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Extremal Problems in Fourier Analysis and Applications to Number Theory

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Chapter 1

Introduction

In this thesis, we investigate a collection of extremal problems centered on the interplay between a function and its Fourier transform. A recurring theme is the study of constraints imposed simultaneously on both a function and its Fourier transform, and the sharp inequalities and rigidity phenomena that arise from such conditions. We also explore applications of some of these extremal problems to questions in analytic number theory.

The thesis is based on the following manuscripts:

[A1] Sign uncertainty and de Branges spaces (*with E. Carneiro and A. P. Ramos*), to appear in *Ann. Inst. Fourier*, preprint available at https://arxiv.org/abs/2408.01186 (2024).

[A2] Sharp sign uncertainty for trigonometric polynomials, preprint available at https://arxiv.org/abs/2606.02299 (2026).

[A3] Maximizers of the $L^2 \to L^4$ Fourier extension inequality for cones in finite fields (*with C. González-Riquelme*), *J. Funct. Anal.*, 291(1):111478, 2026. https://doi.org/10.1016/j.jfa.2026.111478.

[A4] Fourier optimization and pair correlation problems (*with M. K. Das and A. P. Ramos*), to appear in *Rev. Mat. Iberoam.*, preprint available at https://arxiv.org/abs/2502.05106 (2025).

[A5] Littlewood's estimates for L-functions in the hyperelliptic ensemble (*with E. Carneiro, P. Darbar, M. K. Das, and A. P. Ramos*), *Bull. Lond. Math. Soc.*, 58 (5): e70388, 2026. https://doi.org/10.1112/blms.70388, also available at https://arxiv.org/abs/2509.21019.

The thesis is organized around three themes. The first theme, *sign uncertainty phenomena*, comprises [A1] and [A2], presented in Chapters 2 and 3. The second theme, *sharp Fourier restriction over finite fields*, is based on [A3] and presented in Chapter 4. The third theme, *Fourier optimization and number theory*, comprises [A4] and [A5], presented in Chapters 5 and 6. We give an overview of each theme in the sections that follow. The notation and conventions used throughout are collected in Section 1.4.

1.1 Sign uncertainty phenomena

The Fourier transform $\widehat{f}$ of a Schwartz function $f \in \mathscr{S}(\mathbb{R}^d)$ is defined by

$$\mathscr{F}[f](\boldsymbol{\xi}) \coloneqq \widehat{f}(\boldsymbol{\xi}) \coloneqq \int_{\mathbb{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\boldsymbol{x}} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \quad \boldsymbol{\xi} \in \mathbb{R}^d.$$

This operator extends to $L^p(\mathbb{R}^d)$, $1 \leqslant p \leqslant 2$, (cf. [111, Chapter I]) and is a fundamental tool across PDEs, probability, number theory, and beyond. One of its most striking features is the *Fourier uncertainty principle*. Broadly speaking, it refers to the idea that one cannot control a function and its Fourier transform arbitrarily well at the same time. A classical quantitative form is the Heisenberg inequality [68, 79, 122]: for $f \in L^2(\mathbb{R}^d)$,

$$\left(\int_{\mathbb{R}^d} |\boldsymbol{x}|^2 |f(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\right)\left(\int_{\mathbb{R}^d} |\boldsymbol{\xi}|^2 |\widehat{f}(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi}\right) \geqslant \frac{d^2}{16\pi^2}\left(\int_{\mathbb{R}^d} |f(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\right)^2.$$

The Heisenberg inequality is only one of many incarnations of Fourier uncertainty; other classical manifestations include the Hardy [67], Amrein–Berthier [4], and Donoho–Stark [43] uncertainty principles; see [48] for a survey. The manifestation that is most relevant to this thesis is the *sign uncertainty principle*, which we now describe.

The sign uncertainty principle, first observed by Bourgain, Clozel, and Kahane [13], states that the negative parts of a function and its Fourier transform cannot both be arbitrarily concentrated near the origin. More precisely, $f\colon \mathbb{R}^d \to \mathbb{R}$ is *eventually non-negative* if $f(\boldsymbol{x}) \geqslant 0$ for $|\boldsymbol{x}| \geqslant r$ for some $r > 0$; the infimum of such r is the *radius of last sign change* $\mathrm{r}(f)$, i.e.,

$$\mathrm{r}(f) \coloneqq \inf\{r > 0 \;:\; f(\boldsymbol{x}) \geqslant 0 \text{ for } |\boldsymbol{x}| \geqslant r\}.$$

If $f \in L^1(\mathbb{R}^d)$ is even, $f(\mathbf{0}) \leqslant 0$, $\widehat{f}(\mathbf{0}) \leqslant 0$, and both f and $\widehat{f}$ are eventually non-negative, then

$$\sqrt{\mathrm{r}(f)\cdot \mathrm{r}(\widehat{f})} \geqslant \mathbb{A}(d) > 0,$$

where $\mathbb{A}(d) > 0$ depends only on the dimension. Determining the sharp constant $\mathbb{A}(d)$ is a delicate problem and the only known value is $\mathbb{A}(12) = \sqrt{2}$, obtained by Cohn and Gonçalves [39] using the machinery of Viazovska [119] on modular forms.

The sign uncertainty principle has since been studied in various settings and generalized in several directions [23, 61–63]. The works [A1] and [A2], presented in the following chapters, approach this line of research from a different perspective: in these works we assume an a priori restriction on $\widehat{f}$ and seek the minimal possible radius $\mathrm{r}(f)$ alone. In [A1], the functions are *bandlimited* (their Fourier transforms are compactly supported; equivalently, by the Paley–Wiener theorem, f is an entire function of finite exponential type). In [A2], the analogous problem on the circle $\mathbb{T}$ is studied for trigonometric polynomials of a prescribed degree N. Both works also treat the problem with respect to a general measure; we outline how this is done in the following sections.

1.1.1 Sign uncertainty principle for entire functions of exponential type

Chapter 2, based on [A1], treats the sign uncertainty principle for bandlimited functions ($\mathrm{supp}(\widehat{f})$ is compact). Recall that a non-identically-zero entire function $F:\mathbb{C}\to\mathbb{C}$ is said to be of *exponential type* if

$$\tau(F) := \limsup_{|z|\to\infty} \frac{\log|F(z)|}{|z|} < \infty,$$

and for such functions we call $\tau(F)$ the exponential type of F. By the Paley–Wiener theorem, bandlimited functions are identified with entire functions of exponential type. Recall that the constraint $\widehat{f}(0) \leqslant 0$ in the framework of Bourgain–Clozel–Kahane is exactly $\int_{\mathbb{R}} f(x)\,\mathrm{d}x \leqslant 0$, that is, integration of f against the Lebesgue measure. Replacing the Lebesgue measure by an arbitrary Borel measure μ, locally finite and even, leads to the following problem.

Extremal problem 1.1 **(EP1.1)**. For a locally finite, even Borel measure μ on $\mathbb{R}$ and $\delta \geqslant 0$, determine

$$\Lambda(\mu,\delta) := \inf_{0\not\equiv F\in\mathcal{E}(\mu,\delta)} \mathrm{r}\big(F|_{\mathbb{R}}\big),$$

where $\mathcal{E}(\mu,\delta)$ is the class of entire functions $F:\mathbb{C}\to\mathbb{C}$ such that:

(i) F is of exponential type at most δ, i.e., $\tau(F)\leqslant\delta$;

(ii) F maps $\mathbb{R}$ to $\mathbb{R}$ and $F|_{\mathbb{R}}$ is eventually non-negative;

(iii) $F|_{\mathbb{R}} \in L^1(\mathbb{R},\mu)$ and $\displaystyle\int_{\mathbb{R}} F(x)\,\mathrm{d}\mu(x) \leqslant 0$.

We pose (EP1.1) on the real line for clarity; the analogous problem for functions on $\mathbb{R}^d$, with respect to a rotation-invariant measure, is also treated in Chapter 2. To describe our result we recall necessary notions. An entire function $E:\mathbb{C}\to\mathbb{C}$ is said to be *Hermite–Biehler* if it satisfies

$$|E(\bar{z})| < |E(z)|$$

for all $z\in\mathbb{C}$ such that $\mathrm{Im}(z) > 0$.

Theorem 1.1 (see Theorem 2.2)**.** *Let E be a Hermite–Biehler function of exponential type τ such that*

(i) *$E(ix) \in \mathbb{R}$ for all $x \in \mathbb{R}$;*

(ii) *E has no real zeros;*

(iii) $\displaystyle\int_{\mathbb{R}} \frac{\log^+ |E(t)|}{1+t^2}\,\mathrm{d}t < \infty.$

Then

$$\Lambda\big(|E(x)|^{-2}\mathrm{d}x, 2\tau\big) = \xi_1,$$

where ξ_1 is the smallest positive zero of $A(z) \coloneqq \big(E(z)+\overline{E(\bar z)}\big)/2$. The unique extremizer (up to a positive multiplicative constant) is

$$\frac{A(z)^2}{z^2 - \xi_1^2}.$$

As a particular case, taking $E(z) = e^{-i\delta z}$ for each $\delta > 0$ allows us to solve the problem (EP1.1) for the Lebesgue measure in the full range of $\delta > 0$, that is

$$\Lambda(\mathrm{d}x, 2\delta) = \frac{\pi}{2\delta}.$$

In fact, with the framework we develop in Chapter 2, we can also treat the family of power-weight measures $\mathrm{d}\mu_\nu(x) = |x|^{2\nu+1}\mathrm{d}x$ for $\nu > -1$. For these measures, the sharp constant is expressed in terms of the first positive zero of the Bessel function J_ν of the first kind, and the extremizer involves the classical entire function $A_\nu(z) = \Gamma(\nu+1)(z/2)^{-\nu}J_\nu(z)$. More generally, any measure whose density is asymptotically comparable to a power weight $|x|^{2\nu+1}$ with $\nu > -1$ falls within the scope of the theorem. We refer the reader to Chapter 2 for the precise statement.

The mechanism underlying these results is a family of *quadrature formulas* for the integral $\int_{\mathbb{R}} F\,\mathrm{d}\mu$: exact identities expressing this integral as a weighted sum of point evaluations of F, valid for every $F \in \mathcal{E}(\mu, \delta)$. Such formulas turn the integral constraint in (EP1.1) into a condition at countably many points, which is what pins down $\Lambda(\mu, \delta)$ exactly. These quadrature formulas arise from the theory of reproducing kernel Hilbert spaces, specifically from the de Branges spaces of entire functions associated with E; we develop this framework in Chapter 2.

In the following subsection, we study the sign uncertainty problem in the periodic setting of the unit circle, where bandlimited functions are replaced by trigonometric polynomials and a complete solution is obtained for all finite even measures.

1.1.2 Optimal sign uncertainty for trigonometric polynomials

Analogues of the Bourgain–Clozel–Kahane sign uncertainty principle on the circle were established by Gonçalves, Oliveira e Silva, and Ramos [62, 63]. In Chapter 3, we adapt the framework of [A1] to the periodic setting, replacing bandlimited functions by trigonometric polynomials of prescribed degree N and working with an even finite Borel measure μ on $\mathbb{T}$ (i.e., $\mu(A) = \mu(-A)$ for all $A \subseteq \mathbb{T}$).

We adopt the following terminology. Let $\mathbb{X}$ be the Euclidean space $\mathbb{R}^d$, the sphere $\mathbb{S}^d$, the unit circle $\mathbb{T}$, or the unit interval $[0,1]$, equipped with its standard metric $\mathsf{d}(\cdot,\cdot)$, and let $x^* \in \mathbb{X}$ be a distinguished point (the origin, or the north pole for $\mathbb{S}^d$). A function $f : \mathbb{X} \to \mathbb{R}$ is *eventually non-negative* if there exists $r > 0$ such that $f(x) \geqslant 0$ for all $x \in \mathbb{X} \setminus B_r(x^*)$, where $B_r(x^*) = \{y \in \mathbb{X} : \mathsf{d}(x^*, y) < r\}$. The smallest such r is the *radius of last sign change*

$$\mathrm{r}(f) \coloneqq \inf\{r > 0 : f(x) \geqslant 0 \text{ for all } x \in \mathbb{X} \setminus B_r(x^*)\}. \tag{1.1}$$

When $\mathbb{X} = \mathbb{R}^d$ and x^* is the origin, this coincides with the usual definition.

As in [A1], we restrict to bandlimited functions, which in this periodic setting are trigonometric polynomials $f(x) = \sum_{n=-N}^{N} \widehat{f}(n)e^{2\pi i n x}$ (where $\widehat{f}(n) = \int_{\mathbb{T}} e^{-2\pi i n x} f(x)\,\mathrm{d}x$ are the Fourier coefficients), and study the smallest radius of last sign change with the following formulation:

Extremal problem 1.2 **(EP1.2)**. Given a finite even measure μ on $\mathbb{T}$ and $N \geqslant 1$, determine

$$\rho(\mu, N) \coloneqq \inf_{0 \not\equiv f \in \mathcal{T}(\mu,N)} \mathrm{r}(f),$$

where the infimum is taken over the following family of trigonometric polynomials

$$\mathcal{T}(\mu, N) \coloneqq \left\{ f(x) = \sum_{n=-N}^{N} \widehat{f}(n)\, e^{2\pi i n x} \;\middle|\; \begin{array}{l} f : \mathbb{T} \to \mathbb{R} \text{ is eventually non-negative} \\ \text{and } \displaystyle\int_{\mathbb{T}} f(x)\,\mathrm{d}\mu(x) \leqslant 0 \end{array} \right\}.$$

In contrast to [A1], where the solution is known only for certain classes of measures, the problem (EP1.2) admits a complete solution for *all* finite even Borel measures on $\mathbb{T}$, yielding a full understanding of the sign uncertainty phenomenon in the framework of trigonometric polynomials.

As in the setting of the real line, the value $\rho(\mu, N)$ is identified by means of quadrature formulas for the integral $\int_{\mathbb{T}} f\,\mathrm{d}\mu$ valid for every $f \in \mathcal{T}(\mu, N)$. The statement of our main result is most naturally expressed in terms of the classical theory of orthogonal polynomials on the unit circle (OPUC), which plays the role here that de Branges spaces play in Chapter 2. Specifically, for a probability measure μ on $\mathbb{T}$, equivalently on $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$, let $\varphi_n(z; \mathrm{d}\mu)$ be the orthonormal polynomials obtained from $1, z, z^2, \ldots$ by Gram–Schmidt in $L^2(\partial\mathbb{D}, \mu)$, and let $\varphi_n^*(z; \mathrm{d}\mu) = z^n\overline{\varphi_n(1/\bar{z}; \mathrm{d}\mu)}$ be the associated reversed polynomials. We then define the auxiliary polynomials

$$A_n(z; \mathrm{d}\mu) \coloneqq \frac{\varphi_n(z; \mathrm{d}\mu) + \varphi_n^*(z; \mathrm{d}\mu)}{2},$$

whenever $\varphi_n(z; \mathrm{d}\mu)$ exists; in the borderline case $\#\mathrm{supp}(\mu) = n$ we set instead $A_n(z; \mathrm{d}\mu) \coloneqq (z\varphi_{n-1}(z; \mathrm{d}\mu) + \varphi_{n-1}^*(z; \mathrm{d}\mu))/2$. Since $|\varphi_n(z)| < |\varphi_n^*(z)|$ for all $z \in \mathbb{D}$

(see [108, §1.7]), the polynomial A_n has no zeros in the unit disk, and the symmetry $A_n = A_n^*$ then forces all its zeros to lie on $\partial\mathbb{D}$. With these notions in hand, we can state our main result, splitting it into two cases according to the support of μ.

Theorem 1.2 (see Theorem 3.1)**.** *Let μ be a finite even Borel measure on $\mathbb{T}$.*

(i) *If $1 \leqslant N < \#\mathrm{supp}(\mu)$, then*

$$\rho(\mu, N) = \theta_1,$$

where $0 < \theta_1 \leqslant 1/2$ is the smallest positive zero of $A_{N+1}(e^{2\pi i x}; \mathrm{d}\mu)$, i.e.,

$$\theta_1 = \min\{\theta > 0 \,:\, A_{N+1}(e^{2\pi i\theta}; \mathrm{d}\mu) = 0\}.$$

Moreover, the unique extremizer (up to a constant factor) is given by

$$f(x) = \frac{|A_{N+1}(e^{2\pi i x}; \mathrm{d}\mu)|^2}{\cos(2\pi\theta_1) - \cos(2\pi x)}.$$

(ii) *If μ has finite support and $\#\mathrm{supp}(\mu) \leqslant N$, then $\rho(\mu, N) = 0$.*

As an illustration, for the Lebesgue measure $\mathrm{d}x$ on $\mathbb{T}$ the theorem yields the exact value

$$\rho(\mathrm{d}x, N) = \frac{1}{2N+2},$$

a special case addressed earlier by Babenko [7], which also appeared in the work of Yudin [123].

In Chapter 3 we also resolve the analogues of problem (EP1.2) in higher dimensions and on the unit interval. For *polar measures* on the sphere $\mathbb{S}^d$ (measures invariant under rotations fixing the north pole), the problem reduces, via the polar part of the measure, to the one-dimensional problem solved in Theorem 1.2, and an explicit extremizer is exhibited as a zonal function. We also give a complete solution to the analogous problem on the interval $[0, 1]$, for eventually non-negative polynomials of a prescribed degree; here the relevant tools are orthogonal polynomials on the real line and Gauss quadrature.

1.2 Sharp Fourier restriction over finite fields

1.2.1 The Fourier restriction problem

The Fourier restriction problem was introduced by Stein in the late 1960s [109] and asks when the Fourier transform of a function can be meaningfully restricted to hypersurfaces $S \subset \mathbb{R}^{d+1}$. In its adjoint (extension) formulation the problem is equivalent to asking for which pairs of exponents (r, s) the *extension operator*

$$(\mathrm{E}_S\, f)(\boldsymbol{\xi}) \coloneqq \int_S e^{2\pi i\, \boldsymbol{\xi}\cdot\boldsymbol{x}}\, f(\boldsymbol{x})\, \mathrm{d}\sigma(\boldsymbol{x}), \qquad \boldsymbol{\xi} \in \mathbb{R}^{d+1},$$

satisfies the estimate

$$\|\mathrm{E}_S f\|_{L^s(\mathbb{R}^{d+1})} \leqslant C\, \|f\|_{L^r(S,\sigma)}, \tag{1.2}$$

for some constant C independent of f, where $S \subset \mathbb{R}^{d+1}$ is a smooth hypersurface equipped with a positive Borel measure σ. Stein's key observation was that the curvature of S forces decay of $\widehat{\mathrm{d}\sigma}$ at infinity, improving the mapping properties of E_S beyond what is possible for flat surfaces. This is made precise by the Stein–Tomas estimates [110, 117], which establish (1.2) for $S = \mathbb{S}^d$ with $r = 2$ and $s \geqslant 2(d+2)/d$. For a comprehensive treatment of the restriction conjecture and the recent progress on it, we refer to the survey [115].

Among all quadratic surfaces, the paraboloid $\mathcal{P}^d \coloneqq \{(\boldsymbol{\xi}', |\boldsymbol{\xi}'|^2) : \boldsymbol{\xi}' \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1}$, where $|\boldsymbol{\xi}'|$ denotes the Euclidean length of $\boldsymbol{\xi}' \in \mathbb{R}^d$, and the cone $\Upsilon^d \coloneqq \{\boldsymbol{\xi} \in \mathbb{R}^{d+1} : \xi_{d+1}^2 = \sum_{j=1}^d \xi_j^2\} \subset \mathbb{R}^{d+1}$ have attracted a great deal of attention due to their direct connection to dispersive PDEs. Strichartz [112] established estimates (1.2) for quadratic surfaces, including $\mathcal{P}^d$ and Υ^d, with the exponent $r = 2$ and natural measures σ, yielding Strichartz estimates for the Schrödinger and wave equations respectively. Moreover, in the same work, Strichartz established extension estimates for the following surfaces

$$\Upsilon^d_{(a,b)} \coloneqq \Big\{\boldsymbol{x} \in \mathbb{R}^{d+1} : \sum_{i=1}^{a} x_i^2 = \sum_{j=a+1}^{d+1} x_j^2\Big\}, \tag{1.3}$$

where $a, b \geqslant 1$ with $a + b = d + 1$. The set $\Upsilon^d_{(a,b)}$ is invariant under the scaling $\boldsymbol{x} \mapsto \lambda \boldsymbol{x}$ for every $\lambda > 0$, hence is a cone; since it is the zero set of a quadratic form of signature (a, b), we call it the *cone with signature* (a, b) in $\mathbb{R}^{d+1}$.

1.2.2 Sharp extension estimates in the Euclidean setting

Beyond establishing (1.2), a deeper question is to determine the *optimal constant*

$$\mathbf{R}^*_S(r \to s) \coloneqq \sup_{0 \not\equiv f \in L^r(S,\sigma)} \frac{\|\mathrm{E}_S f\|_{L^s(\mathbb{R}^{d+1},\mathrm{d}\boldsymbol{x})}}{\|f\|_{L^r(S,\sigma)}}$$

and to characterize the *extremizers*, i.e., functions at which the supremum is attained. The problem is most tractable when r and s are even integers: one can then rewrite $\|\mathrm{E}_S f\|^s_{L^s(\mathbb{R}^{d+1},\mathrm{d}\boldsymbol{x})}$ as the $L^2(\mathbb{R}^{d+1}, \mathrm{d}\boldsymbol{x})$ norm of an $(s/2)$-fold convolution of the measure $f\,\mathrm{d}\sigma$ with itself, reducing the problem to an algebraic estimate on S via Plancherel's theorem.

The first sharp constants in this context were determined by Foschi [49], who computed $\mathbf{R}^*_{\mathcal{P}^2}(2 \to 4)$, $\mathbf{R}^*_{\mathcal{P}^1}(2 \to 6)$, and $\mathbf{R}^*_{\Upsilon^3_{(3,1)}}(2 \to 4)$, identifying the class of extremizers in each case. An independent proof for the values of $\mathbf{R}^*_{\mathcal{P}^1}(2 \to 6)$ and $\mathbf{R}^*_{\mathcal{P}^2}(2 \to 4)$ can be found in [9, 18, 60, 71]. Further sharp constants have been obtained for the sphere $\mathbb{S}^d$ in dimensions $2 \leqslant d \leqslant 6$ [22, 34, 50], for hyperboloids [29, 101], and a sharpened form of the cone inequality was obtained in [98]. For a comprehensive survey we refer to [51].

1.2.3 The restriction problem over finite fields

The study of the restriction problem over finite fields was initiated by Mockenhaupt and Tao [93]. Let $\mathbb{F}_q$ denote the finite field with $q = p^n$ elements (p an odd prime). The analogue of the extension operator on a set $\mathcal{S} \subset \mathbb{F}_q^{d+1}$, equipped with the normalized counting measure σ ($= 1/\#\mathcal{S}$ per point), is

$$(f\sigma)^\vee(\boldsymbol{x}) = \frac{1}{\#\mathcal{S}} \sum_{\boldsymbol{\eta}\in\mathcal{S}} f(\boldsymbol{\eta})\, e(\boldsymbol{\eta}\cdot\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{F}_q^{d+1},$$

where $e : \mathbb{F}_q \to \mathbb{C}^\times$ is a fixed non-principal additive character. The corresponding extension inequality is

$$\|(f\sigma)^\vee\|_{L^s(\mathbb{F}_q^{d+1},\mathrm{d}\boldsymbol{x})} \leqslant \mathfrak{R}^*_{\mathcal{S}}(r\to s)\, \|f\|_{L^r(\mathcal{S},\sigma)}, \tag{1.4}$$

where $\mathrm{d}\boldsymbol{x}$ is the usual counting measure on $\mathbb{F}_q^{d+1}$ and the sharp constant is obtained by

$$\mathfrak{R}^*_{\mathcal{S}}(r\to s) := \sup_{f\not\equiv 0} \frac{\|(f\sigma)^\vee\|_{L^s(\mathbb{F}_q^{d+1},\mathrm{d}\boldsymbol{x})}}{\|f\|_{L^r(\mathcal{S},\sigma)}}.$$

We use $\mathfrak{R}^*$ (instead of $\mathbf{R}^*$) for the finite field setting to distinguish the two. Note the dependence of this constant on the underlying field $\mathbb{F}_q$. One studies estimates (1.4) for algebraic sets $\mathcal{S}$, which are zero sets of polynomials in $\mathbb{Z}[x_1, x_2, \ldots, x_{d+1}]$ (see [93]). We say that the surface $\mathcal{S}$ has *extension property with exponents* (r, s)[1] if $\mathfrak{R}^*_{\mathcal{S}}(r \to s)$ remains bounded independently of q. In $\mathbb{F}_q^{d+1}$, the *cone with signature* (a, b) ($a + b = d + 1$, $a, b \geqslant 1$) is the analogue of (1.3):

$$\Upsilon^d_{(a,b)} := \Big\{\boldsymbol{\eta} \in \mathbb{F}_q^{d+1} : \sum_{i=1}^{a} \eta_i^2 = \sum_{j=a+1}^{d+1} \eta_j^2\Big\},$$

and the *punctured cone* $\Gamma^d_{(a,b)} := \Upsilon^d_{(a,b)} \setminus \{\mathbf{0}\}$, obtained by deleting the origin. In [93], the $2 \to 4$ restriction property was established for paraboloids $\mathcal{P}^d := \{(\boldsymbol{\eta}, \boldsymbol{\eta}\cdot\boldsymbol{\eta}) : \boldsymbol{\eta} \in \mathbb{F}_q^d\} \subset \mathbb{F}_q^{d+1}$ in low dimensions $d = 1, 2$ and for the cone $\Upsilon^2_{(2,1)}$; the $2 \to 4$ restriction property for the $(3, 1)$-cone in $\mathbb{F}_q^4$ was subsequently obtained by Koh, Lee, and Pham [81].

