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Toeplitz Matrices and Toeplitz Determinants under the Impetus of the Ising Model: Some History and Some Recent Results

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Toeplitz matrices and Toeplitz determinants under the impetus of
the Ising model. Some history and some recent results

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In memory of Bruria Kaufman, 1918–2010

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Acronyms: RHP (p. 4), SSLT (p. 5), OPUC (p. 14), FH (p. 29), CMV (p. 56)

1 Setting the problem

An $n \times n$ *Toeplitz matrix* T_n is a matrix with coefficients of the form $(T_n)_{jk} = c_{j-k}$, $0 \leq j, k \leq n-1$, for some given sequence $\{c_\ell\}_{\ell \in \mathbb{Z}}$. An $n \times n$ *Toeplitz determinant* $D_n(T)$ is the determinant of some given Toeplitz matrix $T = T_n$. If $\varphi = \varphi(e^{i\theta})$ is an integrable function on the unit circle $S^1 = \{z = e^{i\theta}\}$ (oriented in the positive direction) with Fourier coefficients

$$\varphi_\ell = \int_{-\pi}^{\pi} e^{-i\ell\theta} \varphi(e^{i\theta}) \frac{d\theta}{2\pi}, \quad \ell \in \mathbb{Z}$$

the Toeplitz matrices $T_n(\varphi)$ and Toeplitz determinants $D_n(\varphi)$ associated with φ are given by

$$(1) \quad T_n(\varphi) = \{\varphi_{j-k}\}_{0 \leq j, k \leq n-1}$$

and

$$(2) \quad D_n(\varphi) = \det T_n(\varphi)$$

respectively. Note that if $c_{-\ell} = \bar{c}_\ell$, then T_n is self-adjoint. This is true, in particular, if φ is real valued, and it follows that for such functions φ , $D_n(\varphi)$ and the eigenvalues $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ of $T_n(\varphi)$ are real.

Toeplitz matrices and determinants are named for Otto Toeplitz, who, in his Habilitationsschrift in 1907 (see [Toep1] [Toep2]), initiated the study of quadratic forms $\sum \varphi_{jk} x_j y_k$ with coefficients of special type $\varphi_{jk} = \varphi_{j-k}$. At the time Toeplitz was a young researcher in Göttingen where Hilbert was developing his general and abstract theory of functional analysis. Toeplitz introduced the forms $\sum \varphi_{j-k} x_j y_k$, together with the associated matrices $T_n(\varphi) = \{\varphi_{j-k}\}_{0 \leq j, k \leq n-1}$ and determinants $D_n(\varphi) = \det(T_n(\varphi))$, in order to give concrete examples of Hilbert's general theory which could be analyzed in great detail. Toeplitz called the forms $\sum \varphi_{j-k} x_j y_k$ "*L-forms*" because of their connection, via Fourier theory, with Laurent series $\sum_{-\infty}^{\infty} \varphi_k z^k$.

A striking example of the connection between Laurent series, and more broadly analytic function theory, on the one hand, and *L-forms* on the other, is given by the following problem studied by Carathéodory: For which sequences c_0, c_1, \dots, c_n of complex numbers does the polynomial

$$p(z) = c_0 + c_1 z + \dots + c_n z^n$$

have an extension to an analytic function $\varphi(z)$ in the unit disk $\mathbb{D} = \{|z| < 1\}$

$$(3) \quad \begin{aligned} \varphi(z) &= d_0 + d_1 z + \dots + d_n z^n + d_{n+1} z^{n+1} + \dots, \\ d_j &= c_j, \quad 0 \leq j \leq n \end{aligned}$$

such that $\Re \varphi(z) \geq 0$ in \mathbb{D} ? In [Car1] in 1907, Carathéodory characterized such sequences in terms of Minkowski's theory of convex bodies, but a few years later in 1911 Toeplitz [Toep3] gave the following algebraic characterization (see also [Car2], [CarFej]): $p(z) = c_0 + c_1 z + \dots + c_n z^n$ has an extension $\varphi(z)$ as above with $\Re \varphi(z) \geq 0$ in \mathbb{D} if and only if the Toeplitz matrix

$$(4) \quad \begin{pmatrix} 2c_0 & c_1 & c_2 & \dots & c_{n-1} & c_n \\ c_{-1} & 2c_0 & c_1 & \dots & & c_{n-1} \\ c_{-2} & c_{-1} & 2c_0 & & & \\ \vdots & & & & & \\ c_{-n+1} & \dots & & & & \\ c_{-n} & c_{-n+1} & \dots & & & 2c_0 \end{pmatrix}$$

with $c_{-k} \equiv \bar{c}_k$, $0 \leq k \leq n$, is non-negative definite. Over the years, up till the present, there have been many applications of this beautiful result, both theoretical and applied (see e.g. [Weg]). Many of the results and applications in this paper, however, rely on the connection between Toeplitz matrices/determinants and another problem in analytic function theory, the Riemann–Hilbert Problem (RHP) — see, in particular, [CG] [LS] and the references therein. We note that Wiener-Hopf factorization theory and the Wiener-Hopf method fall under the general rubric of RHP.

An extraordinary variety of problems in mathematics, physics and engineering, can be expressed in terms of Toeplitz matrices and determinants. This is a remarkable fact that could hardly have been anticipated in 1907, and there is a vast literature on the subject. A classical reference is the text by Grenander and Szegő¹ [GreSz] in which the authors describe applications of Toeplitz matrices to problems in analytic function theory, stationary stochastic processes and linear estimation theory in statistics. In addition to problems in probability theory and statistics, Fisher and Hartwig [FisHart1] present a long and varied list of applications of Toeplitz determinants to problems in statistical mechanics, including the ground-breaking work of Kaufman and Onsager on the Ising model. Many results in the theory of Toeplitz determinants can be found, for example, in the seminal papers of Harold Widom (see [Wid1] [Wid2] [Wid3] [Wid4], amongst many others) and in the books of Böttcher, Silbermann, and Grudsky (see [BottSilb1] [BottSilb2] [BottSilb3] [BottGr], amongst others). For a review of some more recent developments in the theory of Toeplitz determinants, see [Kr].

Closely related to Toeplitz matrices are Toeplitz operators, defined as follows (see e.g. [Doug], [BottSilb3]). Let S^1 denote the unit circle as above with Lebesgue measure $\frac{d\theta}{2\pi}$, and let $L^2(S^1)$ denote the associated Hilbert space. The Hardy space H^2 is defined as the closed subspace

$$\left\{ f \in L^2(S^1) : \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{in\theta} d\theta = 0, \quad n = 1, 2, \dots \right\}.$$

The orthogonal projection P_+ of $L^2(S^1)$ onto H^2 is given by

$$L^2(S^1) \ni f = \sum_{-\infty}^{\infty} f_k e^{ik\theta} \longmapsto P_+ f = \sum_0^{\infty} f_k e^{ik\theta} \in H^2.$$

Now let $\varphi \in L^\infty(S^1)$. Then the *Toeplitz operator* $T(\varphi) : H^2 \rightarrow H^2$ associated with φ is given by

$$(5) \quad T(\varphi)f = P_+ \varphi f, \quad f \in H^2.$$

The operator $T(\varphi)$ is bounded in H^2 , and with respect to the standard basis $\{e^{ik\theta} : k \geq 0\}$, $T(\varphi)$ is represented by the semi-infinite matrix $\{\varphi_{j-k}\}_{j,k \geq 0}$ acting in $\ell_+^2 = \{u = (u_0, u_1, \dots) : \sum_0^\infty |u_j|^2 < \infty\}$. Thus the Toeplitz matrices $T_n(\varphi)$ are *finite sections* of the Toeplitz operator $T(\varphi)$. The function φ is frequently referred to as the *symbol* of the operator $T(\varphi)$ and the sequences $\{T_n(\varphi)\}_{n \geq 1}$, $\{D_n(\varphi)\}_{n \geq 1}$.

Matrices with coefficients of the form $\{c_{j+k}\}_{0 \leq j, k \leq n-1}$ are called *Hankel matrices*. These matrices are in turn finite sections of the semi-infinite *Hankel operators* with matrix coefficients $\{c_{j+k}\}_{j, k \geq 0}$ acting in ℓ_+^2 . The theory of Hankel matrices and operators is parallel to, and closely related to the theory of Toeplitz matrices and operators (see e.g. (166) et seq. below). Operators

¹For many years “Szegő” was mis-spelled in the literature as “Szegő”. Knuth first introduced the Hungarian umlaut into TeX specifically so that names like Szegő, Erdős and König would be spelled correctly. (Private communication to the authors from D. Knuth.)

acting in $L^2(0, \infty)$ with kernels of the form $K(x - y)$, or of the form $K(x + y)$, $x, y \geq 0$, may be viewed as continuum limits of Toeplitz matrices, or Hankel matrices, and are called Wiener-Hopf and Hankel operators respectively (see e.g. [BottSilb3] [Pel]).

Our goal in this paper is not to review the above applications and developments in a systematic way, but rather to focus on **one central problem** in the theory of Toeplitz determinants, viz., the Szegő Strong Limit Theorem (SSLT) (see Theorem 3 below). We will, in particular, trace the history of this theorem and its later generalizations, in response to a succession of questions raised in the analysis of the Ising model. An earlier, and very informative, description of the influence of the Ising model on SSLT and related developments, was given by Böttcher in [Bott1].

The story begins in 1915 when Szegő [Sz1], at the very start of his career, proved the following result conjectured earlier by Polya.

Theorem 1. *Let $\varphi(e^{i\theta}) > 0$ be a continuous, positive function on the unit circle S^1 and let $D_n(\varphi)$ be the associated Toeplitz determinant. Then*

$$(6) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log D_n(\varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \varphi(e^{i\theta}) d\theta.$$

Note that $D_n(\varphi) > 0$. Indeed for $u = (u_0, u_1, \dots, u_{n-1})^T \in \mathbb{C}^n$, a simple calculation shows that

$$(u, T_n(\varphi) u) = \int_{-\pi}^{\pi} \varphi(e^{i\theta}) \left| \sum_{k=0}^{n-1} u_k e^{ik\theta} \right|^2 \frac{d\theta}{2\pi}$$

and as $\varphi(e^{i\theta}) > 0$, we conclude that $T_n(\varphi)$ is strictly positive definite, $T_n(\varphi) > 0$. Thus $D_n(\varphi) = \det T_n(\varphi) > 0$. In addition, the eigenvalues $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ of $T_n(\varphi)$ are positive and so (6) can be rewritten in the form

$$(7) \quad \lim_{n \rightarrow \infty} \frac{\log \lambda_1^{(n)} + \dots + \log \lambda_n^{(n)}}{n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \varphi(e^{i\theta}) d\theta.$$

Szegő recognized that (7) is a special case of equidistribution for the $\lambda_j^{(n)}$ in the sense of Weyl, and in 1920 [Sz2] Szegő proved the following result (see also [GreSz]).

Theorem 2. *Let $\varphi(e^{i\theta})$ be a real-valued function in $L^\infty(S^1)$ with $m \leq \varphi(e^{i\theta}) \leq M$ a.e. If $F(\lambda)$ is any continuous function defined on the interval $m \leq \lambda \leq M$, we have*

$$(8) \quad \lim_{n \rightarrow \infty} \frac{F(\lambda_1^{(n)}) + \dots + F(\lambda_n^{(n)})}{n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\varphi(e^{i\theta})) d\theta.$$

Taking $F(\lambda) = \log \lambda$, we clearly recover (7). We consider more detailed asymptotic properties of the eigenvalues below (see Section 9).

Relation (6) may be rewritten in the form

$$(9) \quad D_n(\varphi) = \exp \left[n \int_{-\pi}^{\pi} \log \varphi(e^{i\theta}) \frac{d\theta}{2\pi} + o(n) \right]$$

as $n \rightarrow \infty$. The issue of determining the precise nature of the term $o(n)$ came to the fore in the following way.

The two-dimensional Ising model, the central model in statistical mechanics (see [McWu1] [Pal] [Lie] [Dom1]), named for E. Ising (see [Is]), concerns the interaction of random spins $\sigma_{i,j} = \pm 1$ at sites (i, j) in \mathbb{Z}^2 . Of great interest is the situation in which only nearest-neighbor spins interact and the interaction energy is given by

$$-J_1 \sigma_{i,j} \sigma_{i,j+1} - J_2 \sigma_{i,j} \sigma_{i+1,j}$$

where the vertical and horizontal interaction constants, J_1 and J_2 respectively, are translation invariant, and in addition, J_1 and J_2 are positive, so that the system is *ferromagnetic*, i.e. parallel spins have lower energy than anti-parallel spins. As is standard in statistical mechanics, one analyzes the infinite system on \mathbb{Z}^2 by first considering the spins in a finite rectangular box $\Lambda \subset \mathbb{Z}^2$ of size $M \times N$ (see below for a discussion of boundary conditions) and then letting $M, N \rightarrow \infty$. For such a box the total interaction energy of the spins $\sigma = (\sigma_{i,j})$ is given by

$$(10) \quad E_\Lambda(\sigma) = - \sum_{\Lambda} [J_1 \sigma_{i,j} \sigma_{i,j+1} + J_2 \sigma_{i,j} \sigma_{i+1,j}]$$

where $J_1 \sigma_{i,j} \sigma_{i,j+1}$ is included in the sum if (i, j) or $(i, j + 1) \in \Lambda$ and $J_2 \sigma_{i,j} \sigma_{i+1,j}$ is included if (i, j) or $(i + 1, j) \in \Lambda$. The associated normalized *Gibbs measure* is then

$$(11) \quad Pr_\Lambda(\sigma) = \frac{1}{Z_\Lambda} e^{-E_\Lambda(\sigma)/k_B T}$$

where k_B is Boltzmann's constant and T is the temperature. Here the *partition function* Z_Λ is

$$(12) \quad Z_\Lambda = \sum_{\sigma} e^{-E_\Lambda(\sigma)/k_B T}$$

where the sum is taken over all possible spin configurations σ in Λ . For fixed finite sets $A \subset \Lambda$, one defines the *correlation functions*

$$(13) \quad \left\langle \prod_{(i,j) \in A} \sigma_{i,j} \right\rangle_{\Lambda} = \sum_{\sigma} \prod_{(i,j) \in A} \sigma_{i,j} Pr_\Lambda(\sigma).$$

The *thermodynamic limits* of these correlation functions as $\Lambda \uparrow \mathbb{Z}^2$, i.e. $M, N \rightarrow \infty$,

$$(14) \quad \left\langle \prod_{(i,j) \in A} \sigma_{i,j} \right\rangle \equiv \lim_{\Lambda \uparrow \mathbb{Z}^2} \left\langle \prod_{(i,j) \in A} \sigma_{i,j} \right\rangle_{\Lambda}$$

are the objects of principal physical interest. In particular such correlations $\left\langle \prod_{(i,j) \in A} \sigma_{i,j} \right\rangle$ should model physical phenomena such as phase transitions. For example, a bar magnet has a critical temperature T_c called the Curie point: for temperatures $T < T_c$, the magnet exhibits spontaneous magnetization, but for $T > T_c$ the magnetization is zero (in zero external field). This behavior cannot be described by a function such as $\left\langle \prod_{(i,j) \in A} \sigma_{i,j} \right\rangle_{\Lambda}$ in (13), which is real analytic for $T > 0$: in the thermodynamic limit, however, analyticity can be, and indeed is, destroyed.

In the presence of an external magnetic field, one must add a term of the form $h \sum_{(i,j) \in \Lambda} \sigma_{i,j}$ to $E_\Lambda(\sigma)$, $h \in \mathbb{R}$, but the most complete results have been obtained only when the field is absent, and

we will always assume $h = 0$. It remains an outstanding problem in the theory of the Ising model to find the analogs of the explicit formulae below for the correlation functions in the case $h \neq 0$.

We note the following: In order to compute $E_\Lambda(\sigma)$ in (10), we need to know $\sigma_{i,j}$ not only at the points $(i, j) \in \Lambda$, but also at the points adjacent to Λ . In particular, if M and N are integers and $\Lambda = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i \leq M, 0 \leq j \leq N\}$, then we need to know $\sigma_{M+1,j}$, $\sigma_{-1,j}$, $0 \leq j \leq N$ and $\sigma_{i,N+1}$, $\sigma_{i,-1}$, $0 \leq i \leq M$. In practice, this issue is addressed by imposing boundary conditions on Λ . One common choice is, for example, periodic boundary conditions, $\sigma_{i+M+1,j} = \sigma_{i,j}$, $\sigma_{i,j+N+1} = \sigma_{i,j}$. Another choice is the fully magnetized boundary conditions where $\sigma_{i,j} = +1$ (alternatively $\sigma_{i,j} = -1$) at all points (i, j) adjacent to Λ . General boundary conditions are discussed, for example, in [Gal], and also from a different point of view, in [LebM-Lof]. For each choice of boundary conditions, provided the thermodynamic limits in (14) exist, we obtain a family of functions

$$(15) \quad \langle \sigma_{i,j} \rangle, \quad \langle \sigma_{i,j} \sigma_{i',j'} \rangle, \quad \dots$$

In the spirit of the classical moment problem, one says that the family in (15) determines a (Gibbs) equilibrium state (see e.g. [Sim1] [Ru]). A priori, different boundary conditions can give rise to different families (15), and hence to different equilibrium states. For example (cf [Pal]), for closed boundary conditions $\sigma_{ij} = +1$ adjacent to Λ , one finds $\langle \sigma_{0,0} \rangle > 0$, but for periodic boundary conditions, by symmetry, $\langle \sigma_{0,0} \rangle = 0$. Nevertheless, it suffices for our purposes to note that for the two-point correlation function that we discuss below, the thermodynamic limit is independent of the boundary conditions (cf [BGJ-LS]). The same is true for the free energy F per unit spin in (17) below.

In 1 dimension, Ising [Is] showed that the Ising model does not exhibit a phase transition for any temperature $T > 0$. What about 2 or 3 dimensions? In a landmark paper in 1936, Peierls [Pei] gave a simple argument asserting that in 2 or 3 dimensions, the Ising model does indeed exhibit spontaneous magnetization at some temperature $T_c > 0$. It turns out that Peierls' argument involved an incorrect step², which was corrected only many years later in 1964 by Griffiths [Grif]. Griffiths confirmed Peierls' conclusion and "Peierls argument" remains a standard tool in many statistical situations. The first exact quantitative result for the 2-dimensional Ising model was obtained by Kramers and Wannier in 1941 [KraWan] when they wrote down the following formula for T_c . In the case $J_1 = J_2 = J$,

$$(16) \quad \sinh\left(\frac{2J}{k_B T_c}\right) = 1.$$

In 1942, Wannier gave a talk on this work at a meeting of the New York Academy of Sciences. In an extraordinary denouement, at the end of the talk Onsager announced that he had obtained an exact solution for the two-dimensional Ising model (without magnetic field). Onsager³ published the proof of his result two years later in [Ons1]. More precisely, Onsager had obtained a formula for the partition function Z_Λ , and, in particular, in the case of a square lattice Λ_N of size $N \times N$, with $J_1 = J_2 = J$, he showed that in the thermodynamic limit the free energy per unit spin F

²First discovered by M. Fisher (see [Grif]).

³All of Onsager's published papers, including his PhD dissertation, can be found in "The Collected Works of Lars Onsager, Eds. P. C. Hemmer, H. Holden and S. K. Rattkje, World Sci. Ser. in 20'th Century Phys., Vol. 17, World Scientific, Singapore, 1996". An extensive collection of Onsager's hand-written notes, unpublished manuscripts, etc., can be found on the Onsager archive at Trondheim, <http://www.ntnu.no/ub/spesialsamlingene/tekark/tek5/arkiv5.php>: much of this material was scanned from the "Guide to the Lars Onsager Papers MS794" at the Yale library.

exists and is given by

$$\begin{aligned}
 (17) \quad F &= -k_B T \lim_{N \rightarrow \infty} \frac{1}{N^2} \log Z_{\Lambda_N} \\
 &= -k_B T \left[\log \left(2 \cosh \frac{2J}{k_B T} \right) + \frac{1}{2\pi^2} \int_0^\pi d\phi_1 \int_0^\pi d\phi_2 \log \left(1 - \frac{\kappa}{2} (\cos \phi_1 + \cos \phi_2) \right) \right] \\
 &= -k_B T \left[\log \left(2 \cosh \frac{2J}{k_B T} \right) + \frac{1}{2\pi} \int_0^\pi \log \frac{1}{2} \left(1 + \sqrt{1 - \kappa^2 \sin^2 \phi} \right) d\phi \right]
 \end{aligned}$$

where $\kappa = 2 \sinh(2J/k_B T) / \cosh^2(2J/k_B T)$. It follows from this formula that the specific heat $C = -T \frac{\partial^2 F}{\partial T^2}$ diverges logarithmically as $T \rightarrow T_c \pm 0$, where T_c is precisely the critical temperature obtained by Kramers and Wannier (note that for T_c we have $\kappa = 1$, and the argument of the logarithm in the double integral vanishes at the integration limit $\phi_1 = \phi_2 = 0$). Onsager's analysis allows for different interaction constants $J_1 \neq J_2$, in which case equation (16) for T_c must be replaced by

$$(18) \quad \sinh \left(\frac{2J_1}{k_B T_c} \right) \sinh \left(\frac{2J_2}{k_B T_c} \right) = 1.$$

For the description of the events that followed, we refer to the informative, and entertaining, recent papers by Baxter [Bax] and McCoy [McC1]. In the mid-1940's, Kaufman, a doctoral student at Columbia University, began working with Onsager, who was then at Yale, on the 2-D Ising model. Onsager's computation of the partition function had stunned the physics community by its ingenuity and complexity, and Kaufman was determined to find a simpler approach. As Onsager relates [Ons2], Kaufman "...decided to look for a possible connection with spinor theory. Why not? In fact, it seemed like very good sense, and so it was ... By the summer of 1946 she had a beautifully compact computation of the partition function, bypassing all tedious details."

The methods of Onsager and Kaufman for the partition function are based on the calculation of the eigenvalues of the transfer matrix [KraWan] associated with the problem. Another simplified solution based on this approach was given in 1964 by Schultz, Mattis, and Lieb [ScML]. On the other hand, it was observed in 1941 by van der Waerden [vanderW] that the calculation of the Ising partition function is equivalent to the problem of counting closed polygons on the square lattice. This observation was the origin of an alternative (and more direct) combinatorial approach to the Ising model. A solution using such an approach was given by Kac and Ward in 1952 [KacW] (see also [PotWar] where the calculation of the correlation functions we discuss below is addressed by this method). Kac and Ward did not intend to present a rigorous derivation; the missing proofs were given later by Sherman [Sher] and Burgoyne [Bur] following a conjecture by Feynman. A rigorous combinatorial solution by a related method was given by Hurst and Green [HurGr]. Another combinatorial solution was obtained by Kasteleyn [Kas1]. A useful modification of Kasteleyn's approach is due to Fisher [Fis3] (see [McWu1]). One more simple solution based on the method of Kac and Ward was given by Vdovichenko [Vdov] (see [LanLif]) although [Vdov] omits the discussion of the boundary conditions. In [Kas1], Kasteleyn maps the Ising problem onto the dimer covering problem (i.e. the problem of counting all possible coverings by dimers) on a special lattice, slightly more complex than the square lattice. It is interesting to note that the dimer problem on the square lattice, which thus can be regarded as a simpler version of the Ising problem, was solved by Kasteleyn himself [Kas2] and, independently, by Temperley and Fisher [TemFis] in 1961 (see also [LieLoss] for a later alternative, particularly simple, solution of the dimer problem on the square lattice).

We now return to 1946. Kaufman’s method and results were eventually published three years later in [Kau1]. Onsager continues in [Ons2]: “By itself that was only a more elegant derivation of an old result; but the approach looked powerful enough to produce a few new ones. Very well, how about correlations?” As noted by Baxter, the history of the Ising model from that time forth has been the study of these correlations.

In 1948, at a conference at Cornell, and again in 1949 at a conference in Florence, Onsager astounded the audience by announcing that he and Kaufman had recently obtained an exact formula for the spontaneous magnetization M_0 of the 2-D Ising model. He gave the result as

$$(19) \quad M_0 = (1 - k_{\text{ons}}^2)^{1/8}, \quad k_{\text{ons}} \equiv \left(\sinh \frac{2J_1}{k_B T} \sinh \frac{2J_2}{k_B T} \right)^{-1}$$

for $0 < k_{\text{ons}} < 1$, corresponding to $T < T_c$. And for $k_{\text{ons}} > 1$, corresponding to $T > T_c$, $M_0 = 0$.

Onsager and Kaufman never published a proof of this result. The situation was as follows. In 1949 they published their famous paper [KauOns], based on Kaufman’s spinor (free-fermion) approach, in which they found, in particular, an explicit expression (see (21) below) for the two-point correlation function along a row in the thermodynamic limit, i.e. $\langle \sigma_{1,1} \sigma_{1,1+n} \rangle$. By physical arguments

$$(20) \quad M_0 = \lim_{n \rightarrow \infty} \langle \sigma_{1,1} \sigma_{1,1+n} \rangle^{\frac{1}{2}}$$

so that in order to derive (19), what they had to do was to control the limit in (20). As Onsager describes the situation [Ons3]: “And then, finally, it became apparent that we had a last problem, of the degree of the order, and that turned out to need more invention.”

2 Toeplitz meets Ising

What precisely was this “last problem” that Kaufman and Onsager had to face in order to compute the limit in (20)? It is here that we make contact with the theory of Toeplitz matrices. In [KauOns] Kaufman and Onsager expressed $\langle \sigma_{1,1} \sigma_{1,1+n} \rangle$ as a sum of two Toeplitz determinants as follows (in the notation of [MPW]):

$$(21) \quad \begin{aligned} (-1)^n \langle \sigma_{1,1} \sigma_{1,1+n} \rangle &= c_2^{*2} \det \begin{pmatrix} b_{-1} & b_{-2} & b_{-3} & \dots & b_{-n} \\ b_0 & b_{-1} & b_{-2} & \dots & b_{1-n} \\ b_1 & b_0 & b_{-1} & \dots & b_{2-n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n-2} & b_{n-3} & b_{n-4} & \dots & b_{-1} \end{pmatrix} \\ &- s_2^{*2} \det \begin{pmatrix} b_1 & b_2 & b_3 & \dots & b_n \\ b_0 & b_1 & b_2 & \dots & b_{n-1} \\ b_{-1} & b_0 & b_1 & \dots & b_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{2-n} & b_{3-n} & b_{4-n} & & b_1 \end{pmatrix} \end{aligned}$$

where

$$(22) \quad c_2^* = \cosh K_2^*, \quad s_2^* = \sinh K_2^*, \quad e^{-2K_2^*} = \tanh \frac{J_2}{k_B T}$$

$$(23) \quad b_j = \frac{1}{\pi} \int_0^\pi \cos(\hat{\delta}(\theta) - j\theta) d\theta$$

$$(24) \quad \cos \hat{\delta}(\theta) = \sin(\delta^*(\theta)) \sin \theta \cosh 2K_2^* - \cos(\delta^*(\theta)) \cos \theta$$

and $\varphi_{\text{ons}}(e^{i\theta}) = e^{i\delta^*(\theta)}$ is a function introduced previously by Onsager in [Ons1]

$$(25) \quad \varphi_{\text{ons}}(e^{i\theta}) = \left[\left(\frac{1 - \gamma_1 e^{i\theta}}{1 - \gamma_1 e^{-i\theta}} \right) \left(\frac{1 - \gamma_2 e^{-i\theta}}{1 - \gamma_2 e^{i\theta}} \right) \right]^{\frac{1}{2}}$$

$$(26) \quad \begin{aligned} &\text{where} \quad \gamma_1 = z_1 z_2^*, \quad \gamma_2 = z_2^*/z_1 \\ &z_1 = \tanh \frac{J_1}{k_B T}, \quad z_2 = \tanh \frac{J_2}{k_B T}, \quad z_2^* = \frac{1 - z_2}{1 + z_2}, \end{aligned}$$

The branch in (25) is chosen so that $\varphi_{\text{ons}}(e^{i\pi}) = e^{i\delta^*(\pi)} > 0$.

