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Non-local models in homogenization and phase transitions

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*“Un bel giorno, senza dire niente a nessuno,
me ne andai a Genova e mi imbarcai
su un cargo battente bandiera liberiana...”*

Abstract

In this thesis work, we collect some results about models for homogenization and phase transitions. These classical topics in the Calculus of Variations are investigated via Γ -convergence in different non-local frameworks or with the use of higher-order singular perturbations.

In the first chapter we address the multiscale analysis of periodically oscillating energies of fractional and convolution-type and of general non-local energies on perforated domains, with particular emphasis in determining regimes for which the non-local-to-local passage occurs before the homogenization process. In spite of the similarities, the study of the energies of fractional and convolution-type requires different methods and their asymptotic analysis displays distinct outcomes. The problem of perforated domains is studied at the critical exponent: this instance incorporates several technical difficulties due to scaling invariance and leads to the emergence of a convex *strange term*.

The second chapter is devoted to the study of some diffuse-interface models. First, we investigate equi-coerciveness with respect to the Hausdorff convergence of the graphs in a phase-field model involving a fractional singular perturbation. Then, we turn our attention to the study of higher-order singular perturbations models for phase transitions. We obtain an approximation of the perimeter in the vein of Modica and Mortola replacing the rescaled Dirichlet energy with a singular perturbation of arbitrarily high integer order. We further extend the analysis by accounting for several anisotropic perturbations of intermediate order that contribute negatively to the energy.

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Introduction

This thesis work concerns some results about homogenization in non-local settings and models for phase transitions. The common ground on which these topics are investigated is De Giorgi's Γ -convergence. This tool, introduced in [64, 65], allows to describe the asymptotic behaviour of a family of minimum problems in a unified and flexible framework and it is therefore well suited to rigorously treat the passage from non-local to local models, to describe the macroscopic behavior of inhomogeneous media, or to study problems with discontinuities.

The first chapter of this thesis is devoted to the study of homogenization problems in non-local settings. The finite-differences approximation of continuous models is a prominent branch of modern Calculus of Variations. In the late 1990s, a particular effort has been devoted to the approximation of free-discontinuity problems inspired by Computer Vision, such as the problem of image segmentation in connection with the Mumford-Shah functional. In 1995, a discrete approximation based on the weak membrane model by Blake and Zisserman [24] was obtained by Chambolle [53] (see also [54]) and, two years later, Braides and Dal Maso proposed the introduction of non-local integral functionals depending on the average of the gradient on small balls [33]. In 1998, Gobbino obtained an approximation by means of double integrals where finite-differences are weighted by an interaction kernel that prevents anisotropies [84], proving an earlier conjecture by De Giorgi.

A new impetus to this research line, which continues to thrive, was given in 2001 by a seminal paper by Bourgain, Brezis, and Mironescu [27], who studied the asymptotic behaviour of suitable double integrals in the vein of [84] for the approximation of the Dirichlet p -energy. This result, later extended by Ponce [115, 116], has paved the way to the approximation of many other models involving integral energies on Sobolev spaces such as Ginzburg-Landau energy [7, 125], thin films [16, 40], micromagnetics [62, 83], perforated domains [10, 38, 39], and disconnected sets [41].

Along the same lines, in this thesis we address some problems concerning elliptic homogenization and periodically perforated domains at the critical exponent. A common feature of such problems is the presence of two different interacting scales: one accounting for the non-local-to-local passage and the other accounting for a periodic structure. We treat the problem of elliptic homogenization for two prototypical choices of the interaction kernels; namely, fractional kernels [32] and convolution-type kernels [44]. We discuss how the interplay of the small scales affects the limit energy with particular emphasis in determining a regime for which the passage from non-local to local takes place “before” homogenization, so that the effective energy is the one typically described by the cell-problem formula [34]. Despite the similarities, the analysis of oscillating energies of fractional and convolution-type requires the use of very different strategies and techniques: in the former case we employ a complex discretization argument based on the Kuhn's decomposition of a cube, in the latter we use a direct approach starting from the variational theory developed in [5] for general convolution-type energies. The non-local Dirichlet problem in perforated domains [50] has a different flavor: here, the relevant regimes for the asymptotic analysis are hinted in previous

works [10, 123] and therefore, the relevance of our study lies in the analysis at the critical exponent; that is, the one that makes the problem scaling-invariant. Despite the long story of the problem and the vast number of contributions related to it, the literature for the scaling-invariant version is quite scarce. The resulting multiscale analysis presents several technical issues but leads to an interesting observation: the energy density of the *strange term*, the one accounting for the effect of the perforations in the limit, is always convex.

The framework discussed in [27] is related to the theory of peridynamics, a non-local reformulation of Continuum Mechanics proposed by Silling [124] that has been studied in a variational framework, e.g., in [21, 22, 100], and also shares some similarities with the variational theory of non-local gradients. The latter takes into account classical (local) integral energies in the Calculus of Variations replacing the gradient by a non-local counterpart accounting for long-range interactions such as the fractional gradient, that is obtained through the convolution of the usual gradient with a Riesz kernel, see [120, 121, 128]. Recent works have shed some light on the intimate connection with the classical theory, highlighting some remarkable analogies concerning relaxation and Γ -convergence [60, 92, 93], and have inspired similar adaptations in the framework of H -convergence [45, 51].

The second chapter of this thesis is devoted to the study of some models for phase transitions involving fractional or higher-order singular perturbations. In 1977, Modica and Mortola exhibited “*un esempio di Γ -convergenza*” [lit. “an example of Γ -convergence”] [103] to approximate the total variation in terms of elliptic functionals where a potential with infinitely many zeros (the so-called wells) is perturbed through a regularizing term; that is, a penalization term involving the Dirichlet energy. Such a result has been subsequently adapted by Modica [102] to the case of a double-well potential in order to approximate the perimeter of a sufficiently regular set by means of a “diffuse interface”. This observation confirmed the minimal-interface criterion conjectured by Gurtin [86] for the Van der Waals-Cahn-Hilliard model for phase transitions: in such a situation, a first-order singular perturbation allows one to rule out irregular sets, that correspond to physically unreasonable minimal configurations, so that minimizers of the effective energy (usually subject to volumetric constraints) minimize the area of the separating sharp interface.

Building on these ideas, several phase-field models have been developed to deal with problems with discontinuities. Notable examples are the Ambrosio-Tortorelli model [13] for image segmentation or the Kobayashi-Warren-Carter model for polycrystals [91, 90]. Inspired by the latter and by a recent paper by Giga, Okamoto, and Uesaka [82], we present an equi-coerciveness result (with respect to the convergence of the graphs) concerning a single-well potential energy perturbed by a fractional-order term with fractional parameter close to one [46]. Our analysis is based on standard interpolation inequalities and is robust enough to address also the case of first-order singular perturbations, providing then an alternative proof of the corresponding result in [82]. From a variational perspective, our work is related to [118], where fractional-order singular perturbations (of any fractional parameter less than one) of a double-well potential are investigated (see also [112]) as well as to [126], which extends this analysis to fractional parameters greater than one.

The study of higher-order singular perturbations models naturally arises in the context of Materials Science. A first example is provided by the description of solid-to-solid phase transitions in the theory of strain gradient in plasticity, whose variational analysis has been carried out by Conti, Fonseca, and Leoni [59]. In a previous paper [74], in order to better understand the role of a perturbation of higher-order, Fonseca and Mantegazza investigated a simplified model similar to the one studied by Modica obtained replacing the Dirichlet energy with a second-order term. They proved analogous compactness and Γ -convergence results and

mentioned the necessity of employing possibly different arguments in order to perform the asymptotic analysis for perturbations of integer order $k > 2$. The second main result of this chapter addresses this problem and states that, for any $k > 2$, the limit is a sharp-interface energy that is determined by an optimal-profile problem on the real line [48]. In accordance with the one-dimensional nature of the problem, we carry out a delicate analysis on intervals and we employ a new *local* interpolation technique, adapting an argument due to Solci [127], in order to detect small *transition subintervals* where the interfacial energy is stored. The flexibility of this method is such that we extend the results in [74] for $k = 2$ to a wider class of double-well potentials.

Models involving perturbations of the second order also occur in the description of second-order materials studied, e.g., by Coleman, Marcus, and Mizel [58]. Compared to the one analyzed in [74], the free energy examined here also entails a negative multiple of a first-order term that may favor the formation of periodic patterns of alternated phases. The asymptotic analysis of this model has been independently performed by Chermisi, Dal Maso, Fonseca, and Leoni [55] and by Cicalese, Spadaro, and Zeppieri [56] who proved that, if the negative *stiffness coefficient* satisfies a suitable lower bound, a sharp-interface energy is obtained. The third main result of this chapter extends this analysis to the case of k -th order materials for $k > 2$, addressing the presence of possibly several intermediate-order singular perturbations [49]. To carry out our analysis, a key step is establishing a *global* non-linear interpolation inequality that, up to assuming suitable growth conditions on the double-well potential, provides a priori bounds on the energy. A further feature of this problem is the presence of general tensor norms that generate anisotropies and prevent the use of a one-dimensional argument. This difficulty is overcome by an application of the blow-up method of Fonseca and Müller [75].

It is worth mentioning that, in [6], Alicandro, Braides, and Gelli successfully introduced a second-order singular perturbation for a local variant of the above mentioned non-local approximation of the Mumford-Shah functional [33]. Such an example, recently generalized by Solci for higher-order singular perturbations [127], illustrates how the effect of the non-locality may sometimes be emulated by the presence of singular perturbations of higher-order since the latter account for longer-range interactions. Higher-order singular perturbations models have been recently employed in diverse contexts and with different scopes such as to the study of the Perona-Malik model [20, 35, 85], of phase-field models for free-discontinuity problems [17, 25, 73], and to provide numerical approximation for high-order phase-field models for brittle fracture [109].

Structure of the thesis and summary of contributions

The thesis is made up of two independent chapters. The first one deals with non-local problems in homogenization and perforated domains; the contributions presented therein are as follows:

- *Another look at elliptic homogenization*, in collaboration with Andrea Braides and Davide Donati, published [32];
- *Multiscale homogenization of non-local energies of convolution-type*, submitted [44];
- *Emergence of a convex strange term via homogenization of non-local energies at the critical exponent*, in collaboration with Giuliana Fusco, submitted [50].

In the second chapter we collect the results about models for phase transitions:

- *Graph convergence in a fractional phase-field model*, in collaboration with Francesco Cassandra, in preparation [46];
- *Higher-order singular perturbation models for phase transitions*, in collaboration with Davide Donati and Margherita Solci, published [48];
- *Singular perturbations models in phase transitions for anisotropic higher-order materials*, in collaboration with Davide Donati and Chiara Trifone, published [49].

Each chapter is essentially self-contained and presents the main notation adopted therein as a first section; see Sections 1.1 and 2.1, respectively. In Sections 1.2 and 1.3 we discuss the asymptotic behaviour of periodically oscillating non-local energies of fractional and convolution-type, respectively, and in Section 1.4 we address the problem of perforated domains at the critical exponent in a fairly general non-local setting. Section 2.2 is devoted to the study of equi-coerciveness for a fractional phase-field model. In Section 2.3 we present a result about a sharp-interface model involving a singular perturbation of arbitrarily high order and in Section 2.4 we extend the analysis considering several intermediate-order derivatives that may contribute negatively to the energy.

Further contributions

The following works, carried out during the PhD program, are not presented in this thesis:

- *A note on non-local Sobolev spaces and non-local perimeters*, in collaboration with Konstantinos Bessas, published [23];
- *H-convergence and Γ -convergence in the Riesz fractional setting: the nonlinear case*, in collaboration with Maicol Caponi, Alessandro Carbotti, Alberto Maione, and Fabio Paronetto, submitted [45];
- *Γ -convergence of convolution-type functionals for free discontinuity problems*, in collaboration with Davide Donati, Sergio Scalabrino, Chiara Trifone, and Edoardo Voglino, submitted [47].

Even though some of the results obtained in these works are related to non-local homogenization and non-local problems with discontinuities, we prefer not to include them to keep our exposition as coherent and unified as possible.

Chapter 1

Multiscale homogenization of non-local energies

In their celebrated paper [27], Bourgain, Brezis, and Mironescu studied the asymptotic behaviour of the double integrals

$$\int_{\Omega} \int_{\Omega} \rho_{\varepsilon}(y-x) \left| \frac{u(y) - u(x)}{\varepsilon} \right|^p dx dy, \quad p \in (1, +\infty), \quad (1.0.1)$$

with $\{\rho_{\varepsilon}\}_{\varepsilon}$ a family of radially symmetric non-negative kernels satisfying

$$\frac{1}{\varepsilon^p} \int_{\mathbb{R}^d} \rho_{\varepsilon}(\xi) |\xi|^p d\xi = 1 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^p} \int_{\mathbb{R}^d \setminus B_r} \rho_{\varepsilon}(\xi) |\xi|^p d\xi = 0 \quad (1.0.2)$$

for all $\varepsilon, r > 0$. They proved that functionals (1.0.1) converge to a (multiple of the) Dirichlet p -energy

$$\kappa \int_{\Omega} |\nabla u|^p dx,$$

with domain the usual Sobolev space $W^{1,p}(\Omega)$, where the constant κ is determined by the kernels $\{\rho_{\varepsilon}\}_{\varepsilon}$. Subsequently, their results have been extended by Davila [69] to the case $p = 1$ and by Ponce [115, 116] to more general double integrals, also in the sense of Γ -convergence. Very recently, sharp conditions on the family of kernels $\{\rho_{\varepsilon}\}_{\varepsilon}$ for the validity of such approximation have been detected [63, 78].

Kernels satisfying assumptions (1.0.2) favor short-range interactions in $\Omega \times \Omega$ as $\varepsilon \rightarrow 0$ so that relevant pairwise interactions occur near the diagonal of the domain of integration. This can be regarded as a *concentration* phenomenon that, in the limit, leads to an energy that is described as a single integral on Ω and which is finite on $W^{1,p}(\Omega)$, such as the Dirichlet p -energy. Provided that the functions have a fixed mean value on Ω , these functionals are equi-coercive in $L^p(\Omega)$ (see [115, Theorems 1.1, 1.2]); hence, by the properties of convergence of minima of Γ -convergence, minimum problems involving the Dirichlet p -energy can be approximated by problems involving double integrals, upon suitably defining boundary-value problems if necessary.

This approximation can be extended to other problems involving integral energies on Sobolev spaces in a regular domain $\Omega \subset \mathbb{R}^d$, such as homogenization problems, two of the most classical of which are the asymptotic analysis of (isotropic) oscillating energies

$$\int_{\Omega} a\left(\frac{x}{\delta}\right) |\nabla u|^p dx, \quad (1.0.3)$$

where a is a 1-periodic function, and of the Dirichlet p -energy subject to boundary conditions on a perforated domain

$$\int_{\Omega} |\nabla u|^p dx, \quad u \in W_0^{1,p}(\Omega_{\delta}), \quad (1.0.4)$$

where the set Ω_{δ} (the perforated domain) is obtained removing from Ω tiny perforations arranged on an array of period δ .

In the case of the oscillating energies (1.0.3), the Γ -limit as $\delta \rightarrow 0$ is the *homogenized* functional

$$\int_{\Omega} f_{\text{hom}}(\nabla u) dx, \quad (1.0.5)$$

where the density f_{hom} is characterized by the *homogenization formula*

$$f_{\text{hom}}(z) = \min \left\{ \int_{(0,1)^d} a(x) |z + \nabla v(x)|^p dx : v \text{ is 1-periodic} \right\}; \quad (1.0.6)$$

see, e.g., [34, Theorem 14.7]. By the result of Bourgain, Brezis and Mironescu, we can formally define an approximation of the limit functional (1.0.5) using the energies

$$\int_{\Omega} \int_{\Omega} \rho_{\varepsilon}(y-x) a\left(\frac{x}{\delta}\right) \left| \frac{u(y) - u(x)}{\varepsilon} \right|^p dx dy.$$

Indeed, for fixed $\delta > 0$, the Γ -limit as $\varepsilon \rightarrow 0$ is

$$\kappa \int_{\Omega} a\left(\frac{x}{\delta}\right) |\nabla u|^p dx,$$

which then Γ -converge as $\delta \rightarrow 0$ to a multiple of (1.0.5).

As for the constrained energies (1.0.4), the Γ -limit as $\delta \rightarrow 0$ presents an additional term of order zero, the so-called *strange term*, and equals

$$\int_{\Omega} |\nabla u|^p dx + C_{p,d} \int_{\Omega} |u|^p dx, \quad u \in W_0^{1,p}(\Omega), \quad (1.0.7)$$

for a positive constant $C_{p,d}$ that is expressed in terms of a capacitary problem in accordance with the values of p and d , see [57]. The same heuristic argument can be employed, so that the functionals

$$\int_{\Omega} \int_{\Omega} \rho_{\varepsilon}(y-x) \left| \frac{u(y) - u(x)}{\varepsilon} \right|^p dx dy, \quad \text{if } u = 0 \text{ on } \Omega \setminus \Omega_{\delta}$$

Γ -converge to a multiple of (1.0.7) letting first $\varepsilon \rightarrow 0$ and then $\delta \rightarrow 0$.

Up to incorporating boundary conditions in the case of oscillating energies (such as prescribing the mean value of the functions under consideration), the functionals defined by double integrals are equi-coercive in $L^p(\Omega)$; hence, we can formally use a diagonal argument (see [61] for the metrizable of Γ -convergence) to deduce that there exists a scale $\delta = \delta_{\varepsilon}$ such that $\delta \rightarrow 0$ as $\varepsilon \rightarrow 0$ along which the Γ -limit is the same that is obtained passing to the limit separately in the two parameters, as previously illustrated.

One of the main goals of this chapter is investigating the scales $\delta = \delta_{\varepsilon}$ at which these limits hold, giving rise to a *separation of the scales* δ and ε . As for the oscillating energies, we address this problem for two remarkable classes of interaction kernels: fractional and convolution-type kernels. In the latter case, we are actually able to compute the limit energy in all of the possible regimes of ε, δ . The problem of perforated domain is set in the framework

recently considered in [5], where a general variational theory for non-local energies comparable to (1.0.1) has been developed and which also encompasses convolution-type kernels as a special case. We underscore that the main challenge here does not reside in the identification of the scale δ_ε , but rather in the multiscale analysis of the problem at the critical exponent $p = d$.

This chapter is organized as follows: in Section 1.1 we introduce the main notation. Section 1.2 is devoted to the analysis of oscillating fractional energies and contains the results in [32]. Section 1.3 contains the results presented in [44] about the homogenization of convolution-type energies. Finally, in Section 1.4 we illustrate the results in [50], where we study the problem of perforated domains at the critical exponent in a general non-local framework.

1.1 Notation

We let d, m be two positive integers and we identify $\mathbb{R}^d$ with the ambient space containing the domain Ω , a bounded open set with Lipschitz boundary, and $\mathbb{R}^m$ with the target space of the vector-valued functions we consider. Our results mainly concern functions in $L^p(\Omega; \mathbb{R}^m)$ or in the Sobolev space $W^{1,p}(\Omega; \mathbb{R}^m)$ (see [96]). If $p = 2$ we let $H^s(\Omega; \mathbb{R}^m)$, $s \in (0, 1]$, denote the corresponding (possibly fractional) Sobolev space (see [67, 97]).

For $k \in \{d, m\}$, we let $|\xi|$ denote the Euclidean norm of $\xi \in \mathbb{R}^k$ and set $\mathbb{S}^{k-1} := \{x \in \mathbb{R}^k : |x| = 1\}$. We let $B_r(x)$ denote the d -dimensional Euclidean ball of center x and radius r and set $B_r := B_r(0)$. We use different notation for d -dimensional cubes: we set

$$Q_r := (0, r)^d$$

and we set

$$Q(x, r) := x + \left(-\frac{r}{2}, \frac{r}{2}\right)^d.$$

with the convention that $Q(r) = Q(0, r)$. This last notation will be widely used in Section 1.4 and we point out that a special notation shall be introduced in Section 1.2 to consider cubes with sides that are not parallel to the coordinate axes (see Remark 1.2.4).

For $M \in \mathbb{R}^{m \times d}$ we consider the Frobenius norm

$$|M| := \left(\sum_{i=1}^d \sum_{j=1}^m M_{ij}^2 \right)^{\frac{1}{2}}.$$

We denote the product of a matrix $M \in \mathbb{R}^{m \times d}$ with $x \in \mathbb{R}^d$ either by Mx or $\langle M, x \rangle$ and we denote the product of two matrices $M, M' \in \mathbb{R}^{m \times d}$ by $\langle M, M' \rangle$.

Given $t \in \mathbb{R}$, we let $[t]$ and $\lceil t \rceil$ denote the lower and the upper integer part of t , respectively. Analogously, given $\xi \in \mathbb{R}^d$, we set $\lfloor \xi \rfloor := (\lfloor \xi_1 \rfloor, \dots, \lfloor \xi_d \rfloor) \in \mathbb{Z}^d$ and $\lceil \xi \rceil := (\lceil \xi_1 \rceil, \dots, \lceil \xi_d \rceil) \in \mathbb{Z}^d$.

We let $\mathcal{A}_{\text{reg}}(\Omega)$ denote the family of open subsets of Ω with Lipschitz boundary and let

$$A_\varepsilon(\xi) := \{x \in A : x + \varepsilon \xi \in A\} = A \cap (A - \varepsilon \xi). \quad (1.1.1)$$

for $\varepsilon > 0$ and $\xi \in \mathbb{R}^d$. If $A \subset \mathbb{R}^d$, we set $A^c := \mathbb{R}^d \setminus A$ and $\text{dist}(x, A) := \inf\{|y - x| : y \in A\}$ for all $x \in \mathbb{R}^d$.

We say that a set $E \subseteq \mathbb{R}^d$ is measurable provided that it is measurable with respect to the d -dimensional Lebesgue measure and, in this case, we let $|E|$ denote its measure. We say that a property holds almost everywhere (a.e. for short) provided that this is valid for almost

every point with respect to the Lebesgue measure on $\mathbb{R}^d$. We let $\mathcal{H}^k$ denote the k -dimensional Hausdorff measure on $\mathbb{R}^h$ for some $k \in (0, h]$ and we set $\sigma_{d-1} := \mathcal{H}^{d-1}(\mathbb{S}^{d-1})$.

Let us fix $p \in (1, +\infty)$. In the following, we shall deal with the minimization of integral energies subject to Dirichlet boundary conditions. To treat such a problem in a non-local framework, it is useful to introduce suitable classes of functions. Given $g : \mathbb{R}^d \rightarrow \mathbb{R}^m$ a Lipschitz continuous function, we let

$$\mathcal{D}_{r,g}(A; \mathbb{R}^m) := \{u \in L^p(A; \mathbb{R}^m) : u = g \text{ for a.e. } x \in A, \text{dist}(x, A^c) < r\}, \quad (1.1.2)$$

with A an open subset of $\mathbb{R}^d$, $M \in \mathbb{R}^{m \times d}$, and $r > 0$. If $g(x) = Mx$, we simply set $\mathcal{D}_{r,M}(A; \mathbb{R}^m) = \mathcal{D}_{r,g}(A; \mathbb{R}^m)$. We introduce an analogous class where boundary conditions are imposed close to the boundary of a reference set with respect to the norm

$$|x|_\infty := \max\{|x_i| : i \in \{1, \dots, d\}\};$$

namely, we set

$$\mathcal{D}_{r,g}^\infty(A; \mathbb{R}^m) := \{u \in L^p(A; \mathbb{R}^m) : u = g \text{ for a.e. } x \in A, \text{dist}_\infty(x, A^c) < r\}, \quad (1.1.3)$$

with $\text{dist}_\infty(x, A) := \inf\{|y - x|_\infty : y \in A\}$. We point out that the above classes can be equivalently considered when dealing with the asymptotics of minimum problems that are defined on large reference sets (such as cubes with diverging side-lengths) or on a fixed reference set when r tends to zero.

We introduce further classes of functions subject to periodic boundary conditions; we consider

$$L_{\#,M}^p(Q_1; \mathbb{R}^m) := \{u \in L_{\text{loc}}^p(\mathbb{R}^d; \mathbb{R}^m) : u(x) - Mx \text{ is 1-periodic}\} \quad (1.1.4)$$

and

$$W_{\#,M}^{1,p}(Q_1; \mathbb{R}^m) := \{u \in W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^m) : u(x) - Mx \text{ is 1-periodic}\}, \quad (1.1.5)$$

where we say that a function $u : \mathbb{R}^d \rightarrow \mathbb{R}^m$ is 1-periodic if, letting $\{e_1, \dots, e_d\}$ denote the canonical basis of $\mathbb{R}^d$, it holds

$$u(x + e_i) = u(x)$$

for almost every $x \in \mathbb{R}^d$ and for all $i \in \{1, \dots, d\}$.

Finally, we recall the definition of Γ -convergence and mention the fundamental property on the convergence of minima. For the purpose of this chapter, it suffices to provide the definition of with respect to the L^p -topology. For further details we refer to [29, 34, 61].

Definition 1.1.1 (Γ -convergence). Let $p \in [1, +\infty)$ and let $F_\varepsilon : L^p(\Omega; \mathbb{R}^m) \rightarrow [-\infty, +\infty]$ for every $\varepsilon > 0$ and $F : L^p(\Omega; \mathbb{R}^m) \rightarrow [-\infty, +\infty]$. We say that F is the Γ -limit of $\{F_\varepsilon\}_\varepsilon$ with respect to the convergence in $L^p(\Omega; \mathbb{R}^m)$ as $\varepsilon \rightarrow 0$ if, given any positive sequence $\{\varepsilon_j\}_j$ converging to 0, the following hold:

- (i) for every $u \in L^p(\Omega; \mathbb{R}^m)$ and every $\{u_j\}_j \subset L^p(\Omega; \mathbb{R}^m)$ such that $u_j \rightarrow u$ in $L^p(\Omega; \mathbb{R}^m)$ as $j \rightarrow +\infty$, it holds

$$\liminf_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) \geq F(u) \quad (\liminf \text{ inequality}),$$

- (ii) for every $u \in L^p(\Omega; \mathbb{R}^m)$ there exists $\{v_j\}_j \subset L^p(\Omega; \mathbb{R}^m)$ such that $v_j \rightarrow u$ in $L^p(\Omega; \mathbb{R}^m)$ as $j \rightarrow +\infty$ and

$$\limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(v_j) \leq F(u) \quad (\limsup \text{ inequality}).$$

Theorem 1.1.2 (Convergence of minima). *Let $\{F_\varepsilon\}_\varepsilon$ and F as in Definition 1.1.1 be such that $\{F_\varepsilon\}_\varepsilon$ Γ -converge to F and $\{F_\varepsilon\}_\varepsilon$ is an equi-coercive family on $L^p(\Omega; \mathbb{R}^m)$. Then, given any positive sequence $\{\varepsilon_j\}_j$ converging to 0, it holds that*

$$\exists \min F = \lim_{j \rightarrow +\infty} \inf F_{\varepsilon_j}.$$

Moreover, if $\{u_j\}_j$ is a converging sequence such that $\lim_j F_{\varepsilon_j}(u_j) = \lim_j \inf F_{\varepsilon_j}$, then its limit is a minimizer for F .

1.2 Oscillating energies of fractional-type

We consider quadratic inhomogeneous energies modelled on (1.0.1). In a fractional setting, these reduce to properly rescaled oscillating Gagliardo seminorms of fractional parameter $1 - \varepsilon$ of the form

$$F_{\varepsilon, \delta}^{\text{frac}}(u) := \varepsilon \int_{\Omega} \int_{\Omega} a\left(\frac{x}{\delta}\right) \frac{|u(y) - u(x)|^2}{|y - x|^{d+2(1-\varepsilon)}} dx dy, \quad (1.2.1)$$

where $\Omega \subset \mathbb{R}^d$. Such energies are naturally defined on the fractional Sobolev space $H^{1-\varepsilon}(\Omega; \mathbb{R}^m)$, which is obtained as

$$H^{1-\varepsilon}(\Omega; \mathbb{R}^m) := \{u \in L^2(\Omega; \mathbb{R}^m) : [u]_{H^{1-\varepsilon}(\Omega)} < +\infty\},$$

where the Gagliardo seminorm $[u]_{H^{1-\varepsilon}(\Omega)}$ is given by

$$[u]_{H^{1-\varepsilon}(\Omega)} := \left(\int_{\Omega} \int_{\Omega} \frac{|u(y) - u(x)|^2}{|y - x|^{d+2(1-\varepsilon)}} dx dy \right)^{\frac{1}{2}}.$$

In order to determine a critical scale $\delta_\varepsilon \rightarrow 0$ such that the Γ -limit of $\{F_{\varepsilon, \delta_\varepsilon}^{\text{frac}}\}_\varepsilon$ coincides with the limit obtained letting first $\varepsilon \rightarrow 0$ and then $\delta \rightarrow 0$, we employ the following heuristic argument.

First, we reduce the computation of (1.2.1) only to points x, y sufficiently close to each other. This amounts to detecting a scale r_ε that determines the minimal range of relevant pairwise interactions. If we consider a family of equi-lipschitz continuous functions $\{u_\varepsilon\}_\varepsilon$, there exists C a positive constant independent of ε such that

$$\begin{aligned} \varepsilon \iint_{\Omega \times \Omega \cap \{|y-x| > r_\varepsilon\}} a\left(\frac{x}{\delta_\varepsilon}\right) \frac{|u_\varepsilon(y) - u_\varepsilon(x)|^2}{|y - x|^{d+2(1-\varepsilon)}} dx dy &\leq C\varepsilon \iint_{\Omega \times \Omega \cap \{|y-x| > r_\varepsilon\}} \frac{1}{|x - y|^{d-2\varepsilon}} dx dy \\ &\leq C|\Omega| \sigma_{d-1} \varepsilon \int_{r_\varepsilon}^{\text{diam } \Omega} t^{-1+2\varepsilon} dt \\ &= \frac{C|\Omega| \sigma_{d-1}}{2} [(\text{diam } \Omega)^{2\varepsilon} - r_\varepsilon^{2\varepsilon}], \end{aligned}$$

which tends to 0 as $\varepsilon \rightarrow 0$ provided that

$$r_\varepsilon^{2\varepsilon} \rightarrow 1. \quad (1.2.2)$$

Then, working for simplicity in the scalar case $m = 1$, we let $z \in \mathbb{R}^d$ be fixed and let v be as in the formula characterizing $f_{\text{hom}}(z)$ in (1.0.6). We can use $u_\varepsilon(x) = \langle z, x \rangle + \delta_\varepsilon v(x/\delta_\varepsilon)$ as test functions for the Γ -limit of $F_{\varepsilon, \delta_\varepsilon}$. We may suppose v to be twice continuously differentiable,

so that, using the Taylor development of v at x/δ_ε and the above reduction argument, we have

$$\begin{aligned}
& \varepsilon \iint_{\Omega \times \Omega} a\left(\frac{x}{\delta_\varepsilon}\right) \frac{|u_\varepsilon(y) - u_\varepsilon(x)|^2}{|y - x|^{d+2(1-\varepsilon)}} dx dy \\
& \sim \varepsilon \iint_{\Omega \times \Omega \cap \{|y-x| \leq r_\varepsilon\}} a\left(\frac{x}{\delta_\varepsilon}\right) \frac{|u_\varepsilon(y) - u_\varepsilon(x)|^2}{|y - x|^{d+2(1-\varepsilon)}} dx dy \\
& \sim \varepsilon \iint_{\Omega \times \Omega \cap \{|y-x| \leq r_\varepsilon\}} a\left(\frac{x}{\delta_\varepsilon}\right) \frac{|\langle z + \nabla v(\frac{x}{\delta_\varepsilon}), y - x \rangle|^2 + O(\frac{|y-x|^4}{\delta_\varepsilon^2})}{|y - x|^{d+2(1-\varepsilon)}} dx dy \\
& \sim \frac{\sigma_{d-1}}{2d} \int_{\Omega} a\left(\frac{x}{\delta_\varepsilon}\right) \left|z + \nabla v\left(\frac{x}{\delta_\varepsilon}\right)\right|^2 dx + O\left(\varepsilon \left(\frac{r_\varepsilon}{\delta_\varepsilon}\right)^2\right).
\end{aligned}$$

This argument suggests that when

$$\varepsilon \left(\frac{r_\varepsilon}{\delta_\varepsilon}\right)^2 \rightarrow 0 \quad (1.2.3)$$

we may construct recovery sequences optimizing the oscillations of a , and actually is a proof of the upper bound in this case for the target function $u(x) = \langle z, x \rangle$. The combination of (1.2.2) and (1.2.3) determines the regime

$$\delta_\varepsilon^\varepsilon \rightarrow 1. \quad (1.2.4)$$

We prove that, under assumption (1.2.4), the expected separation of the scales δ and ε occurs. The main argument in the proof of the lower bound is obtained by a discretization procedure (which as a byproduct also gives a different proof of the results in [116]) based on the use of Kuhn's decomposition [95] and an integration argument on the set of orthogonal bases in $\mathbb{R}^d$ borrowed from a recent paper by Solci [125]. Thanks to (1.2.4), we can then reduce the computation to studying the limit of oscillating energies (1.2.1) on piecewise-affine interpolations at a scale much smaller than δ , for which the known homogenization result can be applied.

1.2.1 Statement of the result and preliminaries

Let a be a 1-periodic continuous function such that

$$0 < \alpha \leq a(x) \leq \beta < +\infty \quad (1.2.5)$$

for every $x \in \mathbb{R}^d$. For $\varepsilon > 0$ we let $\delta_\varepsilon > 0$ be a parameter that vanishes as $\varepsilon \rightarrow 0$ and we introduce the functional $F_\varepsilon^{\text{frac}} : L^2(\Omega; \mathbb{R}^m) \rightarrow [0, +\infty]$ defined as

$$F_\varepsilon^{\text{frac}}(u) := \begin{cases} \varepsilon \iint_{\Omega} a\left(\frac{x}{\delta_\varepsilon}\right) \frac{|u(y) - u(x)|^2}{|y - x|^{d+2(1-\varepsilon)}} dx dy & \text{if } u \in H^{1-\varepsilon}(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases} \quad (1.2.6)$$

where $\Omega \subset \mathbb{R}^d$ is a bounded open set with Lipschitz boundary. The main result of this section is the following.

Theorem 1.2.1. *Let $F_\varepsilon^{\text{frac}}$ be defined by (1.2.6) and assume that*

$$\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^\varepsilon = 1.$$

Then, the family $\{F_\varepsilon^{\text{frac}}\}_\varepsilon$ Γ -converges with respect to the convergence in $L^2(\Omega; \mathbb{R}^m)$ to the functional $F_0^{\text{frac}} : L^2(\Omega; \mathbb{R}^m) \rightarrow [0, +\infty]$ defined as

$$F_0^{\text{frac}}(u) := \begin{cases} \frac{\sigma_{d-1}}{2d} \int_{\Omega} \langle A_{\text{hom}} \nabla u, \nabla u \rangle dx & \text{if } u \in H^1(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases} \quad (1.2.7)$$

where A_{hom} is a quadratic form on $\mathbb{R}^{m \times d}$ given by

$$\langle A_{\text{hom}} M, M \rangle = \min \left\{ \int_{Q_1} a(x) |\nabla u(x)|^2 dx : u \in W_{\#, M}^{1,2}(Q_1; \mathbb{R}^m) \right\}, \quad (1.2.8)$$

with $W_{\#, M}^{1,2}(Q_1; \mathbb{R}^m)$ as in (1.1.5).

Remark 1.2.2. In the original paper [32] the above result is proved under the stronger assumption

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\delta_\varepsilon^2} = 0.$$

Here, the proof for the regime $\delta_\varepsilon^\varepsilon \rightarrow 1$ is obtained with some minor adaptations, mainly needed in the upper bound. The regimes that differ from (1.2.4) are not dealt with in this thesis and are the object of a work in preparation by Braides, Mannella, and Schiavoni Piazza on which we shall comment in the forthcoming section.

We state a lemma that allows us to take into account only those interactions due to pairs of sufficiently close points in Ω .

Lemma 1.2.3. Assume that $\delta_\varepsilon^\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$ and let $\{u_\varepsilon\}_\varepsilon \subset L^2(\Omega; \mathbb{R}^m)$ be a family of equi-lipschitz continuous functions. Then

$$\lim_{\varepsilon \rightarrow 0} \left(F_\varepsilon^{\text{frac}}(u_\varepsilon) - \varepsilon \iint_{\Omega \times \Omega \cap \{(x,y): |x-y| \leq \delta_\varepsilon^q\}} a\left(\frac{x}{\delta_\varepsilon}\right) \frac{|u_\varepsilon(y) - u_\varepsilon(x)|^2}{|y-x|^{d+2(1-\varepsilon)}} dx dy \right) = 0$$

for every $q \in \mathbb{R}$.

Proof. Using (1.2.5) and the fact that $\{u_\varepsilon\}_\varepsilon$ are equi-lipschitz continuous functions, there exists a positive constant C independent of ε such that

$$\begin{aligned} & \varepsilon \iint_{\Omega \times \Omega \cap \{(x,y): |y-x| > \delta_\varepsilon^q\}} a\left(\frac{x}{\delta_\varepsilon}\right) \frac{|u_\varepsilon(y) - u_\varepsilon(x)|^2}{|y-x|^{d+2(1-\varepsilon)}} dx dy \\ & \leq C\varepsilon \iint_{\Omega \times \Omega \cap \{(x,y): |y-x| > \delta_\varepsilon^q\}} \frac{1}{|y-x|^{d-2\varepsilon}} dx dy \\ & \leq C|\Omega| \sigma_{d-1} \varepsilon \int_{\delta_\varepsilon^q}^{\text{diam } \Omega} t^{-1+2\varepsilon} dt = \frac{C|\Omega| \sigma_{d-1}}{2} ((\text{diam } \Omega)^{2\varepsilon} - \delta_\varepsilon^{2q\varepsilon}). \end{aligned}$$

The thesis follows letting $\varepsilon \rightarrow 0$ as $\delta_\varepsilon^\varepsilon \rightarrow 1$ by assumption. $\square$

In the proof of the liminf inequality we take advantage of a discretization argument based on the construction of proper lattices, which can be parameterized on orthogonal bases.

Remark 1.2.4 (Notation for the set of orthogonal bases). Following the notation of [125] we define the set of orthonormal bases of $\mathbb{R}^d$

$$V := \{\bar{\nu} = (\nu_1, \dots, \nu_d) : \nu_j \in \mathbb{S}^{d-1} \text{ such that } \langle \nu_i, \nu_j \rangle = 0 \text{ for } i \neq j\}$$

and observe that V has Hausdorff dimension equal to $k_d := d(d-1)/2$. For every $n \in \{1, \dots, d\}$ and fixed $\nu \in \mathbb{S}^{d-1}$ we define the set

$$V_n^\nu := \{\bar{\nu} \in V \text{ such that } \nu_n = \nu\},$$

whose Hausdorff dimension is $k_d - (d-1)$. Note that we have

$$\mathcal{H}^{k_d-(d-1)}(V_n^\nu) = \frac{\mathcal{H}^{k_d}(V)}{\sigma_{d-1}}, \quad (1.2.9)$$

and that the formula

$$\int_V f(\bar{\nu}) d\mathcal{H}^{k_d}(\bar{\nu}) = \int_{\mathbb{S}^{d-1}} \int_{V_n^\nu} f^\nu(\bar{\nu}_n) d\mathcal{H}^{k_d-(d-1)}(\bar{\nu}_n) d\mathcal{H}^{d-1}(\nu) \quad (1.2.10)$$

holds with $\bar{\nu}_n := (\nu_1, \dots, \nu_{n-1}, \nu_{n+1}, \dots, \nu_d)$ and $f^\nu(\bar{\nu}_n) := f(\bar{\nu})$ for every $n \in \{1, \dots, d\}$ and f non negative measurable function.

Given $t > 0$ and $\bar{\nu} \in V$, we define the lattice

$$\mathbb{Z}_{t\bar{\nu}}^d := \{t\nu_1 z_1 + t\nu_2 z_2 + \dots + t\nu_d z_d : (z_1, \dots, z_d) \in \mathbb{Z}^d\}$$

and let $Q_{t\bar{\nu}}$ denote the cube described by the orthogonal basis $\{t\nu_1, \dots, t\nu_d\}$.

Remark 1.2.5 (Kuhn's decomposition). A cube $Q_{t\bar{\nu}}$ defined as in the previous remark can be further decomposed into $d!$ d -simplices of measure $t^d/d!$ through *Kuhn's decomposition* in the following way: for every τ permutation of d indices, we define $\Delta_{t\bar{\nu}}^\tau$ as the simplex described by the vertices

$$t\nu_{\tau(1)}, t\nu_{\tau(1)} + t\nu_{\tau(2)}, \dots, t\nu_{\tau(1)} + t\nu_{\tau(2)} + \dots + t\nu_{\tau(d)}.$$

As shown in [95, Lemma 1], as τ vary among all the permutations, we get a family of $d!$ simplices which constitutes the desired partition. We denote the vertices of each simplex by

$$\Delta_{t\bar{\nu}}^{\tau,0} = 0, \quad \Delta_{t\bar{\nu}}^{\tau,\ell} := t\nu_{\tau(1)} + t\nu_{\tau(2)} + \dots + t\nu_{\tau(\ell)} \quad \text{for } \ell \in \{1, \dots, d\}$$

and observe that, with this choice, it holds that

$$\Delta_{t\bar{\nu}}^{\tau,\ell} - \Delta_{t\bar{\nu}}^{\tau,\ell-1} = t\nu_{\tau(\ell)} \quad \text{for } \ell \in \{1, \dots, d\} \quad (1.2.11)$$

and that

$$\{\Delta_{t\bar{\nu}}^{\tau,\ell} - \Delta_{t\bar{\nu}}^{\tau,\ell-1} : \ell \in \{1, \dots, d\}\} = \{t\nu_1, \dots, t\nu_d\}. \quad (1.2.12)$$

1.2.2 Lower bound

Throughout this section we fix positive sequences $\{\varepsilon_j\}_j, \{\delta_j := \delta_{\varepsilon_j}\}_j$ tending to 0 as $j \rightarrow +\infty$ and such that

$$\lim_{j \rightarrow +\infty} \delta_j^{\varepsilon_j} = 1. \quad (1.2.13)$$

Let $u \in L^2(\Omega; \mathbb{R}^m)$ and consider a sequence $\{u_j\}_j$ converging to u in $L^2(\Omega; \mathbb{R}^m)$. Without loss of generality, we may assume that $\sup_j F_{\varepsilon_j}(u_j) < +\infty$, which implies that $u_j \in H^{1-\varepsilon_j}(\Omega; \mathbb{R}^m)$ for every $j \in \mathbb{N}$.

We aim at proving that $u \in H^1(\Omega'; \mathbb{R}^m)$ for every Ω' open subset well contained in Ω and that

$$\liminf_{j \rightarrow +\infty} \varepsilon_j \int_{\Omega} \int_{\Omega} a\left(\frac{x}{\delta_j}\right) \frac{|u_j(y) - u_j(x)|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy \geq \frac{\sigma_{d-1}}{2d} \int_{\Omega'} \langle A_{\text{hom}} \nabla u, \nabla u \rangle dx.$$

As the left-hand side is independent of Ω' , in particular this implies that $u \in H^1(\Omega; \mathbb{R}^m)$.

Lemma 1.2.3 states that interactions of points at distance larger than $\delta_j^q, q > 1$, are negligible. We set

$$r_j := \delta_j^q \quad \text{for some } q > 1, \quad (1.2.14)$$

and, recalling the notation (1.1.1), we apply the change of variables $\xi := (y - x)/r_j$ and the coarea formula to get

$$\begin{aligned} F_{\varepsilon_j}(u_j) &\geq \varepsilon_j \iint_{\Omega \times \Omega \cap \{(x,y): |x-y| \leq r_j\}} a\left(\frac{x}{\delta_j}\right) \frac{|u_j(y) - u_j(x)|^2}{|y-x|^{d+2(1-\varepsilon_j)}} dx dy \\ &= \varepsilon_j \int_{B_1} \frac{r_j^{2\varepsilon_j}}{|\xi|^{d-2\varepsilon_j}} \int_{\Omega_{r_j}(\xi)} a\left(\frac{x}{\delta_j}\right) \frac{|u_j(x+r_j\xi) - u_j(x)|^2}{|r_j\xi|^2} dx d\xi \\ &= \varepsilon_j \int_0^1 t^{d-1} \int_{\mathbb{S}^{d-1}} \frac{r_j^{2\varepsilon_j}}{t^{d-2\varepsilon_j}} \\ &\quad \int_{\Omega_{r_j t}(\nu)} a\left(\frac{x}{\delta_j}\right) \frac{|u_j(x+r_j t\nu) - u_j(x)|^2}{|r_j t|^2} dx d\mathcal{H}^{d-1}(\nu) dt. \end{aligned}$$

Taking into account (1.2.9), we rewrite the above term as

$$\begin{aligned} &\varepsilon_j \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \int_{\mathbb{S}^{d-1}} \frac{1}{d} \sum_{n=1}^d \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \\ &\quad \int_{V_n^\nu} \int_{\Omega_{r_j t}(\nu)} a\left(\frac{x}{\delta_j}\right) \frac{|u_j(x+r_j t\nu) - u_j(x)|^2}{|r_j t|^2} dx d\mathcal{H}^{k_d-(d-1)}(\bar{\nu}_n) d\mathcal{H}^{d-1}(\nu) dt, \end{aligned}$$

then we apply formula (1.2.10) to get

$$\begin{aligned} &\frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \\ &\quad \sum_{n=1}^d \int_{\Omega_{r_j t}(\nu_n)} a\left(\frac{x}{\delta_j}\right) \frac{|u_j(x+r_j t\nu_n) - u_j(x)|^2}{|r_j t|^2} dx d\mathcal{H}^{k_d}(\bar{\nu}) dt, \end{aligned} \quad (1.2.15)$$

ν_n being the n -th element of the orthonormal basis $\bar{\nu}$.

For $\bar{\nu} \in V$ (recall the notation of Remark 1.2.4) we set

$$\mathcal{I}_{t\bar{\nu}}^{r_j} := \{k \in \mathbb{Z}_{t\bar{\nu}}^d : r_j k + r_j Q_{t\bar{\nu}} \subset \subset \Omega^{r_j t}\},$$

where, given $\eta > 0$, we set

$$\Omega^\eta := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \eta\}.$$

Note that for every $\bar{\nu} \in V$ and for every $n \in \{1, \dots, d\}$ it holds that

$$\bigcup_{k \in \mathcal{I}_{t\bar{\nu}}^{r_j}} r_j k + r_j Q_{t\bar{\nu}} \subseteq \Omega^{r_j t} \subseteq \Omega_{r_j t}(\nu_n);$$

hence, (1.2.15) is bounded from below by

$$\begin{aligned} &\frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \\ &\quad \sum_{n=1}^d \int_{\bigcup_{k \in \mathcal{I}_{t\bar{\nu}}^{r_j}} (r_j k + r_j Q_{t\bar{\nu}})} a\left(\frac{x}{\delta_j}\right) \frac{|u_j(x+r_j t\nu_n) - u_j(x)|^2}{|r_j t|^2} dx d\mathcal{H}^{k_d}(\bar{\nu}) dt. \end{aligned} \quad (1.2.16)$$

We now exploit the uniform continuity of the function a to factor it out from the inner integral; for this reason, we introduce a modulus of continuity $\omega : [0, +\infty) \rightarrow [0, +\infty)$; that is, an increasing continuous function such that $\omega(0) = 0$ and $|a(x_1) - a(x_2)| \leq \omega(|x_1 - x_2|)$ for every $x_1, x_2 \in \mathbb{R}^d$.

We rewrite the right-hand side of (1.2.16) as

$$\begin{aligned} & \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \\ & \quad \int_V \sum_{n=1}^d \sum_{k \in \mathcal{I}_{t\bar{\nu}}^{r_j}} a\left(\frac{r_j k}{\delta_j}\right) \int_{r_j k + r_j Q_{t\bar{\nu}}} \frac{|u_j(x + r_j t \nu_n) - u_j(x)|^2}{|r_j t|^2} dx d\mathcal{H}^{k_d}(\bar{\nu}) dt \\ & + \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \\ & \quad \int_V \sum_{n=1}^d \sum_{k \in \mathcal{I}_{t\bar{\nu}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{\nu}}} \left(a\left(\frac{x}{\delta_j}\right) - a\left(\frac{r_j k}{\delta_j}\right)\right) \frac{|u_j(x + r_j t \nu_n) - u_j(x)|^2}{|r_j t|^2} dx d\mathcal{H}^{k_d}(\bar{\nu}) dt \end{aligned}$$

and we note that the last term is negligible as $j \rightarrow +\infty$ since, using (1.2.5), its absolute value is estimated from above by

$$\begin{aligned} & \omega\left(\frac{r_j \sqrt{d}}{\delta_j}\right) \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \\ & \quad \int_V \sum_{n=1}^d \int_{\Omega_{r_j t}(\nu_n)} \frac{|u_j(x + r_j t \nu_n) - u_j(x)|^2}{|r_j t|^2} dx d\mathcal{H}^{k_d}(\bar{\nu}) dt \\ & \leq \omega\left(\frac{r_j \sqrt{d}}{\delta_j}\right) \frac{1}{\alpha} F_j(u_j), \end{aligned}$$

which tends to 0 since $\sup_j F_j(u_j) < +\infty$ and since $r_j/\delta_j = \delta_j^{q-1} \rightarrow 0$ by (1.2.14).

We obtain the inequality

$$\begin{aligned} F_{\varepsilon_j}(u_j) & \geq \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \\ & \quad \int_V \sum_{n=1}^d \sum_{k \in \mathcal{I}_{t\bar{\nu}}^{r_j}} a\left(\frac{r_j k}{\delta_j}\right) \int_{r_j k + r_j Q_{t\bar{\nu}}} \frac{|u_j(x + r_j t \nu_n) - u_j(x)|^2}{|r_j t|^2} dx d\mathcal{H}^{k_d}(\bar{\nu}) dt. \end{aligned} \quad (1.2.17)$$

To produce a lower bound for (1.2.17), we take advantage of a discretization argument based on piecewise-affine auxiliary functions.

Given ε_j sufficiently small, $t \in (0, 1)$ and $\bar{\nu} \in V$ we define the function $u_j^{t\bar{\nu}}$ in two steps:

- first, we assign values on the lattice $\mathcal{I}_{t\bar{\nu}}^{r_j}$ setting

$$u_j^{t\bar{\nu}}(r_j k) = \frac{1}{|r_j t|^d} \int_{r_j k + r_j Q_{t\bar{\nu}}} u_j dx \quad \text{for every } k \in \mathcal{I}_{t\bar{\nu}}^{r_j};$$

- then, given k in the subset of indexes defined as

$$\tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j} := \{k \in \mathbb{Z}_{t\bar{\nu}}^d : r_j k + 2r_j Q_{t\bar{\nu}} \subset \subset \Omega^{r_j t}\},$$

consider the cube $r_j k + r_j Q_{t\bar{\nu}}$, τ a permutation of the indices $\{1, \dots, d\}$, and $r_j k + r_j \Delta_{t\bar{\nu}}^\tau$ the corresponding simplex in Kuhn's decomposition; on this simplex we define $u_j^{t\bar{\nu}}$ being the affine interpolation of the previously defined values

$$u_j^{t\bar{\nu}}(r_j k) = u_j^{t\bar{\nu}}(r_j k + r_j \Delta_{t\bar{\nu}}^{\tau,0}), u_j^{t\bar{\nu}}(r_j k + r_j \Delta_{t\bar{\nu}}^{\tau,1}), \dots, u_j^{t\bar{\nu}}(r_j k + r_j \Delta_{t\bar{\nu}}^{\tau,d}).$$

We remark that, in the second step, our definition is well posed since $k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}$ ensures that $r_j k + r_j \Delta_{t\bar{\nu}}^{\tau,\ell}$ belongs to $\mathcal{I}_{t\bar{\nu}}^{r_j}$ for every τ permutation of d indexes and $\ell \in \{0, \dots, d\}$; that is, on the simplex denoted by τ , to each vertex has already been assigned a value in the first step of the construction. For the same reason, $u_j^{t\bar{\nu}}$ cannot be defined in this way on the cubes $r_j k + r_j Q_{t\bar{\nu}}$ for which $k \notin \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}$. This means that $u_j^{t\bar{\nu}}$ is not defined on each cell of the above form contained in Ω , but these exceptions only occur near the boundary of Ω , and for our purpose this issue is overcome simply extending to 0 the function $u_j^{t\bar{\nu}}$.

By construction, we have that $u_j^{t\bar{\nu}}$ is well defined as a continuous piecewise-affine function on

$$\bigcup_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} r_j k + r_j Q_{t\bar{\nu}};$$

hence, on each simplex $r_j k + r_j \Delta_{t\bar{\nu}}^\tau, k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}$, its gradient is constant and by (1.2.11) and (1.2.12) it holds

$$\begin{aligned} & \int_{r_j k + r_j \Delta_{t\bar{\nu}}^\tau} |\nabla u_j^{t\bar{\nu}}|^2 dx \\ &= \sum_{\ell=1}^d \int_{r_j k + r_j \Delta_{t\bar{\nu}}^\tau} \frac{|u_j^{t\bar{\nu}}(r_j k + r_j \Delta_{t\bar{\nu}}^{\tau,\ell}) - u_j^{t\bar{\nu}}(r_j k + r_j \Delta_{t\bar{\nu}}^{\tau,\ell-1})|^2}{|r_j t|^2} dx \\ &= \frac{1}{|r_j t|^2} \sum_{\ell=1}^d \int_{r_j k + r_j \Delta_{t\bar{\nu}}^\tau} \left| \frac{1}{|r_j t|^d} \left(\int_{r_j k + r_j \Delta_{t\bar{\nu}}^{\tau,\ell} + r_j Q_{t\bar{\nu}}} u_j dy - \int_{r_j k + r_j \Delta_{t\bar{\nu}}^{\tau,\ell-1} + r_j Q_{t\bar{\nu}}} u_j dy \right) \right|^2 dx \\ &= \frac{1}{|r_j t|^2} \sum_{\ell=1}^d \int_{r_j k + r_j \Delta_{t\bar{\nu}}^\tau} \left| \frac{1}{|r_j t|^d} \left(\int_{r_j k + r_j \Delta_{t\bar{\nu}}^{\tau,\ell-1} + r_j Q_{t\bar{\nu}}} u_j(y + r_j t \nu_{\tau(\ell)}) - u_j(y) dy \right) \right|^2 dx \\ &\leq \frac{1}{|r_j t|^2} \sum_{\ell=1}^d \int_{r_j k + r_j \Delta_{t\bar{\nu}}^\tau} \frac{1}{|r_j t|^d} \int_{r_j k + r_j \Delta_{t\bar{\nu}}^{\tau,\ell-1} + r_j Q_{t\bar{\nu}}} |u_j(y + r_j t \nu_{\tau(\ell)}) - u_j(y)|^2 dy dx \\ &= \frac{1}{|r_j t|^d} \sum_{\ell=1}^d \int_{r_j k + r_j \Delta_{t\bar{\nu}}^\tau} \int_{r_j k + r_j \Delta_{t\bar{\nu}}^{\tau,\ell-1} + r_j Q_{t\bar{\nu}}} \frac{|u_j(y + r_j t \nu_{\tau(\ell)}) - u_j(y)|^2}{|r_j t|^2} dy dx \\ &= \frac{1}{d!} \sum_{\ell=1}^d \int_{r_j k + r_j \Delta_{t\bar{\nu}}^{\tau,\ell-1} + r_j Q_{t\bar{\nu}}} \frac{|u_j(y + r_j t \nu_{\tau(\ell)}) - u_j(y)|^2}{|r_j t|^2} dy, \end{aligned}$$

where we used Jensen's inequality and the fact that $|r_j k + r_j \Delta_{t\bar{\nu}}^\tau| = |r_j t|^d/d!$.

Summing over the permutations τ and $k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}$ and reinstating the coefficient a , we get

$$\begin{aligned}
& \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} a\left(\frac{r_j k}{\delta_j}\right) \int_{r_j k + r_j Q_{t\bar{\nu}}} |\nabla u_j^{t\bar{\nu}}|^2 dx \\
&= \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} a\left(\frac{r_j k}{\delta_j}\right) \sum_{\tau} \int_{r_j k + r_j \Delta_{t\bar{\nu}}^{\tau}} |\nabla u_j^{t\bar{\nu}}|^2 dx \\
&\leq \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} a\left(\frac{r_j k}{\delta_j}\right) \sum_{\tau} \frac{1}{d!} \sum_{\ell=1}^d \int_{r_j k + r_j \Delta_{t\bar{\nu}}^{\tau, \ell-1} + r_j Q_{t\bar{\nu}}} \frac{|u_j(y + r_j t \nu_{\tau(\ell)}) - u_j(y)|^2}{|r_j t|^2} dy \\
&= \frac{1}{d!} \sum_{n=1}^d \sum_{\{\tau, \ell: \tau(\ell)=n\}} \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} a\left(\frac{r_j k}{\delta_j}\right) \int_{r_j k + r_j \Delta_{t\bar{\nu}}^{\tau, \ell-1} + r_j Q_{t\bar{\nu}}} \frac{|u_j(y + r_j t \nu_n) - u_j(y)|^2}{|r_j t|^2} dy. \quad (1.2.18)
\end{aligned}$$

Keeping n, ℓ, τ fixed, we put $h := k + \Delta_{t\bar{\nu}}^{\tau, \ell-1}$ and note that $k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}$ implies $h \in \mathcal{I}_{t\bar{\nu}}^{r_j}$, so that (1.2.18) is less than or equal to

$$\begin{aligned}
& \frac{1}{d!} \sum_{n=1}^d \sum_{\{\tau, \ell: \tau(\ell)=n\}} \sum_{h \in \mathcal{I}_{t\bar{\nu}}^{r_j}} a\left(\frac{r_j h}{\delta_j}\right) \int_{r_j h + r_j Q_{t\bar{\nu}}} \frac{|u_j(y + r_j t \nu_n) - u_j(y)|^2}{|r_j t|^2} dy + E(\bar{\nu}) \\
&= \sum_{n=1}^d \sum_{h \in \mathcal{I}_{t\bar{\nu}}^{r_j}} a\left(\frac{r_j h}{\delta_j}\right) \int_{r_j h + r_j Q_{t\bar{\nu}}} \frac{|u_j(y + r_j t \nu_n) - u_j(y)|^2}{|r_j t|^2} dy + E(\bar{\nu}),
\end{aligned}$$

where we used that the cardinality of the set $\{\tau, \ell : \tau(\ell) = n\}$ is $d!$ for every $n = 1, \dots, d$, and we set

$$E(\bar{\nu}) := \sum_{n=1}^d \sum_{h \in \mathcal{I}_{t\bar{\nu}}^{r_j}} \left(a\left(\frac{r_j h - r_j \Delta_{t\bar{\nu}}^{\tau, \ell-1}}{\delta_j}\right) - a\left(\frac{r_j h}{\delta_j}\right) \right) \int_{r_j h + r_j Q_{t\bar{\nu}}} \frac{|u_j(y + r_j t \nu_n) - u_j(y)|^2}{|r_j t|^2} dy.$$

Summarizing, we have proved that for every $j, t, \bar{\nu}$, it holds

$$\begin{aligned}
& \sum_{n=1}^d \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} a\left(\frac{r_j k}{\delta_j}\right) \int_{r_j k + r_j Q_{t\bar{\nu}}} \frac{|u_j(y + r_j t \nu_n) - u_j(y)|^2}{|r_j t|^2} dy \\
&\geq \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} a\left(\frac{r_j k}{\delta_j}\right) \int_{r_j k + r_j Q_{t\bar{\nu}}} |\nabla u_j^{t\bar{\nu}}|^2 dx - E(\bar{\nu}), \quad (1.2.19)
\end{aligned}$$

and therefore, having achieved an estimate in terms of the inner integral in (1.2.17), we get

$$\begin{aligned}
F_{\varepsilon_j}(u_j) &\geq \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \\
&\quad \int_V \sum_{n=1}^d \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} a\left(\frac{r_j k}{\delta_j}\right) \int_{r_j k + r_j Q_{t\bar{\nu}}} \frac{|u_j(x + r_j t \nu_n) - u_j(x)|^2}{|r_j t|^2} dx d\mathcal{H}^{k_d}(\bar{\nu}) dt \quad (1.2.20)
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \int_V \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} a\left(\frac{r_j k}{\delta_j}\right) \int_{r_j k + r_j Q_{t\bar{\nu}}} |\nabla u_j^{t\bar{\nu}}|^2 dx d\mathcal{H}^{k_d}(\bar{\nu}) dt \quad (1.2.21)
\end{aligned}$$

$$- \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \int_V E(\bar{\nu}) d\mathcal{H}^{k_d}(\bar{\nu}) dt. \quad (1.2.22)$$

Note that (1.2.22) vanishes as $j \rightarrow +\infty$; indeed, considering the modulus of continuity, the bound from above in (1.2.5) and the bound from above involving (1.2.20), we have

$$\begin{aligned}
& \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \int_V E(\bar{\nu}) d\mathcal{H}^{k_d}(\bar{\nu}) dt \\
& \leq \omega\left(\frac{r_j\sqrt{d}}{\delta_j}\right) \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \\
& \quad \int_V \sum_{n=1}^d \sum_{h \in \mathcal{I}_{t\bar{\nu}}^{r_j}} \int_{r_j h + r_j Q_{t\bar{\nu}}} \frac{|u_j(y + r_j t \nu_n) - u_j(y)|^2}{|r_j t|^2} dy d\mathcal{H}^{k_d}(\bar{\nu}) dt \\
& \leq \omega\left(\frac{r_j\sqrt{d}}{\delta_j}\right) \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \frac{1}{\alpha} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \\
& \quad \int_V \sum_{n=1}^d \sum_{h \in \mathcal{I}_{t\bar{\nu}}^{r_j}} a\left(\frac{r_j h}{\delta_j}\right) \int_{r_j h + r_j Q_{t\bar{\nu}}} \frac{|u_j(y + r_j t \nu_n) - u_j(y)|^2}{|r_j t|^2} dy d\mathcal{H}^{k_d}(\bar{\nu}) dt \\
& \leq \omega\left(\frac{r_j\sqrt{d}}{\delta_j}\right) \frac{1}{\alpha} F_{\varepsilon_j}(u_j),
\end{aligned}$$

which tends to 0 since $\sup_j F_{\varepsilon_j}(u_j) < +\infty$ and since $r_j/\delta_j \rightarrow 0$ by (1.2.14).

As for (1.2.21), we can reinsert the coefficient a in the integral; in particular, taking into account the modulus of continuity and using (1.2.5), (1.2.19), the inequality leading to (1.2.20), and the fact that (1.2.22) vanishes, we have

$$\begin{aligned}
& \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \left| \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \int_V \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{\nu}}} \left(a\left(\frac{r_j k}{\delta_j}\right) - a\left(\frac{x}{\delta_j}\right) \right) |\nabla u_j^{t\bar{\nu}}|^2 dx d\mathcal{H}^{k_d}(\bar{\nu}) dt \right| \\
& \leq \omega\left(\frac{r_j\sqrt{d}}{\delta_j}\right) \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \int_V \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{\nu}}} |\nabla u_j^{t\bar{\nu}}|^2 dx d\mathcal{H}^{k_d}(\bar{\nu}) dt \\
& \leq \omega\left(\frac{r_j\sqrt{d}}{\delta_j}\right) \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \frac{1}{\alpha} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \\
& \quad \int_V \sum_{n=1}^d \sum_{k \in \mathcal{I}_{t\bar{\nu}}^{r_j}} a\left(\frac{r_j k}{\delta_j}\right) \int_{r_j k + r_j Q_{t\bar{\nu}}} \frac{|u(y + r_j t \nu_n) - u(y)|^2}{|r_j t|^2} dy + E(\bar{\nu}) d\mathcal{H}^{k_d}(\bar{\nu}) dt + o_j(1) \\
& \leq \omega\left(\frac{r_j\sqrt{d}}{\delta_j}\right) \frac{1}{\alpha} F_{\varepsilon_j}(u_j) + o_j(1),
\end{aligned}$$

and we conclude as above.

We finally obtain

$$\begin{aligned}
F_{\varepsilon_j}(u_j) & \geq \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \int_V \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{\nu}}} a\left(\frac{x}{\delta_j}\right) |\nabla u_j^{t\bar{\nu}}|^2 dx d\mathcal{H}^{k_d}(\bar{\nu}) dt + o_j(1) \\
& \geq \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \int_V \int_{\Omega'} a\left(\frac{x}{\delta_j}\right) |\nabla u_j^{t\bar{\nu}}|^2 dx d\mathcal{H}^{k_d}(\bar{\nu}) dt + o_j(1), \tag{1.2.23}
\end{aligned}$$

where Ω' is any open set well contained in Ω that we assume satisfies

$$\Omega' \subset \bigcup_{k \in \tilde{\mathcal{I}}_{t\bar{v}}^j} r_j k + r_j Q_{t\bar{v}}$$

for ε_j sufficiently small.

In the final part of the proof, we average the auxiliary functions obtained by discretization weighting them with the appropriate kernel.

For every ε_j sufficiently small we define the probability measure μ_{ε_j} on $[0, 1] \times V$ by

$$d\mu_{\varepsilon_j}(t, \bar{v}) := \frac{2\varepsilon_j}{\mathcal{H}^{k_d}(V)} t^{2\varepsilon_j-1} \mathcal{L}^1 \otimes \mathcal{H}^{k_d};$$

then we estimate the leading term in (1.2.23) as

$$\begin{aligned} & \frac{\sigma_{d-1}}{2d} \int_{[0,1] \times V} r_j^{2\varepsilon_j} \int_{\Omega'} a\left(\frac{x}{\delta_j}\right) |\nabla u_j^{t\bar{v}}|^2 dx d\mu_{\varepsilon_j}(t, \bar{v}) \\ &= \frac{\sigma_{d-1}}{2d} \int_{\Omega'} r_j^{2\varepsilon_j} a\left(\frac{x}{\delta_j}\right) \int_{[0,1] \times V} |\nabla u_j^{t\bar{v}}|^2 d\mu_{\varepsilon_j}(t, \bar{v}) dx \\ &\geq \frac{\sigma_{d-1}}{2d} \int_{\Omega'} r_j^{2\varepsilon_j} a\left(\frac{x}{\delta_j}\right) \left| \int_{[0,1] \times V} \nabla u_j^{t\bar{v}} d\mu_{\varepsilon_j}(t, \bar{v}) \right|^2 dx \end{aligned} \quad (1.2.24)$$

by Jensen's inequality.

We set

$$\bar{u}_j(x) := \int_{[0,1] \times V} u_j^{t\bar{v}}(x) d\mu_{\varepsilon_j}(t, \bar{v}),$$

and note that, since $u_j^{t\bar{v}}$ is continuous on Ω' , this function belongs to $H^1(\Omega'; \mathbb{R}^m)$ with

$$\nabla \bar{u}_j(x) = \int_{[0,1] \times V} \nabla u_j^{t\bar{v}}(x) d\mu_{\varepsilon_j}(t, \bar{v})$$

for almost every $x \in \Omega'$. As a consequence, (1.2.24) reduces to

$$\frac{\sigma_{d-1}}{2d} \int_{\Omega'} r_j^{2\varepsilon_j} a\left(\frac{x}{\delta_j}\right) |\nabla \bar{u}_j(x)|^2 dx$$

and, since $r_j^{\varepsilon_j} \rightarrow 1$ as $j \rightarrow +\infty$ in virtue of (1.2.14) and the assumption (1.2.13), the proof of the lower bound is concluded if we show that

$$\liminf_{j \rightarrow +\infty} \frac{\sigma_{d-1}}{2d} \int_{\Omega'} a\left(\frac{x}{\delta_j}\right) |\nabla \bar{u}_j(x)|^2 dx \geq \frac{\sigma_{d-1}}{2d} \int_{\Omega'} \langle A_{\text{hom}} \nabla u, \nabla u \rangle dx.$$

In turn, to this end it is sufficient to prove that $\bar{u}_j \rightarrow u$ in $L^2(\Omega'; \mathbb{R}^m)$. Indeed, this implies that u is the $L^2(\Omega'; \mathbb{R}^m)$ -limit of a bounded sequence in $H^1(\Omega'; \mathbb{R}^m)$, so that $u \in H^1(\Omega'; \mathbb{R}^m)$ and the conclusion follows by the lower bound provided in [34, Theorem 14.7].

First we prove the convergence under the additional assumption that $\{u_j\}_j$ is bounded in $L^\infty(\Omega'; \mathbb{R}^m)$. If this is the case, it suffices to prove that $\bar{u}_j - u_j \rightarrow 0$ in $L^1(\Omega'; \mathbb{R}^m)$.

As $\Omega' \subseteq \bigcup_{k \in \tilde{\mathcal{I}}_{t\bar{v}}^{r_j}}$, we have

$$\begin{aligned}
\|\bar{u}_j - u_j\|_{L^1(\Omega'; \mathbb{R}^m)} &= \int_{\Omega'} \left| u_j(x) - \int_{[0,1] \times V} u_j^{t\bar{v}}(x) d\mu_{\varepsilon_j}(t, \bar{v}) \right| dx \\
&\leq \int_{[0,1] \times V} \int_{\Omega'} |u_j(x) - u_j^{t\bar{v}}(x)| dx d\mu_{\varepsilon_j}(t, \bar{v}) \\
&\leq \int_{[0,1] \times V} \sum_{k \in \tilde{\mathcal{I}}_{t\bar{v}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{v}}} |u_j(x) - u_j^{t\bar{v}}(x)| dx d\mu_{\varepsilon_j}(t, \bar{v}) \\
&\leq \int_{[0,1] \times V} \sum_{k \in \tilde{\mathcal{I}}_{t\bar{v}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{v}}} |u_j(x) - u_j^{t\bar{v}}(r_j k)| dx d\mu_{\varepsilon_j}(t, \bar{v}) \quad (1.2.25)
\end{aligned}$$

$$+ \int_{[0,1] \times V} \sum_{k \in \tilde{\mathcal{I}}_{t\bar{v}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{v}}} |u_j^{t\bar{v}}(x) - u_j^{t\bar{v}}(r_j k)| dx d\mu_{\varepsilon_j}(t, \bar{v}). \quad (1.2.26)$$

We first prove that (1.2.25) tends to 0 as ε_j vanishes. Applying Hölder's inequality and a rescaled version of the Poincaré-Wirtinger inequality (see [97, Theorem 6.33]) we get

$$\begin{aligned}
&\sum_{k \in \tilde{\mathcal{I}}_{t\bar{v}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{v}}} |u_j(x) - u_j^{t\bar{v}}(r_j k)| dx \\
&\leq |r_j t|^{\frac{d}{2}} \sum_{k \in \tilde{\mathcal{I}}_{t\bar{v}}^{r_j}} \left(\int_{r_j k + r_j Q_{t\bar{v}}} |u_j(x) - u_j^{t\bar{v}}(r_j k)|^2 dx \right)^{\frac{1}{2}} \\
&\leq P |r_j t|^{\frac{d}{2} + (1-\varepsilon_j)} \sum_{k \in \tilde{\mathcal{I}}_{t\bar{v}}^{r_j}} \left(\iint_{(r_j k + r_j Q_{t\bar{v}}) \times (r_j k + r_j Q_{t\bar{v}})} \frac{|u_j(y) - u_j(x)|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy \right)^{\frac{1}{2}},
\end{aligned}$$

where P is the Poincaré-Wirtinger constant for the d -dimensional unit cube. Then we make use of the concavity of $x \mapsto x^{\frac{1}{2}}$ so that

$$\begin{aligned}
&P |r_j t|^{\frac{d}{2} + (1-\varepsilon_j)} \# \tilde{\mathcal{I}}_{t\bar{v}}^{r_j} \sum_{k \in \tilde{\mathcal{I}}_{t\bar{v}}^{r_j}} \frac{1}{\# \tilde{\mathcal{I}}_{t\bar{v}}^{r_j}} \left(\iint_{(r_j k + r_j Q_{t\bar{v}}) \times (r_j k + r_j Q_{t\bar{v}})} \frac{|u_j(y) - u_j(x)|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy \right)^{\frac{1}{2}} \\
&\leq P |r_j t|^{\frac{d}{2} + (1-\varepsilon_j)} \# \tilde{\mathcal{I}}_{t\bar{v}}^{r_j} \left(\sum_{k \in \tilde{\mathcal{I}}_{t\bar{v}}^{r_j}} \frac{1}{\# \tilde{\mathcal{I}}_{t\bar{v}}^{r_j}} \iint_{(r_j k + r_j Q_{t\bar{v}}) \times (r_j k + r_j Q_{t\bar{v}})} \frac{|u_j(y) - u_j(x)|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy \right)^{\frac{1}{2}} \\
&\leq P |r_j t|^{\frac{d}{2} + (1-\varepsilon_j)} (\# \tilde{\mathcal{I}}_{t\bar{v}}^{r_j})^{\frac{1}{2}} \left(\iint_{\Omega \times \Omega} \frac{|u_j(y) - u_j(x)|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy \right)^{\frac{1}{2}} \\
&\leq P |r_j t|^{\frac{d}{2} + (1-\varepsilon_j)} |2r_j t|^{-\frac{d}{2}} |\Omega|^{\frac{1}{2}} [u_j]_{H^{1-\varepsilon_j}(\Omega)} \\
&\leq C |r_j t|^{1-\varepsilon_j} [u_j]_{H^{1-\varepsilon_j}(\Omega)}
\end{aligned}$$

for a positive constant C independent of j . After integration we obtain that (1.2.25) is bounded from above by

$$C r_j^{1-\varepsilon_j} \varepsilon_j [u_j]_{H^{1-\varepsilon_j}(\Omega)} \int_0^1 t^{\varepsilon_j} dt \leq C r_j^{1-\varepsilon_j} \varepsilon_j [u_j]_{H^{1-\varepsilon_j}(\Omega)}$$

for a possibly different constant C . Then we use that $\varepsilon_j[u_j]_{H^{1-\varepsilon_j}(\Omega)}$ is uniformly bounded in j by assumption, and that $r_j^{1-\varepsilon_j} \rightarrow 0$ to deduce that the above term vanishes.

We treat (1.2.26) with a similar argument. By the Cauchy-Schwarz inequality we have

$$\begin{aligned} & \int_{[0,1] \times V} \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{\nu}}} |u_j^{t\bar{\nu}}(x) - u_j^{t\bar{\nu}}(r_j k)| dx d\mu_{\varepsilon_j}(t, \bar{\nu}) \\ & \leq \int_{[0,1] \times V} \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{\nu}}} |\nabla u_j^{t\bar{\nu}}(x)| |x - r_j k| dx d\mu_{\varepsilon_j}(t, \bar{\nu}) \\ & \leq \sqrt{d} \int_{[0,1] \times V} |r_j t| \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{\nu}}} |\nabla u_j^{t\bar{\nu}}(x)| dx d\mu_{\varepsilon_j}(t, \bar{\nu}) \end{aligned}$$

and by Hölder's inequality, the concavity of $x \mapsto x^{\frac{1}{2}}$ and Jensen's inequality, this is bounded from above by

$$\begin{aligned} & \sqrt{d} \int_{[0,1] \times V} |r_j t|^{\frac{d}{2}+1} \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} \left(\int_{r_j k + r_j Q_{t\bar{\nu}}} |\nabla u_j^{t\bar{\nu}}(x)|^2 dx \right)^{\frac{1}{2}} d\mu_{\varepsilon_j}(t, \bar{\nu}) \\ & = \sqrt{d} \int_{[0,1] \times V} |r_j t|^{\frac{d}{2}+1} \# \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j} \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} \frac{1}{\# \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} \left(\int_{r_j k + r_j Q_{t\bar{\nu}}} |\nabla u_j^{t\bar{\nu}}(x)|^2 dx \right)^{\frac{1}{2}} d\mu_{\varepsilon_j}(t, \bar{\nu}) \\ & \leq \sqrt{d} \int_{[0,1] \times V} |r_j t|^{\frac{d}{2}+1} (\# \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j})^{\frac{1}{2}} \left(\sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{\nu}}} |\nabla u_j^{t\bar{\nu}}(x)|^2 dx \right)^{\frac{1}{2}} d\mu_{\varepsilon_j}(t, \bar{\nu}) \\ & \leq C r_j \int_{[0,1] \times V} \left(\sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{\nu}}} |\nabla u_j^{t\bar{\nu}}(x)|^2 dx \right)^{\frac{1}{2}} d\mu_{\varepsilon_j}(t, \bar{\nu}) \\ & \leq C r_j \left(\int_{[0,1] \times V} \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{\nu}}} |\nabla u_j^{t\bar{\nu}}(x)|^2 dx d\mu_{\varepsilon_j}(t, \bar{\nu}) \right)^{\frac{1}{2}} \end{aligned} \tag{1.2.27}$$

for a positive constant C independent of j . Now we write

$$\begin{aligned} & \int_{[0,1] \times V} \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{\nu}}} |\nabla u_j^{t\bar{\nu}}(x)|^2 dx d\mu_{\varepsilon_j}(t, \bar{\nu}) \\ & = \frac{2\varepsilon_j}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{1}{t^{1-2\varepsilon_j}} \int_V \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} \int_{r_j k + r_j Q_{t\bar{\nu}}} |\nabla u_j^{t\bar{\nu}}|^2 dx d\mathcal{H}^{k_d}(\bar{\nu}) dt \\ & \leq \left(\frac{2d}{\alpha \sigma_{d-1} r_j^{2\varepsilon_j}} \right) \frac{\varepsilon_j}{d} \frac{\sigma_{d-1}}{\mathcal{H}^{k_d}(V)} \int_0^1 \frac{r_j^{2\varepsilon_j}}{t^{1-2\varepsilon_j}} \int_V \sum_{k \in \tilde{\mathcal{I}}_{t\bar{\nu}}^{r_j}} a\left(\frac{r_j k}{\delta_j}\right) \int_{r_j k + r_j Q_{t\bar{\nu}}} |\nabla u_j^{t\bar{\nu}}|^2 dx d\mathcal{H}^{k_d}(\bar{\nu}) dt \end{aligned}$$

and we note that this last term equals (1.2.21) (up to a factor that is bounded for every j large enough since $r_j^{\varepsilon_j} \rightarrow 1$), therefore, as a byproduct of the previous computations, it is bounded above by a multiple of $F_{\varepsilon_j}(u_j)$, which is uniformly bounded for small ε_j . We conclude that (1.2.27) vanishes since $r_j \rightarrow 0$.

Finally, we remove the boundedness condition by a truncation argument. Given $u_j \in H^{1-\varepsilon_j}(\Omega; \mathbb{R}^m)$, we denote its components as $(u_j^1, \dots, u_j^m)$ and we let

$$(u_j^i)^S := \max\{-S, \min\{S, u_j^i\}\}, i \in \{1, \dots, m\}, \quad u_j^S := ((u_j^1)^S, \dots, (u_j^m)^S)$$

for every $S > 0$. Since $u_j \rightarrow u$ in $L^2(\Omega; \mathbb{R}^m)$ implies that $(u_j)^S \rightarrow u^S$ in $L^2(\Omega; \mathbb{R}^m)$ and since $F_{\varepsilon_j}(u_j) \geq F_{\varepsilon_j}((u_j)^S)$, we have

$$\liminf_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) \geq \liminf_{j \rightarrow +\infty} F_{\varepsilon_j}((u_j)^S) \geq \frac{\sigma_{d-1}}{2d} \int_{\Omega} \langle A_{\text{hom}} \nabla(u^S), \nabla(u^S) \rangle dx,$$

and then

$$\sup_S \|\nabla(u^S)\|_{L^2(\Omega; \mathbb{R}^{m \times d})} \leq \frac{2d}{\alpha \sigma_{d-1}} \sup_j F_{\varepsilon_j}(u_j) < +\infty,$$

so that $u \in H^1(\Omega; \mathbb{R}^m)$ with $\nabla(u^S) \rightarrow \nabla u$. The proof is concluded once we let $S \rightarrow +\infty$.

1.2.3 Upper bound

Also in this section we fix positive sequences $\{\varepsilon_j\}_j, \{\delta_j := \delta_{\varepsilon_j}\}_j$ tending to 0 as $j \rightarrow +\infty$ and such that

$$\lim_{j \rightarrow +\infty} \delta_j^{\varepsilon_j} = 1. \quad (1.2.28)$$

To prove the Γ -lim sup inequality, we first study the case where u is piecewise-affine. We then recover the inequality on the whole space $H^1(\Omega; \mathbb{R}^m)$ by means of a density argument. For the sake of clarity, we first deal with the case where u is affine.

Step 1. We first suppose that $u(x) = Mx$, for some $M \in \mathbb{R}^{m \times d}$. Recalling (1.2.8), for fixed $\theta > 0$ we consider $v \in W_{\#,0}^{1,2}(Q_1; \mathbb{R}^m) \cap C^\infty(\mathbb{R}^d; \mathbb{R}^m)$ a 1-periodic function such that

$$\int_{Q_1} a(x) |M + \nabla v(x)|^2 dx < \langle A_{\text{hom}} M, M \rangle + \theta; \quad (1.2.29)$$

we claim that $u_j(x) = Mx + \delta_j v(\frac{x}{\delta_j})$ is an approximate recovery sequence for u , i.e., it is such that

$$\limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) \leq \frac{\sigma_{d-1}}{2d} |\Omega| (\langle A_{\text{hom}} M, M \rangle + \theta). \quad (1.2.30)$$

This, combined with a diagonal argument and the arbitrariness of $\theta > 0$, will lead to the upper bound.

In order to prove (1.2.30), we take advantage of Lemma 1.2.3 and consider the first-order Taylor expansion of v to get

$$\begin{aligned} F_{\varepsilon_j}(u_j) &= \varepsilon_j \int_{\Omega} \int_{\Omega} a\left(\frac{x}{\delta_j}\right) \frac{|M(y-x) + \delta_j v(\frac{y}{\delta_j}) - \delta_j v(\frac{x}{\delta_j})|^2}{|y-x|^{d+2(1-\varepsilon_j)}} dx dy \\ &= \varepsilon_j \iint_{\Omega \times \Omega \cap \{(x,y): |x-y| \leq \delta_j\}} a\left(\frac{x}{\delta_j}\right) \frac{|(M + \nabla v(\frac{x}{\delta_j}))(y-x) + \frac{1}{\delta_j} R(y-x)|^2}{|y-x|^{d+2(1-\varepsilon_j)}} dx dy + o_j(1), \end{aligned}$$

where the function $R(\xi)$ is the remainder in Lagrange form of v and it is such that

$$|R(\xi)| \leq C|\xi|^2, \quad (1.2.31)$$

for a positive constant C depending on $\nabla^2 v$.

By the inequality $|a + b|^2 \leq (1 + \eta)a^2 + (1 + 1/\eta)b^2$, with $\eta > 0$, and by the change of variables $\xi := y - x$, we deduce

$$F_{\varepsilon_j}(u_j) + o_j(1) \leq (1 + \eta)\varepsilon_j \int_{\Omega} \int_{B_{\delta_j}} a\left(\frac{x}{\delta_j}\right) \frac{|(M + \nabla v(\frac{x}{\delta_j}))\xi|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi \quad (1.2.32)$$

$$+ \left(1 + \frac{1}{\eta}\right) \frac{\varepsilon_j}{\delta_j^2} \int_{\Omega} \int_{B_{\delta_j}} a\left(\frac{x}{\delta_j}\right) \frac{|R(\xi)|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi. \quad (1.2.33)$$

For a fixed $\eta > 0$, the term in (1.2.33) vanishes as ε_j tends to zero. Indeed, by (1.2.5) and (1.2.31), we have that

$$\begin{aligned} \int_{\Omega} \int_{B_{\delta_j}} a\left(\frac{x}{\delta_j}\right) \frac{|R(\xi)|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi &\leq C\beta|\Omega| \int_{B_{\delta_j}} \frac{1}{|\xi|^{d-2\varepsilon_j-2}} d\xi \\ &= C\beta|\Omega|\sigma_{d-1} \int_0^{\delta_j} t^{1+2\varepsilon_j} dt \\ &= \frac{C\beta|\Omega|\sigma_{d-1}}{2 + 2\varepsilon_j} \delta_j^{2+2\varepsilon_j}. \end{aligned} \quad (1.2.34)$$

Therefore, (1.2.33) is bounded from above by

$$C\beta|\Omega|\sigma_{d-1} \left(1 + \frac{1}{\eta}\right) \frac{\delta_j^{2+2\varepsilon_j}}{\delta_j^2} \frac{\varepsilon_j}{2 + 2\varepsilon_j} = C\beta|\Omega|\sigma_{d-1} \left(1 + \frac{1}{\eta}\right) \delta_j^{2\varepsilon_j} \frac{\varepsilon_j}{2 + 2\varepsilon_j},$$

which vanishes as $j \rightarrow +\infty$ by (1.2.28).

