Received: 18 August 2023 | Revised: 18 October 2024 | Accepted: 7 November 2024

DOI: 10.1112/jlms.70045

RESEARCH ARTICLE

Journal of the London Mathematical Society

Central limit theorem for smooth statistics of one-dimensional free fermions

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Funding information
Swedish Research Council, Grant/Award Number: 2022-04882; Ragnar Söderbergs stiftelse, Grant/Award Number: SG230501-290430; CNRS

Abstract

We consider the determinantal point processes associated with the spectral projectors of a Schrödinger operator on $\mathbb{R}$, with a smooth confining potential. In the semiclassical limit, where the number of particles tends to infinity, we obtain a Szegő-type central limit theorem for the fluctuations of smooth linear statistics. More precisely, the Laplace transform of any statistic converges without renormalisation to a Gaussian limit with a $H^{1/2}$-type variance, which depends on the potential. In the one-well (one-cut) case, using the quantum action-angle theorem and additional micro-local tools, we reduce the problem to the asymptotics of Fredholm determinants of certain approximately Toeplitz operators. In the multi-cut case, we show that for generic potentials, a similar result holds and the contributions of the different wells are independent in the limit.

MSC 2020
60B20, 81Q20, 60F05 (primary)

1 | INTRODUCTION

Free fermions. This article is a follow-up on [15] on the semiclassical analysis of free fermions or determinantal point processes associated with spectral projectors of Schrödinger operators [33]. To define the model, let $V \in C^\infty(\mathbb{R}, \mathbb{R})$ be a function such that $V(x) \to +\infty$ as $x \to \pm\infty$. Then, for $\hbar \in (0, 1]$, the operator $-\hbar^2\Delta + V$ is essentially self-adjoint with compact resolvent on $L^2(\mathbb{R})$ [22].

Consequently, given $\mu \in \mathbb{R}$, the orthogonal projector

$$\Pi_\hbar(\mu) = \mathbb{1}_{(-\infty,\mu]}(-\hbar^2\Delta + V) \tag{1}$$

is well defined by the spectral theorem, and has finite rank $N = N_\hbar(\mu)$. This operator defines a determinantal point process on $\mathbb{R}$, denoted by $\mathrm{X} := \sum_{j=1}^N \delta_{x_j}$, which describes the joint spatial distribution of N fermions occupying the low-energy states of $-\hbar^2\Delta + V$. This point process is characterised by the property that for any function $f \in L^\infty(\mathbb{R}, \mathbb{C})$, the Laplace transform of the linear statistic $\mathrm{X}(f)$ is given by the Fredholm determinant

$$\mathbb{E}[e^{\mathrm{X}(f)}] = \det(\mathrm{I} + (e^f - 1)\Pi_\hbar), \tag{2}$$

where the function $e^f - 1$ is interpreted as a bounded operator on $L^2(\mathbb{R})$ acting by multiplication. We refer to the surveys [26, 40] and the appendix of [15] for different reviews about determinantal processes. Such free fermions point processes have also been extensively studied in the physics literature, for example, [10–13, 18, 38] and for the harmonic oscillator in [1].

These point processes can be studied for a general smooth (confining) potential V in the semiclassical limit $\hbar \to 0$. By Weyl's law, the number of particles $N \sim \frac{1}{\pi\hbar}\int(\mu - V)_+^{1/2}$ tends to infinity, while the point process concentrates on a deterministic measure supported on the set $\{V \leqslant \mu\}$, which is sometimes called *the droplet*, see, for example, [15, Theorem I.1]. The main goal of [15] was to establish that the (microscopic) scaling limits of these point processes in the bulk and at the edge of the droplet of particles are given by the Sine, respectively, the Airy, point processes from random matrix theory [13, 41]. These results rely on the asymptotics of the kernel associated to $\Pi_\hbar$ and they are generalised to free fermions on $\mathbb{R}^n$ for any $n \geqslant 1$; the scaling limits are independent of the potential V. There is, however, a major difference in the behaviour of linear statistics in dimension 1 versus $n \geqslant 2$. Indeed, one expects that for any (non-constant) smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with suitable growth, the corresponding linear statistic satisfies $\operatorname{var}\mathrm{X}(f) \asymp \hbar^{1-n}$ as $\hbar \to 0$. It is an open problem to obtain exact asymptotics for these variance in dimension $n \geqslant 2$. We proved in [15] that if $f \in C_c^\infty(\{V < \mu\}, \mathbb{R})$, then $\operatorname{var}\mathrm{X}_\hbar(f) \leqslant C(f)\hbar^{1-n}$, while for any (non-constant) function $f \in H^1(\mathbb{R}^n)$, we have $\hbar^{2-n}\operatorname{var}\mathrm{X}_\hbar(f) \to \infty$ as $\hbar \to 0$. This lower bound is not sharp, but it does imply a central limit theorem (CLT) for linear statistics, cf. [15, Theorem I.3] because the variance tends to infinity. In contrast, in dimension $n = 1$, the variance stays bounded and a more precise study is needed to establish a CLT, if any.

Main results. In this article, we therefore focus on one-dimensional free fermions point processes and our goal is to show that for any test function $f \in C_c^\infty(\mathbb{R}, \mathbb{R})$, the fluctuations of the linear statistics $\mathrm{X}(f)$ are Gaussian in the limit $\hbar \to 0$; we also determine the limit of $\operatorname{var}\mathrm{X}(f)$ and how it depends on the Newtonian dynamics for the potential V at energy μ. To this end, we directly analyse the determinant (2) by using the *quantum action-angle theorem* to reduce the problem to the harmonic oscillator; this procedure is specific to the 1D case. This strategy works directly in the *one-cut* case where $\{V \leqslant \mu\}$ is connected; we can also treat *multi-cut* situations under some generic assumptions.

Theorem 1 (One-cut case). *Let $V \in C^\infty(\mathbb{R}, \mathbb{R})$, with $V(x) \to +\infty$ as $x \to \pm\infty$. Let $\mu \in \mathbb{R}$ be such that $\{V = \mu\} = \{x_0^-, x_0^+\}$ with $x_0^- < x_0^+$ and $V'(x_0^\pm) \neq 0$. Then, for any $f \in C_c^\infty(\mathbb{R}, \mathbb{R})$, uniformly for*

$\eta \in \mathbb{C}$ sufficiently small,

$$\log\det(\mathrm{I} + (e^{\eta f} - 1)\Pi_\hbar) = \eta\,\mathrm{tr}(f\Pi_\hbar) + \frac{\eta^2}{2}\Sigma^2_{(V,\mu)}(f) + \underset{\hbar\to 0}{o}(1).$$

This implies that in distribution, as the number of particles $N \to \infty$, the random variable

$$\mathrm{X}(f) - \mathbb{E}\mathrm{X}(f) \;\to\; \Sigma_{(V,\mu)}(f)\,\mathcal{N}, \tag{3}$$

where $\mathcal{N}$ is a standard Gaussian. The limiting variance is a weighted $H^{1/2}$-seminorm, meaning that there is a map $\psi : [0, 2\pi] \to [x_0^-, x_0^+]$ so that

$$\Sigma^2_{(V,\mu)}(f) = \sum_{k\in\mathbb{N}} k\left|\int_0^{2\pi} e^{\mathbf{i}k\theta} f(\psi(\theta))\frac{\mathrm{d}\theta}{2\pi}\right|^2. \tag{4}$$

The map ψ depends on the Newtonian dynamics with potential V at energy μ and it is constructed in Section 2.1. (See also Section A for an interpretation of (4) in terms of the Gaussian free field.)

By a standard truncation, a version of Theorem 1 holds for continuous test functions with at most exponential growth and which are C^∞ in a neighbourhood of the droplet $\{V \leqslant \mu\}$. In this case, the distributional limit (3) still holds and all centred moments of the linear statistic $\mathrm{X}(f)$ also converge; we clarify this in Proposition 2.16. Formula (4) can be recovered from the asymptotics of the covariance kernel of the counting function $\mathbb{R} \ni x \mapsto \mathrm{X}(\mathbb{1}_{(-\infty,x]})$ of the fermions point process. These asymptotics are well known for the harmonic oscillator (see [7] for a thorough study of the GUE counting function) and have been derived using physical arguments in [38], for general one-cut potential, based on WKB asymptotics of the Schrödinger operator eigenfunctions. We will report on the relationship with [38] and the Gaussian free field interpretation of Theorem 1 in the Appendix A.

Example 1.1. For the *quantum harmonic oscillator*, $V(x) = x^2$ on $\mathbb{R}$ and $\mu = 1$, the Schrödinger operator $-\hbar^2\Delta + V$ is diagonalised by the Hermite functions with respect to the weight $x \mapsto e^{-x^2/\hbar}$. Then, the free fermions point process corresponds to the eigenvalues of the Gaussian unitary ensemble (GUE) and the equilibrium measure is the Wigner semi-circle law $\rho(x) = \frac{2}{\pi}\sqrt{1-x^2}$. Moreover, according to Section 2.1, one has $\psi(\theta) = \cos\theta$ for $\theta \in [0, 2\pi]$, so that $\Sigma^2(f) = \sum_{k\in\mathbb{N}} k|\hat{f}_k|^2$ where $(\hat{f}_k)_{k\in\mathbb{N}_0}$ denotes the Fourier–Chebyshev coefficients of f. In this case, Theorem 1 corresponds to [28, Thm 2.4] with $\beta = 2$; see also [41, Chap. 3]. Note that in this limit, one can also replace the mean $\mathbb{E}\mathrm{X}(f)$ by $N\int f\mathrm{d}\rho$ up to a vanishing error.

The results of [28] hold for general β-ensembles (or 1d log-gases) in the one-cut case and the strategy there relies on the so-called loop equation method. In contrast to Theorem 1, the asymptotic variance for one-cut β-matrix models is universal in the sense that it depends on the potential V only via the support of the equilibrium measure. We refer to [2, 4, 31] and the last version of [3] for further recent developments on the loop equation method for β-ensembles.

It is also of interest to compare Theorem 1 to the (strong) *Szegő limit theorem* which arises in a slightly different geometric context, considering free fermions on a one-dimensional torus (circular unitary ensemble (CUE)).

Example 1.2. Consider the determinantal process associated with the operator $\mathbb{1}_{(-\infty,1]}(-\hbar^2\Delta)$ where Δ is the Laplacian on $\mathbb{T} = [0, 2\pi]$ with periodic boundary conditions so that the number of particles is $N = 2\hbar^{-1} + 1$. The eigenfunctions of this operator are just Fourier modes and this point process corresponds to the eigenvalues of the classical *CUE* from random matrix theory. In particular, one can rewrite (2) as

$$\mathbb{E}[e^{\mathrm{X}(f)}] = \det(A^{(N)}),$$

where A is a Toeplitz matrix, $\left(A^{(N)}_{ij} = \hat{g}_{i-j}\right)_{i,j\in[N]}$ and $\left(\hat{g}_k\right)_{k\in\mathbb{Z}}$ denotes the usual Fourier coefficient of the function $g = e^f - 1$. The asymptotics of such *Toeplitz determinants* are a classical subject in analysis and are known as Szegő limit theorems. In particular, if $f \in H^{1/2}(\mathbb{T},\mathbb{C})$ (i.e if $\sum_{k\in\mathbb{Z}} \sqrt{1+k^2}|\hat{f}_k|^2 < \infty$), then

$$\log\det(A^{(N)}) = N\hat{f}_0 + \frac{1}{2}\sum_{k\in\mathbb{Z}} |k|\hat{f}_k\hat{f}_{-k} + \underset{N\to\infty}{o}(1).$$

This result is discussed in details in [36, Chap. 6] with several different proofs, including the combinatorial approach developed by Kac [29] and Soshnikov [39] that we follow in Section 2.5. We also refer to the survey [14] for further proofs and applications of the strong Szegő limit theorem and to [6, 32] for the relationship with the GUE eigenvalue fluctuations and log-correlated Gaussian fields.

In the *multi-cut* case, that is, when the droplet $\{V \leqslant \mu\}$ has several disconnected components, we can also formulate a CLT under an additional genericity condition on the potential. This genericity condition ensures that the different cuts are non-resonant (their eigenvalues are sufficiently far away from each other), so that eigenfunctions are all localised in exactly one of the cuts. This genericity holds along any discrete set of values of $\hbar$ accumulating at 0 with at least polynomial speed. Such subsequences are relevant in the context of quantisation on a compact phase space (where $\hbar_m \asymp 1/m$); they are also relevant since the number of particles is expected to grow linearly (since $\hbar N$ is roughly constant by the Weyl law).

Theorem 2 (Multi-cut case). *Let $V \in C^\infty(\mathbb{R},\mathbb{R})$ with $V(x) \to +\infty$ as $x \to \pm\infty$, and assume that $\mu \in \mathbb{R}$ satisfies, for some $\ell \geqslant 2$,*

$$\{V \leqslant \mu\} = \bigcup_{j=1}^{\ell} I_j$$

with disjoint (non-empty) intervals I_j. Let $(\chi_k)_{1\leqslant k\leqslant \ell}$ be a family of $C^\infty_c(\mathbb{R},\mathbb{R}_+)$ functions with disjoint supports such that $\chi_k = 1$ on I_k and fix a small $\epsilon > 0$. Let $(\hbar_m)_{m\in\mathbb{N}}$ be a sequence of semiclassical parameters such that for some $\alpha > 0$, $\hbar_m \leqslant m^{-\alpha}$ if m is large enough, and define

$$\Pi_m = \mathbb{1}_{(-\infty,\mu]}\left(-\hbar_m^2\Delta + V + \sum_{j=1}^{\ell} w_j\chi_j\right), \qquad w \in [-\epsilon,\epsilon]^\ell.$$

Then, for almost every $w \in [-\epsilon,\epsilon]^\ell$, for any $f \in C^\infty_c(\mathbb{R},\mathbb{R})$, it holds uniformly for $\eta \in \mathbb{C}$ sufficiently small,

$$\log\det(I + (e^{\eta f} - 1)\Pi_m) = \eta\,\mathrm{tr}(f\Pi_m) + \frac{\eta^2}{2}\sum_{j=1}^{\ell} \Sigma^2_{(W_j,\mu)}(f) + \underset{m\to\infty}{o}(1),$$

where $W_j = V + w_j$ on a neighbourhood of I_j, is strictly larger than μ outside and $\Sigma^2_{(W_j,\mu)}$ is as in Theorem 1.

This behaviour is in sharp contrast with that of multi-cut random matrix models. Indeed, it was pointed out by Pastur [34] that because orthonormal polynomials are delocalised on every cut in the semi-classical regime, the numbers of eigenvalues in the different cuts of the equilibrium measure has non-trivial order 1 fluctuations for large N. Then, for real-analytic potentials, it was established in [35] and [3, Theorem 1.6] that these fluctuations are given by an explicit multivariate discrete Gaussian law. This phenomenon entails that the fluctuations of a generic linear statistic are also uniformly bounded in N, but they are generically non-Gaussian. For unitary-invariant Hermitian random matrices ($\beta = 2$), this problem was revisited in [6] using Riemann–Hilbert asymptotics for orthogonal polynomials in the multi-cut regime. In particular, the connection with the Gaussian free field on a Riemann surface is discussed in detail in [6, Section 1.5]. In our situation, the eigenfunctions of $-\hbar^2\Delta + V + \sum w_j\chi_j$ are, generically, localised on exactly one of the intervals I_j, which we use to prove that the fluctuations are Gaussian.

