Nonequilibrium Full Counting Statistics and Symmetry-Resolved Entanglement from Space-Time Duality

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Owing to its probabilistic nature, a measurement process in quantum mechanics produces a distribution of possible outcomes. This distribution—or its Fourier transform known as full counting statistics (FCS)—contains much more information than say the mean value of the measured observable, and accessing it is sometimes the only way to obtain relevant information about the system. In fact, the FCS is the limit of an even more general family of observables—the *charged moments*—that characterize how quantum entanglement is split in different symmetry sectors in the presence of a global symmetry. Here we consider the evolution of the FCS and of the charged moments of a U(1) charge truncated to a finite region after a global quantum quench. For large scales these quantities take a simple large-deviation form, showing two different regimes as functions of time: while for times much larger than the size of the region they approach a stationary value set by the local equilibrium state, for times shorter than region size they show a nontrivial dependence on time. We show that, whenever the initial state is also U(1) symmetric, the leading order in time of FCS and charged moments in the out-of-equilibrium regime can be determined by means of a space-time duality. Namely, it coincides with the stationary value in the system where the roles of time and space are exchanged. We use this observation to find some general properties of FCS and charged moments out of equilibrium, and to derive an exact expression for these quantities in interacting integrable models. We test this expression against exact results in the Rule 54 quantum cellular automaton and exact numerics in the XXZ spin-1/2 chain.

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Introduction.—The connection between symmetries and conservation laws—which culminates in the celebrated Noether's theorem and the Ward identities [1–3]—is arguably the most fundamental aspect of our understanding of the physical world. Loosely stated, this connection implies that for any continuous symmetry of a physical system there is an associated conserved quantity, or charge, that remains invariant during the time evolution. An immediate consequence of this fact is that—even when the system is out of equilibrium—the presence of a symmetry implies that the value of the associated charge is fixed. A conserved charge, however, can still show nontrivial fluctuations when restricted to a subsystem [4–13]. In fact, whenever the system is prepared in an out-of-equilibrium state, these charge fluctuations evolve nontrivially in time even in the presence of translational invariance [14–16].

Because of the special nature of the conserved charge, one can expect the time evolution of its fluctuations to give universal information about the system's dynamics. To make this statement more quantitative let us consider a one-dimensional quantum many-body system enjoying a global U(1) symmetry generated by a charge $\hat{Q}$ that can be split as a direct sum $\hat{Q} = \hat{Q}_A \oplus \hat{Q}_{\bar{A}}$ for any spatial bipartition $A\bar{A}$. We then prepare the system in some low-entangled non-equilibrium initial state $|\Psi_0\rangle$, let it evolve according to its own unitary dynamics, and look at the time evolution of the full-counting statistics (FCS) at time t

$$Z_\beta(A,t) = \langle \Psi_t | e^{i\beta \hat{Q}_A} | \Psi_t \rangle = \mathrm{tr}\big(\hat{\rho}_A(t) e^{i\beta \hat{Q}_A}\big). \qquad (1)$$

Here A is a contiguous block and $\hat{\rho}_A(t) = \mathrm{tr}_{\bar{A}} |\Psi_t\rangle\langle\Psi_t|$ is the reduced density matrix of the subsystem A. This quantity characterizes the full probability distribution of $\hat{Q}_A$ in $|\Psi_t\rangle$. Indeed, considering its derivatives in $\beta = 0$ one can reproduce all the moments of the reduced charge.

Because of the generic phenomenon of local relaxation [17–23] we expect the FCS [Eq. (1)] to show qualitatively different behaviors in the two regimes, (i) $t \gg |A|$ and (ii) $t \ll |A|$,where $|A|$ denotes the size of A. Specifically, for $t \gg |A|$ we expect the subsystem A to relax to a stationary state $\hat{\rho}_{\mathrm{st},A}$ and, therefore, the FCS to become time independent at leading order in time

$$Z_\beta(A,t) \simeq \mathrm{tr}[\hat{\rho}_{\mathrm{st},A} e^{i\beta\hat{Q}_A}]. \tag{2}$$

For this reason we refer to (i) as the equilibrium regime. Conversely, in the regime (ii) the FCS generically shows a nontrivial time dependence even at leading order in time, and we hence refer to it as the out-of-equilibrium regime.