The study of *sharp restriction estimates* in the finite field setting is very recent and was inaugurated by González-Riquelme and Oliveira e Silva [65]. They determined $\mathfrak{R}^*_{\mathcal{P}^2}(2 \to 4)$ and $\mathfrak{R}^*_{\mathcal{P}^1}(2 \to 6)$ for all q, and $\mathfrak{R}^*_{\Gamma^3_{(3,1)}}(2 \to 4)$ when $q \equiv 3 \pmod 4$, showing constant functions are extremizers for the estimates (1.4). Sharp endpoint extension inequalities for the moment curve were subsequently established by Biswas,

[1] this is equivalent to saying it has the restriction property with exponents (s', r'), where $r' = r/(r-1)$ is the conjugate Lebesgue exponent for r.

Carneiro, Flock, Oliveira e Silva, Stovall, and Tautges [12]. The case $q \equiv 1 \pmod 4$ for the cone $\Gamma^3_{(3,1)}$, where -1 is a square in $\mathbb{F}_q$ and the cone acquires a richer additive structure, was left open in [65].

The work [A3], presented in Chapter 4, resolves this remaining case by studying the $(2,2)$-cone $\Gamma^3_{(2,2)}$ for all odd prime powers q. Since $\Gamma^3_{(2,2)} = \Gamma^3_{(3,1)}$ when $q \equiv 1 \pmod 4$ (as -1 is then a square in $\mathbb{F}_q$), this simultaneously completes the analysis of all sharp $L^2 \to L^4$ extension estimates for punctured cones in $\mathbb{F}_q^4$. Writing $\Gamma^3 := \Gamma^3_{(2,2)}$, the main result is:

Theorem 1.3 (see Theorems 4.1 and 4.2)**.** *For all odd prime powers q, the optimal constant for the $L^2 \to L^4$ extension inequality* (1.4) *is*

$$\mathfrak{R}^*_{\Gamma^3}(2 \to 4) = q \left(\frac{q^5 + 4q^4 - 4q^3 - 6q^2 + 3q + 3}{(q+1)^6 \, (q-1)^3} \right)^{1/4}, \tag{1.5}$$

and $f : \Gamma^3 \to \mathbb{C}$ is an extremizer if and only if

$$f(\eta_1, \eta_2, \eta_3, \eta_4) = \lambda \cdot \exp\!\left(\frac{2\pi i}{p} \operatorname{Tr}_n(a_1\eta_1 + a_2\eta_2 + a_3\eta_3 + a_4\eta_4) \right) \tag{1.6}$$

for some $\lambda \in \mathbb{C}^\times$ and $(a_1, a_2, a_3, a_4) \in \mathbb{F}_q^4$, where $\operatorname{Tr}_n : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map.

We emphasize that the Euclidean analogues of both claims in Theorem 1.3 for the $(2,2)$-cone in $\mathbb{R}^4$ remain entirely unknown. Since $\Gamma^3_{(2,2)} = \Gamma^3_{(3,1)}$ when $q \equiv 1 \pmod 4$, Theorem 1.3 also completes the missing case from [65]:

Corollary 1.4 (see Corollary 4.3)**.** *When $q \equiv 1 \pmod 4$, the sharp constant $\mathfrak{R}^*_{\Gamma^3_{(3,1)}}(2 \to 4)$ is given by* (1.5)*, and the extremizers are precisely the functions of the form* (1.6)*.*

The argument is a combination of combinatorial, geometric and algebraic observations. Since the exponent $s = 4$ is an even integer, by [65, Proposition 2.1] the extension inequality (1.4) is equivalent to a sharp bound for a weighted additive-energy functional on Γ^3, obtained by expanding $\|(f\sigma)^\vee\|^4_{L^4}$ as the squared L^2-norm of the self-convolution of f over the cone. The analysis then reduces to understanding, for each $\boldsymbol{\xi} \in \mathbb{F}_q^4$, the *decomposition set* $\Sigma_{\boldsymbol{\xi}} := \{(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in (\Gamma^3)^2 : \boldsymbol{\eta}_1 + \boldsymbol{\eta}_2 = \boldsymbol{\xi}\}$. Based on this geometric information we apply Cauchy–Schwarz repeatedly and reduce the problem into an inequality for elements of the planar decomposition of the cone Γ^3. The classification in Theorem 1.3 follows by tracking the equality cases in this argument, the rigidity of the planar decomposition forcing every extremizer into the character form (1.6).

We now turn to the third and final theme of the thesis, in which Fourier-analytic extremal problems are brought to bear not on geometric inequalities but on questions in analytic number theory.

1.3 Fourier optimization and number theory

The use of harmonic analysis to study arithmetic problems has a long history; the monograph of Montgomery [95] is a classical reference for this interface. The third theme of this thesis concerns exactly this type of problems, i.e., extremal problems arising from number theory that concern a function and its Fourier transform. Chapters 5 and 6, based on [A4] and [A5] respectively, illustrate two different facets of this: the former studies the *average* behavior of pair correlation form factors for zeros of L-functions, while the latter studies the *extreme values* of L-functions and their associated argument functions on the critical circle in the function field setting.

1.3.1 Fourier optimization and pair correlation problems

Let $\zeta(s)$ denote the Riemann zeta function. The Riemann hypothesis (RH) is a statement about the horizontal distribution of the non-trivial zeros of $\zeta(s)$. A natural follow-up question concerns the vertical distribution of these zeros, i.e., how the ordinates of non-trivial zeros of $\zeta(s)$ are distributed. Assuming RH, Montgomery [94] studied this question and conjectured the following: for any fixed $\beta > 0$,

$$N(\beta, T) := \sum_{\substack{0<\gamma,\gamma'\leqslant T \\ 0<\gamma-\gamma'\leqslant \frac{2\pi\beta}{\log T}}} 1 \sim \frac{T\log T}{2\pi}\int_0^\beta \left\{1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right\} \mathrm{d}u, \quad \text{as } T \to \infty,$$

where the double sum runs over the ordinates of non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. This is known as *Montgomery's pair correlation conjecture* (PCC). This conjecture has several implications regarding both the distribution of primes and the non-trivial zeros of $\zeta(s)$. Gallagher and Mueller [55] showed that (PCC) and (RH) imply that almost all non-trivial zeros of $\zeta(s)$ are simple, and that one obtains better error terms for the prime counting function.

To study (PCC) Montgomery introduced the form factor

$$F(\alpha, T) := \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma'\leqslant T} T^{i\alpha(\gamma-\gamma')} w(\gamma - \gamma'),$$

where $\alpha \in \mathbb{R}$, $T \geqslant 2$, and $w(u) = 4/(4 + u^2)$. In 1988, Goldston [57] showed that (PCC) is equivalent to the asymptotic identity

$$\frac{1}{\ell}\int_b^{b+\ell} F(\alpha, T)\,\mathrm{d}\alpha \sim 1, \quad \text{as } T \to \infty,$$

for any fixed $\ell > 0$ and $b \geqslant 1$. Since (PCC) remains a difficult open problem, it is natural to ask for upper and lower bounds for these averages. Carneiro, Chandee, Chirre, and Milinovich [30] introduced a Fourier optimization framework to provide such bounds, improving the earlier results of Goldston and Gonek [58]. Subsequently, the numerical constants were refined by Carneiro, Milinovich, and Ramos [32], where, for sufficiently large ℓ, the following estimates were established:

$$0.9303 + o(1) \leqslant \frac{1}{\ell}\int_b^{b+\ell} F(\alpha, T)\,\mathrm{d}\alpha \leqslant 1.3208 + o(1), \quad \text{as } T \to \infty.$$

In Chapter 5, based on [A4], we develop a unified framework for obtaining such estimates for a wide class of sequences.

Rather than working specifically with the Riemann zeta function, we adopt an axiomatic approach applying to any sequence satisfying the following conditions. Let $\Gamma(T) = (\gamma_n(T))$ be a sequence of real numbers that may depend on a parameter $T > 0$. We assume:

(H1) there is a $\lambda > 0$ such that

$$\liminf_{T\to\infty} \frac{\#\{n : \gamma_n(T) \in (0, T]\}}{\frac{\lambda T}{2\pi}\log T} = 1;$$

(H2) there exists $\Delta > 0$ such that the associated form factor

$$\mathfrak{F}_\Gamma(\alpha, T) := \left(\frac{\lambda T}{2\pi}\log T\right)^{-1} \sum_{\gamma,\gamma' \in (0,T]\cap\Gamma(T)} T^{i\lambda\alpha(\gamma-\gamma')}\, w(\gamma - \gamma')$$

converges in the weak* topology to a finite Borel measure ν on $(-\Delta, \Delta)$, with $\nu(\{\pm\Delta\}) = 0$, as $T \to \infty$.

The bounds for the long-term averages of the form factor are expressed in terms of the following Fourier optimization problem.

Extremal problem 1.3 **(EP1.3)**. Given $\Delta > 0$ and a finite Borel measure ν on $[-\Delta, \Delta]$, determine

$$\mathcal{C}_\nu := \inf_{\substack{g\in\mathcal{A}_\Delta \\ g(0)>0}} \frac{1}{g(0)}\int_{-\Delta}^{\Delta} \widehat{g}(\alpha)\,\mathrm{d}\nu(\alpha),$$

where $\mathcal{A}_\Delta$ is the class of continuous even functions $g : \mathbb{R} \to \mathbb{R}$ satisfying:

(i) g and $\widehat{g} \in L^1(\mathbb{R})$;

(ii) $\widehat{g}(\alpha) \leqslant 0$ for $\alpha \in \mathbb{R} \setminus [-\Delta, \Delta]$;

(iii) $g(\alpha) \geqslant 0$ for all $\alpha \in \mathbb{R}$.

The class $\mathcal{A}_\Delta$ was introduced by Cohn and Elkies [38] to study sphere packing problems. The main result of [A4] yields explicit upper and lower bounds for the long-term averages of the form factor in terms of $\mathcal{C}_\nu$ and $s_0 := \min_{x\in\mathbb{R}} \frac{\sin x}{x} = -0.2172\ldots$. The main result of Chapter 5 is the following.

Theorem 1.5 (see Theorems 5.1 and 5.2)**.** *Let Γ satisfy* (H1) *and* (H2) *with limiting measure ν in $[-\Delta,\Delta]$. Given $\varepsilon>0$, for any $b\geqslant 1$ and sufficiently large ℓ,*

$$\left(\frac{1}{2}+s_0(\mathcal{C}_\nu-1)\right)_+ +\frac{1}{2}-\varepsilon+o(1)<\frac{1}{\ell}\int_b^{b+\ell}\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha<\mathcal{C}_\nu+\varepsilon+o(1),$$

as $T\to\infty$, with $\ell\geqslant\ell_0(\varepsilon)$ for the upper bound and $\ell\geqslant\ell_0(b,\varepsilon)$ for the lower bound, where $\mathcal{C}_\nu$ and s_0 are as above.

Remark. The lower bound above combines Theorems 5.1 and 5.2: since the quantity $\left(\frac{1}{2}+s_0(\mathcal{C}_\nu-1)\right)_+ +\frac{1}{2}$ equals the larger of the two lower bounds $1+s_0(\mathcal{C}_\nu-1)$ and $\frac{1}{2}$.

In other words, Theorem 1.5 converts local information about $\mathfrak{F}_\Gamma$ near $\alpha=0$, encoded in the limiting measure ν, into two-sided bounds for its long-term averages, with all of the measure-dependence concentrated in the single quantity $\mathcal{C}_\nu$ of (EP1.3). The usefulness of the result therefore hinges on estimating $\mathcal{C}_\nu$. This can be done in two ways: numerically, by optimizing over the full class $\mathcal{A}_\Delta$; or analytically, by restricting to the subclass of functions whose Fourier transforms are supported in $[-\Delta,\Delta]$. We pursue the analytic route, which renders $\mathcal{C}_\nu$ amenable to the theory of reproducing kernel Hilbert spaces.

More specifically, Lemma 5.4 establishes that

$$\mathcal{C}_\nu\leqslant\frac{1}{K_\nu(0,0)},$$

where $K_\nu(\cdot,\cdot)$ is the reproducing kernel of a Hilbert space $\mathcal{H}_\nu$, the space of entire functions of exponential type at most $\pi\Delta$ whose restriction to $\mathbb{R}$ is square-integrable against $\widehat{\nu}(\alpha)\mathrm{d}\alpha$, equipped with the inner product

$$\langle F,G\rangle_\nu:=\int_{\mathbb{R}}F(\alpha)\,\overline{G(\alpha)}\widehat{\nu}(\alpha)\,\mathrm{d}\alpha.$$

This bound is obtained explicitly for measures on $[-\Delta,\Delta]$ of the form

$$\mathrm{d}\nu(\alpha)=c_1\boldsymbol{\delta}(\alpha)+c_2|\alpha|e^{-c_3|\alpha|}\,\mathrm{d}\alpha,$$

where $c_1>0$, $c_2,c_3\geqslant 0$, and $c_2\Delta^2/c_1\leqslant 5/3$. The limiting measures arising from the pair correlation of zeros of natural families of L-functions — among them primitive elements of the Selberg class [96, 107], Dedekind zeta functions of number fields [42], and the zeros of $\operatorname{Re}\zeta$ and $\operatorname{Im}\zeta$ (for which Assumptions (H1) and (H2) follow from work of Garaev [56], Ki [80], and Gonek–Ki [64]) — all take this form, so our results apply to each of them. In a particular situation we establish these explicit bounds for the averages of $\mathfrak{F}_\Gamma$ associated with the zeros of $\operatorname{Re}\zeta$ or $\operatorname{Im}\zeta$; see Corollary 5.10. Moreover, this result allows us to say that a conjecture by Gonek and Ki [64] concerning the asymptotic behavior of $\mathfrak{F}_\Gamma$ cannot be true as stated in [64]. The explicit estimates are carried out in Chapter 5.

1.3.2 Littlewood's estimates for L-functions in the hyperelliptic ensemble

We again let $\zeta(s)$ denote the Riemann zeta function, and we let $N(t)$ be the number of non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with ordinate $\gamma \in (0, t]$, a zero of ordinate exactly t being counted with weight $\frac{1}{2}$. For $t \geqslant 2$, one has

$$N(t) = \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O\left(\frac{1}{t}\right).$$

The *argument function* $S(t)$ that appears in this classical formula is defined by

$$S(t) \coloneqq \frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2} + it\right),$$

if t is not the ordinate of a zero of $\zeta(s)$. If t is the ordinate of a zero of $\zeta(s)$, we set $S(t) \coloneqq \frac{1}{2}\lim_{\varepsilon\to 0}\left(S(t-\varepsilon) + S(t+\varepsilon)\right)$ [116]. Useful information on the oscillatory behavior of $S(t)$ is encoded in its antiderivatives. It is convenient to consider a specific sequence of antiderivatives by setting $S_0(t) = S(t)$ and, for $n \geqslant 1$ and $t > 0$,

$$S_n(t) \coloneqq \int_0^t S_{n-1}(\tau)\,\mathrm{d}\tau + \delta_n,$$

where, at each step, the constant δ_n is chosen so that S_n has mean value zero for all $n \geqslant 0$, that is

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S_n(\tau)\,\mathrm{d}\tau = 0. \tag{1.7}$$

Under the Riemann hypothesis (RH), Littlewood proved in [88] that

$$\log\left|\zeta\left(\tfrac{1}{2} + it\right)\right| \ll \frac{\log t}{\log\log t} \qquad \text{and} \qquad |S_n(t)| \ll \frac{\log t}{(\log\log t)^{n+1}}.$$

The current best estimates, all valid for sufficiently large t under RH, are

$$\log\left|\zeta\left(\tfrac{1}{2} + it\right)\right| \leqslant \left(\frac{\log 2}{2} + o(1)\right)\frac{\log t}{\log\log t}, \tag{1.8}$$

due to Chandee and Soundararajan [35],

$$|S(t)| \leqslant \left(\frac{1}{4} + o(1)\right)\frac{\log t}{\log\log t}, \tag{1.9}$$

due to Carneiro, Chandee, and Milinovich [25, 27], and, more generally, for $n \geqslant 1$,

$$-\left(C_n^- + o(1)\right)\frac{\log t}{(\log\log t)^{n+1}} \leqslant S_n(t) \leqslant \left(C_n^+ + o(1)\right)\frac{\log t}{(\log\log t)^{n+1}}, \tag{1.10}$$

where, for $n = 4k \pm 1$,

$$C_n^{\mp} = \frac{\zeta(n+1)}{\pi\cdot 2^{n+1}} \qquad \text{and} \qquad C_n^{\pm} = \frac{\left(1 - 2^{-n}\right)\zeta(n+1)}{\pi\cdot 2^{n+1}}, \tag{1.11}$$

and, for $n \geqslant 2$ even,

$$\begin{aligned} C_n^+ = C_n^- &= \left[\frac{2(C_{n+1}^+ + C_{n+1}^-)C_{n-1}^+ C_{n-1}^-}{C_{n-1}^+ + C_{n-1}^-}\right]^{1/2} \\ &= \frac{\sqrt{2}}{\pi\, 2^{n+1}} \left[\frac{(1-2^{-n-2})(1-2^{-n+1})\,\zeta(n)\,\zeta(n+2)}{1-2^{-n}}\right]^{1/2}. \end{aligned} \tag{1.12}$$

The case $n = 1$ of (1.10) is due to Carneiro, Chandee, and Milinovich [25], while the cases $n \geqslant 2$ are due to Carneiro and Chirre [19]. In Chapter 6, based on work [A5], we establish unconditional function field analogues of the estimates (1.8)–(1.10). Before stating our results, we recall the function field setting following the classical reference [104].

Fix a finite field $\mathbb{F}_q$ of odd cardinality and let $\mathbb{F}_q[x]$ be the polynomial ring over $\mathbb{F}_q$. If f is a non-zero polynomial in $\mathbb{F}_q[x]$ then the norm of f is defined as $|f| := q^{\deg(f)}$. If $f = 0$, we set $|f| = 0$. A monic irreducible polynomial P, of degree at least 1, is called a *prime polynomial* or simply a *prime*. For odd $d = 2g+1$, the *hyperelliptic ensemble* is

$$\mathcal{H}_d := \{D \in \mathbb{F}_q[x] : D \text{ monic, square-free, } \deg D = d\}.$$

Each $D \in \mathcal{H}_d$ gives rise to a quadratic Dirichlet character $\chi_D(f) := \left(\frac{D}{f}\right)$, where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Jacobi symbol in $\mathbb{F}_q[x]$, and to an L-function defined as

$$L(s,\chi_D) = \sum_{f \text{ monic}} \frac{\chi_D(f)}{|f|^s} = \prod_{P \text{ prime}} \left(1 - \chi_D(P)\,|P|^{-s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

This function can be written in terms of $u = q^{-s}$ as

$$\mathcal{L}(u,\chi_D) = \sum_{f \text{ monic}} \chi_D(f)\, u^{\deg f} = \prod_{P \text{ prime}} \left(1 - \chi_D(P)\, u^{\deg P}\right)^{-1}, \qquad |u| < q^{-1}.$$

By work of Weil [121], $\mathcal{L}(u,\chi_D)$ is a polynomial of degree $2g$ all of whose zeros lie unconditionally on the circle $|u| = q^{-1/2}$, the analogue of the critical line. Setting $\mathsf{e}(x) := e^{2\pi i x}$, we write these zeros as $u_j = q^{-1/2}\mathsf{e}(\theta_j)$, $j = 1, \ldots, 2g$, and parametrize the critical circle as $\{q^{-1/2}\mathsf{e}(\theta) : \theta \in \mathbb{R}/\mathbb{Z}\}$.

With the zeros of $\mathcal{L}(u,\chi_D)$ located on the critical circle, we can now introduce the function-field analogue of the argument function $S(t)$. When $q^{-1/2}\mathsf{e}(\theta)$ is not a zero of $\mathcal{L}(u,\chi_D)$, let Γ_θ be the path consisting of the positively oriented circular arc from q^{-2} to $q^{-2}\mathsf{e}(\theta)$ followed by the radial segment from $q^{-2}\mathsf{e}(\theta)$ to $q^{-1/2}\mathsf{e}(\theta)$, and set

$$S(\theta,\chi_D) := -\frac{1}{\pi}\,\Delta_{\Gamma_\theta} \arg \mathcal{L}(u,\chi_D),$$

where $\Delta_{\Gamma_\theta}\arg$ is the variation of the argument along Γ_θ; at a zero of $\mathcal{L}(u,\chi_D)$ one defines $S(\theta,\chi_D)$ by a symmetric limit. The function $\theta \mapsto S(\theta,\chi_D)$ is 1-periodic and has mean value zero. Its antiderivatives are defined, for $n \geqslant 1$, by

$$S_n(\theta, \chi_D) \coloneqq \int_0^\theta S_{n-1}(\alpha, \chi_D)\, \mathrm{d}\alpha + c_n,$$

where the constant $c_n = c_n(\chi_D)$ is chosen so that $\int_{\mathbb{R}/\mathbb{Z}} S_n(\theta, \chi_D)\, \mathrm{d}\theta = 0$. These functions are the direct analogues of $S_n(t)$ from (1.7). Finally, the bounds on $S_n(\theta, \chi_D)$ are expressed in terms of the Bernoulli polynomials $B_n(x)$, defined by

$$\frac{t\, e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!}\, t^n, \qquad |t| < 2\pi,$$

via the quantities $M_n \coloneqq \max_{x\in[0,1]} B_n(x)$ and $m_n \coloneqq \min_{x\in[0,1]} B_n(x)$.

Theorem 1.6 (see Theorems 6.1, 6.2 and 6.3)**.** *For $d = 2g+1$, $D \in \mathcal{H}_d$, $n \geqslant 1$, and $\theta \in \mathbb{R}/\mathbb{Z}$, as $d \to \infty$,*

(i) $\log\left|\mathcal{L}(q^{-1/2}\mathsf{e}(\theta), \chi_D)\right| \leqslant \left(\frac{\log 2}{2} + o(1)\right) \frac{d}{\log_q d}$,

(ii) $S(\theta, \chi_D) \leqslant \left(\frac{1}{4} + o(1)\right) \frac{d}{\log_q d}$

(iii) $-\left(\frac{A_n^-}{(2\pi)^n} + o(1)\right) \frac{d}{(\log_q d)^{n+1}} \leqslant S_n(\theta, \chi_D) \leqslant \left(\frac{A_n^+}{(2\pi)^n} + o(1)\right) \frac{d}{(\log_q d)^{n+1}}$

where $A_n^- \coloneqq \frac{\pi^n}{2} \cdot \frac{M_{n+1}}{(n+1)!}$ and $A_n^+ \coloneqq -\frac{\pi^n}{2} \cdot \frac{m_{n+1}}{(n+1)!}$.

Note the similarities of the constants in the above theorem with the constants in the estimates (1.8) and (1.9). The constants $A_n^\pm$ in Theorem 1.6 exactly match the best known classical constants in (1.11) for the Riemann zeta function when n is odd; this follows from classical evaluations, due to Lehmer [83], of the extreme values of B_{n+1} on $[0,1]$, which are attained at 0 or $\frac{1}{2}$ and can be expressed in terms of $\zeta(n+1)$. For even $n \geqslant 2$, however, the extreme values of B_{n+1} are not attained at these special points, and one finds constants strictly smaller than constants $C_n^\pm$ in the bound (1.12) for $S_n(t)$ under RH. This improvement is possible because, in the periodic setting, the associated extremal problem in trigonometric polynomial approximation is completely solved, whereas the analogous Beurling–Selberg extremal problem on the real line, which governs the classical estimates, remains open for even n. The proofs of all three bounds in Theorem 1.6 follow a unified strategy: each quantity is expressed as a sum of translates of a special periodic function over the angles θ_j — the function $\log(2|\sin(\pi\theta)|)$ for the modulus, and the Bernoulli periodic function $\mathcal{B}_{n+1}(\theta) \coloneqq B_{n+1}(\theta - \lfloor\theta\rfloor)$ for S_n — which is then one-sidedly approximated by a trigonometric polynomial of degree N, using the sharp constructions of [17, 24].

1.4 Notation and conventions

We collect here some conventions used throughout the thesis.

- For a field $\mathbb{F}$, vectors in $\mathbb{F}^d$ are denoted here with bold font (e.g., $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$) and numbers in $\mathbb{F}$ with regular font (e.g., x, y, z). For any $\boldsymbol{\rho} \in \mathbb{F}^d$, we use the notation $\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_d)$.
- For a field $\mathbb{F}$, we denote by $\mathbb{F}^\times := \mathbb{F} \setminus \{0\}$ its multiplicative group; particular examples we use are $\mathbb{C}^\times$ and $\mathbb{F}_q^\times$.
- For $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, we write $|\boldsymbol{x}|$ for its Euclidean length, $|\boldsymbol{x}| := (\boldsymbol{x} \cdot \boldsymbol{x})^{1/2} = (x_1^2 + \cdots + x_d^2)^{1/2}$.
- We write $\mathbb{S}^d := \{\boldsymbol{x} \in \mathbb{R}^{d+1} : |\boldsymbol{x}| = 1\}$ for the unit d-dimensional sphere, and ω_d for its total mass with respect to d-dimensional Hausdorff measure $\mathcal{H}^d$.
- We denote by $\mathscr{S}(\mathbb{R}^d)$ the space of Schwartz functions.
- We use the following definition of the Fourier transform for a given function $f \in L^1(\mathbb{R}^d)$:
$$\mathscr{F}[f](\boldsymbol{\xi}) := \widehat{f}(\boldsymbol{\xi}) := \int_{\mathbb{R}^d} e^{-2\pi i \boldsymbol{\xi} \cdot \boldsymbol{x}} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}.$$
- We also adopt this normalization for the Fourier transform of a finite Borel measure μ on $\mathbb{R}$, setting
$$\widehat{\mu}(\xi) := \int_{\mathbb{R}} e^{-2\pi i \xi x} \, \mathrm{d}\mu(x).$$
- $\boldsymbol{\delta}$ denotes the Dirac mass centered at the origin.
- We say that $A \ll B$ or $A = O(B)$ if there is a positive constant C such that $|A| \leqslant CB$. We say that $A \asymp B$ when $A \ll B$ and $B \ll A$. Sometimes, to make a dependence of the constant C on a parameter α explicit, we write $A \ll_\alpha B$ or $A = O_\alpha(B)$.
- For a real number $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ the largest integer not exceeding x.
- If $\mathcal{F}$ is a finite set of variables, then $C(\mathcal{F})$ denotes a quantity that only depends on elements of $\mathcal{F}$.
- The characteristic (indicator) function of a set S is denoted by $\mathbf{1}_S$. Moreover, $\#S$ denotes the cardinality of a set S.
- We denote by $\mathcal{P}_n(\mathbb{F})$ the set of all polynomials of degree at most n with coefficients in $\mathbb{F}$.
- We set $\mathsf{e}(x) := e^{2\pi i x}$ and $\exp(x) := e^x$.
- If X, Y are two metric spaces, and μ is a measure on X, and $T : X \to Y$ is a Borel map, we denote by $T_\# \mu$ the *push-forward of μ through T*, defined by
$$T_\# \mu(B) := \mu(T^{-1}(B)),$$

for all Borel sets $B \subseteq Y$.