Remark 2.1. As noted in [MPW], there is a sign error in formula (21) in [KauOns].

So the problem that Kaufman and Onsager had to face to compute M_0 via (20), was the purely mathematical problem of computing the asymptotics of $n \times n$ Toeplitz determinants as $n \rightarrow \infty$. At the time, however, all that was known was Szegő's result (9) with unknown error term $o(n)$. But to compute (20), this error term is precisely what one needs to know!

Note that

$$(27) \quad 0 < \gamma_1 < 1 \quad \text{and} \quad \gamma_1 < \gamma_2.$$

If $\gamma_2 \neq 1$, we see that

$$(28) \quad \varphi_{\text{ons}}(e^{i\theta}) = \frac{(1 - \gamma_1 e^{i\theta})(1 - \gamma_2 e^{-i\theta})}{|(1 - \gamma_1 e^{-i\theta})(1 - \gamma_2 e^{i\theta})|}$$

and hence $\varphi_{\text{ons}}(e^{i\theta})$ is a smooth, single-valued function on the unit circle S^1 . On the other hand, if $\gamma_2 = 1$, then

$$(29) \quad \varphi_{\text{ons}}(e^{i\theta}) = i e^{-i\theta/2} \frac{(1 - \gamma_1 e^{i\theta})}{|1 - \gamma_1 e^{-i\theta}|}, \quad 0 < \theta < 2\pi.$$

Simple algebra shows that for $\hat{\chi}_i = J_i/k_B T$, $i = 1, 2$,

$$\gamma_2 \geq 1 \iff \sinh(\hat{\chi}_1 + \hat{\chi}_2) - \cosh(\hat{\chi}_1 - \hat{\chi}_2) \leq 0$$

and

$$\sinh^2(\hat{\chi}_1 + \hat{\chi}_2) - \cosh^2(\hat{\chi}_1 - \hat{\chi}_2) = k_{\text{ons}}^{-1} - 1$$

where $k_{\text{ons}} = (\sinh 2\hat{\chi}_1 \sinh 2\hat{\chi}_2)^{-1}$ as in (19). Thus

$$(30) \quad \gamma_2 \geq 1 \iff k_{\text{ons}} \geq 1.$$

In other words, sub-critical (resp. super-critical) temperatures, $T < T_c$ (resp. $T > T_c$) correspond to $z_2^* < z_1$ (resp. $z_2^* > z_1$). And, of course, $T = T_c \iff z_2^* = z_1$.

It turns out that in their initial (unpublished) calculations to compute M_0 , Kaufman and Onsager did not actually use (21). What happened is described by Onsager in [Ons2] and [Ons3] (see also [Bax] for more details, particularly concerning the Wiener-Hopf calculation below). In addition to (21), Kaufman and Onsager had derived, but apparently did not publish (see private communication to C. Domb [Dom2, p. 201]), an expression for the 2-point correlation function along a diagonal $\langle \sigma_{1,1}, \sigma_{1+n,1+n} \rangle$, which is much simpler than (21). Only one Toeplitz determinant is involved,

$$(31) \quad \langle \sigma_{1,1} \sigma_{1+n,1+n} \rangle = D_n(\varphi_{\text{diag}})$$

where⁴

$$(32) \quad \varphi_{\text{diag}}(e^{i\theta}) = \left(\frac{1 - k_{\text{ons}} e^{-i\theta}}{1 - k_{\text{ons}} e^{i\theta}} \right)^{\frac{1}{2}}$$

and again k_{ons} is given as in (19) and $\varphi_{\text{diag}}(e^{i\pi}) > 0$. The same physical arguments as in (20) yield

$$(33) \quad M_0 = \lim_{n \rightarrow \infty} \langle \sigma_{1,1} \sigma_{1+n,1+n} \rangle^{\frac{1}{2}}.$$

Onsager then realized that the eigenvalue equation for the Toeplitz matrix $T(\varphi_{\text{diag}})$,

$$u_k = \frac{1}{\lambda} \sum_{j=0}^{n-1} (\varphi_{\text{diag}})_{k-j} u_j, \quad 0 \leq k \leq n-1$$

was a discrete analog of the Milne integral equation

$$(34) \quad f(x) = \int_0^\infty A(x-y) f(y) dy, \quad x > 0$$

which arises, with a particular, explicit function $A(\cdot)$, in radiative equilibrium theory (see, for example [DymMc]). Milne's equation can be solved using the Wiener-Hopf method, a technique with which Onsager was familiar from lectures that Wiener had given earlier in the Mathematics Department at Yale. So Onsager tried the Wiener-Hopf technique on the above eigenvalue equation for $T(\varphi_{\text{diag}})$, to obtain the eigenvalues $\lambda = \lambda_\ell^{(n)}$, $1 \leq \ell \leq n$, and then computed $D_n(\varphi_{\text{diag}}) = \prod_{\ell=1}^n \lambda_\ell^{(n)}$. Letting $n \rightarrow \infty$, formula (19) for M_0 then emerged. This was the basis for the first announcements of the result in 1948 and 1949. So why didn't Kaufman and Onsager publish the details of their calculation? The situation was as follows. As in the case of Milne's equation, the Wiener-Hopf method requires very detailed knowledge of the kernel $(\varphi_{\text{diag}})_{j-k}$, in the case of Ising. Onsager sensed that there was another prize to be had: What about Toeplitz determinants with general symbols φ ? And so, before Kaufman and Onsager could get around to publishing their calculation, Onsager began looking for a method to evaluate asymptotically Toeplitz determinants with general symbols. We quote Onsager [Ons3]: "...and lo and behold I found it. It was a general formula for the evaluation of Toeplitz matrices. The only thing I did not know was how

⁴In [Dom2], the symbol is $\overline{\varphi_{\text{diag}}}$, not φ_{diag} . However, the determinant is the same, $D_n(\overline{\varphi_{\text{diag}}}) = \det T_n(\overline{\varphi_{\text{diag}}}) = \det T_n(\varphi_{\text{diag}})^T = \det T_n(\varphi_{\text{diag}}) = D_n(\varphi_{\text{diag}})$.

to fill out the holes in the mathematics and show the epsilons and deltas and all of that, and the limiting processes; I did not know just how it should be done and what mathematicians really knew about limiting processes in that ball park.” As it turned out, by that point, mathematicians knew a lot! Before it was clear what conditions to place on the symbol, Kaufman and Onsager spoke to Kakutani and then Kakutani spoke to Szegő. Szegő then revisited his calculations from 1915, and in 1952 [Sz3] evaluated the $o(n)$ term in (9) for a very general class of symbols φ . This celebrated result is the Szegő Strong Limit Theorem mentioned above. Faced with Szegő’s general result, Kaufman and Onsager published neither their first, nor their second method. As Onsager noted [Ons2], “. . . the mathematicians got there first.” In the long history of mathematics and physics, it is most unusual for a physicist to be scooped out of a formula by a mathematician!

There is, however, more to the story. On the web (see [Bax]) there is the draft of a paper titled “Long-Range Order” from 1950 (never published), without names attached but almost certainly by Onsager and Kaufman, that describes their more general method and obtains the result (19) for M_0 . Here the authors use formulae (20) and (21). There is also an earlier letter [Kau2] written on April 12, 1950, from Onsager to Kaufman containing some of the calculations from the above draft. Presumably to allay any concerns about priority, Onsager assures Kaufman at the end of the letter that “There will be time for all these things.” As one reads [Ons2] [Ons3], one senses the faint hint of regret that Onsager must have felt for such optimism. In a letter from Kaufman to Onsager, written on May 12, 1950, a month after the letter [Kau2], Kaufman says “Here is a draft of Crystal Statistics IV”. In [Bax], Baxter makes a strong case that the above paper from 1950 is indeed Kaufman’s draft of Crystal Statistics IV. However, this may not be so. On the Onsager archive in Trondheim (see footnote 3) under “Selected research material and writings”, item 17.121, one finds the paper “Long-Range order” discussed by Baxter. Although, again, no names are attached, this establishes the provenance of the paper beyond any reasonable doubt, as suspected by Baxter. Item 17.121, however, also contains a second, completely separate paper called “Crystal statistics IV. Long-Range Order in a Binary Crystal”, and both Kaufman’s and Onsager’s names are attached.⁵ The “Long-range” paper describes, as noted above, the second method of Kaufman-Onsager for general symbols, and the second paper describes their first Wiener-Hopf type method. However, it is unfinished and it ends abruptly. So when Kaufman wrote to Onsager “Here is the draft of Crystal Statistics IV”, which of these two drafts did she mean? Either way, the evidence that Kaufman/Onsager knew how to evaluate the asymptotics of Toeplitz determinants (in two ways!) is now clear and written down in their own hand.

3 Szegő’s Theorem

Here is Szegő’s result [Sz3] [GreSz].

Theorem 3 (Szegő Strong Limit Theorem). *Let $\varphi(e^{i\theta})$ be a positive, $C^{1+\varepsilon}$, $\varepsilon > 0$, function on S^1 . Let $(\log \varphi)_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\theta} \log \varphi(e^{i\theta}) d\theta$, $k \in \mathbb{Z}$, denote the Fourier coefficients of $\log \varphi(e^{i\theta})$. Then*

$$(35) \quad \lim_{n \rightarrow \infty} \frac{D_n(\varphi)}{e^{n(\log \varphi)_0}} = e^{E(\varphi)}$$

where

$$(36) \quad E(\varphi) = \sum_{k=1}^{\infty} k |(\log \varphi)_k|^2.$$

⁵The authors thank H. Holden for locating this paper on the Trondheim archive.

□

Thus the $o(n)$ term in (9) is given by $E(\varphi) + o(1)$.

Remark 3.1. Note that

$$(37) \quad e^{(\log \varphi)_0} = \exp \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \varphi(e^{i\theta}) d\theta \right]$$

is the geometric mean of φ .

In contrast to (35), Szegő's earlier result Theorem 1, $\lim_{n \rightarrow \infty} (D_n(\varphi))^{\frac{1}{n}} = e^{(\log \varphi)_0}$, is known simply as Szegő's First Theorem, or sometimes just Szegő's Theorem.

In the years that followed, there was considerable effort by many mathematicians to weaken the assumptions in Theorem 3. Along the way, many new methods, quite different one from the other, were discovered to prove (35). These developments are described in the outstanding recent monographs by Barry Simon [Sim2] and Böttcher and Silbermann [BottSilb3]: also see [McC2] and [Bott1] for earlier summaries. In [Sim2], the contributions of Kac [Kac], G. Baxter [BaxG1], Hirschman [Hir], and Devinatz [Dev], in particular, are discussed, leading up through successively weaker assumptions to the definitive result of Ibragimov [Ibr] in 1968 giving necessary and sufficient conditions on $\varphi(e^{i\theta})$ for (35) to hold under the assumption of the positivity of φ .

Theorem 4 (Ibragimov). *Let $\varphi(e^{i\theta}) \frac{d\theta}{2\pi}$ be a probability measure on S^1 and suppose that $\log \varphi(e^{i\theta})$ is integrable. Then (35) is always true in the following sense: $\lim_{n \rightarrow \infty} \frac{D_n(\varphi)}{e^{n(\log \varphi)_0}}$ always exists and equals $e^{E(\varphi)}$, including the case where one, and hence both, are infinite.* □

One can consider Toeplitz matrices $T_n(d\mu)$ for general probability measures $d\mu(\theta) = \varphi(e^{i\theta}) \frac{d\theta}{2\pi} + d\mu_s(\theta)$ on S^1 , where $d\mu_s$ denotes the singular part of $d\mu$. We have $T_n(d\mu) = \{\mu_{j-k}\}_{0 \leq j, k \leq n-1}$, where $\mu_\ell = \int e^{-i\ell\theta} d\mu(\theta)$, $\ell \in \mathbb{Z}$ and $D_n(d\mu) = \det T_n(d\mu)$. The following remarkable result generalizes Theorem 1.

Theorem 5 ([Sz1][Sz2][Sz4][Ver]). *Let $d\mu(\theta) = \varphi(e^{i\theta}) \frac{d\theta}{2\pi} + d\mu_s(\theta)$ be a probability measure as above. Then Szegő's Limit Theorem is always true in the following sense: $\lim_{n \rightarrow \infty} D_n(d\mu)^{\frac{1}{n}}$ always exists and equals $\exp \left[\int_{-\pi}^{\pi} \log \varphi(e^{i\theta}) \frac{d\theta}{2\pi} \right]$, including the case where one, and hence both, are zero.* □

Szegő proved the result when $d\mu_s = 0$, whereas Verblunsky was able to handle the case $d\mu_s \neq 0$. Of course, as $\varphi(e^{i\theta}) \in L^1 \left(\frac{d\theta}{2\pi} \right)$, $\int_{-\pi}^{\pi} \log \varphi(e^{i\theta}) \frac{d\theta}{2\pi} < \infty$, and the integral can diverge, possibly, only to $-\infty$. The striking feature of Theorem 5 is that $\lim_{n \rightarrow \infty} D_n(d\mu)^{\frac{1}{n}}$ is independent of $d\mu_s$.

Is it possible that (35) remains true when $d\mu_s \neq 0$? Here the definitive result is due to Golinskii and Ibragimov.

Theorem 6 ([GolIbr]). *Let $d\mu(\theta) = \varphi(e^{i\theta}) \frac{d\theta}{2\pi} + d\mu_s(\theta)$ be a probability measure as above, and suppose $\log \varphi(e^{i\theta}) \in L^1 \left(\frac{d\theta}{2\pi} \right)$. If $d\mu_s \neq 0$, then*

$$(38) \quad \lim_{n \rightarrow \infty} \frac{D_n(d\mu)}{\exp \left[n \int_{-\pi}^{\pi} \log \varphi(e^{i\theta}) \frac{d\theta}{2\pi} \right]} = +\infty$$

□

In other words, if (35) were to hold with $E(\varphi) = \sum_{k=1}^{\infty} k |(\log \varphi)_k|^2 < \infty$, then necessarily $d\mu_s = 0$.

In [Sim2], Simon presents six different proofs of (35). Some of these proofs require stronger conditions on φ (see [Sim2]). We now describe these methods in some detail (but without giving precise conditions on φ in each case) in order to illustrate the extraordinary variety of mathematical areas and techniques that inter-relate with Szegő's Strong Limit Theorem. Our presentation follows [Sim2]. After this presentation, we will also describe three additional proofs not covered in [Sim2], together with some brief comments on Szegő's original proof (see Remark 3.2).

The first proof uses ideas and results from the theory of orthogonal polynomials on the unit circle (OPUC's). These polynomials were introduced by Szegő in the early 1920's [Sz4] [Sz5] in the course of his investigation of the eigenvalues $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ of Toeplitz matrices T_n . The classical references for OPUC's are [Sz5] [Ge]. The OPUC's $p_k(z) = \chi_k z^k + \dots, \chi_k > 0$, associated with a probability measure $d\mu(\theta)$ on S^1 are formed by orthonormalizing $1, z, z^2, \dots, z^k, \dots$ with respect to $d\mu(\theta)$ in $L^2(S^1, d\mu(\theta))$,

$$(39) \quad \int_{S^1} \overline{p_k(e^{i\theta})} p_j(e^{i\theta}) d\mu(\theta) = \delta_{k,j}, \quad k, j \geq 0.$$

OPUC's are intimately related to Toeplitz determinants. For example

$$(40) \quad \frac{D_{n+1}(d\mu)}{D_n(d\mu)} = \frac{1}{\chi_n^2} = \frac{1}{\chi_0^2} \prod_{j=0}^{n-1} (1 - |\xi_j|^2), \quad n \geq 1$$

where $\xi_j \equiv -(\chi_{j+1})^{-1} \overline{p_{j+1}(0)}$, $j \geq 0$, are the so-called *Verblunsky* coefficients. The ξ_j 's have modulus less than 1, and so $F(d\mu) \equiv \lim_{n \rightarrow \infty} D_{n+1}(d\mu)/D_n(d\mu)$ always exists and equals $\chi_0^{-2} \prod_{j=0}^{\infty} (1 - |\xi_j|^2)$. But then $\lim_{n \rightarrow \infty} (D_n(d\mu))^{\frac{1}{n}}$ always exists, and has the same limit. So the proof of Szegő's Limit Theorem, boils down to showing that

$$(41) \quad \frac{1}{\chi_0^2} \prod_{j=0}^{\infty} (1 - |\xi_j|^2) = \exp \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \varphi(e^{i\theta}) d\theta \right]$$

where $d\mu(\theta) = \varphi(e^{i\theta}) \frac{d\theta}{2\pi} + d\mu_s(\theta)$. In turn, one finds that

$$(42) \quad \lim_{n \rightarrow \infty} \frac{D_n(d\mu)}{(F(d\mu))^n} = \lim_{n \rightarrow \infty} \frac{\prod_{j=0}^{n-2} (1 - |\xi_j|^2)^{-j-1}}{\prod_{j=n-1}^{\infty} (1 - |\xi_j|^2)^n}$$

and the proof of SSLT for $d\mu_s = 0$ reduces to showing that the limit on the RHS is just $e^{E(\varphi)}$. As we will see below, OPUC's play a crucial role in analyzing the Fisher-Hartwig conjecture for Toeplitz determinants with singular symbols $\varphi(e^{i\theta})$.

The second proof utilizes a remarkable identity which gives an exact formula for $D_n(\varphi)$,

$$(43) \quad D_n(\varphi) = e^{n(\log \varphi)_0 + E(\varphi)} \det(1 - Q_n H(b) H(\tilde{c}) Q_n), \quad n \geq 1$$

where \det denotes the determinant in $\ell_2^+ = \ell_2(\mathbb{Z}_+)$, $\mathbb{Z}_+ = \{0, 1, \dots\}$, Q_n is the orthogonal projection in ℓ_2^+ onto $\ell_{2,n}^+ = \{u = (u_0, u_1, \dots) \in \ell_2^+ : u_i = 0 \text{ for } 0 \leq i < n\}$, and $H(b)$, $H(\tilde{c})$ are the Hankel

operators with kernels $\{b_{i+j+1}\}_{i,j \geq 0}$, $\{c_{-i-j-1}\}_{i,j \geq 0}$. Here $b(e^{i\theta}) = \sum_j b_j e^{ij\theta}$, $c(e^{i\theta}) = \sum_j c_j e^{ij\theta}$ are defined in terms of the *Szegő function* $\mathcal{D}(z) = \exp\left(\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \varphi(e^{i\theta}) d\theta\right)$,

$$(44) \quad b \equiv \frac{\overline{\mathcal{D}}}{\mathcal{D}}, \quad c \equiv \frac{\mathcal{D}}{\overline{\mathcal{D}}}$$

where $\mathcal{D} = \mathcal{D}(e^{i\theta}) = \lim_{z \rightarrow e^{i\theta}, |z| < 1} \mathcal{D}(z)$. The point is that $H(b)H(\tilde{c})$ is a trace-class operator in $\ell_2^+ = \ell_{2,0}^+$, and hence $Q_n H(b)H(\tilde{c})Q_n \rightarrow 0$ in trace-norm by abstract arguments as $n \rightarrow \infty$. But then $\det(1 - Q_n H(b)H(\tilde{c})Q_n) \rightarrow 1$ as $n \rightarrow \infty$, by the continuity properties of the determinant. (For more information about the trace class and determinants, see [BottSilb3] [Sim3].) Thus SSLT is immediate, once one has proved (43). Formula (43) is known as the Borodin-Okounkov formula and was proved [BorOk] in 2000 in the context of combinatorics and random matrix theory. The formula evoked great interest, and meanwhile several new proofs and generalizations have been given (see, e.g., [BasWid] [BottWid1] and [Sim2]). However, it turns out that it was already proven many years earlier by Geronimo and Case in 1979 in their work on inverse scattering theory [GerCase]. The broader significance of this paper was not appreciated at the time. But more to the point, in a separate section in their paper titled ‘‘Szegő’s Theorem’’, Geronimo and Case actually used (43) to prove SSLT. It is unfortunate that these developments were overlooked by the experts in the field.

The third and fourth methods utilize the following Heine-type multi-integral representation for $D_n(\varphi)$ (see, e.g., [BottSilb3] [Sim2])

$$(45) \quad D_n(\varphi) = \frac{1}{n!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{0 \leq j < k \leq n-1} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=0}^{n-1} \varphi(e^{i\theta_j}) \frac{d\theta_j}{2\pi}.$$

In the third method, due to Bump and Diaconis [BumpDia], the authors observe that the RHS of (45) can be re-written via Weyl’s integration formula as $\int_{\mathcal{U}(n)} e^{F_n(g)} dg$, where dg denotes Haar measures on the unitary group $\mathcal{U}(n)$, and $F_n(g) = \sum_{j=0}^{n-1} \log \varphi(e^{i\theta_j(g)})$, where $\{e^{i\theta_j(g)}\}$ are the eigenvalues of g . Substituting the Fourier expansion $\log \varphi(e^{i\theta}) = \sum_{k=-\infty}^{\infty} (\log \varphi)_k e^{ik\theta}$, one obtains

$$(46) \quad D_n(\varphi) = e^{n(\log \varphi)_0} \int_{\mathcal{U}(n)} \exp\left(\sum_{k \neq 0} (\log \varphi)_k \operatorname{tr}(g^k)\right) dg$$

and after expanding out the exponential, one ends up with the representation

$$(47) \quad D_n(\varphi) = e^{n(\log \varphi)_0} \int_{\mathcal{U}(n)} \overline{\left(\sum_t \eta_t T_t(g)\right)} \left(\sum_s \eta_s T_s(g)\right) dg.$$

Here the sums are over all k -tuples of non-negative integers $t = (t_1, \dots, t_k)$, $s = (s_1, \dots, s_k)$, and for any k , $T_t(g) = \prod_{j=1}^k [\operatorname{tr}(g^j)]^{t_j}$, and the η_t ’s, η_s ’s are explicit constants depending on $\{(\log \varphi)_\ell\}$. The magic of the method is that one can use the theory of characters together with Frobenius-Schur-Weyl duality to compute $\int_{\mathcal{U}(n)} \overline{T_t(g)} T_s(g) dg$ explicitly, obtaining

$$(48) \quad \int_{\mathcal{U}(n)} \overline{T_t(g)} T_s(g) dg = \delta_{ts} W(t)$$

where $W(t) = \prod_{j=1}^k j^{t_j} t_j!$, provided $\sum_{j=1}^k j t_j$ and $\sum_{j=1}^k j s_j$ are $\leq n$. Substituting (48) into (47) and controlling the errors as $n \rightarrow \infty$, one immediately obtains (35).

In the fourth proof, due to Johansson [Joh1], one rewrites (45) with $\varphi(e^{i\theta}) = e^{V(\theta)}$ in the form

$$(49) \quad D_n(e^V) = \frac{1}{n!} \int e^{-2H_n(\theta_0, \dots, \theta_{n-1})} \frac{d\theta_0}{2\pi} \dots \frac{d\theta_{n-1}}{2\pi}$$

where

$$(50) \quad H_n = -\frac{1}{2} \sum_{j=0}^{n-1} V(\theta_j) + \frac{1}{2} \sum_{0 \leq k < j \leq n-1} W(\theta_k - \theta_j)$$

and

$$(51) \quad W(\theta) = -\log |e^{i\theta} - 1|^2.$$

This expression exhibits $n!D_n(e^V)$ as the partition function of a gas of 1-dimensional particles with 2-dimensional Coulomb interactions — a “log-gas” in the terminology of Dyson. Dyson introduced this terminology in the context of his analysis [Dy1] of Circular Ensembles of $n \times n$ random matrices. For such ensembles Dyson found that the distributions of the eigenvalues $\{e^{i\theta_j}\}_{j=0}^{n-1}$, $0 \leq \theta_j < 2\pi$, of the matrices were given by

$$(52) \quad \frac{1}{Z_{n,\beta}} \prod_{0 \leq j < k \leq n-1} |e^{i\theta_j} - e^{i\theta_k}|^\beta \frac{d\theta_0}{2\pi} \dots \frac{d\theta_{n-1}}{2\pi}.$$

Here $\beta = 1$ for the orthogonal, $\beta = 2$ for the unitary, and $\beta = 4$ for the symplectic ensemble, and

$$(53) \quad Z_{n,\beta} = \int_0^{2\pi} \frac{d\theta_0}{2\pi} \dots \int_0^{2\pi} \frac{d\theta_{n-1}}{2\pi} \prod_{0 \leq j < k \leq n-1} |e^{i\theta_j} - e^{i\theta_k}|^\beta$$

is the normalization constant. Dyson then noted that the distribution of the eigenvalues $\{e^{i\theta_j}\}_{j=0}^{n-1}$ was identical with the distribution of the positions $0 \leq \theta_j < 2\pi$, $j = 0, \dots, n-1$ of charges in a finite gas with 2-D Coulomb interactions at inverse temperature β and with Gibbs measure $\frac{1}{Z_{n,\beta}} \exp\{-\beta H_n(\theta_0, \dots, \theta_{n-1})\} \frac{d\theta_0}{2\pi} \dots \frac{d\theta_{n-1}}{2\pi}$, where $H_n(\theta_0, \dots, \theta_{n-1}) = -\sum_{0 \leq j < k \leq n-1} \log |e^{i\theta_j} - e^{i\theta_k}|$, and the partition function is given by (53). As emphasized by Dyson [Dy1], a consequence of this identification is that “...the thermodynamic notions of entropy, specific heat, etc., can be transferred from the Coulomb gas to the eigenvalue series. This will prove very useful, as it gives us a precise and well-understood language in which to describe the statistical properties of the eigenvalues”. If the matrices U in the ensembles carry a weight $e^{-\beta \text{tr} Q(U)}$, then the energy in the corresponding Coulomb gas must be replaced by

$$(54) \quad H_n(\theta_0, \dots, \theta_{n-1}) = -\sum_{0 \leq j < k \leq n-1} \log |e^{i\theta_j} - e^{i\theta_k}| + \sum_{j=0}^{n-1} Q(e^{i\theta_j})$$

where Q now plays the role of an external force acting on the charges.