As for the term in (1.2.32), note that by the symmetry of the integrand it holds

$$\begin{aligned} \int_{B_{\delta_j}} \frac{|(M + \nabla v(\frac{x}{\delta_j}))\xi|^2}{|\xi|^{d+2(1-\varepsilon_j)}} d\xi &= \int_0^{\delta_j} t^{-1+2\varepsilon_j} \int_{\mathbb{S}^{d-1}} \left| \left(M + \nabla v\left(\frac{x}{\delta_j}\right) \right) \nu \right|^2 d\mathcal{H}^{d-1}(\nu) dt \\ &= \int_0^{\delta_j} t^{-1+2\varepsilon_j} \frac{1}{d} \int_{\mathbb{S}^{d-1}} \left| M + \nabla v\left(\frac{x}{\delta_j}\right) \right|^2 d\mathcal{H}^{d-1}(\nu) dt \\ &= \frac{\sigma_{d-1}}{d} \left| M + \nabla v\left(\frac{x}{\delta_j}\right) \right|^2 \int_0^{\delta_j} t^{-1+2\varepsilon_j} dt \\ &= \frac{\sigma_{d-1}}{d} \left| M + \nabla v\left(\frac{x}{\delta_j}\right) \right|^2 \frac{\delta_j^{2\varepsilon_j}}{2\varepsilon_j}. \end{aligned} \quad (1.2.35)$$

Recalling that v and a are 1-periodic, we may substitute (1.2.35) in (1.2.32) to get

$$\begin{aligned} (1 + \eta)\varepsilon_j \int_{\Omega} \int_{B_{\delta_j}} a\left(\frac{x}{\delta_j}\right) \frac{|(M + \nabla v(\frac{x}{\delta_j}))\xi|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi \\ &= (1 + \eta) \frac{\sigma_{d-1}}{2d} \delta_j^{2\varepsilon_j} \int_{\Omega} a\left(\frac{x}{\delta_j}\right) \left| M + \nabla v\left(\frac{x}{\delta_j}\right) \right|^2 dx \\ &\leq (1 + \eta) \frac{\sigma_{d-1}}{2d} \delta_j^{2\varepsilon_j} \frac{|\Omega|}{\delta_j^d} \int_{Q_{\delta_j}} a\left(\frac{x}{\delta_j}\right) \left| M + \nabla v\left(\frac{x}{\delta_j}\right) \right|^2 dx \\ &= (1 + \eta) \frac{\sigma_{d-1}}{2d} \delta_j^{2\varepsilon_j} |\Omega| \int_{Q_1} a(y) |M + \nabla v(y)|^2 dy. \end{aligned} \quad (1.2.36)$$

In view of (1.2.29), recalling that $\delta_j^{\varepsilon_j} \rightarrow 1$ we infer

$$\limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) \leq (1 + \eta) \frac{\sigma_{d-1}}{2d} |\Omega| (\langle A_{\text{hom}} M, M \rangle dx + \theta),$$

and we conclude by letting η to 0.

Step 2. We now suppose the function u is piecewise-affine in Ω . More precisely, we suppose that there exists a finite family of d -simplices $\{\Delta_i\}_{i \in I}$ covering Ω , $\{M_i\}_{i \in I} \subseteq \mathbb{R}^{m \times d}$, and $\{m_i\}_{i \in I} \subseteq \mathbb{R}^m$ such that $u(x) = M_i x + m_i$ for every $x \in \Delta_i \cap \Omega$ and for every $i \in I$. To simplify the notation, from now on we will always write Δ_i in place of $\Delta_i \cap \Omega$.

Similarly to what we have done in the previous case, for a fixed $\theta > 0$ and for every $i \in I$, we consider $v_i \in W_{\#,0}^{1,2}(Q_1; \mathbb{R}^m) \cap C^\infty(\mathbb{R}^d; \mathbb{R}^m)$ a 1-periodic function satisfying

$$\int_{Q_1} a(x) |M_i + \nabla v_i(x)|^2 dx < \langle A_{\text{hom}} M_i, M_i \rangle + \frac{\theta}{\#I |\Delta_i|}. \quad (1.2.37)$$

Having in mind the construction of the recovery sequence performed in the previous case, we aim at constructing an approximate recovery sequence for u by perturbing, in a suitable way, u with the functions v_i . To this end, for every Δ_i , we consider the subset defined by

$$\Delta_i^{\delta_j} := \{x \in \Omega : \text{dist}(x, \Delta_i^c) > \delta_j\}.$$

For fixed $j \in \mathbb{N}$ and $i \in I$, we consider a function $\psi_{\delta_j}^i \in C_c^\infty(\mathbb{R}^d)$, satisfying the following conditions

$$\begin{cases} \psi_{\delta_j}^i(x) = 0 & \text{for } x \notin \Delta_i^{\delta_j}, \\ \psi_{\delta_j}^i(x) = 1 & \text{for } x \in \Delta_i^{2\delta_j}, \\ 0 \leq \psi_{\delta_j}^i(x) \leq 1 & \text{for } x \in \mathbb{R}^d, \\ |\nabla \psi_{\delta_j}^i(x)| \leq \frac{2}{\delta_j} & \text{for } x \in \mathbb{R}^d, \\ |\nabla^2 \psi_{\delta_j}^i(x)| \leq \frac{4}{\delta_j^2} & \text{for } x \in \mathbb{R}^d, \end{cases} \quad (1.2.38a)$$

$$|\nabla \psi_{\delta_j}^i(x)| \leq \frac{2}{\delta_j} \quad \text{for } x \in \mathbb{R}^d, \quad (1.2.38b)$$

$$|\nabla^2 \psi_{\delta_j}^i(x)| \leq \frac{4}{\delta_j^2} \quad \text{for } x \in \mathbb{R}^d, \quad (1.2.38c)$$

where $\nabla^2 \psi_{\delta_j}^i$ is the Hessian matrix of $\psi_{\delta_j}^i$.

We claim that an approximate recovery sequence for u is given by

$$u_j(x) = \sum_{i \in I} \left(M_i x + \delta_j v_i\left(\frac{x}{\delta_j}\right) \psi_{\delta_j}^i(x) + m_i \right).$$

Note that u_j is a continuous function since it coincides with u on the boundary of each Δ_i . Moreover, in light of (1.2.38a) and (1.2.38b), we have

$$\sup_j \|\nabla u_j\|_{L^\infty(\Omega; \mathbb{R}^{m \times d})} < +\infty.$$

Therefore, we are in position to apply Lemma 1.2.3, so that it is enough to show that

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon_j \iint_A a\left(\frac{x}{\delta_j}\right) \frac{|u_j(y) - u_j(x)|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy \leq \frac{\sigma_{d-1}}{2d} \int_\Omega \langle A_{\text{hom}} \nabla u, \nabla u \rangle dx + \theta,$$

where we let $A := \Omega \times \Omega \cap \{|x - y| \leq \delta_j\}$. To make the computations more manageable, we consider a suitable partition of A . For $i \in I$, we set

$$B_i = \Delta_i^{2\delta_j} \times \Delta_i^{2\delta_j} \cap \{|x - y| \leq \delta_j\},$$

$$C_i = \Delta_i \times \Delta_i \cap \{|x - y| \leq \delta_j \text{ and either } x \text{ or } y \in \Delta_i \setminus \Delta_i^{2\delta_j}\},$$

and for $j \in I, j \neq i$, we set

$$D_{ij} = \Delta_i \times \Delta_j \cap \{|x - y| \leq \delta_j\},$$

so that $A = \bigcup_{i \in I} (B_i \cup C_i) \cup \bigcup_{i \neq j} D_{ij}$.

Taking advantage of this decomposition and of the subadditivity of the lim sup, we have

$$\begin{aligned} \limsup_{j \rightarrow +\infty} \varepsilon_j \iint_A a\left(\frac{x}{\delta_j}\right) \frac{|u_j(y) - u_j(x)|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy \\ \leq \limsup_{j \rightarrow +\infty} \sum_{i \in I} \varepsilon_j \iint_{B_i} a\left(\frac{x}{\delta_j}\right) \frac{|u_j(y) - u_j(x)|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy \end{aligned} \quad (1.2.39)$$

$$+ \sum_{i \in I} \limsup_{j \rightarrow +\infty} \varepsilon_j \iint_{C_i} a\left(\frac{x}{\delta_j}\right) \frac{|u_j(y) - u_j(x)|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy \quad (1.2.40)$$

$$+ \sum_{i \in I} \sum_{j \in I, j \neq i} \limsup_{j \rightarrow +\infty} \varepsilon_j \iint_{D_{ij}} a\left(\frac{x}{\delta_j}\right) \frac{|u_j(y) - u_j(x)|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy. \quad (1.2.41)$$

We begin by estimating (1.2.39). Since for fixed $i \in I$ we have $\psi_{\delta_j}^i = 1$ on B_i , by the first-order Taylor expansion of v_i , for a fixed $\eta > 0$ we get

$$\begin{aligned} \sum_{i \in I} \varepsilon_j \iint_{B_i} a\left(\frac{x}{\delta_j}\right) \frac{|(M_i + \nabla v_i(\frac{x}{\delta_j}))(y - x) + \frac{1}{\delta_j} R_i(y - x)|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy \\ \leq \sum_{i \in I} (1 + \eta) \varepsilon_j \int_{\Delta_i^{2\delta_j}} \int_{B_{\delta_j}} a\left(\frac{x}{\delta_j}\right) \frac{|(M_i + \nabla v_i(\frac{x}{\delta_j}))\xi|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi \end{aligned} \quad (1.2.42)$$

$$+ \sum_{i \in I} \left(1 + \frac{1}{\eta}\right) \frac{\varepsilon_j}{\delta_j^2} \int_{\Delta_i^{2\delta_j}} \int_{B_{\delta_j}} a\left(\frac{x}{\delta_j}\right) \frac{|R_i(\xi)|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi, \quad (1.2.43)$$

where R_i is the remainder in Lagrange form of the first-order Taylor expansion of v_i , which satisfies

$$|R_i(\xi)| \leq C|\xi|^2 \quad (1.2.44)$$

for every $i \in I$. Thus, we may estimate (1.2.43) with

$$\beta C^2 \sum_{i \in I} \left(1 + \frac{1}{\eta}\right) \frac{\varepsilon_j}{\delta_j^2} \int_{\Delta_i} \int_{B_{\delta_j}} \frac{|\xi|^4}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi,$$

which vanishes for $\varepsilon_j \rightarrow 0$ in view of (1.2.34) (with Ω replaced by Δ_i).

As for (1.2.42), taking advantage of the computations leading to (1.2.36), we see that

$$\begin{aligned} \sum_{i \in I} (1 + \eta) \varepsilon_j \int_{\Delta_i^{2\delta_j}} \int_{B_{\delta_j}} a\left(\frac{x}{\delta_j}\right) \frac{|(M_i + \nabla v_i(\frac{x}{\delta_j}))\xi|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi \\ \leq \sum_{i \in I} (1 + \eta) \varepsilon_j \int_{\Delta_i} \int_{B_{\delta_j}} a\left(\frac{x}{\delta_j}\right) \frac{|(M_i + \nabla v_i(\frac{x}{\delta_j}))\xi|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi \\ \leq (1 + \eta) \frac{\sigma_{d-1}}{2d} \sum_{i \in I} |\Delta_i| \int_{Q_1} a(x) |M_i + \nabla v_i(x)|^2 dx. \end{aligned}$$

Recalling (1.2.37), we finally conclude

$$\begin{aligned}
& \limsup_{\varepsilon \rightarrow 0} \sum_{i \in I} (1 + \eta) \varepsilon_j \iint_{B_i} a\left(\frac{x}{\delta_j}\right) \frac{|u_j(y) - u_j(x)|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy \\
& \leq \limsup_{j \rightarrow +\infty} (1 + \eta) \frac{\sigma_{d-1}}{2d} \sum_{i \in I} \left(\int_{\Delta_i} \langle A_{\text{hom}} M_i, M_i \rangle dx + \frac{\theta}{\#I} \right) \\
& \leq (1 + \eta) \frac{\sigma_{d-1}}{2d} \int_{\Omega} \langle A_{\text{hom}} \nabla u, \nabla u \rangle dx + \theta,
\end{aligned}$$

so that letting $\theta \rightarrow 0$, (1.2.39) is estimated.

We now show that (1.2.40) vanishes for $\varepsilon_j \rightarrow 0$. As $\#I$ is finite and independent on ε_j , it suffices to take into account a single term in the sum. By the convexity of $x \mapsto x^2$, we have

$$\begin{aligned}
& \varepsilon_j \iint_{C_i} a\left(\frac{x}{\delta_j}\right) \frac{|u_j(y) - u_j(x)|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy \\
& \leq 3\beta \varepsilon_j \iint_{C_i} \frac{|M_i|^2 |y - x|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy
\end{aligned} \tag{1.2.45}$$

$$+ 3\beta \varepsilon_j \iint_{C_i} \frac{|\delta_j \psi_{\delta_j}^i(y) (v_i(\frac{y}{\delta_j}) - v_i(\frac{x}{\delta_j}))|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy \tag{1.2.46}$$

$$+ 3\beta \varepsilon_j \iint_{C_i} \frac{|\delta_j v_i(\frac{x}{\delta_j}) (\psi_{\delta_j}^i(y) - \psi_{\delta_j}^i(x))|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy. \tag{1.2.47}$$

We start with (1.2.45). Via a change of variables, we see that

$$\begin{aligned}
\varepsilon_j \iint_{C_i} \frac{|M_i|^2 |y - x|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy & \leq \varepsilon_j |M_i|^2 |\Delta_i \setminus \Delta_i^{2\delta_j}| \int_{B_{\delta_j}} \frac{|\xi|^2}{|\xi|^{d+2(1-\varepsilon_j)}} d\xi \\
& \leq \varepsilon_j |M_i|^2 |\Delta_i \setminus \Delta_i^{2\delta_j}| \sigma_{d-1} \int_0^1 t^{-1+2\varepsilon_j} dt \\
& = \frac{\sigma_{d-1}}{2} |M_i|^2 |\Delta_i \setminus \Delta_i^{2\delta_j}|,
\end{aligned} \tag{1.2.48}$$

which converges to zero as $\varepsilon_j \rightarrow 0$.

As for (1.2.46), using the first-order Taylor expansions of v_i , (1.2.38a), and recalling (1.2.44), one gets

$$\begin{aligned}
& \varepsilon_j \iint_{C_i} \frac{|\delta_j \psi_{\delta_j}^i(y) (v_i(\frac{y}{\delta_j}) - v_i(\frac{x}{\delta_j}))|^2}{|y - x|^{d+2(1-\varepsilon_j)}} dx dy \\
& \leq \varepsilon_j \int_{\Delta_i \setminus \Delta_i^{2\delta_j}} \int_{B_{\delta_j}} \frac{|\nabla v_i(\frac{x}{\delta_j}) \xi + \frac{1}{\delta_j} R_i(\xi)|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi \\
& \leq 2\varepsilon_j \left(\int_{\Delta_i \setminus \Delta_i^{2\delta_j}} \int_{B_{\delta_j}} \frac{|\nabla v_i(\frac{x}{\delta_j}) \xi|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi + \frac{C^2}{\delta_j^2} |\Delta_i \setminus \Delta_i^{2\delta_j}| \int_{B_{\delta_j}} \frac{|\xi|^4}{|\xi|^{d+2(1-\varepsilon_j)}} d\xi \right).
\end{aligned}$$

The first term can be estimated with

$$\varepsilon_j \int_{\Delta_i \setminus \Delta_i^{2\delta_j}} \int_{B_{\delta_j}} \frac{|\nabla v_i(\frac{x}{\delta_j}) \xi|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi \leq C \varepsilon_j \|\nabla v_i\|_{L^\infty(B_1)}^2 \int_{\Delta_i \setminus \Delta_i^{2\delta_j}} \int_{B_{\delta_j}} \frac{|\xi|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi$$

and therefore vanishes as in (1.2.48), while for the second term we have

$$\frac{\varepsilon_j}{\delta_j^2} |\Delta_i \setminus \Delta_i^{2\delta_j}| \int_{B_{\delta_j}} \frac{|\xi|^4}{|\xi|^{d+2(1-\varepsilon_j)}} d\xi \quad (1.2.49)$$

that vanishes as in (1.2.34).

In order to estimate (1.2.47), we denote by $\tilde{R}_i$ the Lagrange remainders of the first-order Taylor expansion of $\psi_{\delta_j}^i$ and we note that by (1.2.38c) it follows that

$$|\tilde{R}_i(\xi)| \leq \frac{C}{\delta_j^2} |\xi|^2.$$

for every $i \in I$ and $j \in \mathbb{N}$. Using the first-order Taylor expansion of $\psi_{\delta_j}^i$, we conclude that

$$\begin{aligned} \varepsilon_j \iint_{C_i} \frac{|\delta_j v_i(\frac{x}{\delta_j})(\psi_{\delta_j}^i(y) - \psi_{\delta_j}^i(x))|^2}{|y-x|^{d+2(1-\varepsilon_j)}} dx dy \\ \leq \varepsilon_j \|v_i\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^m)}^2 \int_{\Delta_i \setminus \Delta_i^{2\delta_j}} \int_{B_{\delta_j}} \frac{|\delta_j \nabla \psi_{\delta_j}^i(x) \xi + \delta_j \tilde{R}_i(\xi)|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi \\ \leq 2\varepsilon_j \|v_i\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^m)}^2 \left(\int_{\Delta_i \setminus \Delta_i^{2\delta_j}} \int_{B_{\delta_j}} \frac{|\delta_j \nabla \psi_{\delta_j}^i(x) \xi|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi \right. \end{aligned} \quad (1.2.50)$$

$$\left. + C^2 |\Delta_i \setminus \Delta_i^{2\delta_j}| \frac{1}{\delta_j^2} \int_{B_{\delta_j}} \frac{|\xi|^4}{|\xi|^{d+2(1-\varepsilon_j)}} d\xi \right). \quad (1.2.51)$$

In view of (1.2.49), we infer that the term in (1.2.51) converges to zero.

As for (1.2.50), we may use (1.2.38b) to obtain that

$$\begin{aligned} \varepsilon_j \int_{\Delta_i \setminus \Delta_i^{2\delta_j}} \int_{B_{\delta_j}} \frac{|\delta_j \nabla \psi_{\delta_j}^i(x) \xi|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi \\ \leq 2\varepsilon_j |\Delta_i \setminus \Delta_i^{2\delta_j}| \int_{B_1} \frac{|\xi|^2}{|\xi|^{d+2(1-\varepsilon_j)}} d\xi = \sigma_{d-1} |\Delta_i \setminus \Delta_i^{2\delta_j}|, \end{aligned}$$

so that we have finally shown that (1.2.40) tends to 0 as $\varepsilon_j \rightarrow 0$.

We conclude by proving that (1.2.41) vanishes as $\varepsilon_j \rightarrow 0$. Also in this case, since $\#I$ is finite and independent of ε_j , it suffices to take into account just one summand.

Note that if $(x, y) \in D_{ij}$, then $x \in \Delta_i \setminus \Delta_i^{\delta_j}$ and $y \in \Delta_j \setminus \Delta_j^{\delta_j}$, so that $\psi_{\delta_j}^i(x) = \psi_{\delta_j}^j(y) = 0$. Thus, taking advantage of the Lipschitz continuity of u , we get

$$\begin{aligned} \varepsilon_j \iint_{D_{ij}} a\left(\frac{x}{\delta_j}\right) \frac{|u_j(y) - u_j(x)|^2}{|y-x|^{d+2(1-\varepsilon_j)}} &\leq \beta \varepsilon_j \int_{\Delta_i \setminus \Delta_i^{\delta_j}} \int_{B_{\delta_j}} \frac{|u(x+\xi) - u(x)|^2}{|\xi|^{d+2(1-\varepsilon_j)}} dx d\xi \\ &\leq \beta \|\nabla u\|_{L^\infty(\Omega; \mathbb{R}^{m \times d})} \varepsilon_j |\Delta_i \setminus \Delta_i^{\delta_j}| \int_{B_1} \frac{|\xi|^2}{|\xi|^{d+2(1-\varepsilon_j)}} d\xi \\ &= \beta \|\nabla u\|_{L^\infty(\Omega; \mathbb{R}^{m \times d})} \varepsilon_j |\Delta_i \setminus \Delta_i^{\delta_j}| \sigma_{d-1} \frac{1}{2\varepsilon_j} \\ &= \beta \|\nabla u\|_{L^\infty(\Omega; \mathbb{R}^{m \times d})} \frac{\sigma_{d-1}}{2} |\Delta_i \setminus \Delta_i^{\delta_j}| \end{aligned}$$

and this last term vanishes for $\varepsilon_j \rightarrow 0$ deducing that (1.2.41) tends to 0.

Step 3. The functional F_0^{frac} defined in (1.2.7), which has been proven to be the desired Γ -limit on piecewise-affine functions, is continuous in the strong topology of $H^1(\Omega; \mathbb{R}^m)$. Since Ω is an open bounded set with Lipschitz boundary, the class of piecewise-affine functions, namely, functions which are piecewise-affine on a finite family of simplices covering Ω , is dense in $H^1(\Omega; \mathbb{R}^m)$ (see [70, Proposition 2.1]). Hence, the Γ -lim sup inequality holds for any $u \in H^1(\Omega; \mathbb{R}^m)$ by a density argument.

1.3 Oscillating energies of convolution-type

A class of kernels $\{\rho_\varepsilon\}_\varepsilon$ that satisfies (1.0.2) can be obtained starting from a non-negative ρ having p -moment on $\mathbb{R}^d$ equal to 1 and letting

$$\rho_\varepsilon(\xi) := \frac{1}{\varepsilon^d} \rho\left(\frac{\xi}{\varepsilon}\right).$$

Because of this rescaling property, the resulting functionals (1.0.1) are called of *convolution-type*, and the corresponding oscillating energies are of the form

$$F_{\varepsilon, \delta}^{\text{conv}}(u) := \frac{1}{\varepsilon^d} \int_{\Omega} \int_{\Omega} \rho\left(\frac{y-x}{\varepsilon}\right) f\left(\frac{x}{\delta}, \frac{y}{\delta}, \frac{u(y)-u(x)}{\varepsilon}\right) dx dy. \quad (1.3.1)$$

In this setting we are able to encompass a wider generality compared to the fractional case. In particular, we are able to deal with a more involved inhomogeneity (which is not necessarily described by an oscillating coefficient and that depends on both spatial variables) and with more general p -growth conditions on f (which is as assumed to be convex in the last variable), so that the quadratic case can be treated as a particular instance.

Contrary to the fractional case, the asymptotic analysis of (1.3.1) can be directly set as a homogenization-concentration problem regardless of the rates at which ε and δ vanish and can be dealt with through the direct method of Γ -convergence resorting to Γ -compactness and integral representation results presented in [5]. These allow to obtain as limit energy (depending on an extracted subsequence) a local homogenized functional of the form

$$\int_{\Omega} \varphi(\nabla u) dx. \quad (1.3.2)$$

Exploiting the quasiconvexity of φ and the convergence of boundary-value problems for non-local functionals, one obtains that

$$\varphi(M) = \lim_{j \rightarrow +\infty} \inf \{F_{\varepsilon, \delta}^{\text{conv}}(u, Q) : u = Mx \text{ close to } \partial Q\},$$

where we have ‘localized’ the functional $F_{\varepsilon, \delta}^{\text{conv}}$ in a cube Q contained in Ω ; note that, here, the term ‘localization’ is meant in the sense of the ‘localization method’ for Γ -convergence in which integral functionals are regarded as set functions as well (see [34, Chapter 9] or [5, Chapter 5] for the non-local counterpart). Upon manipulating the localized energies, we obtain that $\varphi(M)$ is determined in terms of the ‘non-local cell-problem formulas’

$$\inf \left\{ \int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f\left(x, x + \frac{\varepsilon}{\delta} \xi, \frac{u(x + \frac{\varepsilon}{\delta} \xi) - u(x)}{\frac{\varepsilon}{\delta}}\right) dx d\xi : u(x) - Mx \text{ is 1-periodic} \right\}. \quad (1.3.3)$$

If $\varepsilon/\delta = \lambda \in (0, +\infty)$, a change of variables yields that the energy density is characterized by a *non-local homogenization formula*

$$f_\lambda(M) = \inf \left\{ \int_{\mathbb{R}^d} \int_{Q_1} \frac{1}{\lambda^d} \rho\left(\frac{y-x}{\lambda}\right) f\left(x, y, \frac{u(y)-u(x)}{\lambda}\right) dx dy : u(x) - Mx \text{ is 1-periodic} \right\},$$

see [5, Theorem 6.2]. This hints that the limit behaviour may be dictated by the value of

$$\lambda := \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\delta} \in [0, +\infty] \quad (1.3.4)$$

and in particular that the same kind of result presented in Section 1.2 for fractional energies may be obtained in the subcritical regime $\varepsilon \ll \delta$.

When $\varepsilon \ll \delta$, passing formally to the limit in (1.3.3) using the results in [27, 116] we obtain an energy density f_0 given by

$$f_0(M) = \inf \left\{ \int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f(x, x, \nabla u(x)\xi) dx d\xi : u(x) - Mx \text{ is 1-periodic} \right\}.$$

This can be seen as a (local) cell-problem formula comparable to (1.0.6) in which the non-local nature of the original problem is portrayed by the necessity of optimizing the oscillations of u taking into account all the directions ξ and the corresponding weight $\rho(\xi)$ determined by the interaction kernel.

Besides the analysis of the subcritical case, it is now rather natural to address all the possible values of λ , and especially a *supercritical regime* occurring when $\lambda = +\infty$. In this instance, the heuristics based on a separation of scales argument implies that the Γ -limit should be obtained letting first $\delta \rightarrow 0$ and then $\varepsilon \rightarrow 0$. These two limits can be computed taking advantage of the Riemann-Lebesgue lemma and of the results in [27, 116], respectively, so that the Γ -limit is as in (1.3.2) with an energy density $\varphi(M)$ given by

$$\varphi(M) = \int_{\mathbb{R}^d} \int_{Q_1} \int_{Q_1} \rho(\xi) f(x, y, M\xi) dx dy d\xi,$$

which is a mere average of the oscillations.

The core of our proof lies in the asymptotic analysis of the minima (1.3.3). The technical difficulty here is detecting sufficient compactness properties on a sequence of minimizers as these become weaker as λ increases. If $\lambda = 0$ one can use the results in [5, 116] to infer convergence to a Sobolev function (up to subsequences). The situation gets more complicated already in the case $\lambda \in (0, +\infty)$, where we only obtain weak compactness in L^p . Finally, no compactness properties seem to be available when $\lambda = +\infty$, we then rely on a more delicate argument in the vein of the two-scale convergence [11, 110].

1.3.1 Setting of the problem and statement of the result

For $\varepsilon > 0$ we let $\delta_\varepsilon > 0$ be a parameter that vanishes as $\varepsilon \rightarrow 0$ and, for $p \in (1, +\infty)$, we introduce the functional $F_\varepsilon^{\text{conv}} : L^p(\Omega; \mathbb{R}^m) \rightarrow [0, +\infty]$ defined as

$$F_\varepsilon^{\text{conv}}(u) := \frac{1}{\varepsilon^d} \int_{\Omega} \int_{\Omega} \rho\left(\frac{y-x}{\varepsilon}\right) f\left(\frac{x}{\delta_\varepsilon}, \frac{y}{\delta_\varepsilon}, \frac{u(y)-u(x)}{\varepsilon}\right) dx dy. \quad (1.3.5)$$

We suppose that $\rho : \mathbb{R}^d \rightarrow [0, +\infty]$ is a non-negative Borel function (not necessarily radially symmetric) such that there exist $c_0, r_0 > 0$ with the property that

$$\rho(\xi) \geq c_0 \quad \text{for almost every } \xi \in B_{r_0}, \quad (\rho_1)$$

and

$$\int_{\mathbb{R}^d} \rho(\xi) |\xi|^p d\xi < +\infty, \quad (\rho_2)$$

then we suppose that $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow [0, +\infty)$ is a Borel function such that

$$\begin{aligned} f(\cdot, y, z) \text{ and } f(x, \cdot, z) \text{ are } Q_1\text{-periodic for every } z \in \mathbb{R}^m \\ \text{and for almost every } y, x \in \mathbb{R}^d, \text{ respectively,} \end{aligned} \quad (\text{P})$$

$$f(x, y, \cdot) \text{ is convex for almost every } x, y \in \mathbb{R}^d, \quad (\text{C})$$

and that fulfills the growth conditions

$$\alpha|z|^p \leq f(x, y, z) \leq \beta|z|^p \text{ for almost every } x, y \in \mathbb{R}^d \text{ and for every } z \in \mathbb{R}^m \quad (\text{GC})$$

for some strictly positive numbers α, β .

Besides the above standing hypotheses, according to the value of the parameter λ defined in (1.3.4), we shall assume that

$$f(x, \cdot, z) \text{ is continuous for almost every } x \in \mathbb{R}^d \text{ and for every } z \in \mathbb{R}^m, \quad (\text{H0})$$

or

$$\begin{cases} \rho(\xi) = \rho(-\xi) & \text{for almost every } \xi \in \mathbb{R}^d, \\ f(x, y, z) = f(x, y, -z) & \text{for almost every } x, y \in \mathbb{R}^d \text{ and for every } z \in \mathbb{R}^m, \end{cases} \quad (\text{H1})$$

or that

$$\int_{\mathbb{R}^d} \rho(\xi) d\xi = +\infty. \quad (\text{H2})$$

Note that if $f(x, y, z) = f(y, x, z)$ and $f(x, y, z) = f(x, y, -z)$ for a.e. $x, y \in \mathbb{R}^d$ and for all $z \in \mathbb{R}^m$, it is not restrictive to assume that $\rho(\xi) = \rho(-\xi)$ for a.e. $\xi \in \mathbb{R}^d$; see [5, Remark 2.1].

The main result of this section is the following.

Theorem 1.3.1. *Let $F_\varepsilon^{\text{conv}}$ be defined by (1.3.5). Assume that*

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\delta_\varepsilon} = \lambda \in [0, +\infty]$$

and moreover, if $\lambda \in [0, +\infty)$ assume that (H0) is satisfied and, if $\lambda = +\infty$ assume that either (H1) or (H2) is satisfied. Then, the family $\{F_\varepsilon^{\text{conv}}\}_\varepsilon$ Γ -converges with respect to the convergence in $L^p(\Omega; \mathbb{R}^m)$ to the functional $F_0^{\text{conv}} : L^p(\Omega; \mathbb{R}^m) \rightarrow [0, +\infty]$ defined as

$$F_0^{\text{conv}}(u) := \begin{cases} \int_{\Omega} f_\lambda(\nabla u) dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases}$$

In particular, the following hold for every $M \in \mathbb{R}^{m \times d}$:

(i) *if $\lambda = 0$ and (H0) holds, then*

$$f_0(M) = \inf \left\{ \int_{\mathbb{R}^d} \int_{Q_1} \rho(\xi) f(x, x, (\nabla u(x))\xi) dx d\xi : u \in W_{\#,M}^{1,p}(Q_1; \mathbb{R}^m) \right\},$$

with $W_{\#,M}^{1,p}(Q_1; \mathbb{R}^m)$ as in (1.1.5);

(ii) *if $\lambda \in (0, +\infty)$ and (H0) holds, then*

$$f_\lambda(M) = \inf \left\{ \int_{\mathbb{R}^d} \int_{Q_1} \frac{1}{\lambda^d} \rho\left(\frac{y-x}{\lambda}\right) f\left(x, y, \frac{u(y) - u(x)}{\lambda}\right) dx dy : u \in L_{\#,M}^p(Q_1; \mathbb{R}^m) \right\},$$

with $L_{\#,M}^p(Q_1; \mathbb{R}^m)$ as in (1.1.4);

(iii) if $\lambda = +\infty$ and either (H1) or (H2) holds, then

$$f_{+\infty}(M) = \int_{\mathbb{R}^d} \int_{Q_1} \int_{Q_1} \rho(\xi) f(x, y, M\xi) dx dy d\xi.$$

As anticipated in Remark 1.2.2, the identification of the limit of oscillating fractional energies in all the possible regimes is investigated in an ongoing work by Braides, Mannella, and Schiavoni Piazza. It is worth mentioning the main result contained therein in order to have a complete picture for both the fractional and convolution-type energies and to highlight similarities and differences.

Theorem 1.3.2. *Let $F_\varepsilon^{\text{frac}}$ be defined by (1.2.6) and assume that*

$$\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^\varepsilon = \lambda \in [0, 1].$$

Then, the family $\{F_\varepsilon^{\text{frac}}\}_\varepsilon$ Γ -converges with respect to the convergence in $L^2(\Omega; \mathbb{R}^m)$ to the functional $F_0^{\text{frac}} : L^2(\Omega; \mathbb{R}^m) \rightarrow [0, +\infty]$ defined as

$$F_0^{\text{frac}}(u) := \begin{cases} \frac{\sigma_{d-1}}{2d} \left[\lambda \int_{\Omega} \langle A_{\text{hom}} \nabla u, \nabla u \rangle dx + (1 - \lambda) \int_{\Omega} \bar{a} |\nabla u|^2 dx \right] & \text{if } u \in H^1(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases} \quad (1.3.6)$$

where A_{hom} is as defined in (1.2.8) and $\bar{a}$ is the mean value of a on the unit cube.

In the case of convolution-type energies, the regime $\varepsilon \sim \delta$; i.e., $\lambda \in (0, +\infty)$, can be regarded as a critical case as it determines a gap between the limit regimes $\lambda = 0$ and $\lambda = +\infty$, where a separation of scales occurs. In the fractional case, to some extent, we have a separation of scales also for the intermediate regimes $\lambda \in (0, 1)$. Here, λ represents a threshold below which the concentration effect takes place before the homogenization phenomenon and above which the converse occurs.

It is easily seen how the intermediate limit energies (1.3.6) continuously depend on λ and interpolate the extreme cases. The same behaviour can be detected for the energy densities in Theorem 1.3.1 as it is possible to infer that the function

$$\lambda \mapsto f_\lambda(M)$$

is continuous in $[0, +\infty]$ for all $M \in \mathbb{R}^{m \times d}$. We point out that the continuity at $\lambda = 0$ is formally obtained adapting the equi-coerciveness and Γ -convergence results in [115, 116] and can be regarded as a localization of the non-local cell-problem formulas defining f_λ to the local cell-problem formula for f_0 .

1.3.2 Preliminary results

Throughout this section, we shall always assume that the kernel $\rho : \mathbb{R}^d \rightarrow [0, +\infty]$ is a Borel function that satisfies (ρ_1) and (ρ_2) , and that the density $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow [0, +\infty)$ is a Borel function that satisfies (P), (C), and (GC). Under these assumptions, the non-local energies we investigate fit in a very general framework that has been studied in [5]. We then draw on this reference and mention several preliminary results, some of which will be of use also in the forthcoming Section 1.4. We point out that in each of the following statements, the Γ -limit is meant with respect to the strong convergence in $L^p(\Omega; \mathbb{R}^m)$.

Remark 1.3.3. This section is entirely devoted to the study of functional of convolution-type (1.3.5). For this reason, and since no ambiguity may arise, we use the shorthand

$$F_\varepsilon^{\text{conv}} = F_\varepsilon.$$

We introduce a version of the functionals (1.3.5) where the pairwise interactions are confined to points with finite distance. To this end, for all $\varepsilon > 0$ and $T > 0$, we define ‘truncated’ and ‘localized’ functionals $F_\varepsilon^T : L^p(\Omega; \mathbb{R}^m) \times \mathcal{A}_{\text{reg}}(\Omega) \rightarrow [0, +\infty]$ as

$$F_\varepsilon^T(u, A) := \frac{1}{\varepsilon^d} \iint_{A \times A \cap \{(x,y): |y-x| \leq \varepsilon T\}} \rho\left(\frac{y-x}{\varepsilon}\right) f\left(\frac{x}{\delta_\varepsilon}, \frac{y}{\delta_\varepsilon}, \frac{u(y)-u(x)}{\varepsilon}\right) dx dy.$$

Performing the change of variables $\xi := (y-x)/\varepsilon$, we rewrite the above functionals in a more handy form that we will widely use in what follows; that is,

$$F_\varepsilon^T(u, A) = \int_{B_T} \rho(\xi) \int_{A_\varepsilon(\xi)} f\left(\frac{x}{\delta_\varepsilon}, \frac{x+\varepsilon\xi}{\delta_\varepsilon}, \frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\right) dx d\xi, \quad (1.3.7)$$

where $A_\varepsilon(\xi)$ is given in (1.1.1). If $A = \Omega$ we simply write $F_\varepsilon^T(u)$ in place of $F_\varepsilon^T(u, \Omega)$. The notation for the localized functionals is immediately adapted to the non-truncated energies $\{F_\varepsilon\}_\varepsilon$ in (1.3.5).

According to the following result (see [5, Lemma 5.1]), the asymptotic analysis of general non-local energies can be simplified by considering first finite-range interactions kernels.

Lemma 1.3.4. *Let $\{T_h\}_h$ be a sequence monotonically increasing to $+\infty$ and let $F_\varepsilon^{T_h}$ be defined by (1.3.7). Assume that there exist functionals $\{F^{T_h}\}_h$ such that for all $h \in \mathbb{N}$ we have*

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon^{T_h}(u, A) = F^{T_h}(u, A)$$

for every $u \in L^p(\Omega; \mathbb{R}^m)$ and $A \in \mathcal{A}_{\text{reg}}(\Omega)$. Then

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u, A) = \lim_{h \rightarrow +\infty} F^{T_h}(u, A)$$

for every $u \in L^p(\Omega; \mathbb{R}^m)$ and $A \in \mathcal{A}_{\text{reg}}(\Omega)$.

Following [5, Proposition 6.1], we prove that if an integral representation is in place for the Γ -limit of $\{F_\varepsilon^T\}_\varepsilon$ with $T > 0$, then, as a consequence of the assumption (P), it holds that the integrand only depends on the gradient variable.

Lemma 1.3.5. *Let $T > 0$ and let F_ε^T be defined by (1.3.7). Assume that there exists a Carathéodory function $\varphi : \Omega \times \mathbb{R}^{m \times d} \rightarrow [0, +\infty)$ such that*

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon^T(u, A) = \int_A \varphi(x, \nabla u) dx$$

for every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and $A \in \mathcal{A}_{\text{reg}}(\Omega)$. Then, φ is independent of the first variable.

Proof. We set

$$F(u, A) := \int_A \varphi(x, \nabla u) dx$$

for $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and $A \in \mathcal{A}_{\text{reg}}(\Omega)$ and we note that the conclusion follows if we prove that

$$F(Mx, B_r(y)) = F(Mx, B_r(y'))$$

for every $M \in \mathbb{R}^{m \times d}$ and for every $y, y' \in \mathbb{R}^d, r > 0$ such that $B_r(y)$ and $B_r(y')$ are contained in Ω . To prove this claim, it suffices to show that

$$F(Mx, B_{r'}(y)) \leq F(Mx, B_r(y'))$$

for every M, y, y', r as above and $r' \in (0, r)$.

Let us fix a positive sequence $\{\varepsilon_j\}_j$ converging to 0 and set $\{\delta_j := \delta_{\varepsilon_j}\}_j$. Consider $\{u_j\}_j$ such that $u_j \rightarrow Mx$ in $L^p(B_r(y'); \mathbb{R}^m)$ as $j \rightarrow +\infty$ and

$$\lim_{j \rightarrow +\infty} F_{\varepsilon_j}^T(u_j, B_r(y')) = F(Mx, B_r(y')). \quad (1.3.8)$$

Note that, upon assuming that j is large enough, we have

$$B_{r'}(y) - \delta_j \left\lfloor \frac{y - y'}{\delta_j} \right\rfloor \subseteq B_r(y')$$

and then, the functions

$$v_j(x) := u_j\left(x - \delta_j \left\lfloor \frac{y - y'}{\delta_j} \right\rfloor\right) + \delta_j M \left\lfloor \frac{y - y'}{\delta_j} \right\rfloor, \quad j \in \mathbb{N},$$

are well defined on $B_{r'}(y)$ and such that $v_j \rightarrow Mx$ in $L^p(B_{r'}(y); \mathbb{R}^m)$ as $j \rightarrow +\infty$. Moreover, by a change of variables and assumption (P), it holds

$$\begin{aligned} F_{\varepsilon_j}^T(v_j, B_{r'}(y)) &= \int_{B_T} \rho(\xi) \int_{(B_{r'}(y))_{\varepsilon_j}(\xi)} f\left(\frac{x}{\delta_j}, \frac{x + \varepsilon_j \xi}{\delta_j}, \frac{v_j(x + \varepsilon_j \xi) - v_j(x)}{\varepsilon_j}\right) dx d\xi \\ &= \int_{B_T} \rho(\xi) \int_{(B_{r'}(y) - \delta_j \left\lfloor \frac{y - y'}{\delta_j} \right\rfloor)_{\varepsilon_j}(\xi)} f\left(\frac{x}{\delta_j}, \frac{x + \varepsilon_j \xi}{\delta_j}, \frac{u_j(x + \varepsilon_j \xi) - u_j(x)}{\varepsilon_j}\right) dx d\xi \\ &\leq F_{\varepsilon_j}^T(u_j, B_r(y')). \end{aligned}$$

Therefore, recalling (1.3.8), we infer

$$F(Mx, B_r(y')) = \lim_{j \rightarrow +\infty} F_{\varepsilon_j}^T(u_j, B_r(y')) \geq \liminf_{j \rightarrow +\infty} F_{\varepsilon_j}^T(v_j, B_{r'}(y)) \geq F(Mx, B_{r'}(y)),$$

which concludes the proof. $\square$

The following compactness and integral representation result is a consequence of [5, Theorem 5.1] and Lemma 1.3.5, and constitutes the starting point of our analysis.

Theorem 1.3.6. *Let $T > 0$ and let F_{ε}^T be defined by (1.3.7). Then, given any positive sequence $\{\varepsilon_j\}_j$ converging to 0, there exist a (not relabelled) subsequence and a quasiconvex function $\varphi : \mathbb{R}^{m \times d} \rightarrow [0, +\infty)$ such that*

$$\Gamma\text{-}\lim_{j \rightarrow +\infty} F_{\varepsilon_j}^T(u, A) = \begin{cases} \int_A \varphi(\nabla u) dx & \text{if } u \in W^{1,p}(A; \mathbb{R}^m), \\ +\infty & \text{if } u \in L^p(\Omega; \mathbb{R}^m) \setminus W^{1,p}(A; \mathbb{R}^m) \end{cases}$$

for every $A \in \mathcal{A}_{\text{reg}}(\Omega)$.

We recall [5, Proposition 5.4], which establishes the stability of Γ -convergence for fixed boundary conditions.

Proposition 1.3.7. *Let $T, s > 0$, $M \in \mathbb{R}^{m \times d}$, and $\{\varepsilon_j\}_j$ be a positive sequence converging to 0. Assume that there exists a functional F such that*

$$\Gamma\text{-}\lim_{j \rightarrow +\infty} F_{\varepsilon_j}^T(u, A) = F(u, A)$$

for every $u \in L^p(\Omega; \mathbb{R}^m)$ and $A \in \mathcal{A}_{\text{reg}}(\Omega)$. Then

$$\lim_{j \rightarrow +\infty} \inf \{F_{\varepsilon_j}^T(u, A) : u \in \mathcal{D}_{s\varepsilon_j, M}(A; \mathbb{R}^m)\} = \inf \{F(u, A) : u(x) - Mx \in W_0^{1,p}(A; \mathbb{R}^m)\}$$

for every $A \in \mathcal{A}_{\text{reg}}(\Omega)$, where $\mathcal{D}_{s\varepsilon_j, M}(A; \mathbb{R}^m)$ is as defined in (1.1.2).

To conclude, we state a simplified version of [72, Theorem 7.5], a useful result about the lower semicontinuity of integral functionals with respect to the strong-weak convergence in $L^p \times L^p$.

Theorem 1.3.8. *Let $p \in (1, +\infty)$, $\ell \in \mathbb{N}^+$, $E \subseteq \mathbb{R}^\ell$ be measurable, and $\Psi : E \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow [0, +\infty)$ be a Borel function such that*

$$\Psi(s, \cdot, \cdot) \text{ is continuous for almost every } s \in E$$

and

$$\Psi(s, t, \cdot) \text{ is convex for almost every } s \in E \text{ and for every } t \in \mathbb{R}^d.$$

Then, the functional defined on $L^p(E; \mathbb{R}^d) \times L^p(E; \mathbb{R}^m)$ as

$$(v, u) \mapsto \int_E \Psi(s, v(s), u(s)) ds$$

is sequentially weakly lower semicontinuous with respect to the strong-weak convergence in $L^p(E; \mathbb{R}^d) \times L^p(E; \mathbb{R}^m)$.

1.3.3 Asymptotic analysis of functionals with finite-range interactions

This subsection is devoted to the asymptotic analysis of functionals (1.3.7) for which the interaction kernel ρ is supported on a ball. This analysis, in combination with Lemma 1.3.4, shall allow us to prove the main result. We begin with two preliminary lemmas.

Lemma 1.3.9. *Let $M \in \mathbb{R}^{m \times d}$, $\{u_j\}_j \subset L_{\#, M}^p(Q_1; \mathbb{R}^m)$, $\{\lambda_j\}_j \subset (0, +\infty)$, and $v_j(x) := u_j(x) - Mx, j \in \mathbb{N}$. For every $E \subset \mathbb{R}^d$ measurable bounded set and $r > 0$, there exists a positive constant C depending on d, p, r , and E such that*

$$\int_E \int_{Q_1} \left| \frac{v_j(x + \lambda_j \xi) - v_j(x)}{\lambda_j} \right|^p dx d\xi \leq C \int_{B_r} \int_{Q_1} \left| \frac{v_j(x + \lambda_j \xi) - v_j(x)}{\lambda_j} \right|^p dx d\xi$$

for every $j \in \mathbb{N}$. In particular, if we assume that

$$\sup_j \int_{B_r} \int_{Q_1} \left| \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j} \right|^p dx d\xi < +\infty$$

for some $r > 0$, then

$$\sup_j \int_E \int_{Q_1} \left| \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j} \right|^p dx d\xi < +\infty.$$

Proof. It suffices to prove the statement in the case $E = B_R$. If $R \leq r$ this is obviously true; therefore, we prove the statement in the case $R = 2r$, so that the general case follows by a simple dyadic argument.

Using the changes of variables $\xi' := \xi/2$ and $x' := x + \lambda_j \xi'$, for all $j \in \mathbb{N}$ we have

$$\begin{aligned} \int_{B_{2r}} \int_{Q_1} \left| \frac{v_j(x + \lambda_j \xi) - v_j(x)}{\lambda_j} \right|^p dx d\xi &= 2^d \int_{B_r} \int_{Q_1} \left| \frac{v_j(x + 2\lambda_j \xi') - v_j(x)}{\lambda_j} \right|^p dx d\xi' \\ &\leq 2^{d+p-1} \left\{ \int_{B_r} \int_{Q_1} \left| \frac{v_j(x + 2\lambda_j \xi') - v_j(x + \lambda_j \xi')}{\lambda_j} \right|^p dx d\xi' \right. \\ &\quad \left. + \int_{B_r} \int_{Q_1} \left| \frac{v_j(x + \lambda_j \xi') - v_j(x)}{\lambda_j} \right|^p dx d\xi' \right\} \\ &= 2^{d+p-1} \left\{ \int_{B_r} \int_{Q_1 + \lambda_j \xi'} \left| \frac{v_j(x' + \lambda_j \xi') - v_j(x')}{\lambda_j} \right|^p dx' d\xi' \right. \\ &\quad \left. + \int_{B_r} \int_{Q_1} \left| \frac{v_j(x + \lambda_j \xi') - v_j(x)}{\lambda_j} \right|^p dx d\xi' \right\}. \end{aligned}$$

Since v_j is Q_1 -periodic, the function

$$x' \mapsto \left| \frac{v_j(x' + \lambda_j \xi') - v_j(x')}{\lambda_j} \right|^p$$

is Q_1 -periodic as well for every $\xi' \in \mathbb{R}^d$, hence,

$$\int_{Q_1 + \lambda_j \xi'} \left| \frac{v_j(x' + \lambda_j \xi') - v_j(x')}{\lambda_j} \right|^p dx' = \int_{Q_1} \left| \frac{v_j(x' + \lambda_j \xi') - v_j(x')}{\lambda_j} \right|^p dx'$$

and then

$$\int_{B_{2r}} \int_{Q_1} \left| \frac{v_j(x + \lambda_j \xi) - v_j(x)}{\lambda_j} \right|^p dx d\xi \leq 2^{d+p} \int_{B_r} \int_{Q_1} \left| \frac{v_j(x + \lambda_j \xi) - v_j(x)}{\lambda_j} \right|^p dx d\xi,$$

which proves the first part of the statement.

The second part of the statement is an immediate consequence. $\square$

The following lemma provides some precompactness results for sequences having equi-bounded energy.

Lemma 1.3.10. *Let $M \in \mathbb{R}^{m \times d}$, $\{u_j\}_j \subset L^p_{\#,M}(Q_1; \mathbb{R}^m)$, $\{\lambda_j\}_j \subset \mathbb{R}^+$ such that $\lambda_j \rightarrow \lambda \in [0, +\infty)$ as $j \rightarrow +\infty$, and assume that there exists A a bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary such that $Q_1 \subseteq A$ and*

$$\sup_j \left\{ \int_{Q_1} u_j dx + \int_{B_r} \int_A \left| \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j} \right|^p dx d\xi \right\} < +\infty$$

for some $r > 0$. The following hold:

- (i) if $\lambda = 0$, there exist a subsequence $\{u_{j_k}\}_k$ and a function $u \in W^{1,p}(A; \mathbb{R}^m)$ such that $u_{j_k} \rightarrow u$ strongly in $L^p(A; \mathbb{R}^m)$ as $k \rightarrow +\infty$;
- (ii) if $\lambda \in (0, +\infty)$, there exist a subsequence $\{u_{j_k}\}_k$ and a function $u \in L^p(A; \mathbb{R}^m)$ such that $u_{j_k} \rightharpoonup u$ weakly in $L^p(A; \mathbb{R}^m)$ as $k \rightarrow +\infty$;

in both cases, it holds that $\sup_j \|u_j\|_{L^p(Q_1; \mathbb{R}^m)}$ is finite.

Proof. The statement (i) readily follows by [5, Corollary 4.2].

As for the proof of (ii), we observe that, by the translation invariance of the functional, it is not restrictive to suppose that

$$\int_{Q_1} u_j dx = 0 \quad (1.3.9)$$

and that $\lambda/2 \leq \lambda_j \leq 2\lambda$ for every $j \in \mathbb{N}$; moreover, since $\{u_j\}_j \subset L^p_{\#,M}(Q_1; \mathbb{R}^m)$, we may further assume that $A = Q_1$.

Using Lemma 1.3.9 with $E = Q_{2/\lambda}$, we get

$$S := \sup_j \int_{Q_{2/\lambda}} \int_{Q_1} \left| \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j} \right|^p dx d\xi < +\infty;$$

hence, applying the change of variables $\xi' := \lambda_j \xi$, we obtain

$$\begin{aligned} (2\lambda)^p S &\geq \sup_j \int_{Q_{2/\lambda}} \int_{Q_1} |u_j(x + \lambda_j \xi) - u_j(x)|^p dx d\xi \\ &= \sup_j \lambda_j^{-d} \int_{Q_{\frac{2\lambda_j}{\lambda}}} \int_{Q_1} |u_j(x + \xi') - u_j(x)|^p dx d\xi' \\ &\geq (2\lambda)^{-d} \sup_j \int_{Q_1} \int_{Q_1} |u_j(x + \xi') - u_j(x)|^p dx d\xi'. \end{aligned} \quad (1.3.10)$$

Resorting to (1.3.9) and to the fact that $u_j(x) - Mx$ is Q_1 -periodic, we apply Jensen's inequality to infer

$$\begin{aligned} \int_{Q_1} |u_j(x)|^p dx &= \int_{Q_1} \left| u_j(x) - \int_{Q_1} u_j(\xi) d\xi \right|^p dx \\ &= \int_{Q_1} \left| \int_{Q_1} u_j(x) - u_j(x + \xi) + u_j(x + \xi) - u_j(\xi) d\xi \right|^p dx \\ &= \int_{Q_1} \left| \int_{Q_1} u_j(x) - u_j(x + \xi) + Mx d\xi \right|^p dx \\ &\leq 2^{p-1} \left\{ \int_{Q_1} \int_{Q_1} |u_j(x + \xi) - u_j(x)|^p dx d\xi + d^{\frac{p}{2}} |M|^p \right\}, \end{aligned}$$

which, combined with (1.3.10), leads to

$$\sup_j \int_{Q_1} |u_j(x)|^p dx \leq 2^{p-1} \{ (2\lambda)^{d+p} S + d^{\frac{p}{2}} |M|^p \} < +\infty,$$

that implies the thesis. $\square$

From now on, we let $\{\varepsilon_j\}_j$ and $\{\delta_j := \delta_{\varepsilon_j}\}_j$ be positive sequences such that $\varepsilon_j \rightarrow 0$ and $\delta_j \rightarrow 0$ as $j \rightarrow +\infty$, we set

$$\lambda_j := \frac{\varepsilon_j}{\delta_j}, \quad j \in \mathbb{N},$$

and we assume that there exists

$$\lambda := \lim_{j \rightarrow +\infty} \lambda_j \in [0, +\infty].$$

We let T be fixed so that

$$T > r_0 \quad \text{and} \quad Q_1 \subset B_T,$$

with r_0 the positive constant appearing in (ρ_1) ; both these conditions are not restrictive since, eventually, we will let $T \rightarrow +\infty$.

We consider the truncated functionals

$$F_{\varepsilon_j}^T(u, A) := \int_{B_T} \rho(\xi) \int_{A_{\varepsilon_j}(\xi)} f\left(\frac{x}{\delta_j}, \frac{x + \varepsilon_j \xi}{\delta_j}, \frac{u(x + \varepsilon_j \xi) - u(x)}{\varepsilon_j}\right) dx d\xi$$

for $u \in L^p(\Omega; \mathbb{R}^m)$ and $A \in \mathcal{A}_{\text{reg}}(\Omega)$, and we recall that the kernel ρ satisfies the assumptions (ρ_1) and (ρ_2) , and that the density f satisfies the assumptions (P), (C), and (GC). We prove that if the Γ -limit of the truncated functionals $\{F_{\varepsilon_j}^T\}_j$ exists and equals an integral functional, then its energy density φ can be characterized through an ‘asymptotic formula’ and, when λ is finite, also by means of certain ‘non-local cell-problem formulas’ involving the parameters $\{\lambda_j\}_j$.

Proposition 1.3.11. *Assume that there exists a quasiconvex function $\varphi : \mathbb{R}^{m \times d} \rightarrow [0, +\infty)$ such that*

$$\Gamma\text{-}\lim_{j \rightarrow +\infty} F_{\varepsilon_j}^T(u, A) = \begin{cases} \int_A \varphi(\nabla u) dx & \text{if } u \in W^{1,p}(A; \mathbb{R}^m), \\ +\infty & \text{if } u \in L^p(\Omega; \mathbb{R}^m) \setminus W^{1,p}(A; \mathbb{R}^m) \end{cases}$$

for every $A \in \mathcal{A}_{\text{reg}}(\Omega)$, where the Γ -convergence is meant with respect to the convergence in $L^p(\Omega; \mathbb{R}^m)$. For every $M \in \mathbb{R}^{m \times d}$ the following hold:

(i) for every $\lambda \in [0, +\infty]$ we have

$$\varphi(M) = \lim_{j \rightarrow +\infty} \inf \left\{ \left(\frac{\delta_j}{r} \right)^d \int_{B_T} \rho(\xi) \int_{(Q_{r/\delta_j})_{\lambda_j}(\xi)} f\left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j}\right) dx d\xi : \right. \\ \left. u \in \mathcal{D}_{T\lambda_j, M}(Q_{r/\delta_j}; \mathbb{R}^m) \right\}$$

for some $r > 0$;

(ii) for every $\lambda \in [0, +\infty]$ we have

$$\varphi(M) \geq \limsup_{j \rightarrow +\infty} \inf \left\{ \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j}\right) dx d\xi : u \in L_{\#, M}^p(Q_1; \mathbb{R}^m) \right\};$$

(iii) for every $\lambda \in [0, +\infty)$ we have

$$\varphi(M) = \lim_{j \rightarrow +\infty} \inf \left\{ \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j}\right) dx d\xi : u \in L_{\#, M}^p(Q_1; \mathbb{R}^m) \right\}.$$

Proof. Fix $M \in \mathbb{R}^{m \times d}$ and let Q be a cube of side-length r which is contained in Ω . Using the quasiconvexity of φ and the convergence of boundary-value problems established in Proposition 1.3.7 with $A = Q$ and $s = T$, we have

$$\begin{aligned} \varphi(M) &= \inf \left\{ \frac{1}{r^d} \int_Q \varphi(\nabla u) dx : u - Mx \in W_0^{1,p}(Q; \mathbb{R}^m) \right\} \\ &= \lim_{j \rightarrow +\infty} \inf \left\{ \frac{1}{r^d} F_j^T(u, Q) : u \in \mathcal{D}_{T\varepsilon_j, M}(Q; \mathbb{R}^m) \right\}. \end{aligned}$$

For the sake of exposition, we assume that $Q = Q_r = (0, r)^d$, the general case being analogous. Consider $u \in \mathcal{D}_{T\varepsilon_j, M}(Q_r; \mathbb{R}^m)$, and set $v(x) := u(\delta_j x)/\delta_j$. We have that $v \in \mathcal{D}_{T\lambda_j, M}(Q_{r/\delta_j}; \mathbb{R}^m)$ and, by a change of variables,

$$\begin{aligned} F_{\varepsilon_j}^T(u, Q_r) &= \int_{B_T} \rho(\xi) \int_{(Q_r)_{\varepsilon_j}(\xi)} f\left(\frac{x}{\delta_j}, \frac{x + \varepsilon_j \xi}{\delta_j}, \frac{u(x + \varepsilon_j \xi) - u(x)}{\varepsilon_j}\right) dx d\xi \\ &= \int_{B_T} \rho(\xi) \int_{(Q_r)_{\varepsilon_j}(\xi)} f\left(\frac{x}{\delta_j}, \frac{x + \varepsilon_j \xi}{\delta_j}, \frac{v(\frac{x}{\delta_j} + \frac{\varepsilon_j}{\delta_j} \xi) - v(\frac{x}{\delta_j})}{\varepsilon_j/\delta_j}\right) dx d\xi \\ &= \delta_j^d \int_{B_T} \rho(\xi) \int_{(Q_{r/\delta_j})_{\lambda_j}(\xi)} f\left(x, x + \lambda_j \xi, \frac{v(x + \lambda_j \xi) - v(x)}{\lambda_j}\right) dx d\xi. \end{aligned}$$

By the arbitrariness of u , we obtain that

$$\begin{aligned} \varphi(M) &= \lim_{j \rightarrow +\infty} \inf \left\{ \left(\frac{\delta_j}{r}\right)^d \int_{B_T} \rho(\xi) \int_{(Q_{r/\delta_j})_{\lambda_j}(\xi)} f\left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j}\right) dx d\xi : \right. \\ &\quad \left. u \in \mathcal{D}_{T\lambda_j, M}(Q_{r/\delta_j}; \mathbb{R}^m) \right\}, \end{aligned}$$

which proves (i).

Now we let

$$\varphi_j(M) := \inf \left\{ \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j}\right) dx d\xi : u \in L_{\#, M}^p(Q_1; \mathbb{R}^m) \right\}$$

and prove that

$$\varphi(M) \geq \limsup_{j \rightarrow +\infty} \varphi_j(M). \quad (1.3.11)$$

Given $u \in \mathcal{D}_{T\lambda_j, M}(Q_{r/\delta_j}; \mathbb{R}^m)$, we define

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in Q_{r/\delta_j}, \\ Mx & \text{if } x \in Q_{\lceil r/\delta_j \rceil} \setminus Q_{r/\delta_j}, \end{cases}$$

we note that $\tilde{u} \in \mathcal{D}_{T\lambda_j, M}(Q_{\lceil r/\delta_j \rceil}; \mathbb{R}^m)$, and we write

$$\begin{aligned} &\left(\frac{\delta_j}{r}\right)^d \int_{B_T} \rho(\xi) \int_{(Q_{\lceil r/\delta_j \rceil})_{\lambda_j}(\xi)} f\left(x, x + \lambda_j \xi, \frac{\tilde{u}(x + \lambda_j \xi) - \tilde{u}(x)}{\lambda_j}\right) dx d\xi \\ &= \left(\frac{\delta_j}{r}\right)^d \int_{B_T} \rho(\xi) \int_{(Q_{r/\delta_j})_{\lambda_j}(\xi)} f\left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j}\right) dx d\xi \\ &\quad + \left(\frac{\delta_j}{r}\right)^d \int_{B_T} \rho(\xi) \int_{(Q_{\lceil r/\delta_j \rceil})_{\lambda_j}(\xi) \setminus (Q_{r/\delta_j})_{\lambda_j}(\xi)} f\left(x, x + \lambda_j \xi, \frac{\tilde{u}(x + \lambda_j \xi) - \tilde{u}(x)}{\lambda_j}\right) dx d\xi. \end{aligned}$$

If $\xi \in B_T$ and $x \in (Q_{\lceil r/\delta_j \rceil})_{\lambda_j}(\xi) \setminus (Q_{r/\delta_j})_{\lambda_j}(\xi)$, then $\tilde{u}(x + \lambda_j \xi) - \tilde{u}(x) = \lambda_j M \xi$, hence, using the upper bound in (GC) we get

$$\begin{aligned} &\left(\frac{\delta_j}{r}\right)^d \int_{B_T} \rho(\xi) \int_{(Q_{\lceil r/\delta_j \rceil})_{\lambda_j}(\xi) \setminus (Q_{r/\delta_j})_{\lambda_j}(\xi)} f\left(x, x + \lambda_j \xi, \frac{\tilde{u}(x + \lambda_j \xi) - \tilde{u}(x)}{\lambda_j}\right) dx d\xi \\ &= \left(\frac{\delta_j}{r}\right)^d \int_{B_T} \rho(\xi) \int_{(Q_{\lceil r/\delta_j \rceil})_{\lambda_j}(\xi) \setminus (Q_{r/\delta_j})_{\lambda_j}(\xi)} f(x, x + \lambda_j \xi, M \xi) dx d\xi \\ &\leq \left(\frac{\delta_j}{r}\right)^d |E_j| \beta |M|^p \int_{B_T} \rho(\xi) |\xi|^p d\xi \\ &=: \theta_j, \end{aligned}$$

where we have set

$$E_j := Q_{\lceil r/\delta_j \rceil} \setminus Q_{r/\delta_j - T\lambda_j}, \quad j \in \mathbb{N}, \quad (1.3.12)$$

and we have used that $(Q_{\lceil r/\delta_j \rceil})_{\lambda_j}(\xi) \setminus (Q_{r/\delta_j})_{\lambda_j}(\xi) \subset E_j$ for all $\xi \in B_T$. Recalling (ρ_2) we infer there exists a positive constant C , independent of j , such that

$$\theta_j = C \left(\frac{\delta_j}{r} \right)^d \left(\left\lceil \frac{r}{\delta_j} \right\rceil^d - \left(\frac{r}{\delta_j} - T\lambda_j \right)^d \right) = C \left[\left(\frac{\lceil r/\delta_j \rceil}{r/\delta_j} \right)^d - \left(1 - \frac{T}{r} \varepsilon_j \right)^d \right],$$

which tends to 0 as $j \rightarrow +\infty$. As a consequence of these observations, we get that

$$\begin{aligned} & \left(\frac{\delta_j}{r} \right)^d \int_{B_T} \rho(\xi) \int_{(Q_{r/\delta_j})_{\lambda_j}(\xi)} f\left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j}\right) dx d\xi \\ & \geq \left(\frac{\delta_j}{r} \right)^d \int_{B_T} \rho(\xi) \int_{(Q_{\lceil r/\delta_j \rceil})_{\lambda_j}(\xi)} f\left(x, x + \lambda_j \xi, \frac{\tilde{u}(x + \lambda_j \xi) - \tilde{u}(x)}{\lambda_j}\right) dx d\xi - \theta_j, \end{aligned}$$

which, by the arbitrariness of u , yields

$$\begin{aligned} \varphi(M) &= \lim_{j \rightarrow +\infty} \inf \left\{ \left(\frac{\delta_j}{r} \right)^d \int_{B_T} \rho(\xi) \int_{(Q_{r/\delta_j})_{\lambda_j}(\xi)} f\left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j}\right) dx d\xi : \right. \\ & \quad \left. u \in \mathcal{D}_{T\lambda_j, M}(Q_{r/\delta_j}; \mathbb{R}^m) \right\} \\ &\geq \limsup_{j \rightarrow +\infty} \inf \left\{ \left(\frac{\delta_j}{r} \right)^d \int_{B_T} \rho(\xi) \int_{(Q_{\lceil r/\delta_j \rceil})_{\lambda_j}(\xi)} f\left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j}\right) dx d\xi : \right. \\ & \quad \left. u \in \mathcal{D}_{T\lambda_j, M}(Q_{\lceil r/\delta_j \rceil}; \mathbb{R}^m) \right\}. \end{aligned}$$

Consider now $u \in \mathcal{D}_{T\lambda_j, M}(Q_{\lceil r/\delta_j \rceil}; \mathbb{R}^m)$, let $\tilde{w}$ be the $Q_{\lceil r/\delta_j \rceil}$ -periodic extension of $u(x) - Mx$, let $\tilde{u}(x) := \tilde{w}(x) + Mx$, and define the function

$$w(x) := \frac{1}{\lceil r/\delta_j \rceil^d} \sum_{i \in \mathbb{Z}^d \cap [0, \lceil r/\delta_j \rceil]^d} \tilde{u}(x + i).$$

Since $w \in L^p_{\#, M}(Q_1; \mathbb{R}^m)$, by the assumptions (C) and (P) we obtain

$$\begin{aligned} \varphi_j(M) &\leq \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j \xi, \frac{w(x + \lambda_j \xi) - w(x)}{\lambda_j}\right) dx d\xi \\ &\leq \frac{1}{\lceil r/\delta_j \rceil^d} \sum_{i \in \mathbb{Z}^d \cap [0, \lceil r/\delta_j \rceil]^d} \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j \xi, \frac{\tilde{u}(x + i + \lambda_j \xi) - \tilde{u}(x + i)}{\lambda_j}\right) dx d\xi \\ &= \frac{1}{\lceil r/\delta_j \rceil^d} \sum_{i \in \mathbb{Z}^d \cap [0, \lceil r/\delta_j \rceil]^d} \int_{B_T} \rho(\xi) \int_{i+Q_1} f\left(x, x + \lambda_j \xi, \frac{\tilde{u}(x + \lambda_j \xi) - \tilde{u}(x)}{\lambda_j}\right) dx d\xi \\ &= \frac{1}{\lceil r/\delta_j \rceil^d} \int_{B_T} \rho(\xi) \int_{Q_{\lceil r/\delta_j \rceil}} f\left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j}\right) dx d\xi. \end{aligned} \quad (1.3.13)$$

Moreover, observing that $Q_{\lceil r/\delta_j \rceil} \setminus (Q_{\lceil r/\delta_j \rceil})_{\lambda_j}(\xi) \subset E_j$ for all $\xi \in B_T$ and using (GC), we get

$$\begin{aligned} & \frac{1}{\lceil r/\delta_j \rceil^d} \int_{B_T} \rho(\xi) \int_{Q_{\lceil r/\delta_j \rceil} \setminus (Q_{\lceil r/\delta_j \rceil})_{\lambda_j}(\xi)} f\left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j}\right) dx d\xi \\ & \leq \frac{1}{\lceil r/\delta_j \rceil^d} |E_j| |\beta| M^p \int_{B_T} \rho(\xi) |\xi|^p d\xi \\ & = \frac{(r/\delta_j)^d}{\lceil r/\delta_j \rceil^d} \theta_j. \end{aligned} \quad (1.3.14)$$

Therefore, combining (1.3.13) with (1.3.14), we obtain

$$\left(\frac{\delta_j}{r}\right)^d \int_{B_T} \rho(\xi) \int_{(Q_{\lceil r/\delta_j \rceil})_{\lambda_j}(\xi)} f\left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j}\right) dx d\xi \geq \left(\frac{\lceil r/\delta_j \rceil}{r/\delta_j}\right)^d \varphi_j(M) - \theta_j;$$

hence, taking into account the arbitrariness of u and that $\theta_j \rightarrow 0$, we infer (1.3.11) passing to the limit, concluding the proof of (ii).

Finally, we assume that $\lambda \in [0, +\infty)$ and we prove that

$$\varphi(M) \leq \liminf_{j \rightarrow +\infty} \varphi_j(M). \quad (1.3.15)$$

Let $\{u_j\}_j \subset L^p_{\#,M}(Q_1; \mathbb{R}^m)$ be such that

$$\liminf_{j \rightarrow +\infty} \varphi_j(M) = \liminf_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j \xi, \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j}\right) dx d\xi,$$

and, by the translation invariance of the functional, suppose that

$$\int_{Q_1} u_j dx = 0$$

for every $j \in \mathbb{N}$. Since $T > r_0$, by (ρ_1) , the lower bound in (GC), and the previous step, we have

$$\varphi(M) \geq \alpha c_0 \liminf_{j \rightarrow +\infty} \int_{B_{r_0}} \int_{Q_1} \left| \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j} \right|^p dx d\xi;$$

therefore, upon extracting a not relabeled subsequence, we apply Lemma 1.3.10 with $r = r_0$ and $A = Q_1$ to obtain that $\sup_j \|u_j\|_{L^p(Q_1; \mathbb{R}^m)} < +\infty$, and then

$$\sup_j \|u_j - Mx\|_{L^p(Q_1; \mathbb{R}^m)} < +\infty. \quad (1.3.16)$$

Now we set $w_j(x) := \delta_j u_j(x/\delta_j)$, $j \in \mathbb{N}$, and note that $w_j \rightarrow Mx$ in $L^p(Q_1; \mathbb{R}^m)$ as $j \rightarrow +\infty$. To see this, we also let $v_j(x) := u_j(x) - Mx$, $j \in \mathbb{N}$, and observe that, since v_j is Q_1 -periodic,

$$\begin{aligned} \int_{Q_1} |w_j(x) - Mx|^p dx &= \int_{Q_1} \left| \delta_j v_j\left(\frac{x}{\delta_j}\right) \right|^p dx \\ &= \delta_j^{d+p} \int_{Q_{1/\delta_j}} |v_j(x)|^p dx \\ &\leq \frac{\lceil 1/\delta_j \rceil^d}{(1/\delta_j)^d} \delta_j^p \int_{Q_1} |v_j(x)|^p dx, \end{aligned}$$

which tends to 0 by (1.3.16) and the fact that $\delta_j \rightarrow 0$. Using the definition of Γ -limit and a

change of variables, we obtain

$$\begin{aligned}
\varphi(M) &= \int_{Q_1} \varphi(M) dx \leq \liminf_{j \rightarrow +\infty} F_{\varepsilon_j}^T(w_j, Q_1) \\
&= \liminf_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) \int_{(Q_1)_{\varepsilon_j}(\xi)} f\left(\frac{x}{\delta_j}, \frac{x + \varepsilon_j \xi}{\delta_j}, \frac{w_j(x + \varepsilon_j \xi) - w_j(x)}{\varepsilon_j}\right) dx d\xi \\
&= \liminf_{j \rightarrow +\infty} \delta_j^d \int_{B_T} \rho(\xi) \int_{(Q_{\frac{1}{\delta_j}})_{\lambda_j}(\xi)} f\left(x, x + \lambda_j \xi, \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j}\right) dx d\xi \\
&\leq \liminf_{j \rightarrow +\infty} \delta_j^d \int_{B_T} \rho(\xi) \int_{Q_{[1/\delta_j]}} f\left(x, x + \lambda_j \xi, \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j}\right) dx d\xi \\
&= \liminf_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j \xi, \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j}\right) dx d\xi \\
&= \liminf_{j \rightarrow +\infty} \varphi_j(M),
\end{aligned}$$

where we also used (P) and the 1-periodicity of the functions

$$x \mapsto \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j}, \quad j \in \mathbb{N}.$$

This leads to (1.3.15) and proves (iii). $\square$

Now we explicitly determine the integrand φ for each value of $\lambda \in [0, +\infty]$. We devote the remaining part of this subsection to the proof of the following proposition.

Proposition 1.3.12. *Let r_0 be the positive constant appearing in (ρ_1) , let T be such that $T > r_0$ and $Q_1 \subset B_T$, and let $F_{\varepsilon_j}^T$ be defined by (1.3.7). Assume that there exist*

$$\lambda = \lim_{j \rightarrow +\infty} \frac{\varepsilon_j}{\delta_j} \in [0, +\infty]$$

and a quasiconvex function $\varphi : \mathbb{R}^{m \times d} \rightarrow [0, +\infty)$ such that

$$\Gamma\text{-}\lim_{j \rightarrow +\infty} F_{\varepsilon_j}^T(u, A) = \begin{cases} \int_A \varphi(\nabla u) dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^m), \\ +\infty & \text{if } u \in L^p(\Omega; \mathbb{R}^m) \setminus W^{1,p}(\Omega; \mathbb{R}^m), \end{cases} \quad (1.3.17)$$

for every $A \in \mathcal{A}_{\text{reg}}(\Omega)$, where the Γ -convergence is meant with respect to the convergence in $L^p(\Omega; \mathbb{R}^m)$. For every $M \in \mathbb{R}^{m \times d}$ the following hold :

(i) if $\lambda = 0$ and (H0) holds, then

$$\varphi(M) = \inf \left\{ \int_{B_T} \int_{Q_1} \rho(\xi) f(x, x, (\nabla u)\xi) dx d\xi : u \in W_{\#,M}^{1,p}(Q_1; \mathbb{R}^m) \right\};$$

(ii) if $\lambda \in (0, +\infty)$ and (H0) holds, then

$$\varphi(M) = \inf \left\{ \int_{B_T} \int_{Q_1} \rho(\xi) f\left(x, x + \lambda \xi, \frac{u(x + \lambda \xi) - u(x)}{\lambda}\right) dx d\xi : u \in L_{\#,M}^p(Q_1; \mathbb{R}^m) \right\};$$

(iii) if $\lambda = +\infty$ and either (H1) or (H2) holds, then

$$\varphi(M) = \int_{B_T} \int_{Q_1} \int_{Q_1} \rho(\xi) f(x, y, M\xi) dx dy d\xi.$$

The subcritical and critical cases, $\lambda \in [0, +\infty)$. The cases $\lambda = 0$ and $\lambda \in (0, +\infty)$ are treated with similar arguments. We recall that, in both these instances, we assume that

$$f(x, \cdot, z) \text{ is continuous for almost every } x \in \mathbb{R}^d \text{ and for every } z \in \mathbb{R}^m. \quad (\text{H0})$$

For the sake of exposition, we first establish the corresponding lower bounds, and then we prove their optimality.

Let $M \in \mathbb{R}^{m \times d}$ be fixed. By (1.3.17) and Proposition 1.3.11(iii), there exists $\{u_j\}_j \subset L^p_{\#,M}(Q_1; \mathbb{R}^m)$ such that

$$\varphi(M) = \lim_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j \xi, \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j}\right) dx d\xi,$$

and which, by the translation invariance of the functional, it is not restrictive to suppose satisfy

$$\int_{Q_1} u_j dx = 0 \quad (1.3.18)$$

for every $j \in \mathbb{N}$. Using the lower bound in (GC) and (ρ_1) , we get that

$$\sup \int_{B_{r_0}} \int_{Q_1} \left| \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j} \right|^p dx d\xi < +\infty,$$

and then, by Lemma 1.3.9 applied with $r = r_0$ and $E = B_T$, we obtain

$$\sup_j \int_{B_T} \int_{Q_1} \left| \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j} \right|^p dx d\xi < +\infty. \quad (1.3.19)$$

Therefore, we set

$$U_j(x, \xi) := \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j}, \quad j \in \mathbb{N},$$

and infer that the sequence of functions $\{U_j\}_j$ is equi-bounded in $L^p(Q_1 \times B_T; \mathbb{R}^m)$. Upon extracting a (not relabelled) converging subsequence, we may assume there exists U such that

$$U_j(x, \xi) \rightharpoonup U(x, \xi) \quad \text{as } j \rightarrow +\infty \quad (1.3.20)$$

weakly in $L^p(Q_1 \times B_T; \mathbb{R}^m)$; then, we apply Theorem 1.3.8 with $\ell = d \times d$, $s = (\xi, x)$, and

$$E = B_T \times Q_1, \quad \Psi((\xi, x), t, q) = \rho(\xi) f(x, t, q),$$

to infer

$$\begin{aligned} \varphi(M) &= \lim_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j \xi, \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j}\right) dx d\xi \\ &\geq \int_{B_T} \rho(\xi) \int_{Q_1} f(x, x + \lambda \xi, U(x, \xi)) dx d\xi. \end{aligned} \quad (1.3.21)$$

At this point, we identify the weak limit U in accordance with the value of the parameter λ and obtain different lower bounds.

Lower bound in the subcritical case, $\lambda = 0$. By the 1-periodicity of $u_j - Mx$, it is easy to observe that (1.3.19) implies

$$\sup_j \int_{B_T} \int_A \left| \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j} \right|^p dx d\xi < +\infty \quad (1.3.22)$$

for A any bounded open set. Resorting to (1.3.18) and (1.3.22), we apply Lemma 1.3.10(i) with A any bounded open set containing the closure of Q_1 and $r = T$ in order to obtain a (not relabeled) subsequence $\{u_j\}_j$ and a function $u \in W_{\#,M}^{1,p}(Q_1; \mathbb{R}^m)$ such that $u_j \rightarrow u$ strongly in $L^p(Q_1; \mathbb{R}^m)$. Note that the direct use of Lemma 1.3.10(i) with $A = Q_1$ would only imply $u - Mx \in W^{1,p}(Q_1; \mathbb{R}^m)$, which, extended by periodicity on $\mathbb{R}^d$, does not necessarily belong to $W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^m)$.

We claim that

$$U(x, \xi) = (\nabla u(x))\xi \quad (1.3.23)$$

for a.e. $x \in Q_1$ and $\xi \in B_T$. By the equi-boundedness of $\{U_j\}_j$, it is sufficient testing the weak convergence stated in (1.3.20) with a class of functions whose linear span is dense in the dual space of $L^p(Q_1 \times B_T; \mathbb{R}^m)$. Let then $\psi \in C_c^\infty(Q_1; \mathbb{R}^m)$ and $E \subseteq B_T$ be measurable; by a change of variables we have

$$\begin{aligned} \int_E \int_{Q_1} \psi(x) \cdot U_j(x, \xi) dx d\xi &= \int_E \int_{Q_1} \psi(x) \cdot \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j} dx d\xi \\ &= \int_E \int_{\mathbb{R}^d} u_j(x) \cdot \frac{\psi(x - \lambda_j \xi) - \psi(x)}{\lambda_j} dx d\xi, \end{aligned}$$

where we let $a \cdot b$ denote the scalar product in $\mathbb{R}^m$. Since $u_j - Mx$ is 1-periodic and $u_j \rightarrow u$ strongly in $L^p(Q_1; \mathbb{R}^m)$, we have that $u_j \rightarrow u$ strongly in $L^p(K; \mathbb{R}^m)$ for K any compact subset of $\mathbb{R}^d$. Recalling that ψ has compact support, it is possible to pass to the limit using the Dominated Convergence Theorem to get

$$\begin{aligned} \int_E \int_{Q_1} \psi(x) \cdot U(x, \xi) dx d\xi &= - \int_E \int_{\mathbb{R}^d} u(x) \cdot (\nabla \psi(x))\xi dx d\xi \\ &= \int_E \int_{Q_1} \psi(x) \cdot (\nabla u(x))\xi dx d\xi, \end{aligned}$$

and this proves (1.3.23). Recalling (1.3.21), we get

$$\varphi(M) \geq \inf \left\{ \int_{B_T} \rho(\xi) \int_{Q_1} f(x, x, (\nabla u)\xi) dx d\xi : u \in W_{\#,M}^{1,p}(Q_1; \mathbb{R}^m) \right\},$$

that is the lower bound for the case $\lambda = 0$.

Lower bound in the critical case, $\lambda \in (0, +\infty)$. Using (1.3.18), (1.3.19), and Lemma 1.3.10(ii), we obtain a (not relabelled) subsequence $\{u_j\}_j$ and a function $u \in L_{\#,M}^p(Q_1; \mathbb{R}^m)$ such that $u_j \rightharpoonup u$ weakly in $L^p(Q_1; \mathbb{R}^m)$. Using an argument similar to that employed in the subcritical case, it is then immediate to verify that

$$U(x, \xi) = \frac{u(x + \lambda \xi) - u(x)}{\lambda},$$

which, combined with (1.3.21), implies

$$\varphi(M) \geq \inf \left\{ \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda\xi, \frac{u(x + \lambda\xi) - u(x)}{\lambda}\right) dx d\xi : u \in L_{\#,M}^p(Q_1; \mathbb{R}^m) \right\},$$

achieving the lower bound in the case $\lambda \in (0, +\infty)$.

Now we prove that the lower bounds previously obtained are optimal. Although the proofs for each regime are similar, we illustrate them separately for the sake of clarity.

Upper bound in the subcritical case, $\lambda = 0$. Let $\eta > 0$ and let $w \in W_{\#,M}^{1,p}(Q_1; \mathbb{R}^m)$ be such that

$$\begin{aligned} \inf \left\{ \int_{B_T} \rho(\xi) \int_{Q_1} f(x, x, (\nabla u)\xi) dx d\xi : u \in W_{\#,M}^{1,p}(Q_1; \mathbb{R}^m) \right\} + \eta \\ \geq \int_{B_T} \rho(\xi) \int_{Q_1} f(x, x, (\nabla w)\xi) dx d\xi. \end{aligned}$$

We observe that the functional that appears in the minimum problem is continuous with respect to the strong convergence in $W_{\text{loc}}^{1,p}(\mathbb{R}^d; \mathbb{R}^m)$. This follows applying the Dominated Convergence Theorem whose use is justified by (C), by the estimate

$$\rho(\xi) f(x, x, (\nabla u(x))\xi) \leq \beta \rho(\xi) |\xi|^p |\nabla u(x)|^p \quad \text{for a.e. } (x, \xi) \in Q_1 \times B_T, \quad (1.3.24)$$

which follows by (GC), and by (ρ_2) . Therefore, it is not restrictive to suppose that $w \in C^\infty(\mathbb{R}^d; \mathbb{R}^m)$.

Applying Proposition 1.3.11(iii) we have

$$\begin{aligned} \varphi(M) &= \lim_{j \rightarrow +\infty} \inf \left\{ \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j}\right) dx d\xi : u \in L_{\#,M}^p(Q_1; \mathbb{R}^m) \right\} \\ &\leq \lim_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j \xi, \frac{w(x + \lambda_j \xi) - w(x)}{\lambda_j}\right) dx d\xi. \end{aligned}$$

In order to conclude, we use the Dominated Convergence Theorem. The pointwise convergence is ensured by the fact that f satisfies (C) and (H0), and that

$$\frac{w(x + \lambda_j \xi) - w(x)}{\lambda_j} \rightarrow (\nabla w(x))\xi$$

for all $x, \xi \in \mathbb{R}^d$ as $j \rightarrow +\infty$; a uniform bound for the integrands

$$\rho(\xi) f\left(x, x + \lambda_j \xi, \frac{w(x + \lambda_j \xi) - w(x)}{\lambda_j}\right), \quad j \in \mathbb{N},$$

holds in virtue of (GC), the inequality

$$\left| \frac{w(x + \lambda_j \xi) - w(x)}{\lambda_j} \right| \leq \|\nabla w\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^{m \times d})} |\xi| \quad (1.3.25)$$

for all $x, \xi \in \mathbb{R}^d$, and (ρ_2) . Finally, we get

$$\begin{aligned} \varphi(M) &\leq \int_{B_T} \rho(\xi) \int_{Q_1} f(x, x, (\nabla w(x))\xi) dx d\xi \\ &\leq \inf \left\{ \int_{B_T} \rho(\xi) \int_{Q_1} f(x, x, (\nabla u)\xi) dx d\xi : u \in W_{\#,M}^{1,p}(Q_1; \mathbb{R}^m) \right\} + \eta, \end{aligned}$$

which, by the arbitrariness of η , concludes the proof.

Upper bound in the critical case, $\lambda \in (0, +\infty)$. Let $\eta > 0$ and let $w \in L^p_{\#,M}(Q_1; \mathbb{R}^m) \cap C^\infty(\mathbb{R}^d; \mathbb{R}^m)$ be such that

$$\begin{aligned} \inf \left\{ \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda\xi, \frac{u(x + \lambda\xi) - u(x)}{\lambda}\right) dx d\xi : u \in L^p_{\#,M}(Q_1; \mathbb{R}^m) \right\} + \eta \\ \geq \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda\xi, \frac{w(x + \lambda\xi) - w(x)}{\lambda}\right) dx d\xi. \end{aligned}$$

Note indeed that a similar argument to that used for $\lambda = 0$ proves the continuity of the functional with respect to the strong L^p -convergence.

Once again, we apply Proposition 1.3.11(iii), use w as a test function for the corresponding minimum problems, and apply the Dominated Convergence Theorem (whose use is made possible also by (C), (H0), (GC), (1.3.25), and (ρ_2)) to infer

$$\begin{aligned} \varphi(M) &\leq \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda\xi, \frac{w(x + \lambda\xi) - w(x)}{\lambda}\right) dx d\xi \\ &\leq \inf \left\{ \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda\xi, \frac{u(x + \lambda\xi) - u(x)}{\lambda}\right) dx d\xi : u \in L^p_{\#,M}(Q_1; \mathbb{R}^m) \right\} + \eta, \end{aligned}$$

which, by the arbitrariness of η , concludes the proof.

