $[M, G/O] \to [M, G/TOP]$. Using the surgery exact sequence [40, Theorems 10.3 and 10.5]

$$\to L_5(\mathbb{Z}[\pi_1(M)], w_1(M)) \to \mathcal{S}^{TOP}(M) \to [M, G/TOP] \to \tag{2.24}$$

one concludes that there is a topological s-cobordism between these two 4-manifolds. Given that $\pi_1(M) = \mathbb{Z}/2$, the surgery group vanishes [27, section 4] and results of Freedman imply that M is homeomorphic to $M\#_{S^1}M_A$ (cf.[17, chapter 11], [16]).

2.7 | An invariant to distinguish smooth structures

We collect results of Stolz [38] that are employed in this note to distinguish two smooth structures. Let (M, ϕ^M) be a closed smooth 4-manifold with Pin^+-structure ϕ^M and let $\eta(M, \phi^M)$ be the η-invariant of the twisted Dirac operator on M [38, section 2].

Theorem 11 (Stolz [38, Propositions 4.3 and 7.3]). *Let (M, ϕ^M) be a closed smooth 4-manifold with* Pin^+*-structure* ϕ^M.

- *The invariant* $\eta(M, \phi^M) \mod 2\mathbb{Z}$ *is a complete* Pin^+*-bordism invariant.*

- *The value of the η-invariant of the mapping torus (2.22) is*

$$\eta(M_A, \pm\phi^{M_A}) = 1 \mod 2\mathbb{Z} \tag{2.25}$$

  for both Pin$^+$*-structures* $(M_A, \pm\phi^{M_A})$*.*
- *Let* $\alpha_1 \subset M_1$ *be an orientation-reversing simple loop in a closed 4-manifold* M_1 *with* Pin$^+$*-structure* ϕ^{M_1}*, which restricts to the* Pin$^+$ *structure on the submanifold* α_1 *that extends to the 2-disk. There is a* Pin$^+$*-structure* ϕ^M *on the circle sum*

$$M := M_1 \#_{S^1} M_A = (M_1 \setminus \nu(\alpha_1)) \cup (M_A \setminus \nu_0) \tag{2.26}$$

  for which the η-invariant is

$$\eta(M, \phi^M) = \eta(M_1, \phi^{M_1}) + 1 \mod 2\mathbb{Z}. \tag{2.27}$$

A consequence of Theorem 11 is the following result. We say that two smooth 4-manifolds X and Y are stably diffeomorphic if there is an $n \in \mathbb{N}$ such that the connected sums $X\#n(S^2 \times S^2)$ and $Y\#n(S^2 \times S^2)$ are diffeomorphic.

Theorem 12 (Stolz [38, Theorems B and 7.4]). *Let* $\{M_i : i = 1, 2\}$ *be closed smooth nonorientable 4-manifolds that admit a* Pin$^+$*-structure* ϕ^{M_i} *for* $i = 1, 2$*. Suppose* $H^1(M_i; \mathbb{Z}/2) = \mathbb{Z}/2$ *for* $i = 1, 2$*. If*

$$\eta(M_1, \phi^{M_1}) \neq \pm\eta(M_2, \phi^{M_2}) \mod 2\mathbb{Z}, \tag{2.28}$$

then M_1 *and* M_2 *are neither diffeomorphic nor stably diffeomorphic.*

Remark 13. Stolz showed that the condition (2.28) is an obstruction to the existence of a diffeomorphism $f : M_1 \to M_2$ such that $f^*\phi^{M_2} = \phi^{M_1}$. The hypothesis $H^1(M_i; \mathbb{Z}/2) = \mathbb{Z}/2$ of Theorem 12 implies that there are only two possible choices of Pin$^+$-structures on M_i for $i = 1, 2$ [38, Corollary 6.4] and these are mutual inverse elements in the fourth Pin$^+$-cobordism group [33, p. 190].

We now record the changes of the η-invariant under the cut-and-paste constructions discussed in Subsection 2.5. Let $X_0 = (X \setminus \nu(R)) \cup (D^3 \widetilde{\times} S^1)$ be the 4-manifold obtained by blowing down a smoothly embedded real projective plane $R \subset X$ as in [3, p. 94] and let X_R be the 4-manifold obtained by twisting it as in Subsection 2.5.