- If a measure μ is defined on $\mathbb{T}$, then we set $\breve{\mu}$ as the "upper-half" of μ which is a measure on the interval $[0, 1/2]$ and is given by

$$\breve{\mu}(A) \coloneqq \mu(A \cap (0, 1/2)) + \frac{1}{2}\mu(A \cap \{0, 1/2\}),$$

for every $A \subseteq [0, 1/2]$.

- For a Lebesgue exponent $1 < p < \infty$, we set its conjugate exponent $p' = p/(p-1)$. For the end points, this definition extends by passing to the appropriate limit.

- In a metric space $(\mathbb{X}, \mathsf{d})$ the radius of last sign change of a function $f : \mathbb{X} \to \mathbb{R}$ is denoted by

$$\mathrm{r}(f) \coloneqq \inf\{r > 0 : f(x) \geqslant 0 \text{ for all } x \in \mathbb{X} \setminus B_r(x^*)\},$$

where $B_r(x^*) \coloneqq \{y \in \mathbb{X} : \mathsf{d}(x^*, y) < r\}$ is the open ball of radius r centered at x^*, a distinguished fixed point of the space $\mathbb{X}$.

- Real and imaginary parts of a given complex number $z \in \mathbb{C}$ are denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

- The principal value of the argument is $\mathrm{Arg}(z) \in (-\pi, \pi]$.

- We denote by $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the open upper half-plane.

- We denote by s_0 the quantity

$$s_0 \coloneqq \min_{x \in \mathbb{R}} \frac{\sin x}{x} = -0.217233\ldots.$$

Chapter 2

Sign uncertainty for entire functions of exponential type

2.1 Introduction

In Section 1.1 we recalled that the sign uncertainty principle of Bourgain, Clozel, and Kahane [13] asserts $\sqrt{\mathrm{r}(F)\cdot\mathrm{r}(\widehat{F})} \geqslant \mathbb{A}(d) > 0$ for suitable F. More precisely, let $\mathcal{A}(d)$ denote the class of continuous, even functions $F:\mathbb{R}^d\to\mathbb{R}$ such that (i) $F,\widehat{F}\in L^1(\mathbb{R}^d)$, (ii) $F(\mathbf{0})\leqslant 0$ and $\widehat{F}(\mathbf{0})\leqslant 0$, and (iii) F and $\widehat{F}$ are eventually non-negative, and set

$$\mathbb{A}(d) := \inf_{0\not\equiv F\in\mathcal{A}(d)} \sqrt{\mathrm{r}(F)\cdot\mathrm{r}(\widehat{F})}\,.$$

The product $\mathrm{r}(F)\cdot\mathrm{r}(\widehat{F})$ is invariant under the rescaling $F(\boldsymbol{x})\mapsto F(\lambda\boldsymbol{x})$, $\lambda>0$, so $\mathbb{A}(d)$ is a well-defined dimensional constant. Bourgain, Clozel, and Kahane established

$$\sqrt{\frac{d}{2\pi e}} \;\leqslant\; \mathbb{A}(d) \;\leqslant\; \sqrt{\frac{d+2}{2\pi}}, \qquad d\in\mathbb{N},$$

so that $\mathbb{A}(d)>0$ already follows, while the sharp value of $\mathbb{A}(d)$ remains unknown except in dimension $d=12$, where Cohn and Gonçalves [39] proved $\mathbb{A}(12)=\sqrt{2}$. Subsequent works have extended this framework to an abstract operator setting [63], to a setting in which the signs of F and $\widehat{F}$ are required to resonate with a prescribed function [23], and to further regularity and mass-concentration questions [61, 62].

This chapter, based on [A1], develops a different approach already announced in Section 1.1.1: instead of constraining both F and $\widehat{F}$ as above, we fix the Fourier support of F and ask how small $\mathrm{r}(F)$ alone can be made, with respect to a general measure μ. We now state this problem in full generality, for arbitrary dimension d, and reduce it to the one-dimensional case stated in Section 1.1.1.

2.1.1 A general extremal problem in dimension d and its reduction

For $\boldsymbol{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d$, write $|\boldsymbol{z}| := (|z_1|^2 + \cdots + |z_d|^2)^{1/2}$ for the Euclidean norm, and define a second norm by

$$\|\boldsymbol{z}\| := \sup\left\{\left|\sum_{n=1}^{d} z_n\, t_n\right| \;:\; \boldsymbol{t} \in \mathbb{R}^d,\ |\boldsymbol{t}| \leqslant 1\right\}.$$

When $d = 1$ the two norms coincide. A non-identically-zero entire function $F : \mathbb{C}^d \to \mathbb{C}$ has *exponential type* if

$$\tau(F) := \limsup_{\|\boldsymbol{z}\| \to \infty} \frac{\log |F(\boldsymbol{z})|}{\|\boldsymbol{z}\|} < \infty,$$

in which case $\tau(F)$ is called the exponential type of F. For $d = 1$ this is the classical definition; for $d \geqslant 2$, $\tau(F)$ is the exponential type of F with respect to the Euclidean unit ball, a special case of the more general notion of type with respect to a compact, convex, symmetric set $K \subset \mathbb{R}^d$ (cf. [111, pp. 111–112]). With this normalization, the Paley–Wiener theorem extends as expected: for $F \in L^2(\mathbb{R}^d)$, F agrees a.e. with the restriction to $\mathbb{R}^d$ of an entire function of exponential type at most δ if and only if $\widehat{F}$ is supported in the ball of radius $\delta/2\pi$ centred at the origin.

To pass between dimensions, we lift measures on $\mathbb{R}$ to radial measures on $\mathbb{R}^d$. Given a locally finite, even, non-negative Borel measure μ on $\mathbb{R}$, let μ^d be defined in polar coordinates $\boldsymbol{x} = r\omega$, with $r > 0$ and $\omega \in \mathbb{S}^{d-1}$, by

$$\mathrm{d}\mu^d(\boldsymbol{x}) = \mathrm{d}\sigma_{d-1}(\omega)\, \mathrm{d}\mu(r), \tag{2.1}$$

where σ_{d-1} is the surface measure on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, and set $\mu^1 := \mu$. For example, if $\mathrm{d}\mu(x) = W(x)\,\mathrm{d}x$, then $\mathrm{d}\mu^d(\boldsymbol{x}) = |\boldsymbol{x}|^{1-d} W(|\boldsymbol{x}|)\,\mathrm{d}\boldsymbol{x}$. Finally, we say that an entire function $F : \mathbb{C}^d \to \mathbb{C}$ is *real entire* if its restriction to $\mathbb{R}^d$ is real-valued. With these conventions, (EP1.1) admits the following generalization.

Extremal problem 2.1 **(EP2.1)**. Given a locally finite, even, non-negative Borel measure μ on $\mathbb{R}$, $\delta \geqslant 0$, and $d \in \mathbb{N}$, determine

$$\Lambda(\mu, \delta, d) := \inf_{0 \not\equiv F \in \mathcal{E}(\mu,\delta,d)} \mathrm{r}\big(F|_{\mathbb{R}^d}\big),$$

where $\mathcal{E}(\mu, \delta, d)$ is the class of real entire functions $F : \mathbb{C}^d \to \mathbb{C}$ of exponential type at most δ such that $F|_{\mathbb{R}^d} \in L^1(\mathbb{R}^d, \mu^d)$, $\int_{\mathbb{R}^d} F\,\mathrm{d}\mu^d \leqslant 0$, and $F|_{\mathbb{R}^d}$ is eventually non-negative.

Two remarks clarify the scope of (EP2.1). First, the problem has the same tension as in dimension one: $F|_{\mathbb{R}^d}$ must be eventually non-negative while having non-positive μ^d-integral, and $\Lambda(\mu, \delta, d)$ measures how small can be the radius of last sign change under this tension. Second, the measure μ is essentially arbitrary, subject only to local finiteness; in fact, this problem came to our attention motivated by particular choices of measures μ arising in applications in number theory, see Section 2.1.4. The condition of μ being locally finite avoids the situation where all functions in the class $\mathcal{E}(\mu,\delta,d)$ need to vanish in a certain region. The class $\mathcal{E}(\mu,\delta,d)$ can nevertheless be trivial for other reasons: if $\mathrm{d}\mu(x) = e^{|x|^4}\,\mathrm{d}x$, then $\mathcal{E}(\mu,\delta,d) = \{0\}$ as a consequence of Hardy's uncertainty principle. Problems of the type (EP2.1), restricted to bandlimited functions of constant sign at infinity, have also been studied in [66, 90, 91].

Our first result reduces (EP2.1) to the one-dimensional case.

Theorem 2.1 (Dimension shifts)**.** *For every μ, δ, and $d \in \mathbb{N}$ as in* (EP2.1)*,*

$$\Lambda(\mu,\delta,d) = \Lambda(\mu,\delta,1).$$

Theorem 2.1 is proved in Section 2.2 by a radial symmetrization argument. In light of it, we write

$$\mathcal{E}(\mu,\delta) := \mathcal{E}(\mu,\delta,1), \qquad \Lambda(\mu,\delta) := \Lambda(\mu,\delta,1),$$

so that (EP2.1) with $d = 1$ recovers exactly (EP1.1) from Section 1.1.1, and all results in the remainder of this chapter are stated for this one-dimensional problem, with the higher-dimensional statements following from Theorem 2.1.

2.1.2 Sign uncertainty and de Branges spaces

In the situations of interest to us, (EP1.1) connects to the beautiful theory of de Branges spaces of entire functions [41]. We recall here the definitions from complex analysis needed to state our main theorem; a more detailed discussion of de Branges spaces is given in Section 2.3.1.1.

For an entire function $F : \mathbb{C} \to \mathbb{C}$, define the entire function $F^* : \mathbb{C} \to \mathbb{C}$ by $F^*(z) := \overline{F(\bar{z})}$; note that $F^* = F$ precisely when F is real entire. We write $\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ for the open upper half-plane. Recall from Section 1.1.1 that an entire function $E : \mathbb{C} \to \mathbb{C}$ is *Hermite–Biehler* if $|E(\bar{z})| < |E(z)|$ for all $z \in \mathbb{C}^+$; equivalently, in terms of E^*,

$$|E^*(z)| < |E(z)|, \qquad z \in \mathbb{C}^+. \tag{2.2}$$

We say that an analytic function $F : \mathbb{C}^+ \to \mathbb{C}$ has *bounded type* in $\mathbb{C}^+$ if it can be written as the quotient of two functions that are analytic and bounded in $\mathbb{C}^+$. If $F : \mathbb{C}^+ \to \mathbb{C}$ has bounded type in $\mathbb{C}^+$, its Nevanlinna factorization [41, Theorems 9 and 10] gives

$$v(F) := \limsup_{y\to\infty} \frac{\log|F(iy)|}{y} < \infty, \tag{2.3}$$

and $v(F)$ is called the *mean type* of F. If a Hermite–Biehler function E has bounded type in $\mathbb{C}^+$, then so does E^*: writing $E = P/Q$ with P, Q analytic and bounded in $\mathbb{C}^+$, we have $E^* = ((E^*/E)P)/Q$, since $|E^*/E| < 1$ in $\mathbb{C}^+$ by (2.2). By a result of Krein (Lemma 2.8 below), E then has exponential type, with

$$\tau(E) = \max\{v(E), v(E^*)\} = v(E),$$

the last equality following from (2.2) and (2.3).

Associated to a Hermite–Biehler function $E : \mathbb{C} \to \mathbb{C}$ is the pair of real entire functions A, B such that $E = A - iB$, given explicitly by

$$A(z) := \frac{1}{2}(E(z) + E^*(z)), \qquad B(z) := \frac{i}{2}(E(z) - E^*(z)); \tag{2.4}$$

by (2.2), A and B can only have real zeros. We work with Hermite–Biehler functions E satisfying:

(C1) E has bounded type in $\mathbb{C}^+$;

(C2) the function $z \mapsto E(iz)$ is real entire;

(C3) E has no real zeros.

Condition (C2) is equivalent to A being even and B odd, and (C3) is equivalent to A and B having no common zero (in particular $A(0) \neq 0$). We consider measures μ on $\mathbb{R}$ that are related to the measure $|E(x)|^{-2}\mathrm{d}x$ in an integral sense given by the following condition:

(C4) for every real entire $F : \mathbb{C} \to \mathbb{C}$ of exponential type at most $2\tau(E)$ that is non-negative on $\mathbb{R}$,

$$\int_{\mathbb{R}} F(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}} F(x)\,|E(x)|^{-2}\,\mathrm{d}x. \tag{2.5}$$

If

$$\mathrm{d}\mu(x) = |E(x)|^{-2}\,\mathrm{d}x, \tag{2.6}$$

then (C4) holds trivially. As we shall see in Section 2.1.4, it is important for many applications to have the flexibility allowed by the integral condition (C4) rather than the pointwise condition (2.6).

We can now state the main result of this chapter, which fully solves (EP1.1) whenever μ satisfies (C4) for some E as above.

Theorem 2.2 (Sharp constants)**.** *Let $E : \mathbb{C} \to \mathbb{C}$ be a Hermite–Biehler function satisfying* (C1), (C2), *and* (C3), *and let μ be a locally finite, even, non-negative Borel measure on $\mathbb{R}$ satisfying* (C4). *Let $A := \frac{1}{2}(E + E^*)$ and let ξ_1 be the smallest positive real zero of A. Then*

$$\Lambda(\mu, 2\tau(E)) = \xi_1,$$

and the unique extremizer (up to multiplication by a positive constant) is

$$F(z) = \frac{A(z)^2}{z^2 - \xi_1^2}. \tag{2.7}$$

Taking $\mu = |E(x)|^{-2}\,\mathrm{d}x$ as in (2.6), for which (C4) is automatic, Theorem 2.2 recovers Theorem 1.1: by Krein's theorem (Lemma 2.8), the conditions (C1), (C2), and (C3) imposed on E here correspond exactly to the hypotheses of Theorem 1.1.

Remark. (i) If A has no zeros, then $A(z) = a$ and $B(z) = bz$ for some $a, b \in \mathbb{R}$ with $ab > 0$, so $E(z) = a - ibz$; in this case $\mathcal{E}(\mu, 0) = \{0\}$.

(ii) From the proof of Theorem 2.1 we also get, for $d > 1$, that the unique extremizer (up to multiplication by a positive constant) of $\Lambda(\mu, 2\tau(E), d)$, that is radial on $\mathbb{R}^d$, is the lift of the function (2.7), as discussed in Section 2.2.

(iii) Recall from Section 1.1.1 that the function $E(z) = e^{-i\delta z}$ is of type $\tau(E) = \delta$, and it gives $\Lambda(\mathrm{d}x, 2\delta) = \pi/(2\delta)$ for the Lebesgue measure. Theorem 2.2 further identifies the extremizer:

$$F(z) = \frac{\cos^2(\delta z)}{z^2 - \big(\pi/(2\delta)\big)^2}.$$

(iv) For a given μ, several Hermite–Biehler functions E may satisfy (C4); see the discussion in Section 2.5.1. The smallest positive zero ξ_1 of $A = \frac{1}{2}(E + E^*)$ is the same for all such E.

We shall present two different proofs for Theorem 2.2 in Section 2.3. The first proof uses a basic polynomial lemma to reduce the search for extremizers to even functions with only one sign change in the positive half-line, and in such a setup one can connect matters to a related extremal problem in de Branges spaces, concerning the multiplication by z operator. The second proof makes use of certain quadrature formulas in this context of de Branges spaces. The latter approach can be seen as a general framework for problems of this type: it is the key idea behind the proof of Theorem 1.2, which solves the periodic problem.

2.1.3 A negative result

In Section 2.1.2 we have been interested in a broad situation where we can actually find the sharp version of the uncertainty principle in (EP1.1). In general, it is an interesting (and seemingly non-trivial) problem to find necessary and sufficient conditions on the measure μ that would guarantee a qualitative sign uncertainty principle (i.e. the fact that $\Lambda(\mu, \delta) > 0$). Our next result shows that, if the measure μ has a certain amount of mass around the origin and an exponential decay, the sign uncertainty does not hold.

Theorem 2.3. *Let μ be a locally finite, even and non-negative Borel measure on $\mathbb{R}$ such that:*

(i) $\mu([-r, r]) > 0$ *for each* $r > 0$.

(ii) $\int_{\mathbb{R}} e^{\varepsilon|x|}\, \mathrm{d}\mu(x) < \infty$ *for some* $\varepsilon > 0$.

Then $\Lambda(\mu, \delta) = 0$ *for any* $\delta \geqslant 0$.

In particular, if the measure μ verifies (i) and has compact support, we fall under the hypotheses of Theorem 2.3. The conditions in Theorem 2.3 are sufficient but not necessary, as the following example shows: if one considers $\mathrm{d}\mu(x) = \sum_{n\in\mathbb{Z}} \boldsymbol{\delta}(x-n)$, where $\boldsymbol{\delta}$ is the usual Dirac delta at the origin, the function $F(z) = \sin^2(\pi z) \in \mathcal{E}(\mu, 2\pi)$ and verifies $\mathrm{r}(F) = 0$. If one prefers a function that is not identically zero μ-a.e. one can consider instead $F(z) = (z^2 - \varepsilon^2)\sin^2(\pi z)/(\pi z)^2$ for $\varepsilon > 0$ small.

Remark. The proof of Theorem 2.3 relies only on the fact that polynomials are dense in $L^2(\mathbb{R}, \mu)$. Hypothesis (ii) (exponential decay) is a simple sufficient condition for this density. By a classical criterion of M. Riesz [103], this density is equivalent to the determinacy of the moment problem for μ, that is, the moment sequence $\left(\int_{\mathbb{R}} x^n\, \mathrm{d}\mu(x)\right)_{n\geqslant 0}$ determines μ uniquely among all measures with the same moments (for a thorough treatment of the moment problem, see [1]). Thus the theorem actually holds for the wider class of measures with a determinate moment problem; we keep the more concrete exponential decay formulation because it avoids additional terminology and still covers many cases of interest.

2.1.4 Applications

We discuss here three applications of Theorem 2.2.

2.1.4.1 Bounds for low-lying zeros in families of L-functions

(EP1.1) is related to the problem of bounding the height of the first zero in families of L-functions. Let us very briefly discuss the setup for this problem; the reader can consult the references presented in this subsection for the technical details and a variety of examples.

Let $\mathcal{F}$ be a set of number theoretical objects (one can think of Dirichlet characters, for example) and assume that to each $f \in \mathcal{F}$ one can associate an L-function

$$L(s, f) = \sum_{n=1}^{\infty} \lambda_f(n)\, n^{-s},$$

where the coefficients $\lambda_f(n) \in \mathbb{C}$. Assume that $L(s, f)$ admits an analytic continuation to an entire function. Let us denote the analytic conductor of $L(s, f)$ by c_f and write its non-trivial zeros as $\rho_f = \frac{1}{2} + i\gamma_f$. Assume also the validity of the Generalized Riemann Hypothesis (GRH) for such L-functions; i.e., that $\gamma_f \in \mathbb{R}$. The density conjecture of Katz and Sarnak [77, 78] asserts that for each natural family $\{L(s, f),\ f \in \mathcal{F}\}$

of L-functions there is an associated symmetry group $G = G(\mathcal{F})$, where G is either: unitary U, symplectic Sp, orthogonal O, even orthogonal SO(even), or odd orthogonal SO(odd). Here one wishes to consider averages over $f \in \mathcal{F}$ ordered by the conductor. Following the notation of Iwaniec, Luo and Sarnak [74], let $\mathcal{F}(Q) := \{f \in \mathcal{F} : c_f = Q\}$[1] and let $\#\mathcal{F}(Q)$ denote its cardinality. If $\phi : \mathbb{R} \to \mathbb{R}$ is an even Schwartz function whose Fourier transform has compact support, Katz and Sarnak [77, 78] conjectured that

$$\lim_{Q\to\infty} \frac{1}{\#\mathcal{F}(Q)} \sum_{f\in\mathcal{F}(Q)} \sum_{\gamma_f} \phi\left(\gamma_f \frac{\log c_f}{2\pi}\right) = \int_{\mathbb{R}} \phi(x)\, W_G(x)\, \mathrm{d}x, \tag{2.8}$$

where the sum over γ_f counts multiplicity, and W_G is a function (or density) depending on the symmetry group $G(\mathcal{F})$. This is the so-called 1-level density of the low-lying zeros of the family. For these five symmetry groups, Katz and Sarnak determined the density functions:

$$\begin{aligned} W_{\mathrm{U}}(x) = 1 \quad ; \quad W_{\mathrm{Sp}}(x) = 1 - \frac{\sin 2\pi x}{2\pi x} \quad ; \quad W_{\mathrm{O}}(x) = 1 + \tfrac{1}{2}\boldsymbol{\delta}(x)\,; \\ W_{\mathrm{SO(even)}}(x) = 1 + \frac{\sin 2\pi x}{2\pi x} \quad ; \quad W_{\mathrm{SO(odd)}}(x) = 1 - \frac{\sin 2\pi x}{2\pi x} + \boldsymbol{\delta}(x). \end{aligned} \tag{2.9}$$

The simplest example would perhaps be when $\mathcal{F}$ is the set of non-principal Dirichlet characters χ (and here the analytic conductor of $L(s,\chi)$ is just the modulus q of χ). In this case, Hughes and Rudnick [70] proved the validity of (2.8), with unitary symmetry $G(\mathcal{F}) = \mathrm{U}$, for even Schwartz functions $\phi : \mathbb{R} \to \mathbb{R}$ with $\mathrm{supp}(\widehat{\phi}\,) \subset [-2,2]$. There are several other results in the literature establishing the validity of (2.8) for different families of L-functions and different sizes of the support of $\widehat{\phi}$; see, for instance, [31, 44, 52, 70, 74, 99] and the references therein.

In this context, the analysis problem that arises is the following: assume the validity of (2.8) for even Schwartz functions $\phi : \mathbb{R} \to \mathbb{R}$ with $\mathrm{supp}(\widehat{\phi}\,) \subset [-\Delta, \Delta]$, with $\Delta > 0$ fixed; what is the sharpest upper bound that one can get for the height of the first zero in the family as $Q \to \infty$ with this information? Under these assumptions, if ϕ is such that $\phi(x) \geqslant 0$ for $|x| \geqslant r$ and $\int_{\mathbb{R}} \phi(x)\, W_G(x)\, \mathrm{d}x < 0$, relation (2.8) plainly yields

$$\limsup_{Q\to\infty} \min_{\substack{\gamma_f \\ f\in\mathcal{F}(Q)}} \left|\frac{\gamma_f \log c_f}{2\pi}\right| \leqslant r.$$

Hence, one is led to the problem of finding the infimum over all such possible values of r, which is exactly (EP1.1) in the case of the five measures μ with densities given by (2.9)[2]. By the Paley–Wiener theorem one seeks here the constant $\Lambda(\mu, 2\pi\Delta)$.

There are three previous works in the literature that address the problem of estimating the height of the first zero within this framework. The first one, by Hughes and Rudnick [70, Theorem 8.1], deals with the case of $G(\mathcal{F}) = \mathrm{U}$. The other two, by

[1] One may also consider $\mathcal{F}(Q) := \{f \in \mathcal{F} : c_f \leqslant Q\}$ in some situations.

[2] The fact that the functions are even and Schwartz, and that the integral is strictly less than 0, are harmless due to standard approximation arguments.

Bernard [11, Theorem 1 and Proposition 2] and by Carneiro, Chirre, and Milinovich [31, Theorem 9], deal with the other measures. These papers have a common feature: they work under the additional assumption that ϕ is of the form

$$\phi(x) = (x^2 - r^2)\,\psi(x)\,\psi^*(x), \tag{2.10}$$

where ψ is such that $\mathrm{supp}(\widehat{\psi}) \subset [-\Delta/2, \Delta/2]$. Within this more restrictive class, they arrive at the extremal solution. However, in principle, it is not obvious that one can restrict the search in the original problem to functions as in (2.10), which have only one sign change. The cases of these five measures with densities given by (2.9) fall under the general umbrella of Theorem 2.2, and this provides a subtle, yet conceptually interesting, contribution to this framework, namely:

Corollary 2.4. *The additional assumption* (2.10) *brings no loss of generality in the search for* $\Lambda(\mu, 2\pi\Delta)$ *in the cases of the measures* μ *with densities given by* (2.9).

Hence, the results obtained in [11, 31, 70] are in fact the extremal solutions in the larger class $\mathcal{E}(\mu, 2\pi\Delta)$.

2.1.4.2 Power weights

An interesting situation in our framework is the following. Let $\nu > -1$ and consider the locally finite, even and non-negative Borel measure μ_ν on $\mathbb{R}$ given by

$$\mathrm{d}\mu_\nu(x) = |x|^{2\nu+1}\,\mathrm{d}x. \tag{2.11}$$

Let J_ν be the classical Bessel function of the first kind [120]. Then

$$A_\nu(z) := \Gamma(\nu+1)\left(\tfrac{1}{2}z\right)^{-\nu} J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{1}{2}z\right)^{2n}}{n!(\nu+1)(\nu+2)\cdots(\nu+n)} \tag{2.12}$$

defines an even entire function of exponential type 1. When $\nu = -1/2$ (which is the case of the Lebesgue measure on $\mathbb{R}$) we simply have $A_{-1/2}(z) = \cos z$. The following result comes as a corollary of Theorem 2.2.