The supercritical case, $\lambda = +\infty$. The supercritical regime requires a different and more complex argument. Here, we assume that either

$$\begin{cases} \rho(\xi) = \rho(-\xi) & \text{for almost every } \xi \in \mathbb{R}^d, \\ f(x, y, z) = f(x, y, -z) & \text{for almost every } x, y \in \mathbb{R}^d \text{ and for every } z \in \mathbb{R}^m, \end{cases} \quad (\text{H1})$$

or that

$$\int_{\mathbb{R}^d} \rho(\xi) d\xi = +\infty. \quad (\text{H2})$$

Lower bound in the supercritical case, $\lambda = +\infty$. Let $M \in \mathbb{R}^{m \times d}$ be fixed. By Proposition 1.3.11(ii), there exists $\{u_j\}_j \subset L^p_{\#,M}(Q_1; \mathbb{R}^m)$ such that

$$\varphi(M) \geq \limsup_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j \xi, \frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j}\right) dx d\xi.$$

In the supercritical regime, there are no compactness properties that we are able to infer for the sequence $\{u_j\}_j$. We consider the 1-periodic functions $v_j(x) := u_j(x) - Mx$, $j \in \mathbb{N}$, and we let

$$V_j(x, \xi) := \frac{v_j(x + \xi) - v_j(x)}{\lambda_j}, \quad j \in \mathbb{N},$$

so that

$$\varphi(M) \geq \limsup_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) \int_{Q_1} f(x, x + \lambda_j \xi, V_j(x, \lambda_j \xi) + M\xi) dx d\xi. \quad (1.3.26)$$

In order to describe the idea behind our proof, we note that if we had $V_j(x, \xi) = V(x, \xi)$ for all $j \in \mathbb{N}$, then

$$\begin{aligned} \varphi(M) &\geq \limsup_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) \int_{Q_1} f(x, x + \lambda_j \xi, V(x, \lambda_j \xi) + M\xi) dx d\xi \\ &= \limsup_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) g(\xi, \lambda_j \xi) d\xi, \end{aligned}$$

where we set

$$g(\xi, y) := \int_{Q_1} f(x, x + y, V(x, y) + M\xi) dx.$$

Since g is periodic in the second variable and $\lambda_j \rightarrow +\infty$, we may use a known result, which can be regarded as a consequence of a two-scale convergence, to obtain that

$$g(\xi, \lambda_j \xi) \rightharpoonup \int_{Q_1} g(\xi, y) dy$$

in a weak sense; and then, formally passing to the limit, we would get

$$\varphi(M) \geq \int_{B_T} \rho(\xi) \int_{Q_1} \int_{Q_1} f(x, x + y, V(x, y) + M\xi) dx dy d\xi.$$

Our aim is to mimic this reasoning taking into account that $\{V_j\}_j$ is not a constant sequence. This requires the study of the equi-integrability of an auxiliary sequence of functions defined similarly to the function g above, and some additional care has to be used since in general ρ does not belong to any L^p -space. These issues seem to prevent the direct use of the two-scale convergence and, also in our setting, allows us to pass to the limit only upon having truncated ‘vertically’ the kernel.

From now on, we shall further assume that

$$\inf_j \lambda_j > 2;$$

this condition is not restrictive since we are assuming that $\lambda_j \rightarrow +\infty$.

We note that each V_j is 1-periodic in both variables. Moreover, resorting to Lemma 1.3.9, we have that (1.3.19) holds, which implies

$$\sup_j \|V_j(x, \lambda_j \xi)\|_{L^p(Q_1 \times B_T; \mathbb{R}^m)} < +\infty. \quad (1.3.27)$$

The following observation is then a simple consequence.

Lemma 1.3.13. *It holds that*

$$\sup_j \|V_j(x, \xi)\|_{L^p(Q_1 \times Q_1; \mathbb{R}^m)} < +\infty.$$

Proof. Since $Q_1 \subset B_T$, we combine the periodicity of V_j in the second variable with a change of variables to get

$$\begin{aligned} \int_{B_T} \int_{Q_1} |V_j(x, \lambda_j \xi)|^p dx d\xi &\geq \int_{Q_1} \int_{Q_1} |V_j(x, \lambda_j \xi)|^p dx d\xi \\ &= \lambda_j^{-d} \int_{Q_{\lambda_j}} \int_{Q_1} |V_j(x, \xi)|^p dx d\xi \\ &\geq \lambda_j^{-d} [\lambda_j]^d \int_{Q_1} \int_{Q_1} |V_j(x, \xi)|^p dx d\xi \\ &\geq \frac{1}{2^d} \int_{Q_1} \int_{Q_1} |V_j(x, \xi)|^p dx d\xi, \end{aligned}$$

where the last inequality follows since $\lambda_j > 2$ for all $j \in \mathbb{N}$. By (1.3.27), the proof is concluded. $\square$

Now we let

$$g_j(\xi) := \int_{Q_1} f(x, x + \lambda_j \xi, V_j(x, \lambda_j \xi) + M\xi) dx, \quad \xi \in \mathbb{R}^d,$$

$$\tilde{g}_j(\xi) := \int_{Q_1} \int_{Q_1} f(x, x + y, V_j(x, y) + M\xi) dx dy, \quad \xi \in \mathbb{R}^d,$$

and observe the following.

Lemma 1.3.14. *The sequences $\{g_j\}_j$ and $\{\tilde{g}_j\}_j$ admit a weakly converging subsequence in $L^1(B_T)$.*

Proof. Consider first the sequence $\{g_j\}_j$. We prove that there exists a positive constant C such that for any cube $Q \subset B_T$ it holds

$$\limsup_{j \rightarrow +\infty} \int_Q g_j(\xi) d\xi \leq C|Q|. \quad (1.3.28)$$

To this end, we first observe that by (GC) we have

$$\begin{aligned} \int_Q g_j(\xi) d\xi &= \int_Q \int_{Q_1} f(x, x + \lambda_j \xi, V_j(x, \lambda_j \xi) + M\xi) dx d\xi \\ &\leq \beta \int_Q \int_{Q_1} |V_j(x, \lambda_j \xi) + M\xi|^p dx d\xi \\ &\leq \beta 2^{p-1} \left\{ \int_Q \int_{Q_1} |V_j(x, \lambda_j \xi)|^p dx d\xi + |M|^p |T|^p |Q| \right\}. \end{aligned} \quad (1.3.29)$$

Now we note that, by (P),

$$\begin{aligned} \int_Q \int_{Q_1} |V_j(x, \lambda_j \xi)|^p dx d\xi &= \lambda_j^{-d} \int_{\lambda_j Q} \int_{Q_1} |V_j(x, \xi)|^p dx d\xi \\ &\leq \lambda_j^{-d} \sum_{i \in \mathcal{I}_j} \int_{Q_1} \int_{Q_1} |V_j(x, \xi)|^p dx d\xi, \end{aligned}$$

where $\mathcal{I}_j := \{i \in \mathbb{Z}^d : i + Q_1 \cap \lambda_j Q \neq \emptyset\}$. Since $\lambda_j^{-d} \#\mathcal{I}_j \rightarrow |Q|$, we deduce

$$\limsup_{j \rightarrow +\infty} \int_Q \int_{Q_1} |V_j(x, \lambda_j \xi)|^p dx d\xi \leq |Q| \limsup_{j \rightarrow +\infty} \|V_j(x, \xi)\|_{L^p(Q_1 \times Q_1; \mathbb{R}^m)}^p,$$

which, together with (1.3.29), proves (1.3.28) in light of Lemma 1.3.13. By a covering argument, we then have

$$\limsup_{j \rightarrow +\infty} \int_E g_j(\xi) d\xi \leq C|E|. \quad (1.3.30)$$

for every $E \subset B_T$ measurable and with the same constant C .

In light of (1.3.27), we have that $\{g_j\}_j$ is a bounded sequence in $L^1(B_T)$. Hence, by the Biting Lemma (see [71]), there exist a not relabelled subsequence $\{g_j\}_j$, a function $g \in L^1(B_T)$, and a family $\{E_\eta\}_\eta$, $\eta > 0$, of measurable subsets of B_T such that

$$g_j \rightharpoonup g \text{ weakly in } L^1(B_T \setminus E_\eta) \quad \text{and} \quad |E_\eta| \leq \eta$$

for every $\eta > 0$. Therefore, letting $u \in L^\infty(B_T)$, we have

$$\left| \int_{B_T} (g_j - g)u dx \right| \leq \left| \int_{B_T \setminus E_\eta} (g_j - g)u dx \right| + \|u\|_{L^\infty(B_T)} \left(\int_{E_\eta} g dx + \int_{E_\eta} g_j dx \right).$$

Passing to the limit and resorting to (1.3.30) we obtain

$$\limsup_{j \rightarrow +\infty} \left| \int_{B_T} (g_j - g)u dx \right| \leq \|u\|_{L^\infty(B_T)} \left(\int_{E_\eta} g dx + C|E_\eta| \right),$$

and by the arbitrariness of η we infer that $\{g_j\}_j$ weakly converges to g in $L^1(B_T)$.

As for the sequence $\{\tilde{g}_j\}_j$, by (GC) we simply have

$$\begin{aligned} \int_A \tilde{g}_j(\xi) d\xi &= \int_A \int_{Q_1} \int_{Q_1} f(x, x + y, V_j(x, y) + M\xi) dx dy d\xi \\ &\leq \beta 2^{p-1} \left\{ \|V_j(x, y)\|_{L^p(Q_1 \times Q_1; \mathbb{R}^m)}^p |A| + |M|^p |T|^p |A| \right\} \end{aligned}$$

for every $A \subseteq B_T$ measurable and for every $j \in \mathbb{N}$. By Lemma 1.3.13, we infer that there exists a positive constant C such that

$$\int_A \tilde{g}_j(\xi) d\xi \leq C|A|$$

for every $A \subseteq B_T$ measurable and for every $j \in \mathbb{N}$. Therefore, $\{\tilde{g}_j\}_j$ is an equi-integrable sequence in $L^1(B_T)$ and the conclusion follows. $\square$

From now on we assume that both $\{g_j\}_j$ and $\{\tilde{g}_j\}_j$ are weakly converging in $L^1(B_T)$, without considering a subsequence for simplicity of notation. We state our main lemma.

Lemma 1.3.15. *It holds that*

$$g_j(\xi) - \tilde{g}_j(\xi) \rightharpoonup 0$$

weakly in $L^1(B_T)$ as $j \rightarrow +\infty$.

Proof. Due to the equi-integrability of the sequence in $L^1(B_T)$ established in Lemma 1.3.14, it suffices to test the weak convergence with the characteristic functions of cubes contained in B_T . Let Q denote such a cube, we have

$$\begin{aligned} \int_Q g_j(\xi) - \tilde{g}_j(\xi) d\xi &= \lambda_j^{-d} \int_{\lambda_j Q} g_j(\xi/\lambda_j) - \tilde{g}_j(\xi/\lambda_j) d\xi \\ &= \lambda_j^{-d} \sum_{i \in \mathcal{I}_j} \int_{i+Q_1} g_j(\xi/\lambda_j) - \tilde{g}_j(\xi/\lambda_j) d\xi + \lambda_j^{-d} \int_{E_j} g_j(\xi/\lambda_j) - \tilde{g}_j(\xi/\lambda_j) d\xi, \end{aligned}$$

where we set $\mathcal{I}_j := \{i \in \mathbb{Z}^d : i + Q_1 \subset \lambda_j Q\}$ and E_j is obtained removing from $\lambda_j Q$ all the unit cubes of the fundamental lattice. Since there exists $c > 0$ such that $|E_j| \leq c\lambda_j^{d-1}$ for all $j \in \mathbb{N}$, we have that

$$\lambda_j^{-d} \int_{E_j} g_j(\xi/\lambda_j) - \tilde{g}_j(\xi/\lambda_j) d\xi = \int_{\lambda_j^{-1} E_j} g_j(\xi) - \tilde{g}_j(\xi) d\xi$$

vanishes as $j \rightarrow +\infty$ by Lemma 1.3.14; therefore, applying a change of variables, the periodicity of each V_j , and (P), the claim follows if we prove that

$$\begin{aligned} \lim_{j \rightarrow +\infty} \lambda_j^{-d} \sum_{i \in \mathcal{I}_j} \int_{Q_1} \left\{ \int_{Q_1} f\left(x, x + \xi, V_j(x, \xi) + M \frac{i + \xi}{\lambda_j}\right) dx \right. \\ \left. - \int_{Q_1} \int_{Q_1} f\left(x, x + y, V_j(x, y) + M \frac{i + \xi}{\lambda_j}\right) dx dy \right\} d\xi = 0. \end{aligned} \quad (1.3.31)$$

To this end, we first prove that

$$\begin{aligned} \lim_{j \rightarrow +\infty} \lambda_j^{-d} \sum_{i \in \mathcal{I}_j} \int_{Q_1} \int_{Q_1} f\left(x, x + \xi, V_j(x, \xi) + M \frac{i + \xi}{\lambda_j}\right) dx d\xi \\ - \int_Q \int_{Q_1} \int_{Q_1} f(x, x + \xi, V_j(x, \xi) + Mz) dx d\xi dz = 0. \end{aligned} \quad (1.3.32)$$

Indeed, we have

$$\begin{aligned}
& \lim_{j \rightarrow +\infty} \left| \lambda_j^{-d} \sum_{i \in \mathcal{I}_j} \int_{Q_1} \int_{Q_1} f\left(x, x + \xi, V_j(x, \xi) + M \frac{i + \xi}{\lambda_j}\right) dx d\xi \right. \\
& \quad \left. - \int_Q \int_{Q_1} \int_{Q_1} f(x, x + \xi, V_j(x, \xi) + Mz) dx d\xi dz \right| \\
&= \lim_{j \rightarrow +\infty} \left| \sum_{i \in \mathcal{I}_j} \int_{\frac{i+Q_1}{\lambda_j}} \left\{ \int_{Q_1} \int_{Q_1} f\left(x, x + \xi, V_j(x, \xi) + M \frac{i + \xi}{\lambda_j}\right) dx d\xi \right. \right. \\
& \quad \left. \left. - \int_{Q_1} \int_{Q_1} f(x, x + \xi, V_j(x, \xi) + Mz) dx d\xi \right\} dz \right| \\
&\leq \lim_{j \rightarrow +\infty} \sum_{i \in \mathcal{I}_j} \int_{\frac{i+Q_1}{\lambda_j}} \int_{Q_1} \int_{Q_1} \left| f\left(x, x + \xi, V_j(x, \xi) + M \frac{i + \xi}{\lambda_j}\right) \right. \\
& \quad \left. - f(x, x + \xi, V_j(x, \xi) + Mz) \right| dx d\xi dz, \tag{1.3.33}
\end{aligned}$$

where we used that, by Lemma 1.3.14,

$$\lim_{j \rightarrow +\infty} \int_{Q \setminus \bigcup_{i \in \mathcal{I}_j} \frac{i+Q_1}{\lambda_j}} \left\{ \int_{Q_1} \int_{Q_1} f(x, x + \xi, V_j(x, \xi) + Mz) dx d\xi \right\} dz = \lim_{j \rightarrow +\infty} \int_{Q \setminus \bigcup_{i \in \mathcal{I}_j} \frac{i+Q_1}{\lambda_j}} \widetilde{g}_j(z) dz = 0.$$

We recall that, by (C), the function f is locally Lipschitz continuous in the last variable and, in particular, resorting also to (GC), it holds that

$$|f(x, y, z_1) - f(x, y, z_2)| \leq C(1 + |z_1|^{p-1} + |z_2|^{p-1})|z_1 - z_2| \tag{1.3.34}$$

for a.e. $x, y \in \mathbb{R}^d$ and for every $z_1, z_2 \in \mathbb{R}^m$, where C is a constant that depends on p, d , and the constant β appearing in (GC). Let then $i \in \mathcal{I}_j$ and $z \in \frac{i+Q_1}{\lambda_j}$; using (1.3.34) we have

$$\begin{aligned}
& \left| f\left(x, x + \xi, V_j(x, \xi) + M \frac{i + \xi}{\lambda_j}\right) - f(x, x + \xi, V_j(x, \xi) + Mz) \right| \\
& \leq C \left(1 + \left| V_j(x, \xi) + M \frac{i + \xi}{\lambda_j} \right|^{p-1} + |V_j(x, \xi) + Mz|^{p-1} \right) \left| M \left(\frac{i + \xi}{\lambda_j} - z \right) \right| \\
& \leq C'(1 + |V_j(x, \xi)|^{p-1}) \lambda_j^{-1}
\end{aligned}$$

for a.e. $x, \xi \in Q_1$, where C' is a positive constant depending on p, d, β , and M . Combining this estimate with (1.3.33) and using Hölder's inequality, we infer

$$\begin{aligned}
& \lim_{j \rightarrow +\infty} \left| \lambda_j^{-d} \sum_{i \in \mathcal{I}_j} \int_{Q_1} \int_{Q_1} f\left(x, x + \xi, V_j(x, \xi) + M \frac{i + \xi}{\lambda_j}\right) dx d\xi \right. \\
& \quad \left. - \int_Q \int_{Q_1} \int_{Q_1} f(x, x + \xi, V_j(x, \xi) + Mz) d\xi dx dz \right| \\
& \leq \lim_{j \rightarrow +\infty} C' \lambda_j^{-1} \int_Q \int_{Q_1} \int_{Q_1} 1 + |V_j(x, \xi)|^{p-1} dx d\xi dz \\
& \leq C' |Q| \lim_{j \rightarrow +\infty} \lambda_j^{-1} (1 + \|V_j(x, \xi)\|_{L^p(Q_1 \times Q_1; \mathbb{R}^m)}^p),
\end{aligned}$$

which equals 0 by Lemma 1.3.13. Then, (1.3.32) is proved.

A similar argument shows that

$$\begin{aligned}
& \lim_{j \rightarrow +\infty} \lambda_j^{-d} \sum_{i \in \mathcal{I}_j} \int_{Q_1} \int_{Q_1} \int_{Q_1} f\left(x, x + y, V_j(x, y) + M \frac{i + \xi}{\lambda_j}\right) dx dy d\xi \\
& \quad - \int_Q \int_{Q_1} \int_{Q_1} f(x, x + y, V_j(x, y) + Mz) dx dy dz = 0. \tag{1.3.35}
\end{aligned}$$

Indeed, arguing as before, we have

$$\begin{aligned}
& \lim_{j \rightarrow +\infty} \left| \lambda_j^{-d} \sum_{i \in \mathcal{I}_j} \int_{Q_1} \int_{Q_1} \int_{Q_1} f\left(x, x+y, V_j(x, y) + M \frac{i+\xi}{\lambda_j}\right) dx dy d\xi \right. \\
& \quad \left. - \int_Q \int_{Q_1} \int_{Q_1} f(x, x+y, V_j(x, y) + Mz) dx dy dz \right| \\
&= \lim_{j \rightarrow +\infty} \left| \sum_{i \in \mathcal{I}_j} \int_{\frac{i+Q_1}{\lambda_j}} \left\{ \int_{Q_1} \int_{Q_1} \int_{Q_1} f\left(x, x+y, V_j(x, y) + M \frac{i+\xi}{\lambda_j}\right) dx dy d\xi \right. \right. \\
& \quad \left. \left. - \int_{Q_1} \int_{Q_1} f(x, x+y, V_j(x, y) + Mz) dx dy \right\} dz \right| \\
&\leq \lim_{j \rightarrow +\infty} \sum_{i \in \mathcal{I}_j} \int_{\frac{i+Q_1}{\lambda_j}} \left\{ \int_{Q_1} \left[\int_{Q_1} \int_{Q_1} \left| f\left(x, x+y, V_j(x, y) + M \frac{i+\xi}{\lambda_j}\right) \right. \right. \right. \\
& \quad \left. \left. - f(x, x+y, V_j(x, y) + Mz) \right| dx dy \right] d\xi \right\} dz \\
&\leq C' |Q| \lim_{j \rightarrow +\infty} \lambda_j^{-1} (1 + \|V_j(x, y)\|_{L^p(Q_1 \times Q_1; \mathbb{R}^m)}^p),
\end{aligned}$$

which equals 0, and this proves (1.3.35).

Gathering (1.3.32) and (1.3.35) we obtain (1.3.31), which is the thesis. $\square$

We are now in position to prove the lower bound. Recalling (1.3.26) and the definitions of g_j and $\tilde{g}_j$, $j \in \mathbb{N}$, we fix any $R > 0$ and apply Lemma 1.3.15 to obtain

$$\begin{aligned}
\varphi(M) &\geq \limsup_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) g_j(\xi) d\xi \\
&\geq \liminf_{j \rightarrow +\infty} \int_{B_T} \min\{\rho(\xi), R\} (g_j(\xi) - \tilde{g}_j(\xi)) d\xi + \int_{B_T} \min\{\rho(\xi), R\} \tilde{g}_j(\xi) d\xi \\
&= \liminf_{j \rightarrow +\infty} \int_{B_T} \min\{\rho(\xi), R\} \tilde{g}_j(\xi) d\xi \\
&= \liminf_{j \rightarrow +\infty} \int_{B_T} \min\{\rho(\xi), R\} \int_{Q_1} \int_{Q_1} f(x, x+y, V_j(x, y) + M\xi) dx dy d\xi.
\end{aligned}$$

Using Lemma 1.3.13, it is not restrictive to suppose that $V_j(x, y) \rightharpoonup V(x, y)$ weakly in $L^p(Q_1 \times Q_1; \mathbb{R}^m)$ and, setting

$$\begin{aligned}
\mathcal{C} := \Big\{ U \in L_{\text{loc}}^p(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^m) : & U(\cdot, y) \text{ is } Q_1\text{-periodic for a.e. } y \in \mathbb{R}^d, \\
& U(x, \cdot) \text{ is } Q_1\text{-periodic for a.e. } x \in \mathbb{R}^d, \\
& \text{and } \int_{Q_1} U(x, y) dx = 0 \text{ for a.e. } y \in \mathbb{R}^d \Big\},
\end{aligned}$$

by the definition of $V_j(x, y)$ and the periodicity of $v_j(x)$, it is easily seen that $V \in \mathcal{C}$. Then, we apply Theorem 1.3.8 with $\ell = d \times d$, $s = (x, y)$ and

$$E = Q_1 \times Q_1, \quad \Psi((x, y), t, q) = f(x, x+y, q + M\xi),$$

and use Fatou's Lemma to infer

$$\begin{aligned}
\varphi(M) &\geq \int_{B_T} \min\{\rho(\xi), R\} \liminf_{j \rightarrow +\infty} \int_{Q_1} \int_{Q_1} f(x, x+y, V_j(x, y) + M\xi) dx dy d\xi \\
&\geq \int_{B_T} \min\{\rho(\xi), R\} \int_{Q_1} \int_{Q_1} f(x, x+y, V(x, y) + M\xi) dx dy d\xi,
\end{aligned}$$

which, by the arbitrariness of $R > 0$ and $V \in \mathcal{C}$, implies

$$\varphi(M) \geq \inf \left\{ \int_{B_T} \rho(\xi) \int_{Q_1} \int_{Q_1} f(x, x+y, V(x, y) + M\xi) dx dy d\xi : V \in \mathcal{C} \right\}.$$

For the sake of notation, we set

$$F(V) := \int_{B_T} \rho(\xi) \int_{Q_1} \int_{Q_1} f(x, x+y, V(x, y) + M\xi) dx dy d\xi$$

so that

$$\varphi(M) \geq \inf\{F(V) : V \in \mathcal{C}\}.$$

Let us suppose first that

$$\begin{cases} \rho(\xi) = \rho(-\xi) & \text{for almost every } \xi \in \mathbb{R}^d, \\ f(x, y, z) = f(x, y, -z) & \text{for almost every } x, y \in \mathbb{R}^d \text{ and for every } z \in \mathbb{R}^m, \end{cases} \quad (\text{H1})$$

Using a change of variables and (H1), we have that

$$\begin{aligned} F(V) &= \int_{B_T} \rho(\xi) \int_{Q_1} \int_{Q_1} f(x, x+y, V(x, y) + M\xi) dx dy d\xi \\ &= \int_{B_T} \rho(-\xi) \int_{Q_1} \int_{Q_1} f(x, x+y, V(x, y) - M\xi) dx dy d\xi \\ &= \int_{B_T} \rho(\xi) \int_{Q_1} \int_{Q_1} f(x, x+y, -V(x, y) + M\xi) dx dy d\xi \\ &= F(-V); \end{aligned}$$

therefore, recalling (C), we infer

$$F(0) \leq \frac{F(V) + F(-V)}{2} = F(V).$$

The arbitrariness of V yields

$$\varphi(M) \geq F(0) = \int_{B_T} \rho(\xi) \int_{Q_1} \int_{Q_1} f(x, x+y, M\xi) dx dy d\xi, \quad (1.3.36)$$

which, using Fubini's Theorem and (P), yields the desired lower bound.

Assume now that

$$\int_{\mathbb{R}^d} \rho(\xi) d\xi = +\infty \quad (\text{H2})$$

and let $V \in \mathcal{C}$. The lower bound in (GC) implies that

$$\rho(\xi)f(x, x+y, V(x, y) + M\xi) \geq \alpha\rho(\xi)|V(x, y) + M\xi|^p$$

for a.e. $\xi, x, y \in \mathbb{R}^d$ and for every $z \in \mathbb{R}^m$, and since there exist positive constants c_1, c_2 such that $|z_1 + z_2|^p \geq c_1|z_1|^p - c_2|z_2|^p$ for every $z_1, z_2 \in \mathbb{R}^m$, we infer that

$$\rho(\xi)f(x, x+y, V(x, y) + M\xi) + \alpha c_2 \rho(\xi)|M\xi|^p \geq \alpha c_1 \rho(\xi)|V(x, y)|^p.$$

Integrating this inequality, we get

$$F(V) + \alpha c_2 \int_{B_T} \rho(\xi)|M\xi|^p d\xi \geq \alpha c_1 \left(\int_{B_T} \rho(\xi) d\xi \right) \left(\int_{Q_1} \int_{Q_1} |V(x, y)|^p dx dy \right);$$

therefore, by (ρ_2) and (H2), we get that $F(V)$ is finite only if $V = 0$, which implies (1.3.36).

Upper bound in the supercritical case, $\lambda = +\infty$. In order to prove the optimality of the previous lower bound we recall that, according to Proposition 1.3.11(i), we have

$$\begin{aligned} \varphi(M) &= \lim_{j \rightarrow +\infty} \inf \left\{ \left(\frac{\delta_j}{r} \right)^d \int_{B_T} \rho(\xi) \int_{(Q_{r/\delta_j})_{\lambda_j}(\xi)} f \left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j} \right) dx d\xi : \right. \\ &\quad \left. u \in \mathcal{D}_{T\lambda_j, M}(Q_{r/\delta_j}; \mathbb{R}^m) \right\} \end{aligned}$$

for some $r > 0$. We use $u(x) = Mx$ as a test function for these minimum problems to obtain

$$\begin{aligned} \varphi(M) &\leq \liminf_{j \rightarrow +\infty} \left(\frac{\delta_j}{r} \right)^d \int_{B_T} \rho(\xi) \int_{(Q_{r/\delta_j})_{\lambda_j}(\xi)} f(x, x + \lambda_j \xi, M\xi) dx d\xi \\ &\leq \liminf_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) \int_{Q_1} f(x, x + \lambda_j \xi, M\xi) dx d\xi, \end{aligned} \quad (1.3.37)$$

where the last inequality follows by (P).

On the one hand, if we apply Lemma 1.3.15 with $V_j = 0$ for every $j \in \mathbb{N}$, we have that

$$g_j(\xi) = \int_{Q_1} f(x, x + \lambda_j \xi, M\xi) dx \rightharpoonup \int_{Q_1} \int_{Q_1} f(x, y, M\xi) dx dy$$

weakly in $L^1(B_T)$.

On the other hand, we observe that $\{\rho g_j\}_j$ is an equi-integrable sequence in $L^1(B_T)$ since, by (GC), we have

$$\int_A \rho(\xi) g_j(\xi) d\xi \leq \beta |M|^p \int_A \rho(\xi) |\xi|^p d\xi$$

for every $A \subseteq B_T$ measurable and ρ has finite p -th moment according to (ρ_2) . Then, it is not restrictive to suppose that $\rho g_j \rightharpoonup \Theta$ weakly in $L^1(B_T)$ as $j \rightarrow +\infty$.

We claim that

$$\rho(\xi) \int_{Q_1} \int_{Q_1} f(x, y, M\xi) dx dy = \Theta(\xi) \quad (1.3.38)$$

for a.e. $\xi \in B_T$. Let $\chi_R := \chi_{\{\xi \in B_T : \rho(\xi) < R\}}$ and $\psi \in C_c^\infty(B_T)$. Since $\rho \chi_R \psi \in L^\infty(B_T)$, by the weak convergence of $\{g_j\}_j$ we infer

$$\lim_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) \chi_R(\xi) \psi(\xi) g_j(\xi) d\xi = \int_{B_T} \rho(\xi) \chi_R(\xi) \psi(\xi) \left(\int_{Q_1} \int_{Q_1} f(x, y, M\xi) dx dy \right) d\xi,$$

and analogously, since $\chi_R \psi \in L^\infty(B_T)$, we have

$$\lim_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) \chi_R(\xi) \psi(\xi) g_j(\xi) d\xi = \int_{B_T} \Theta(\xi) \chi_R(\xi) \psi(\xi) d\xi.$$

We deduce that

$$\rho(\xi) \int_{Q_1} \int_{Q_1} f(x, y, M\xi) dx dy = \Theta(\xi)$$

for a.e. $\xi \in B_T$ such that $\rho(\xi) < R$ and for every $R > 0$; and since the set $\{\xi \in \mathbb{R}^d : \rho(\xi) = +\infty\}$ has measure zero in virtue of (ρ_2) , we infer (1.3.38).

Recalling (1.3.37) we conclude

$$\begin{aligned} \varphi(M) &\leq \liminf_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) \int_{Q_1} f(x, x + \lambda_j \xi, M\xi) dx d\xi \\ &= \lim_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) g_j(\xi) d\xi \\ &= \int_{B_T} \rho(\xi) \int_{Q_1} \int_{Q_1} f(x, y, M\xi) dx dy d\xi, \end{aligned}$$

which proves the upper bound.

Remark 1.3.16. As a byproduct, the above arguments show that

$$\begin{aligned} &\int_{B_T} \int_{Q_1} \int_{Q_1} \rho(\xi) f(x, y, M\xi) dx dy d\xi \\ &= \lim_{j \rightarrow +\infty} \inf \left\{ \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j}\right) dx d\xi : u \in L_{\#, M}^p(Q_1; \mathbb{R}^m) \right\}, \end{aligned}$$

and therefore, although in an indirect way, we obtain that Proposition 1.3.11(iii) is valid also for $\lambda = +\infty$. We point out that, in this case, we also had to also resort to the assumption (H1) or (H2).

Further comments on the supercritical case. We discuss our necessity to assume (H1) or (H2) in order to treat the supercritical case $\lambda = +\infty$. We just proved that if a certain φ is a density for the Γ -limit, then

$$\varphi(M) \geq \inf \left\{ \int_{B_T} \rho(\xi) \int_{Q_1} \int_{Q_1} f(x, x+y, V(x, y) + M\xi) dx dy d\xi : V \in \mathcal{C} \right\}$$

for all $M \in \mathbb{R}^{m \times d}$. Inspecting the proof, we note that our argument is only based on the equi-boundedness of the finite differences

$$V_j(x, \xi) := \frac{v_j(x + \xi) - v_j(x)}{\lambda_j}, \quad j \in \mathbb{N},$$

which implies that (up to subsequences)

$$\frac{v_j(x + \xi) - v_j(x)}{\lambda_j} \rightharpoonup V(x, \xi)$$

weakly in $L^p(Q_1 \times Q_1; \mathbb{R}^m)$ for some $V \in \mathcal{C}$, but it does not exploit the fact that $\{u_j(x) = v_j(x) + Mx\}_j$ is a sequence of minimizers for the problems appearing in Proposition 1.3.11(ii). This may lead us to believe that there is some room for an improvement: for instance, we may suspect that, incorporating the optimality of $\{u_j\}_j$, it may be possible to directly prove that

$$\frac{v_j(x + \xi) - v_j(x)}{\lambda_j} \rightarrow 0, \tag{1.3.39}$$

which would allow us to prove Theorem 1.3.1(iii) without any additional assumptions. According to the next proposition, this may not be the case, even for a simple choice of the density f .

Let $\{\varepsilon_j\}_j, \{\delta_j\}_j$ be vanishing sequences such that $\lambda_j := \varepsilon_j/\delta_j \rightarrow +\infty$ as $j \rightarrow +\infty$ and consider

$$F_{\varepsilon_j}(u) = \int_{\Omega} \int_{\Omega} \frac{1}{\varepsilon_j^d} \rho\left(\frac{y-x}{\varepsilon_j}\right) a\left(\frac{x}{\delta_j}\right) \left| \frac{u(y) - u(x)}{\varepsilon_j} \right|^2 dx dy,$$

with ρ a kernel in $L^\infty(\mathbb{R}^d)$ supported on B_T for some $T > r_0$ and fulfilling (ρ_1) and (ρ_2) , and a a non-constant Q_1 -periodic function such that $0 < \alpha \leq a(x) \leq \beta < +\infty$ for a.e. $x \in \mathbb{R}^d$. With this choice, we have that $f(x, y, z) = a(x)|z|^2$, which implies that $f(x, y, z) = f(x, y, -z)$ for a.e. $x, y \in \mathbb{R}^d$ and for every $z \in \mathbb{R}$, where we work in the scalar case $m = 1$ for the sake of simplicity. Therefore, we have that (H1) is equivalent to

$$\rho(\xi) = \rho(-\xi) \tag{\widetilde{H1}}$$

for a.e. $\xi \in B_T$ and that (H2) fails.

Proposition 1.3.17. *Let $M \in \mathbb{R}^d, M \neq 0$, and, for all $j \in \mathbb{N}$, let u_j be a minimizer for*

$$\inf \left\{ \int_{B_T} \rho(\xi) \int_{Q_1} a(x) \left| \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j} \right|^2 dx d\xi : u \in L^2_{\#, M}(Q_1) \right\}, \tag{1.3.40}$$

let $v_j(x) := u_j(x) - Mx$, and let

$$V_j(x, \xi) := \frac{v_j(x + \xi) - v_j(x)}{\lambda_j}.$$

If $V_j \rightharpoonup 0$ weakly in $L^2(Q_1 \times Q_1)$ as $j \rightarrow +\infty$, then, letting $x \cdot y$ denote the scalar product in $\mathbb{R}^d$, we have

$$\int_{B_T} \rho(\xi) M \cdot \xi d\xi = 0.$$

Proof. Since $\rho \in L^1(\mathbb{R}^d)$, it is possible to compute the Euler-Lagrange equations for the minimum problems (1.3.40) obtaining that for every $j \in \mathbb{N}$ we have

$$\int_{B_T} \rho(\xi) \int_{Q_1} a(x) \left(\frac{u_j(x + \lambda_j \xi) - u_j(x)}{\lambda_j} \right) \left(\frac{w(x + \lambda_j \xi) - w(x)}{\lambda_j} \right) dx d\xi = 0$$

for every $w \in L^2_{\#,0}(Q_1)$.

By first testing this equation with $w(x) = v_j(x)$, we infer

$$\begin{aligned} \int_{B_T} \rho(\xi) \int_{Q_1} a(x) \left| \frac{v_j(x + \lambda_j \xi) - v_j(x)}{\lambda_j} \right|^2 dx d\xi \\ = - \int_{B_T} \rho(\xi) M \cdot \xi \int_{Q_1} a(x) \left(\frac{v_j(x + \lambda_j \xi) - v_j(x)}{\lambda_j} \right) dx d\xi; \end{aligned} \quad (1.3.41)$$

then, testing with $w(x) = \lambda_j \tilde{w}(x)$ for some $\tilde{w} \in L^2_{\#,0}(Q_1)$, we obtain

$$\begin{aligned} \int_{B_T} \rho(\xi) \int_{Q_1} a(x) \left(\frac{v_j(x + \lambda_j \xi) - v_j(x)}{\lambda_j} \right) (\tilde{w}(x + \lambda_j \xi) - \tilde{w}(x)) dx d\xi \\ = - \int_{B_T} \rho(\xi) M \cdot \xi \int_{Q_1} a(x) (\tilde{w}(x + \lambda_j \xi) - \tilde{w}(x)) dx d\xi. \end{aligned} \quad (1.3.42)$$

Now we note that $V_j(x, \lambda_j \xi) \rightharpoonup 0$ weakly in $L^2(Q_1 \times B_T)$. Indeed, by (1.3.27), this sequence is equi-bounded in $L^2(Q_1 \times B_T)$, hence, it suffices to test the weak convergence with the characteristic functions of measurable sets $E \times Q \subset Q_1 \times B_T$, with Q a cube. Also using the equi-integrability of $\{V_j\}_j$ in $L^1(Q_1 \times Q_1)$, it is immediate to observe that

$$\begin{aligned} \lim_{j \rightarrow +\infty} \int_Q \int_E V_j(x, \lambda_j \xi) dx d\xi &= \lim_{j \rightarrow +\infty} \lambda_j^{-d} \int_{\lambda_j Q} \int_E V_j(x, \xi) dx d\xi \\ &= |Q| \lim_{j \rightarrow +\infty} \int_{Q_1} \int_E V_j(x, \xi) dx d\xi, \end{aligned}$$

which equals 0 since $V_j \rightharpoonup 0$ by assumption. Passing to the limit in (1.3.41), we get

$$\lim_{j \rightarrow +\infty} \int_{B_T} \rho(\xi) \int_{Q_1} a(x) \left| \frac{v_j(x + \lambda_j \xi) - v_j(x)}{\lambda_j} \right|^2 dx d\xi = 0,$$

which, by (ρ_1) and the fact that $\inf a \geq \alpha$, implies

$$\lim_{j \rightarrow +\infty} \int_{B_{r_0}} \int_{Q_1} \left| \frac{v_j(x + \lambda_j \xi) - v_j(x)}{\lambda_j} \right|^2 dx d\xi = 0,$$

and then, by Lemma 1.3.9,

$$\lim_{j \rightarrow +\infty} \int_{B_T} \int_{Q_1} \left| \frac{v_j(x + \lambda_j \xi) - v_j(x)}{\lambda_j} \right|^2 dx d\xi = 0.$$

Upon assuming that $\int_{Q_1} \tilde{w} = 0$, by the Riemann-Lebesgue Lemma we have

$$\tilde{w}(x + \lambda_j \xi) - \tilde{w}(x) \rightharpoonup -\tilde{w}(x)$$

weakly in $L^2(Q_1 \times B_T)$ as $j \rightarrow +\infty$; hence, we pass to the limit in (1.3.42) using the strong-weak convergence to obtain

$$0 = \int_{B_T} \rho(\xi) M \cdot \xi d\xi \int_{Q_1} a(x) \tilde{w}(x) dx.$$

Finally, properly choosing $\tilde{w}$, we get

$$\int_{B_T} \rho(\xi) M \cdot \xi d\xi = 0,$$

which is the thesis. $\square$

The above proposition asserts that if we consider a bounded kernel ρ supported on B_T and $\widetilde{M} \neq 0$ such that

$$\int_{B_T} \rho(\xi) \widetilde{M} \cdot \xi \, d\xi \neq 0,$$

then (1.3.39) fails for the sequence $\{v_j\}_j$ corresponding to such $\widetilde{M}$. This illustrates how, if we neglect both the assumptions (H1) and (H2), our argument does not provide the expected lower bound.

1.3.4 Proof of the main result

Finally, we prove Theorem 1.3.1 also for kernels with support that is not compact.

Proof of Theorem 1.3.1. Let $\{T_h\}_h$ be any sequence monotonically increasing to $+\infty$ and assume that $T_h > r_0$ and $Q_1 \subset B_{T_h}$ for all $h \in \mathbb{N}$, where r_0 is as in (ρ_1) . By Theorem 1.3.6 and a diagonal argument, there exists a subsequence $\{j_k\}_k$ such that, for every $h \in \mathbb{N}$, the sequence of truncated functionals $\{F_{j_k}^{T_h}\}_k$ Γ -converges to the functional

$$F^{T_h}(u, A) := \begin{cases} \int_A \varphi_h(\nabla u) \, dx & \text{if } u \in W^{1,p}(A; \mathbb{R}^m), \\ +\infty & \text{if } u \in L^p(\Omega; \mathbb{R}^m) \setminus W^{1,p}(A; \mathbb{R}^m) \end{cases} \quad (1.3.43)$$

for all $A \in \mathcal{A}_{\text{reg}}(\Omega)$ and for some quasiconvex $\varphi_h : \mathbb{R}^{m \times d} \rightarrow [0, +\infty)$. By Lemma 1.3.4, we have that

$$\Gamma\text{-}\lim_{k \rightarrow +\infty} F_{j_k}(u, A) = \lim_{h \rightarrow +\infty} F^{T_h}(u, A)$$

for every $u \in L^p(\Omega; \mathbb{R}^m)$ and $A \in \mathcal{A}_{\text{reg}}(\Omega)$. Clearly, $\{\varphi_h\}_h$ is an increasing sequence of functions; therefore, by (1.3.43) and the Monotone Convergence Theorem, we have

$$\Gamma\text{-}\lim_{k \rightarrow +\infty} F_{j_k}(u, A) = \begin{cases} \int_A \lim_{h \rightarrow +\infty} \varphi_h(\nabla u) \, dx & \text{if } u \in W^{1,p}(A; \mathbb{R}^m), \\ +\infty & \text{if } u \in L^p(\Omega; \mathbb{R}^m) \setminus W^{1,p}(A; \mathbb{R}^m) \end{cases}$$

for all $A \in \mathcal{A}_{\text{reg}}(\Omega)$. In order to conclude, we prove that

$$\lim_{h \rightarrow +\infty} \varphi_h(M) = f_\lambda(M)$$

for all $M \in \mathbb{R}^{m \times d}$, where f_λ is defined as in the statement of Theorem 1.3.1 in accordance with the value of $\lambda \in [0, +\infty]$. Indeed, since f_λ is independent of the subsequence $\{j_k\}_k$, the conclusion follows by the Urysohn property of the Γ -convergence (see [61]).

Subcritical case, $\lambda = 0$. Let $M \in \mathbb{R}^{m \times d}$ be fixed. By Proposition 1.3.12, we have that

$$\begin{aligned} \lim_{h \rightarrow +\infty} \varphi_h(M) &= \lim_{h \rightarrow +\infty} \inf \left\{ \int_{B_{T_h}} \rho(\xi) \int_{Q_1} f(x, x, (\nabla u)\xi) \, dx \, d\xi : u \in W_{\#, M}^{1,p}(Q_1; \mathbb{R}^m) \right\} \\ &\leq \inf \left\{ \int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f(x, x, (\nabla u)\xi) \, dx \, d\xi : u \in W_{\#, M}^{1,p}(Q_1; \mathbb{R}^m) \right\} \\ &= f_0(M). \end{aligned}$$

Conversely, let $\{u_h\}_h \subset W_{\#, M}^{1,p}(Q_1; \mathbb{R}^m)$ be such that

$$\lim_{h \rightarrow +\infty} \varphi_h(M) = \lim_{h \rightarrow +\infty} \int_{B_{T_h}} \rho(\xi) \int_{Q_1} f(x, x, (\nabla u_h)\xi) \, dx \, d\xi, \quad (1.3.44)$$

and, without loss of generality, assume

$$\int_{Q_1} u_h \, dx = 0 \quad (1.3.45)$$

for every $h \in \mathbb{N}$. Combining (1.3.44) with (ρ_1) and (GC), we have

$$\sup_h \int_{Q_1} \int_{B_{r_0}} |(\nabla u_h)\xi|^p d\xi dx < +\infty;$$

and since there exists a positive constant C such that

$$\int_{B_{r_0}} |L\xi|^p d\xi \geq C|L|^p$$

for every $L \in \mathbb{R}^{m \times d}$, we infer

$$\sup_h \int_{Q_1} |\nabla u_h|^p dx < +\infty.$$

This, together with (1.3.45), implies that there exists a subsequence $\{u_{h_i}\}_i$ that converges to a function $u \in W_{\#,M}^{1,p}(Q_1; \mathbb{R}^m)$ weakly in $W^{1,p}(Q_1; \mathbb{R}^m)$. As a consequence, we get that, for any $R > 0$ fixed,

$$(\nabla u_{h_i}(x))\xi \rightharpoonup (\nabla u(x))\xi$$

weakly in $L^p(Q_1 \times B_R; \mathbb{R}^m)$ as $i \rightarrow +\infty$.

Since f satisfies (C) and (H0), we apply Theorem 1.3.8 with $\ell = d \times d, s = (\xi, x)$, and

$$E = Q_1 \times B_R, \quad \Psi((\xi, x), y, z) = \rho(\xi)f(x, y, z),$$

to obtain

$$\begin{aligned} \lim_{h \rightarrow +\infty} \varphi_h(M) &= \lim_{i \rightarrow +\infty} \int_{B_{T_{h_i}}} \rho(\xi) \int_{Q_1} f(x, x, (\nabla u_{h_i})\xi) dx d\xi \\ &\geq \int_{B_R} \rho(\xi) \int_{Q_1} f(x, x, (\nabla u)\xi) dx d\xi, \end{aligned}$$

and, by the arbitrariness of $R > 0$,

$$\begin{aligned} \lim_{h \rightarrow +\infty} \varphi_h(M) &\geq \int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f(x, x, (\nabla u)\xi) dx d\xi \\ &\geq \inf \left\{ \int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f(x, x, (\nabla u)\xi) dx d\xi : u \in W_{\#,M}^{1,p}(Q_1; \mathbb{R}^m) \right\} \\ &= f_0(M), \end{aligned}$$

concluding the proof in the subcritical case.

Critical case, $\lambda \in (0, +\infty)$. Fix $M \in \mathbb{R}^{m \times d}$. By Proposition 1.3.12, we have that

$$\begin{aligned} \lim_{h \rightarrow +\infty} \varphi_h(M) &= \lim_{h \rightarrow +\infty} \inf \left\{ \int_{B_{T_h}} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda\xi, \frac{u(x + \lambda\xi) - u(x)}{\lambda}\right) dx d\xi : u \in L_{\#,M}^p(Q_1; \mathbb{R}^m) \right\} \\ &\leq \inf \left\{ \int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda\xi, \frac{u(x + \lambda\xi) - u(x)}{\lambda}\right) dx d\xi : u \in L_{\#,M}^p(Q_1; \mathbb{R}^m) \right\}. \end{aligned}$$

To prove the converse, let $\{u_h\}_h \subset L_{\#,M}^p(Q_1; \mathbb{R}^m)$ be such that

$$\lim_{h \rightarrow +\infty} \varphi_h(M) = \lim_{h \rightarrow +\infty} \int_{B_{T_h}} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda\xi, \frac{u_h(x + \lambda\xi) - u_h(x)}{\lambda}\right) dx d\xi$$

and suppose that (1.3.45) holds for every $h \in \mathbb{N}$. Reasoning as in the subcritical case, we have

$$\sup_h \int_{B_{r_0}} \int_{Q_1} \left| \frac{u_h(x + \lambda\xi) - u_h(x)}{\lambda} \right|^p dx d\xi < +\infty,$$

and then, applying Lemma 1.3.10(ii) with $A = Q_1$ and $r = r_0$, we obtain that there exists a subsequence $\{u_{h_i}\}_i$ and a function $u \in L^p_{\#,M}(Q_1; \mathbb{R}^m)$ such that

$$\frac{u_{h_i}(x + \lambda\xi) - u_{h_i}(x)}{\lambda} \rightharpoonup \frac{u(x + \lambda\xi) - u(x)}{\lambda}$$

weakly in $L^p(Q_1 \times B_R; \mathbb{R}^m)$ as $i \rightarrow +\infty$, for any $R > 0$ fixed. The conclusion now follows as for the subcritical case. Indeed, we apply Theorem 1.3.8 in the same way to obtain

$$\begin{aligned} \lim_{h \rightarrow +\infty} \varphi_h(M) &= \lim_{i \rightarrow +\infty} \int_{B_{T_{h_i}}} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda\xi, \frac{u_{h_i}(x + \lambda\xi) - u_{h_i}(x)}{\lambda}\right) dx d\xi \\ &\geq \int_{B_R} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda\xi, \frac{u(x + \lambda\xi) - u(x)}{\lambda}\right) dx d\xi, \end{aligned}$$

and, by the arbitrariness of R and u , we conclude that

$$\lim_{h \rightarrow +\infty} \varphi_h(M) \geq \inf \left\{ \int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda\xi, \frac{u(x + \lambda\xi) - u(x)}{\lambda}\right) dx d\xi : u \in L^p_{\#,M}(Q_1; \mathbb{R}^m) \right\}.$$

Finally, using the change of variables $y := x + \lambda\xi$, we get

$$\inf \left\{ \int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda\xi, \frac{u(x + \lambda\xi) - u(x)}{\lambda}\right) dx d\xi : u \in L^p_{\#,M}(Q_1; \mathbb{R}^m) \right\} = f_\lambda(M),$$

that is the thesis.

Supercritical case, $\lambda = +\infty$. In this case the proof is immediate; indeed, by Proposition 1.3.12 and the Monotone Convergence Theorem,

$$\begin{aligned} \lim_{h \rightarrow +\infty} \varphi_h(M) &= \lim_{h \rightarrow +\infty} \int_{B_{T_h}} \int_{Q_1} \int_{Q_1} \rho(\xi) f(x, y, M\xi) dx dy d\xi \\ &= \int_{\mathbb{R}^d} \int_{Q_1} \int_{Q_1} \rho(\xi) f(x, y, M\xi) dx dy d\xi \\ &= f_{+\infty}(M) \end{aligned}$$

for every $M \in \mathbb{R}^{m \times d}$, concluding the proof. $\square$

As a corollary, we extend Proposition 1.3.11(iii) to kernels that are not necessarily supported on a ball, and also to the case $\lambda = +\infty$. Consequently, we obtain that the densities $\{f_\lambda\}_\lambda$ vary continuously with respect to λ .

Corollary 1.3.18. *Let $M \in \mathbb{R}^{m \times d}$ and let $\{\lambda_j\}_j$ be a positive sequence converging to $\lambda \in [0, +\infty]$. Then*

$$f_\lambda(M) = \lim_{j \rightarrow +\infty} \inf \left\{ \int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j\xi, \frac{u(x + \lambda_j\xi) - u(x)}{\lambda_j}\right) dx d\xi : u \in L^p_{\#,M}(Q_1; \mathbb{R}^m) \right\}.$$

Moreover, the function $\lambda \mapsto f_\lambda(M)$ is continuous in $[0, +\infty]$.

Proof. According to Theorem 1.3.1(ii) and a change of variables, we need to prove that

$$f_\lambda(M) = \lim_{j \rightarrow +\infty} f_{\lambda_j}(M).$$

Let us first suppose $\lambda \in [0, +\infty)$ and let $\{u_j\}_j \subset L^p_{\#,M}(Q_1; \mathbb{R}^m)$ be such that

$$\liminf_{j \rightarrow +\infty} f_{\lambda_j}(M) = \liminf_{j \rightarrow +\infty} \int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j\xi, \frac{u_j(x + \lambda_j\xi) - u_j(x)}{\lambda_j}\right) dx d\xi.$$

Upon assuming that the mean value of u_j on Q_1 is zero for all $j \in \mathbb{N}$, we obtain compactness properties for the sequence $\{u_j\}_j$ reasoning exactly as in Subsection 1.3.3 for the proofs of the lower bounds in the subcritical and critical cases, and also resorting to Theorem 1.3.8 we get that

$$\begin{aligned} \liminf_{j \rightarrow +\infty} f_{\lambda_j}(M) &\geq \inf \left\{ \int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f(x, x, (\nabla u)\xi) dx d\xi : u \in W_{\#,M}^{1,p}(Q_1; \mathbb{R}^m) \right\} \\ &= f_0(M) \end{aligned}$$

if $\lambda = 0$, and that

$$\begin{aligned} \liminf_{j \rightarrow +\infty} f_{\lambda_j}(M) &\geq \inf \left\{ \int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda\xi, \frac{u(x + \lambda\xi) - u(x)}{\lambda}\right) dx d\xi : u \in L_{\#,M}^p(Q_1; \mathbb{R}^m) \right\} \\ &= f_\lambda(M) \end{aligned}$$

if $\lambda \in (0, +\infty)$. Conversely, to prove that

$$\limsup_{j \rightarrow +\infty} f_{\lambda_j}(M) \leq f_\lambda(M),$$

we observe that

$$\limsup_{j \rightarrow +\infty} f_{\lambda_j}(M) \leq \limsup_{j \rightarrow +\infty} \int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j\xi, \frac{w(x + \lambda_j\xi) - w(x)}{\lambda_j}\right) dx d\xi \quad (1.3.46)$$

for any $w \in L_{\#,M}^p(Q_1; \mathbb{R}^m)$, and that

$$\int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j\xi, \frac{w(x + \lambda_j\xi) - w(x)}{\lambda_j}\right) dx d\xi$$

tends to

$$\int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f(x, x, (\nabla w)\xi) dx d\xi$$

if $\lambda = 0$ and $w \in W_{\#,M}^{1,p}(Q_1; \mathbb{R}^m) \cap C^\infty(\mathbb{R}^d; \mathbb{R}^m)$, and tends to

$$\int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda\xi, \frac{w(x + \lambda\xi) - w(x)}{\lambda}\right) dx d\xi$$

if $\lambda \in (0, +\infty)$ and $w \in L_{\#,M}^p(Q_1; \mathbb{R}^m) \cap C^\infty(\mathbb{R}^d; \mathbb{R}^m)$. Then, letting w be a (almost) minimizer for the minimum problem corresponding to $f_0(M)$ or to $f_\lambda(M)$, we infer the desired inequality by (1.3.46).

To conclude, we consider the case $\lambda = +\infty$. On the one hand, letting $T > 0$ sufficiently large so that Remark 1.3.16 is valid, we have

$$\begin{aligned} \liminf_{j \rightarrow +\infty} f_{\lambda_j}(M) &\geq \lim_{j \rightarrow +\infty} \inf \left\{ \int_{B_T} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j\xi, \frac{u(x + \lambda_j\xi) - u(x)}{\lambda_j}\right) dx d\xi : u \in L_{\#,M}^p(Q_1; \mathbb{R}^m) \right\} \\ &= \int_{B_T} \int_{Q_1} \int_{Q_1} \rho(\xi) f(x, y, M\xi) dx dy d\xi, \end{aligned}$$

and then, letting $T \rightarrow +\infty$,

$$\liminf_{j \rightarrow +\infty} f_{\lambda_j}(M) \geq f_{+\infty}(M).$$

Conversely, we recall that, in order to prove the upper bound for the supercritical case in Subsection 1.3.3, we proved that

$$\rho(\xi) \int_{Q_1} f(x, x + \lambda_j\xi, M\xi) dx \rightharpoonup \rho(\xi) \int_{Q_1} \int_{Q_1} f(x, y, M\xi) dx dy$$

weakly in $L^1(B_T)$ for any $T > 0$ large enough. By (GC) and (ρ_2) , we have that for a.e. $\xi \in \mathbb{R}^d$ and for every $j \in \mathbb{N}$ it holds

$$\rho(\xi) \int_{Q_1} f(x, x + \lambda_j\xi, M\xi) dx \leq \beta |M|^p \rho(\xi) |\xi|^p \in L^1(\mathbb{R}^d);$$

and therefore, we deduce that

$$\rho(\xi) \int_{Q_1} f(x, x + \lambda_j \xi, M\xi) dx \rightharpoonup \rho(\xi) \int_{Q_1} \int_{Q_1} f(x, y, M\xi) dx dy$$

weakly in $L^1(\mathbb{R}^d)$. We obtain

$$\begin{aligned} f_{+\infty}(M) &= \int_{\mathbb{R}^d} \int_{Q_1} \int_{Q_1} \rho(\xi) f(x, y, M\xi) dx dy d\xi \\ &= \lim_{j \rightarrow +\infty} \int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f(x, x + \lambda_j \xi, M\xi) dx d\xi \\ &\geq \limsup_{j \rightarrow +\infty} \inf \left\{ \int_{\mathbb{R}^d} \rho(\xi) \int_{Q_1} f\left(x, x + \lambda_j \xi, \frac{u(x + \lambda_j \xi) - u(x)}{\lambda_j}\right) dx d\xi : u \in L^p_{\#, M}(Q_1; \mathbb{R}^m) \right\} \\ &= \limsup_{j \rightarrow +\infty} f_{\lambda_j}(M), \end{aligned}$$

where we have used $u(x) = Mx$ as a test function for the minimum problems. This concludes the proof. $\square$

Remark 1.3.19. A milder growth condition from above on f can be required upon enhancing the integrability of ρ at the origin. In particular, we may require in addition that

$$\int_{\mathbb{R}^d} \rho(\xi) d\xi < +\infty, \quad (1.3.47)$$

in order to replace (GC) with

$$\alpha|z|^p \leq f(x, y, z) \leq \beta(1 + |z|^p) \text{ for almost every } x, y \in \mathbb{R}^d \text{ and for every } z \in \mathbb{R}^m. \quad (1.3.48)$$

Indeed, there are only a few points in our proofs where a growth condition from above on f is employed. We mention some of them and briefly illustrate how to modify the proofs according to the new set of assumptions.

In the first part of the proof of Proposition 1.3.11 we observed that

$$\left(\frac{\delta_j}{r}\right)^d \int_{B_T} \rho(\xi) \int_{(Q_{\lceil r/\delta_j \rceil})_{\lambda_j}(\xi) \setminus (Q_{r/\delta_j})_{\lambda_j}(\xi)} f(x, x + \lambda_j \xi, M\xi) dx d\xi \quad (1.3.49)$$

tends to 0. Recalling (1.3.12) and using (1.3.48), it is easily seen that (1.3.49) is estimated from above by

$$\left(\frac{\delta_j}{r}\right)^d |E_j| \beta \int_{B_T} \rho(\xi) (1 + |M|^p |\xi|^p) d\xi;$$

and since we already proved that

$$\left(\frac{\delta_j}{r}\right)^d |E_j| \beta |M|^p \int_{B_T} \rho(\xi) |\xi|^p d\xi$$

tends to 0 as $j \rightarrow +\infty$, it suffices to observe that

$$\left(\frac{\delta_j}{r}\right)^d |E_j| \beta \int_{B_T} \rho(\xi) d\xi$$

vanishes by (1.3.47) and the fact that $(\delta_j/r)^d |E_j| \rightarrow 0$.

In the upper bounds for the case $\lambda \in [0, +\infty)$ in Subsection 1.3.3, (GC) is employed to obtain some uniform bounds from above useful to apply the Dominated Convergence Theorem. As an example, we observe that inequality (1.3.24) is now replaced by

$$\rho(\xi) f(x, x, (\nabla u(x))\xi) \leq \beta \rho(\xi) (1 + |\xi|^p |\nabla u(x)|^p) \quad \text{for a.e. } (x, \xi) \in Q_1 \times B_T,$$

and the desired estimates are now obtained also resorting to (1.3.47).

In the supercritical case $\lambda = +\infty$, similar simple adaptations are needed in the proofs of Lemma 1.3.14 and Lemma 1.3.15. As for the proof of the upper bound, it suffices to observe that

$$\rho(\xi) \int_{Q_1} f(x, x + \lambda_j \xi, M\xi) dx, \quad j \in \mathbb{N},$$

is still an equi-integrable sequence since, by (1.3.48), we have that, for every $j \in \mathbb{N}$ and $A \subseteq B_T$ measurable, it holds

$$\int_A \rho(\xi) \int_{Q_1} f(x, x + \lambda_j \xi, M\xi) dx d\xi \leq \beta \left\{ \int_A \rho(\xi) d\xi + \int_A \rho(\xi) |M\xi|^p d\xi \right\}$$

so that the conclusion follows by (1.3.47). Clearly, in this case, the main result holds assuming (H1).

1.4 Perforated domains at the critical exponent

In a celebrated paper [57], Cioranescu and Murat investigated the asymptotic behaviour of Dirichlet problems in periodically perforated domains

$$\Omega_\delta := \Omega \setminus \bigcup_{i \in \mathbb{Z}^d} Q(i\delta, r_\delta), \quad (1.4.1)$$

where δ is the period of the array of perforations and the side-length of the cubic perforations is

$$r_\delta \sim \begin{cases} \exp(-\delta^{-2}) & \text{if } d = 2, \\ \delta^{\frac{d}{d-2}} & \text{if } d \geq 3, \end{cases}$$

(see also the earlier paper by Marchenko and Khruslov [99]). This turns out to be a critical choice for the size of the perforations, in the sense that the limit analysis gets trivial for different scales. It is proved that, for fixed $\phi \in H^{-1}(\Omega)$, the family of solutions $\{u_\delta\}_\delta$ to the problems

$$-\Delta u_\delta = \phi, \quad u_\delta \in H_0^1(\Omega_\delta),$$

weakly converges to a function $u \in H_0^1(\Omega)$ that solves a different limit problem; namely,

$$-\Delta u + C_d u = \phi, \quad u \in H_0^1(\Omega),$$

where an additional zero-order term, the so-called *strange term*, originates from the presence of the perforations. The constant C_d that appears in the limit equation is characterized by a capacity problem and, similarly to the scale r_δ , its nature varies in accordance with the dimension d . When $d > 2$, the constant is given by the 2-capacity of the unit cube in $\mathbb{R}^d$. If $d = 2$, such a characterization cannot hold anymore since, by scaling-invariance, the 2-capacity of every bounded set in $\mathbb{R}^2$ is zero; therefore, a more involved description in terms of the asymptotics of the ‘relative’ capacity of two concentric cubes is needed. This singular behaviour is not peculiar to the quadratic case. Dealing with the p -capacity in $\mathbb{R}^d$ for $p \in (1, d)$, $d \geq 2$, one can always observe that the relative capacity of concentric cubes has a polynomial behaviour with respect to the side-lengths, while a logarithmic behaviour is exhibited in the case $p = d$. For this reason we shall address the case $p \in (1, d)$ as a subcritical exponent and the case $p = d$ as the *critical exponent*.

In [15], Ansini and Braides extended this result to a vectorial non-linear setting for subcritical exponents $p \in (1, d)$ by computing the Γ -limit of the functionals $F_\delta : L^p(\Omega; \mathbb{R}^m) \rightarrow [0, +\infty]$ defined as

$$F_\delta(u) := \begin{cases} \int_\Omega f(\nabla u) dx & \text{if } u \in W_0^{1,p}(\Omega_\delta; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

for $f : \mathbb{R}^{m \times d} \rightarrow [0, +\infty)$ a quasiconvex p -homogeneous integrand satisfying p -growth conditions. Conformally with the work by Cioranescu and Murat, they assumed that $r_\delta \sim \delta^{d/(d-p)}$ and proved that the Γ -limit of $\{F_\delta\}_\delta$ equals

$$\int_{\Omega} f(\nabla u) dx + \int_{\Omega} \varphi_p(u) dx, \quad u \in W_0^{1,p}(\Omega; \mathbb{R}^m),$$

where $\varphi_p : \mathbb{R}^m \rightarrow [0, +\infty)$ is given by a non-linear capacity formula:

$$\varphi_p(z) := \min \left\{ \int_{\mathbb{R}^d} f(\nabla u) dx : u - z \in W^{1,p}(\mathbb{R}^d; \mathbb{R}^m), u = 0 \text{ on } Q(1) \right\}.$$

The extension of the above analysis to the critical exponent $p = d$ required a different proof provided by Sigalotti [122], who showed that, upon assuming that $r_\delta \sim \exp(-\delta^{d/(1-d)})$, the limit energy is given by

$$\int_{\Omega} f(\nabla u) dx + \int_{\Omega} \varphi_d(u) dx, \quad u \in W_0^{1,d}(\Omega; \mathbb{R}^m), \quad (1.4.2)$$

where the density $\varphi_d : \mathbb{R}^m \rightarrow [0, +\infty)$ is now described by a *homogenization-type formula*:

$$\varphi_d(z) := \lim_{L \rightarrow +\infty} (\log L)^{d-1} \min \left\{ \int_{Q(L)} f(\nabla u) dx : u - z \in W_0^{1,d}(Q(L); \mathbb{R}^m), u = 0 \text{ on } Q(1) \right\}. \quad (1.4.3)$$

In a similar vein, in this section we extend the analysis of Dirichlet problems on periodically perforated domains for non-local energies to the critical exponent, the subcritical case being recently investigated by Alicandro, Gelli, and Leone [10]. We consider non-local energies subject to boundary conditions

$$F_{\varepsilon,\delta}(u) := \begin{cases} \frac{1}{\varepsilon^d} \int_{\Omega} \int_{\Omega} f\left(\frac{y-x}{\varepsilon}, \frac{u(y)-u(x)}{\varepsilon}\right) dx dy & \text{if } u = 0 \text{ on } \Omega \setminus \Omega_\delta, \\ +\infty & \text{otherwise,} \end{cases}$$

with Ω_δ as in (1.4.1) and $r_\delta \sim \exp(-\delta^{\frac{d}{1-d}})$, and we prove that, in the regime $\varepsilon \ll r_\delta$, a separation of scales occurs yielding a limit energy comparable to (1.4.2).

The problem of perforated domains, together with its many variants, is a classical topic that still draws considerable attention. Recent results have been obtained in the context of stochastic homogenization by Scardia, Zemas, and Zeppieri [119] and by Deangelis, Focardi, and Zeppieri [66] in the case of fractional energies. Very recently, Palatucci has thoroughly discussed the phenomenon of capacity screening in periodically perforated domains for fractional energies defined through the Gagliardo seminorm on the whole $\mathbb{R}^d$, see [111]. These works, as well as many others in the literature, deal with the case of subcritical exponents (that, in a fractional setting, are characterized by the inequality $p < d/s$, being s the fractional parameter).

The necessity of treating a scaling-invariant problem leads us to carry out a delicate study, that substantially departs from the ones performed in the subcritical cases. Our strategy is partly inspired by the works [31, 43], where an extension to the critical exponent for the problem of perforated domains in heterogeneous media originally studied in [14] is obtained.

A technical insight of our analysis is that the energy density φ_d in (1.4.3) is convex. This feature is relevant (and non-trivial) in the vector-valued case $m > 1$ and displays a striking similarity with the Ginzburg-Landau model. Indeed, the convexity of the strange term reflects the emergence of an optimal multiscale microstructure since configurations of minimal energies are obtained distributing equally the energy on every scale from r_δ (the side-length of the perforations) to δ (a much larger scale) by means of a dyadic construction. This observation is not related to the non-local nature of the problem and actually applies to the original Dirichlet problem on perforated domains. However, it is worth noting that the convexity of φ_d has not been the starting point of our investigation; from our perspective, it is more appropriate to regard it as an outcome of the multiscale analysis as we shall explain in the forthcoming subsection.

1.4.1 Setting of the problem and statement of the result

In this subsection we describe the setting of our problem and we state our main result.

We let $f : \mathbb{R}^d \times \mathbb{R}^m \rightarrow [0, +\infty)$ be a function satisfying the following assumptions:

- (H) $f(\xi, \cdot)$ is d -homogeneous for every $\xi \in \mathbb{R}^d$; i.e., $f(\xi, tz) = t^d f(\xi, z)$ for every $t \geq 0, \xi \in \mathbb{R}^d$, and $z \in \mathbb{R}^m$;
- (G) the functions $m(\xi) := \inf_{z \in \mathbb{S}^{m-1}} f(\xi, z)$ and $M(\xi) := \sup_{z \in \mathbb{S}^{m-1}} f(\xi, z)$ defined for every $\xi \in \mathbb{R}^d$ satisfy:
 - (G0) there exist positive constants r_0, C such that $m(\xi) \geq C$ for a.e. $\xi \in B_{r_0}$,
 - (G1) $\int_{\mathbb{R}^d} M(\xi)(1 + |\xi|^d) d\xi < +\infty$;
- (L) there exists a positive constant C such that

$$|f(\xi, z) - f(\xi, z')| \leq CM(\xi)(|z|^{d-1} + |z'|^{d-1})|z - z'|$$

for every $\xi \in \mathbb{R}^d$ and for every $z, z' \in \mathbb{R}^m$.

Remark 1.4.1. We comment on the above hypotheses:

- (i) assumption (H) yields $m(\xi)|z|^p \leq f(\xi, z) \leq M(\xi)|z|^p$ for every $(\xi, z) \in \mathbb{R}^d \times \mathbb{R}^m$;
- (ii) hypothesis (L) is satisfied if f fulfills (H), (G), and $f(\xi, \cdot)$ is convex for every $\xi \in \mathbb{R}^d$.

We introduce the family of unconstrained functionals and, for later convenience, we also highlight the dependence on the fixed domain. To make our notation more flexible for later use, here we let σ denote a small positive parameter. For fixed $\sigma > 0$ and $A \subseteq \mathbb{R}^d$ measurable set, we let $\mathcal{F}_\sigma(\cdot, A) : L^d(A; \mathbb{R}^m) \rightarrow [0, +\infty]$ be defined as

$$\mathcal{F}_\sigma(u, A) := \begin{cases} \frac{1}{\sigma^d} \int_A \int_A f\left(\frac{y-x}{\sigma}, \frac{u(y)-u(x)}{\sigma}\right) dx dy & \text{if } u \in L^d(A; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases}$$

We will mainly use the following equivalent form that is obtained performing the change of variables $\xi := (y-x)/\sigma$,

$$\mathcal{F}_\sigma(u, A) := \begin{cases} \int_{\mathbb{R}^d} \int_{A_\sigma(\xi)} f\left(\xi, \frac{u(x+\sigma\xi)-u(x)}{\sigma}\right) dx d\xi & \text{if } u \in L^d(A; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases} \quad (1.4.4)$$

where $A_\sigma(\xi)$ is as in (1.1.1). For the sake of notation, if $A = \Omega$ we simply write $\mathcal{F}_\sigma(u)$ in place of $\mathcal{F}_\sigma(u, \Omega)$. The asymptotic analysis of the above energies and the convergence of minimum problems are obtained in [5, Theorem 6.1, Proposition 6.4].

Theorem 1.4.2. *Let A be a bounded open set with Lipschitz boundary and let $\mathcal{F}_\sigma(\cdot, A)$ be defined by (1.4.4). Then, the family $\{\mathcal{F}_\sigma(\cdot, A)\}_\sigma$ Γ -converges with respect to the convergence in $L^d(A; \mathbb{R}^m)$ -topology as $\sigma \rightarrow 0$ to the functional $\mathcal{F}(\cdot, A) : L^d(A; \mathbb{R}^m) \rightarrow [0, +\infty]$ defined as*

$$\mathcal{F}(u, A) := \begin{cases} \int_A f_{\text{hom}}(\nabla u) dx & \text{if } u \in W^{1,d}(A; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

where $f_{\text{hom}} : \mathbb{R}^{m \times d} \rightarrow [0, +\infty)$ is given by

$$f_{\text{hom}}(M) := \lim_{h \rightarrow +\infty} \frac{1}{h^d} \inf \left\{ \int_{Q(h)} \int_{Q(h)} f(y-x, v(y)-v(x)) dx dy : v \in \mathcal{D}_{1,M}^\infty(Q(h); \mathbb{R}^m) \right\}, \quad (1.4.5)$$

with $\mathcal{D}_{1,M}^\infty(Q(h); \mathbb{R}^m)$ as in (1.1.3) for every $M \in \mathbb{R}^{m \times d}$. In particular, f_{hom} is a d -homogeneous quasiconvex function and there exists a positive constant C such that, for every $M \in \mathbb{R}^{m \times d}$,

$$\frac{1}{C}|M|^d \leq f_{\text{hom}}(M) \leq C|M|^d. \quad (1.4.6)$$

Moreover, for any $g : \mathbb{R}^d \rightarrow \mathbb{R}^m$ Lipschitz continuous function and $T > 0$, there holds

$$\liminf_{\sigma \rightarrow 0} \{\mathcal{F}_\sigma(u, A) : u \in \mathcal{D}_{\sigma T, g}^\infty(A; \mathbb{R}^m)\} = \min \left\{ \int_A f_{\text{hom}}(\nabla u) dx : u - g \in W_0^{1,d}(A; \mathbb{R}^m) \right\},$$

with $\mathcal{D}_{\sigma T, g}^\infty(A; \mathbb{R}^m)$ as in (1.1.3).

Now we define the functionals of our interest by imposing Dirichlet boundary conditions on an array of perforations. We let ε be a small parameter denoting the length-scale of the non-local interactions and we introduce two further positive parameters assuming that both are ruled by ε : $\delta = \delta_\varepsilon$ denotes the period of the centers of the perforations, and $r = r_\varepsilon$ denotes the side-length of the cubic perforations. We define the array of cubic perforations as

$$P_\varepsilon := \bigcup_{i \in \mathbb{Z}^d} Q(i\delta_\varepsilon, r_\varepsilon)$$

and we set

$$\mathcal{P}_\varepsilon(\Omega; \mathbb{R}^m) := \{u \in L^d(\Omega; \mathbb{R}^m) : u(x) = 0 \text{ for a.e. } x \in P_\varepsilon\}.$$

Then, we define functionals $F_\varepsilon : L^d(\Omega; \mathbb{R}^m) \rightarrow [0, +\infty]$ as

$$F_\varepsilon(u) := \begin{cases} \frac{1}{\varepsilon^d} \int_\Omega \int_\Omega f\left(\frac{y-x}{\varepsilon}, \frac{u(y)-u(x)}{\varepsilon}\right) dx dy & \text{if } u \in \mathcal{P}_\varepsilon(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

or equivalently, by the change of variables $\xi := (y-x)/\varepsilon$,

$$F_\varepsilon(u) := \begin{cases} \int_{\mathbb{R}^d} \int_{\Omega_\varepsilon(\xi)} f\left(\xi, \frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\right) dx d\xi & \text{if } u \in \mathcal{P}_\varepsilon(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases} \quad (1.4.7)$$

Our main result is as follows.

Theorem 1.4.3. *Let F_ε be defined by (1.4.7) and assume that*

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{r_\varepsilon} = 0$$

and that there exists $\gamma > 0$ such that

$$\lim_{\varepsilon \rightarrow 0} \frac{\exp(-(\gamma\delta_\varepsilon^d)^{\frac{1}{1-d}})}{r_\varepsilon} = 1.$$

Then, the family $\{F_\varepsilon\}_\varepsilon$ Γ -converges with respect to the convergence in $L^d(\Omega; \mathbb{R}^m)$ to the functional $F : L^d(\Omega; \mathbb{R}^m) \rightarrow [0, +\infty]$ defined as

$$F(u) := \begin{cases} \int_\Omega f_{\text{hom}}(\nabla u) dx + \gamma \int_\Omega \varphi(u) dx & \text{if } u \in W^{1,d}(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases} \quad (1.4.8)$$

where f_{hom} is given by (1.4.5) and φ is a convex function that, for every $z \in \mathbb{R}^m$, is given by

$$\varphi(z) := \lim_{L \rightarrow +\infty} (\log L)^{d-1} \min \left\{ \int_{Q(L)} f_{\text{hom}}(\nabla v) dx : v - z \in W_0^{1,d}(Q(L); \mathbb{R}^m), v = 0 \text{ on } Q(1) \right\}.$$

In [50], the above result is proved in a unified framework that also accounts for discrete energies, for which a variational theory has been developed by Alicandro and Cicalese [8]. We extend to the critical exponent and in the regime $\varepsilon \ll r_\varepsilon$ the results in [77, 123] by investigating the problem of perforated domains for pairwise interaction energies

$$\int_{\Omega} \int_{\Omega} f\left(\frac{y-x}{\varepsilon}, u(y) - u(x)\right) d\mu_\varepsilon(x) d\mu_\varepsilon(y),$$

where the measure μ_ε is selected among

$$\mu_\varepsilon := \begin{cases} \frac{1}{\varepsilon^d} \mathcal{L}^d & (\text{continuous case}), \\ \sum_{\alpha \in \varepsilon \mathbb{Z}^d} \delta_\alpha & (\text{discrete case}). \end{cases}$$

In this regard, we point out that we opt for cubic perforations to simplify the argument in the discrete setting, where cubic lattices are considered. However, the shape of the perforations does not affect our result as a standard comparison argument yields that the same conclusion holds true replacing the cube $Q(1)$ with any bounded set with non-empty interior.

Outline of the proof and the convexity of the strange term. We conclude this subsection by illustrating in broad terms the idea of our proof and how this led us to observe, somewhat indirectly, the convexity of the strange term. Our argument is based on a new variant of the ‘joining lemma on varying domains’ (Lemma 1.4.17) devised by Ansini and Braides in [15]. It allows one to modify a converging family $\{u_\varepsilon\}_\varepsilon$ (at a small energetic cost) by imposing constant boundary conditions on several dyadic concentric annuli surrounding the perforations. In this way, one can isolate the contributions to the energy coming from the perforations and the remaining region. The latter is essentially the unconstrained energy (1.4.4) and contributes to the limit energy yielding the first order term in (1.4.8).

To estimate the energy due to a single perforation, that we assume to be centered at 0 for simplicity, we perform a dyadic construction: we fix a large $L \in \mathbb{N}$ and cover the reference cell $Q(\delta_\varepsilon)$ by means of concentric cubes

$$Q(r_\varepsilon), Q(r_\varepsilon L), Q(r_\varepsilon L^2), \dots, Q(r_\varepsilon L^S) \sim Q(\delta_\varepsilon), \quad S \sim \frac{\log(\delta_\varepsilon/r_\varepsilon)}{\log L}.$$

Thanks to an argument that allows us to consider only finite-range interactions, we decompose the energy close to the perforation on each “frame” as follows:

$$F_\varepsilon(u_\varepsilon, Q(\delta_\varepsilon)) \sim \sum_{h=1}^S \mathcal{F}_\varepsilon(u, Q(r_\varepsilon L^h) \setminus Q(r_\varepsilon L^{h-1})).$$

By the joining lemma, we can assume that there exist constants $\{u_\varepsilon^{0,h} : h \in \{0, \dots, S\}\}$ such that

$$u_\varepsilon = u_\varepsilon^{0,h} \text{ close to } \partial Q(r_\varepsilon L^h) \quad \text{for every } h \in \{0, \dots, S\},$$

with $u_\varepsilon^{0,0} = 0$ and, taking into account such boundary conditions, we can optimize the energy on $Q(\delta_\varepsilon)$ to get

$$\begin{aligned} F_\varepsilon(u_\varepsilon, Q(\delta_\varepsilon)) &\gtrsim \sum_{h=1}^S \min\{\mathcal{F}_\varepsilon(u, Q(r_\varepsilon L^h) \setminus Q(r_\varepsilon L^{h-1})) : \\ &\quad u = u_\varepsilon^{0,h} \text{ close to } \partial Q(r_\varepsilon L^h), u = u_\varepsilon^{0,h-1} \text{ close to } \partial Q(r_\varepsilon L^{h-1})\}. \end{aligned}$$

To estimate the energy on the single frame $Q(r_\varepsilon L^h) \setminus Q(r_\varepsilon L^{h-1})$, we use the scaling invariance of the energies and the convergence of minima in Theorem 1.4.2 to get that

$$\begin{aligned} &\min\{\mathcal{F}_\varepsilon(u, Q(r_\varepsilon L^h) \setminus Q(r_\varepsilon L^{h-1})) : u = u_\varepsilon^{0,h} \text{ close to } \partial Q(r_\varepsilon L^h), u = u_\varepsilon^{0,h-1} \text{ close to } \partial Q(r_\varepsilon L^{h-1})\} \\ &= \min\{\mathcal{F}_{\varepsilon/(r_\varepsilon L^{h-1})}(u, Q(L) \setminus Q(1)) : u = u_\varepsilon^{0,h} \text{ close to } \partial Q(L), u = u_\varepsilon^{0,h-1} \text{ close to } \partial Q(1)\} \\ &\sim \min\left\{\int_{Q(L)} f_{\text{hom}}(\nabla u) dx : u - (u_\varepsilon^{0,h} - u_\varepsilon^{0,h-1}) \in W_0^{1,d}(Q(L); \mathbb{R}^m), u = 0 \text{ on } Q(1)\right\} \end{aligned} \quad (1.4.9)$$

as $\varepsilon \sim 0$. Note that here we made use of the assumption $\varepsilon/r_\varepsilon \rightarrow 0$ and of a uniform-convergence argument that allows us to keep the boundary conditions unaltered in spite of the fact that these depend on ε . Recalling (1.4.3), we have that (1.4.9) is approximately

$$(\log L)^{1-d} \varphi(u_\varepsilon^{0,h} - u_\varepsilon^{0,h-1}).$$

Since $S \sim \frac{\log(\delta_\varepsilon/r_\varepsilon)}{\log L}$ and $r_\varepsilon \sim \exp(-\delta_\varepsilon^{d/(1-d)})$, we have that $(\log L)^{1-d} \sim \delta_\varepsilon^d S^{d-1}$; hence, summing up over the frames, we have

$$F_\varepsilon(u_\varepsilon, Q(\delta_\varepsilon)) \gtrsim \delta_\varepsilon^d S^{d-1} \sum_{h=1}^S \varphi(u_\varepsilon^{0,h} - u_\varepsilon^{0,h-1}).$$

Upon observing that

$$\sum_{h=1}^S (u_\varepsilon^{0,h} - u_\varepsilon^{0,h-1}) = u_\varepsilon^{0,S} - u_\varepsilon^{0,0} = u_\varepsilon^{0,S},$$

we can further optimize the previous sum and use the d -homogeneity of φ to get

$$F_\varepsilon(u_\varepsilon, Q(\delta_\varepsilon)) \gtrsim \delta_\varepsilon^d \min \left\{ S^{d-1} \sum_{h=1}^S \varphi(y_h) : \sum_{h=1}^S y_h = u_\varepsilon^{0,S} \right\} = \delta_\varepsilon^d \min \left\{ \frac{1}{S} \sum_{h=1}^S \varphi(y_h) : \frac{1}{S} \sum_{h=1}^S y_h = u_\varepsilon^{0,S} \right\}.$$

The latter term is seen to be approximately $\delta_\varepsilon^d \varphi^{**}(u_\varepsilon^{0,S})$ as $S \sim +\infty$, where φ^{**} denotes the convex envelope of φ . Then, summing up over all the perforations and using that $u_\varepsilon \rightarrow u$, we infer that

$$F_\varepsilon \left(u_\varepsilon, \bigcup_{i \in \mathbb{Z}^d \cap \delta_\varepsilon^{-1} \Omega} Q(i\delta_\varepsilon, \delta_\varepsilon) \right) \gtrsim \sum_{i \in \mathbb{Z}^d \cap \delta_\varepsilon^{-1} \Omega} \delta_\varepsilon^d \varphi^{**}(u_\varepsilon^{i,S}) \sim \int_{\Omega} \varphi^{**}(u) dx$$

as $\varepsilon \sim 0$ and $S \sim +\infty$.

The heuristic that yields this lower bound (that is seen to be optimal) seems to be in contrast with the separation of scales argument leading to (1.4.8) and, to some extent, hints that φ is convex. Conscious of this fact, we directly prove that φ coincides with its convex envelope starting from its very definition (see Proposition 1.4.13).

1.4.2 Preliminary results

We collect results from the literature that are instrumental for our analysis. Throughout the rest of this section, we let C denote a generic positive constant that may change from line to line, but which is independent of the relevant parameters.

A class of reference energies. Given r_0 as in (G0), for every $\sigma > 0$, $A \subseteq \mathbb{R}^d$ measurable set, and $u \in L^d(A; \mathbb{R}^m)$, we define the reference energies

$$G_\sigma(u, A) := \int_{B_{r_0}} \int_{A_\sigma(\xi)} \left| \frac{u(x + \sigma\xi) - u(x)}{\sigma} \right|^d dx d\xi. \quad (1.4.10)$$

Concerning these auxiliary functionals, we recall a lemma that allows us to control interactions of any range with short-range interactions only. The following is a consequence of [5, Lemma 4.1].

Lemma 1.4.4. *There exists a positive constant C such that, for every Ω open set, for every $A \subseteq \Omega$ bounded open set with Lipschitz boundary, $T > 0$, $\xi \in B_T$, and $\sigma > 0$ such that*

$$\text{dist}_\infty(A, \Omega^c) > (T + r_0)\sigma,$$

it holds

$$\int_A \left| \frac{u(x + \sigma\xi) - u(x)}{\sigma} \right|^d dx \leq C(1 + |\xi|^d) G_\sigma(u, A + Q((T + r_0)\sigma))$$

for every $u \in L^d(\Omega; \mathbb{R}^m)$.

Remark 1.4.5. The above lemma also applies with $\Omega = \mathbb{R}^d$. In particular, there exists a positive constant C such that, for every $\xi \in \mathbb{R}^d$ and $\sigma > 0$, it holds

$$\int_{\mathbb{R}^d} \left| \frac{u(x + \sigma\xi) - u(x)}{\sigma} \right|^d dx \leq C(1 + |\xi|^d) G_\sigma(u, \mathbb{R}^d) \quad (1.4.11)$$

for every $u \in L^d(\mathbb{R}^d; \mathbb{R}^m)$.

The following proposition is the analogue of the classical Poincaré-Wirtinger inequality. In the non-local case this can be inferred from [5, Proposition 4.2].