Proposition 14. *Let X be a closed smooth 4-manifold that contains a smoothly embedded real projective plane* $R \subset X$ *with tubular neighborhood* $\nu(R) = D^2 \widetilde{\times} \mathbb{R}P^2$*. Suppose there is a* Pin$^+$*-structure* (X, ϕ^X) *that restricts to a* Pin$^+$*-structure on* $(\nu(R), \phi^{\nu(R)})$ *that extends to the* Pin$^+$*-structure* $(\mathbb{R}P^4, \pm\phi^{\mathbb{R}P^4})$ *with the convention introduced in Example 7. There are* Pin$^+$*-structures* (X_R, ϕ^{X_R}) *and* (X_0, ϕ^{X_0}) *such that the identities*

$$\eta\left(X_0, \phi^{X_0}\right) = \eta(X, \phi^X) \mp \frac{1}{8} \mod 2\mathbb{Z}, \tag{2.29}$$

$$\eta\left(X_0, \phi^{X_0}\right) = \eta\left(X_R, \phi^{X_R}\right) \pm \frac{1}{8} \mod 2\mathbb{Z} \tag{2.30}$$

and

$$\eta\left(X_R, \phi^{X_R}\right) = \eta(X, \phi^X) \mp \frac{2}{8} \mod 2\mathbb{Z} \tag{2.31}$$

hold.

Proof. Assemble together (X_0, ϕ^{X_0}) and (X_R, ϕ^{X_R}) from Pin$^+$-structures on the building blocks that match on the common boundaries. The Pin$^+$-structure ϕ^X induces Pin$^+$-structures $(X \setminus \nu(R), \phi^X|)$ and $(\nu(R), \phi^{\nu(R)})$ as these are codimension zero submanifolds of X [33], and both these Pin$^+$-structures coincide on $\partial(X \setminus \nu(R)) = \partial\nu(R)$. Recall that the self-diffeomorphism (2.10) of $S^2\widetilde{\times}S^1$ identifies the two possible choices of Pin$^+$-structures (cf. [32, p. 35]). The existence of the Pin$^+$-structure (X_R, ϕ^{X_R}) follows. Moreover, either choice of Pin$^+$-structure $(D^3\widetilde{\times}S^1, \pm\phi)$ yields a Pin$^+$-structure (X_0, ϕ^{X_0}). To see that the identities (2.29) and (2.30) hold, one constructs a Pin$^+$-cobordism as in the proof of Lemma 2. Such cobordism is obtained by patching together $X_0 \times [0,1]$ and $\mathbb{R}P^4 \times [0,1]$ along $X_0 \times \{1\}$ and $\mathbb{R}P^4 \times \{1\}$. We obtain Pin$^+$ cobordisms $(V; X, X_0 \sqcup \mathbb{R}P^4)$ and $(V_R; X_R, X_0 \sqcup \mathbb{R}P^4)$ inducing different Pin$^+$-structures on $\mathbb{R}P^4$. The η-invariant is a complete Pin$^+$-cobordism invariant and it is additive with respect to disjoint unions. It follows that (2.29) and (2.30) hold. These two values imply (2.31). □

2.8 | An inequivalent smooth structure and an embedded 2-sphere

With the purpose of unveiling some of its properties, we now describe in detail the construction of the 4-manifold Y of Theorem A given in [39].

Proposition 15 (cf. [39, section 3.2]). *There is a smooth 4-manifold Y that satisfies the following properties.*

- *It is homeomorphic to $S(2\gamma \oplus \mathbb{R})$.*
- *The value of its η-invariant is*

$$\eta(Y, \phi_i^Y) = 1 \mod 2\mathbb{Z} \tag{2.32}$$

  for both Pin$^+$*-structures $\{(Y, \phi_i^Y) : i = 1, 2\}$, and Y is neither diffeomorphic nor stably diffeomorphic to $S(2\gamma \oplus \mathbb{R})$.*
- *There is a smoothly embedded 2-sphere $\Sigma \subset Y$ with trivial tubular neighborhood $\nu(\Sigma) = D^2 \times S^2$ and nontrivial homology class $[\Sigma] \neq 0 \in H_2(Y; \mathbb{Z}/2)$.*

The computation of (2.32) appeared in [39, section 3.2, Proposition 4]. We provide the argument in order to make this paper more self-contained.