Corollary 2.5. *For* $\nu > -1$ *let* μ_ν *be given by* (2.11). *Let* $\mathrm{j}_{\nu,1}$ *be the first positive zero of the Bessel function* J_ν. *Then, for* $\delta > 0$,

$$\Lambda(\mu_\nu, \delta) = \frac{2\,\mathrm{j}_{\nu,1}}{\delta}.$$

Moreover, the unique extremizer for $\Lambda(\mu_\nu, \delta)$ *(up to multiplication by a positive constant) is*

$$F(z) = \frac{A_\nu(\delta z/2)^2}{z^2 - 4\mathrm{j}_{\nu,1}^2/\delta^2}. \tag{2.13}$$

Remark. (i) For $\delta = 0$ we have $\mathcal{E}(\mu_\nu, 0) = \{0\}$.

(ii) By Theorem 2.1 and its proof, for $d > 1$ the unique extremizer (up to multiplication by a positive constant) of $\Lambda(\mu_\nu, \delta, d)$, that is radial on $\mathbb{R}^d$, is the lift of the function (2.13), as discussed in Section 2.2.

2.1.4.3 Measures asymptotically comparable to power weights

One may wonder how broad is the reach of Theorem 2.2. In fact, fairly general measures would fall under the scope of this theorem (and, in particular, the sign uncertainty principle holds), as our next result illustrates.

Corollary 2.6. *Let μ be a locally finite, even and non-negative Borel measure on $\mathbb{R}$, given by*

$$\mathrm{d}\mu(x) = W(x)\,\mathrm{d}x$$

on the set $\{x \in \mathbb{R} : |x| > R\}$ for some $R > 0$, where W is a non-negative Borel measurable function. Assume further that

$$W(x) \asymp |x|^{2\nu+1}$$

on the set $\{x \in \mathbb{R} : |x| > R\}$, for some $\nu > -1$. Then, for any $\delta > 0$, the problem of computing $\Lambda(\mu, \delta)$ falls under the framework of Theorem 2.2. In particular, $\Lambda(\mu, \delta) > 0$.

Remark. In the setting of Corollary 2.6, for $\delta = 0$ we have $\mathcal{E}(\mu, 0) = \{0\}$.

2.2 Lifts and symmetrization: proof of Theorem 2.1

2.2.1 Preliminaries

The radial symmetry of the problem plays an important role in this argument, and we start by recalling two suitable constructions from the work of Holt and Vaaler [69, Section 6].

If $G : \mathbb{C} \to \mathbb{C}$ is an even entire function with power series representation

$$G(z) = \sum_{k=0}^{\infty} c_k z^{2k}\,,$$

we define its *lift* to $\mathbb{C}^d$, namely the function $\mathcal{L}_d(G) : \mathbb{C}^d \to \mathbb{C}$, by

$$\mathcal{L}_d(G)(\boldsymbol{z}) := \sum_{k=0}^{\infty} c_k (z_1^2 + \ldots + z_d^2)^k.$$

If $d > 1$ and $F : \mathbb{C}^d \to \mathbb{C}$ is an entire function, we define its *radial symmetrization* $\widetilde{F} : \mathbb{C}^d \to \mathbb{C}$ by

$$\widetilde{F}(\boldsymbol{z}) := \int_{\mathrm{SO}(d)} F(R\boldsymbol{z})\,\mathrm{d}\eta(R),$$

where $\mathrm{SO}(d)$ denotes the compact topological group of real orthogonal $d \times d$ matrices with determinant 1, with associated Haar measure η normalized so that $\eta(\mathrm{SO}(d)) = 1$. If $d = 1$, we simply set $\widetilde{F}(z) := \frac{1}{2}\{F(z) + F(-z)\}$.

The following result is a compilation of [69, Lemmas 18 and 19], and is concerned with the relation between these symmetrization mechanisms and the exponential type of the underlying functions.

Lemma 2.7. *The following propositions hold:*

(i) *Let $G : \mathbb{C} \to \mathbb{C}$ be an even entire function. Then G is of exponential type if and only if $\mathcal{L}_d(G)$ is of exponential type, and $\tau(G) = \tau(\mathcal{L}_d(G))$.*

(ii) *Let $F : \mathbb{C}^d \to \mathbb{C}$ be an entire function. Then $\widetilde{F} : \mathbb{C}^d \to \mathbb{C}$ is an entire function with power series expansion of the form*

$$\widetilde{F}(\boldsymbol{z}) = \sum_{k=0}^{\infty} c_k (z_1^2 + \ldots + z_d^2)^k.$$

Moreover, if F is of exponential type, then $\widetilde{F}$ is of exponential type and

$$\tau(\widetilde{F}) \leqslant \tau(F).$$

2.2.2 Proof of Theorem 2.1

Let $F \in \mathcal{E}(\mu, \delta, d)$. Then, from Lemma 2.7 (ii) we have that $\tau(\widetilde{F}) \leqslant \tau(F) \leqslant \delta$. The fact that F is real entire, not identically zero and eventually non-negative implies that $\widetilde{F}$ is also real entire, not identically zero and eventually non-negative, and we plainly have $\mathrm{r}(\widetilde{F}) \leqslant \mathrm{r}(F)$. Using Fubini's theorem and the fact that μ^d is rotationally invariant, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |\widetilde{F}(\boldsymbol{x})| \,\mathrm{d}\mu^d(\boldsymbol{x}) &\leqslant \int_{\mathbb{R}^d} \int_{\mathrm{SO}(d)} |F(R\boldsymbol{x})| \,\mathrm{d}\eta(R) \,\mathrm{d}\mu^d(\boldsymbol{x}) \\
&= \int_{\mathrm{SO}(d)} \int_{\mathbb{R}^d} |F(R\boldsymbol{x})| \,\mathrm{d}\mu^d(\boldsymbol{x}) \,\mathrm{d}\eta(R) \\
&= \int_{\mathrm{SO}(d)} \int_{\mathbb{R}^d} |F(\boldsymbol{x})| \,\mathrm{d}\mu^d(\boldsymbol{x}) \,\mathrm{d}\eta(R) \\
&= \int_{\mathbb{R}^d} |F(\boldsymbol{x})| \,\mathrm{d}\mu^d(\boldsymbol{x}).
\end{aligned} \tag{2.14}$$

Hence $\widetilde{F} \in L^1(\mathbb{R}^d, \mu^d)$. An identical computation to (2.14), without the absolute values, yields

$$\int_{\mathbb{R}^d} \widetilde{F}(\boldsymbol{x}) \,\mathrm{d}\mu^d(\boldsymbol{x}) = \int_{\mathbb{R}^d} \int_{\mathrm{SO}(d)} F(R\boldsymbol{x}) \,\mathrm{d}\eta(R) \,\mathrm{d}\mu^d(\boldsymbol{x}) = \int_{\mathbb{R}^d} F(\boldsymbol{x}) \,\mathrm{d}\mu^d(\boldsymbol{x}) \leqslant 0.$$

We then conclude that $\widetilde{F} \in \mathcal{E}(\mu, \delta, d)$ (in short, we can restrict ourselves to real entire functions that are radial on $\mathbb{R}^d$ without loss of generality).

Using Lemma 2.7 we see that $\widetilde{F} = \mathcal{L}_d(G)$ for a certain $G : \mathbb{C} \to \mathbb{C}$ real entire, even, eventually non-negative, and with $\tau(G) = \tau(\widetilde{F}\,) \leqslant \delta$. Moreover, from (2.1) we have

$$\begin{aligned}\int_{\mathbb{R}^d} |\widetilde{F}(\boldsymbol{x})| \, \mathrm{d}\mu^d(\boldsymbol{x}) &= \int_0^\infty \int_{\mathbb{S}^{d-1}} |G(r)| \, \mathrm{d}\sigma_{d-1}(\omega) \, \mathrm{d}\mu(r) \\ &= \frac{\omega_{d-1}}{2} \int_{\mathbb{R}} |G(x)| \, \mathrm{d}\mu(x)\,,\end{aligned} \tag{2.15}$$

where ω_{d-1} is the surface area of $\mathbb{S}^{d-1}$, and hence $G \in L^1(\mathbb{R}, \mu)$. Similarly,

$$\frac{\omega_{d-1}}{2} \int_{\mathbb{R}} G(x) \, \mathrm{d}\mu(x) = \int_0^\infty \int_{\mathbb{S}^{d-1}} G(r) \, \mathrm{d}\sigma_{d-1}(\omega) \, \mathrm{d}\mu(r) = \int_{\mathbb{R}^d} \widetilde{F}(\boldsymbol{x}) \, \mathrm{d}\mu^d(\boldsymbol{x}) \leqslant 0, \tag{2.16}$$

and we conclude that $G \in \mathcal{E}(\mu, \delta)$. Since $\mathrm{r}(G) = \mathrm{r}\left(\widetilde{F}\,\right)$ we arrive at the inequality

$$\Lambda(\mu, \delta) \leqslant \Lambda(\mu, \delta, d). \tag{2.17}$$

Conversely, starting now with $H \in \mathcal{E}(\mu, \delta)$, since the measure μ is even, we can replace it by $\widetilde{H}(z) := \frac{1}{2}\{H(z) + H(-z)\}$ that verifies $\mathrm{r}\left(\widetilde{H}\,\right) \leqslant \mathrm{r}(H)$. We then consider the lift $\mathcal{L}_d(\widetilde{H})$, and note that Lemma 2.7 and computations as in (2.15) and (2.16) show that $\mathcal{L}_d(\widetilde{H}) \in \mathcal{E}(\mu, \delta, d)$. Since $\mathrm{r}\left(\widetilde{H}\right) = \mathrm{r}\left(\mathcal{L}_d(\widetilde{H})\right)$, we arrive at the inequality

$$\Lambda(\mu, \delta, d) \leqslant \Lambda(\mu, \delta). \tag{2.18}$$

The result now follows from (2.17) and (2.18).

2.3 De Branges spaces and the proof of Theorem 2.2

2.3.1 Preliminaries

We start by reviewing some of the basic elements from the theory of de Branges spaces, and collecting a few lemmata from the literature that will be important for our purposes.

2.3.1.1 De Branges spaces

We provide a brief overview, building up on the material introduced in Section 2.1.2. For a detailed exposition we refer the reader to [41, Chapters I and II].

Given a Hermite–Biehler function $E : \mathbb{C} \to \mathbb{C}$, the de Branges space $\mathcal{H}(E)$ associated to E is the space of entire functions $F : \mathbb{C} \to \mathbb{C}$ such that

$$\|F\|^2_{\mathcal{H}(E)} := \int_{\mathbb{R}} |F(x)|^2 \, |E(x)|^{-2} \, \mathrm{d}x < \infty\,, \tag{2.19}$$

and such that F/E and F^*/E have bounded type and non-positive mean type in $\mathbb{C}^+$. This is a Hilbert space with inner product given by

$$\langle F, G\rangle_{\mathcal{H}(E)} := \int_{\mathbb{R}} F(x)\,\overline{G(x)}\,|E(x)|^{-2}\,\mathrm{d}x.$$

In fact, this is a reproducing kernel Hilbert space. That is, for each $w \in \mathbb{C}$ the evaluation functional $F \mapsto F(w)$ is bounded on $\mathcal{H}(E)$ or, equivalently, by the Riesz representation theorem, there exists a function $K(w,\cdot) \in \mathcal{H}(E)$ such that

$$F(w) = \langle F, K(w,\cdot)\rangle_{\mathcal{H}(E)}$$

for all $F \in \mathcal{H}(E)$. Writing $E(z) = A(z) - iB(z)$ with A and B real entire as in (2.4), the reproducing kernel $K(w,\cdot)$ is given by (see [41, Theorem 19])

$$K(w,z) = \frac{B(z)A(\overline{w}) - A(z)B(\overline{w})}{\pi(z-\overline{w})}\,, \tag{2.20}$$

and, when $z = \overline{w}$,

$$K(\overline{z}, z) = \frac{B'(z)A(z) - A'(z)B(z)}{\pi}.$$

We shall only be interested in the situation where $K(w,w) = \|K(w,\cdot)\|^2_{\mathcal{H}(E)} > 0$ for all $w \in \mathbb{C}$, which is equivalent (see [69, Lemma 11]) to the statement that E has no real zeros. A phase function $\varphi : \mathbb{R} \to \mathbb{R}$ associated with E is a continuous function such that $E(x)e^{i\varphi(x)}$ is real for all values of x. In this case,

$$\varphi'(x) = \frac{\pi\,K(x,x)}{|E(x)|^2} > 0. \tag{2.21}$$

If α is a given real number, the solutions of $\varphi(\xi) \equiv \alpha \pmod{\pi}$ correspond to the zeros (which must be real and simple) of the real entire function

$$T_\alpha(z) := \frac{e^{i\alpha}E(z) - e^{-i\alpha}E^*(z)}{2i}. \tag{2.22}$$

Note that the zeros of any two T_α and T_β interlace. With this notation, observe that $T_0 = -B$ and $T_{\pi/2} = A$. Observe also that there is at most one real number α modulo π such that $T_\alpha \in \mathcal{H}(E)$ (otherwise we would have $E \in \mathcal{H}(E)$, which would contradict (2.19)). One of the cornerstones of the theory is the following orthogonality result [41, Theorem 22]: the set of functions $\Gamma_\alpha := \{K(\xi,\cdot)\,; T_\alpha(\xi) = 0\}$ is always an orthogonal set in $\mathcal{H}(E)$; if $T_\alpha \notin \mathcal{H}(E)$, the set Γ_α is an orthogonal basis of $\mathcal{H}(E)$ and, if $T_\alpha \in \mathcal{H}(E)$, the only elements of $\mathcal{H}(E)$ that are orthogonal to Γ_α are the constant multiples of T_α. In particular, if $T_\alpha \notin \mathcal{H}(E)$, for every $F \in \mathcal{H}(E)$ we have

$$F(z) = \sum_{T_\alpha(\xi)=0} \frac{F(\xi)}{K(\xi,\xi)}\,K(\xi,z) \quad \text{and} \quad \|F\|^2_{\mathcal{H}(E)} = \sum_{T_\alpha(\xi)=0} \frac{|F(\xi)|^2}{K(\xi,\xi)}. \tag{2.23}$$

2.3.1.2 Krein's memoir

We recall a result of M. G. Krein [82] that allows us to relate the condition of bounded type in $\mathbb{C}^+$ to the condition of exponential type in $\mathbb{C}$. Recall that $\log^+|x| := \max\{0, \log|x|\}$.

Lemma 2.8 (Krein). *Let $F : \mathbb{C} \to \mathbb{C}$ be an entire function. The following conditions are equivalent:*

(i) *F and F^* have bounded type in the open upper half-plane $\mathbb{C}^+$.*

(ii) *F has exponential type and*

$$\int_{\mathbb{R}} \frac{\log^+|F(x)|}{1+x^2}\,\mathrm{d}x < \infty.$$

If either and therefore both of these conditions hold then $\tau(F) = \max\{v(F), v(F^*)\}$.

Let $\mathcal{B}$ denote the set of entire functions F which satisfy one and therefore both of the conditions (i) or (ii) in Lemma 2.8. The next result can be proved using Lemma 2.8 and is contained in [69, Lemma 12]. It allows us to easily characterize the de Branges space $\mathcal{H}(E)$ when E has bounded type in $\mathbb{C}^+$.

Lemma 2.9. *Let E be a Hermite–Biehler function of bounded type in $\mathbb{C}^+$, and F be an entire function. Then E belongs to $\mathcal{B}$ and the following are equivalent:*

(i) *$F \in \mathcal{H}(E)$.*

(ii) *$F \in \mathcal{B}$, $\max\{v(F), v(F^*)\} \leqslant v(E)$ and (2.19) holds.*

(iii) *F has exponential type, $\tau(F) \leqslant \tau(E)$ and (2.19) holds.*

Moreover, if ξ is real and $0 < K(\xi,\xi)$, the entire function $K(\xi,\cdot) \in \mathcal{H}(E)$ verifies $\tau(K(\xi,\cdot)) = \tau(E)$.

Our final preliminary lemma in this subsection is [21, Lemma 14], a factorization result for real entire functions of exponential type that are non-negative on $\mathbb{R}$. This is an important tool to connect the L^1 and the L^2 theories in our context.

Lemma 2.10. *Let E be a Hermite–Biehler function of bounded type in $\mathbb{C}^+$ with exponential type $\tau(E)$. Let $F : \mathbb{C} \to \mathbb{C}$ be a real entire function of exponential type at most $2\tau(E)$ that satisfies*

$$F(x) \geqslant 0$$

for all $x \in \mathbb{R}$ and

$$\int_{\mathbb{R}} F(x)\,|E(x)|^{-2}\,\mathrm{d}x < \infty.$$

Then there exists $U \in \mathcal{H}(E)$ such that

$$F(z) = U(z)\,U^*(z)$$

for all $z \in \mathbb{C}$.

2.3.1.3 Polynomial reductions

We consider in this subsection a basic auxiliary result that allows us to reduce the search for extremizers in (EP1.1) to functions that have only one sign change. The next proposition is related to [100, Problem 45, Part VI] and we provide here a short proof for the reader's convenience.

Proposition 2.11. *Let* $n \geqslant 1$ *be an integer and let* $0 < t_1 < t_2 < \ldots < t_n$ *be distinct positive real numbers. If* P *is the monic polynomial defined by*

$$P(x) = \prod_{k=1}^{n} (x^2 - t_k^2),$$

then, for any positive number $r > t_n$*, there exist even polynomials* Q *and* R*, both non-negative on* $\mathbb{R}$*, satisfying the following properties:*

(i) $P(x) = (x^2 - r^2)Q(x) + R(x)$ *for all* $x \in \mathbb{R}$.

(ii) $\deg(Q) = \begin{cases} 2n-2, & \text{if } n \text{ is odd;} \\ 2n-4, & \text{if } n \text{ is even;} \end{cases}$ *and* $\deg(R) = \begin{cases} 2n-2, & \text{if } n \text{ is odd;} \\ 2n, & \text{if } n \text{ is even.} \end{cases}$

Proof. We prove this by induction on n. At each step we show how to construct the polynomials $Q = Q_n$ and $R = R_n$ (it should be clear to the reader that these polynomials depend not only on n but on $t_1, t_2, \ldots, t_n$ and r as well). If $n = 1$ and $0 < t_1 < r$, note that

$$P(x) = x^2 - t_1^2 = (x^2 - r^2) \cdot 1 + (r^2 - t_1^2).$$

In this case we can take $Q_1(x) \coloneqq 1$ and $R_1(x) \coloneqq r^2 - t_1^2$, which are polynomials of degree 0.

Now let $n = 2$ and $0 < t_1 < t_2 < r$. Since $P(x) = (x^2 - t_1^2)(x^2 - t_2^2)$, we can apply the previous case to each of the factors as follows

$$\begin{aligned} P(x) &= \left((x^2 - r^2)\, Q_1(x) + R_1(x)\right) \left((x^2 - r^2) \cdot 1 + (r^2 - t_2^2)\right) \\ &= (x^2 - r^2) \left\{R_1(x) + (r^2 - t_2^2)\, Q_1(x)\right\} + \left\{(x^2 - r^2)^2\, Q_1(x) + (r^2 - t_2^2)\, R_1(x)\right\}. \end{aligned}$$

In this case we can take $Q_2(x) \coloneqq R_1(x) + (r^2 - t_2^2)\, Q_1(x)$ and $R_2(x) \coloneqq (x^2 - r^2)^2\, Q_1(x) + (r^2 - t_2^2)\, R_1(x)$. Note that since $\deg(Q_1) = \deg(R_1) = 0$ we have $\deg(Q_2) = 0$ and $\deg(R_2) = 4$.

Assume that the assertion holds for all $n \leqslant N-1$, for some integer $N \geqslant 3$. Consider the case $n = N$, with $0 < t_1 < t_2 < \ldots < t_{N-1} < t_N < r$ and write

$$P(x) = (x^2 - t_N^2) \prod_{k=1}^{N-1} (x^2 - t_k^2). \tag{2.24}$$

By the induction hypothesis, there exist even polynomials $Q_{N-1}(x)$ and $R_{N-1}(x)$, both non-negative on $\mathbb{R}$, such that

$$\prod_{k=1}^{N-1} (x^2 - t_k^2) = (x^2 - r^2)\, Q_{N-1}(x) + R_{N-1}(x). \tag{2.25}$$

On the other hand, the factor $(x^2 - t_N^2)$ falls into the scope of the case $n = 1$; i.e.,

$$x^2 - t_N^2 = (x^2 - r^2) \cdot 1 + (r^2 - t_N^2). \tag{2.26}$$

Combining (2.24), (2.25) and (2.26) we get

$$\begin{aligned} P(x) =& \left((x^2 - r^2)\, Q_{N-1}(x) + R_{N-1}(x)\right) \left((x^2 - r^2) \cdot 1 + (r^2 - t_N^2)\right) \\ =& (x^2 - r^2) \left\{ R_{N-1}(x) + (r^2 - t_N^2)\, Q_{N-1}(x) \right\} \\ & \qquad + \left\{ (x^2 - r^2)^2\, Q_{N-1}(x) + (r^2 - t_N^2)\, R_{N-1}(x) \right\} \\ =& (x^2 - r^2)\, Q_N(x) + R_N(x), \end{aligned}$$

where $Q_N(x) := R_{N-1}(x) + (r^2 - t_N^2)\, Q_{N-1}(x)$ and $R_N(x) := (x^2 - r^2)^2\, Q_{N-1}(x) + (r^2 - t_N^2)\, R_{N-1}(x)$. Note that Q_N and R_N are both even and non-negative on $\mathbb{R}$. Also, by the induction hypothesis,

$$\begin{aligned} \deg(Q_N) = \max\{\deg(Q_{N-1}), \deg(R_{N-1})\} &= \deg(R_{N-1}) \\ &= \begin{cases} 2N - 4, & \text{if } N \text{ is even;} \\ 2N - 2, & \text{if } N \text{ is odd;} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \deg(R_N) = \max\{4 + \deg(Q_{N-1}), \deg(R_{N-1})\} &= 4 + \deg(Q_{N-1}) \\ &= \begin{cases} 2N, & \text{if } N \text{ is even;} \\ 2N - 2, & \text{if } N \text{ is odd.} \end{cases} \end{aligned}$$

This concludes the proof. $\square$

With the aid of Proposition 2.11 we establish the following reduction in the search for extremizers in (EP1.1).

Proposition 2.12. *Let $F \in \mathcal{E}(\mu, \delta)$ be an even function and set $r := \mathrm{r}(F)$. Then there exists $G \in \mathcal{E}(\mu, \delta)$ of the form $G(z) = (z^2 - r^2)H(z)$, where H is an even and real* *entire function of exponential type at most δ that is non-negative on $\mathbb{R}$. In particular, note that* $\mathrm{r}(G) = \mathrm{r}(F)$.

Proof. Since F is even and belongs to $\mathcal{E}(\mu, \delta)$, it has only finitely many sign changes on the positive axis. If it has only one sign change, we are done. If not, we label these as $0 < t_1 < t_2 < \ldots < t_n < r$ and write

$$F(x) = \left(\prod_{k=1}^{n} (x^2 - t_k^2) \right) (x^2 - r^2)\, F_1(x)\,, \tag{2.27}$$

where F_1 is an even and real entire function of exponential type at most δ that is non-negative on $\mathbb{R}$. We apply Proposition 2.11 to the product from $k = 1$ to n in (2.27) and observe that

$$\begin{aligned} F(x) &= \Big((x^2 - r^2) Q(x) + R(x) \Big) (x^2 - r^2)\, F_1(x) \\ &= (x^2 - r^2)^2\, Q(x) F_1(x) + (x^2 - r^2) R(x) F_1(x) \\ &\geqslant (x^2 - r^2) R(x) F_1(x) \end{aligned} \tag{2.28}$$

for all $x \in \mathbb{R}$. Let $G(z) := (z^2 - r^2) R(z) F_1(z)$. By Proposition 2.11 we have $\deg(R) \leqslant 2n$ and hence

$$|R(x)| \ll \left| \prod_{k=1}^{n} (x^2 - t_k^2) \right| \tag{2.29}$$

for $|x|$ large. Since $F \in L^1(\mathbb{R}, \mu)$, from (2.29) and the fact that μ is locally finite we conclude that $G \in L^1(\mathbb{R}, \mu)$. Moreover, from (2.28) we plainly have

$$0 \geqslant \int_{\mathbb{R}} F(x)\, \mathrm{d}\mu(x) \geqslant \int_{\mathbb{R}} G(x)\, \mathrm{d}\mu(x).$$

Hence, $G \in \mathcal{E}(\mu, \delta)$, as desired. □

2.3.2 Proof of Theorem 2.2

We are now in position to present our first proof of Theorem 2.2. Assume now that we are considering a measure μ and a Hermite–Biehler function E that verify the conditions (C1), (C2), (C3) and (C4).

2.3.2.1 Reduction to an L^2-extremal problem

As already observed in the proof of Theorem 2.1, if we start with $F \in \mathcal{E}(\mu, 2\tau(E))$, we can consider $\widetilde{F}(z) := \frac{1}{2}\{F(z) + F(-z)\}$ that verifies $\widetilde{F} \in \mathcal{E}(\mu, 2\tau(E))$ (since μ is even) and $\mathrm{r}\left(\widetilde{F}\right) \leqslant \mathrm{r}(F)$. Hence, for the purposes of finding the sharp constant, we can restrict our search to even functions. We have seen in Proposition 2.12 that we can further restrict our search to functions $G \in \mathcal{E}(\mu, 2\tau(E))$ of the form $G(z) = (z^2 - r^2)H(z)$, with H even, real entire of exponential type at most $2\tau(E)$, and non-negative on $\mathbb{R}$.