It is not clear what happens in the resonant case, such as for the symmetric double well potential $V(x) = x^4 - x^2$ at energies $\mu < 0$. From the results of [25], eigenvalues appear in exponentially close pairs and if μ is not in between them (so that N is even), then again, the projector exponentially closes to the sum of the two one-well projectors. In contrast, if N is odd, we do not know whether a CLT should hold.

Proof strategy and additional comments

Our proof of Theorem 1 relies crucially on the *complete integrability* of one-dimensional Schrödinger operators. Namely, under the assumptions of Theorem 1, the eigenfunctions of the operator $-\hbar^2\Delta + V$ with eigenvalues in a neighbourhood of μ have all order asymptotic expansions. This is usually performed using Wentzel–Kramers–Brillouin (WKB) approximations in quantum mechanics [8]. Rather than using directly these approximations, our method exploits the *quantum action-angle theorem* (Proposition 2.4) to reduce (by an approximate conjugation of the Fredholm determinant (2)) the problem to the case of the harmonic oscillator (Example 1.1), at the price of working with a general *pseudo-differential operator* instead of a multiplication operator. In this context, we obtain Szegő-type asymptotics (Proposition 2.2) for certain Fredholm determinants of pseudo-differential operators.

Our proof involves the fact that pseudo-differential operators have an *approximate Toeplitz* structure in the eigenbasis of the harmonic oscillator. This reduces the problem to Szegő-type asymptotics for which one can apply classical methods such as the cumulant method and the Dyson–Hunt–Kac (DHK) combinatorial lemma, see [29] and [36, Chap. 6.5]. This method has already been successfully used in another generalisation of the strong Szegő limit theorem using microlocal tools [21].

When using the DHK combinatorial approach, there is no need for a symmetry hypothesis (either the self-adjointness used in [29] or the time reversibility used in [21]) to obtain a strong Szegő theorem; we clarify this by reviewing this approach.

Szegő-type limit theorems also hold for general one-cut unitary-invariant Hermitian random matrix ensembles, and we refer to [5, 30] for different approaches to Szegő-type asymptotics using the connection with orthogonal polynomials. In particular, different forms of the DHK formula are discussed in [30, App A].

The action-angle strategy requires the non-degeneracy 'Airy-type edge' generic condition $\nabla V \neq 0$ on $\{V = \mu\}$. It would be interesting to consider the case of hyperbolic critical points (such as appearing in Landau-type potentials), where the asymptotic behaviour of eigenfunctions is also known [9].

The proof of Theorem 2 relies on well-known localisation techniques (see, e.g. [23, 24]) to show that, generically (in the sense of Theorem 2), the projector $\Pi_\hbar$ can be approximated by a commuting sum of projectors which correspond to the potential localised on each well. Hence, in this case, we can reduce the asymptotics of linear statistics to the case of Theorem 1.

Organisation of the paper

The proof of Theorem 1 (and its generalisation to pseudo-differential operators, Proposition 2.2) is given in Section 2 and is organised as follows. In Sections 2.1 and 2.2, we review the classical action-angle theorem for certain one-dimensional Hamilton flow and its quantum analogue for Schrödinger operators. This also gives us the opportunity to introduce several important notation and basic results from semiclassical analysis.

In Section 2.3, we establish that as a consequence of the *quantum angle-action theorem* and the asymptotic properties of the Hermite functions, a certain class of pseudo-differential operators have an approximately Toeplitz structure (as $\hbar \to 0$) in the eigenbasis of $-\hbar^2\Delta + V$. In Section 2.4, we gather some results concerning the Taylor expansion of Fredholm determinants and its continuity property with respect to perturbations. Finally, in Section 2.5, we combine these results to complete the proof of Theorem 1. The core of the argument relies on the approximately Toeplitz property of pseudo-differential operators and a combinatorial approach to the Strong Szegő theorem known as the Dyson–Kac–Hunt (DHK) formula.

The proof of Theorem 2 is given in Section 3. It relies on Theorem 1, localisation techniques to isolate the contribution from each well and the genericity condition to guarantee that there are no *resonances* between the wells.

2 | ONE-CUT CASE — PROOF OF THEOREM 1

Throughout this section, we assume that the potential $V \in C^\infty(\mathbb{R}, \mathbb{R})$ is such that $V \to +\infty$ when $|x| \to +\infty$ and we fix $\mu \in \mathbb{R}$ such that $\{V \leqslant \lambda\} = [x_0^-(\lambda), x_0^+(\lambda)]$ with $x_0^-(\lambda) < x_0^+(\lambda)$ for λ in a neighbourhood of μ. Furthermore, we assume a non-degeneracy condition $V'(x_0^\pm(\lambda)) \neq 0$ for every λ in this neighbourhood of μ. Under these hypotheses, the maps $\lambda \mapsto x_0^\pm(\lambda)$ are C^∞-diffeomorphisms. Later on, we will assume more technical assumptions about the behaviour of V near infinity, but they can be lifted (see [15, Section 2.2]).

2.1 | Classical action-angle theorem

Let $(\phi_t)_{t\in\mathbb{R}}$ denote the Hamilton flow of $\mathcal{H} : (x, \xi) \mapsto \xi^2 + V(x)$, that is, for every $(x_0, \xi_0) \in \mathbb{R}^2$, the function $\mathbb{R}^2 \ni (x_0, \xi_0) \mapsto (x_t, \xi_t) = \phi_t(x_0, \xi_0) \in \mathbb{R}^2$ solves the evolution equation

$$\begin{cases} \partial_t x_t = 2\xi_t \\ \partial_t \xi_t = -V'(x_t). \end{cases} \tag{5}$$

A remarkable property of one-dimensional Hamiltonian dynamics is *complete integrability*. The dynamics preserve the energy $\mathcal{H}$, and the flow is periodic along the curves

$$C_\lambda := \left\{(x,\xi) \in \mathbb{R}^2, \mathcal{H}(x,\xi) = \lambda\right\}, \tag{6}$$

with (shortest) period $T(\lambda)$. By assumptions, these curves are simply connected and $T(\lambda) < \infty$ for λ in a neighbourhood of μ.

The flow ϕ_t is, on a neighbourhood of C_μ, conjugated to the Hamiltonian flow of a function of the harmonic oscillator; letting

$$\mathrm{g}(\lambda) := \frac{1}{2\pi}\mathrm{Area}\big(\{(x,\xi) \in \mathbb{R}^2, \mathcal{H}(x,\xi) \leqslant \lambda\}\big), \tag{7}$$

then the formula

$$\kappa : (\phi_t(x_0^+(\lambda), 0)) \mapsto \sqrt{2\mathrm{g}(\lambda)}\Big(\cos\tfrac{2\pi t}{T(\lambda)}, -\sin\tfrac{2\pi t}{T(\lambda)}\Big)$$

defines a C^∞-diffeomorphism from a ϕ_t-invariant neighbourhood of C_μ to an annular neighbourhood of $\{x^2 + \xi^2 = 2\mathrm{g}(\mu)\}$. In fact, κ is area-preserving and $\mathrm{g}\circ\mathcal{H} = \frac{|\kappa|^2}{2}$ on this neighbourhood.

Given $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$ and I in a neighbourhood of $\mathrm{g}(\mu)$, we can now define the Fourier coefficients of f along the Hamilton flow as

$$\hat{f}_j(I) := \frac{1}{T(\lambda)}\int_0^{T(\lambda)} e^{-\frac{2i\pi jt}{T(\lambda)}} f(\phi_t(x_0^+(\lambda), 0))\mathrm{d}t \qquad \lambda = \mathrm{g}^{-1}(I), \quad k \in \mathbb{Z}. \tag{8}$$

An equivalent definition is

$$\hat{f}_j(I) := \frac{1}{2\pi}\int_0^{2\pi} f\circ\kappa^{-1}(\sqrt{2I}\cos(\theta), \sqrt{2I}\sin(\theta))e^{ij\theta}\mathrm{d}\theta, \qquad k \in \mathbb{Z}. \tag{9}$$

Then, in the context of Theorem 1, the asymptotic variance Σ is defined by, for any $f \in C_c^\infty(\mathbb{R}, \mathbb{R})$,

$$\Sigma^2_{(V,\mu)}(f) := \sum_{k\in\mathbb{N}} k\big|\hat{f}_k(\mathrm{g}(\mu))\big|^2, \tag{10}$$

where we view f as a (smooth) function on $\mathbb{R}^2$ independent of the second variable. This amounts to the $H^{\frac{1}{2}}$ seminorm of the periodic function $f\circ\psi$, where according to the Hamiltonian flow (5),

$$\psi(\theta) = x_{\frac{\theta T(\mu)}{2\pi}}(x_0^+(\mu), 0), \qquad \theta \in [0, 2\pi].$$

Choosing the action I as the parameter for $\hat{f}_j$, rather than the energy μ, will simplify the computations in Section 2.3.

2.2 | Semiclassical tools

Here, we review several techniques from semiclassical analysis, notably a quantum version of the considerations of Subsection 2.1 and we give a generalisation of Theorem 1 (Proposition 2.2) in this context. We begin with a quick overview of Weyl quantisation, pseudo-differential operators and their application to the spectral theory of Schrödinger operators, referring to [16, 45] for more detailed introductions. Then, we collect some useful estimates for the eigenfunctions of one-dimensional Schrödinger operators.

Definition 2.1. We consider the symbol class $S^0(\mathbb{R}^2, \mathbb{C})$; an element of S^0 is a function $a \in C^\infty(\mathbb{R}^2, \mathbb{C})$ such that

$$\forall (j,k) \in \mathbb{N}^2,\ \exists C_{j,k} \in \mathbb{R}_+, \qquad \sup_{(x,\xi)\in\mathbb{R}^2} |\partial_x^j \partial_\xi^k a(x,\xi)| \leqslant C_{j,k}.$$

Let $\hbar > 0$. The Weyl quantisation of $a \in \mathcal{S}(\mathbb{R}^2, \mathbb{C})$ is the operator with kernel

$$Op_\hbar(a)(x,y) = \frac{1}{2\pi\hbar} \int e^{i\frac{\xi(x-y)}{\hbar}} a(\tfrac{x+y}{2}, \xi) \mathrm{d}\xi.$$

If $a \in \mathcal{S}(\mathbb{R}^2, \mathbb{C})$, this defines a trace class operator on $L^2(\mathbb{R})$, whose trace norm is controlled by a finite number of seminorms of a in $\mathcal{S}$. This definition can be extended to symbols in the class $S^0(\mathbb{R}^2, \mathbb{C})$, in which case $Op_\hbar(a)$ is still a bounded operator; its operator norm satisfies the Gårding inequality ([45], Theorem 4.32)

$$\|Op_\hbar(a)\|_{L^2\to L^2} = \|a\|_{L^\infty} + \mathcal{O}(\hbar). \tag{11}$$

In this paper, symbols will be allowed to depend on the semiclassical parameter $\hbar$. We will only use *classical symbols*: an element of S^0_{cl} is a function $a : \mathbb{R}^2 \times (0, +\infty) \to \mathbb{C}$ such that there exists a sequence $(a_k)_{k\in\mathbb{N}}$ of elements of S^0 with

$$\forall (j,k,\ell) \in \mathbb{N}^2,\ \exists C_{j,k,\ell} \in \mathbb{R}_+, \qquad \sup_{(x,\xi,\hbar)\in\mathbb{R}^2\times\mathbb{R}_+} \left\{ \hbar^{-\ell} \left| \partial_x^j \partial_\xi^k [a(x,\xi,\hbar) - \textstyle\sum_{m=0}^{\ell-1} \hbar^m a_m(x,\xi)] \right| \right\} \leqslant C_{j,k,\ell}.$$

The first term a_0 is called the *principal symbol* of a.

In this framework, we will prove the following generalisation of Theorem 1.

Proposition 2.2 (Szegő asymptotics). *Let $\mu > 0$ and $\Pi_\hbar = \Pi_\hbar(\mu)$ be given by (1) under the assumptions of Theorem 1. Let $a \in S^0_{\mathrm{cl}}(\mathbb{R}^2, \mathbb{C})$ with principal symbol $\|a_0\|_{L^\infty} \leqslant c < 1$. Then,*

$$\log\det(\mathrm{I} + Op_\hbar(a)\Pi_\hbar) = \mathrm{tr}(\log(1 + Op_\hbar(a))\Pi_\hbar) + \frac{1}{2}\sum_{\ell\in\mathbb{N}} |\ell| \widehat{f}_\ell \widehat{f}_{-\ell} + \underset{\hbar\to 0}{o}(1),$$

where $(\widehat{f}_\ell)_{\ell\in\mathbb{Z}}$ are the 'Fourier coefficients' of the function $f = -\log(1 + a_0)$ given by formula (8) with $I = \mathrm{g}(\mu)$.

Obviously, if $f \in C_c^\infty(\mathbb{R}, \mathbb{R})$, the function $a : (x,\xi) \mapsto e^{\eta f(x)} - 1$ belongs to $S^0(\mathbb{R}^2, \mathbb{C})$ with $\|a\|_{L^\infty} < 1$ if the parameter $\eta \in \mathbb{C}$ is small enough. It is independent of ξ and $Op_\hbar(a) = a$ in this case. Then, by Proposition 2.2 (rescaling f to ηf), we conclude that if η is sufficiently small, as $\hbar \to 0$,

$$\log\det(1 + a\Pi_\hbar) = \eta\, \mathrm{tr}(f\Pi_\hbar) + \frac{\eta^2}{2}\sum_{\ell\in\mathbb{N}} |\ell| \widehat{f}_\ell \widehat{f}_{-\ell} + o(1).$$

Hence, by definition of the variance $\Sigma^2_{(V,\mu)}$ (10), this proves Theorem 1 in the one-cut case.

Pseudo-differential operators are well-adapted to the study of Schrödinger operators under some conditions on V at infinity.[†] In the rest of this article, we will always assume that there exists $m > 0$ such that

$$\forall k \in \mathbb{N}, \exists C_k, \forall x \in \mathbb{R}, |\partial^k V(x)| \leqslant C_k(1+|x|)^m$$

and

$$\exists c_0, R > 0, \ \forall |x| > R, \ |V(x)| \geqslant c_0(1+|x|)^m.$$

Under these conditions, not only is the operator $-\hbar^2\Delta + V$ bounded from below with compact resolvent, but smooth compactly supported spectral functions of $-\hbar^2\Delta + V$ are given by pseudo-differential operators with compactly supported symbol ([45], Theorem 14.9).

Fredholm determinants are defined for trace-class operators or other *Schatten class* of operators after a suitable modification, see [37] for an introduction. We review the associated notion of Schatten norms; for a bounded operator A on a Hilbert space, define for $1 \leqslant p < +\infty$,

$$\|A\|_{J^p} = \mathrm{tr}((A^*A)^{\frac{p}{2}})^{\frac{1}{p}}.$$

If this quantity is finite $A \in J^p$. Plainly, finite rank operators belong to J^p for any $p \geqslant 1$. We will use the cases $p = 1$ (trace norm) and $p = 2$ (Hilbert–Schmidt norm).

It turns out that the Fredholm determinant in Proposition 2.2, beyond the leading order, depends on commutators such as $[\Pi_\hbar, Op_\hbar(a)]$. The next lemma implies that we can also assume that the symbol a in Proposition 2.2 is supported on a small neighbourhood of the curve C_μ (6).

