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$\bar\partial$-problem for the focusing nonlinear Schrödinger equation and soliton shielding

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We consider soliton gas solutions of the focusing nonlinear Schrödinger (NLS) equation, where the point spectrum of the Zakharov–Shabat linear operator condenses in a bounded domain $\mathcal{D}$ in the upper half-plane. We show that the corresponding inverse scattering problem can be formulated as a $\bar\partial$-problem on the complex plane. We prove that the τ-function of the N soliton solution converges in the limit $N\to\infty$ to the τ-function (a Fredholm determinant) of the $\bar\partial$-problem. Furthermore, we prove that such a τ-function is non-vanishing for all values of x and t, thus showing the existence of a solution of the $\bar\partial$-problem. Then we show that, when the domain $\mathcal{D}$ is an ellipse and the soliton gas spectral data are analytic, the inverse problem reduces to the soliton spectra concentrating on the segment connecting the foci of the ellipse (soliton shielding). The NLS solution for fixed times is asymptotically step-like oscillatory, and it is described by a periodic elliptic function as $x\to-\infty$ while it vanishes exponentially fast as $x\to+\infty$.

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1. Introduction

In this paper, we consider the focusing nonlinear Schrödinger equation (NLS)

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0, \tag{1.1}$$

with non-standard initial data that originate from an infinite soliton limit. The NLS equation is an integrable equation and the Cauchy problem can be solved via the inverse scattering transform of the Zakharov–Shabat linear operator [1] when the initial data satisfy zero boundary conditions at infinity, it is periodic [2], or the initial data have asymmetric boundary conditions [3,4], or also when one considers an initial boundary value problem instead of a Cauchy problem (e.g. [5,6]). In all these cases, the inverse problem can be formulated as a Riemann–Hilbert (RH) problem for a 2×2 matrix function defined on the complex plane with discontinuities along paths and with a certain number of poles. These paths and poles correspond to the support of the spectral scattering data. Here we consider instead the case in which the inverse problem is cast as a $\overline{\partial}$-problem because the spectral scattering data have a two-dimensional support in the complex spectral plane. Such spectral data originate by letting the number of solitons go to infinity, and the soliton spectra converge to the uniform measure of a bounded domain of the complex plane. The $\overline{\partial}$-method in inverse scattering was developed by Fokas–Ablowitz [7] and Beals–Coifman [8,9] to study the Cauchy problem of integrable, dispersive nonlinear equations in two or more space dimensions. Its application to inverse scattering has been studied by many authors (see for example the monograph [10–12]).

To formulate the $\overline{\partial}$-problem for the NLS equation, let $\mathcal{D} \subseteq \mathbb{C}_+$ denote a compact domain with smooth boundary and contained in the upper half-plane $\mathbb{C}_+$, and let $\beta : \mathcal{D} \to \mathbb{C}$ be a smooth, non-zero and bounded function continuous up to the boundary of $\mathcal{D}$, that describes the spectral data of the problem. The $\overline{\partial}$-problem is to find a 2×2 matrix function $\Gamma(z; x, t)$ depending on the complex variable $z \in \mathbb{C}$, the complex conjugate $\bar{z}$ and $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ that satisfies the conditions

$$\left.\begin{aligned} &\overline{\partial}\Gamma(z;x,t) = \Gamma(z;x,t)J_\Gamma(z;x,t) \\ \text{and}\qquad &\Gamma(z;x,t) = \mathbb{I} + \mathcal{O}(z^{-1}), \quad \text{as } z \to \infty, \end{aligned}\right\} \tag{1.2}$$

where $\overline{\partial} := \partial/\partial\bar{z}$. The matrix J_Γ takes the form

$$J_\Gamma(z;x,t) := \begin{cases} \begin{pmatrix} 0 & -\beta^*(z)^2\,\mathrm{e}^{-2\theta(z;x,t)} \\ 0 & 0 \end{pmatrix} & \text{for } z \in \overline{\mathcal{D}} \\[2ex] \begin{pmatrix} 0 & 0 \\ \beta(z)^2\,\mathrm{e}^{2\theta(z;x,t)} & 0 \end{pmatrix} & \text{for } z \in \mathcal{D} \end{cases} \tag{1.3}$$

and

$$\theta(z;x,t) = \mathrm{i}(z^2 t + zx), \tag{1.4}$$

where here and below $\beta^*(z) := \overline{\beta(\bar{z})}$ and $\overline{\mathcal{D}} = \{z \in \mathbb{C} \,|\, \bar{z} \in \mathcal{D}\}$. We write for brevity $\beta(z)^2$ where it is understood that the function is smooth and not necessarily analytic.

If the solution $\Gamma(z; x, t)$ of the $\overline{\partial}$-problem (equation (1.2)) exists, the solution ψ, of the NLS equation, is recovered from the solution $\Gamma(z; x, t)$ by the formula

$$\psi(x,t) = 2\mathrm{i} \lim_{z\to\infty} z(\Gamma(z;x,t))_{12}. \tag{1.5}$$

A $\overline{\partial}$-problem with non-zero boundary conditions has recently been considered in [13] in the framework of the NLS equation. The $\overline{\partial}$-problem of the form of equation (1.2) was included in [14] in the setting of the generalization of the theory of integrable operators developed by Its–Izergin–Korepin–Slavnov (IIKS) [15] (see section below). In particular, the solvability of the $\overline{\partial}$-problem was connected with the non-vanishing of a suitable defined τ-function of $\overline{\partial}$-problem.

The motivation to consider the $\overline{\partial}$-problem of the form of equation (1.2) comes from the study of solutions of the NLS equation in the limit of an infinite number of solitons initiated in [16,17] for the Korteweg de Vries (KdV) and modified KdV equations, and in [18,19] for the NLS equation. Such a limiting configuration of solitons is named soliton gas and it refers to both a random and deterministic configuration of solitons. The theory of soliton gas is attracting increasing attention because of the rich dynamical behaviour first spotted by Zakharov [20] in his formulation of the soliton gas kinetic equations for the KdV equation, later extended to the NLS equation by El & Tovbis [21] and also for the connections with statistical mechanics (see e.g. the review [22]).

For general $\mathcal{D}$ and smooth β, the class of initial data $\psi(x,0)$ described by the $\overline{\partial}$-problem (equation (1.2)) is in generally unknown and requires the asymptotic analysis of $\overline{\partial}$-problems and the extension of the $\overline{\partial}$-steepest descent method developed by McLaughlin & Miller [23] and Dieng *et al.* [24].

In [18], it was shown that, if the domain $\mathcal{D}$ is a suitable quadrature domain and $\beta(z)=\bar{z}^{m-1}r(z)$ with the integer m depending on the domain and $r(z)$ analytic, then the solution of the NLS equation with an infinite number of solitons behaves as a finite m soliton solution. Furthermore, if β is analytic in $\mathcal{D}$ and the domain is a 'generalized quadrature domain' [25], the $\overline{\partial}$-problem (equation (1.2)) can be reduced to a classical RH problem with discontinuities along the mother-body $\mathcal{L}$ of the domain $\mathcal{D}$. We have called this reduction *soliton shielding* because the effective soliton charge can be reduced from a two-dimensional domain to a collection of points (in the case of reduction to a finite number of solitons) or to a collection of arcs inside the domain.

An example of generalized quadrature domain is an ellipse where the mother body $\mathcal{L}$ is the segment joining the foci. The goals of this paper are:

— to show that the τ-function of the $\overline{\partial}$-problem (equation (1.2)) according to the definition in [14] coincides with the τ-function obtained from the infinite soliton limit (see theorems 3.2 and 3.3). Such a τ-function is not vanishing for all $x\in\mathbb{R}$ and $t\in\mathbb{R}^+$ thus proving solvability of the $\overline{\partial}$-problem (equation (1.2)).

— to show that when $\mathcal{D}$ is an ellipse and $\beta(z)$ is analytic, the solution of the $\overline{\partial}$-problem (equation (1.2)) gives a solution $\psi(x,t)$ of the NLS equation that is classical and step-like oscillatory. In particular, we show in theorem 4.1 and proposition 4.7 that there is a $\delta>0$ sufficiently small so that for any fixed $T>0$, for all $t\in[0,T)$ and $|x|>T/\delta$,

$$\psi_N(x,t)=\begin{cases}-\mathrm{i}\mathrm{e}^{\mathrm{i}\chi(x,t)}(\alpha_2+\alpha_1)\,\mathrm{dn}((\alpha_2+\alpha_1)(x-x_0);m)+\mathcal{O}(\mathrm{e}^{c_- x}), & \text{as } x\to-\infty,\\ \mathcal{O}(\mathrm{e}^{-c_+ x}), & \text{as } x\to+\infty,\end{cases} \tag{1.6}$$

where $\mathrm{dn}(z;m)$ is the Jacobi elliptic function of modulus $m=4\alpha_2\alpha_1/(\alpha_2+\alpha_1)^2$, where α_1 and α_2 are the positions of the foci of the ellipse $\mathcal{D}$ on the positive imaginary axis, $c_\pm$ are positive constants, x_0 is a constant that depends on $\beta(z)$ and the geometry of the problem, the phase $\chi(x,t)=2g_\infty x+2f_\infty+(\alpha_1^2+\alpha_2^2)t$, with g_∞ and f_∞ real constants.

This paper is organized as follows. In §2, we show how to obtain the $\overline{\partial}$-problem (equation (1.2)) from a N-soliton solution of the NLS equation in the limit $N\to\infty$. In §3, we review the theory of integrable operators and show that the τ-function of the $\overline{\partial}$-problem (equation (1.2)) coincides with the limiting τ-function of the N soliton solution. In §4, we give an asymptotic characterization of $\psi(x,t)$ for fixed t when $\mathcal{D}$ is an ellipse and the function β is analytic in $\mathcal{D}$. In particular, we show that $\psi(x,t)$ is step-like oscillatory like the soliton gas configuration for the KdV equation and the modified KdV equation considered in [16,17], where the soliton spectrum is uniformly distributed on a segment of the imaginary axis.

2. Infinite soliton limit and $\overline{\partial}$-problem

The $\overline{\partial}$-problem (equation (1.2)) appears when considering the limit of infinitely many solitons of the NLS initial data. The one-soliton solution is a travelling wave and is given by

$$\psi(x,t)=2b\,\mathrm{sech}[2b(x+2at-x_0)]\,\mathrm{e}^{-2\mathrm{i}[ax+(a^2-b^2)t+(\phi_0/2)]}, \tag{2.1}$$

where x_0 is the initial peak position of the soliton, ϕ_0 is the initial phase, $2b$ is the modulus of the wave maximal amplitude and $-2a$ is the soliton velocity. The N soliton solution can be obtained from the inverse scattering problem for the Zhakarov–Shabat linear spectral problem [1] with only a discrete spectrum $\{z_j\}_{j=1}^N$ and norming constants $\{c_j\}_{j=1}^N \in \mathbb{C}^N \setminus \{0\}$. When the solitons are far apart, each spectral point $z_j = a_j + \mathrm{i}b_j$ corresponds to a wave profile of the form of equation (2.1) with speed $-2a_j \in \mathbb{R}$ and amplitude $2b_j > 0$. When the N solitons are interacting, the dynamic is strongly nonlinear and it can be recovered via inverse scattering from the spectral data $\{z_j, c_j\}_{j=1}^N$, where we consider the generic case, that is, there are no higher-order solitons [8]. This goal is accomplished by formulating the inverse scattering problem as a RH problem for a 2×2 matrix function Γ_N.

RH problem 2.1. *Find a holomorphic matrix function* $\Gamma_N : \mathbb{C}\backslash(\mathcal{Z} \cup \overline{\mathcal{Z}}) \to SL_2(\mathbb{C})$, $\mathcal{Z} = \{z_1, \ldots, z_N\}$, *with the following properties*:

(*i*) $\Gamma_N(z; x, t) = \mathbb{I} + \mathcal{O}(z^{-1})$ *as* $z \to \infty$.