Since $x^2H(x) \in L^1(\mathbb{R},\mu)$ (recall that μ is locally finite), condition (C4) yields

$$\begin{aligned} 0 \geqslant \int_{\mathbb{R}} G(x)\,\mathrm{d}\mu(x) &= \left(\int_{\mathbb{R}} x^2H(x)\,\mathrm{d}\mu(x)\right) - r^2\left(\int_{\mathbb{R}} H(x)\,\mathrm{d}\mu(x)\right) \\ &= \left(\int_{\mathbb{R}} x^2H(x)\,|E(x)|^{-2}\,\mathrm{d}x\right) - r^2\left(\int_{\mathbb{R}} H(x)\,|E(x)|^{-2}\,\mathrm{d}x\right) \\ &= \int_{\mathbb{R}} G(x)\,|E(x)|^{-2}\,\mathrm{d}x. \end{aligned} \tag{2.30}$$

Since $H \in L^1(\mathbb{R},|E(x)|^{-2}\,\mathrm{d}x)$ (from condition (C4)), we may use Lemma 2.10 to decompose $H = UU^*$, with $U \in \mathcal{H}(E)$. Hence, as in (2.30), the fact that $\int_{\mathbb{R}} G(x)\,\mathrm{d}\mu(x) \leqslant 0$ is equivalent to the fact that

$$r^2 \geqslant \frac{\int_{\mathbb{R}} x^2\,|U(x)|^2\,|E(x)|^{-2}\,\mathrm{d}x}{\int_{\mathbb{R}} |U(x)|^2\,|E(x)|^{-2}\,\mathrm{d}x} = \frac{\|z\,U\|^2_{\mathcal{H}(E)}}{\|U\|^2_{\mathcal{H}(E)}}.$$

As our task is to minimize r, we have reduced (EP1.1) to the following extremal problem related to multiplication by z in a general de Branges space.

Extremal problem 2.2 **(EP2.2)**. Let E be a Hermite–Biehler function. Determine

$$\mathbb{M}_z(E) := \inf_{0\not\equiv U\in\mathcal{H}(E)} \frac{\|z\,U\|_{\mathcal{H}(E)}}{\|U\|_{\mathcal{H}(E)}}.$$

It turns out that this problem has recently been addressed in the work of Carneiro, Chirre, and Milinovich [31, Theorem 14], for E verifying conditions (C2) and (C3) (note that condition (C1) was already used above when we invoked Lemma 2.10). We discuss its solution in the next subsection.

2.3.2.2 Solution to the problem (EP2.2)

We recall here the result in [31, Theorem 14] and briefly reproduce its proof for the reader's convenience. Extensions of this result, computing the analogous infimum related to multiplication by $(z-a)^k$, for some $a \in \mathbb{R}$, and $k \in \mathbb{N}$, was obtained in [33, 102].

Proposition 2.13 (cf. [31])**.** *Let E be a Hermite–Biehler function verifying conditions* (C2) *and* (C3)*. Let $A := \frac{1}{2}(E+E^*)$ and let ξ_1 be the smallest positive real zero of A. Then*

$$\mathbb{M}_z(E) = \xi_1.$$

The unique extremizer (up to multiplication by a non-zero complex constant) is

$$U(z) = \frac{A(z)}{z^2-\xi_1^2}. \tag{2.31}$$

Remark. If A has no zeros then, from the discussion in Section 2.3.1.1, the space $\mathcal{H}(E)$ has dimension at most 1 and there are no functions $U \in \mathcal{H}(E)$ such that $zU \in \mathcal{H}(E)$.

Proof. Let $\mathcal{X}(E) := \{U \in \mathcal{H}(E) \,:\, zU \in \mathcal{H}(E)\}$, and recall that $B = \frac{i}{2}(E - E^*)$. The proof considers two cases: when $A \notin \mathcal{H}(E)$ (a typical situation), and when $A \in \mathcal{H}(E)$ (an exceptional situation).

We first consider the case $A \notin \mathcal{H}(E)$. In this case, if $0 \neq U \in \mathcal{X}(E)$, from (2.23) we have

$$\|U\|^2_{\mathcal{H}(E)} = \sum_{A(\xi)=0} \frac{|U(\xi)|^2}{K(\xi,\xi)} \leqslant \frac{1}{\xi_1^2} \sum_{A(\xi)=0} \frac{|\xi|^2\,|U(\xi)|^2}{K(\xi,\xi)} = \frac{1}{\xi_1^2}\,\|zU\|^2_{\mathcal{H}(E)}. \tag{2.32}$$

This shows that $\mathbb{M}_z(E) \geqslant \xi_1$. In order to have equality in (2.32) one must have $U(\xi) = 0$ for each zero ξ of A with $\xi \neq \pm\xi_1$. By the interpolation formula in (2.23) (recall that $K(\xi_1,\xi_1) = K(-\xi_1,-\xi_1)$ in our setup) we get

$$U(z) = \frac{1}{K(\xi_1,\xi_1)}\big(U(\xi_1)\,K(\xi_1,z) + U(-\xi_1)\,K(-\xi_1,z)\big). \tag{2.33}$$

From (2.33) and (2.20) one can write

$$\begin{aligned} zU(z) = \frac{A(z)B(\xi_1)}{\pi K(\xi_1,\xi_1)}\big(U(-\xi_1) - U(\xi_1)\big) &+ \frac{\xi_1\,U(\xi_1)}{K(\xi_1,\xi_1)}K(\xi_1,z) \\ &+ \frac{(-\xi_1)\,U(-\xi_1)}{K(\xi_1,\xi_1)}K(-\xi_1,z). \end{aligned} \tag{2.34}$$

From (2.34), one now observes that $zU \in \mathcal{H}(E)$ if and only if $U(\xi_1) = U(-\xi_1)$. This leads to the conclusion that $\mathbb{M}_z(E) = \xi_1$, with extremizers as in (2.31).

We now consider the exceptional case $A \in \mathcal{H}(E)$. By [41, Theorem 29] a function $G \in \mathcal{H}(E)$ is orthogonal to the subspace $\mathcal{X}(E)$ if and only if it is of the form $G(z) = \alpha A(z) + \beta B(z)$ for constants $\alpha, \beta \in \mathbb{C}$. If there is such a function G with $\beta \neq 0$, then $B \in \mathcal{H}(E)$, and hence $E \in \mathcal{H}(E)$, contradicting (2.19). Therefore $\mathcal{X}(E)^\perp = \operatorname{span}\{A\}$. Recalling the discussion at the end of Section 2.3.1.1, that $T_{\pi/2} = A$ and $\Gamma_{\pi/2} \cup \{A\}$ is an orthogonal basis of $\mathcal{H}(E)$, if $0 \neq U \in \mathcal{X}(E)$ we have

$$\|U\|^2_{\mathcal{H}(E)} = \sum_{A(\xi)=0} \frac{|U(\xi)|^2}{K(\xi,\xi)} \leqslant \frac{1}{\xi_1^2} \sum_{A(\xi)=0} \frac{|\xi|^2\,|U(\xi)|^2}{K(\xi,\xi)} \leqslant \frac{1}{\xi_1^2}\,\|zU\|^2_{\mathcal{H}(E)}\,, \tag{2.35}$$

which shows that $\mathbb{M}_z(E) \geqslant \xi_1$. Equality in (2.35) occurs if and only if $U(\xi) = 0$ for each zero ξ of A with $\xi \neq \pm\xi_1$, and $zU \perp A$. As we are assuming that $U \in \mathcal{X}(E)$, we have that $U \perp A$, which implies that U has a representation as in (2.33). From (2.34) one sees that $zU \perp A$ if and only if $U(\xi_1) = U(-\xi_1)$, which leads us to the conclusion that $\mathbb{M}_z(E) = \xi_1$, with extremizers as in (2.31). $\square$

In light of the discussion in Section 2.3.2.1 and Proposition 2.13, we arrive at the conclusion that

$$\Lambda(\mu, 2\tau(E)) = \mathbb{M}_z(E) = \xi_1,$$

as desired.

2.3.2.3 Classification of the extremizers

The discussion in Section 2.3.2.1 and Section 2.3.2.2 also leads us to the following conclusion: if $G \in \mathcal{E}(\mu, 2\tau(E))$ is an even extremizer that has only one sign change on the positive real axis at ξ_1, then G must be of the form (2.7).

Observe that any even function $F \in \mathcal{E}(\mu, 2\tau(E))$ verifies

$$\int_{\mathbb{R}} F(x)\, \mathrm{d}\mu(x) = \int_{\mathbb{R}} F(x)\, |E(x)|^{-2}\, \mathrm{d}x. \tag{2.36}$$

This follows from condition (C4), by writing F as a difference of two functions that are non-negative on $\mathbb{R}$ and belong to $L^1(\mathbb{R}, \mu)$ (looking at the factorization (2.27), this plainly follows by writing an even polynomial as a difference of two polynomials that are non-negative on $\mathbb{R}$). Now assume that $F \in \mathcal{E}(\mu, 2\tau(E))$ is an *even extremizer that has more than one sign change on the positive real axis* (i.e. $n \geqslant 1$ in the factorization (2.27)). One then notices that the inequality $F(x) \geqslant G(x)$ in (2.28) is strict for a.e. $x \in \mathbb{R}$. In light of (2.36), we are led to the fact that

$$\begin{aligned} 0 \geqslant \int_{\mathbb{R}} F(x)\, \mathrm{d}\mu(x) &= \int_{\mathbb{R}} F(x)\, |E(x)|^{-2}\, \mathrm{d}x \\ &> \int_{\mathbb{R}} G(x)\, |E(x)|^{-2}\, \mathrm{d}x = \int_{\mathbb{R}} G(x)\, \mathrm{d}\mu(x). \end{aligned} \tag{2.37}$$

Recall that G has the form $G(z) = (z^2 - \xi_1^2)H(z)$, with H non-negative on $\mathbb{R}$. In the situation (2.37), the function $G_\varepsilon(z) = (z^2 - (\xi_1 - \varepsilon)^2)H(z)$ would belong to $\mathcal{E}(\mu, 2\tau(E))$ for small ε, contradicting the minimality of ξ_1. We conclude that the unique even extremizer is indeed the one given by (2.7).

Finally, we need to show that all extremizers must be even. If $F \in \mathcal{E}(\mu, 2\tau(E))$ is a generic extremizer, we have seen that $\widetilde{F}(z) \coloneqq \frac{1}{2}\{F(z) + F(-z)\}$ is an even extremizer, and hence given by (2.7). So, we want to show that if

$$\frac{1}{2}\{F(z) + F(-z)\} = \frac{A(z)^2}{z^2 - \xi_1^2}\,, \tag{2.38}$$

then $F(z) = \frac{A(z)^2}{z^2 - \xi_1^2}$. Since F is an extremizer, $F(x) \geqslant 0$ for $|x| \geqslant \xi_1$. Then, by (2.38), for each zero ξ of A we must have $F(\xi) = 0$ (with even multiplicity if $|\xi| > \xi_1$). This tells us that F factors as

$$F(z) = \frac{A(z)^2}{z^2 - \xi_1^2}\, R(z)\,, \tag{2.39}$$

with R real entire and non-negative on $\{x \in \mathbb{R} \,:\, |x| \geqslant \xi_1\}$. Then (2.38) and (2.39) imply that

$$\frac{1}{2}\{R(z) + R(-z)\} = 1,$$

and we find that R is bounded on $\mathbb{R}$. Recall that

$$K(\xi_1, z) = -\frac{A(z)B(\xi_1)}{\pi(z-\xi_1)} \quad \text{and} \quad K(-\xi_1, z) = \frac{A(z)B(\xi_1)}{\pi(z+\xi_1)},$$

and by Lemma 2.9 we have $\tau(K(\pm\xi_1,\cdot)) = v(K(\pm\xi_1,\cdot)) = \tau(E)$. Hence

$$v\left(\frac{A(z)^2}{z^2-\xi_1^2}\right) = v\left(\frac{A(z)}{z-\xi_1}\right) + v\left(\frac{A(z)}{z+\xi_1}\right) = 2\tau(E). \tag{2.40}$$

Since F is real entire, by Lemma 2.9, $v(F) = \tau(F) \leqslant 2\tau(E)$. Since $R \in \mathcal{B}$ as in Lemma 2.8, from (2.39), (2.40), and the additive property of the mean type, we arrive at

$$2\tau(E) \geqslant v(F) = v\left(\frac{A(z)^2}{z^2-\xi_1^2}\right) + v(R) = 2\tau(E) + v(R)\,,$$

which leads us to the conclusion that $\tau(R) = v(R) = 0$. Now, an application of the Phragmén–Lindelöf principle [41, Theorem 1] tells us that R has to be identically 1, as desired.

2.3.3 Alternative proof of Theorem 2.2

This second proof makes use of certain quadrature formulas in our context of de Branges spaces. It is slightly more direct than our first proof (although we are going to invoke some elements from the proof of Proposition 2.13). On the other hand, the polynomial reduction used in the first proof has the advantage that it can also be used for general measures μ. Here we again consider a measure μ and a Hermite–Biehler function E that verify the conditions (C1), (C2), (C3) and (C4). We also assume that A has at least one zero.

2.3.3.1 Quadrature formulas

First observe that there exists a real entire function M of exponential type at most $2\tau(E)$ that is strictly positive on $\mathbb{R}$ and verifies $M \in L^1(\mathbb{R}, |E(x)|^{-2}\mathrm{d}x)$ (which, by (C4), is equivalent to $M \in L^1(\mathbb{R}, \mu)$). Take, for instance,

$$M(z) = K(\xi_1, z)^2 + K(0, z)^2 = \frac{B(\xi_1)^2 A(z)^2}{\pi^2(z-\xi_1)^2} + \frac{A(0)^2 B(z)^2}{\pi^2 z^2}.$$

Now, for each $F \in \mathcal{E}(\mu, 2\tau(E))$, there is a positive constant $c = c_F$ such that $F + cM$ is non-negative on $\mathbb{R}$. Then $F + cM \in L^1(\mathbb{R}, \mu)$ and, by (C4), $F + cM \in$

$L^1(\mathbb{R}, |E(x)|^{-2}\mathrm{d}x)$. In particular, $F \in L^1(\mathbb{R}, |E(x)|^{-2}\mathrm{d}x)$. Using (C4) again, we have

$$\begin{aligned}\int_{\mathbb{R}} F(x)\,\mathrm{d}\mu(x) &= \int_{\mathbb{R}} (F(x) + cM(x))\,\mathrm{d}\mu(x) - \int_{\mathbb{R}} cM(x)\,\mathrm{d}\mu(x)\\ &= \int_{\mathbb{R}} (F(x) + cM(x))\,|E(x)|^{-2}\,\mathrm{d}x - \int_{\mathbb{R}} cM(x)\,|E(x)|^{-2}\,\mathrm{d}x \qquad (2.41)\\ &= \int_{\mathbb{R}} F(x)\,|E(x)|^{-2}\,\mathrm{d}x.\end{aligned}$$

Using Lemma 2.10, we can write $F + cM = UU^*$ and $cM = VV^*$ with $U, V \in \mathcal{H}(E)$, thus arriving at the representation

$$F = UU^* - VV^*. \qquad (2.42)$$

For $\alpha \in \mathbb{R}$, let T_α be as in (2.22). If $T_\alpha \notin \mathcal{H}(E)$, from (2.42) and (2.23) we get

$$\begin{aligned}\int_{\mathbb{R}} F(x)\,|E(x)|^{-2}\,\mathrm{d}x &= \int_{\mathbb{R}} |U(x)|^2\,|E(x)|^{-2}\,\mathrm{d}x - \int_{\mathbb{R}} |V(x)|^2\,|E(x)|^{-2}\,\mathrm{d}x\\ &= \sum_{T_\alpha(\xi)=0} \frac{|U(\xi)|^2}{K(\xi,\xi)} - \sum_{T_\alpha(\xi)=0} \frac{|V(\xi)|^2}{K(\xi,\xi)} \qquad (2.43)\\ &= \sum_{T_\alpha(\xi)=0} \frac{F(\xi)}{K(\xi,\xi)}.\end{aligned}$$

Hence, from (2.41) and (2.43), we arrive at the quadrature formula

$$\int_{\mathbb{R}} F(x)\,\mathrm{d}\mu(x) = \sum_{T_\alpha(\xi)=0} \frac{F(\xi)}{K(\xi,\xi)}. \qquad (2.44)$$

2.3.3.2 Finding the sharp constant

Assume that there exists $F \in \mathcal{E}(\mu, 2\tau(E))$ such that $\mathrm{r}(F) < \xi_1$. Then there is an interval $(a, b) \subset (\mathrm{r}(F), \xi_1)$ in which F is strictly positive. As discussed in Section 2.3.1.1, choose α such that $T_\alpha \notin \mathcal{H}(E)$ and T_α has a zero $\zeta \in (a, b)$. Note that such a T_α will have no other zero in the interval $(-\xi_1, \xi_1)$ by the interlacing property of the zeros. This plainly contradicts (2.44), since the left-hand side is non-positive by hypothesis, while the right-hand side is strictly positive. Hence, $\Lambda(\mu, 2\tau(E)) \geqslant \xi_1$.

In order to see that the function F given by (2.7) verifies $\int_{\mathbb{R}} F(x)\,\mathrm{d}\mu(x) = 0$, we can argue as in the proof of Proposition 2.13. In the generic case when $A = T_{\pi/2} \notin \mathcal{H}(E)$ we can, alternatively, use (2.44) directly. This shows that $\Lambda(\mu, 2\tau(E)) = \xi_1$.

2.3.3.3 Classification of the extremizers

Let $F \in \mathcal{E}(\mu, 2\tau(E))$ be an extremizer for our problem (EP1.1). Arguing as in Section 2.3.3.2, we conclude that F cannot take strictly positive values in the interval $(-\xi_1, \xi_1)$. In particular, since F is continuous, we must have $F(\pm\xi_1) = 0$. This means that F can be factored as $F(z) = (z^2 - \xi_1^2)H(z)$, with H real entire, of exponential type at most $2\tau(E)$, and non-negative on $\mathbb{R}$. From condition (C4), we may use Lemma 2.10 to decompose $H = UU^*$, with $U \in \mathcal{H}(E)$. Then, arguing as in the proof of Proposition 2.13 we arrive at the conclusion that F must be given by (2.7).

2.4 Absence of sign uncertainty: proof of Theorem 2.3

2.4.1 Density of polynomials

The following result is classical in approximation theory; see, e.g., [10, Theorem 6] or [53, Chapter II, Theorems 4.2 and 5.2]. We include a brief proof for the convenience of the reader.

Lemma 2.14. *Let μ be a locally finite, even and non-negative Borel measure on $\mathbb{R}$ such that*
$$\int_{\mathbb{R}} e^{\varepsilon|x|}\,\mathrm{d}\mu(x) < \infty \tag{2.45}$$
for some $\varepsilon > 0$. Then the space of polynomials is dense in $L^2(\mathbb{R}, (1+x^2)\,\mathrm{d}\mu(x))$.

Proof. Let $f \in L^2(\mathbb{R}, (1+x^2)\,\mathrm{d}\mu(x))$ be such that
$$\int_{\mathbb{R}} f(x)\, x^k (1+x^2)\,\mathrm{d}\mu(x) = 0 \tag{2.46}$$
for each $k = 0, 1, 2, \ldots$. From (2.45) and an application of the Cauchy–Schwarz inequality we have
$$\int_{\mathbb{R}} |f(x)(1+x^2)|\,\mathrm{d}\mu(x) \leqslant \left(\int_{\mathbb{R}} (1+x^2)\,\mathrm{d}\mu(x)\right)^{\frac12} \left(\int_{\mathbb{R}} |f(x)|^2(1+x^2)\,\mathrm{d}\mu(x)\right)^{\frac12} < \infty.$$
Hence $f(x)(1+x^2) \in L^1(\mathbb{R}, \mu)$. For $\varepsilon > 0$ as in (2.45), let Ω_ε be the region in the complex plane given by $\Omega_\varepsilon := \{z \in \mathbb{C} : |\operatorname{Im}(z)| < \frac{\varepsilon}{8\pi}\}$, and define
$$F(z) := \int_{\mathbb{R}} e^{-2\pi i z x} f(x)(1+x^2)\,\mathrm{d}\mu(x), \quad \text{for} \ \ z \in \Omega_\varepsilon.$$
Note that for each $z \in \Omega_\varepsilon$ we have
$$\begin{aligned} |F(z)| &\leqslant \int_{\mathbb{R}} \left|e^{-2\pi i z x}\right| |f(x)|\,(1+x^2)\,\mathrm{d}\mu(x) \leqslant \int_{\mathbb{R}} e^{\frac{\varepsilon}{4}|x|}|f(x)|\,(1+x^2)\,\mathrm{d}\mu(x) \\ &\ll \int_{\mathbb{R}} e^{\frac{\varepsilon}{2}|x|}|f(x)|\sqrt{1+x^2}\,\mathrm{d}\mu(x) \\ &\leqslant \left(\int_{\mathbb{R}} e^{\varepsilon|x|}\,\mathrm{d}\mu(x)\right)^{\frac12} \left(\int_{\mathbb{R}} |f(x)|^2(1+x^2)\,\mathrm{d}\mu(x)\right)^{\frac12} < \infty. \end{aligned} \tag{2.47}$$
Hence, F defines an analytic function in Ω_ε by Morera's theorem.

Using (2.47) (for the absolute convergence), Fubini's theorem, and (2.46), we have, for each $z \in \Omega_\varepsilon$ with $|z| < \frac{\varepsilon}{8\pi}$,

$$\begin{aligned} F(z) &= \int_{\mathbb{R}} \left(\sum_{k=0}^{\infty} \frac{(-2\pi i z x)^k}{k!} \right) f(x)(1+x^2)\, \mathrm{d}\mu(x) \\ &= \sum_{k=0}^{\infty} \frac{(-2\pi i z)^k}{k!} \int_{\mathbb{R}} f(x)\, x^k (1+x^2)\, \mathrm{d}\mu(x) = 0. \end{aligned}$$

Since F is analytic on Ω_ε, we conclude that $F = 0$ in this region. Since F is the Fourier transform of the finite Borel measure $f(x)(1+x^2)\,\mathrm{d}\mu(x)$, we conclude that this measure must be identically zero. This in turn implies that $f = 0$ in $L^2(\mathbb{R}, (1+x^2)\,\mathrm{d}\mu(x))$, as desired. $\square$

Remark. The rest of the argument relies on the previous lemma. As we mentioned in Section 2.1.3 the density of polynomials is characterized using the moment problem [103].

2.4.2 Proof of Theorem 2.3

Let μ_1 be the measure given by $\mathrm{d}\mu_1(x) = (1+x^2)\,\mathrm{d}\mu(x)$. Fix $0 < r < 2$ and let $\eta = \eta(r) > 0$ to be chosen later. Using Lemma 2.14, let P be a polynomial such that

$$\|P - \mathbf{1}_{[-r,r]}\|_{L^2(\mathbb{R},\,\mu_1)} \leqslant \eta. \tag{2.48}$$

Note that

$$\begin{aligned} &\int_{\mathbb{R}} (x^2 - r^2)\, |P(x)|^2\, \mathrm{d}\mu(x) \\ &\quad\leqslant \int_{[-r,r]^c} (x^2+1)\, |P(x)|^2\, \mathrm{d}\mu(x) + \int_{-r/2}^{r/2} (x^2 - r^2)\, |P(x)|^2\, \mathrm{d}\mu(x) \\ &\quad\leqslant \|P - \mathbf{1}_{[-r,r]}\|^2_{L^2(\mathbb{R},\,\mu_1)} + \left(\left(\frac{r}{2}\right)^2 - r^2 \right) \int_{-r/2}^{r/2} |P(x)|^2\, \mathrm{d}\mu(x) \\ &\quad\leqslant \eta^2 - \left(\frac{3r^2}{4}\right) \frac{1}{2} \int_{-r/2}^{r/2} (1+x^2)|P(x)|^2\, \mathrm{d}\mu(x). \end{aligned} \tag{2.49}$$

From the triangle inequality and (2.48) we get

$$\begin{aligned} \big| \|P\|_{L^2([-\frac{r}{2},\frac{r}{2}],\,\mu_1)} - \|\mathbf{1}\|_{L^2([-\frac{r}{2},\frac{r}{2}],\,\mu_1)} \big| &\leqslant \|P - \mathbf{1}\|_{L^2([-\frac{r}{2},\frac{r}{2}],\,\mu_1)} \\ &\leqslant \|P - \mathbf{1}_{[-r,r]}\|_{L^2(\mathbb{R},\,\mu_1)} \leqslant \eta, \end{aligned}$$

and hence

$$\|P\|_{L^2([-\frac{r}{2},\frac{r}{2}],\,\mu_1)} \geqslant \|\mathbf{1}\|_{L^2([-\frac{r}{2},\frac{r}{2}],\,\mu_1)} - \eta = \sqrt{\mu_1([-\tfrac{r}{2},\tfrac{r}{2}])} - \eta. \tag{2.50}$$

From (2.49) and (2.50) we obtain

$$\int_{\mathbb{R}} (x^2 - r^2)\, |P(x)|^2 \, \mathrm{d}\mu(x) \leqslant \eta^2 - \frac{3r^2}{8} \left(\sqrt{\mu_1([-\tfrac{r}{2}, \tfrac{r}{2}])} - \eta \right)^2 . \tag{2.51}$$

Choosing η sufficiently small, we can make the right-hand side of (2.51) be negative. Then the function $F(z) = (z^2 - r^2)P(z)P^*(z)$ belongs to the class $\mathcal{E}(\mu, \delta)$ and $\mathrm{r}(F) = r$. Since r was arbitrary, we conclude that $\Lambda(\mu, \delta) = 0$.

2.5 Applications

2.5.1 Preliminaries

The following alternative characterization of de Branges spaces is given by [41, Theorem 23].