Proposition 1.4.6. *Let $A \subset \mathbb{R}^d$ be a connected bounded open set with Lipschitz boundary and let $A' \subseteq A$ be a subset of positive measure. There exist positive constants C, σ_0 depending on A such that, having set*

$$(u)_E := \frac{1}{|E|} \int_E u dx,$$

it holds

$$\int_{\tau A} |u - (u)_{\tau A'}|^d dx \leq C \frac{|A|}{|A'|} \tau^d G_\sigma(u, \tau A)$$

for every $\tau > 0$, $u \in L^d(\tau A; \mathbb{R}^m)$ and $\sigma < \sigma_0 \tau$.

Functionals with finite-range interactions. As done in Subsection 1.3.2, we introduce “truncated” versions of the functionals in (1.4.4). For every $T > r_0$ and for every $z \in \mathbb{R}^m$ we let

$$f^T(\xi, z) := \begin{cases} f(\xi, z) & \text{if } |\xi| \leq T, \\ 0 & \text{otherwise,} \end{cases}$$

and, given $\sigma > 0$ and $A \subseteq \mathbb{R}^d$ measurable, we let $\mathcal{F}_\sigma^T(\cdot, A) : L^d(A; \mathbb{R}^m) \rightarrow [0, +\infty]$ be defined as

$$\mathcal{F}_\sigma^T(u, A) := \begin{cases} \int_{\mathbb{R}^d} \int_{A_\sigma(\xi)} f^T\left(\xi, \frac{u(x + \sigma\xi) - u(x)}{\sigma}\right) dx d\xi & \text{if } u \in L^d(A; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases} \quad (1.4.12)$$

We write $\mathcal{F}_\sigma(u)$ in place of $\mathcal{F}_\sigma(u, \Omega)$ when $A = \Omega$.

Remark 1.4.7. Since $T > r_0$, by Remark 1.4.1 it follows

$$CG_\sigma(u, A) \leq \mathcal{F}_\sigma^T(u, A). \quad (1.4.13)$$

As for the constrained functionals (1.4.7), for $\varepsilon > 0$ we define $F_\varepsilon^T : L^d(\Omega; \mathbb{R}^m) \rightarrow [0, +\infty]$ as

$$F_\varepsilon^T(u) := \begin{cases} \int_{\mathbb{R}^d} \int_{\Omega_\varepsilon(\xi)} f^T\left(\xi, \frac{u(x + \varepsilon\xi) - u(x)}{\varepsilon}\right) dx d\xi & \text{if } u \in \mathcal{P}_\varepsilon(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases} \quad (1.4.14)$$

The Γ -limit of the functionals $\{\mathcal{F}_\sigma^T\}_\sigma$ in (1.4.12) can be computed making use of Theorem 1.4.2. According to the following proposition, the corresponding energy densities, denoted by f_{hom}^T , converge as $T \rightarrow +\infty$ to f_{hom} , the energy density of the Γ -limit of the functionals $\{\mathcal{F}_\sigma\}_\sigma$ defined by (1.4.4).

Proposition 1.4.8. *Assume $T > r_0$ and let*

$$f_{\text{hom}}^T(M) := \lim_{h \rightarrow +\infty} \frac{1}{h^d} \min \left\{ \int_{Q(h)} \int_{Q(h)} f^T(y - x, v(y) - v(x)) dx dy : v \in \mathcal{D}_{1,M}^\infty(Q(h); \mathbb{R}^m) \right\} \quad (1.4.15)$$

for every $M \in \mathbb{R}^{m \times d}$. Given f_{hom} as in (1.4.5), it holds that

$$\lim_{T \rightarrow +\infty} f_{\text{hom}}^T = f_{\text{hom}} \quad (1.4.16)$$

monotonically and uniformly on compact subsets of $\mathbb{R}^{m \times d}$.

Proof. The proof of the pointwise convergence in (1.4.16) is contained in the proof of [5, Theorem 6.1]. As for the uniform convergence on compact subsets of $\mathbb{R}^{m \times d}$, we note that $f_{\text{hom}}^T \leq f_{\text{hom}}$ for every T . By (1.4.6), this implies that there exists a positive constant C such that

$$\frac{1}{C}|M|^d \leq f_{\text{hom}}^T(M) \leq f_{\text{hom}}(M) \leq C|M|^d \quad (1.4.17)$$

for every $T > r_0$ and $M \in \mathbb{R}^{m \times d}$. Since f_{hom}^T is a d -homogeneous quasiconvex function, as a consequence of [34, Remark 4.13] and (1.4.17), there exists a positive constant C such that

$$|f_{\text{hom}}^T(M) - f_{\text{hom}}^T(M')| \leq C(|M|^{d-1} + |M'|^{d-1})|M - M'|$$

for every $T > r_0$ and $M, M' \in \mathbb{R}^{m \times d}$. Hence, the pointwise convergence and Ascoli-Arzelà Theorem allow to obtain the claim. $\square$

The main advantage in dealing with functionals accounting for finite-range interactions only is that, in broad terms, they behave quite similarly to “local” functionals. It turns out that, for the constrained energies $\{F_\varepsilon\}_\varepsilon$ defined in (1.4.7), it is possible to perform the asymptotic analysis considering first functionals with finite-range interactions (1.4.14), and then recovering the general case by letting $T \rightarrow +\infty$. This behaviour is analogous to that highlighted for the unconstrained energies in Lemma 1.3.4 in the previous section. Also in this case, the proof is obtained arguing as in [5, Lemma 5.1]; see also [77, Section 5, Step 1] for the case of discrete energies subject to Dirichlet boundary conditions on a periodic array of perforations.

Lemma 1.4.9. *Let $\{T_h\}_h$ be a sequence monotonically increasing to $+\infty$ and let $F_\varepsilon^{T_h}$ be defined by (1.4.14). Assume that there exist functionals $\{F^{T_h}\}_h$ such that for all $h \in \mathbb{N}$ we have*

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon^{T_h}(u) = F^{T_h}(u)$$

for every $u \in L^d(\Omega; \mathbb{R}^m)$. Then

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u) = \lim_{h \rightarrow +\infty} F^{T_h}(u)$$

for every $u \in L^d(\Omega; \mathbb{R}^m)$.

The next result accounts for the compactness in the strong L^d -topology of families of functions with uniformly bounded energies. We provide a more general statement for non-truncated energies (1.4.4). The proof of this fact follows combining [5, Theorem 4.2] with (1.4.13).

Proposition 1.4.10. *Let $\mathcal{F}_\varepsilon$ be defined by (1.4.4) and let $\{u_\varepsilon\}_\varepsilon \subset L^d(\Omega; \mathbb{R}^m)$ be such that*

$$\sup_\varepsilon (\mathcal{F}_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_{L^d(\Omega; \mathbb{R}^m)}) < +\infty.$$

Then, there exist a positive sequence $\{\varepsilon_j\}_j$ converging to 0 and a function $u \in W^{1,d}(\Omega; \mathbb{R}^m)$ such that $u_{\varepsilon_j} \rightarrow u$ in $L^d(\Omega; \mathbb{R}^m)$ as $j \rightarrow +\infty$.

Capacity and homogenization-type capacity formula. We recall the definition of the relative d -capacity for subsets of $\mathbb{R}^d$. We refer to [87]. Given open sets $U \subset V \subseteq \mathbb{R}^d$, we define the d -capacity of U relative to V as

$$\text{Cap}_d(U, V) := \min \left\{ \int_V |\nabla v|^d dx : v \in W_0^{1,d}(V) : v = 1 \text{ on } U \right\}.$$

If $U = B_r$ and $V = B_R$ for some $0 < r < R$, the explicit computation of the corresponding relative capacity can be performed by the reduction to a 1-dimensional problem. In particular, it holds that

$$\text{Cap}_d(B_r, B_R) = \sigma_{d-1} \left(\log \left(\frac{R}{r} \right) \right)^{1-d}.$$

Such logarithmic behaviour differs from that exhibited in the case $p < d$, where the capacity depends on r and R with polynomial growth. For this reason, we address the exponent d as the *critical exponent* for the capacity. Through a comparison argument, it is possible to infer that

$$\lim_{L \rightarrow +\infty} (\log L)^{d-1} \text{Cap}_d(Q(1), Q(L)) = \sigma_{d-1},$$

so that, in particular, there exists a positive constant C such that

$$\frac{1}{C} \leq (\log L)^{d-1} \text{Cap}_d(Q(1), Q(L)) \leq C \quad (1.4.18)$$

for every $L > 4$.

Analogous capacity problems can be set in a vectorial framework. Given $z \in \mathbb{R}^m$ and U, V as above, we may consider the minimum problem

$$\min \left\{ \int_V |\nabla v|^d dx : v \in W_0^{1,d}(V; \mathbb{R}^m) : v = z \text{ on } U \right\}.$$

Due to the scaling property and rotational invariance, it is easily seen that the above minimum coincides with

$$\min \left\{ \int_V |\nabla v|^d dx : v \in W_0^{1,d}(V; \mathbb{R}^m) : v = (1, 0, \dots, 0) \text{ on } U \right\} |z|^d = \text{Cap}_d(U, V) |z|^d.$$

We mention some useful facts about the density φ appearing in the *strange term* in our main result Theorem 1.4.3. The following proposition, which coincides with [122, Proposition 5.1] up to replacing spherical perforations with cubic ones, establishes the asymptotics of nonlinear capacities modelled on d -homogeneous densities and makes the function φ well defined.

Proposition 1.4.11. *Let $g : \mathbb{R}^{m \times d} \rightarrow [0, +\infty)$ be a quasiconvex d -homogeneous function and assume that there exists a positive constant C such that, for every $M \in \mathbb{R}^{m \times d}$, it holds*

$$\frac{1}{C} |M|^d \leq g(M) \leq C |M|^d.$$

Then, there exist

$$\psi(z) := \lim_{R \rightarrow +\infty} (\log R)^{d-1} \min \left\{ \int_{Q(R)} g(\nabla v) dx : v - z \in W_0^{1,d}(Q(R); \mathbb{R}^m) : v = 0 \text{ on } Q(1) \right\},$$

and a positive constant C such that, for every $z \in \mathbb{R}^m$, it holds

$$\frac{1}{C} |z|^d \leq \psi(z) \leq C |z|^d.$$

One of our main observations is that the function ψ in Proposition 1.4.11 is convex. To see this, we first prove the following lemma.

Lemma 1.4.12. *Let $\psi : \mathbb{R}^m \rightarrow [0, +\infty)$ be a coercive continuous function and let ψ^{**} denote its convex envelope. For every $z \in \mathbb{R}^m$ it holds*

$$\psi^{**}(z) = \lim_{S \rightarrow +\infty} \min \left\{ \frac{1}{S} \sum_{h=1}^S \psi(z_h) : \frac{1}{S} \sum_{h=1}^S z_h = z \right\}. \quad (1.4.19)$$

Proof. Let $\tilde{\psi}$ denote the right-hand side of (1.4.19). We only prove $\psi^{**} \geq \tilde{\psi}$ as the converse inequality is immediately satisfied.

By Carathéodory's theorem (see [117, Theorem 17.1]), there exist $\{\lambda_1, \dots, \lambda_{m+1}\} \subset [0, 1]$ and $\{z_1, \dots, z_{m+1}\} \subset \mathbb{R}^m$ such that

$$\sum_{i=1}^{m+1} \lambda_i \psi(z_i) = \psi^{**}(z), \quad \sum_{i=1}^{m+1} \lambda_i = 1, \quad \text{and} \quad \sum_{i=1}^{m+1} \lambda_i z_i = z.$$

Consider S a positive integer and let

$$(y_1, \dots, y_S) := \left(\underbrace{\frac{\lambda_1 S}{[\lambda_1 S]} z_1, \dots, \frac{\lambda_1 S}{[\lambda_1 S]} z_1}_{[\lambda_1 S] \text{ times}}, \dots, \underbrace{\frac{\lambda_{m+1} S}{[\lambda_{m+1} S]} z_{m+1}, \dots, \frac{\lambda_{m+1} S}{[\lambda_{m+1} S]} z_{m+1}}_{[\lambda_{m+1} S] \text{ times}}, \underbrace{0, \dots, 0}_{S - \sum_{i=1}^{m+1} [\lambda_i S] \text{ times}} \right).$$

We have that

$$\frac{1}{S} \sum_{h=1}^S y_h = \frac{1}{S} \sum_{i=1}^{m+1} [\lambda_i S] \frac{\lambda_i S}{[\lambda_i S]} z_i = z$$

and

$$\frac{1}{S} \sum_{h=1}^S \psi(y_h) = \frac{1}{S} \sum_{i=1}^{m+1} [\lambda_i S] \psi\left(\frac{\lambda_i S}{[\lambda_i S]} z_i\right) + \left(1 - \frac{1}{S} \sum_{i=1}^{m+1} [\lambda_i S]\right) \psi(0);$$

therefore, passing to the limit as $S \rightarrow +\infty$, we obtain

$$\tilde{\psi}(z) \leq \sum_{i=1}^{m+1} \lambda_i \psi(z_i) = \psi^{**}(z),$$

which is the thesis. $\square$

Proposition 1.4.13. *Let g and ψ be as in Proposition 1.4.11. Then, ψ is convex.*

Proof. We prove that $\psi \leq \psi^{**}$, the other inequality being trivially satisfied. Let $z \in \mathbb{R}^m$. In light of Lemma 1.4.12 and the d -homogeneity of ψ , we have

$$\psi^{**}(z) = \lim_{S \rightarrow +\infty} \min \left\{ \frac{1}{S} \sum_{h=1}^S \psi(z_h) : \frac{1}{S} \sum_{h=1}^S z_h = z \right\} = \lim_{S \rightarrow +\infty} S^{d-1} \min \left\{ \sum_{h=1}^S \psi(z_h) : \sum_{h=1}^S z_h = z \right\}. \quad (1.4.20)$$

Let L, S be fixed positive integers. Consider $(z_1^*, \dots, z_S^*)$ such that $\sum_{h=1}^S z_h^* = z$ and

$$\sum_{h=1}^S \psi(z_h^*) = \min \left\{ \sum_{h=1}^S \psi(z_h) : \sum_{h=1}^S z_h = z \right\}, \quad (1.4.21)$$

and, for every $h \in \{1, \dots, S\}$, consider a function v_h such that $v_h - z_h^* \in W_0^{1,d}(Q(L); \mathbb{R}^m)$, $v_h = 0$ on $Q(1)$ and

$$\min \left\{ \int_{Q(L)} g(\nabla v) dx : v - z_h^* \in W_0^{1,d}(Q(L); \mathbb{R}^m) : v = 0 \text{ on } Q(1) \right\} = \int_{Q(L)} g(\nabla v_h) dx. \quad (1.4.22)$$

Now we define the function $w : Q(L^S) \rightarrow \mathbb{R}^m$ as

$$w(x) := \begin{cases} v_1(x) & \text{if } x \in Q(L), \\ v_h\left(\frac{x}{L^{h-1}}\right) + \sum_{l=1}^{h-1} z_l^* & \text{if } x \in Q(L^h) \setminus Q(L^{h-1}), h \in \{2, \dots, S\}, \end{cases}$$

and we note that $w - z \in W_0^{1,d}(Q(L^S); \mathbb{R}^m)$ and $w = 0$ on $Q(1)$. Using the d -homogeneity of g and (1.4.22), we have

$$\begin{aligned} \int_{Q(L^S)} g(\nabla w) dx &= \sum_{h=1}^S \int_{Q(L^h) \setminus Q(L^{h-1})} \frac{1}{(L^{h-1})^d} g\left(\nabla v_h\left(\frac{x}{L^{h-1}}\right)\right) dx \\ &= \sum_{h=1}^S \int_{Q(L)} g(\nabla v_h) dx \\ &= \sum_{h=1}^S \min \left\{ \int_{Q(L)} g(\nabla v) dx : v - z_h^* \in W_0^{1,d}(Q(L); \mathbb{R}^m) : v = 0 \text{ on } Q(1) \right\}. \end{aligned}$$

Therefore, multiplying by $(\log L^S)^{d-1}$ we obtain

$$\begin{aligned} & (\log L^S)^{d-1} \int_{Q(L^S)} g(\nabla w) dx \\ &= S^{d-1} \sum_{h=1}^S (\log L)^{d-1} \min \left\{ \int_{Q(L)} g(\nabla u) dx : u - z_h^* \in W_0^{1,d}(Q(L); \mathbb{R}^m) : u = 0 \text{ on } Q(1) \right\}, \end{aligned}$$

and passing to the limit as $L \rightarrow +\infty$, we get

$$\psi(z) \leq S^{d-1} \sum_{h=1}^S \psi(z_h^*) = S^{d-1} \min \left\{ \sum_{h=1}^S \psi(z_h) : \sum_{h=1}^S z_h = z \right\}$$

in virtue of (1.4.21). Recalling (1.4.20) and letting $S \rightarrow +\infty$ we have $\psi(z) \leq \psi^{**}(z)$, which concludes the proof. $\square$

1.4.3 Approximating capacity-type energy densities

In this subsection we introduce some auxiliary functions that, asymptotically, shall constitute our energy density of capacity-type.

We let T, ℓ, L, σ be positive parameters satisfying

$$T \in \mathbb{N}, T > r_0, \quad \ell \leq 1, \quad L > 4, \quad \sigma T < 1,$$

and, recalling the definitions of $\mathcal{F}_\sigma^T$, f_{hom}^T , and f_{hom} in (1.4.12), (1.4.15), and (1.4.5), respectively, for every $z \in \mathbb{R}^m$ we let

$$\varphi_{\ell,L,\sigma}^T(z) := \inf \{ \mathcal{F}_\sigma^T(v, Q(L) \setminus \overline{Q}(\ell)) : v \in L^d(Q(L); \mathbb{R}^m), \quad (1.4.23)$$

$$v(x) = 0 \text{ for a.e. } x \in \overline{Q}(\ell + \sigma T), v(x) = z \text{ for a.e. } x \in Q(L - \sigma T)^c \},$$

$$\varphi_L^T(z) := \min \left\{ \int_{Q(L)} f_{\text{hom}}^T(\nabla v) dx : v - z \in W_0^{1,d}(Q(L); \mathbb{R}^m), v = 0 \text{ on } Q(1) \right\},$$

$$\varphi^T(z) := \lim_{L \rightarrow +\infty} (\log L)^{d-1} \varphi_L^T(z), \quad (1.4.24)$$

$$\varphi(z) := \lim_{L \rightarrow +\infty} (\log L)^{d-1} \min \left\{ \int_{Q(L)} f_{\text{hom}}(\nabla v) dx : v - z \in W_0^{1,d}(Q(L); \mathbb{R}^m), v = 0 \text{ on } Q(1) \right\}. \quad (1.4.25)$$

Remark 1.4.14. The following properties hold true:

- (i) recalling that energies $\mathcal{F}_\sigma^T$ only account for pairwise interactions with range at most σT , for any function v admissible for $\varphi_{\ell,L,\sigma}^T(z)$ we have

$$\mathcal{F}_\sigma^T(v, Q(L) \setminus \overline{Q}(\ell)) = \mathcal{F}_\sigma^T(v, Q(L)) = \mathcal{F}_\sigma^T(v - z, \mathbb{R}^d),$$

upon setting $v = 0$ on $Q(\ell)$ and $v = z$ on $Q(L)^c$;

- (ii) Functions φ_L^T and φ are well-defined as minima since f_{hom}^T and f_{hom} are quasiconvex and coerciveness is ensured by the boundary conditions and an application of Poincaré inequality;
- (iii) taking into account (1.4.17) and the fact that f_{hom}^T and f_{hom} are d -homogeneous, we have that φ^T and φ are well defined in light of Proposition 1.4.11 and are convex by Proposition 1.4.13;
- (iv) by (H) and the d -homogeneity of f_{hom}^T and f_{hom} , we have that the functions $\varphi_{\ell,L,\sigma}^T, \varphi_L^T, \varphi^T$, and φ are d -homogeneous;
- (v) combining assumption (H) and a change of variables, it is easily seen that

$$\varphi_{\ell,L,\sigma}^T = \varphi_{c\ell,cL,c\sigma}^T$$

for every $c > 0$;

(vi) for $\ell \leq \ell'$ and $L \leq L'$ we have

$$\varphi_{\ell,L,\sigma}^T \leq \varphi_{\ell',L,\sigma}^T, \quad \varphi_{\ell,L',\sigma}^T \leq \varphi_{\ell,L,\sigma}^T, \quad \text{and} \quad \varphi_{L'}^T \leq \varphi_L^T.$$

We establish uniform bounds for functions in (1.4.23).

Lemma 1.4.15. *Let $\varphi_{\ell,L,\sigma}^T$ be as in (1.4.23), then there exists a positive constant C such that*

$$(\log L)^{d-1} \varphi_{\ell,L,\sigma}^T(z) \leq C|z|^d \quad (1.4.26)$$

and

$$|(\log L)^{d-1} \varphi_{\ell,L,\sigma}^T(z) - (\log L)^{d-1} \varphi_{\ell,L,\sigma}^T(z')| \leq C(|z|^{d-1} + |z'|^{d-1})|z - z'| \quad (1.4.27)$$

for every $T \in \mathbb{N}$, $T > r_0$, $\ell \leq 1$, $L > 4$ and $\sigma < 1/T$, and for every $z, z' \in \mathbb{R}^m$.

Proof. Let $z \in \mathbb{R}^m$ and $u \in C^1(\mathbb{R}^d; \mathbb{R}^m)$ such that $u - z \in C_c^1(Q(L - \sigma T); \mathbb{R}^m)$ and $u = 0$ in $\overline{Q}(\ell + \sigma T)$. We show that there exists a constant C independent of z, u , such that

$$G_\sigma(u - z, \mathbb{R}^d) \leq C \int_{Q(L - \sigma T)} |\nabla u|^d dx, \quad (1.4.28)$$

where $G_\sigma(u - z, \mathbb{R}^d)$ is defined as in (1.4.10) with $A = \mathbb{R}^d$.

Using that

$$\frac{u(x + \sigma\xi) - u(x)}{\sigma} = \int_0^1 \partial_\xi u(x + t\sigma\xi) dt,$$

by Jensen's inequality and Fubini's Theorem we obtain

$$\begin{aligned} G_\sigma(u - z, \mathbb{R}^d) &= \int_{B_{r_0}} \int_{\mathbb{R}^d} \left| \int_0^1 \partial_\xi u(x + t\sigma\xi) dt \right|^d dx d\xi \\ &\leq \int_{B_{r_0}} \int_{\mathbb{R}^d} \int_0^1 |\partial_\xi u(x + t\sigma\xi)|^d dt dx d\xi \\ &\leq \int_{B_{r_0}} |\xi|^d \int_0^1 \int_{\mathbb{R}^d} |\nabla u(x)|^d dx dt d\xi \leq C \int_{Q(L - \sigma T)} |\nabla u|^d dx, \end{aligned}$$

which proves (1.4.28).

Taking into account Remark 1.4.1(i) we apply (G1) and (1.4.11) to get

$$\begin{aligned} \mathcal{F}_\sigma^T(u, Q(L) \setminus \overline{Q}(\ell)) &= \mathcal{F}_\sigma^T(u - z, \mathbb{R}^d) \\ &\leq \int_{\mathbb{R}^d} M(\xi) \int_{\mathbb{R}^d} \left| \frac{u(x + \sigma\xi) - u(x)}{\sigma} \right|^d dx d\xi \\ &\leq C \left(\int_{\mathbb{R}^d} M(\xi)(1 + |\xi|^d) d\xi \right) G(u - z, \mathbb{R}^d) \leq C \int_{Q(L - \sigma T)} |\nabla u|^d dx, \end{aligned}$$

and then, by the arbitrariness of u , we conclude

$$\begin{aligned} &(\log L)^{d-1} \varphi_{\ell,L,\sigma}^T(z) \\ &\leq C(\log L)^{d-1} \inf \left\{ \int_{Q(L - \sigma T)} |\nabla v|^d dx : v - z \in C_c^1(Q(L - \sigma T); \mathbb{R}^m), v = 0 \text{ on } Q(\ell + \sigma T) \right\} \\ &= C(\log L)^{d-1} \text{Cap}_d(Q(\ell + \sigma T), Q(L - \sigma T)) |z|^d \\ &\leq C(\log L)^{d-1} \left(\log \left(\frac{L - \sigma T}{\ell + \sigma T} \right) \right)^{1-d} |z|^d \\ &\leq C(\log L)^{d-1} \left(\log \left(\frac{L - 1}{2} \right) \right)^{1-d} |z|^d \leq C|z|^d, \end{aligned}$$

which proves (1.4.26).

Now, we prove (1.4.27). We first observe that, if $z = 0$ or $z' = 0$, the inequality follows by (1.4.26). Then, we can suppose z and z' are both different from 0 and consider $\Theta : \mathbb{R}^m \rightarrow \mathbb{R}^m$ defined by

$$\Theta(\zeta) = \frac{|z'|}{|z|} \mathcal{R}_z^{z'}(\zeta), \quad \zeta \in \mathbb{R}^m,$$

where $\mathcal{R}_z^{z'}$ is a rotation that maps z into $\frac{|z|}{|z'|}z'$. Note that $\Theta(0) = 0$, $\Theta(z) = z'$ and

$$|\Theta(\zeta)| = \frac{|z'|}{|z|} |\zeta|, \quad |(\Theta - I)(\zeta)| \leq C \frac{|z - z'|}{|z|} |\zeta|, \quad (1.4.29)$$

for every $\zeta \in \mathbb{R}^m$, where we let I denote the identity matrix of $\mathbb{R}^{m \times m}$. Let v_z be (almost) optimal for (1.4.23); i.e., $\mathcal{F}_\sigma^T(v_z, Q(L) \setminus \overline{Q}(\ell)) \leq (1 + \eta) \varphi_{\ell, L, \sigma}^T(z)$. We set $v_{z'} := \Theta \circ v_z$ and we note that $v_{z'}$ is admissible for the minimum problem (1.4.23). Then,

$$\begin{aligned} \varphi_{\ell, L, \sigma}^T(z') &\leq \mathcal{F}_\sigma^T(v_{z'}, Q(L) \setminus \overline{Q}(\ell)) \\ &\leq (1 + \eta) \varphi_{\ell, L, \sigma}^T(z) + \mathcal{F}_\sigma^T(v_{z'}, Q(L) \setminus \overline{Q}(\ell)) - \mathcal{F}_\sigma^T(v_z, Q(L) \setminus \overline{Q}(\ell)). \end{aligned} \quad (1.4.30)$$

We set $\Delta_\sigma^\xi w(x) := \frac{w(x + \sigma \xi) - w(x)}{\sigma}$. By assumption (L) and (1.4.29) we get

$$\begin{aligned} &|f^T(\xi, \Delta_\sigma^\xi v_{z'}(x)) - f^T(\xi, \Delta_\sigma^\xi v_z(x))| \\ &\leq CM(\xi) (|\Delta_\sigma^\xi v_{z'}(x)|^{d-1} + |\Delta_\sigma^\xi v_z(x)|^{d-1}) |\Delta_\sigma^\xi v_{z'}(x) - \Delta_\sigma^\xi v_z(x)| \\ &= CM(\xi) (|\Theta(\Delta_\sigma^\xi v_z(x))|^{d-1} + |\Delta_\sigma^\xi v_z(x)|^{d-1}) |(\Theta - I)(\Delta_\sigma^\xi v_z(x))| \\ &\leq CM(\xi) \left[\left(\frac{|z'|}{|z|} \right)^{d-1} |\Delta_\sigma^\xi v_z(x)|^{d-1} + |\Delta_\sigma^\xi v_z(x)|^{d-1} \right] \frac{|z - z'|}{|z|} |\Delta_\sigma^\xi v_z(x)| \\ &\leq CM(\xi) \frac{|z|^{d-1} + |z'|^{d-1}}{|z|^d} |z - z'| |\Delta_\sigma^\xi v_z(x)|^d \end{aligned}$$

for every $x \in \mathbb{R}^d$ up to extending $v_z, v_{z'}$ on $\mathbb{R}^d$ according to their boundary conditions. Then, we apply (1.4.11) to get

$$\begin{aligned} &|\mathcal{F}_\sigma^T(v_{z'}, Q(L) \setminus \overline{Q}(\ell)) - \mathcal{F}_\sigma^T(v_z, Q(L) \setminus \overline{Q}(\ell))| \\ &= |\mathcal{F}_\sigma^T(v_{z'} - v_z, \mathbb{R}^d) - \mathcal{F}_\sigma^T(v_z - v_z, \mathbb{R}^d)| \\ &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f^T(\xi, \Delta_\sigma^\xi v_{z'}(x)) - f^T(\xi, \Delta_\sigma^\xi v_z(x))| dx d\xi \\ &\leq C \left(\int_{\mathbb{R}^d} M(\xi) \int_{\mathbb{R}^d} |\Delta_\sigma^\xi v_z(x)|^d dx d\xi \right) \frac{|z|^{d-1} + |z'|^{d-1}}{|z|^d} |z - z'| \\ &\leq C \left(\int_{\mathbb{R}^d} M(\xi) (1 + |\xi|^d) d\xi \right) G_\sigma(v_z - v_z, \mathbb{R}^d) \frac{|z|^{d-1} + |z'|^{d-1}}{|z|^d} |z - z'|. \end{aligned}$$

Since $T > r_0$, inequality (1.4.13) holds, and by (G1) we obtain

$$\begin{aligned} |\mathcal{F}_\sigma^T(v_{z'}, Q(L) \setminus \overline{Q}(\ell)) - \mathcal{F}_\sigma^T(v_z, Q(L) \setminus \overline{Q}(\ell))| &\leq C \mathcal{F}_\sigma^T(v_z, Q(L) \setminus \overline{Q}(\ell)) \frac{|z|^{d-1} + |z'|^{d-1}}{|z|^d} |z - z'| \\ &\leq C(1 + \eta) \varphi_{\ell, L, \sigma}^T(z) \frac{|z|^{d-1} + |z'|^{d-1}}{|z|^d} |z - z'|. \end{aligned}$$

Multiplying by $(\log L)^{d-1}$ on both sides of (1.4.30) and using (1.4.26) we infer

$$\begin{aligned} (\log L)^{d-1} \varphi_{\ell, L, \sigma}^T(z') &\leq (1 + \eta) (\log L)^{d-1} \varphi_{\ell, L, \sigma}^T(z) \\ &\quad + (\log L)^{d-1} |\mathcal{F}_\sigma^T(v_{z'}, Q(L) \setminus \overline{Q}(\ell)) - \mathcal{F}_\sigma^T(v_z, Q(L) \setminus \overline{Q}(\ell))| \\ &\leq (1 + \eta) \left[(\log L)^{d-1} \varphi_{\ell, L, \sigma}^T(z) + C (\log L)^{d-1} \varphi_{\ell, L, \sigma}^T(z) \frac{|z|^{d-1} + |z'|^{d-1}}{|z|^d} |z - z'| \right] \\ &\leq (1 + \eta) \left[(\log L)^{d-1} \varphi_{\ell, L, \sigma}^T(z) + C(|z|^{d-1} + |z'|^{d-1}) |z - z'| \right], \end{aligned}$$

and letting $\eta \rightarrow 0$ and reversing the roles of z and z' we obtain (1.4.27), which concludes the proof. $\square$

Proposition 1.4.16. *The following hold:*

(i) *For every $T \in \mathbb{N}, T > r_0, \ell \leq 1$, and $L > 4$ we have*

$$\lim_{\sigma \rightarrow 0} \frac{\varphi_{\ell, L, \sigma}^T}{\varphi_{L/\ell}^T} = 1$$

uniformly on $\mathbb{R}^m \setminus \{0\}$;

(ii) *for every $T \in \mathbb{N}, T > r_0$ we have*

$$\lim_{L \rightarrow +\infty} \frac{(\log L)^{d-1} \varphi_L^T}{\varphi^T} = 1$$

uniformly on $\mathbb{R}^m \setminus \{0\}$;

(iii) *we have*

$$\lim_{T \rightarrow +\infty} \varphi^T = \varphi$$

monotonically and uniformly on compact subsets of $\mathbb{R}^m$.

Proof. To prove (i), we first show that

$$\lim_{\sigma \rightarrow 0} \varphi_{\ell, L, \sigma}^T = \varphi_{L/\ell}^T \quad (1.4.31)$$

uniformly on compact subsets of $\mathbb{R}^m$. By (1.4.27) it is enough to prove the pointwise convergence. We note that

$$\{v \in L^d(Q(L); \mathbb{R}^m), v = 0 \text{ a.e. on } Q(\ell + \sigma T), v = z \text{ a.e. on } x \in Q(L - \sigma T)^c\} = \mathcal{D}_{\sigma T, g}^\infty(Q(L); \mathbb{R}^m),$$

where $g : \mathbb{R}^d \rightarrow \mathbb{R}^m$ is a Lipschitz function such that

$$g(x) = \begin{cases} 0 & \text{if } x \in Q(\ell + 1), \\ z & \text{if } x \in Q(L - 1)^c. \end{cases}$$

Then, we can rewrite $\varphi_{\ell, L, \sigma}^T(z)$ as

$$\inf\{\mathcal{F}_\sigma^T(v, Q(L) \setminus \overline{Q}(\ell)) : v \in \mathcal{D}_{\sigma T, g}^\infty(Q(L); \mathbb{R}^m)\}$$

and, applying the second part of Theorem 1.4.2, we obtain

$$\lim_{\sigma \rightarrow 0} \varphi_{\ell, L, \sigma}^T(z) = \min\left\{\int_{Q(L) \setminus Q(\ell)} f_{\text{hom}}^T(\nabla v) dx : v - g \in W_0^{1,d}(Q(L) \setminus Q(\ell); \mathbb{R}^m)\right\}.$$

Upon setting $v = 0$ on $Q(\ell)$ and recalling that $f_{\text{hom}}^T(0) = 0$, we get

$$\begin{aligned} \lim_{\sigma \rightarrow 0} \varphi_{\ell, L, \sigma}^T(z) &= \min\left\{\int_{Q(L)} f_{\text{hom}}^T(\nabla v) dx : v - z \in W_0^{1,d}(Q(L); \mathbb{R}^m), v = 0 \text{ on } Q(\ell)\right\} \\ &= \min\left\{\int_{Q(L/\ell)} f_{\text{hom}}^T(\nabla v) dx : v - z \in W_0^{1,d}(Q(L/\ell); \mathbb{R}^m), v = 0 \text{ on } Q(1)\right\} = \varphi_{L/\ell}^T(z), \end{aligned}$$

where the second equality follows by a change of variables and the d -homogeneity of f_{hom}^T , and this proves (1.4.31).

We observe that there exists a positive constant C such that

$$\frac{1}{C}|z|^d \leq (\log(L/\ell))^{d-1} \varphi_{L/\ell}^T(z) \leq C|z|^d \quad (1.4.32)$$

for every $z \in \mathbb{R}^m$. The upper bound readily follows passing to the limit in (1.4.26) by (1.4.31). For the lower bound, it suffices to note that $f_{\text{hom}}^T(z) \geq |z|^d/C$ by (1.4.17), so that the desired estimate follows by (1.4.18).

We conclude the proof of (i). By Remark 1.4.14(iv), the function $\varphi_{\ell,L,\sigma}^T/\varphi_{L/\ell}^T$ is 0-homogeneous for every $\sigma \in (0, 1/T)$. Letting then $\zeta := z/|z|$ for every $z \in \mathbb{R}^m \setminus \{0\}$ and using the lower bound in (1.4.32), we have

$$\left| \frac{\varphi_{\ell,L,\sigma}^T(z)}{\varphi_{L/\ell}^T(z)} - 1 \right| = \left| \frac{\varphi_{\ell,L,\sigma}^T(\zeta)}{\varphi_{L/\ell}^T(\zeta)} - 1 \right| = \left| \frac{\varphi_{\ell,L,\sigma}^T(\zeta) - \varphi_{L/\ell}^T(\zeta)}{\varphi_{L/\ell}^T(\zeta)} \right| \leq C(\log(L/\ell))^{d-1} |\varphi_{\ell,L,\sigma}^T(\zeta) - \varphi_{L/\ell}^T(\zeta)|,$$

and the conclusion follows since the convergence in (1.4.31) is uniform on $\mathbb{S}^{m-1}$.

The proof of (ii) follows the same line of (i) upon observing that for every T we have

$$\lim_{L \rightarrow +\infty} (\log L)^{d-1} \varphi_L^T = \varphi^T$$

uniformly on compact subsets of $\mathbb{R}^m$. To see this, we apply (1.4.31) with $\ell = 1$ in order to pass to the limit in (1.4.27) as $\sigma \rightarrow 0$ and get

$$|(\log L)^{d-1} \varphi_L^T(z) - (\log L)^{d-1} \varphi_L^T(z')| \leq C(|z|^{d-1} + |z'|^{d-1})|z - z'| \quad (1.4.33)$$

for every $T > r_0, L > 4$, and for every $z, z' \in \mathbb{R}^m$. As a consequence, $\{(\log L)^{d-1} \varphi_L^T\}_L$ is a family of locally equi-lipschitz continuous functions on $\mathbb{R}^m$, and since the pointwise convergence follows by Proposition 1.4.11, the uniform convergence on compact subsets of $\mathbb{R}^m$ is proved.

Finally, we prove (iii). Let $z \in \mathbb{R}^m$. Recalling the definitions of $\varphi^T(z)$ and $\varphi(z)$ in (1.4.24) and (1.4.25), respectively, we aim at proving that

$$\begin{aligned} & \lim_{T \rightarrow +\infty} \left[\lim_{L \rightarrow +\infty} (\log L)^{d-1} \min \left\{ \int_{Q(L)} f_{\text{hom}}^T(\nabla v) dx : v - z \in W_0^{1,d}(Q(L); \mathbb{R}^m), v = 0 \text{ on } Q(1) \right\} \right] \\ &= \lim_{L \rightarrow +\infty} (\log L)^{d-1} \min \left\{ \int_{Q(L)} f_{\text{hom}}(\nabla v) dx : v - z \in W_0^{1,d}(Q(L); \mathbb{R}^m), v = 0 \text{ on } Q(1) \right\} \end{aligned}$$

uniformly in $z \in \mathbb{R}^m$.

We claim that there exists a positive constant C with the property that for every $\eta > 0$ there exists $T_\eta > 0$ such that

$$\begin{aligned} & (\log L)^{d-1} \left| \min \left\{ \int_{Q(L)} f_{\text{hom}}(\nabla v) dx : v - z \in W_0^{1,d}(Q(L); \mathbb{R}^m), v = 0 \text{ on } Q(1) \right\} \right. \\ & \quad \left. - \left\{ \int_{Q(L)} f_{\text{hom}}^T(\nabla v) dx : v - z \in W_0^{1,d}(Q(L); \mathbb{R}^m), v = 0 \text{ on } Q(1) \right\} \right| \leq C|z|^d \eta \quad (1.4.34) \end{aligned}$$

for every $L > 4, T > T_\eta$, and $z \in \mathbb{R}^m$.

Let $\eta > 0$. By Proposition 1.4.8, there exists T_η such that

$$|f_{\text{hom}}(M) - f_{\text{hom}}^T(M)| \leq \eta \quad (1.4.35)$$

for every $M \in \mathbb{R}^{m \times d}$ with $|M| = 1$ and for every $T > T_\eta$. Let now $\bar{v}$ be a minimizer for $\varphi_L^T(z)$; i.e., for

$$\min \left\{ \int_{Q(L)} f_{\text{hom}}^T(\nabla v) dx : v - z \in W_0^{1,d}(Q(L); \mathbb{R}^m), v = 0 \text{ on } Q(1) \right\}.$$

Recalling that f_{hom}^T and f_{hom} are d -homogeneous and that $f_{\text{hom}}^T(0) = f_{\text{hom}}(0) = 0$, we use (1.4.17),

the upper bound in (1.4.32) with $\ell = 1$, and (1.4.35) to get

$$\begin{aligned}
& (\log L)^{d-1} \left[\min \left\{ \int_{Q(L)} f_{\text{hom}}(\nabla v) dx : v - z \in W_0^{1,d}(Q(L); \mathbb{R}^m), v = 0 \text{ on } Q(1) \right\} - \varphi_L^T(z) \right] \\
& \leq (\log L)^{d-1} \left[\int_{Q(L)} f_{\text{hom}}(\nabla \bar{v}) dx - \int_{Q(L)} f_{\text{hom}}^T(\nabla \bar{v}) dx \right] \\
& = (\log L)^{d-1} \int_{Q(L) \setminus \{\nabla \bar{v}=0\}} \left[f_{\text{hom}}\left(\frac{\nabla \bar{v}}{|\nabla \bar{v}|}\right) - f_{\text{hom}}^T\left(\frac{\nabla \bar{v}}{|\nabla \bar{v}|}\right) \right] |\nabla \bar{v}|^d dx \\
& \leq (\log L)^{d-1} \eta \int_{Q(L)} |\nabla \bar{v}|^d dx \\
& \leq C(\log L)^{d-1} \eta \int_{Q(L)} f_{\text{hom}}^T(\nabla \bar{v}) dx \\
& = C\eta(\log L)^{d-1} \varphi_L^T(z) \leq C\eta|z|^d.
\end{aligned}$$

At this point we can repeat a similar argument choosing $\bar{v}$ minimizer for

$$\min \left\{ \int_{Q(L)} f_{\text{hom}}(\nabla v) dx : v - z \in W_0^{1,d}(Q(L); \mathbb{R}^m), v = 0 \text{ on } Q(1) \right\},$$

which proves the other inequality in order to get our claim. Finally, passing to the limit as $L \rightarrow +\infty$ in (1.4.34) we get

$$|\varphi(z) - \varphi^T(z)| \leq C|z|^d \eta$$

for every $T > T_\eta$ and $z \in \mathbb{R}^m$; then the conclusion follows. Since, by definition, $\varphi^T \leq \varphi^{T+1}$, the convergence is monotone and the proof is concluded. $\square$

1.4.4 Asymptotic analysis

This subsection is devoted to the proof of Theorem 1.4.3. We fix $\{\varepsilon_j\}_j, \{\delta_j := \delta_{\varepsilon_j}\}_j, \{r_j := r_{\varepsilon_j}\}_j$ positive sequences tending to 0 as $j \rightarrow +\infty$ and such that

$$\lim_{j \rightarrow +\infty} \frac{\varepsilon_j}{r_j} = 0, \quad \lim_{j \rightarrow +\infty} \frac{\exp(-(\gamma \delta_j^d)^{\frac{1}{1-d}})}{r_j} = 1 \quad (1.4.36)$$

for some $\gamma > 0$. We immediately observe that (1.4.36) implies

$$\lim_{j \rightarrow +\infty} \frac{|\log r_j|^{1-d}}{\delta_j^d} = \gamma \quad \text{and} \quad \lim_{j \rightarrow +\infty} \frac{(\log(\frac{\delta_j}{2r_j}))}{|\log r_j|} = 1,$$

which, in turn, yields

$$\lim_{j \rightarrow +\infty} \frac{(\log(\frac{\delta_j}{2r_j}))^{1-d}}{\delta_j^d} = \gamma. \quad (1.4.37)$$

The main ingredient of our proof is the following ‘joining lemma on varying domains’, which is a variant of that originally devised in [15]. This is used to modify a sequence with equi-bounded energy so that Dirichlet boundary conditions are imposed on several concentric dyadic annuli surrounding the perforations.

Lemma 1.4.17. *Let T, k, N be fixed integers such that $T > r_0$, $2 \leq k \leq N - 3$, and let $\{S_j\}_j$ be a sequence of positive integers such that*

$$r_j 2^{NS_j} < \frac{\delta_j}{2} \quad (1.4.38)$$

for every $j \in \mathbb{N}$. Let $\{u_j\}_j \subset L^d(\Omega; \mathbb{R}^m)$ be a sequence such that $u_j \rightarrow u$ in $L^d(\Omega; \mathbb{R}^m)$ for some $u \in W^{1,d}(\Omega; \mathbb{R}^m)$ as $j \rightarrow +\infty$ and let

$$Z_j(\Omega) := \left\{ i \in \mathbb{Z}^d \cap \frac{1}{\delta_j} \Omega : \text{dist}_\infty(i\delta_j, \Omega^c) > \delta_j \right\} \quad (1.4.39)$$

for every $j \in \mathbb{N}$. Assume that

$$\lim_{j \rightarrow +\infty} \frac{\varepsilon_j}{r_j} = 0 \quad (1.4.40)$$

and

$$\sup_j \mathcal{F}_{\varepsilon_j}^T(u_j) < +\infty. \quad (1.4.41)$$

Then, there exists a sequence $\{w_j\}_j \subset L^d(\Omega; \mathbb{R}^m)$ converging to u in $L^d(\Omega; \mathbb{R}^m)$ as $j \rightarrow +\infty$ with the property that for every j large enough, for every $i \in Z_j(\Omega)$, and for every $h \in \{1, \dots, S_j\}$ there exists $k_{i,h} \in \{0, \dots, k-1\}$ such that, having set

$$C_j^{i,h} := Q(i\delta_j, r_j 2^{Nh-k_{i,h}}) \setminus \overline{Q}(i\delta_j, r_j 2^{Nh-k_{i,h}-1}), \quad (1.4.42)$$

$$\rho_j^{i,h} := \frac{3}{4} r_j 2^{Nh-k_{i,h}},$$

$$u_j^{i,h} := \frac{1}{|C_j^{i,h}|} \int_{C_j^{i,h}} u_j \, dx,$$

the following hold:

$$w_j = u_j \quad \text{a.e. on } \Omega \setminus \left[\bigcup_{\substack{i \in Z_j(\Omega) \\ h \in \{1, \dots, S_j\}}} C_j^{i,h} \right], \quad (1.4.43)$$

$$w_j = u_j^{i,h} \quad \text{a.e. on } Q(i\delta_j, \rho_j^{i,h} + \varepsilon_j T) \setminus Q(i\delta_j, \rho_j^{i,h} - \varepsilon_j T), \quad (1.4.44)$$

$$|\mathcal{F}_{\varepsilon_j}^T(u_j) - \mathcal{F}_{\varepsilon_j}^T(w_j)| \leq \frac{C}{k} \quad (1.4.45)$$

for some $C > 0$ independent of j .

Proof. For every $j \in \mathbb{N}$, $i \in Z_j(\Omega)$, $h \in \{1, \dots, S_j\}$, and $l \in \{0, \dots, k-1\}$, consider

$$C_{j,l}^{i,h} := Q(i\delta_j, r_j 2^{Nh-l}) \setminus \overline{Q}(i\delta_j, r_j 2^{Nh-l-1}),$$

$$\rho_{j,l}^h := \frac{3}{4} r_j 2^{Nh-l},$$

$$u_{j,l}^{i,h} := \frac{1}{|C_{j,l}^{i,h}|} \int_{C_{j,l}^{i,h}} u_j \, dx.$$

We define the open sets $C_{j,l,T}^{i,h}$ as

$$C_{j,l,T}^{i,h} := Q(i\delta_j, r_j 2^{Nh-l} - \varepsilon_j T) \setminus \overline{Q}(i\delta_j, r_j 2^{Nh-l-1} + \varepsilon_j T),$$

and cut-off functions $\chi_{j,l}^{i,h} \in C_c^\infty(C_{j,l,T}^{i,h})$ such that

$$\begin{cases} 0 \leq \chi_{j,l}^{i,h} \leq 1 & \text{on } \mathbb{R}^d, \\ \chi_{j,l}^{i,h} = 1 & \text{on } \overline{Q}(i\delta_j, \rho_{j,l}^h + \varepsilon_j T) \setminus Q(i\delta_j, \rho_{j,l}^h - \varepsilon_j T), \\ |\nabla \chi_{j,l}^{i,h}| \leq \frac{C}{\rho_{j,l}^h} & \text{on } \mathbb{R}^d. \end{cases}$$

We remark that, if we assume that j is large enough, then $C_{j,l,T}^{i,h}$ and $\chi_{j,l}^{i,h}$ are well defined for every $i \in Z_j(\Omega)$, $h \in \{1, \dots, S_j\}$, and $l \in \{0, \dots, k-1\}$. It suffices to note that, by a straightforward computation, the inequalities

$$r_j 2^{Nh-l} - \varepsilon_j T > \rho_{j,l}^h + \varepsilon_j T \quad \text{and} \quad \rho_{j,l}^h - \varepsilon_j T > r_j 2^{Nh-l-1} + \varepsilon_j T$$

are satisfied for every h, l provided that $r_j/\varepsilon_j > 4(2T+1)$, which holds for j sufficiently large in light of the assumption (1.4.40). Similarly, the estimate on $|\nabla \chi_{j,l}^{i,h}|$ follows by

$$|\nabla \chi_{j,l}^{i,h}| \leq \frac{C}{\left(\left[\frac{r_j}{\varepsilon_j} 2^{Nh-l}\right] \varepsilon_j - \varepsilon_j T\right) - (\rho_{j,l}^h + \varepsilon_j T)} \leq \frac{C}{\rho_{j,l}^h}.$$

Let then

$$w_{j,l}^{i,h}(x) := \chi_{j,l}^{i,h}(x) u_{j,l}^{i,h} + (1 - \chi_{j,l}^{i,h}(x)) u_j(x) \quad \text{for a.e. } x \in \Omega.$$

We aim at estimating $\mathcal{F}_{\varepsilon_j}^T(w_{j,l}^{i,h}, C_{j,l}^{i,h})$. Using Remark 1.4.1(i), we get

$$\begin{aligned} \mathcal{F}_{\varepsilon_j}^T(w_{j,l}^{i,h}, C_{j,l}^{i,h}) &= \int_{\mathbb{R}^d} \int_{(C_{j,l}^{i,h})_{\varepsilon_j}(\xi)} f^T\left(\xi, \frac{w_{j,l}^{i,h}(x + \varepsilon_j \xi) - w_{j,l}^{i,h}(x)}{\varepsilon_j}\right) dx d\xi \\ &\leq \int_{B_T} M(\xi) \int_{(C_{j,l}^{i,h})_{\varepsilon_j}(\xi)} \left| \frac{w_{j,l}^{i,h}(x + \varepsilon_j \xi) - w_{j,l}^{i,h}(x)}{\varepsilon_j} \right|^d dx d\xi. \end{aligned} \quad (1.4.46)$$

By definition, we have

$$\begin{aligned} w_{j,l}^{i,h}(x + \varepsilon_j \xi) - w_{j,l}^{i,h}(x) &= (u_{j,l}^{i,h} - u_j(x + \varepsilon_j \xi))(\chi_{j,l}^{i,h}(x + \varepsilon_j \xi) - \chi_{j,l}^{i,h}(x)) + (1 - \chi_{j,l}^{i,h}(x))(u_j(x + \varepsilon_j \xi) - u_j(x)), \end{aligned}$$

and then it holds that

$$\left| \frac{w_{j,l}^{i,h}(x + \varepsilon_j \xi) - w_{j,l}^{i,h}(x)}{\varepsilon_j} \right|^d \leq C |u_{j,l}^{i,h} - u_j(x + \varepsilon_j \xi)|^d \frac{|\xi|^d}{(\rho_{j,l}^h)^d} + C \left| \frac{u_j(x + \varepsilon_j \xi) - u_j(x)}{\varepsilon_j} \right|^d.$$

Substituting in (1.4.46) we obtain

$$\begin{aligned} \mathcal{F}_{\varepsilon_j}^T(w_{j,l}^{i,h}, C_{j,l}^{i,h}) &\leq C \frac{1}{(\rho_{j,l}^h)^d} \int_{B_T} M(\xi) |\xi|^d \int_{(C_{j,l}^{i,h})_{\varepsilon_j}(\xi)} |u_{j,l}^{i,h} - u_j(x + \varepsilon_j \xi)|^d dx d\xi \\ &\quad + C \int_{B_T} M(\xi) \int_{(C_{j,l}^{i,h})_{\varepsilon_j}(\xi)} \left| \frac{u_j(x + \varepsilon_j \xi) - u_j(x)}{\varepsilon_j} \right|^d dx d\xi \\ &\leq C \frac{1}{(\rho_{j,l}^h)^d} \int_{B_T} M(\xi) |\xi|^d \int_{C_{j,l}^{i,h}} |u_{j,l}^{i,h} - u_j(x)|^d dx d\xi \\ &\quad + C \int_{B_T} M(\xi) \int_{C_{j,l}^{i,h}} \left| \frac{u_j(x + \varepsilon_j \xi) - u_j(x)}{\varepsilon_j} \right|^d dx d\xi, \end{aligned} \quad (1.4.47)$$

where in the last inequality we used that, upon assuming j large enough,

$$\text{dist}_{\infty}(C_{j,l}^{i,h}, \Omega^c) > \frac{\delta_j}{2} > (T + r_0)\varepsilon_j \quad (1.4.48)$$

in light of (1.4.38), (1.4.39), and the fact that $\varepsilon_j/\delta_j \rightarrow 0$; and therefore $C_{j,l}^{i,h} + \varepsilon_j \xi \subset \Omega$ for every $\xi \in B_T, i \in Z_j(\Omega), h \in \{1, \dots, S_j\}$, and $l \in \{0, \dots, k-1\}$.

For the sake of convenience, we introduce the sets

$$E_{j,l}^{i,h} := Q(i\delta_j, 2r_j 2^{Nh-l}) \setminus \overline{Q}\left(i\delta_j, \frac{1}{2}r_j 2^{Nh-l-1}\right), \quad (1.4.49)$$

and we note that by (1.4.40), upon assuming that j is large enough, it holds that

$$C_{j,l}^{i,h} + Q((T + r_0)\varepsilon_j) \subset E_{j,l}^{i,h} \subset \Omega \quad (1.4.50)$$

for every i, h, l .

To estimate the first term in (1.4.47), we apply the rescaled version of the Poincaré-Wirtinger inequality in Proposition 1.4.6 with

$$\sigma = \varepsilon_j, \quad \tau = r_j 2^{Nh-l-1},$$

$$A = \frac{1}{r_j 2^{Nh-l-1}} (E_{j,l}^{i,h} - i\delta_j) = Q(4) \setminus \overline{Q}(1/2), \quad A' = \frac{1}{r_j 2^{Nh-l-1}} (C_{j,l}^{i,h} - i\delta_j).$$

Note indeed that, given σ_0 as in Proposition 1.4.6, for j large enough we have $\varepsilon_j < \sigma_0 r_j 2^{Nh-l-1}$ for every h, l since

$$\frac{\varepsilon_j}{r_j 2^{Nh-l-1}} \leq \frac{\varepsilon_j}{r_j},$$

which tends to 0 by (1.4.40). Upon choosing j large enough, it also holds that $r_j 2^{Nh-l-1} \leq 2\rho_{j,l}^h$ for every h, l ; then, combining this observation with (1.4.13) we infer that

$$\begin{aligned} \int_{C_{j,l}^{i,h}} |u_{j,l}^{i,h} - u_j(x)|^d dx &\leq \int_{E_{j,l}^{i,h}} |u_{j,l}^{i,h} - u_j(x)|^d dx \\ &\leq C \frac{|E_{j,l}^{i,h}|}{|C_{j,l}^{i,h}|} (r_j 2^{Nh-l-1})^d G_{\varepsilon_j}(u_j, E_{j,l}^{i,h}) \\ &\leq C (\rho_{j,l}^h)^d G_{\varepsilon_j}(u_j, E_{j,l}^{i,h}) \leq C (\rho_{j,l}^h)^d \mathcal{F}_{\varepsilon_j}^T(u_j, E_{j,l}^{i,h}), \end{aligned} \quad (1.4.51)$$

where C depends on the Poincaré constant of $Q(4) \setminus \overline{Q}(1/2)$ and is independent of all the involved indexes.

Now we estimate the second term in (1.4.47). By (1.4.48), we are in position to apply Lemma 1.4.4 with

$$\sigma = \varepsilon_j, \quad A = C_{j,l}^{i,h},$$

and this, combined with (1.4.50), (G1), and (1.4.13), yields

$$\begin{aligned} \int_{B_T} M(\xi) \int_{C_{j,l}^{i,h}} \left| \frac{u_j(x + \varepsilon_j \xi) - u_j(x)}{\varepsilon_j} \right|^d dx d\xi &\leq C \left(\int_{\mathbb{R}^d} M(\xi) (1 + |\xi|^d) d\xi \right) G_{\varepsilon_j}(u_j, E_{j,l}^{i,h}) \\ &\leq C \mathcal{F}_{\varepsilon_j}^T(u_j, E_{j,l}^{i,h}), \end{aligned} \quad (1.4.52)$$

where, also in this case, the constant C depends on $Q(4) \setminus \overline{Q}(1/2)$ but not on the involved indexes.

We substitute (1.4.51) and (1.4.52) in (1.4.47) to obtain

$$\mathcal{F}_{\varepsilon_j}^T(w_{j,l}^{i,h}, C_{j,l}^{i,h}) \leq C \mathcal{F}_{\varepsilon_j}^T(u_j, E_{j,l}^{i,h}),$$

and we deduce

$$|\mathcal{F}_{\varepsilon_j}^T(u_j, C_{j,l}^{i,h}) - \mathcal{F}_{\varepsilon_j}^T(w_{j,l}^{i,h}, C_{j,l}^{i,h})| \leq C \mathcal{F}_{\varepsilon_j}^T(u_j, E_{j,l}^{i,h}). \quad (1.4.53)$$

For every j large enough, it holds that

$$r_j 2^{Nh-k} \leq r_j 2^{Nh-l-1} \leq \rho_{j,l}^h \leq r_j 2^{Nh-l} \leq r_j 2^{Nh}$$

for every $h \in \{1, \dots, S_j\}$ and $l \in \{0, \dots, k-1\}$; therefore, recalling that $k \leq N-3$ by assumption, the sets $C_{j,l}^{i,h}$ are pairwise disjoint. By construction, it follows that the sets $E_{j,l}^{i,h}$ mutually intersect at most four at a time when i, h are fixed and l ranges in $\{0, \dots, k-1\}$, and

$$\bigcup_{l=0}^{k-1} E_{j,l}^{i,h} \subseteq Q(i\delta_j, r_j 2^{Nh+1}) \setminus \overline{Q}(i\delta_j, r_j 2^{Nh-k-1} - \varepsilon_j/2) =: E_j^{i,h}.$$

Hence, summing over $l \in \{0, \dots, k-1\}$ in (1.4.53), we obtain

$$\sum_{l=0}^{k-1} |\mathcal{F}_{\varepsilon_j}^T(u_j, C_{j,l}^{i,h}) - \mathcal{F}_{\varepsilon_j}^T(w_{j,l}^{i,h}, C_{j,l}^{i,h})| \leq C \mathcal{F}_{\varepsilon_j}^T(u_j, E_j^{i,h}),$$

and therefore, there exists $k_{i,h} \in \{0, \dots, k-1\}$ such that

$$|\mathcal{F}_{\varepsilon_j}^T(u_j, C_{j,k_{i,h}}^{i,h}) - \mathcal{F}_{\varepsilon_j}^T(w_j^{i,h}, C_{j,k_{i,h}}^{i,h})| \leq \frac{C}{k} \mathcal{F}_{\varepsilon_j}^T(u_j, E_j^{i,h}). \quad (1.4.54)$$

We obtain that conditions (1.4.43), (1.4.44), and (1.4.45) are satisfied choosing

$$C_j^{i,h} := C_{j,k_{i,h}}^{i,h}, \quad u_j^{i,h} := u_{j,k_{i,h}}^{i,h}, \quad \rho_j^{i,h} := \rho_{j,k_{i,h}}^h,$$

and

$$w_j(x) := \begin{cases} w_{j,k_{i,h}}^{i,h}(x) & \text{for a.e. } x \in C_j^{i,h}, i \in Z_j(\Omega), h \in \{1, \dots, S_j\}, \\ u_j(x) & \text{otherwise.} \end{cases}$$

Indeed, (1.4.43) is satisfied by definition and since $\chi_{j,k_{i,h}}^{i,h} = 1$ on $\overline{Q}(i\delta_j, \rho_j^{i,h} + \varepsilon_j T) \setminus Q(i\delta_j, \rho_j^{i,h} - \varepsilon_j T)$, (1.4.44) follows. Since $k \leq N-3$ by assumption, we have that

$$r_j 2^{Nh-k-1} - \varepsilon_j/2 > r_j 2^{N(h-1)+1}$$

for every h provided that $\varepsilon_j/r_j < 1/2$; hence, for j large and for fixed $i \in Z_j(\Omega)$, the sets $E_j^{i,h}$, $h \in \{1, \dots, S_j\}$, are pairwise disjoint. Moreover, by assumption (1.4.38), we have that $E_j^{i,h} \subseteq Q(i\delta_j, \delta_j)$ for every j, i, h , and therefore, the sets $E_j^{i,h}$ are pairwise disjoint also when i ranges in $Z_j(\Omega)$. Resorting to (1.4.43) and (1.4.54), we obtain

$$\begin{aligned} |\mathcal{F}_{\varepsilon_j}^T(u_j) - \mathcal{F}_{\varepsilon_j}^T(w_j)| &\leq \sum_{i \in Z_j(\Omega)} \sum_{h=1}^{S_j} |\mathcal{F}_{\varepsilon_j}^T(u_j, C_j^{i,h}) - \mathcal{F}_{\varepsilon_j}^T(w_j, C_j^{i,h})| \\ &\leq \frac{C}{k} \sum_{i \in Z_j(\Omega)} \sum_{h=1}^{S_j} \mathcal{F}_{\varepsilon_j}^T(u_j, E_j^{i,h}) \\ &\leq \frac{C}{k} \mathcal{F}_{\varepsilon_j}^T(u_j) \leq \frac{C}{k} \end{aligned}$$

in virtue of the assumption (1.4.41), which proves (1.4.45).

Finally, we prove that $w_j \rightarrow u$ in $L^d(\Omega; \mathbb{R}^m)$. Since $u_j \rightarrow u$, we prove that $u_j - w_j \rightarrow 0$. By (1.4.43) and the definition of w_j , we have

$$\lim_{j \rightarrow +\infty} \int_{\Omega} |u_j - w_j|^d dx \leq \lim_{j \rightarrow +\infty} \sum_{i \in Z_j(\Omega)} \sum_{h=1}^{S_j} \int_{C_j^{i,h}} |u_j - u_j^{i,h}|^d dx. \quad (1.4.55)$$

Recalling (1.4.49), we set $\tilde{E}_j^{i,h} := E_{j,k_{i,h}}^{i,h}$. Using (1.4.51) and (1.4.38), we obtain

$$\int_{C_j^{i,h}} |u_j - u_j^{i,h}|^d dx \leq C(\rho_{j,k_{i,h}}^h)^d \mathcal{F}_{\varepsilon_j}^T(u_j, \tilde{E}_j^{i,h}) \leq C\delta_j^d \mathcal{F}_{\varepsilon_j}^T(u_j, \tilde{E}_j^{i,h}),$$

and then, summing over $i \in Z_j(\Omega)$ and $h \in \{1, \dots, S_j\}$ and substituting in (1.4.55), we infer

$$\lim_{j \rightarrow +\infty} \int_{\Omega} |u_j - w_j|^d dx \leq C \lim_{j \rightarrow +\infty} \delta_j^d \mathcal{F}_{\varepsilon_j}^T(u_j),$$

which equals 0 by (1.4.41). This concludes the proof. $\square$

The following lemma allows us to obtain our *strange term* as the sum of the contributions to the total energy occurring near the perforations.

Lemma 1.4.18. *Let $\{u_j\}_j \subset L^d(\Omega; \mathbb{R}^m)$ be a sequence such that $u_j \rightarrow u$ in $L^d(\Omega; \mathbb{R}^m)$ as $j \rightarrow +\infty$ with $u \in W^{1,d}(\Omega; \mathbb{R}^m)$, and let $T, k, N, \{S_j\}_j$, and $\{u_j^{i,S_j}\}_j$ be as in Lemma 1.4.17 for every j large enough and for every $i \in Z_j(\Omega)$. Assume that*

$$r_j 2^{NS_j} < \frac{\delta_j}{2} \leq r_j 2^{N(S_j+1)} \quad (1.4.56)$$

for every $j \in \mathbb{N}$ and that

$$\sup_j (\mathcal{F}_{\varepsilon_j}^T(u_j) + \|u_j\|_{L^\infty(\Omega; \mathbb{R}^m)}) < +\infty. \quad (1.4.57)$$

Then, letting φ^T be as defined in (1.4.24), it holds that

$$\lim_{j \rightarrow +\infty} \sum_{i \in Z_j(\Omega)} \delta_j^d \varphi^T(u_j^{i,S_j}) = \int_{\Omega} \varphi^T(u) dx.$$

Proof. To simplify the notation we set $Q_j^i := Q(i\delta_j, \delta_j)$. The following estimate holds:

$$\begin{aligned} & \left| \sum_{i \in Z_j(\Omega)} \delta_j^d \varphi^T(u_j^{i,S_j}) - \int_{\Omega} \varphi^T(u) dx \right| \\ &= \left| \sum_{i \in Z_j(\Omega)} \int_{Q_j^i} (\varphi^T(u_j^{i,S_j}) - \varphi^T(u)) dx - \int_{\Omega \setminus \bigcup_{i \in Z_j(\Omega)} Q_j^i} \varphi^T(u) dx \right| \\ &\leq \sum_{i \in Z_j(\Omega)} \int_{Q_j^i} |\varphi^T(u_j^{i,S_j}) - \varphi^T(u_j)| dx + \sum_{i \in Z_j(\Omega)} \int_{Q_j^i} |\varphi^T(u_j) - \varphi^T(u)| dx \\ &\quad + \int_{\Omega \setminus \bigcup_{i \in Z_j(\Omega)} Q_j^i} \varphi^T(u) dx =: A_j^1 + A_j^2 + A_j^3. \end{aligned}$$

Letting $L \rightarrow +\infty$ in (1.4.33), we get

$$|\varphi^T(z) - \varphi^T(z')| \leq C(|z|^{d-1} + |z'|^{d-1})|z - z'|.$$

Hence, by (1.4.57) we obtain that

$$|\varphi^T(u_j) - \varphi^T(u)| \leq C|u_j - u|; \quad (1.4.58)$$

and we have

$$A_j^2 \leq C\|u_j - u\|_{L^1(\Omega; \mathbb{R}^m)},$$

which tends to 0 as $j \rightarrow +\infty$ by assumption. Moreover, using the fact that

$$\left| \Omega \setminus \bigcup_{i \in Z_j(\Omega)} Q_j^i \right| \rightarrow 0$$

as $j \rightarrow +\infty$, we infer that $A_j^3 \rightarrow 0$ as well.

To establish the claim, it remains to prove that $A_j^1 \rightarrow 0$ as $j \rightarrow +\infty$. Using again (1.4.58) and (1.4.57), it suffices to prove that

$$\lim_{j \rightarrow +\infty} \sum_{i \in Z_j(\Omega)} \int_{Q_j^i} |u_j^{i,S_j} - u_j| dx = 0. \quad (1.4.59)$$

By Hölder's inequality we get

$$\sum_{i \in Z_j(\Omega)} \int_{Q_j^i} |u_j^{i,S_j} - u_j| dx \leq \sum_{i \in Z_j(\Omega)} \delta_j^{d-1} \left(\int_{Q_j^i} |u_j^{i,S_j} - u_j|^d dx \right)^{\frac{1}{d}}. \quad (1.4.60)$$

By the first inequality in (1.4.56) we have $C_j^{i,S_j} \subset Q_j^i$, where C_j^{i,S_j} is as in (1.4.42). Recalling that

$$u_j^{i,S_j} = \frac{1}{|C_j^{i,S_j}|} \int_{C_j^{i,S_j}} u_j \, dx,$$

we apply the rescaled version of Poincaré-Wirtinger inequality in Proposition 1.4.6 with

$$\sigma = \varepsilon_j, \quad \tau = \delta_j, \quad A = \frac{1}{\delta_j}(Q_j^i - i\delta_j) = Q(1), \quad A' = \frac{1}{\delta_j}(C_j^{i,S_j} - i\delta_j).$$

Note indeed that, given σ_0 as in Proposition 1.4.6, we have $\varepsilon_j < \sigma_0 \delta_j$ for j large enough since $\varepsilon_j/\delta_j \rightarrow 0$ as $j \rightarrow +\infty$. We obtain that

$$\begin{aligned} \int_{Q_j^i} |u_j^{i,S_j} - u_j|^d \, dx &\leq C \delta_j^d \frac{|Q_j^i|}{|C_j^{i,S_j}|} G_{\varepsilon_j}(u_j, Q_j^i) \\ &= C \delta_j^d \frac{\delta_j^d}{(2^d - 1)(r_j 2^{NS_j - k_{i,S_j} - 1})^d} G_{\varepsilon_j}(u_j, Q_j^i) \\ &\leq C 2^{(2N-2)d} \delta_j^d G_{\varepsilon_j}(u_j, Q_j^i), \end{aligned}$$

where C depends on d and on the Poincaré constant of $Q(1)$ but is independent of j, i and we used that

$$r_j 2^{NS_j - k_{i,S_j} - 1} \geq \delta_j 2^{-2N+2}$$

in light of the second inequality in (1.4.56) and the fact that $0 \leq k_{i,S_j} \leq N - 4$. Hence, we obtain

$$\int_{Q_j^i} |u_j^{i,S_j} - u_j|^d \, dx \leq C \delta_j^d G_{\varepsilon_j}(u_j, Q_j^i),$$

with the constant C independent of j, i . Substituting this inequality in (1.4.60) and using the concavity of the function $x \mapsto x^{\frac{1}{d}}$, (1.4.13) and (1.4.57), we conclude that

$$\begin{aligned} \delta_j^{d-1} \sum_{i \in Z_j(\Omega)} \left(\int_{Q_j^i} |u_j^{i,S_j} - u_j|^d \, dx \right)^{\frac{1}{d}} &\leq C \delta_j^{d-1} \sum_{i \in Z_j(\Omega)} \delta_j (G_{\varepsilon_j}(u_j, Q_j^i))^{\frac{1}{d}} \\ &\leq C \delta_j^d (\#Z_j(\Omega))^{1-1/d} \left(\sum_{i \in Z_j(\Omega)} G_{\varepsilon_j}(u_j, Q_j^i) \right)^{\frac{1}{d}} \\ &\leq C \delta_j (G_{\varepsilon_j}(u_j, \Omega))^{\frac{1}{d}} \\ &\leq C \delta_j (\mathcal{F}_{\varepsilon_j}^T(u_j))^{\frac{1}{d}} \leq C \delta_j, \end{aligned}$$

and (1.4.59) follows as $\delta_j \rightarrow 0$. □

Finally, we recall a lemma that asserts that the asymptotic analysis can be reduced to sequences of functions that are uniformly bounded in L^∞ , see [10, Lemma 6.2].

Lemma 1.4.19. *Let $T > r_0$ and $\{u_j\}_j \subset L^d(\Omega; \mathbb{R}^m)$ be a sequence such that*

$$\sup_j (\mathcal{F}_\varepsilon^T(u_j) + \|u_j\|_{L^d(\Omega; \mathbb{R}^m)}) < +\infty.$$

Then, for every $\eta > 0$ and $L \in \mathbb{N}$, there exist a subsequence (not relabelled), a constant $R_L > L$ and a 1-Lipschitz function $t_L : \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that $t_L(z) = z$ if $|z| < R_L$, $t_L(z) = 0$ if $|z| > 2R_L$, and

$$\liminf_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}^T(t_L(u_j)) \leq \liminf_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}^T(u_j) + \eta.$$

Finally, we pass to the proof of our main result.

Proof of Theorem 1.4.3. For the sake of exposition, we divide the proof into three steps.

Step 1. Let $T > r_0$ be a fixed natural. Let $u \in W^{1,d}(\Omega; \mathbb{R}^m)$, $u_j \rightarrow u$ in $L^d(\Omega; \mathbb{R}^m)$ and, without loss of generality, suppose that $\sup_j F_{\varepsilon_j}^T(u_j) < +\infty$. We claim that

$$\liminf_{j \rightarrow +\infty} F_{\varepsilon_j}^T(u_j) \geq \int_{\Omega} f_{\text{hom}}^T(\nabla u) dx + \gamma \int_{\Omega} \varphi^T(u) dx, \quad (1.4.61)$$

where f_{hom}^T and φ^T are defined by (1.4.15) and (1.4.24), respectively.

In a first instance, we further assume that

$$\sup_j \|u_j\|_{L^\infty(\Omega; \mathbb{R}^m)} < +\infty. \quad (1.4.62)$$

Fix $k \geq 2$ natural and $\eta \in (0, 1)$. Applying Proposition 1.4.16(ii) with $L = 2^{N+k}$, we have that

$$\frac{(\log 2^{N+k})^{d-1} \varphi_{2^{N+k}}^T}{\varphi^T} \rightarrow 1 \quad \text{as } N \rightarrow +\infty$$

uniformly on $\mathbb{R}^m \setminus \{0\}$. Recalling that $\varphi_{2^{N+k}}^T(0) = \varphi^T(0) = 0$, we deduce that there exists N^* such that

$$(\log 2^N)^{d-1} \varphi_{2^{N+k}}^T(z) \geq (1 - \eta) \varphi^T(z) \quad \text{if } N \geq N^* \quad (1.4.63)$$

for every $z \in \mathbb{R}^m$. Then, we fix $N > \max\{N^*, k + 3\}$. Similarly, applying Proposition 1.4.16(i), with $\ell = 1$ and $L = 2^{N+k}$, we have

$$\frac{\varphi_{1, 2^{N+k}, \sigma}^T}{\varphi_{2^{N+k}}^T} \rightarrow 1 \quad \text{as } \sigma \rightarrow 0$$

uniformly on $\mathbb{R}^m \setminus \{0\}$, and therefore there exists $\sigma^* < 1/T$ such that

$$\varphi_{1, 2^{N+k}, \sigma}^T(z) \geq (1 - \eta) \varphi_{2^{N+k}}^T(z) \quad \text{if } \sigma \leq \sigma^* \quad (1.4.64)$$

for every $z \in \mathbb{R}^m$. Since k is fixed and $\varepsilon_j/r_j \rightarrow 0$, we assume that $2^k \varepsilon_j/r_j < \sigma^*$ for every $j \in \mathbb{N}$.

We apply Lemma 1.4.17 with T, k, N fixed as above and a sequence of naturals $\{S_j\}_j$ such that

$$r_j 2^{NS_j} < \frac{\delta_j}{2} \leq r_j 2^{N(S_j+1)} \quad (1.4.65)$$

for every $j \in \mathbb{N}$. Note that, upon considering j large enough, condition (1.4.65) is compatible with the choice of the parameter N since $\delta_j/r_j \rightarrow +\infty$ as $j \rightarrow +\infty$. We obtain a sequence $\{w_j\}_j \subset L^d(\Omega; \mathbb{R}^m)$ converging to u in $L^d(\Omega; \mathbb{R}^m)$ such that, for every j large enough, we have

$$w_j = u_j \quad \text{a.e. on } \Omega \setminus \left[\bigcup_{\substack{i \in Z_j(\Omega) \\ h \in \{1, \dots, S_j\}}} C_j^{i,h} \right], \quad (1.4.66)$$

$$w_j = u_j^{i,h} \quad \text{a.e. on } Q(i\delta_j, \rho_j^{i,h} + \varepsilon_j T) \setminus Q(i\delta_j, \rho_j^{i,h} - \varepsilon_j T), \quad (1.4.67)$$

$$F_{\varepsilon_j}^T(u_j) \geq \mathcal{F}_{\varepsilon_j}^T(w_j) - \frac{C}{k} \quad (1.4.68)$$

for every $i \in Z_j(\Omega)$, $h \in \{1, \dots, S_j\}$, and some $C > 0$ independent of j .

We set

$$E_j := \bigcup_{i \in Z_j(\Omega)} Q(i\delta_j, \rho_j^{i, S_j})$$

and we first estimate from below the contribution to the energy of the region $\Omega \setminus E_j$; that is, far from the perforations that are strictly contained in Ω .

Analysis far from the perforations. We define $v_j \in L^d(\Omega; \mathbb{R}^m)$, $j \in \mathbb{N}$, as

$$v_j(x) := \begin{cases} u_j^{i, S_j} & \text{for a.e. } x \in Q(i\delta_j, \rho_j^{i, S_j}), i \in Z_j(\Omega), \\ w_j(x) & \text{otherwise,} \end{cases}$$

and we claim that $v_j \rightarrow u$ in $L^d(\Omega; \mathbb{R}^m)$. To this end, we note that, by (1.4.67) and (1.4.68), it holds

$$\mathcal{F}_{\varepsilon_j}^T(v_j) = \mathcal{F}_{\varepsilon_j}^T(w_j, \Omega \setminus E_j) \leq F_{\varepsilon_j}^T(u_j) + C$$

so that $\sup_j \mathcal{F}_{\varepsilon_j}^T(v_j) < +\infty$. Moreover, by (1.4.62) and the construction of Lemma 1.4.17, we have

$$\sup_j \|v_j\|_{L^\infty(\Omega; \mathbb{R}^m)} < +\infty;$$

and then, by applying Proposition 1.4.10, we infer that a (not relabelled) subsequence of $\{v_j\}_j$ converges in $L^d(\Omega; \mathbb{R}^m)$ to a certain v . Let χ denote the periodic extension of the characteristic function of the cube $Q(2/3)$. In virtue of (1.4.65), we have

$$\bigcup_{h=1}^{S_j} C_j^{i,h} \subset Q\left(i\delta_j, \frac{\delta_j}{2}\right)$$

for every $i \in Z_j(\Omega)$; hence, recalling (1.4.66), we get

$$u_j(x) \left(1 - \chi\left(\frac{x}{\delta_j}\right)\right) = v_j(x) \left(1 - \chi\left(\frac{x}{\delta_j}\right)\right) \quad \text{for a.e. } x \in \Omega. \quad (1.4.69)$$

Since $\chi(\cdot/\delta_j)$ weakly*-converges to the constant $(2/3)^d$ in $L^\infty(\Omega; \mathbb{R}^m)$ and since $u_j \rightarrow u, v_j \rightarrow v$ in $L^d(\Omega; \mathbb{R}^m)$, we pass to the limit in (1.4.69) to infer that $u = v$ a.e. on Ω .

Using now that $v_j \rightarrow u$ and resorting to Theorem 1.4.2, we obtain

$$\liminf_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}^T(w_j, \Omega \setminus E_j) = \liminf_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}^T(v_j) \geq \int_{\Omega} f_{\text{hom}}^T(\nabla u) dx, \quad (1.4.70)$$

which is the desired lower bound.

Analysis near the perforations. Now we perform the asymptotic analysis on E_j .

Fix $i \in Z_j(\Omega)$ and let

$$w_j^{i,h}(x) := \begin{cases} 0 & \text{if } x \in Q(\rho_j^{i,h} + \varepsilon_j T), \\ w_j(x + i\delta_j) - u_j^{i,h} & \text{if } x \in Q(\rho_j^{i,h+1} - \varepsilon_j T) \setminus Q(\rho_j^{i,h} + \varepsilon_j T), \\ u_j^{i,h+1} - u_j^{i,h} & \text{if } x \in Q(\rho_j^{i,h+1} - \varepsilon_j T)^c, \end{cases}$$

for every $h \in \{1, \dots, S_j - 1\}$ and for a.e. $x \in \mathbb{R}^d$. To deal with the Dirichlet boundary condition imposed on the perforation, we let

$$w_j^{i,0}(x) := \begin{cases} 0 & \text{if } x \in Q(r_j), \\ w_j(x + i\delta_j) & \text{if } x \in Q(\rho_j^{i,1} - \varepsilon_j T) \setminus Q(r_j), \\ u_j^{i,1} & \text{if } x \in Q(\rho_j^{i,1} - \varepsilon_j T)^c, \end{cases}$$

for a.e. $x \in \mathbb{R}^d$. We observe that $r_j 2^{-k} + \varepsilon_j T \leq r_j$ for j large enough; hence, it also holds

$$w_j^{i,0} = \begin{cases} 0 & \text{for a.e. } x \in Q(r_j 2^{-k} + \varepsilon_j T), \\ u_j^{i,1} & \text{for a.e. } x \in Q(\rho_j^{i,1} - \varepsilon_j T)^c. \end{cases}$$

Then, we have

$$\begin{aligned} \mathcal{F}_{\varepsilon_j}^T(w_j, Q(i\delta_j, \rho_j^{i,S_j})) &\geq \mathcal{F}_{\varepsilon_j}^T(w_j, Q(i\delta_j, \rho_j^{i,1}) \setminus \overline{Q}(i\delta_j, r_j)) + \sum_{h=1}^{S_j-1} \mathcal{F}_{\varepsilon_j}^T(w_j, Q(i\delta_j, \rho_j^{i,h+1}) \setminus Q(i\delta_j, \rho_j^{i,h})) \\ &= \mathcal{F}_{\varepsilon_j}^T(w_j^{i,0}, Q(\rho_j^{i,1}) \setminus Q(r_j 2^{-k})) + \sum_{h=1}^{S_j-1} \mathcal{F}_{\varepsilon_j}^T(w_j^{i,h}, Q(\rho_j^{i,h+1}) \setminus Q(\rho_j^{i,h})) \\ &\geq \varphi_{r_j 2^{-k}, \rho_j^{i,1}, \varepsilon_j}^T(u_j^{i,1}) + \sum_{h=1}^{S_j-1} \varphi_{\rho_j^{i,h}, \rho_j^{i,h+1}, \varepsilon_j}^T(u_j^{i,h+1} - u_j^{i,h}). \end{aligned}$$

For every j large enough, it holds

$$r_j 2^{Nh-k} \leq \rho_j^{i,h} \leq r_j 2^{Nh}$$

for every $i \in Z_j(\Omega)$ and $h \in \{1, \dots, S_j\}$; hence, using Remark 1.4.14(vi) we infer

$$\mathcal{F}_{\varepsilon_j}^T(w_j, Q(i\delta_j, \rho_j^{i,S_j})) \geq \sum_{h=0}^{S_j-1} \varphi_{r_j 2^{Nh-k}, r_j 2^{N(h+1)}, \varepsilon_j}^T(u_j^{i,h+1} - u_j^{i,h}),$$

where we let $u_j^{i,0} := 0$. Upon setting

$$\sigma_j^h := \frac{\varepsilon_j}{r_j} 2^{-Nh+k}, \quad j \in \mathbb{N}, \quad h \in \{0, \dots, S_j-1\},$$

we apply Remark 1.4.14(v) to get

$$\varphi_{r_j 2^{Nh-k}, r_j 2^{N(h+1)}, \varepsilon_j}^T(u_j^{i,h+1} - u_j^{i,h}) = \varphi_{1, 2^{N+k}, \sigma_j^h}^T(u_j^{i,h+1} - u_j^{i,h})$$

for every $h \in \{0, \dots, S_j-1\}$. As a consequence, we obtain

$$\mathcal{F}_{\varepsilon_j}^T(w_j, Q(i\delta_j, \rho_j^{i,S_j})) \geq \sum_{h=0}^{S_j-1} \varphi_{1, 2^{N+k}, \sigma_j^h}^T(u_j^{i,h+1} - u_j^{i,h}).$$

We note that $\sigma_j^h \leq 2^k \varepsilon_j / r_j < \sigma^*$ for every $j \in \mathbb{N}$ and $h \in \{0, \dots, S_j-1\}$. Then, using (1.4.63) and (1.4.64), we get

$$\begin{aligned} \mathcal{F}_{\varepsilon_j}^T(w_j, Q(i\delta_j, \rho_j^{i,S_j})) &\geq (1-\eta) \sum_{h=0}^{S_j-1} \varphi_{2^{N+k}}^T(u_j^{i,h+1} - u_j^{i,h}) \\ &\geq (1-\eta)^2 (\log 2^N)^{1-d} \sum_{h=0}^{S_j-1} \varphi^T(u_j^{i,h+1} - u_j^{i,h}). \end{aligned} \quad (1.4.71)$$

Now we observe that

$$\sum_{h=0}^{S_j-1} (u_j^{i,h+1} - u_j^{i,h}) = u_j^{i,S_j} - u_j^{i,0} = u_j^{i,S_j},$$

and therefore, using that the function φ^T is convex and d -homogeneous (see Remark 1.4.14(iii),(iv)), we have

$$\sum_{h=0}^{S_j-1} \varphi^T(u_j^{i,h+1} - u_j^{i,h}) \geq S_j \varphi^T\left(\frac{u_j^{i,S_j}}{S_j}\right) = S_j^{1-d} \varphi^T(u_j^{i,S_j}). \quad (1.4.72)$$

We substitute (1.4.72) in (1.4.71) and recall that $S_j \log 2^N \leq \log(\delta_j/(2r_j))$ by the first inequality in (1.4.65). We obtain

$$\mathcal{F}_{\varepsilon_j}^T(w_j, Q(i\delta_j, \rho_j^{i,S_j})) \geq (1-\eta)^2 (S_j \log 2^N)^{1-d} \varphi^T(u_j^{i,S_j}) \geq (1-\eta)^2 \left(\log\left(\frac{\delta_j}{2r_j}\right)\right)^{1-d} \varphi^T(u_j^{i,S_j}).$$

Summing over $i \in Z_j(\Omega)$ we get

$$\begin{aligned} \liminf_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}^T(w_j, E_j) &\geq \liminf_{j \rightarrow +\infty} \sum_{i \in Z_j(\Omega)} \mathcal{F}_{\varepsilon_j}^T(w_j, Q(i\delta_j, \rho_j^{i,S_j})) \\ &\geq (1-\eta)^2 \liminf_{j \rightarrow +\infty} \sum_{i \in Z_j(\Omega)} \left(\log\left(\frac{\delta_j}{2r_j}\right)\right)^{1-d} \varphi^T(u_j^{i,S_j}) \\ &= (1-\eta)^2 \liminf_{j \rightarrow +\infty} \sum_{i \in Z_j(\Omega)} \gamma \delta_j^d \varphi^T(u_j^{i,S_j}), \end{aligned}$$

where the last equality follows by (1.4.37). Since $\{u_j\}_j$ is bounded in $L^\infty(\Omega; \mathbb{R}^m)$ by (1.4.62), we apply Lemma 1.4.18 and we infer

$$\liminf_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}^T(w_j, E_j) \geq (1 - \eta)^2 \gamma \int_{\Omega} \varphi^T(u) dx. \quad (1.4.73)$$

Substituting (1.4.70) and (1.4.73) in (1.4.68), we obtain

$$\liminf_{j \rightarrow +\infty} F_{\varepsilon_j}^T(u_j) \geq \int_{\Omega} f_{\text{hom}}^T(\nabla u) dx + (1 - \eta)^2 \gamma \int_{\Omega} \varphi^T(u) dx - \frac{C}{k}$$

and we infer (1.4.61) by letting first $\eta \rightarrow 0$ and then $k \rightarrow +\infty$.