(*ii*) $\Gamma_N(z; x, t)$ *has simple poles at each* $z_k \in \mathcal{Z}$ *and* $\overline{z_k} \in \overline{\mathcal{Z}}$ *(where* $\overline{z_k}$ *stands for complex conjugate) at which*

$$\left.\begin{aligned} Res_{z_k}\Gamma_N(z;x,t) &= \lim_{z\to z_k} \Gamma_N(z;x,t)\begin{pmatrix} 0 & 0 \\ c_k\,\mathrm{e}^{2\theta} & 0 \end{pmatrix} \\ Res_{\overline{z_k}}\Gamma_N(z;x,t) &= \lim_{z\to \overline{z_k}} \Gamma_N(z;x,t)\begin{pmatrix} 0 & -\overline{c_k}\,\mathrm{e}^{-2\theta} \\ 0 & 0 \end{pmatrix}; \end{aligned}\right\} \tag{2.2}$$

(*iii*) *the symmetry condition*

$$\Gamma_N(z;x,t) = \sigma_2\overline{\Gamma_N(\overline{z};x,t)}\sigma_2, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.3}$$

It is a consequence of Liouville's theorem that if a solution exists, it is unique. The existence of solutions of the RH problem 2.1 for all x, t follows by means of Zhou's vanishing lemma argument [26], after replacing the poles by jumps along small circular contours around the poles. This can also be achieved by introducing a single counterclockwise close curve γ_+ in the upper half space $\mathbb{C}_+$ that encircles all poles $\mathcal{Z}$ and similarly $\gamma_- = \{z \in \mathbb{C} | \overline{z} \in \gamma_+\}$ that is oriented counterclockwise and encircles all poles $\overline{\mathcal{Z}}$. We define a new matrix function $Y_N(z; x, t)$ as

$$Y_N(z;x,t) = \Gamma_N(z;x,t)A_N(z) \tag{2.4}$$

and

$$\left.\begin{aligned} A_N(z) &= \begin{pmatrix} 1 & C_N^*(z)\mathbf{1}_{D_{\gamma_-}}(z) \\ -C_N(z)\mathbf{1}_{D_{\gamma_+}}(z) & 1 \end{pmatrix} \\ C_N(z) &= \sum_{j=1}^N \frac{c_j\,\mathrm{e}^{2\theta(z,x,t)}}{z - z_j}, \end{aligned}\right\} \tag{2.5}$$

where $\mathbf{1}_{D_{\gamma_\pm}}(z)$ are the characteristic functions of the domains $D_{\gamma_\pm}$ bounded by $\gamma_\pm$, respectively, and $C_N^*(z) = \overline{C_N(\overline{z})}$. In this way, the matrix $Y_N(z; x, t)$ is analytic for $z \in \mathbb{C}\backslash\{\gamma_+ \cup \gamma_-\}$ and with boundary values

$$(Y_N(z;x,t))_+ = (Y_N(z;x,t))_- \begin{pmatrix} 1 & C_N^*(z)\mathbf{1}_{\gamma_-}(z) \\ -C_N(z)\mathbf{1}_{\gamma_+}(z) & 1 \end{pmatrix}, \tag{2.6}$$

for $z \in \gamma_+ \cup \gamma_-$; the notation $(Y_N(z; x, t))_\pm$ indicates the (non-tangential) boundary values from the left, $+$, and right, $-$, sides of the oriented contour. The matrix $Y_N(z; x, t)$ in the region outside $\gamma_\pm$, coincides with the meromorphic matrix $\Gamma_N(z; x, t)$. The normalization of the problem for $Y_N(z; x, t)$

is therefore the same as the one for $\varGamma_N(z;x,t)$. The solution of the RH problem 2.1 gives a matrix $\varGamma_N$ of the form

$$\varGamma_N(z;x,t) = \mathbb{I} + \sum_{j=0}^{N-1} \frac{\begin{pmatrix} l_j(x,t) & 0 \\ m_j(x,t) & 0 \end{pmatrix}}{z - z_j} + \sum_{j=0}^{N-1} \frac{\begin{pmatrix} 0 & -\overline{m_j(x,t)} \\ 0 & \overline{l_j(x,t)} \end{pmatrix}}{z - \overline{z_j}}, \tag{2.7}$$

where the coefficients m_j and l_j are obtained by solving a determined linear system obtained by imposing the residue conditions equation (2.2). Alternatively by the definition $Y_N(z;x,t) = \varGamma_N(z;x,t)A_N(z)$ and the requirement that this expression gives an analytic matrix in the interior regions $D_{\gamma_\pm}$ (without poles at the spectral points $z = z_j$ and $z = \overline{z_j}$) yields an equivalent determined linear system of equations for the coefficients l_j and m_j. From the inverse scattering problem, one finds that as $z \to \infty$

$$\varGamma_N(z;x,t) = Y_N(z;x,t) = I + \frac{1}{2\mathrm{i}z} \begin{pmatrix} -\int_x^\infty |\psi(s,t)|^2 \, \mathrm{d}s & \psi_N(x,t) \\ \overline{\psi_N(x,t)} & \int_x^\infty |\psi(s,t)|^2 \, \mathrm{d}s \end{pmatrix} + O(z^{-2}), \tag{2.8}$$

so that (e.g. [10,27])

$$\left.\begin{aligned} \psi_N(x,t) &= 2\mathrm{i} \lim_{z\to\infty} z(Y_N(z;x,t))_{12} = -2\mathrm{i} \sum_{j=1}^N \overline{m_j(x,t)} \\ |\psi_N(x,t)|^2 &= 2\mathrm{i}\partial_x \sum_{j=1}^N l_j(x,t) = \partial_x^2 \log(\det(\mathbb{I}_{\mathbb{C}^N} + \varPhi_N(x,t)\,\overline{\varPhi_N(x,t)})), \end{aligned}\right\} \tag{2.9}$$

where $\varPhi_N$ is an $N \times N$ matrix with elements

$$(\varPhi_N(x,t))_{jk} := \frac{\sqrt{c_j \overline{c_k}}\, \mathrm{e}^{\theta(z_j,x,t) - \theta(\overline{z_k},x,t)}}{i(z_j - \overline{z_k})} \quad \text{and} \quad \varPhi_N(x,t)^\dagger = \varPhi_N(x,t). \tag{2.10}$$

We consider the limit $N \to \infty$ as in [18] and in the spirit of [16,17]. Let $\mathcal{D}$ be a simply connected and close domain strictly contained in D_{γ_+}. We assume that the poles z_j accumulate in $\mathcal{D}$ as $N \to \infty$ to a uniform limiting measure. We choose the norming constants c_j to be interpolated by a smooth non-vanishing function $\beta^2(z)$ supported on $\mathcal{D}$:

$$c_j = \frac{\mathcal{A}}{\pi N} \beta^2(z_j), \tag{2.11}$$

where $\mathcal{A}$ is the area of $\mathcal{D}$. Upon taking the limit $N \to +\infty$, we get

$$\begin{aligned} C_N(z) &= \sum_{j=1}^N \frac{c_j \, \mathrm{e}^{2\theta(z,x,t)}}{z - z_j} \\ &= \mathrm{e}^{2\theta(z,x,t)} \sum_{j=1}^N \frac{\mathcal{A}}{\pi N} \frac{\beta^2(z_j)}{(z - z_j)} \xrightarrow[N\to\infty]{} C(z) := \mathrm{e}^{2\theta(z,x,t)} \iint_{\mathcal{D}} \frac{\beta^2(w)}{z - w} \frac{\mathrm{d}^2 w}{\pi}, \end{aligned} \tag{2.12}$$

where $\mathrm{d}^2 w = \mathrm{d}x\mathrm{d}y$ is the usual area element. Since $\mathcal{D}$ has finite distance from γ_+ (and so does $\overline{\mathcal{D}}$ from $\gamma_- = \overline{\gamma_+}$) the convergence is uniform on $\gamma_\pm$. Note that we could consider a non-uniform eigenvalue distribution by re-defining the function β. Consequently, the jump condition equation (2.6) for $Y_N(z;x,t)$ in the limit $N \to \infty$, leads to the following RH problem.

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RH problem 2.2. *To find* 2×2 *matrix* $Y(z;x,t)$ *analytic in* $\mathbb{C}\backslash\{\gamma_+ \cup \gamma_-\}$ *such that*

$$\left.\begin{aligned} &Y_+(z;x,t) = Y_-(z;x,t)J_Y(z;x,t), \quad z \in \gamma_+ \cup \gamma_- \\ \text{and} \quad &Y(z;x,t) = \mathbb{I} + \mathcal{O}(z^{-1}), \quad \text{as } z \to \infty. \end{aligned}\right. \tag{2.13}$$

The matrix J_Y *is called the 'jump matrix', and is given by*

$$J_Y(z;x,t) = \begin{pmatrix} 1 & C^*(z)\mathbf{1}_{\gamma_-}(z) \\ -C(z)\mathbf{1}_{\gamma_+}(z) & 1 \end{pmatrix}, \tag{2.14}$$

where C is defined in equation (2.12) *and* $C^*(z) := \overline{C(\overline{z})} = \iint_{\overline{\mathcal{D}}} e^{-2\theta(z,x,t)}\beta^*(w)^2 d^2w/\pi(z-w)$.

Similarly to Γ_N the matrix Y satisfies the symmetry equation (2.3). Before moving on, using Zhou vanishing lemma [26] one can prove the existence of solutions of the above RH problem for $Y(z;x,t)$ and its differentiability in x (see appendix in [19]).

Proposition 2.1 ([19]). *There is a unique solution* $Y(z;x,t)$ *to the RH problem 2.2, differentiable in* x*, which determines a solution* $\psi(x,t)$ *to the NLS equation via*

$$\left.\begin{aligned} \psi(x,t) &= 2\mathrm{i}\lim_{z\to\infty} z(Y(z;x,t))_{12} = -2\mathrm{i}\,\overline{\lim_{z\to\infty} z(Y(z;x,t))_{21}} \\ |\psi(x,t)|^2 &= -2\mathrm{i}\lim_{z\to\infty} z\partial_x(Y(z;x,t))_{22} = 2\mathrm{i}\lim_{z\to\infty} z\partial_x(Y(z;x,t))_{11}. \end{aligned}\right\} \tag{2.15}$$

Moreover, $\psi(x,t)$ *is a classical solution to the NLS equation, which belongs to the class* $C^\infty(\mathbb{R}\times\mathbb{R}^+)$.

We can relate the RH problem for Y to a $\overline{\partial}$-problem as we now explain. We have transformed Γ_N to Y_N via $\Gamma_N = Y_N A_N(z)$ (see equation (2.4)). In this way, we have removed the poles by introducing a jump to Y_N. Now we can define $A(z) := \lim_{N\to\infty} A_N(z)$ where the matrix $A_N(z)$ is defined in equation (2.4). We now transform $Y \to \Gamma$ with

$$\Gamma(z;x,t) := Y(z;x,t)A^{-1}(z;x,t), \tag{2.16}$$

where the matrix $A(z;x,t)$ is

$$A(z;x,t) = \begin{pmatrix} 1 & C^*(z)\mathbf{1}_{D_{\gamma_-}}(z) \\ -C(z)\mathbf{1}_{D_{\gamma_+}}(z) & 1 \end{pmatrix}, \tag{2.17}$$

and $C(z)$ is defined in equation (2.12). Clearly, Γ can be viewed as the limit of Γ_N as $N\to\infty$. Note that now $\Gamma(z;x,t)$ is continuous (and analytic) across $\gamma_\pm$, and continuous throughout the complex plane. It fails to be analytic on $\mathcal{D}\cup\overline{\mathcal{D}}$, which is the point we are heading to next. We apply the $\overline{\partial}$ operator to equation (2.16)

$$\overline{\partial}\Gamma(z;x,t) = Y(z;x,t)\overline{\partial}A^{-1}(z;x,t) = \Gamma(z;x,t)J_\Gamma(z;x,t), \tag{2.18}$$

where $J_\Gamma(z;x,t) = A(z;x,t)\overline{\partial}A^{-1}(z;x,t)$. A simple computation using the identity $\overline{\partial}C(z) = e^{2\theta(z,x,t)}\beta^2(z)$ for $z\in\mathcal{D}$, shows that J_Γ coincides with equation (1.3) so that equation (2.18) is exactly the $\overline{\partial}$-problem (equation (1.2)). Existence of solution of the $\overline{\partial}$-problem for Γ follows from the existence of solution of Y as proved in proposition 2.1. Since the matrix $A(z;x,t) = \mathbb{I}$ at $z=\infty$ and $Y(z;x,t)$ is equal to $\Gamma(z;x,t)$ for large z we have

$$\Gamma(z) = \mathbb{I} + \frac{\Gamma^{(1)}(x,t)}{z} + \mathcal{O}(z^{-2}) \quad \text{as } z\to\infty. \tag{2.19}$$

By proposition 2.1 it follows that $\Gamma^{(1)}(x,t)$ is differentiable in x so that the NLS solution is recovered by the formulae

$$\psi(x,t) = 2\mathrm{i}\lim_{z\to\infty} z(\Gamma(z;x,t))_{12} = 2\mathrm{i}\Gamma^{(1)}_{12} \quad \text{and} \quad |\psi(x,t)|^2 = 2\mathrm{i}\partial_x\Gamma^{(1)}_{11}(x,t). \tag{2.20}$$

Remark 2.2. It would be interesting to consider a similar limit for higher-order solitons or even breathers in the spirit of [28–30].

3. Fredholm determinant for the soliton gas

In [14], it was shown that the $\overline{\partial}$-problems of the form of equation (1.2) with J_Γ as in equation (1.3) extend the theory of integrable operators of the IIKS framework [15]. The gist of [14] is to consider a Hilbert–Schmidt operator of integrable type acting on $L^2(\mathcal{D}, \mathrm{d}^2 w) \otimes \mathbb{C}^n$ to itself:

$$\mathcal{K}[v](z,\overline{z}) = \iint_{\mathcal{D}} \hat{\mathcal{K}}(z,\overline{z},w,\overline{w}) v(w) \mathrm{d}^2 w, \quad z \in \mathcal{D}, \tag{3.1}$$

with kernel of the form

$$\hat{\mathcal{K}}(z,\overline{z},w,\overline{w}) = \frac{p(z,\overline{z})^{\mathrm{T}} q(w,\overline{w})}{(z-w)} \tag{3.2}$$

and

$$p(z,\overline{z})^{\mathrm{T}} q(z,\overline{z}) \equiv 0 \equiv (\overline{\partial} p(z,\overline{z}))^{\mathrm{T}} q(z,\overline{z}) \quad p,q \in \mathcal{C}^\infty(\mathcal{D}, \mathrm{Mat}(r \times n, \mathbb{C})). \tag{3.3}$$

Here $\mathcal{D}$ is just an arbitrary compact set in the complex plane (consisting possibly of several connected components): as mentioned earlier, we drop explicit notation of the $\overline{z}$ dependence when indicating a smooth function. The result of [14] is that the resolvent operator $\mathcal{R} = (\mathrm{Id} - \mathcal{K})^{-1} - \mathrm{Id}$ of $\mathcal{K}$ exists if and only if the following $\overline{\partial}$-problem for the matrix $\Gamma(z) \in GL_r(\mathbb{C})$ admits a solution:

$$\left.\begin{aligned} &\overline{\partial}\Gamma(z) = \pi \Gamma(z) J_\Gamma(z), \quad J_\Gamma(z) = p(z) q(z)^{\mathrm{T}} \mathbf{1}_{\mathcal{D}}(z) \\ \text{and} \quad &\Gamma(z) \underset{z\to\infty}{\to} \mathbb{I}. \end{aligned}\right\} \tag{3.4}$$

The resolvent operator $\mathcal{R}$ has kernel $\hat{\mathcal{R}}(z,w)$ expressed via Γ, p and q by the relation

$$\hat{\mathcal{R}}(z,w) = \frac{p(z)^{\mathrm{T}} \Gamma^{\mathrm{T}}(z) \Gamma^{-\mathrm{T}}(w) q(w)}{z-w}. \tag{3.5}$$

We assume that $J_\Gamma(z)$ depends analytically on some parameters $\{t_j\}_{j=1}^k$, and we define the operator δ of exterior total differentiation with respects to these parameters, namely, $\delta f = \sum_{j=1}^k \partial_{t_j} f \,\mathrm{d}t_j$, for some analytic function f, we then showed the following.