Proof. The 4-manifold in the statement of the proposition is the circle sum

$$Y = S(2\gamma \oplus \mathbb{R})\#_{S^1} M_A = (S(2\gamma \oplus \mathbb{R})\backslash\nu(\alpha)) \cup (M_A\backslash\nu_0), \tag{2.33}$$

where $\alpha \subset S(2\gamma \oplus \mathbb{R})$ is a loop whose homotopy class generates the fundamental group $\pi_1(S(2\gamma \oplus \mathbb{R})) = \mathbb{Z}/2$, and $M_A \setminus \nu_0$ is the compact 4-manifold constructed in Subsection 2.6. Theorem 10 implies that Y is homeomorphic to $S(2\gamma \oplus \mathbb{R})$. There are two Pin$^+$-structures related to the η-

invariant that we need to consider as $H^1(Y;\mathbb{Z}/2) = \mathbb{Z}/2$ [38, Corollary 6.4]. We have that

$$\eta(S(2\gamma \oplus \mathbb{R}), \phi_i) = 0 \mod 2\mathbb{Z} \tag{2.34}$$

for $i = 1, 2$ given that $S(2\gamma \oplus \mathbb{R})$ is a Pin$^+$-boundary of a D^3-bundle over $\mathbb{R}P^2$. Applying Theorem 11, we determine the value

$$\eta(Y, \phi_i^Y) = 1 \mod 2\mathbb{Z} \tag{2.35}$$

for $i = 1, 2$. As the values of the η-invariants are different for all Pin$^+$-structures, we conclude that Y is neither diffeomorphic nor stably diffeomorphic to $S(2\gamma \oplus \mathbb{R})$ by Theorem 12. The 2-sphere fiber $\Sigma \hookrightarrow S(2\gamma \oplus \mathbb{R})$ has a tubular neighborhood $\nu(\Sigma) = D^2 \times S^2$ and it is disjoint from the seam of the gluing along loops (2.33) in the assembly of Y. We abuse notation and conclude that there is an embedded 2-sphere $\Sigma \subset Y$ with the desired properties. □

2.9 | Inequivalent smooth structures on nonorientable 4-manifolds

Cappell–Shaneson constructed a 4-manifold

$$Q := \mathbb{R}P^4 \#_{S^1} M_A = (D^2 \widetilde{\times} \mathbb{R}P^2) \cup (M_A \setminus \nu_0) \tag{2.36}$$

that is homeomorphic, but not diffeomorphic to $\mathbb{R}P^4$ [12, 17, 37]. Let $\gamma \subset Q$ be a loop whose homotopy class generates the group $\pi_1 Q$. A close look to Cappell–Shaneson's proof reveals their construction of a compact 4-manifold

$$N := Q \setminus \nu(\gamma) \tag{2.37}$$

that is homeomorphic but not diffeomorphic to $D^2 \widetilde{\times} \mathbb{R}P^2$ and whose boundary is $\partial N = S^2 \widetilde{\times} S^1$ [3, section 2]. A handlebody of (2.37) was drawn by Akbulut in [3, figs. 1.26 and 1.33]. The second author of this note observed in [39, Theorem 2] that (2.37) along with Theorems 10 and 12 can be used to produce an inequivalent smooth structure on a myriad of closed nonorientable 4-manifolds, which include

$$Y = S(2\gamma \oplus \mathbb{R}) \#_{S^1} M_A. \tag{2.38}$$

and

$$Z = (\mathbb{R}P^4 \#_{S^1} \mathbb{R}P^4) \#_{S^1} M_A = \mathbb{R}P^4 \#_{S^1} Q. \tag{2.39}$$

2.10 | A result of Akbulut

For nonorientable 4-manifolds with a Pin$^+$-structure, the building block (2.37) and the η-invariant are effective tools to produce inequivalent smooth structures (cf. [14, 15, 34]). The situation where there is no Pin$^+$-structure available is strikingly different as the following theorem suggests.

Theorem 16 (See Akbulut [3, section 2]). *Let $\alpha \subset \mathbb{R}P^2 \times S^2$ be a simple loop whose homotopy class generates the group $\pi_1(\mathbb{R}P^2 \times S^2)$. The 4-manifold*

$$Y' := ((\mathbb{R}P^2 \times S^2) \setminus \nu(\alpha)) \cup (M_A \setminus \nu_0) \tag{2.40}$$

is diffeomorphic to $\mathbb{R}P^2 \times S^2$.

Theorem 16 is one of the two key ingredients in the proof of Theorem A.