To conclude, we prove the claim removing the assumption (1.4.62). Let $u \in W^{1,d}(\Omega; \mathbb{R}^m)$, $u_j \rightarrow u$ in $L^d(\Omega; \mathbb{R}^m)$ and suppose that $\sup_j \mathcal{F}_{\varepsilon_j}^T(u_j) < +\infty$. By Lemma 1.4.19 and the lower bound we just proved for a sequence bounded in $L^\infty(\Omega; \mathbb{R}^m)$, up to extracting a not relabelled subsequence we have

$$\begin{aligned} \liminf_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}^T(u_j) &\geq \liminf_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}^T(t_L(u_j)) - \eta \\ &\geq \int_{\Omega} f_{\text{hom}}^T(t_L(u)) dx + \gamma \int_{\Omega} \varphi^T(t_L(u)) dx - \eta, \end{aligned}$$

and then, letting $L \rightarrow +\infty$, we get

$$\liminf_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}^T(u_j) \geq \int_{\Omega} f_{\text{hom}}^T(u) dx + \gamma \int_{\Omega} \varphi^T(u) dx - \eta.$$

The conclusion follows by the arbitrariness of η .

Step 2. Let $T > r_0$ be a fixed natural and let $u \in W^{1,d}(\Omega; \mathbb{R}^m)$. We prove that there exists a sequence $u_j \rightarrow u$ in $L^d(\Omega; \mathbb{R}^m)$ such that

$$\limsup_{j \rightarrow +\infty} F_{\varepsilon_j}^T(u_j) \leq \int_{\Omega} f_{\text{hom}}^T(\nabla u) dx + \gamma \int_{\Omega} \varphi^T(u) dx. \quad (1.4.74)$$

Fix $\tilde{\Omega}$ a bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary such that $\bar{\Omega} \subset \tilde{\Omega}$ and extend u to a function in $W^{1,d}(\tilde{\Omega}; \mathbb{R}^m)$. By Theorem 1.4.2, consider a sequence $\{\tilde{u}_j\}_j \subset L^d(\tilde{\Omega}; \mathbb{R}^m)$ converging to u in $L^d(\tilde{\Omega}; \mathbb{R}^m)$ such that

$$\lim_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}^T(\tilde{u}_j, \tilde{\Omega}) = \int_{\tilde{\Omega}} f_{\text{hom}}^T(\nabla u) dx. \quad (1.4.75)$$

It is not restrictive to assume that u is bounded and then, resorting to Lemma 1.4.19, we may also suppose that

$$\sup_j \|\tilde{u}_j\|_{L^\infty(\tilde{\Omega}; \mathbb{R}^m)} < +\infty. \quad (1.4.76)$$

As in the first step, we fix $k \geq 2$ natural and $\eta \in (0, 1)$, and we use Proposition 1.4.16(ii) with $L = 2^{N-k}$ to obtain that there exists N^* such that

$$(\log 2^N)^{d-1} \varphi_{2^{N-k}}^T(z) \leq (1 + \eta) \varphi^T(z) \quad \text{if } N \geq N^* \quad (1.4.77)$$

for every $z \in \mathbb{R}^m$. Then, we fix $N > \max\{N^*, k + 3\}$. We apply Proposition 1.4.16(i) with $\ell = 1$ and $L = 2^{N-k}$ to obtain that there exists $\sigma^* < 1/T$ such that

$$\varphi_{1, 2^{N-k}, \sigma}^T(z) \leq (1 + \eta) \varphi_{2^{N-k}}^T(z) \quad \text{if } \sigma \leq \sigma^* \quad (1.4.78)$$

for every $z \in \mathbb{R}^m$, and we assume that $\varepsilon_j/r_j < \sigma^*$ for every $j \in \mathbb{N}$.

We apply Lemma 1.4.17 to the sequence $\{\tilde{u}_j\}_j$ on $\tilde{\Omega}$, with T, k, N fixed as above and $\{S_j\}_j$ a sequence of naturals such that

$$r_j 2^{NS_j} \leq \frac{\delta_j}{2} < r_j 2^{N(S_j+1)} \quad (1.4.79)$$

for every $j \in \mathbb{N}$. We obtain a sequence $w_j \subset L^d(\tilde{\Omega}; \mathbb{R}^m)$, converging to u in $L^d(\tilde{\Omega}; \mathbb{R}^m)$ such that, for every j large enough, the following hold:

$$w_j = \tilde{u}_j \quad \text{a.e. on } \tilde{\Omega} \setminus \left[\bigcup_{\substack{i \in Z_j(\tilde{\Omega}) \\ h \in \{1, \dots, S_j\}}} C_j^{i,h} \right],$$

$$w_j = \tilde{u}_j^{i, S_j} \quad \text{a.e. on } Q(i\delta_j, \rho_j^{i, S_j} + \varepsilon_j T) \setminus Q(i\delta_j, \rho_j^{i, S_j} - \varepsilon_j T), \quad (1.4.80)$$

$$\mathcal{F}_{\varepsilon_j}^T(w_j, \tilde{\Omega}) \leq \mathcal{F}_{\varepsilon_j}^T(\tilde{u}_j, \tilde{\Omega}) + \frac{C}{k} \quad (1.4.81)$$

for every $i \in Z_j(\tilde{\Omega})$, $h \in \{1, \dots, S_j\}$, and some $C > 0$ independent of j .

In analogy with the first step, we let

$$E_j := \bigcup_{i \in Z_j(\Omega)} Q(i\delta_j, \rho_j^{i, S_j})$$

and

$$\tilde{E}_j := \bigcup_{i \in Z_j(\tilde{\Omega})} Q(i\delta_j, \rho_j^{i, S_j})$$

and we start by defining our recovery sequence on $\Omega \setminus \tilde{E}_j$.

Analysis far from the perforations. We set

$$u_j(x) := w_j(x) \quad \text{for a.e. } x \in \Omega \setminus \tilde{E}_j,$$

and observe that by (1.4.81) we have

$$\mathcal{F}_{\varepsilon_j}^T(u_j, \Omega \setminus \tilde{E}_j) \leq \mathcal{F}_{\varepsilon_j}^T(w_j, \tilde{\Omega}) \leq \mathcal{F}_{\varepsilon_j}^T(\tilde{u}_j, \tilde{\Omega}) + \frac{C}{k}.$$

Hence, by (1.4.75) we obtain

$$\limsup_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}^T(u_j, \Omega \setminus \tilde{E}_j) \leq \int_{\tilde{\Omega}} f_{\text{hom}}^T(\nabla u) dx + \frac{C}{k}, \quad (1.4.82)$$

which is the desired upper bound.

Analysis near the perforations. Now we further modify the functions $\{w_j\}_j$ near the perforations that are far from $\partial\tilde{\Omega}$. Recalling the definition of $Z_j(\tilde{\Omega})$ in (1.4.39), upon assuming that j is large enough we have that

$$\mathbb{Z}^d \cap \frac{1}{\delta_j} \Omega \subset Z_j(\tilde{\Omega}),$$

hence, this amounts to modifying the functions $\{w_j\}_j$ on all the perforations of Ω , including those that intersect $\partial\Omega$.

For every $j \in \mathbb{N}$, $i \in Z_j(\tilde{\Omega})$, and $h \in \{0, \dots, S_j - 2\}$, consider $\bar{v}_j^{i,h}$ an admissible function for the minimum problem $\varphi_{r_j 2^{N(h)}, r_j 2^{N(h+1)}, \varepsilon_j}^T(\tilde{u}_j^{i, S_j} / S_j)$ such that

$$\mathcal{F}_{\varepsilon_j}^T(\bar{v}_j^{i,h}, Q(r_j 2^{N(h+1)}) \setminus Q(r_j 2^{N(h)})) \leq (1 + \eta) \varphi_{r_j 2^{N(h)}, r_j 2^{N(h+1)}, \varepsilon_j}^T(\tilde{u}_j^{i, S_j} / S_j). \quad (1.4.83)$$

For $h = S_j - 1$, consider $\bar{v}_j^{i, S_j-1}$ an admissible function for the minimum problem

$$\varphi_{r_j 2^{N(S_j-1)}, \rho_j^{i, S_j}, \varepsilon_j}^T(\tilde{u}_j^{i, S_j} / S_j)$$

such that

$$\mathcal{F}_{\varepsilon_j}^T(\bar{v}_j^{i, S_j-1}, Q(\rho_j^{i, S_j}) \setminus Q(r_j 2^{N(S_j-1)})) \leq (1 + \eta) \varphi_{r_j 2^{N(S_j-1)}, \rho_j^{i, S_j}, \varepsilon_j}^T(\tilde{u}_j^{i, S_j} / S_j). \quad (1.4.84)$$

For every $h \in \{0, \dots, S_j - 1\}$ we let

$$v_j^{i,h}(x) := \bar{v}_j^{i,h}(x - i\delta_j) + \frac{h}{S_j} \tilde{u}_j^{i,S_j} \quad \text{for a.e. } x \in Q(i\delta_j, r_j 2^{N(h+1)}),$$

and we note that, by definition,

$$v_j^{i,h} = \begin{cases} \frac{h}{S_j} \tilde{u}_j^{i,S_j} & \text{a.e. on } Q(i\delta_j, r_j 2^{N(h+1)} + \varepsilon_j T), \\ \frac{h+1}{S_j} \tilde{u}_j^{i,S_j} & \text{a.e. on } Q(i\delta_j, r_j 2^{N(h+1)} - \varepsilon_j T)^c, \end{cases}$$

for every $h \in \{0, \dots, S_j - 2\}$ and

$$v_j^{i,S_j-1} = \begin{cases} \frac{S_j-1}{S_j} \tilde{u}_j^{i,S_j} & \text{a.e. on } Q(i\delta_j, r_j 2^{N(S_j-1)} + \varepsilon_j T), \\ \tilde{u}_j^{i,S_j} & \text{a.e. on } Q(i\delta_j, r_j 2^{N(S_j-1)} - \varepsilon_j T)^c. \end{cases}$$

For every $i \in Z_j(\tilde{\Omega})$, we set

$$u_j(x) := \begin{cases} v_j^{i,0}(x) & \text{if } x \in Q(i\delta_j, r_j 2^N), \\ v_j^{i,h}(x) & \text{if } x \in (Q(i\delta_j, r_j 2^{N(h+1)}) \setminus Q(i\delta_j, r_j 2^{N(h)})), h \in \{1, \dots, S_j - 2\}, \\ v_j^{i,S_j-1}(x) & \text{if } x \in (Q(i\delta_j, \rho_j^{i,S_j}) \setminus Q(i\delta_j, r_j 2^{N(S_j-1)})), \end{cases}$$

where the equality is valid for a.e. $x \in Q(i\delta_j, \rho_j^{i,S_j})$, and we observe that

$$u_j(x) = v_j^{i,0}(x) = 0, \quad \text{for a.e. } x \in Q(i\delta_j, r_j). \quad (1.4.85)$$

We now estimate the energy near the perforations that are far from $\partial\tilde{\Omega}$.

Recalling (1.4.83) and (1.4.84), we get

$$\begin{aligned} \mathcal{F}_{\varepsilon_j}^T(u_j, Q(i\delta_j, \rho_j^{i,S_j})) &= \sum_{h=0}^{S_j-2} \mathcal{F}_{\varepsilon_j}^T(v_j^{i,h}, Q(i\delta_j, r_j 2^{N(h+1)}) \setminus Q(i\delta_j, r_j 2^{N(h)})) \\ &\quad + \mathcal{F}_{\varepsilon_j}^T(v_j^{i,S_j-1}, Q(i\delta_j, \rho_j^{i,S_j}) \setminus Q(i\delta_j, r_j 2^{N(S_j-1)})) \\ &= \sum_{h=0}^{S_j-2} \mathcal{F}_{\varepsilon_j}^T(\bar{v}_j^{i,h}, Q(r_j 2^{N(h+1)}) \setminus Q(r_j 2^{N(h)})) \\ &\quad + \mathcal{F}_{\varepsilon_j}^T(\bar{v}_j^{i,S_j-1}, Q(\rho_j^{i,S_j}) \setminus Q(r_j 2^{N(S_j-1)})) \\ &\leq (1+\eta) \sum_{h=0}^{S_j-2} \varphi_{r_j 2^{N(h)}, r_j 2^{N(h+1)}, \varepsilon_j}^T(\tilde{u}_j^{i,S_j}/S_j) \\ &\quad + (1+\eta) \varphi_{r_j 2^{N(S_j-1)}, \rho_j^{i,S_j}, \varepsilon_j}^T(\tilde{u}_j^{i,S_j}/S_j) \\ &\leq (1+\eta) \sum_{h=0}^{S_j-1} \varphi_{r_j 2^{N(h)}, r_j 2^{N(h+1)-k}, \varepsilon_j}^T(\tilde{u}_j^{i,S_j}/S_j), \end{aligned} \quad (1.4.86)$$

where the last inequality follows by Remark 1.4.14(vi) and by the fact that, for j large enough,

$$\rho_j^{i,S_j} = \frac{3}{4} r_j 2^{NS_j - k_{i,S_j}} \geq r_j 2^{NS_j - k}$$

for every i since $k_{i,S_j} \leq k - 1$. Upon setting

$$\sigma_j^h := \frac{\varepsilon_j}{r_j} 2^{-Nh}, \quad j \in \mathbb{N}, h \in \{0, \dots, S_j - 1\},$$

Remark 1.4.14(v) yields

$$\varphi_{r_j 2^{N h}, r_j 2^{N(h+1)-k}, \varepsilon_j}^T(\tilde{u}_j^{i, S_j} / S_j) = \varphi_{1, 2^{N-k}, \sigma_j^h}^T(\tilde{u}_j^{i, S_j} / S_j),$$

and then (1.4.86) can be rewritten as

$$\mathcal{F}_{\varepsilon_j}^T(u_j, Q(i\delta_j, \rho_j^{i, S_j})) \leq (1 + \eta) \sum_{h=0}^{S_j-1} \varphi_{1, 2^{N-k}, \sigma_j^h}^T(\tilde{u}_j^{i, S_j} / S_j).$$

As $\sigma_j^h \leq \varepsilon_j / r_j < \sigma^*$ for every $j \in \mathbb{N}$ and $h \in \{0, \dots, S_j - 1\}$, we use (1.4.77) and (1.4.78) and we employ the d -homogeneity of φ^T to get

$$\begin{aligned} \mathcal{F}_{\varepsilon_j}^T(u_j, Q(i\delta_j, \rho_j^{i, S_j})) &\leq (1 + \eta)^2 S_j \varphi_{2^{N-k}}^T(\tilde{u}_j^{i, S_j} / S_j) \\ &\leq (1 + \eta)^3 S_j (\log 2^N)^{1-d} \varphi^T(\tilde{u}_j^{i, S_j} / S_j) \\ &= (1 + \eta)^3 (S_j \log 2^N)^{1-d} \varphi^T(\tilde{u}_j^{i, S_j}). \end{aligned}$$

By the second inequality in (1.4.79), we have

$$S_j \log 2^N \geq \log\left(\frac{\delta_j}{2r_j}\right) - \log 2^N$$

for every $j \in \mathbb{N}$; hence, taking into account that $\delta_j / r_j \rightarrow +\infty$ as $j \rightarrow +\infty$ and that N and η are fixed, it is not restrictive to suppose that

$$S_j \log 2^N \geq (1 + \eta)^{\frac{1}{1-d}} \log\left(\frac{\delta_j}{2r_j}\right)$$

for every $j \in \mathbb{N}$. We obtain that

$$\mathcal{F}_{\varepsilon_j}^T(u_j, Q(i\delta_j, \rho_j^{i, S_j})) \leq (1 + \eta)^4 \left(\log\left(\frac{\delta_j}{2r_j}\right)\right)^{1-d} \varphi^T(\tilde{u}_j^{i, S_j}).$$

Finally, we sum over $i \in Z_j(\tilde{\Omega})$. By (1.4.76), we are in position to apply Lemma 1.4.18 and then, also using (1.4.37), we get

$$\begin{aligned} \limsup_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}^T(u_j, \tilde{E}_j) &= \limsup_{j \rightarrow +\infty} \sum_{i \in Z_j(\tilde{\Omega})} \mathcal{F}_{\varepsilon_j}^T(u_j, Q(i\delta_j, \rho_j^{i, S_j})) \\ &\leq (1 + \eta)^4 \limsup_{j \rightarrow +\infty} \sum_{i \in Z_j(\tilde{\Omega})} \left(\log\left(\frac{\delta_j}{2r_j}\right)\right)^{1-d} \varphi^T(\tilde{u}_j^{i, S_j}) \\ &= (1 + \eta)^4 \limsup_{j \rightarrow +\infty} \sum_{i \in Z_j(\tilde{\Omega})} \gamma \delta_j^d \varphi^T(\tilde{u}_j^{i, S_j}) \\ &= (1 + \eta)^4 \gamma \int_{\tilde{\Omega}} \varphi^T(u) dx. \end{aligned} \tag{1.4.87}$$

Using (1.4.80) and combining (1.4.82) and (1.4.87), we conclude that

$$\begin{aligned} \limsup_{j \rightarrow +\infty} \mathcal{F}_{\varepsilon_j}^T(u_j) &= \limsup_{j \rightarrow +\infty} (\mathcal{F}_{\varepsilon_j}^T(u_j, \Omega \setminus \tilde{E}_j) + \mathcal{F}_{\varepsilon_j}^T(u_j, \tilde{E}_j)) \\ &\leq \int_{\tilde{\Omega}} f_{\text{hom}}^T(\nabla u) dx + (1 + \eta)^4 \gamma \int_{\tilde{\Omega}} \varphi^T(u) dx + \frac{C}{k}, \end{aligned}$$

and (1.4.74) follows by the arbitrariness of η, k and $\tilde{\Omega}$ and the fact that, in virtue of (1.4.85), $\{u_j\}_j$ is an admissible sequence for $\{F_{\varepsilon_j}^T\}_j$.

It remains to verify that $u_j \rightarrow u$ in $L^d(\Omega; \mathbb{R}^m)$. By construction, we have

$$\begin{aligned} \lim_{j \rightarrow +\infty} \int_{\tilde{\Omega}} |u_j - w_j|^d dx &= \lim_{j \rightarrow +\infty} \int_{\tilde{E}_j} |u_j - w_j|^d dx \\ &= \lim_{j \rightarrow +\infty} \sum_{i \in Z_j(\tilde{\Omega})} \int_{Q(i\delta_j, \rho_j^{i, S_j})} |u_j - w_j|^d dx. \end{aligned}$$

By (1.4.80) we have that $u_j - w_j = 0$ a.e. on $Q(i\delta_j, \rho_j^{i, S_j}) \setminus Q(i\delta_j, \rho_j^{i, S_j} - \varepsilon_j T)$ for every $i \in Z_j(\tilde{\Omega})$. Therefore, we apply a rescaled version of Poincaré inequality (see [5, Proposition 4.1]) to infer

$$\begin{aligned} \lim_{j \rightarrow +\infty} \sum_{i \in Z_j(\tilde{\Omega})} \int_{Q(i\delta_j, \rho_j^{i, S_j})} |u_j - w_j|^d dx &\leq C \lim_{j \rightarrow +\infty} \sum_{i \in Z_j(\tilde{\Omega})} (\rho_j^{i, S_j})^d G_{\varepsilon_j}(u_j - w_j, Q(i\delta_j, \rho_j^{i, S_j})) \\ &\leq C \lim_{j \rightarrow +\infty} \delta_j^d \sum_{i \in Z_j(\tilde{\Omega})} (G_{\varepsilon_j}(u_j, Q(i\delta_j, \rho_j^{i, S_j})) + G_{\varepsilon_j}(w_j, Q(i\delta_j, \rho_j^{i, S_j}))) \\ &\leq C \lim_{j \rightarrow +\infty} \delta_j^d (\mathcal{F}_{\varepsilon_j}^T(u_j, \tilde{E}_j) + \mathcal{F}_{\varepsilon_j}^T(w_j, \tilde{\Omega})) = 0 \end{aligned}$$

in light of (1.4.75), (1.4.81), and (1.4.87). Taking into account that $w_j \rightarrow u$ by Lemma 1.4.17, we infer that $u_j \rightarrow u$.

Step 3. In the first two steps we proved that, for any $T > r_0$ natural, it holds

$$\Gamma\text{-}\lim_{j \rightarrow +\infty} F_{\varepsilon_j}^T(u) = \int_{\Omega} f_{\text{hom}}^T(\nabla u) dx + \gamma \int_{\Omega} \varphi^T(u) dx, \quad u \in W^{1,d}(\Omega; \mathbb{R}^m).$$

Hence, resorting to Proposition 1.4.8, Proposition 1.4.16(iii), and Lemma 1.4.9, we obtain

$$\Gamma\text{-}\lim_{j \rightarrow +\infty} F_{\varepsilon_j}(u) = \int_{\Omega} f_{\text{hom}}(\nabla u) dx + \gamma \int_{\Omega} \varphi(u) dx,$$

which concludes the proof of Theorem 1.4.3. □

Chapter 2

Models for phase transitions

Diffuse-interface models are widely used to describe discontinuities occurring in image segmentation problems or inherent in physical processes, such as phase separations. A by-now classical example is the Ambrosio-Tortorelli functional (see [13])

$$\int_{\Omega} v^2 |\nabla u|^2 dx + \int_{\Omega} \frac{1}{\varepsilon} V(v) + \varepsilon |\nabla v|^2 dx.$$

Here, V is a single-well potential that vanishes at 1, u is the reconstructed image, and v is a phase-field variable that describes a diffuse interface on the edges of the image, allowing for discontinuities of u as $\varepsilon \rightarrow 0$. A comparable model is the one proposed by Kobayashi, Warren, and Carter for polycrystals (see [91, 90]) that, in a simplified form, may be expressed in terms of the functional

$$\int_{\Omega} v^2 d|Du| + \int_{\Omega} \frac{1}{\varepsilon} V(v) + \varepsilon |\nabla v|^2 dx,$$

where $|Du|$ denotes the total variation of Du and u describes the orientation of the grains. In [82], Giga, Okamoto, and Uesaka studied the following one-dimensional example: they considered $\Omega = (-1, 1)$ and u a piecewise constant function with a single discontinuity at 0, obtaining as resulting energy

$$bv(0)^2 + \int_{-1}^1 \frac{1}{\varepsilon} V(v) + \varepsilon |v'|^2 dx, \quad b > 0.$$

They observed that minimizers of the above energies develop a cusp-like behaviour at the origin as $\varepsilon \rightarrow 0$, which led them to perform an asymptotic analysis with respect to the convergence of graphs.

In the first section of this chapter, we address a related problem by studying the equi-coerciveness with respect to the graph convergence of the fractional energies

$$\frac{1}{\varepsilon} \int_{-1}^1 V(v) dx + \varepsilon^{2s-1} \int_{-1}^1 \int_{-1}^1 \frac{|v(y) - v(x)|^2}{|y - x|^{1+2s}} dx dy,$$

for $s \in (1/2, 1)$. We prove that the graphs of a sequence with equi-bounded energies approach the graph of a set-valued function that attains everywhere the value of the well of V with countably many exceptions. Our approach easily extends to the case $s = 1$; i.e., when the squared Gagliardo seminorm is replaced by the Dirichlet energy, and provides an alternative and more elementary proof of the corresponding result in [82].

Another widely used model for phase transitions (e.g., for liquid-to-liquid phase separations) can be formulated as a minimum principle for double-well energies perturbed by a gradient term, the so-called Van der Waals-Cahn-Hilliard energy

$$\int_{\Omega} W(u) dx + \varepsilon^2 \int_{\Omega} |\nabla u|^2 dx, \tag{2.0.1}$$

where Ω is a bounded open set in $\mathbb{R}^d$ with Lipschitz boundary, $u : \Omega \rightarrow \mathbb{R}$, and W is a non-negative function with only two zeroes; say, -1 and 1 . The variable u is usually subject to an additional

integral constraint, which forbids trivial minimizers. The asymptotic analysis of such energies can be stated as the computation of a Γ -limit. To this end, the energies have to be scaled by $\frac{1}{\varepsilon}$ in order not to have a trivial result. The computation of this Γ -limit was essentially carried out in the seminal paper by Modica and Mortola [103] (for functionals explicitly in the form (2.0.1) we refer to a subsequent paper [102], where the result in [103] was restated for double-well problems). In particular, from their results we have that the domain of the Γ -limit is the set $BV(\Omega; \{-1, 1\})$ of the functions of bounded variations u which only attain the values 1 and -1 . In terms of the set of essential discontinuity points $S(u)$, which can be interpreted as the interface between the regions where $u = 1$ and $u = -1$, the Γ -limit takes the form

$$m_1 \mathcal{H}^{d-1}(S(u) \cap \Omega) = m_1 P(\{u = 1\}; \Omega),$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$ -dimensional Hausdorff measure and $P(E; \Omega)$ denotes the perimeter of a set E in Ω in the sense of sets of finite perimeter, see [98]. The constant m_1 can be characterized by an *optimal-profile problem* (see, e.g., [76])

$$m_1 = \inf \left\{ \int_{-\infty}^{+\infty} [W(u) + |u'|^2] dt : u \in H_{\text{loc}}^1(\mathbb{R}), \lim_{t \rightarrow \pm\infty} u(t) = \pm 1 \right\}.$$

Hence, after showing that the integral constraint for u is compatible with Γ -convergence and rereading it as a constraint for the measure $|\{u = 1\}|$, the Fundamental Theorem of Γ -convergence allows one to interpret the limit of minimizers as a set of minimal perimeter, thus confirming a *minimal-interface criterion* as conjectured by Gurtin [86]. It is worth remarking that the problem defining m_1 is set on the real line, and depends only on W . This is due to the fact that minimizing functions at given ε have a one-dimensional structure, following scaled optimal profiles in the direction orthogonal to the limit interface. This highlights that in fact the relevant analysis is in dimension $d = 1$, the higher-dimensional case being derived from that one using by-now-standard slicing arguments [28].

In the second section of this chapter, we describe the behaviour of the functionals in (2.0.1) when the gradient is substituted by a suitably-scaled higher-order derivative; more precisely, of functionals

$$\int_{\Omega} W(u) dx + \varepsilon^{2k} \int_{\Omega} \|\nabla^{(k)} u\|_k^2 dx, \quad (2.0.2)$$

where $\|\nabla^{(k)} u\|_k$ denotes the operatorial norm of the symmetric k -tensor $\nabla^{(k)} u$. Besides the motivations related to the fine study of models of phase transition, for which we refer to the detailed introduction of the paper by Fonseca and Mantegazza [74], where they analyze the case $k = 2$, higher-order perturbations have also been used to approximate free-discontinuity energies [6, 9, 26, 108, 127], in the study of multi-well problems in elasticity (see, e.g., [59, 88]) and have been examined in relation to other limit procedures, such as dimension reduction (see, e.g., [89]).

Here, we consider functionals (2.0.2) for arbitrary $k \geq 3$, for which the relevant analysis by Γ -convergence is that of the scaled one-dimensional functionals

$$\frac{1}{\varepsilon} \int_I W(u) dx + \varepsilon^{2k-1} \int_I |u^{(k)}|^2 dx,$$

defined for $u \in H^k(I)$, where I is an open interval. We show that their Γ -limit both in the $L^1(I)$ topology and with respect to the convergence in measure is still of the form

$$m_k \#(S(u) \cap I), \quad u \in BV(I; \{-1, 1\}),$$

where m_k is the value given by the optimal-profile problem

$$m_k = \inf \left\{ \int_{-\infty}^{+\infty} [W(u) + |u^{(k)}|^2] dt : u \in H_{\text{loc}}^k(\mathbb{R}), \lim_{t \rightarrow \pm\infty} u(t) = \pm 1 \right\},$$

in accordance with the results in [103] for $k = 1$ and [74] for $k = 2$.

In the third section of this chapter, we study a wider family of functionals in which possibly several singular perturbations, and accordingly several orders of derivatives, appear and more general tensor norms are involved. We study the asymptotic behaviour of functionals given by

$$\int_{\Omega} \left[\frac{1}{\varepsilon} W(u) + \sum_{\ell=1}^k q_{\ell} \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_{\ell}^2 \right] dx, \quad u \in H^k(\Omega).$$

Here, $q_1, \dots, q_{k-1}$ are possibly negative constant coefficients, $q_k = 1$, and $|\cdot|_\ell$ is any norm on the space of symmetric ℓ -tensors for every $\ell \in \{1, \dots, k\}$. In this situation, the negativity of the coefficients $q_1, \dots, q_{k-1}$ yields the lack of a priori bounds on the energies, and therefore the asymptotic analysis can be carried out provided that such coefficients satisfy suitable lower bounds that prevent the energy from diverging negatively. Due to the presence of general tensor norms, the Γ -limit equals an anisotropic perimeter

$$\int_{\partial^* A \cap \Omega} g(\nu_A) d\mathcal{H}^{d-1}, \quad u \in BV(\Omega; \{-1, 1\}), \quad \text{with } u = 2\chi_A - 1,$$

where A is a set of finite perimeter in Ω , $\partial^* A$ is its reduced boundary, ν_A is its inner unit normal, and the energy density g is defined through an asymptotic cell-problem formula. Our analysis extends those performed independently by Chermisi, Dal Maso, Fonseca, and Leoni [55] and by Cicalese, Spadaro, and Zeppieri [56] for the case $k = 2$. In the case of second-order materials, problems involving the homogenization of energies modelling phase transitions have been considered, e.g., in [18, 68]. In a one-dimensional setting, where anisotropies are irrelevant, the outcome of our analysis is comparable with the one in the recent work [42], as the effective energy reduces to

$$m\#(S(u) \cap I), \quad u \in BV(I; \{-1, 1\}),$$

where m is the value given by the corresponding one-dimensional optimal-profile problem

$$m = \inf \left\{ \int_{-\infty}^{+\infty} \left[W(u) + \sum_{\ell=1}^k q_\ell |u^{(\ell)}|^2 \right] dt : u \in H_{\text{loc}}^k(\mathbb{R}), \lim_{t \rightarrow \pm\infty} u(t) = \pm 1 \right\}.$$

The chapter is organized as follows: in Section 2.1 we introduce the main notation and collect some results about functions of bounded variation and sets of finite perimeter. Section 2.2 is devoted to the study of the fractional phase-field model [46]. Section 2.3 contains the results in [48] about the higher-order singular perturbation model. Finally, in Section 2.4 we illustrate the results obtained in [49], where several singular perturbations of different orders are involved.

2.1 Notation and general preliminaries

We let Ω be a bounded open subset of $\mathbb{R}^d$ with Lipschitz or C^1 boundary, for $d \geq 1$. For $s \in (1/2, 1)$ we consider the fractional-order Sobolev space $H^s(\Omega) = W^{s,2}(\Omega)$ (see [67, 97]). Analogously, we let k denote a positive integer, usually strictly larger than 1, and we consider the integer-order Sobolev space $H^k(\Omega) = W^{k,2}(\Omega)$ (see [96]). We let $\{e_1, \dots, e_d\}$ denote the canonical basis of $\mathbb{R}^d$, we let $|x|$ denote the Euclidean norm of $x \in \mathbb{R}^d$ and we let $x \cdot y$ denote the scalar product of $x, y \in \mathbb{R}^d$. We set $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ and, given $\nu \in \mathbb{S}^{d-1}$, we let the symbol $\nu^\perp$ denote the hyperplane $\{x \in \mathbb{R}^d : x \cdot \nu = 0\}$. To any $\nu \in \mathbb{S}^{d-1}$, we arbitrarily associate an orthonormal basis of $\mathbb{R}^d$ $\{\nu_1, \dots, \nu_{d-1}, \nu\}$ and, given $r > 0$, we let $Q_r^\nu(x)$ denote the open cube of side length r centered at x and with two faces orthogonal to ν described by such a basis; i.e.,

$$Q_r^\nu(x) := \left\{ y \in \mathbb{R}^d : |(y-x) \cdot \nu| < \frac{r}{2}, |(y-x) \cdot \nu_i| < \frac{r}{2} \text{ for every } i \in \{1, \dots, d-1\} \right\}.$$

We let $B_r(x)$ denote the open ball of center x and radius r . Both for cubes and balls, we omit to specify the center if this coincides with the origin.

We let $[t]$ and $\lceil t \rceil$ denote the lower and upper integer part of a real number t , respectively.

We say that a set $E \subseteq \mathbb{R}^d$ is measurable provided that it is measurable with respect to the d -dimensional Lebesgue measure and, in this case, we let $|E|$ denote its measure. We say that a property holds almost everywhere (a.e. for short) provided that this is valid for almost every point with respect to the Lebesgue measure on $\mathbb{R}^d$. Sometimes, in order to emphasize the dimension of the ambient space, we shall denote the h -dimensional Lebesgue measure on $\mathbb{R}^h$ by $\mathcal{L}^h$. We let $\mathcal{H}^{d-1}$ denote the $(d-1)$ -dimensional Hausdorff measure on $\mathbb{R}^d$ and we set $\sigma_{d-1} := \mathcal{H}^{d-1}(\mathbb{S}^{d-1})$. Given a Borel measure μ on Ω , we let $|\mu|$ denote its total variation and, given B a Borel subset of Ω , we let $\mu \llcorner B$ denote the restriction of μ to B ; that is, the measure defined by $\mu \llcorner B(A) := \mu(A \cap B)$.

For every positive integer ℓ , we let $\|\cdot\|_\ell$ denote the operatorial norm of a symmetric ℓ -tensor on $\mathbb{R}^d$ defined by

$$\|T\|_\ell := \sup\{|T(\underbrace{\xi, \dots, \xi}_\ell)| : \xi \in \mathbb{S}^{d-1}\},$$

while a generic norm will be denoted by $|\cdot|_\ell$.

Given $u \in L^1(\Omega)$, the variation of u in Ω is defined as

$$V(u; \Omega) := \sup\left\{\int_\Omega u \operatorname{div} \varphi \, dx : \varphi \in C_c^1(\Omega; \mathbb{R}^d), |\varphi| \leq 1\right\}.$$

We say that u has bounded variation, and we write $u \in BV(\Omega)$, if $V(u; \Omega) < +\infty$. For such a function u , the singular set $S(u)$ is defined as the complement in Ω of the set of Lebesgue points; i.e.,

$$S(u) := \Omega \setminus \left\{x \in \Omega : \exists z \in \mathbb{R} \text{ such that } \lim_{r \rightarrow 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u(y) - z| \, dy = 0\right\}.$$

The distributional derivative of u in Ω is the sum of three mutually singular vector measures,

$$Du = \mu^a + \mu^j + \mu^c.$$

Here, μ^a is absolutely continuous with respect to the Lebesgue measure, μ^j is a $(d-1)$ -dimensional measure concentrated on $S(u)$ with density given by $(u^+ - u^-)\nu_u$, where u^-, u^+ are the bilateral traces of u on $S(u)$ defined in accordance with the unit vector field ν_u , while μ^c is the so-called Cantor part. We refer to [12] for further details.

Given a measurable set $A \subseteq \Omega$, the perimeter of A in Ω is defined as

$$P(A; \Omega) := \sup\left\{\int_A \operatorname{div} \varphi \, dx : \varphi \in C_c^1(\Omega; \mathbb{R}^d), |\varphi| \leq 1\right\}.$$

We say that A has finite perimeter in Ω if $P(A; \Omega) < +\infty$. Since Ω is assumed to be bounded, the set A has finite perimeter in Ω if and only if its characteristic function χ_A is a function of bounded variation in Ω ; i.e., $\chi_A \in BV(\Omega)$. Therefore, for such a set A , the distributional derivative of χ_A on Ω is a vector measure μ_A . Denoting by $|\mu_A|$ the total variation of this measure, the reduced boundary of A is defined as

$$\partial^* A := \left\{x \in \mathbb{R}^d : \text{there exists } \nu_A(x) := \lim_{r \rightarrow 0} \frac{\mu_A(B_r(x))}{|\mu_A|(B_r(x))}, \text{ with } |\nu_A(x)| = 1\right\},$$

and the unit vector $\nu_A(x)$ is the inner unit normal (in the sense of measure theory) to A at x .

We summarize the main properties we need about sets of finite perimeter in the following statements.

Theorem 2.1.1. *If A is a set of finite perimeter in Ω , the distributional derivative of χ_A in Ω is*

$$\mu_A = \nu_A \mathcal{H}^{d-1} \llcorner (\partial^* A \cap \Omega),$$

and $P(A; \Omega) = |\mu_A|(\Omega) = \mathcal{H}^{d-1}(\partial^* A \cap \Omega)$. Moreover, setting $H_\nu := \{y \in \mathbb{R}^d : y \cdot \nu > 0\}$, for every point $x \in \partial^* A \cap \Omega$, it holds that

$$\begin{aligned} \chi_{\frac{A-x}{r}} &\rightarrow \chi_{H_{\nu_A(x)}} \text{ in } L_{\text{loc}}^1(\mathbb{R}^d) && \text{as } r \rightarrow 0, \\ \frac{\mathcal{H}^{d-1}(\partial^* A \cap Q_r^{\nu_A(x)}(x))}{r^{d-1}} &\rightarrow 1 && \text{as } r \rightarrow 0. \end{aligned}$$

We remark that a function $u \in BV(\Omega; \{-1, 1\})$ is of the form $u = 2\chi_A - 1$ for a certain A set of finite perimeter in Ω . In particular, it holds that $S(u) = \partial^* A \cap \Omega$ up to $\mathcal{H}^{d-1}$ -negligible sets and the distributional derivative of u in Ω is the measure $2\nu_A \mathcal{H}^{d-1} \llcorner (\partial^* A \cap \Omega)$. In a one-dimensional setting, a function $u \in BV(\Omega; \{-1, 1\})$, with Ω an open interval, is a piecewise constant function with a finite number of jump-discontinuity points collected in $S(u)$.

We also recall a well-known result of approximation of sets of finite perimeter via polyhedral sets, see [1, Lemma 3.6] and [19, Lemma 3.1].

Theorem 2.1.2. *Let A be a set of finite perimeter in $\Omega \subset \mathbb{R}^d$, $d \geq 2$. There exists a sequence $\{A_n\}_n$ of polyhedral sets satisfying*

- (i) $\chi_{A_n} \rightarrow \chi_A$ in $L^1(\Omega)$ as $n \rightarrow \infty$,
- (ii) $P(A_n; \Omega) \rightarrow P(A; \Omega)$ as $n \rightarrow +\infty$,
- (iii) $\mathcal{H}^{d-1}(\partial^* A_n \cap \partial\Omega) = 0$ for every $n \in \mathbb{N}$.

Moreover, for every non-negative continuous function g on $\mathbb{S}^{d-1}$, there holds

$$\lim_{n \rightarrow +\infty} \int_{\partial^* A_n \cap \Omega} g(\nu_{A_n}) d\mathcal{H}^{d-1} = \int_{\partial^* A \cap \Omega} g(\nu_A) d\mathcal{H}^{d-1}. \quad (2.1.1)$$

Given an open set $\Omega \subseteq \mathbb{R}^d$, $\xi \in \mathbb{S}^{d-1}$, and $y \in \xi^\perp$, we define the slice $\Omega^{\xi, y}$ as

$$\Omega^{\xi, y} := \{t \in \mathbb{R} : y + t\xi \in \Omega\},$$

and we define Ω^ξ the orthogonal projection of Ω on $\xi^\perp$ as

$$\Omega^\xi := \{y \in \xi^\perp : \Omega^{\xi, y} \neq \emptyset\}.$$

For every function $u : \Omega \rightarrow \mathbb{R}$ and for every $y \in \Omega^\xi$, we define the function $u^{\xi, y} : \Omega^{\xi, y} \rightarrow \mathbb{R}$ by

$$u^{\xi, y}(t) := u(y + t\xi).$$

Using Theorem 11.45 in [96] one can see that, given a function $u \in H^k(\Omega)$, for every $\xi \in \mathbb{S}^{d-1}$ and for $\mathcal{H}^{d-1}$ -a.e. $y \in \xi^\perp$ it holds that

$$\nabla^{(\ell)} u(y + t\xi) \underbrace{(\xi, \dots, \xi)}_{\ell \text{ times}} = \frac{d^\ell u^{\xi, y}}{dt^\ell}(t) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \Omega^{\xi, y} \text{ and for all } \ell \in \{1, \dots, k\}.$$

We also mention a characterization of functions of bounded variation with values in $\{-1, 1\}$ in terms of their slices; see [12, Remark 3.104].

Theorem 2.1.3. *A function $u \in L^1(\Omega; \{-1, 1\})$ belongs to $BV(\Omega; \{-1, 1\})$ if and only if for every $m \in \{1, \dots, d\}$ the one-dimensional slices of the function u given by $u^{e_m, y}$ have bounded variation in the open set $\Omega^{e_m, y}$ for $\mathcal{H}^{d-1}$ -a.e. $y \in e_m^\perp$, and*

$$\int_{\Omega^{e_m}} \#(S(u^{e_m, y}) \cap \Omega^{e_m, y}) d\mathcal{H}^{d-1}(y) < +\infty.$$

We recall that Γ -convergence is defined in Definition 1.1.1 and we anticipate that in Section 2.3 we shall discuss Γ -convergence with respect to the convergence in measure, which is defined analogously.

Finally, we recall the definitions of Γ -lim inf and Γ -lim sup with respect to the convergence in $L^2(\Omega)$ that will be of use in Section 2.4. Given functionals $F_\varepsilon : L^2(\Omega) \rightarrow [-\infty, +\infty]$ for $\varepsilon > 0$, and $\{\varepsilon_j\}_j$ a positive sequence converging to 0, we set

$$\begin{aligned} \Gamma\text{-}\liminf_{j \rightarrow +\infty} F_{\varepsilon_j}(u) &:= \inf\{\liminf_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) : u_j \rightarrow u \text{ in } L^2(\Omega)\}; \\ \Gamma\text{-}\limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(u) &:= \inf\{\limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) : u_j \rightarrow u \text{ in } L^2(\Omega)\}. \end{aligned} \quad (2.1.2)$$

Both functionals are lower semicontinuous with respect to the convergence in $L^2(\Omega)$, they coincide if and only if the Γ -limit of $\{F_{\varepsilon_j}\}_j$ exists and, in this case, the Γ -limit equals their common value.

2.2 Graph convergence in a fractional phase-field model

In this section we present some results that will be expanded in a forthcoming paper [46] on the asymptotic analysis of single-well potential energies singularly perturbed by fractional terms, namely, proper rescaling of the Gagliardo seminorms. We address the problem of equi-coerciveness in a suitable topology of the energies

$$\frac{1}{\varepsilon} \int_I V(u) dx + \varepsilon^{2s-1} \int_I \int_I \frac{|u(y) - u(x)|^2}{|y - x|^{1+2s}} dx dy, \quad u \in H^s(I), \quad (2.2.1)$$

where I is a bounded open interval that we may assume to be $(-1, 1)$, V a single-well potential vanishing at 1, and $s \in (1/2, 1)$. The interest in such a study stems from a slightly different problem that is related to a significantly simplified version of the Kobayashi-Warren-Carter model. If we add the penalization term $u(0)^2$ to the energies (2.2.1), it is possible to infer that minimizers u_ε subject to periodic boundary conditions develop a singular behaviour at 0. Such minimizers converge locally uniformly to the constant 1, the pure phase, on $I \setminus \{0\}$ and attain their minimum point at 0, with $\{u_\varepsilon(0)\}_\varepsilon$ converging to a constant σ strictly smaller than 1. For this reason, the graph of u_ε performs a double transition (between values close to 1 and values close to σ and vice versa) around the origin that gets sharper as $\varepsilon \rightarrow 0$. The graphs of the minimizers $\{u_\varepsilon\}_\varepsilon$ converge, in the Hausdorff distance, to the graph of a multi-valued function that is everywhere equal to 1 except at the origin, where a vertical segment is generated.

This behaviour was first detected in [82] (see also [81] for the extension in higher dimensions) in the case $s = 1$, when the singular perturbation is a rescaling of the Dirichlet energy. In that work, among other results, the authors proved that the corresponding energies are equi-coercive with respect to the Hausdorff-convergence of the graphs and that the limit is a multi-valued function which is always equal to 1 with countably many exceptions for which the attained value is a compact interval. Here, we present an analogous result for the fractional case (2.2.1). We point out that our argument applies to the case $s = 1$ with some minor adaptations and, in fact, with some simplifications. Indeed, it relies on the continuity of H^s -functions and on standard interpolation inequalities only. This allows us to bypass the “unfolding argument” employed in [82] that is based on the absolute continuity of (integer-order) Sobolev functions. It is worth mentioning that the Kobayashi-Warren-Carter model has been drawing some attention over the last years (see [2, 79, 94, 104, 105, 106, 107]) and, very recently, has been proposed also for image segmentation (see [80]).

It would be interesting to extend our analysis to different fractional orders, and in particular to the critical case $s = 1/2$ (see [3, 36, 113, 114] for double-well potentials) and to the case $s > 1$ by adapting the technique of Solci [126] (see also [30, 52]).

2.2.1 Setting of the problem and statement of the result

Let $V : \mathbb{R} \rightarrow [0, +\infty)$ be a nonnegative continuous function satisfying:

(V1) $V(u) = 0$ if and only if $u = 1$;

(V2) there exist positive constants c_1, c_2 such that $V(u) \geq c_1|u - 1|^2$ if $|u - 1| > c_2$.

Let $I \subset \mathbb{R}$ be a bounded open interval and let $s \in (1/2, 1)$. For every $\varepsilon > 0$, we define functionals $F_\varepsilon : H^s(I) \rightarrow [0, +\infty)$ as

$$F_\varepsilon(u) := \frac{1}{\varepsilon} \int_I V(u) dx + \varepsilon^{2s-1} [u]_{H^s(I)}^2, \quad (2.2.2)$$

where, for any I open interval, we set

$$[u]_{H^s(I)}^2 := \int_I \int_I \frac{|u(y) - u(x)|^2}{|y - x|^{1+2s}} dx dy,$$

and the fractional Sobolev space $H^s(I)$ is defined as

$$H^s(I) := \{u \in L^2(I) : [u]_{H^s(I)} < +\infty\}.$$

We further introduce the *homogeneous* fractional Sobolev space on $\mathbb{R}$ as

$$\dot{H}^s(\mathbb{R}) := \{u \in L^2_{\text{loc}}(\mathbb{R}) : [u]_{H^s(\mathbb{R})} < +\infty\}.$$

By Morrey's embedding, we always identify elements of $H^s(I)$ with functions that are continuous up to the boundary of I . As a consequence, the graph of a function $u \in H^s(I)$ defined as

$$\text{Graph}(u) = \{(x, u(x)) : x \in \bar{I}\}$$

is a compact subset of $\bar{I} \times \mathbb{R}$.

Our result concerns the asymptotic analysis of energies (2.2.2) with respect to the Hausdorff convergence of the graphs. We introduce some classes of interest for our investigation and we collect some basic facts about the Hausdorff convergence of compact sets.

Letting $\mathcal{P}(\mathbb{R})$ denote the power set of $\mathbb{R}$, we define the graph of a set-valued function $\Gamma : \bar{I} \rightarrow \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ as

$$\text{Graph}(\Gamma) = \{(x, y) \in \bar{I} \times \mathbb{R} : y \in \Gamma(x)\}.$$

By definition, Γ is bounded if its graph is a bounded subset of $\bar{I} \times \mathbb{R}$ and is upper semicontinuous if its graph is a closed subset of $\bar{I} \times \mathbb{R}$. We denote by $\mathcal{B}$ the class of bounded upper semicontinuous set-valued functions from $\bar{I}$ to $\mathbb{R}$; i.e.,

$$\mathcal{B} := \{\Gamma : \bar{I} \rightarrow \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\} : \text{Graph}(\Gamma) \text{ is compact in } \bar{I} \times \mathbb{R}\}$$

and we identify functions in $H^s(I)$ as elements of $\mathcal{B}$ through their graphs. We endow $\mathcal{B}$ with the metric d_g defined as

$$d_g(\Gamma_1, \Gamma_2) := d_H(\text{Graph}(\Gamma_1), \text{Graph}(\Gamma_2)), \quad \Gamma_1, \Gamma_2 \in \mathcal{B},$$

where $d_H(A, B)$ denotes the Hausdorff distance of nonempty compact sets A and B ; that is,

$$d_H(A, B) := \max \left\{ \max_{a \in A} \text{dist}(a, B), \max_{b \in B} \text{dist}(b, A) \right\}.$$

We refer to the convergence with respect to the metric d_g as the graph convergence and, given a sequence $\{\Gamma_j\}_j \subset \mathcal{B}$ and $\Gamma \in \mathcal{B}$, we write $\Gamma_j \xrightarrow{g} \Gamma$ as $j \rightarrow +\infty$ if $d_g(\Gamma_j, \Gamma) \rightarrow 0$ as $j \rightarrow +\infty$.

It is useful to recall that the Hausdorff convergence and the Kuratowski convergence are equivalent in a fixed compact set. This means that, given $\{\Gamma_j\}_j \subset \mathcal{B}$ such that $\Gamma_j \subset \bar{I} \times [-\Lambda, \Lambda]$ for every $j \in \mathbb{N}$ and for some $\Lambda > 0$, and given $\Gamma \in \mathcal{B}$, it holds that $\Gamma_j \xrightarrow{g} \Gamma$ as $j \rightarrow +\infty$ if and only if the following conditions are satisfied:

- (K1) if $p_j \in \Gamma_j, j \in \mathbb{N}$, is a sequence such that $p_j \rightarrow p$ as $j \rightarrow +\infty$, then $p \in \Gamma$,
- (K2) if $p \in \Gamma$, then there exists $p_j \in \Gamma_j$ for every $j \in \mathbb{N}$ such that $p_j \rightarrow p$ as $j \rightarrow +\infty$.

Given $\Gamma \in \mathcal{B}$, we let $\mathcal{E}(\Gamma)$ denote the set of *exceptional points* of Γ which is defined as

$$\mathcal{E}(\Gamma) := \{x \in \bar{I} : \Gamma(x) \text{ is a nondegenerate compact interval containing } 1\}.$$

Then we introduce a special subset of $\mathcal{B}$ as

$$\mathcal{B}_c := \{\Gamma \in \mathcal{B} : \mathcal{E}(\Gamma) \text{ is countable and } \Gamma(x) = \{1\} \text{ if } x \notin \mathcal{E}(\Gamma)\}.$$

For every $\Gamma \in \mathcal{B}_c$, we set

$$\begin{aligned} \Gamma_-(x) &:= \min\{y \in \mathbb{R} : (x, y) \in \Gamma(x)\}, \\ \Gamma_+(x) &:= \max\{y \in \mathbb{R} : (x, y) \in \Gamma(x)\}. \end{aligned}$$

Note that, if $x \notin \mathcal{E}(\Gamma)$, then $\Gamma_-(x) = \Gamma_+(x) = 1$ and, if $x \in \mathcal{E}(\Gamma)$, we have $\Gamma_-(x) \leq 1 \leq \Gamma_+(x)$ with at least one of the inequalities being strict.

We can now state our main result.

Theorem 2.2.1 (Compactness). *Let F_ε be defined by (2.2.2). Then, the family $\{F_\varepsilon\}_\varepsilon$ is equi-coercive with respect to the graph convergence; that is, given any family $\{u_\varepsilon\}_\varepsilon \subset H^s(I)$ such that*

$$\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty,$$

there exist a positive sequence $\{\varepsilon_j\}_j$ converging to 0 and a set-valued function $\Gamma \in \mathcal{B}_c$ such that $u_{\varepsilon_j} \xrightarrow{g} \Gamma$ as $j \rightarrow +\infty$.

2.2.2 Preliminary results

In this subsection we collect some preliminary results. It is convenient to localize functionals (2.2.2) by letting

$$F_\varepsilon(u, J) = \frac{1}{\varepsilon} \int_J V(u) dx + \varepsilon^{2s-1} [u]_{H^s(J)}^2$$

for any open interval $J \subseteq I$.

Throughout this section we will always assume that $s \in (1/2, 1)$. Under this assumption, we can resort to Morrey's inequality and the corresponding continuous embedding (see [67, 97, 101]).

Theorem 2.2.2. *Let J be a bounded interval. Then, every element $v \in H^s(J)$ has a representative in $C^{0, s-\frac{1}{2}}(\bar{J})$, and there exists a constant $C_s > 0$ depending only on s such that*

$$|v(x) - v(y)| \leq C_s |x - y|^{s-\frac{1}{2}} [v]_{H^s(J)} \quad (2.2.3)$$

for every $x, y \in \bar{J}$.

Given a function $u \in H^s(J)$ with J a bounded open interval, we set

$$\text{osc}_J v := \max_J v - \min_J v.$$

Morrey's inequality allows us to control the oscillation as follows.

Lemma 2.2.3. *Let J be a bounded open interval. Then, there exists a constant $C_s > 0$ depending only on s such that*

$$(\text{osc}_J v)^2 \leq C_s \left(\int_J |v|^2 dx \right)^{1-\frac{1}{2s}} [v]_{H^s(J)}^{\frac{1}{s}}$$

for every $v \in H^s(J)$.

Proof. Let w_J denote the mean value of v on the interval J and let $w := v - w_J$. We note that

$$w_J = 0, \quad [w]_{H^s(J)} = [v]_{H^s(J)}, \quad \text{osc}_J w = \text{osc}_J v, \quad \text{and} \quad \|w\|_{L^2(J)} \leq \|v\|_{L^2(J)}. \quad (2.2.4)$$

Without loss of generality, let us assume that $M := \|w\|_{L^\infty(J)} > 0$ and that $\bar{x} \in \bar{J}$ is such that $w(\bar{x}) = M$. Since $w_J = 0$, there exists $\bar{y} \in \bar{J}$ such that $w(\bar{y}) < 0$. Hence, by (2.2.3) we have

$$M < w(\bar{x}) - w(\bar{y}) \leq C_s |\bar{x} - \bar{y}|^{s-\frac{1}{2}} [w]_{H^s(J)},$$

from which it follows that

$$\rho := \left(\frac{M}{2C_s [w]_{H^s(J)}} \right)^{\frac{2}{2s-1}} < |\bar{x} - \bar{y}|.$$

Therefore, either the interval $(\bar{x}, \bar{x} + \rho)$ or the interval $(\bar{x} - \rho, \bar{x})$ is contained in J and we let $I(\bar{x}, \rho)$ denote such an interval. Resorting again to (2.2.3), we have that

$$w(x) \geq w(\bar{x}) - C_s [w]_{H^s(J)} \rho^{s-\frac{1}{2}} = M - C_s [w]_{H^s(J)} \rho^{s-\frac{1}{2}} = \frac{M}{2},$$

for every $x \in I(\bar{x}, \rho)$. Upon relabelling the constant C_s , we obtain the estimate

$$\int_J |w|^2 dx \geq \int_{I(\bar{x}, \rho)} |w|^2 dx \geq \frac{M^2}{4} \left(\frac{M}{2C_s [w]_{H^s(J)}} \right)^{\frac{2}{2s-1}} \geq C_s M^{\frac{4s}{2s-1}} [w]_{H^s(J)}^{-\frac{2}{2s-1}}$$

from which the following inequality follows

$$\|w\|_{L^\infty(J)} = M \leq C_s \left(\int_J |w|^2 dx \right)^{\frac{2s-1}{4s}} [w]_{H^s(J)}^{\frac{1}{2s}} = C_s \|w\|_{L^2(J)}^{1-\frac{1}{2s}} [w]_{H^s(J)}^{\frac{1}{2s}}.$$

Using that $\text{osc}_J w \leq 2\|w\|_{L^\infty(J)}$ and recalling (2.2.4), we obtain that

$$(\text{osc}_J v)^2 \leq C_s \left(\int_J |v|^2 dx \right)^{1-\frac{1}{2s}} [v]_{H^s(J)}^{\frac{1}{s}},$$

which concludes the proof. $\square$

Lemma 2.2.3 implies the following.

Proposition 2.2.4. *Let J be a bounded open interval and let C_s be positive constant in Lemma 2.2.3. Then, it holds that*

$$(\operatorname{osc}_J v)^2 \leq C_s \left\{ \frac{1}{\varepsilon} \int_J |v|^2 dx + \varepsilon^{2s-1} [v]_{H^s(J)}^2 \right\}$$

for every $v \in H^s(J)$ and for every $\varepsilon > 0$.

Proof. Let us set

$$A := \frac{1}{\varepsilon} \int_J |v|^2 dx \quad \text{and} \quad B := \varepsilon^{2s-1} [v]_{H^s(J)}^2.$$

By the Young's inequality

$$A^{1-\frac{1}{2s}} B^{\frac{1}{2s}} \leq \left(1 - \frac{1}{2s}\right) A + \left(\frac{1}{2s}\right) B$$

and the fact that $1/2 < s < 1$, we get that

$$\left(\int_J |v|^2 dx \right)^{1-\frac{1}{2s}} [v]_{H^s(J)}^{\frac{1}{s}} \leq \frac{1}{\varepsilon} \int_J |v|^2 dx + \varepsilon^{2s-1} [v]_{H^s(J)}^2;$$

therefore, the thesis follows by Lemma 2.2.3. $\square$

The following lemma, together with the subsequent proposition, provides a useful tool in order to estimate from below our energies.

Lemma 2.2.5. *Let J be a bounded open interval and let $w \in H^s(J)$. Then $w^+, w^- \in H^s(J)$, where $w^+(x) := \max\{w(x), 0\}$ and $w^-(x) = \max\{-w(x), 0\}$ are the positive and negative parts of w . Moreover,*

$$[w]_{H^s(J)}^2 \geq [w^+]_{H^s(J)}^2 + [w^-]_{H^s(J)}^2. \quad (2.2.5)$$

Proof. The fact that $w^+, w^- \in H^s(J)$ follows from the fact that the maps $x \mapsto x^+$ and $x \mapsto x^-$ are Lipschitz continuous.

For $a, b \in \mathbb{R}$, we have that

$$\begin{aligned} |a - b|^2 &= |a^+ - b^+|^2 + |a^- - b^-|^2 - 2(a^+ - b^+)(a^- - b^-) \\ &\geq |a^+ - b^+|^2 + |a^- - b^-|^2 \end{aligned}$$

since $(a^+ - b^+)(a^- - b^-) \leq 0$. Letting $a = w(x)$ and $b = w(y)$ and integrating this inequality on $J \times J$, inequality (2.2.5) follows. $\square$

Proposition 2.2.6. *Let J be a bounded open interval. It holds that*

$$F_\varepsilon(w, J) \geq F_\varepsilon(1 + (w - 1)^+, J) + F_\varepsilon(1 - (w - 1)^-, J)$$

for every $w \in H^s(J)$ and $\varepsilon > 0$.

Proof. We note that

$$\int_J V(w) dx = \int_J V(1 + (w - 1)^+) dx + \int_J V(1 - (w - 1)^-) dx.$$

Then, we apply Lemma 2.2.5 to infer

$$\begin{aligned} F_\varepsilon(w, J) &= \frac{1}{\varepsilon} \int_J V(1 + (w - 1)^+) dx + \frac{1}{\varepsilon} \int_J V(1 - (w - 1)^-) dx + \varepsilon^{2s-1} [w - 1]_{H^s(J)}^2 \\ &\geq \frac{1}{\varepsilon} \int_J V(1 + (w - 1)^+) dx + \frac{1}{\varepsilon} \int_J V(1 - (w - 1)^-) dx \\ &\quad + \varepsilon^{2s-1} [(w - 1)^+]_{H^s(J)}^2 + \varepsilon^{2s-1} [(w - 1)^-]_{H^s(J)}^2 \\ &= \frac{1}{\varepsilon} \int_J V(1 + (w - 1)^+) dx + \frac{1}{\varepsilon} \int_J V(1 - (w - 1)^-) dx \\ &\quad + \varepsilon^{2s-1} [1 + (w - 1)^+]_{H^s(J)}^2 + \varepsilon^{2s-1} [1 - (w - 1)^-]_{H^s(J)}^2 \\ &= F_\varepsilon(1 + (w - 1)^+, J) + F_\varepsilon(1 - (w - 1)^-, J), \end{aligned}$$

which proves the claim. $\square$

We conclude this subsection recalling a compactness result in H^s .

Lemma 2.2.7. *Let $R > 0$ and consider a sequence of functions $\{w_j\}_j \subset H^s(-R, R)$ such that*

$$\sup_j (|w_j(0)| + [w_j]_{H^s(-R, R)}) < +\infty. \quad (2.2.6)$$

Then, $\{w_j\}_j$ is an equi-bounded sequence in $H^s(-R, R)$ and in particular it admits a subsequence weakly converging in $H^s(J)$ and uniformly converging in $[-R, R]$.

Proof. It suffices to prove that $\{w_j\}_j$ is an equi-bounded sequence in $L^2(-R, R)$. To this end, we apply Morrey's inequality (2.2.3) on the interval $(-R, R)$ and we get

$$\begin{aligned} \int_{-R}^R |w_j|^2 dx &\leq 2 \int_{-R}^R |w_j - w_j(0)|^2 dx + 4R |w_j(0)|^2 \\ &\leq 2C_s [w_j]_{H^s(-R, R)}^2 \int_{-R}^R |x|^{2s-1} dx + 4R |w_j(0)|^2 \\ &\leq 4R^{2s} C_s [w_j]_{H^s(-R, R)}^2 + 4R |w_j(0)|^2. \end{aligned}$$

Then, the thesis follows recalling (2.2.6). $\square$

2.2.3 Proof of the main result

In order to prove our compactness result, we consider optimal-profile problems determined by the functions $\varphi_s^{T, \eta}, \varphi_s : \mathbb{R} \rightarrow [0, +\infty)$ defined as

$$\varphi_s^{T, \eta}(z) := \inf \left\{ \int_{-T}^T V(w) dx + [w]_{H^s(-T, T)}^2 : w \in H^s(-T, T), |w(0) - z| < \eta \right\}$$

and

$$\varphi_s(z) := \inf \left\{ \int_{-\infty}^{+\infty} V(w) dx + [w]_{H^s(\mathbb{R})}^2 : w \in \dot{H}^s(\mathbb{R}), w(0) = z \right\}.$$

Proposition 2.2.8. *It holds that*

$$\lim_{T \rightarrow +\infty} \lim_{\eta \rightarrow 0} \varphi_s^{T, \eta}(z) = \varphi_s(z)$$

for every $z \in \mathbb{R}$. Moreover, the function φ_s is a well defined minimum, is lower semicontinuous and such that $\varphi_s(z) > 0$ for every $z \neq 1$.

Proof. We observe that $\varphi_s^{T, \eta}$ is decreasing in η and increasing in T ; hence, the limit exists. Let us set

$$\varphi_s^T(z) := \lim_{\eta \rightarrow 0} \varphi_s^{T, \eta}(z).$$

We prove that

$$\varphi_s^T(z) = \inf \left\{ \int_{-T}^T V(w) dx + [w]_{H^s(-T, T)}^2 : w \in H^s(-T, T), w(0) = z \right\}.$$

It suffices to prove that $\varphi_s^T(z) \leq \lim_{\eta \rightarrow 0} \varphi_s^{T, \eta}(z)$. Let us consider $\{\eta_j\}_j$ a positive sequence converging to 0 and, for every $j \in \mathbb{N}$, let w_j be an admissible function for $\varphi_s^{T, \eta_j}(z)$ such that $|w_j(0) - z| < \eta_j$ and

$$\lim_{j \rightarrow +\infty} \varphi_s^{T, \eta_j}(z) = \lim_{j \rightarrow +\infty} \int_{-T}^T V(w_j) dx + [w_j]_{H^s(-T, T)}^2.$$

Assuming without loss of generality that $\eta_j < 1$, we apply Lemma 2.2.7 on $(-T, T)$ to obtain that there exists $w \in H^s(-T, T)$ such that $w_j \rightarrow w$ uniformly on $[-T, T]$, so that $w(z) = 0$. Since w is an admissible function for $\varphi_s^T(z)$, by Fatou's lemma we have

$$\begin{aligned} \lim_{j \rightarrow +\infty} \varphi_s^{T, \eta_j}(z) &= \lim_{j \rightarrow +\infty} \int_{-T}^T V(w_j) dx + [w_j]_{H^s(-T, T)}^2 \\ &\geq \int_{-T}^T V(w) dx + [w]_{H^s(-T, T)}^2 \geq \varphi_s^T(z), \end{aligned}$$

which proves the claim.

Now we prove that $\lim_{T \rightarrow +\infty} \varphi_s^T(z) = \varphi_s(z)$. It suffices to prove that $\varphi_s(z) \leq \lim_{T \rightarrow +\infty} \varphi_s^T(z)$. Let $\{T_j\}_j$ be a diverging sequence and, for every $j \in \mathbb{N}$, let w_j be an admissible function for $\varphi_s^{T_j}(z)$ such that $w_j(0) = z$ and

$$\lim_{j \rightarrow +\infty} \varphi_s^{T_j}(z) = \lim_{j \rightarrow +\infty} \int_{-T_j}^{T_j} V(w_j) dx + [w_j]_{H^s(-T_j, T_j)}^2.$$

Applying now Lemma 2.2.7 on any fixed interval $(-R, R)$, we infer that there exists $w \in \dot{H}^s(\mathbb{R})$ such that, up to subsequences, $w_j \rightarrow w$ uniformly on $[-R, R]$ for every $R > 0$; hence,

$$\begin{aligned} \lim_{j \rightarrow +\infty} \varphi_s^{T_j}(z) &= \lim_{j \rightarrow +\infty} \int_{-T_j}^{T_j} V(w_j) dx + [w_j]_{H^s(-T_j, T_j)}^2 \\ &\geq \int_{-R}^R V(w) dx + [w]_{H^s(-R, R)}^2, \end{aligned}$$

and the conclusion follows by the arbitrariness of R .

To prove the last part of the statement, let us consider $\{w_j\}_j \subset \dot{H}^s(\mathbb{R})$ such that $w_j(z) = 0$ for every $j \in \mathbb{N}$ and

$$\varphi_s(z) = \lim_{j \rightarrow +\infty} \int_{-\infty}^{+\infty} V(w_j) dx + [w_j]_{H^s(\mathbb{R})}^2.$$

By the usual compactness argument we obtain w uniform limit on compact sets of $\{w_j\}_j$, and we get

$$\varphi_s(z) \geq \int_{-\infty}^{+\infty} V(w) dx + [w]_{H^s(\mathbb{R})}^2.$$

Since $w(z) = 0$, it follows that $\varphi_s(z)$ admits a minimizer. The lower semicontinuity of φ_s follows analogously.

Finally we observe that, if $\varphi_s(z) = 0$, there exists a minimizer w such that

$$\int_{-\infty}^{+\infty} V(w) dx + [w]_{H^s(\mathbb{R})}^2 = 0,$$

which readily implies $w = 1$ on $\mathbb{R}$, and therefore, $z = 1$. □

Now we turn to the proof of our main result.

Proof of Theorem 2.2.1. We divide the proof into four steps. For the sake of notation, we let

$$\Gamma_\varepsilon := \text{Graph}(u_\varepsilon), \quad \varepsilon > 0,$$

and

$$M := \sup_{\varepsilon} F_\varepsilon(u_\varepsilon) < +\infty. \tag{2.2.7}$$

Step 1. We prove that there exist a subsequence $\{\varepsilon_j\}_j$ tending to 0 and a compact set Γ such that $\Gamma_{\varepsilon_j} \xrightarrow{H} \Gamma$ as $j \rightarrow +\infty$.

We claim that

$$\sup_{\varepsilon} \|u_{\varepsilon}\|_{L^{\infty}(I)} < +\infty. \quad (2.2.8)$$

To see this, we recall that by (V2) it holds

$$V(u_{\varepsilon}) \geq c_1 |u_{\varepsilon} - 1|^2 \quad (2.2.9)$$

if $|u_{\varepsilon} - 1| > c_2$. Consider the open set $\{x \in I : |u_{\varepsilon}(x) - 1| > c_2\}$. If such a set is empty, then $\|u_{\varepsilon}\|_{L^{\infty}(I)} \leq 1 + c_2$. Let us assume that this is not the case. By (2.2.7) and (2.2.9) we have

$$M\varepsilon \geq \int_{\{x \in I : |u_{\varepsilon}(x) - 1| > c_2\}} V(u_{\varepsilon}) dx \geq c_1 c_2^2 |\{x \in I : |u_{\varepsilon}(x) - 1| > c_2\}|$$

and therefore, up to considering ε small enough, $\{x \in I : |u_{\varepsilon}(x) - 1| > c_2\}$ is strictly contained in I . We write the open set $\{x \in I : |u_{\varepsilon}(x) - 1| > c_2\}$ as a countable union of disjoint open intervals

$$\{x \in I : |u_{\varepsilon}(x) - 1| > c_2\} = \bigcup_{i=1}^{+\infty} (a_{\varepsilon}^i, b_{\varepsilon}^i).$$

For every $i \in \mathbb{N}$ we have that at least one of a_{ε}^i and b_{ε}^i is contained in I ; i.e., does not belong to ∂I , and then we have that $|u_{\varepsilon}(a_{\varepsilon}^i) - 1| = c_2$ or $|u_{\varepsilon}(b_{\varepsilon}^i) - 1| = c_2$. We apply Proposition 2.2.4 and use (2.2.9) and (2.2.7) to infer

$$\begin{aligned} \|u_{\varepsilon} - 1\|_{L^{\infty}(a_{\varepsilon}^i, b_{\varepsilon}^i)} &\leq c_2 + \operatorname{osc}_{(a_{\varepsilon}^i, b_{\varepsilon}^i)} |u_{\varepsilon} - 1| \\ &\leq c_2 + C_s^{1/2} \left\{ \frac{1}{\varepsilon} \int_{a_{\varepsilon}^i}^{b_{\varepsilon}^i} |u_{\varepsilon} - 1|^2 dx + \varepsilon^{2s-1} [u_{\varepsilon}]_{H^s(a_{\varepsilon}^i, b_{\varepsilon}^i)}^2 \right\}^{1/2} \\ &\leq c_2 + C_s^{1/2} \left\{ \frac{1}{c_1 \varepsilon} \int_{a_{\varepsilon}^i}^{b_{\varepsilon}^i} V(u_{\varepsilon}) dx + \varepsilon^{2s-1} [u_{\varepsilon}]_{H^s(a_{\varepsilon}^i, b_{\varepsilon}^i)}^2 \right\}^{1/2} \\ &\leq c_2 + (C_s(1 + 1/c_1) F_{\varepsilon}(u_{\varepsilon}))^{1/2} \\ &\leq c_2 + (C_s(1 + 1/c_1) M)^{1/2}. \end{aligned}$$

Therefore, we have

$$\sup_{\varepsilon} \|u_{\varepsilon} - 1\|_{L^{\infty}(\{x \in I : |u_{\varepsilon}(x) - 1| > c_2\})} \leq c_2 + (C_s(1 + 1/c_1) M)^{1/2}$$

and then,

$$\sup_{\varepsilon} \|u_{\varepsilon} - 1\|_{L^{\infty}(I)} \leq c_2 + (C_s(1 + 1/c_1) M)^{1/2}.$$

In turn, this implies

$$\sup_{\varepsilon} \|u_{\varepsilon}\|_{L^{\infty}(I)} \leq 1 + c_2 + (C_s(1 + 1/c_1) M)^{1/2} =: \Lambda, \quad (2.2.10)$$

which proves (2.2.8).

We obtain that

$$\Gamma_{\varepsilon} \subset \bar{I} \times [-\Lambda, \Lambda] \quad (2.2.11)$$

for every $\varepsilon > 0$; hence, there exist a positive sequence $\{\varepsilon_j\}_j$ converging to 0 and a compact set $\Gamma \subseteq \bar{I} \times [-\Lambda, \Lambda]$ such that

$$d_H(\Gamma_{\varepsilon_j}, \Gamma) \rightarrow 0 \quad \text{as } j \rightarrow +\infty,$$

and the first step is concluded.

By (2.2.11), the Hausdorff convergence and the Kuratowski convergence are equivalent. We shall make use of this equivalence in the subsequent steps.

Step 2. We prove that $\bar{I} \times \{1\} \subset \Gamma$ and, as a consequence, that $\Gamma \in \mathcal{B}$.

Since

$$M\varepsilon \geq \int_I V(u_\varepsilon) dx,$$

by (V1) we have that $u_{\varepsilon_j} \rightarrow 1$ in measure; hence, upon extracting a not relabelled subsequence, it is not restrictive to assume that there exists $E \subset I$ such that $|E| = 0$ and $u_{\varepsilon_j}(x) \rightarrow 1$ for every $x \in I \setminus E$. As $(x, u_{\varepsilon_j}(x)) \rightarrow (x, 1)$ for $x \in I \setminus E$, by (K2) we infer that $I \setminus E \times \{1\} \subset \Gamma$. Since Γ is closed and $I \setminus E$ is dense in $\bar{I}$, we obtain that $\bar{I} \times \{1\} \subset \Gamma$. It follows that Γ can be identified with a set-valued function in $\mathcal{B}$ such that $1 \in \Gamma(x)$ for every $x \in \bar{I}$, which is the claim.

Step 3. We prove that $\Gamma(x)$ is a possibly degenerate compact interval for every $x \in \bar{I}$.

If $\Gamma(x) = \{1\}$ the proof is immediate. Fix $x \in \bar{I}$ and suppose that $y_1, y_2 \in \Gamma(x)$ with $y_1 < y_2$. Let $y_1 < y < y_2$. Since $y_1, y_2 \in \Gamma(x)$, the points (x, y_1) and (x, y_2) belong to Γ and therefore, by (K1), there exist sequences $\{(x_j^1, u_{\varepsilon_j}(x_j^1))\}_j$ and $\{(x_j^2, u_{\varepsilon_j}(x_j^2))\}_j$ such that

$$x_j^1, x_j^2 \rightarrow x, \quad u_{\varepsilon_j}(x_j^1) \rightarrow y_1 \quad \text{and} \quad u_{\varepsilon_j}(x_j^2) \rightarrow y_2.$$

Hence, for every j sufficiently large, we have that $u_{\varepsilon_j}(x_j^1) < y < u_{\varepsilon_j}(x_j^2)$ and then there exists a point x_j such that

$$\min\{x_j^1, x_j^2\} \leq x_j \leq \max\{x_j^1, x_j^2\} \quad \text{and} \quad u_j(x_j) = y.$$

As $(x_j, u_{\varepsilon_j}(x_j)) \in \Gamma_{\varepsilon_j}$ for every $j \in \mathbb{N}$ and $(x_j, u_{\varepsilon_j}(x_j)) = (x_j, y) \rightarrow (x, y)$, by (K1) it follows that $(x, y) \in \Gamma$; i.e., $y \in \Gamma(x)$. So $\Gamma(x)$ is an interval, and being Γ compact it follows that $\Gamma(x)$ is compact.

Step 4. Finally, we prove that $\mathcal{E}(\Gamma)$ is a countable set; namely, that $\Gamma(x) = \{1\}$ for all but at most countably many $x \in \bar{I}$.

For every positive integer n , we set

$$\mathcal{E}_n(\Gamma) := \{x \in \bar{I} : \Gamma_+(x) - \Gamma_-(x) > 1/n\}.$$

By definition,

$$\mathcal{E}(\Gamma) = \{x \in \bar{I} : \Gamma_+(x) > \Gamma_-(x)\};$$

and therefore we have

$$\mathcal{E}(\Gamma) = \bigcup_{n=1}^{+\infty} \mathcal{E}_n(\Gamma).$$

The assertion follows if we show that $\mathcal{E}_n(\Gamma)$ is a finite set for all n .

Fix n a positive integer and let $\{z_1, \dots, z_N\}$ be a finite subset of $\mathcal{E}_n(\Gamma)$. It is not restrictive to assume that $N \geq 2$ and that $z_i \notin \partial I$ for every $i \in \{1, \dots, N\}$. Let r be such that

$$r < \frac{1}{2}|z_i - z_h| \quad \text{and} \quad r < \frac{1}{2}\text{dist}(z_i, \partial I) \quad \text{for every } i, h \in \{1, \dots, N\}$$

and define open subintervals of I as

$$J_i := (z_i - r, z_i + r), \quad i \in \{1, \dots, N\}.$$

Note that these intervals are pairwise disjoint and do not intersect ∂I .

Fix $\eta \in (0, 1/2n)$ and $T > 0$ in such a way that, by Proposition 2.2.8, it holds

$$\varphi_s^{T, \eta}(\Gamma_+(z_i)) \geq \frac{1}{2}\varphi_s(\Gamma_+(z_i)) \quad \text{and} \quad \varphi_s^{T, \eta}(\Gamma_-(z_i)) \geq \frac{1}{2}\varphi_s(\Gamma_-(z_i)) \quad (2.2.12)$$

for every $i \in \{1, \dots, N\}$. Assume that j is sufficiently large so that

$$d_H(\Gamma_{\varepsilon_j}, \Gamma) < \min\{r/2, \eta\}.$$

For every $i \in \{1, \dots, N\}$, let $(x_j^{i,+}, u_{\varepsilon_j}(x_j^{i,+})) \in \bar{I} \times \mathbb{R}$ be such that

$$\text{dist}(\Gamma_{\varepsilon_j}, (z_i, \Gamma_+(z_i))) = |(x_j^{i,+}, u_{\varepsilon_j}(x_j^{i,+})) - (z_i, \Gamma_+(z_i))|.$$

We observe that

$$\begin{aligned} \min\{r/2, \eta\} &> d_H(\Gamma_{\varepsilon_j}, \Gamma) \\ &\geq \text{dist}(\Gamma_{\varepsilon_j}, (z_i, \Gamma_+(z_i))) \\ &= |(x_j^{i,+}, u_{\varepsilon_j}(x_j^{i,+})) - (z_i, \Gamma_+(z_i))| \\ &\geq \max\{|x_j^{i,+} - z_i|, |u_{\varepsilon_j}(x_j^{i,+}) - \Gamma_+(z_i)|\}; \end{aligned}$$

therefore,

$$x_j^{i,+} \in \left(z_i - \frac{r}{2}, z_i + \frac{r}{2}\right) \subset J_i, \quad |u_{\varepsilon_j}(x_j^{i,+}) - \Gamma_+(z_i)| < \eta < \frac{1}{2n}. \quad (2.2.13)$$

With an analogous reasoning, we obtain points $x_j^{i,-}$, $j \in \mathbb{N}$, such that

$$x_j^{i,-} \in \left(z_i - \frac{r}{2}, z_i + \frac{r}{2}\right) \subset J_i, \quad |u_{\varepsilon_j}(x_j^{i,-}) - \Gamma_-(z_i)| < \eta < \frac{1}{2n}. \quad (2.2.14)$$

Considering j large enough so that $\varepsilon_j \leq \frac{r}{2T}$, we have that

$$(x_j^{i,\pm} - \varepsilon_j T, x_j^{i,\pm} + \varepsilon_j T) \subset J_i, \quad i \in \{1, \dots, N\}.$$

We let

$$w_{\varepsilon_j}^+(x) := 1 + (u_{\varepsilon_j}(\varepsilon_j x + x_j^{i,+}) - 1)^+, \quad w_{\varepsilon_j}^-(x) := 1 - (u_{\varepsilon_j}(\varepsilon_j x + x_j^{i,-}) - 1)^-$$

and we note that $w_{\varepsilon_j}^{\pm} \in H^s(-T, T)$ and

$$w_{\varepsilon_j}^+(0) = \max\{u_{\varepsilon_j}(x_j^{i,+}), 1\}, \quad w_{\varepsilon_j}^-(0) = \min\{u_{\varepsilon_j}(x_j^{i,-}), 1\}.$$

These observations, combined with (2.2.13) and (2.2.14), yield

$$|w_{\varepsilon_j}^+(0) - \Gamma_+(z_i)| < \eta, \quad |w_{\varepsilon_j}^-(0) - \Gamma_-(z_i)| < \eta.$$

Using Proposition 2.2.6 and a change of variables, for every j sufficiently large we have

$$\begin{aligned} F_{\varepsilon_j}(u_{\varepsilon_j}, J_i) &\geq F_{\varepsilon_j}(1 + (u_{\varepsilon_j} - 1)^+, J_i) + F_{\varepsilon_j}(1 - (u_{\varepsilon_j} - 1)^-, J_i) \\ &\geq F_{\varepsilon_j}(1 + (u_{\varepsilon_j} - 1)^+, (x_j^{i,+} - \varepsilon_j T, x_j^{i,+} + \varepsilon_j T)) \\ &\quad + F_{\varepsilon_j}(1 - (u_{\varepsilon_j} - 1)^-, (x_j^{i,-} - \varepsilon_j T, x_j^{i,-} + \varepsilon_j T)) \\ &= \int_{-T}^T V(w_{\varepsilon_j}^+) dx + [w_{\varepsilon_j}^+]_{H^s(-T, T)}^2 \\ &\quad + \int_{-T}^T V(w_{\varepsilon_j}^-) dx + [w_{\varepsilon_j}^-]_{H^s(-T, T)}^2 \\ &\geq \varphi_s^{T, \eta}(\Gamma_+(z_i)) + \varphi_s^{T, \eta}(\Gamma_-(z_i)) \\ &\geq \frac{1}{2} \varphi_s(\Gamma_+(z_i)) + \frac{1}{2} \varphi_s(\Gamma_-(z_i)), \end{aligned} \quad (2.2.15)$$

where the last inequality follows by (2.2.12).

Since $\Gamma_+(z_i) - \Gamma_-(z_i) > 1/n$, we infer that $\min\{|\Gamma_+(z_i) - 1|, |\Gamma_-(z_i) - 1|\} \geq 1/2n$, hence, we obtain

$$F_{\varepsilon_j}(u_{\varepsilon_j}, J_i) \geq \frac{1}{2} \min\left\{\varphi_s(t) : t \in \left[-\Lambda, 1 - \frac{1}{2n}\right] \cup \left[1 + \frac{1}{2n}, \Lambda\right]\right\} =: \mu_n,$$

with Λ as in (2.2.10). By Proposition 2.2.8, we have that μ_n is a strictly positive constant; therefore,

$$M \geq \sum_{i=1}^N F_{\varepsilon_j}(u_{\varepsilon_j}, J_i) \geq N\mu_n$$

and we conclude that $N \leq M/\mu_n$. Since we proved that the cardinality of any finite subset of $\mathcal{E}_n(\Gamma)$ is bounded by a constant independent of the choice of such a subset, we infer that $\mathcal{E}_n(\Gamma)$ is finite as well, and the proof is complete. $\square$

Through the proof of the compactness, we infer a lower bound for our energies on the interior of the domain I . The behaviour at the boundary points is more delicate and will be included in the final version of our work.