Theorem 3.1. *[14] The operator* $\mathrm{Id} - \mathcal{K}$, *with* $\mathcal{K}$ *as in equation (3.1) with kernel* $\hat{\mathcal{K}}(z,w)$ *of the form of equation (3.2), is invertible in* $L^2(\mathcal{D}, \mathrm{d}^2 z) \otimes \mathbb{C}^n$ *if and only if the* $\overline{\partial}$-*problem 3.4 admits a solution. Furthermore,*

$$\delta \log \det(\mathrm{Id} - \mathcal{K}) = -\iint_{\mathcal{D}} \mathrm{Tr}(\Gamma^{-1}(z) \partial_z \Gamma(z) \delta J_\Gamma(z)) \frac{\mathrm{d}\overline{z} \wedge \mathrm{d}z}{2\pi i} - \delta \mathrm{Tr}(\mathcal{K}), \tag{3.6}$$

where $\Gamma(z)$ *is the solution of the* $\overline{\partial}$-*problem (equation (3.4)) with jump matrix* $J_\Gamma(z)$.

Note that applying the standard variational formulae to the log of the Fredholm determinant one gets $\delta \log \det(\mathrm{Id} - \mathcal{K}) = -\mathrm{Tr}((\mathrm{Id} - \mathcal{K})^{-1} \circ \delta\mathcal{K})$. The integration of equation (3.6) with respect to the parameters, by definition, gives the τ-function of the $\overline{\partial}$-problem that has been identified in [14] with the Hilbert–Carleman determinant (e.g. [31]) of the Hilbert–Schmidt operator $\mathcal{K}$

$$\tau := \det{}_2[\mathrm{Id} - \mathcal{K}] = \det[(\mathrm{Id} - \mathcal{K}) \mathrm{e}^{\mathcal{K}}], \tag{3.7}$$

defined on the space $L^2(\mathcal{D}, \mathrm{d}^2 z) \otimes \mathbb{C}^n$.

The goal of this section is to show that the τ-function of the N soliton solution in the limit as $N \to \infty$ converges to the τ-function of the $\overline{\partial}$-problem (equation (3.4)) and such τ-function is non-vanishing for all $x \in \mathbb{R}$ and $t > 0$. This gives an alternative proof to the fact that the $\overline{\partial}$-problem (equation (3.4)) has a solution.

To proceed with our analysis, we first revisit the theory of N-soliton solutions.

(a) Review of the N-soliton solution

The N-soliton solution of the NLS equation (1.1) with scattering data $\{z_j, c_j\}_{j=1}^N$, $\mathrm{Im}(z_j) > 0$, $c_j \in \mathbb{C}\backslash\{0\}$ can be recovered by a type of Kay–Moses formula (equation (2.9)) where Φ is an $N \times N$

matrix with elements (equation (2.10)). We claim and prove in appendix A that the τ-function of the N-soliton solution (equation (2.6)), according to the IIKS theory, is

$$\tau_N(x,t) = \det(\mathbb{I}_{\mathbb{C}^N} + \Phi_N(x,t)\overline{\Phi_N(x,t)}). \tag{3.8}$$

We can represent the matrix Φ_N as a composition of an operator with its adjoint: define $\mathcal{M}_N : L^2([x,+\infty)) \to \mathbb{C}^N$ and its adjoint $\mathcal{M}_N^\dagger : \mathbb{C}^N \to L^2([x,+\infty))$ as

$$\left.\begin{aligned} \mathcal{M}_N[u]_j &:= \int_x^{+\infty} \sqrt{c_j}\, \mathrm{e}^{\mathrm{i}(z_j s + z_j^2 t)} u(s)\mathrm{d}s \\ \mathcal{M}_N^\dagger[\boldsymbol{v}](y) &:= \sum_{k=1}^N \sqrt{\overline{c_k}}\, \mathrm{e}^{-\mathrm{i}(\overline{z_k} y + \overline{z_k}^2 t)} v_k, \quad y \in [x,\infty). \end{aligned}\right\} \tag{3.9}$$

Note that the integrals in equation (3.9) converge since $\mathrm{Im}(z_j) > 0$. A direct computation shows that $\Phi_N = -\mathcal{M}_N \circ \mathcal{M}_N^\dagger$. Therefore Φ_N is not only a Hermitian matrix, but a negative-definite one. Similarly the complex conjugate matrix $\overline{\Phi_N}$ (no transposition), is also a Hermitian, negative definite matrix and hence the (determinant (3.8)) is necessarily strictly positive.

We now consider the limit $N \to \infty$ of the τ-function equation (3.8).

Theorem 3.2. *Let $\{z_j, c_j\}_{j=1}^N$ be the spectral data of the N soliton solution. Let us assume that the constants $\{c_j\}_{j=1}^N$ are interpolated by a smooth non-vanishing function β as in equation (2.11) and that the points $\{z_j\}_{j=1}^N$ converge, as $N \to \infty$, to a uniform limiting density in the bounded region $\mathcal{D} \subset \mathbb{C}_+$ Then the function $\tau_N(x,t)$ defined in (equation (3.8)) converges, for (x,t) in a compact set, to the τ-function*

$$\tilde{\tau}(x,t) := \det(\mathrm{Id}_{L^2([x,+\infty))} + \mathcal{B} \circ \mathcal{B}^\dagger), \tag{3.10}$$

where $\mathcal{B}^\dagger$ is the hermitian conjugate and where the operator $\mathcal{B} : L^2([x,+\infty)) \to L^2([x,+\infty))$ is defined by

$$\mathcal{B}[u](y) := \int_x^\infty \widehat{\mathcal{B}}(y+s)u(s)\mathrm{d}s \quad \text{and} \quad \widehat{\mathcal{B}}(s) := \iint_{\mathcal{D}} \beta(w)^2\, \mathrm{e}^{\mathrm{i}(ws + 2w^2 t)} \frac{\mathrm{d}^2 w}{\pi}. \tag{3.11}$$

Moreover, $\tilde{\tau}(x,t) > 0$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.

Proof. Consider the operator $\mathcal{M}_N$ defined in equation (3.9). It is convenient to introduce the two new operators $\mathcal{B}_N := (\overline{\mathcal{M}_N^\dagger} \circ \mathcal{M}_N)$ and $\overline{\mathcal{B}}_N := (\mathcal{M}_N^\dagger \circ \overline{\mathcal{M}}_N)$ on $L^2([x,+\infty))$ to itself which then have the explicit form

$$\mathcal{B}_N[u](y) = \int_x^{+\infty} \widehat{\mathcal{B}}_N(y+s)u(s)\mathrm{d}s \quad \text{and} \quad \widehat{\mathcal{B}}_N(s) = \sum_{k=1}^N c_k \mathrm{e}^{\mathrm{i}z_k s + 2\mathrm{i}z_k^2 t}. \tag{3.12}$$

Then the τ-function (equation (3.8)) reads

$$\begin{aligned} \tau_N &= \det[\mathbb{I}_{\mathbb{C}^N} + \Phi_N \overline{\Phi_N}] = \det[\mathbb{I}_{\mathbb{C}^N} + \mathcal{M}_N \mathcal{M}_N^\dagger \overline{\mathcal{M}}_N \overline{\mathcal{M}_N^\dagger}] \\ &= \det[\mathrm{Id}_{L^2([x,+\infty))} + \overline{\mathcal{M}_N^\dagger} \mathcal{M}_N \mathcal{M}_N^\dagger \overline{\mathcal{M}}_N] = \det[\mathrm{Id}_{L^2([x,+\infty))} + \mathcal{B}_N \overline{\mathcal{B}_N}]. \end{aligned} \tag{3.13}$$

We now rescale the constants in the same way as in equation (2.11) and send $N \to +\infty$. For (x,t) in a compact set, the kernels $\widehat{\mathcal{B}}_N$ and $\overline{\widehat{\mathcal{B}}}_N$, defined in equation (3.12), converge uniformly as $N \to +\infty$ to the kernels $\widehat{\mathcal{B}}$ and $\overline{\widehat{\mathcal{B}}}$ defined as

$$\widehat{\mathcal{B}}(s) := \iint_{\mathcal{D}} \beta(w)^2\, \mathrm{e}^{\mathrm{i}(ws+2w^2 t)} \frac{\mathrm{d}^2 w}{\pi} \quad \text{and} \quad \overline{\widehat{\mathcal{B}}}(s) := \iint_{\mathcal{D}} \overline{\beta(w)}^2 \mathrm{e}^{-\mathrm{i}(\overline{w}s + 2\overline{w}^2 t)} \frac{\mathrm{d}^2 w}{\pi}. \tag{3.14}$$

The corresponding convolution operators $\mathcal{B}$ and $\overline{\mathcal{B}}$ from $L^2([x,+\infty))$ to itself

$$\mathcal{B}[u](y) := \int_x^\infty \widehat{\mathcal{B}}(y+s)u(s)\mathrm{d}s \quad \text{and} \quad \overline{\mathcal{B}[u]}(y) := \int_x^\infty \overline{\widehat{\mathcal{B}}}(y+s)u(s)\mathrm{d}s$$

are, in fact, one the adjoint of the other, $\overline{\mathcal{B}} = \mathcal{B}^\dagger$, (being that the convolution kernel is a Hankel operator). Note that $\mathcal{B}$ is a Hilbert–Schmidt operator since its kernel is square integrable in

$L^2([x,\infty))$ and $\mathcal{B}\circ\mathcal{B}^\dagger$ is a trace-class operator because it is the product of two Hilbert–Schmidt operators. Then, from standard results on the convergence of operators in trace-class norm and continuity of the Fredholm determinant [31], the τ-functions $\tau_N(x,t)$ converge as $N\to+\infty$ and for x,t in compact sets, to the Fredholm determinant $\tilde{\tau}(x,t)$ defined in equation (3.10). Since $\mathrm{Id}_{L^2([x,+\infty))}+\mathcal{B}\circ\mathcal{B}^\dagger$ is a positive trace class operator, then its Fredholm determinant $\tilde{\tau}(x,t)>0$ for all $(x,t)\in\mathbb{R}\times\mathbb{R}^+$. ∎

The next goal is to identify the limiting function $\tilde{\tau}$ in equation (3.10) with the τ-function (equation (3.7)) of the $\overline{\partial}$-problem (equation (1.2)). In our case, we can factorize the matrix $J_\Gamma(z)$, defined in equation (1.3), in the matrix product of 2×1 vectors $p(z)$ and $q(z)$ as in equation (3.4):

$$p(z)=\frac{\mathrm{e}^{-\theta(z,x,t)\sigma_3}}{\sqrt{\pi}}\begin{bmatrix}-\beta^*(z)\mathbf{1}_{\overline{\mathcal{D}}}(z)\\ \beta(z)\mathbf{1}_{\mathcal{D}}(z)\end{bmatrix}\quad\text{and}\quad q(z)=\frac{\mathrm{e}^{\theta(z,x,t)\sigma_3}}{\sqrt{\pi}}\begin{bmatrix}\beta(z)\mathbf{1}_{\mathcal{D}}(z)\\ \beta^*(z)\mathbf{1}_{\overline{\mathcal{D}}}(z)\end{bmatrix}. \tag{3.15}$$

From equation (3.2), the corresponding operator $\mathcal{K}$ acting on $L^2(\mathcal{D}\cup\overline{\mathcal{D}})$ has the following kernel:

$$\begin{aligned}\widehat{\mathcal{K}}(z,w)=&\frac{\beta(z)\beta^*(w)\mathrm{e}^{\theta(z,x,t)-\theta(w,x,t)}\mathbf{1}_{\mathcal{D}}(z)\mathbf{1}_{\overline{\mathcal{D}}}(w)}{\pi(z-w)}\\ &-\frac{\beta^*(z)\beta(w)\mathrm{e}^{\theta(w,x,t)-\theta(z,x,t)}\mathbf{1}_{\mathcal{D}}(w)\mathbf{1}_{\overline{\mathcal{D}}}(z)}{\pi(z-w)}.\end{aligned} \tag{3.16}$$

Theorem 3.3. *The function $\tilde{\tau}$ in equation (3.10) coincides with the τ-function (equation (3.7)) of the $\overline{\partial}$-problem (equation (1.2)), namely,*

$$\tilde{\tau}(x,t)=\tau(x,t)=\det{}_2(\mathrm{Id}_{L^2(\mathcal{D}\cup\overline{\mathcal{D}})}-\mathcal{K})=\det[(\mathrm{Id}_{L^2(\mathcal{D}\cup\overline{\mathcal{D}})}-\mathcal{K})\mathrm{e}^{\mathcal{K}}], \tag{3.17}$$

where $\mathcal{K}$ is a trace class integrable operator acting on $L^2(\mathcal{D}\cup\overline{\mathcal{D}})$ with kernel $\widehat{\mathcal{K}}$ given by the equation (3.16). Moreover, $\tau(x,t)>0$ for all $(x,t)\in\mathbb{R}\times\mathbb{R}^+$.