2.11 | Connected sums with $\mathbb{CP}^2$

The closed smooth 4-manifolds with fundamental group $\mathbb{Z}/2$ and Euler characteristic equal to two that are considered in this note compose three homeomorphism classes and five diffeomorphism classes. All these 4-manifolds become diffeomorphic under the connected sum of a single copy of the complex projective plane. The following proposition addresses the situation regarding the representatives of the three homeomorphism classes.

Proposition 17. *There are diffeomorphisms*

$$f_1 : S(2\gamma \oplus \mathbb{R})\#\mathbb{CP}^2 \to (\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2 \tag{2.41}$$

and

$$f_2 : (\mathbb{R}P^4\#_{S^1}\mathbb{R}P^4)\#\mathbb{CP}^2 \to (\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2. \tag{2.42}$$

Proof. The existence of the diffeomorphism (2.41) follows from Lemma 1 and the known property that Gluck twists come undone by taking a connected sum with a copy of $\mathbb{CP}^2$ (see [22, Exercise 5.2.7(b)] and [11, Theorem 1.1]). The existence of the diffeomorphism (2.42) follows from Example 4, Lemma 6, and (2.41). □

The following result implies that the inequivalent smooth structures that we have considered in this paper become diffeomorphic after adding a copy of $\mathbb{CP}^2$.

Theorem 18 (See Akbulut [3, Theorem 2]). *Let Q be Cappell–Shaneson's exotic real projective 4-space that was constructed in (2.36). There is a diffeomorphism between $Q\#\mathbb{CP}^2$ and $\mathbb{R}P^4\#\mathbb{CP}^2$.*

The existence of topologically equivalent and smoothly inequivalent 2- and 3-spheres in the nontrivial $\mathbb{R}P^2$-bundle over S^2 is an immediate consequence of Theorem 18 and work of Cappell–Shaneson [12].

Example 19 (Akbulut, Cappell–Shaneson). There is a pair $\{S_1^k, S_2^k\}$ of k-spheres for $k \in \{2, 3\}$ that are smoothly embedded in $\mathbb{R}P^4\#\mathbb{CP}^2$, whose complements have fundamental group of order two and such that there is a homeomorphism of pairs

$$\left(\mathbb{R}P^4\#\mathbb{CP}^2, S_1^k\right) \to \left(\mathbb{R}P^4\#\mathbb{CP}^2, S_2^k\right) \tag{2.43}$$

but no such diffeomorphism of pairs exists.

To prove Theorem 18, Akbulut shows that there is a diffeomorphism

$$N\#\mathbb{CP}^2 \to (D^2\tilde{\times}\mathbb{R}P^2)\#\mathbb{CP}^2 \tag{2.44}$$

for the compact 4-manifold (2.37). Corollary 20 follows immediately from Akbulut's Theorem 18 by noticing that both constructions (2.38) and (2.39) contain a copy of N. The latter is responsible for the inequivalent smooth structure. For example,

$$N \subset Z = ((\mathbb{R}P^4 \#_{S^1} \mathbb{R}P^4)\#_{S^1} M_A) = \mathbb{R}P^4 \#_{S^1} Q. \tag{2.45}$$

We finish this section by stating a consequence of Theorem 18 that is one of the main ingredients in the proof of Theorem E: the inequivalent smooth structures (2.38) and (2.39) are $\mathbb{CP}^2$-stable in the following sense.

Corollary 20. *There are diffeomorphisms*

$$f_3 : Y\#\mathbb{CP}^2 \to (\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2 \tag{2.46}$$

and

$$f_4 : Z\#\mathbb{CP}^2 \to (\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2. \tag{2.47}$$

3 | PROOFS OF MAIN RESULTS

3.1 | Knotted 2-spheres: Proof of Theorem A

Construct

$$Y' := (Y \setminus \nu(\Sigma)) \cup_\varphi (D^2 \times S^2) \tag{3.1}$$

by performing a Gluck twist to the 2-sphere $\Sigma \subset Y$ of Proposition 15. Let $\alpha \subset \mathbb{R}P^2 \times S^2$ be a simple loop that generates $\pi_1(\mathbb{R}P^2 \times S^2) = \mathbb{Z}/2$. Another construction of (3.1) is $Y' = ((\mathbb{R}P^2 \times S^2) \setminus \nu(\alpha)) \cup (M_A \setminus \nu_0)$, and Theorem 16 says that it is diffeomorphic to $\mathbb{R}P^2 \times S^2$. From (3.1), we see that there is a smoothly embedded 2-sphere $S \subset \mathbb{R}P^2 \times S^2$ such that one recovers Y by performing a Gluck twist on it. To conclude that the exteriors are not homotopy equivalent, we look at their fundamental groups. The exterior of the factor 2-sphere $(\mathbb{R}P^2 \times S^2) \setminus \nu(\{pt\} \times S^2)$ is $Mb \times S^2$ and $(\mathbb{R}P^2 \times S^2) \setminus \nu(S) = (Mb \times S^2)\#_{S^1} M_A$, where the circle sum is taken with matching Pin$^+$-structures. These two compact 4-manifolds have nonisomorphic fundamental groups [12, Proposition 2.4].