In this Letter we consider the evolution of Eq. (1) in the out-of-equilibrium regime and obtain two main results. First, we show that, unexpectedly, whenever the state $|\Psi_0\rangle$ is an eigenstate of the charge $\hat{Q}$ the FCS in the out-of-equilibrium regime can be written in terms of an *equilibrium quantity* for the "dual system" where the roles of space and time have been exchanged. This allows us to find a number of general features of its evolution in any locally interacting systems. Second, we use our observation to find an *exact prediction* for the nonequilibrium dynamics of Eq. (1) in interacting integrable models treatable by thermodynamic Bethe ansatz (TBA) [24,25]. To the best of our knowledge, this represents the first closed form expression of the FCS for interacting systems in the out-of-equilibrium regime, and complements existing results on the dynamics of FCS in the local equilibrium state [12,26–31].

In fact, our arguments are not limited to charge fluctuations in a single replica and can be extended to entanglement-related quantities. Namely, they also apply for the more general family of observables known as charged moments (CM) [15,32,33]

$$Z_{\alpha,\beta}(A,t) = \mathrm{tr}\big(\hat{\rho}_A(t)^\alpha e^{i\beta\hat{Q}_A}\big), \qquad \alpha,\beta \in \mathbb{R}. \tag{3}$$

These quantities measure how the entanglement between A and $\bar{A}$ is decomposed in different charge sectors—their Fourier transforms in β are the symmetry resolved entanglement entropies (SREEs) [32–36]—and, remarkably, they are accessible in ion-trap experiments [37–41].

Space-time duality.—To explain our reasoning it is convenient to begin by considering the case in which the system of interest is a *brickwork quantum circuit*. Namely, it is composed of a collection of $2L$ qudits with d internal states arranged on a discrete lattice, and its time evolution is implemented by discrete applications of the unitary operator

$$\hat{\mathbb{U}} = \hat{\Pi}^\dagger \hat{U}^{\otimes L} \hat{\Pi} \hat{U}^{\otimes L}. \tag{4}$$

Here $\hat{U}$ acts on two neighboring sites, and $\hat{\Pi}$ is the periodic shift by one site. Brickwork quantum circuits dispose of most features of real-world quantum matter but retain spatial locality and unitarity. Therefore, they are regarded as the simplest possible extended quantum systems [42–45]. Importantly, these systems emerge naturally in the context of both classical [46,47] and quantum [48] simulation of quantum dynamics.

In a quantum circuit the conservation of the charge $\hat{Q}$ can be implemented locally via a traceless operator $\hat{q}$ that together with $\hat{U}$ satisfies

$$(e^{i\beta\hat{q}} \otimes e^{i\beta\hat{q}})\hat{U} = \hat{U}(e^{i\beta\hat{q}} \otimes e^{i\beta\hat{q}}), \quad \forall\ \beta \in \mathbb{R}. \tag{5}$$

This ensures that $\hat{Q} = \sum_j \hat{q}_j$—where $\hat{q}_j$ acts as $\hat{q}$ at site j and as the identity elsewhere—is conserved and can be split as a direct sum for any spatial bipartition.