Remark 3.4. The operator $\mathcal{K}$ is a trace class operator (see equation (3.20) and the following discussion) and $\mathrm{Tr}\,\mathcal{K}=\int_{\mathcal{D}\cup\overline{\mathcal{D}}}\widehat{\mathcal{K}}(z,z)d^2z=0$. We conclude that the Hilbert–Carleman determinant (equation (3.17)) coincides with the standard Fredholm determinant. More pragmatically, the series defining the two determinants coincide because $\mathcal{K}$ is identically zero on the diagonal.

Remark 3.5. The existence of a non-vanishing τ-function guarantees, by theorem 3.6, an alternative proof of the solvability of the $\overline{\partial}$-problem (equation (1.2)).

Proof of theorem 3.3. To connect the Fredholm determinant (equation (3.10)) with the Hilbert–Carleman determinant (equation (3.17)) for the operator $\mathcal{K}$ defined in equation (3.16), we decompose the operator $\mathcal{B}$ in two Hilbert–Schmidt operators

$$\mathcal{B}[u]=(\mathcal{L}\circ\mathcal{F})[u],$$

where $\mathcal{L}:L^2(\mathcal{D})\to L^2([x,+\infty))$ and $\mathcal{F}:L^2([x,+\infty))\to L^2(\mathcal{D})$ are defined in the following way:

$$\mathcal{L}[\varphi](s):=\iint_{\mathcal{D}}\beta(w)\mathrm{e}^{\mathrm{i}(ws+w^2t)}\varphi(w)\frac{\mathrm{d}^2w}{\pi},\quad \varphi(w)\in L^2(\mathcal{D}),\ s\in[x,+\infty) \tag{3.18}$$

and

$$\mathcal{F}[f](w):=\beta(w)\int_x^{+\infty}\mathrm{e}^{\mathrm{i}(ws+w^2t)}f(s)\mathrm{d}s,\quad w\in\mathcal{D},\ f(s)\in L^2([x,+\infty)). \tag{3.19}$$

Note that $\mathcal{L}$ is a Hilbert–Schmidt operator since its kernel is square integrable in $L^2(\mathcal{D}\times[x,\infty))$, thanks to the fact that $\mathcal{D}$ is bounded away from the real axis and in the upper half plane. The same is valid also for the operator $\overline{\mathcal{B}} = \mathcal{B}^\dagger$. Now, using cyclicity of the determinant, we can rewrite equation (3.10) in the form

$$\tilde{\tau}(x,t) = \det(\mathrm{Id}_{L^2([x,+\infty))} + \mathcal{B}\circ\overline{\mathcal{B}}) = \det(\mathrm{Id}_{L^2(\mathcal{D})} + \overline{\mathcal{P}}\circ\mathcal{P}),$$

where $\mathcal{P} = \mathcal{L}^\dagger \circ \mathcal{L} : L^2(\mathcal{D}) \to L^2(\mathcal{D})$ and $\overline{\mathcal{P}} = \mathcal{F}\circ\mathcal{F}^\dagger : L^2(\mathcal{D}) \to L^2(\mathcal{D})$ are given, after a short computation, by (recall that $\theta(z;x,t) = \mathrm{i}xz + \mathrm{i}tz^2$)

$$\mathcal{P}[\varphi](z) = \mathrm{i}\iint_{\mathcal{D}} \frac{\overline{\beta(z)}\beta(w)\mathrm{e}^{-\theta(\overline{z};x,t)+\theta(w;x,t)}}{(w-\overline{z})}\varphi(w)\frac{\mathrm{d}^2w}{\pi} \tag{3.20}$$

and

$$\overline{\mathcal{P}}[\varphi](z) = -\mathrm{i}\iint_{\mathcal{D}} \frac{\beta(z)\overline{\beta(w)}\mathrm{e}^{\theta(z;x,t)-\theta(\overline{w};x,t)}}{(\overline{w}-z)}\varphi(w)\frac{\mathrm{d}^2w}{\pi}. \tag{3.21}$$

Both the operators $\mathcal{P}$ and $\overline{\mathcal{P}}$ are trace class, being the product of two Hilbert–Schmidt operators. The spaces $L^2(\mathcal{D})$ and $L^2(\overline{\mathcal{D}})$ are clearly isometric and we can interpret $\mathcal{P}, \overline{\mathcal{P}}$ as maps $\mathcal{P} : L^2(\mathcal{D}) \to L^2(\overline{\mathcal{D}})$ and $\overline{\mathcal{P}} : L^2(\overline{\mathcal{D}}) \to L^2(\mathcal{D})$ (we use the same symbols) given by the similar formulae

$$\mathcal{P}[\varphi](z) = \mathrm{i}\iint_{\mathcal{D}} \frac{\beta^*(z)\beta(w)\mathrm{e}^{-\theta(z;x,t)+\theta(w;x,t)}}{(w-z)}\varphi(w)\frac{\mathrm{d}^2w}{\pi}, \quad z\in\overline{\mathcal{D}},\ \varphi\in L^2(\mathcal{D}) \tag{3.22}$$

and

$$\overline{\mathcal{P}}[\psi](z) = -\mathrm{i}\iint_{\overline{\mathcal{D}}} \frac{\beta(z)\beta^*(w)\mathrm{e}^{\theta(z;x,t)-\theta(w;x,t)}}{(w-z)}\psi(w)\frac{\mathrm{d}^2w}{\pi}, \quad z\in\mathcal{D},\ \psi\in L^2(\overline{\mathcal{D}}). \tag{3.23}$$

We thus have

$$\begin{aligned}\tilde{\tau}(x,t) &= \det(\mathrm{Id}_{L^2(\mathcal{D})} + \overline{\mathcal{P}}\circ\mathcal{P}) = \det\begin{bmatrix}\mathrm{Id}_{L^2(\mathcal{D})} + \overline{\mathcal{P}}\circ\mathcal{P} & \mathrm{i}\overline{\mathcal{P}}\\ 0 & \mathrm{Id}_{L^2(\overline{\mathcal{D}})}\end{bmatrix}\\ &= \det\left(\begin{bmatrix}\mathrm{Id}_{L^2(\mathcal{D})} & \mathrm{i}\overline{\mathcal{P}}\\ \mathrm{i}\mathcal{P} & \mathrm{Id}_{L^2(\overline{\mathcal{D}})}\end{bmatrix}\begin{bmatrix}\mathrm{Id}_{L^2(\mathcal{D})} & 0\\ -\mathrm{i}\mathcal{P} & \mathrm{Id}_{L^2(\overline{\mathcal{D}})}\end{bmatrix}\right) = \det\begin{bmatrix}\mathrm{Id}_{L^2(\mathcal{D})} & \mathrm{i}\overline{\mathcal{P}}\\ \mathrm{i}\mathcal{P} & \mathrm{Id}_{L^2(\overline{\mathcal{D}})}\end{bmatrix}.\end{aligned} \tag{3.24}$$

With this understanding, we define the trace class operator $\mathcal{K}$ on $L^2(\mathcal{D}\cup\overline{\mathcal{D}}) \simeq L^2(\mathcal{D}) \oplus L^2(\overline{\mathcal{D}})$ as

$$\mathcal{K} := -\mathrm{i}\mathcal{P} - \mathrm{i}\overline{\mathcal{P}}, \tag{3.25}$$

that has kernel $\widehat{\mathcal{K}}(z,w)$ defined in equation (3.16). Note that $\widehat{\mathcal{K}}(z,z) = 0$ and therefore $\mathrm{Tr}\mathcal{K} = 0$. Then the Hilbert–Carleman determinant of $\mathcal{K}$ is a Fredholm determinant and now we show that it coincides with $\tilde{\tau}(x,t)$ in equation (3.24) as follows:

$$\begin{aligned}\tau(x,t) &= \det(\mathrm{Id}_{L^2(\mathcal{D}\cup\overline{\mathcal{D}})} - \mathcal{K}) = \det\begin{pmatrix}\mathrm{Id}_{L^2(\mathcal{D})} & \mathrm{i}\overline{\mathcal{P}}\\ i\mathcal{P} & \mathrm{Id}_{L^2(\overline{\mathcal{D}})}\end{pmatrix}\\ &= \det(\mathrm{Id}_{L^2(\mathcal{D})} + \overline{\mathcal{P}}\circ\mathcal{P}) = \tilde{\tau}(x,t) > 0.\end{aligned}$$

■

Next, for completeness, we show that the limiting τ-function $\tau(x,t)$ gives a solution of the NLS equation not only for (x,t) in compact sets but for all values of $(x,t) \in \mathbb{R}\times\mathbb{R}^+$.

Lemma 3.6. *The τ-function (equation (3.17)) of the $\overline{\partial}$-problem 1.2 gives a solution of the NLS equation (1.1) via the formula*

$$|\psi(x,t)|^2 = \partial_x^2 \ln\tau(x,t). \tag{3.26}$$

Proof. From theorem 3.1, equations (3.6) and (1.3), we can write

$$\partial_x \ln\tau(x,t) = -\iint_{\mathcal{D}\cup\overline{\mathcal{D}}} \mathrm{i}z\mathrm{Tr}(\Gamma^{-1}(z)\partial_z\Gamma(z)[J_\Gamma(z),\sigma_3])\frac{\mathrm{d}^2z}{2\pi\mathrm{i}}. \tag{3.27}$$

We expand the commutator in the integrand and, from the cyclic invariance of the trace and the fact that $\Gamma(z)$ solves equation (1.2), we get

$$\mathrm{Tr}(\Gamma^{-1}(z)\partial_z\Gamma(z)[J_\Gamma(z),\sigma_3]) = \partial_{\bar z}\mathrm{Tr}(\sigma_3(\Gamma(z))^{-1}\partial_z\Gamma(z)) - \mathrm{Tr}(\partial_z J_\Gamma(z)\sigma_3). \tag{3.28}$$

Since J_Γ is diagonal free, the second term is identically zero and equation (3.27) becomes

$$\begin{aligned}\partial_x \ln\tau(x,t) &= -\iint_{\mathcal{D}\cup\overline{\mathcal{D}}} \partial_{\bar z}(\mathrm{i}z\mathrm{Tr}(\sigma_3(\Gamma(z))^{-1}\partial_z\Gamma(z)))\frac{\mathrm{d}^2z}{2\pi\mathrm{i}}\\ &= \oint_{-\partial\mathcal{D}\cup\partial\overline{\mathcal{D}}} \mathrm{i}z(\Gamma_{22}(z)\partial_z\Gamma_{11}(z) - \Gamma_{11}(z)\partial_z\Gamma_{22}(z))\frac{\mathrm{d}z}{2\pi\mathrm{i}} \qquad (3.29)\\ &\quad + \oint_{-\partial\mathcal{D}\cup\partial\overline{\mathcal{D}}} \mathrm{i}z(\Gamma_{21}(z)\partial_z\Gamma_{12}(z) - \Gamma_{12}(z)\partial_z\Gamma_{21}(z))\frac{\mathrm{d}z}{2\pi\mathrm{i}}, \qquad (3.30)\end{aligned}$$

where we have used Stokes' theorem and $\partial\mathcal{D}$, $\partial\overline{\mathcal{D}}$ are the anti-clockwise oriented boundaries of the domains $\mathcal{D}$ and $\overline{\mathcal{D}}$, respectively. From the $\bar\partial$-problem (equation (1.2)) and the form of the matrix J_Γ in equation (1.3), we deduce that Γ is analytic in $\mathbb{C}\backslash\{\mathcal{D}\cup\overline{\mathcal{D}}\}$ and so we can apply the Cauchy theorem to the above integrals. In particular, the integrand in equation (3.30) is of order $\mathcal{O}(z^{-2})$, so the residue at infinity is identically zero, while the integrand in equation (3.29) gives

$$\mathrm{Res}_{z=\infty}(\mathrm{i}z(\Gamma_{22}(z)\partial_z\Gamma_{11}(z) - \Gamma_{11}(z)\partial_z\Gamma_{22}(z))\,\mathrm{d}z) = i(\Gamma_{11}^{(1)} - \Gamma_{22}^{(1)}) \tag{3.31}$$

where the matrix $\Gamma^{(1)}$ is the first moment at infinity of the matrix Γ, defined in equation (2.19). Then equation (3.27) becomes

$$\partial_x \ln\tau(x,t) = \mathrm{i}(\Gamma_{11}^{(1)} - \Gamma_{22}^{(1)}), \tag{3.32}$$

and, from the relations (2.15) and (2.20) we obtain the statement of the lemma. ■

Remark 3.7. The class of solutions (equation (2.20)) of the NLS equation is in general different from the class studied in [32] and represented via a Fredholm determinant of a kernel acting on contours. From equations (3.10) and (3.11), the kernel of the τ-function is the composition of Hankel operators acting on domains of the complex plane. Recently, Bothner [33] and Krajenbrink–Doussal [34] enlarged, in a different direction with respect to our case, the class of Hankel composition operators to obtain new classes of solutions of the modified KdV equation. Applications are obtained in [35,36].