Lemma 2.3 (Microlocalisation). *Let $(\phi_j)_{j\in\mathbb{N}_0}$ be a Hilbert basis of eigenfunctions of $-\hbar^2\Delta + V$ with eigenvalues $(\lambda_j)_{j\in\mathbb{N}_0}$. Let $a \in S^0(\mathbb{R}^2, \mathbb{C})$ such that* $\mathrm{supp}(a) \cap C_\mu = \emptyset$. *There exists a (small) $\delta > 0$ so that for every $j \in \mathbb{N}_0$,*

$$Op_\hbar(a)\phi_j = \mathcal{O}_{L^2}(\hbar^\infty) \qquad \textit{uniformly for } |\lambda_j - \mu| \leqslant \delta.$$

Then, as $\hbar \to 0$,

$$\big\|[\Pi_\hbar, Op_\hbar(a)]\big\|_{J^1} = \mathcal{O}(\hbar^\infty).$$

Proof. The first claim is exactly [45, Theorem 6.4]. To obtain the second claim, we decompose

$$\Pi_\hbar = \mathbb{1}_{(-\infty,\mu]}(-\hbar^2\Delta + V) = \vartheta(-\hbar^2\Delta + V) + \chi\mathbb{1}_{(-\infty,\mu]}(-\hbar^2\Delta + V), \tag{12}$$

where $\chi \in C_c^\infty(\mathbb{R}, [0,1])$ is supported on a neighbourhood of μ, and $\vartheta \in C_c^\infty(\mathbb{R}, [0,1])$ is supported on $\{\xi^2 + V(x) < \mu\}$. By the functional calculus ([45], Theorem 14.9), one has

$$\vartheta(-\hbar^2\Delta + V) = Op_\hbar(b) + \mathcal{O}_{J^1}(\hbar^\infty),$$

[†] By elementary (Agmon-type) techniques, changing V away from $\{V \leqslant \mu\}$ has negligible impact on the quantities studied here; see [15, Section 2.2] for details.

where $b_\hbar \in \mathcal{S}$ satisfies $b_\hbar \in \{0,1\}$ on supp(a). Then, using the composition formula for Weyl quantisation (e.g. [45, Theorem 4.11 and 4.12]),

$$[Op_\hbar(b), Op_\hbar(a)] = Op_\hbar(b\#a - a\#b) \qquad \text{where} \quad b\#a = a\#b + \mathcal{O}_{\mathcal{S}}(\hbar^\infty).$$

Consequently, by [45, Theorem 4.21], $\|Op_\hbar(b\#a - a\#b)\|_{L^2\to L^2} = \mathcal{O}(\hbar^\infty)$. Using the Weyl-type bound rank$(\vartheta(-\hbar^2\Delta + V)) = \mathcal{O}(\hbar^{-1})$, we finally obtain

$$\|[\vartheta(-\hbar^2\Delta + V), Op_\hbar(a)]\|_{J^1} = \mathcal{O}(\hbar^\infty). \tag{13}$$

Moreover, we also have

$$\|\chi(-\hbar^2\Delta + V)Op_\hbar(a)\|_{J^2} = \sum_{j\in\mathbb{N}_0} \chi(\lambda_j)^2 \|Op_\hbar(a)\phi_j\|^2,$$

where every term is $\mathcal{O}(\hbar^\infty)$ (by the first claim as χ is supported in a δ-neighbourhood of μ). Since there are $\mathcal{O}(\hbar^{-1})$ non-zero terms, this shows that

$$\|\chi(-\hbar^2\Delta + V)Op_\hbar(a)\|_{J^2} = \mathcal{O}(\hbar^\infty). \tag{14}$$

Since both operators on the RHS of (12) have rank $\mathcal{O}(\hbar^{-1})$, the claim follows by combining the estimates (13) and (14). □

Like the classical Hamiltonian dynamics in two-dimensional phase space, the quantum evolution problem and the eigenvalue problem are *integrable* for one-dimensional pseudo-differential operators. This basic structure has been thoroughly exploited, notably through WKB expansions for eigenfunctions, and it lies at the foundation of quantum mechanics. The modern formulation and generalisation of this structure is the existence of a unitary conjugation of a Schrödinger operator into a function of the harmonic oscillator. This conjugation is *semi-global*, that is, holds in the vicinity of a given energy level μ, which is either regular (no critical point) or elliptic (critical points are Morse and the Hessian has positive determinant).

Proposition 2.4 (Quantum action-angle theorem). *Recall that* g *is the area map given by (7). There exists a bounded operator $U_\hbar$ on $L^2(\mathbb{R})$ such that for every $\chi \in C_c^\infty(\mathbb{R},\mathbb{R})$ supported in a small neighbourhood of μ,*

$$U_\hbar \chi(H_\hbar) U_\hbar^* = \vartheta\Big(-\tfrac{\hbar^2}{2}\Delta + \tfrac{x^2}{2}\Big) + \mathcal{O}_{J^1}(\hbar^\infty),$$

where $\vartheta \in S^0_{\mathrm{cl}}$ with compact support (in a small neighbourhood of the support of ϑ_0) and principal symbol $\vartheta_0 = \chi \circ \mathrm{g}^{-1}$. Moreover, $U_\hbar$ is microlocally unitary near the spectral curve C_μ:

- *for every $a \in C_c^\infty(\mathbb{R}^2,\mathbb{R})$ supported in a small neighbourhood of C_μ,*

$$Op_\hbar(a)U_\hbar^* U_\hbar = Op_\hbar(a) + \mathcal{O}_{J^1}(\hbar^\infty),$$

- *for every $b \in C_c^\infty(\mathbb{R}^2,\mathbb{R})$ in a small neighbourhood of the circle $\kappa(C_\mu)$,*

$$Op_\hbar(b)U_\hbar U_\hbar^* = Op_\hbar(b) + \mathcal{O}_{J^1}(\hbar^\infty),$$

- *for every* $a \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$ *supported in a small neighbourhood* C_μ*, there exists* $b \in S^0_{\mathrm{cl}}(\mathbb{R}^2, \mathbb{R})$ *with compact support and principal symbol* $b_0 = a \circ \kappa^{-1}$ *such that*

$$U_\hbar Op_\hbar(a) U_\hbar^* = Op_\hbar(b) + \mathcal{O}_{J^1}(\hbar^\infty),$$

A good presentation (in French) of the quantum action-angle theorem near regular trajectories can be found in [44]; the statement above corresponds to Théorème 4.1.8 (see also [8, 43]). There are several versions of this theorem; usually, one conjugates the operator into a function of $-i\hbar\partial/\partial\theta$ acting on $L^2(S^1)$, but in our context, it is simpler to conjugate to the harmonic oscillator, because of the absence of eigenvalue shifts (vanishing of the Maslov index), as will come into play below.

We begin with a description of the model case.

Proposition 2.5 [45, Chap. 6.1]. *The operator* $H_\hbar = -\frac{\hbar^2}{2}\Delta + \frac{x^2}{2}$ *is the (quantum) harmonic oscillator. Its eigenfunctions* $(\psi_j)_{j\in\mathbb{N}_0}$ *are the (semi-classical) Hermite functions. In particular, we have* $H_\hbar = A_\hbar^* A_\hbar + \hbar/2$ *where* $A_\hbar^* = \frac{1}{\sqrt{-2}}(\hbar\partial_x + x)$ *is the creation operator. We have* $\psi_0(x) \propto e^{-x^2/2\hbar}$ *for* $x \in \mathbb{R}$ *and* $\psi_j \propto A_\hbar^{*j}\psi_0$ *for* $j \in \mathbb{N}$ *so that* $H_\hbar\psi_j = (j + 1/2)\hbar\psi_j$ *for* $j \in \mathbb{N}_0$.

Remark 2.6. Let $\mathcal{N}$ be a small (but independent of $\hbar$) neighbourhood of μ. Propositions 2.4 and 2.5 imply that for any $\chi \in C_c^\infty(\mathcal{N}, \mathbb{R})$, the spectrum of $\chi(H_\hbar)$ lies $\mathcal{O}(\hbar^\infty)$-close to $\{\vartheta((n + \frac{1}{2})\hbar), n \in \mathbb{N}\}$, see, for instance, [22, Theorem 8.20]. This implies that the eigenvalues of $H_\hbar$ in $\mathcal{N}$ are simple and separated from each other by about $\hbar$ (one can see this by considering the spectra of $\chi(H_\hbar)$ for different functions χ).

In our context, using the conventions from Proposition 2.5, the relevant consequence of Proposition 2.4 is the following result.

Proposition 2.7. *There exists a Hilbert basis* $(\phi_j)_{j\in\mathbb{N}}$ *of eigenfunctions of* $-\hbar^2\Delta + V$ *such that the following holds. For every* $a \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$ *supported in a small neighbourhood of the curve* C_μ*, there exists* $b \in S^0_{\mathrm{cl}}(\mathbb{R}^2, \mathbb{R})$ *with compact support and principal symbol* $b_0 = a \circ \kappa^{-1}$ *so that, uniformly for* $j, k \in \mathbb{N}_0$,

$$\langle \phi_j, Op_\hbar(a)\phi_k\rangle = \langle \psi_j, Op_\hbar(b)\psi_k\rangle + \mathcal{O}(\hbar^\infty). \tag{15}$$

In particular, $\mathrm{rank}(\mathbb{1}_{(-\infty,\mu]}(-\hbar^2\Delta + V)) = \hbar^{-1}\mathrm{g}(\mu) + \mathcal{O}(1)$.

Proof. Since the eigenvalues of $H_\hbar$ lying in a neighbourhood of μ are separated by about $\hbar$ (Remark 2.6), the operator $U_\hbar$ from Proposition 2.4 maps the eigenfunction of $H_\hbar$ onto the corresponding Hermite function;[†] for every $j \in \mathbb{N}$ such that $\lambda_j \in [\mu - \epsilon, \mu + \epsilon]$, there exists $k = k(j) \in$

[†] The proof of this well-known fact goes as follows: suppose a bounded self-adjoint operator H_0 has an isolated simple eigenvalue λ_0 with $\mathrm{dist}(\lambda_0, \sigma(H_0) \setminus \lambda_0) = \epsilon_0 > 0$. If ψ is normalised in L^2 and approximately solves the eigenvalue equation: $\|(H_0 - \lambda_0)\psi\| = \delta \ll \epsilon_0$, then $\psi = \langle \psi_0, \psi\rangle\psi_0 + \Pi\psi$ where Π is the spectral projector onto the orthogonal of ψ_0 and $\|(H_0 - \lambda_0)\Pi\psi\| = \delta$. Hence, since $(H_0 - \lambda_0)$ is invertible on the range of Π, with inverse bounded by ϵ_0^{-1}, we conclude that $\|\Pi\psi\| \leqslant \delta/\epsilon_0$.

$\mathbb{N}$ and a phase α_j such that

$$\phi_j = e^{i\alpha_j} U_\hbar \psi_k + \mathcal{O}(\hbar^\infty). \tag{16}$$

In the last equation, one has, in fact, $j = k$. This follows by computing the Maslov index of $U_\hbar$ (see [8, 42]). A direct way to see that indices match is to use a deformation argument and a stronger action-angle formula, which we now explain.

Let $p : \mathbb{R}^2 \to \mathbb{R}$ be such that $p(x,\xi) = V(x) + \xi^2$ near C_μ and such that, on $\{p < 2\mu\}$, there is only one critical point which is a non-degenerate local minimum. In this situation, we can extend the action-angle theorem to the whole bottom of the spectrum (as in [44], Theorem 4.22), and since the bottom of the spectra must match, for the eigenfunctions $\tilde{\phi}_j$ of $Op_\hbar(p)$, one has exactly $U_\hbar \tilde{\phi}_j = e^{i\alpha_j}\psi_j + \mathcal{O}(\hbar^\infty)$ for all j with $\tilde{\lambda}_j < 2\mu$.

Now, by microlocalisation arguments (Lemma 2.3), the eigenvalues and eigenfunctions of $Op_\hbar(p)$ and $-\hbar^2\Delta + V$ are identical, up to $\mathcal{O}(\hbar^\infty)$, for all eigenvalues in $[\mu - \epsilon, \mu + \epsilon]$ and there is no index shift in this region because one can construct a homotopy between these two operators which fixes the number of eigenvalues in $\{\lambda < \mu - \epsilon\}$ and $\{\lambda \leqslant \mu + \epsilon\}$. Thus, if $\lambda_j \in [\mu - \epsilon, \mu + \epsilon]$,

$$\tilde{\phi}_j = \phi_j + \mathcal{O}(\hbar^\infty)$$

without index shift, which allows us to conclude.

To conclude the proof of Proposition 2.7, we can replace ϕ_k by $e^{-i\alpha_k}\phi_k$ and apply Proposition 2.4 if both $\lambda_j, \lambda_k \in [\mu - \epsilon, \mu + \epsilon]$, we obtain

$$\langle \phi_j, Op_\hbar(a)\phi_k\rangle = \langle \psi_j, Op_\hbar(b)\psi_k\rangle + \mathcal{O}(\hbar^\infty).$$

Otherwise, by Lemma 2.3, say ϕ_j is microlocalised away from the support of a, while ψ_j is microlocalised away from the support of $a \circ \kappa^{-1}$; thus, both sides of (15) are $\mathcal{O}(\hbar^\infty)$; which concludes the proof. □

2.3 | Approximate Toeplitz structure

It is possible to derive an asymptotic formula for $\langle \psi_j, Op_\hbar(b)\psi_k\rangle$ using either the induction formula on the Hermite functions $(\psi_k)_{k\in\mathbb{N}_0}$ or their WKB asymptotics as $\hbar \to 0$. Similar asymptotic formulas also appear in the literature [21] where the proof is quite involved, notably using the special form of the Guillemin–Wodzicki residue in the case of a periodic classical flow. We use a different and simpler approach in this article; a unitary conjugation to Berezin–Toeplitz quantisation on the Bargmann–Fock space.

Proposition 2.8 ([19, Propositions 2.96 and 2.97]; see also [45, Chapter 13]). *Recall the notation from Proposition 2.5. Let $\mathcal{B}_\hbar$ be the Bargmann transform (or wavepacket transform) defined by the integral kernel*

$$\mathbb{C} \times \mathbb{R} \ni (z, x) \mapsto \mathcal{B}_\hbar(z, x) = \frac{1}{(\pi\hbar)^{1/4}} \exp\left(-\frac{z^2 + |z|^2 - 2\sqrt{2}zx + x^2}{2\hbar}\right).$$

Then, $\mathcal{B}_\hbar$ is a unitary transformation from $L^2(\mathbb{R})$ into

$$\mathcal{H}_\hbar = \{f \in L^2(\mathbb{C}), z \mapsto e^{\frac{|z|^2}{2\hbar}} f(z) \text{ is holomorphic}\}$$

such that for all $j \in \mathbb{N}_0$,

$$\mathcal{B}_\hbar \psi_j : z \mapsto \frac{\hbar^{-\frac{1+j}{2}}}{\sqrt{\pi j!}} z^j e^{-\frac{|z|^2}{2\hbar}}.$$

Then, for every $b \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$, there exists $f \in S^0_{\mathrm{cl}}(\mathbb{C}, \mathbb{R})$ with compact support and principal symbol $f_0(x + i\xi) = b_0(x, \xi)$ such that uniformly as $\hbar \to 0$, for all $u, v \in \mathcal{H}_\hbar$ with $\|u\| = \|v\| = 1$,

$$\langle u, \mathcal{B}_\hbar Op_\hbar(b) \mathcal{B}_\hbar^* v\rangle = \int u(z)\overline{v(z)} f_\hbar(\sqrt{2}z)\mathrm{d}z + \mathcal{O}(\hbar^\infty). \tag{17}$$

Remark 2.9. $\mathcal{B}_\hbar A_\hbar^* \mathcal{B}_\hbar^* : u \mapsto zu$ corresponds to the usual creation operator on $\mathcal{H}_\hbar$. In particular, formula (17) is exact in this case and the normalisation of $\mathcal{B}_\hbar$ is consistent with the physics literature.