In operational terms, the identification implies, in particular, (see [Dy2]) that the free energy $F \equiv -\beta^{-1} \log Z_{n,\beta}$ is approximated to leading order as $n \rightarrow \infty$ by the classical thermodynamic recipe, $F \sim F_T$, where

$$(55) \quad \beta F_T = G_2 + G_1 + G_0.$$

Here G_2 is the macroscopic Coulomb energy

$$(56) \quad G_2 = -\frac{\beta}{2} \int \int \sigma(e^{i\theta})\sigma(e^{i\theta'}) \log |e^{i\theta} - e^{i\theta'}| d\theta d\theta' + \beta \int Q(e^{i\theta})\sigma(e^{i\theta})d\theta$$

and $G_1 + G_0$ is the local free energy term

$$(57) \quad G_1 = \left(1 - \frac{\beta}{2}\right) \int \sigma(e^{i\theta}) \log \sigma(e^{i\theta}) d\theta,$$

$$(58) \quad G_0 = n \left[\left(1 - \frac{\beta}{2}\right) \log \frac{2\pi}{n} + \log \Gamma \left(1 + \frac{\beta}{2}\right) \right] - \log \Gamma \left(1 + \frac{\beta}{2}n\right).$$

The function $\sigma(e^{i\theta}) \geq 0$ is the macroscopic density of the gas and is chosen to minimize the functional on the right-hand side of (55) subject to the constraint

$$(59) \quad \int \sigma(e^{i\theta}) d\theta = n.$$

Dyson first applied these thermodynamic notions to compute [Dy2] the probability P_s that there are no eigenvalues for a random matrix in the interval $(0, 2s/\pi)$, in the bulk scaling limit where the average distance between the eigenvalues is rescaled to 1, for the Gaussian Unitary Ensemble (see [Meh]). Using these notions, he derived, for the first time, the leading order of P_s , $P_s \sim \exp\{-s^2/2\}$ as $s \rightarrow \infty$. (We return to this problem further on (see (225) et seq).) A year later in [Dy3], Dyson showed how these notions play out in the context of SSLT. First Dyson noted, as explained above, that the Heine-type representation (45) immediately displayed $n!D_n(e^V)$ as the partition function of a log-gas as in (49)–(51) above. This connection between Toeplitz determinants and the Coulomb gas was first pointed out in an unpublished manuscript by G. Kreisel, which was made available to Dyson by courtesy of E. Wigner.⁶

For $\beta = 2$ the term $G_1 = 0$ and the constant $G_0 = -\log n!$, and we have

$$(60) \quad F_T = -\frac{1}{2} \int \int \sigma(e^{i\theta})\sigma(e^{i\theta'}) \log |e^{i\theta} - e^{i\theta'}| d\theta d\theta' + \int Q(e^{i\theta})\sigma(e^{i\theta})d\theta - \frac{1}{2} \log n!$$

The constrained minimum is obtained when

$$(61) \quad Q(e^{i\theta}) - \int \sigma(e^{i\theta'}) \log |e^{i\theta} - e^{i\theta'}| d\theta' = \text{const}$$

on the unit circle. Solving this equation together with (59), Dyson finds that for $Q(e^{i\theta})$ sufficiently smooth,

$$(62) \quad F_T = -\frac{1}{2} \log n! + nQ_0 - \sum_{k=-\infty}^{\infty} |k|Q_kQ_{-k}$$

where $\{Q_k\}$ are the Fourier coefficients of $Q(e^{i\theta})$. For $V = -\beta Q = -2Q$, we obtain

$$(63) \quad -\frac{1}{2} \log(n!D_n(e^V)) = F \sim F_T = -\frac{1}{2} \log n! + nQ_0 - \sum_{k=-\infty}^{\infty} |k|Q_kQ_{-k}$$

⁶We note that already in the 1880's Stieltjes gave an electrostatic interpretation of the zeros of certain classical orthogonal polynomials (see [Sz5], Section 6.7). For a discussion of logarithmic potentials from the viewpoint of potential theory, see the book of Saff and Totik [SaTo].

which is precisely SSLT, $D_n(e^V) \sim \exp\{nV_0 + \sum_1^\infty k|V_k|^2\}$. There is an irony in this argument: on the one hand, SSLT was proved in response to a problem in statistical mechanics, but on the other hand, SSLT is obtained here by the methods of statistical mechanics. The above calculation is based on the (unproven) validity of the thermodynamic recipe (55)–(58), (59). Johansson’s proof in [Joh1] does not rely on the thermodynamic recipe. The proof in [Joh1] starts with the identity $D_n(1) = 1$ and develops a perturbation theory around $\varphi(z) = 1$. Johansson notices that the minimum of the double sum in (50), and thus the maximum of the integrand in (49) for $V \equiv 0$ is reached when the points $e^{i\theta_j}$, $j = 0, \dots, n-1$, are uniformly distributed on the unit circle, i.e. when they are the vertices of a regular n -gon. For $\varphi(z) = e^{V(z)} \neq 1$, normalize φ by requiring $V_0 = 0$. If one adds the corresponding first sum with $V \neq 0$ to (50), the position of the maximum of the integrand in (49) becomes displaced, but in a sense only slightly if n is large. For large n Johansson introduces the following change of variables in (49):

$$(64) \quad \theta_j = \psi_j - \frac{1}{n}h(e^{i\psi_j}), \quad j = 0, \dots, n-1,$$

where $h(z) = -i \sum_{k=1}^\infty (V_k z^k - V_{-k} z^{-k})$ is the conjugate function to $V(z)$. The main contribution to the integral comes from a small neighborhood of a configuration where ψ_j ’s are uniformly distributed on the unit circle. Due to this crucial fact, all the sums which appear (as a result of the change of variables (64)) in the exponent, apart from the main one, $\sum_{0 \leq k < j \leq n-1} W(\psi_k - \psi_j)$, can be replaced by integrals with good precision and produce, as $n \rightarrow \infty$, the constant (36), while the remaining $\frac{1}{n!} \int \exp(-\sum_{0 \leq k < j \leq n-1} W(\psi_k - \psi_j)) \prod_{k=0}^{n-1} \frac{d\psi_k}{2\pi} = D_n(1) = 1$.

The fifth method is combinatorial and is due basically to Kac [Kac]. At the start one renormalizes $\varphi(e^{i\theta})$ so that $\sup_\theta \varphi(e^{i\theta}) = 1$. Setting $h = 1 - \varphi$ and $w_\lambda = 1 - \lambda h$, $0 \leq \lambda \leq 1$, we have $w_1 = \varphi$. Under the additional assumption that $\inf_\theta \varphi(e^{i\theta}) > 0$, we clearly have $0 \leq h(\theta) \leq \sup_\theta h(\theta) < 1$. Using the identity $\log \det A = \text{tr} \log A$, one expands $\log D_n(1 - \lambda h)$ for $0 \leq \lambda \leq 1$ in a power series (note that $\|T_n(h)\| \leq \|h\|_\infty < 1$)

$$(65) \quad \log D_n(1 - \lambda h) = - \sum_{m=1}^\infty \frac{\lambda^m}{m} \text{tr} ([T_n(h)]^m)$$

and also

$$(66) \quad \int \log(1 - \lambda h) \frac{d\theta}{2\pi} = - \sum_{m=1}^\infty \frac{\lambda^m}{m} \int h^m(\theta) \frac{d\theta}{2\pi}.$$

On the other hand

$$(67) \quad (\log(1 - \lambda h))_k = - \int e^{-ik\theta} \sum_{m=1}^\infty \frac{\lambda^m}{m} h^m(\theta) \frac{d\theta}{2\pi}$$

and so

$$(68) \quad \sum_{k=1}^\infty k |(\log(1 - \lambda h))_k|^2 = \sum_{m=1}^\infty s_m \frac{\lambda^m}{m}$$

where

$$s_m = \sum_{k=1}^\infty k \sum_{\ell=1}^{m-1} \frac{m}{\ell(m-\ell)} \left(\int e^{-ik\theta} h^\ell(\theta) \frac{d\theta}{2\pi} \right) \left(\int e^{ik\theta} h^{m-\ell}(\theta) \frac{d\theta}{2\pi} \right).$$

Thus the strategy is to show

$$(69) \quad \lim_{n \rightarrow \infty} - \left[\text{tr} ([T_n(h)])^m - n \int h^m(\theta) \frac{d\theta}{2\pi} \right] = s_m.$$

Combinatorial issues arise when one tries to control the complicated multilinear sums in $\text{tr} [T_n(h)]^m$.

The sixth method is based on a formula of Baik, Deift, McLaughlin, and Zhou (unpublished) for relative determinants

$$(70) \quad \log \frac{D_n(\varphi\psi)}{D_n(\psi)} = \int_0^1 dt \int_0^{2\pi} \left[\frac{d}{dt} \log \varphi_t \right] K_t(\theta) \varphi_t(\theta) \frac{d\theta}{2\pi}$$

where φ and ψ are two functions on S^1 and $\varphi_t = (1-t) + t\varphi$ and $K_t(\theta)$ is the *Christoffel-Darboux* kernel

$$(71) \quad K_t(\theta) = \sum_{j=0}^{n-1} \left| p_j(e^{i\theta}, \varphi_t \psi) \right|^2$$

and $p_j(e^{i\theta}, \varphi_t \psi)$ are the OPUC's associated with the weight $\varphi_t(e^{i\theta}) \psi(e^{i\theta}) \frac{d\theta}{2\pi}$. SSLT then follows by computing the (weak) limit of $\frac{1}{n} K_t(\theta)$ as $n \rightarrow \infty$. Thus, as in the first method, SSLT is seen here as following from properties of OPUC's $p_j(e^{i\theta})$.

The mathematical challenge that is posed by SSLT can be viewed as follows. In functional analysis one commonly considers continuous maps F , say, from one fixed topological space X to another fixed topological space Y . Then in order to verify that $\lim_{n \rightarrow \infty} F(x_n) = F(x)$ all one has to do is verify that $x_n \rightarrow x$ as $n \rightarrow \infty$ in the topology of X . In particular, if X is the space of trace class operators on a separable Hilbert space \mathcal{H} , and $F(x) = \det(1+x)$, then $F: X \rightarrow Y = \mathbb{C}$ and $\lim_{n \rightarrow \infty} \det(1+x_n) = \det(1+x)$ if $x_n \rightarrow x$ in trace norm. But SSLT is not of this standard type. Instead, one has $F: M(n, \mathbb{C}) \rightarrow \mathbb{C}$ taking $A_n \in M(n, \mathbb{C})$ to $F(A_n) = \det A_n$. So the initial space $X = X_n = M(n, \mathbb{C})$ is changing with n and it is not at all clear how to topologize the situation so that " $\lim_n F(A_n) = F(\lim A_n)$ ". If, for example, we let $X = M(\infty, \mathbb{C})$ be the inductive limit of the $M(n, \mathbb{C})$'s, then $T_n(\varphi)$ converges to the Toeplitz operator $T(\varphi)$ in the associated topology, but $T(\varphi)$ is not of the form $1 + \text{trace class}$, and so $F(T(\varphi)) = \det(T(\varphi))$ is not defined. In such situations when $X = X_n$, or $Y = Y_n$, is varying, there is no general procedure to follow and each problem must be addressed on an ad hoc basis. And when there is no preferred path to the solution of a problem, there are many paths, as we see from the six very different methods above. In this regard, the method of Borodin-Okounkov-Geronimo-Case is singled out: after factoring out $e^{n(\log \varphi)_0 + E(\varphi)}$, one has

$$(72) \quad \frac{\det(T_n(\varphi))}{e^{n(\log \varphi)_0 + E(\varphi)}} = \det(1 - B_n)$$

where $B_n = Q_n H(b) H(\tilde{c}) Q_n$ is trace class in ℓ_2^+ and $B_n \rightarrow 0$ in trace norm as $n \rightarrow \infty$. In other words, one of the paths is to renormalize the problem so that it assumes the standard form.

This pathway, to reduce SSLT to standard form, goes back to Harold Widom, who was the first to apply operator-theoretic methods to the problem of Toeplitz asymptotics. In [Wid4], Widom gave yet another proof of (35). This proof, which was found more than 35 years ago, has not lost its charm. We follow the presentation in [Bott1]. It turns out that by a simple trick, the determinants of finite sections of the inverse Toeplitz operator $T(\varphi)^{-1}$ are easy to compute,

$$(73) \quad \det P_n T(\varphi)^{-1} P_n = e^{-n(\log \varphi)_0}$$

where P_n projects $u = (u_0, u_1, \dots, u_{n-1}, u_n, \dots) \in \ell_2^+$ to $(u_0, u_1, \dots, u_{n-1}) \in \mathbb{C}^n$. Simple algebra shows that

$$(74) \quad \det T_n(\varphi) = \det P_n T(\varphi) P_n = \det(P_n T^{-1}(\varphi^{-1}) P_n) \\ \times \det\left(1 + (P_n T^{-1}(\varphi^{-1}) P_n)^{-1} P_n K P_n\right)$$

where $K \equiv T(\varphi) - T^{-1}(\varphi^{-1})$ is trace-class. By (73),

$$(75) \quad \det P_n T^{-1}(\varphi^{-1}) P_n = e^{n(\log \varphi)_0}.$$

But $(P_n T^{-1}(\varphi^{-1}) P_n)^{-1} P_n$ can be shown to converge strongly as $n \rightarrow \infty$ in ℓ_2^+ to $(T^{-1}(\varphi^{-1}))^{-1} = T(\varphi^{-1})$, and as K is trace class, $R_n \equiv (P_n T^{-1}(\varphi^{-1}) P_n)^{-1} P_n K P_n \rightarrow T(\varphi^{-1}) K$. We conclude that as $n \rightarrow \infty$

$$(76) \quad \frac{\det T_n(\varphi)}{e^{n(\log \varphi)_0}} = \det(1 + R_n) \rightarrow \det(1 + T(\varphi^{-1}) K) = \det T(\varphi^{-1}) T(\varphi).$$

Using the remarkable Helton-Howe-Pincus formula (see [HeltHow]), $\det(e^A e^B e^{-A} e^{-B}) = e^{\text{tr}[A, B]}$, if A, B are bounded, and $[A, B] = AB - BA$ is trace class, one then shows that $\det(T(\varphi^{-1}) T(\varphi)) = e^{E(\varphi)}$, and (35) is proved.

Of all the methods to prove SSLT, perhaps the simplest and most direct was introduced by Basor and Helton in [BasHelt]. As in [Wid4], the method also proceeds by reducing SSLT to standard form. We follow [BasHelt]: Suppose φ has a Wiener-Hopf factorization $\varphi = \varphi_+ \varphi_-$, where $\varphi_+, \varphi_+^{-1}$ are bounded analytic functions in $\{|z| < 1\}$, and $\varphi_-, \varphi_-^{-1}$ are bounded analytic functions in $\{|z| > 1\}$. Then by direct calculation, the Toeplitz operator $T(\varphi)$ has a factorization $T(\varphi) = T(\varphi_-) T(\varphi_+)$, furthermore,

$$(77) \quad P_n T(\varphi_+) = P_n T(\varphi_+) P_n, \quad T(\varphi_-) P_n = P_n T(\varphi_-) P_n.$$

As $T(\varphi_{\pm}) T(\varphi_{\pm}^{-1}) = 1$, we see that $T(\varphi_{\pm})^{-1}$ exist and $T(\varphi_{\pm})^{-1} = T(\varphi_{\pm}^{-1})$. We thus have

$$(78) \quad P_n T(\varphi) P_n = P_n T(\varphi_+) P_n T(\varphi_+)^{-1} T(\varphi_-) T(\varphi_+) T(\varphi_-)^{-1} P_n T(\varphi_-) P_n$$

so that

$$(79) \quad D_n(\varphi) = \det(P_n T(\varphi_+) P_n) \det(P_n \{T(\varphi_+)^{-1}, T(\varphi_-)\} P_n) \det(P_n T(\varphi_-) P_n) \\ = ((\varphi_+)_0 (\varphi_-)_0)^n \det(P_n \{T(\varphi_+)^{-1}, T(\varphi_-)\} P_n)$$

where $\{T(\varphi_+)^{-1}, T(\varphi_-)\} = T(\varphi_+)^{-1} T(\varphi_-) T(\varphi_+) T(\varphi_-)^{-1}$ is the multiplicative commutator. However $\{T(\varphi_+)^{-1}, T(\varphi_-)\} = 1 + T(\varphi_+)^{-1} [T(\varphi_-), T(\varphi_+)] T(\varphi_-)^{-1}$ and under mild smoothness conditions on $\varphi(e^{i\theta})$, the commutator $[T(\varphi_-), T(\varphi_+)] = T(\varphi_-) T(\varphi_+) - T(\varphi_+) T(\varphi_-)$ is trace class. For such φ , $\det(P_n \{T(\varphi_+)^{-1}, T(\varphi_-)\} P_n)$ is equal to the Fredholm determinant

$$\det(1 + P_n T(\varphi_+)^{-1} [T(\varphi_-), T(\varphi_+)] T(\varphi_-)^{-1} P_n)$$

which converges by abstract theory as $n \rightarrow \infty$ to

$$(80) \quad \det(1 + T(\varphi_+)^{-1} [T(\varphi_-), T(\varphi_+)] T(\varphi_-)^{-1}) = \det\{T(\varphi_+)^{-1}, T(\varphi_-)\} \\ = \det(T(\varphi_+)^{-1} T(\varphi_-) T(\varphi_+) T(\varphi_-)^{-1}) \\ = \det T(\varphi^{-1}) T(\varphi).$$

Again by the Helton-Howe-Pincus formula, we have

$$(81) \quad \lim_{n \rightarrow \infty} \frac{D_n(\varphi)}{e^{n(\log \varphi)_0}} = \det T(\varphi^{-1})T(\varphi) = e^{E(\varphi)}$$

which proves SSLT under mild assumptions on φ (see [BasHelt] Theorem 2.1.).

In [Dei1], the author gives yet another proof of (35) by expressing $D_n(\varphi)$ as a Fredholm determinant of the following form:

$$(82) \quad D_n(\varphi) = \det(1 - V_n)$$

where V_n is a trace class operator acting on $L^2(S^1)$

$$(83) \quad V_n f(z) = \int_{|z'|=1} V_n(z, z') f(z') dz', \quad |z| = 1,$$

with

$$(84) \quad V_n(z, z') = \frac{z^n (z')^{-n} - 1}{z - z'} \frac{1 - \varphi(z')}{2\pi i}.$$

The operator V_n is oscillatory and does not converge as $n \rightarrow \infty$ in the trace-norm topology. However, V_n has the special form of an *integrable operator*, that is, operators with kernels of the form $\sum_{j=1}^m \frac{f_j(z)g_j(z')}{z-z'}$, with z, z' on some contour Γ in \mathbb{C} , $m < \infty$, and f_j, g_j are given functions on Γ . Such operators were first singled out as a distinguished class by Its, Izergin, Korepin and Slavnov⁷ (see [Dei1] for a pedagogic presentation). Associated with the integrable operators there is a canonical Riemann-Hilbert Problem, which, in this case, is of oscillatory type. Such problems can be analyzed asymptotically by applying the nonlinear steepest descent method for Riemann-Hilbert Problems introduced by Deift and Zhou in [DZ], and further developed in [DVZ], and SSLT then follows (see [Dei1]).

Remark 3.2. In the methods of Widom and Basor-Helton the existence of the limit of $\frac{D_n(\varphi)}{e^{n(\log \varphi)_0}}$ as $n \rightarrow \infty$ follows directly from general theorems in functional analysis: the identification of the limit with $e^{E(\varphi)}$ is a separate matter and uses other means (Helton-Howe-Pincus). The situation is similar regarding Szegő's original proof of SSLT (see [Sz3] [GreSz]): as $\varphi(e^{i\theta}) > 0$, the ratio $\frac{D_n(\varphi)}{e^{n(\log \varphi)_0}}$ is monotone increasing in n , so the existence of the limit (which may, a priori, be infinite) is immediate. Again the evaluation of the limit is a separate matter. Indeed, Szegő first computes the limit for a dense class of symbols $\{\tilde{\varphi}\}$ for which $D_n(\tilde{\varphi})$ can be evaluated explicitly for n sufficiently large, $n > n_{\tilde{\varphi}}$, and then obtains the general result by an approximation argument $\tilde{\varphi} \rightarrow \varphi$.

By contrast, all the other methods we have described above prove the existence of the limit, and compute its value, both at the same time.

We return now briefly to the 2-dimensional Ising model. One notes that Szegő's Theorem 3 does not actually apply to $\varphi_{\text{diag}}(\theta)$, say, in (32), because the symbol is not real valued. Moreover, Szegő's proof of (35) depends in a crucial way on the positivity of $\varphi(e^{i\theta})$ (see Remark 3.2 above), and it was not at all clear at the time how to extend Theorem 3 to non-real symbols φ . So Onsager's lament that "...the mathematicians got there first", was in fact not true! The result in the 1950 unpublished draft of Kaufman and Onsager mentioned above, without "the epsilons and deltas and

⁷ Some of the elements of the theory of integrable operators were already present in the earlier work [Sakh].

all of that”, but allowing for non-real φ , was in fact the first to break the tape! Now although the symbol $\varphi_{\text{diag}}(\theta)$ is not real, for $0 < k_{\text{ons}} < 1$ however, $\log \varphi_{\text{diag}}(\theta)$ is still smooth on S^1 , in particular, the winding number of $\varphi_{\text{diag}}(\theta)$ on S^1 is zero. The first result for such symbols was proved by G. Baxter in 1963 [BaxG2]. He showed that under certain technical conditions, if $\varphi(e^{i\theta})$ is smooth, non-zero, and has zero winding number on S^1 , then

$$(85) \quad \lim_{n \rightarrow \infty} \frac{D_n(\varphi)}{e^{n(\log \varphi)_0}} = e^{\sum_{k=1}^{\infty} k(\log \varphi)_k (\log \varphi)_{-k}}.$$

Note that for $\varphi_{\text{diag}}(\theta)$ in (32), one has for $0 < k_{\text{ons}} < 1$, $(\log \varphi_{\text{diag}})_\ell = (\text{sgn } \ell) k_{\text{ons}}^{|\ell|} / 2|\ell|$ for $\ell \neq 0$ and $(\log \varphi_{\text{diag}})_0 = 0$. Hence $\sum_{\ell=1}^{\infty} \ell (\log \varphi)_\ell (\log \varphi)_{-\ell} = -\frac{1}{4} \sum_1^{\infty} \frac{(k_{\text{ons}}^2)^\ell}{\ell} = \frac{1}{4} \log(1 - k_{\text{ons}}^2)$, which yields (19) via (33).

In retrospect, many of the proofs of SSLT do extend directly to complex-valued symbols with smooth logarithm. This is true, in particular, for Kac’s proof of (35) in 1954 [Kac], as noted by Montroll, Potts and Ward in [MPW] (see footnote 17, p. 316). Moreover, the proofs in [Wid4] and in the text [BottSilb3], for example, are all designed from the very beginning to cover complex-valued functions.

If $\varphi(e^{i\theta})$ is non-zero and continuous, say, on S^1 , and has zero winding number, then $\varphi(e^{i\theta}) = e^{V(e^{i\theta})}$ for some continuous, periodic function $V(e^{i\theta}) = \log \varphi(e^{i\theta})$. The strongest version of SSLT with complex symbols $\varphi(e^{i\theta})$ is the following.

Theorem 7. *Let $V(e^{i\theta}) \in L^1(S^1)$ be a (possibly complex-valued) function on S^1 with Fourier coefficients $(V_k)_{k \in \mathbb{Z}}$ satisfying*

$$(86) \quad \sum_{k=-\infty}^{\infty} |k| |V_k|^2 < \infty,$$

then

$$(87) \quad \lim_{n \rightarrow \infty} \frac{D_n(e^V)}{e^{nV_0}} = e^{\sum_1^{\infty} k V_k V_{-k}}.$$

□

Note that for $V(e^{i\theta})$ satisfying (86), $e^{V(e^{i\theta})} \in L^p(S^1, \frac{d\theta}{2\pi})$ for $1 \leq p < \infty$ (see [Sim2]). The proof of Theorem 7 is given by Johansson [Joh1]. In particular, his argument for the extension of the result to complex-valued $V(z)$ proceeds by considering the analytic functions

$$g_n(z) = D_n\left(e^{(\Re V + z \Im V)}\right) / e^{n[(\Re V)_0 + z(\Im V)_0]}$$

and applying Vitali’s Theorem. When z is real, $\lim_{n \rightarrow \infty} g_n(z)$ is given by (85), and to obtain (87) one just sets $z = i$.

Under the additional assumption that V is in $L^\infty(S^1)$, Theorem 7 was already established in [Wid4]. We also note that condition (86) can be replaced by certain “non-symmetric” requirements on the decay of the Fourier coefficients V_k and V_{-k} , $k \geq 0$; see, for example, Theorem 10.32 and Corollary 10.42 of [BottSilb3].

Remark 3.3. In [Joh1], Johansson observes the following very interesting consequence of Theorem 7 (see also [Joh2]). Let $V(e^{i\theta})$ be a real-valued function on S^1 with Fourier coefficients $\{V_k\}$ satisfying (86) and suppose $V_0 = \int V(e^{i\theta}) \frac{d\theta}{2\pi} = 0$. As in the Bump-Diaconis method for SSLT described above, equip the unitary group $\mathcal{U}(n)$ with Haar measure dg and consider the random variable $F_n(g) = \sum_{j=0}^{n-1} V(e^{i\theta_j(g)}) = \text{tr } V(g)$, where $\{e^{i\theta_j(g)}\}$ are the eigenvalues of $g \in \mathcal{U}(n)$. Then we have the following Central Limit Theorem:

$$(88) \quad F_n \text{ converges in distribution to } N(0, \sigma^2), \quad \sigma = \left(2 \sum_{k=1}^{\infty} k |V_k|^2 \right)^{\frac{1}{2}}$$

where $N(0, \sigma^2)$ denotes the normal distribution with mean zero and variance σ . Indeed, for any real t we have

$$(89) \quad \int_{\mathcal{U}(n)} e^{it F_n(g)} dg = D_n(e^{itV})$$

and so by Theorem 7, as $n \rightarrow \infty$

$$(90) \quad \int_{\mathcal{U}(n)} e^{it F_n(g)} dg \rightarrow e^{\sum_{k=1}^{\infty} k(itV)_k (itV)_{-k}} = e^{-\frac{t^2}{2} [2(\sum_{k=1}^{\infty} k |V_k|^2)]}$$

from which we conclude (88). One notes that as opposed to the standard Central Limit Theorem for independent random variables, the θ_j 's are very strongly uniformly distributed and no $n^{-1/2}$ scaling is needed. A suitable scaling is needed in the more general ‘‘interpolating’’ situation considered by Duits and Johansson [DJ] where $V(z)$ depends on n (cf. Section 11). In [DJ], for each $n \geq 1$, the authors consider sequences $\{k_j(n)\}_{j \geq 1}$ of mutually distinct positive integers, and they show that for each $m \geq 1$, the random variables $\frac{1}{\sqrt{\min(k_j(n), n)}} \text{tr } g^{k_j(n)}$, $j = 1, \dots, m$, converge to independent standard complex normals. (This connects earlier results of Diaconis and Shahshahani for fixed k_j , and of Rains for $k_j(n) > n$: see [DJ] for references.) Note an interesting difference in scaling for $k_j(n) \leq n$ and for $k_j(n) > n$. In the latter case, the scaling is $n^{-1/2}$. The authors in [DJ] obtained their results by proving an analogue of SSLT for symbols depending on n in a special way, as follows. If

$$V(n; z) = \sum_{|j| > 0} \frac{\gamma_j z^{k_j(n)}}{\sqrt{\min(|k_j(n)|, n)}}$$

where $\sum_j |\gamma_j| < \infty$, the sequences $\{k_j(n)\}_{j \geq 1}$ are as above, and $k_{-j}(n) = -k_j(n)$, then

$$\lim_{n \rightarrow \infty} D_n(e^{V(n; z)}) = e^{\sum_{j=1}^{\infty} \gamma_j \gamma_{-j}}.$$

4 Back to Ising

After Onsager announced formula (19) for M_0 in 1948, and again in 1949, but did not provide a proof, three years went by until a derivation of the formula was found by Yang in a tour de force in 1952 [Yan1]. Yang’s method is based on results of Kaufman and Onsager, but his approach is different, and he does not explicitly use Toeplitz determinants. In his Selected Papers in 2005 ([Yan2]), Yang described the long and winding road that led him to the Kaufman-Onsager formula (19). The story has much charm:

“I was thus led to a long calculation, the longest in my career. Full of local, tactical tricks, the calculation proceeded by twists and turns. There were many obstructions. But always, after a few days, a new trick was somehow found that pointed to a new path. The trouble was that I soon felt I was in a maze and was not sure whether in fact, after so many turns, I was anywhere nearer the goal than when I began. This kind of strategic overview was very depressing, and several times I almost gave up. But each time something drew me back, usually a new tactical trick that brightened the scene, even though only locally.

Finally, after about six months of work off and on, all the pieces suddenly fitted together, producing miraculous cancellation, and I was staring at the amazingly simple final result.”