4. Step-like oscillatory initial data

For certain class of domains $\mathcal{D}$ called generalized quadrature domains and β analytic, the $\bar\partial$-problem can be reduced to the RH problem. A generalized quadrature domain $\mathcal{D}$ is simply connected and the boundary of $\mathcal{D}$ can be described by the so-called Schwarz function $S(z)$ of the domain $\mathcal{D}$ through the equation

$$\bar z = S(z),$$

where $S(z)$ is analytic in a neighbourhood of $\partial\mathcal{D}$. The condition to be a generalized quadrature domain is that the Schwarz function of the domain [25] admits an analytic extension to an open and dense maximal sub-domain $\mathcal{D}^0 \subset \mathcal{D}$ and $\mathcal{L} := \mathcal{D}\setminus\mathcal{D}^0$ consists of a *mother-body*, i.e. a collection of smooth arcs. Using Stokes' theorem and the Cauchy theorem, the $\bar\partial$-problem can be reduced to a Riemann problem with discontinuities along $\mathcal{L}$. We consider the case when $\mathcal{D}$ is an ellipse. In this case, we can reduce the $\bar\partial$-problem to an RH problem on two segments, one connecting the foci of the ellipse and the other its Schwartz reflection as we now explain. The degeneration of this case to the circle was already considered in [18]. For the sake of simplicity, we assume that the

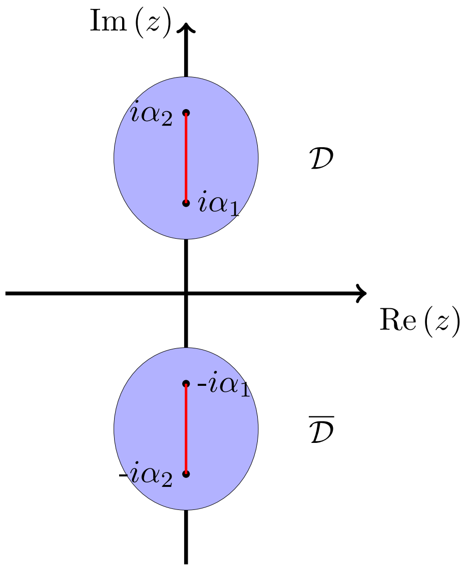

Figure 1. The domains $\mathcal{D}$ and $\overline{\mathcal{D}}$.

focal points of the ellipse E_1 and E_2 are situated on the imaginary axis, i.e. $E_1 = \mathrm{i}\alpha_1$ and $E_2 = \mathrm{i}\alpha_2$ with $\alpha_2 > \alpha_1 > 0$. The equation of the ellipse is

$$\sqrt{\mathrm{Re}(z)^2 + (\mathrm{Im}(z) - \alpha_1)^2} + \sqrt{\mathrm{Re}(z)^2 + (\mathrm{Im}(z) - \alpha_2)^2} = 2\rho > 0,$$

where ρ is chosen sufficiently small so that $\mathcal{D}$ lies in the upper half plane (figure 1).

We can apply Green's theorem for $z \notin \mathcal{D}$ to the entries of the matrix J_Y in equation (2.14) and obtain

$$\iint_{\mathcal{D}} \frac{\beta(w)^2 \mathrm{d}^2 w}{\pi(z-w)} = \oint_{\partial\mathcal{D}} \frac{\beta(w)^2 \overline{w}}{z-w} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \tag{4.1}$$

and similarly for the integral over $\overline{\mathcal{D}}$. We consider the Schwartz function S of the ellipse

$$\bar{z} = S(z) = \left(1 - 2\frac{\rho^2}{c^2}\right)(z - \mathrm{i}y_0) + 2\frac{\rho}{c^2}\sqrt{\rho^2 - c^2}\,\tilde{S}(z), \tag{4.2}$$

where $\tilde{S}(z) := \sqrt{(z-\mathrm{i}\alpha_1)(z-\mathrm{i}\alpha_2)}$, $y_0 := (\alpha_1 + \alpha_2)/2$ and $c := (\alpha_2 - \alpha_1)/2$. The Schwartz function S is analytic in $\mathbb{C} \setminus \mathcal{I}$, where $\mathcal{I} := [\mathrm{i}\alpha_1, \mathrm{i}\alpha_2]$, with boundary values $S_\pm(w)$. For $z \notin \mathcal{D}$ (or $z \notin \overline{\mathcal{D}}$), the integral along the boundary $\partial\mathcal{D}$ ($\partial\overline{\mathcal{D}}$) of the ellipse in equation (4.1) can be deformed to a line integral on the segment $\mathcal{I} = [\mathrm{i}\alpha_1, \mathrm{i}\alpha_2]$ ($\overline{\mathcal{I}} := [-\mathrm{i}\alpha_2, -\mathrm{i}\alpha_1]$), that is,

$$\int_{\partial\mathcal{D}} \frac{\beta(w)^2 \overline{w}}{z-w} \frac{\mathrm{d}w}{2\pi \mathrm{i}} = \oint_{\partial\mathcal{D}} \frac{\beta(w)^2 S(w)}{z-w} \frac{\mathrm{d}w}{2\pi \mathrm{i}} = \int_{\mathcal{I}} \frac{\beta(w)^2 \delta S(w)}{z-w} \frac{\mathrm{d}w}{2\pi \mathrm{i}},$$

where

$$\delta S(z) = S_+(z) - S_-(z) = \frac{4\rho}{c^2}\sqrt{\rho^2 - c^2}\sqrt{(z-\mathrm{i}\alpha_1)(z-\mathrm{i}\alpha_2)}. \tag{4.3}$$

We define the matrix $T(z)$ as

$$T(z) := \begin{cases} Y(z), & z \in \mathbb{C} \backslash \{D_{\gamma_+} \cup D_{\gamma_-}\} \\ Y(z)J(z), & z \in D_{\gamma_+} \cup D_{\gamma_-} \end{cases} \tag{4.4}$$

with Y as in equation (2.13), $D_{\gamma_\pm}$ the interior regions of the contours $\gamma_\pm$, respectively, and

$$J(z)=\begin{pmatrix} 1 & \displaystyle\int_{\overline{\mathcal{I}}}\frac{(\beta^*(w))^2\delta S^*(w)\mathrm{e}^{-2\theta(z;x,t)}}{w-z}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\mathbf{1}_{D_{\gamma_-}}(z) \\ \displaystyle\int_{\mathcal{I}}\frac{\beta(w)^2\delta S(w)\mathrm{e}^{2\theta(z;x,t)}}{z-w}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\mathbf{1}_{D_{\gamma_+}}(z) & 1\end{pmatrix}.$$

This new matrix $T(z)$ extends analytically across $\gamma_+\cup\gamma_-$. The matrix $J(z)$ has a discontinuity on $\mathcal{I}\cup\overline{\mathcal{I}}$ so that $T(z)$ has to be determined from the solution of the RH problem described below:

RH problem 4.1. *To find a matrix $T(z)$ analytic and analytically invertible in $\mathbb{C}\backslash\{\mathcal{I}\cup\overline{\mathcal{I}}\}$ with jump conditions*

$$\left.\begin{aligned} &T_+(z;x,t)=T_-(z;x,t)J_-^{-1}J_+=T_-(z)V_0(z;x,t)\quad z\in\mathcal{I}\cup\overline{\mathcal{I}}\\ &V_0(z;x,t)=\begin{pmatrix}1 & \mathbf{1}_{\overline{\mathcal{I}}}(z)r^*(z)\mathrm{e}^{-2\theta(z;x,t)}\\ -\mathbf{1}_{\mathcal{I}}(z)r(z)\mathrm{e}^{2\theta(z;x,t)} & 1\end{pmatrix},\\ &r(z):=\delta S(z)\beta(z)^2\end{aligned}\right\}\tag{4.5}$$

and $T(z;x,t)=\mathbb{I}+O(z^{-1})$, as $z\to\infty$.

We can find the same RH problem (equation (4.1)) also while studying the infinite soliton limit when the spectral points have a suitable density on the segments $\mathcal{I}\cup\overline{\mathcal{I}}$. The solution of the NLS equation is recovered from $T(z;x,t)$ by the relation $\psi(x,t)=2\mathrm{i}\lim_{z\to\infty}z(T(z;x,t))_{12}$.

(a) Characterization of the initial datum

We now focus on the RH problem 4.1 at $t=0$. To describe our result, we introduce the elliptic curve

$$\mathcal{C}=\{(w,z)\in\mathbb{C}^2|w^2=R(z)=(z^2+\alpha_1^2)(z^2+\alpha_2^2)\}.\tag{4.6}$$

The projection $\pi:\mathcal{C}\to\mathbb{C}$, $\pi((z,w))=z$, realizes $\mathcal{C}$ as a two sheeted covering of the complex plane. To make the curve $\mathcal{C}$ a compact Riemann surface $\mathcal{S}$, we add the two points at infinity $\infty^{1,2}$ where ∞^1 is on the first sheet (+ sign of the radical) and ∞^2 is the second sheet (− sign). When indicating a point on the surface as z, it is understood that $(z,w)\in\mathcal{C}$ is a point on the first sheet. On the first sheet of the Riemann surface the function $\sqrt{(z^2+\alpha_1^2)(z^2+\alpha_2^2)}$ is real and positive on the real axis and real on the imaginary axis minus the segments $\mathcal{I}\cup\overline{\mathcal{I}}$ where the function is discontinuous. We consider the normalized holomorphic one form

$$\omega=\left(\oint_\alpha\frac{\mathrm{d}z}{w}\right)^{-1}\frac{\mathrm{d}z}{w}=\frac{\mathrm{i}(\alpha_2+\alpha_1)}{4K(m)}\frac{\mathrm{d}z}{w},\quad m=\frac{4\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2},\tag{4.7}$$

where $K(m)=\int_0^1\mathrm{d}s/\sqrt{1-s^2}\sqrt{1-ms^2}$ is the complete elliptic integral of the first kind. We define the modulus

$$\tau:=\oint_\beta\omega=\frac{\mathrm{i}(\alpha_1+\alpha_2)}{2K(m)}\int_{\mathrm{i}\alpha_2}^{\mathrm{i}\alpha_1}\frac{\mathrm{d}z}{\sqrt{R(z)_+}}=\frac{\mathrm{i}K(m')}{K(m)},\quad m'=1-m,$$

where the α cycle consists of a close oriented contour from $\mathrm{i}\alpha_1$ to $-\mathrm{i}\alpha_1$ of the first sheet and from $-\mathrm{i}\alpha_1$ to $\mathrm{i}\alpha_1$ on the second sheet, while β cycle is a close anti-clockwise contour around the segment $[\mathrm{i}\alpha_1,\mathrm{i}\alpha_2]$. We recall the definition of the Jacobi theta function $\vartheta:\mathbb{C}\to\mathbb{C}$ with modulus τ:

$$\vartheta(z,\tau):=\sum_{n\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}\pi n^2\tau+2\mathrm{i}\pi nz}.\tag{4.8}$$

The goal of the section is to prove the following theorem.

Theorem 4.1. *The solution of the RH problem (equation (4.5)) generates an initial datum $\psi_0(x)$ of the NLS equation (1.1) that is step-like oscillatory with the following behaviours at $x \to \pm\infty$:*

$$\psi_0(x) = \begin{cases} \mathcal{O}(\mathrm{e}^{-c_+x}) \text{ as } x \to +\infty \\ -\mathrm{i}\mathrm{e}^{2\mathrm{i}g_\infty x+2\mathrm{i}f_\infty}(\alpha_2-\alpha_1)\frac{\vartheta(0;\tau)}{\vartheta(1/2;\tau)}\frac{\vartheta((\Omega x-\mathrm{i}\Delta)/2\pi+(1/2);\tau)}{\vartheta((\Omega x-\mathrm{i}\Delta)/2\pi)} \\ \quad +\mathcal{O}(\mathrm{e}^{c_-x}) \text{ as } x \to -\infty, \end{cases} \tag{4.9}$$

where $c_\pm$ are positive constants, $g_\infty, f_\infty \in \mathbb{R}$ are constants, $\vartheta(z;\tau)$ is the Jacobi theta function defined in equation (4.8) and

$$\Omega = -\frac{\pi(\alpha_1+\alpha_2)}{K(m)} \in \mathbb{R}, \tag{4.10}$$

with $K(m)$ the complete elliptic integral of the first kind with modulus $m = 4\alpha_1\alpha_2/(\alpha_1+\alpha_2)^2$ and

$$\Delta = -\frac{\mathrm{i}(\alpha_2+\alpha_1)}{2K(m)}\left[\int_{\mathrm{i}\alpha_1}^{\mathrm{i}\alpha_2}\frac{\log r(\zeta)}{\sqrt{R(\zeta)}_+}d\zeta - \int_{-\mathrm{i}\alpha_2}^{-\mathrm{i}\alpha_1}\frac{\log r^*(\zeta)}{\sqrt{R(\zeta)}_+}\mathrm{d}\zeta\right] \in \mathrm{i}\mathbb{R}, \tag{4.11}$$

where $\sqrt{R(z)}$ is a multivalued complex function, analytic in $\mathbb{C}\setminus\{\mathcal{I}\cup\overline{\mathcal{I}}\}$ and positive in the interval $(-i\alpha_1, i\alpha_1)$ and $\sqrt{R(z)}_\pm$ denotes the value on the positive/negative side of the oriented intervals $\mathcal{I}$ and $\overline{\mathcal{I}}$.

To prove the above theorem, we follow the established procedure of the Deift–Zhou steepest descent analysis [37], (following the steps of [16]) which involves several reformulations of the RH problem 4.1 for $t = 0$ as detailed in the next few paragraphs.