With this conjugation, for any $b \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$, one can easily obtain the asymptotics of matrix elements for $Op_\hbar(b)$ in the Hermite basis (see Proposition 2.5).

Proposition 2.10. *Let $b \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$. For every $\epsilon > 0$, as $\hbar \to 0$, uniformly for $j, k \geqslant \epsilon\hbar^{-1}$,*

$$\langle \psi_j, Op_\hbar(b)\psi_k\rangle = \hat{b}_{k-j}(\tfrac{j+k}{2}\hbar) + \mathcal{O}_\epsilon(\hbar|j-k|^{-\infty}) + \mathcal{O}(\hbar^\infty).$$

where, for $\lambda > 0$, $\big(\hat{b}_k(\lambda)\big)_{k\in\mathbb{Z}}$ denotes the Fourier coefficients of the function $\theta \in [0, 2\pi] \mapsto f_0(\sqrt{2\lambda}e^{\mathrm{i}\theta})$ with $f_0(x + i\xi) = b_0(x, \xi)$.

Proof. We have for $j, k \in \mathbb{N}_0$,

$$\langle \psi_j, Op_\hbar(b)\psi_k\rangle = \int_{\mathbb{C}} u_j(z)\overline{u_k(z)} f(\sqrt{2}z)\mathrm{d}z + \mathcal{O}(\hbar^\infty),$$

where $u_j = \mathcal{B}_\hbar \psi_j$ and $f \in S^0_{\mathrm{cl}}(\mathbb{C}, \mathbb{R})$ is as in Proposition 2.8. In particular, $u_k(z) = \gamma_k z^k u_0(z)$ with $\gamma_k = \sqrt{\frac{\hbar^{-k}}{k!}}$ for $k \in \mathbb{N}_0$, so that

$$\int_{\mathbb{C}} u_j(z)\overline{u_k(z)} f_\hbar(\sqrt{2}z)\mathrm{d}z = \frac{\gamma_j\gamma_k}{\hbar} \int_0^\infty \hat{f}_{k-j}(\lambda) e^{-\lambda/\hbar} \lambda^{\frac{j+k}{2}} \mathrm{d}\lambda, \tag{18}$$

where $\big(\hat{f}_k(\lambda)\big)_{k\in\mathbb{Z}}$ denotes the Fourier coefficients of the function $\theta \in [0, 2\pi] \mapsto f(\sqrt{2\lambda}e^{\mathrm{i}\theta})$ for $\lambda > 0$.

Note that for $\alpha > 0$, the function $\lambda \mapsto \lambda - \alpha \log \lambda$ is strictly convex on $\mathbb{R}_+$ with a unique minimum $\varpi(\alpha)$ for $\lambda = \alpha$. Let $\chi \in C_c^\infty(\mathbb{R})$ be a smooth cut-off which equals to 1 on a neighbourhood of α. By the Laplace method, since $\hat{f}_{k-j} \in C^2(\mathbb{R}_+)$ is bounded, it holds as $\hbar \to 0$,

$$\int_0^\infty \hat{f}_{k-j}(\lambda) e^{-(\lambda - \alpha \log \lambda)/\hbar} \mathrm{d}\lambda = \int \hat{f}_{k-j}(\lambda)\chi(\lambda) e^{-(\lambda-\alpha\log\lambda)/\hbar}\mathrm{d}\lambda + \mathcal{O}(\hbar^\infty)$$
$$= 2\sqrt{\pi\hbar}\, e^{-\varpi(\alpha)/\hbar}\big(\hat{f}_{k-j}(\alpha) + \mathcal{O}(\hbar)\big),$$

where the errors are controlled by $\|\hat{f}_{k-j}\|_{C^0}$ and $\|\chi\hat{f}_{k-j}\|_{C^2}$, respectively. Moreover, this estimate is locally uniform for $\alpha \in \mathbb{R}_+$.

Now, if $f \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$, the Fourier coefficients $\hat{f}_k$ have rapid decay as $k \to +\infty$ (in the C^∞ topology). To be precise, let $K \subset (0,\infty)$ be a compact set and $\ell \in \mathbb{N}$ By repeated integrations by part, for every $m \in \mathbb{N}$,

$$\sup_{r\in K} |\hat{f}_k^{(\ell)}(r)| = \sup_{r\in K}\left|\int_0^{2\pi} e^{-ik\theta}\partial_r^\ell f(\sqrt{2r}e^{\mathrm{i}\theta})\mathrm{d}\theta\right|$$
$$= k^{-m}\sup_{r\in K}\left|\int_0^{2\pi} e^{-ik\theta}\partial_r^\ell\partial_\theta^m f(\sqrt{2r}e^{\mathrm{i}\theta})\mathrm{d}\theta\right|$$
$$\leqslant k^{-m} C(K,\ell,m)\|f\|_{C^{m+\ell}}.$$

Hence, applying the Laplace method as above, we conclude that if the indices $j,k \geqslant \epsilon\hbar^{-1}$,

$$\int_0^\infty \hat{f}_{k-j}(\lambda)e^{-\lambda/\hbar}\lambda^{\frac{j+k}{2}}\mathrm{d}\lambda = \int_0^\infty \hat{f}_{k-j}(\lambda)e^{-(\lambda-\alpha\log\lambda)/\hbar}\mathrm{d}\lambda, \qquad \alpha = \tfrac{j+k}{2}\hbar,$$
$$= 2\sqrt{\pi\hbar}\, e^{-\varpi(\alpha)/\hbar}\big(\hat{f}_{k-j}(\alpha) + \mathcal{O}_\epsilon(\hbar|k-j|^{-\infty})\big).$$

In particular, we can replace f by its principal symbol f_0 while computing the Fourier coefficients, up to a similar error.

Now, going back to (18) with $f_\hbar = 1$ and $j = k$, we have for $k \geqslant \epsilon\hbar^{-1}$,

$$1 = \int_{\mathbb{C}} |u_k(z)|^2\mathrm{d}z = 2\gamma_k^2\sqrt{\pi/\hbar}\, e^{-\varpi(k\hbar)/\hbar}(1+\mathcal{O}(\hbar)).$$

This implies that for $j,k \geqslant \epsilon\hbar^{-1}$,

$$\int_{\mathbb{C}} u_j(z)\overline{u_k(z)} f_\hbar(\sqrt{2}z)\mathrm{d}z = e^{\left(\varpi(k\hbar)+\varpi(j\hbar)-2\varpi(\frac{j+k}{2}\hbar)\right)/2\hbar}\left(\hat{f}_{k-j}(\tfrac{j+k}{2}\hbar) + \mathcal{O}_\epsilon(\hbar|k-j|^{-\infty})\right).$$

Since $\varpi : \mathbb{R}_+ \ni \alpha \to \alpha(1-\log\alpha)$ is a smooth strictly concave function with $\|\varpi\|_{C^2(\epsilon,\infty)} \leqslant C/\epsilon$, we conclude that

$$e^{\left(\varpi(k\hbar)+\varpi(j\hbar)-2\varpi(\frac{j+k}{2}\hbar)\right)/2\hbar} = 1 - \mathcal{O}_\epsilon\big(\hbar|k-j|^2\big),$$

which together with the decay in the variable $k-j$ above, yields the final claim. □

Propositions 2.7 and 2.10 together imply the following approximate Toeplitz formula for the matrix elements of a pseudo-differential operator.

Proposition 2.11. *Recall that* $(\phi_j)_{j\in\mathbb{N}_0}$ *is a Hilbert basis of eigenfunctions of* $-\hbar^2\Delta + V$*. For every symbol* $a \in S^0_{\rm cl}(\mathbb{R}^2,\mathbb{R})$ *with compact support in a small neighbourhood of the curve* C_μ*, it holds uniformly for* $0 \leqslant j,k \leqslant 2N(\hbar)$*,*

$$\langle \phi_j, Op_\hbar(a)\phi_k\rangle = \hat{a}_{k-j}(\tfrac{j+k}{2}\hbar) + \mathcal{O}(\hbar|j-k|^{-\infty}),$$

where $\big(\hat{a}_k(I)\big)_{k\in\mathbb{Z}}$ *are the Fourier coefficients of the principle symbol* a_0*, given by formula (9). Moreover, the coefficients* $\hat{a}_k(I)$ *are* $\mathcal{O}(|k|^{-\infty})$ *uniformly for* $I > 0$*.*

Proof. Proposition 2.7 (a consequence of the action-angle theorem) allows to compute the matrix elements in the Hermite basis after replacing the symbol a by b, up to a uniform error $\mathcal{O}(\hbar^\infty)$ for the appropriate range of indices. Then, Proposition 2.10 yields the asymptotics of these coefficients with the appropriate off-diagonal decay. We note that the leading terms only depend on the principal part $b_0 = a\circ\kappa^{-1}$ and they correspond to the Fourier coefficients (9) (or equivalently (8)) with $I = \hbar\frac{j+k}{2}$, while the errors $\mathcal{O}(\hbar^\infty)$ are negligible compared to $\mathcal{O}(\hbar|j-k|^{-\infty})$ in the appropriate range. □

Lemma 2.12. *For every symbol* $a \in S^0(\mathbb{R}^2,\mathbb{C})$*, as* $\hbar \to 0$*,*

$$\|[\Pi_\hbar, Op_\hbar(a)]\|_{J^2} = \mathcal{O}(1).$$

Proof. Proposition 2.11 can be used to determine the limit of $\|[\Pi_\hbar, Op_\hbar(a)]\|_{J^2}$ as $\hbar \to 0$ for any symbol $a \in S^0(\mathbb{R}^2,\mathbb{C})$. By Lemma 2.3, one can assume that the symbol is supported on a small neighbourhood of the curve C_μ. Then, letting $A_{ij} = \langle\phi_j, Op_\hbar(a)\phi_i\rangle$ for $i,j\in\mathbb{N}_0$, by definition of the Hilbert–Schmidt norm,

$$\|[\Pi_\hbar, Op_\hbar(a)]\|_{J^2}^2 = \sum_{i,j\in\mathbb{N}_0} (\mathbb{1}\{j<N, i\geqslant N\} + \mathbb{1}\{i<N, j\geqslant N\})|A_{ij}|^2,$$

where we recall that $N = \mathrm{rank}(\mathbb{1}_{(-\infty,\mu]}(-\hbar^2\Delta + V))$. By Proposition 2.11, there is a constant C depending only on the symbol such that the coefficients $|A_{ij}| \leqslant C|i-j|^{-3}$ and $A_{ij} = \hat{a}_{j-i}(\frac{i+j}{2}\hbar) + \mathcal{O}(\hbar)$ as $\hbar \to 0$. In particular, using that the functions $I \in (0,\infty) \mapsto \hat{a}_k(I)$ are C^1 (uniformly for $k\in\mathbb{Z}$), if $i,j \in [N-L, N+L]$ for some L independent of $\hbar$, then $A_{ij} = \hat{a}_{j-i}(N\hbar) + \mathcal{O}(L\hbar)$ as $\hbar \to 0$. Moreover, by Weyl law $\hbar N \sim \mathrm{g}(\mu)$ in terms of the area map (7), so that in this range $A_{ij} = \hat{a}_{j-i}(\mathrm{g}(\mu)) + o_L(1)$ as $\hbar \to 0$ (here the error terms depends on L). This implies that for any given $L\in\mathbb{N}$,

$$\begin{aligned}\sum_{i,j\in\mathbb{N}_0} \mathbb{1}\{j<N\leqslant i\}|A_{ij}|^2 &= \sum_{i,j\in\mathbb{N}_0} \mathbb{1}\{N-L\leqslant j<N\}\mathbb{1}\{N\leqslant i\leqslant N+L\})|A_{ij}|^2 + \mathcal{O}(1/L)\\ &= \sum_{i,j\in\mathbb{Z}} \mathbb{1}\{j\in[-L,0)\}\mathbb{1}\{i\in[0,L]\}|\hat{a}_{i-j}(\mathrm{g}(\mu))|^2 + \underset{\hbar\to0}{o_L(1)} + \underset{L\to\infty}{o(1)},\end{aligned}$$

where the second error is independent of $\hbar$, and the same estimate holds for the other sum. Taking the limit as $\hbar \to 0$ (for L fixed so that the sum is finite) and then as $L \to \infty$ (by monotone convergence), we obtain

$$\sum_{i,j\in\mathbb{N}_0} \mathbb{1}\{j < N \leqslant i\}|A_{ij}|^2 \to \sum_{i,j\in\mathbb{Z}} \mathbb{1}\{j < 0 \leqslant i\}|\hat{a}_{i-j}(\mathrm{g}(\mu))|^2.$$

The sum on the RHS converges absolutely because of the decay of the Fourier coefficients $\{\hat{a}_k(I)\}_{k\in\mathbb{Z}}$ (the symbol a is smooth) so, by rearranging the sum, we obtain that as $\hbar \to 0$,

$$\sum_{i,j\in\mathbb{N}_0} \mathbb{1}\{j < N \leqslant i\}|A_{ij}|^2 \to \sum_{k\geqslant 1} k|\hat{a}_k(\mathrm{g}(\mu))|^2,$$

and similarly for the other sum. Combining these limits, we conclude that

$$\lim_{\hbar\to 0} \|[\Pi_\hbar, Op_\hbar(a)]\|_{J^2}^2 = \sum_{k\in\mathbb{Z}} |k||\hat{a}_k(\mathrm{g}(\mu))|^2.$$

This proves the claim and we note that, by symmetry, the limit is $2\Sigma^2_{(V,\mu)}(a)$ according to formula (10). We also record that the Szegő limit in Theorem 1 is $\Sigma^2_{(V,\mu)}(f)$ with $a = e^{\eta f} - 1$. In particular, as expected, we recover that $\Sigma^2_{(V,\mu)}(a) \sim \eta^2\Sigma^2_{(V,\mu)}(f)$ as $\eta \to 0$. Theorem 1 is a generalisation of this J_2 norm convergence, by computing the limit of higher order terms (cumulants) in the Taylor expansion of the Laplace transform $\log \mathbb{E}[e^{\eta \mathrm{X}(f)}]$ (2).

Alternative arguments (valid in any dimension) can be found in [15, Section 5.4] when a is a function of x with compact support inside the bulk (by Lemma 2.3, we may assume that a is supported on a small neighbourhood of C_μ) and also in the proof of [20]. □

2.4 | Fredholm determinants

The proof of Theorem 1 relies on a Taylor expansion of the Fredholm determinant on the RHS of (2), or rather of a renormalised determinant (obtained by removing the mean of $\mathrm{X}(f)$); this requires still a few preparatory results.