Remark 4.1. Let $E_\Lambda(\sigma)$ denote the interaction energy in a rectangular box Λ of size $M \times N$ as in (10) above, and let $E_{\Lambda,h}(\sigma) = E_\Lambda(\sigma) - h \sum_{(i,j) \in \Lambda} \sigma_{i,j}$ denote the energy in the presence of a magnetic field of strength h . Let $Z_{\Lambda,h} = \sum_\sigma e^{-E_{\Lambda,h}(\sigma)/k_B T}$ denote the associated partition function, and let $F_{\Lambda,h} = -\frac{k_B T}{MN} \log Z_{\Lambda,h}$ denote the associated free energy in the box Λ . What Yang proved in [Yan1] was that, for $T < T_c$,

$$(91) \quad \begin{aligned} M_Y &\equiv - \lim_{h \downarrow 0} \lim_{|\Lambda| \rightarrow \infty} \frac{F_{\Lambda,h/N} - F_{\Lambda,0}}{h/N} \\ &= (1 - k_{\text{ons}}^2)^{\frac{1}{8}} = M_0 \equiv \lim_{n \rightarrow \infty} \langle \sigma_{1,1} \sigma_{1,n} \rangle^{\frac{1}{2}}. \end{aligned}$$

(In [Yan1], Yang only considered the case $J_1 = J_2$. Yang’s method was extended to the general case $J_1 \neq J_2$ by Chang [Cha].)

As noted and emphasized by Schultz, Mattis and Lieb [ScML] in 1964, neither M_Y nor M_0 coincide, a priori, with the physically fundamental definition of the spontaneous magnetization M_{phys} , viz.,

$$(92) \quad \begin{aligned} M_{\text{phys}} &\equiv - \lim_{h \downarrow 0} \lim_{|\Lambda| \rightarrow \infty} \frac{\partial F_{\Lambda,h}}{\partial h} \\ &= \lim_{h \downarrow 0} \lim_{|\Lambda| \rightarrow \infty} \frac{1}{MN} \left[\frac{\sum_\sigma \left(\sum_{(i,j) \in \Lambda} \sigma_{i,j} \right) e^{-\frac{1}{k_B T} E_{\Lambda,h}(\sigma)}}{\sum_\sigma e^{-\frac{1}{k_B T} E_{\Lambda,h}(\sigma)}} \right]. \end{aligned}$$

Only some years later was it shown that indeed for $T < T_c$

$$(93) \quad M_{\text{phys}} = M_Y = M_0 = (1 - k_{\text{ons}}^2)^{\frac{1}{8}}$$

(see, in particular, [LebM-Lof], [BGJ-LS]).

In 1955 an important technical advance was made by Potts and Ward [PotWar] when they showed, in particular, that the correlation function $\langle \sigma_{1,1} \sigma_{1,n} \rangle$ along a row for the 2-D Ising model could be expressed in terms of a *single* Toeplitz determinant, rather than a sum of two Toeplitz determinants as in (21). They were unable at the time to verify that their solution agreed with (21): This was done later in 1963 [MPW] by Montroll, Potts and Ward, who also used recent ideas of Kasteleyn on the Pfaffian approach to the dimer and Ising problems to give a new and simpler proof of the formula in [PotWar]. The formula in [PotWar] [MPW] is the following

$$(94) \quad \langle \sigma_{1,1} \sigma_{1,1+n} \rangle = D_n(\varphi_{\text{ons}})$$

where $\varphi_{\text{ons}}(e^{i\theta})$ is the Onsager function defined in (25). Through discussions with Mark Kac, Montroll et al. were aware of SSLT. Furthermore they knew, as noted above, how to extend Szegő's result via Kac's method to the case where the symbol is complex. For $T < T_c$, Montroll, Potts and Ward noted that $0 < \gamma_1, \gamma_2 < 1$ by (30), and so $\varphi_{\text{ons}}^{\pm 1}(e^{i\theta}) \neq 0$ and φ_{ons} has no winding on S^1 . As in the computation for $\varphi_{\text{diag}}(e^{i\theta})$ above, we find

$$(95) \quad (\log \varphi_{\text{ons}})_\ell = - \left(\frac{\gamma_1^\ell}{2\ell} - \frac{\gamma_2^\ell}{2\ell} \right), \quad \ell \geq 1$$

$$(96) \quad (\log \varphi_{\text{ons}})_{-\ell} = \left(\frac{\gamma_1^\ell}{2\ell} - \frac{\gamma_2^\ell}{2\ell} \right), \quad \ell \geq 1$$

and $(\log \varphi_{\text{ons}})_0 = 0$. This implies $\sum_{\ell=1}^{\infty} \ell (\log \varphi_{\text{ons}})_\ell (\log \varphi_{\text{ons}})_{-\ell} = \frac{1}{4} \log \frac{(1-\gamma_1^2)(1-\gamma_2^2)}{(1-\gamma_1\gamma_2)^2}$ and after some elementary algebra using (26), we again obtain (19), but now via (20) (cf. the discussion following (85)).

We now consider the spontaneous magnetization for the Ising model for temperatures $T \geq T_c$. As $k_{\text{ons}} \uparrow 1$ when $T \uparrow T_c$, one anticipates from (19) that at $T = T_c$ the spontaneous magnetization M_0 is zero. In [KauOns], Kaufman and Onsager proceed in the following way. They derive an approximate formula for b_k in (23) ($b_k = \sum_{-k}$ in the notation of [KauOns]) at $T = T_c$

$$(97) \quad b_k \sim \frac{2}{\pi(2k+1)} \quad \text{as} \quad k \rightarrow \infty.$$

They then take (97) as an equality for all $k \geq 0$, and substitute this relation into (21) to obtain an approximate asymptotic formula for $\langle \sigma_{1,1} \sigma_{1,1+n} \rangle$ in the form of a linear combination $c_2^{*2} W_n - s_2^* \widetilde{W}_n$ of two Cauchy determinants W_n and $\widetilde{W}_n = W_{-n}$. Such determinants can be evaluated explicitly in terms of gamma functions⁸

$$(98) \quad |W_n| = \frac{2}{\pi} \prod_{q=1}^{n-1} \frac{\Gamma(q+1)\Gamma(q+1)}{\Gamma(q+\frac{1}{2})\Gamma(q+\frac{3}{2})},$$

$$(99) \quad |\widetilde{W}_n| = \frac{2}{3\pi} \prod_{q=1}^{n-1} \frac{\Gamma(q+1)\Gamma(q+1)}{\Gamma(q-\frac{1}{2})\Gamma(q+\frac{5}{2})}$$

and Kaufman and Onsager observe that both W_n and $\widetilde{W}_n \rightarrow 0$ as $n \rightarrow \infty$, with \widetilde{W}_n decaying at a much faster rate than W_n . They conclude that at $T = T_c$

$$(100) \quad \langle \sigma_{1,1} \sigma_{1,1+n} \rangle \sim (-1)^n c_2^{*2} W_n \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

and so $M_0 = 0$. In 1959, Fisher [Fis1] picks up on these calculations, and using Stirling's formula in (98) he shows via (100) that as $n \rightarrow \infty$

$$(101) \quad \langle \sigma_{1,1} \sigma_{1,1+n} \rangle \sim C n^{-1/4}, \quad T = T_c$$

for some (undetermined) constant C . This is the first derivation of an explicit decay rate for the correlation function at T_c . Fisher argues further that off the lattice axes, (101) may be generalized (in the case $J_1 = J_2$) to

$$(102) \quad \langle \sigma_{1,1} \sigma_{1+\ell,1+m} \rangle \sim C(\theta) k^{-1/4}, \quad T = T_c$$

⁸ Formulae (98) (99) differ slightly from the corresponding formulae in [KauOns].

where $\tan \theta = \ell/m$ and $k = (\ell^2 + m^2)^{\frac{1}{2}} \rightarrow \infty$, and $C(\theta)$ depends mildly on θ (see more below). As Fisher learned later from Onsager himself (see reference 8 in [Fis1]; also, cf. the reference preceding (31) above to [Dom2]), the analogue of formula (100) is in fact *exact* for correlations along the diagonal:

$$(103) \quad \langle \sigma_{1,1} \sigma_{1+n,1+n} \rangle = (-1)^n W_n, \quad T = T_c.$$

Indeed from (31) , (32), for $k_{\text{ons}} = 1$ at T_c ,

$$(104) \quad \varphi_{\text{diag}}(\theta) = \frac{1 - e^{-i\theta}}{|1 - e^{i\theta}|} = e^{i(\pi-\theta)/2}, \quad 0 < \theta < 2\pi$$

and so

$$(105) \quad (\varphi_{\text{diag}})_k = \frac{2}{\pi} \frac{1}{2k+1}, \quad k \in \mathbb{Z}$$

which agrees with (97) (note from (21) that W_n is a Toeplitz determinant with shifted entries b_{k-1} , and to observe (103) one can write W_n as a determinant of the transposed matrix). If one repeats Fisher's calculation and keeps track of the constant, one obtains the exact asymptotics at $T = T_c$,

$$(106) \quad \langle \sigma_{1,1} \sigma_{1+n,1+n} \rangle = e^{\frac{1}{4}} A^{-3} 2^{\frac{1}{12}} n^{-1/4} \left(1 - \frac{1}{64} \frac{1}{n^2} + \dots \right) \quad \text{as } n \rightarrow \infty$$

where

$$(107) \quad A = e^{\frac{1}{12}} e^{-\zeta'(-1)}$$

is Glaisher's constant, $\zeta =$ Riemann's zeta function. Formula (106) was first obtained by T. T. Wu in [Wu] in serendipitous circumstances, as we describe below. The full expansion of (106) to all orders was first given in [AuYPer1], together with a similar result, in the symmetric case $J_1 = J_2$, for the next-to-diagonal correlation functions.

For $T > T_c$, Fisher notes in [Fis1] that one can use the matrix viewpoint in [Ons1] to show that as $k = \sqrt{\ell^2 + m^2} \rightarrow \infty$

$$(108) \quad \langle \sigma_{1,1} \sigma_{1+\ell,1+m} \rangle \sim B k^{-\frac{1}{4}} e^{-f_k(T)}$$

for some θ -dependent constant $B = B(\theta)$, where to leading order as $k \rightarrow \infty$

$$(109) \quad f_k(T) \sim k g(T).$$

As $T \downarrow T_c$, $g(T) \downarrow 0$. These calculations show, in particular, that $M_0 = 0$ for $T > T_c$. If one includes logarithmic correction terms in (109), one obtains (see [Fis2]) for $T > T_c$

$$(110) \quad \langle \sigma_{1,1} \sigma_{1+\ell,1+m} \rangle \sim C(\theta, T) \frac{e^{-k g(T)}}{k^{1/2}} (1 + O(1))$$

as $k \rightarrow \infty$. This is the first derivation of an explicit decay rate for the correlation function at $T > T_c$. In [KauOns] the authors show how the conclusion $M_0 = 0$ also follows from their determinantal formulae, at least for $T > T_c$ sufficiently large. In [Kad], Kadanoff obtains a decay rate for $T > T_c$ of the same form as in (110), but only in the double scaling limit $T - T_c \downarrow 0$, $k \rightarrow \infty$ such that

$|T - T_c| k = c = \text{constant}$. In [Rya], Ryazanov obtains the $k^{-1/4}$ decay rate at $T = T_c$, and he also obtains the general form (110), at least in the case that T is large, $T \gg T_c$. The double scaling limit, $T \rightarrow T_c$ and $k \rightarrow \infty$, for the Ising model is an important issue, to which we will return at various points in this paper. The importance of this issue rests in the fact that many believe in universality for spin models in the sense that the double scaling limit for the spin-spin correlation function in the Ising model also gives precisely the same limit (modulo lattice-dependent factors) for a wide class of “non-integrable” two-dimensional models with short range interactions.

Remark 4.2. The result that M_{phys} , the physically fundamental spontaneous magnetization defined above, is in fact zero for all $T > T_c$, was proven rigorously only many years later by Lebowitz in [Leb] in (1972). The proof in [Leb] uses a variety of techniques from statistical mechanics, but as shown in [LebM-Lof] [BGJ-LS], $M_{\text{phys}} = M_0 = \lim_{n \rightarrow \infty} \langle \sigma_{1,1} \sigma_{1,1+n} \rangle^{\frac{1}{2}}$ for all $T > 0$, and so the proof that $M_{\text{phys}} = 0$ for $T > T_c$ (in fact, for $T \geq T_c$) follows, a posteriori, from a proof that $\lim_{n \rightarrow \infty} D_n(\varphi_{\text{ons}}) = 0$, $T \geq T_c$.

Remark 4.3. In the isotropic case, $J_1 = J_2$, explicit expressions for the correlation functions $\langle \sigma_{1,1} \sigma_{1+m,1+\ell} \rangle$ for values of (m, ℓ) other than $(0, \ell)$, $(m, 0)$, or (m, m) , were derived by Shrock and Ghosh [ShGh]. In particular, they obtained expressions in the cases $(m, \ell) = (2, 1)$, $(3, 1)$, $(3, 2)$, $(4, 1)$, $(4, 2)$, and $(4, 3)$, in terms of complete elliptic integrals K and E . They also inferred a general structural formula for arbitrary $\langle \sigma_{1,1} \sigma_{1+m,1+\ell} \rangle$ in terms of these elliptic integrals K and E .

In further developments, Au-Yang, Jin, and Perk [AuYPer2] [AuYJPer] showed that the next-to-diagonal correlations $\langle \sigma_{1,1} \sigma_{1+n,n} \rangle$ have a “bordered” Toeplitz determinant form, that is, an $n \times n$ Toeplitz determinant with the last column replaced by a certain explicit vector $(b_{n-1}, b_{n-2}, \dots, b_0)^T$. This representation is then used to express $\langle \sigma_{1,1} \sigma_{1+n,n} \rangle$ in terms of a solution to the Painlevé VI equation by Witte in [Wi].

5 New questions for SSLT

In view of the above calculations for $T \geq T_c$, the Ising problem raised new questions and challenges in Toeplitz theory. From (94), $\langle \sigma_{1,1} \sigma_{1,1+n} \rangle = D_n(\varphi_{\text{ons}})$ (similarly, from (31), $\langle \sigma_{1,1} \sigma_{1+n,1+n} \rangle = D_n(\varphi_{\text{diag}})$), and so we want to know precisely how $D_n(\varphi_{\text{ons}})$ behaves as $n \rightarrow \infty$ for $T \geq T_c$. But for $T > T_c$, we have $0 < \gamma_1 < 1 < \gamma_2$ (see (27), (30)), and so in contrast to the case $T < T_c$, the winding number of φ_{ons} is $-1 \neq 0$. And at $T = T_c$, we see from (29) that $\varphi_{\text{ons}}(e^{i\theta})$ has a jump discontinuity at $\theta = 0$. For such discontinuities the Fourier coefficients $(\log \varphi_{\text{ons}})_k$ decay as k^{-1} and so $\sum_{k=1}^{\infty} k |(\log \varphi_{\text{ons}})_k|^2 = \infty$. In both cases SSLT, or more properly Theorem 7, does not apply.

In 1966, in a tour de force in [Wu], Wu gave the first derivation of the precise asymptotics of $\langle \sigma_{1,1}, \sigma_{1,1+n} \rangle = D_n(\varphi_{\text{ons}})$ as $n \rightarrow \infty$, for $T \geq T_c$. For $T > T_c$, Wu showed that for $n \rightarrow \infty$

$$(111) \quad \begin{aligned} \langle \sigma_{1,1} \sigma_{1,1+n} \rangle &= (\pi n)^{-\frac{1}{2}} \gamma_2^{-n} (1 - \gamma_1^2)^{\frac{1}{4}} (1 - \gamma_2^{-2})^{-\frac{1}{4}} (1 - \gamma_1 \gamma_2)^{-\frac{1}{2}} \\ &\times \left(1 + \frac{A_1}{n} + \frac{A_2}{n^2} + \frac{A_3}{n^3} + \dots \right) \end{aligned}$$

with explicit expressions for A_1 , A_2 and A_3 in terms of $\gamma_1 = z_1 z_2^*$ and $\gamma_2 = z_2^*/z_1$, defined in (26). As $T > T_c \iff \gamma_2 > 1$, (111) agrees in the order of decay with (110) for $\ell = 0$ and $m = n$, but now we have the precise rate and constants. For $T = T_c$, Wu obtained

$$(112) \quad \begin{aligned} \langle \sigma_{1,1} \sigma_{1,1+n} \rangle &= e^{\frac{1}{4}} 2^{\frac{1}{12}} A^{-3} n^{-\frac{1}{4}} (1 + \gamma_1)^{\frac{1}{4}} (1 - \gamma_1)^{-\frac{1}{4}} \\ &\times (1 + B_1 n^{-2} + O(n^{-3})) \end{aligned}$$

with an explicit expression for B_1 in terms of γ_1 , and A is again Glaisher's constant (107). For $T < T_c$, Wu also obtained higher order terms in (19) (20), i.e., as $n \rightarrow \infty$

$$(113) \quad \langle \sigma_{1,1} \sigma_{1,1+n} \rangle = (1 - k_{\text{ons}}^2)^{\frac{1}{4}} \left(1 + (2\pi n^2)^{-1} \gamma_2^{2n} (\gamma_2^{-1} - \gamma_2)^{-2} \left[1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \dots \right] \right)$$

with explicit expression for c_1 and c_2 in terms of γ_1 and γ_2 . In particular as $\gamma_2 < 1$, we see that $\langle \sigma_{1,1} \sigma_{1,1+n} \rangle \rightarrow (1 - k_{\text{ons}}^2)^{\frac{1}{4}} = M_0$ exponentially fast.

There is an interesting twist to the story of Wu's computation of (112) at T_c . As a warm-up problem at T_c , $\sinh \frac{2J_1}{k_B T_c} \sinh \frac{2J_2}{k_B T_c} = 1$, he considered the limiting case when $J_1 \rightarrow 0$ and $J_2 \rightarrow \infty$ in such a way that $\gamma_1 \downarrow 0$ and $\gamma_2 \uparrow 1$ in (25), and hence

$$\varphi_{\text{ons}}(e^{i\theta}) = \left(\frac{1 - e^{-i\theta}}{1 - e^{i\theta}} \right)^{\frac{1}{2}} = e^{i(\pi-\theta)/2}, \quad 0 < \theta < 2\pi.$$

Wu recognized $\det \left((\varphi_{\text{ons}})_{j-k} \right)_{0 \leq j, k \leq n-1}$ as a Cauchy determinant and computed the asymptotics which we have recorded above in (106). Wu was unaware at the time, just like Fisher before him, that he had serendipitously computed the correlation function along the diagonal at T_c ! Had Kaufman and Onsager published their formula for $\langle \sigma_{1,1} \sigma_{1+n,1+n} \rangle$, this comedy of errors would probably have been avoided.

In the symmetric situation, $J_1 = J_2$, one finds that at T_c , $\gamma_1 = 3 - 2\sqrt{2}$ and so $\left(\frac{1+\gamma_1}{1-\gamma_1} \right)^{\frac{1}{4}} = 2^{\frac{1}{8}}$. Substituting this relation into (112), we obtain as $n \rightarrow \infty$

$$(114) \quad \langle \sigma_{1,1} \sigma_{1,1+n} \rangle = e^{\frac{1}{4}} 2^{5/24} A^{-3} n^{-\frac{1}{4}} \left(1 + O\left(\frac{1}{n^2}\right) \right), \quad J_1 = J_2.$$

This relation and (106) are consistent with the formula (cf. (102))

$$(115) \quad \langle \sigma_{1,1} \sigma_{1+\ell,1+m} \rangle = e^{\frac{1}{4}} 2^{5/24} A^{-3} k^{-\frac{1}{4}} \left(1 + O\left(\frac{1}{k^2}\right) \right), \quad J_1 = J_2$$

where $k = \sqrt{\ell^2 + m^2} \rightarrow \infty$. This asymptotic formula was derived by Cheng and Wu [ChenWu] in 1967 (they also consider $J_1 \neq J_2$), but it was only established rigorously in the last year by H. Pinson [Pin] using a Phragmen-Lindelof type argument to interpolate between (106) and (114). Recently two additional independent proofs of the rotational symmetry of $\langle \sigma_{1,1} \sigma_{1+\ell,1+m} \rangle$ as $k = \sqrt{\ell^2 + m^2} \rightarrow \infty$ have been given by Dubédat [Dub] and by Chelkak, Hongler, and Izyurov [CheHonIzy]. In [Dub] the author proceeds by equating the product of two Ising correlators with a free field (bosonic) correlator. In [CheHonIzy], the proof is based on convergence results for discrete holomorphic spinor variables pioneered by S. Smirnov. In earlier work, Smirnov used such variables to prove conformal invariance and to detect Schramm-Loewner evolution within the Ising model, in an appropriate scaling limit (for references, see [CheHonIzy]). If $J_1 \neq J_2$, one expects elliptical rather than rotational symmetry in the analog of (115) (see [ChenWu] and also (199) below, which is taken from [WMTB]).

The full expansion of $\langle \sigma_{1,1} \sigma_{1+\ell,1+m} \rangle$ as $\ell^2 + m^2 \rightarrow \infty$ at $T \neq T_c$ in terms of so-called form factors $f_{M,N}^{(j)}$, is discussed in detail in [McC1],

$$(116) \quad \langle \sigma_{1,1} \sigma_{1+m,1+\ell} \rangle = (1-t)^{\frac{1}{4}} \left(1 + \sum_{n=1}^{\infty} f_{m,\ell}^{(2n)} \right) \quad \text{for } T < T_c$$

and

$$(117) \quad \langle \sigma_{1,1} \sigma_{1+m,1+\ell} \rangle = (1-t)^{\frac{1}{4}} \sum_{n=0}^{\infty} f_{m,\ell}^{(2n+1)} \quad \text{for } T > T_c.$$

For $T < T_c$, $t = k_{\text{ons}}^2$ and for $T > T_c$, $t = k_{\text{ons}}^{-2}$. These expansions were first obtained by Wu, McCoy, Tracy and Barouch [WMTB] in 1976, but explicit expressions for the form factors were only given in 1999 and 2000 by Nickel [Nick1] [Nick2].

The derivation of the form factor expansions (116) (117) for $\langle \sigma_{1,1} \sigma_{1+m,1+\ell} \rangle$ was obtained in [WMTB] using the Fredholm determinant representations of Cheng and Wu [ChenWu]. In the special cases $m = 0$, $\ell \rightarrow \infty$, or $m = \ell \rightarrow \infty$, Lyborg and McCoy [LyMcC] used the finite Toeplitz determinant representations to derive form factor expansions as in (116) (117), but the detailed form of the expansions they obtained differs from the general expansions obtained in [WMTB]. It is an interesting, and still unresolved, question to reconcile the form factor expansions obtained in these two different ways. For recent results on the form factor expansion of the diagonal correlations $\langle \sigma_{1,1} \sigma_{1+m,1+m} \rangle$ which utilize results from [BorOk] and [GerCase], see [WiFo].

6 Fisher-Hartwig singularities. Fisher-Hartwig conjecture

In addition to the new questions and challenges raised in Toeplitz theory by the Ising model for $T \geq T_c$, other kinds of singularities also began to appear. For example, in 1964, in his work on the ground state of the one-dimensional system of impenetrable bosons [Len1] Lenard expressed the one-particle density matrix for the system as a Toeplitz matrix with a symbol $\varphi(e^{i\theta})$ that vanished at some points on S^1 (see (125) and particularly Remark 6.1 below for further discussion). In 1968, in a major step in the development and generalization of SSLT, Fisher and Hartwig [FisHart1] introduced a class of singular symbols for Toeplitz determinants, that allowed for zeros, (integrable) singularities, discontinuities and non-zero winding numbers.

The symbols of Fisher-Hartwig (FH) class have the following form (we use the notation in [DIK1])

$$(118) \quad f(z) = e^{V(z)} z^{\sum_{j=0}^m \beta_j} \prod_{j=0}^m |z - z_j|^{2\alpha_j} g_{z_j, \beta_j}(z) z_j^{-\beta_j}, \quad z = e^{i\theta}, \quad 0 \leq \theta < 2\pi,$$

for some $m = 0, 1, 2, \dots$, where

$$(119) \quad z_j = e^{i\theta_j}, \quad j = 0, 1, \dots, m, \quad 0 = \theta_0 < \theta_1 < \dots < \theta_m < 2\pi,$$

$$(120) \quad g_{z_j, \beta_j}(z) \equiv g_{\beta_j}(z) = \begin{cases} e^{i\pi\beta_j}, & 0 \leq \arg z < \theta_j, \\ e^{-i\pi\beta_j}, & \theta_j \leq \arg z < 2\pi, \end{cases}$$

$$(121) \quad \Re\alpha_j > -\frac{1}{2}, \quad \beta_j \in \mathbb{C}, \quad j = 0, 1, \dots, m,$$

and $V(e^{i\theta})$ is a sufficiently smooth function on S^1 . Here the condition on $\Re\alpha_j$ insures integrability. Note that a Fisher-Hartwig singularity at z_j , $j = 1, \dots, m$, consists of a root-type singularity

$$(122) \quad |z - z_j|^{2\alpha_j} = \left| 2 \sin \frac{\theta - \theta_j}{2} \right|^{2\alpha_j}$$

and a jump singularity $z^{\beta_j} g_{\beta_j}(z)$ at z_j (observe that $z^{\beta_j} g_{\beta_j}(z)$ is continuous at $z = 1$ for $j \neq 0$). For $j = 0$, $z_0 = 1$, in addition to the root-type singularity (122) at $\theta_0 = 0$, one has $g_{z_0, \beta_0}(z) = e^{-i\pi\beta_0}$, $0 \leq \theta < 2\pi$ and so $z^{\beta_0} g_{z_0, \beta_0}(z) = e^{i(\theta-\pi)\beta_0}$, $0 \leq \theta < 2\pi$, has a jump at $z = z_0 = 1$. The factors $z_j^{-\beta_j}$ are singled out to simplify comparison with the existing literature, in particular, [FisHart1].

Here are some examples of FH symbols:

- If $f(e^{i\theta}) = -i$ for $0 \leq \theta < \pi$ and $+i$ for $\pi \leq \theta < 2\pi$ (cf [BasTr]), then $z_0 = 1$, $z_1 = -1 = e^{i\pi}$, $\alpha_0 = \alpha_1 = 0$, $\beta_0 = \frac{1}{2}$ and $\beta_1 = -\frac{1}{2}$, and

$$(123) \quad \begin{aligned} f(e^{i\theta}) &= z^{\beta_0+\beta_1} g_{1, \frac{1}{2}}(z) g_{-1, -\frac{1}{2}}(z) z_0^{-\beta_0} z_1^{-\beta_1} \\ &= g_{1, \frac{1}{2}}(z) g_{-1, -\frac{1}{2}}(z) e^{i\pi/2} \end{aligned}$$

- If $f(e^{i\theta})$ is a sufficiently smooth function on S^1 (in particular, a trigonometric polynomial) with two zeros at $0 < \theta_1 < \theta_2 < 2\pi$ (cf [BGrM], [DIK2]), then with $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2}$, $\alpha_1 = \alpha_2 = 1/2$, $\beta_1 = \frac{1}{2}$, $\beta_2 = -\frac{1}{2}$,

$$(124) \quad \begin{aligned} f(z) &= e^{V(z)} z^{\beta_1+\beta_2} |z - z_1| |z - z_2| g_{z_1, \frac{1}{2}}(z) g_{z_2, -\frac{1}{2}}(z) \left(\frac{z_1}{z_2}\right)^{-\frac{1}{2}} \\ &= e^{V(z)} |z - z_1| |z - z_2| g_{z_1, \frac{1}{2}}(z) g_{z_2, -\frac{1}{2}}(z) (z_1/z_2)^{-\frac{1}{2}} \end{aligned}$$

for a suitable function $V(e^{i\theta})$ on S^1 .

- The following symbol arose in Lenard's work in [Len1] on impenetrable bosons mentioned above,

$$(125) \quad f(e^{i\theta}) = \left| z - e^{i\theta_1} \right| \left| z - e^{i\theta_2} \right|$$

where $\theta_1 \neq \theta_2 \pmod{2\pi}$. Here $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2}$, $\alpha_1 = \alpha_2 = 1/2$ and $\beta_1 = \beta_2 = 0$.