(b) The g-function and the f-function

We introduce the functions g and f defined as follows:

$$g(z) := -z + \int_{\mathrm{i}\alpha_2}^{z}\frac{\zeta^2+\kappa}{\sqrt{R(\zeta)}}\mathrm{d}\zeta \tag{4.12}$$

and

$$\kappa := -\frac{\int_{-\mathrm{i}\alpha_1}^{\mathrm{i}\alpha_1}\zeta^2\mathrm{d}\zeta/\sqrt{R(\zeta)}}{\int_{-\mathrm{i}\alpha_1}^{\mathrm{i}\alpha_1}\mathrm{d}\zeta/\sqrt{R(\zeta)}} = \alpha_2^2\left(1-\frac{E(m_1)}{K(m_1)}\right), \quad m_1 = \frac{\alpha_1^2}{\alpha_2^2}, \tag{4.13}$$

where $E(m_1) = \int_0^1\sqrt{(1-m_1s^2)/(1-s^2)}\mathrm{d}s$ is the complete elliptic integral of the second kind. The phase Ω in equation (4.10) is simply

$$\left.\begin{aligned} \Omega &:= \oint_\beta \frac{\zeta^2+\kappa}{\sqrt{R(\zeta)}}\mathrm{d}\zeta = -\frac{\pi(\alpha_1+\alpha_2)}{K(m)} \in \mathbb{R}; \\ f(z) &:= \exp\left[\frac{R(z)}{2\pi\mathrm{i}}\left(-\int_{\mathrm{i}\alpha_1}^{\mathrm{i}\alpha_2}\frac{\log r(\zeta)}{\sqrt{R(\zeta)}_+(\zeta-z)}\mathrm{d}\zeta\right.\right. \\ &\quad \left.\left.+\int_{-\mathrm{i}\alpha_2}^{-\mathrm{i}\alpha_1}\frac{\log r^*(\zeta)}{\sqrt{R(\zeta)}_+(\zeta-z)}\mathrm{d}\zeta + \int_{-\mathrm{i}\alpha_1}^{\mathrm{i}\alpha_1}\frac{\Delta}{\sqrt{R(\zeta)}(\zeta-z)}\mathrm{d}\zeta\right)\right]. \end{aligned}\right\} \tag{4.14}$$

The following properties of the functions f, g follow from the above definition.

Lemma 4.2. *The functions g and f defined in equations (4.12) and (4.14) satisfy*

(*i*) *Schwartz symmetry:* $\overline{g(\overline{z})} = g(z) \quad \overline{f(\overline{z})} = f^{-1}(z)$;
(*ii*) *as* $z \to \infty$

$$\left.\begin{aligned} g(z) &= g_\infty + \left[\frac{\alpha_1^2+\alpha_2^2}{2} - \kappa\right]\frac{1}{z} + \mathcal{O}(z^{-2}) \\ f(z) &= \mathrm{e}^{\mathrm{i}f_\infty} + \mathcal{O}(z^{-2}); \end{aligned}\right\} \tag{4.15}$$

where $g_\infty \in \mathbb{R}$ and $f_\infty \in \mathbb{R}$ are constants;

(*iii*) *g is bounded near the endpoints $\pm i\alpha_j$, for $j=1,2$;*
(*iv*) *the function $g(z)$ solves the following scalar RH problem:*

$$\left.\begin{aligned} &g_+(z)+g_-(z)=-2z \quad \text{for } z\in\mathcal{I}\cup\overline{\mathcal{I}} \\ \text{and}\quad &g_+(z)-g_-(z)=\Omega \quad \text{for } z\in[-i\alpha_1;i\alpha_1], \end{aligned}\right\} \tag{4.16}$$

with the real constant Ω as in equation (4.10);
(*v*) *the function $f(z)$ satisfies the scalar RH problem*

$$\left.\begin{aligned} &f_-(z)f_+(z)=r^{-1}(z) \quad \text{for } z\in\mathcal{I} \\ &f_-(z)f_+(z)=r^*(z) \quad \text{for } z\in\overline{\mathcal{I}} \\ \text{and}\quad &\frac{f_+(z)}{f_-(z)}=e^{\Delta} \quad \text{for } z\in[-i\alpha_1;i\alpha_1], \end{aligned}\right\} \tag{4.17}$$

with Δ defined in equation (4.11).

(c) First transformation: $T(z;x)\longrightarrow T^{(1)}(z;x)$

The entry 21 of the jump matrix $V_0(z;x,0)$ of the RH problem 4.5 grows exponentially fast as $x\to-\infty$. To take control of the behaviour of $T(z;x)$ as $x\to-\infty$, we need to do a series of transformations that change the jump matrix V_0. This is accomplished by defining a new matrix using the function f and g that have been suitably constructed above:

$$T^{(1)}(z;x)=e^{-i(g_\infty x+f_\infty)\sigma_3}T(z;x)\,e^{ig(z)x\sigma_3}f(z)^{\sigma_3}. \tag{4.18}$$

As a consequence of the transformation (4.18) and of the properties of f,g established in lemma 4.2, we obtain

$$\psi_0(x)=2ie^{i(g_\infty x+f_\infty)}\lim_{z\to\infty}[zT^{(1)}_{12}(z;x)\,e^{ig(z)x}f(z)], \tag{4.19}$$

and the matrix $T^{(1)}(z;x)$ satisfies the following RH problem.

RH problem 4.2. *To find a 2×2 matrix $T^{(1)}(z;x,t)$ analytic and invertible in $\mathbb{C}\setminus[-i\alpha_2,i\alpha_2]$ with the following boundary value conditions:*

$$T^{(1)}_+(z,x)=T^{(1)}_-(z;x)V^{(1)}(z;x), \tag{4.20}$$

where

$$V^{(1)}(z;x)=\begin{cases}\begin{pmatrix} e^{ix(g_+-g_-)}\dfrac{f_+}{f_-} & \mathbf{1}_{\overline{\mathcal{I}}} \\ -\mathbf{1}_{\mathcal{I}} & e^{-ix(g_+-g_-)}\dfrac{f_-}{f_+}\end{pmatrix} & \text{for } z\in\mathcal{I}\cup\overline{\mathcal{I}} \\ \begin{pmatrix} e^{ix\Omega+\Delta} & 0 \\ 0 & e^{-ix\Omega-\Delta}\end{pmatrix} & \text{for } z\in[-i\alpha_1;i\alpha_1] \end{cases} \tag{4.21}$$

and

$$T^{(1)}(z;x)=\mathbb{I}+O(z^{-1}),\quad \text{as } z\to\infty.$$

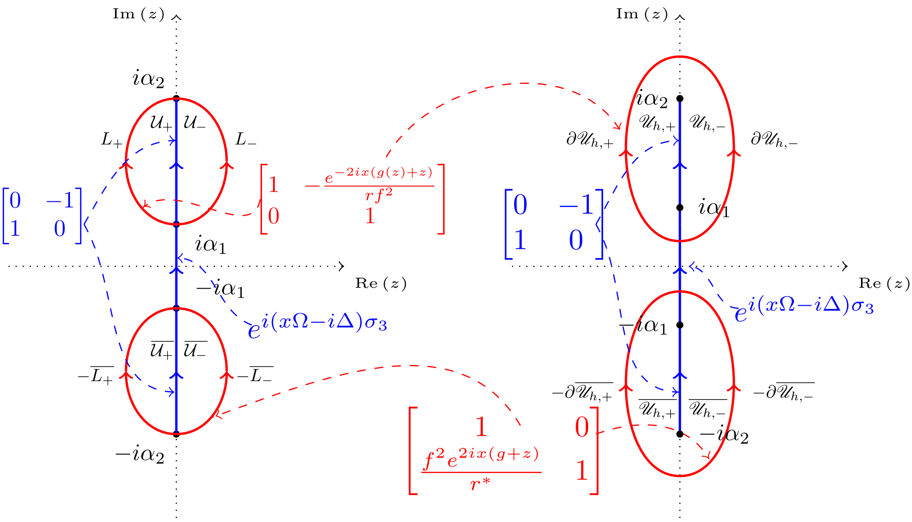

Figure 2. On the left: the lenses $\mathcal{U}_+$ and $\mathcal{U}_-$. On the right: the modified new lenses $\mathscr{U}_{h,\pm}$.

(d) The opening of the lenses

The goal is to deform the jump contours and jump matrix in such a way that the off diagonal entries become constant up to exponentially small terms. We factorize the jump matrices (equation (4.21)) for $z \in \mathcal{I} \cup \overline{\mathcal{I}}$ as follows:

— for $z \in \overline{\mathcal{I}}$

$$V^{(1)}(z;x) = \begin{pmatrix} 1 & 0 \\ e^{2ix(g_-+z)}\dfrac{(f_-(z))^2}{r(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{2ix(g_++z)}\dfrac{(f_+(z))^2}{r(z)} & 1 \end{pmatrix}$$

— for $z \in \mathcal{I}$

$$V^{(1)}(z;x) = \begin{pmatrix} 1 & -\dfrac{e^{-2ix(g_-+z)}}{r^*(z)(f_-(z))^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\dfrac{e^{-2ix(g_++z)}}{r^*(z)(f_+(z))^2} \\ 0 & 1 \end{pmatrix}.$$

We define with $\mathcal{U}_\pm$ the open set on the left $(+)$ and the right $(-)$ of the segment $\mathcal{I}$ as shown on the left of figure 2. In the same way, we define $\overline{\mathcal{U}_\pm}$ as the complex conjugate of $\mathcal{U}_\pm$, respectively.

The function $\delta S(z) = S_+(z) - S_-(z)$ is in principle defined only on $\mathcal{I}$; however, one can define an analytic extension to the left and in the right of $\mathcal{I}$. The function $\beta^2(z)$ is assumed analytic in the neighbourhood of $\mathcal{I}$ and thus we can analytically extend $r(z)$ to the regions $\mathcal{U}_\pm$. We denote by $\hat{r}(z)$ the analytic extension of $r(z)$. This procedure can be repeated by Schwartz symmetry also to $r^*(z)$.

(e) Second transformation: $T^{(1)}(z;x) \longrightarrow T^{(2)}(z;x)$

We introduce a further transformation of the problem:

$$T^{(2)}(z;x) = T^{(1)}(z;x)G(z;x), \tag{4.22}$$

where

$$G(z;x)=\begin{cases}\begin{pmatrix}1 & \dfrac{e^{-2ix(g(z)+z)}}{\hat{r}(z)f^2(z)}\\ 0 & 1\end{pmatrix} & \text{for } z\in\mathcal{U}_+;\\ \begin{pmatrix}1 & -\dfrac{e^{-2ix(g(z)+z)}}{\hat{r}(z)f^2(z)}\\ 0 & 1\end{pmatrix} & \text{for } z\in\mathcal{U}_-;\\ \begin{pmatrix}1 & 0\\ -e^{2ix(g(z)+z)}\dfrac{f^2(z)}{\hat{r}^*(z)} & 1\end{pmatrix} & \text{for } z\in\overline{\mathcal{U}_+};\\ \begin{pmatrix}1 & 0\\ e^{2ix(g(z)+z)}\dfrac{f^2(z)}{\hat{r}^*(z)} & 1\end{pmatrix} & \text{for } z\in\overline{\mathcal{U}_-};\\ \mathbb{I} & \text{otherwise.}\end{cases} \tag{4.23}$$

As a consequence of the above transformation, we obtain from equation (4.19) that

$$\psi_0(x)=2ie^{i(g_\infty x+f_\infty)}\lim_{z\to\infty}[zT^{(2)}_{12}(z;x)\,e^{ig(z)x}f(z)], \tag{4.24}$$

and the matrix $T^{(2)}$ satisfies a new RH problem with discontinuities not only on $\mathcal{I}\cup\overline{\mathcal{I}}\cup[-i\alpha_1,i\alpha_1]$ but also on the boundaries of the lenses $\mathcal{U}_\pm$ and $\overline{\mathcal{U}_\pm}$, which we denote with $L_\pm$ and $\overline{L_\pm}$, namely,

$$\left.\begin{aligned}&T^{(2)}_+(z;x)=T^{(2)}_-(z;x)V^{(2)}(z;x)\\ \text{and}\quad &T^{(2)}(z;x)=\mathbb{I}+O(z^{-1}),\quad \text{as } z\to\infty,\end{aligned}\right\} \tag{4.25}$$

where the matrix $V^{(2)}$ takes the following form:

$$V^{(2)}(z;x)=\begin{cases}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} & \text{for } z\in\mathcal{I}\cup\overline{\mathcal{I}}\\ \begin{pmatrix}e^{ix\Omega+\Delta} & 0\\ 0 & e^{-ix\Omega-\Delta}\end{pmatrix} & \text{for } z\in[-i\alpha_1;i\alpha_1]\end{cases}$$

while on the contours $L_\pm$ and $\overline{L_\pm}$

$$V^{(2)}(z;x)=\begin{cases}\begin{pmatrix}1 & -\dfrac{e^{-2ix(g(z)+z)}}{\hat{r}(z)f^2(z)}\\ 0 & 1\end{pmatrix} & \text{for } z\in L_+\cup L_-,\\ \begin{pmatrix}1 & 0\\ e^{2ix(g(z)+z)}\dfrac{f^2(z)}{\hat{r}^*(z)} & 1\end{pmatrix} & \text{for } z\in\overline{L_+}\cup\overline{L_-};\end{cases}$$

so that the z-dependent oscillatory part appears only along the outer boundary of the lenses.

Now we need to study the sign of $\mathrm{Im}(g(z)+z)$ around the lenses.

Lemma 4.3. *The function $g(z)$ satisfy the following inequalities*:

$$\left.\begin{aligned}&\mathrm{Im}(g(z)+z)>0\quad \textit{for } z\in\mathbb{C}_+\backslash\{i\alpha_1;i\alpha_2\}\\ \textit{and}\quad &\mathrm{Im}(g(z)+z)<0\quad \textit{for } z\in\mathbb{C}_-\backslash\{-i\alpha_2;-i\alpha_1\}.\end{aligned}\right\} \tag{4.26}$$

Proof. The expression we are studying is (see equation (4.12))

$$\Phi(z) := \operatorname{Im} \int_{\mathrm{i}\alpha_2}^{z} \frac{\zeta^2+\kappa}{\sqrt{R(\zeta)}} \mathrm{d}\zeta, \tag{4.27}$$

with κ defined in equation (4.13). One verifies that, while the integral is not single-valued on $\mathbb{C} \setminus \mathcal{I} \cup \overline{\mathcal{I}}$, its periods are purely real thanks to the definition of κ. Therefore, Φ is a harmonic function on $\mathbb{C} \setminus \mathcal{I} \cup \overline{\mathcal{I}}$. Next, one observes that $\Phi(z)$ is zero on $\mathcal{I}$ because both boundary values of the integral are real and in particular Φ is continuous across $\mathcal{I} \cup \overline{\mathcal{I}}$ (but not differentiable). Moreover for $z \in \mathbb{R}$, we also have $\Phi(z) \equiv 0$ because the integral is purely real. Finally, since the integrand is $1 + \mathcal{O}(\zeta^{-2})$), the integral behaves as $z + \mathcal{O}(1)$ as $z \to \infty$ and hence $\Phi(z)$ has the same sign $\operatorname{Im} z$ for large z. By the extremum principle of harmonic functions we deduce that Φ is strictly positive in $\mathbb{C}_+ \setminus \mathcal{I}$ and negative in $\mathbb{C}_- \setminus \overline{\mathcal{I}}$. ■

Let U_j be a neighbourhood of $\mathrm{i}\alpha_j$, and similarly $\overline{U_j}$ be a neighbourhood of $-\mathrm{i}\alpha_j$ with $j = 1, 2$. Let $\gamma = L_+ \cup L_- \cup \overline{L_+} \cup \overline{L_-}$. Lemma 4.26, implies that for $x \to -\infty$ the jump matrix $V^{(2)}(z;x)$ converges to the identity exponentially fast for $z \in \hat{\gamma}$ where $\hat{\gamma} = \gamma \backslash \{\gamma \cap \{U_1 \cup U_2 \cup \overline{U_1} \cup \overline{U_2}\}\}$. We arrive to the model problem for a matrix $X(z;x)$.