Let Π be a finite rank projection. Recall that if A is an operator with $\|A\| < 1$, then the Fredholm determinant $\det(1 + A\Pi)$ is well defined and positive. Moreover, the operator $\log(1 + A)$ is bounded and we have

$$\Upsilon_\Pi(A) := \log\det(1 + A\Pi) - \mathrm{tr}(\log(1 + A)\Pi) = \sum_{n\geqslant 2} \frac{(-1)^n}{n} \mathrm{tr}(\Pi A^n \Pi - (\Pi A \Pi)^n). \tag{19}$$

This series is (absolutely) convergent since we have the trivial bound $\|(\Pi A \Pi)^n - \Pi A^n \Pi\|_{J^1} \leqslant 2N\|A\|^n$ where $N = \mathrm{tr}\,\Pi$. In particular, we write $\Upsilon_\Pi(zA) = \sum_{n\geqslant 2} \frac{(-z)^n}{n} \Upsilon^n_\Pi(A)$ for $|z| < 1$. Our next result provides a uniform bound for (19) and a perturbative expansion of its coefficients in terms of the Hilbert–Schmidt norm of certain commutators.

Lemma 2.13. *For any operator A with $\|A\| \leqslant \rho$, one has $|\Upsilon^n_\Pi(A)| \leqslant \frac{n(n-1)}{2}\|[\Pi, A]\|^2_{J^2}\,\rho^{n-2}$ for any $n \geqslant 2$. Moreover, if $\rho < 1$, there is a constant C depending only on ρ and $\|[\Pi, A]\|_{J^2}$ so that for any* *operator R with $\|R\| \leqslant \rho$, one has for any $n \geqslant 2$.*

$$|\Upsilon^n_\Pi(A+R) - \Upsilon^n_\Pi(A)| \leqslant C2^n \max_{1\leqslant k<n} \|[\Pi, A^{k-1}R]\|_{J^2}.$$

Proof. Firstly observe that for $n \geqslant 2$,

$$\Pi A^n \Pi - (\Pi A \Pi)^n = \big(\Pi[\Pi, A][A^{n-1}, \Pi]\Pi\big) + (\Pi A \Pi)(\Pi A^{n-1}\Pi - (\Pi A \Pi)^{n-1})$$

so that by induction,

$$\begin{aligned}\|\Pi A^n \Pi - (\Pi A \Pi)^n\|_{J^1} &\leqslant \|[\Pi, A][A^{n-1}, \Pi]\|_{J^1} + \rho\|\Pi A^{n-1}\Pi - (\Pi A \Pi)^{n-1}\|_{J^1} \\ &\leqslant \|[\Pi, A]\|_{J^2} \sum_{1\leqslant k<n} \rho^{k-1}\|[A^{n-k}, \Pi]\|_{J^2},\end{aligned}$$

where we used that $\|BQ\|_{J^1} \leqslant \|B\|_{J^2}\|Q\|_{J^2}$ for Hilbert–Schmidt operators B, Q. Moreover, we also have for $n \geqslant 2$,

$$\|[A^n, \Pi]\|_{J^2} \leqslant \sum_{1\leqslant k\leqslant n} \|A^{k-1}[A, \Pi]A^{n-k}\|_{J^2} \leqslant n\rho^{n-1}\|[\Pi, A]\|_{J^2}.$$

This implies that for $n \geqslant 2$,

$$\|\Pi A^n \Pi - (\Pi A \Pi)^n\|_{J^1} \leqslant \tfrac{n(n-1)}{4}\|[\Pi, A]\|^2_{J^2}\rho^{n-2},$$

which yields the first bound.

Let us now turn to the second bound. Expanding each term in

$$\Upsilon^n_\Pi(A+R) := \mathrm{tr}(\Pi(A+R)^n\Pi - (\Pi(A+R)\Pi)^n),$$

we obtain

$$\Upsilon^n_\Pi(A+R) - \Upsilon^n_\Pi(A) = \sum_{\Gamma\in\Omega_n} \mathrm{tr}\,(\Pi\Gamma_1 \cdots \Gamma_n\Pi - \Pi\Gamma_1\Pi \cdots \Gamma_n\Pi), \tag{20}$$

where $\Omega_n := \big\{\Gamma \in \{A, R\}^n : \Gamma \neq (A, \dots, A)\big\}$.

We are going to study the sum (20) by induction on n. and to this end, we separate cases depending on Γ_1. Let $\Gamma' \in \Omega_{n-1}$ be given by

$$\Gamma' = \begin{cases} (\Gamma_1, \dots, \Gamma_{n-1}) & \text{if } \Gamma_1 = R \\ (\Gamma_2, \dots, \Gamma_n) & \text{if } \Gamma_1 = A; \end{cases}$$

in other words, we construct Γ' from Γ by removal of the *last* letter if the first letter is an R, and by removal of the *first* letter if the first letter is an A. The point is that one has always $\Gamma' \in \Omega_{n-1}$, which would not be the case if one indiscriminately removed the first or the last letter of Γ.

Let us also denote

$$\pi(\Gamma) = \Gamma_1 \cdots \Gamma_n \qquad \text{and} \qquad Q(\Gamma) = \Pi\Gamma_1 \cdots \Gamma_n\Pi - \Pi\Gamma_1\Pi \cdots \Gamma_n\Pi.$$

The point of our definition of Γ' is the following: for all $\Gamma \in \Omega_n$ with $n \geqslant 2$, we claim that

$$\|Q(\Gamma)\|_{J^1} \leqslant C\|[\Pi, \pi(\Gamma')]\|_{J^2} + \rho\|Q(\Gamma')\|_{J^1}, \tag{21}$$

where $\|[A, \Pi]\|_{J^2}, \|[R, \Pi]\|_{J^2} \leqslant C$ and we recall $\rho < 1$.

To prove (21) in case $\Gamma_1 = R$, we decompose

$$\begin{aligned} Q(\Gamma) &= \Pi[\Pi, \Gamma_1 \cdots \Gamma_{n-1}]\Gamma_n\Pi + (\Pi\Gamma_1 \cdots \Gamma_{n-1}\Pi - \Pi\Gamma_1\Pi \cdots \Pi\Gamma_{n-1}\Pi)\Pi\Gamma_n\Pi \\ &= [\Pi, \pi(\Gamma')](1 - \Pi)[\Gamma_n, \Pi] + Q(\Gamma')\Pi\Gamma_n\Pi. \end{aligned}$$

The argument is analogous in the other case $\Gamma_1 = A$.

Then, by induction on (21), there is a sequence $\Gamma^k \in \Omega_k$ for $k = 1, \dots, n$ such that $\Gamma^n = \Gamma, \Gamma^{k-1} = (\Gamma^k)'$ and

$$\|Q(\Gamma)\|_{J^1} \leqslant C \sum_{1 \leqslant k < n} \rho^{n-k-1}\|[\Pi, \pi(\Gamma^k)]\|_{J^2}.$$

Now we claim that there is a constant $c(\rho)$ so that for any $n \geqslant 2$ and $\Gamma \in \Omega_n$,

$$\|[\Pi, \pi(\Gamma)]\|_{J^2} \leqslant c(\rho) \max_{0 \leqslant j < n} \|[A^j R, \Pi]\|_{J^2}. \tag{22}$$

This proves the claim: indeed, from the two previous bounds, there is a constant $C(\rho)$ such that for any $n \geqslant 2$ and $\Gamma \in \Omega_n$,

$$\|\Pi\Gamma_1 \cdots \Gamma_n\Pi - \Pi\Gamma_1\Pi \cdots \Pi\Gamma_n\Pi\|_{J^1} \leqslant C(\rho) \max_{0 \leqslant j < n-1} \|[A^j R, \Pi]\|_{J^2}.$$

Going back to formula (20), with $|\Omega_n| \leqslant 2^n$, this proves the claim.

Now, to prove (22), note that for $\Gamma \in \Omega_n$, we can always write $\pi(\Gamma)$ in the form

$$\pi(\Gamma) = A^{\ell-1}R\,\Gamma_{\ell+1} \cdots \Gamma_n \qquad \text{for} \quad \ell \in \{1, \dots, n\},$$

so that

$$\begin{aligned} [\Pi, \pi(\Gamma)] &= [\Pi, A^{\ell-1}R]\Gamma_{\ell+1} \cdots \Gamma_n + A^{\ell-1}R[\Pi, \Gamma_{\ell+1} \cdots \Gamma_n] \\ \|[\Pi, \pi(\Gamma)]\|_{J^2} &\leqslant \|[\Pi, A^{\ell-1}R]\|_{J^2} + \|R[\Pi, \Gamma_{\ell+1} \cdots \Gamma_n]\|_{J^2}. \end{aligned}$$

If $\Gamma_{\ell+1} = \cdots = \Gamma_n = A$, we can bound

$$\begin{aligned} R[\Pi, \Gamma_{\ell+1} \cdots \Gamma_n] &= [R, \Pi]A^{n-\ell} + [\Pi, RA^{n-\ell}] \\ \|R[\Pi, \Gamma_{\ell+1} \cdots \Gamma_n]\|_{J^2} &\leqslant \|[R, \Pi]\|_{J^2} + \|[\Pi, A^{n-\ell}R]\|_{J^2} \end{aligned}$$

using that $[\Pi, RA^\ell]^* = -[\Pi, A^\ell R]$ for $\ell \geqslant 0$ since A, R are self-adjoint; then we are done. Otherwise,

$$\|[\Pi, \pi(\Gamma)]\|_{J^2} \leqslant \|[\Pi, A^{\ell-1}R]\|_{J^2} + \rho\|[\Pi, \pi(\Gamma_{\ell+1}, \ldots, \Gamma_n)]\|_{J^2},$$

and, by induction (using that $\rho < 1$), we obtain(22). □

Proposition 2.14. *Let $a \in S^0_{\mathrm{cl}}(\mathbb{R}^2, \mathbb{C})$ and suppose $\|a_0\|_{L^\infty} \leqslant c < 1$. Define*

$$\Upsilon_\hbar(a) := \Upsilon_{\Pi_\hbar}(Op_\hbar(a)) = \sum_{n=2}^{+\infty} \frac{(-1)^n}{n} \underbrace{\mathrm{tr}(\Pi Op_\hbar(a)^n\Pi - (\Pi Op_\hbar(a)\Pi)^n)}_{\Upsilon^n_\hbar(a)}.$$

The coefficients $\Upsilon^n_\hbar(a)$ are bounded by constants C_n (independent of $\hbar$) with $\sum_{n\geqslant 2} C_n < \infty$. Moreover, for any cutoff $\chi \in C^\infty_c(\mathbb{R}^2, [0,1])$ which is equal to 1 on a neighbourhood of the curve C_μ, as $\hbar \to 0$,

$$\Upsilon_\hbar(a) = \Upsilon_\hbar(a\chi) + o(1)$$

and $\Upsilon^n_\hbar(a) = \Upsilon^n_\hbar(a\chi) + O_n(\hbar^\infty)$ for $n \geqslant 2$.

Proof. Let $B = Op_\hbar(a)$, $A = Op_\hbar(\chi a)$, $R = Op_\hbar((1-\chi)a)$ and $\Pi = \Pi_\hbar$. By Gårding's inequality ([45, Theorem 4.30]),[†] there exists a fixed $\rho < 1$ such that for all $\hbar > 0$ sufficiently small,

$$\|A\|, \|B\|, \|R\| \leqslant \rho.$$

Moreover, $\|[\Pi, A]\|_{J^2}, \|[\Pi, B]\|_{J^2} \leqslant C$ for a constant C independent of $\hbar$ (by Lemma 2.12), so using the first estimate from Lemma 2.13, $|\Upsilon^n_\hbar(a)| \leqslant Cn^2\rho^n$ and these constants are summable.

Now we claim that

$$\|[\Pi, A^k R]\|_{J^2} = \mathcal{O}_k(\hbar^\infty). \tag{23}$$

Indeed, $A^k R = Op_\hbar(r_k)$ where $r_0 = (1-\chi)a$ and $r_{k,\hbar} = (\chi a)\# r_{k-1,\hbar}$ for $k \in \mathbb{N}$; see [45, Theorem 4.11]. Then, by [45, Theorem 4.12], $r_{k,\hbar} = r'_{k,\hbar} + \mathcal{O}_k(\hbar^\infty)$ where $\mathrm{supp}(r'_{k,\hbar}) = \mathrm{supp}(r_0)$ for $k \in \mathbb{N}$ and since $\mathrm{supp}(r_0) \cap C_\mu = \emptyset$, applying Lemma 2.3, this proves the estimate (23).

Then, using the second bound from Lemma 2.13, this implies that for any $n \geqslant 2$,

$$|\Upsilon^n_\hbar(a) - \Upsilon^n_\hbar(a\chi)| = \mathcal{O}_n(\hbar^\infty).$$

Hence, we also have $\Upsilon_\hbar(a) = \Upsilon_\hbar(a\chi) + o(1)$ as $\hbar \to 0$ by the dominated convergence theorem. □

Proposition 2.14 has the following important consequence; to compute the asymptotics of $\Upsilon^n_\hbar(a)$ for a general symbol $a \in S^0_{\mathrm{cl}}$, one can replace a by its principal symbol and assume that a_0 is supported in a small neighbourhood of the curve C_μ. This assumption is crucial in order to apply

[†] Under our assumptions, $1 - a \geqslant 1 - c > 0$ so that $Op_\hbar(1-a) = \mathrm{I} - B \geqslant \gamma\mathrm{I}$ as an operator if $\hbar \leqslant \hbar_\gamma$, for any $\gamma < 1 - c$.

the *quantum version of the action-angle theorem* in the next section. We can further replace the operator $Op_\hbar(\chi a)$ by a finite rank operator A via a truncation.

Proposition 2.15. *Let $a \in S^0_{\mathrm{cl}}(\mathbb{R}^2, \mathbb{C})$ with $\|a_0\|_{L^\infty} \leqslant c < 1$. Let $\Pi = \mathbb{1}_{(-\infty,\mu]}(-\hbar^2\Delta + V)$ and $\mathrm{Q} = \vartheta(-\hbar^2\Delta + V)$ where $\vartheta \in C^\infty(\mathbb{R}, [0, 1])$ is such that $\vartheta \geqslant \mathbb{1}_{(-\infty,\mu+\delta]}$ for a small $\delta > 0$. Then, for any cutoff $\chi \in C^\infty_c(\mathbb{R}^2, [0, 1])$ supported on a small neighbourhood of the curve C_μ and equal to 1 on a smaller neighbourhood of this curve, as $\hbar \to 0$,*

$$\Upsilon_\hbar(a) = \log\det(1 + Op_\hbar(a)\Pi) - \mathrm{tr}(\log(1 + Op_\hbar(a))\Pi) = \Upsilon_\Pi(A) + o(1),$$

where $A = \mathrm{Q}Op_\hbar(\chi a_0)\mathrm{Q}$.