Analogously, considering the family of two-site translational invariant pair-product states

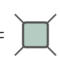
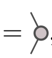
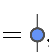
$$|\Psi_0\rangle = |\psi_0\rangle^{\otimes L}, \quad |\psi_0\rangle = \sum_{i,j=1}^{d} m_{ij}|i,j\rangle, \quad \mathrm{tr}[mm^\dagger] = 1, \tag{6}$$

where $\{|i\rangle\}$ is a basis of the Hilbert space of a single qudit, we have that if and only if

$$e^{i\beta\hat{q}}\hat{m} = e^{i\beta\bar{q}}\hat{m}e^{-i\beta\hat{q}^T}, \quad \forall\ \beta \in \mathbb{R}, \tag{7}$$

with $\bar{q}$ a scalar and $(\cdot)^T$ denoting transposition, then $\hat{Q}|\Psi_0\rangle = L\bar{q}|\Psi_0\rangle$.

Introducing the following tensor-network inspired [49] diagrammatic representation

$\hat{U} =$ , $\hat{m} =$ , $e^{i\beta\hat{q}} =$ , $e^{-i\beta\hat{q}^T} =$ , (8)

we can depict $e^{i\beta\hat{Q}_A}|\Psi_t\rangle$ as in Fig. 1(a). Note that the matrices in Eq. (8) act from bottom to top, and for convenience we define $|A|$ as the number of qudits in the subsystem divided by 2. Our first step is to show that, using Eqs. (5) and (7), we can "deform" the string of red circles in the diagram passing from Figs. 1(a) to 1(c).

To see this we first repeatedly use the diagrammatic representation of Eq. (5), reported in Fig. 1(d1), and obtain Fig. 1(b) from Fig. 1(a). Next, we use the relations (5) and (7) to propagate the circles "sideways," i.e., in the space direction. Specifically, Eq. (8) implies that $\hat{m}$ already acts in the space direction while Eq. (5) gives

$$\tilde{\hat{U}}(e^{i\beta\hat{q}} \otimes e^{-i\beta\hat{q}^T}) = (e^{-i\beta\hat{q}^T} \otimes e^{i\beta\hat{q}})\tilde{\hat{U}}, \tag{9}$$

where we introduced the reshuffled local gate with elements $\tilde{U}^{ij}_{kl} = U^{lj}_{ki}$ [50]. The two relations (7) and (9) are represented diagrammatically in Figs. 1(d2) and 1(d3) respectively. In particular, $\tilde{\hat{U}}$ is still represented by the green tensor in Eq. (8) but now the latter is seen as a matrix propagating from left to right. A repeated application of Figs. 1(d2) and 1(d3) brings us from Figs. 1(b) to 1(c).