Lemma 2.15 (cf. [41])**.** *Let H be a Hilbert space of entire functions that contains a non-zero element and verifies the following hypotheses:*

(i) *If $F \in H$ and has a non-real zero w, the function $z \mapsto F(z)(z - \overline{w})/(z - w)$ also belongs to H and has the same norm of F.*

(ii) *For every non-real number w, the linear functional defined on H by $F \mapsto F(w)$ is continuous.*

(iii) *If $F \in H$, the function F^* also belongs to H and has the same norm of F.*

Then H is equal isometrically to some de Branges space $\mathcal{H}(E)$.

We have seen in Section 2.3.1.1 how to construct the K from E. An observation that is important for applications is the fact that we can revert this process and construct E from K (not necessarily in a unique way). This works as follows: if K is the reproducing kernel of a Hilbert space H verifying the conditions of Lemma 2.15, define the function

$$L(w, z) \coloneqq 2\pi i(\overline{w} - z)K(w, z)\,, \tag{2.52}$$

and the entire function

$$E(z) \coloneqq \frac{L(i, z)}{L(i, i)^{\frac{1}{2}}}. \tag{2.53}$$

One can show that E is a Hermite–Biehler function such that

$$L(w, z) = E(z)E^*(\overline{w}) - E^*(z)E(\overline{w})\,,$$

and that H is equal isometrically to the de Branges space $\mathcal{H}(E)$. See [28, Appendix A] for details.

2.5.2 Proof of Corollary 2.4

For $\Delta > 0$ and W one of the density functions given by (2.9), one can consider the Hilbert space H of entire functions F of exponential type at most Δ such that

$$\|F\|_H := \left(\int_{\mathbb{R}} |F(x)|^2 \, W(x) \, \mathrm{d}x \right)^{1/2} < \infty.$$

The usual Paley–Wiener space $PW(\Delta)$ is the (reproducing kernel) Hilbert space of entire functions F of exponential type at most Δ such that

$$\|F\|_{PW(\Delta)} := \|F\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |F(x)|^2 \, \mathrm{d}x \right)^{1/2} < \infty.$$

One sees that H and $PW(\Delta)$ are equal as sets (since the density function W is asymptotically 1). Moreover, due to classical uncertainty principles for the Fourier transform, one can show that

$$\|F\|_H \asymp \|F\|_{L^2(\mathbb{R})}.$$

See [31, Section 3] for details. We then conclude that H is also a reproducing kernel Hilbert space, and fits into the hypotheses of Lemma 2.15.

It is shown in [31, §4.1] that the reproducing kernel K of H verifies the following properties, for all $w, z \in \mathbb{C}$: (i) $K(w, w) > 0$; (ii) $K(-w, -z) = K(w, z)$; (iii) $K(z, w) = \overline{K(w, z)}$; (iv) $\overline{K(\overline{w}, \overline{z})} = K(w, z)$. It then follows that E defined by (2.52) - (2.53) verifies conditions (C2) and (C3) (if $E(w) = 0$ for $w \in \mathbb{R}$, one would have $K(w, w) = 0$). Since $K(i, z)$ has exponential type, so does E. Also, since $K(i, z)$ belongs to $L^2(\mathbb{R})$, we may use Jensen's inequality to obtain

$$\begin{aligned} \frac{1}{\pi} \int_{\mathbb{R}} \frac{\log^+ |E(x)|}{1 + x^2} \, \mathrm{d}x &\leqslant \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\log \left(1 + |E(x)|\right)^2}{1 + x^2} \, \mathrm{d}x \\ &\leqslant \frac{1}{2} \log \left(1 + \frac{1}{\pi} \int_{\mathbb{R}} \frac{|E(x)|^2}{1 + x^2} \, \mathrm{d}x \right) < \infty. \end{aligned} \tag{2.54}$$

Hence, E verifies the conditions of Lemma 2.8, which implies it has bounded type in $\mathbb{C}^+$ (condition (C1)). From Lemma 2.9 we have $\tau(E) = \tau(K(0, \cdot))$, which is at most Δ. In fact, we must have $\tau(E) = \Delta$, since by Lemma 2.9 the space $H = \mathcal{H}(E)$ admits only functions F with $\tau(F) \leqslant \tau(E)$, and we know from the start that there exist functions in $PW(\Delta)$ with exponential type equal to Δ. Finally, condition (C4) follows from the classical Krein's factorization in $PW(\Delta)$ or Lemma 2.10, depending if we start with the left-hand side or the right-hand side of (2.5) being finite, and the fact that H is isometric to $\mathcal{H}(E)$.

The conclusion is that this situation falls in the framework of Theorem 2.2, as claimed.

2.5.3 Proof of Corollary 2.5

Let $\nu > -1$. A de Branges space $\mathcal{H}(E)$ is said to be homogeneous of order ν if, for all $0 < a < 1$ and all $f \in \mathcal{H}(E)$, the function $z \mapsto a^{\nu+1} f(az)$ belongs to $\mathcal{H}(E)$ and has the same norm as f. Such spaces were characterized by L. de Branges in [40] (see also [41, Section 50]).

Let A_ν be the even real entire function defined in (2.12). We consider an odd real entire function B_ν given by

$$B_\nu(z) = \Gamma(\nu+1)\left(\tfrac{1}{2}z\right)^{-\nu} J_{\nu+1}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{1}{2}z\right)^{2n+1}}{n!(\nu+1)(\nu+2)\ldots(\nu+n+1)}.$$

Then the function

$$E_\nu(z) \coloneqq A_\nu(z) - iB_\nu(z)$$

turns out to be a Hermite–Biehler function of bounded type in $\mathbb{C}^+$, with no real zeros, and $\mathcal{H}(E_\nu)$ is homogeneous of order ν. Moreover we have $\tau(A_\nu) = \tau(B_\nu) = \tau(E_\nu) = 1$. Note also that $A_\nu, B_\nu \notin \mathcal{H}(E_\nu)$ (for this one can look at the behaviour of the Bessel function J_ν at infinity). We refer the reader to [41, Section 50], [69, Section 5] or [21, Sections 3 and 4] for details. Some properties of the spaces $\mathcal{H}(E_\nu)$ that are relevant for our purposes are collected in the next lemma, which is contained in [69, Eqs. (5.1), (5.2) and Lemma 16].

Lemma 2.16. *Let $\nu > -1$. The following properties hold:*

(i) *There exist positive constants a_ν, b_ν such that*

$$a_\nu |x|^{2\nu+1} \leqslant |E_\nu(x)|^{-2} \leqslant b_\nu |x|^{2\nu+1}$$

for all $x \in \mathbb{R}$ with $|x| \geqslant 1$.

(ii) *For $F \in \mathcal{H}(E_\nu)$ we have the identity*

$$\int_{\mathbb{R}} |F(x)|^2\, |E_\nu(x)|^{-2}\, \mathrm{d}x = c_\nu \int_{\mathbb{R}} |F(x)|^2\, |x|^{2\nu+1}\, \mathrm{d}x\,,$$

with $c_\nu = \pi\, 2^{-2\nu-1}\,\Gamma(\nu+1)^{-2}$.

(iii) *An entire function F belongs to $\mathcal{H}(E_\nu)$ if and only if F has exponential type at most 1 and*

$$\int_{\mathbb{R}} |F(x)|^2\, |x|^{2\nu+1}\, \mathrm{d}x < \infty.$$

This leads us to the conclusion that E_ν verifies conditions (C1), (C2), (C3) and, with the aid of Lemma 2.16 and Lemma 2.10, one also verifies (C4). We then fall in the framework of Theorem 2.2, and the corollary follows (note that one can go from exponential type 1 to an arbitrary exponential type $\delta > 0$ with a dilation argument).

2.5.4 Proof of Corollary 2.6

We proceed with the proof for $\delta = 1$. For a generic exponential type $\delta > 0$, one simply needs to consider the function $z \mapsto E_\nu(\delta z)$ in the proof below.

Let H be the space of entire functions F of exponential type at most 1 such that

$$\|F\|_H \coloneqq \left(\int_{\mathbb{R}} |F(x)|^2 \, \mathrm{d}\mu(x)\right)^{1/2} < \infty.$$

From Lemma 2.16 and the hypotheses of Corollary 2.6, one sees that H and $\mathcal{H}(E_\nu)$ are equal as sets. The next piece of information required for this proof is the following result.

Proposition 2.17. *Under the hypotheses of* Corollary 2.6 *we have*

$$\|F\|_H \asymp \|F\|_{\mathcal{H}(E_\nu)}. \tag{2.55}$$

Proof. Let $K_\nu(w, z)$ be the reproducing kernel of the space $\mathcal{H}(E_\nu)$. If $F \in H = \mathcal{H}(E_\nu)$ one has the following pointwise bound, as a consequence of the reproducing kernel property and the Cauchy–Schwarz inequality,

$$\begin{aligned} |F(x)|^2 =& \left|\langle F, K_\nu(x, \cdot)\rangle_{\mathcal{H}(E_\nu)}\right|^2 \\ &\leqslant \|F\|^2_{\mathcal{H}(E_\nu)} \|K_\nu(x, \cdot)\|^2_{\mathcal{H}(E_\nu)} = \|F\|^2_{\mathcal{H}(E_\nu)} K_\nu(x, x). \end{aligned} \tag{2.56}$$

Then, from (2.56), the hypotheses of Corollary 2.6, and Lemma 2.16,

$$\begin{aligned} \int_{\mathbb{R}} |F(x)|^2 \, \mathrm{d}\mu(x) &= \int_{|x|\leqslant R} |F(x)|^2 \, \mathrm{d}\mu(x) + \int_{|x|>R} |F(x)|^2 \, \mathrm{d}\mu(x) \\ &\ll \|F\|^2_{\mathcal{H}(E_\nu)} \int_{|x|\leqslant R} K_\nu(x, x) \, \mathrm{d}\mu(x) + \int_{|x|>R} |F(x)|^2 \, |x|^{2\nu+1} \, \mathrm{d}x \\ &\ll \|F\|^2_{\mathcal{H}(E_\nu)}. \end{aligned}$$

This accounts for one of the inequalities needed to prove (2.55).

For the other inequality we argue as follows. If $\nu \geqslant -1/2$, we have that $F \in H = \mathcal{H}(E_\nu)$ implies $F \in L^2(\mathbb{R})$ and, from the Paley–Wiener theorem, $\mathrm{supp}(\widehat{F}) \subset [-\frac{1}{2\pi}, \frac{1}{2\pi}]$. Then, from the classical Amrein–Berthier–Nazarov uncertainty principle for the Fourier transform [4, 97] (see, alternatively, [33, Lemma 9]),

$$\int_{|x|\leqslant R} |F(x)|^2 \, \mathrm{d}x \ll \int_{|x|>R} |F(x)|^2 \, \mathrm{d}x. \tag{2.57}$$

From (2.57) we get

$$\begin{aligned} \int_{\mathbb{R}} |F(x)|^2 \, |x|^{2\nu+1} \, \mathrm{d}x &\leqslant R^{2\nu+1} \int_{|x|\leqslant R} |F(x)|^2 \, \mathrm{d}x + \int_{|x|>R} |F(x)|^2 \, |x|^{2\nu+1} \, \mathrm{d}x \\ &\ll R^{2\nu+1} \int_{|x|>R} |F(x)|^2 \, \mathrm{d}x + \int_{|x|>R} |F(x)|^2 \, |x|^{2\nu+1} \, \mathrm{d}x \end{aligned}$$

$$\begin{aligned}&\ll \int_{|x|>R} |F(x)|^2\, |x|^{2\nu+1}\, \mathrm{d}x\\ &\ll \int_{\mathbb{R}} |F(x)|^2\, \mathrm{d}\mu(x),\end{aligned}$$

which is the desired inequality due to Lemma 2.16 (ii).

Now we consider the case $-1 < \nu < -1/2$. For $0 < \varepsilon < 1$, using (2.56) we get

$$\begin{aligned}\int_{|x|<\varepsilon} |F(x)|^2\, |x|^{2\nu+1}\, \mathrm{d}x &\leqslant \|F\|^2_{\mathcal{H}(E_\nu)} \int_{|x|<\varepsilon} K_\nu(x,x)\, |x|^{2\nu+1}\, \mathrm{d}x\\ &\leqslant \left(\max_{|x|\leqslant 1} K_\nu(x,x) \int_{|x|<\varepsilon} |x|^{2\nu+1}\, \mathrm{d}x\right) \|F\|^2_{\mathcal{H}(E_\nu)}.\end{aligned} \tag{2.58}$$

Hence, from Lemma 2.16 (ii) and (2.58), we can choose $\varepsilon > 0$ sufficiently small, independently of F, such that

$$\int_{|x|<\varepsilon} |F(x)|^2\, |x|^{2\nu+1}\, \mathrm{d}x \leqslant \int_{|x|\geqslant\varepsilon} |F(x)|^2\, |x|^{2\nu+1}\, \mathrm{d}x. \tag{2.59}$$

Now consider the entire function E_α, with $\alpha = -\nu - 1$, that has exponential type at most 1. Note from Lemma 2.16 (i) that

$$|E_\alpha(x)|^2 \asymp |x|^{2\nu+1}$$

for $|x| \geqslant \varepsilon$ (we extended the range from $|x| \geqslant 1$ to $|x| \geqslant \varepsilon$ simply by noting that E_α has no real zeros). We then note that $F \in H = \mathcal{H}(E_\nu)$ implies that $F\, E_\alpha \in L^2(\mathbb{R})$ and has exponential type at most 2. An application of the Amrein–Berthier–Nazarov uncertainty principle [4, 97] yields

$$\begin{aligned}\int_{\varepsilon\leqslant|x|\leqslant R} |F(x)|^2\, |x|^{2\nu+1}\, \mathrm{d}x &\ll \int_{\varepsilon\leqslant|x|\leqslant R} |F(x)E_\alpha(x)|^2\, \mathrm{d}x\\ &\ll \int_{|x|>R} |F(x)E_\alpha(x)|^2\, \mathrm{d}x\\ &\ll \int_{|x|>R} |F(x)|^2\, |x|^{2\nu+1}\, \mathrm{d}x.\end{aligned} \tag{2.60}$$

Then, (2.59) and (2.60) plainly lead to

$$\int_{\mathbb{R}} |F(x)|^2\, |x|^{2\nu+1}\, \mathrm{d}x \ll \int_{|x|>R} |F(x)|^2\, |x|^{2\nu+1}\, \mathrm{d}x \ll \int_{\mathbb{R}} |F(x)|^2\, \mathrm{d}\mu(x)\,,$$

as desired. This concludes the proof. □

From Proposition 2.17 we conclude that H is a reproducing kernel Hilbert space and the conditions to invoke Lemma 2.15 are in place. Letting K be the reproducing kernel of the Hilbert space H, one can show (as done in [31, §4.1]), since the measure μ is even, that K verifies the following properties, for all $w, z \in \mathbb{C}$:

(i) $K(w,w) > 0$;

(ii) $K(-w,-z) = K(w,z)$;

(iii) $K(z,w) = \overline{K(w,z)}$;

(iv) $\overline{K(\overline{w},\overline{z})} = K(w,z)$.

From this it follows that E defined by (2.52) - (2.53) verifies conditions (C2) and (C3). Since $\log^+|ab| \leqslant \log^+|a| + \log^+|b|$, we have $\log^+|E(x)| \leqslant \log^+\big|E(x)|x|^{\nu+\frac{1}{2}}\big| + \log^+\big||x|^{-\nu-\frac{1}{2}}\big|$ and an application of Jensen's inequality as in (2.54) yields

$$\int_{\mathbb{R}} \frac{\log^+|E(x)|}{1+x^2}\,\mathrm{d}x < \infty.$$

From Lemma 2.8 we see that E has bounded type in $\mathbb{C}^+$ (condition (C1)), and moreover, since $H = \mathcal{H}(E_\nu)$ as sets, with the aid of Lemma 2.9 we conclude that $\tau(E) = \tau(K(0,\cdot)) \leqslant 1$. In fact, we must have $\tau(E) = 1$, since by Lemma 2.9 the space $H = \mathcal{H}(E_\nu) = \mathcal{H}(E)$ admits only functions F with $\tau(F) \leqslant \tau(E)$, and we know from the start that there exist functions $F \in \mathcal{H}(E_\nu)$ with exponential type equal to 1. Finally, condition (C4) follows from Lemma 2.10 applied to E_ν or E, depending if we start with the left-hand side or the right-hand side of (2.5) being finite, and the fact that H is isometric to $\mathcal{H}(E)$.

Chapter 3

Optimal sign uncertainty for trigonometric polynomials

3.1 Introduction

The sign uncertainty principle, its sharp constants, and the critical role of the measure were discussed in Section 1.1 and Chapter 2. In particular, Chapter 2 showed that the measure against which the non-positive integral condition is imposed can completely erase the sign uncertainty phenomenon; yet for a large class of measures the problem (EP2.1) connects to the theory of de Branges spaces and yields sharp constants with explicit extremizers. The present chapter, based on [A2], carries this philosophy to the compact setting of the unit circle $\mathbb{T}$. Here bandlimited functions become trigonometric polynomials of bounded degree, and the measure is a symmetric finite Borel measure on $\mathbb{T}$. We obtain a complete characterization of when sign uncertainty holds and determine the exact constants in the affirmative cases.

3.1.1 The extremal problem and main result

Let μ be a finite Borel measure on $\mathbb{T}$ satisfying the symmetry condition

$$\mu(A) = \mu(-A), \quad \text{for all } A \subseteq \mathbb{T}. \tag{3.1}$$

The main object of this chapter is the quantity $\rho(\mu, N)$ introduced in (EP1.2): for each $N \geqslant 1$, one seeks

$$\rho(\mu, N) := \inf_{0 \not\equiv f \in \mathcal{T}(\mu,N)} \mathrm{r}(f),$$

where $\mathrm{r}(f)$ is the radius of last sign change (defined with the metric of $\mathbb{T}$; see Section 1.1.2) and

$$\mathcal{T}(\mu, N) := \left\{ f(x) = \sum_{n=-N}^{N} \widehat{f}(n)\, e^{2\pi i n x} \;\middle|\; \begin{array}{l} f : \mathbb{T} \to \mathbb{R} \text{ is eventually non-negative} \\ \text{and } \displaystyle\int_{\mathbb{T}} f(x)\, \mathrm{d}\mu(x) \leqslant 0 \end{array} \right\}.$$

Note that the metrics on $\mathbb{S}^1$ and on $\mathbb{T}$ differ by a factor of 2π. For the problem (EP1.2), $\mathrm{r}(f)$ is measured with respect to the metric of $\mathbb{T}$. The following elementary reductions allow us to restrict to a smaller subclass without loss of generality.

REMARK.

(i) $\rho(c\mu, N) = \rho(\mu, N)$ for any $c > 0$ and $N \geqslant 1$, so it suffices to work with probability measures.

(ii) The even part $f_e(x) := \frac{f(x)+f(-x)}{2}$ of any $f \in \mathcal{T}(\mu, N)$ satisfies $f_e \in \mathcal{T}(\mu, N)$ and $\mathrm{r}(f_e) \leqslant \mathrm{r}(f)$.

(iii) If $f \in \mathcal{T}(\mu, N)$, then $f_0(x) := f(x) - \int_{\mathbb{T}} f(t)\,\mathrm{d}\mu(t)$ belongs to $\mathcal{T}(\mu, N)$ and satisfies $\mathrm{r}(f_0) \leqslant \mathrm{r}(f)$.

Items (ii) and (iii) together show that the infimum in (EP1.2) can be taken over the subfamily of even trigonometric polynomials with zero μ-integral.

For the Lebesgue measure $\mathrm{d}x$ the sign uncertainty phenomenon is already apparent. We now give an elementary computation that shows $\rho(\mathrm{d}x, N) > 0$. Let $f \in \mathcal{T}(\mathrm{d}x, N)$ be non-trivial, and let $f_\pm(x) := \max\{\pm f(x), 0\}$ denote the positive and negative parts of f. Since f has non-positive integral, $\|f\|_{L^1} \leqslant 2\int_{\mathbb{T}} f_-(x)\,\mathrm{d}x$. As f is eventually non-negative, the support of f_- is contained in $(-\mathrm{r}(f), \mathrm{r}(f))$, so

$$\begin{aligned}\|f\|_{L^1} \leqslant 2\int_{-\mathrm{r}(f)}^{\mathrm{r}(f)} f_-(x)\,\mathrm{d}x \leqslant 4\,\mathrm{r}(f)\|f\|_{L^\infty} \\ \leqslant 4(2N+1)\,\mathrm{r}(f)\max_{|n|\leqslant N}|\widehat{f}(n)| \leqslant 4(2N+1)\,\mathrm{r}(f)\|f\|_{L^1}.\end{aligned}$$

Dividing by $\|f\|_{L^1} > 0$ gives $\mathrm{r}(f) \geqslant 1/(4(2N+1))$, whence $\rho(\mathrm{d}x, N) > 0$. This estimate is not sharp; the exact value $\rho(\mathrm{d}x, N) = 1/(2N+2)$ (announced in Section 1.1.2) is a consequence of a result of Babenko [7] and will be recovered as a special case of Theorem 3.1 below. Related problems for this particular case of the Lebesgue measure have been studied in [73, 124].

To state our main results, we recall several important concepts from the classical theory of orthogonal polynomials on the unit circle (OPUC). The theory of OPUC together with the theory of reproducing kernel Hilbert spaces of polynomials (see Section 3.3.1) play here the role that the theory of de Branges spaces [41] played in the proof of Theorem 2.2. We follow the classic reference [108] for OPUC; a more detailed discussion is given in Section 3.3.2.

3.1.1.1 Setup

We denote by $\mathbb{D}$ the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and by $\partial\mathbb{D}$ the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Throughout this chapter, we use the parametrization $\{e^{2\pi i\theta} : \theta \in \mathbb{T}\}$ of the unit circle $\partial\mathbb{D}$. Let μ be a probability measure on $\partial\mathbb{D}$ (equivalently on $\mathbb{T}$) and set

$$\int_{\partial\mathbb{D}} f(z)\,\mathrm{d}\mu(z) \coloneqq \int_{\mathbb{T}} f(e^{2\pi i\theta})\,\mathrm{d}\mu(\theta),$$

for any measurable function $f : \partial\mathbb{D} \to \mathbb{C}$. We say a probability measure μ on $\partial\mathbb{D}$ is *trivial* if its support is finite, i.e. $\#\mathrm{supp}(\mu) < \infty$.

3.1.1.2 Orthogonal polynomials on the unit circle

For a probability measure μ on $\partial\mathbb{D}$ consider the Hilbert space $L^2(\partial\mathbb{D},\mu)$ equipped with the inner product

$$\langle f, g\rangle \coloneqq \int_{\partial\mathbb{D}} f(z)\overline{g(z)}\,\mathrm{d}\mu(z).$$

Since any bounded function on $\partial\mathbb{D}$ belongs to $L^2(\partial\mathbb{D},\mu)$, the polynomials $1, z, z^2, \dots$ lie in this Hilbert space. As long as the measure μ is non-trivial, the Gram–Schmidt process yields the unique *monic orthogonal polynomials* $\Phi_n(z) \coloneqq \Phi_n(z;\mathrm{d}\mu)$, for all $n \geqslant 0$, in $L^2(\partial\mathbb{D},\mu)$. By definition $\Phi_0(z) \equiv 1$, and $\Phi_n(z) = z^n +$ lower order terms satisfies

$$\int_{\partial\mathbb{D}} \Phi_n(z)\,\overline{z^j}\,\mathrm{d}\mu(z) = 0, \quad 0 \leqslant j < n.$$

We also consider the sequence of *orthonormal polynomials* $\varphi_n(z) \coloneqq \Phi_n(z)/\|\Phi_n\|_2$.

Note that for a trivial measure μ, with $\#\mathrm{supp}(\mu) = M$, one can construct the polynomials $\Phi_0, \Phi_1, \dots, \Phi_M$ uniquely by the same procedure, and the process terminates exactly at Φ_M, since it vanishes at every point in the support of μ. Thus, the only well-defined orthonormal polynomials are $\varphi_0, \varphi_1, \dots, \varphi_{M-1}$.

3.1.1.3 Auxiliary polynomials

Let $P(z)$ be a polynomial of degree $n \geqslant 0$, given by

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0.$$

For an integer $m \geqslant n$ define the *reversed polynomial* $P^{*,m}(z)$ by

$$P^{*,m}(z) \coloneqq z^m\overline{P(1/\overline{z})} = \overline{a_0}z^m + \overline{a_1}z^{m-1} + \ldots + \overline{a_{n-1}}z^{m-n+1} + \overline{a_n}z^{m-n}.$$

For simplicity we denote $P^* = P^{*,\deg(P)}$.

Whenever μ is a probability measure and the unique orthonormal polynomials $\varphi_n(z;\mathrm{d}\mu)$ exist, one has

$$|\varphi_n(z)| < |\varphi_n^*(z)| \tag{3.2}$$

for all $z \in \mathbb{D}$ (see [108, §1.7]). Based on these facts, we define the following auxiliary polynomials. Whenever $\varphi_n(z;\mathrm{d}\mu)$ is well-defined as in the procedure described above, we set

$$A_n(z;\mathrm{d}\mu) = \frac{\varphi_n(z;\mathrm{d}\mu) + \varphi_n^*(z;\mathrm{d}\mu)}{2}.$$

In the particular case of a trivial measure μ with $\#\text{supp}(\mu) = n$, as $\varphi_n(z; \mathrm{d}\mu)$ is not well-defined, we use the following definition instead

$$A_n(z; \mathrm{d}\mu) = \frac{z\varphi_{n-1}(z; \mathrm{d}\mu) + \varphi_{n-1}^*(z; \mathrm{d}\mu)}{2}.$$

Later, in Section 3.3.1, we provide a complete justification for these polynomials and for their role in (EP1.2). Note that (3.2) implies that $A_n(z)$ has no zeros in the unit disk, while the symmetry $A_n = A_n^*$ implies that all zeros of A_n lie on $\partial\mathbb{D}$.

3.1.1.4 Statement of the main theorem

Having introduced all the components, we restate our main result on (EP1.2) (announced as Theorem 1.2 in the introduction) and prove it in Section 3.3. We split the statement into two cases according to the support of the measure.