- For the Ising model at $T < T_c$, the winding number of $\varphi_{\text{ons}}(e^{i\theta})$ is zero and so $\varphi_{\text{ons}}(z) = e^{V(z)}$ for a suitable smooth function $V(e^{i\theta})$ on S^1 . For $T = T_c$, we have from (29) with $0 < \gamma_1 < 1$,

$$(126) \quad \begin{aligned} \varphi_{\text{ons}}(e^{i\theta}) &= \frac{1 - \gamma_1 e^{i\theta}}{|1 - \gamma_1 e^{-i\theta}|} i e^{-i\theta/2}, \quad 0 \leq \theta < 2\pi, \\ &= e^{V(z)} z^{\beta_0} g_{z_0, \beta_0}(z) z_0^{-\beta_0} \end{aligned}$$

where $z_0 = 1$, $\alpha_0 = 0$, $\beta_0 = -\frac{1}{2}$ and $V(e^{i\theta})$ is smooth on S^1 . For $T > T_c$, we have from (28) with $0 < \gamma_1 < 1 < \gamma_2$,

$$(127) \quad \begin{aligned} \varphi_{\text{ons}}(e^{i\theta}) &= \frac{(1 - \gamma_1 e^{i\theta})(\gamma_2 - e^{i\theta})}{|1 - \gamma_1 e^{-i\theta}| |1 - \gamma_2 e^{i\theta}|} (-e^{-i\theta}) \\ &= e^{V(z)} z^{\beta_0} g_{z_0, \beta_0}(z) z_0^{-\beta_0} \end{aligned}$$

where $z_0 = 1$, $\alpha_0 = 0$, $\beta_0 = -1$ and again $V(z)$ is smooth on S^1 .

From the point of view of Toeplitz determinants with FH singularities, Wu was the first to obtain [Wu] detailed asymptotic results for symbols with discontinuities as in (126) and for symbols with non-zero winding as in (127). Lenard was the first to obtain [Len1] [Len2] detailed asymptotic results for symbols which have zeros on S^1 as in (125). For symbols (118) with $\beta_j = 0$ and α_j real, $\alpha_j > -\frac{1}{2}$, $j = 0, \dots, m$, Lenard made a general conjecture [Len2]⁹ about the asymptotic form of $D_n(f)$,

$$(128) \quad D_n(f) \sim E(e^V, \alpha_0, \dots, \alpha_m, \theta_0, \dots, \theta_m) n^{\sum_{j=0}^m \alpha_j^2} e^{nV_0}$$

as $n \rightarrow \infty$, where $V_0 = \frac{1}{2\pi} \int_0^{2\pi} V(e^{i\theta}) d\theta$ as in (37) (87), and $E(f, \alpha_0, \dots, \alpha_m, \theta_0, \dots, \theta_m)$ is some (unspecified) constant. Moreover he was able to verify this conjecture for the case $f(e^{i\theta}) = |z - e^{i\theta_1}| |z - e^{i\theta_2}|$ in (125).

Remark 6.1. As noted above, the symbol (125) arose in Lenard's study of impenetrable bosons [Len1]. More precisely, Lenard considered N free bosons in one dimension confined to a box of length L with periodic boundary conditions and lying in the (symmetric) ground state of the system. According to the criterion of Penrose and Onsager [PenOns], Bose-Einstein condensation takes place if the largest eigenvalue $\lambda_{N,L}^{(\max)}$ of the density matrix $\rho_{N,L}(x, x')$ for the system is of order N as $N \sim L \rightarrow \infty$. For the system at hand $\rho_{N,L}$ is translation invariant, $\rho_{N,L}(x, x') = \rho_{N,L}(x - x')$, so that the eigenvalues of $\rho_{N,L}$ are just the Fourier coefficients $\rho_{N,L}^{(n)}$ of $\rho_{N,L}(\cdot)$. A simple calculation shows that $\lambda_{N,L}^{(\max)} = \rho_{N,L}^{(0)}$, the zeroth Fourier coefficient. Writing $\rho_{N,L}(\xi) = \frac{1}{L} R_N(2\pi\xi/L)$, we have that

$$(129) \quad \frac{\lambda_{N,L}^{(\max)}}{N} = \frac{1}{2\pi N} \int_{-\pi}^{\pi} R_N(t) dt.$$

(Note that in this case $\lambda_{N,L}^{(\max)}$ is actually independent of L .) Now it turns out that

$$(130) \quad R_N(t) = D_{N-1}(f_t)$$

where f_t is precisely a symbol of type (125),

$$(131) \quad f_t(e^{i\theta}) = |e^{i\theta} - e^{it/2}| |e^{i\theta} - e^{-it/2}|.$$

(Formula (130) was also obtained independently by Dyson, see [Len1].) At that point (April 10, 1963, according to a letter Lenard has kindly made available to the authors) Lenard asked Szegő for help in evaluating the Toeplitz determinant $D_{N-1}(f_t)$. On July 10, three months later, Szegő wrote back to Lenard with the information that, although he had not been able to obtain precise asymptotics, he had obtained the bound

$$(132) \quad |R_N(t)| \leq \left| \frac{eN}{\sin(t/2)} \right|^{1/2}.$$

Substituting (132) into (129), we see that

$$(133) \quad \frac{\lambda_{N,L}^{(\max)}}{N} = O(N^{-1/2}) \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

⁹ Note that although [Len2] was published in 1972, a preliminary unpublished version of the paper was already in existence in 1968 (see [FisHart1]).

In a letter of thanks to Szegő on July 23, Lenard describes the situation as follows: “This has great physical interest: It may be expressed as saying that for the model considered there is no Bose-Einstein condensation in momentum space”.¹⁰

Following a further suggestion contained in the above correspondence with Szegő in 1963, Lenard considered, in particular, symbols of the form

$$(134) \quad f^{(\lambda, \mu)}(e^{i\theta}) = |e^{i\theta} - e^{i\pi/2}|^\lambda |e^{i\theta} - e^{-i\pi/2}|^\mu, \quad \lambda, \mu > 0.$$

He started with the general formula for positive symbols $f(e^{i\theta}) > 0$

$$(135) \quad D_n(f) = \prod_{k=0}^{n-1} \chi_k^{-2}$$

where $\chi_k > 0$ is the leading coefficient of the orthonormal polynomial $p_k(z) = \chi_k z^k + \dots, k \geq 0$, with respect to the weight of $d\mu = f(e^{i\theta}) d\theta/(2\pi)$ on S^1 as in (39). For $f^{(\lambda, \mu)}(e^{i\theta})$ in (134), the polynomials $p_k(z)$ can be expressed in terms of classical Jacobi polynomials on the interval $[-1, 1]$. This then leads to explicit formulae for the χ_k 's in terms of Euler's gamma functions.

After Lenard published the bound (132) for the impenetrable boson problem (a Plasma Physics Laboratory preprint of [Len1] appeared in 1963), Dyson [Dy3] took up the problem of computing the asymptotics of (129) precisely as $N \rightarrow \infty$. In order to control the singularity in the symbol $f_t(e^{i\theta}) = |e^{i\theta} - e^{it/2}| |e^{i\theta} - e^{-it/2}|$, Dyson resorted to his interpretation of $D_{N-1}(f_t)$ as a Coulomb gas at inverse temperature $\beta = 2$ with $N - 1$ unit charges free to move around on the unit circle, but now with two additional half-charges. The probability of finding the half-charges at an angular distance t apart is then proportional to $P(t) = |2 \sin \frac{t}{2}|^{1/2} R_N(t)$. On physical grounds, Dyson then argued that for distances t much greater than the “Debye length” $\lambda = 2\pi/N$, the two half-charges are effectively screened off from each other, and so $P(t) = \text{const.}$ for $t \gg \lambda$. In other words, $R_N(t) = k_N (\sin \frac{t}{2})^{-1/2}$ for $t \gg N^{-1}$, where k_N is a constant. However, clearly $k_N = R_N(\pi)$, which can be evaluated explicitly as $N \rightarrow \infty$ since

$$f_t(e^{i\theta})|_{t=\pi} = f^{(\lambda, \mu)}(e^{i\theta})|_{\lambda=\mu=1}$$

(see discussion of (134) above). Substituting the result into (129) and neglecting the contribution from $|t| < N^{-1}$, Dyson obtained, as $N \rightarrow \infty$,

$$(136) \quad \frac{\lambda_{N,L}^{(\max)}}{N} \sim CN^{-1/2}, \quad C = \left(\frac{e}{\pi}\right)^{1/2} 2^{-5/6} A^{-6} \Gamma\left(\frac{1}{4}\right)^2$$

where A is again Glaisher's constant (107). The crux of the above calculation is clearly the explicit evaluation of $R_N(\pi)$ as $N \rightarrow \infty$.

More detailed analysis of (129) requires further study of the double-scaling limit as $t \rightarrow 1, N \rightarrow \infty$. We return to this issue in Section 11.

Inspired by the work of Wu in [Wu], and perhaps also influenced by the calculations of Lenard in [Len1] [Len2], Fisher and Hartwig [FisHart1] made a conjecture about the asymptotic form of $D_n(f)$ for general symbols (118),

$$(137) \quad D_n(f) = E(e^V, \alpha_0, \dots, \alpha_m, \beta_0, \dots, \beta_m, \theta_0, \dots, \theta_m) \\ \times n^{\sum_{j=0}^m (\alpha_j^2 - \beta_j^2)} e^{nV_0} (1 + o(1)), \quad V_0 = \frac{1}{2\pi} \int_0^{2\pi} V(e^{i\theta}) d\theta$$

¹⁰This confirms an earlier result of Schultz [Sch] proving the absence of condensation. Schultz used other methods which produced the (weaker) bound $O(N^{-4/\pi^2})$.

as $n \rightarrow \infty$, together with a conjecture on the general form of the constant E . In [FisHart2], Hartwig and Fisher present various examples, heuristics and theorems in support of their conjecture.

In the remarkable paper [Wid2], representing “a jump of several quanta in depth and sophistication” (see MathSciNet MR0331107), Widom verified Lenard’s conjecture for complex α_j with $\Re\alpha_j > -\frac{1}{2}$, $j = 0, \dots, m$, and $V(e^{i\theta})$ suitably smooth, and obtained an explicit expression for the constant

$$(138) \quad E(e^V, \alpha_0, \dots, \alpha_m, \theta_0, \dots, \theta_m) = E(e^V) \prod_{0 \leq j < k \leq m} \left| e^{i\theta_j} - e^{i\theta_k} \right|^{-2\alpha_j \alpha_k} \\ \times \prod_{j=0}^m e^{-\alpha_j \widehat{V}(e^{i\theta_j})} \times \prod_{j=0}^m E_{\alpha_j}$$

where

$$(139) \quad E(e^V) = e^{\sum_{k=1}^{\infty} k V_k V_{-k}}, \quad V_k = \text{Fourier coefficient of } V(e^{i\theta}),$$

$$(140) \quad \widehat{V}(e^{i\theta}) = V(e^{i\theta}) - V_0$$

and

$$(141) \quad E_{\alpha} = G^2(1 + \alpha)/G(1 + 2\alpha), \quad G(z) = \text{Barnes G-function (see [Bar])}.$$

Formula (138) is consistent with and confirms the general form of the conjecture [FisHart1] for the constant term in the case $\beta_0 = \beta_1 = \dots = \beta_m = 0$.

In [Bas1] Basor considered complex α_j with $\Re\alpha_j > -\frac{1}{2}$, β_j pure imaginary, $j = 0, \dots, m$, and $V(e^{i\theta})$ suitably smooth, and verified (137) with constant term, again consistent with the general conjecture in [FisHart1],

$$(142) \quad E(e^V, \alpha_0, \dots, \alpha_m, \beta_0, \dots, \beta_m, \theta_0, \dots, \theta_m) \\ = E(e^V) \prod_{j=0}^m \left[b_+(z_j)^{-\alpha_j + \beta_j} b_-(z_j)^{-\alpha_j - \beta_j} \right] \\ \times \prod_{0 \leq j < k \leq m} \left[|z_j - z_k|^{2(\beta_j \beta_k - \alpha_j \alpha_k)} \left(\frac{z_k}{z_j e^{i\pi}} \right)^{\alpha_j \beta_k - \alpha_k \beta_j} \right] \\ \times \prod_{j=0}^m \frac{G(1 + \alpha_j + \beta_j) G(1 + \alpha_j - \beta_j)}{G(1 + 2\alpha_j)}.$$

with $E(e^V)$ and $G(z)$ as above, and

$$(143) \quad b_+(z) = e^{\sum_{k=1}^{\infty} V_k z^k}, \quad b_-(z) = e^{\sum_{k=-1}^{-\infty} V_k z^k}.$$

The branches in (142) are determined naturally, $b_+(z_j)^{-\alpha_j + \beta_j} = e^{(-\alpha_j + \beta_j) \sum_{k=1}^{\infty} V_k z_j^k}$ etc, and

$$\left(\frac{z_k}{z_j e^{i\pi}} \right)^{(\alpha_j \beta_k - \alpha_k \beta_j)} = e^{i(\theta_k - \theta_j - \pi)(\alpha_j \beta_k - \alpha_k \beta_j)}, \quad 0 \leq \theta_j < \theta_k < 2\pi.$$

In the paper [BasHelt] discussed above, Basor and Helton also introduce the first example of a so-called “separation” theorem of the following form: For symbols φ and ψ of a certain prescribed type, including FH symbols with disjoint singularities, they show that

$$(144) \quad \lim_{n \rightarrow \infty} \frac{D_n(\varphi\psi)}{D_n(\varphi)D_n(\psi)} \equiv L(\varphi, \psi)$$

exists. Moreover, they provide an explicit form for $L(\varphi, \psi)$. Iterating this result, they are able to reduce the problem for general FH symbols to adding in pure FH singularities of the form $z^\beta |z - \widehat{z}|^{2\alpha} g_{\widehat{z}, \beta}(z) \widehat{z}^{-\beta}$, $|\widehat{z}| = 1$, one at a time. They analyze the asymptotics of Toeplitz determinants with such pure (i.e., with $V \equiv 0$) FH singularities in the following way. In addition to the results described above, in [Wid2] Widom also verified (137) for a single FH singularity with $|\Re \alpha| < \frac{1}{2}$ and $|\Re \beta| < \frac{1}{2}$, but without determining the exact value of the constant E . But then Basor’s formula (142) for E obtained in [Bas1] for $\Re \alpha_j > -\frac{1}{2}$ and β_j pure imaginary, together with Vitali’s theorem, directly yields the desired explicit asymptotics for a single pure FH singularity with $|\Re \alpha| < \frac{1}{2}$ and $|\Re \beta| < \frac{1}{2}$. (In fact, as discovered later by Böttcher and Silbermann, see (145) below, there exists an exact formula for a Toeplitz determinant with a single pure FH singularity in terms of the Barnes G -functions, which easily yields the asymptotics). In the end, Basor and Helton are able to verify (137) (142) for α_j, β_k in the open set $\{\max_{j,k} (|\alpha_j|, |\beta_k|) < \delta\} \subset \mathbb{C}^{2n}$ for some (small) $\delta > 0$.

In [Bas3], Basor showed using a separation theorem that (137) (142) also hold in the case $\alpha_j = 0$, $|\Re \beta_j| < \frac{1}{2}$, $j = 0, \dots, m$.

Böttcher and Silbermann [BottSilb5] finally proved (137) with Basor’s constant (142) under the assumption that $|\Re \alpha_j| < 1/2$ and $|\Re \beta_j| < 1/2$ for all j . This is the most general setting in which the conjecture (137) is true in its original form, that is, with the exponent of n conjectured to be $\sum (\alpha_j^2 - \beta_j^2)$, without the need to take care of degenerate situations and of the ambiguity in the β ’s (see (154) below). We note that in [BottSilb6] it had already been shown that (137) does not hold in general if $\alpha_j + \beta_j$ and $\alpha_j - \beta_j$ are nonnegative integers for all $j = 0, \dots, m$, and a corrected version of the conjecture was proved in this case (this is a particular case of a formula conjectured by Basor and Tracy: see next section). The developments up to the 1990’s are described in [Bott1] and [Ehr] and also in the books [BottSilb2] [BottSilb3]. We note that certain separation theorems also play a key role in the work of Böttcher and Silbermann. Another important element in their proofs is an explicit formula, noted above, for a Toeplitz determinant with a single, pure FH singularity found by the authors in [BottSilb5] (see also [BottWid2] [BasChe] for later alternative derivations). Namely, if $V(z) \equiv 0$, $m = 0$, then with $\alpha_0 \equiv \alpha$, $\beta_0 \equiv \beta$,

$$(145) \quad D_n(f) = \frac{G(1 + \alpha + \beta)G(1 + \alpha - \beta)}{G(1 + 2\alpha)} \frac{G(n + 1)G(n + 1 + 2\alpha)}{G(n + 1 + \alpha + \beta)G(n + 1 + \alpha - \beta)},$$

$$\Re \alpha > -\frac{1}{2}, \quad \alpha \pm \beta \neq -1, -2, \dots, \quad n \geq 1.$$

The condition on $\alpha \pm \beta$ is needed because the G -function has zeros at $z = 0, -1, -2, \dots$. The asymptotics for (145) follow from the asymptotics of the G -function (see [Bar])

$$(146) \quad \log G(t + a + 1) = \frac{1}{12} - \log A - \frac{3t^2}{4} - at + \frac{t + a}{2} \log(2\pi)$$

$$+ \left(\frac{t^2}{2} + at + \frac{a^2}{2} - \frac{1}{12} \right) \log t + o(1), \quad \text{as } t \rightarrow \infty$$

where again A is Glaisher’s constant (107). The resulting asymptotics are consistent with (137) (142).

In view of the Heine representation (45), it is not surprising that the formula (145) is related to the Selberg integral. In fact, independently of [BottSilb5], a more general formula was obtained by Selberg himself:

$$(147) \quad \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{0 \leq j < k \leq n-1} |e^{i\theta_j} - e^{i\theta_k}|^{2\gamma} \prod_{j=0}^{n-1} e^{i(\theta_j - \pi)\beta} |1 - e^{i\theta_j}|^{2\alpha} \frac{d\theta_j}{2\pi} \\ = \prod_{j=0}^{n-1} \frac{\Gamma(1 + 2\alpha + j\gamma)\Gamma(1 + (j+1)\gamma)}{\Gamma(1 + \alpha + \beta + j\gamma)\Gamma(1 + \alpha - \beta + j\gamma)\Gamma(1 + \gamma)}$$

where

$$\Re\alpha > -\frac{1}{2}, \quad \Re\gamma > -\min\left\{\frac{1}{n}, \frac{\Re(2\alpha + 1)}{n-1}\right\}.$$

In a published form, this formula first appeared in the thesis of Morris in 1982 (see [ForWar] for a historic account). The interest in (147) was due to the efforts to prove one of Dyson's conjectures in the theory of random matrices in [Dy1]. If we set $\gamma = 1$ in (147), and use the relation $G(z+1) = \Gamma(z)G(z)$, we recover (145) from (45).

The Barnes' function $G(z)$ is a basic object in analysis, bearing the same relation to the gamma function as the gamma function bears to the function z , $\Gamma(z+1) = z\Gamma(z)$. Just as $\log\Gamma(z)$ is a sum of values of $\log z$, so $\log G(z)$ is the iterated sum. It is of interest to delineate some of the analytic pathways by which $G(z)$ appears in the asymptotics of $D_n(f)$.

The most direct example, a particular case of (145), arises for the diagonal correlation function $\langle \sigma_{1,1} \sigma_{1+n,1+n} \rangle$ at T_c , where $\varphi_{\text{diag}}(\theta) = e^{i(\pi-\theta)/2}$, i.e. $V \equiv 0$, $m = 0$, $\alpha_0 = 0$, $\beta_0 = -1/2$, in (104) gives rise to the Cauchy determinant (103) which can be evaluated as in (98) in terms of a product of gamma functions, and hence in terms of G -functions, $\prod_{q=1}^{n-1} \Gamma(q+1) = G(n+1)/G(2)$, etc.

For a system of polynomials orthonormal with respect to a function f on the unit circle, the leading coefficients χ_k are related to $D_n(f)$ by the product formula (135). As we discussed above, in the case of f given by (134) the coefficients χ_k are expressed in terms of gamma functions, and therefore G -functions appear in the product. More generally, there is a certain universality in the asymptotic behavior of orthogonal polynomials associated with FH symbols on S^1 (see [DIK1] [DIK3]), and, mutatis mutandis, for general FH symbols f , the G -functions enter the arena through the same stage door.

As noted above in (126), at $T = T_c$, $\varphi_{\text{ons}}(e^{i\theta})$ has a single FH singularity at $z_0 = 1$ with $\alpha_0 = 0$ and $\beta_0 = -\frac{1}{2}$. The first confirmation of (137) (142) which allowed for these values was given in [BottSilb4], where the authors proved the result for $m = 0$, $\Re\alpha_0 \geq 0$, $\Re(\alpha_0 + \beta_0) > -1$ and $\Re(\alpha_0 - \beta_0) > -1$. It is instructive to recover Wu's asymptotic formula (112) from (137) (142). Here $V(e^{i\theta}) = \frac{1}{2} \log\left(\frac{1-\gamma_1 e^{i\theta}}{1-\gamma_1 e^{-i\theta}}\right)$ and we find $V_0 = 0$ and $V_k = -\gamma_1^{|k|}/2k$ for $k \neq 0$. Thus $E(e^V) = e^{-\frac{1}{4} \sum_{k=1}^{\infty} k \gamma_1^{2k}/k^2} = (1 - \gamma_1^2)^{\frac{1}{4}}$. On the other hand $(b_+(1))^{-\frac{1}{2}} (b_-(1))^{\frac{1}{2}} = e^{\frac{1}{4} \sum_{k=1}^{\infty} \gamma_1^k/k} e^{\frac{1}{4} \sum_{k=1}^{-\infty} \gamma_1^{|k|}/|k|} = (1 - \gamma_1)^{-\frac{1}{2}}$. But $G(\frac{1}{2}) = 2^{\frac{1}{24}} e^{\frac{1}{8}} \pi^{-1/4} A^{-3/2}$, where again A is Glaisher's constant (see [Bar]), and we have $G(\frac{3}{2}) = \Gamma(\frac{1}{2})G(\frac{1}{2}) = 2^{\frac{1}{24}} e^{\frac{1}{8}} \pi^{\frac{1}{4}} A^{-3/2}$. Thus $G(\frac{1}{2})G(\frac{3}{2}) = 2^{\frac{1}{12}} e^{\frac{1}{4}} A^{-3}$. Substituting these relations into (142), we obtain $E(e^V, 0, \frac{1}{2}, \theta_0 = 0) = 2^{\frac{1}{12}} e^{\frac{1}{4}} A^{-3} (1 + \gamma_1)^{\frac{1}{4}} (1 - \gamma_1)^{-\frac{1}{4}}$, and as $n^{-\beta_0^2} = n^{-\frac{1}{4}}$, we arrive at precisely the leading order term in (112).

The **definitive result** for the leading order asymptotics of Toeplitz determinants with FH singularities was obtained by Ehrhardt in his PhD thesis at TU Chemnitz in 1997 (see [Ehr]). Let

$|||\beta|||$ denote the seminorm

$$(148) \quad |||\beta||| = \max_{j,k} |\Re\beta_j - \Re\beta_k|$$

where $1 \leq j, k \leq m$ if $\alpha_0 = \beta_0 = 0$, and $0 \leq j, k \leq m$ otherwise. If $m = 0$, set $|||\beta||| = 0$. Note that in the case of a single singularity, we always have $|||\beta||| = 0$. If $V(e^{i\theta})$ is C^∞ on S^1 , $|||\beta||| < 1$, $\Re\alpha_k > -\frac{1}{2}$ and $\alpha_j \pm \beta_j \neq -1, -2, \dots$ for $j, k = 0, 1, \dots, m$, then Ehrhardt proved the Fisher-Hartwig conjecture (137) with Basor's constant (142). As above, the conditions on $\alpha_j \pm \beta_j$ ensure that Basor's constant is non-zero and (142) indeed provides the leading order asymptotics for $D_n(f)$ as $n \rightarrow \infty$. A key element in Ehrhardt's proof is a suitably generalized version of the separation theorem in [BasHelt].¹¹

An independent proof of Ehrhardt's result was given recently in [DIK3] where only a finite degree of smoothness is needed for $V(e^{i\theta})$. In [DIK3], estimates for the error term in the asymptotics are also provided. If $V(z) \in C^\infty$ on S^1 , the asymptotics (137) (142) hold with error term

$$(149) \quad o(1) = O(n^{|||\beta|||-1}).$$

Note that the smallness of the error term breaks down if $|||\beta||| \geq 1$. We will consider this situation in detail in the next section. The main steps of the method of [DIK3] are as follows. First, the authors derive differential identities for the logarithmic derivatives of $D_n(f)$, in particular, for

$$(150) \quad \frac{\partial}{\partial\alpha_j} \log D_n(f), \quad \frac{\partial}{\partial\beta_j} \log D_n(f),$$

in terms of the associated OPUC's of degrees n and $n+1$ only. Second, the authors obtain the asymptotics of these OPUC's using their Riemann-Hilbert representation and the steepest descent analysis of the Riemann-Hilbert problem for large n . Third, the asymptotic formulas so obtained are substituted into the RHS of the differential identities, and the final result (137) (142) is obtained by integration.¹² In [DIK1], [DIK3] the authors also note various uniformity properties of the large n asymptotics for Toeplitz determinants in the parameters, such as α_j, β_j , on which f depends. This issue becomes important in discussions of double-scaling limits (cf Section 11).

For the Ising model at $T > T_c$, we see from (127) that $z_0 = 1$, $\alpha_0 = 0$ and $\beta_0 = -1$, and so $\alpha_0 + \beta_0 = -1$. Thus (137) (142) do not provide the leading order asymptotics for $\langle \sigma_{1,1} \sigma_{1,1+n} \rangle$ in this case. One must analyze lower order terms, in fact go "beyond all orders" in the language of Kruskal, to capture the exponential decay in Wu's result (111) (see Remark 7.1 below).

7 Basor-Tracy conjecture

As already observed in [BottSilb5] [BottSilb6], Ehrhardt's condition on the seminorm, $|||\beta||| < 1$, is not just a technical matter. In a critical calculation in 1991, Basor and Tracy [BasTr] considered a particular symbol $f(e^{i\theta})$ which has two FH singularities at $z_0 = 1$ and $z_1 = e^{i\pi} = -1$ respectively

$$(151) \quad f(e^{i\theta}) = g_{1, \frac{1}{2}}(z) g_{-1, -\frac{1}{2}}(z) e^{i\pi/2}$$

¹¹Note that the RHS of (137) and (142) make sense for any $\alpha_j, 2\alpha_j \neq -1, -2, \dots$. In fact, (137) (142) still hold in this case. This extension to the left of the lines $\Re\alpha_j = -1/2$ is explained in [Ehr] and corresponds to replacing the symbol f by an appropriate distribution.