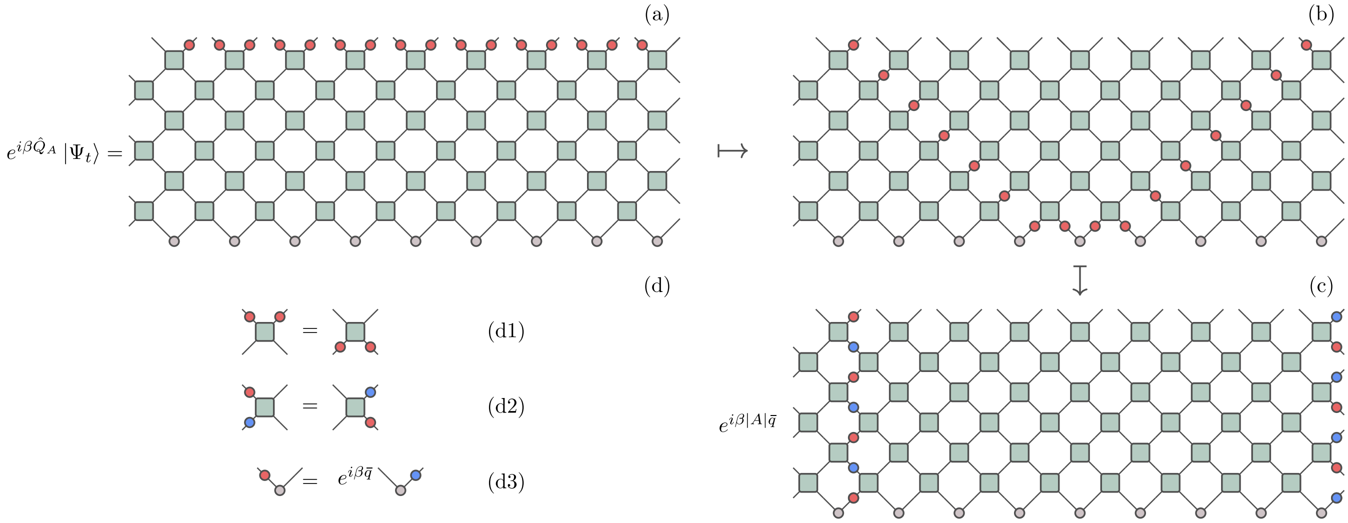


FIG. 1. Diagrammatic representation of $e^{i\beta Q_A}|\Psi_t\rangle$. Starting with the diagrammatic representation in (a), one can use the local continuity relations (d) to first deform the string of red dots to a triangle (b), and then move it completely to the two sides (c). We adopted the convention that when they are acting sideways [cf. the diagrams (d2) (d3)] the matrices act from left to right.

To conclude, we use the representation in Fig. 1(c) to compute the FCS via "space propagation" [50–60]. Namely, we represent Eq. (1) as in Fig. 2(a) and contract it from left to right using the transfer matrix $\mathbb{W}_t$ highlighted in the figure. Translating it into an equation we have

$$Z_\beta(A,t) = e^{i\beta\bar{q}A}\mathrm{tr}[\hat{\mathbb{W}}_t^{|\bar{A}|}(e^{i\beta\hat{\tilde{Q}}_t}\otimes_r \mathbb{1})\mathbb{W}_t^{|A|}(e^{-i\beta\hat{\tilde{Q}}_t}\otimes_r \mathbb{1})], \quad (10)$$

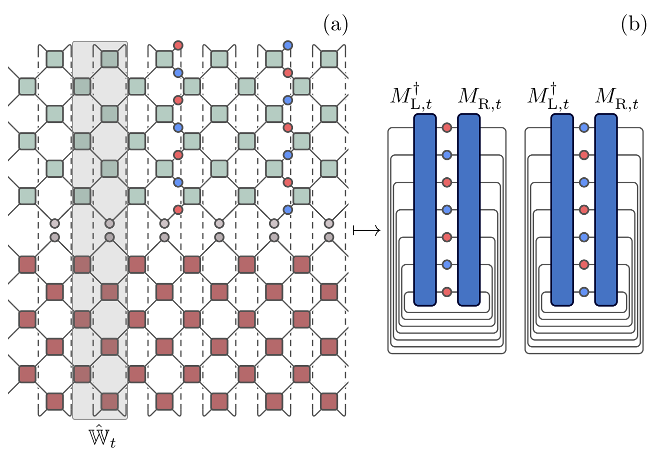


FIG. 2. Diagrammatic representation of $e^{-i\beta|A|t}Z_\beta(A,t)$ for (a) generic choices of t, $|A|$, $|\bar{A}|$, and (b) in the regime $|A|, |\bar{A}| > 2t$. The diagram in the left panel follows directly from the definition of time evolution (and subsequent manipulations in Fig. 1), but it can be equivalently understood as a result of *space propagation* by identifying the shaded part as a transfer matrix $\mathbb{W}_t$ that acts on the *vertical* lattice of $2t$ qudits [cf. Eq. (10)]. Whenever the sizes of the subsystem and the system are large enough compared with the time t, the action of $\mathbb{W}_t$ in each subsystem can be replaced by fixed points $M_{\mathrm{R},t}$ and $M_{\mathrm{L},t}^\dagger$, which gives the diagram in (b).