Proposition 2.2.9 (Lower bound on the interior). *Let $\{\varepsilon_j\}_j$ be a positive sequence converging to 0 and let F_{ε_j} be defined by (2.2.2). Then, for every $\Gamma \in \mathcal{B}_c$ and $\{u_j\}_j \subset H^s(I)$ such that $u_j \xrightarrow{g} \Gamma$ it holds*

$$\liminf_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) \geq \sum_{z \in \mathcal{E}(\Gamma) \cap I} (\varphi_s(\Gamma_+(z)) + \varphi_s(\Gamma_-(z))).$$

Proof. Without loss of generality we assume that $\sup_j F_{\varepsilon_j}(u_j)$ is finite. By Step 4 in the proof of Theorem 2.2.1, we have that $\mathcal{E}_n(\Gamma) := \{x \in \bar{I} : \Gamma_+(x) - \Gamma_-(x) > 1/n\}$ is a finite set for every n positive integer. Let then $\mathcal{E}_n(\Gamma) \cap I = \{z_1, \dots, z_N\}$ and consider pairwise disjoint intervals $J_i := (z_i - r, z_i + r)$, $i \in \{1, \dots, N\}$, contained in I that do not intersect ∂I . Following the computations leading to (2.2.15), we infer that

$$F_{\varepsilon_j}(u_j, J_i) \geq \varphi_s^{T,\eta}(\Gamma_+(z_i)) + \varphi_s^{T,\eta}(\Gamma_-(z_i))$$

for fixed $\eta, T > 0$ and j sufficiently large. Then, we obtain

$$\begin{aligned} \liminf_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) &\geq \sum_{i=1}^N \liminf_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j, J_i) \\ &\geq \sum_{i=1}^N (\varphi_s^{T,\eta}(\Gamma_+(z_i)) + \varphi_s^{T,\eta}(\Gamma_-(z_i))) \\ &= \sum_{z \in \mathcal{E}_n(\Gamma) \cap I} (\varphi_s^{T,\eta}(\Gamma_+(z_i)) + \varphi_s^{T,\eta}(\Gamma_-(z_i))), \end{aligned}$$

and letting $\eta \rightarrow 0$ and $T \rightarrow +\infty$ we get

$$\liminf_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) \geq \sum_{z \in \mathcal{E}_n(\Gamma) \cap I} (\varphi_s(\Gamma_+(z_i)) + \varphi_s(\Gamma_-(z_i))).$$

The thesis follows by the arbitrariness of n . □

Finally, we remark that the above argument can be adapted to the case studied in [82] for a first-order singular perturbation. In particular, our lower bound on the interior of I coincides with the optimal one established in that work as we have

$$\varphi_1(z) := \inf \left\{ \int_{-\infty}^{+\infty} [V(w) + |w'|^2] dx : w \in \dot{H}^1(\mathbb{R}), w(0) = z \right\} = 2 \left| \int_z^1 \sqrt{V(x)} dx \right|.$$

2.3 Higher-order singular perturbation models

The aim of this section is to perform the asymptotic analysis of energies

$$\int_{\Omega} \left[\frac{1}{\varepsilon} W(u) + \varepsilon^{2k-1} \|\nabla^{(k)} u\|_k^2 \right] dx,$$

where k is a positive integer and W is a double-well potential that vanishes at $-1, 1$. We shall focus our analysis on the one-dimensional case where $\Omega = I$ is a bounded open interval, so that the above energies are of the form

$$\int_I \left[\frac{1}{\varepsilon} W(u) + \varepsilon^{2k-1} |u^{(k)}|^2 \right] dt.$$

The case $k > 2$ requires a delicate analysis of families $\{u_\varepsilon\}_\varepsilon$ with equi-bounded energy. To this end, as for $k \leq 2$, it is necessary to analyze the property of the sets

$$\{t \in I : |u_\varepsilon(t) - 1| < \eta\}, \quad \{t \in I : |u_\varepsilon(t) + 1| < \eta\};$$

that is, of the set of points with u_ε close to the bottom of the wells. In the case $k = 1$, we can limit the analysis to intervals $(-T, T)$ whose endpoints belong to these sets. The energetic behaviour of the family can then be described through the approximate optimal-profile problems

$$m_1(\eta, T) = \inf \left\{ \int_{-T}^T [W(u) + |u'|^2] dt : u \in H^1(-T, T), |u(-T) + 1| < \eta, |u(T) - 1| < \eta \right\}.$$

Heuristically, once it is observed that for η sufficiently small $m_1(\eta, T) \geq c > 0$ for all T , every transition of u_ε between values close to 1 and values close to -1 or the converse requires a fixed amount of energy, from which we can conclude the pre-compactness of the sequence. By optimizing in T and η we can finally prove the lower estimate with m_1 .

In the case $k = 2$, Fonseca and Mantegazza [74] have noted that it is important to study points with u_ε close to the bottom of the wells and of small derivative. They have studied the set of such points using a bound on the $L^{4/3}$ -norm of u'_ε obtained by an interpolation inequality between the L^1 -norm of u_ε and the L^2 -norm of u''_ε . The properties of such points allow one to carry out the argument above with

$$m_2(\eta, T) = \inf \left\{ \int_{-T}^T [W(u) + |u''|^2] dt : u \in H^2(-T, T), \right. \\ \left. |u(-T) + 1| < \eta, |u(T) - 1| < \eta, |u'(\pm T)| < \eta \right\}$$

in the place of $m_1(\eta, T)$. As it is remarked in [74], this interpolation argument requires at least linear growth for W at infinity, and is not easily extended to $k \geq 3$.

We use a different and direct strategy, which in particular does not use any growth hypothesis on W . Its main point is the analysis of the properties of the sets

$$A_\varepsilon^{\eta,+} = \{t \in I : |u_\varepsilon(t) - 1| < \eta, |u_\varepsilon^{(\ell)}(t)| < \varepsilon^\ell \eta \text{ for all } \ell \leq k-1\},$$

and the corresponding $A_\varepsilon^{\eta,-}$ for which the first condition is replaced by $|u_\varepsilon(t) + 1| < \eta$; that is, of the set of points with u_ε close to the bottom of the wells and with small derivatives up to order $k-1$. The main feature of such sets is that all elements of any family of intervals I_ε of length much larger than ε contain points of either $A_\varepsilon^{\eta,+}$ or $A_\varepsilon^{\eta,-}$. This is obtained by using L^2 estimates on all derivatives $u_\varepsilon^{(\ell)}$ for $\ell \in \{1, \dots, k-1\}$, which are achieved by simultaneously using interpolations between the L^2 -norm of $u_\varepsilon - 1$ (or $u_\varepsilon + 1$) and the L^2 -norm of $u_\varepsilon^{(k)}$. The argument is an adaptation of a technique recently devised by Solci [127] and used to obtain free-discontinuity functionals by singular perturbation, generalizing to higher-order derivatives a result by Alicandro, Braides and Gelli [6] (see also Bouchitté, Dubs and Seppecher [26]). In the present situation the argument is simpler since in the limit we may only have discontinuities with a jump size equal to 2, and not of arbitrary size. Note that the use of the L^2 -norm of $u_\varepsilon \pm 1$ does not require any growth condition on W since it is computed only on intervals in $A_\varepsilon^{\eta,\pm}$, on which we have an L^∞ bound on $u_\varepsilon \pm 1$. The argument for $k \geq 2$ is then carried out by the use of the approximate optimal-profile problems

$$m_k(\eta, T) = \inf \left\{ \int_{-T}^T [W(u) + |u^{(k)}|^2] dt : u \in H^k(-T, T), |u(-T) + 1| < \eta, \right. \\ \left. |u(T) - 1| < \eta, |u^{(\ell)}(\pm T)| < \eta \text{ for all } \ell \leq k-1 \right\},$$

which lead to m_k . An additional issue with respect to the cases $k = 1$ and $k = 2$ here is to show that transitions between the bottom of the wells where we do not have a bound on the derivatives at the endpoints are negligible.

2.3.1 Statement of the results

Let $W : \mathbb{R} \rightarrow [0, +\infty)$ be a continuous function such that $W(-1) = W(1) = 0$ and

$$W(u) \geq \alpha_W \min\{(u+1)^2, (u-1)^2, \beta_W\}$$

for some $\alpha_W, \beta_W > 0$. Let $k \geq 1$ be an integer and $a < b$ be real numbers. For every $\varepsilon > 0$ we define the functional $E_\varepsilon^1 : L^1(a, b) \rightarrow [0, +\infty]$ as

$$E_\varepsilon^1(u) := \begin{cases} \int_a^b \left[\frac{1}{\varepsilon} W(u) + \varepsilon^{2k-1} |u^{(k)}|^2 \right] dt & \text{if } u \in H^k(a, b), \\ +\infty & \text{otherwise.} \end{cases} \quad (2.3.1)$$

We will prove the following compactness and Γ -convergence theorem. The result if $k = 1$ and $k = 2$ has been shown in [103] and in [74], respectively. In the latter a condition of linear growth from below for W is required; our proofs allows to remove that assumption also in the case $k = 2$.

Theorem 2.3.1 (Compactness and Γ -convergence). *Let E_ε^1 be defined by (2.3.1). Then, the family $\{E_\varepsilon^1\}_\varepsilon$ is equi-coercive with respect to the convergence in measure; that is, given any family $\{u_\varepsilon\}_\varepsilon \subset H^k(a, b)$ such that*

$$\sup_\varepsilon E_\varepsilon^1(u_\varepsilon) < +\infty,$$

there exist a positive sequence $\{\varepsilon_j\}_j$ converging to 0 and a function $u \in BV((a, b); \{-1, 1\})$ such that $u_{\varepsilon_j} \rightarrow u$ in measure as $j \rightarrow +\infty$. Moreover, the family $\{E_\varepsilon^1\}_\varepsilon$ Γ -converges with respect to the convergence in measure and in $L^1(a, b)$ to the functional $E^1 : L^1(a, b) \rightarrow [0, +\infty]$ defined as

$$E^1(u) := \begin{cases} m_k \# S(u) & \text{if } u \in BV((a, b); \{-1, 1\}), \\ +\infty & \text{otherwise,} \end{cases} \quad (2.3.2)$$

where

$$m_k = \inf \left\{ \int_{-\infty}^{+\infty} [W(u) + |u^{(k)}|^2] dt : u \in H_{\text{loc}}^k(\mathbb{R}), \lim_{t \rightarrow \pm\infty} u(t) = \pm 1 \right\} \quad (2.3.3)$$

and $S(u)$ denotes the singular set of u .

From the one-dimensional result, we will recover an analog in dimension $d > 1$ by applying blow-up and localization techniques. We consider $\Omega \subset \mathbb{R}^d$ an open and bounded set with Lipschitz boundary, and, for $\varepsilon > 0$, we define the functional $E_\varepsilon : H^k(\Omega) \rightarrow [0, +\infty]$ as

$$E_\varepsilon(u) := \int_\Omega \left[\frac{1}{\varepsilon} W(u) + \varepsilon^{2k-1} \|\nabla^{(k)} u\|_k^2 \right] dx, \quad (2.3.4)$$

where $W : \mathbb{R} \rightarrow [0, +\infty)$ is a function as above and $\|\nabla^{(k)} u\|_k$ denotes the operator norm of the tensor $\nabla^{(k)} u$. In accordance with Theorem 2.3.1, the domain of the Γ -limit will be the space $BV(\Omega; \{-1, 1\})$.

Theorem 2.3.2. *Let E_ε be defined by (2.3.4). Then, the family $\{E_\varepsilon\}_\varepsilon$ Γ -converges with respect to the convergence in $L^1(\Omega)$ to the functional $E : BV(\Omega; \{-1, 1\}) \rightarrow [0, +\infty)$ defined as*

$$E(u) := m_k \mathcal{H}^{d-1}(S(u)) = m_k P(\{u = 1\}; \Omega),$$

where m_k is as defined in (2.3.3), $S(u)$ denotes the singular set of u , and $P(\{u = 1\}; \Omega)$ denotes the perimeter of the set $\{u = 1\}$ in Ω .

Remark 2.3.3. If we strengthen the growth assumptions on the double-well potential W , we may deduce that a sequence $\{u_\varepsilon\}_\varepsilon$ such that $\sup_\varepsilon E_\varepsilon(u_\varepsilon) < +\infty$ is strongly precompact in $L^1(\Omega)$. This fact shall be proved in the subsequent section (see Proposition 2.4.18) and follows by a result of Alberti, Bouchitté, and Seppecher [4, Theorem 6.6].

To prove the results, it is useful to introduce the one-dimensional localized functionals

$$E_\varepsilon^1(u, I) = \int_I \left[\frac{1}{\varepsilon} W(u) + \varepsilon^{2k-1} |u^{(k)}|^2 \right] dt$$

for any interval $I \subset (a, b)$. Finally, to simplify the notation, in the following we only consider $(a, b) = (0, 1)$, the changes in the proofs for a general interval being obvious.

2.3.2 Analysis of oscillations close to 1 and -1

In the proofs of the compactness and of the lower bound, we will make use of some optimal-profile problems in order to estimate the number of transitions between wells of sequence of functions with equi-bounded energy. In particular, we will make use of the family of minimum problems given by

$$\begin{aligned} m_k(\eta, N, T) := \inf \Big\{ & \int_{-T}^T [W(u) + |u^{(k)}|^2] dt : u \in H^k(-T, T), \\ & |u(-T) + 1| \leq \eta, |u(T) - 1| \leq \eta, \\ & |u^{(\ell)}(\pm T)| \leq \frac{1}{N} \text{ for all } \ell \in \{1, \dots, k-1\} \Big\}. \end{aligned} \quad (2.3.5)$$

In this subsection, we will study the properties of intervals of points where functions u_ε with equi-bounded energy are close to either -1 or 1 , and their derivatives up to order $k-1$ are small. We will show that every sufficiently large interval (with respect to ε) intersects such sets. The technical tool used to prove this result is the simultaneous application of interpolation inequalities for all derivatives in the intervals where u_ε is close to either -1 or 1 .

Let $\{u_\varepsilon\}_\varepsilon$ be a family of $H^k(0, 1)$ functions with equi-bounded energy, and let

$$S := \sup_\varepsilon E_\varepsilon^1(u_\varepsilon). \quad (2.3.6)$$

For any fixed $\eta \in (0, 1)$ and $\varepsilon > 0$, we define the set

$$A_\varepsilon^\eta = \{t \in (0, 1) : |u_\varepsilon(t) - 1| < \eta\}. \quad (2.3.7)$$

Recalling that $W(z) \geq \alpha_W \min\{(z-1)^2, (z+1)^2, \beta_W\}$ for some $\alpha_W, \beta_W > 0$, we will consider $\eta \in (0, \sqrt{\beta_W})$, so that $W(z) \geq \alpha_W(z-1)^2$ if $|z-1| \leq \eta$ and $W(z) \geq \alpha_W(z+1)^2$ if $|z+1| \leq \eta$.

We now fix an integer $N \geq 1$, and prove an upper estimate for the length of an interval I such that at all points of $I \cap A_\varepsilon^\eta$ there exists at least one $\ell \in \{1, \dots, k-1\}$ such that $|u_\varepsilon^{(\ell)}| \geq \frac{1}{N\varepsilon^\ell}$.

Lemma 2.3.4. *Let $\eta \in (0, \sqrt{\beta_W})$ and $N \geq 1$ be fixed, and let $\{u_\varepsilon\}_\varepsilon$ be such that (2.3.6) holds. Then, there exists a constant $R_k(\eta, N, S) > 0$ such that, for all $\varepsilon > 0$ and for any interval $I \subseteq (0, 1)$ satisfying the hypotheses:*

- (i) $I \subset A_\varepsilon^\eta$;
- (ii) $\left| \left\{ t \in I : |u_\varepsilon^{(\ell)}(t)| < \frac{1}{N\varepsilon^\ell} \text{ for all } \ell \in \{1, \dots, k-1\} \right\} \right| = 0$;

it follows that

$$|I| \leq R_k(\eta, N, S)\varepsilon.$$

Proof. The proof is based on a classical interpolation estimate. There exists a constant $r_k > 0$ such that for any interval I and for any $v \in H^k(I)$ the following estimate holds

$$\|v^{(\ell)}\|_{L^2(I)} \leq r_k \left(\|v\|_{L^2(I)}^\theta \|v^{(k)}\|_{L^2(I)}^{1-\theta} + |I|^{-\ell} \|v\|_{L^2(I)} \right) \quad (2.3.8)$$

for all $\ell \in \{1, \dots, k-1\}$, with $\theta = \frac{k-\ell}{k}$ (see e. g. [96, Theorem 7.41]). Setting

$$a = \varepsilon^{-\theta} \|v\|_{L^2(I)}^{2\theta} \quad \text{and} \quad b = \varepsilon^{(2k-1)(1-\theta)} \|v^{(k)}\|_{L^2(I)}^{2(1-\theta)},$$

by using the convexity inequality $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$ with $p = \frac{1}{\theta}$ and $q = \frac{1}{1-\theta}$, we then deduce that there exists a constant $R_k > 0$ such that

$$\varepsilon^{2\ell-1} \|v^{(\ell)}\|_{L^2(I)}^2 \leq R_k \left(\frac{1}{\varepsilon} \|v\|_{L^2(I)}^2 + \varepsilon^{2k-1} \|v^{(k)}\|_{L^2(I)}^2 + \frac{\varepsilon^{2\ell-1}}{|I|^{2\ell}} \|v\|_{L^2(I)}^2 \right). \quad (2.3.9)$$

Let now I be as in the hypothesis. It is not restrictive to suppose that $|u_\varepsilon - 1| \leq \eta$ in I , the case $|u_\varepsilon + 1| \leq \eta$ being completely analogous. Choosing $v = u_\varepsilon - 1$ in (2.3.9), and recalling that $W(u_\varepsilon) \geq \alpha_W(u_\varepsilon - 1)^2$ in the interval I , we get

$$\varepsilon^{2\ell-1} \|u_\varepsilon^{(\ell)}\|_{L^2(I)}^2 \leq \tilde{R}_k \left(E_\varepsilon^1(u_\varepsilon, I) + \varepsilon^{2\ell-1} |I|^{-2\ell} \|u_\varepsilon - 1\|_{L^2(I)}^2 \right), \quad (2.3.10)$$

for any $\varepsilon > 0$, where $\tilde{R}_k = R_k \max\{\frac{1}{\alpha_W}, 1\}$.

Note that hypothesis (ii) implies that for at least one $\ell \in \{1, \dots, k-1\}$

$$\left| \left\{ t \in I : |u_\varepsilon^{(\ell)}(t)| \geq \frac{1}{N\varepsilon^\ell} \right\} \right| \geq \frac{|I|}{k+1}.$$

By using this lower estimate in (2.3.10), recalling that $(u_\varepsilon - 1)^2 \leq \eta^2$ and that $E_\varepsilon^1(u_\varepsilon, I) \leq S$ we obtain

$$\varepsilon^{-1} \frac{1}{(k+1)N^2} |I| \leq \tilde{R}_k \left(S + \eta^2 \varepsilon^{2\ell-1} |I|^{1-2\ell} \right)$$

for any $\varepsilon > 0$. We then conclude that

$$|I| \leq \tilde{R}_k(k+1)N^2 S \varepsilon + \tilde{R}_k(k+1)N^2 \eta^2 \varepsilon^{2\ell} |I|^{1-2\ell}.$$

Hence, if $|I| > 2\tilde{R}_k(k+1)N^2 S \varepsilon$, it follows that

$$|I| \leq 2\tilde{R}_k(k+1)N^2 \eta^2 \varepsilon^{2\ell} |I|^{1-2\ell},$$

so that $|I|^{2\ell} \leq 2\tilde{R}_k(k+1)N^2 \eta^2 \varepsilon^{2\ell}$. Hence,

$$|I| \leq \max \left\{ 2\tilde{R}_k(k+1)N^2 S, \max_{1 \leq \ell \leq k-1} \left\{ (2\tilde{R}_k(k+1)N^2 \eta^2)^{\frac{1}{2\ell}} \right\} \right\} \varepsilon.$$

With a little abuse of notation, by letting $R_k(\eta, N, S)$ denote the constant in this inequality, we obtain the claim. $\square$

We set

$$A_\varepsilon^\eta(N) = \left\{ t \in A_\varepsilon^\eta : |u_\varepsilon^{(\ell)}(t)| < \frac{1}{N\varepsilon^\ell} \text{ for all } \ell \in \{1, \dots, k-1\} \right\}, \quad (2.3.11)$$

where we recall that A_ε^η is defined in (2.3.7) as the set of t such that $||u_\varepsilon(t)| - 1| \leq \eta$. The sets $A_\varepsilon^\eta(N)$ describe the regions close to $\{\pm 1\}$ where all derivatives of u_ε up to $k-1$ are “small”. We now show that points in such sets cannot have mutual distance much larger than ε .

Lemma 2.3.5. *Let $\eta \in (0, \sqrt{\beta_W})$, $N \geq 1$ and let $\{u_\varepsilon\}_\varepsilon$ be such that (2.3.6) holds. Let $\{\varepsilon_j\}_j$ be a positive sequence converging to 0, and let $\{I_{\varepsilon_j}\}_j$ be a family of subintervals of $(0, 1)$ such that*

$$\lim_{j \rightarrow +\infty} \frac{|I_{\varepsilon_j}|}{\varepsilon_j} = +\infty.$$

Then, there exists $j_0 \in \mathbb{N}$ such that

$$|I_{\varepsilon_j} \cap A_{\varepsilon_j}^\eta(N)| > 0$$

for all $j \geq j_0$.

Proof. Let $\{I_{\varepsilon_j}\}_j$ be the family of intervals in the hypothesis. By contradiction, we suppose that $|I_{\varepsilon_j} \cap A_{\varepsilon_j}^\eta(N)| = 0$ for a (not relabelled) subsequence $\{\varepsilon_j\}_j$ converging to 0 as $j \rightarrow +\infty$. The proof of the contradiction is divided in two steps.

Step 1. Let j be fixed, and let $J \subset I_{\varepsilon_j}$ be an interval such that

$$|J| \geq \varepsilon_j \left(\frac{2S}{\alpha_W \eta^2} + (2k+5)R_k(\eta, N, S) \right),$$

where $R_k(\eta, N, S) > 0$ is the constant given in Lemma 2.3.4. Then for all $\ell \in \{1, \dots, k-1\}$ there exists a point in J where $u_{\varepsilon_j}^{(\ell)}$ vanishes. This can be proved as follows.

We first note that $|\{|u_{\varepsilon_j}| - 1| \geq \eta\}| \leq \varepsilon_j \frac{S}{\alpha_W \eta^2}$. Since $|I_{\varepsilon_j} \cap A_{\varepsilon_j}^\eta(N)| = 0$, we can apply Lemma 2.3.4 deducing that J contains at least $2k+3$ intervals that are well contained in $(0, 1)$ and such that in each one of them $|u_{\varepsilon_j} - 1| < \eta$, with the equality holding at the endpoints, or $|u_{\varepsilon_j} + 1| < \eta$, again

with the equality holding at the endpoints. Let $\{I_n = (a_n, b_n)\}_{n=1}^{2k+3}$ denote the set of these intervals, with a_n and b_n increasing.

Now, we show that in each interval (a_n, a_{n+2}) with $n \leq 2k+1$ there exists at least a point where u'_{ε_j} vanishes. Let $n \leq 2k+1$ and suppose that $|u_{\varepsilon_j} - 1| < \eta$ in I_n . If $u'_{\varepsilon_j} \neq 0$ in the whole I_n , then u_{ε_j} is strictly monotone. We first consider the case u_{ε_j} strictly increasing in I_n , with $u_{\varepsilon_j}(b_n) = 1 + \eta$. If $u'_{\varepsilon_j}(b_n) = 0$ the claim is proved. If $u'_{\varepsilon_j}(b_n) \neq 0$, we note that u_{ε_j} cannot be strictly increasing in the interval (b_n, a_{n+1}) since the value $u_{\varepsilon_j}(a_{n+1})$ is at most $1 + \eta$; then, there exists a local maximizer in (b_n, a_{n+1}) , and the claim follows. If u_{ε_j} is strictly decreasing in I_n , we note that u_{ε_j} cannot be decreasing in the interval (b_n, a_{n+2}) ; indeed, in this case $u_{\varepsilon_j}(b_n) = 1 - \eta$, and if u_{ε_j} is strictly decreasing in (b_n, b_{n+1}) , then $u_{\varepsilon_j}(a_{n+1}) = -1 + \eta$ and $u_{\varepsilon_j}(b_{n+1}) = -1 - \eta$. Since $u_{\varepsilon_j}(a_{n+2}) \geq -1 - \eta$, this shows that u_{ε_j} can not be strictly decreasing in both (b_n, b_{n+1}) and (b_{n+1}, a_{n+2}) . Then, $[b_n, a_{n+2})$ contains a local minimizer, proving the claim. The corresponding argument applies if $|u_{\varepsilon_j} + 1| < \eta$ in I_n .

Hence, there exist at least k internal points in J where $u'_{\varepsilon_j} = 0$. By an iteration of the application of Lagrange Theorem, this implies that for each $\ell \in \{1, \dots, k-1\}$ there exists at least a point where $u_{\varepsilon_j}^{(\ell)} = 0$.

Step 2. Let N_j be the integer part of $\frac{|I_{\varepsilon_j}|}{\varepsilon_j L}$, where

$$L = L_k(\eta, N, S) = \frac{2S}{\alpha_W \eta^2} + (2k+5)R_k(\eta, N, S),$$

so that each interval I_{ε_j} contains N_j disjoint intervals with length $\varepsilon_j L$ and by hypothesis $N_j \rightarrow +\infty$. Then, for any j there exists one of these intervals, denoted by $J_j = (a_j, a_j + \varepsilon_j L)$, such that

$$E_{\varepsilon_j}^1(u_{\varepsilon_j}, J_j) \leq \frac{S}{N_j}.$$

We can then write

$$\begin{aligned} \frac{S}{N_j} &\geq \int_{a_j}^{a_j + \varepsilon_j L} \left[\frac{1}{\varepsilon_j} W(u_{\varepsilon_j}) + \varepsilon_j^{2k-1} |u_{\varepsilon_j}^{(k)}|^2 \right] dt \\ &= \int_0^L [W(v_j) + |v_j^{(k)}|^2] ds, \end{aligned} \quad (2.3.12)$$

where $v_j(t) = u_{\varepsilon_j}(\varepsilon_j t + a_j)$. By Step 1, for any $\ell \in \{1, \dots, k-1\}$ there exists a point t_j^ℓ in J_j where $u_{\varepsilon_j}^{(\ell)}$ vanishes, hence $v_j^{(\ell)} = 0$ in $s_j^\ell = \frac{t_j^\ell - a_j}{\varepsilon_j} \in (0, L)$. An iterated application of the Fundamental Theorem of Calculus then gives

$$\|v_j\|_{H^k(0,L)} \leq C(L) \left(\min_{[0,L]} |v_j|^2 + \|v_j^{(k)}\|_{L^2(0,L)}^2 \right),$$

where $C(L) > 0$ depends on L (and then on η, N, S and k). Moreover, note that there exists at least a point in J_j such that $|u_{\varepsilon_j} - 1| = \eta$, and the same holds for v_j at a point in $(0, L)$. Then, since $N_j \rightarrow +\infty$, by (2.3.12) we deduce that v_j is equi-bounded in $H^k(0, L)$, thus $v_j \rightharpoonup v$ in $H^k(0, L)$ as $j \rightarrow +\infty$. Since $v_j \rightarrow v$ uniformly, by (2.3.12) we deduce that

$$0 = \liminf_{j \rightarrow +\infty} \int_0^L W(v_j) ds \geq \int_0^L W(v) ds,$$

so that either $v = 1$ in $(0, L)$ or $v = -1$ in $(0, L)$. This gives a contradiction since for any j there exists a point $s_j \in (0, L)$ such that $||v_{\varepsilon_j}(s_j)| - 1| = \eta$. $\square$

2.3.3 Optimal-profile problems

In the proof of the equi-coerciveness of the family $\{E_\varepsilon^1\}_\varepsilon$ and of the lower estimate for the Γ -limit, we will make use of the optimal-profile problem

$$m_k = \inf \left\{ \int_{-\infty}^{+\infty} [W(u) + |u^{(k)}|^2] dt : u \in H_{\text{loc}}^k(\mathbb{R}), \lim_{t \rightarrow \pm\infty} u(t) = \pm 1 \right\}.$$

It is convenient to interpret this optimal-profile problem as an optimization on intervals and functions; to this end, we define

$$\begin{aligned}\tilde{m}_k &:= \inf_{T>0} \inf \left\{ \int_{-\infty}^{+\infty} [W(u) + |u^{(k)}|^2] dt : u \in H_{\text{loc}}^k(\mathbb{R}), \right. \\ &\quad \left. u(t) = -1 \text{ for } t \leq -T, u(t) = 1 \text{ for } t \geq T \right\} \\ &= \inf_{T>0} m_k(T) = \lim_{T \rightarrow +\infty} m_k(T),\end{aligned}\tag{2.3.13}$$

where

$$\begin{aligned}m_k(T) &:= \inf \left\{ \int_{-T}^T [W(u) + |u^{(k)}|^2] dt : u \in H^k(-T, T), u(\pm T) = \pm 1, \right. \\ &\quad \left. u^{(\ell)}(\pm T) = 0 \text{ for all } \ell \in \{1, \dots, k-1\} \right\}.\end{aligned}$$

As a consequence of Lemma 2.3.5, we can prove that these two definitions coincide and that the resulting infima are strictly positive.

Proposition 2.3.6. *The equality $m_k = \tilde{m}_k$ holds.*

Proof. Since all test functions for $\tilde{m}_k$ are test functions for m_k , one inequality is trivial.

We then consider a function $u \in H_{\text{loc}}^k(\mathbb{R})$ such that

$$\lim_{t \rightarrow \pm\infty} u(t) = \pm 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} [W(u) + |u^{(k)}|^2] dt < +\infty.$$

With fixed η , N and $\{\varepsilon_j\}_j$ vanishing, we apply Lemma 2.3.5 twice, to the sequences $u_j(t) = u(\frac{t-1}{\varepsilon_j})$ and $u_j(t) = u(\frac{t}{\varepsilon_j})$, respectively, and with $I_{\varepsilon_j} = (0, \frac{1}{2})$ and $I_{\varepsilon_j} = (\frac{1}{2}, 1)$, respectively. In terms of u , we deduce that there are $t_j^- \in (-\frac{1}{\varepsilon_j}, 0)$ and $t_j^+ \in (0, \frac{1}{\varepsilon_j})$ such that

$$|u(t_j^-) + 1| < \eta_j, \quad |u(t_j^+) - 1| < \eta_j, \quad \text{and} \quad |u^{(\ell)}(t_j^\pm)| \leq \frac{1}{N_j} \quad \text{for all } \ell \in \{1, \dots, k-1\},$$

where we chose $\eta_j \rightarrow 0$ and $N_j \rightarrow +\infty$.

We want to extend the restriction of the test function u to (t_j^-, t_j^+) in order to obtain an admissible test function u_j for $\tilde{m}_k$ with an energy not greater than that of u up to an infinitesimal term. To this end, we consider a family of auxiliary minimum problems defined as follows. Given z_j^ℓ for $\ell \in \{0, \dots, k-1\}$, let v_j be a minimizer of the problem

$$\theta_j = \min \left\{ \int_0^1 |v^{(k)}|^2 dt : v^{(\ell)}(0) = z_j^\ell, v^{(\ell)}(1) = 0 \text{ for all } \ell \in \{0, \dots, k-1\} \right\},$$

where we used the notation $v^{(0)} = v$. By choosing $z_j^0 = u(t_j^+) - 1$, $z_j^\ell = u^{(\ell)}(t_j^+)$ for $\ell \in \{1, \dots, k-1\}$, we have $\theta_j \rightarrow 0$, and moreover, by the uniform convergence of the corresponding minimizers v_j to 0, we also have

$$\lim_{j \rightarrow +\infty} \int_0^1 W(v_j(t) + 1) dt = 0.$$

We then define u_j^+ on $(t_j^+, +\infty)$ as

$$u_j^+(t) = \begin{cases} 1 + v_j(t - t_j^+) & \text{if } t \in (t_j^+, t_j^+ + 1), \\ 1 & \text{if } t \geq t_j^+ + 1. \end{cases}$$

Note that

$$\int_{t_j^+}^{+\infty} [W(u_j^+) + |(u_j^+)^{(k)}|^2] dt = \theta_j + \int_0^1 W(v_j(t) + 1) dt = o_j(1)$$

as $j \rightarrow +\infty$. We define symmetrically u_j^- on $(-\infty, t_j^-)$, so that the functions

$$u_j(t) = \begin{cases} u_j^-(t) & \text{if } t \leq t_j^-, \\ u(t) & \text{if } t_j^- < t < t_j^+, \\ u_j^+(t) & \text{if } t \geq t_j^+ \end{cases}$$

are test functions for $\tilde{m}_k$ and

$$\tilde{m}_k \leq \int_{-\infty}^{+\infty} [W(u_j) + |u_j^{(k)}|^2] dt \leq \int_{-\infty}^{+\infty} [W(u) + |u^{(k)}|^2] dt + o_j(1)$$

as $j \rightarrow +\infty$. Hence,

$$\tilde{m}_k \leq \int_{-\infty}^{+\infty} [W(u) + |u^{(k)}|^2] dt.$$

By the arbitrariness of u this proves the claim. $\square$

Proposition 2.3.7. *It holds that $m_k > 0$.*

Proof. Since $m_k = \tilde{m}_k$, we equivalently prove $\tilde{m}_k > 0$.

Let $\{v_j\}$ be a sequence of admissible functions for $\tilde{m}_k$ such that

$$\lim_{j \rightarrow +\infty} \int_{-\infty}^{+\infty} [W(v_j) + |v_j^{(k)}|^2] dt = \tilde{m}_k.$$

For each $j \geq 1$, let T_j be a positive number such that $v_j(t) = -1$ for $t < -T_j$ and $v_j(t) = 1$ for $t > T_j$.

In view of the application of Lemma 2.3.5, we fix $\eta < \min\{1, \sqrt{\beta_W}\}$, and consider the family of open subintervals (a, b) of $(-T_j - 1, T_j + 1)$ such that $|v_j(a) + 1| \leq \eta$, $|v_j(b) - 1| \leq \eta$, and the derivatives of v_j at the endpoints satisfy $|v_j^{(\ell)}| \leq 1$ for all $\ell \in \{1, \dots, k-1\}$. Let (a_j, b_j) be a minimal such interval; that is, any subinterval of (a_j, b_j) does not belong to this family. Hence, (a_j, b_j) does not intersect the set of the points t such that $||v_j(t)| - 1| \leq \eta$ and $|v_j^{(\ell)}(t)| \leq 1$ for all $\ell \in \{1, \dots, k-1\}$.

In order to apply Lemma 2.3.5 to prove that $b_j - a_j$ is equi-bounded, we use the scaling factor $\varepsilon_j = \frac{1}{2T_j}$ to define $u_j(t) = v_j(\frac{t}{\varepsilon_j} - T_j)$. We then have

$$\int_0^1 \left[\frac{1}{\varepsilon_j} W(u_j) + \varepsilon_j^{2k-1} |u_j^{(k)}|^2 \right] dt = \int_{-T_j}^{T_j} [W(v_j) + |v_j^{(k)}|^2] dt,$$

which is equi-bounded. Assume by contradiction that there exists a (not relabelled) subsequence $b_j - a_j \rightarrow +\infty$. Then, we can apply Lemma 2.3.5 to the sequence of scaled intervals $(a_j \varepsilon_j, b_j \varepsilon_j)$ and deduce that there exists $j_0 \in \mathbb{N}$ such that

$$|(a_j \varepsilon_j, b_j \varepsilon_j) \cap A_{\varepsilon_j}^\eta(1)| > 0$$

for all $j \geq j_0$, where we recall that $A_\varepsilon^\eta(N)$ is defined as in (2.3.11). By scaling again to $(-T_j, T_j)$, this gives a contradiction.

Up to subsequences and up to translations, we can then assume that there exists $T^* > 0$ such that $a_j \rightarrow -T^*$ and $b_j \rightarrow T^*$ as $j \rightarrow +\infty$. Up to an asymptotically negligible change of variable, it is not restrictive to assume that $a_j = -T^*$ and $b_j = T^*$. By the equi-boundedness of the boundary condition and by the Fundamental Theorem of Calculus we deduce that $\{v_j\}$ is equi-bounded in $H^k(-T^*, T^*)$. Hence, we extract a (not relabeled) subsequence $\{v_j\}$ weakly converging to a function v in $H^k(-T^*, T^*)$.

Assume now by contradiction that $\tilde{m}_k = 0$. Since $\{v_j\}$ is a minimizing sequence, by the lower semicontinuity of the L^2 -norm, we have

$$0 = \lim_{j \rightarrow +\infty} \int_{-T^*}^{T^*} [W(v_j) + |v_j^{(k)}|^2] dt \geq \liminf_{j \rightarrow +\infty} \int_{-T^*}^{T^*} |v_j^{(k)}|^2 dt \geq \int_{-T^*}^{T^*} |v^{(k)}|^2 dt.$$

Hence, we obtain that $v^{(k)} = 0$ on $(-T^*, T^*)$, so that v is a polynomial of degree at most $k - 1$. Moreover, by the uniform convergence of $v_j \rightarrow v$, we have

$$\int_{-T^*}^{T^*} W(v) dt = \lim_{j \rightarrow +\infty} \int_{-T^*}^{T^*} W(v_j) dt = 0,$$

so that $W(v) = 0$ almost everywhere, and then, since $\{W = 0\} = \{-1, 1\}$, we deduce that v is either identically -1 or 1 on $(-T^*, T^*)$. Since $|v(-T^*) + 1| \leq \eta$ and $|v(T^*) - 1| \leq \eta$, and $\eta < 1$, this gives a contradiction, and the proof is complete. $\square$

As it is usual in phase-transition problems, we will need some auxiliary results involving optimal profile problems approximating m_k defined in through the minimization of $\int_{I_T} (W(v) + (v^{(k)})^2) dt$, where I_T is an interval with length $2T$ and v in $H^k(I_T)$ satisfies suitable boundary conditions; in particular, we will minimize over sets of functions with “small” values of the derivatives at the end-points of the interval. To this end, for all $\eta \in (0, \sqrt{\beta_W})$ and for all integers $N \geq 1$, we define an approximating constant given by

$$m_k(\eta, N) := \inf_{T > 0} m_k(\eta, N, T), \quad (2.3.14)$$

where $m_k(\eta, N, T)$ is the minimum defined in (2.3.5).

Lemma 2.3.8. *Let $\tilde{m}_k$ and $m_k(\eta, N)$ be defined by (2.3.13) and (2.3.14), respectively. Then*

$$\lim_{\eta \rightarrow 0} \lim_{N \rightarrow +\infty} m_k(\eta, N) = \tilde{m}_k.$$

Proof. Since $m_k(\eta, N)$ is increasing as $\eta \rightarrow 0$ and as $N \rightarrow +\infty$, the limits exist. For all $T > 0$, η and N , we have $m_k(T) \geq m_k(\eta, N, T)$, so that $\tilde{m}_k \geq m_k(\eta, N)$ for all η and N . Hence,

$$\tilde{m}_k \geq \lim_{\eta \rightarrow 0} \lim_{N \rightarrow +\infty} m_k(\eta, N).$$

We now prove the converse inequality. Let $\eta_j \rightarrow 0$ and $N_j \rightarrow +\infty$ be such that

$$\lim_{j \rightarrow +\infty} m_k(\eta_j, N_j) = \lim_{\eta \rightarrow 0} \lim_{N \rightarrow +\infty} m_k(\eta, N).$$

For any j , let $T_j > 0$ and $v_j \in H^k(-T_j, T_j)$ be an admissible test function for $m_k(\eta_j, N_j, T_j)$ such that

$$\lim_{j \rightarrow +\infty} m_k(\eta_j, N_j) = \lim_{j \rightarrow +\infty} \int_{-T_j}^{T_j} [W(v_j) + |v_j^{(k)}|^2] dt.$$

To define an admissible test function u_j for $\tilde{m}_k$, we can proceed as in the proof of Proposition 2.3.6 with $-T_j$ in the place of t_j^- , T_j in the place of t_j^+ , and v_j in the place of the restriction of u to (t_j^-, t_j^+) .

In this way, the resulting u_j is a test function for $\tilde{m}_k$ and

$$\lim_{j \rightarrow +\infty} m_k(\eta_j, N_j) = \lim_{j \rightarrow +\infty} \int_{-\infty}^{+\infty} [W(u_j) + |u_j^{(k)}|^2] dt \geq \tilde{m}_k,$$

proving the equality

$$\lim_{\eta \rightarrow 0} \lim_{N \rightarrow +\infty} m_k(\eta, N) = \tilde{m}_k,$$

and the claim. $\square$

We introduce some quantities that will allow us to estimate oscillations of order η . For all $0 < \eta < \min\{1, \sqrt{\beta_W}\}$, integers $N \geq 1$ and $\bar{T} > 0$, we define

$$\begin{aligned} m_k^+(\eta, N, \bar{T}) &= \inf_{0 < T \leq \bar{T}} \min \left\{ \int_0^T [W(u) + |u^{(k)}|^2] dt : u \in H^k(0, T), \right. \\ &\quad |u(0) - 1| \leq \eta, |u(T) - 1| = 2\eta, \\ &\quad \left. \text{and } |u^{(\ell)}(0)| \leq \frac{1}{N} \text{ for all } \ell \in \{1, \dots, k-1\} \right\}, \end{aligned} \quad (2.3.15)$$

and

$$\begin{aligned} m_k^-(\eta, N, \bar{T}) &= \inf_{0 < T \leq \bar{T}} \min \left\{ \int_0^T [W(u) + |u^{(k)}|^2] dt : u \in H^k(0, T), \right. \\ &\quad |u(0) + 1| \leq \eta, |u(T) + 1| = 2\eta, \\ &\quad \left. \text{and } |u^{(\ell)}(0)| \leq \frac{1}{N} \text{ for all } \ell \in \{1, \dots, k-1\} \right\}. \end{aligned}$$

Lemma 2.3.9. *For every $\bar{T} > 0$, it holds*

$$m_k^*(\eta, N, \bar{T}) := \min\{m_k^+(\eta, N, \bar{T}), m_k^-(\eta, N, \bar{T})\} > 0.$$

Proof. We only prove that $m_k^+(\eta, N, \bar{T}) > 0$, the other case being analogous. We argue by contradiction.

Suppose that there exists a sequence $\{T_j\}_j \subset (0, \bar{T}]$ and functions $v_j \in H^k(0, T_j)$ which are admissible for (2.3.15) and such that

$$\lim_{j \rightarrow +\infty} \int_0^{T_j} [W(v_j) + |v_j^{(k)}|^2] dt = 0. \quad (2.3.16)$$

Without loss of generality, we may suppose that $T_j \rightarrow T \in [0, \bar{T}]$ as $j \rightarrow +\infty$. We distinguish the two cases $T = 0$ and $T > 0$.

Case 1: ($T = 0$). By (2.3.16) it holds $\|v_j^{(k)}\|_{L^2(0, T_j)} \rightarrow 0$ as $j \rightarrow +\infty$; hence, recalling that v_j satisfies the boundary conditions appearing in (2.3.15), we apply the Fundamental Theorem of Calculus to deduce that $\|v_j\|_{H^k(0, T_j)} \leq C$ for every j . We then extend each function v_j to the whole interval $(0, 1)$ by setting it equal to p_j , the polynomial of degree at most $k-1$ obtained by imposing $p_j^{(\ell)}(T_j) = v_j^{(\ell)}(T_j)$ for $\ell \in \{0, \dots, k-1\}$. The sequence $\{v_j\}_j$ is then equi-bounded in $H^k(0, 1)$ so that there exists a subsequence, not relabelled, which converges uniformly to some $v \in H^k(0, 1)$. This produces a contradiction since $T_j \rightarrow 0$ and $|v_j(T_j) - v_j(0)| \geq \eta$.

Case 2: ($T > 0$). Up to an asymptotically negligible change of variable we may suppose $T_j = T$ for every j . Then, by (2.3.16), we have that $\{v_j\}_j$ is equi-bounded in $H^k(0, T)$ so that a certain subsequence converges uniformly on $[0, T]$ to a function v in $H^k(0, T)$. Again by (2.3.16) and by the lower semicontinuity of the L^2 -norm, we have that either $v = -1$ or $v = 1$ on $(0, T)$. But $|v_j(T_j) - 1| = 2\eta$ for every j , hence, $|v(T) - 1| = 2\eta$, which is a contradiction since $\eta > 0$. $\square$

2.3.4 Proof of the main result in dimension 1

We divide the proof of Theorem 2.3.1 in proving separately compactness, lower and upper bounds. We start by proving the equi-coerciveness of the family $\{E_\varepsilon^1\}_\varepsilon$.

Proposition 2.3.10 (Compactness). *Let E_ε^1 be defined by (2.3.1). Then, the family $\{E_\varepsilon^1\}_\varepsilon$ is equi-coercive with respect to the convergence in measure; that is, given any family $\{u_\varepsilon\}_\varepsilon \subset H^k(a, b)$ such that*

$$\sup_\varepsilon E_\varepsilon^1(u_\varepsilon) < +\infty,$$

there exist a positive sequence $\{\varepsilon_j\}_j$ converging to 0 and a function $u \in BV((a, b); \{-1, 1\})$ such that $u_{\varepsilon_j} \rightarrow u$ in measure as $j \rightarrow +\infty$.

Proof. Up to subsequences, we may assume that $\sup_\varepsilon E_\varepsilon^1(u_\varepsilon) = S < +\infty$. We consider for any fixed $0 < \eta < \min\{1, \beta_W\}$ and $\varepsilon > 0$ the countable sets of indices $\mathcal{I}_\varepsilon^{1, \eta}$ and $\mathcal{I}_\varepsilon^{-1, \eta}$ such that

$$\begin{aligned} \{t \in (0, 1) : |u_\varepsilon(t) - 1| < \eta\} &= \bigcup_{h \in \mathcal{I}_\varepsilon^{1, \eta}} I_h, \\ \{t \in (0, 1) : |u_\varepsilon(t) + 1| < \eta\} &= \bigcup_{h \in \mathcal{I}_\varepsilon^{-1, \eta}} I_h, \end{aligned}$$

where each $I_h = (a_h, b_h)$ is a maximal open interval (that is, if $a_h \neq 0$ then $|u_\varepsilon(a_h)| - 1| = \eta$, and if $b_h \neq 1$ then $|u_\varepsilon(b_h)| - 1| = \eta$, respectively) and they are pairwise disjoint.

Let $N \in \mathbb{N}$ be fixed. A first remark is that for ε small enough the set $(0, 1) \cap A_\varepsilon^\eta(N)$ of the points where $|u_\varepsilon| - 1| < \eta$ and $|u_\varepsilon^{(\ell)}| < \frac{1}{N\varepsilon^\ell}$ is not empty. This follows by the application of Lemma 2.3.5 to the constant sequence of intervals $I_{\varepsilon_j} = (0, 1)$ and $\{\varepsilon_j\}_j$ any infinitesimal sequence. We can then define

$$a_\varepsilon = \inf((0, 1) \cap A_\varepsilon^\eta(N)) \quad \text{and} \quad b_\varepsilon = \sup((0, 1) \cap A_\varepsilon^\eta(N)),$$

noting that

$$|(0, 1) \setminus (a_\varepsilon, b_\varepsilon)| = o_\varepsilon(1).$$

This can again be proved by applying Lemma 2.3.5 taking $\{I_{\varepsilon_j}\}_j = \{(0, a_{\varepsilon_j})\}_j$, which actually implies that $a_\varepsilon = O(\varepsilon)$, since otherwise the claim of Lemma 2.3.5 would contradict the definition of a_ε . Similarly one proves that $1 - b_\varepsilon = O(\varepsilon)$.

Let $\mathcal{I}_\varepsilon^{1,\eta}(N)$ denote the subset of indices of $\mathcal{I}_\varepsilon^{1,\eta}$ for which there exists an open interval $I = (a'_h, b'_h) \subseteq I_h = (a_h, b_h)$ such that

$$|u^{(\ell)}(a'_h)| \leq \frac{1}{N\varepsilon^\ell} \quad \text{and} \quad |u^{(\ell)}(b'_h)| \leq \frac{1}{N\varepsilon^\ell} \quad \text{for } \ell \in \{1, \dots, k-1\}. \quad (2.3.17)$$

Analogously, we define $\mathcal{I}_\varepsilon^{-1,\eta}(N)$.

For each $h \in \mathcal{I}_\varepsilon^{\pm 1,\eta}(N)$, we consider the maximal open interval (a_h^N, b_h^N) for which (2.3.17) holds true; that is,

$$\begin{aligned} a_h^N &:= \inf \left\{ t \in [a_h, b_h] : |u^{(\ell)}(t)| \leq \frac{1}{N\varepsilon^\ell} \text{ for all } \ell \in \{1, \dots, k-1\} \right\}, \\ b_h^N &:= \sup \left\{ t \in (a_h, b_h] : |u^{(\ell)}(t)| \leq \frac{1}{N\varepsilon^\ell} \text{ for all } \ell \in \{1, \dots, k-1\} \right\}. \end{aligned}$$

For any $t \in (a_\varepsilon, b_\varepsilon) \setminus \bigcup_{h \in \mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N)} (a_h^N, b_h^N)$, we set

$$\begin{aligned} \tau(t) &= \sup \{ b_h^N : b_h^N \leq t, h \in \mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N) \}, \\ \sigma(t) &= \inf \{ a_h^N : a_h^N \geq t, h \in \mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N) \}. \end{aligned}$$

Note that $\tau(t)$ and $\sigma(t)$ may coincide. For any t the following properties hold:

- (i) $[\tau(t), \sigma(t)] \cap \bigcup_{h \in \mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N)} (a_h^N, b_h^N) = \emptyset$, so that, recalling that $A_\varepsilon^\eta(N)$ denotes the set of points t such that $|u_\varepsilon(t)| - 1| < \eta$ and $|u_\varepsilon^{(\ell)}(t)| < \frac{1}{N\varepsilon^\ell}$ for every $\ell \in \{1, \dots, k-1\}$, it follows

$$|(\tau(t), \sigma(t)) \cap A_\varepsilon^\eta(N)| = 0;$$

- (ii) if $I_h \cap [\tau(t), \sigma(t)] \neq \emptyset$ for $h \in (\mathcal{I}_\varepsilon^{1,\eta} \cup \mathcal{I}_\varepsilon^{-1,\eta}) \setminus (\mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N))$, then $I_h \subset [\tau(t), \sigma(t)]$;
- (iii) if $(a_h, a_h^N] \cap [\tau(t), \sigma(t)] \neq \emptyset$, then $(a_h, a_h^N] \subset [\tau(t), \sigma(t)]$; the same property holds for $[b_h^N, b_h)$;
- (iv) by the continuity of u_ε and of its derivatives,

$$|u_\varepsilon(\tau(t))| - 1| \leq \eta \quad \text{and} \quad |u_\varepsilon^{(\ell)}(\tau(t))| \leq \frac{1}{N\varepsilon^\ell} \quad \text{for all } \ell \in \{1, \dots, k-1\},$$

and the same properties hold for $\sigma(t)$.

Moreover, for different t the intervals $(\tau(t), \sigma(t))$ either coincide or are disjoint; indeed, if $s \in (\tau(t), \sigma(t))$ for some t , then $(\tau(t), \sigma(t)) = (\tau(s), \sigma(s))$.

Then, we can write the set $(a_\varepsilon, b_\varepsilon) \setminus \bigcup_{h \in \mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N)} (a_h^N, b_h^N)$ as a disjoint union of intervals of the form $[\tau(t), \sigma(t)]$; that is, there exists a set of indices $\Lambda_\varepsilon(\eta, N)$ such that

$$(a_\varepsilon, b_\varepsilon) \setminus \bigcup_{h \in \mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N)} (a_h^N, b_h^N) = \bigcup_{\lambda \in \Lambda_\varepsilon(\eta, N)} [\tau_\lambda, \sigma_\lambda],$$

where the interior of intervals are pairwise disjoint and the endpoints τ_λ and σ_λ belong to the sets $\{\tau(t) : t \in (a_\varepsilon, b_\varepsilon) \setminus \bigcup_{h \in \mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N)} (a_h^N, b_h^N)\}$ and $\{\sigma(t) : t \in (a_\varepsilon, b_\varepsilon) \setminus \bigcup_{h \in \mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N)} (a_h^N, b_h^N)\}$, respectively.

In light of (iv), we have that

$$\bigcup_{\lambda \in \Lambda_\varepsilon(\eta, N)} \{\tau_\lambda, \sigma_\lambda\} \subseteq \{t \in (0, 1) : |u_\varepsilon(t) - 1| \geq \eta\}.$$

Then we have that $|\bigcup_{\lambda \in \Lambda_\varepsilon(\eta, N)} \{\tau_\lambda, \sigma_\lambda\}| \leq \varepsilon \frac{S}{\alpha_W \eta^2}$ and we deduce that there exists a countable sub-family $\Lambda_\varepsilon^*(\eta, N) \subset \Lambda_\varepsilon(\eta, N)$ such that

$$(a_\varepsilon, b_\varepsilon) \setminus \bigcup_{h \in \mathcal{I}_\varepsilon^{1,\eta}(N) \cup \mathcal{I}_\varepsilon^{-1,\eta}(N)} (a_h^N, b_h^N) = J_\varepsilon \cup \bigcup_{\lambda \in \Lambda_\varepsilon^*(\eta, N)} (\tau_\lambda, \sigma_\lambda),$$

where $|J_\varepsilon| \leq \varepsilon \frac{S}{\alpha_W \eta^2}$.

We now prove that there exists a constant $T_k(\eta, N, S) > 0$ such that

$$|(\tau_\lambda, \sigma_\lambda)| \leq T_k(\eta, N, S)\varepsilon \quad (2.3.18)$$

for any $\varepsilon > 0$ and $\lambda \in \Lambda_\varepsilon^*(\eta, N)$.

We reason by contradiction. If the claim does not hold, then for any $j \in \mathbb{N}$ there exist $\varepsilon_j > 0$ and $\lambda_j \in \Lambda_{\varepsilon_j}^*(\eta, N)$ such that

$$|(\tau_{\lambda_j}, \sigma_{\lambda_j})| > j\varepsilon_j.$$

Since $(\tau_{\lambda_j}, \sigma_{\lambda_j}) \subset (0, 1)$, this implies that $\varepsilon_j \rightarrow 0$ as $j \rightarrow +\infty$. We can then apply Lemma 2.3.5 to the family of intervals $I_{\varepsilon_j} = (\tau_{\lambda_j}, \sigma_{\lambda_j})$, deducing that there exists $j_0 \in \mathbb{N}$ such that

$$|A_{\varepsilon_j}^\eta(N) \cap (\tau_{\lambda_j}, \sigma_{\lambda_j})| > 0$$

for any $j \geq j_0$. This gives a contradiction, since property (i) holds for any $(\tau_{\lambda_j}, \sigma_{\lambda_j})$.

We say that an interval $(\tau_\lambda, \sigma_\lambda)$ with $\lambda \in \Lambda_\varepsilon^*(\eta, N)$ is an *effective transition interval* if either $|u_\varepsilon(\tau_\lambda) - 1| \leq \eta$ and $|u_\varepsilon(\sigma_\lambda) + 1| \leq \eta$, or $|u_\varepsilon(\tau_\lambda) + 1| \leq \eta$ and $|u_\varepsilon(\sigma_\lambda) - 1| \leq \eta$. We now prove that the number of such intervals is equi-bounded. Indeed, setting $v_\varepsilon(s) = u_\varepsilon(\varepsilon s + \frac{\sigma_\lambda + \tau_\lambda}{2})$, by a change of variable we obtain

$$E_\varepsilon^1(u_\varepsilon, (\tau_\lambda, \sigma_\lambda)) = \int_{-\frac{\sigma_\lambda - \tau_\lambda}{2\varepsilon}}^{\frac{\sigma_\lambda - \tau_\lambda}{2\varepsilon}} [W(v_\varepsilon) + |v_\varepsilon^{(k)}|^2] ds \geq m_k(\eta, N) \geq \frac{m_k}{2}, \quad (2.3.19)$$

where the last inequality follows by Lemma 2.3.8. Since $m_k > 0$, the number of intervals of effective transitions is equi-bounded by a constant which is independent on ε, η and N ; more precisely, by $\frac{2S}{m_k}$.

We now consider intervals $(\tau_\lambda, \sigma_\lambda)$ which are not intervals of effective transitions; that is, either $|u_\varepsilon(\tau_\lambda) - 1| \leq \eta$ and $|u_\varepsilon(\sigma_\lambda) - 1| \leq \eta$ or $|u_\varepsilon(\tau_\lambda) + 1| \leq \eta$ and $|u_\varepsilon(\sigma_\lambda) + 1| \leq \eta$. Let $(\tau_\lambda, \sigma_\lambda)$ be such that $|u_\varepsilon(\tau_\lambda) - 1| \leq \eta$ and $|u_\varepsilon(\sigma_\lambda) - 1| \leq \eta$, and suppose that there exists a point t in $(\tau_\lambda, \sigma_\lambda)$ such that $|u_\varepsilon(t) - 1| \geq 2\eta$. Then, setting $t' = \inf\{t : |u_\varepsilon(t) - 1| \geq 2\eta\}$, by a change of variable we have that

$$\begin{aligned} E_\varepsilon^1(u_\varepsilon, (\tau_\lambda, \sigma_\lambda)) &\geq E_\varepsilon^1(u_\varepsilon, (\tau_\lambda, t')) = \int_{\tau_\lambda}^{t'} \left[\frac{1}{\varepsilon} W(u_\varepsilon) + \varepsilon^{2k-1} |u_\varepsilon^{(k)}|^2 \right] dt \\ &= \int_0^{\frac{t' - \tau_\lambda}{\varepsilon}} (W(v_\varepsilon) + (v_\varepsilon^{(k)})^2) ds \\ &\geq \inf_{T \leq T_k(\eta, N, S)} \min_v \left\{ \int_0^T [W(v_\varepsilon) + |v_\varepsilon^{(k)}|^2] ds \right\}, \end{aligned}$$

where the minimum is taken over $v \in H^k(0, T)$ such that $|v(T) - 1| = 2\eta$, $|v(0) - 1| \geq \eta$, $v^{(\ell)}(0) \leq \frac{1}{N}$ for all $\ell \in \{1, \dots, k-1\}$, and we used the fact that $|(\tau_\lambda, \sigma_\lambda)| \leq T_k(\eta, N, S)\varepsilon$ by (2.3.18). Then, by applying Lemma 2.3.9 with $\eta > 0$ and $\bar{T} = T_k(\eta, N, S)$, we obtain

$$E_\varepsilon^1(u_\varepsilon, (\tau_\lambda, \sigma_\lambda)) \geq m_k^*(\eta, N, T_k(\eta, N, S)) > 0.$$

Since a corresponding argument holds if $|u_\varepsilon(\tau_\lambda) + 1| \leq \eta$ and $|u_\varepsilon(\sigma_\lambda) + 1| \leq \eta$, we can conclude that

$$||u_\varepsilon(t) - 1| \leq 2\eta \quad \text{in } (\tau_\lambda, \sigma_\lambda)$$

except for a finite number of indices λ (which of course depends on η , N and S).

Let now $\{(\tau_i, \sigma_i)\}_{i=1}^{M_\varepsilon}$ with $\tau_i < \tau_{i+1}$ denote the set of the effective transition intervals. Since $M_\varepsilon \leq \frac{2S}{m_k}$, we may suppose M_ε is independent of ε , up to passing to a vanishing subsequence $\{\varepsilon_j\}_j$, and we set $M_{\varepsilon_j} = M$. We construct a sequence of functions $u_{\varepsilon_j}^{\eta, N}$ in $(0, 1)$ as follows. To give a definition in the intervals (σ_i, τ_{i+1}) , we note that, by the definitions of $\tau(t)$ and $\sigma(t)$, there exists an index $h \in \mathcal{I}_{\varepsilon_j}^{1, \eta}(N) \cup \mathcal{I}_{\varepsilon_j}^{-1, \eta}(N)$ such that $(a_h^N, b_h^N) \subseteq (\sigma_i, \tau_{i+1})$. Moreover, if there exists an index $h \in \mathcal{I}_{\varepsilon_j}^{-1, \eta}(N)$ such that $(a_h^N, b_h^N) \subseteq (\sigma_i, \tau_{i+1})$, then $(\sigma_i, \tau_{i+1}) \cap \bigcup_{h' \in \mathcal{I}_{\varepsilon_j}^{1, \eta}(N)} (a_{h'}^N, b_{h'}^N) = \emptyset$. Indeed, otherwise (σ_i, τ_{i+1}) would contain an effective transition interval. The analogous property holds if there exists an index $h \in \mathcal{I}_{\varepsilon_j}^{1, \eta}(N)$ such that $(a_h^N, b_h^N) \subseteq (\sigma_i, \tau_{i+1})$. We then set

$$u_{\varepsilon_j}^{\eta, N} = -1 \quad \text{in } (\sigma_i, \tau_{i+1})$$

if there exists $h \in \mathcal{I}_{\varepsilon_j}^{-1, \eta}(N)$ such that $(a_h^N, b_h^N) \subseteq (\sigma_i, \tau_{i+1})$, and

$$u_{\varepsilon_j}^{\eta, N} = 1 \quad \text{in } (\sigma_i, \tau_{i+1})$$

if there exists $h \in \mathcal{I}_{\varepsilon_j}^{1, \eta}(N)$ such that $(a_h^N, b_h^N) \subseteq (\sigma_i, \tau_{i+1})$. Finally, we extend the function $u_{\varepsilon_j}^{\eta, N}$ on the whole interval $(0, 1)$ with values $-1, 1$ in such a way that

$$\#S(u_{\varepsilon_j}^{\eta, N}) = M, \quad \text{for every } j, \eta, N. \quad (2.3.20)$$

Moreover, by (2.3.19) we have that

$$\liminf_{j \rightarrow +\infty} E_{\varepsilon_j}^1(u_{\varepsilon_j}) \geq M m_k(\eta, N). \quad (2.3.21)$$

Let us now consider $N = N(\eta) \rightarrow +\infty$ as $\eta \rightarrow 0$ and set $u_{\varepsilon_j}^{\eta, N(\eta)} = u_{\varepsilon_j}^\eta$. By (2.3.20) we can suppose that $u_{\varepsilon_j}^\eta \rightarrow u^\eta$ in L^1 as $j \rightarrow +\infty$ and that, up to subsequences, the limit u^η takes values in $\{-1, 1\}$, and its singular set $S(u^\eta)$ is composed of no more than M points. Analogously, we can suppose that $u^\eta \rightarrow u$ in $L^1(0, 1)$ as $\eta \rightarrow 0$ with u satisfying the same conditions.

By construction, there exists a positive constant $C_k(\eta, S)$ such that

$$|\{t \in (0, 1) : |u_{\varepsilon_j}(t) - u_{\varepsilon_j}^\eta(t)| > 2\eta\}| \leq C_k(\eta, S)\varepsilon_j.$$

Then, for fixed $\delta > 6\eta$, we have

$$\begin{aligned} |\{|u_{\varepsilon_j} - u| > \delta\}| &\leq |\{|u_{\varepsilon_j} - u_{\varepsilon_j}^\eta| > \delta/3\}| + |\{|u_{\varepsilon_j}^\eta - u^\eta| > \delta/3\}| + |\{|u^\eta - u| > \delta/3\}| \\ &\leq |\{|u_{\varepsilon_j} - u_{\varepsilon_j}^\eta| > 2\eta\}| + |\{|u_{\varepsilon_j}^\eta - u^\eta| > \delta/3\}| + |\{|u^\eta - u| > \delta/3\}| \\ &\leq C_k(\eta, S)\varepsilon_j + |\{|u_{\varepsilon_j}^\eta - u^\eta| > \delta/3\}| + |\{|u^\eta - u| > \delta/3\}|, \end{aligned}$$

and therefore, letting first $j \rightarrow +\infty$ and then $\eta \rightarrow 0$, we infer that $u_{\varepsilon_j} \rightarrow u$ in measure in light of the arbitrariness of δ . $\square$

Finally, we prove the Γ -convergence result.

Proposition 2.3.11 (Lower bound). *Let $\{\varepsilon_j\}_j$ be a positive sequence converging to 0 and let $E_{\varepsilon_j}^1$ be defined by (2.3.1). Then, for every $u \in L^1(0, 1)$ and $\{u_j\}_j \subset H^k(0, 1)$ such that $u_j \rightarrow u$ in measure it holds*

$$\liminf_{j \rightarrow +\infty} E_{\varepsilon_j}^1(u_j) \geq E^1(u),$$

where E^1 is as defined in (2.3.2).

Proof. It suffices to consider the case $\sup_j E_{\varepsilon_j}^1(u_j) < +\infty$ and then, by Proposition 2.3.10, we may assume that $u \in BV((0, 1); \{-1, 1\})$. By (2.3.21), we deduce that

$$\liminf_{j \rightarrow +\infty} E_{\varepsilon_j}^1(u_j) \geq m_k(\eta, N) \# S(u).$$

We can then let $N \rightarrow +\infty$ and $\eta \rightarrow 0$ and apply Lemma 2.3.8 and Proposition 2.3.6. $\square$

Proposition 2.3.12 (Upper bound). *Let $\{\varepsilon_j\}_j$ be a positive sequence converging to 0 and let $E_{\varepsilon_j}^1$ be defined by (2.3.1). Then, for every $u \in L^1(0, 1)$ there exists a sequence $\{u_j\}_j \subset H^k(0, 1)$ such that $u_j \rightarrow u$ in $L^1(0, 1)$ and it holds*

$$\limsup_{j \rightarrow +\infty} E_{\varepsilon_j}^1(u_j) \leq E^1(u),$$

where E^1 is as defined in (2.3.2).

Proof. Let $u \in BV((0, 1); \{-1, 1\})$ and denote by $x_1, \dots, x_r$ its jump points. We set $J_1 := \{n \in \{1, \dots, r\} : u(x_n^+) - u(x_n^-) = 2\}$, $J_2 := \{n \in \{1, \dots, r\} : u(x_n^+) - u(x_n^-) = -2\}$, and for convenience, we also set $x_0 = 0$ and $x_{r+1} = 1$. Fix a positive $\delta < \min\{x_n - x_{n-1} : n = 1, \dots, r+1\}$ and, by Proposition 2.3.6, consider a function v admissible for $\tilde{m}_k$ such that

$$\int_{-\infty}^{+\infty} [W(v) + |v^{(k)}|^2] dt \leq m_k + \frac{\delta}{r}. \quad (2.3.22)$$

Recall that there exists $T > 0$ such that $v = -1$ in $(-\infty, -T)$ and $v = 1$ in $(T, +\infty)$.

Suppose that $\frac{\delta}{2\varepsilon_j} > T$ for every j . For $n \in \{1, \dots, r\}$, define intervals

$$I_n := \left[\frac{x_{n-1} + x_n}{2}, \frac{x_n + x_{n+1}}{2} \right]$$

and consider the sequence defined as

$$u_j(t) := \begin{cases} v\left(\frac{t - x_n}{\varepsilon_j}\right) & \text{if } t \in I_n \text{ and } n \in J_1, \\ v\left(-\frac{t - x_n}{\varepsilon_j}\right) & \text{if } t \in I_n \text{ and } n \in J_2, \\ u(t) & \text{otherwise.} \end{cases}$$

We note that the condition $\frac{\delta}{2\varepsilon_j} > T$ implies that $\{u_j\} \subset H^k(0, 1)$; and moreover, $u_j \rightarrow u$ in $L^1(0, 1)$ as $j \rightarrow +\infty$. To conclude, we estimate

$$\begin{aligned} \lim_{j \rightarrow +\infty} E_{\varepsilon_j}^1(u_j) &= \lim_{j \rightarrow +\infty} \sum_{n=1}^r \int_{I_n} \left[\frac{1}{\varepsilon_j} W(u_j(t)) + \varepsilon_j^{2k-1} |u_j^{(k)}(t)|^2 \right] dt \\ &= \lim_{j \rightarrow +\infty} \sum_{n \in J_1} \int_{\frac{x_{n-1}-x_n}{2\varepsilon_j}}^{\frac{x_{n+1}-x_n}{2\varepsilon_j}} [W(v(s)) + |v^{(k)}(s)|^2] ds \\ &\quad + \lim_{j \rightarrow +\infty} \sum_{n \in J_2} \int_{\frac{x_{n-1}-x_n}{2\varepsilon_j}}^{\frac{x_{n+1}-x_n}{2\varepsilon_j}} [W(v(-s)) + |v^{(k)}(-s)|^2] ds \\ &= \sum_{n \in J_1} \int_{-\infty}^{+\infty} [W(v(s)) + |v^{(k)}(s)|^2] ds \\ &\quad + \sum_{n \in J_2} \int_{-\infty}^{+\infty} [W(v(-s)) + |v^{(k)}(-s)|^2] ds \\ &\leq m_k \# S(u) + \delta, \end{aligned}$$

where in the last inequality we have used (2.3.22). The arbitrariness of δ and a diagonal argument allow us to conclude. $\square$

2.3.5 Γ -limit in dimension $d > 1$

In this subsection we prove Theorem 2.3.2. We first prove the lower bound using a variant of the blow-up technique by Fonseca and Müller [75] adapted to Γ -converging families of functionals in the spirit of [37], combined with a slicing argument that allows us to reduce to the one-dimensional case in Theorem 2.3.1.

Proposition 2.3.13 (Lower bound). *Let $\{\varepsilon_j\}_j$ be a positive sequence converging to 0 and let E_{ε_j} be defined by (2.3.4). Then, for every $u \in BV((0, 1); \{-1, 1\})$ and $\{u_j\}_j \subset H^k(0, 1)$ such that $u_j \rightarrow u$ in $L^1(\Omega)$ it holds*

$$\liminf_{j \rightarrow +\infty} E_{\varepsilon_j}(u_j) \geq m_k \mathcal{H}^{d-1}(S(u)),$$

where m_k is as defined in (2.3.3).

Proof. Without loss of generality, we assume $\sup_j E_{\varepsilon_j}(u_{\varepsilon_j}) < +\infty$ and, up to extracting a subsequence, that the lower limit is actually a limit. Since $u \in BV(\Omega; \{-1, 1\})$, we can write $u = 2\chi_A - 1$, where $A = \{x \in \Omega : u(x) = 1\}$ is a set of finite perimeter in Ω . The set $S(u)$ of discontinuity points of u coincides in Ω with $\partial^* A$, the reduced boundary of A , up to an $\mathcal{H}^{d-1}$ -negligible set. The set $\partial^* A$ is a $(d-1)$ -rectifiable set, and for every $x \in \partial^* A$ there exists $\nu_A(x)$, the inner unit normal to A in the measure-theoretical sense.

For every positive integer j , we introduce the measure μ_{ε_j} defined for $B \subseteq \Omega$ Borel as

$$\mu_{\varepsilon_j}(B) := \int_B \left[\frac{1}{\varepsilon_j} W(u_{\varepsilon_j}) + \varepsilon_j^{2k-1} \|\nabla^{(k)} u_{\varepsilon_j}\|_k^2 \right] dx.$$

Since $\sup_j \mu_{\varepsilon_j}(\Omega) < +\infty$, up to a not relabeled subsequence, we may assume that $\{\mu_{\varepsilon_j}\}$ weakly*-converges to a finite Borel measure μ . By the Besicovitch Derivation Theorem, for $\mathcal{H}^{d-1}$ -almost every x in $\partial^* A \cap \Omega$, it holds

$$\frac{d\mu}{d(\mathcal{H}^{d-1} \llcorner (\partial^* A \cap \Omega))}(x) = \lim_{r \rightarrow 0} \frac{\mu(Q_r^\nu(x))}{\mathcal{H}^{d-1}(Q_r^\nu(x) \cap \partial^* A)} = \lim_{r \rightarrow 0} \frac{\mu(Q_r^\nu(x))}{r^{d-1}}, \quad (2.3.23)$$

where $\nu = \nu_A(x)$ is the inner unit normal to A at x and in the last equality we have used that the $(d-1)$ -dimensional density of $\partial^* A$ is equal to 1 for $\mathcal{H}^{d-1}$ -almost every point in the reduced boundary of A . Since μ is a finite measure, we also have

$$\lim_{j \rightarrow +\infty} \mu_{\varepsilon_j}(Q_r^\nu(x)) = \mu(Q_r^\nu(x)) \quad (2.3.24)$$

for all $r > 0$ but a countable set. Hence, if we fix a point $x \in \partial^* A \cap \Omega$ such that (2.3.23) holds, a diagonal argument and (2.3.24) imply that there exists $r_j = r(\varepsilon_j) \rightarrow 0^+$ such that

$$\frac{d\mu}{d(\mathcal{H}^{d-1} \llcorner (\partial^* A \cap \Omega))}(x) = \lim_{j \rightarrow +\infty} \frac{\mu_{\varepsilon_j}(Q_{r_j}^\nu(x))}{r_j^{d-1}} \quad (2.3.25)$$

and $\frac{\varepsilon_j}{r_j} \rightarrow 0$.

Up to a translation and rotation argument, we may suppose that $x = 0 \in \partial^* A \cap \Omega$ is such that (2.3.25) holds and that $\nu_A(0) = e_d$. In this case, we set $Q_{r_j} := Q_{r_j}^{e_d}(0) = (-\frac{r_j}{2}, \frac{r_j}{2})^d$ and note that the following blow-up condition holds:

$$\chi_{\frac{A}{r_j}} \xrightarrow{L^1_{\text{loc}}(\mathbb{R}^d)} H \quad \text{as } j \rightarrow +\infty, \quad (2.3.26)$$

where H is the characteristic function of the half-space given by $\{x \in \mathbb{R}^d : x_d > 0\}$.

Applying the change of variable $y := x/r_j$ and setting $v_{\varepsilon_j}(x) := u_{\varepsilon_j}(r_j x)$, we get

$$\begin{aligned} \frac{\mu_{\varepsilon_j}(Q_{r_j})}{r_j^{d-1}} &= \frac{1}{r_j^{d-1}} \int_{Q_{r_j}} \left[\frac{1}{\varepsilon_j} W(u_{\varepsilon_j}(x)) + \varepsilon_j^{2k-1} \|\nabla^{(k)} u_{\varepsilon_j}(x)\|_k^2 \right] dx \\ &= \int_{Q_1} \left[\frac{1}{\delta_j} W(v_{\varepsilon_j}(y)) + \delta_j^{2k-1} \|\nabla^{(k)} v_{\varepsilon_j}(y)\|_k^2 \right] dy, \end{aligned}$$

where $\delta_j := \frac{\varepsilon_j}{r_j}$ and $Q_1 = (-\frac{1}{2}, \frac{1}{2})^d$. We set $y = (\xi, t)$, with $\xi \in [-1/2, 1/2]^{d-1}$, $t \in [-\frac{1}{2}, \frac{1}{2}]$ and $v_{\varepsilon_j}^\xi(t) := v_{\varepsilon_j}(\xi, t)$, and we note that

$$\|\nabla^{(k)} v_{\varepsilon_j}\|_k \geq \left| \frac{d^k}{dt^k} v_{\varepsilon_j}^\xi(t) \right|.$$

Therefore, we have

$$\frac{\mu_{\varepsilon_j}(Q_{r_j})}{r_j^{d-1}} \geq \int_{[-\frac{1}{2}, \frac{1}{2}]^{d-1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{1}{\delta_j} W(v_{\varepsilon_j}^\xi(t)) + \delta_j^{2k-1} \left| \frac{d^k}{dt^k} v_{\varepsilon_j}^\xi(t) \right|^2 \right] dt d\xi$$

and by Fatou's Lemma

$$\liminf_{j \rightarrow +\infty} \frac{\mu_{\varepsilon_j}(Q_{r_j})}{r_j^{d-1}} \geq \int_{[-\frac{1}{2}, \frac{1}{2}]^{d-1}} \liminf_{j \rightarrow +\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{1}{\delta_j} W(v_{\varepsilon_j}^\xi(t)) + \delta_j^{2k-1} \left| \frac{d^k}{dt^k} v_{\varepsilon_j}^\xi(t) \right|^2 \right] dt d\xi.$$

Since $\{u_{\varepsilon_j}\}_j$ converges to the function $2\chi_E - 1$ in $L^1(\Omega)$, by (2.3.26) we have that $v_{\varepsilon_j} \rightarrow 2H - 1$ in $L^1(Q_1)$, and in turn, $v_{\varepsilon_j}^\xi \rightarrow 2\chi_{[0, \frac{1}{2})} - 1 =: v^\xi$ for $\mathcal{H}^{d-1}$ -a.e. $\xi \in [-\frac{1}{2}, \frac{1}{2}]^{d-1}$. We apply the lower estimate in Proposition 2.3.13 to infer that

$$\liminf_{j \rightarrow +\infty} \frac{\mu_{\varepsilon_j}(Q_{r_j})}{r_j^{d-1}} \geq \int_{[-\frac{1}{2}, \frac{1}{2}]^{d-1}} m_k \# S(v^\xi) d\xi = m_k.$$

Since this argument is valid for all $x \in \partial^* A$, recalling (2.3.25), we obtain

$$\begin{aligned} \mu(\Omega) &\geq \int_{\partial^* A \cap \Omega} \frac{d\mu}{d(\mathcal{H}^{d-1} \llcorner (\partial^* A \cap \Omega))} d\mathcal{H}^{d-1} \\ &\geq m_k \mathcal{H}^{d-1}(\partial^* A \cap \Omega) = m_k \mathcal{H}^{d-1}(S(u)), \end{aligned}$$

and, since the weak*-convergence of $\{\mu_{\varepsilon_j}\}_j$ to μ implies that $\liminf_{j \rightarrow +\infty} \mu_{\varepsilon_j}(\Omega) \geq \mu(\Omega)$, finally we get

$$\liminf_{j \rightarrow +\infty} \mu_{\varepsilon_j}(\Omega) = \lim_{j \rightarrow +\infty} \mu_{\varepsilon_j}(\Omega) \geq \mu(\Omega) \geq m_k \mathcal{H}^{d-1}(S(u)),$$

and the proof is complete. $\square$

We conclude the proof of the Γ -convergence by showing the existence of a recovery sequence. In what follows, C denotes a generic positive constant which may vary from line to line.

Proposition 2.3.14 (Upper bound). *Let $\{\varepsilon_j\}_j$ be a positive sequence converging to 0 and let E_{ε_j} be defined by (2.3.4). Then, for every $u \in BV(\Omega; \{-1, 1\})$ there exists a sequence $\{u_j\}_j \subset H^k(\Omega)$ such that $u_j \rightarrow u$ in $L^1(\Omega)$ and it holds*

$$\limsup_{j \rightarrow +\infty} E_{\varepsilon_j}(u_j) \leq m_k \mathcal{H}^{d-1}(S(u)),$$

where m_k is as defined in (2.3.3).

Proof. Let A be the set of finite perimeter in Ω such that $u = 2\chi_A - 1$. Then, there exists a sequence $\{A_n\}_n$ of bounded open sets in $\mathbb{R}^d$ with C^∞ boundary such that $\chi_{A_n \cap \Omega} \rightarrow \chi_A$ in $L^1(\Omega)$ and $P(A_n; \Omega) \rightarrow P(A; \Omega)$. Therefore, since $\mathcal{H}^{d-1}(S(u)) = P(A; \Omega)$, it is enough to prove the upper estimate only for functions $u = 2\chi_A - 1$ when $A = \tilde{A} \cap \Omega$ for some smooth bounded set $\tilde{A}$ in $\mathbb{R}^d$. We set $M := \partial \tilde{A}$.

Resorting to Proposition 2.3.6, let us consider $\{v_j\}_j$ a sequence of functions in $H_{\text{loc}}^k(\mathbb{R})$ such that

$$\lim_{j \rightarrow +\infty} \int_{-\infty}^{+\infty} [W(v_j) + |v_j^{(k)}|^2] dt = m_k \quad (2.3.27)$$

and

$$v_j(t) = \begin{cases} -1 & \text{if } t \leq -\frac{1}{\sqrt{\varepsilon_j}}, \\ 1 & \text{if } t \geq \frac{1}{\sqrt{\varepsilon_j}}. \end{cases}$$

Since M is a smooth manifold in $\mathbb{R}^d$, there exists $\delta_0 > 0$ such that for every $\delta < \delta_0$ there exists a unique smooth projection from U_δ on M , where $U_\delta := \{x \in \mathbb{R}^d : \text{dist}(x, M) < \delta\}$ is a tubular neighborhood of M . Therefore, denoting by $\tilde{d}_M$ the signed distance function from M which is negative inside of E , we have that $\tilde{d}_M \in C^\infty(U_\delta)$ for every $\delta < \delta_0$.

We suppose that $\sqrt{\varepsilon_j} < \delta_0/2$ for every j and define

$$u_j(x) = \begin{cases} -1 & \text{if } x \in \Omega \setminus (E \cup U_j), \\ v_j\left(\frac{\tilde{d}_M(x)}{\varepsilon_j}\right) & \text{if } x \in \Omega \cap U_j, \\ 1 & \text{if } x \in E \setminus U_j, \end{cases} \quad (2.3.28)$$

where $U_j := U_{\sqrt{\varepsilon_j}}$.

In order to estimate $F_{\varepsilon_j}(u_j)$, we compute the derivatives of order k of the function u_j . Let α be a multi-index of order k . We observe that

$$\partial^\alpha u_j = \frac{1}{\varepsilon_j^k} v_j^{(k)}\left(\frac{\tilde{d}_M}{\varepsilon_j}\right) (\partial_{\alpha_1} \tilde{d}_M) \cdots (\partial_{\alpha_k} \tilde{d}_M) + \sum_{\ell=1}^{k-1} \frac{1}{\varepsilon_j^\ell} v_j^{(\ell)}\left(\frac{\tilde{d}_M}{\varepsilon_j}\right) P_\alpha^\ell(A(\partial^\alpha \tilde{d}_M)),$$

where $A(\partial^\alpha \tilde{d}_M)$ represents the ordered vector of all the derivatives $\partial^\beta \tilde{d}_M$ with $\beta \subset \alpha, |\beta| \geq 1$, and $P_\alpha^\ell : \mathbb{R}^{2^{|\alpha|}-1} \rightarrow \mathbb{R}$ are polynomials of degree at most k . Therefore, the k -tensor $\nabla^{(k)} u_{\varepsilon_j}$ can be written in the form

$$\nabla^{(k)} u_j = \frac{1}{\varepsilon_j^k} v_j^{(k)}\left(\frac{\tilde{d}_M}{\varepsilon_j}\right) (\nabla \tilde{d}_M \otimes \cdots \otimes \nabla \tilde{d}_M) + \sum_{\ell=1}^{k-1} \frac{1}{\varepsilon_j^\ell} v_j^{(\ell)}\left(\frac{\tilde{d}_M}{\varepsilon_j}\right) T^\ell, \quad (2.3.29)$$

where $T^\ell, \ell \in \{1, \dots, k-1\}$, are k -tensors having equi-bounded operatorial norm. Indeed, the coefficients of T^ℓ only depend on the derivatives of $\tilde{d}_M$ up to order k , and these are uniformly bounded on U_j since we assumed $\sqrt{\varepsilon_j} < \delta_0/2$, which implies the existence of a compact set K such that $U_j \subset \subset K \subset U_{\delta_0}$.

By (2.3.28) and (2.3.29) we obtain

$$\begin{aligned} \limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) &= \limsup_{j \rightarrow +\infty} \left(\int_{U_j} \frac{1}{\varepsilon_j} W\left(v_j\left(\frac{\tilde{d}_M}{\varepsilon_j}\right)\right) dx \right. \\ &\quad \left. + \varepsilon_j^{2k-1} \int_{U_j} \left\| \frac{1}{\varepsilon_j^k} v_j^{(k)}\left(\frac{\tilde{d}_M}{\varepsilon_j}\right) (\nabla \tilde{d}_M \otimes \cdots \otimes \nabla \tilde{d}_M) + \sum_{\ell=1}^{k-1} \frac{1}{\varepsilon_j^\ell} v_j^{(\ell)}\left(\frac{\tilde{d}_M}{\varepsilon_j}\right) T^\ell \right\|^2 dx \right). \end{aligned}$$

Let $\Phi : M \times (-\delta_0/2, \delta_0/2) \rightarrow U_{\delta_0/2}$ be the diffeomorphism defined by $\Phi(y, t) := y + t\nu(y)$, where $\nu(y)$ is the outer unit normal to M at y , and denote by $J(y, t)$ its Jacobian. Applying this change of variable, we get

$$\begin{aligned} \limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) &= \limsup_{j \rightarrow +\infty} \int_{M \cap \Omega} \int_{-\sqrt{\varepsilon_j}}^{\sqrt{\varepsilon_j}} \left(\frac{1}{\varepsilon_j} W\left(v_j\left(\frac{t}{\varepsilon_j}\right)\right) \right. \\ &\quad \left. + \varepsilon_j^{2k-1} \left\| \frac{1}{\varepsilon_j^k} v_j^{(k)}\left(\frac{t}{\varepsilon_j}\right) (\nabla \tilde{d}_M \otimes \cdots \otimes \nabla \tilde{d}_M)(\Phi(y, t)) \right. \right. \\ &\quad \left. \left. + \sum_{\ell=1}^{k-1} \frac{1}{\varepsilon_j^\ell} v_j^{(\ell)}\left(\frac{t}{\varepsilon_j}\right) T^\ell(\Phi(y, t)) \right\|^2 \right) J(y, t) dt d\mathcal{H}^{d-1}(y). \end{aligned} \quad (2.3.30)$$

We set

$$I_j^k := \int_{M \cap \Omega} \int_{-\sqrt{\varepsilon_j}}^{\sqrt{\varepsilon_j}} \left[\frac{1}{\varepsilon_j} W\left(v_j\left(\frac{t}{\varepsilon_j}\right)\right) + \frac{1}{\varepsilon_j} \left| v_j^{(k)}\left(\frac{t}{\varepsilon_j}\right) \right|^2 \right] J(y, t) dt d\mathcal{H}^{d-1}(y)$$

$$I_j^\ell := \int_{-\sqrt{\varepsilon_j}}^{\sqrt{\varepsilon_j}} \varepsilon_j^{2k-1-2\ell} \left| v_j^{(\ell)}\left(\frac{t}{\varepsilon_j}\right) \right|^2 dt \quad \text{if } \ell < k,$$

and

$$I_j^{r,\ell} := \int_{-\sqrt{\varepsilon_j}}^{\sqrt{\varepsilon_j}} \varepsilon_j^{2k-1-r-\ell} \left| v_j^{(r)}\left(\frac{t}{\varepsilon_j}\right) v_j^{(\ell)}\left(\frac{t}{\varepsilon_j}\right) \right| dt \quad \text{if } r < \ell \leq k.$$

Taking into account that the Jacobian and the operatorial norm of all T^ℓ are bounded on $M \times (-\delta_0/2, \delta_0/2)$, expanding the square in the right-hand side of (2.3.30) we have

$$\begin{aligned} \limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) &\leq \limsup_{j \rightarrow +\infty} \left(I_j^k + C \left(\sum_{\ell=1}^{k-1} I_j^\ell + \sum_{\ell=1}^{k-1} I_j^{\ell,k} + \sum_{\ell=1}^{k-1} \sum_{r=1}^{\ell-1} I_j^{r,\ell} \right) \right) \\ &= \limsup_{j \rightarrow +\infty} \left(I_j^k + C \left(\sum_{\ell=1}^{k-1} I_j^\ell + \sum_{\ell=1}^k \sum_{r=1}^{\ell-1} I_j^{r,\ell} \right) \right). \end{aligned}$$

We have also used that $\|\nabla \tilde{d}_M \otimes \cdots \otimes \nabla \tilde{d}_M\| \leq |\nabla \tilde{d}_M|^k = 1$ since $\tilde{d}_M$ is a 1-Lipschitz function.