¹²Note that if one just used the formula (135) instead of the differential identities, then the product $\prod_{k=0}^{n_0} \chi_k^{-2}$ for small n_0 , and hence the constant factor E in (137), would remain undetermined.

with $\beta_0 = \frac{1}{2}$, $\beta_1 = -\frac{1}{2}$ so that $\|\beta\| = 1$. By direct computation they found that as $n \rightarrow \infty$

$$(152) \quad D_n(f) = \frac{1 + (-1)^n}{2} \sqrt{\frac{2}{n}} G\left(\frac{1}{2}\right)^2 G\left(\frac{3}{2}\right)^2 (1 + O(1))$$

which is not of the general FH asymptotic form. Basor and Tracy observed, however, that $f(e^{i\theta})$ had another FH representation with $\beta_0 = -\frac{1}{2}$ and $\beta_1 = \frac{1}{2}$,

$$(153) \quad f(e^{i\theta}) = e^{i\pi} g_{1, -\frac{1}{2}}(z) g_{-1, \frac{1}{2}}(z) e^{-i\pi/2}.$$

If one substitutes the values $V = 0$, $\alpha_0 = \alpha_1 = 0$, $\beta_0 = \frac{1}{2}$, $\beta_1 = -\frac{1}{2}$ into (137) (142), one obtains for (151)

$$\begin{aligned} n^{-(\frac{1}{4} + \frac{1}{4})} |1 - e^{i\pi}|^{-\frac{1}{2}} \frac{G(\frac{3}{2}) G(\frac{1}{2})}{G(1)} \frac{G(\frac{1}{2}) G(\frac{3}{2})}{G(1)} (1 + o(1)) \\ = (2n)^{-\frac{1}{2}} G\left(\frac{3}{2}\right)^2 G\left(\frac{1}{2}\right)^2 (1 + o(1)) \end{aligned}$$

as $G(1) = 1$. On the other hand, if one substitutes the values $V = i\pi$, $\alpha_0 = \alpha_1 = 0$, $\beta_0 = -\frac{1}{2}$, $\beta_1 = \frac{1}{2}$, one obtains for (153)

$$e^{in\pi} (2n)^{-\frac{1}{2}} G^2\left(\frac{1}{2}\right) G^2\left(\frac{3}{2}\right) (1 + o(1)).$$

Just adding these two asymptotic forms, we obtain precisely the Basor-Tracy result (152). So we see that if a symbol $f(e^{i\theta})$ has more than one FH representation, then the asymptotics of $D_n(f)$ is given, at least in this case, by some combination of the Fisher-Hartwig asymptotics for each of them. Based on this example, Basor and Tracy [BasTr] made the following bold, general conjecture. First note the following. Let $f(e^{i\theta})$ be a Fisher-Hartwig symbol as in (118). Let n_0, n_1, \dots, n_m be a set of integers with $\sum_{j=0}^m n_j = 0$, and let $f(z; \hat{\beta})$ denote the FH symbol obtained by replacing β_j with $\hat{\beta}_j = \beta_j + n_j$, $0 \leq j \leq m$, and $e^{V(z)}$ by $\left(\prod_{j=0}^m z_j^{n_j}\right) e^V = e^{V + i \sum_{j=0}^m n_j \theta_j}$. Then $f(\hat{\beta}) = f(z; \hat{\beta})$ gives another FH representation for $f(z)$,

$$(154) \quad f(z) = f(z; \hat{\beta}), \quad \hat{\beta}_j = \beta_j + n_j, \quad \sum_{j=0}^m n_j = 0.$$

Moreover, it is easy to see that all FH representations of $f(z)$ arise in this way. Given β , we call

$$(155) \quad O_\beta = \left\{ \hat{\beta} : \hat{\beta}_j = \beta_j + n_j, \quad \sum_{j=0}^m n_j = 0 \right\}$$

the *orbit* of β . We consider the discrete minimization problem

$$(156) \quad F_\beta = \min_{\hat{\beta} \in O_\beta} \left(\sum_{j=0}^m (\Re \hat{\beta}_j)^2 \right).$$

It is clear that the minimum is indeed obtained in O_β . Let $\mathcal{M}_\beta = \left\{ \hat{\beta} \in O_\beta : \sum_{j=0}^m (\Re \hat{\beta}_j)^2 = F_\beta \right\}$.

We say \mathcal{M}_β is *non-degenerate* if $\alpha_i \pm \hat{\beta}_j \neq -1, -2, \dots$ for all $j = 0, \dots, m$ and all $\hat{\beta} \in \mathcal{M}_\beta$.

Conjecture 8. [BasTr] Let $f(e^{i\theta})$ be given as in (118) with $V(e^{i\theta})$ sufficiently smooth on S^1 and $\Re\alpha_j > -\frac{1}{2}$, $0 \leq j \leq m$. Suppose \mathcal{M}_β is non-degenerate. Then as $n \rightarrow \infty$,

$$(157) \quad D_n(f) = \sum_{\widehat{\beta} \in \mathcal{M}_\beta} \left[R_n \left(f(\widehat{\beta}) \right) (1 + o(1)) \right].$$

Each $R_n \left(f(\widehat{\beta}) \right)$ stands for the FH asymptotic form (137) (142) with $\widehat{\beta}$ and $\left(\prod_{j=0}^m z_j^{n_j} \right) e^V$ in place of β and e^V respectively.

Note that by the definition of \mathcal{M}_β , all the terms $R_n \left(f(\widehat{\beta}) \right)$ in (157) have the same order of magnitude. Indeed for all $\widehat{\beta} \in \mathcal{M}_\beta$, $\Re \sum_{j=0}^m \widehat{\beta}_j^2 = F_\beta - \sum_{j=0}^m (\Im \beta_j)^2$.

To see how this conjecture works for $f(e^{i\theta})$ in (151), where $\beta_0 = \frac{1}{2}$ and $\beta_1 = -\frac{1}{2}$, and $n_0 + n_1 = 0$, consider

$$\sum_{j=0}^1 (\Re \beta_j + n_j)^2 = \left(\frac{1}{2} + n_0 \right)^2 + \left(-\frac{1}{2} - n_0 \right)^2 = 2 \left(\frac{1}{2} + n_0 \right)^2$$

which clearly achieves its minimum at $n_0 = 0 = -n_1$ and also at $n_0 = -1 = -n_1$. These values correspond to $\widehat{\beta} = \beta = \left(\frac{1}{2}, -\frac{1}{2} \right)$ and $\widehat{\beta} = \left(\frac{1}{2} - 1, -\frac{1}{2} + 1 \right) = \left(-\frac{1}{2}, \frac{1}{2} \right)$, respectively. Thus the leading asymptotics of $D_n(f)$ is a sum of contributions from $f(z) = f(z; \beta)$ and $f(z) = f(z; \widehat{\beta})$, as in (152).

Conjecture 8 was proved recently in [DIK1].

Theorem 9. [DIK1] *Conjecture 8 holds.*

The relation of the semi-norm $|||\beta|||$ to the minimization problem is given by the following elementary result.

Lemma 10. [DIK1] *Given β , there are only two, mutually exclusive, possibilities: Either*

- (i) *There exists $\widehat{\beta} \in O_\beta$ such that $|||\widehat{\beta}||| < 1$. Then such a $\widehat{\beta}$ is unique and it is the unique element of $\mathcal{M}_\beta = \{ \widehat{\beta} \}$*

or

- (ii) *There exists $\widehat{\beta} \in O_\beta$ such that $|||\widehat{\beta}||| = 1$. Then there are at least two such $\widehat{\beta}$'s and all of them are obtained from each other by a repeated application of the following rule: add 1 to a $\widehat{\beta}_j$ with the smallest real part, $\Re \widehat{\beta}_j = \min_k \Re \widehat{\beta}_k$, and subtract 1 from a $\widehat{\beta}_{j'}$ with the largest real part, $\Re \widehat{\beta}_{j'} = \max_k \Re \widehat{\beta}_k$. Moreover $\mathcal{M}_\beta = \{ \widehat{\beta} \in O_\beta : |||\widehat{\beta}||| = 1 \}$.*

Clearly for $f(e^{i\theta})$ in (151), $|||\widehat{\beta}||| = 1$, if and only if $\widehat{\beta} = \beta$ or $\widehat{\beta} = \beta + (-1, 1)$. Also we see from (i) that the case $|||\beta||| < 1$ in Ehrhardt's Theorem corresponds to a unique minimizer in (156), and hence to a single term in (157).

The proof of Conjecture 8 in [DIK1] proceeds as follows. Given β , if there exists $\widehat{\beta} \in O_\beta$ with $|||\widehat{\beta}||| < 1$, then $\widehat{\beta}$ is unique and as $n \rightarrow \infty$, $D_n(f) = R_n \left(f(\widehat{\beta}) \right) (1 + o(1))$ (Ehrhardt's Theorem). On the other hand, if there exists $\widehat{\beta} \in O_\beta$ with $|||\widehat{\beta}||| = 1$, then $\mathcal{M}_\beta = \{ \widehat{\beta} \in O_\beta : |||\widehat{\beta}||| = 1 \}$ has at least two elements. Then for some b ,

$$(158) \quad b \leq \Re \widehat{\beta}_j \leq b + 1$$

for all $\widehat{\beta}_j$, and for some $p, \ell > 0$ there are p values of j such that $\Re \widehat{\beta}_j = b$ and ℓ values for which $\Re \widehat{\beta}_j = b + 1$. Let $\widetilde{f}(e^{i\theta})$ be the FH symbol (*not* a FH-representation of $f(e^{i\theta})$) with α_j, V as before but $\widetilde{\beta}_j = \widehat{\beta}_j$ if $\Re \widehat{\beta}_j < b + 1$ and $\widetilde{\beta}_j = \widehat{\beta}_j - 1$ if $\Re \widehat{\beta}_j = b + 1$. Then clearly

$$(159) \quad f(z) = c z^\ell \widetilde{f}(z)$$

for some simple constant c , and by a general relation

$$(160) \quad D_n(f) = \frac{c^n (-1)^{\ell n} F_n D_n(\widetilde{f})}{\prod_{j=0}^{\ell-1} j!}$$

where

$$(161) \quad F_n = \det \left(\frac{d^j \widetilde{\pi}_{n+k}}{dz^j} (0) \right)_{0 \leq j, k \leq \ell-1}$$

and $\widetilde{\pi}_q(z) = z^q + \dots, q \geq 0$, are the monic orthogonal polynomials with respect to the (in general non-real) weight $\widetilde{f}(e^{i\theta}) \frac{d\theta}{2\pi}$ on S^1 ,

$$(162) \quad \int_0^{2\pi} \widetilde{\pi}_q(e^{i\theta}) e^{-ir\theta} \widetilde{f}(e^{i\theta}) \frac{d\theta}{2\pi} = 0, \quad 0 \leq r < q.$$

(A part of the proof of Conjecture 8 consists in showing that the $\widetilde{\pi}_q$'s exist and are unique for q sufficiently large, $q \geq q_{\widetilde{f}}$.) Now clearly $|||\widetilde{\beta}||| < 1$, and so we can evaluate $D_n(\widetilde{f})$ as $n \rightarrow \infty$ using Ehrhardt's Theorem. The proof of the conjecture then reduces by (160) (161) to evaluating the polynomials $\widetilde{\pi}_n(z), \dots, \widetilde{\pi}_{n+\ell-1}(z)$ and their derivatives at $z = 0$, as $n \rightarrow \infty$. This is effected by using the steepest-descent method of Deift and Zhou for Riemann-Hilbert problems mentioned in Section 3 above (see also, for example, [DKMVZ1], [DKMVZ2]). The fact that orthogonal polynomials with respect to a weight on the line, can be expressed in terms of a Riemann-Hilbert problem is due to Fokas, Its and Kitaev in [FIK]: the extension of this result to orthogonal polynomials on the circle is given in [BDJ].

Remark 7.1. As noted above, for the Ising model, at $T > T_c$, $\varphi_{\text{ons}}(e^{i\theta})$ in (127) corresponds to a degenerate case for Ehrhardt's Theorem. However $\varphi_{\text{ons}}(e^{i\theta}) = -e^{-i\theta} \widetilde{\varphi}(e^{i\theta})$ where $\widetilde{\varphi}(e^{i\theta}) = \frac{(1-\gamma_1 e^{i\theta})(\gamma_2 - e^{i\theta})}{|1-\gamma_1 e^{i\theta}| |1-\gamma_2 e^{i\theta}|}$ is smooth, nonzero, and has no winding on S^1 . Hence $D_n(\widetilde{\varphi})$ can be computed as $n \rightarrow \infty$ using SSLT as in (85). Analogous to (159), (160), we have the formula

$$(163) \quad D_n(\varphi_{\text{ons}}) = \widehat{\pi}_n(0) D_n(\widetilde{\varphi})$$

where $\widehat{\pi}_n(z) = z^n + \dots, n \geq 0$ are the complementary orthogonal polynomials to the $\widetilde{\pi}_n$'s (see [DIK1])

$$(164) \quad \int_0^{2\pi} \widehat{\pi}_n(e^{-i\theta}) e^{ij\theta} \widetilde{\varphi}(e^{i\theta}) \frac{d\theta}{2\pi} = 0, \quad 0 \leq j < n.$$

Applying the steepest descent method to the Riemann-Hilbert Problem for $\widehat{\pi}_n$, and taking into account the analyticity of $\widetilde{\varphi}(z)$ in an annulus around $\{|z| = 1\}$, we obtain the precise decay rate of $\widehat{\pi}_n(0)$ as $n \rightarrow \infty$, and hence via (163), the precise leading order exponential decay rate $cn^{-1/2} \gamma_2^{-n} (1 + o(1))$ as in (111). The steepest descent method also yields the higher order terms in

the expression, to any desired order. An alternative, and very elegant, proof of (111) using a contour deformation argument, is given in [FF]. For $\varphi_{\text{diag}}(e^{i\theta})$ and the diagonal correlation function, the situation is similar, and one obtains:

$$(165) \quad \langle \sigma_{1,1} \sigma_{1+n,1+n} \rangle = \frac{k_{\text{ons}}^{-n}}{\sqrt{n}} \frac{\sqrt{\pi}}{(1 - k_{\text{ons}}^{-2})^{1/4}} (1 + o(1)), \quad T > T_c.$$

We also note that in Section 10.7 of [BottSilb3] and in the papers [BottWid1] [Wid7], the authors also analyze the Toeplitz determinants arising in the case $T > T_c$.

Remark 7.2. Formulae (160) and (163) should be compared respectively with formulae (108) and (97) in [FisHart1].

8 Hankel and Toeplitz+Hankel determinants

In [DIK1] the authors also consider Hankel and Toeplitz + Hankel determinants with FH-type singularities. For a weight $w = w(x) \geq 0$ on $[-1, 1]$, the associated Hankel determinant is given by [Sz5]

$$(166) \quad D_n^{(H)}(w) = \det \left(\int_{-1}^1 x^{j+k} w(x) dx \right)_{j,k=0}^{n-1}.$$

The Hankel determinant $D_n^{(H)}(w)$ is related to the polynomials orthogonal with respect to the weight w on the interval $[-1, 1]$ in a similar way as a Toeplitz determinant is related to OPUC's.

For fixed $r = 0, 1, 2, \dots$ we consider $w(x)$ of the following form:

$$(167) \quad w(x) = e^{U(x)} \prod_{j=0}^{r+1} |x - \lambda_j|^{2\alpha_j} \omega_j(x)$$

where

$$\begin{aligned} 1 = \lambda_0 &> \lambda_1 > \dots > \lambda_{r+1} = -1, \\ \omega_j(x) &= \begin{cases} e^{i\pi\beta_j}, & -1 \leq x \leq \lambda_j, \\ e^{-i\pi\beta_j}, & 1 \geq x > \lambda_j \end{cases} \quad \Re\beta_j \in \left(-\frac{1}{2}, \frac{1}{2}\right] \\ \beta_0 = \beta_{r+1} &= 0, \quad \Re\alpha_j > -\frac{1}{2}, \quad j = 0, 1, \dots, r+1 \end{aligned}$$

and where $U(x)$ is a sufficiently smooth function on $[-1, 1]$.

Remark 8.1. Note that there is no loss of generality in setting $\beta_0 = \beta_{r+1} = 0$ and $\Re\beta_j \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ as the functions $\omega_0(x), \omega_{r+1}(x)$ are just constant on $(-1, 1)$ and $\omega_j(x; \beta_j + k_j) = (-1)^{k_j} \omega_j(x; \beta_j)$ for $k_j \in \mathbb{Z}$.

In [DIK1] the authors prove that for w as in (167) with $\Re\beta_j \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, $j = 1, \dots, r$, as $n \rightarrow \infty$,

$$(168) \quad D_n^{(H)}(w) = 2^{-n^2} G_H^n n^{-\frac{1}{4} + 2(\alpha_0^2 + \alpha_r^2) + \sum_{j=1}^r (\alpha_j^2 - \beta_j^2)} E_H (1 + o(1))$$

for certain explicit constants G_H and E_H depending on $U(x)$, λ_j and β_j (see (1.37) in [DIK1]). This result is derived by introducing the function

$$(169) \quad f(e^{i\theta}) = f(e^{i(2\pi-\theta)}) \equiv w(\cos \theta) |\sin \theta|, \quad 0 \leq \theta < 2\pi$$

on S^1 . With w as in (167), f has FH singularities of the form

$$(170) \quad f(e^{i\theta}) = c e^{V(e^{i\theta})} |z-1|^{4\alpha_0+1} |z+1|^{4\alpha_{r+1}+1} \prod_{j=1}^r |z-z_j|^{2\alpha_j} |z-z'_j|^{2\alpha_j} \\ \times \prod_{j=1}^r \left[g_{z_j, -\beta_j}(z) z_j^{\beta_j} g_{z'_j, \beta_j}(z) (z'_j)^{-\beta_j} \right], \quad 0 \leq \theta < 2\pi$$

where

$$\begin{cases} \cos \theta_j = \lambda_j, & z_j = e^{i\theta_j}, & j = 0, 1, \dots, r+1, & 0 = \theta_0 < \theta_1 < \dots < \theta_{r+1} = \pi \\ z'_j = e^{i(2\pi-\theta_j)}, & & j = 0, 1, \dots, r+1 \\ c = 2^{-2(\sum_{j=0}^{r+1} \alpha_j) - 1} e^{2i \sum_{j=1}^r \beta_j \arcsin \lambda_j} \end{cases}$$

and

$$(171) \quad V(e^{i\theta}) = U(\cos \theta).$$

The Hankel determinant $D_n^{(H)}(w)$ and the Toeplitz determinant $D_{2n}(f)$ are related in the following way:

$$(172) \quad D_n^{(H)}(w)^2 = \frac{\pi^{2n}}{4^{(n-1)^2}} \frac{(\pi_{2n}(0) + 1)^2}{\pi_{2n}(1) \pi_{2n}(-1)} D_{2n}(f)$$

where $\pi_k(z) = z^k + \dots, k \geq 0$, are the monic orthogonal polynomials with respect to the weight $f(e^{i\theta}) \frac{d\theta}{2\pi}$ on S^1 as in (162),

$$\int_0^{2\pi} \pi_k(e^{i\theta}) e^{-ij\theta} f(e^{i\theta}) \frac{d\theta}{2\pi} = 0, \quad 0 \leq j < k.$$

As the semi-norm $|||\beta||| < 1$ for $f(e^{i\theta})$, we may use Ehrhardt's result to evaluate $D_{2n}(f)$, $n \rightarrow \infty$, and the result (168) follows from (172) by computing the asymptotics for $\pi_{2n}(z)$ for $z = 0, \pm 1$. Note that if $\Re \beta_j = \frac{1}{2}$ for some $j \in \{0, 1, \dots, r+1\}$, then $|\Re \beta_j - \Re(-\beta_j)| = 1$ and so $|||\beta||| = 1$. In this case we would have had to use the Basor-Tracy form (157) for $D_{2n}(f)$: we have restricted our attention to the case $\Re \beta_j \in (-\frac{1}{2}, \frac{1}{2})$ only for simplicity.

Remark 8.2. For a discussion of results of various authors, particularly Basor and Ehrhardt, related to (168), see [DIK1], Remark 1.24.

For even FH symbols f , i.e., $f(e^{i\theta}) = f(e^{i(2\pi-\theta)})$, the authors in [DIK1] also consider Toeplitz + Hankel determinants of four types defined in terms of the Fourier coefficients f_k of f ,

$$(173) \quad \det(f_{j-k} + f_{j+k})_{j,k=0}^{n-1}, \quad \det(f_{j-k} - f_{j+k+2})_{j,k=0}^{n-1}, \quad \det(f_{j-k} \pm f_{j+k+1})_{j,k=0}^{n-1}.$$

Such matrices arise, in particular, in the theory of classical groups and its application to random matrix theory and statistical mechanics, and there are simple relations between these determinants and Hankel determinants on $[-1, 1]$ with added singularities at the end-points (see [DIK1] and the references therein). For example, for $f(e^{i\theta}) = f(e^{i(2\pi-\theta)})$,

$$(174) \quad \det(f_{j-k} + f_{j+k})_{j,k=0}^{n-1} = \frac{2^{n^2-2n+2}}{\pi^n} D_n^{(H)}(v)$$

where $D_n^{(H)}(v)$ is the Hankel determinant with symbol $v(x) = f(e^{i\theta(x)})/\sqrt{1-x^2}$, $\theta(x) = \arccos x$, on $[-1, 1]$, with similar formulae for the other three determinants. Let $f(e^{i\theta}) = f(e^{i(2\pi-\theta)})$ with FH singularities at $0 = \theta_0 < \theta_1 < \dots < \theta_r < \theta_{r+1} = \pi$ and complementary singularities at $2\pi - \theta_j$, $1 \leq j \leq r$. Suppose that $\Re\beta_j \in (-\frac{1}{2}, \frac{1}{2})$, $j = 1, \dots, r$, and $\beta_0 = \beta_{r+1} = 0$. Then utilizing (174) and (168), the authors show that as $n \rightarrow \infty$

$$(175) \quad \det(f_{j-k} + f_{j+k})_{j,k=0}^{n-1} = G_{T+H}^n n^{\sum_{j=1}^r (\alpha_j^2 - \beta_j^2) + \frac{1}{2}(\alpha_0^2 + \alpha_{r+1}^2 - \alpha_0 - \alpha_{r+1})} \times E_{T+H} (1 + o(1))$$

for certain explicit constants G_{T+H} , E_{T+H} , with similar results for the other three Toeplitz+Hankel determinants in (173).

Remark 8.3. For a discussion of the related results of Basor and Ehrhardt for determinants of type (173), see [DIK1], Remark 1.27.

Remark 8.4. In [FF] the authors discuss a large number of conjectures and results for Toeplitz, Hankel, and Toeplitz+Hankel determinants with FH-type singularities. The determinants arise in turn from problems in statistical physics. Many of the conjectures in [FF] can now be resolved using results from [DIK1] such as (168) and (175).

9 Some applications

As advertised at the outset, the goal of this paper has been to show how SSLT developed and was generalized in response to questions arising in the analysis of the Ising model. Although the case $\|\beta\| = 1$ in [DIK1] does not arise in the Ising model (the same is true for the Hankel and Toeplitz + Hankel determinants (166) and (173) respectively), this case does arise in many other problems in statistical physics. For example, the probability $P_E(n)$ of a string of length n of ferromagnetically aligned spins in the antiferromagnetic ground state in the XY spin chain, is given, for a certain range of parameters, by a Toeplitz determinant with 2 β -singularities such that $\|\beta\| = 1$. Thus (157) verifies the result on $P_E(n)$, $n \rightarrow \infty$, presented in [FrAb], which the authors based on the Basor-Tracy conjecture. In a similar way, the results in [DIK1] can be used to justify the asymptotic results for correlations arising in the theory of the impenetrable Bose gas that were obtained in the work of Ovchinnikov [Ov] on the basis of the Basor-Tracy conjecture. In another direction in [GGM], the authors use (157) to evaluate the long-time behavior of a variety of non-equilibrium 1D many-body problems. In particular, in the Fermi edge singularity and tunneling spectroscopy problems considered in [GGM], the authors show that the various contributions to (157) have a very transparent physical meaning: they correspond to contributions to the Green's functions for these problems from multiple Fermi edges.

Theorem 9 can also be used to analyze the asymptotic behavior of the eigenvalues $\lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \dots \leq \lambda_n^{(n)}$, $n \rightarrow \infty$, of a Toeplitz matrix $T_n(\varphi)$ with a real symbol φ . At the beginning of our story (see Theorem 2) we interpreted Szegő's early results as saying that the $\lambda_j^{(n)}$'s were equidistributed as $n \rightarrow \infty$. But what about the behavior of individual eigenvalues, $\lambda_k^{(n)}$, $n \rightarrow \infty$? Early work on this question considered the so-called extreme eigenvalues $\lambda_k^{(n)}$ and $\lambda_{n-k}^{(n)}$ with k fixed as $n \rightarrow \infty$ (see [GreSz], [BGrM] and the references therein). The important case for $\lambda_k^{(n)}$ with $n \rightarrow \infty$ and $k/n \rightarrow x \in (0, 1)$ was only considered for the first time recently in [BGrM]. Recall that by general principles (see e.g. [GreSz]), for bounded, real-valued functions $f(e^{i\theta})$, the eigenvalues $\{\lambda_k^{(n)}\}$ lie

in the interval $[L, M]$, $L = \inf f(e^{i\theta})$, $M = \sup f(e^{i\theta})$, and as $n \rightarrow \infty$ the $\lambda_k^{(n)}$'s fill out the interval in the sense that if $\lambda \in [L, M]$, then there exist $k_n(\lambda)$ such that $\lambda_{k_n(\lambda)}^{(n)} \rightarrow \lambda$. In [BGrM] the authors consider function $f(e^{i\theta})$ that are unimodal, real-valued trigonometric polynomials, i.e., for some $0 < \theta_0 < 2\pi$, $\frac{d}{d\theta} f(e^{i\theta}) > 0$ for $0 < \theta < \theta_0$ and $\frac{d}{d\theta} f(e^{i\theta}) < 0$ for $\theta_0 < \theta < 2\pi$, with the additional property that $\frac{d^2}{d\theta^2} f(e^{i\theta}) \neq 0$ at $\theta = 0$ and $\theta = \theta_0$. For any $\lambda \in (L, M)$, there exist unique $0 < \theta_1(\lambda) < \theta_0 < \theta_2(\lambda) < 2\pi$ such that $f(e^{i\theta_j(\lambda)}) - \lambda = 0$, $j = 1, 2$. The function

$$\psi(\lambda) = \frac{1}{2} [\theta_1(\lambda) - \theta_2(\lambda)] + \pi$$

extends to a homeomorphism from $[L, M]$ onto $[0, \pi]$. In [BGrM] the authors prove, in particular, that for any $x \in (0, 1)$

$$(176) \quad \lambda_k^{(n)} = \lambda_x + o(1)$$

as $n \rightarrow \infty$ and $k/n \rightarrow x$, where $\psi(\lambda_x) = \pi x$, and

$$(177) \quad \lambda_{k+1}^{(n)} - \lambda_k^{(n)} = \frac{\pi}{\psi'(\lambda_x)} \frac{1}{n+1} + o\left(\frac{1}{n}\right)$$

if $|k/n - x| = O\left(\frac{1}{n}\right)$ as $n \rightarrow \infty$.

The estimate (177) brings precision to the equidistribution result in Theorem 2. In [DIK2] the authors extend the result in [BGrM] to C^∞ unimodal, real-valued functions on S^1 , and they also consider piecewise-constant symbols of the form

$$(178) \quad \begin{aligned} f(e^{i\theta}) &= 1, & \theta &\in [0, \widehat{\theta}_1) \cup [\widehat{\theta}_2, 2\pi) \\ &= e^{2\pi\gamma}, & \theta &\in [\widehat{\theta}_1, \widehat{\theta}_2) \end{aligned}$$

where $0 \leq \widehat{\theta}_1 < \widehat{\theta}_2 < 2\pi$ and $\gamma > 0$ are given. For symbols such as (178), the eigenvalues $\lambda_k^{(n)}$ cannot be spaced $1/n$ apart as in (177). This is clear, for example, from (8) in Theorem 2, if we let $F(\lambda)$ be any non-negative continuous function with support in $(1, e^{2\pi\gamma})$. And indeed, the general result in [Bas2] asserts that in the case (178), for any small $\epsilon > 0$, the interval $(1 + \epsilon, e^{2\pi\gamma} - \epsilon)$ contains order $\log n$ eigenvalues $\lambda_k^{(n)}$, spaced at distance $1/\log n$ apart, as $n \rightarrow \infty$: The bulk of the eigenvalues accumulate at 1 and at $e^{2\pi\gamma}$. On the basis of numerical computations, the authors in [LeSh] (see also [LeSoSo]) conjectured that in the case

$$(179) \quad \widehat{\theta}_2 - \widehat{\theta}_1 = 2\pi p/q, \quad p, q \in \mathbb{Z}, \quad 0 < p < q$$

there is a “near periodicity” in the spectrum of $T_n(f)$. In [DIK2] the authors prove this conjecture in the following sense: for n sufficiently large, if $\lambda_k^{(n)}$ is an eigenvalue of $T_n(f)$ in $(1 + \epsilon, e^{2\pi\gamma} - \epsilon)$, then there is an eigenvalue $\lambda_j^{(n+q)}$ of T_{n+q} such that $|\lambda_k^{(n)} - \lambda_j^{(n+q)}| \leq c(n \log n)^{-1}$. Of course $1/(n \log n) \ll 1/\log n$, the spacing of the $\lambda_k^{(n)}$'s, $k = 1, \dots, n$.