where the tensor product $\otimes_r$ is between forward and backward time sheets [top and bottom part of Fig. 2(a)] and we introduced the charge of the space-time swapped model $\hat{\tilde{Q}}_t = \sum_{j=1}^{t}(\hat{q}_{2j-1} - \hat{q}_{2j}^T)$. Using now that for $x \geq 2t$ the matrix $\hat{\mathbb{W}}_t^x$ becomes a projector onto its unique fixed points parametrized by the matrices $M_{L,t}$ and $M_{R,t}$ (see Fig. 2(b), and, e.g., Ref. [56] for more details), we find that for $|A|, |\bar{A}| \geq 2t$ [61],

$$Z_\beta(A,t) = e^{i\beta\bar{q}|A|}\mathrm{tr}[\hat{\tilde{\rho}}_{\mathrm{st},t}e^{i\beta\hat{\tilde{Q}}_t}]\mathrm{tr}[\hat{\tilde{\rho}}_{\mathrm{st},t}e^{-i\beta\hat{\tilde{Q}}_t}], \quad (11)$$

where we introduced the pseudodensity matrix $\hat{\tilde{\rho}}_{\mathrm{st},t} = M_{L,t}^\dagger M_{R,t}$ [56]. We now follow Ref. [56] and interpret $\hat{\tilde{\rho}}_{\mathrm{st},t}$ as the stationary state of the "space-time swapped" circuit—i.e., the quantum circuit obtained from the starting one by exchanging the roles of space and time. Although this matrix is not Hermitian in the usual sense, i.e., $\hat{\tilde{\rho}}_{\mathrm{st},t} \neq \hat{\tilde{\rho}}_{\mathrm{st},t}^\dagger$, it is diagonalizable. Moreover, its eigenvalues are real, non-negative, and sum to 1 [56]. This means that it can be interpreted as a thermal state of a system with a non-Hermitian, yet positive, Hamiltonian [62].

A comparison between Eqs. (2) and (11) reveals that the FCS in the nonequilibrium regime is written in terms of equilibrium FCS for the space-time swapped model. This means that the FCS in the nonequilibrium regime can be *written in terms of equilibrium quantities*. This observation constitutes our first main result.

General properties.—Before showing how Eq. (11) can be used to produce quantitative predictions we make three important observations. (A) The analog of Eq. (11) also holds for the CM [Eq. (3)].Indeed, applying the above reasoning we find

$$Z_{\alpha,\beta}(A,t) = e^{i\beta\bar{q}|A|}\mathrm{tr}[\hat{\tilde{\rho}}_{\mathrm{st},t}^\alpha e^{i\beta\hat{\tilde{Q}}_t}]\mathrm{tr}[\hat{\tilde{\rho}}_{\mathrm{st},t}^\alpha e^{-i\beta\hat{\tilde{Q}}_t}], \quad (12)$$

for $|A|, |\bar{A}| \geq 2t$. (B) Equations (11) and (12) immediately imply that SREEs display a *delay time for activation*, i.e., the entanglement entropies of a sector with charge $Q = |A|\bar{q} + \Delta Q$ is identically zero up to a time $t_{\mathrm{D}} \propto |\Delta Q|$. This observation generalizes the free-fermion result of Refs. [15,16] to generic quantum circuits. To prove it we note that it suffices to show that $\hat{\rho}_{A,Q}(t)$—the density matrix reduced to the block of charge Q—has zero trace for $t \leq t_{\mathrm{D}}$. Indeed, since $\hat{\rho}_{A,Q}(t)$ is positive semidefinite, it has zero trace precisely when it is zero. Using now Eqs. (1) and (11) and considering the physically relevant case of charge operators with integer spectrum we have

$$\mathrm{tr}[\hat{\rho}_{A,Q}(t)] \int_{-\pi}^{\pi} \frac{\mathrm{d}\beta}{2\pi} \mathrm{tr}\left[\hat{\tilde{\rho}}_{\mathrm{st},t} e^{-i\beta \hat{\tilde{Q}}_t}\right] \mathrm{tr}\left[\hat{\tilde{\rho}}_{\mathrm{st},t} e^{i\beta \hat{\tilde{Q}}_t}\right] e^{i\beta\Delta Q}. \quad (13)$$