We first prove that

$$\limsup_{j \rightarrow +\infty} I_j^k \leq m_k \mathcal{H}^{d-1}(S(u)).$$

By a change of variable, we have

$$\begin{aligned} I_j^k &= \int_{M \cap \Omega} \int_{-\sqrt{\varepsilon_j}}^{\sqrt{\varepsilon_j}} \left[\frac{1}{\varepsilon_j} W\left(v_j\left(\frac{t}{\varepsilon_j}\right)\right) + \frac{1}{\varepsilon_j} \left| v_j^{(k)}\left(\frac{t}{\varepsilon_j}\right) \right|^2 \right] J(y, t) dt d\mathcal{H}^{d-1}(y) \\ &= \int_{M \cap \Omega} \int_{-\frac{1}{\sqrt{\varepsilon_j}}}^{\frac{1}{\sqrt{\varepsilon_j}}} [W(v_j(s)) + |v_j^{(k)}(s)|^2] J(y, s\varepsilon_j) ds d\mathcal{H}^{d-1}(y) \\ &\leq \left(\sup_{\substack{y \in M \\ t \in (-\sqrt{\varepsilon_j}, \sqrt{\varepsilon_j})}} J(y, t) \right) \int_{M \cap \Omega} \int_{-\infty}^{+\infty} [W(v_j(s)) + |v_j^{(k)}(s)|^2] ds d\mathcal{H}^{d-1}(y). \end{aligned}$$

Since M is compact, $J(y, t)$ converges uniformly to 1 as $t \rightarrow 0$; hence, by (2.3.27) we obtain

$$\limsup_{j \rightarrow +\infty} I_j^k \leq m_k \mathcal{H}^{d-1}(M \cap \Omega) = m_k \text{Per}(E; \Omega) = m_k \mathcal{H}^{d-1}(S(u)).$$

We now prove that for fixed $\ell \in \{1, \dots, k-1\}$ we have $\limsup_{j \rightarrow +\infty} I_j^\ell = 0$. To this end, we set $w_j(t) := v_j(t/\varepsilon_j)$ and $A_j := (-\sqrt{\varepsilon_j}, \sqrt{\varepsilon_j})$, and write

$$I_j^\ell = \varepsilon_j^{2k-1-2\ell} \int_{A_j} \left| v_j^{(\ell)}\left(\frac{t}{\varepsilon_j}\right) \right|^2 dt = \varepsilon_j^{2k-1} \int_{A_j} |w_j^{(\ell)}(t)|^2 dt. \quad (2.3.31)$$

By a change of variable, we have

$$\int_{A_j} \left[\frac{1}{\varepsilon_j} W(w_j) + \varepsilon_j^{2k-1} |w_j^{(k)}|^2 \right] dt = \int_{-\infty}^{+\infty} [W(v_j) + (v_j^{(k)})^2] dt;$$

hence, by (2.3.27) and by the fact that $W \geq 0$, we infer that

$$\|w_j^{(k)}\|_{L^2(A_j)} \leq C \varepsilon_j^{\frac{1-2k}{2}}. \quad (2.3.32)$$

Note that for every positive integer j and for any $h \in \{0, \dots, k-1\}$ there exists $z_j^h \in [-\sqrt{\varepsilon_j}, \sqrt{\varepsilon_j}]$ such that $w_j^{(h)}(z_j^h) = 0$. Indeed, this holds with $z_j^h = \sqrt{\varepsilon_j}$ for $h \in \{1, \dots, k-1\}$, while the existence of z_j^0 follows by the continuity of w_j . Hence, applying the Fundamental Theorem of Calculus and using (2.3.32), we get

$$|w_j(t)| \leq \varepsilon_j^{\frac{2k-1}{4}} \|w_j^{(k)}\|_{L^2(A_j)} \leq C \varepsilon_j^{\frac{1-2k}{4}} \quad \text{for every } t \in A_j,$$

so that

$$\|w_j\|_{L^2(A_j)} \leq C \varepsilon_j^{\frac{1-k}{2}}. \quad (2.3.33)$$

By the interpolation inequality (2.3.8), it holds

$$\|w_j^{(\ell)}\|_{L^2(A_j)} \leq C \left(\|w_j^{(k)}\|_{L^2(A_j)}^{\frac{\ell}{k}} \|w_j\|_{L^2(A_j)}^{\frac{k-\ell}{k}} + |A_j|^{-\ell} \|w_j\|_{L^2(A_j)} \right).$$

Hence, combining (2.3.32) and (2.3.33) we get

$$\|w_j^{(\ell)}\|_{L^2(A_j)} \leq C \left(\varepsilon_j^{\left(\frac{1-2k}{2}\right)\left(\frac{\ell}{k}\right)} \varepsilon_j^{\left(\frac{1-k}{2}\right)\left(\frac{k-\ell}{k}\right)} + \varepsilon_j^{-\frac{\ell}{2} + \frac{1-k}{2}} \right) = C \varepsilon_j^{\frac{-k-\ell+1}{2}}. \quad (2.3.34)$$

Note that the above formula, up to a constant factor that is independent of j , reduces to (2.3.32) when $\ell = k$. Recalling (2.3.31) we conclude

$$I_j^\ell = \varepsilon_j^{2k-1} \|w_j^{(\ell)}\|_{L^2(A_j)}^2 \leq C \varepsilon_j^{k-\ell},$$

which tends to 0 since $\ell < k$.

Finally, we prove that for fixed $r < \ell$, we have $\limsup_{j \rightarrow +\infty} I_j^{r,\ell} = 0$.

Recalling that $w_j(t) = v_j(t/\varepsilon_j)$, by Hölder's inequality we have

$$\begin{aligned} I_j^{r,\ell} &= \varepsilon_j^{2k-1-r-\ell} \int_{A_j} \left| v_j^{(r)}\left(\frac{t}{\varepsilon_j}\right) \right| \left| v_j^{(\ell)}\left(\frac{t}{\varepsilon_j}\right) \right| dt \\ &= \varepsilon_j^{2k-1} \int_{A_j} |w_j^{(r)}(t)| |w_j^{(\ell)}(t)| dt \\ &\leq \varepsilon_j^{2k-1} \|w_j^{(r)}\|_{L^2(A_j)} \|w_j^{(\ell)}\|_{L^2(A_j)}. \end{aligned}$$

Since (2.3.34) holds for every $\ell \in \{1, \dots, k\}$, we get

$$I_j^{r,\ell} \leq C \varepsilon_j^{2k-1} \varepsilon_j^{\frac{-k-r+1}{2}} \varepsilon_j^{\frac{-k-\ell+1}{2}} = C \varepsilon_j^{k-\frac{r+\ell}{2}},$$

which tends to 0 since $r + \ell < 2k$, concluding the proof. $\square$

2.4 Models with singular perturbations of different orders

This section is devoted to the analysis of the energies

$$\int_{\Omega} \left[\frac{1}{\varepsilon} W(u) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2 \right] dx, \quad u \in H^k(\Omega),$$

where $q_1, \dots, q_{k-1}$ are (possibly negative) constant coefficients, $q_k = 1$, and $|\cdot|_\ell$ is any norm on the space of symmetric ℓ -tensors for every $\ell \in \{1, \dots, k\}$. Here, Ω is a bounded open subset of $\mathbb{R}^d$ with C^1 boundary, and the double-well potential W satisfies stronger growth conditions at infinity compared to the previous section.

One of the difficulties in dealing with general tensor norms is the emergence of possible anisotropies, which makes slicing techniques inadequate; for this reason, we rely on a blow-up argument [75]. But the main issue of our study concerns the presence of negative terms in the energy. To overcome this difficulty we combine the approaches of Chermisi, Dal Maso, Fonseca, and Leoni in [55], and of

Cicalese, Spadaro, and Zeppieri in [56], where the case $k = 2$ is independently treated: in particular, in the former work the analysis is performed also in higher dimensions (for energies involving the operatorial norm of the Hessian), while the latter gives a detailed description of the one dimensional case.

Our first step is proving a non-linear interpolation inequality that determines lower bounds on the constants $q_1, \dots, q_{k-1}$ for which our asymptotic analysis can be carried out. Compared to the previous section, the strategy adopted here requires stronger assumptions on W , such as quadratic growth at infinity, as our argument relies on the use of standard interpolation inequalities on the whole domain, and not just close to the bottom of the wells.

Having such an interpolation estimate at our disposal allows us to find a priori bounds on the functionals, provided that the coefficients $q_1, \dots, q_{k-1}$ are not too small. This is the key ingredient to get our compactness result: indeed, in this situation, the above energies can be directly compared with those studied in Section 2.3 (that are obtained for $q_1 = \dots = q_{k-1} = 0$) and, consequently, a compactness result can be immediately inferred by the corresponding extension to the multi-dimensional case of Theorem 2.3.1 (see also Remark 2.3.3). We point out that the quadratic growth of W yields now precompactness in L^2 , and not just in measure and that, clearly, the interpolation argument is not needed if $q_\ell \geq 0$ for every $\ell \in \{1, \dots, k-1\}$.

Finally, we compute the Γ -limit of the family $\{F_\varepsilon\}_\varepsilon$ with respect to the strong convergence in $L^2(\Omega)$. Having already observed that this is finite on $BV(\Omega; \{-1, 1\})$, we prove that it equals an anisotropic perimeter.

2.4.1 Statement of the results

Let $W : \mathbb{R} \rightarrow [0, +\infty)$ be a continuous function satisfying the following assumptions:

- (H1) $W(u) = 0$ if and only if $u \in \{-1, 1\}$;
- (H2) $W(u) \geq \min\{(u+1)^2, (u-1)^2\}$ for every $s \in \mathbb{R}$;
- (H3) $W(u) \leq W(v) + 1$ for every $v \in \mathbb{R}$ and for every $u \in \mathbb{R}$ with $|u| \leq |v|$.

Let $k > 1$ be an integer and Ω be a bounded open subset of $\mathbb{R}^d$ with C^1 boundary. For every $\varepsilon > 0$ we define the functionals $F_\varepsilon : L^2(\Omega) \rightarrow [-\infty, +\infty]$ as

$$F_\varepsilon(u) := \begin{cases} \int_\Omega \left[\frac{1}{\varepsilon} W(u) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2 \right] dx & \text{if } u \in H^k(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \quad (2.4.1)$$

where the $q_1, \dots, q_{k-1}$ are constant coefficients, $q_k = 1$, and $|\cdot|_\ell$ is any norm on the space of symmetric ℓ -tensors for every $\ell \in \{1, \dots, k\}$.

Our first main result is the following non-linear interpolation inequality.

Theorem 2.4.1 (Interpolation inequality). *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with C^1 boundary, let $k > 1$ be an integer, and assume that (H2) is satisfied. Then, for every $\ell \in \{1, \dots, k-1\}$, there exists a positive constant q_ℓ^* independent of Ω such that for every $q < q_\ell^*$ there exists $\varepsilon_0 = \varepsilon_0(q, \Omega)$ such that it holds*

$$q \varepsilon^{2\ell} \int_\Omega \|\nabla^{(\ell)} u\|_\ell^2 dx \leq \int_\Omega \left[W(u) + \varepsilon^{2k} \|\nabla^{(k)} u\|_k^2 \right] dx$$

for every $\varepsilon \in (0, \varepsilon_0)$ and $u \in H^k(\Omega)$. Moreover, for every $\ell \in \{1, \dots, k-1\}$, there exists a positive constant $\tilde{q}_\ell^*$ independent of Ω such that for every $q < \tilde{q}_\ell^*$ there exists $\varepsilon_0 = \varepsilon_0(q, \Omega)$ such that it holds

$$q \varepsilon^{2\ell} \int_\Omega |\nabla^{(\ell)} u|_\ell^2 dx \leq \int_\Omega \left[W(u) + \varepsilon^{2k} |\nabla^{(k)} u|_k^2 \right] dx$$

for every $\varepsilon \in (0, \varepsilon_0)$ and $u \in H^k(\Omega)$. In particular, $\tilde{q}_\ell^*$ depends on q_ℓ^* , $|\cdot|_\ell$, and $|\cdot|_k$.

The above estimate inequality is proved by, first, adapting the arguments in [56], which yield the result in the case $d = 1$, and then by applying the slicing method as in [55] to recover the general case.

The main coerciveness result reads as follows.

Theorem 2.4.2 (Compactness). *Let F_ε be defined by (2.4.1) and let $\mathcal{N} := \{\ell \in \{1, \dots, k-1\} : q_\ell \leq 0\}$. Assume that (H1) and (H2) are satisfied and that one of the following conditions holds:*

- (i) $\mathcal{N} = \emptyset$;
- (ii) $\mathcal{N} \neq \emptyset$ and $q_\ell > -\bar{q}_\ell$ for every $\ell \in \mathcal{N}$ for suitable positive constants $\{\bar{q}_\ell : \ell \in \mathcal{N}\}$.

Then, the family $\{F_\varepsilon\}_\varepsilon$ is equi-coercive with respect to the convergence in $L^2(\Omega)$; that is, given any family $\{u_\varepsilon\}_\varepsilon \subset H^k(\Omega)$ such that

$$\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty,$$

there exist a positive sequence $\{\varepsilon_j\}_j$ converging to 0 and a function $u \in BV(\Omega; \{-1, 1\})$ such that $u_{\varepsilon_j} \rightarrow u$ in $L^2(\Omega)$ as $j \rightarrow +\infty$. In particular, if $\mathcal{N} \neq \emptyset$, letting $\tilde{q}_1^*, \dots, \tilde{q}_{k-1}^*$ denote the same positive constants appearing in Theorem 2.4.1 and given $\{\alpha_\ell : \ell \in \mathcal{N}\}$ such that $\alpha_\ell > 0$ for all $\ell \in \mathcal{N}$ and $\sum_{\ell \in \mathcal{N}} \alpha_\ell = 1$, it holds that $\bar{q}_\ell = \alpha_\ell \tilde{q}_\ell^*$ for all $\ell \in \mathcal{N}$; hence, $\{\bar{q}_\ell : \ell \in \mathcal{N}\}$ are independent of Ω .

To state our result of Γ -convergence, we introduce some notation. We fix a function $\bar{u} \in H_{\text{loc}}^k(\mathbb{R})$ such that $\bar{u}(t) = -1$ if $t \leq -1/8$, $\bar{u}(t) = 1$ if $t \geq 1/8$, $|\bar{u}| \leq 1$, and

$$0 < \int_{-\infty}^{+\infty} \left[W(\bar{u}) + \sum_{\ell=1}^k |q_\ell| |\bar{u}^{(\ell)}|^2 \right] dt < +\infty,$$

then, for fixed $\nu \in \mathbb{S}^{d-1}$, we define the function

$$\bar{u}^\nu(x) := \bar{u}(x \cdot \nu) \quad \text{for all } x \in \mathbb{R}^d,$$

and for every $\varepsilon > 0$ we further define

$$\bar{u}_\varepsilon^\nu(x) := \bar{u}^\nu\left(\frac{x}{\varepsilon}\right) = \bar{u}\left(\frac{x \cdot \nu}{\varepsilon}\right) \quad \text{for all } x \in \mathbb{R}^d.$$

We introduce a class of admissible functions on unit cubes as

$$\mathcal{A}_\varepsilon^\nu := \{v \in H^k(Q_1^\nu) : v = \bar{u}_\varepsilon^\nu \text{ near } \partial Q_1^\nu\}, \quad (2.4.2)$$

where by ‘near ∂Q_1^ν ’, we mean ‘in $Q_1^\nu \setminus Q_r^\nu$ for some $r \in (0, 1)$ ’. Finally, for every $\nu \in \mathbb{S}^{d-1}$ we introduce the function

$$g(\nu) := \inf \left\{ \int_{Q_1^\nu} \left[\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \right] dx : v \in \mathcal{A}_\varepsilon^\nu, \varepsilon \in (0, 1) \right\}. \quad (2.4.3)$$

We state our last main result.

Theorem 2.4.3 (Γ -convergence). *Let F_ε be defined by (2.4.1) and let $\mathcal{N} := \{\ell \in \{1, \dots, k-1\} : q_\ell \leq 0\}$. Assume that (H1)-(H3) are satisfied and that one of the following conditions holds:*

- (i) $\mathcal{N} = \emptyset$;
- (ii) $\mathcal{N} \neq \emptyset$ and $q_\ell > -\bar{q}_\ell$ for every $\ell \in \mathcal{N}$ with the same positive constants $\{\bar{q}_\ell : \ell \in \mathcal{N}\}$ independent of Ω appearing in Theorem 2.4.2.

Then, the family $\{F_\varepsilon\}_\varepsilon$ Γ -converges with respect to the convergence in $L^2(\Omega)$ to the functional $F : L^2(\Omega) \rightarrow [0, +\infty]$ defined as

$$F(u) := \begin{cases} \int_{\partial^* A \cap \Omega} g(\nu_A) d\mathcal{H}^{d-1} & \text{if } u \in BV(\Omega; \{-1, 1\}), \text{ with } u = 2\chi_A - 1, \\ +\infty & \text{otherwise,} \end{cases} \quad (2.4.4)$$

where A is a set of finite perimeter in Ω , $\partial^* A$ is its reduced boundary, ν_A is its inner unit normal, and g is as defined in (2.4.3).

In the special case $q_1 = \dots = q_{k-1} = 0$, this result improves the corresponding one in Section 2.3 as it deals with arbitrary norms on the spaces of tensors, upon assuming stronger hypotheses on the double-well potential.

In order to compute the Γ -limit, we follow the strategy presented in [55]. The lower bound is proved by using the blow-up argument and relying on a method due to De Giorgi that allows to vary boundary conditions on converging sequences (Lemma 2.4.20). The construction of a recovery sequence is confined to the case of polyhedra, which are seen to be dense in energy for the limit F , being the surface energy density g upper semicontinuous (Proposition 2.4.6).

Remark 2.4.4. We point out that Theorem 2.4.1 is not available on cubes. In [55], where $k = 2$, the corresponding result is obtained by slicing. There, such method is well suited because, roughly speaking, the operatorial norm of a 1-tensor (the gradient) coincides with its Euclidean norm as a vector of $\mathbb{R}^d$. Because of the higher order of derivatives that our interpolation estimates involve, such argument seems to be difficult to adapt. As a consequence of the lack of interpolation estimates on cubes, the proofs and the statements of the lower bound (Proposition 2.4.19) and of a fundamental auxiliary result (Lemma 2.4.20) present several differences with respect to their counterparts in [55]. In particular, we will need to apply our interpolation inequalities on a C^1 subset of the cube Q_1^ν that coincides with such a cube inside the strip $\{x \in \mathbb{R}^d : |x \cdot \nu| \leq \frac{1}{4}\}$ which is close to the interface on which most of the energy is stored.

Remark 2.4.5. With some minor adaptations in the computations, we could also assume more in general that, for some positive constants α, β , the following conditions hold:

$$(H2') \quad W(s) \geq \alpha \min\{(s+1)^2, (s-1)^2\} \text{ for every } s \in \mathbb{R};$$

$$(H3') \quad W(s) \leq \beta W(t) + \beta \text{ for every } t \in \mathbb{R} \text{ and for every } s \in \mathbb{R} \text{ with } |s| \leq |t|,$$

having our main results still holding unchanged.

We conclude this subsection proving a characterization of the density introduced in (2.4.3). In view of the asymptotic analysis, the following proposition makes irrelevant the choice of the orthonormal basis $\{\nu_1, \dots, \nu_{d-1}, \nu\}$ fixed in the definition of Q_1^ν in Section 2.1 and then also in (2.4.2). In the following we let $SO(d)$ denote the group of rotations in $\mathbb{R}^d$.

Proposition 2.4.6. *For every $\nu \in \mathbb{S}^{d-1}$ and for every $R \in SO(d)$ such that $R\nu = \nu$ it holds*

$$g(\nu) = \inf \left\{ \int_{R(Q_1^\nu)} \left[\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \right] dx : v \circ R \in \mathcal{A}_\varepsilon^\nu, \varepsilon \in (0, 1) \right\} \quad (2.4.5)$$

$$= \inf \left\{ \int_{R'(Q_1^\nu)} \left[\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \right] dx : v \circ R' \in \mathcal{A}_\varepsilon^\nu, \varepsilon \in (0, 1), \right. \\ \left. R' \in SO(d) \text{ with } R'\nu = \nu \right\}. \quad (2.4.6)$$

Moreover, g is upper semicontinuous.

Proof. We first prove (2.4.6). Fix $\nu \in \mathbb{S}^{d-1}$ and let $\varepsilon \in (0, 1)$, $R' \in SO(d)$ satisfying $R'\nu = \nu$, and $v \circ R \in \mathcal{A}_\varepsilon^\nu$. We claim that for every $\eta \in (0, \varepsilon)$, there exists a function $w_\eta \in \mathcal{A}_\eta^\nu$ such that

$$\limsup_{\eta \rightarrow 0} \int_{Q_1^\nu} \left[\frac{1}{\eta} W(w_\eta) + \sum_{\ell=1}^k q_\ell \eta^{2\ell-1} |\nabla^{(\ell)} w_\eta|_\ell^2 \right] dx \leq \int_{R'(Q_1^\nu)} \left[\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \right] dx. \quad (2.4.7)$$

For the sake of exposition, we set $T := 1/\varepsilon$ and $\tilde{v}(x) := v(\frac{1}{T}x)$ for all $x \in R'(Q_T^\nu)$. For $S > T$, we introduce

$$\mathcal{Z}(T, S) := \{z = T(z_1\nu_1 + \dots + z_{d-1}\nu_{d-1}) : z + R'(Q_T^\nu) \subset Q_S^\nu, (z_1, \dots, z_{d-1}) \in \mathbb{Z}^{d-1}\},$$

where $\{\nu_1, \dots, \nu_{d-1}, \nu\}$ is the orthonormal basis that we arbitrarily fixed in the definitions (2.1) and (2.4.2), and

$$E(T, S) := \bigcup_{z \in \mathcal{Z}(T, S)} (z + R'(Q_T^\nu)).$$

Then, we define the function

$$\tilde{w}_S(x) := \begin{cases} \tilde{v}(x - z) & \text{if } x \in z + R'(Q_T^\nu) \text{ for some } z \in \mathcal{Z}(T, S), \\ \bar{u}^\nu(x) & \text{otherwise in } Q_S^\nu, \end{cases}$$

and we note that $\tilde{w} \in H^k(Q_S^\nu)$ with $\tilde{w}_S = \bar{u}^\nu$ close to ∂Q_S^ν . Upon introducing the localized functionals

$$F_1(u, B) := \int_B \left[W(u) + \sum_{\ell=1}^k q_\ell |\nabla^{(\ell)} u|_\ell^2 \right] dx$$

for $B \subset \mathbb{R}^d$ a Borel set, we have

$$\begin{aligned} \frac{F_1(\tilde{w}_S, Q_S^\nu)}{S^{d-1}} &= \frac{F_1(\tilde{w}_S, E(T, S)) + F_1(\tilde{w}_S, Q_S^\nu \setminus E(T, S))}{S^{d-1}} \\ &= \left(\sum_{z \in \mathcal{Z}(T, S)} \frac{F_1(\tilde{w}_S, z + R'(Q_T^\nu))}{S^{d-1}} \right) + \frac{F_1(\tilde{w}_S, Q_S^\nu \setminus E(T, S))}{S^{d-1}} \\ &= \#\mathcal{Z}(T, S) \frac{F_1(\tilde{v}, R'(Q_T^\nu))}{S^{d-1}} + \frac{\mathcal{H}^{d-1}(\pi^\nu(Q_S^\nu \setminus E(T, S)))}{S^{d-1}} c_0, \end{aligned}$$

having set $c_0 := \int_{-\infty}^{+\infty} [W(\bar{u}) + \sum_{\ell=1}^k q_\ell |\bar{u}^{(\ell)}|^2] dt$ and letting π^ν denote the orthogonal projection on the hyperplane $\nu^\perp$. We observe that

$$\lim_{S \rightarrow +\infty} \frac{\#\mathcal{Z}(T, S)}{(S/T)^{d-1}} = 1,$$

and

$$\mathcal{H}^{d-1}(\pi^\nu(Q_S^\nu \setminus E(T, S))) = S^{d-1} - \#\mathcal{Z}(T, S) T^{d-1},$$

therefore,

$$\limsup_{S \rightarrow +\infty} \frac{F_1(\tilde{w}_S, Q_S^\nu)}{S^{d-1}} \leq \frac{F_1(\tilde{v}, R(Q_T^\nu))}{T^{d-1}};$$

that is

$$\limsup_{S \rightarrow +\infty} \frac{1}{S^{d-1}} \int_{Q_S^\nu} \left[W(\tilde{w}_S) + \sum_{\ell=1}^k q_\ell |\nabla^{(\ell)} \tilde{w}_S|_\ell^2 \right] dx \leq \frac{1}{T^{d-1}} \int_{R(Q_T^\nu)} \left[W(\tilde{v}) + \sum_{\ell=1}^k q_\ell |\nabla^{(\ell)} \tilde{v}|_\ell^2 \right] dx. \quad (2.4.8)$$

Finally, we set

$$\eta := \frac{1}{S}, \quad w_\eta(x) := \tilde{w}_{\frac{1}{\eta}}\left(\frac{1}{\eta}x\right) \text{ for all } x \in Q_1^\nu,$$

so that $w_\eta \in \mathcal{A}_\eta^\nu$. By applying the change of variables $y := \eta x$ on the left-hand side and $y := \varepsilon x$ on the right-hand side, (2.4.8) can be written as

$$\limsup_{\eta \rightarrow 0} \int_{Q_1^\nu} \left[\frac{1}{\eta} W(w_\eta) + \sum_{\ell=1}^k q_\ell \eta^{2\ell-1} |\nabla^{(\ell)} w_\eta|_\ell^2 \right] dy \leq \int_{R'(Q_1^\nu)} \left[\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \right] dy,$$

which is (2.4.7). By the arbitrariness of ε, R' , and v , inequality (2.4.7) and a diagonal argument yield (2.4.6).

As for the proof of (2.4.5), consider $R_1, R_2 \in SO(d)$ satisfying $R_1\nu = R_2\nu = \nu$ and $v \circ R_1 \in \mathcal{A}_\varepsilon^\nu$. With a similar argument to the one employed above, we may prove that for every $\eta \in (0, \varepsilon)$, there exists a function w_η such that $w_\eta \circ R_2 \in \mathcal{A}_\eta^\nu$ and

$$\limsup_{\eta \rightarrow 0} \int_{R_2(Q_1^\nu)} \left[\frac{1}{\eta} W(w_\eta) + \sum_{\ell=1}^k q_\ell \eta^{2\ell-1} |\nabla^{(\ell)} w_\eta|_\ell^2 \right] dx \leq \int_{R_1(Q_1^\nu)} \left[\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \right] dx,$$

which, by the arbitrariness of the rotations R_1, R_2 and of the function v , proves (2.4.5).

Finally, we prove that g is upper semicontinuous. Let $\nu \in \mathbb{S}^{d-1}$ and $\{\nu_n\}_n \subset \mathbb{S}^{d-1}$ such that $\nu_n \rightarrow \nu$. We aim to show that

$$\limsup_{n \rightarrow +\infty} g(\nu_n) \leq g(\nu).$$

Fix $\eta > 0$ and let $\varepsilon \in (0, 1)$ and $v \in \mathcal{A}_\varepsilon^\nu$ such that

$$g(\nu) + \eta \geq \int_{Q_1^\nu} \left[\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \right] dx. \quad (2.4.9)$$

For each $n \in \mathbb{N}$ we can find $R_n = (R_n^{i,j})_{i,j=1}^d \in SO(d)$ such that $R_n\nu = \nu_n$ and $R_n \rightarrow I_d$, where we let I_d denote the identity matrix in $\mathbb{R}^{d \times d}$. Consider the cube $R_n(Q_1^\nu)$ and, for each n , a rotation $S_n \in SO(d)$ such that

$$S_n\nu_n = \nu_n \text{ and } R_n(Q_1^\nu) = S_n(Q_1^{\nu_n}).$$

Define $v_n(x) := v(R_n^T x)$ for every $x \in R_n(Q_1^\nu) = S_n(Q_1^{\nu_n})$ and note that $v_n \circ S_n \in \mathcal{A}_\varepsilon^{\nu_n}$. By (2.4.6) and a change of variables

$$\begin{aligned} g(\nu_n) &\leq \int_{S_n(Q_1^{\nu_n})} \left[\frac{1}{\varepsilon} W(v_n(x)) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v_n(x)|_\ell^2 \right] dx \\ &= \int_{R_n(Q_1^\nu)} \left[\frac{1}{\varepsilon} W(v_n(x)) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v_n(x)|_\ell^2 \right] dx \\ &= \int_{Q_1^\nu} \left[\frac{1}{\varepsilon} W(v(x)) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v_n(R_n x)|_\ell^2 \right] dx. \end{aligned} \quad (2.4.10)$$

We show that, for each $\ell \in \{1, \dots, k\}$ and for a.e. $x \in Q_1^\nu$, we have $\nabla^{(\ell)} v_n(R_n x) \rightarrow \nabla^{(\ell)} v(x)$ in the space of ℓ -tensors as $n \rightarrow +\infty$. By the equivalence of the norms, to prove this it is sufficient to show that for a.e. $x \in Q_1^\nu$ it holds

$$\lim_{n \rightarrow +\infty} \nabla^{(\ell)} v_n(R_n x)(e_{i_1}, \dots, e_{i_\ell}) = \nabla^{(\ell)} v(x)(e_{i_1}, \dots, e_{i_\ell}) \text{ for every } (i_1, \dots, i_\ell) \in \{1, \dots, d\}^\ell. \quad (2.4.11)$$

Let us fix $(i_1, \dots, i_\ell) \in \{1, \dots, d\}^\ell$. For a.e. $x \in Q_1^\nu$ we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \nabla^{(\ell)} v_n(R_n x)(e_{i_1}, \dots, e_{i_\ell}) &= \lim_{n \rightarrow +\infty} \frac{\partial^\ell v_n}{\partial x_{i_1} \dots \partial x_{i_\ell}}(R_n x) \\ &= \lim_{n \rightarrow +\infty} \sum_{j_1=1}^d \dots \sum_{j_\ell=1}^d R_n^{i_1, j_1} \dots R_n^{i_\ell, j_\ell} \frac{\partial^\ell v}{\partial x_{j_1} \dots \partial x_{j_\ell}}(x) \\ &= \sum_{j_1=1}^d \dots \sum_{j_\ell=1}^d \delta_{i_1 j_1} \dots \delta_{i_\ell j_\ell} \frac{\partial^\ell v}{\partial x_{j_1} \dots \partial x_{j_\ell}}(x) \\ &= \frac{\partial^\ell v}{\partial x_{i_1} \dots \partial x_{i_\ell}}(x) = \nabla^{(\ell)} v(x)(e_{i_1}, \dots, e_{i_\ell}), \end{aligned}$$

and this equality proves (2.4.11).

To conclude, we first observe that, since $R_n \rightarrow I_d$, we may assume that $|R_n^{i,j}| \leq 2$ for all $i, j \in \{1, \dots, d\}$ and $n \in \mathbb{N}$ so that

$$\begin{aligned} |\nabla^{(\ell)} v_n(R_n x)|_\ell &\leq C \max \left\{ \left| \frac{\partial^\ell v}{\partial x_{i_1} \dots \partial x_{i_\ell}}(R_n x) \right| : (i_1, \dots, i_\ell) \in \{1, \dots, d\}^\ell \right\} \\ &= C \max \left\{ \left| \sum_{j_1=1}^d \dots \sum_{j_\ell=1}^d R_n^{i_1, j_1} \dots R_n^{i_\ell, j_\ell} \frac{\partial^\ell v}{\partial x_{j_1} \dots \partial x_{j_\ell}}(x) \right| : (i_1, \dots, i_\ell) \in \{1, \dots, d\}^\ell \right\} \\ &\leq C \max \left\{ \left| \frac{\partial^\ell v}{\partial x_{i_1} \dots \partial x_{i_\ell}}(x) \right| : (i_1, \dots, i_\ell) \in \{1, \dots, d\}^\ell \right\} \leq C |\nabla^{(\ell)} v(x)|_\ell, \end{aligned} \quad (2.4.12)$$

where C is a positive constant that may vary from line to line which only depends on d and ℓ as it arises by the equivalence of the tensor norms. Finally, combining (2.4.11) with (2.4.12), we apply the Dominated Convergence Theorem in (2.4.10), so that by (2.4.9) we obtain

$$\begin{aligned} \limsup_{n \rightarrow +\infty} g(\nu_n) &\leq \lim_{n \rightarrow +\infty} \int_{Q_1^\nu} \left[\frac{1}{\varepsilon} W(v(x)) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v_n(R_n x)|_\ell^2 \right] dx \\ &= \int_{Q_1^\nu} \left[\frac{1}{\varepsilon} W(v(x)) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v(x)|_\ell^2 \right] dx \leq g(\nu) + \eta. \end{aligned}$$

The arbitrariness of η concludes the proof. $\square$

Remark 2.4.7. With the previous proof, we also showed that

$$g(\nu) = \lim_{\varepsilon \rightarrow 0} \inf \left\{ \int_{Q_1^\nu} \left[\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \right] dx : v \in \mathcal{A}_\varepsilon^\nu \right\}.$$

This observation will be useful in the last subsection in order to study the density g in the one-dimensional case.

2.4.2 Interpolation inequalities

This subsection is devoted to the proof of Theorem 2.4.1. Furthermore, we obtain a useful result (Corollary 2.4.14) that we will use in the proof of Theorem 2.4.2. Throughout this subsection, we keep track of the dependence of the constants on the involved parameters with the only exceptions of d , the dimension of the ambient Euclidean space, and k , the highest order of derivation.

We start by mentioning a classical interpolation estimate on intervals (see [96, Theorem 7.41]).

Theorem 2.4.8. *Let $I \subset \mathbb{R}$ be a bounded open interval and let k be a positive integer. Then there exists a positive constant c independent of I such that*

$$\int_I |u^{(k-1)}|^2 dt \leq c |I|^{-2(k-1)} \int_I u^2 dt + c \int_I |u^{(k)}|^2 dt$$

for every $u \in H^k(I)$.

We adapt the argument of Lemma 3.1 in [56] to bound the square of the L^2 -norm of the derivative of order $k-1$ by means of the integral of the double-well potential and the square of the L^2 -norm of the derivative of order k . Then, we deduce the same kind of estimate for any derivative of intermediate order $\ell \in \{1, \dots, k-1\}$.

Lemma 2.4.9. *Let $I \subset \mathbb{R}$ be a bounded open interval, let $k > 1$ be an integer, and assume that (H2) is satisfied. Then there exists a positive constant q'_{k-1} independent of I such that*

$$q \int_I |u^{(k-1)}(t)|^2 dt \leq |I|^{-2(k-1)} \int_I W(u) dt + |I|^2 \int_I |u^{(k)}|^2 dt$$

for every $q < q'_{k-1}$ and $u \in H^k(I)$.

Proof. It is sufficient to prove the statement for $I = (0, 1)$, the general case being obtained by translating and rescaling. We set

$$z := \int_0^1 u^{(k-1)} dt,$$

and by the Fundamental Theorem of Calculus we infer that

$$|u^{(k-1)}(t) - z| \leq \int_0^1 |u^{(k)}| dt \quad \text{for all } t \in (0, 1). \quad (2.4.13)$$

As a consequence

$$\int_0^1 |u^{(k-1)}|^2 dt \leq 2 \int_0^1 |u^{(k)}|^2 dt + 2z^2;$$

therefore, the proof is complete if we prove that

$$z^2 \leq C \int_0^1 [W(u) + |u^{(k)}|^2] dt \quad (2.4.14)$$

for some $C > 0$. If $z^2 \leq 4 \int_0^1 |u^{(k)}|^2 dt$, then (2.4.14) is immediately satisfied. Otherwise, by Jensen's inequality, we have that $|z| > 2 \int_0^1 |u^{(k)}| dt$, which together with (2.4.13), yields

$$|u^{(k-1)}(t)| \geq \frac{|z|}{2} > 0 \quad \text{for all } t \in (0, 1). \quad (2.4.15)$$

Consider the intervals $(\frac{n}{k}, \frac{n+1}{k})$, $n \in \{0, \dots, k-1\}$. We observe that there exists $n^* \in \{0, \dots, k-1\}$ such that $u \neq 0$ on $(\frac{n^*}{k}, \frac{n^*+1}{k})$. To see this, assume by contradiction that u vanishes on each of these intervals; then, u has at least k zeros on I . By Lagrange's Theorem, this implies that u' vanishes at $k-1$ points of I , and then, iterating this argument, we obtain that $u^{(k-1)}$ vanishes at a point of I , contradicting (2.4.15). We suppose $u > 0$ in $(\frac{n^*}{k}, \frac{n^*+1}{k})$, the case $u < 0$ being analogous. Applying Theorem 2.4.8 to the function $u-1$ on the interval $(\frac{n^*}{k}, \frac{n^*+1}{k})$ and resorting to (H2), we obtain

$$\begin{aligned} \int_{\frac{n^*}{k}}^{\frac{n^*+1}{k}} |u^{(k-1)}|^2 dt &\leq ck^{2(k-1)} \int_{\frac{n^*}{k}}^{\frac{n^*+1}{k}} (u-1)^2 dt + c \int_{\frac{n^*}{k}}^{\frac{n^*+1}{k}} |u^{(k)}|^2 dt \\ &\leq ck^{2(k-1)} \int_{\frac{n^*}{k}}^{\frac{n^*+1}{k}} W(u) dt + c \int_{\frac{n^*}{k}}^{\frac{n^*+1}{k}} |u^{(k)}|^2 dt, \end{aligned}$$

therefore, taking into account (2.4.15),

$$\begin{aligned} z^2 &\leq 4k \int_{\frac{n^*}{k}}^{\frac{n^*+1}{k}} |u^{(k-1)}|^2 dt \\ &\leq 4ck^{2k-1} \int_{\frac{n^*}{k}}^{\frac{n^*+1}{k}} W(u) dt + 4ck \int_{\frac{n^*}{k}}^{\frac{n^*+1}{k}} |u^{(k)}|^2 dt, \end{aligned}$$

which, by the positivity of W , proves (2.4.14) and concludes the proof. $\square$

Corollary 2.4.10. *Let $I \subset \mathbb{R}$ be a bounded open interval, let $k > 1$ be an integer, and assume that (H2) is satisfied. Then, for every $\ell \in \{1, \dots, k-1\}$, there exists a positive constant q'_ℓ independent of I such that*

$$q \int_I |u^{(\ell)}|^2 dt \leq |I|^{-2\ell} \int_I W(u) dt + |I|^{2(k-\ell)} \int_I |u^{(k)}|^2 dt$$

for every $q < q'_\ell$ and $u \in H^k(I)$.

Proof. The thesis follows by applying iteratively Lemma 2.4.9. $\square$

We adapt the interpolation inequality at scale ε , in view of its application to functionals (2.4.1).

Proposition 2.4.11. *Let $I \subset \mathbb{R}$ be a bounded open interval, let $k > 1$ be an integer, and assume that (H2) is satisfied. Then, for every $\ell \in \{1, \dots, k-1\}$, there exists a positive constant q_ℓ'' independent of I such that, for every $\varepsilon \in (0, |I|/2)$, it holds*

$$q\varepsilon^{2\ell} \int_I |u^{(\ell)}|^2 dt \leq \int_I \left[W(u) + \varepsilon^{2k} |u^{(k)}|^2 \right] dt$$

for every $q < q_\ell''$ and $u \in H^k(I)$.

Proof. We set $v(t) := u(\varepsilon t)$ for $t \in \frac{1}{\varepsilon}I$ so that

$$\varepsilon^{2\ell-1} \int_I |u^{(\ell)}|^2 dt = \int_{\frac{1}{\varepsilon}I} |v^{(\ell)}|^2 dt.$$

Then, we set $i_\varepsilon := \lfloor \frac{|I|}{\varepsilon} \rfloor$ and we subdivide the rescaled interval $\frac{1}{\varepsilon}I$ in i_ε pairwise disjoint open intervals $I_\varepsilon^n, n \in \{1, \dots, i_\varepsilon\}$, each of length $\frac{|I|}{\varepsilon i_\varepsilon}$. Then we apply Corollary 2.4.10 on each subinterval I_ε^n with $q < q_\ell''$ and we obtain

$$\begin{aligned} \varepsilon^{2\ell-1} \int_I |u^{(\ell)}|^2 dt &= \sum_{n=1}^{i_\varepsilon} \int_{I_\varepsilon^n} |v^{(\ell)}|^2 dt \\ &\leq \sum_{n=1}^{i_\varepsilon} \frac{1}{q} \left\{ |I_\varepsilon^n|^{-2\ell} \int_{I_\varepsilon^n} W(v) dt + |I_\varepsilon^n|^{2(k-\ell)} \int_{I_\varepsilon^n} |v^{(k)}|^2 dt \right\} \\ &\leq \frac{1}{q} \left\{ 2^{2\ell} \int_{\frac{1}{\varepsilon}I} W(v) dt + \left(\frac{3}{2}\right)^{2(k-\ell)} \int_{\frac{1}{\varepsilon}I} |v^{(k)}|^2 dt \right\} \\ &\leq \frac{\max\{2^{2\ell}, (3/2)^{2(k-\ell)}\}}{q} \int_I \left[\frac{W(u)}{\varepsilon} + \varepsilon^{2k-1} |u^{(k)}|^2 \right] dt, \end{aligned}$$

where the second inequality follows by the fact $|I_\varepsilon^n| \in (\frac{1}{2}, \frac{3}{2})$ if $\varepsilon \in (0, |I|/2)$.

We multiply the above inequality by ε to obtain that

$$\tilde{q}\varepsilon^{2\ell} \int_I |u^{(\ell)}|^2 dt \leq \int_I \left[W(u) + \varepsilon^{2k} |u^{(k)}|^2 \right] dt$$

for every $\tilde{q} < q_\ell''$ and $\varepsilon \in (0, |I|/2)$, where we set

$$q_\ell'' := \frac{q_\ell'}{\max\{2^{2\ell}, (3/2)^{2(k-\ell)}\}};$$

therefore, the proof is complete. $\square$

The following corollary proves Theorem 2.4.1 in dimension 1.

Corollary 2.4.12. *Let $\Omega \subset \mathbb{R}$ be the union of finitely many bounded open disjoint intervals, let $k > 1$ be an integer, and assume that (H2) is satisfied. Then, denoting by $|I_o|$ the length of the shortest connected component of Ω , for every $\ell \in \{1, \dots, k-1\}$ and for every $\varepsilon \in (0, |I_o|/2)$, it holds*

$$q\varepsilon^{2\ell} \int_\Omega |u^{(\ell)}|^2 dt \leq \int_\Omega \left[W(u) + \varepsilon^{2k} |u^{(k)}|^2 \right] dt$$

for every $q < q_\ell''$ and $u \in H^k(\Omega)$.

Proof. The proof follows by an immediate application of Proposition 2.4.11. $\square$

Finally, we prove the main result of this subsection. Below, we will make use of some notation about slicing that has been previously introduced.

Proof of Theorem 2.4.1. It suffices to prove the part of the statement concerning the operatorial norms, since the other case follows by the equivalence of the norms. We follow the proof of Theorem 1.2 in [55]. For $r > 0$, $\xi \in \mathbb{S}^{d-1}$, and $y \in \Omega^\xi$, we define $\Omega_r^{\xi,y}$ as the finite union of the connected components of the slice $\Omega^{\xi,y}$ having length larger than r . Then we set $\Omega(\xi, r) := \{y + t\xi \in \mathbb{R}^d : y \in \Omega^\xi, t \in \Omega_r^{\xi,y}\}$ and we observe that $(\Omega(\xi, r))^{\xi,y} = \Omega_r^{\xi,y}$.

Applying Corollary 2.4.12 we obtain

$$\tilde{q}\varepsilon^{2\ell} \int_{\Omega_r^{\xi,y}} |(u^{\xi,y})^{(\ell)}|^2 dt \leq \int_{\Omega_r^{\xi,y}} \left[W(u^{\xi,y}) + \varepsilon^{2k} |(u^{\xi,y})^{(k)}|^2 \right] dt$$

for all $\tilde{q} < q_\ell''$ and $\varepsilon < r/2$. We integrate the above inequality on $\xi^\perp$ to get

$$\tilde{q}\varepsilon^{2\ell} \int_{\Omega(\xi,r)} |(\nabla^{(\ell)}u)(\xi, \dots, \xi)|^2 dx \leq \int_{\Omega} \left[W(u) + \varepsilon^{2k} \|\nabla^{(k)}u\|_k^2 \right] dx,$$

and setting $A(\xi, r) := \Omega \setminus \Omega(\xi, r)$ we obtain

$$\tilde{q}\varepsilon^{2\ell} \int_{\Omega} |(\nabla^{(\ell)}u)(\xi, \dots, \xi)|^2 dx - \tilde{q}\varepsilon^{2\ell} \int_{A(\xi,r)} \|\nabla^{(\ell)}u\|_\ell^2 dx \leq \int_{\Omega} \left[W(u) + \varepsilon^{2k} \|\nabla^{(k)}u\|_k^2 \right] dx.$$

Averaging this inequality on $\mathbb{S}^{d-1}$ yields

$$\begin{aligned} \varepsilon^{2\ell} \frac{\tilde{q}}{\sigma_{d-1}} \int_{\mathbb{S}^{d-1}} \int_{\Omega} |(\nabla^{(\ell)}u)(\xi, \dots, \xi)|^2 dx d\mathcal{H}^{d-1}(\xi) - \varepsilon^{2\ell} \frac{\tilde{q}}{\sigma_{d-1}} \int_{\mathbb{S}^{d-1}} \int_{A(\xi,r)} \|\nabla^{(\ell)}u\|_\ell^2 dx d\mathcal{H}^{d-1}(\xi) \\ \leq \int_{\Omega} \left[W(u) + \varepsilon^{2k} \|\nabla^{(k)}u\|_k^2 \right] dx. \end{aligned}$$

By a simple compactness argument, there exists a positive constant $\tau(\ell)$ such that

$$\frac{1}{\sigma_{d-1}} \int_{\mathbb{S}^{d-1}} |T(\xi, \dots, \xi)|^2 d\mathcal{H}^{d-1}(\xi) \geq \tau(\ell) \|T\|_\ell^2$$

for every symmetric ℓ -tensor T . Therefore, by Fubini's Theorem,

$$\frac{1}{\sigma_{d-1}} \int_{\mathbb{S}^{d-1}} \int_{\Omega} |(\nabla^{(\ell)}u)(\xi, \dots, \xi)|^2 dx d\mathcal{H}^{d-1}(\xi) \geq \tau(\ell) \int_{\Omega} \|\nabla^{(\ell)}u\|_\ell^2 dx. \quad (2.4.16)$$

Again by Fubini's Theorem, we have

$$\frac{1}{\sigma_{d-1}} \int_{\mathbb{S}^{d-1}} \int_{A(\xi,r)} \|\nabla^{(\ell)}u\|_\ell^2 dx d\mathcal{H}^{d-1}(\xi) = \frac{1}{\sigma_{d-1}} \int_{\Omega} \|\nabla^{(\ell)}u\|_\ell^2 \mathcal{H}^{d-1}(D(x, r)) dx,$$

where $D(x, r) := \{\xi \in \mathbb{S}^{d-1} : x \in A(\xi, r)\}$; hence, to conclude the proof, we have to select r small enough so that $\mathcal{H}^{d-1}(D(x, r))$ is small uniformly in x . Given $x \in \partial\Omega$, we let $C_R(x)$ denote the cylinder having height $2R$ and radius R with axis parallel to the normal $\nu(x)$ to $\partial\Omega$ at x .

Let $q < \tau(\ell)q_\ell''$, and fix $\tilde{q}$ such that $\frac{q}{\tau(\ell)} < \tilde{q} < q_\ell''$. Fix $\eta = \eta(q) > 0$ such that

$$q < \tilde{q}(\tau(\ell) - \eta); \quad (2.4.17)$$

since Ω has C^1 boundary, there exists R such that, for every $x \in \partial\Omega$, it holds $C_R(x) \cap \partial\Omega$ is the graph of a C^1 function with respect to the base of the cylinder, and moreover,

$$\mathcal{H}^{d-1}(\{\xi \in \mathbb{S}^{d-1} : \xi \in T_{\partial\Omega}(y), y \in C_R(x) \cap \partial\Omega\}) \leq \eta\sigma_{d-1}.$$

Employing the same argument presented in the proof of Theorem 1.2 in [55], it holds that,

$$D(x, r) \subseteq \{\xi \in \mathbb{S}^{d-1} : \xi \in T_{\partial\Omega}(y), y \in C_R(x) \cap \partial\Omega\}$$

up to choosing $r < R/2$; as a consequence, $\mathcal{H}^{d-1}(D(x, r)) \leq \eta$ for all $x \in \partial\Omega$ if $r < R/2$, and then

$$\frac{1}{\sigma_{d-1}} \int_{\mathbb{S}^{d-1}} \int_{A(\xi, r)} \|\nabla^{(\ell)} u\|_\ell^2 dx d\mathcal{H}^{d-1}(\xi) \leq \eta \int_\Omega \|\nabla^{(\ell)} u\|_\ell^2 dx, \quad (2.4.18)$$

for all $r < R/2$.

Gathering (2.4.16) and (2.4.18), we obtain that

$$\tilde{q}(\tau(\ell) - \eta) \varepsilon^{2\ell} \int_\Omega \|\nabla^{(\ell)} u\|_\ell^2 dx \leq \int_\Omega \left[W(u) + \varepsilon^{2k} \|\nabla^{(k)} u\|_k^2 \right] dx \quad (2.4.19)$$

for all $\tilde{q} < q_\ell''$ and $\varepsilon < R/2$, where $R = R(\eta, \Omega)$.

Recalling (2.4.17), by (2.4.19) we get

$$q \varepsilon^{2\ell} \int_\Omega \|\nabla^{(\ell)} u\|_\ell^2 dx \leq \int_\Omega \left[W(u) + \varepsilon^{2k} \|\nabla^{(k)} u\|_k^2 \right] dx.$$

The above inequality is valid for all $q < \tau(\ell) q_\ell''$ and $\varepsilon < R(\eta, \Omega)/2$. Then, the thesis holds with $q_\ell^* := \tau(\ell) q_\ell''$ and $\varepsilon_0 := R(\eta, \Omega)/2$, which in turn, since η depends on q , only depends on q and Ω . $\square$

Remark 2.4.13. We are not able to prove the interpolation inequality on an open cube. The same slicing arguments employed in [55] cannot be applied in our case because of the high order of the derivatives that we want to estimate. In what follows, it will be important to apply Theorem 2.4.1 on a class of C^1 sets $\{\Omega_\rho : \rho \in (\rho_1, \rho_2)\}$ for some $\rho_1 < \rho_2$ for which the threshold $\varepsilon_0 = \varepsilon_0(q, \Omega_\rho)$ can be chosen independently of $\rho \in (\rho_1, \rho_2)$. Inspecting the proof of Theorem 2.4.1, this amounts to prove that for each properly fixed q , given $\eta > 0$ such that

$$q < \tilde{q}(\tau(\ell) - \eta),$$

there exists $R > 0$ such that, for every $\rho \in (\rho_1, \rho_2)$ and for every $x \in \partial\Omega_\rho$, it holds that

$$C_R(x) \cap \partial\Omega_\rho \text{ is the graph of a } C^1 \text{ function} \quad (2.4.20)$$

and

$$\mathcal{H}^{d-1}(\{\xi \in \mathbb{S}^{d-1} : \xi \in T_{\partial\Omega_\rho}(y), y \in C_R(x) \cap \partial\Omega_\rho\}) \leq \eta \sigma_{d-1}. \quad (2.4.21)$$

We claim that the above conditions are satisfied if we consider Ω a bounded open set with C^2 boundary and

$$\Omega_\rho := \{x \in \Omega : \text{dist}(x, \partial\Omega) < \rho\}$$

with $\rho \in (0, \rho_0)$, for a certain ρ_0 to be determined.

First, we fix ρ^* small enough so that $\{x \in \Omega : \text{dist}(x, \partial\Omega) = \rho\}$ is a $(d-1)$ -manifold of class C^1 and there exists a C^1 projection from Ω_ρ onto $\partial\Omega$ for every $\rho < 2\rho^*$.

Let $\eta' = \eta'(\eta, d)$ be such that

$$\mathcal{H}^{d-1}(\{\xi \in \mathbb{S}^{d-1} : |\xi_d| \leq \eta'\}) \leq \eta \sigma_{d-1}. \quad (2.4.22)$$

Since $\partial\Omega$ is of class C^1 , for every $x \in \partial\Omega$ there exist $R > 0$ and a C^1 function ψ defined on the base of $C_R(x)$ such that

$$C_R(x) \cap \partial\Omega = C_R(x) \cap \text{graph } \psi.$$

By a rotation argument, we can further assume that $\nabla\psi(x) = 0$, and upon taking a smaller R , that

$$\|\nabla\psi\|_{L^\infty(C_R(x))} \leq \eta'.$$

Moreover, since $\partial\Omega$ is compact, we can select such R as above independent of $x \in \partial\Omega$. Without loss of generality, we further assume that $R < \rho^*$.

Now, let $x \in \partial\Omega$ and $0 < \rho < R/4$ and note that

$$\{y \in C_{R-\rho}(x) : \text{dist}(y, \text{graph } \psi) = \rho\} = C_{R-\rho}(x) \cap \partial\Omega_\rho.$$

Then, letting $x_\rho \in \partial\Omega_\rho$ denote the unique point such that $|x - x_\rho| = \rho$, we have that $C_{R-2\rho}(x_\rho) \subset C_{R-\rho}(x)$, which implies

$$\{y \in C_{R-2\rho}(x_\rho) : \text{dist}(y, \text{graph } \psi) = \rho\} = C_{R-2\rho}(x_\rho) \cap \partial\Omega_\rho.$$

We also note that there exists a C^1 function ψ_ρ such that

$$\{y \in C_{R-2\rho}(x_\rho) : \text{dist}(y, \text{graph } \psi) = \rho\} = C_{R-2\rho}(x_\rho) \cap \text{graph } \psi_\rho,$$

and moreover

$$\|\nabla\psi_\rho\|_{L^\infty(C_{R-2\rho}(x_\rho))} \leq \|\nabla\psi\|_{L^\infty(C_{R-2\rho}(x_\rho))} \leq \|\nabla\psi\|_{L^\infty(C_R(x))}.$$

Recalling that there exists a C^1 projection from Ω_ρ onto $\partial\Omega$ for every $\rho \in (0, 2\rho^*)$, the above observations prove that for every $\rho \in (0, R/4)$ and for every $x \in \partial\Omega_\rho$, it holds that

$$C_{R-2\rho}(x) \cap \partial\Omega_\rho = C_{R-2\rho}(x) \cap \text{graph } \psi_\rho,$$

where ψ_ρ is a C^1 function such that

$$\|\nabla\psi_\rho\|_{L^\infty(C_{R-2\rho}(x))} \leq \|\nabla\psi\|_{L^\infty(C_{R-2\rho}(x))} \leq \eta'.$$

As ρ ranges in $(0, R/4)$, in particular we obtain

$$C_{R/2}(x) \cap \partial\Omega_\rho = C_{R/2}(x) \cap \text{graph } \psi_\rho \quad (2.4.23)$$

and

$$\|\nabla\psi_\rho\|_{L^\infty(C_{R/2}(x))} \leq \eta'. \quad (2.4.24)$$

We conclude that our claim is proved with $\rho_0 = R/4$. Indeed, (2.4.20) follows by (2.4.23), and we are left with proving that (2.4.24) implies (2.4.21). To see this, note that for fixed $\rho \in (0, R/4)$ and $x \in \partial\Omega_\rho$, (2.4.24) implies that

$$\begin{aligned} & \{\xi \in \mathbb{S}^{d-1} : \xi \in T_{\partial\Omega_\rho}(y), y \in C_{R/2}(x) \cap \partial\Omega_\rho\} \\ & \subseteq \{\xi \in \mathbb{S}^{d-1} : |\xi_d| = |\nabla\varphi_\rho(y_1, \dots, y_{d-1}) \cdot (\xi_1, \dots, \xi_{d-1})|, y \in C_{R/2}(x) \cap \partial\Omega_\rho\} \\ & \subseteq \{\xi \in \mathbb{S}^{d-1} : |\xi_d| \leq \eta'\}; \end{aligned}$$

therefore, by (2.4.22) we conclude that

$$\mathcal{H}^{d-1}(\{\xi \in \mathbb{S}^{d-1} : \xi \in T_{\partial\Omega_\rho}(y), y \in C_{R/2}(x) \cap \partial\Omega_\rho\}) \leq \eta \mathcal{H}^{d-1}(\mathbb{S}^{d-1}).$$

Being the above estimates uniform in ρ and x , (2.4.21) is satisfied.

The following corollary will come into play in the forthcoming subsections.

Corollary 2.4.14. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with C^1 boundary, let $k > 1$ be an integer, and let $\mathcal{N} := \{\ell \in \{1, \dots, k-1\} : q_\ell \leq 0\}$. Assume that $\mathcal{N} \neq \emptyset$ and that (H2) is satisfied. There exist positive constants $\{\bar{q}_\ell : \ell \in \mathcal{N}\}$ independent of Ω such that if $q_\ell > -\bar{q}_\ell$ for every $\ell \in \mathcal{N}$ there exist positive constants $\varepsilon_0 = \varepsilon_0(\{q_\ell : \ell \in \mathcal{N}\}, \Omega)$ and $\delta = \delta(\{q_\ell : \ell \in \mathcal{N}\})$ such that it holds*

$$\int_{\Omega} \left[\frac{1}{\varepsilon} W(u) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2 \right] dx \geq \delta \int_{\Omega} \left[\frac{1}{\varepsilon} W(u) + \sum_{\ell=1}^k \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2 \right] dx$$

for every $\varepsilon \in (0, \varepsilon_0)$ and $u \in H^k(\Omega)$. In particular, letting $\tilde{q}_1^*, \dots, \tilde{q}_{k-1}^*$ denote the same positive constants appearing in Theorem 2.4.1 and given $\{\alpha_\ell : \ell \in \mathcal{N}\}$ such that $\alpha_\ell > 0$ for all $\ell \in \mathcal{N}$ and $\sum_{\ell \in \mathcal{N}} \alpha_\ell = 1$, it holds that $\bar{q}_\ell = \alpha_\ell \tilde{q}_\ell^*$ for all $\ell \in \mathcal{N}$.

Proof. By Theorem 2.4.1, for every $\ell \in \{1, \dots, k-1\}$ there exists a positive constant $\tilde{q}_\ell^*$ independent of Ω such that for every $q > -\tilde{q}_\ell^*$ there exists $\varepsilon_0 = \varepsilon_0(q, \Omega)$ such that it holds

$$\int_{\Omega} \left[\frac{1}{\varepsilon} W(u) + \varepsilon^{2k-1} |\nabla^{(k)} u|_k^2 \right] dx + q \int_{\Omega} \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2 dx \geq 0 \quad (2.4.25)$$

for every $\varepsilon \in (0, \varepsilon_0)$ and $u \in H^k(\Omega)$.

Let $\mathcal{N} := \{\ell \in \{1, \dots, k-1\} : q_\ell \leq 0\}$ and consider $\{\alpha_\ell : \ell \in \mathcal{N}\}$ such that $\alpha_\ell > 0$ for all $\ell \in \mathcal{N}$ and $\sum_{\ell \in \mathcal{N}} \alpha_\ell = 1$. Suppose that $q_\ell > -\alpha_\ell \tilde{q}_\ell^*$ for every $\ell \in \mathcal{N}$ and let δ be a positive real number smaller than 1 such that

$$\frac{1}{\alpha_\ell} \frac{q_\ell - \delta}{1 - \delta} > -\tilde{q}_\ell^* \quad (2.4.26)$$

for all $\ell \in \mathcal{N}$. Note that

$$\begin{aligned} & \alpha_\ell \frac{W(u)}{\varepsilon} + q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell + \alpha_\ell \varepsilon^{2k-1} |\nabla^{(k)} u|_k \\ &= (1 - \delta) \left(\alpha_\ell \frac{W(u)}{\varepsilon} + \frac{q_\ell - \delta}{1 - \delta} \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell + \alpha_\ell \varepsilon^{2k-1} |\nabla^{(k)} u|_k \right) \\ & \quad + \delta \left(\alpha_\ell \frac{W(u)}{\varepsilon} + \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell + \alpha_\ell \varepsilon^{2k-1} |\nabla^{(k)} u|_k \right). \end{aligned}$$

We integrate the above equality and, taking into account (2.4.26), we apply (2.4.25) with $q = \frac{1}{\alpha_\ell} \frac{q_\ell - \delta}{1 - \delta}$ to obtain

$$\begin{aligned} & \int_{\Omega} \left[\alpha_\ell \frac{W(u)}{\varepsilon} + q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell + \alpha_\ell \varepsilon^{2k-1} |\nabla^{(k)} u|_k \right] dx \\ & \geq \delta \int_{\Omega} \left[\alpha_\ell \frac{W(u)}{\varepsilon} + \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell + \alpha_\ell \varepsilon^{2k-1} |\nabla^{(k)} u|_k \right] dx \end{aligned} \quad (2.4.27)$$

for every $\ell \in \mathcal{N}$, $\varepsilon < \varepsilon_0(q_\ell, \Omega)$.

Clearly, for $\ell \notin \mathcal{N}$ the coefficient q_ℓ is strictly positive; hence it holds

$$\int_{\Omega} q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2 dx \geq \delta \int_{\Omega} q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2 dx \quad (2.4.28)$$

since $\delta < 1$. Therefore, summing over $\ell \in \{1, \dots, k-1\}$ and gathering (2.4.27) and (2.4.28), upon considering a smaller ε_0 depending on $\{q_\ell : \ell \in \mathcal{N}\}$ and Ω , we get

$$\int_{\Omega} \left[\frac{W(u)}{\varepsilon} + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2 \right] dx \geq \delta \int_{\Omega} \left[\frac{W(u)}{\varepsilon} + \sum_{\ell=1}^k \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_\ell^2 \right] dx$$

for every $\varepsilon < \varepsilon_0$, completing the proof. $\square$

Remark 2.4.15. The above statement clearly holds (for every ε) if $\mathcal{N} = \emptyset$ with $\delta := \min\{q_\ell : \ell \in \{1, \dots, k\}\}$, which is strictly positive.

2.4.3 Compactness

In this subsection we prove Theorem 2.4.2. By strengthening the assumptions on W , we first improve the compactness and Γ -convergence result in Theorem 2.3.1 concerning the one-dimensional energies

$$E_\varepsilon^1(u, I) := \begin{cases} \int_I \left[\frac{1}{\varepsilon} W(u) + \varepsilon^{2k-1} |u^{(k)}|^2 \right] dt, & \text{if } u \in H^k(I), \\ +\infty & \text{if } u \in L^2(I) \setminus H^k(I), \end{cases} \quad (2.4.29)$$

for $I \subset \mathbb{R}$ finite union of bounded open disjoint intervals.

Corollary 2.4.16. *Let E_ε^1 be defined by (2.4.29). Then, the family $\{E_\varepsilon^1\}_\varepsilon$ is equi-coercive with respect to the convergence in $L^2(I)$; that is, given any family $\{u_\varepsilon\}_\varepsilon \subset H^k(I)$ such that*

$$\sup_\varepsilon E_\varepsilon^1(u_\varepsilon, I) < +\infty,$$

there exist a positive sequence $\{\varepsilon_j\}_j$ converging to 0 and a function $u \in BV(I; \{-1, 1\})$ such that $u_{\varepsilon_j} \rightarrow u$ in $L^2(I)$ as $j \rightarrow +\infty$. Moreover, the family $\{E_\varepsilon^1\}_\varepsilon$ Γ -converges with respect to the convergence in $L^2(I)$ to the functional

$$E^1(u, I) := \begin{cases} m_k \# S(u) & \text{if } u \in BV(I; \{-1, 1\}), \\ +\infty & \text{if } u \in L^2(I) \setminus BV(I; \{-1, 1\}), \end{cases}$$

where m_k is as defined in (2.3.3).

Proof. It suffices to prove the statement in the case that I is an interval. Let $\{u_j\}_j \subset H^k(I)$ be a sequence such that $\sup_j E_{\varepsilon_j}^1(u_j, I) < +\infty$. Since (H2) holds, we are in position to apply Theorem 2.3.1 to deduce that, up to subsequences, $u_j \rightarrow u$ in measure for some $u \in BV(I; \{-1, 1\})$. By (H2) we also have that

$$|u_j|^2 \leq 1 + 2(W(u_j) + 1) = 2W(u_j) + 3$$

and since $\sup_j E_{\varepsilon_j}^1(u_j, I) < +\infty$, it holds that $W(u_j) \rightarrow 0$ in $L^1(I)$ as $j \rightarrow +\infty$. Therefore, $\{|u_j|^2\}_j$ is bounded from above by an equi-integrable sequence in $L^1(I)$, and then the sequence $\{|u_j|^2\}_j$ is equi-integrable as well. This observation, combined with the convergence in measure, implies that $u_j \rightarrow u$ in $L^2(I)$.

As for the Γ -limit, the lower bound follows by Proposition 2.3.13, while a recovery sequence for the $L^2(I)$ -convergence is obtained exactly as in Proposition 2.3.14. $\square$

At this point, we prove an analogous compactness result for functionals (2.4.29), replacing I with $\Omega \subset \mathbb{R}^d$, $d > 1$; that is,

$$E_\varepsilon(u, \Omega) := \begin{cases} \int_\Omega \left[\frac{1}{\varepsilon} W(u) + \varepsilon^{2k-1} \|\nabla^{(k)} u\|_k^2 \right] dx, & \text{if } u \in H^k(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus H^k(\Omega), \end{cases} \quad (2.4.30)$$

for $\Omega \subset \mathbb{R}^d$ a bounded open set with Lipschitz boundary. This is possible by applying a compactness criterion by slicing that appears in [4]. To illustrate the argument in a simplified framework, we introduce some specific notation: here, Ω denotes a 2-dimensional rectangle which is the product of the bounded open intervals I, J , and a point $x \in \Omega$ is denoted by (y, z) with $y \in I, z \in J$. Given a function u defined on Ω , for every $y \in I$ we let u^y denote the function $u^y(z) := u(y, z)$ for every $z \in J$, and for every $z \in J$ we let u^z denote the function $u^z(y) := u(y, z)$ for every $y \in I$. We say that two sequences $\{u_j\}_j$ and $\{v_j\}_j$ are δ -close if $\|u_j - v_j\|_{L^1(\Omega)} \leq \delta$ for every $j \in \mathbb{N}$.

Proposition 2.4.17. *Assume that a sequence $\{u_j\}_j \subset L^1(\Omega)$ is equi-integrable, and that for every $\delta > 0$ there exist sequences $\{v_j\}_j$ and $\{w_j\}_j$ that are δ -close to $\{u_j\}_j$, and such that $\{v_j^y\}_j$ is precompact in $L^1(J)$ for a.e. $y \in I$, and $\{w_j^z\}_j$ is precompact in $L^1(I)$ for a.e. $z \in J$. Then $\{u_j\}_j$ is precompact in $L^1(\Omega)$.*

We employ this result exactly as in [74]. For the convenience of the reader, we sketch the proof, and for simplicity, we consider only the case $d = 2$ when needed.

Proposition 2.4.18. *Let E_ε be defined by (2.4.30). Then, the family $\{E_\varepsilon\}_\varepsilon$ is equi-coercive with respect to the convergence in $L^2(I)$; that is, given any family $\{u_\varepsilon\}_\varepsilon \subset H^k(I)$ such that*

$$\sup_\varepsilon E_\varepsilon(u_\varepsilon) < +\infty,$$

there exist a positive sequence $\{\varepsilon_j\}_j$ converging to 0 and a function $u \in BV(\Omega; \{-1, 1\})$ such that $u_{\varepsilon_j} \rightarrow u$ in $L^2(\Omega)$ as $j \rightarrow +\infty$.

Proof. For simplicity, in the first part of the proof we suppose $d = 2$ and that Ω is the rectangle $I \times J$, otherwise, it suffices to cover a general Ω by essentially disjoint rectangles in order to reduce the proof to our simpler case. Let us consider a positive sequence $\{\varepsilon_j\}_j$ converging to 0 and, for the sake of notation, let us set $u_j := u_{\varepsilon_j}$, $j \in \mathbb{N}$. Using the notation previously introduced, we recall that $\{u_j^y\}_j \subset H^k(J)$ for a.e. $y \in I$, and, analogously, that $\{u_j^z\}_j \subset H^k(I)$ for a.e. $z \in J$. In particular, it holds

$$\frac{d^k u_j^y}{dz^k}(z) = \frac{\partial^k u_j}{\partial z^k}(x), \quad \frac{d^k u_j^z}{dy^k}(y) = \frac{\partial^k u_j}{\partial y^k}(x)$$

for a.e. $x = (y, z) \in \Omega$ and for all j . As a consequence, by the definition of the operatorial norm, we have $\|\nabla^{(k)} u_j\|_k \geq \max\left\{\left|\frac{d^k u_j^y}{dz^k}\right|, \left|\frac{d^k u_j^z}{dy^k}\right|\right\}$, and then

$$E_{\varepsilon_j}(u_j, \Omega) \geq \int_I E_{\varepsilon_j}^1(u_j^y, J) dy, \quad E_{\varepsilon_j}(u_j, \Omega) \geq \int_J E_{\varepsilon_j}^1(u_j^z, I) dz. \quad (2.4.31)$$

Arguing as in the proof of Corollary 2.4.16, we find that $\{u_j\}_j$ and $\{|u_j|^2\}_j$ are equi-integrable sequences; therefore, we fix $\delta > 0$ and we find $\delta' \in (0, \delta)$ such that

$$\mathcal{L}^2(A) \leq \delta' \mathcal{L}^1(J) \implies \sup_j \int_A (|u_j| + 1) dx \leq \delta. \quad (2.4.32)$$

Letting S denote $\sup_j E_{\varepsilon_j}(u_j, \Omega)$, for every $j \in \mathbb{N}$ we set

$$v_j^y(z) := \begin{cases} u_j^y(z) & \text{if } E_{\varepsilon_j}^1(u_j^y, J) \leq S/\delta', \\ 1 & \text{otherwise.} \end{cases}$$

We also put $Y_j := \{y \in I : u_j^y - v_j^y \text{ is not identically } 0\}$ and we observe that

$$\mathcal{L}^1(Y_j) \leq \mathcal{L}^1(\{y \in I : E_{\varepsilon_j}^1(u_j^y, J) > S/\delta'\}) \leq \frac{\delta'}{S} \int_I E_{\varepsilon_j}^1(u_j^y, J) dy \leq \delta',$$

where the last inequality follows by (2.4.31). This inequality, combined with (2.4.32), implies that

$$\|u_j - v_j\|_{L^1(\Omega)} \leq \int_{Y_j \times J} (|u_j| + 1) dx \leq \delta;$$

that is, $\{u_j\}_n$ and $\{v_j\}_n$ are δ -close. Moreover, by construction and by (H1), we have that for every $y \in I$ the sequence $\{v_j^y\}_j$ is such that $\sup_j E_{\varepsilon_j}^1(v_j^y, J) \leq S/\delta'$. Hence, by Corollary 2.4.16, this sequence is precompact in $L^2(J)$, and thus in $L^1(J)$.

A similar argument allows to find a sequence $\{w_j\}_j$ that is δ -close to $\{u_j\}_j$ and such that, for a.e. $z \in J$, the sequence $\{w_j^z\}_j$ is precompact in $L^1(I)$. Therefore, by applying Proposition 2.4.17 we deduce that $\{u_j\}_j$ is precompact in $L^1(\Omega)$. Finally, since $\{|u_n|^2\}_n$ is equi-integrable, we deduce the precompactness in $L^2(\Omega)$.

To conclude, we prove, without further assumptions on d and Ω , that if the (not relabelled) subsequence $\{u_j\}_j$ converges to u in $L^2(\Omega)$ as $j \rightarrow +\infty$, then $u \in BV(\Omega; \{-1, 1\})$. To see this, using the notation introduced in Section 2.1, we prove that for every $m \in \{1, \dots, d\}$, the one-dimensional slice $u^{e_m, y}$ has bounded variation on the open set $\Omega^{e_m, y}$ for $\mathcal{H}^{d-1}$ -a.e. $y \in \Omega^{e_m}$, and

$$\int_{\Omega^{e_m}} \#(S(u^{e_m, y}) \cap \Omega^{e_m, y}) d\mathcal{H}^{d-1}(y) < +\infty.$$

Arguing as previously done for (2.4.31), we have that

$$E_{\varepsilon_j}(u_j, \Omega) \geq \int_{\Omega^{e_m}} E_{\varepsilon_j}^1(u_j^{e_m, y}, \Omega^{e_m, y}) d\mathcal{H}^{d-1}(y).$$

Moreover, upon extracting a further subsequence that we do not relabel, it holds that $u_j^{e_m, y} \rightarrow u^{e_m, y}$ in $L^2(\Omega^{e_m, y})$ for $\mathcal{H}^{d-1}$ -a.e. $y \in \Omega^{e_m}$. Hence, by Fatou's lemma and Corollary 2.4.16 applied with $I = \Omega^{e_m, y}$ and $u = u^{e_m, y}$, we get

$$\begin{aligned} +\infty > S &\geq \liminf_{j \rightarrow +\infty} E_{\varepsilon_j}(u_j, \Omega) \geq \int_{\Omega^{e_m}} \liminf_{j \rightarrow +\infty} E_{\varepsilon_j}^1(u_j^{e_m, y}, \Omega^{e_m, y}) d\mathcal{H}^{d-1}(y) \\ &\geq \int_{\Omega^{e_m}} E^1(u^{e_m, y}, \Omega^{e_m, y}) d\mathcal{H}^{d-1}(y). \end{aligned}$$

As a consequence, $E^1(u^{e_m, y}, \Omega^{e_m, y})$ is finite for $\mathcal{H}^{d-1}$ -a.e. $y \in \Omega^{e_m}$, which implies that

$$\begin{aligned} +\infty > S &\geq \int_{\Omega^{e_m}} E^1(u^{e_m, y}, \Omega^{e_m, y}) d\mathcal{H}^{d-1}(y) \\ &\geq m_k \int_{\Omega^{e_m}} \#(S(u^{e_m, y}) \cap \Omega^{e_m, y}) d\mathcal{H}^{d-1}(y). \end{aligned}$$

Then, since $m_k > 0$, for such y it holds that $u^{e_m, y} \in BV(\Omega^{e_m, y}; \{-1, 1\})$, hence, the conclusion follows applying Theorem 2.1.3. $\square$

Finally, the proof our main result is an immediate consequence.

Proof of Theorem 2.4.2. Let S denote $\sup_{\varepsilon} F_{\varepsilon}(u_{\varepsilon})$. If $\mathcal{N} = \emptyset$, by the equivalence of the norms there exists $c > 0$ such that $F_{\varepsilon}(u_{\varepsilon}) \geq cE_{\varepsilon}(u_{\varepsilon}, \Omega)$ for every ε so that

$$S/c \geq \sup_{\varepsilon} E_{\varepsilon}(u_{\varepsilon}, \Omega)$$

and the conclusion follows by Proposition 2.4.18.

If $\mathcal{N} \neq \emptyset$, again by the equivalence of the norms and by Corollary 2.4.14, upon restricting to ε small enough, it holds

$$S/(c\delta) \geq \sup_{\varepsilon} E_{\varepsilon}(u_{\varepsilon}, \Omega);$$

therefore, the conclusion follows by Proposition 2.4.18. $\square$

2.4.4 Lower bound

In this subsection we prove the lower bound of Theorem 2.4.3. In particular, we prove the following result.

Proposition 2.4.19 (Lower bound). *Let $\{\varepsilon_j\}_j$ be a positive sequence converging to 0, let F_{ε_j} be defined by (2.4.1), and let $\mathcal{N} := \{\ell \in \{1, \dots, k-1\} : q_{\ell} \leq 0\}$. Assume that (H1)-(H3) are satisfied and that one of the following conditions holds:*

- (i) $\mathcal{N} = \emptyset$;
- (ii) $\mathcal{N} \neq \emptyset$ and $q_{\ell} > -\bar{q}_{\ell}$ for every $\ell \in \mathcal{N}$ with the same positive constants $\{\bar{q}_{\ell} : \ell \in \mathcal{N}\}$ independent of Ω appearing in Theorem 2.4.2.

Then, for every $u \in L^2(\Omega)$ and $\{u_j\}_j \subset H^k(\Omega)$ such that $u_j \rightarrow u$ in $L^2(\Omega)$ it holds

$$\liminf_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) \geq F(u),$$

where F is as defined in (2.4.4)

In the following, C denotes a generic positive constant that may change from line to line, but which is independent of the relevant parameters. It is useful to introduce a localized version of the functionals (2.4.1). Given $B \subseteq \Omega$ a Borel set and $\varepsilon > 0$, we define

$$F_{\varepsilon}(u, B) := \begin{cases} \int_B \left[\frac{1}{\varepsilon} W(u) + \sum_{\ell=1}^k q_{\ell} \varepsilon^{2\ell-1} |\nabla^{(\ell)} u|_{\ell}^2 \right] dx & \text{if } u \in H^k(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \quad (2.4.33)$$

We recall that $\bar{u}$ is a function in $H_{\text{loc}}^k(\mathbb{R})$ such that $\bar{u}(t) = -1$ if $t \leq -1/8$ and $\bar{u}(t) = 1$ if $t \geq 1/8$, $|\bar{u}| \leq 1$, and

$$0 < \int_{-\infty}^{+\infty} \left[W(\bar{u}) + \sum_{\ell=1}^k |q_\ell| |\bar{u}^{(\ell)}|^2 \right] dt < +\infty \quad (2.4.34)$$

and that, for every $\varepsilon > 0$, we have set $\bar{u}_\varepsilon^\nu(x) := \bar{u}(\frac{x \cdot \nu}{\varepsilon})$ for all $\nu \in \mathbb{S}^{d-1}$ and $x \in \mathbb{R}^d$. We observe that, thanks to (2.4.34), the definition of $\bar{u}_\varepsilon^\nu$, and Fubini's Theorem, for every open set $V \subseteq \Omega$ there exists a constant $C > 0$ such that for every $\nu \in \mathbb{S}^{d-1}$ and $\varepsilon > 0$ we have that

$$\begin{aligned} F_\varepsilon(\bar{u}_\varepsilon, V) &\leq C \int_{\pi^\nu(V)} \int_{-\infty}^{+\infty} \left[W(\bar{u}) + \sum_{\ell=1}^k |q_\ell| |\bar{u}^{(\ell)}|^2 \right] dt d\mathcal{H}^{d-1}(y) \\ &\leq C \int_{-\infty}^{+\infty} \left[W(\bar{u}) + \sum_{\ell=1}^k |q_\ell| |\bar{u}^{(\ell)}|^2 \right] dt \leq C, \end{aligned} \quad (2.4.35)$$

where π^ν denotes the orthogonal projection onto $\nu^\perp$ and where we have also used that $(\bar{u}_\varepsilon^\nu)^{\nu, y}(t) = \bar{u}(\frac{t}{\varepsilon})$ for every $\varepsilon > 0$, $y \in \nu^\perp$, and $t \in \mathbb{R}$.

We further set

$$u_0^\nu(x) := \text{sgn}(x \cdot \nu), \quad (2.4.36)$$

where sgn denotes the sign function and remark that, for every $K \subset \mathbb{R}^d$ compact, $\{\bar{u}_\varepsilon^\nu\}_\varepsilon$ converges to u_0^ν in $L^2(K)$ as $\varepsilon \rightarrow 0$.

We start by proving a lemma which allows to modify boundary conditions on a converging sequence at the cost of an asymptotically negligible amount of energy.

For every $x \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, we introduce $\tilde{Q}^\nu(x)$ an open subset of $Q^\nu(x)$ having C^2 boundary and such that

$$\tilde{Q}^\nu(x) \cap \{y \in \mathbb{R}^d : |y \cdot \nu| \leq \frac{1}{4}\} = Q^\nu(x) \cap \{y \in \mathbb{R}^d : |y \cdot \nu| \leq \frac{1}{4}\}.$$

We suppose that all the above sets are congruent, namely, that they are obtained starting by $\tilde{Q}^{e_d}(0)$ by a translation and a rotation. Also for the *rounded* cube $\tilde{Q}^\nu(x)$ we use the convention that when $x = 0$ we simply write $\tilde{Q}^\nu$.

Lemma 2.4.20. *Let $\{\varepsilon_j\}_j$ be a positive sequence converging to 0, $\nu \in \mathbb{S}^{d-1}$, and let $\{v_j\}_j \subset H^k(\tilde{Q}^\nu)$ be a sequence converging to u_0^ν in $L^2(\tilde{Q}^\nu)$. Assume that (H1)-(H3) are satisfied and that one of the two conditions (i) and (ii) of Proposition 2.4.19 holds. Then, there exists a sequence $\{w_j\}_j \subset H^k(\tilde{Q}^\nu)$ such that*

$$(i) \quad w_j \rightarrow u_0^\nu \text{ in } L^2(\tilde{Q}^\nu) \text{ as } j \rightarrow +\infty,$$

$$(ii) \quad \text{the support of } w_j - \bar{u}_j \text{ is compactly contained in } \tilde{Q}^\nu \text{ for every } j \in \mathbb{N},$$

(iii) it holds

$$\liminf_{j \rightarrow +\infty} F_{\varepsilon_j}(v_j, \tilde{Q}^\nu) \geq \liminf_{j \rightarrow +\infty} F_{\varepsilon_j}(w_j, \tilde{Q}^\nu).$$

Proof. For the sake of exposition, we only consider the case $\nu = e_d$, the others being analogous. For simplicity of notation, throughout the rest of the proof we set $\bar{u}_j := \bar{u}_{\varepsilon_j}^{e_d}$ and $\tilde{Q} := \tilde{Q}^{e_d}(0)$. It is not restrictive to assume that

$$\|v_j - \bar{u}_j\|_{L^2(\tilde{Q})} > 0 \quad \text{for every } j \in \mathbb{N}$$

since, if $v_j = \bar{u}_j$ for infinitely many $j \in \mathbb{N}$, it is enough to set $w_j := \bar{u}_j$ to conclude. Up to extracting a not relabelled subsequence, we also assume that

$$\liminf_{j \rightarrow +\infty} F_{\varepsilon_j}(v_j, \tilde{Q}) = \lim_{j \rightarrow +\infty} F_{\varepsilon_j}(v_j, \tilde{Q}) \leq C < +\infty. \quad (2.4.37)$$

By Corollary 2.4.14, (2.4.35), and (2.4.37) there exists a constant $\delta > 0$, independent of n and $\tilde{Q}$, such that

$$\int_{\tilde{Q}} \left[\frac{W(v_j)}{\varepsilon_j} + \sum_{\ell=1}^k \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} v_j|_\ell^2 \right] dx \leq \frac{1}{\delta} F_{\varepsilon_j}(v_j, \tilde{Q}) \leq C, \quad (2.4.38)$$

$$\int_{\tilde{Q}} \left[\frac{W(\bar{u}_j)}{\varepsilon_j} + \sum_{\ell=1}^k \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} \bar{u}_j|_\ell^2 \right] dx \leq \frac{1}{\delta} F_{\varepsilon_j}(\bar{u}_j, \tilde{Q}) \leq C, \quad (2.4.39)$$

for $j \in \mathbb{N}$ large enough depending on $\{q_\ell : q_\ell \leq 0\}$ and $\tilde{Q}$. Additionally, as both v_j and $\bar{u}_j$ converge to u_0^{ed} in $L^2(\tilde{Q})$ as $j \rightarrow +\infty$, we have that

$$\int_{\tilde{Q}} |v_j - \bar{u}_j|^2 dx \leq C \quad (2.4.40)$$

for every $j \in \mathbb{N}$.

Since the set $\tilde{Q}$ is C^2 , there exists $\rho_0 > 0$ such that Remark 2.4.13 applies and, for every $0 < \rho < \rho_0$, there is a unique C^1 projection from $\{x \in \tilde{Q} : \text{dist}(x, \partial\tilde{Q}) < \rho\}$ onto $\partial\tilde{Q}$. In particular, for every $0 < \rho < \rho_0$ the sets $\{x \in \tilde{Q} : \text{dist}(x, \partial\tilde{Q}) = \rho\}$ are $(d-1)$ -dimensional manifolds in $\mathbb{R}^d$ of class C^1 .