RH problem 4.3. *Find a matrix $X(z) : \mathbb{C} \to GL(2, \mathbb{C})$, analytic and analytical invertible in $z \in \mathbb{C} \setminus [-\mathrm{i}\alpha_2, \mathrm{i}\alpha_2]$, with jump conditions*

$$\left.\begin{aligned} &X_+(z;x) = X_-(z;x) V_X(z;x) \\ \textit{and} \qquad &V_X(z;x) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \textit{for } z \in \mathcal{I} \cup \overline{\mathcal{I}} \\ \begin{pmatrix} \mathrm{e}^{\mathrm{i}x\Omega + \Delta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}x\Omega - \Delta} \end{pmatrix} & \textit{for } z \in [-\mathrm{i}\alpha_1; \mathrm{i}\alpha_1] \end{cases} \end{aligned}\right\} \tag{4.28}$$

with boundary condition at infinity

$$X(z;x) = \mathbb{I} + \mathcal{O}(z^{-1}), \quad \textit{as } z \to \infty.$$

The RH problem 4.3 is solved using the Jacobi theta function (equation (4.8)) (e.g. [38–40]). We define the *Abel map* $u : \mathcal{C} \to \mathbb{C}$ as

$$u(z, z_0) = \frac{\mathrm{i}(\alpha_1 + \alpha_2)}{4K(m)} \int_{(z_0, w_0)}^{(z,w)} \frac{\mathrm{d}\zeta}{\sqrt{R(\zeta)}}, \tag{4.29}$$

where each point (z, w) and (z_0, w_0) belong to the elliptic curve $\mathcal{C}$ in equation (4.6). By setting $(z_0, w_0) = \infty^1$, the Abel map equation (4.29) has the following jump conditions along the segment $[-\mathrm{i}\alpha_2, \mathrm{i}\alpha_2]$:

$$\left.\begin{aligned} &u_+(z;\infty^1) + u_-(z;\infty^1) = -\frac{1}{2} \quad \text{for } z \in \mathcal{I}; \\ &u_+(z;\infty^1) - u_-(z;\infty^1) = \tau \quad \text{for } z \in [-\mathrm{i}\alpha_1; \mathrm{i}\alpha_1]; \\ \text{and} \qquad &u_+(z;\infty^1) + u_-(z;\infty^1) = \frac{1}{2} \quad \text{for } z \in \overline{\mathcal{I}}. \end{aligned}\right\} \tag{4.30}$$

Here and below the point (z, w) is simply referred to as z, and it is understood to belong to the first sheet of the surface $\mathcal{C}$. Using the property $\vartheta(z + j + l\tau; \tau) = \vartheta(z;\tau)\mathrm{e}^{-2\mathrm{i}\pi z l - \mathrm{i}\pi l^2 \tau}$, for $l, j \in \mathbb{Z}$ and the jump conditions of $u(z, \infty^1)$ equation (4.30), we can show that the solution of the model problem 4.3 has the following form:

$$X(z;x) = \frac{\vartheta(0;\tau)}{2\vartheta(\epsilon;\tau)} \begin{pmatrix} \dfrac{\vartheta(u(z^{(1)}, \infty^1) - \epsilon;\tau)}{\vartheta(u(z^{(1)}, \infty^1);\tau)} (\phi(z) + \phi(z)^{-1}) & -\mathrm{i}\dfrac{\vartheta(u(z^{(2)}, \infty^1) - \epsilon;\tau)}{\vartheta(u(z^{(2)}, \infty^1);\tau)} (\phi(z) - \phi(z)^{-1}) \\ \mathrm{i}\dfrac{\vartheta(u(z^{(1)}, \infty^2) - \epsilon;\tau)}{\vartheta(u(z^{(1)}, \infty^2)\tau)} (\phi(z) - \phi(z)^{-1}) & \dfrac{\vartheta(u(z^{(2)}, \infty^2) - \epsilon;\tau)}{\vartheta(u(z^{(2)}, \infty^2);\tau)} (\phi(z) + \phi(z)^{-1}) \end{pmatrix}, \tag{4.31}$$

where

$$\phi(z) = \left(\frac{(z+\mathrm{i}\alpha_1)(z-\mathrm{i}\alpha_2)}{(z+\mathrm{i}\alpha_2)(z-\mathrm{i}\alpha_1)}\right)^{1/4} \quad \text{and} \quad \epsilon = \frac{x\Omega - i\Delta}{2\pi},$$

and we referred with $z^{(1)}, z^{(2)}$ the point $(z, \pm w) \in \mathcal{C}$ in the first and second sheet, respectively.

Such solution is well defined. Indeed, the Jacobi elliptic function $\vartheta(z;\tau)$ has only one zero located in $z = \frac{1}{2} + (\tau/2)$. Since Ω is real while Δ is imaginary, it follows that $\vartheta(\epsilon;\tau) \neq 0$ for all $x \in \mathbb{R}$. Furthermore, as it is explained in [38], the ratio $(\phi(z) + \phi(z)^{-1})/\vartheta(u(z^{(1)}, \infty^1);\tau)$ in the 11 entry does not have poles but only fourth root singularities at the points $\pm\mathrm{i}\alpha_1$ and $\pm\mathrm{i}\alpha_2$. The same considerations apply to the other entries of the matrix $X(z;x)$.

Since $\phi(z) - \phi(z)^{-1} \to 0$ and $u(z^{(2)}, \infty^1) \to -\frac{1}{2}$ as $z \to \infty$, it is immediate to verify that $X(z;x) \to \mathbb{I} + \mathcal{O}(z^{-1})$ as $z \to \infty$.

(f) The error parametrix near the endpoints $\pm\mathrm{i}\alpha_1, \pm\mathrm{i}\alpha_2$

The last step of the nonlinear steepest descent analysis is the definition and study of the error matrices around the end points of the segments $\mathcal{I}$ and $\overline{\mathcal{I}}$. Before analysing the error parametrix, we should consider some assumptions about the behaviours of $r(z)$ near the points $\pm\mathrm{i}\alpha_1, \pm\mathrm{i}\alpha_2$. Indeed, Girotti *et al.* [17] studied an RH problem similar to equation (4.5) for the modified KdV and they proved that if the function $r(z)$ has a local behaviour near the end points $\pm\mathrm{i}\alpha_1, \pm\mathrm{i}\alpha_2$ of the form $r(z) \sim |z \pm \mathrm{i}\alpha_j|^{\pm 1/2} Q(z)$ for $j = 1, 2$, with $Q(z)$ an analytic function locally bounded and non-zero in a neighbourhood of the end points, then it is possible to modify the lenses of the opening factorization so that the error matrices tends to the identity exponentially fast uniformly in $z \in \mathbb{C}$. Specifically, we consider the following assumption.

Assumption 4.4. *Let $h > 0$ and let us consider the open set $\mathscr{U}_{h,+}$ defined as*

$$\mathscr{U}_{h,+} := \left\{ z \in \mathbb{C} |\, \mathrm{Re}(z) \in (0, h] \text{ and } \alpha_1 - \sqrt{h^2 - \mathrm{Re}(z)^2} \leq \mathrm{Im}(z) \leq \alpha_2 + \sqrt{h^2 - \mathrm{Re}(z)^2} \right\}, \tag{4.32}$$

with some $0 < h < \alpha_1$. The open set $\mathscr{U}_{h,-}$ is defined by symmetry, $\mathscr{U}_{h,-} = \{z | -\bar{z} \in \mathscr{U}_{h,+}\}$.

When $r(z)|z - \mathrm{i}\alpha_j|^{\pm 1/2}$ is bounded and non-zero on $\mathcal{I}$, we assume that $r(z)$ admits an analytical continuation to $\mathscr{U}_{h,-} \cup \mathscr{U}_{h,+}$

$$\hat{r}(z) \text{ analytic in } \mathscr{U}_{h,-} \cup \mathscr{U}_{h,+} \quad \hat{r}(z)|_{z\in\mathcal{I}} = r(z) \tag{4.33}$$

and

$$\hat{r}_+(z) + \hat{r}_-(z) = 0 \quad z \in [\mathrm{i}\alpha_2, \mathrm{i}(\alpha_2 + h)] \cup [\mathrm{i}(\alpha_1 - h), \mathrm{i}\alpha_1]. \tag{4.34}$$

In our case, we have that $r(z) = \delta S(z)\beta^2(z)$, with $\beta^2(z)$ bounded in the original domain $\mathcal{D}$ and $\delta S(z)$ defined in $\mathcal{I}$ with behaviour at the end point of the form $\delta S(z) \sim |z - \mathrm{i}\alpha_j|^{1/2}$. So, we are in the main hypothesis of assumption 4.4 and we consider $\mathscr{U}_{h,+}$ and $\mathscr{U}_{h,-}$ as the new lenses (figure 2 right). From this assumption, after we apply the transformation (4.22), we have jumps in the segments $[\mathrm{i}\alpha_2, \mathrm{i}(\alpha_2 + h)]$ and $[\mathrm{i}(\alpha_1 - h), \mathrm{i}\alpha_1]$

— $z \in [\mathrm{i}\alpha_2, \mathrm{i}(\alpha_2 + h)]$

$$(T_-^{(2)}(z;x))^{-1} T_+^{(2)}(z;x) = \begin{pmatrix} 1 & (\hat{r}_+(z)^{-1} + \hat{r}_-(z)^{-1}) \dfrac{\mathrm{e}^{-2\mathrm{i}(g(z)+z)}}{(f(z))^2} \\ 0 & 1 \end{pmatrix} = \mathbb{I}; \tag{4.35}$$

— $z \in [\mathrm{i}(\alpha_1 - h), \mathrm{i}\alpha_1]$

$$(T_-^{(2)}(z;x))^{-1} T_+^{(2)}(z;x) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}x\Omega + \Delta} & (\hat{r}_+(z)^{-1} + \hat{r}_-(z)^{-1}) \dfrac{\mathrm{e}^{-2\mathrm{i}(g(z)+z)}}{(f(z))^2} \\ 0 & \mathrm{e}^{-\mathrm{i}x\Omega - \Delta} \end{pmatrix} = \mathrm{e}^{(\mathrm{i}x\Omega + \Delta)\sigma_3}. \tag{4.36}$$

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This means that we can detach the lenses from the points $i\alpha_1$ and $i\alpha_2$ so that they fully enclose the interval $\mathcal{I}$. The same applies to $\hat{r}^*(z)$ and with the complex conjugate $\overline{\mathscr{U}_{h+}}$ and $\overline{\mathscr{U}_{h-}}$ that enclose $\overline{\mathcal{I}}$.

We now define the error matrix

$$\mathcal{E}(z;x) := T^{(2)}(z;x)(X(z;x))^{-1}, \tag{4.37}$$

which is analytic for $z \in \mathbb{C} \setminus \partial(\mathscr{U}_{h,+} \cup \mathscr{U}_{h,-} \cup \overline{\mathscr{U}_{h,+}} \cup \overline{\mathscr{U}_{h,-}})$ and it has the jump condition

$$\mathcal{E}_+(z;x) = \mathcal{E}_-(z;x) \begin{cases} \mathbb{I} + \dfrac{e^{-2ix(g(z)+z)}}{\hat{r}(z)f^2(z)} X(z)\sigma_+(X(z))^{-1} & \text{for } z \in \partial\mathscr{U}_{h+}; \\ \mathbb{I} + \dfrac{e^{-2ix(g(z)+z)}}{\hat{r}(z)f^2(z)} X(z)\sigma_+(X(z))^{-1} & \text{for } z \in \partial\mathscr{U}_{h-}; \\ \mathbb{I} + \dfrac{e^{2ix(g(z)+z)}f^2(z)}{\hat{r}^*(z)} X(z)\sigma_-(X(z))^{-1} & \text{for } z \in \partial\overline{\mathscr{U}_{h+}}; \\ \mathbb{I} + \dfrac{e^{2ix(g(z)+z)}f^2(z)}{\hat{r}^*(z)} X(z)\sigma_-(X(z))^{-1} & \text{for } z \in \partial\overline{\mathscr{U}_{h-}}; \end{cases} \tag{4.38}$$

where the matrix

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Since $X(z)$ is bounded in x, the jump matrices tends to the identity exponentially fast with respect to the matrix norm.