Using that the integrand is analytic and 2π periodic we have that the integration contour can be shifted along the imaginary axis. Therefore, if the integrand vanishes at either $\pm i\infty$ the integral is zero. As is shown in the Supplemental Material [63], this happens for $t \leq t_{\mathrm{D}} \coloneqq |\Delta Q|/2q_{\mathrm{diff}}$ where q_{diff} is the difference between the largest and smallest eigenvalues of $\hat{q}$ and is equal to the maximal eigenvalue of $\hat{\tilde{Q}}_t/t$ [63]. Moreover, using the continuity equation for $\hat{Q}_A$ it is possible to interpret $\hat{\tilde{Q}}_t$ as the associated current operator integrated in time, up to t at the left (right) boundary of A [68]. Thus the time delay is the shortest possible time in which the charge $|\Delta Q|$ can be transported through the boundaries of the system. (C) Interpreting $\hat{\tilde{\rho}}_{\mathrm{st},t}$ as a (generalized) Gibbs state one can use general arguments of statistical mechanics to show that the "number entropy" $-\sum_Q \mathrm{tr}[\hat{\rho}_{A,Q}(t)] \log \mathrm{tr}[\hat{\rho}_{A,Q}(t)]$ grows in time as $(1/2)\log t$ [63]. This observation once again generalizes the free-fermion result of Refs. [15,16] to generic systems.

Integrable models.—Let us now proceed to show that the general observations above can be used to find predictions in interacting integrable quantum many-body systems. To this aim, we begin by recalling few basic facts about the latter systems. The spectrum of an integrable model generically consists of a number of stable quasiparticle species, parametrized by a species index n and a rapidity λ. Their properties are described through a compact set of kinematic data: energy $\varepsilon_n(\lambda)$, momentum $p_n(\lambda)$, and charge q_n of a quasiparticle, as well as the two-particle scattering kernel $T_{nm}(\lambda)$, and the density of the charge q_0 in the reference state without quasiparticles. In the equilibrium regime, for a large subsystem $|A| \to \infty$ we can use the TBA framework along with the quench action method [69,70] to find the asymptotic logarithmic density of charged moments:

$$\begin{aligned} d_{\alpha,\beta} &= \lim_{|A|\to\infty} \frac{1}{|A|} \log \mathrm{tr}\left[\rho^{\alpha}_{\mathrm{st},A} e^{i\beta Q_A}\right] \\ &= i\beta q_0 + \sum_n \int \frac{\mathrm{d}\lambda}{2\pi} p_n'(\lambda) \mathcal{K}_n^{(\alpha,\beta)}(\lambda), \end{aligned} \quad (14)$$

with the functions $\mathcal{K}_n^{(\alpha,\beta)}(\lambda)$ satisfying a set of coupled integral equations,

$$\begin{aligned} \mathcal{K}_n^{(\alpha,\beta)} &= \mathrm{sgn}[p_n'] \log\left[(1-\vartheta_n)^{\alpha} + \frac{\vartheta_n^{\alpha}}{x_n^{\mathrm{sgn}[p_n']}}\right] \\ \log[x_n(\lambda)] &= -i\beta q_n + \sum_m \int \mathrm{d}\mu T_{nm}(\lambda-\mu) \mathcal{K}_m^{(\alpha,\beta)}(\mu). \end{aligned} \quad (15)$$

Here $\vartheta_n(\lambda)$ are the occupation functions of the quasiparticles in the long time steady state which can be determined exactly for certain combinations of initial states and models [71–85]. This reproduces the results of Ref. [86] obtained using the Gartner-Ellis theorem.