Let $0 < \rho < \rho_0/2$ and $0 < \sigma < \rho_0/2$, and let $\lceil \varepsilon_j^{-1} \rceil$ be the smallest integer larger than or equal to ε_j^{-1} , so that

$$\frac{\varepsilon_j}{2} \leq \lceil \varepsilon_j^{-1} \rceil^{-1} \leq \varepsilon_j \quad (2.4.41)$$

In light of the observations above, the set of class C^1 given by

$$\{x \in \tilde{Q} : \text{dist}(x, \partial\tilde{Q}) < \rho + \sigma\}$$

can be subdivided into $\lceil \varepsilon_j^{-1} \rceil$ open subsets with C^1 boundary defined as

$$L_{j,\sigma,\rho}^i := \left\{ x \in \tilde{Q} : (i-1) \frac{\sigma}{\lceil \varepsilon_j^{-1} \rceil} + \rho < \text{dist}(x, \partial\tilde{Q}) < i \frac{\sigma}{\lceil \varepsilon_j^{-1} \rceil} + \rho \right\} \quad \text{for } i \in \{1, \dots, \lceil \varepsilon_j^{-1} \rceil\}.$$

We observe that

$$|L_{j,\sigma,\rho}^i| \leq C \frac{\sigma}{\lceil \varepsilon_j^{-1} \rceil}. \quad (2.4.42)$$

From (2.4.38)-(2.4.40) we deduce that

$$\begin{aligned} & \sum_{i=1}^{\lceil \varepsilon_j^{-1} \rceil} \left(\int_{L_{j,\sigma,\rho}^i} \left[\frac{W(v_j)}{\varepsilon_j} + \sum_{\ell=1}^k \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} v_j|_\ell^2 \right] dx \right. \\ & \quad \left. + \int_{L_{j,\sigma,\rho}^i} \left[\frac{W(\bar{u}_j)}{\varepsilon_j} + \sum_{\ell=1}^k \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} \bar{u}_j|_\ell^2 \right] dx + \int_{L_{j,\sigma,\rho}^i} \frac{|v_j - \bar{u}_j|^2}{\|v_j - \bar{u}_j\|_{L^2(\tilde{Q})}^2} dx \right) \leq C, \end{aligned}$$

for every j large enough. Hence, by (2.4.41) we find $i^* = i^*(j, \sigma, \rho) \in \{1, \dots, \lceil \varepsilon_j^{-1} \rceil\}$ such that

$$\begin{aligned} & \int_{L_{j,\sigma,\rho}^{i^*}} \left[\frac{W(v_j)}{\varepsilon_j} + \sum_{\ell=1}^k \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} v_j|_\ell^2 \right] dx \\ & \quad + \int_{L_{j,\sigma,\rho}^{i^*}} \left[\frac{W(\bar{u}_j)}{\varepsilon_j} + \sum_{\ell=1}^k \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} \bar{u}_j|_\ell^2 \right] dx + \int_{L_{j,\sigma,\rho}^{i^*}} \frac{|v_j - \bar{u}_j|^2}{\|v_j - \bar{u}_j\|_{L^2(\tilde{Q})}^2} dx \leq C\varepsilon_j. \end{aligned}$$

In particular, for every $\ell \in \{1, \dots, k\}$ we have

$$\varepsilon_j^{2\ell-1} \|\nabla^{(\ell)} v_j\|_{L^2(L_{j,\sigma,\rho}^{i*})}^2 \leq C\varepsilon_j, \quad (2.4.43)$$

$$\varepsilon_j^{2\ell-1} \|\nabla^{(\ell)} \bar{u}_j\|_{L^2(L_{j,\sigma,\rho}^{i*})}^2 \leq C\varepsilon_j, \quad (2.4.44)$$

$$\frac{1}{\varepsilon_j} \int_{L_{j,\sigma,\rho}^{i*}} W(v_j) dx + \frac{1}{\varepsilon_j} \int_{L_{j,\sigma,\rho}^{i*}} W(\bar{u}_j) dx \leq C\varepsilon_j, \quad (2.4.45)$$

$$\int_{L_{j,\sigma,\rho}^{i*}} |v_j - \bar{u}_j|^2 dx \leq C\varepsilon_j \|v_j - \bar{u}_j\|_{L^2(\tilde{Q})}^2. \quad (2.4.46)$$

We now introduce the sets

$$A_{j,\sigma,\rho} := \{x \in \tilde{Q} : \text{dist}(x, \partial\tilde{Q}) \leq \rho + (i^* - 1) \frac{\sigma}{|\varepsilon_j^{-1}|}\},$$

$$B_{j,\sigma,\rho} := \{x \in \tilde{Q} : \text{dist}(x, \partial\tilde{Q}) > \rho + i^* \frac{\sigma}{|\varepsilon_j^{-1}|}\},$$

and cut-off functions $\phi_{j,\sigma,\rho} \in C_c^\infty(\tilde{Q}; [0, 1])$ such that

$$\begin{cases} \phi_{j,\sigma,\rho}(x) = 0 & \text{if } x \in A_{j,\sigma,\rho}, \\ \phi_{j,\sigma,\rho}(x) = 1 & \text{if } x \in B_{j,\sigma,\rho}, \\ \|\nabla^{(\ell)} \phi_{j,\sigma,\rho}\|_{L^\infty(\tilde{Q})} \leq C(\sigma\varepsilon_j)^{-\ell} & \text{for every } \ell \in \{1, \dots, k\}. \end{cases} \quad (2.4.47)$$

For $x \in \tilde{Q}$ we define

$$w_j(x) = w_j^{\sigma,\rho}(x) := \phi_{j,\sigma,\rho}(x) v_j(x) + (1 - \phi_{j,\sigma,\rho}(x)) \bar{u}_j(x), \quad (2.4.48)$$

and we observe that $w_j \in H^k(\tilde{Q})$ and that $w_j - \bar{u}_j$ is compactly supported in $\tilde{Q}$. Also note that, since both $\{v_j\}_j$ and $\{\bar{u}_j\}_j$ converge to $u_0^{e_d}$ in $L^2(\tilde{Q})$, for every choice of σ and ρ the sequence $\{w_j\}_j$ converges in $L^2(\tilde{Q})$ to $u_0^{e_d}$ as well.

By (2.4.47) and (2.4.48), we have

$$\begin{aligned} F_{\varepsilon_j}(w_j, \tilde{Q}) &= F_{\varepsilon_j}(v_j, B_{j,\sigma,\rho}) + F_{\varepsilon_j}(w_j, L_{j,\sigma,\rho}^{i*}) + F_{\varepsilon_j}(\bar{u}_j, A_{j,\sigma,\rho}) \\ &\leq F_{\varepsilon_j}(v_j, B_{j,\sigma,\rho}) + \int_{L_{j,\sigma,\rho}^{i*}} \left[\frac{W(w_j)}{\varepsilon_j} + \sum_{\ell=1}^k |q_\ell| \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} w_j|^2 \right] dx \\ &\quad + \int_{A_{j,\sigma,\rho}} \left[\frac{W(\bar{u}_j)}{\varepsilon_j} + \sum_{\ell=1}^k |q_\ell| \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} \bar{u}_j|^2 \right] dx =: I_j^1 + I_j^2 + I_j^3. \end{aligned} \quad (2.4.49)$$

We now study the asymptotic behaviour of these three quantities.

We claim that

$$\limsup_{j \rightarrow +\infty} I_j^1 = \limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(v_j, B_{j,\sigma,\rho}) \leq \lim_{j \rightarrow +\infty} F_{\varepsilon_j}(v_j, \tilde{Q}) \quad (2.4.50)$$

for every fixed σ, ρ fixed as above. This inequality follows from an application of Remark 2.4.13. To prove (2.4.50), we set

$$\tilde{Q}_\tau := \{x \in \tilde{Q} : \text{dist}(x, \partial\tilde{Q}) < \tau\},$$

for $\tau \in (0, \rho_0)$. Then, by Remark 2.4.13, for every $\ell \in \{1, \dots, k-1\}$ we find $\varepsilon_0 = \varepsilon_0(q_\ell, \tilde{Q}) > 0$ such that

$$\int_{\tilde{Q}_\tau} \left[\frac{1}{\varepsilon_j} W(v_j) + q_\ell \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} v_j|^2 \right] dx \geq 0$$

holds for every j sufficiently large so that $\varepsilon_j \leq \varepsilon_0$ and for every $\tau \in (0, \rho_0)$. Arguing as in the proof of Corollary 2.4.14, we deduce that there exists a possibly smaller $\varepsilon_0 = \varepsilon_0(\{q_\ell : q_\ell \leq 0\}, \tilde{Q})$ such that

$$\int_{\tilde{Q}_\tau} \left[\frac{1}{\varepsilon_j} W(v_j) + q_\ell \sum_{\ell=1}^k \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} v_j|^2 \right] dx \geq 0 \quad (2.4.51)$$

for every j sufficiently large so that $\varepsilon_j \leq \varepsilon_0$ and for every $\tau \in (0, \rho_0)$. We now apply (2.4.51) with $\tau = \rho + i^* \frac{\sigma}{\lceil \varepsilon_j^{-1} \rceil}$, which is less than ρ_0 , so that $\tilde{Q}_\tau = \tilde{Q} \setminus B_{j,\sigma,\rho}$ and we obtain that

$$F_{\varepsilon_j}(v_j, \tilde{Q}) = F_{\varepsilon_j}(v_j, \tilde{Q} \setminus B_{j,\sigma,\rho}) + F_{\varepsilon_j}(v_j, B_{j,\sigma,\rho}) \geq F_{\varepsilon_j}(v_j, B_{j,\sigma,\rho}),$$

which implies (2.4.50).

We now show that I_j^2 is asymptotically negligible. We begin by studying the term involving the double well potential. We remark that the continuity of W and (H3) imply that there exists $C > 0$ such that

$$W(s) \leq W(t) + C \quad (2.4.52)$$

for every $s = (1 - \theta)t + \theta t_0$ with $\theta \in [0, 1)$ and $|t_0| \leq 1$. Indeed, if $|t| \leq 1$, then $|s| \leq 1$, so that $W(s) \leq \sup_{\tau \in [-1, 1]} W(\tau)$. If instead $|t| > 1$, then $s \leq (1 - \theta)|t| + \theta|t_0| \leq |t|$, so that (2.4.52) follows from (H3). Thus, recalling that $|\bar{u}_j| \leq 1$, from (2.4.52), (2.4.45), (2.4.42), and (2.4.41) we deduce that

$$\begin{aligned} \frac{1}{\varepsilon_j} \int_{L_{j,\sigma,\rho}^{i^*}} W(w_j) dx &\leq \frac{1}{\varepsilon_j} \left(\int_{L_{j,\sigma,\rho}^{i^*}} W(v_j) dx + C|L_{j,\sigma,\rho}^{i^*}| \right) \\ &\leq C\varepsilon_j + \frac{C\sigma}{\varepsilon_j \lceil \varepsilon_j^{-1} \rceil} \leq C(\varepsilon_j + \sigma). \end{aligned} \quad (2.4.53)$$

We now investigate the remaining terms in I_j^2 . We note that for every $\ell \in \{1, \dots, k\}$ and multi-index α of order $|\alpha| = \ell$ we have

$$\begin{aligned} \partial^\alpha w_j &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left(\partial^\beta \phi_{j,\sigma,\rho} \partial^{\alpha-\beta} v_j + \partial^\beta (1 - \phi_{j,\sigma,\rho}) \partial^{\alpha-\beta} \bar{u}_j \right) \\ &= \phi_{j,\sigma,\rho} \partial^\alpha v_j + (1 - \phi_{j,\sigma,\rho}) \partial^\alpha \bar{u}_j + \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta \phi_{j,\sigma,\rho} \partial^{\alpha-\beta} (v_j - \bar{u}_j)) \end{aligned}$$

so that, by (2.4.47), we deduce that for every $\ell \in \{1, \dots, k\}$ the following inequality holds

$$\begin{aligned} |\nabla^{(\ell)} w_j|_\ell^2 &\leq C \left(|\nabla^{(\ell)} v_j|_\ell^2 + |\nabla^{(\ell)} \bar{u}_j|_\ell^2 \right. \\ &\quad \left. + (\sigma \varepsilon_j)^{-2\ell} |v_j - \bar{u}_j|^2 + \sum_{h=1}^{\ell-1} (\sigma \varepsilon_j)^{-2(\ell-h)} |\nabla^{(h)} (v_j - \bar{u}_j)|_h^2 \right). \end{aligned}$$

Combining this inequality with (2.4.43), (2.4.44), (2.4.46), and (2.4.41) we obtain

$$\begin{aligned} \varepsilon_j^{2\ell-1} \|\nabla^{(\ell)} w_j\|_{L^2(L_{j,\sigma,\rho}^{i^*})}^2 &\leq C\varepsilon_j + C\sigma^{-2\ell} \varepsilon_j^{-1} \int_{L_{j,\sigma,\rho}^{i^*}} |v_j - \bar{u}_j|^2 dx \\ &\quad + C \sum_{h=1}^{\ell-1} \frac{\varepsilon_j^{2h-1}}{\sigma^{2(\ell-h)}} \|\nabla^{(h)} (v_j - \bar{u}_j)\|_{L^2(L_{j,\sigma,\rho}^{i^*})}^2 \\ &\leq C\varepsilon_j + C\sigma^{-2\ell} \|v_j - \bar{u}_j\|_{L^2(\tilde{Q})}^2 + C \left(\sum_{h=1}^{\ell-1} \frac{1}{\sigma^{2(\ell-h)}} \right) \varepsilon_j. \end{aligned}$$

Finally, this last inequality, together with (2.4.53), yields

$$I_j^2 \leq C\varepsilon_j + C\sigma + C \left(\sum_{\ell=1}^k \sigma^{-2\ell} \|v_j - \bar{u}_j\|_{L^2(\tilde{Q})}^2 + C \left(\sum_{\ell=1}^k \sum_{h=1}^{\ell-1} \frac{1}{\sigma^{2(\ell-h)}} \right) \varepsilon_j \right). \quad (2.4.54)$$

As for the term I_j^3 , we use Fubini's Theorem, the definition of $\bar{u}_j$, a change of variable, and (2.4.34) to get

$$I_j^3 \leq C\mathcal{H}^{d-1}(A_{j,\sigma,\rho} \cap \nu^\perp) \left(\int_{-\infty}^{+\infty} [W(\bar{u}(t)) + \sum_{\ell=1}^k |q_\ell| |\bar{u}^{(\ell)}|^2] dt \right) \leq C(\rho + \sigma),$$

so that

$$\limsup_{j \rightarrow +\infty} I_j^3 \leq C(\rho + \sigma). \quad (2.4.55)$$

Since $\{v_j - \bar{u}_j\}_j$ converges to 0 in $L^2(\tilde{Q})$ as $j \rightarrow +\infty$, by (2.4.49), (2.4.50), (2.4.54), and (2.4.55), we infer that

$$\limsup_{\rho \rightarrow 0} \limsup_{\sigma \rightarrow 0} \limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(w_j, \tilde{Q}) \leq \lim_{j \rightarrow +\infty} F_{\varepsilon_j}(v_j, \tilde{Q}).$$

Recalling that for every fixed σ and ρ the sequence $\{w_j\}_j = \{w_j^{j, \sigma, \rho}\}_j$ converges to u_0^ν in $L^2(\tilde{Q})$ and is such that the support of $w_j - \bar{u}_j$ is compactly contained in $\tilde{Q}$ for all $j \in \mathbb{N}$, we let σ and ρ vary in two positive vanishing sequences and use a diagonal argument to conclude the proof. $\square$

We are now ready to prove the liminf inequality. The proof is based on the blow-up argument of Fonseca and Müller [75].

Proof of Proposition 2.4.19. Let us fix a positive sequence $\{\varepsilon_j\}_j$ converging to 0 as $j \rightarrow +\infty$. Up to extracting a not relabelled subsequence, we may assume that

$$\liminf_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) = \lim_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) \leq C < +\infty. \quad (2.4.56)$$

In light of Theorem 2.4.2, this implies that $u \in BV(\Omega; \{-1, 1\})$ so that $u = 2\chi_A - 1$ for A a set of finite perimeter in Ω and $S(u) = \partial^* A \cap \Omega$, where we recall that $\partial^* A$ denotes the reduced boundary of A .

For every $j \in \mathbb{N}$, we consider the measure μ_{ε_j} defined by

$$\mu_{\varepsilon_j}(B) := F_{\varepsilon_j}(u_j, B) \quad \text{for every } B \text{ Borel subset of } \Omega.$$

Note that by Corollary 2.4.14 and (2.4.56), the sequence $\{f_j := \frac{W(u_j)}{\varepsilon_j} + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} u_j|^2\}_j$ is equi-bounded in $L^1(\Omega)$. Therefore, we have $\sup_j |\mu_{\varepsilon_j}|(\Omega) \leq C$, so that, up to passing to a not relabelled subsequence, we may assume that $\{\mu_{\varepsilon_j}\}_j$ and $\{|\mu_{\varepsilon_j}|\}_j$ weakly*-converge to two finite Radon measures μ and λ , respectively. We now show that $\mu \geq 0$. To this aim, we use the Besicovitch Derivation Theorem (see [72, Theorem 1.153]) to write for $|\mu|$ -a.e. $x \in \Omega$

$$\frac{d\mu}{d|\mu|}(x) = \lim_{r \rightarrow 0^+} \frac{\mu(B_r(x))}{|\mu|(B_r(x))},$$

where $d\mu/d|\mu|$ is the Radon-Nikodým derivative of μ with respect to its total variation $|\mu|$. Let us fix a point x such that the previous equality holds. Since the set $\{r \in (0, +\infty) : \lambda(\partial B_r(x)) > 0\}$ is at most countable, we may choose a sequence of positive real numbers $\{r_n\}_n$, vanishing as $n \rightarrow +\infty$, such that

$$\frac{d\mu}{d|\mu|}(x) = \lim_{n \rightarrow +\infty} \lim_{j \rightarrow +\infty} \frac{1}{|\mu|(B_{r_n}(x))} \int_{B_{r_n}(x)} f_j(y) dy.$$

For every $n \in \mathbb{N}$ fixed, by Corollary 2.4.14 the integral in the right-hand side of the previous equality is non-negative for $j \in \mathbb{N}$ large enough, hence, we conclude that $d\mu/d|\mu|$ is non-negative $|\mu|$ -a.e. and then $\mu \geq 0$.

We now show that the Radon-Nikodým derivative of μ with respect to $\mathcal{H}^{d-1} \llcorner (\partial^* A \cap \Omega)$, denoted by f , can be estimated from below in terms of the function g defined by (2.4.3). To see this, for every $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{d-1}$ we consider $\tilde{Q}^\nu(x)$ an open subset of $Q^\nu(x)$ having C^2 boundary and such that

$$\tilde{Q}^\nu(x) \cap \{y \in \mathbb{R}^d : |y \cdot \nu| \leq \tfrac{1}{4}\} = Q^\nu(x) \cap \{y \in \mathbb{R}^d : |y \cdot \nu| \leq \tfrac{1}{4}\}. \quad (2.4.57)$$

For every $x \in \partial^* A \cap \Omega$ we consider $\nu = \nu_A(x)$ the inner unit normal to $\partial^* A \cap \Omega$ at x . By (2.4.57), using Theorem 2.1.1 one can show that for $\mathcal{H}^{d-1}$ -a.e. $x \in \partial^* A \cap \Omega$ we have

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^{d-1}(\tilde{Q}_r^\nu(x) \cap \partial^* A)}{r^{d-1}} = 1, \quad (2.4.58)$$

where we used the notation $\tilde{Q}_r^\nu(x) = \{y \in \mathbb{R}^n : y/r \in \tilde{Q}^\nu(x)\}$. For $\mathcal{H}^{d-1}$ -a.e. $x \in \partial^* A \cap \Omega$ we have

$$f(x) = \lim_{r \rightarrow 0^+} \frac{\mu(\tilde{Q}_r^\nu(x))}{\mathcal{H}^{d-1}(\tilde{Q}_r^\nu(x) \cap \partial^* A)}; \quad (2.4.59)$$

additionally, given $x \in \partial^* A \cap \Omega$ such that (2.4.59) holds, since $\{\mu_{\varepsilon_j}\}_j$ weakly*-converges to μ , it follows from (2.4.58) and a diagonal argument that there exists a vanishing sequence $\{r_j\}_j$, with $\varepsilon_j \ll r_j$, such that

$$f(x) = \lim_{j \rightarrow +\infty} \frac{\mu_{\varepsilon_j}(\tilde{Q}_{r_j}^\nu(x))}{r_j^{d-1}}. \quad (2.4.60)$$

Upon translating the point x , it is not restrictive to assume that $x = 0 \in \partial^* A \cap \Omega$; moreover, for the sake of notation and simplicity of exposition, we only discuss the case and $\nu_E(x) = e_d$, the other cases being analogous.

We write $\tilde{Q}$ in place of $\tilde{Q}^{e_d}(0)$ and $\tilde{Q}_j$ in place of $\tilde{Q}_{r_j}^{e_d}(0)$. Then, we set $v_j(y) := u_j(r_j y)$ for every $y \in \tilde{Q}$ and, by Theorem 2.1.1, we have that $\{v_j\}_j$ converges in $L^2(\tilde{Q})$ to the function $u_0^{e_d}$ defined by (2.4.36). We further set $\delta_j := \varepsilon_j/r_j$ and use the change of variables $y = x/r_j$ to obtain

$$\frac{\mu_{\varepsilon_j}(\tilde{Q}_j)}{r_j^{d-1}} = \frac{F_{\varepsilon_j}(u_j, \tilde{Q}_j)}{r_j^{d-1}} = \int_{\tilde{Q}} \left[\frac{1}{\delta_j} W(v_j) + \sum_{\ell=1}^k q_\ell \delta_j^{2\ell-1} |\nabla^{(\ell)} v_j|_\ell^2 \right] dy. \quad (2.4.61)$$

We apply Lemma 2.4.20 to $\{v_j\}_j$ in order to obtain a sequence $\{w_j\}_j \subset H^k(\tilde{Q})$ which satisfies (i)-(iii). Noting that $\delta_j \rightarrow 0$ as $j \rightarrow +\infty$, by (2.4.60), (2.4.61) and Lemma 2.4.20(iii), it follows

$$f(0) \geq \liminf_{j \rightarrow +\infty} \int_{\tilde{Q}} \left[\frac{1}{\delta_j} W(w_j) + \sum_{\ell=1}^k q_\ell \delta_j^{2\ell-1} |\nabla^{(\ell)} w_j|_\ell^2 \right] dy. \quad (2.4.62)$$

Now we observe that, by Lemma 2.4.20(ii), putting

$$w_j := \bar{u}_{\delta_j}^{e_d} \quad \text{on } Q_1^{e_d}(0) \setminus \tilde{Q}$$

for all $j \in \mathbb{N}$, we have $\{w_j\}_j \subset \mathcal{A}_{\delta_j}^{e_d}$. Moreover, since $|\bar{u}^{e_d}(y)| = 1$ if $|y \cdot e_d| > \frac{1}{8}$, it follows that $|\bar{u}_{\delta_j}^{e_d}(y)| = 1$ if $|y \cdot e_d| > \frac{\delta_j}{8}$. Therefore, by (2.4.57), we have

$$f(0) \geq \liminf_{j \rightarrow +\infty} \int_{Q_1^{e_d}(0)} \left[\frac{1}{\delta_j} W(w_j) + \sum_{\ell=1}^k q_\ell \delta_j^{2\ell-1} |\nabla^{(\ell)} w_j|_\ell^2 \right] dy,$$

and by the definition of g in (2.4.3), we conclude

$$f(0) \geq g(e_d).$$

Since the above argument holds for $\mathcal{H}^{d-1}$ -a.e. $x \in \partial^* A \cap \Omega$ and every $\nu = \nu_A(x)$, we obtain

$$f(x) \geq g(\nu_A(x)). \quad (2.4.63)$$

Finally, by the inequality $\liminf_{j \rightarrow +\infty} \mu_{\varepsilon_j}(\Omega) \geq \mu(\Omega)$ and (2.4.63) we infer that

$$\lim_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) = \lim_{j \rightarrow +\infty} \mu_{\varepsilon_j}(\Omega) \geq \mu(\Omega) \geq \int_{\partial^* A \cap \Omega} f(x) d\mathcal{H}^{d-1}(x) \geq \int_{\partial^* A \cap \Omega} g(\nu_A(x)) d\mathcal{H}^{d-1}(x),$$

which concludes the proof. $\square$

Remark 2.4.21. Thanks to (2.4.62), for every $\{\varepsilon_j\}_j$ positive sequence converging to 0, we have also have that

$$\Gamma\text{-}\liminf_{j \rightarrow +\infty} F_{\varepsilon_j}(u) \geq \begin{cases} \int_{\partial^* A \cap \Omega} \tilde{g}(\nu_A) d\mathcal{H}^{d-1} & \text{if } u \in BV(\Omega; \{-1, 1\}), \text{ with } u = 2\chi_A - 1, \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega; \{-1, 1\}), \end{cases}$$

where

$$\tilde{g}(\nu) := \inf \left\{ \int_{\tilde{Q}^\nu} \left[\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \right] dx : v \in \tilde{\mathcal{A}}_\varepsilon^\nu, \varepsilon \in (0, 1) \right\}$$

for every $\nu \in \mathbb{S}^{d-1}$ and we have set

$$\tilde{\mathcal{A}}_\varepsilon^\nu := \{v \in H^k(\tilde{Q}^\nu) : v = \bar{u}_\varepsilon^\nu \text{ near } \partial\tilde{Q}^\nu\}.$$

Such observation will be useful in order to prove that the Γ -limit is strictly positive.

2.4.5 Upper bound

In this subsection we conclude the proof of Theorem 2.4.3 by proving the upper estimate. As in Subsection 2.4.4, we let C denote a generic positive constant that may change from line to line, but which is independent of the relevant parameters. Also here we shall make use of the localized functionals introduced in (2.4.33). We now state the main result.

Proposition 2.4.22. *Let $\{\varepsilon_j\}_j$ be a positive sequence converging to 0 and let F_{ε_j} be defined by (2.4.1). Then, for every $u \in L^2(\Omega)$ there exists a sequence $\{u_j\}_j \subset H^k(\Omega)$ such that $u_j \rightarrow u$ in $L^2(\Omega)$ and it holds*

$$\limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) \leq F(u),$$

where F is as defined in (2.4.4).

Proof. We subdivide the proof into three steps.

Step 1. First, we consider the case where Ω has Lipschitz boundary (and not necessarily C^1) and $u = 2\chi_A - 1$, where

$$A := \{x \in \Omega : (x - x_0) \cdot \nu > 0\}$$

for some $x_0 \in \Omega$ and $\nu \in \mathbb{S}^{d-1}$ such that

$$\nu_\Omega(x) \cdot \nu = 0 \quad \text{for every } x \in \partial\Omega \text{ with } |(x - x_0) \cdot \nu| \text{ small enough,} \quad (2.4.64)$$

recalling that $\nu_\Omega(x)$ is the inner unit normal to Ω at x . For the sake of exposition, we assume $x_0 = 0$ and $\nu = e_d$, the other cases being analogous, so that for every $x \in \Omega$ we have $u(x) = u_0^{e_d}(x)$, where by definition $u_0^{e_d}(x) = \text{sgn}(x \cdot e_d)$, and

$$F(u) = g(e_d) \mathcal{H}^{d-1}(\Omega \cap \{x_d = 0\}).$$

For every $\eta > 0$, there exist $0 < \varepsilon_0 < 1$ and $v \in \mathcal{A}_{\varepsilon_0}^{e_d}$ such that $v = \bar{u}_{\varepsilon_0}^{e_d}$ in a neighborhood of $\partial Q_1^{e_d}$ and

$$\int_{Q_1^{e_d}} \left[\frac{1}{\varepsilon_0} W(v) + \sum_{\ell=1}^k q_\ell \varepsilon_0^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \right] dx \leq g(e_d) + \eta. \quad (2.4.65)$$

By Proposition 2.4.6, upon rotating $Q_1^{e_d}$ leaving e_d fixed, we may assume that $Q_1^{e_d} = Q := (-\frac{1}{2}, \frac{1}{2})^d$. We then extend v to a function in $H_{\text{loc}}^k(\mathbb{R}^d)$ which is periodic in the first $d-1$ variables; that is, such that

$$v(x) = v(x + e_i), \quad (2.4.66)$$

for all $i \in \{1, \dots, d-1\}$. We also set $\Omega_j := \{x \in \Omega : |x_d| \leq \frac{\varepsilon_j}{2\varepsilon_0}\}$ for every $j \in \mathbb{N}$.

We claim that a recovery sequence for u is given by the sequence of functions $\{u_j\}_j \subset H^k(\Omega)$ defined by

$$u_j(x) := \begin{cases} v\left(\frac{\varepsilon_0 x}{\varepsilon_j}\right) & \text{if } x \in \Omega_j, \\ u(x) & \text{if } x \in \Omega \setminus \Omega_j. \end{cases} \quad (2.4.67)$$

To prove the claim, we first show that

$$\|u_j - u\|_{L^2(\Omega)} \rightarrow 0 \text{ as } j \rightarrow +\infty. \quad (2.4.68)$$

To this aim, we set $\Omega' := \{x' \in \mathbb{R}^{d-1} : (x', 0) \in \Omega\}$ and observe that by (2.4.64) it holds

$$\Omega \cap \Omega_j = \Omega' \times \left[-\frac{\varepsilon_j}{2\varepsilon_0}, \frac{\varepsilon_j}{2\varepsilon_0}\right]$$

for j sufficiently large.

Using the change of variables $(x', x_d) \rightarrow (x', \frac{\varepsilon_j}{\varepsilon_0} x_d)$, by Fubini's Theorem and (2.4.67) we have

$$\begin{aligned} \int_{\Omega_j} |u_j|^2 dx &= \int_{\Omega'} \int_{-\frac{\varepsilon_j}{2\varepsilon_0}}^{\frac{\varepsilon_j}{2\varepsilon_0}} \left| v\left(\frac{\varepsilon_0 x'}{\varepsilon_j}, \frac{\varepsilon_0 x_d}{\varepsilon_j}\right) \right|^2 dx_d dx' \\ &= \frac{\varepsilon_j}{\varepsilon_0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\Omega'} \left| v\left(\frac{\varepsilon_0 x'}{\varepsilon_j}, x_d\right) \right|^2 dx' dx_d. \end{aligned}$$

We observe that, since $v \in H_{\text{loc}}^k(\mathbb{R}^d)$, the function $f: \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ defined by

$$f(y) = \int_{-\frac{1}{2}}^{\frac{1}{2}} |v(y, x_d)|^2 dx_d \quad \text{for every } y \in \mathbb{R}^{d-1}$$

is integrable on Ω' and that, by definition of v , it is periodic. We may then apply the Riemann-Lebesgue Lemma to the function f to obtain

$$\begin{aligned} \lim_{j \rightarrow +\infty} \int_{\Omega'} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| v\left(\frac{\varepsilon_0 x'}{\varepsilon_j}, x_d\right) \right|^2 dx_d dx' &= \int_{\Omega'} \int_{(-\frac{1}{2}, \frac{1}{2})^{d-1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |v(y', x_d)|^2 dx_d dy' dx' \\ &= \mathcal{L}^{d-1}(\Omega') \|v\|_{L^2(Q)}^2, \end{aligned} \quad (2.4.69)$$

which implies

$$\lim_{j \rightarrow +\infty} \int_{\Omega_j} |u_j|^2 dx = \mathcal{L}^{d-1}(\Omega') \lim_{j \rightarrow +\infty} \frac{\varepsilon_j}{\varepsilon_0} \|v\|_{L^2(Q)}^2 = 0. \quad (2.4.70)$$

Using also (2.4.67) and the fact that u takes values only in $\{\pm 1\}$, we deduce that

$$\|u_j - u\|_{L^2(\Omega)}^2 \leq 2 \int_{\Omega_j} |u_j|^2 dx + 2|\Omega_j|$$

tends to 0 as $j \rightarrow +\infty$, which proves (2.4.68).

We now show that $\{u_j\}_j$ is a recovery sequence for u . To this aim, we first note that for every $x \in \Omega \setminus \Omega_j$ we have

$$u_j(x) = u(x) \in \{-1, 1\} \quad \text{and} \quad \nabla^{(\ell)} u_j(x) = 0 \quad \text{for every } \ell \in \{1, \dots, k\},$$

while, if $x \in \Omega_j$, we have

$$\nabla^{(\ell)} u_j(x) = \left(\frac{\varepsilon_0}{\varepsilon_j}\right)^\ell \nabla^{(\ell)} v\left(\frac{\varepsilon_0 x}{\varepsilon_j}\right) \quad \text{for every } \ell \in \{1, \dots, k\}.$$

Then, we use (H1), (2.4.67), Fubini's Theorem, and a change of variable to obtain

$$\begin{aligned} F_{\varepsilon_j}(u_j) &= \int_{\Omega_j} \left[\frac{1}{\varepsilon_j} W(u_j) + \sum_{\ell=1}^k q_\ell \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} u_j|_\ell^2 \right] dx \\ &= \int_{\Omega'} \int_{-\frac{\varepsilon_j}{2\varepsilon_0}}^{\frac{\varepsilon_j}{2\varepsilon_0}} \left[\frac{1}{\varepsilon_j} W\left(v\left(\frac{\varepsilon_0 x'}{\varepsilon_j}, \frac{\varepsilon_0 x_d}{\varepsilon_j}\right)\right) + \sum_{\ell=1}^k q_\ell \varepsilon_j^{2\ell} \left| \nabla^{(\ell)} v\left(\frac{\varepsilon_0 x'}{\varepsilon_j}, \frac{\varepsilon_0 x_d}{\varepsilon_j}\right) \right|_\ell^2 \right] dx_d dx' \\ &= \int_{\Omega'} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{1}{\varepsilon_0} W\left(v\left(\frac{\varepsilon_0 x'}{\varepsilon_j}, x_d\right)\right) + \sum_{\ell=1}^k q_\ell \varepsilon_0^{2\ell-1} \left| \nabla^{(\ell)} v\left(\frac{\varepsilon_0 x'}{\varepsilon_j}, x_d\right) \right|_\ell^2 \right] dx_d dx' \end{aligned} \quad (2.4.71)$$

for every $j \in \mathbb{N}$. Since v satisfies (2.4.65), the function $h: \mathbb{R}^{d-1} \rightarrow \mathbb{R}^d$ defined for $y \in \mathbb{R}^{d-1}$ by

$$h(y) := \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{1}{\varepsilon_0} W(v(y, x_d)) + \sum_{\ell=1}^k q_\ell \varepsilon_0^{2\ell-1} |\nabla^{(\ell)} v(y, x_d)|_\ell^2 \right] dx_d$$

is integrable and, by (2.4.66), is also periodic. Therefore, we may apply again the Riemann-Lebesgue Lemma to obtain

$$\begin{aligned} \lim_{j \rightarrow +\infty} \int_{\Omega'} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{1}{\varepsilon_j} W\left(v\left(\frac{\varepsilon_0 x'}{\varepsilon_j}, x_d\right)\right) + \sum_{\ell=1}^k q_\ell \varepsilon_0^{2\ell-1} \left|\nabla^{(\ell)} v\left(\frac{\varepsilon_0 x'}{\varepsilon_j}, x_d\right)\right|_\ell^2 \right] dx_d dx' \\ = \mathcal{L}^{d-1}(\Omega') \int_Q \left[\frac{1}{\varepsilon_0} W(v) + \sum_{\ell=1}^k q_\ell \varepsilon_0^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \right] dx. \end{aligned}$$

Combining this equality with (2.4.71) and (2.4.65), we conclude that

$$\lim_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) = \mathcal{L}^{d-1}(\Omega') \int_Q \left[\frac{1}{\varepsilon_0} W(v) + \sum_{\ell=1}^k q_\ell \varepsilon_0^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \right] dx \leq F(u) + \mathcal{L}^{d-1}(\Omega') \eta.$$

Since η is arbitrary, a diagonal argument allows to prove the thesis in this special case.

Step 2. We now assume that $u = 2\chi_A - 1$, with $A = P \cap \Omega$ and P polyhedral. In particular we assume that $\partial P = \bigcup_{i=1}^M H_i \cup F$, where the sets $\{H_i\}_{i=1}^M$ are pairwise disjoint and relatively open convex polyhedra of dimension $d-1$, while F is the union of a finite number of convex polyhedra of dimension $d-2$. In this way, there exist $\{\nu_1, \dots, \nu_M\} \subset \mathbb{S}^{d-1}$ and $\{x_1, \dots, x_M\} \subset \Omega$ such that

$$H_i \subseteq \{x \in \mathbb{R}^d : (x - x_i) \cdot \nu_i = 0\},$$

and each ν_i is the inner unit normal to ∂P on H_i . For simplicity, we may also assume that $\partial\Omega \cap \partial P$ is the union of a finite number of C^1 manifolds of dimension $d-2$.

Let us fix $0 < \sigma < 1$ and let $H'_1, \dots, H'_M$ be relatively open subsets of $H_1, \dots, H_M$ with $(d-2)$ -dimensional boundary of class C^1 , such that

$$\{x \in H_i \cap \Omega : \text{dist}(x, \partial\Omega \cup F) \geq \frac{\sigma}{2}\} \subseteq H'_i \subseteq \overline{H'_i} \subseteq H_i.$$

Fix $0 < \eta < \frac{\sigma}{2}$ and define for each $i \in \{1, \dots, M\}$

$$\Omega_i := \{x + t\nu_i : x \in H'_i, t \in (-\eta, \eta)\}.$$

We can choose η so small that the sets $\Omega_1, \dots, \Omega_M$ are pairwise disjoint. Since each Ω_i satisfies (2.4.64), by the previous step, we obtain sequences $\{u_j^i\}_j \subset H^k(\Omega_i)$ such that

$$u_j^i \rightarrow u \text{ in } L^2(\Omega_i), \quad (2.4.72)$$

$$u_j^i(x) = u(x) \text{ for every } x \in \left\{x \in \overline{\Omega_i} : \text{dist}(x, H_i) \geq \frac{\varepsilon_j}{2\varepsilon_0^i}\right\},$$

and

$$\limsup_{j \rightarrow \infty} F_{\varepsilon_j}(u_j^i, \Omega_i) \leq (g(\nu_i) + \eta) \mathcal{H}^{d-1}(\Omega_i \cap H_i), \quad (2.4.73)$$

where for every $x \in \Omega_i$ it holds

$$u_j^i(x) := \begin{cases} v^i\left(\frac{\varepsilon_0^i}{\varepsilon_j}(x - x_i)\right) & \text{if } |(x - x_i) \cdot \nu_i| \leq \frac{\varepsilon_j}{2\varepsilon_0^i}, \\ u(x) & \text{if } |(x - x_i) \cdot \nu_i| > \frac{\varepsilon_j}{2\varepsilon_0^i}, \end{cases}$$

and, for each $i \in \{1, \dots, M\}$, v^i and ε_0^i are such that

$$\int_{Q_1^{\nu_i}} \left[\frac{1}{\varepsilon_0^i} W(v) + \sum_{\ell=1}^k q_\ell (\varepsilon_0^i)^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \right] dx \leq g(\nu_i) + \eta.$$

In order to present the candidate recovery sequence, we let $\{\Psi_\varepsilon\}_\varepsilon$ be a family of radial smooth convolution kernels, with $\text{supp}\Psi_\varepsilon \subset B_\varepsilon$ and $\int_{\mathbb{R}^d} \Psi_\varepsilon dx = 1$ for every $\varepsilon > 0$. We extend u to the whole $\mathbb{R}^d$ by setting $u := 2\chi_P - 1$ and introduce smooth functions $\tilde{u}_j$ defined for every $j \in \mathbb{N}$ by the convolution

$$\tilde{u}_j := u * \Psi_{\varepsilon_j}.$$

Observe that these functions satisfy the following conditions:

$$\tilde{u}_j(x) = u(x) \quad \text{for every } x \in \{x \in \Omega : \text{dist}(x, \partial P) \geq \varepsilon_j\}, \quad (2.4.74)$$

$$\|\tilde{u}_j\|_{L^\infty(\mathbb{R}^d)} \leq 1 \quad \text{and} \quad \|\nabla^{(\ell)} \tilde{u}_j\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{\varepsilon_j^\ell} \quad \text{for every } \ell \in \{1, \dots, k\}. \quad (2.4.75)$$

For $\tau \in (0, 1)$ we set

$$U_\tau := \{x \in \Omega : \text{dist}(x, F \cup (\partial\Omega \cap \partial P)) \leq \tau\}$$

and choose cut-off functions $\phi_\sigma \in C_c^\infty(\mathbb{R}^d; [0, 1])$ such that

$$\begin{cases} \phi_\sigma(x) = 1 & \text{if } x \in U_\sigma, \\ \phi_\sigma(x) = 0 & \text{if } x \in \mathbb{R}^d \setminus U_{2\sigma}, \\ \|\nabla^{(\ell)} \phi_\sigma\|_{L^\infty(\mathbb{R}^d)} \leq C\sigma^{-\ell} & \text{for every } \ell \in \{1, \dots, k\}. \end{cases} \quad (2.4.76)$$

We claim that the sequence of functions $\{u_j\}_j \subset H^k(\Omega)$ defined by

$$u_j := \begin{cases} (1 - \phi_\sigma)u_j^i + \phi_\sigma \tilde{u}_j & \text{in } \bar{\Omega}_i \text{ for } i \in \{1, \dots, M\}, \\ \tilde{u}_j & \text{in } \Omega' \end{cases}$$

is a recovery sequence for u , with

$$\Omega' := \Omega \setminus \bigcup_{i=1}^M \bar{\Omega}_i.$$

The fact that $\{u_j\}_j \subset H^k(\Omega)$ follows from the same argument used in [55, Theorem 1.3, Substep 2B]. Moreover, (2.4.72) and the fact that $\tilde{u}_j \rightarrow u$ in $L^2(\Omega)$ imply that $u_j \rightarrow u$ in $L^2(\Omega)$ as $j \rightarrow +\infty$.

We now study the asymptotic behaviour of the energy of $\{u_j\}_j$. First, we show that $F_{\varepsilon_j}(u_j, \Omega')$ is negligible. To this aim, for every $j \in \mathbb{N}$ we set

$$\mathcal{R}_j := \left\{x \in \Omega : \text{dist}(x, \partial P) \leq \max\left\{\varepsilon_j, \frac{\varepsilon_j}{2\varepsilon_0^1}, \dots, \frac{\varepsilon_j}{2\varepsilon_0^M}\right\}\right\}. \quad (2.4.77)$$

We observe that $\Omega' \cap \mathcal{R}_j \subset U_\sigma$ for j large enough and that $\mathcal{H}^{d-1}(\partial P \cap U_\sigma) \leq C\sigma$. Hence, by the definition of Ω' , (2.4.74), and (2.4.75), we have

$$F_{\varepsilon_j}(u_j, \Omega') = F_{\varepsilon_j}(u_j, \Omega' \cap \mathcal{R}_j) = \int_{\Omega' \cap \mathcal{R}_j} \left[\frac{1}{\varepsilon_j} W(\tilde{u}_j) + \sum_{\ell=1}^k |q_\ell| \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} \tilde{u}_j|_\ell^2 \right] dx \leq C\sigma. \quad (2.4.78)$$

We now study the energetic contribution of the remaining terms. We fix $i \in \{1, \dots, M\}$ and compute

$$\begin{aligned} F_{\varepsilon_j}(u_j, \Omega_i) &= F_{\varepsilon_j}(u_j^i, \Omega_i \setminus U_{2\sigma}) + F_{\varepsilon_j}(u_j, \Omega_i \cap (U_{2\sigma} \setminus U_\sigma)) + F_{\varepsilon_j}(\tilde{u}_j, \Omega_i \cap U_\sigma) \\ &\leq F_{\varepsilon_j}(u_j^i, \Omega_i \setminus U_{2\sigma}) + \int_{\Omega_i \cap (U_{2\sigma} \setminus U_\sigma)} \left[\frac{1}{\varepsilon_n} W(u_j) + \sum_{\ell=1}^k |q_\ell| \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} u_j|_\ell^2 \right] dx \\ &\quad + \int_{\Omega_i \cap U_\sigma} \left[\frac{1}{\varepsilon_j} W(\tilde{u}_j) + \sum_{\ell=1}^k |q_\ell| \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} \tilde{u}_j|_\ell^2 \right] dx \\ &=: \mathcal{K}_j^1 + \mathcal{K}_j^2 + \mathcal{K}_j^3. \end{aligned}$$

We separately investigate the three terms $\mathcal{K}_j^1$, $\mathcal{K}_j^2$, and $\mathcal{K}_j^3$, starting from $\mathcal{K}_j^1$. Recalling that $q_k = 1$, we observe that

$$\begin{aligned}
\mathcal{K}_j^1 &= \int_{\Omega_i \setminus U_{2\sigma}} \left[\frac{1}{\varepsilon_j} W(u_j^i) + \varepsilon_j^{2k-1} |\nabla^{(k)} u_j^i|_k^2 \right] dx + \int_{\Omega_i \setminus U_{2\sigma}} \left[\sum_{\ell=1}^{k-1} q_\ell \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} u_j^i|_\ell^2 \right] dx \\
&\leq \int_{\Omega_i} \left[\frac{1}{\varepsilon_j} W(u_j^i) + \varepsilon_j^{2k-1} |\nabla^{(k)} u_j^i|_k^2 \right] dx + \int_{\Omega_i} \left[\sum_{\ell=1}^{k-1} q_\ell \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} u_j^i|_\ell^2 \right] dx \\
&\quad - \int_{\Omega_i \cap U_{2\sigma}} \left[\sum_{\ell=1}^{k-1} q_\ell \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} u_j^i|_\ell^2 \right] dx \\
&\leq F_{\varepsilon_j}(u_j^i, \Omega_i) + \int_{\Omega_i \cap U_{2\sigma}} \left[\sum_{\ell=1}^{k-1} |q_\ell| \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} u_j^i|_\ell^2 \right] dx.
\end{aligned} \tag{2.4.79}$$

By the periodicity of u_j^i with respect to the variables tangential to H_i , we may argue as in *Step 1*, using Fubini's Theorem and the Riemann-Lebesgue Lemma to deduce that

$$\lim_{j \rightarrow +\infty} \int_{\Omega_i \cap U_{2\sigma}} \left[\sum_{\ell=1}^{k-1} |q_\ell| \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} u_j^i|_\ell^2 \right] dx \leq C\sigma \int_{Q_1^{\nu_i}} \left[\sum_{\ell=1}^{k-1} |\nabla^{(\ell)} v^i|_\ell^2 \right] dx \leq C\sigma.$$

Hence, it follows from (2.4.79) and (2.4.73) that

$$\limsup_{j \rightarrow +\infty} \mathcal{K}_j^1 \leq (g(\nu_i) + \eta) \mathcal{H}^{d-1}(H_i \cap \Omega_i) + C\sigma. \tag{2.4.80}$$

We now investigate the term $\mathcal{K}_j^2$. We begin observing that for every $\ell \in \{1, \dots, k\}$ and multi-index α of order $|\alpha| = \ell$, we have

$$\begin{aligned}
\partial^\alpha u_j &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left(\partial^\beta (1 - \phi_\sigma) \partial^{\alpha-\beta} u_j^i + \partial^\beta \phi_\sigma \partial^{\alpha-\beta} \tilde{u}_j \right) \\
&= (1 - \phi_\sigma) \partial^\alpha u_j^i + \phi_\sigma \partial^\alpha \tilde{u}_j + \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \left(\partial^\beta \phi_\sigma \partial^{\alpha-\beta} (\tilde{u}_j - u_j^i) \right),
\end{aligned}$$

which by (2.4.76) implies

$$\begin{aligned}
|\nabla^{(\ell)} u_j|_\ell^2 &\leq C \left(|\nabla^{(\ell)} u_j^i|_\ell^2 + |\nabla^{(\ell)} \tilde{u}_j|_\ell^2 \right. \\
&\quad \left. + \sigma^{-2\ell} |\tilde{u}_j - u_j^i|^2 + \sum_{h=1}^{\ell-1} \sigma^{-2(\ell-h)} |\nabla^{(h)} (\tilde{u}_j - u_j^i)|_h^2 \right).
\end{aligned} \tag{2.4.81}$$

Since by (H3) we have that

$$W(u_j) \leq W(u_j^i) + C, \tag{2.4.82}$$

it follows from (2.4.77), (2.4.81), and (2.4.82) that

$$\begin{aligned}
\mathcal{K}_j^2 &\leq C \int_{\Omega_i \cap (U_{2\sigma} \setminus U_\sigma) \cap \mathcal{R}_j} \left[\frac{1}{\varepsilon_j} W(u_j^i) + \frac{1}{\varepsilon_j} + \sum_{\ell=1}^k |q_\ell| \varepsilon_j^{2\ell-1} \left(|\nabla^{(\ell)} u_j^i|_\ell^2 + \sum_{h=1}^{\ell-1} \sigma^{-2(\ell-h)} |\nabla^{(h)} u_j^i|_h^2 \right) \right] dx \\
&\quad + C \int_{\Omega_i \cap (U_{2\sigma} \setminus U_\sigma) \cap \mathcal{R}_j} \left[\sum_{\ell=1}^k |q_\ell| \varepsilon_j^{2\ell-1} \left(|\nabla^{(\ell)} \tilde{u}_j|_\ell^2 + \sum_{h=1}^{\ell-1} \sigma^{-2(\ell-h)} |\nabla^{(h)} \tilde{u}_j|_h^2 + \sigma^{-2\ell} |u_j^i - \tilde{u}_j|^2 \right) \right] dx.
\end{aligned}$$

Arguing as in (2.4.69) and (2.4.71), we obtain that

$$\begin{aligned}
\lim_{j \rightarrow +\infty} \int_{\Omega_i \cap (U_{2\sigma} \setminus U_\sigma) \cap \mathcal{R}_j} &\left[\frac{1}{\varepsilon_j} W(u_j^i) + \frac{1}{\varepsilon_j} + \sum_{\ell=1}^k |q_\ell| \varepsilon_j^{2\ell-1} \left(|\nabla^{(\ell)} u_j^i|_\ell^2 + \sum_{h=1}^{\ell-1} \sigma^{-2(\ell-h)} |\nabla^{(h)} u_j^i|_h^2 \right) \right] dx \\
&\leq C \mathcal{H}^{d-1}(H_i \cap U_{2\sigma}) \leq C\sigma.
\end{aligned} \tag{2.4.83}$$

Then, we observe that using (2.4.70), with u_j replaced by u_j^i , v replaced by v^i , and Ω replaced by Ω_i , together with (2.4.75), we have

$$\begin{aligned}
& \int_{\Omega_i \cap (U_{2\sigma} \setminus U_\sigma) \cap \mathcal{R}_j} \left[\sum_{\ell=1}^k |q_\ell| \varepsilon_j^{2\ell-1} \left(|\nabla^{(\ell)} \tilde{u}_j|_\ell^2 + \sum_{h=1}^{\ell-1} \sigma^{-2(\ell-h)} |\nabla^{(h)} \tilde{u}_j|_h^2 + \sigma^{-2\ell} |\tilde{u}_j - u_j^i|^2 \right) \right] dx \\
& \leq \frac{C}{\varepsilon_j} \left(\sum_{\ell=1}^k \left(1 + \sum_{h=1}^{\ell-1} \frac{\varepsilon_j^{2(\ell-h)}}{\sigma^{2(\ell-h)}} \right) \right) |\Omega_i \cap (U_{2\sigma} \setminus U_\sigma) \cap \mathcal{R}_j| \\
& \quad + C \sum_{\ell=1}^k \left(\frac{\varepsilon_j^{2\ell+1}}{\sigma^{2\ell}} \|v^i\|_{L^2(\Omega_i)}^2 + \frac{\varepsilon_j^{2\ell-1}}{\sigma^{2\ell}} |\Omega_i \cap (U_{2\sigma} \setminus U_\sigma) \cap \mathcal{R}_j| \right) \\
& \leq C \left(\sum_{\ell=1}^k \left(1 + \sum_{h=1}^{\ell-1} \frac{\varepsilon_j^{2(\ell-h)}}{\sigma^{2(\ell-h)}} \right) \right) \sigma + C \sum_{\ell=1}^k \left(\frac{\varepsilon_j^{2\ell+1}}{\sigma^{2\ell}} + \frac{\varepsilon_j^{2\ell}}{\sigma^{2\ell-1}} \right). \tag{2.4.84}
\end{aligned}$$

Combining (2.4.83) with (2.4.84), we obtain that

$$\limsup_{j \rightarrow +\infty} \mathcal{K}_j^2 \leq C\sigma. \tag{2.4.85}$$

To conclude, we use (2.4.75) to estimate the remaining term $\mathcal{K}_j^3$ as

$$\begin{aligned}
\mathcal{K}_j^3 & \leq \int_{U_\sigma \cap \{x: |(x-x_i) \cdot \nu_i| \leq \varepsilon_j\}} \left[\frac{1}{\varepsilon_j} W(\tilde{u}_j) + \sum_{\ell=1}^k |q_\ell| \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} \tilde{u}_j|_\ell^2 \right] dx \\
& \leq \frac{C}{\varepsilon_j} |\{x \in U_\sigma : |(x-x_i) \cdot \nu_i| \leq \varepsilon_j\}| \leq C \mathcal{H}^{d-1}(U_\sigma \cap H_i) \leq C\sigma. \tag{2.4.86}
\end{aligned}$$

Finally, gathering (2.4.78), (2.4.80), (2.4.85), (2.4.86), and summing over $i \in \{1, \dots, M\}$ we obtain that

$$\begin{aligned}
\limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j) & \leq \sum_{i=1}^M \limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j, \Omega_i) + \limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j, \Omega') \\
& = \sum_{i=1}^M (g(\nu_i) + \eta) \mathcal{H}^{d-1}(\Omega_i \cap H_i) + C\sigma \leq F(u) + C(\eta + \sigma).
\end{aligned}$$

Since $\eta > 0$ and $\sigma > 0$ are arbitrary, a diagonal arguments allows to conclude.

Step 3. Finally, we deal with the general case $u = 2\chi_A - 1$, for $A \subseteq \Omega$ a set of finite relative perimeter in Ω . By Theorem 2.1.2, we can find a sequence of polyhedral sets $\{A_n\}_n$ such that $\chi_{A_n} \rightarrow \chi_A$ in $L^1(\Omega)$. By the previous step, setting $u_n := 2\chi_{A_n} - 1$, we have

$$\Gamma\text{-}\limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(u_n) \leq F(u_n) \quad \text{for every } n \in \mathbb{N},$$

where we recall that the Γ -limsup is the functional defined by (2.1.2). By the lower semicontinuity of the Γ -limsup we then infer

$$\Gamma\text{-}\limsup_{j \rightarrow +\infty} F_{\varepsilon_j}(u) \leq \liminf_{n \rightarrow \infty} F(u_n).$$

Since g is upper semicontinuous on $\mathbb{S}^{d-1}$ (see Proposition 2.4.6), by [61, Remark 9.11, Theorem 9.15] there exists a sequence $\{g_m\}_m$ of Lipschitz continuous functions such that

$$g \leq g_{m+1} \leq g_m \leq \max_{\mathbb{S}^{d-1}} g$$

and

$$g(\nu) = \inf_m g_m(\nu) \quad \text{for every } \nu \in \mathbb{S}^{d-1}.$$

Using (2.1.1) we obtain for every $m \in \mathbb{N}$ that

$$\begin{aligned} \liminf_{n \rightarrow \infty} F(u_n) &= \liminf_{n \rightarrow \infty} \int_{\partial^* A_n \cap \Omega} g(\nu_{A_n}) d\mathcal{H}^{d-1} \\ &\leq \lim_{n \rightarrow \infty} \int_{\partial^* A_n \cap \Omega} g_m(\nu_{A_n}) d\mathcal{H}^{d-1} \\ &= \int_{\partial^* A \cap \Omega} g_m(\nu_A) d\mathcal{H}^{d-1}. \end{aligned}$$

Finally, we obtain that

$$\liminf_{n \rightarrow \infty} F(u_n) \leq \lim_{m \rightarrow \infty} \int_{\partial^* A \cap \Omega} g_m(\nu_A) d\mathcal{H}^{d-1} = \int_{\partial^* A \cap \Omega} g(\nu_A) d\mathcal{H}^{d-1} = F(u),$$

which concludes the proof. $\square$

We conclude this subsection proving that the surface energy density g is strictly positive. This observation makes the Γ -limit non trivial.

Proposition 2.4.23. *Let g be the function defined in (2.4.3) with $k > 1$ an integer and let $\mathcal{N} := \{\ell \in \{1, \dots, k-1\} : q_\ell \leq 0\}$. Assume that (H1)-(H3) are satisfied, and, if $\mathcal{N} \neq \emptyset$, further assume that $q_\ell > -\bar{q}_\ell$ for all $\ell \in \mathcal{N}$, where $\{\bar{q}_\ell : \ell \in \mathcal{N}\}$ denote the same positive constants appearing in Corollary 2.4.14. Then $\inf_{\mathbb{S}^{d-1}} g > 0$.*

Proof. Similarly to Subsection 2.4.4, we introduce $\tilde{Q}^\nu$ an open set with C^2 boundary contained in Q^ν such that

$$\tilde{Q}^\nu \cap \{y \in \mathbb{R}^d : |y \cdot \nu| \leq \tfrac{1}{4}\} = Q^\nu \cap \{y \in \mathbb{R}^d : |y \cdot \nu| \leq \tfrac{1}{4}\}$$

and, letting $\{\nu_1, \dots, \nu_{d-1}, \nu\}$ denote an orthonormal basis describing Q^ν , with the additional property that

$$\tilde{Q}^\nu \cap \bigcap_{i=1}^{d-1} \{y \in \mathbb{R}^d : |y \cdot \nu_i| \leq \tfrac{1}{4}\} = Q^\nu \cap \bigcap_{i=1}^{d-1} \{y \in \mathbb{R}^d : |y \cdot \nu_i| \leq \tfrac{1}{4}\} =: R^\nu. \quad (2.4.87)$$

By the proof of Proposition 2.4.19, the function $\tilde{g} : \mathbb{S}^{d-1} \rightarrow \mathbb{R}$ given by

$$\tilde{g}(\nu) := \inf \left\{ \int_{\tilde{Q}^\nu} \left[\frac{1}{\varepsilon} W(v) + \sum_{\ell=1}^k q_\ell \varepsilon^{2\ell-1} |\nabla^{(\ell)} v|_\ell^2 \right] dx : v \in \tilde{\mathcal{A}}_\varepsilon^\nu, \varepsilon \in (0, 1) \right\}$$

introduced in Remark 2.4.21 satisfies $\tilde{g}(\nu) \geq g(\nu)$ for every $\nu \in \mathbb{S}^{d-1}$. We combine this observation with Remark 2.4.21 and Proposition 2.4.22 to infer that

$$\begin{aligned} \Gamma\text{-}\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u) &\geq \begin{cases} \int_{\partial^* A \cap \Omega} \tilde{g}(\nu_A) d\mathcal{H}^{d-1} & \text{if } u \in BV(\Omega; \{-1, 1\}), \text{ with } u = 2\chi_A - 1, \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega; \{-1, 1\}), \end{cases} \\ &\geq \begin{cases} \int_{\partial^* A \cap \Omega} g(\nu_A) d\mathcal{H}^{d-1} & \text{if } u \in BV(\Omega; \{-1, 1\}), \text{ with } u = 2\chi_A - 1, \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega; \{-1, 1\}) \end{cases} \\ &\geq \Gamma\text{-}\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u), \end{aligned}$$

which, in turn, implies that $\tilde{g}(\nu) = g(\nu)$ for every $\nu \in \mathbb{S}^{d-1}$. For this reason, we equivalently prove that $\inf_{\mathbb{S}^{d-1}} \tilde{g} > 0$.

Fix $\nu \in \mathbb{S}^{d-1}$, and let $\{\varepsilon_j\}_j$, $v_j \in \tilde{\mathcal{A}}_{\varepsilon_j}^\nu$, $j \in \mathbb{N}$, such that

$$\tilde{g}(\nu) = \lim_{j \rightarrow +\infty} \int_{\tilde{Q}^\nu} \left[\frac{1}{\varepsilon_j} W(v_j) + \sum_{\ell=1}^k q_\ell \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} v_j|_\ell^2 \right] dx.$$

Arguing as in Remark 2.4.7, it is not restrictive to suppose $\varepsilon_j \rightarrow 0$ so that we may further assume that for all $j \in \mathbb{N}$ it holds $\varepsilon_j < \varepsilon_0(\{q_\ell : \ell \in \mathcal{N}\}, \tilde{Q}^\nu)$, the constant appearing in Corollary 2.4.14.

Clearly, the set R^ν defined in (2.4.87) is contained in $\tilde{Q}^\nu$; then, we apply Corollary 2.4.14 with $\Omega = \tilde{Q}^\nu$ and, by the equivalence of the norms, we obtain that

$$\begin{aligned} \int_{\tilde{Q}^\nu} \left[\frac{1}{\varepsilon_j} W(v_j) + \sum_{\ell=1}^k q_\ell \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} v_j|^2 \right] dx &\geq \delta \int_{\tilde{Q}^\nu} \left[\frac{1}{\varepsilon_j} W(v_j) + \sum_{\ell=1}^k \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} v_j|^2 \right] dx \\ &\geq \delta \int_{R^\nu} \left[\frac{1}{\varepsilon_j} W(v_j) + \sum_{\ell=1}^k \varepsilon_j^{2\ell-1} |\nabla^{(\ell)} v_j|^2 \right] dx \\ &\geq \delta' \int_{R^\nu} \left[\frac{1}{\varepsilon_j} W(v_j) + \varepsilon_j^{2k-1} \|\nabla^{(k)} v_j\|_k^2 \right] dx \end{aligned}$$

for every $j \in \mathbb{N}$ and some $\delta' > 0$ which is independent of $\tilde{Q}^\nu$ and j . Applying Fubini's Theorem, the definition of operatorial norm, a change of variables, and (2.4.87), we obtain

$$\begin{aligned} \int_{R^\nu} \left[\frac{1}{\varepsilon_j} W(v_j) + \varepsilon_j^{2k-1} \|\nabla^{(k)} v_j\|_k^2 \right] dx &\geq \int_{R^\nu \cap \nu^\perp} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{1}{\varepsilon_j} W(v_j^{\nu,y}) + \varepsilon_j^{2k-1} |(v_j^{\nu,y})^{(k)}|^2 \right] dt d\mathcal{H}^{d-1}(y) \\ &= \int_{R^\nu \cap \nu^\perp} \int_{-\frac{1}{2\varepsilon_j}}^{\frac{1}{2\varepsilon_j}} [W(w_j^{\nu,y}) + |(w_j^{\nu,y})^{(k)}|^2] dt d\mathcal{H}^{d-1}(y) \\ &\geq \frac{1}{2^{d-1}} \inf \left\{ \int_{-\frac{1}{2\varepsilon_j}}^{\frac{1}{2\varepsilon_j}} [W(w) + |w^{(k)}|^2] dt, w \in H^k\left(-\frac{1}{2\varepsilon_j}, \frac{1}{2\varepsilon_j}\right), \right. \\ &\quad \left. w(t) = \operatorname{sgn}(t) \text{ if } |t| > T \text{ for some } T \in \left(0, \frac{1}{2\varepsilon_j}\right) \right\}, \end{aligned}$$

where we set $v_j^{\nu,y}(t) := v_j^{\nu,y}(\varepsilon_j t)$ and in the last inequality we used that $v_j \in \tilde{\mathcal{A}}_{\varepsilon_j}^\nu$ for all $j \in \mathbb{N}$. As (H1) and (H2) are satisfied, we are in position to apply Propositions 2.3.6 and 2.3.7 deducing that

$$\tilde{g}(\nu) \geq \delta' \liminf_{j \rightarrow +\infty} \int_{R^\nu} \left[\frac{1}{\varepsilon_j} W(v_j) + \varepsilon_j^{2k-1} \|\nabla^{(k)} v_j\|_k^2 \right] dx \geq \delta' m_k > 0,$$

m_k being as in (2.3.3), which concludes the proof. $\square$

2.4.6 Analysis of the cell problem in dimension 1

In this final subsection we analyze the surface energy density g defined in (2.4.3) when $d = 1$. In this instance, such density clearly is a constant that we prove being determined by an optimal-profile problem on the real line. It is not restrictive to assume that $|\cdot|_\ell$ coincides with the absolute value for every $\ell \in \{1, \dots, k\}$ as any norm on $\mathbb{R}$ is a multiple of the Euclidean one.

By Remark 2.4.7 and by a change of variable, we find that

$$\begin{aligned} g &= \inf_{T > \frac{1}{2}} \inf \left\{ \int_{-T}^T \left[W(v) + \sum_{\ell=1}^k q_\ell |v^{(\ell)}|^2 \right] dt : v \in H^k(-T, T), \right. \\ &\quad \left. v(t) = \operatorname{sgn}(t) \text{ if } |t| > M \text{ for some } M \in (0, T) \right\} \\ &= \lim_{T \rightarrow +\infty} \inf \left\{ \int_{-\infty}^{+\infty} \left[W(v) + \sum_{\ell=1}^k q_\ell |v^{(\ell)}|^2 \right] dt : v \in H_{\text{loc}}^k(\mathbb{R}), v(t) = \operatorname{sgn}(t) \text{ if } |t| > T \right\}, \end{aligned}$$

hence, for $T > 1/2$, we set

$$m(T) := \inf \left\{ \int_{-\infty}^{+\infty} \left[W(v) + \sum_{\ell=1}^k q_\ell |v^{(\ell)}|^2 \right] dt : v \in H_{\text{loc}}^k(\mathbb{R}), v(t) = \operatorname{sgn}(t) \text{ if } |t| > T \right\}$$

in such a way that

$$g = \lim_{T \rightarrow +\infty} m(T).$$

We set

$$m := \inf \left\{ \int_{-\infty}^{+\infty} \left[W(u) + \sum_{\ell=1}^k q_\ell |u^{(\ell)}|^2 \right] dt : u \in H_{\text{loc}}^k(\mathbb{R}), \lim_{t \rightarrow \pm\infty} u(t) = \pm 1 \right\}. \quad (2.4.88)$$

The main result of this subsection states that, under the by now usual suitable assumptions on the coefficients $q_1, \dots, q_{k-1}$, the equality $g = m$ holds. In order to prove this, we state the one-dimensional version of Corollary 2.4.14 at scale $\varepsilon = 1$.

Corollary 2.4.24. *Let $I \subset \mathbb{R}$ be an open interval of length 1, let $k > 1$ be an integer, and let $\mathcal{N} := \{\ell \in \{1, \dots, k-1\} : q_\ell \leq 0\}$. Assume that (H2) is satisfied, and, if $\mathcal{N} \neq \emptyset$, further assume that $q_\ell > -\bar{q}'_\ell$ for all $\ell \in \mathcal{N}$, where, letting $\{q'_\ell : \ell \in \mathcal{N}\}$ denote the same positive constants appearing in Corollary 2.4.10 and given $\{\alpha_\ell : \ell \in \mathcal{N}\}$ such that $\alpha_\ell > 0$ for all $\ell \in \mathcal{N}$ and $\sum_{\ell \in \mathcal{N}} \alpha_\ell = 1$, it holds that $\bar{q}'_\ell = \alpha_\ell q'_\ell$ for all $\ell \in \mathcal{N}$. Then, there exists a positive constant $\delta = \delta(\{q_\ell : \ell \in \mathcal{N}\})$ such that*

$$\int_I \left[W(u) + \sum_{\ell=1}^k q_\ell |u^{(\ell)}|^2 \right] dt \geq \delta \int_I \left[W(u) + \sum_{\ell=1}^k |u^{(\ell)}|^2 \right] dt$$

for every $u \in H^k(I)$.

Proof. The proof is obtained by following the same line of the one of Corollary 2.4.14, with Corollary 2.4.10 used for every $\ell \in \{1, \dots, k-1\}$ in place of Theorem 2.4.1. $\square$

Proposition 2.4.25. *Let $k > 1$ be an integer, assume that $q_\ell > -\bar{q}'_\ell$ for all $\ell \in \mathcal{N}$, with $\mathcal{N}$ and $\{\bar{q}'_\ell : \ell \in \mathcal{N}\}$ as in Corollary 2.4.24, and assume that (H1) and (H2) are satisfied. Then, it holds*

$$m = \lim_{T \rightarrow +\infty} m(T).$$

Proof. Clearly, $m(T) \geq m$ for every T ; hence, $\lim_{T \rightarrow +\infty} m(T) \geq m$.

To prove the converse, let $u \in H_{\text{loc}}^k(\mathbb{R})$ be such that

$$\lim_{t \rightarrow \pm\infty} u(t) = \pm 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} \left[W(u) + \sum_{\ell=1}^k q_\ell |u^{(\ell)}|^2 \right] dt < +\infty \quad (2.4.89)$$

and let $T_o > 0$ be such that $||u(t)| - 1| < 1/2$ if $|t| > T_o$.

Consider a function $\phi \in C^\infty(\mathbb{R}; [0, 1])$ that is supported in $(-\infty, 1/2)$ and such that $\phi(t) = 1$ for every $t < 0$. For every $j \in \mathbb{N}$ we set

$$v_j(t) := \phi(t - j)u(t) + 1 - \phi(t - j),$$

and we observe that v_j coincides with u on $(-\infty, j)$ and it is constantly equal to 1 on $(j + 1/2, +\infty)$. As a consequence

$$\begin{aligned} \int_{-\infty}^{+\infty} \left[W(v_j) + \sum_{\ell=1}^k q_\ell |v_j^{(\ell)}|^2 \right] dt &= \int_{-\infty}^{j-\frac{1}{2}} \left[W(u) + \sum_{\ell=1}^k q_\ell |u^{(\ell)}|^2 \right] dt \\ &\quad + \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \left[W(v_j) + \sum_{\ell=1}^k q_\ell |v_j^{(\ell)}|^2 \right] dt. \end{aligned}$$

We estimate the energy of the function v_j on the interval $(j - 1/2, j + 1/2)$; that is, we estimate

$$\int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \left[W(v_j) + \sum_{\ell=1}^k q_\ell |v_j^{(\ell)}|^2 \right] dt. \quad (2.4.90)$$

First, we observe that $\min\{u(t), 1\} \leq v_j(t) \leq \max\{u(t), 1\}$ for all $t \in \mathbb{R}$ and $j \in \mathbb{N}$; therefore, since $u(t) \rightarrow 1$ as $t \rightarrow +\infty$, for every $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that $|v_j(t) - 1| < \varepsilon$ for all $t > T_\varepsilon$ and $j \in \mathbb{N}$. Hence, the continuity of W and (H1) imply that

$$\lim_{j \rightarrow +\infty} \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} W(v_j) dt = 0.$$

As for the derivatives of v_j , we observe that for all $\ell \in \{1, \dots, k-1\}$ there exists a positive constant C depending on k, ℓ , and the function ϕ such that

$$|v_j^{(\ell)}(t)|^2 \leq C \left\{ |u(t) - 1|^2 + \sum_{h=1}^{\ell} |u^{(h)}(t)|^2 \right\} \quad \text{for all } t \in \mathbb{R},$$

therefore,

$$\begin{aligned} \left| \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \left[\sum_{\ell=1}^k q_\ell |v_j^{(\ell)}|^2 \right] dt \right| &\leq q_0 \sum_{\ell=1}^k \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \left[C \left\{ |u-1|^2 + \sum_{h=1}^{\ell} |u^{(h)}|^2 \right\} \right] dt \\ &\leq q_0 C k \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \left[W(u) + \sum_{\ell=1}^k |u^{(\ell)}|^2 \right] dt, \end{aligned}$$

where $q_0 := \max\{|q_1|, \dots, |q_{k-1}|, 1\}$, and the last inequality follows by (H2) assuming that $j-1 > T_0$. By (2.4.89), this term tends to 0 as $j \rightarrow +\infty$ and then we conclude that (2.4.90) vanishes as $j \rightarrow +\infty$. With a similar reasoning, we obtain functions $\{w_j\}_j \subset H_{\text{loc}}^k(\mathbb{R})$ which coincide with u on $(-j, j)$, which are equal to the sign function outside $(-j - \frac{1}{2}, j + \frac{1}{2})$, and such that

$$\int_{-\infty}^{+\infty} \left[W(w_j) + \sum_{\ell=1}^k q_\ell |w_j^{(\ell)}|^2 \right] dt = \int_{-j-\frac{1}{2}}^{j+\frac{1}{2}} \left[W(u) + \sum_{\ell=1}^k q_\ell |u^{(\ell)}|^2 \right] dt + o_j(1).$$

Since each function w_j is admissible for the minimum problem $m(j+1)$, letting $j \rightarrow +\infty$ the conclusion follows by the arbitrariness of u . $\square$

Finally, we prove the existence of an optimal-profile. We omit some minor details in the proof as the arguments involved are similar to those that appear in the proofs of Propositions 2.3.6 and 2.3.7 and [74, Lemma 2.5].

Proposition 2.4.26. *Let $k > 1$ be an integer, assume that $q_\ell > -\bar{q}'_\ell$ for all $\ell \in \mathcal{N}$, with $\mathcal{N}$ and $\{\bar{q}'_\ell : \ell \in \mathcal{N}\}$ as in Corollary 2.4.24, and assume that (H1) and (H2) are satisfied. Then, the infimum (2.4.88) is a minimum.*

Proof. Consider $\{u_j\}_j$ a minimizing sequence and, by translation invariance, assume that $u_j(0) = 0$ for every $j \in \mathbb{N}$. Applying Corollary 2.4.24 on each interval $(i, i+1)$, $i \in \mathbb{Z}$, and summing over i , we have

$$\int_{-\infty}^{+\infty} \left[W(u_j) + \sum_{\ell=1}^k q_\ell |u_j^{(\ell)}|^2 \right] dt \geq \delta \int_{-\infty}^{+\infty} \left[W(u_j) + \sum_{\ell=1}^k |u_j^{(\ell)}|^2 \right] dt \quad (2.4.91)$$

for all $j \in \mathbb{N}$, and then

$$\sup_j \int_{-\infty}^{+\infty} \left[W(u_j) + \sum_{\ell=1}^k |u_j^{(\ell)}|^2 \right] dt < +\infty. \quad (2.4.92)$$

We apply the Fundamental Theorem of Calculus and Hölder's inequality to obtain

$$|u_j(t)| \leq \int_0^t |u'_j(s)| ds \leq \sqrt{t} \|u'_j\|_{L^2(\mathbb{R})} \quad \text{for all } t \in \mathbb{R},$$

which, combined with (2.4.92), implies that for every $T > 0$ there exists a positive constant $C(T)$ such that

$$\sup_j \|u_j\|_{H^k(-T, T)} \leq C(T).$$

Therefore, there exists $u \in H_{\text{loc}}^k(\mathbb{R})$ such that, up to subsequences, $\{u_j^{(\ell)}\}_j$ converges locally uniformly on compact subsets of $\mathbb{R}$ to $u^{(\ell)}$ for all $\ell \in \{0, \dots, k-1\}$, with the convention that $v^{(0)} = v$, and $u_j^{(k)} \rightharpoonup u^{(k)}$ weakly in $L^2(\mathbb{R})$. Note that, in particular, $u(0) = 0$.

As for the lower semicontinuity of the energy, we first note that, as a consequence of Corollary 2.4.24, we have

$$\int_{-\infty}^{-T} \left[W(u_j) + \sum_{\ell=1}^k q_\ell |u_j^{(\ell)}|^2 \right] dt + \int_T^{+\infty} \left[W(u_j) + \sum_{\ell=1}^k q_\ell |u_j^{(\ell)}|^2 \right] dt \geq 0 \quad (2.4.93)$$

for all $j \in \mathbb{N}$ and $T \in \mathbb{N}$. Then, we observe that Rellich's Theorem implies that $u_j^{(\ell)} \rightarrow u^{(\ell)}$ in $L^2(-T, T)$ as $j \rightarrow +\infty$ for every $\ell \in \{1, \dots, k-1\}$ and $T > 0$, which in turn yields

$$\lim_{j \rightarrow +\infty} \int_{-T}^T \left[\sum_{\ell=1}^{k-1} q_\ell |u_j^{(\ell)}|^2 \right] dt = \int_{-T}^T \left[\sum_{\ell=1}^{k-1} q_\ell |u^{(\ell)}|^2 \right] dt \quad (2.4.94)$$

for every $T > 0$; moreover, by Fatou's lemma and the lower semicontinuity of the L^2 -norm, we have

$$\liminf_{j \rightarrow +\infty} \int_{-\infty}^{+\infty} \left[W(u_j) + |u_j^{(k)}|^2 \right] dt \geq \int_{-\infty}^{+\infty} \left[W(u) + |u^{(k)}|^2 \right] dt. \quad (2.4.95)$$

Combining (2.4.93), (2.4.94), and (2.4.95), and recalling that $q_k = 1$, we obtain

$$\begin{aligned} m &= \lim_{j \rightarrow +\infty} \int_{-\infty}^{+\infty} \left[W(u_j) + \sum_{\ell=1}^k q_\ell |u_j^{(\ell)}|^2 \right] dt \geq \liminf_{j \rightarrow +\infty} \int_{-T}^T \left[W(u_j) + \sum_{\ell=1}^k q_\ell |u_j^{(\ell)}|^2 \right] dt \\ &\geq \int_{-T}^T \left[W(u) + \sum_{\ell=1}^k q_\ell |u^{(\ell)}|^2 \right] dt \end{aligned} \quad (2.4.96)$$

for all $T \in \mathbb{N}$.

Recall that $\{u_j\}_j$ is a minimizing sequence, hence, by (2.4.91) and Fatou's Lemma, we infer

$$m = \liminf_{j \rightarrow +\infty} \int_{-\infty}^{+\infty} \left[W(u_j) + \sum_{\ell=1}^k q_\ell |u_j^{(\ell)}|^2 \right] dt \geq \delta \int_{-\infty}^{+\infty} \left[W(u) + \sum_{\ell=1}^k |u^{(\ell)}|^2 \right] dt;$$

therefore, we can pass to the limit as $T \rightarrow +\infty$ in (2.4.96) by the Dominated Convergence Theorem to obtain

$$m \geq \int_{-\infty}^{+\infty} \left[W(u) + \sum_{\ell=1}^k q_\ell |u^{(\ell)}|^2 \right] dt.$$

If $\lim_{t \rightarrow \pm\infty} u(t) = \pm 1$, the proof is completed; while, if $\lim_{t \rightarrow \pm\infty} u(t) = \mp 1$, a minimizer is $u(-t)$. For this reason, in order to conclude, we prove that both $\lim_{t \rightarrow -\infty} u(t)$ and $\lim_{t \rightarrow +\infty} u(t)$ exist, and that $\{\lim_{t \rightarrow -\infty} u(t), \lim_{t \rightarrow +\infty} u(t)\} = \{-1, 1\}$.

As a first step we prove the existence of the limit at $+\infty$, the other case being analogous. To see this, it is useful to observe that $u' \in H^1(\mathbb{R})$, which implies

$$\lim_{t \rightarrow \pm\infty} u'(t) = 0. \quad (2.4.97)$$

If $\lim_{t \rightarrow +\infty} u(t)$ exists, it equals -1 or 1 , otherwise $\int_{-\infty}^{+\infty} W(u) dt = +\infty$ which is a contradiction. Assume by contradiction that such a limit does not exist. Then there exist real numbers a, b , with $a < b$, and two increasing sequences $\{x_j\}_j, \{y_j\}_j$ such that

$$\lim_{j \rightarrow +\infty} x_j = +\infty, \quad \lim_{j \rightarrow +\infty} y_j = +\infty, \quad \lim_{j \rightarrow +\infty} u(x_j) = a, \quad \lim_{j \rightarrow +\infty} u(y_j) = b.$$

Without loss of generality, we can suppose that $x_j < y_j$ for every $j \in \mathbb{N}$ and, by the continuity of u , we can also assume that both -1 and 1 are not contained in the interval (a, b) .

If there exists $\{j_n\}_n$ such that $|x_{j_n} - y_{j_n}| \leq 2$ for every $n \in \mathbb{N}$, then, by Lagrange's Theorem, there exists $z_{j_n} \in (x_{j_n}, y_{j_n})$ such that

$$|u'(z_{j_n})| \geq \frac{|u(y_{j_n}) - u(x_{j_n})|}{2} \geq \frac{b-a}{2},$$

hence, $\limsup_{t \rightarrow +\infty} |u'(t)| \geq (b-a)/2$, which contradicts (2.4.97).

Then, we may assume that $|x_j - y_j| \geq 2$ for every $j \in \mathbb{N}$. We set

$$\tilde{x}_j := \max\{t \in (x_j, y_j) : u(t) \leq a + \frac{b-a}{4}\}$$

and

$$\tilde{y}_j := \min\{t \in (x_j, y_j) : u(t) \geq b - \frac{b-a}{4}\}.$$

If $|\tilde{x}_j - \tilde{y}_j| \leq 1$ for infinitely many $n \in \mathbb{N}$, the conclusion follows by Lagrange's Theorem as before, otherwise, it holds that $|\tilde{x}_j - \tilde{y}_j| \geq 1$ for all $j \geq J_o$ with some $J_o \in \mathbb{N}$ large enough. Recalling that the interval (a, b) does not contain the points $-1, 1$, the above observation and (H1) imply

$$\int_{-\infty}^{+\infty} W(u) dt \geq \sum_{j \geq J_o} \int_{\tilde{x}_j}^{\tilde{y}_j} W(u) dt \geq \sum_{j \geq J_o} \min\{W(t) : t \in (a + \frac{b-a}{4}, b - \frac{b-a}{4})\} = +\infty,$$

which is a contradiction, and concludes the proof of the existence of the limits at infinity. We finally prove that they are different.

By contradiction suppose $\lim_{t \rightarrow \pm\infty} u(t) = -1$. By Lemma 2.3.5, we find points $\{t_j\}_j$ such that

$$t_j \rightarrow +\infty, \quad u_j(t_j) \rightarrow -1, \quad u_j^{(\ell)}(t_j) \rightarrow 0 \quad \text{for all } \ell \in \{1, \dots, k-1\} \quad (2.4.98)$$

for j large enough. Then we have

$$\begin{aligned} \int_{-\infty}^{+\infty} \left[W(u_j) + \sum_{\ell=1}^k q_\ell |u_j^{(\ell)}|^2 \right] dt &= \int_{-\infty}^{t_j} \left[W(u_j) + \sum_{\ell=1}^k q_\ell |u_j^{(\ell)}|^2 \right] dt + \int_{t_j}^{+\infty} \left[W(u_j) + \sum_{\ell=1}^k q_\ell |u_j^{(\ell)}|^2 \right] dt \\ &\geq \int_{-\infty}^{t_j} \left[W(u_j) + \sum_{\ell=1}^k q_\ell |u_j^{(\ell)}|^2 \right] dt + m - \tilde{m}(u_j(t_j), \dots, u_j^{(k-1)}(t_j)), \end{aligned}$$

where we set

$$\begin{aligned} \tilde{m}(u_0, \dots, u_{k-1}) &:= \inf \left\{ \int_0^1 \left[W(v) + \sum_{\ell=1}^k q_\ell |v^{(\ell)}|^2 \right] dt : v(0) = -1, v^{(\ell)}(0) = 0 \text{ for all } \ell \in \{1, \dots, k-1\}, \right. \\ &\quad \left. v^{(\ell)}(1) = u_\ell \text{ for all } \ell \in \{0, \dots, k-1\} \right\}, \end{aligned}$$

with the convention that $v^{(0)} = v$. Since $\tilde{m}(u_0, u_1, \dots, u_{k-1}) \rightarrow 0$ as $(u_0, u_1, \dots, u_{k-1}) \rightarrow (-1, 0, \dots, 0)$, by (2.4.98) we infer that

$$\begin{aligned} m &\geq \int_{-\infty}^{+\infty} \left[W(u) + \sum_{\ell=1}^k q_\ell |u^{(\ell)}|^2 \right] dt \\ &\geq \liminf_{j \rightarrow +\infty} \int_{-\infty}^{t_j} \left[W(u) + \sum_{\ell=1}^k q_\ell |u^{(\ell)}|^2 \right] dt + m - \tilde{m}(u(t_j), \dots, u^{(k-1)}(t_j)) \\ &\geq \liminf_{j \rightarrow +\infty} \int_{-\infty}^{t_j} \left[W(u) + \sum_{\ell=1}^k q_\ell |u^{(\ell)}|^2 \right] dt + m, \end{aligned}$$

which implies

$$\liminf_{j \rightarrow +\infty} \int_{-\infty}^{t_j} \left[W(u) + \sum_{\ell=1}^k q_\ell |u^{(\ell)}|^2 \right] dt \leq 0.$$

By Corollary 2.4.24 and Fatou's Lemma, it holds that

$$\begin{aligned} \liminf_{j \rightarrow +\infty} \int_{-\infty}^{t_j} \left[W(u) + \sum_{\ell=1}^k q_\ell |u^{(\ell)}|^2 \right] dt &\geq \delta \liminf_{j \rightarrow +\infty} \int_{-\infty}^{t_j} \left[W(u) + \sum_{\ell=1}^k |u^{(\ell)}|^2 \right] dt \\ &\geq \delta \int_{-\infty}^{+\infty} \left[W(u) + \sum_{\ell=1}^k |u^{(\ell)}|^2 \right] dt \geq 0; \end{aligned}$$

therefore, we deduce that that u is identically equal to -1 . This contradicts $u(0) = 0$, which concludes the proof. $\square$

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