3.2 | Surgeries to Cappell–Shaneson's mapping torus and smooth structures: Proof of Theorem B

Let $\alpha_0^2 \subset M_A$ be the simple loop that represents the element $[\alpha_0^2] \in \pi_1(M_A)$ where $\alpha_0 \subset M_A$ is the cross-section of the mapping torus (2.22) that was described in Subsection 2.6. Carve out a tubular neighborhood $\nu(\alpha_0^2) = S^1 \times D^3$ of the loop $\alpha_0^2 \subset M_A$ and build the 4-manifold

$$Y(\psi) := (M_A \setminus \nu(\alpha_0^2)) \cup_\psi (D^2 \times S^2), \tag{3.2}$$

where $\psi = \mathrm{id}$ corresponds to the canonical framing defined in Subsection 2.6, and $\psi = \varphi$ corresponds to the noncanonical one, where the diffeomorphism φ is described in (2.2). Notice that the 4-manifold (3.2) is a boundary component of the cobordism that is obtained from adding a 2-handle to $M_A \times I$. Rephrase (3.2) as

$$Y(\psi) = (M_A \setminus \nu(\alpha_0^2)) \cup_\psi (D^2 \times S^2) = ((Mb \times S^2)\#_{S^1} M_A)) \cup_\psi (D^2 \times S^2). \tag{3.3}$$

Lemma 1 and Theorem 16 imply that $Y(\mathrm{id})$ is diffeomorphic to $\mathbb{R}P^2 \times S^2$; notice that $Y(\mathrm{id})$ is denoted by Y' in Theorem 16. The 4-manifold $Y(\varphi)$ is denoted by Y in Proposition 15 and Theorem A.

Remark 21. A Cappell–Shaneson homotopy 4-sphere is constructed by doing surgery along a cross-section of a certain 3-torus bundle over the circle M_{A^2} with monodromy A^2 for $A \in \mathrm{GL}(3;\mathbb{Z})$ [1, 12]. A myriad of these homotopy 4-spheres are known to be diffeomorphic to S^4 by work of several authors [1, 7, 9, 10, 19–21]. Theorem B addresses the corresponding situation in the nonorientable realm.

3.3 | Surgery on $\mathbb{R}P^2$ and twisted doubles: Proof of Theorem C

Deconstruct the smooth structure on $S(2\gamma \oplus \mathbb{R})$ of Proposition 15 as

$$Y = N \cup_{\mathrm{id}} (D^2 \tilde{\times} \mathbb{R}P^2), \tag{3.4}$$

where N is the 4-manifold (2.37). A tubular neighborhood $\nu(\mathbb{R}P^2) = D^2 \tilde{\times} \mathbb{R}P^2$ of a smoothly embedded $\mathbb{R}P^2 \subset Y$ is visible from (3.4). Twisting this $\mathbb{R}P^2$ yields

$$Z = N \cup_{\varphi'} (D^2 \tilde{\times} \mathbb{R}P^2). \tag{3.5}$$

It is immediate that (3.5) is homeomorphic to $\mathbb{R}P^4 \#_{S^1} \mathbb{R}P^4$. Using Theorem 12 and the additivity of the η-invariant under the operation of circle sum, we conclude that Z is neither diffeomorphic nor stably diffeomorphic to $\mathbb{R}P^4 \#_{S^1} \mathbb{R}P^4$.