To evaluate Eqs. (11) and (12) we need to write the stationary densities of the system where the roles of position and time are swapped. Following Ref. [56] we obtain them from Eqs. (14) and (15) by performing a space-time swap in Fourier space, i.e., exchanging the roles of $p_n(\lambda)$ and $\varepsilon_n(\lambda)$. This leads to

$$\begin{aligned} s_{\alpha,\beta} &= \lim_{t\to\infty} \frac{1}{t} \log \mathrm{tr}\left[\tilde{\rho}^{\alpha}_{\mathrm{st},t} e^{i\beta \tilde{Q}_t}\right] \\ &= i\beta \tilde{q}_0 + \sum_n \int \frac{\mathrm{d}\lambda}{2\pi} \varepsilon_n'(\lambda) \mathcal{L}_n^{(\alpha,\beta)}(\lambda), \end{aligned} \quad (16)$$

where now we have that

$$\begin{aligned} \mathcal{L}_n^{(\alpha,\beta)} &= \mathrm{sgn}[\varepsilon_n'] \log\left[(1-\vartheta_n)^{\alpha} + \frac{\vartheta_n^{\alpha}}{y_n^{\mathrm{sgn}[\varepsilon_n']}}\right], \\ \log[y_n(\lambda)] &= -i\beta \tilde{q}_n + \sum_m \int \mathrm{d}\mu T_{nm}(\lambda-\mu) \mathcal{L}_m^{(\alpha,\beta)}(\mu). \end{aligned} \quad (17)$$

The dual driving term $\tilde{q}_n$ and reference-state density $\tilde{q}_0$ depend upon the form of $\hat{\tilde{Q}}_t$, but can be determined on a case by case basis. We now arrive at our second main result: for interacting integrable models the leading order values of CM in the out-of-equilibrium regime are determined by Eqs. (16) and (17).

We emphasize that Eqs. (14) and (16) predict an exponential decay in time of the charged moments in the nonequilibrium regime and an exponential decay in space in the equilibrium regime. This behavior can be understood intuitively by noting that the logarithm of a charged moment in a stationary state is generically extensive. Interestingly, this phenomenology is in contrast with what is observed in the case of random unitary circuits with conservation laws, which show subexponential decay [87,88]. The latter results are not in contradiction with space-time duality: they merely indicate that for random unitary circuits with conservation laws the logarithms of the charged moments in the space-time swapped stationary state are not extensive, i.e., $s_{\alpha,\beta} = 0$.

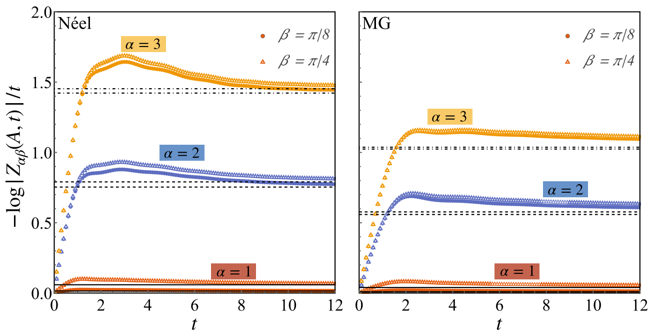


FIG. 3. Logarithmic slope of the charged moments after a quench in the XXZ model with $\Delta = 1/2$, starting from the Néel state (left panel), Majumdar-Gosh state (right panel). Symbols are the iTEBD data computed with $|A| = 50$, straight lines are the asymptotic predictions $i\beta|A|\bar{q} + t(s_{\alpha,\beta} + s_{\alpha,-\beta})$. Different α values correspond to different colors and have been identified with labels; different symbols identify different values of β. In both cases spin-flip symmetry fixes these to be real.