Proof. By Proposition 2.14, on can replace the symbol a by χa_0 in such a way that $\mathrm{supp}(a_0\chi) \cap C_{\mu\pm\delta} = \emptyset$. Then, from the proof of Proposition 2.3 (see (13)), $\|[\mathrm{Q}, Op_\hbar(a)]\|_{J^1} = \mathcal{O}(\hbar^\infty)$. This implies that for $n \geqslant 2$,

$$\Upsilon^n_\Pi(A) = \mathrm{tr}(\Pi A^n\Pi - (\Pi A\Pi)^n) = \mathrm{tr}(\Pi Op_\hbar(a)^n\Pi - (\Pi Op_\hbar(a)\Pi)^n) + \mathcal{O}_n(\hbar^\infty)$$

where the leading term equals $\Upsilon^n_\hbar(a)$ and we used that $\Pi\mathrm{Q} = \mathrm{Q}\Pi = \Pi$. Hence, using again that $\|Op_\hbar(a)\| \leqslant \rho < 1$ and $\|[\Pi, Op_\hbar(a)]\|_{J^2} \leqslant C$ (Lemma 2.12), by Lemma 2.13 and the dominated convergence theorem, we conclude that $\Upsilon_\hbar(a) = \Upsilon_\Pi(A) + o(1)$ as $\hbar \to 0$ as claimed. □

2.5 | Strong Szegő asymptotics

The goal of this section is to prove Proposition 2.2. Starting from Proposition 2.15, we now obtain the asymptotics of $\Upsilon_\Pi(A)$ with $A = \mathrm{Q}Op_\hbar(\chi a_0)\mathrm{Q}$ by computing the traces $\Upsilon^n_\Pi(A)$ using the Hilbert basis $(\phi_\ell)_{\ell\in\mathbb{N}_0}$ from Proposition 2.11. Let $A_{\ell k} = \langle\phi_\ell, Op_\hbar(\chi a_0)\phi_k\rangle$ for $k, \ell \in \mathbb{N}_0$. Using that $\mathrm{Q}\phi_j = \vartheta(\lambda_j)\phi_j$ and $\vartheta(\lambda) = 1$ for $\lambda \leqslant \mu$, every term with $j_0, \dots, j_n < N$ cancels when computing the difference of both traces, so we obtain for $n \geqslant 2$,

$$\Upsilon^n_\Pi(A) = (\mathrm{tr}(\Pi A^n\Pi) - \mathrm{tr}(\Pi A\Pi)^n) = \sum_{j\in\mathbb{N}^{n+1} : j_0=j_n<N} \omega(j)\, A_{j_0j_1}A_{j_1j_2}\cdots A_{j_{n-1}j_n}, \tag{24}$$

where

$$\omega(j) = \textstyle\prod_{k=1}^{n-1}\vartheta(\lambda_{j_k})^2(1 - \prod_{k=1}^{n=1}\mathbb{1}_{j_k\leqslant N}).$$

Moreover, since $\vartheta(\lambda) = 0$ for $\lambda > \mu$ sufficiently large, this sum is finite (with a number of terms with polynomial growth in N).

Then, we can make a change of variables $\{j \in \mathbb{N}^{n+1} : j_0 = j_n\} \to \{\ell \in \mathbb{Z}, i \in \mathbb{Z}^n :, i_1 + \cdots + i_n = 0\}$ given by $\ell = N - j_0$ and $i_1 = j_1 - j_0, i_2 = j_2 - j_1, \dots, i_{n-1} = j_0 - j_{n-1}$, then the condition $\{\exists 1 \leqslant r \leqslant n-1 : j_r \geqslant N\}$ corresponds to

$$\left\{m_*(i) := \max(i_1, i_1 + i_2, \dots, i_1 + i_2 + \cdots + i_n) \geqslant \ell\right\}. \tag{25}$$

By Proposition 2.11, the coefficients $|A_{j,k}| = \mathcal{O}(|j-k|^{-\infty})$, so we can truncate the sum (24) to the subset

$$\{\max\{i_k : k \leqslant n\} \leqslant L, 1 \leqslant \ell \leqslant nL\}$$

for some large $L \in \mathbb{N}$. By Proposition 2.11, in this range, for every $k \leqslant n$,

$$\vartheta(\lambda_{j_k}) = 1, \qquad A_{j_{k-1}j_k} = \hat{a}_{i_k} + \underset{\hbar\to 0}{o(1)},$$

where $\hat{a}_i = \hat{a}_i(\mathrm{g}(\mu))$ for $i \in \mathbb{Z}$ are the Fourier coefficients of a_0 (these coefficients are evaluated at $N\hbar \sim \mathrm{g}(\mu)$ and independent of the cutoff χ; see formulas (7)–(8)). This implies that

$$\mathrm{Y}^n_\Pi(A) = \sum_{\ell\in\mathbb{N}} \sum_{i\in\mathbb{Z}^n} \mathbb{1}\{m_*(i) \geqslant \ell\}\, \hat{a}_{i_1} \dots \hat{a}_{i_n} + \underset{\hbar\to 0, L\to\infty}{o(1)},$$

where the sums are constraint to $\ell \leqslant nL$ and $\{i \in \mathbb{Z}^n : i_1 + \dots + i_n = 0, |i_k| \leqslant L\, \forall k \leqslant n\}$ and the error goes to 0 as $\hbar \to 0$, then $L \to \infty$. Taking the limits (using the decay of the Fourier coefficients), we obtain for any $n \geqslant 2$, as $\hbar \to 0$

$$\mathrm{Y}^n_\Pi(A) \sim \sum_{i\in\mathbb{Z}^n : i_1+\dots+i_n=0} m_*(i)\, \hat{a}_{i_1} \dots \hat{a}_{i_n}$$

after computing the sum over $\ell \in \mathbb{N}$.

At this point, we can apply the Dyson–Hunt–Kac formula ([29], Theorem 4.2), which states that

$$\begin{aligned} \sum_{\sigma\in\mathfrak{S}_n} m^*(i_{\sigma(1)}, \dots, i_{\sigma(n)}) &= \sum_{\sigma\in\mathfrak{S}_n} i_{\sigma(1)} \sum_{r=1}^n \mathbb{1}(i_{\sigma(1)} + \dots + i_{\sigma(r)} > 0) \\ &= \sum_{r=1}^n \frac{1}{2r} \sum_{\sigma\in\mathfrak{S}_n} |i_{\sigma(1)} + \dots + i_{\sigma(r)}|. \end{aligned}$$

By symmetry, we obtain

$$\mathrm{Y}^n_\Pi(A) \sim \sum_{r=1}^n \frac{1}{2r} \sum_{i\in\mathbb{Z}^n : i_1+\dots+i_n=0} |i_1 + \dots + i_r|\, \hat{a}_{i_1} \dots \hat{a}_{i_n} = \sum_{r=1}^n \frac{1}{2r} \sum_{\ell\in\mathbb{Z}} |\ell|\, (\widehat{a_0^r})_\ell (\widehat{a_0^{n-r}})_{-\ell}$$

after grouping together terms with $i_1 + \dots + i_r = \ell$ and using that $(\widehat{a^r})_\ell = \sum_{i\in\mathbb{Z}^r : i_1+\dots+i_r=\ell} \hat{a}_{i_1} \cdots \hat{a}_{i_r}$ for $r \in \mathbb{N}$ and $\ell \in \mathbb{Z}$.

Recall that the coefficients $\mathrm{Y}^n_\Pi(A)$ are bounded uniformly in $\hbar$ in a summable way, by Proposition 2.14. Hence, by dominated convergence, as $\hbar \to 0$,

$$\mathrm{Y}_\Pi(A) \sim \frac{1}{2} \sum_{n>r\geqslant 1} \frac{(-1)^n}{n \cdot r} \sum_{\ell\in\mathbb{Z}} |\ell|\, (\widehat{a^r})_\ell (\widehat{a^{n-r}})_{-\ell}.$$

Introducing $s = n - r$, and averaging under the symmetry $(r, s, \ell) \leftrightarrow (s, r, -\ell)$, we can rearrange this sum

$$
\begin{aligned}
\sum_{n>r\geqslant 1} \frac{(-1)^n}{n \cdot r} \sum_{\ell \in \mathbb{Z}} |\ell|\, (\widehat{a_0^r})_\ell (\widehat{a_0^{n-r}})_{-\ell} &= \sum_{r,s\geqslant 1} \frac{(-1)^{r+s}}{(r+s)\cdot r} \sum_{\ell\in\mathbb{Z}} |\ell| (\widehat{a_0^r})_\ell (\widehat{a_0^s})_{-\ell} \\
&= \frac{1}{2} \sum_{r,s\geqslant 1} (-1)^{r+s} \left(\frac{1}{(r+s)s} + \frac{1}{(r+s)r} \right) \sum_{\ell\in\mathbb{Z}} |\ell| (\widehat{a_0^r})_\ell (\widehat{a_0^s})_{-\ell} \\
&= \frac{1}{2} \sum_{r,s\geqslant 1} \frac{(-1)^{r+s}}{rs} \sum_{\ell\in\mathbb{Z}} |\ell| (\widehat{a_0^r})_\ell (\widehat{a_0^s})_{-\ell}.
\end{aligned}
$$

Recall that $\|a_0\|_{L^\infty} < 1$, so we can expand $f = -\log(1 + a_0) = \sum_{r\geqslant 1} \frac{(-1)^r}{r} a_0^r$ in a convergent power series so that $\widehat{f}_\ell = \sum_{r\geqslant 1} \frac{(-1)^r}{r} (\widehat{a_0^r})_\ell$. Hence, we conclude that as $\hbar \to 0$,

$$
\Upsilon_\Pi(A) \sim \frac{1}{4} \sum_{\ell\in\mathbb{Z}} |\ell| \widehat{f}_\ell \widehat{f}_{-\ell} = \frac{1}{2} \sum_{\ell\in\mathbb{N}} |\ell| |\widehat{f}_\ell|^2,
$$

using that f is real-valued. By Proposition 2.15, this yields the asymptotics of Proposition 2.2 and this completes the proof.

2.6 | Non-compactly supported test functions

The technical condition $f \in C_b^\infty(\mathbb{R}, \mathbb{R})$ is important for our proof of Theorem 1, but it can eventually be relaxed using the *exponential decay* of the kernel of the spectral projector Π_μ in the *forbidden region* $\{V > \mu\}$. However, the hypothesis $\|a_\hbar\|_{L^\infty} \leqslant c < 1$ in Proposition 2.2 is important to obtain the convergence of the Fredholm determinants (or the Laplace transform of the linear statistic $\mathrm{X}(f)$, (2)). By relaxing the mode of convergence, we easily deduce the following result.

Proposition 2.16. *Let $f \in C(\mathbb{R}, \mathbb{R})$ with at most exponential growth and such that $f \in C^\infty(\{V < \varkappa\})$ for some $\varkappa > \mu$. Let $\mathcal{N}$ denote the standard Gaussian. Then, one has convergence in distribution*

$$
\mathrm{X}(f) - \mathbb{E}\mathrm{X}(f) \;\to\; \Sigma_{(V,\mu)}(f)\, \mathcal{N} \qquad \text{as } \hbar \to 0
$$

and convergence of moments: for every $r \in \mathbb{N}$,

$$
\mathbb{E}\left[(\mathrm{X}(f) - \mathbb{E}\mathrm{X}(f))^r \right] \to \Sigma^r_{(V,\mu)}(f)\, \mathbb{E}[\mathcal{N}^r] \qquad \text{as } \hbar \to 0.
$$

Proof. Let $\chi \in C^\infty(\mathbb{R}, [0, 1])$ be equal to 1 on $\{V \leqslant \varkappa\}$. We claim that under the assumptions of Proposition 2.16, for any $r \in \mathbb{N}$,

$$
\mathbb{E}[\mathrm{X}(f)^r] = \mathbb{E}[\mathrm{X}(f\chi)^r] + \mathcal{O}_r(\hbar^\infty). \tag{26}
$$

Indeed, recall that $(\phi_j)_{j\in\mathbb{N}}$ is a Hilbert basis of eigenfunctions of $-\hbar^2\Delta + V$ with eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$. By [15, Proposition 2.3], there is a compact set $\{V \leqslant \mu\} \Subset \mathcal{K} \Subset \{V \leqslant \kappa\}$ and a constant $c > 1$ such that

$$\max_{\lambda_j \leqslant \mu} \|e^{\mathrm{dist}(x,\mathcal{K})/\hbar}\phi_j\|_{L^2} \leqslant c.$$

So, letting $\mathcal{K}_k := \{x \in \mathbb{R}^n : k \leqslant \mathrm{dist}(x,\mathcal{K}) < k+1\}$ for $k \geqslant 0$, we have

$$\max_{\lambda_j \leqslant \mu} \int_{\mathcal{K}_k^c} \phi_j^2 \leqslant c e^{-k/\hbar}.$$

Moreover, since f grows at most exponentially, there is a constant $C \geqslant 0$ so that $|f| \leqslant e^{Ck/2}$ on $\mathcal{K}_k$. In particular, if $f = 0$ on $\mathcal{K}_0$, then

$$\mathbb{E}[\mathrm{X}(f)^{2r}] \leqslant \sum_{k\geqslant 1} e^{Crk}\mathbb{E}[\mathrm{X}(\mathbb{1}_{\mathcal{K}_k})], \qquad \mathbb{E}[\mathrm{X}(\mathbb{1}_{\mathcal{K}_k})] \leqslant \mathbb{E}[\mathrm{X}(\mathbb{1}_{\mathcal{K}_{k-1}^c})] = \int_{\mathcal{K}_{k-1}^c} \Pi_\hbar(x,x)\mathrm{d}x \leqslant N \max_{\lambda_r \leqslant \mu} \int_{\mathcal{K}_{j-1}^c} \phi_r^2,$$

where $\mathcal{K}_k^c = \mathbb{R}^n \setminus \mathcal{K}_k$ for $k \geqslant 0$. Since $N \asymp \hbar^{-n}$, this implies that

$$\mathbb{E}[\mathrm{X}(f)^{2r}] \leqslant \mathcal{O}(\hbar^{-n}) \sum_{k\geqslant 1} e^{Crk-k/\hbar} = \mathcal{O}_r(\hbar^\infty)$$

since this geometric sum is exponentially small as $\hbar \to 0$. This proves (26) since we can choose a cutoff $\chi = 1$ on $\mathcal{K}_0$.

Hence, using (26), it follows immediately from Theorem 1 that in distribution,

$$\mathrm{X}(f) - \mathbb{E}\mathrm{X}(f) \;\to\; \Sigma_{(V,\mu)}(\chi f)\,\mathcal{N}$$

and all moments converge. Since, $\chi = 1$ on $\{V \leqslant \mu\}$, the Fourier coefficients of the functions χf and f coincide in formula (10) so that $\Sigma_{(V,\mu)}(\chi f) = \Sigma_{(V,\mu)}(f)$ as expected. This concludes the proof. □

3 | MULTI-CUT CASE – PROOF OF THEOREM 2

The action-angle theorem (Proposition 2.4) has a microlocal nature, and allows to find quasi-modes for any Schrödinger operator using only a local hypothesis. Thus, given $V : \mathbb{R} \to \mathbb{R}$ confining such that the support of the density of states $\{V \leqslant \mu\}$ consists of several disjoint (compact) intervals, each satisfying the geometric requirements of Section 2, the eigenvalues (up to μ) of $-\hbar^2\Delta + V$ are given, up to a small error, by the union of the eigenvalues of local models $-\hbar^2\Delta + V_j$, where the potential V_j has a single well; the associated quasi-modes will be localised on a single well. Under a generic condition, the eigenvalues of different wells are sufficiently separated from each other, so that the quasi-modes are very close to actual eigenfunctions.