3.4 | Inequivalent smooth structures on nonorientable 4-manifolds with fundamental group $\mathbb{Z}/2 \times \mathbb{Z}/2$: Proof of Theorem D

The 4-manifold M is homotopy equivalent to either $V = \mathbb{R}P^2 \tilde{\times} \mathbb{R}P^2$ or to the 4-manifold W of Example 5 by [23, Theorem 9.3]. Follow (2.13) and (2.14) to construct

$$V' = N \cup_{\mathrm{id}} (Mb \tilde{\times} \mathbb{R}P^2) \text{ and } W' = N \cup_{\varphi'} (Mb \tilde{\times} \mathbb{R}P^2),$$

where N is the 4-manifold (2.37). There are homeomorphisms $V' \to V$ and $W' \to W$ by construction. To see that these 4-manifolds are not diffeomorphic, we show that they are not Pin^+-cobordant by computing the η-invariant for all choices of Pin^+-structure [38] (cf. Theorem 12 and Remark 13). As $H^1(M;\mathbb{Z}/2) = \mathbb{Z}/2\times\mathbb{Z}/2$, there are four Pin$^+$-structures to consider. We see that

$$\eta(V,\phi_i^V) = \eta(W,\phi_i^W) = \pm\frac{1}{8} \mod 2\mathbb{Z}$$

for $i = 1,2,3,4$. These values are computed as follows. Let $\alpha\subset\mathbb{R}P^4$ be a simple loop whose homotopy class generates the group $\pi_1(\mathbb{R}P^4) = \mathbb{Z}/2$ and let $\gamma\subset\mathbb{R}P^3\tilde{\times}S^1$ be a cross-section of the nontrivial $\mathbb{R}P^3$-bundle over S^1. Glue $\mathbb{R}P^4$ and $\mathbb{R}P^3\tilde{\times}S^1$ along these loops and with matching Pin$^+$-structures to produce

$$V = (D^2\tilde{\times}\mathbb{R}P^2)\cup_{\mathrm{id}}(\mathbb{R}P^3\tilde{\times}S^1\setminus\nu(\gamma)) \text{ and } W = (D^2\tilde{\times}\mathbb{R}P^2)\cup_{\varphi'}(\mathbb{R}P^3\tilde{\times}S^1\setminus\nu(\gamma))$$

with all their possible Pin$^+$-structures [32, 33]. Lemma 2 says that there are Pin$^+$-cobordisms between (V,ϕ_i^V) or (W,ϕ_i^W) and the disjoint union $(\mathbb{R}P^4,\pm\phi^{\mathbb{R}P^4})\sqcup(\mathbb{R}P^3\tilde{\times}S^1,\phi_i^{\mathbb{R}P^3\tilde{\times}S^1})$. The additivity of the η-invariant with respect to disjoint unions implies that the values hold. Analogously, we get $\eta(V',\phi_i^{V'}) = \eta(W',\phi_i^{W'}) = \pm\frac{7}{8} \mod 2\mathbb{Z}$ for $i = 1,2,3,4$ by employing $(Q,\pm\phi^Q)$ [38].

Remark 22. Follow the assembly of N in (2.36) to construct

$$V' = (D^2\tilde{\times}\mathbb{R}P^2)\cup_{\mathrm{id}}((Mb\tilde{\times}\mathbb{R}P^2)\cup(M_A\setminus\nu_0))\setminus\nu(\alpha)$$

and

$$W' = (D^2\tilde{\times}\mathbb{R}P^2)\cup_{\varphi'}((Mb\tilde{\times}\mathbb{R}P^2)\cup M_A\setminus\nu_0))\setminus\nu(\alpha),$$

where $\alpha\subset(Mb\tilde{\times}\mathbb{R}P^2)\cup(M_A\setminus\nu_0)$ is a suitably chosen simple loop. Therefore, V' is obtained by twisting a smoothly embedded $\mathbb{R}P^2\subset W'$ and vice versa.