Theorem 3.1. *Let μ be a finite Borel measure on $\mathbb{T}$ which satisfies (3.1). The following holds.*

(i) *If $1 \leqslant N < \#\text{supp}(\mu)$, then*

$$\rho(\mu, N) = \theta_1,$$

where $0 < \theta_1 \leqslant 1/2$ is the smallest positive zero of $A_{N+1}(e^{2\pi i x}; \mathrm{d}\mu)$, i.e.,

$$\theta_1 = \min\{\theta > 0 \,:\, A_{N+1}(e^{2\pi i\theta}; \mathrm{d}\mu) = 0\}.$$

Moreover, the extremizer is unique (up to a constant factor) and is given by

$$f(x) = \frac{|A_{N+1}(e^{2\pi i x}; \mathrm{d}\mu)|^2}{\cos(2\pi\theta_1) - \cos(2\pi x)}. \tag{3.3}$$

(ii) *If $\#\text{supp}(\mu) \leqslant N < \infty$, then $\rho(\mu, N) = 0$.*

3.1.2 Analogue in higher dimensions

Let $d \geqslant 2$ and consider the d-dimensional sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, and let $\boldsymbol{v} \in \mathbb{S}^d$ be a distinguished point. Without loss of generality we assume $\boldsymbol{v} = (0, 0, \ldots, 0, 1)$. We denote by $\mathrm{SO}(d+1)$ the special orthogonal group, consisting of all $(d+1) \times (d+1)$ real orthogonal matrices with determinant 1, and by $\mathrm{SO}_{\boldsymbol{v}}(d+1) \subsetneq \mathrm{SO}(d+1)$ the stabilizer subgroup of $\boldsymbol{v}$, namely the closed subgroup of all rotations in $\mathrm{SO}(d+1)$ that fix $\boldsymbol{v}$.

3.1.2.1 Polar measures

We say a finite Borel measure μ is a *polar measure (with a pole at $\boldsymbol{v}$)* on $\mathbb{S}^d$ if it satisfies the symmetry property

$$\mu(A) = \mu(R(A)), \qquad \text{for all } A \subseteq \mathbb{S}^d \text{ and for all } R \in \mathrm{SO}_{\boldsymbol{v}}(d+1), \tag{3.4}$$

where $R(A) := \{Rx \,|\, x \in A\}$. Define the normalized colatitude $\vartheta : \mathbb{S}^d \to [0, 1/2]$ by

$$\vartheta(\boldsymbol{x}) := \frac{1}{2\pi} \arccos(\boldsymbol{x} \cdot \boldsymbol{v}),$$

i.e., the geodesic distance from $\boldsymbol{v}$ rescaled to the interval $[0, 1/2]$. For a polar measure μ, its *polar part* is the even Borel measure μ_p on $\mathbb{T}$ given by

$$\int_{\mathbb{T}} g(\theta)\, \mathrm{d}\mu_p(\theta) = \int_{\mathbb{S}^d} (g(\vartheta(\boldsymbol{x})) + g(-\vartheta(\boldsymbol{x})))\, \mathrm{d}\mu(\boldsymbol{x})$$

for every bounded Borel function $g : \mathbb{T} \to \mathbb{R}$. Equivalently, for any Borel set $A \subseteq \mathbb{T}$,

$$\mu_p(A) = \mu(\vartheta^{-1}(A)) + \mu(\vartheta^{-1}(-A)), \qquad -A := \{-\theta \mid \theta \in A\}.$$

The set of all polar measures admits a simple characterization in terms of their polar parts; see Proposition 3.7. For example, the polar part of the normalized surface measure σ_d is given by

$$\mathrm{d}\mu_{d-1}(\theta) := \mathrm{d}(\sigma_d)_p(\theta) = C_d \left|\sin(2\pi\theta)\right|^{d-1} \mathrm{d}\theta,$$

where $C_d > 0$ is a dimensional constant. Note that the polar part of a polar measure μ is always an even measure on $\mathbb{T}$.

3.1.2.2 Extremal problem in higher dimensions

Denote by $\mathscr{H}_k$ the space of spherical harmonics of degree k on $\mathbb{S}^d$. Each $\mathscr{H}_k$ is a finite-dimensional subspace of $L^2(\mathbb{S}^d, \sigma_d)$, and the span of all spherical harmonics is dense in $L^2(\mathbb{S}^d, \sigma_d)$; see [111, Chapter IV].

Let μ be a polar measure with a pole at $\boldsymbol{v}$ on $\mathbb{S}^d$, and let $N \geqslant 1$ be an integer. Consider the family $\mathcal{T}(d, \mu, N)$ of functions $f : \mathbb{S}^d \to \mathbb{R}$ satisfying:

(i) $f(\boldsymbol{x}) = \sum_{k=0}^{N} Y_k(\boldsymbol{x})$, where $Y_k \in \mathscr{H}_k$;

(ii) f is eventually non-negative on $\mathbb{S}^d$;

(iii) $\displaystyle\int_{\mathbb{S}^d} f(\boldsymbol{x})\, \mathrm{d}\mu(\boldsymbol{x}) \leqslant 0.$

We seek the function in this class whose negative part is concentrated in the smallest possible spherical cap, that is, in a ball with respect to the geodesic metric of $\mathbb{S}^d$. We formulate this as follows.

Extremal problem 3.1 **(EP3.1)**. For each $N \geqslant 1$, determine

$$\rho(d, \mu, N) := \inf_{0 \not\equiv f \in \mathcal{T}(d,\mu,N)} \mathrm{r}(f),$$

where $\mathrm{r}(f)$ is the radius of last sign change on $\mathbb{S}^d$, taken with respect to its geodesic metric (see Section 1.1.2).

The case $d = 1$ may also be considered; it coincides with (EP1.2) up to a factor of 2π, owing to the difference between the metrics of $\mathbb{T}$ and $\mathbb{S}^1$.

The next result shows that, for polar measures, the nature of (EP3.1) does not depend on the dimension d in the following sense: it reduces to the one-dimensional problem (EP1.2) solved by Theorem 3.1.

Theorem 3.2. *For every polar measure μ on $\mathbb{S}^d$ and every integer $N \geqslant 1$,*

$$\rho(d, \mu, N) = 2\pi\, \rho(\mu_p, N),$$

where μ_p is the polar part of μ. Moreover, the function $\widetilde{f}$ given by

$$\widetilde{f}(\boldsymbol{x}) = f\left(\frac{1}{2\pi}\arccos(\boldsymbol{x}\cdot\boldsymbol{v})\right)$$

is an extremizer for (EP3.1), where $f : \mathbb{T} \to \mathbb{R}$ is the function defined in (3.3).

Theorem 3.2 is proved later in this chapter by reducing the problem to a set of zonal spherical harmonics and exploiting the invariance (3.4) of μ under $\mathrm{SO}_{\boldsymbol{v}}(d+1)$.

3.1.3 Analogue for polynomials

We now turn to the polynomial analogue of the sign uncertainty phenomenon considered in (EP1.2).

3.1.3.1 Setup

Let μ be a finite Borel measure on $[0, 1]$, and fix $N \geqslant 1$. Define the family of real polynomials of degree at most N by

$$\mathcal{P}(\mu, N) := \left\{ p(x) = \sum_{n=0}^{N} a_n x^n \;\middle|\; \begin{array}{l} p : [0,1] \to \mathbb{R} \text{ is eventually non-negative,} \\ \displaystyle\int_0^1 p(x)\,\mathrm{d}\mu(x) \leqslant 0 \end{array} \right\}.$$

Here the distinguished point is $x^* = 0$; thus $\mathcal{P}(\mu, N)$ consists of polynomials with non-positive μ-integral that are non-negative on some left neighborhood of $x = 1$. We again study the smallest possible radius of last sign change.

Extremal problem 3.2 **(EP3.2)**. For a finite Borel measure μ on $[0, 1]$ and an integer $N \geqslant 1$, determine

$$\Omega(\mu, N) := \inf_{0 \not\equiv p \in \mathcal{P}(\mu, N)} \mathrm{r}(p),$$

where $\mathrm{r}(p)$ is given by (1.1).

This problem can again be addressed by the classical theory of orthogonal polynomials, now on the real line.

3.1.3.2 Orthogonal polynomials on the real line (OPRL)

Given a finite Borel measure μ on $[0,1]$, consider $L^2(\mathbb{R},\mu)$ with the inner product $\langle f,g\rangle \coloneqq \int_{\mathbb{R}} f(x)\overline{g(x)}\,\mathrm{d}\mu(x)$. As $1, x, x^2, \dots \in L^2(\mathbb{R},\mu)$, the Gram–Schmidt process yields a unique sequence of *monic orthogonal polynomials* $\Psi_0 \equiv 1, \Psi_1, \Psi_2, \dots, \Psi_n(x) \coloneqq \Psi_n(x;\mathrm{d}\mu)$. As in Section 3.1.1.2, the process terminates at $\Psi_m(x;\mathrm{d}\mu)$ precisely when $\#\mathrm{supp}(\mu) = m$, in which case Ψ_m vanishes on $\mathrm{supp}(\mu)$. For a more detailed treatment of OPRL we refer to [72, 114].

For a finite positive Borel measure μ on $[0,1]$ we define a quantity $d(\mu) \in \{1,2,\dots\} \cup \{+\infty\}$ as follows. If $\#\mathrm{supp}(\mu) = +\infty$, we set $d(\mu) \coloneqq +\infty$. Otherwise μ is supported on finitely many points and can be written as

$$\mu = \sum_{j=1}^{m} w_j\,\boldsymbol{\delta}_{t_j}, \qquad 0 \leqslant t_1 < t_2 < \cdots < t_m \leqslant 1, \quad w_j > 0,$$

where $\boldsymbol{\delta}_{t_j}$ denotes the Dirac measure with a mass at the point t_j. With $\mathbf{1}$ the indicator function, set

$$\gamma_0 \coloneqq \mathbf{1}_{\mathrm{supp}(\mu)}(0) \quad \text{and} \quad \gamma_1 \coloneqq \mathbf{1}_{\mathrm{supp}(\mu)}(1),$$

so that γ_0 (respectively γ_1) equals 1 exactly when the left (respectively right) endpoint of $[0,1]$ belongs to $\mathrm{supp}(\mu)$, and define

$$d(\mu) \coloneqq 2m - \gamma_0 - \gamma_1.$$

The exact value of $\Omega(\mu,N)$ in (EP3.2) is given by the following theorem.

Theorem 3.3. *Let μ be a finite positive Borel measure on $[0,1]$ and let μ_1 be the measure $\mathrm{d}\mu_1(x) \coloneqq (1-x)\,\mathrm{d}\mu(x)$. Then $\Omega(\mu,N) > 0$ for every integer $1 \leqslant N < d(\mu)$, and $\Omega(\mu,N) = 0$ for every integer $N \geqslant d(\mu)$. More precisely, for every integer N with $1 \leqslant N < d(\mu)$ the following holds.*

(i) *If $N = 2n-1$ with $n \geqslant 1$, then $\Omega(\mu,2n-1) = x_{n,1}$, where $x_{n,1}$ is the smallest zero of $\Psi_n(x;\mathrm{d}\mu)$, and every extremizer is of the form*

$$p(x) = C\,\frac{\left(\Psi_n(x;\mathrm{d}\mu)\right)^2}{x - x_{n,1}}, \qquad C > 0.$$

(ii) *If $N = 2n$ with $n \geqslant 1$, then $\Omega(\mu,2n) = \xi_{n,1}$, where $\xi_{n,1}$ is the smallest zero of $\Psi_n(x;\mathrm{d}\mu_1)$, and every extremizer is of the form*

$$p(x) = C\,(1-x)\,\frac{\left(\Psi_n(x;\mathrm{d}\mu_1)\right)^2}{x - \xi_{n,1}}, \qquad C > 0.$$

It was first observed by Szegö [113] that OPUC and OPRL are not entirely independent theories: for measures related in a specific way, one can pass from one setting to the other. In particular, a relation between (EP1.2) and (EP3.2) can be established; see Proposition 3.15 below. We nevertheless present the proofs of Theorem 3.1 and Theorem 3.3 independently, as each argument is self-contained. More importantly, Theorem 3.3 expresses the solution of (EP3.2) directly in terms of μ and μ_1, without first passing through the circle via Proposition 3.15.

3.1.4 Applications

3.1.4.1 The Lebesgue measure on $\mathbb{T}$

Let $\mathrm{d}x$ be the usual Lebesgue measure on $\mathbb{T}$ with the normalization $\int_{\mathbb{T}} \mathrm{d}x = 1$. In this case, it is easy to see that the orthogonal polynomials on the unit circle are given by $\Phi_n(z; \mathrm{d}x) = z^n$, $n \geqslant 0$. Moreover, $\varphi_n = \Phi_n$ for all $n \geqslant 0$. Hence, the value of $\rho(\mathrm{d}x, N)$ is determined by the zeros of the polynomial $A_{N+1}(z) = (z^{N+1} + 1)/2$, and by Theorem 3.1 we have

$$\rho(\mathrm{d}x, N) = \frac{1}{2\pi} \cdot \frac{\pi}{N+1} = \frac{1}{2(N+1)}.$$

The unique extremizer (up to a constant factor) takes the form

$$f(x) = C\,\frac{|e^{2\pi i(N+1)x} + 1|^2}{4(\cos(\pi/(N+1)) - \cos(2\pi x))} = \frac{C\cos^2(\pi(N+1)x)}{\cos(\pi/(N+1)) - \cos(2\pi x)},$$

where $C > 0$. In Figure 3.1 we show plots of some of these functions.

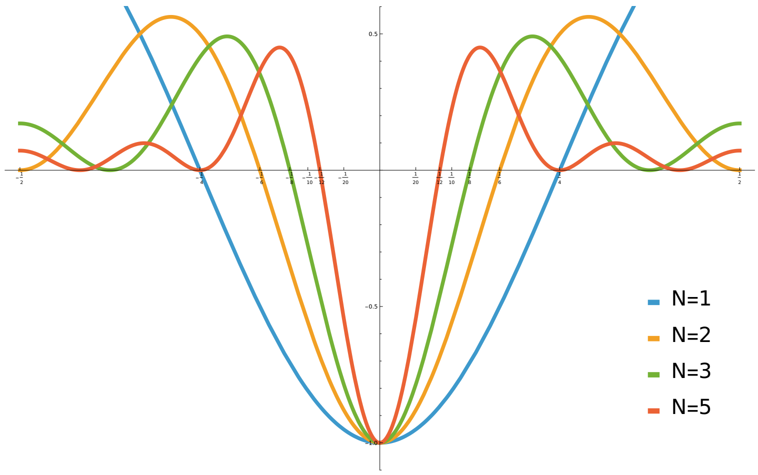

FIGURE 3.1: Extremizers for (EP1.2) when $\mu = \mathrm{d}x$ is the Lebesgue measure, for several values of N.

3.1.4.2 Uniform measures on higher-dimensional spheres

Let σ_d be the normalized surface measure on $\mathbb{S}^d$. By Proposition 3.7, its polar part is the measure μ_{d-1} given by

$$\mathrm{d}\mu_{d-1}(\theta) = C_d|\sin(2\pi\theta)|^{d-1}\,\mathrm{d}\theta. \tag{3.5}$$

As a consequence of Theorems 3.1, 3.2 and 3.3 we conclude the following.

Corollary 3.4. *Let $d \geqslant 2$, $n \geqslant 1$, and set $\alpha = (d-2)/2$. Let σ_d denote the normalized surface measure on $\mathbb{S}^d$. Then*

(i) $\rho(d, \sigma_d, 2n-1) = 2\pi\rho(\mu_{d-1}, 2n-1) = \arccos(x_{n,n})$, *where $x_{n,n}$ is the largest zero of the Jacobi polynomial $P_n^{(\alpha,\alpha)}(x)$;*

(ii) $\rho(d, \sigma_d, 2n) = 2\pi\rho(\mu_{d-1}, 2n) = \arccos(\xi_{n,n})$, *where $\xi_{n,n}$ is the largest zero of the Jacobi polynomial $P_n^{(\alpha,\alpha+1)}(x)$.*

The qualitative statement that $\rho(d, \sigma_d, N) > 0$ for all $N \geqslant 1$ follows also from a general sign uncertainty principle for spherical harmonics established by Gonçalves, Oliveira e Silva, and Ramos [63]; see Theorem 2.3 and §2.1 of that reference. The present corollary sharpens their result by identifying the exact value of $\rho(d, \sigma_d, N)$ in terms of zeros of specific Jacobi polynomials.

3.1.4.3 Application to a problem of Yudin

The previous corollary can be applied to study the problem about the measure of the negativity sets of harmonic polynomials posed by Yudin in [123]. The problem asks the following:

> *If $f(\boldsymbol{x})$ ($\boldsymbol{x} \in \mathbb{R}^{d+1}$) is a harmonic polynomial of degree at most k such that $f(0) = 0$, then*
> $$\int_{\mathbb{S}^d} f(\boldsymbol{x})\, \mathrm{d}\sigma_d(\boldsymbol{x}) = 0$$
> *by the mean value theorem. Hence f assumes both positive and negative values on $\mathbb{S}^d$. What is the smallest possible value of*
> $$\sigma_d(\{\boldsymbol{x} \in \mathbb{S}^d : f(\boldsymbol{x}) > 0\})?$$

Although the original work of Babenko [7] addressed this problem in dimension one ($d = 1$), it is a very delicate problem in higher dimensions. Recent progress on this and other related problems can be found in [6, 125] and references therein.

Corollary 3.5. *Let $0 \leqslant \tau_k(d)$ be the answer to the question of Yudin. Then*

$$\tau_{2n-1}(d) \leqslant \sigma_d\big(\mathcal{C}(x_{n,n})\big) \quad \text{and} \quad \tau_{2n}(d) \leqslant \sigma_d\big(\mathcal{C}(\xi_{n,n})\big),$$

where $\mathcal{C}(t) := \{\boldsymbol{x} \in \mathbb{S}^d : \boldsymbol{x} \cdot \boldsymbol{v} \geqslant t\}$, $-1 \leqslant t \leqslant 1$, is a spherical cap centered at $\boldsymbol{v}$ and $x_{n,n}$, $\xi_{n,n}$ are as in Corollary 3.4.

Corollary 3.5 is an immediate consequence of Corollary 3.4: by the remark on page 50, the extremizer $\widetilde{f}$ of Theorem 3.2 may be normalized so that $\int_{\mathbb{S}^d} \widetilde{f}\, \mathrm{d}\sigma_d = 0$, i.e. $\widetilde{f}(0) = 0$, which makes $\widetilde{f}$ an admissible competitor for Yudin's problem, with positivity set contained in $\mathcal{C}(x_{n,n})$ (resp. $\mathcal{C}(\xi_{n,n})$).

3.1.4.4 The dual problem for quadrature nodes

As before, let μ be an even probability measure on $\mathbb{T}$. A finite set $\xi = \{\xi_1, \dots, \xi_N\} \subsetneq \mathbb{T}$ is a *set of nodes* for a quadrature formula for μ if there exist weights $\lambda_j \in (0, 1]$, $1 \leqslant j \leqslant N$, such that

$$\int_{\mathbb{T}} f(x)\, \mathrm{d}\mu(x) = \sum_{j=1}^{N} \lambda_j\, f(\xi_j) \tag{3.6}$$

holds for every trigonometric polynomial $f(x) = \sum_{|n| \leqslant N-1} \widehat{f}(n) e^{2\pi i n x}$. We then call (3.6) an *N-point quadrature formula* for μ; for a detailed account of such formulas, see [75] and the references therein. The following extremal problem for sets of nodes is dual to (EP1.2).

Extremal problem 3.3 **(EP3.3)**. For a probability measure μ on $\mathbb{T}$ that is even, and an integer $N \geqslant 1$, determine

$$\Xi(\mu, N) := \sup_{\#\xi = N} \min\{|\xi_j| : \xi_j \in \xi\},$$

where the supremum is over all N-point sets of nodes ξ for μ, with the convention $\Xi(\mu, N) = 0$ if no such set exists.

Theorem 3.6. *For every even probability measure μ on $\mathbb{T}$ and every $N \geqslant 1$, the extremal problems* (EP1.2) *and* (EP3.3) *are related by*

$$\rho(\mu, N) = \Xi(\mu, N + 1).$$

3.2 Dimension shift: proof of Theorem 3.2

The following result establishes the complete characterization of polar measures on $\mathbb{S}^d$. We were unable to find an exact reference for it, so for the convenience of the reader, we provide here a short proof.

Proposition 3.7. *If μ is a polar measure on $\mathbb{S}^d$, then it can be decomposed as $\mu = \breve{\mu}_p \otimes \sigma_{d-1}$, in the following sense*

$$\int_{\mathbb{S}^d} f(\boldsymbol{x})\, \mathrm{d}\mu(\boldsymbol{x}) = \int_0^{1/2} \left(\int_{\mathbb{S}^{d-1}} f\left(\boldsymbol{x}' \sin(2\pi\theta), \cos(2\pi\theta)\right) \mathrm{d}\sigma_{d-1}(\boldsymbol{x}') \right) \mathrm{d}\breve{\mu}_p(\theta),$$

where σ_{d-1} is the normalized uniform surface measure on $\mathbb{S}^{d-1}$, i.e., $\sigma_{d-1}(\mathbb{S}^{d-1}) = 1$, and μ_p is the polar part of μ.

Proof. We parametrize $\mathbb{S}^d$ using the polar angle and parallel latitudes:

$$\mathbb{S}^d = \left\{ \boldsymbol{x} = (\boldsymbol{x}' \sin(2\pi\theta), \cos(2\pi\theta)) \;:\; \boldsymbol{x}' \in \mathbb{S}^{d-1},\; 0 \leqslant \theta \leqslant 1/2 \right\}.$$

Consider the colatitude map $\vartheta : \mathbb{S}^d \to [0, 1/2]$ defined by

$$\vartheta(\boldsymbol{x}) = \frac{1}{2\pi} \arccos(\boldsymbol{x} \cdot \boldsymbol{v}).$$

The push-forward measure $\breve{\lambda} := \vartheta_{\#}\mu$ is a finite Borel measure on $[0, 1/2]$. Applying Rokhlin's Disintegration Theorem [3, Theorem 5.3.1], we deduce that for each $\theta \in [0, 1/2]$ there exists a finite measure ν_θ on $\mathbb{S}^{d-1}$ such that $\theta \mapsto \nu_\theta$ is a $\breve{\lambda}$-measurable map and $\nu_\theta(\mathbb{S}^{d-1}) = 1$ for $\breve{\lambda}$-a.e. $\theta \in [0, 1/2]$. Moreover, we have

$$\int_{\mathbb{S}^d} f(\boldsymbol{x})\, \mathrm{d}\mu(\boldsymbol{x}) = \int_0^{1/2} \left(\int_{\mathbb{S}^{d-1}} f(\boldsymbol{x}' \sin(2\pi\theta), \cos(2\pi\theta))\, \mathrm{d}\nu_\theta(\boldsymbol{x}') \right) \mathrm{d}\breve{\lambda}(\theta) \tag{3.7}$$

for all $f \in L^1(\mathbb{S}^d, \mu)$. Since μ is a polar measure, the left-hand side of (3.7) is preserved under the action of $R \in \mathrm{SO}_{\boldsymbol{v}}(d+1)$. On the other hand, this action also preserves the polar angle $\theta \in [0, 1/2]$, thus it must preserve the inner integral on the right-hand side of (3.7) for $\breve{\lambda}$-a.e. $\theta \in [0, 1/2]$. Noting that $\nu_\theta(\mathbb{S}^{d-1}) = 1$ and applying the uniqueness of the uniformly distributed measures on metric spaces [92, Theorem 3.4], we deduce that $\nu_\theta = \sigma_{d-1}$ for $\breve{\lambda}$-a.e. $\theta \in [0, 1/2]$. It remains to identify $\breve{\lambda}$ with $\breve{\mu}_p$. By the definition of the polar part recalled in Section 3.1.2.1, for any Borel set $A \subseteq \mathbb{T}$,

$$\mu_p(A) = \mu(\vartheta^{-1}(A)) + \mu(\vartheta^{-1}(-A)) = \breve{\lambda}(A) + \breve{\lambda}(-A).$$

For $A \subseteq (0, 1/2)$ the reflected set $-A$ lies in $(-1/2, 0)$, which is disjoint from the range $[0, 1/2]$ of ϑ, so $\vartheta^{-1}(-A) = \emptyset$ and hence $\mu_p(A) = \mu(\vartheta^{-1}(A)) = \breve{\lambda}(A)$. For the endpoints, $-\{0\} = \{0\}$ and $-\{1/2\} = \{1/2\}$ on $\mathbb{T}$, so $\mu_p(\{0\}) = 2\mu(\{\boldsymbol{v}\}) = 2\breve{\lambda}(\{0\})$ and $\mu_p(\{1/2\}) = 2\mu(\{-\boldsymbol{v}\}) = 2\breve{\lambda}(\{1/2\})$. Recalling that $\breve{\mu}_p$ assigns half-weight to the poles,

$$\breve{\mu}_p(A) = \mu_p(A \cap (0, 1/2)) + \tfrac{1}{2}\mu_p(A \cap \{0, 1/2\}) = \breve{\lambda}(A),$$

so $\breve{\lambda} = \breve{\mu}_p$ on $[0, 1/2]$.