The proof of the above results in [DIK2] rests on the simple fact that $\det(T_n(f) - \lambda) = \det(T_n(f - \lambda))$, so that the eigenvalue equation for $T_n(f)$ is equivalent to the vanishing of the Toeplitz determinant $D_n(f - \lambda)$. Now if $f(e^{i\theta})$ is a smooth, unimodal function on S^1 , then $f(e^{i\theta}) - \lambda$

is a FH symbol with two singularities at $z_1 = e^{i\theta_1(\lambda)}$, $z_2 = e^{i\theta_2(\lambda)}$ and $\alpha_1 = \alpha_2 = \frac{1}{2}$, $\beta_1 = -\beta_2 = \frac{1}{2}$ (see (124))

$$(180) \quad f(z) - \lambda = e^{V(z)} |z - z_1| |z - z_2| g_{z_1, \frac{1}{2}}(z) g_{z_2, -\frac{1}{2}}(z) \left(\frac{z_1}{z_2}\right)^{-\frac{1}{2}}$$

where $V(e^{i\theta})$ is smooth on S^1 . This is a case of a FH symbol with $|||\beta||| = 1$. Therefore we need to use Theorem 9 to obtain the asymptotics for $D_n(f - \lambda)$. The set M_β here contains 2 elements, and the condition for an eigenvalue is the vanishing of the sum in (157):

$$(181) \quad D_n(f - \lambda) = R_n(f(\beta_1, \beta_2) - \lambda)(1 + o(1)) + R_n(f(\beta_1 - 1, \beta_2 + 1) - \lambda)(1 + o(1)) = 0$$

The asymptotic results on the eigenvalues follow from this formula.

The Toeplitz matrix $T_n(f)$ with the symbol $f(e^{i\theta})$ in (178) is handled similarly. The function $f(z) - \lambda$ in this case has 2 β -type singularities whose positions on the circle are fixed, but the imaginary parts of β 's depend on λ : $\beta_1 = -\beta_2 = \frac{1}{2} + i\gamma(\lambda)$. Note that we again have $|||\beta||| = 1$, and the condition for the eigenvalues is thus given by Theorem 9.

Finally we describe an example where the results in [DIK1] are needed for a Toeplitz+Hankel determinant of type (173). The following problem arises in the framework of the random matrix approach to the theory of the Riemann zeta-function and other L -functions (see [Keat] for a recent survey). Define

$$(182) \quad \phi(e^{i\theta}) = \left| 2 \sin \frac{\theta}{2} \right|^{2k} e^{V(e^{i\theta})}, \quad k \in \mathbb{N}$$

where

$$V(e^{i\theta}) = 2k \left[\int_1^e u(y) \left\{ \sum_{j=-\infty}^{\infty} \text{Ci}(|\theta + 2\pi j| (\log y)(\log X)) \right\} dy - \log \left| 2 \sin \frac{\theta}{2} \right| \right],$$

$$\text{Ci}(z) = - \int_z^{\infty} \frac{\cos t}{t} dt$$

and $u(y)$ is a smooth, non-negative function supported on $[e^{1-X^{-1}}, e]$ and of total mass one. Consider the following average over the orthogonal group $SO(2n)$,

$$(183) \quad \mathbb{E}_{SO(2n)} \left(\prod_{j=1}^n \phi(e^{i\theta_j}) \right).$$

Here $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_n}$ are the eigenvalues of a random matrix in $SO(2n)$: note that ϕ is even, $\phi(e^{i\theta}) = \phi(e^{-i\theta})$. This expectation was introduced by Bui and Keating in [BuKe] (following [GHK]) as the random matrix counterpart of a key term contributing to the mean values of certain Dirichlet L -functions in the Katz-Sarnak orthogonal family. The issue is the large n and large X behavior of this average. The question can be resolved with the help of (175). One observes that

$$(184) \quad \mathbb{E}_{SO(2n)} \left(\prod_{j=1}^n \phi(e^{i\theta_j}) \right) = \frac{1}{2} \det(\phi_{j-k} + \phi_{j+k})_{j,k=0}^{n-1}$$

and that the symbol (182) is of FH type with a single α -singularity at $z_0 = 1$ and $\alpha_0 = k$. Directly from (the precise version of) the asymptotic formula (175) (see Theorem 1.25 in [DIK1]), one obtains, for X large, as $n \rightarrow \infty$

$$(185) \quad \mathbb{E}_{SO(2n)} \left(\prod_{j=1}^n \phi(e^{i\theta_j}) \right) \sim G(1+k) \left(\frac{\Gamma(1+2k)}{G(1+2k) \Gamma(1+k)} \right)^{\frac{1}{2}} \left(\frac{2n}{e^{\gamma_E} \log X} \right)^{\frac{k(k-1)}{2}}$$

where γ_E is Euler's constant. Formula (185) is precisely the asymptotic form conjectured by Bui and Keating in [BuKe].

10 Block Toeplitz matrices

We now consider block Toeplitz matrices $D_n(\varphi)$ of the form $D_n(\varphi) = \det(\varphi_{j-k})_{j,k=0}^{n-1}$ where $\varphi_q \in M(r, \mathbb{C})$ are the Fourier coefficients of an $r \times r$ -matrix-valued integrable function $\varphi(e^{i\theta})$ on S^1 . The earliest asymptotic result for such determinants is the analog of Szegő's Theorem 1 due to Gyires in 1956 [Gy], who showed that if $\varphi(z)$ is continuous and positive definite for $|z| = 1$, then

$$(186) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log D_n(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} \log \det \varphi(e^{i\theta}) d\theta.$$

Block Toeplitz matrices arose in the Ising model in the following way. In their famous paper [LeYa] on the Lee-Yang Theorem, asserting that the zeros of the partition function $Z_{\Lambda, h}$ (cf. Remark 4.1 in Section 4), for the Ising model in the presence of a magnetic field, all lie on the imaginary h -axis, Lee and Yang also proposed, but did not prove, expressions for the free energy $F_h = \lim_{|\Lambda| \rightarrow \infty} -k_B T \log Z_{\Lambda, h}$ and the magnetization $M_h = \lim_{n \rightarrow \infty} \langle \sigma_{1,1} \sigma_{1,1+n} \rangle^{\frac{1}{2}}$ at the particular imaginary value of h , $\frac{h}{k_B T} = \frac{1}{2} i \pi$. In terms of the variable $\rho = e^{-2h/k_B T}$, the zeros of $Z_{\Lambda, h}$ lie on the circle $|\rho| = 1$. As noted in [LeYa], M_h is precisely the density of the zeros of $Z_{\Lambda, h}$ on $\{|\rho| = 1\}$ at $\rho = -1$, in the limit as $|\Lambda| \rightarrow \infty$. In [McWu2], McCoy and Wu present a derivation of these expressions for F_h and M_h , $h/k_B T = \frac{1}{2} i \pi$. Regarding M_h , for this value of h , they first show how to express the correlation function $\langle \sigma_{1,1} \sigma_{1,1+n} \rangle$ in terms of a 2×2 block Toeplitz determinant B_n . Then they show by certain ingenious manipulations how to evaluate B_n in terms of a determinant which is a product of two scalar Toeplitz determinants to which SSLT applies. In this way they are able to compute M_h explicitly and recover the formula in [LeYa]. This is the first example of an SSLT-type result for block Toeplitz matrices.

The general theory of SSLT for block Toeplitz matrices begins with Widom's paper [Wid5]. Under certain smoothness assumptions on the $r \times r$ -matrix-valued function $\varphi(e^{i\theta})$, Widom shows that

$$(187) \quad Y(\varphi) \equiv \lim_{n \rightarrow \infty} D_n(\varphi) / \exp \left\{ \frac{n}{2\pi} \int_0^{2\pi} \log \det \varphi(e^{i\theta}) d\theta \right\}$$

exists, and in [Wid3] he shows that

$$(188) \quad Y(\varphi) = \det [T(\varphi) T(\varphi^{-1})]$$

where $T(\varphi)$, $T(\varphi^{-1})$ are the Toeplitz operators associated with φ and φ^{-1} respectively (cf. (76)). Part of the proof of (188) consists in showing that $T(\varphi) T(\varphi^{-1}) - 1$ is trace-class in ℓ_+^2 , so that the

RHS is well-defined. In the scalar case, as mentioned above, Widom uses the Helton-Howe-Pincus formula to show that (188) reduces to the standard result $Y(\varphi) = e^{\sum_{k=1}^{\infty} k(\log \varphi)_k (\log \varphi)^{-k}}$. As noted above (see (73) et seq), in [Wid4], Widom introduced a “standard” pathway to SSLT. The method extends with only a change of notation to the block Toeplitz case, and so Widom obtains a new direct proof of (187), (188) with no more effort than in the scalar case. The same is true for the method of Basor and Helton in [BasHelt].

Block Toeplitz matrices occur in many physical situations, for example, in the study of the Ising model with next-nearest-neighbor interactions, and also, more recently, in the theory of the dimer model [BasEhr] and in the study of the entanglement spectrum in quantum systems (see [VLRK], [JK]). The use of the general SSLT for block Toeplitz matrices in physical applications poses new challenges vis a vis the scalar case. The new difficulties are due to the fact that formula (188), as elegant as it is, is hard to evaluate in concrete calculations in the case of a matrix symbol φ . The determinant on the right hand side of (188) is the Fredholm determinant of an infinite matrix, and even for 2×2 matrix functions φ an effective evaluation of $Y(\varphi)$ is a highly nontrivial enterprise. Indeed, the authors are aware of only three situations when such effective evaluations have been made so far. The first situation arises when, as in the work of McCoy and Wu mentioned above, it is possible, by using an ad hoc technique, to express the block Toeplitz determinant in question in terms of scalar Toeplitz determinants. The work of Tanaka, Morita, and Hiroike [TMH], and of Böttcher [Bott2] on the Ising model with the next-nearest-neighbor interactions provides another example of this sort. In this example, the relevant matrix symbol φ is described by the formula

$$(189) \quad \varphi(z) = (I - zR) a(z) (I - z^{-1}S)$$

where $a(z)$ is a diagonal matrix function,

$$(190) \quad a(z) = \begin{pmatrix} \lambda(z) & 0 \\ 0 & \lambda(z^{-1}) \end{pmatrix}$$

with $\lambda(z)$ being an elementary algebraic function, and the constant matrices R and S are expressed as follows:

$$R = U \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} U^{-1}, \quad S = U \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} U^{-1}.$$

In these expressions, U is an invertible 2×2 matrix, and α, β are complex numbers, satisfying the conditions: $|\alpha| < 1, |\beta| < 1$. Note that these inequalities guarantee that the first matrix factor in (189) is holomorphic and invertible inside the unit disc while the last factor is holomorphic and invertible outside of the unit disc. The function $\lambda(z)$ (an analog of Onsager’s symbol (25)), the matrix U , and the numbers α and β are determined by the physical parameters of the problem, the temperature T and the anisotropy coefficients. As in the usual Ising model, there exists a critical temperature T_c , and the behavior of the determinant $D_n(\varphi)$ depends on whether $T > T_c$ or $T < T_c$. Böttcher shows that in the case $T > T_c$ (the paramagnetic phase) the Wiener-Hopf factorization of the matrix symbol $\varphi^{-1}(z)$ has nontrivial partial indices which implies that the operator $T(\varphi^{-1})$ is not invertible. In view of (188), this implies that the factor $Y(\varphi)$ is zero. Moreover, it is easily seen that $\int_0^{2\pi} \log \det \varphi(e^{i\theta}) d\theta = 0$, and hence for $T > T_c$

$$\lim_{n \rightarrow \infty} D_n(\varphi) = 0$$

which has the physical interpretation that there is no spontaneous magnetization in the paramagnetic phase. In the ferromagnetic phase, that is when $T < T_c$, an efficient direct analysis of Widom’s

factor $Y(\varphi)$ is not apparent, and Böttcher develops an ad-hoc approach which uses the specifics of the symbol (189). Indeed, in his preprint, by means of ingenious operator-algebra arguments, Böttcher obtains the formula

$$(191) \quad \lim_{n \rightarrow \infty} \frac{D_{n-1}(\varphi)}{D_n(a)} = (\det U)^{-2} (1 - \alpha\beta)^{-2} \left[\frac{U_{11}U_{22}}{\lambda_+(\beta)\lambda_-(1/\alpha)} - \frac{U_{12}U_{21}}{\lambda_+(\alpha)\lambda_-(1/\beta)} \right]^2$$

where $\lambda = \lambda_+\lambda_-$ is a Wiener-Hopf factorization of λ (which, as Böttcher shows, is possible in the case $T < T_c$). Hence the evaluation of $D_{n-1}(\varphi)$ as $n \rightarrow \infty$ is reduced to the asymptotic evaluation of the scalar Toeplitz determinant $D_n(\lambda)$, for which the standard (scalar) SSLT applies. In the critical case, $T = T_c$, the function $\lambda(z)$ acquires a Fisher-Hartwig type singularity. Böttcher then provides a modification of his previous analysis for this case and, using in addition various results obtained earlier with Silbermann (see Section 6), he then derives the leading asymptotics of $D_n(\varphi)$ in the critical case as well.

The two other types of block Toeplitz determinants for which the Widom pre-factor $Y(\varphi)$ can be evaluated are not reducible to the scalar case. The first one corresponds to the matrix symbols φ with a Fourier series that is truncated on at least one side. This class was singled out by Widom himself in [Wid5]. For such matrices φ , the evaluation of $Y(\varphi)$ is reduced to the evaluation of a finite dimensional determinant (in fact, a block Toeplitz determinant of a finite fixed size). This result of Widom has been used in the recent paper [BasEhr] of Basor and Ehrhardt devoted to the dimer model. The matrix function φ appearing in the dimer model is not originally of the truncated form, but it is truncated up to a scalar algebraic factor. Basor and Ehrhardt show that this factor can be accounted for by means of skillful algebraic manipulations which transform the original quantity $Y(\varphi)$ to one with φ replaced by its truncated part.

The second type of block Toeplitz determinants that are not reducible to the scalar case, but for which the large n limit is still computable, are determinants with algebraic matrix symbols. Such determinants appear in the theory of quantum entanglement when one evaluates the *von Neumann entropy*, which is a fundamental measure of the entanglement of a subsystem with the rest of the quantum system. For instance, as shown in [VLRK], [JK], the evaluation of the von Neumann entropy of a subsystem of neighboring spins in the XY quantum spin chain is equivalent to the asymptotic evaluation of the block Toeplitz determinant $D_n(\varphi)$ whose matrix symbol φ is given by the formula (for more details and references see [JK]),

$$(192) \quad \varphi(z) = \begin{pmatrix} i\mu & \psi(z) \\ -\psi^{-1}(z) & i\mu \end{pmatrix}, \quad \text{and} \quad \psi(z) = \sqrt{\frac{(z - z_1)(z - z_2)}{(1 - z_1z)(1 - z_2z)}}.$$

Here, $z_1 \neq z_2$ are complex nonzero numbers not lying on the unit circle and determined by the physical data of the model (the anisotropy parameter and the magnetic field) and μ is a free complex parameter lying outside of the interval $[-1, 1]$. The asymptotic evaluation of $D_n(\varphi)$ is performed in [IJK] and it is based on yet another formula of Widom which he obtained in [Wid5]. This formula is concerned with block Toeplitz determinants depending on an external parameter, say μ , and is used to evaluate the leading asymptotics of the μ -logarithmic derivative of the determinant in terms of the Wiener-Hopf factorizations of the matrix function φ^{-1} , i.e. in terms of the \pm -matrix functions u_{\pm} and v_{\pm} defined by the equations

$$(193) \quad \varphi^{-1}(z) = u_+(z)u_-(z) = v_-(z)v_+(z) \quad u_-(\infty) = v_-(\infty) = I,$$

where, as usual, f_+ (f_-) means that the matrix function is invertible and analytic inside (outside) of the unit circle, respectively. Under some natural smoothness assumptions (see [Wid5], Theorem

4.1), Widom's formula reads:

$$(194) \quad \frac{d}{d\mu} \log D_n(\varphi) = \frac{n}{2\pi i} \int_{S^1} \operatorname{tr} \left(\varphi^{-1}(z) \frac{\partial \varphi(z)}{\partial \mu} \right) \frac{dz}{z} \\ + \frac{i}{2\pi} \int_{S^1} \operatorname{tr} \left(\left(u'_+(z) u_-(z) - v'_-(z) v_+(z) \right) \frac{\partial \varphi(z)}{\partial \mu} \right) dz + o(1), \quad n \rightarrow \infty.$$

(In the case of symbols analytic in an annulus including the unit circle, formula (194) was re-derived with a quantitative estimate of the error term in [IJK] via the Riemann-Hilbert scheme (see also [IMM]).) Widom used this asymptotic formula in the proof of his main result, vis., the asymptotic relation (187). A curious fact is that, up until the work [IJK], equation (194), apparently, had never been used directly for the asymptotic evaluation of the determinants of block Toeplitz matrices. And for a good reason: the efficiency of formula (194) relies on the effectiveness of the Wiener-Hopf factorizations (193) of a given matrix valued function $\varphi(z)$. Usually, this is a transcendental problem equivalent to the solution of a system of singular integral equations on the unit circle which can seldom be effected in terms of known special functions. However, as was observed in [IJK], in the case of algebraic matrix functions $\varphi(z)$ one can take advantage of the algebro-geometric method that had been developed in the late 70s to the early 90s in soliton theory (see e.g. [BBEIM]). Using the Riemann-Hilbert version of this method exploited in [DIZ] (see also [DKMVZ1], [DKMVZ2]), the authors of [IJK] were able to construct the Wiener-Hopf factorizations of the symbol (192) in terms of Jacobi theta-functions. This in turn, with the help of Widom's formula (194), yielded the following asymptotics of the corresponding Toeplitz determinant:

$$(195) \quad D_n(\varphi) \sim \frac{\theta_3 \left(\frac{1}{2\pi i} \log \frac{\mu+1}{\mu-1} + \frac{\sigma\tau}{2} \right) \theta_3 \left(\frac{1}{2\pi i} \log \frac{\mu+1}{\mu-1} - \frac{\sigma\tau}{2} \right)}{\theta_3^2 \left(\frac{\sigma\tau}{2} \right)} (1 - \mu^2)^n, \quad n \rightarrow \infty.$$

Here

$$\theta_3(s) \equiv \theta(s; \tau) = \sum_{k=-\infty}^{\infty} e^{\pi i \tau k^2 + 2\pi i s k}$$

is the Jacobi theta-function, $\sigma = \pm 1$, and τ is the modulus of the elliptic curve

$$(196) \quad w^2(z) = (z - z_1)(z - z_2)(z - z_2^{-1})(z - z_1^{-1}).$$

The concrete choice of σ depends on the way the branch points $z_{1,2}^{\pm 1}$ are located with respect to the unit circle (see [IJK] for details). We refer the reader to the survey [IK] for more on these results and on their applications in the theory of quantum entanglement. The asymptotic formula (195) was extended in [IMM] to the case of more general integrable spin chains introduced in [KM]. The relevant symbol has the same matrix structure as (192), but with scalar function $\psi(z)$ now given by

$$(197) \quad \psi(z) = \sqrt{\frac{p(z)}{z^{2m} p(1/z)}}$$

where $p(z)$ is a polynomial of degree $2m$. The analog of the formula (195) in the case $m > 1$ involves a hyperelliptic curve instead of an elliptic curve, and instead of the Jacobi theta-function one needs $2m - 1$ dimensional Riemann theta-functions. Independently of [IJK] and [IMM], similar results

for block Toeplitz determinants with algebraic symbols appearing in the theory of the Gelfand-Dickey hierarchy, were obtained, again with the help of the algebro-geometric approach, in [Caf]. It is worth noticing that the method of [IJK] can be applied to the problems considered above by Böttcher and Basor-Ehrhardt. Indeed, the symbol of the determinant appearing in the work of Basor and Ehrhardt is in fact algebraic and hence can be factorized within the algebro-geometric approach, while the symbol of Böttcher's determinant is already in the form u_+u_- . The reverse factorization, i.e. representation of the symbol (189) as v_-v_+ involves an elementary algebraic operation. We note finally that Widom's formula (194) can be used to extend Böttcher's results to symbols of the form $\varphi(z) = R(z)a(z)S(z^{-1})$, where $a(z)$ is a diagonal matrix function and $R(z)$ and $S(z)$ are polynomial matrix functions invertible for all z inside the unit circle.

11 Double-scaling limits

Of all the challenges in Toeplitz theory that arise from the Ising model, the deepest is the issue of the double-scaling limit (or in physicists' parlance, simply the scaling limit), $T \rightarrow T_c$ and $n \rightarrow \infty$. We have already mentioned the importance of this problem in physics. At the purely mathematical level, the issue is the following: Suppose one has a Toeplitz determinant $D_n(\varphi)$ with a symbol that depends on some external parameter, say t , $\varphi(e^{i\theta}) = \varphi_t(e^{i\theta})$. For $t > 0$, φ_t is regular, but at $t = 0$, φ_t has a (Fisher-Hartwig) singularity, and hence, as we know, the nature of the asymptotics of $D_n(\varphi_t)$ as $n \rightarrow \infty$ is different for $t > 0$ and $t = 0$. This raises the general question: How does one describe the transition from the one kind of asymptotics to the other as $n \rightarrow \infty$ and $t \downarrow 0$?

In [WMTB], Wu, McCoy, Tracy, and Barouch discovered a remarkable scaling relation for the Ising model. They defined the scaling functions

$$(198) \quad G_{\pm}(r) = \lim_{\ell, m \rightarrow \infty, t \rightarrow \mp 0} |1 - e^{-2t}|^{-\frac{1}{4}} \langle \sigma_{1,1} \sigma_{1+\ell, 1+m} \rangle, \quad t = \log \left(\sinh \frac{2J_1}{k_B T} \sinh \frac{2J_2}{k_B T} \right)$$

in the double-scaling limit

$$(199) \quad \left(\frac{\sinh(2J_1/k_B T_c) \ell^2 + \sinh(2J_2/k_B T_c) m^2}{\sinh(2J_1/k_B T_c) + \sinh(2J_2/k_B T_c)} \right)^{\frac{1}{2}} |t| \equiv r, \text{ fixed.}$$

Here \pm refer to $T > T_c$ and $T < T_c$ respectively. They then showed that $G_{\pm}(r)$ could be expressed in terms of a solution of the Painlevé III equation

$$(200) \quad \frac{d^2 \eta}{d\theta^2} = \frac{1}{\eta} \left(\frac{d\eta}{d\theta} \right)^2 - \frac{1}{\theta} \frac{d\eta}{d\theta} + \eta^3 - \eta^{-1}$$

as

$$(201) \quad G_{\pm}(r) = \frac{1 \mp \eta(\frac{r}{2})}{2\eta(\frac{r}{2})^{1/2}} \exp \left[\frac{1}{4} \int_{r/2}^{\infty} \theta \frac{(1 - \eta^2)^2 - (\eta')^2}{\eta^2} d\theta \right]$$

with the boundary condition $\eta(\theta) \sim 1 - \frac{2}{\pi} K_0(2\theta)$ as $\theta \rightarrow \infty$, where K_0 denotes the modified Bessel function. The above calculation was the first derivation of an explicit scaling law for a two-spin correlation function for any model in statistical physics. As noted before, the scaling functions $G_{\pm}(r)$ are believed to be universal for a wide class of two-dimensional models with short range interactions.

A crucial issue for [WMTB] scaling theory was to show that formula (198) matches the critical $1/4$ behavior (106) as $r \rightarrow 0$. In order to do this, the authors needed to know the $\infty \leftrightarrow 0$ connection formulae for the Painlevé functions $\eta(\theta)$. In [WMTB] they extracted this information from the unpublished thesis of Myers [My] (see below). In a later paper in 1977, McCoy, Tracy, and Wu [MTW] derived $\infty \leftrightarrow 0$ connection formulae for a two-parameter class of bounded solutions $\eta(\theta; \nu, \lambda)$ of the general Painlevé III equation

$$\frac{d^2\eta}{d\theta^2} = \frac{1}{\eta} \left(\frac{d\eta}{d\theta} \right)^2 - \frac{1}{\theta} \frac{d\eta}{d\theta} + \frac{2\nu}{\theta} (\eta^2 - 1) + \eta^3 - \eta^{-1}.$$

We now describe in more detail what was done in [MTW], restricting our discussion to the case $\nu = 0$ which is relevant to [WMTB].

Consider the one parameter family $\eta(\theta) = \eta(\theta; 0, \lambda)$ of the solutions of equation (200) defined by the asymptotic condition

$$(202) \quad \eta(\theta) \sim 1 - 2\lambda K_0(2\theta), \quad \theta \rightarrow \infty.$$

In [MTW], it was shown that, for $0 \leq \lambda < 1/\pi$, as $r \rightarrow 0$,

$$(203) \quad \eta(r/2) = Br^\sigma \left(1 - \frac{1}{16} B^{-2} (1 - \sigma)^{-2} r^{2-2\sigma} + O(r^2) \right)$$

where the constants B and σ are functions of λ given by the explicit formulae:

$$(204) \quad \sigma = \sigma(\lambda) = \frac{2}{\pi} \arcsin(\pi\lambda), \quad B = B(\sigma) = 2^{-3\sigma} \frac{\Gamma((1 - \sigma)/2)}{\Gamma((1 + \sigma)/2)}.$$

In [My], Myers derived the asymptotics for $\eta(r/2; 0, 1/\pi)$ as $r \downarrow 0$,

$$(205) \quad \eta\left(\frac{r}{2}; 0, \frac{1}{\pi}\right) \sim -\frac{r}{2} \left(\log \frac{r}{8} + \gamma_E \right)$$

where γ_E is Euler's constant. In [MTW] the authors recovered (205) by analyzing formally the limit $\lambda \rightarrow 1/\pi$ in (203). Proofs of (205) were given later in [Wid8] and, using the Riemann-Hilbert method, in [Niles]. In turn, the asymptotic formula (205) yields the estimate:

$$(206) \quad G_\pm \sim \text{const } r^{-1/4} \left[1 \pm \frac{r}{2} \left(\ln \frac{r}{8} + \gamma_E \right) \right].$$

In order to see that this estimate matches equation (106), it is enough to notice that (198) and (199) imply that

$$(207) \quad \langle \sigma_{1,1} \sigma_{1+n,1+n} \rangle \sim 2^{1/4} G_\pm(r) |t|^{1/4} = 2^{1/4} G_\pm(r) r^{1/4} n^{-1/4}.$$

The main term in equation (106), up to the multiplicative constant, follows now directly from (206). The evaluation of the constant in (206) that matches the transcendental constant A in (106) was done by Tracy [T] in 1991.

The Painlevé functions, PI, . . . , PVI, (see [In]) were discovered and analyzed intensively at the beginning of the 20th century until the First World War, and then fell into a period of latency. They reappeared in theoretical physics quite unexpectedly in 1965 in the work of Myers [My] mentioned above, who showed that the scattering of electromagnetic radiation from a strip in the plane could

be expressed in terms of a solution of the Painlevé III equation. The paper [WMTB] was a seminal event in mathematical physics, followed soon after by the landmark paper of Jimbo et al [JMMS] on the impenetrable Bose gas, and the Painlevé equations have emerged over the years as the core of modern special function theory, with applications across the board in mathematics, physics and engineering (see, e.g., [FIKN]). In PDE's, in particular, the Painlevé equations (specifically, Painlevé II) appeared for the first time in the analysis of Ablowitz and Segur [AS] of self-similar solutions of the Korteweg-de Vries equation. This pioneering development opened the door to understanding the crucial role that the Painlevé equations play in the theory of integrable dynamical systems (see [AC], [FIKN]). The work [MTW] demonstrated for the first time the possibility to derive explicit connection formulae for families of solutions to a Painlevé equation, a property which had been previously known only for the linear ODEs of hypergeometric type. Similar connection formulae were obtained for the second Painlevé equation in [AS] (only formulae for the amplitude) and in [SA] (complete formulae for the amplitude and the phase). The papers [MTW], [AS], and [SA] generated a surge of activity in the global asymptotic analysis of the Painlevé equations (see again [FIKN] for more history and the state of the art in the area). We view these developments as one more example of the way in which questions arising in the analysis of the Ising model gave rise to developments in Toeplitz theory, and in this case, through Toeplitz theory, to the global theory of the Painlevé equations.