3.5 | Knotted 2-spheres: Proof of Theorem E

We select the five 2-spheres $\{S_i : i = 0,\dots,4\}$ as follows. Denote by S_0 the image of the canonical embedding of $\mathbb{CP}^1\hookrightarrow(\mathbb{R}P^2\times S^2)\#\mathbb{CP}^2$ for $\mathbb{CP}^1\subset\mathbb{CP}^2\setminus D^4$. Set $S_i := f_i(\mathbb{CP}^1)$ where f_i, depending on $i\in\{1,2,3,4\}$, is the corresponding diffeomorphism of Proposition 17 and Corollary 20. Item (1) and Item (3) of Theorem E follow by construction. Item (2) follows from the fact that, by examining the diffeomorphisms f_1 and f_2 at the level of handlebodies, one can explicitly see that the homology classes $[S_1]$ and $[S_2]$ are both equal to $[\mathbb{CP}^1]+[\{pt\}\times S^2]\in H_2((\mathbb{R}P^2\times S^2)\#\mathbb{CP}^2;\mathbb{Z})$. Indeed, a handlebody for $S(2\gamma\oplus\mathbb{R})\#\mathbb{CP}^2$ is obtained by adding a 1-framed unknot to the handlebody of $M_1 = S(2\gamma\oplus\mathbb{R})$ given on top of Figure 1 for $k = 1$. The 2-sphere $\mathbb{CP}^1\subset\mathbb{CP}^2\setminus D^4\subset S(2\gamma\oplus\mathbb{R})\#\mathbb{CP}^2$ arises as the union of the core of the 1-framed 2-handle and a 2-disk in S^3 bounding its attaching circle. In the following, we keep track of the image of this 2-sphere via the diffeomorphism f_1. Slide the (1,0)-framed 2-handle over the 1-framed 2-handle. Slide the 1-framed 2-handle over the 0-framed 2-handle to unlink it from the diagram. These two handleslides yield a handlebody depicted on top of Figure 1 with $k = 2$ and a 1-framed unknot, which is a handlebody for $(\mathbb{R}P^2\times S^2)\#\mathbb{CP}^2$. In particular, the 2-sphere $\mathbb{CP}^1\subset S(2\gamma\oplus\mathbb{R})\#\mathbb{CP}^2$ is mapped onto a 2-sphere that is obtained by tubing a factor 2-sphere in $\mathbb{R}P^2\times S^2$ with $\mathbb{CP}^1\subset(\mathbb{R}P^2\times S^2)\#\mathbb{CP}^2$. This shows that $[S_1] = [S_0]+[\{pt\}\times S^2]\in H_2((\mathbb{R}P^2\times S^2)\#\mathbb{CP}^2;\mathbb{Z})$. The claim about the homology class of S_2 is proven in a similar way. On the other hand, one can check that the map induced in

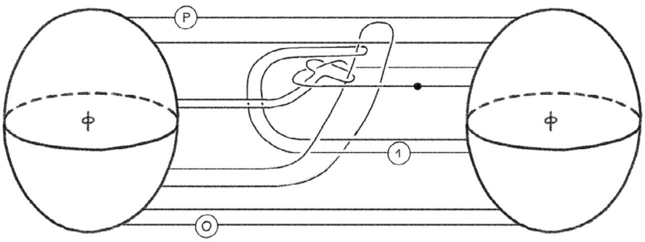

FIGURE 4 Handlebodies of the 4-manifolds Y and Z.

homology by the diffeomorphism $N\#\mathbb{CP}^2 \approx (D^2\widetilde{\times}\mathbb{R}P^2)\#\mathbb{CP}^2$ described in (2.44) sends the class of the exceptional sphere $\mathbb{CP}^1 \subset \mathbb{CP}^2$ onto itself. The decompositions (2.11) and (2.12) hence imply the statement for the homology classes $[S_3], [S_4] \in H_2((\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2; \mathbb{Z})$, by reducing to the case of $[S_1]$ and $[S_2]$, respectively. Item (4) is immediate as

$$(\mathbb{R}P^2 \times S^2) \setminus D^4, S(2\gamma \oplus \mathbb{R}) \setminus D^4 \text{ and } (\mathbb{R}P^4\#_{S^1}\mathbb{R}P^4) \setminus D^4$$

are pairwise non-homotopy equivalent. Item (5) follows from Corollary 20 and Proposition 15. Indeed, a diffeomorphism of pairs

$$((\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2, S_1) \to ((\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2, S_3)$$

would induce a diffeomorphism between the complements of the 2-spheres

$$(\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2 \setminus \nu(S_1) \to (\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2 \setminus \nu(S_3).$$

This is not possible: Theorem A says that the complements $(\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2 \setminus \nu(S_1) \approx S(2\gamma \oplus \mathbb{R}) \setminus D^4$ and $(\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2 \setminus \nu(S_3) \approx Y \setminus D^4$ are not diffeomorphic. In a similar way, the existence of a diffeomorphism of pairs $((\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2, S_2) \to ((\mathbb{R}P^2 \times S^2)\#\mathbb{CP}^2, S_4)$ would contradict Theorem C.

Handlebodies of (3.4) and (3.5) are drawn in Figure 4. The value $p = -1$ corresponds to Y, and $p = 1$ to Z. Figure 4 corrects [39, fig. 2].

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