Proposition 3.1 [23, Lemma 2.3 and Theorem 2.4]. *Let $V \in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ and let $\mu > 0$. Suppose that, for some $\epsilon > 0$, the set $\{V \leqslant \mu + \epsilon\}$ (consisting of disjoint intervals $I'_1, \dots, I'_\ell$) is compact, on which V* *is C^∞. Suppose also that $\{V \leqslant \mu\}$ is a finite union of non-empty intervals $I_1, \dots, I_\ell$ with $I_j \subset I'_j$ for $j = 1, \dots, \ell$.*

For $j = 1, \dots, \ell$, let $V_j \in C^\infty(\mathbb{R}, \mathbb{R})$ be equal to V on I'_j, and greater than $\mu + \epsilon$ on $\mathbb{R} \setminus I_{j'}$. For all $\hbar > 0$ and $1 \leqslant j \leqslant \ell$, consider the eigenpairs $(\phi_{j,k}, \lambda_{j,k})$ of $-\hbar^2\Delta + V_j$ such that $\lambda_{j,k} \leqslant \mu + \epsilon/2$.

1. *There exists $c > 0, C > 0$ such that*

$$\|(-\hbar^2\Delta - V - \lambda_{j,k})\phi_{j,k}\|_{L^2} \leqslant Ce^{-c\hbar^{-1}}.$$

2. *Letting $E = \operatorname{span}(\phi_{j,k} : \lambda_{j,k} \leqslant \mu + \epsilon/2, j = 1, \dots, \ell)$, for all $u \in E^\perp$,*

$$\langle u, (-\hbar^2\Delta - V), u\rangle \geqslant (\mu + \tfrac{\epsilon}{4})\|u\|^2_{L^2}.$$

Thus, the eigenvalue/eigenfunction asymptotics in the case where V has several wells can be reduced to the one-well case if the eigenvalues $\{\lambda_{j,k} \leqslant \mu + \epsilon/2\}$ are well separated. Indeed, under such assumptions, the eigenfunctions $\{\phi_{j,k}\}$ are quasi-orthogonal, and ultimately, one controls the asymptotics of the spectral projector $\Pi_\hbar(\mu)$. Using usual results of spectral stability for self-adjoint operators ([22, Theorem 8.20], see also footnote), we obtain the following result.

Proposition 3.2. *Under the assumptions of Proposition 3.1, let*

$$\Pi_{\hbar;j} = \mathbb{1}_{(-\infty,\mu]}(-\hbar^2\Delta + V_j), \qquad 1 \leqslant j \leqslant \ell.$$

Suppose that the eigenvalues are not exponentially close to each other or to μ: there is a small $\epsilon > 0$ so that for all $\hbar$ sufficiently small,

1. *for every $1 \leqslant j \neq k \leqslant \ell$, $[\mu - \epsilon, \mu + \epsilon] \cap \operatorname{Sp}(-\hbar^2\Delta + V_j)$ lies at distance greater than $e^{-\epsilon\hbar^{-1}}$ from $[\mu - \epsilon, \mu + \epsilon] \cap \operatorname{Sp}(-\hbar^2\Delta + V_k)$,*
2. *for all $1 \leqslant k \leqslant \ell$, $[\mu - e^{-\epsilon\hbar^{-1}}, \mu + e^{-\epsilon\hbar^{-1}}] \cap \operatorname{Sp}(-\hbar^2\Delta + V_k) = \emptyset$.*

Then, the projector

$$\Pi_\hbar = \sum_{j=1}^{\ell} \Pi_{\hbar;j} + \mathcal{O}_{J^1}(\hbar^\infty), \tag{27}$$

and, for any symbol $a \in S^0(\mathbb{R}^2, \mathbb{R})$ with $\|a\|_{L^\infty} < 1$,

$$\log\det(I + Op_\hbar(a)\Pi_\hbar) = \sum_{j=1}^{\ell} \log\det(I + Op_\hbar(a)\Pi_{\hbar;j}) + \mathcal{O}(\hbar^\infty).$$

Proof. Firstly, we can isolate $[\mu - \frac{\epsilon}{2}, \mu + \frac{\epsilon}{2}]$ from the rest of the spectrum. Let $\chi \in C^\infty(\mathbb{R}, [0,1])$ be equal to 1 on a neighbourhood of $(-\infty, \mu - \frac{\epsilon}{4}]$ and equal to 0 on a neighbourhood of $[\mu, +\infty)$. Decompose

$$\Pi_\hbar = \chi(H_\hbar) + (1 - \chi)\mathbb{1}_{(-\infty,\mu]}(H_\hbar)$$

and decompose similarly $\Pi_{\hbar;j}$ for $1 \leqslant j \leqslant \ell$. By the smooth functional calculus of pseudo-differential operators ([16], Theorem 8.7),

$$\chi(H_\hbar) = \sum_{1\leqslant j\leqslant \ell} \chi(H_{\hbar;j}) + \mathcal{O}_{J^1}(\hbar^\infty)$$

since both sides are pseudo-differential operators with the same symbol at any order.

It remains to study the spectrum in $[\mu - \frac{\epsilon}{2}, \mu + \frac{\epsilon}{2}]$, where we use the separation of eigenvalues. For every $j \in [\ell]$, let $\chi_j \in C_c^\infty(\mathbb{R}, [0,1])$ with $\mathbb{1}_{I_j} \leqslant \chi_j \leqslant \mathbb{1}_{I'_j}$ (see Proposition 3.1). By Agmon estimates ([15, Proposition 2.3]), there is a $c > 0$ so that if $u_\hbar$ is a (normalised) eigenfunction of $H_\hbar$ with energy $\lambda \in [\mu - \frac{\epsilon}{2}, \mu + \frac{\epsilon}{2}]$, for every $j \in [\ell]$, $\chi_j u_\hbar$ is an $O(e^{-c\hbar^{-1}})$-quasi-mode for both $H_\hbar$ and $H_{\hbar;j}$ at energy λ and $\sum_{j\leqslant \ell} \|\chi_j u_\hbar\|_{L^2} = 1 - O(e^{-c\hbar^{-1}})$. Moreover, by Remark 2.6, for every $j \in [\ell]$, the eigenvalues of $H_{\hbar;j}$ are simple and separated by about $\hbar$. Hence, by spectral stability ([22, Theorem 8.20], see also footnote), if $\|\chi_j u_\hbar\|_{L^2} \geqslant \frac{1}{2\ell}$, there is an eigenvalue $\lambda_{j,k}$ of $H_{\hbar;j}$ such that $|\lambda - \lambda_{j,k}| = \mathcal{O}(e^{-c\hbar^{-1}/2})$. Using the separation hypothesis 1 with $\epsilon < c/2$, there is a unique such $j \in [\ell]$ and $\chi_j u_\hbar = \phi_{j,k} + O(e^{-c\hbar^{-1}/2})$. This has the following consequences.

(a) The spectrum $H_\hbar$ in $[\mu - \frac{\epsilon}{2}, \mu + \frac{\epsilon}{2}]$ is simple and its eigenvalues are separated by about $e^{-\epsilon\hbar^{-1}}$.
(b) For every eigenpair $(u_\hbar, \lambda)$ of $H_\hbar$, there is a unique (j,k) such that $|\lambda - \lambda_{j,k}| = \mathcal{O}(e^{-c\hbar^{-1}/2})$ and $u_\hbar = \phi_{j,k} + O(e^{-c\hbar^{-1}/2})$.

On the other hand, by Proposition 3.1, $\phi_{j,k}$ are also $O(e^{-c\hbar^{-1}})$-quasi-modes of $H_\hbar$ with energy $\lambda_{j,k}$ so that using that the eigenvalues of $H_\hbar$ are separated (a), there is a 1-1 correspondence between $\mathrm{spec}(H_\hbar) \cap [\mu - \frac{\epsilon}{2}, \mu + \frac{\epsilon}{2}]$ and the relevant part of $\bigcup_{j=1}^{\ell} \mathrm{spec}(H_{\hbar;j})$ as in (b).

Now, (a) and the separation hypothesis 1 with $\epsilon < c/2$ also guarantee that $\mathrm{spec}(H_\hbar) \cap [\mu - \frac{\epsilon}{2}, \mu] = \bigcup_{j=1}^{\ell} \mathrm{spec}(H_{\hbar;j}) \cap [\mu - \frac{\epsilon}{2}, \mu]$, and using (b), we obtain the spectral decomposition;

$$(1-\chi)\mathbb{1}_{(-\infty,\mu]}(H_\hbar) = \sum_{j=1}^{\ell} (1-\chi)\mathbb{1}_{(-\infty,\mu]}(H_{\hbar;j}) + \mathcal{O}_{J^1}(Ne^{-c/2\hbar}),$$

which concludes this part of the proof.

To prove closeness of the log-determinants, we write $A = Op_\hbar(a) \sum_{j=1}^{\ell} \Pi_{\hbar,j}$, $B = Op_\hbar(a)\Pi_\hbar = A + R$, so that

$$\det(I + B) = \det(I + A) \cdot \det(I + R(1 + A)^{-1})$$

and $\|R(1+A)^{-1}\|_{J^1} = \mathcal{O}(\hbar^\infty)$ since $\|R\|_{J^1} = \mathcal{O}(\hbar^\infty)$ and $(1 + A)^{-1}$ is a bounded (by assumption $\|a\|_{L^\infty} < 1$ and by Gårding's inequality $\|Op_\hbar(a)\| < 1$). By continuity of log det with respect to the J^1 norm, we conclude that

$$\log\det(I + B) = \log\det(I + A) + \mathcal{O}(\hbar^\infty).$$

Now, as the projections satisfy $[\Pi_{\hbar;j}, \Pi_{\hbar;j'}] = \mathcal{O}(\hbar^\infty)$ for all $1 \leqslant j, j' \leqslant \ell$, using the Baker–Campbell–Haussdorf formula [17],

$$\log\det(I + A) = \sum_{j=1}^{\ell} \log\det(I + Op_\hbar(a)\Pi_{\hbar;j}) + \mathcal{O}(\hbar^\infty)$$

as claimed. □

We are now ready to complete the proof of Theorem 2 by a Borel–Cantelli argument.

As in the hypothesis of Theorem 2, let $(\chi_k)_{1\leqslant k\leqslant \ell}$ be a family of $C_c^\infty(\mathbb{R},\mathbb{R}_+)$ functions with disjoint supports such that $\chi_k = 1$ on I_k and consider the Schrödinger operator

$$-\hbar_N^2\Delta + W^w, \qquad W^w := V + \textstyle\sum_{j=1}^{\ell} w_j\chi_j \tag{28}$$

for $w \in [-\epsilon,\epsilon]^\ell$ where $\epsilon > 0$ is a small parameter. In particular, for almost every $w \in [-\epsilon,\epsilon]^\ell$, the set $\{W^w \leqslant \mu\}$ still consists of ℓ disjoint (compact) intervals, denoted as $I_1^w, \dots, I_\ell^w$ and $\nabla W^w = \nabla V \neq 0$ on $\{W^w = \mu\}$.

Replacing V by W^w amounts to replacing V_j with $W_j^w = V_j + w_j$ (so that the eigenvalues $\lambda_{k;j}$ change into $\lambda_{k;j} + w_j$ for $1 \leqslant j \leqslant \ell$) in Propositions 3.1 and 3.2. Then, observe that for $\hbar > 0$, the measure of the set

$$\Big\{ w \in [-\epsilon,\epsilon]^\ell \ : \ \exists (k,j),(k',j') \text{ with } j \neq j' \text{ and } \left|\lambda_{k;j} - \lambda_{k';j'} + w_j - w_j'\right| < e^{-c/2\hbar} \text{ or } \left|\lambda_{k;j} + w_j - \mu\right| < e^{-c/2\hbar} \Big\}$$

is $\mathcal{O}(\hbar^{-2}e^{-c/2\hbar})$ where the factor $\mathcal{O}(\hbar^{-2})$ accounts for all possible choices of k, k'. Hence, for any sequence $\hbar_m \to 0$ as $m \to \infty$ sufficiently fast so that $\sum_m \hbar_m^{-2} e^{-c/2\hbar_m} < \infty$ (this is plainly the case if for some $\alpha > 0$, $\hbar_m \leqslant C m^{-\alpha}$), by the Borel–Cantelli lemma, there is a full measure set $E \subset [-\epsilon,\epsilon]^\ell$ such that for $w \in E$, if m is sufficiently large, the operator (28) satisfies both assumptions of Proposition 3.2. Hence, applying this proposition with $\Pi_m = \mathbb{1}_{(-\infty,\mu]}(-\hbar_m^2\Delta + W^w)$ and $\Pi_{m,j} = \mathbb{1}_{(-\infty,\mu]}(-\hbar_m^2\Delta + W_j^w)$ for $1 \leqslant j \leqslant \ell$, we conclude that for any symbol $a \in S^0(\mathbb{R}^2,\mathbb{R})$ with $\|a\|_{L^\infty} < 1$, as $m \to \infty$,

$$\begin{aligned}\log\det(I + Op_\hbar(a)\Pi_m) &= \sum_{j=1}^{\ell} \log\det(I + Op_\hbar(a)\Pi_{m;j}) + o(1) \\ &= \sum_{j=1}^{\ell}\left(\operatorname{tr}(\log(1 + Op_\hbar(a))\Pi_{m;j}) + \frac{1}{2}\sum_{\ell\in\mathbb{Z}} |\ell| \widehat{f}_\ell(I_j)\widehat{f}_{-\ell}(I_j)\right) + o(1),\end{aligned}$$

where we applied the (one-cut) asymptotics of Proposition 2.2 at the second step, $f = -\log(1+a)$ and $I_j = \mathrm{g}^{-1}(\mu - w_j)$ according to the notation of Section 2.1. For a given $f \in C_c^\infty(\mathbb{R},\mathbb{R})$, choosing the symbol $a = e^{\eta f} - 1$ which is in $S^0(\mathbb{R}^2,\mathbb{C})$ with $\|a\|_{L^\infty} < 1$ if the parameter $\eta \in \mathbb{C}$ is sufficiently small, this completes the proof of Theorem 2. Here, we have $\operatorname{tr}(f\Pi_m) = \sum_{j=1}^{\ell}\operatorname{tr}(f\Pi_{m;j}) + o(1)$ (by Proposition 3.2) and the variance equals $\sum_{j=1}^{\ell}\Sigma^2_{(W_j^w,\mu)}(f)$. This concludes the proof.

APPENDIX A: GAUSSIAN FREE FIELD (GFF) INTERPRETATION

We provide a functional interpretation of central limit Theorem 1 in terms of the Gaussian free field. This interpretation of Szegő-type limit theorems is classical in random matrix theory, pioneered in [27] for the CUE (Example 1.2 — we refer, e.g. [7] for recent developments), and this proceeds by considering the counting function

$$h_N : x \mapsto \mathrm{X}(\mathbb{1}_{(-\infty,x]}).$$

In the physics literature, the asymptotics of the correlation kernel of h_N were recently obtained using WKB asymptotics of the Schrödinger operator eigenfunctions and the result is interpreted in terms of the GFF, though Gaussian fluctuations were not established in this paper.