From the small norm lemma (e.g. [39]), we have that the matrix $\mathcal{E}$, uniformly in $z \in \mathbb{C}$, tends exponentially to the identity as $x \to -\infty$, i.e.

$$\mathcal{E}(z;x) = \mathbb{I} + \mathcal{O}(e^{c_- x}), \quad \text{as } x \to -\infty, \tag{4.39}$$

with $c_- > 0$. It follows that the model problem $X(z;x)$ coincides with $T^{(2)}(z;x)$ up to an exponentially small error as $x \to -\infty$. For $x \to +\infty$, the jump matrices of the RH problem (equation (4.5)) tend to the identity matrix exponentially fast, i.e. $(T(z;x))_{12} \sim e^{-c_+ x}$, with $c_+ > 0$. From equation (1.5), we have that

$$\psi_0(x) = \mathcal{O}(e^{-c_+ x}).$$

We now focus on the asymptotic behaviour for $x \to -\infty$. From the knowledge of $X(z;x)$ (exponentially close to the model problem $T^{(2)}(z)$), we can find out the asymptotic behaviour of the initial datum $\psi_0(x)$ for the NLS equation. Indeed from equations (4.24), (4.37) and (4.39) we have

$$\begin{aligned} \psi_0(x) &= 2ie^{i(g_\infty x + f_\infty)} \lim_{z\to\infty} [zT^{(2)}_{12}(z;x)\, e^{ig(z)x} f(z)] \\ &= 2ie^{i(g_\infty x + f_\infty)} \lim_{z\to\infty} [z(\mathcal{E}(z;x)X(z;x))_{12}\, e^{ig(z)x} f(z)] \\ &= 2ie^{i(g_\infty x + f_\infty)} \lim_{z\to\infty} [zX_{12}(z;x)\, e^{ig(z)x} f(z)] + \mathcal{O}(e^{c_- x}), \quad \text{for } x \to -\infty. \end{aligned} \tag{4.40}$$

Expanding $g(z), f(z)$ and $X_{12}(z)$ as $z \to \infty$ in equation (4.40) we obtain equation (4.9).

Remark 4.5. The formula for the elliptic solution (equation (4.9)) can be rewritten in terms of the Jacobi elliptic function $\mathrm{dn}(z;m)$. Let us introduce the Jacobi theta-functions

$$\theta_3(z;\tau) := \vartheta(z;\tau), \quad \theta_4(z;\tau) := \vartheta\left(z + \frac{1}{2};\tau\right) \quad \text{and} \quad 1 - m_1 = \frac{\theta_4^4(z;\tau)}{\theta_3^4(z;\tau)} \tag{4.41}$$

and the Jacobi elliptic function

$$\mathrm{dn}(2K(m)z;m) = \frac{\theta_4(0;\tau)}{\theta_3(0;\tau)} \frac{\theta_3(z;\tau)}{\theta_4(z;\tau)}. \tag{4.42}$$

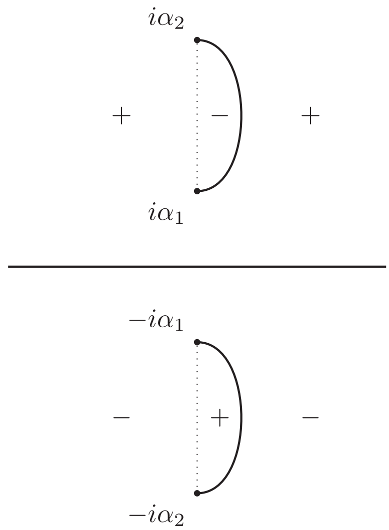

Figure 3. The signs of the function $\mathrm{Im}(\tilde{g}(z,\eta) + \hat{\theta}(z,\eta))$.

Then, using the above relations, one can rewrite the initial datum in equation (4.9) in the form equation (1.6).

Remark 4.6. The initial data described by theorem 4.1 is step-like oscillatory. The long time asymptotic behaviour of the solution of the NLS equation is being considered in [41], and it is inspired by the asymptotic analysis performed for step-like plane wave initial data considered in [42–44].

(g) The stability of the step-like behaviour for finite time

We now focus on the RH problem 4.1 in the case where $t < T$, with $T > 0$ fixed. The asymptotic behaviour of $\psi(x,t)$ as $x \to +\infty$ is still exponentially small in x. To consider the case $x \to -\infty$, we need to do some extra work. We define the parameter $\eta := t/x$ and the function $\hat{\theta}(z,\eta)$ so that

$$\theta(z;x,t) = \mathrm{i}x\hat{\theta}(z,\eta) \quad \text{and} \quad \hat{\theta}(z,\eta) := \eta z^2 + z. \tag{4.43}$$

We aim to show that the behaviour of $\psi(x,t)$ as $x \to -\infty$ and $|\eta| \ll 1$, $\eta < 0$, is the same as the initial datum $\psi_0(x,0)$ up to a phase. This is achieved by performing the steepest descent analysis of the RH problem 4.1 as $x \to -\infty$ where $\theta(z;x,t)$ is replaced by $\mathrm{i}x\hat{\theta}(z,\eta)$ and η is sufficiently small. For the purpose, one needs to consider a deformed g-function that we call $\tilde{g}$ that takes the form

$$\tilde{g}(z,\eta) = \eta(\sqrt{R(z)} - z^2) + g(z), \tag{4.44}$$

where $g(z)$ is defined in equation (4.12). The function $\tilde{g}(z,\eta)$ satisfies the Schwartz symmetry $\tilde{g}(z,\eta) = \tilde{g}^*(z,\eta)$ and the boundary conditions

$$\left.\begin{aligned} &\tilde{g}_+(z) + \tilde{g}_-(z) = -2\hat{\theta}(z,\eta) \quad \text{for } z \in \mathcal{I} \cup \overline{\mathcal{I}} \\ \text{and} \quad &\tilde{g}_+(z) - \tilde{g}_-(z) = \Omega \quad \text{for } z \in [-\mathrm{i}\alpha_1; \mathrm{i}\alpha_1], \end{aligned}\right\} \tag{4.45}$$

where Ω has been defined in equation (4.10). As $z \to \infty$

$$\tilde{g}(z,\eta) = \tilde{g}_\infty(\eta) + \mathcal{O}(z^{-1}) \quad \text{and} \quad \tilde{g}_\infty(\eta) = \frac{\eta}{2}(\alpha_1^2 + \alpha_2^2) + g_\infty,$$

with g_∞ as in equation (4.15). Next we need to study the signs of $\mathrm{Im}(\tilde{g}(z,\eta) + \hat{\theta}(z,\eta))$. The function $\mathrm{Im}(\sqrt{R(z)})$ is positive on the first and third quadrant and negative on the second and fourth quadrant of the complex plane and it equal to zero on the real axis and the imaginary axis except for the segments $\mathcal{I} \cup \overline{\mathcal{I}}$ where it has a discontinuity. Using lemma 4.3, we can easily conclude that there is a $\delta > 0$ sufficiently small so that for all $|\eta| < \delta$, $\eta < 0$ the sign of $\mathrm{Im}(\tilde{g}(z,\eta) + \hat{\theta}(z,\eta))$ takes the form as in figure 3 in a sufficiently large neighbourhood of $\mathcal{I} \cup \overline{\mathcal{I}}$.

It follows that the steepest descent analysis and the opening of the lenses developed in the case $x = 0$ can be applied without changes (except for t-dependence in the exponential phase) for $|\eta| < \delta$. We then arrive to the following proposition.

Proposition 4.7. *There is a $\delta > 0$ sufficiently small so that for any fixed $T > 0$, for all $t \in [0, T)$ and for all $|x| > T/\delta$*

$$\psi_N(x,t) = \begin{cases} \mathrm{e}^{i(\alpha_1^2+\alpha_2^2)t}\psi(x,0) + \mathcal{O}(\mathrm{e}^{c_- x}), & \text{as } x \to -\infty, \\ \mathcal{O}(\mathrm{e}^{-c_+ x}), & \text{as } x \to +\infty, \end{cases} \tag{4.46}$$

where $\psi(x,0)$ has the asymptotic behaviour given in theorem 4.1.

Data accessibility. This article has no additional data.

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All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

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Appendix A. Derivation of equation (3.8)

According to the general IIKS framework, the jump of the RH problem (equation (2.6)) can be written in the form

$$J(z) = \mathbb{I} + 2\pi \mathrm{i} p(z) q^{\mathrm{T}}(z),$$

with

$$p(z) = \begin{pmatrix} \mathbf{1}_{\gamma_-}(z) \\ \mathbf{1}_{\gamma_+}(z) \end{pmatrix} \quad \text{and} \quad q(z) = \begin{pmatrix} -C(z)\mathbf{1}_{\gamma_+}(z) \\ C^\star(z)\mathbf{1}_{\gamma_-}(z) \end{pmatrix}$$

and

$$C(z) := \sum_{j=1}^{N} \frac{c_j \mathrm{e}^{2\theta(z)}}{2\mathrm{i}\pi(z - z_j)} \quad \text{and} \quad C^\star(z) := \sum_{j=1}^{N} \frac{\overline{c_j} \mathrm{e}^{-2\theta(z)}}{2\mathrm{i}\pi(z - \overline{z_j})}.$$

The corresponding integrable operator $\mathcal{T}$ associated with RH problem (equation (2.6)) acts on $L^2(\gamma_+ \cup \gamma_-) \simeq L^2(\gamma_+) \oplus L^2(\gamma_-)$ and is defined by the kernel

$$\hat{\mathcal{T}}(z,w) := \frac{p^{\mathrm{T}}(z) q(w)}{z - w} = -\frac{C(w)\mathbf{1}_{\gamma_+}(w)\mathbf{1}_{\gamma_-}(z) - C^\star(w)\mathbf{1}_{\gamma_-}(w)\mathbf{1}_{\gamma_+}(z)}{z - w}.$$

The corresponding τ-function is

$$\tau = \det(\mathrm{Id}_{L^2(\gamma_+ \cup \gamma_-)} - \mathcal{T}).$$

In the terms of the splitting $L^2(\gamma_+ \cup \gamma_-) \simeq L^2(\gamma_+) \oplus L^2(\gamma_-)$, the Fredholm determinant to compute is

$$\tau = \det \begin{bmatrix} \mathrm{Id}_{L^2(\gamma_+)} & \mathcal{T}_{-+} \\ \mathcal{T}_{+-} & \mathrm{Id}_{L^2(\gamma_-)} \end{bmatrix}, \tag{A 1}$$

where $\mathcal{T}_{-+}: L^2(\gamma_+) \to L^2(\gamma_-)$ and $\mathcal{T}_{+-}: L^2(\gamma_-) \to L^2(\gamma_+)$ are given by

$$\left.\begin{aligned} \mathcal{T}_{-+}[\phi](z) &= \mathbf{1}_{\gamma_-}(z) \oint_{\gamma_+} \frac{\phi(w)C(w)\,\mathrm{d}w}{z-w} \\ \mathcal{T}_{+-}[\psi](z) &= \mathbf{1}_{\gamma_+}(z) \oint_{\gamma_-} \frac{\psi(w)C^\star(w)\,\mathrm{d}w}{w-z}. \end{aligned}\right\} \tag{A 2}$$

By the standard determinantal identities, we have

$$\tau = \det_{L^2(\gamma_+)}[\mathrm{Id} - \mathcal{T}_{+-} \circ \mathcal{T}_{-+}]. \tag{A 3}$$

One can see that the operator is of finite rank. Indeed

$$\mathcal{T}_{+-} \circ \mathcal{T}_{-+}[\phi](z) = \oint_{\gamma_-} \frac{C^\star(s)\,\mathrm{d}s}{s-z} \oint_{\gamma_+} \frac{\phi(w)C(w)\,\mathrm{d}w}{s-w} \tag{A 4}$$

and the s-integration results in a residue evaluation and the poles of $C^\star$ at $\overline{z}_j$'s:

$$\mathcal{T}_{+-} \circ \mathcal{T}_{-+}[\phi](z) = \sum_{j=1}^{N} \frac{\overline{c_j}\mathrm{e}^{-2\theta(\overline{z}_j)}}{\overline{z_j} - z} \oint_{\gamma_+} \frac{\phi(w)C(w)\mathrm{d}w}{\overline{z_j} - w} = \sum_{j=1}^{N} \widetilde{C}(v_j, \phi)v_j(z), \tag{A 5}$$

where $v_k(z) \in L^2(\gamma_+)$, $\widetilde{C}$ is a bilinear form in $L^2(\gamma_+)$ and they are defined as

$$\widetilde{C}(\varphi, \psi) := \oint_{\gamma_+} \varphi(w)C(w)\psi(w)\mathrm{d}w\; \varphi(z), \quad \psi(z) \in L^2(\gamma_+) \tag{A 6}$$

and

$$v_k(z) := \frac{\sqrt{\overline{c}_k}\mathrm{e}^{-\theta(z_k)}}{\overline{z_k} - z} \quad \text{for } k = 1, \ldots, N. \tag{A 7}$$

Therefore, the determinant (A 3) becomes a finite-dimensional Fredholm determinant of the Gram matrix, namely,

$$\begin{aligned} \widetilde{C}(v_j, v_k) &= \oint_{\gamma_+} \frac{\sqrt{\overline{c_j c_k}}\mathrm{e}^{-\theta(\overline{z}_j)-\theta(\overline{z}_k)}}{\overline{z}_j - z} \frac{C(z)}{\overline{z}_k - z}\mathrm{d}z \\ &= \sum_{\ell=1}^{N} \frac{\sqrt{\overline{c_j c_k}}\mathrm{e}^{-\theta(\overline{z}_j)-\theta(\overline{z}_k)}}{\overline{z}_j - z_\ell} \frac{c_\ell \mathrm{e}^{2\theta(z_\ell)}}{\overline{z_k} - z_\ell} = -(\overline{\Phi_N}\Phi_N)_{jk}, \end{aligned} \tag{A 8}$$

where $\Phi_N(x,t)$ is defined in equation (2.10). Then, up to a conjugation, we have rewritten the Fredholm determinant (3.8).

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