An alternative to the representation (201) of $G_{\pm}(r)$ was obtained in 1980 by Jimbo and Miwa [JM]. Their formula for $G_{-}(r)$ reads:

$$(208) \quad G_{-}(r) = \exp \left[- \int_{2r}^{\infty} \frac{\sigma(x)}{x} dx \right],$$

where $\sigma(x)$ is the unique solution of the Painlevé V equation

$$(209) \quad \left(x \frac{d^2 \sigma}{dx^2} \right)^2 = \left(\sigma - x \frac{d\sigma}{dx} + 2 \left(\frac{d\sigma}{dx} \right)^2 \right)^2 - 4 \left(\frac{d\sigma}{dx} \right)^2 \left(\left(\frac{d\sigma}{dx} \right)^2 - \frac{1}{4} \right)$$

satisfying the boundary conditions

$$(210) \quad \sigma(x) = \begin{cases} -1/4 + O(x \log x), & x \rightarrow 0, \\ \frac{1}{2\pi} x^{-1} e^{-x} \left(1 + O\left(\frac{1}{x}\right) \right), & x \rightarrow +\infty. \end{cases}$$

The equivalence of (208) to the original PIII-formula (201) was established in [MP].

Recently in [CIK] the following generalization of the Jimbo-Miwa result was obtained. The authors in [CIK] considered the double-scaling limit for the Toeplitz determinant $D_n(t)$ with the symbol:

$$(211) \quad \varphi(z) = (z - e^t)^{\alpha+\beta} (z - e^{-t})^{\alpha-\beta} z^{-\alpha+\beta} e^{-i\pi(\alpha+\beta)} e^{V(z)},$$

where $t \geq 0$, $\alpha \pm \beta \neq -1, -2, \dots$, $\Re \alpha > -1/2$, and $V(z)$ is analytic in an annulus containing the unit circle. The powers in (211) are defined by requiring the arguments to lie between zero and 2π . The symbol $\varphi(z)$ is analytic in $\mathbb{C} \setminus ([0, e^{-t}] \cup [e^t, +\infty))$. It is regular for $t > 0$ and has a Fisher-Hartwig singularity for $t = 0$ at $z_0 = 1$, $\varphi(z) = |z - 1|^{2\alpha} z^{\beta} e^{-i\pi\beta} e^{V(z)}$. The main result in [CIK] is the following: There exists a finite (perhaps empty) set $\Delta \subset (0, \infty)$ and a (small) number

$t_0 > 0$, such that for $n \rightarrow \infty$ and for all $0 < t < t_0$, $\log D_n(t)$ has an expansion of the form (see [CIK] Theorem 1.4)

$$(212) \quad \log D_n(t) = nV_0 + (\alpha + \beta)nt + \sum_{k=1}^{\infty} k \left[V_k - (\alpha + \beta) \frac{e^{-tk}}{k} \right] \left[V_{-k} - (\alpha - \beta) \frac{e^{-tk}}{k} \right] \\ + \int_0^{2nt} \frac{\sigma(x) - \alpha^2 + \beta^2}{x} dx + (\alpha^2 - \beta^2) \log 2nt + \log \frac{G(1 + \alpha + \beta)G(1 + \alpha - \beta)}{G(1 + 2\alpha)} + o(1),$$

with the error term being uniform for $0 \leq t < t_0$ provided $\text{dist}(2nt, \Delta) \geq \delta > 0$ for some $\delta > 0$, and the path of integration in (212) does not intersect with Δ . (In fact, the asymptotics (212) hold uniformly in a sector of the complex plane $-\pi/2 + \epsilon < \arg x < \pi/2 - \epsilon$, $0 < \epsilon < \pi/2$, away from a finite number of points. These points, which include the points of Δ , are possible poles of the function $\sigma(x)$. A choice of the integration contour corresponds to a branch of the logarithm.)

The function $\sigma(x)$ in (212) is the unique solution of the Painlevé V equation

$$(213) \quad \left(x \frac{d^2 \sigma}{dx^2} \right)^2 = \left(\sigma - x \frac{d\sigma}{dx} + 2 \left(\frac{d\sigma}{dx} \right)^2 + 2\alpha \frac{d\sigma}{dx} \right)^2 \\ - 4 \left(\frac{d\sigma}{dx} \right)^2 \left(\frac{d\sigma}{dx} + \alpha + \beta \right) \left(\frac{d\sigma}{dx} + \alpha - \beta \right)$$

which is real analytic in $(0, +\infty)$ and satisfies the boundary conditions

$$(214) \quad \sigma(x) = \begin{cases} \alpha^2 - \beta^2 + \frac{\alpha^2 - \beta^2}{2\alpha} \{x - x^{1+2\alpha} C(\alpha, \beta)\} (1 + O(x)), & x \rightarrow 0, \quad 2\alpha \notin \mathbb{Z} \\ \alpha^2 - \beta^2 + O(x) + O(x^{1+2\alpha}) + O(x^{1+2\alpha} \log x), & x \rightarrow 0, \quad 2\alpha \in \mathbb{Z} \\ x^{-1+2\alpha} e^{-x} \frac{-1}{\Gamma(\alpha - \beta)\Gamma(\alpha + \beta)} (1 + O(\frac{1}{x})), & x \rightarrow +\infty, \end{cases}$$

with

$$(215) \quad C(\alpha, \beta) = \frac{\Gamma(1 + \alpha + \beta)\Gamma(1 + \alpha - \beta)}{\Gamma(1 - \alpha + \beta)\Gamma(1 - \alpha - \beta)} \frac{\Gamma(1 - 2\alpha)}{\Gamma(1 + 2\alpha)^2} \frac{1}{1 + 2\alpha}.$$

For fixed $t > 0$, equation (212) yields the standard Szegő large n asymptotics for the regular symbol (211). It is interesting to note that this implies the following integral identity for the Painlevé function $\sigma(x)$ (cf. (1.31) in [CIK]): for any $x_0 > 0$,

$$(216) \quad \int_0^{x_0} \frac{\sigma(x) - \alpha^2 + \beta^2}{x} dx + \int_{x_0}^{\infty} \frac{\sigma(x)}{x} dx + (\alpha^2 - \beta^2) \log x_0 \\ = -\log \frac{G(1 + \alpha + \beta)G(1 + \alpha - \beta)}{G(1 + 2\alpha)}.$$

In the case $t = 0$, the equation (212) transforms to a single-point Fisher-Hartwig formulae: To see this, one utilizes the identity $\sum_{k=1}^{\infty} e^{-2kt}/k = -\ln(1 - e^{-2t})$.

In the special case $\alpha = 0$, $\beta = -\frac{1}{2}$, $V(z) \equiv 0$, and with the identification, $e^{-t} = k_{\text{ons}}$, the symbol (211) becomes $e^{-t/2} \varphi_{\text{diag}}(z)$. Therefore, in this case, $D_n(t) = e^{-nt/2} \langle \sigma_{1,1} \sigma_{1+n,1+n} \rangle$ with $t \downarrow 0$ corresponding to $T \uparrow T_c$. Set $2nt \equiv 2r$. Then formula (212), together with the identity (216) and the definition (198) of the scaling functions $G_{\pm}(r)$, yields the Jimbo-Miwa relation (208). With the equivalence of (208) and (201), this means that the *uniform* asymptotics (212) include the double-scaling limit in [WMTB] as a special case corresponding to $2nt \equiv 2r$ fixed (away from Δ), $n \rightarrow \infty$, and $\alpha = 0$, $\beta = -\frac{1}{2}$, $V(z) \equiv 0$.

Remark 11.1. In [JM], the authors used a remarkable relation (which they themselves discovered) between the diagonal Ising correlations and isomonodromy deformations of a certain 2×2 Fuchsian system. This discovery laid the foundation for the Riemann-Hilbert method in the theory of correlation functions, random matrices, and Toeplitz and Hankel determinants (see [DIZ] for more on the history of these developments). Using this link to Fuchsian systems, it was shown in [JM] that the diagonal correlation function $\langle \sigma_{1,1} \sigma_{1+n,1+n} \rangle$, as a function of the variable

$$s = e^{-2t} \equiv \left(\sinh \frac{2J_1}{k_B T} \sinh \frac{2J_2}{k_B T} \right)^{-2}$$

is expressed in terms of a solution to the Painlevé VI equation. The derivation of (208) in [JM] is done by formally performing a scaling transformation taking the inverse monodromy problem associated with this Painlevé VI to the inverse monodromy (Riemann-Hilbert) problem associated with the fifth Painlevé function $\sigma(x)$. In the rigorous setting of [CIK], the latter problem appears as a parametrix during the implementation of the nonlinear steepest-descent method.

Remark 11.2. The Painlevé VI description of the diagonal correlations in [JM] was the first example of the appearance of the Painlevé functions in the theory of Toeplitz determinants before the large n limit is taken. One realizes now that nonlinear differential equations of Painlevé type are always present in situations where the symbol $\varphi(z)$ satisfies the condition:

$$(217) \quad \frac{d \log \varphi(z)}{dz} = \text{rational function.}$$

This fact is quite easy to see within the Riemann-Hilbert formalism (see, e.g., discussion in [ITW]). These “finite n ” Painlevé representations are not universal: the type of the nonlinear ODE depends strongly on the structure of the rational right hand side of (217), i.e. on the symbol.

Double-scaling problems for Toeplitz determinants occur in many different areas. For example, in combinatorics, let $\pi = (\pi(1), \pi(2), \dots, \pi(N))$ be a permutation of the numbers $1, 2, \dots, N$. If $1 \leq i_1 < i_2 < \dots < i_k \leq N$ and $\pi(i_1) < \pi(i_2) < \dots < \pi(i_k)$, we say that $\pi(i_1), \dots, \pi(i_k)$ is an *increasing subsequence* in π of length k . Let $\ell_N(\pi)$ denote the maximum length of all increasing subsequences in π . With uniform distribution on the permutations, one is interested (see [BDJ]) in the distribution function

$$p_N(n) = \text{Prob} \{ \ell_N(\pi) \leq n \} = \frac{\#\{ \pi : \ell_N(\pi) \leq n \}}{N!}$$

as $N, n \rightarrow \infty$. It turns out [Ges] that the Poissonized version $\phi_n(\lambda)$ of the process $p_N(n)$

$$(218) \quad \phi_n(\lambda) = \sum_{N=0}^{\infty} e^{-\lambda} \frac{\lambda^N}{N!} p_N(n), \quad \lambda > 0$$

can be expressed in terms of a Toeplitz determinant

$$(219) \quad \phi_n(\lambda) = e^{-\lambda} D_n(\varphi_\lambda)$$

where $\phi_\lambda(\theta) = e^{2\sqrt{\lambda} \cos \theta}$. In order to recover $p_N(n)$ as $N, n \rightarrow \infty$ from (218) (219) one must analyze the double-scaling limit for the determinant as $\lambda, n \rightarrow \infty$. The main result in [BDJ] is the following:

$$(220) \quad \lim_{N \rightarrow \infty} \text{Prob} \left(\frac{\ell_N - 2\sqrt{N}}{N^{1/6}} \leq t \right) = F_2(t)$$

where $F_2(t)$ is the Tracy-Widom distribution [TW] for the largest eigenvalue of a random matrix in the Gaussian Unitary Ensemble. The distribution $F_2(t)$ can be expressed in terms of Painlevé functions as follows:

$$(221) \quad F_2(t) = e^{-\int_t^\infty (s-t) u^2(s) ds}$$

where $u(s)$ is the unique solution of PII ,

$$(222) \quad u''(s) = 2u^3(s) + su(s)$$

with

$$u(s) \sim -\text{Ai}(s) \quad \text{as} \quad s \rightarrow +\infty.$$

Here $\text{Ai}(s)$ is the standard Airy function. On the other hand, $F_2(t)$ can be expressed as a Fredholm determinant

$$(223) \quad F_2(t) = \det(I - K_{\text{Airy}}^{(s)})$$

where $K_{\text{Airy}}^{(s)}$ is the trace-class operator with kernel

$$K_{\text{Airy}}^{(s)}(x, y) = \frac{\text{Ai}(x) \frac{d}{dy} \text{Ai}(y) - \text{Ai}(y) \frac{d}{dx} \text{Ai}(x)}{x - y}$$

acting on $L^2(s, \infty)$. In random matrix theory, this determinant is the probability that the interval (s, ∞) contains no eigenvalues in the edge scaling limit in the Gaussian Unitary Ensemble.

The result in [BDJ] was the first of many results linking problems in pure and applied mathematics and in physics to the Tracy-Widom distribution $F_2(t)$ and its symplectic and orthogonal analogs (see, for example [Dei2]).

As already noted on several occasions, in his work on impenetrable bosons [Len1], Lenard was led to consider Toeplitz determinants with symbols of type $|e^{i\theta} - e^{i\theta_1}| |e^{i\theta} - e^{i\theta_2}|$ (see (125) above) that vanished on S^1 . In [Len2] he then considered symbols with any finite number of such α -type FH singularities. Such symbols vanish at only a finite number of points, and in [Wid6] Widom began considering symbols which vanished on a full interval. Let $\varphi_\mu(e^{i\theta})$ denote the characteristic function of the interval $(\mu, 2\pi - \mu)$, $0 < \mu < \pi$. In a virtuoso calculation Widom showed that, for μ fixed, as $n \rightarrow \infty$

$$(224) \quad \log D_n(\varphi_\mu) = n^2 \log \cos \frac{\mu}{2} - \frac{1}{4} \log \left(n \sin \frac{\mu}{2} \right) + c_0 + o(1)$$

where $c_0 = \frac{1}{12} \log 2 + 3\zeta'(-1)$ and $\zeta(z)$ is the Riemann zeta-function. A few years later in 1976, Dyson [Dy4] returned to the problem (cf. discussion following (59) above) of computing the probability P_s that there are no eigenvalues for a random matrix in the interval $(0, 2s/\pi)$, in the bulk scaling limit for the Gaussian Unitary Ensemble. He showed that as $s \rightarrow \infty$, $\log P_s$ has a full asymptotic expansion

$$(225) \quad \log P_s = -\frac{s^2}{2} - \frac{1}{4} \log s + a_0 + \frac{a_1}{s} + \frac{a_2}{s^2} + \dots,$$

Dyson identified all the constants a_0, a_1, a_2, \dots , and it turns out that a_0 , which is of particular interest, is precisely the constant c_0 arising in Widom's expansion (224) above. Dyson arrived at the identification $a_0 = c_0$ by noting first that the probability P_s is given by (see, e.g., [Meh], [Dei3])

$$(226) \quad P_s = \det(I - K_s)$$

where K_s is the trace-class operator with kernel

$$(227) \quad K_s(x, y) = \frac{\sin(x - y)}{\pi(x - y)}$$

acting on $L^2(-s, s)$. A simple calculation shows that

$$(228) \quad D_n(\varphi_\mu) = \det \left(\delta_{jk} - \frac{\sin \mu(j - k)}{\pi(j - k)} \right)_{0 \leq j, k \leq n-1}$$

and so for fixed $s > 0$,

$$(229) \quad \lim_{n \rightarrow \infty} D_n \left(\varphi_{\frac{2s}{n}} \right) = P_s$$

and hence, if the error term $o(1)$ in (224) was uniform in the double scaling limit, $n \rightarrow \infty$, $\mu \rightarrow 0$ such that $\mu n = 2s$, one could conclude from (224) (229) that a_0 is indeed equal to c_0 . The uniformity of the error term $o(1)$, however, remained open. There are now three proofs that $a_0 = c_0$. The first two were given simultaneously and independently by Ehrhardt and Krasovsky, and the third was given a little later in [DIKZ]. The proofs by Krasovsky and by Deift, Its, Krasovsky, and Zhou proceed by following Dyson's observation (229) and verifying that the $o(1)$ error term is indeed uniform. We refer the reader to [DIKZ] for some history of the problem P_s , $s \rightarrow \infty$, and, in particular, for the relevant references to the work of Krasovsky and Ehrhardt. A similar asymptotic problem arises when instead of the sine kernel determinant (226) one considers the general so-called confluent hypergeometric kernel determinant which depends on two extra parameters (and is related to the so-called Bessel kernel determinant and also to the determinant (226) for particular choices of the parameters); another related problem is the asymptotics for the Tracy-Widom distribution (223) as $s \rightarrow -\infty$. These problems can also be represented as double scaling problems for Toeplitz (and also Hankel) determinants, and such representations were crucial for finding complete solutions for the asymptotic problems at hand. We refer the reader to [Kr] for a discussion of the results and for the references.

Dyson's identification of a_1, a_2, \dots in [Dy4] involves an ingenious application of inverse scattering theory for Schrödinger operators on the half-line $0 \leq x < \infty$. He begins by noting that $P_s = \det(I - K_s)$ can be factored

$$(230) \quad \det(I - K_s) = D_+(s)D_-(s)$$

where $D_\pm(s) = \det(I - f_\pm)_{L^2(0,s)}$ are Toeplitz+Hankel type determinants on $L^2(0, s)$ with kernels

$$(231) \quad f_\pm(x, y) = \frac{1}{\pi} \left[\frac{\sin(x - y)}{x - y} \pm \frac{\sin(x + y)}{x + y} \right].$$

He then considers the functions

$$(232) \quad W_\pm(s) = -2 \frac{d^2}{ds^2} \log D_\pm(s) - 1$$

as potentials for Schrödinger operators $H_\pm = -\frac{d^2}{ds^2} + W_\pm(s)$ on $L^2(0, \infty)$ with appropriate Robin boundary conditions at $s = 0$. Using the Gelfand-Levitan formalism (see, e.g., [Fad]), Dyson is able to identify the spectral measures $\rho_\pm(\lambda)d\lambda$ for the operators H_\pm . He then obtains the scattering

phase functions $e^{i\eta_{\pm}}$. In the Marchenko formalism (see, e.g., [Fad]) one utilizes the functions $e^{i\eta_{\pm}}$ to express the potentials $W_{\pm}(s)$ in terms of Fredholm determinants of the form

$$(233) \quad W_{\pm}(s) = -\frac{1}{4} \left(s \pm \frac{1}{2} \right)^{-2} - 2 \frac{d^2}{ds^2} \log \Delta_{\pm}(s), \quad s > \frac{1}{2}$$

where $\Delta_{\pm}(s) = \det(I - F_{\pm})_{L^2(s, \infty)}$ and $F_{\pm}(x, y)$ are certain explicit kernels which decay as $s \rightarrow \infty$. Equating (232) and (233), and integrating twice, one obtains the following identity for $s > \frac{1}{2}$:

$$(234) \quad \log P_s = \log(I - K_s) = \log D_+ D_- = -\frac{1}{2} s^2 - \frac{1}{8} \log \left(s^2 - \frac{1}{4} \right) + a_0 + \log(\Delta_+(s) \Delta_-(s))$$

(the terms linear in s drop out). As $F_{\pm}(x, y)$ decay as $x, y \rightarrow \infty$, it is clear that (234) gives rise to a full expansion of the form (225), where the coefficients can be computed in terms of the trace powers $\text{tr} F_{\pm}^m$.

Dyson is able to compute $e^{i\eta_{\pm}}$, and hence F_{\pm} , explicitly, because of the following fundamental fact from inverse scattering theory. The spectral weights have the form $\rho_{\pm}(\lambda) = \sqrt{\lambda}/(\pi|\varphi_{\pm}(\sqrt{\lambda})|^2)$, where $\varphi_{\pm}(k)$, $k = \sqrt{\lambda}$, are certain functions constructed from the Jost solutions $J_{\pm}(s, k)$ for H_{\pm} , $H_{\pm}J_{\pm}(s, k) = k^2 J_{\pm}(s, k)$, $J_{\pm}(s, k) \sim e^{iks}$ as $s \rightarrow \infty$. The functions $\varphi_{\pm}(k)$ are analytic in $\{\Im k > 0\}$ with prescribed asymptotics as $k \rightarrow \infty$, and in the cases at hand, have no zeros in $\{\Im k > 0\}$. Hence, by standard arguments, $\varphi_{\pm}(k)$ are determined by their absolute value $|\varphi(k)|$ for $\Im k = 0$. Thus $\rho_{\pm}(\lambda)$ determine $\varphi_{\pm}(k)$. However, $e^{i\eta_{\pm}}$ can be expressed in terms of φ_{\pm} as $e^{i\eta_{\pm}(k)} = i\varphi_{\pm}(k)/|\varphi_{\pm}(k)|$, and hence $\rho_{\pm}(\lambda)$ determine the scattering phase functions, and hence, eventually, F_{\pm} .

These considerations bring to mind the Borodin-Okounkov-Geronimo-Case formula (43) in which the Szegő function $\mathcal{D}(z) = \exp\left(\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \varphi(e^{i\theta}) d\theta\right)$, which is analytic in $|z| < 1$, plays the role of the functions $\varphi_{\pm}(k)$. Indeed, $|\mathcal{D}(e^{i\theta})|^2 = \varphi(e^{i\theta})$, which is the spectral measure for the associated Cantero-Moral-Velasquez (CMV) unitary operator acting on ℓ_+^2 (see [Sim2]). Thus we may think of the Toeplitz determinant $D_n(\varphi)$ as an expression in the ‘‘Gelfand-Levitan formalism’’. On the other hand, we may think of the Fredholm determinant on the RHS of (43) as an expression in the ‘‘Marchenko formalism’’. Indeed, from (44), $b = \overline{\mathcal{D}}^2/|\mathcal{D}|^2$, which we may view as a scattering phase function $b = e^{2i\eta}$ for the CMV operator, and c is just $\overline{b} = e^{-2i\eta}$. These considerations are very suggestive and invite further investigation. One notes that the original proof of (43) by Geronimo and Case in [GerCase] was developed in the context of scattering theory.

Another example of a double-scaling limit is the following. Recall that the density matrix for the impenetrable bosons is given by $\rho_{N,L}(\xi) = \frac{1}{L} R_N(2\pi\xi/L)$, where R_N is the Toeplitz determinant (130) with symbol (131). For definiteness, fix the length scale so that $L = N$. As we discussed in Remark 6.1 in Section 6, of particular interest is the double-scaling limit $N \rightarrow \infty$, $\xi/N \rightarrow 0$. In [Len1] Lenard considered this limit $\rho(\xi) = \lim_{N \rightarrow \infty} \frac{1}{N} R_N(2\pi\xi/N)$ with ξ fixed. He obtained an expression for $\rho(\xi)$ in terms of a Fredholm determinant with explicit kernel related to the sine kernel (227). He also obtained another representation for $\rho(\xi)$ in terms of a Fredholm minor for the kernel $2 \frac{\sin \pi \xi |(x-y)|}{\pi(x-y)}$.

In order to complete his analysis of the limiting momentum distribution in the gas, Lenard needed to know the large ξ behavior of $\rho(\xi)$, but he did not derive this behavior in [Len1]. The derivation of this limiting behavior was carried out by Vaidya and Tracy [VaiTr] who found that

$$(235) \quad \rho(\xi) = \frac{\rho_{\infty}}{(\pi\xi)^{1/2}} \left(1 + \frac{1}{8(\pi\xi)^2} \left(\cos 2\pi\xi - \frac{1}{4} \right) + \dots \right), \quad \rho_{\infty} = \pi e^{1/2} 2^{-1/3} A^{-6},$$

as $\xi \rightarrow \infty$, where A is again Glaisher's constant (107). Note that if one evaluates $\rho_{N,N}(\xi) = \frac{1}{N} R_N(2\pi\xi/N)$, $N, \xi = \frac{tN}{2\pi} \rightarrow \infty$ using (137) (138), and assumes that the formulae remain valid as $t = 2\pi\xi/N \rightarrow 0$, then one obtains the leading term in (235).

The double-scaling limit $N \rightarrow \infty$, $\xi/N \rightarrow 0$ corresponds to the Toeplitz determinant with two FH singularities merging (at $z = 1$) with $\alpha_1 = \alpha_2 = \frac{1}{2}$, $\beta_1 = \beta_2 = 0$. A general analysis of the double-scaling asymptotics of a Toeplitz determinant with two merging FH singularities (in particular, of Lenard's symbol (130)) is given in [CK].

Dyson also considered [Dy5] the following scaling problem associated with Toeplitz determinants. Let

$$(236) \quad \Delta(z, t) = \det(I - z\widehat{K}_t)$$

where $0 < z < 1$ and \widehat{K}_t acts on $L^2(-t, t)$ with kernel $\widehat{K}_t(x, y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$. With the extra π in the sine function (cf. K_s in (227)), $\Delta(1, t)$ is now the probability that there are no eigenvalues in the interval $(0, 2t)$. Of interest here is the double scaling limit for $\Delta(z, t)$ as $t \rightarrow \infty$ and $z \uparrow 1$. Dyson analyses the problem by interpreting $\Delta(z, t)$ in terms of a Coulomb gas at inverse temperature $\beta = 2$. In this interpretation, z corresponds to an external potential v via the formula $1 - z = e^{-\beta v}$. Then

$$(237) \quad \Delta(z, t) = \frac{Z_2(v, 2t)}{Z_2(0, 2t)}$$

where

$$(238) \quad Z_\beta = \sum e^{-\beta(W+vn)}$$

is the partition function of the gas with external potential v applied to charges in a fixed interval of length $2t$. The sum in (238) is to be interpreted as an infinite-dimensional integral over all configurations of the gas, W represents the sum of all Coulomb interactions $W_{jk} = -\log|x_j - x_k|$, and for any configuration, n is the number of charges in a fixed interval of length $2t$. Dyson does not define the sum rigorously, but he uses it as a heuristic guide in his calculations. The main result in the paper is the calculation of oscillatory factors in $\Delta(z, t)$ as $t \rightarrow \infty$: These factors turn out to be so-called "genuinely non-linear oscillations" and are described by Jacobi elliptic functions. For $0 < z < 1$, $0 < \lambda < 1$, the symbol

$$(239) \quad f_{z,\lambda}(e^{i\theta}) = \chi_{[\pi\lambda, \pi(2-\lambda)]} + (1-z)\chi_{[-\pi\lambda, \pi\lambda]}$$

gives rise to the Toeplitz determinant

$$(240) \quad D_n(f_{z,\lambda}) = \det \left(\delta_{jk} - z \frac{\sin \lambda\pi(j-k)}{\pi(j-k)} \right)_{0 \leq j, k \leq n-1}$$

and so for fixed z, t

$$(241) \quad \lim_{n \rightarrow \infty} D_n \left(f_{z, \frac{2t}{n}} \right) = \Delta(z, t).$$

Thus Dyson's double scaling limit $z \uparrow 1$, $t \rightarrow \infty$ can be viewed as a triple scaling limit for the Toeplitz determinant $D_n(f_{z, 2t/n})$ with $z \uparrow 1$, $t \rightarrow \infty$, and $n \rightarrow \infty$. Note finally that $f_{z,\lambda}$ is precisely the kind of symbol that arises in the Toeplitz eigenvalue problem (178).

Lenard ends his 1972 paper [Len2] with a hope and a prophecy: “It is the author’s hope that a rigorous analysis will someday carry the results to the point where the true role of the zeros of the generating function will be understood. When that day comes a capstone will have been put on a beautiful edifice to whose construction many contributed and whose foundations lie in the studies of Gabor Szegő half a century ago”.

Almost 100 years have passed since Szegő published his first paper on the asymptotics of Toeplitz determinants. We certainly know more now than in 1972, but many new problems continue to arise, and it is still too early, alas, to set the capstone.

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