The connection with Theorem 1 comes from the fact that for a Schwartz function $f \in \mathcal{S}$,

$$\mathrm{X}(f) = -\int f'(x) h_N(x)\mathrm{d}x$$

so that, viewed as a random Schwartz distribution, $\widetilde{h_N} := \sqrt{2}\pi\big(h_N - \mathbb{E}h_N\big)$ converges weakly as $N \to \infty$ (equivalently $\hbar \to 0$) to a (centred) Gaussian random field $h \in \mathcal{S}'$ with covariance kernel;

$$\widetilde{\mathcal{H}} : (x,z) \mapsto \left(\log\left|\sin\tfrac{\theta(x)+\vartheta(z)}{2}\right| - \log\left|\sin\tfrac{\theta(x)-\vartheta(z)}{2}\right|\right), \qquad x,z \in [x_0^-, x_0^+], \tag{A.1}$$

where recall that $x_0^\pm(\mu)$ are such that $\{V < \mu\} = (x_0^-, x_0^+)$, $T = T(\mu)$ is the period of the flow (5) with $(x_0, \xi_0) = (x_0^+, 0)$ and the map

$$\theta(x) = \frac{\pi}{T}\int_x^{x_0^+} \frac{\mathrm{d}u}{\sqrt{\mu - V(u)}}, \qquad x \in [x_0^-, x_0^+]. \tag{A.2}$$

Formula (A.1) is (up to normalisation) [38, Formula (16)] and it has the following interpretation.

Let ξ be the GFF on $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, that is, the restriction of a 2d GFF on the unit circle. Suitably normalised, ξ is a log-correlated field with covariance kernel

$$\mathbb{E}[\xi(z)\xi(w)] = \log|1 - z\overline{w}|^{-1}, \qquad z, w \in \mathbb{U}.$$

Then, (A.1) is the kernel of the pushforward of the GFF by the map (A.2) in the following sense;

$$\widetilde{\mathcal{H}}(x,z) = \mathbb{E}h(x)h(z), \qquad h(x) = \frac{\xi(e^{\mathbf{i}\theta(x)}) - \xi(e^{-\mathbf{i}\theta(x)})}{\sqrt{2}}.$$

Note that the normalisation of $\widetilde{h_N}$ is similar to [7] (GUE case) and the limit h is a standard log-correlated field.

In the remainder of this section, we explain how to obtain (A.1) from formula (4) and the consideration from Section 2.1. Let $\Sigma = \Sigma^2_{(V,\mu)}$, by Devinatz formula [36, Proposition 6.1.10],

$$\begin{aligned}\Sigma(f) &= \frac{1}{2}\iint_{[-\pi,\pi]^2}\left|\frac{f(\psi(\theta)) - f(\psi(\vartheta))}{e^{\mathbf{i}\theta} - e^{\mathbf{i}\vartheta}}\right|^2 \frac{d\theta}{2\pi}\frac{d\vartheta}{2\pi} \\ &= \iint_{[0,\pi]^2} (f(\psi(\theta)) - f(\psi(\vartheta)))^2\left(\frac{1}{|1 - e^{\mathbf{i}(\theta-\vartheta)}|^2} + \frac{1}{|1 - e^{\mathbf{i}(\theta+\vartheta)}|^2}\right)\frac{d\theta}{2\pi}\frac{d\vartheta}{2\pi}.\end{aligned} \tag{A.3}$$

Here, we used that the curve $\mathcal{C}_\mu$ is symmetric with respect to the axis $\{\xi = 0\}$ and the map $\psi(\theta) = x_{\frac{\theta T}{2\pi}}$ in terms of the flow (5) so that $\psi(2\pi - \theta) = \psi(\theta)$ for $\theta \in [0, \pi]$. We can invert the map ψ :

$(0,\pi) \to (x_0^-, x_0^+)$ using (5) to compute its derivative;

$$\psi'(\theta) = \frac{T}{\pi}\xi_{\frac{\theta T}{2\pi}} = \frac{T}{\pi}\sqrt{\mu - V(\psi(\theta))}, \qquad \theta \in (0,\pi).$$

In particular, the period of the flow is $T(\mu) = \int_{x_0^-}^{x_0^+} \frac{dx}{\sqrt{\mu - V(x)}}$ and the inverse map is given by (A.2) using that $\theta'(x) = 1/\psi'(\theta(x))$ for $x \in (x_0^-, x_0^+)$. The kernel on the RHS of (A.3) is

$$\begin{aligned}\mathcal{K} : (\theta,\vartheta) \mapsto \frac{1}{|1-e^{\mathbf{i}(\theta-\vartheta)}|^2} + \frac{1}{|1-e^{\mathbf{i}(\theta+\vartheta)}|^2} &= \frac{1}{2}\frac{2-\cos(\theta-\vartheta)-\cos(\theta+\vartheta)}{(1-\cos(\theta-\vartheta))(1-\cos(\theta+\vartheta))}\\ &= \frac{1-\cos\theta\cos\vartheta}{(\cos\theta-\cos\vartheta)^2}.\end{aligned}$$

Hence, by a change of variables, we obtain an equivalent formula for the variance

$$\Sigma(f) = \frac{1}{(2\pi T)^2}\iint_{\{V<\mu\}^2} (f(x)-f(z))^2 \frac{1-\cos\theta(x)\cos\theta(z)}{(\cos\theta(x)-\cos\theta(z))^2}\frac{\mathrm{d}x}{\sqrt{\mu-V(x)}}\frac{\mathrm{d}z}{\sqrt{\mu-V(z)}}.$$

To obtain (A.1), we observe that

$$\mathcal{K} : (\theta,\vartheta) \mapsto -\partial_\theta\partial_\vartheta \mathcal{H}(\theta,\vartheta), \qquad \mathcal{H} : (\theta,\vartheta) \mapsto \left(\log\left|\sin\tfrac{\theta+\vartheta}{2}\right| - \log\left|\sin\tfrac{\theta-\vartheta}{2}\right|\right)$$

and that $\left[\partial_\vartheta \mathcal{H}(\theta,\vartheta)\right]_{\theta=0}^{\pi} = 0$, $\left[\mathcal{H}(\vartheta,\theta)\right]_{\theta=0}^{\pi} = 0$. Hence, integrating by parts twice,

$$\begin{aligned}\Sigma(f) &= -\iint_{[0,\pi]^2} \partial_\theta\partial_\vartheta (f(\psi(\theta)) - f(\psi(\vartheta)))^2 \mathcal{H}(\theta,\vartheta)\frac{d\theta}{2\pi}\frac{d\vartheta}{2\pi}\\ &= 2\iint_{[0,\pi]^2} f'(\psi(\theta))f'(\psi(\vartheta))\mathcal{H}(\theta,\vartheta)\frac{d\psi(\theta)}{2\pi}\frac{d\psi(\vartheta)}{2\pi}\\ &= \frac{1}{2\pi^2}\iint_{\{V<\mu\}^2} f'(x)f'(z)\widetilde{\mathcal{H}}(x,z)dxdz\end{aligned}$$

with the kernel $\widetilde{\mathcal{H}}$ as in (A.1). Hence, according to Theorem 1, we conclude that for a $f \in \mathcal{S}$, it holds in distribution

$$\sqrt{2}\pi\int f'(x)h_N(x)\mathrm{d}x \to \int f'(x)h(x)\mathrm{d}x.$$

ACKNOWLEDGEMENTS

We thank the authors of [38] for pointing out their work on the counting function of 1d Fermi gas in the one-cut case and the connection with our Theorem 1.

A.D. acknowledges the support of the PEPS 2021 CNRS grant.

G.L. acknowledges the support of the Ambizione grant S-71114-05-01 from the Swiss National Science Foundation (University of Zurich) and of the VR starting grant 2022-04882 from the Swedish Research Council.

JOURNAL INFORMATION

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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REFERENCES

1. G. Akemann, M. Duits, and L. Molag, *The elliptic Ginibre ensemble: a unifying approach to local and global statistics for higher dimensions*, J. Math. Phys. **64** (2023), no. 2.
2. F. Bekerman, T. Leblé, and S. Serfaty, *CLT for fluctuations of β-ensembles with general potential*, Electron. J. Probab. **23** (2018), no. 115, 1–31.
3. G. Borot and A. Guionnet, *Asymptotic expansion of β matrix models in the one-cut regime*, Comm. Math. Phys. **317** (2013), no. 2, 447–483.
4. P. Bourgade, K. Mody, and M. Pain, *Optimal local law and central limit theorem for β-ensembles*, Comm. Math. Phys. **390** (2022), no. 3, 1017–1079.
5. J. Breuer and M. Duits, *Central limit theorems for biorthogonal ensembles and asymptotics of recurrence coefficients*, J. Amer. Math. Soc. **30** (2017), no. 1, 27–66.
6. C. Charlier, B. Fahs, C. Webb, and M. D. Wong, *Asymptotics of Hankel determinants with a multi-cut regular potential and Fisher-Hartwig singularities*, Mem. Amer. Math. Soc. (2023). https://doi.org/10.48550/arXiv.2111.08395
7. T. Claeys, B. Fahs, G. Lambert, and C. Webb, *How much can the eigenvalues of a random Hermitian matrix fluctuate?* Duke Math. J. **170** (2021), no. 9, 2085–2235.
8. Y. Colin de Verdière, *Spectre conjoint d'opérateurs pseudo-différentiels qui commutent II. Le cas intégrable*, Math. Z. **171** (1980), no. 1, 51–73.
9. Y. Colin de Verdière and B. Parisse, *Équilibre instable en régime semi-classique. II. Conditions de Bohr-Sommerfeld*, Ann. Inst. H. Poincaré Physique Théorique **61** (1994), 347–367.
10. D. Dean, P. Le Doussal, S. Majumdar, and G. Schehr, *Impurities in systems of noninteracting trapped fermions*, SciPost Phys. **10** (2021), no. 4, 082.
11. D. S. Dean, P. Le Doussal, S. N. Majumdar, and G. Schehr, *Universal ground-state properties of free fermions in a d-dimensional trap*, Europhys. Lett. **112** (2015), no. 6, 60001.
12. D. S. Dean, P. Le Doussal, S. N. Majumdar, and G. Schehr, *Nonequilibrium dynamics of noninteracting fermions in a trap*, Europhys. Lett. **126** (2019), no. 2, 20006.
13. D. S. Dean, P. Le Doussal, S. N. Majumdar, and G. Schehr, *Noninteracting fermions in a trap and random matrix theory*, J. Phys. A **52** (2019), no. 14, 144006.
14. P. Deift, A. Its, and I. Krasovsky, *Toeplitz matrices and toeplitz determinants under the impetus of the ising model: some history and some recent results*, Comm. Pure Appl. Math. **66** (2013), no. 9, 1360–1438.
15. A. Deleporte and G. Lambert, *Universality for free fermions and the local Weyl law for semiclassical Schrödinger operators*, J. Euro. Math. Soc. (2024). https://doi.org/10.4171/JEMS/1447
16. M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, Number 268, Cambridge University Press, Cambridge, 1999.
17. E. B. Dynkin, *On the representation by means of commutators of the series* $\log(e^x e^y)$ *for noncommutative x and y*, Mat. Sb.(NS) **25** (1949), no. 67, 1.
18. V. Eisler, *Universality in the full counting statistics of trapped fermions*, Phys. Rev. Lett. **111** (2013), no. 8, 080402.
19. G. Folland, *Harmonic analysis in phase space*, Number 122 in Annals of Math. Studies, Princeton University Press, Princeton, NJ, 1989.
20. S. Fournais and S. Mikkelsen, *An optimal semiclassical bound on commutators of spectral projections with position and momentum operators*, Lett. Math. Phys. **110** (2020), no. 12, 3343–3373.

21. V. Guillemin and K. Okikiolu, *Spectral asymptotics of Toeplitz operators on Zoll manifolds*, J. Funct. Anal. **146** (1997), no. 2, 496–516.
22. B. Helffer, *Spectral theory and its applications*, vol. 139, Cambridge University Press, New York, 2013.
23. B. Helffer and J. Sjöstrand, *Multiple wells in the semi-classical limit I*, Comm. Partial Differential Equations **9** (1984), no. 4, 337–408.
24. B. Helffer and J. Sjöstrand, *Multiple wells in the semi-classical limit III: interaction through non-resonant wells*, Math. Nachr. **124** (1985), no. 1, 263–313.
25. B. Helffer and J. Sjöstrand, *Puits multiples en limite semi-classique. II: Interaction moléculaire. Symétries, Perturbation*, Ann. Inst. H. Poincaré Physique Théorique **42** (1985), 127–212.
26. J. B. Hough, M. Krishnapur, Y. Peres, and B. Virág, *Determinantal processes and independence*, Probab. Surv. **3** (2006), 206–229.
27. C. P. Hughes, J. P. Keating, and N. O'Connell, *On the characteristic polynomial of a random unitary matrix*, Comm. Math. Phys. **220** (2001), no. 2, 429–451.
28. K. Johansson, *On fluctuations of eigenvalues of random Hermitian matrices*, Duke Math. J. **91** (1998), no. 1, 151–204.
29. M. Kac, *Toeplitz matrices, translation kernels and a related problem in probability theory*, Duke Math. J. **21** (1954), no. 3, 501–509.
30. G. Lambert, *CLT for biorthogonal ensembles and related combinatorial identities*, Adv. Math. **329** (2018), 590–648.
31. G. Lambert, M. Ledoux, and C. Webb, *Quantitative normal approximation of linear statistics of β-ensembles*, Ann. Probab. **47** (2019), no. 5, 2619–2685.
32. G. Lambert and E. Paquette, *Strong approximation of Gaussian β ensemble characteristic polynomials: the hyperbolic regime*, Ann. Appl. Probab. **33** (2023), no. 1, 549–612.
33. O. Macchi, *The coincidence approach to stochastic point processes*, Adv. Appl. Probab. **7** (1975), 83–122.
34. L. Pastur, *Limiting laws of linear eigenvalue statistics for Hermitian matrix models*, J. Math. Phys. **47** (2006), no. 10, 103303, 22.
35. M. Shcherbina, *Fluctuations of linear eigenvalue statistics of β matrix models in the multi-cut regime*, J. Stat. Phys. **151** (2013), no. 6, 1004–1034.
36. B. Simon, *Orthogonal polynomials on the unit circle*, American Mathematical Society, Providence, RI, 2005.
37. B. Simon, *Trace ideals and their applications*, Number 120, American Mathematical Society, Providence, RI, 2005.
38. N. R. Smith, P. Le Doussal, S. N. Majumdar, and G. Schehr, *Counting statistics for noninteracting fermions in a d-dimensional potential*, Phys. Rev. E **103** (2021), no. 3, L030105.
39. A. Soshnikov, *The central limit theorem for local linear statistics in classical compact groups and related combinatorial identities*, Ann. Probab. **28** (2000), no. 3, 1353–1370.
40. A. Soshnikov, *Determinantal random point fields*, Rossiiskaya Akademiya Nauk. Moskovskoe Matematicheskoe Obshchestvo. Uspekhi Matematicheskikh Nauk **55** (2000), no. 5(335), 107–160.
41. T. Tao, *Topics in random matrix theory*, Graduate Studies in Mathematics, vol. 132, American Mathematical Society, Providence, RI, 2012.
42. S. Vũ Ngọc, *Bohr-Sommerfeld conditions for integrable systems with critical manifolds of focus-focus type*, Comm. Pure Appl. Math. **53** (2000), no. 2, 143–217.
43. S. Vũ Ngọc, *Formes normales semi-classiques des systèmes complètement intégrables au voisinage d'un point critique de l'application moment*, Asymptot. Anal. **24** (2000), no. 3, 4, 319–342.
44. S. Vũ Ngọc, *Systèmes Intégrables Semi-Classiques: Du Local Au Global*, Panoramas et Synthèses, vol. 22, Société mathématique de France, Providence, Rhode Island, 2006.
45. M. Zworski, *Semiclassical analysis*, vol. 138, American Mathematical Society, Providence, RI, 2012.