Tests.—To test this prediction we perform two nontrivial checks, one analytic and one numerical with details on each presented in the Supplemental Material [63]. For the analytic check we employ the so-called Rule 54 quantum cellular automaton [89] which, despite being an interacting and TBA integrable model [90,91], is simple enough to allow for the exact calculation of several nonequilibrium quantities [91–105] (see Ref. [106] for a recent review). Comparing exact results of the charged moments in a quench from a set of solvable initial states we find exact agreement with Eqs. (16) and (17).

For the numerical check we use the paradigmatic example of an interacting integrable model: the XXZ spin chain, $\hat{H}=\sum_{j=1}^{2L}\hat{\sigma}_j^x\hat{\sigma}_{j+1}^x+\hat{\sigma}_j^y\hat{\sigma}_{j+1}^y+\Delta\hat{\sigma}_j^z\hat{\sigma}_{j+1}^z$, quenched from either the Néel state, $|\Psi_{\mathrm{N}}\rangle = |\uparrow\downarrow\rangle^{\otimes L}$, or the Majumdar-Gosh state, $|\Psi_{\mathrm{MG}}\rangle=[(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle)/\sqrt{2}]^{\otimes L}$. We compare Eqs. (16) and (17) against numerical simulations using infinite time-evolving block decimation scheme (iTEBD) [107] directly in the thermodynamic limit and report the results in Fig. 3 finding good agreement.

Finite time dynamics.—In analogy with what happens for Rényi entropies [56], the expressions (14) and (16) show a breakdown of the quasiparticle picture for CM [15,16] in the presence of interactions. More precisely, they imply that a quasiparticle description is only possible if one admits that the quasiparticle velocities depend on both α and β. This contrasts the usual assumption of the quasiparticles being observable independent [108,109]. We also remark that, as for the Rényi entropies [56], combining Eqs. (14) and (16) by assuming abrupt saturation of each mode, one can reconstruct the full dynamics of CM at leading order, i.e.,

$$\log Z_{\alpha,\beta}(A,t)$$
$$\simeq i\beta\bar{q}|A| + \sum_n \int \mathrm{d}\lambda \, \min[|A|, 2tv_n^{(\alpha,\beta)}(\lambda)]d_n^{(\alpha,\beta)}(\lambda), \qquad (18)$$

where the explicit expression of $v_n^{(\alpha,\beta)}(\lambda)$ and $d_n^{(\alpha,\beta)}(\lambda)$ is reported in the Supplemental Material [63]. This means that, upon computing the Fourier transform of the CM via saddle point integration, our result gives access to the full dynamics of SREE at leading order. The FCS are an important example of this as the Fourier transform returns the probability of measuring charge Q in A at time t. We find that it is normally distributed with standard deviation [63],

$$\mathcal{D}(t) = \sum_n \int \mathrm{d}\lambda \, q_{n,\mathrm{eff}}^2 \rho_n (1-\vartheta_n) \, \min[|A|, 2t|v_n|], \qquad (19)$$

where v_n and $q_{n,\mathrm{eff}}$ are the standard quasiparticle velocity and effective charge. This shows a remergence of the quasiparticle picture in the $\alpha \to 1$ limit as is the case for Renyi entropies [56].

Conclusions.—In this Letter we have studied the quench dynamics of full counting statistics and charged moments, which characterize symmetry resolved entanglement, in interacting systems. Upon identifying two dynamical regimes—equilibrium and nonequilibrium—we have shown that both can be analyzed using equilibrium techniques via space-time duality. We used this observation to determine some generic features of symmetry resolved entanglement: the presence of a time delay for activation, and the logarithmic growth of the number entropy. Moreover, we have conjectured a closed-form expression for full counting statistics and charged moments in interacting integrable models and tested it against exact analytical and numerical results. We considered global quantum quenches from symmetric initial states, i.e., eigenstates of the charge, but our method can be directly applied to mixed initial states relevant for transport settings [12,26–31]. An immediate direction for future research is to extend our approach to cases in which the initial state explicitly breaks the U(1) symmetry [110] or, more generally, to full counting statistics of nonconserved observables [111,112].

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