Moreover, we have the identity

$$\begin{aligned}\int_{\mathbb{S}^d} f(\boldsymbol{x})\, \mathrm{d}\mu(\boldsymbol{x}) &= \int_0^{1/2} \left(\int_{\mathbb{S}^{d-1}} f(\boldsymbol{x}' \sin(2\pi\theta), \cos(2\pi\theta))\, \mathrm{d}\sigma_{d-1}(\boldsymbol{x}') \right) \mathrm{d}\breve{\mu}_p(\theta) \\ &= \frac{1}{2} \int_{\mathbb{T}} \left(\int_{\mathbb{S}^{d-1}} f(\boldsymbol{x}' \sin(2\pi\theta), \cos(2\pi\theta))\, \mathrm{d}\sigma_{d-1}(\boldsymbol{x}') \right) \mathrm{d}\mu_p(\theta). \qquad \square\end{aligned}$$

3.2.1 Polar symmetrization

Let μ be a polar measure on $\mathbb{S}^d$ and $N \geqslant 1$ be an integer. We say a function $f : \mathbb{S}^d \to \mathbb{R}$ is a *polar function (with the pole $\boldsymbol{v}$)* if

$$f(R\boldsymbol{x}) = f(\boldsymbol{x})$$

holds for all $\boldsymbol{x} \in \mathbb{S}^d$ and for all $R \in \mathrm{SO}_{\boldsymbol{v}}(d+1)$. We first show that an appropriate symmetrization process, considered in [87], does not increase the radius of the last sign change. Thus, for the problem in higher dimensions, it suffices to consider only the polar functions.

Following the approach of [87], for any function $f : \mathbb{S}^d \to \mathbb{C}$ we define its symmetrization $\widetilde{f} : \mathbb{S}^d \to \mathbb{C}$ by

$$\widetilde{f}(\boldsymbol{x}) := \int_{\mathrm{SO}_{\boldsymbol{v}}(d+1)} f(R\boldsymbol{x})\,\mathrm{d}\eta(R),$$

where η is the normalized Haar measure on $\mathrm{SO}_{\boldsymbol{v}}(d+1)$, so that $\eta(\mathrm{SO}_{\boldsymbol{v}}(d+1)) = 1$. We claim that this symmetrization process does not increase the radius of the last sign change. To justify this, we recall the following facts about spherical harmonics from [111, Chapter IV].

Lemma 3.8 (cf. [111, Chapter IV]). *Let $k \geqslant 0$ and $Y \in \mathscr{H}_k$ be a spherical harmonic on $\mathbb{S}^d$ with $d \geqslant 2$. The following holds:*

(i) *The symmetrization of Y is given by*

$$\widetilde{Y}(\boldsymbol{x}) = \frac{\omega_d}{\dim \mathscr{H}_k} Y(\boldsymbol{v})\, Z_{\boldsymbol{v}}^{(k)}(\boldsymbol{x}),$$

where $Z_{\boldsymbol{v}}^{(k)} \in \mathscr{H}_k$ is the zonal spherical harmonic of degree k with pole $\boldsymbol{v}$.

(ii) *There exists a polynomial P_k of degree k and a constant $c_{k,d}$ such that*

$$Z_{\boldsymbol{v}}^{(k)}(\boldsymbol{x}) = c_{k,d}\, P_k(\boldsymbol{x} \cdot \boldsymbol{v}), \qquad \text{for all } \boldsymbol{x} \in \mathbb{S}^d.$$

Proof. For part (i) see [111, Chapter IV, Theorem 2.12] and [111, Chapter IV, Corollary 2.9]. For part (ii) see [111, Chapter IV, Theorem 2.14]. □

Lemma 3.9. *If $0 \not\equiv f \in \mathcal{T}(d, \mu, N)$ is a non-trivial function, then $\widetilde{f} \in \mathcal{T}(d, \mu, N)$. Moreover,*

$$\mathrm{r}(\widetilde{f}) \leqslant \mathrm{r}(f).$$

Proof. Since $f(\boldsymbol{x}) = \sum_{k=0}^{N} Y_k(\boldsymbol{x})$, using part (i) of Lemma 3.8 we know $\widetilde{f}$ is a linear combination of zonal spherical harmonics of degree at most N. From the definition of

$\widetilde{f}$ it is clear that $\widetilde{f}(\boldsymbol{x}) \geqslant 0$ for all $\boldsymbol{x} \notin B_{\mathrm{r}(f)}(\boldsymbol{v})$, i.e., $\mathrm{r}(\widetilde{f}) \leqslant \mathrm{r}(f)$. Moreover, we have

$$\begin{aligned}\int_{\mathbb{S}^d} \widetilde{f}(\boldsymbol{x})\,\mathrm{d}\mu(\boldsymbol{x}) &= \int_{\mathrm{SO}_{\boldsymbol{v}}(d+1)} \int_{\mathbb{S}^d} f(R\boldsymbol{x})\,\mathrm{d}\mu(\boldsymbol{x})\,\mathrm{d}\eta(R)\\ &= \int_{\mathbb{S}^d} f(\boldsymbol{x}) \int_{\mathrm{SO}_{\boldsymbol{v}}(d+1)} \mathrm{d}\eta(R)\,\mathrm{d}\mu(\boldsymbol{x}) = \int_{\mathbb{S}^d} f(\boldsymbol{x})\,\mathrm{d}\mu(\boldsymbol{x}) \leqslant 0.\end{aligned}$$

Therefore, $\widetilde{f} \in \mathcal{T}(d, \mu, N)$. □

Remark. Note that, for a function $f \in \mathcal{T}(d, \mu, N)$ with $\mathrm{r}(f) < \pi$ and $\widetilde{f} \equiv 0$, we must have $f \equiv 0$. Indeed, $\widetilde{f} \equiv 0$ together with the fact that $\mathrm{r}(\widetilde{f}) < \pi$ implies that $f(\boldsymbol{x}) = 0$ for all points in a small spherical cap centered at $-\boldsymbol{v}$. Composing f with the stereographic projection, we see that the composition is a rational function in $\mathbb{R}^d$ vanishing on an open ball, thus $f \equiv 0$.

Hence, we can conclude that

$$\rho(d, \mu, N) = \inf_{\substack{0 \not\equiv f \in \mathcal{T}(d,\mu,N) \\ f \text{ is polar}}} \mathrm{r}(f).$$

In the rest of the section, by making explicit constructions, we prove the dimension shift result.

3.2.2 Proof of Theorem 3.2

Recall μ_p denotes the polar part of the measure μ.

3.2.2.1 *Upper bound*

Let $g \in \mathcal{T}(\mu_p, N)$ be a trigonometric polynomial that is not identically zero. As noted in the remark on page 50, the even part of the function has a smaller radius of last sign change. Hence, we can assume g is even. Consider the function $f : \mathbb{S}^d \to \mathbb{R}$ given by

$$f(\boldsymbol{x}) = g\left(\frac{1}{2\pi} \arccos\left(\boldsymbol{x} \cdot \boldsymbol{v}\right)\right).$$

We claim that $f \in \mathcal{T}(d, \mu, N)$. First of all, note that g being an even trigonometric polynomial, it only contains cosine terms, thus it can be written as $g(\theta) = P(\cos(2\pi\theta))$ for some polynomial P of degree at most N, and we have $f(\boldsymbol{x}) = P(\boldsymbol{x} \cdot \boldsymbol{v})$. Also note that f is a polar function by definition. Hence, by Lemma 3.8, the spherical harmonics expansion of f can only contain zonal spherical harmonics of degree at most N with the pole $\boldsymbol{v}$. It is clear that $f(\boldsymbol{x}) \geqslant 0$ for all $\boldsymbol{x} \in \mathbb{S}^d$ with $\boldsymbol{x} \cdot \boldsymbol{v} \leqslant \cos(2\pi\,\mathrm{r}(g))$, i.e., $\mathrm{r}(f) \leqslant 2\pi\,\mathrm{r}(g)$[1]. By Proposition 3.7 we know

$$\int_{\mathbb{S}^d} f(\boldsymbol{x})\,\mathrm{d}\mu(\boldsymbol{x}) = \int_0^{1/2} f(\boldsymbol{x}' \sin(2\pi\theta), \cos(2\pi\theta))\,\mathrm{d}\breve{\mu}_p(\theta) = \frac{1}{2}\int_{\mathbb{T}} g(\theta)\,\mathrm{d}\mu_p(\theta) \leqslant 0.$$

[1]Note that on each side we are using different metrics to define $\mathrm{r}(f)$ and $\mathrm{r}(g)$.

Hence, $f \in \mathcal{T}(d, \mu, N)$ and by the choice of g we have $f \not\equiv 0$. Thus

$$\rho(d, \mu, N) \leqslant 2\pi\, \rho(\mu_p, N).$$

3.2.2.2 Lower bound

Without loss of generality, we can assume that there exists a non-trivial $f \in \mathcal{T}(d, \mu, N)$ such that $\mathrm{r}(f) < \pi$, otherwise the upper bound mentioned above and the fact that $\rho(\mu_p, N) \leqslant 1/2$ complete the proof. Fix any such function f, with $\mathrm{r}(f) < \pi$. We have observed in Lemma 3.9 that $\mathrm{r}(\widetilde{f}) \leqslant \mathrm{r}(f)$, hence we can work with $\widetilde{f}$. By definition, $\widetilde{f}$ is a polar function, and by Lemma 3.8 we know its development in spherical harmonics has the following form:

$$\widetilde{f}(\boldsymbol{x}) = \sum_{k=0}^{N} a_k Z_{\boldsymbol{v}}^{(k)}(\boldsymbol{x}) = \sum_{k=0}^{N} a_k\, c_{k,d}\, P_k(\boldsymbol{x} \cdot \boldsymbol{v}).$$

Hence, we see that $\widetilde{f}$ is the lift of the function $g : \mathbb{T} \to \mathbb{R}$ in the sense $\widetilde{f}(\boldsymbol{x}) = g(\arccos(\boldsymbol{x} \cdot \boldsymbol{v})/2\pi)$, where

$$g(\theta) = \sum_{k=0}^{N} a_k\, c_{k,d}\, P_k(\cos(2\pi\theta)).$$

Clearly g is a trigonometric polynomial of degree at most N. Also note that by Section 3.2.1, we know $g \not\equiv 0$. Moreover, $g(\theta) \geqslant 0$ for all $\mathrm{r}(\widetilde{f})/2\pi \leqslant |\theta| \leqslant 1/2$, and by Proposition 3.7 we have

$$0 \geqslant \int_{\mathbb{S}^d} \widetilde{f}(\boldsymbol{x})\, \mathrm{d}\mu(\boldsymbol{x}) = \int_{\mathbb{S}^d} g(\arccos{(\boldsymbol{x} \cdot \boldsymbol{v})}/2\pi)\, \mathrm{d}\mu(\boldsymbol{x}) = \frac{1}{2} \int_{\mathbb{T}} g(\theta)\, \mathrm{d}\mu_p(\theta).$$

Thus we conclude $g \in \mathcal{T}(\mu_p, N)$ and $2\pi\, \mathrm{r}(g) \leqslant \mathrm{r}(\widetilde{f}) \leqslant \mathrm{r}(f)$. Therefore, $2\pi\, \rho(\mu_p, N) \leqslant \rho(d, \mu, N)$. Combining these two inequalities, we conclude the assertion of Theorem 3.2.

3.3 Proof of Theorem 3.1

In this section, we present the proof of our result for trigonometric polynomials on the unit circle. Before giving the proof, we need to recall the machinery of reproducing kernel Hilbert spaces and orthogonal polynomials. In the first part of this section, we will collect relevant facts from [86, 87].

3.3.1 Reproducing kernel Hilbert spaces of polynomials

For each $n \geqslant 0$, denote by $\mathcal{P}_n(\mathbb{C})$ the set of polynomials of degree at most n with complex coefficients. Recall that, for $P \in \mathcal{P}_n(\mathbb{C})$, we denote by $P^{*,m}$ the reversed polynomial given by

$$P^{*,m}(z) = z^m \overline{P(1/\overline{z})}$$

and the shorthand notation $P^* = P^{*,\deg(P)}$.

3.3.1.1 Setup

Let $P(z)$ be a polynomial of degree $n+1$ such that

$$|P^*(z)| < |P(z)| \tag{3.8}$$

holds for all $z \in \mathbb{D}$.

Remark. The condition (3.8) is equivalent to the fact that $P(z)$ has no zeros in $\mathbb{D}$, and it has at least one zero in $\mathbb{C} \setminus \overline{\mathbb{D}}$.

For our purposes, we may assume that P has no zeros on the unit circle. However, this assumption can be dropped with a slight adjustment in several parts of the arguments presented here. Consider the Hilbert space $\mathcal{H}_n(P)$ that consists of polynomials in $\mathcal{P}_n(\mathbb{C})$ equipped with the inner product

$$\langle Q, R \rangle_{\mathcal{H}_n(P)} := \int_{\mathbb{T}} Q(z)\overline{R(z)}\, |P(z)|^{-2}\, \mathrm{d}t,$$

where $z = e^{2\pi i t}$. It is a well-known fact that the space $\mathcal{H}_n(P)$ is a reproducing kernel Hilbert space with this inner product and the function

$$K(w, z) = \frac{P(z)\overline{P(w)} - P^*(z)\overline{P^*(w)}}{1 - \overline{w}z}, \quad \overline{w}z \neq 1, \tag{3.9}$$

is the reproducing kernel[2] of the space; see for instance [85]. Thus, it satisfies the identity

$$\langle Q, K(w, \cdot) \rangle_{\mathcal{H}_n(P)} = Q(w),$$

for all $w \in \mathbb{C}$ and $Q \in \mathcal{H}_n(P)$. Since $K(w, w) = \langle K(w, \cdot), K(w, \cdot) \rangle_{\mathcal{H}_n(P)}$, we know $K(w, w) = 0$ implies $K(w, z) = 0$ for all $z \in \mathbb{C}$; by the expression of $K(w, z)$ we deduce $P(w) = P^*(w) = 0$. However, using the estimate (3.8) we see that $w \in \partial\mathbb{D}$, and by our assumption P has no zeros on $\partial\mathbb{D}$, a contradiction. Thus, $K(w, w) > 0$ for all $w \in \mathbb{C}$. All of these properties are closely paralleled by those of the de Branges spaces discussed in the previous chapter.

Consider the functions $A(z)$ and $B(z)$ given by

$$A(z) = \frac{1}{2}(P(z) + P^*(z)) \quad \text{and} \quad B(z) = \frac{i}{2}(P(z) - P^*(z)).$$

Observe that the estimate (3.8) implies $\deg(A) = \deg(B) = n+1$ and all of their zeros lie on the unit circle. Moreover, $A^*(z) = A(z)$ and $B^*(z) = B(z)$ and $P(z) = A(z) - iB(z)$. The reproducing kernel of the space $\mathcal{H}_n(P)$ can be represented in terms

[2] The values $K(z, z)$, for $z \in \partial\mathbb{D}$, are defined by passing to the limit $w \to z$.

of A and B as follows:

$$K(w,z) = \frac{2}{i}\frac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{1-\overline{w}z}, \quad \overline{w}z \neq 1. \tag{3.10}$$

Since the inner product of any two elements $K(\xi_1,\cdot)$ and $K(\xi_2,\cdot)$ becomes $K(\xi_1,\xi_2)$, the condition $A(\xi_1) = A(\xi_2) = 0$ gives $K(\xi_1,\xi_2) = 0$, i.e., $K(\xi_1,z)$ and $K(\xi_2,z)$ are orthogonal. Hence the set $\{K(\xi,z) : A(\xi) = 0\}$ forms an orthogonal basis for $\mathcal{H}_n(P)$.

Considering $e^{i\theta}P(z)$ instead of $P(z)$, one obtains the same Hilbert space $\mathcal{H}_n(P)$. As a general fact, for every $\theta \in \mathbb{R}$ the family of polynomials $\{K(\xi,z) : T_\theta(\xi) = 0\}$ forms an orthogonal basis for $\mathcal{H}_n(P)$, where

$$T_\theta(z) := e^{i\theta}P(z) - e^{-i\theta}P^*(z).$$

Using this basis and orthogonality, one obtains the following identity

$$\|Q\|^2_{\mathcal{H}_n(P)} = \sum_{T_\theta(\xi)=0} \frac{|Q(\xi)|^2}{K(\xi,\xi)},$$

for all polynomials $Q \in \mathcal{H}_n(P)$. This is a polynomial analogue of Theorem 22 of de Branges [41], which was proved in [86].

3.3.1.2 The phase function

Since the polynomial $P(z)$ has no zeros on the unit circle $\partial\mathbb{D}$, and the inequality (3.8) holds, we are allowed to define an analytic branch of $\log P(z)$ in a small neighborhood of the closed unit disk $\overline{\mathbb{D}}$. Define the *phase function* by

$$\varphi(t) := (n+1)\pi t - \arg P(e^{2\pi i t}),$$

for all $t \in \mathbb{R}$. Note that $\varphi(t)$ is a well-defined continuous function. Since the argument term is 1-periodic, we deduce that $\varphi(t+1) - \varphi(t) = \pi(n+1)$. The following result is polynomial analogue of the identity (2.21).

Lemma 3.10. *For all $t \in \mathbb{R}$ we have*

$$\varphi'(t) = \pi \cdot \frac{K(e^{2\pi i t}, e^{2\pi i t})}{|P(e^{2\pi i t})|^2} > 0.$$

Proof. Recall that $K(w,z)$ is given by (3.9) for all $w \neq z$. Passing to the limit $w \to z \in \partial\mathbb{D}$ we see that

$$K(z,z) = \frac{P(z)\overline{P'(z)} - P^*(z)\overline{(P^*)'(z)}}{-z}.$$

Hence, using the identity $P^*(z) = z^{n+1}\overline{P(1/\overline{z})}$ we deduce

$$K(e^{2\pi i t}, e^{2\pi i t}) = (n+1)|P(e^{2\pi i t})|^2 - 2\,\mathrm{Re}\big[e^{2\pi i t}P'(e^{2\pi i t})\overline{P(e^{2\pi i t})}\big].$$

Using the identity $\arg P(z) = \operatorname{Im} \log P(z)$ and the fact that $\log P(z)$ is analytic in a neighborhood of the unit circle, we conclude the assertion. $\square$

The following lemma collects several properties of the zeros of polynomials T_θ.

Lemma 3.11. *For each real number θ define the polynomial $T_\theta(z) := e^{i\theta} P(z) - e^{-i\theta} P^*(z)$. The following holds:*

(i) $\deg(T_\theta) = n+1$ *and all zeros of T_θ lie on the unit circle;*

(ii) *For a point $\xi = e^{2\pi i t}$ to be a zero of $T_\theta(z)$ it is necessary and sufficient that t satisfy the equation*
$$\varphi(t) \equiv \theta \pmod{\pi}. \tag{3.11}$$
As a consequence, all zeros of T_θ are simple.

(iii) *If $\theta_1 - \theta_2 \not\equiv 0 \pmod{\pi}$, then the zeros of T_{θ_1} and T_{θ_2} interlace on the unit circle.*

Proof. (i). First of all, note that the leading coefficient of T_θ is given by $e^{i\theta}\overline{P^*(0)} - e^{-i\theta}\overline{P(0)}$. Due to the estimate (3.8), this leading coefficient never vanishes, hence $\deg(T_\theta) = n+1$. By the inequality (3.8) we deduce that T_θ has no zeros in the unit disk $\mathbb{D}$. Similarly, the reversed inequality holds in $\mathbb{C} \setminus \overline{\mathbb{D}}$ and it implies that T_θ has no zeros in this set. Thus, part (i) is complete.

(ii). For a point $\xi = e^{2\pi i t}$ to satisfy $T_\theta(\xi) = 0$ it is necessary and sufficient that
$$\arg\{e^{i\theta} P(\xi)\} - \arg\{e^{-i\theta} P^*(\xi)\} \equiv 0 \pmod{2\pi}.$$
Equivalently,
$$2\theta + \arg\{P(\xi)\} - \arg\{\xi^{n+1}\overline{P(\xi)}\} \equiv 2\theta - 2\varphi(t) \equiv 0 \pmod{2\pi}.$$
Hence all the zeros of T_θ are obtained by solving (3.11). In particular, by Lemma 3.10 we deduce that all the zeros of T_θ are simple. Since $\varphi(t) \pmod{\pi}$ is 1-periodic, we can select the points $t_j \in \mathbb{T}$ such that the set of zeros of T_θ is $\{\xi_j = e^{2\pi i t_j} : j = 1, 2, \ldots, n+1\}$.

(iii). Let $t_1 < t_2 < \cdots < t_{n+1} < t_1 + 1$ be real numbers such that $\xi_j = e^{2\pi i t_j}$, $j = 1, 2, \ldots, n+1$, are the zeros of $T_\theta(z)$. By Lemma 3.10 we know that
$$\varphi(t_1) < \varphi(t_2) < \cdots < \varphi(t_{n+1}) < \varphi(t_1 + 1) = \pi(n+1) + \varphi(t_1).$$
By part (ii), $\varphi(t_j) = \theta + \pi m_j$ for some $m_j \in \mathbb{Z}$. Combining these two facts we deduce $m_{j+1} - m_j = 1$, and hence $\varphi(t_{j+1}) - \varphi(t_j) = \pi$. We now prove part (iii) by contradiction. Suppose that at least two zeros of T_{θ_1} lie between two consecutive zeros of T_{θ_2}. Without loss of generality, we may assume these zeros are
$$e^{2\pi i t_1}, \quad e^{2\pi i s_1}, \quad e^{2\pi i s_2}, \quad e^{2\pi i t_2},$$

where $t_1 < s_1 < s_2 < t_2 < t_3 < \cdots < t_{n+1} < t_1 + 1$. By the above and the strict monotonicity of φ, we obtain

$$\pi = \varphi(t_2) - \varphi(t_1) > \varphi(s_2) - \varphi(s_1) = \pi,$$

a contradiction. Hence the desired conclusion follows. □

3.3.2 Orthogonal polynomials on the unit circle

In this subsection, we give a brief overview of the theory of orthogonal polynomials on the unit circle. A more detailed treatment can be found in [108].

3.3.2.1 OPUC for non-trivial measures

Recall from Section 3.1.1.2 that we are working with probability measures on $\partial\mathbb{D}$ and, for a non-trivial probability measure μ, consider the Hilbert space $L^2(\partial\mathbb{D},\mu)$ with the inner product

$$\langle f, g\rangle := \int_{\partial\mathbb{D}} f(z)\overline{g(z)}\,\mathrm{d}\mu(z).$$

Let $\Phi_n(z) = \Phi_n(z; \mathrm{d}\mu)$ be the sequence of monic orthogonal polynomials in $L^2(\partial\mathbb{D},\mu)$ and let $\varphi_n(z) := \Phi_n(z)/\|\Phi_n\|_2$ be the sequence of orthonormal polynomials. These polynomials satisfy a recursion formula: Φ_n satisfies the *Szegö recursion*

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha_n}\Phi_n^*(z), \quad n = 0, 1, \ldots, \tag{3.12}$$

where $\alpha_n = \alpha_n(\mu) \in \mathbb{C}$ are called the *Verblunsky coefficients of* μ and $\Phi_n^* = \Phi_n^{*,n}$ is the reversed polynomial of Φ_n. One can write the recursion (3.12) in terms of reversed polynomials as follows:

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z\Phi_n(z). \tag{3.13}$$

We see that (3.12) implies $|\alpha_n| < 1$. Indeed, $\langle z\Phi_n(z), z\Phi_n(z)\rangle = \langle \Phi_n(z), \Phi_n(z)\rangle$ and

$$\|\Phi_n(z)\|^2 = \|z\Phi_n(z)\|^2 = \|\Phi_{n+1}(z) + \overline{\alpha_n}\Phi_n^*(z)\|^2 = \|\Phi_{n+1}(z)\|^2 + |\alpha_n|^2\|\Phi_n(z)\|^2.$$

Moreover, we have the identity

$$\|\Phi_{n+1}\|^2 = (1 - |\alpha_n|^2)\|\Phi_n\|^2 = \prod_{j=0}^{n}(1 - |\alpha_j|^2). \tag{3.14}$$

3.3.2.2 OPUC for trivial measures

When μ is a probability measure with $\#\mathrm{supp}(\mu) = m$, the Gram–Schmidt process terminates at $\Phi_m(z; \mathrm{d}\mu)$. Thus, we can still define $\Phi_0, \Phi_1, \ldots, \Phi_m$ and $\varphi_0, \varphi_1, \ldots, \varphi_{m-1}$. All the identities (3.12), (3.13), and (3.14) remain valid for these polynomials. The

Verblunsky coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ are well-defined and satisfy $|\alpha_k| < 1$ for $0 \leqslant k \leqslant m-2$ and $|\alpha_{m-1}| = 1$, which matches the fact that $\|\Phi_m\|_2 = 0$.

3.3.2.3 General facts about OPUC

The following lemma collects the relevant results concerning orthogonal polynomials associated with probability measures on the unit circle.

Lemma 3.12. *Let μ be a probability measure on $\partial\mathbb{D}$.*

(i) *If $\mathrm{d}\mu(z) = \mathrm{d}\mu(\overline{z})$, then $\Phi_n(z)$ has real coefficients.*

(ii) *$\varphi_n(z)$ has all its zeros in $\mathbb{D}$ and $\varphi_n^*(z)$ has all its zeros in $\mathbb{C} \setminus \overline{\mathbb{D}}$ for $n \geqslant 1$. Therefore,*

$$|\varphi_n(z)| < |\varphi_n^*(z)| \quad \text{and} \quad |\Phi_n(z)| < |\Phi_n^*(z)|, \quad z \in \mathbb{D}.$$

(iii) *If $\varphi_0, \varphi_1, \ldots, \varphi_n$ exist and are unique, then they satisfy the Christoffel–Darboux identity*

$$\sum_{j=0}^{n-1} \varphi_j(z)\overline{\varphi_j(w)} = \frac{\varphi_n^*(z)\overline{\varphi_n^*(w)} - \varphi_n(z)\overline{\varphi_n(w)}}{1 - \overline{w}z},$$

for all $\overline{w}z \neq 1$. For $\overline{w}z = 1$, the right-hand side is understood as a limit.

Proof. For part (i), see [21, Lemma 25]. For part (ii), see [108, § 1.7]. For part (iii), see [108, Theorem 2.2.7]. □

Remark. If the measure μ has exactly m points in its support, then the conclusion of part (i) is valid for $n \leqslant m$ and part (ii) is valid only for $n \leqslant m-1$. For $n = m$, the polynomial $\Phi_m(z)$ has all of its zeros on $\partial\mathbb{D}$. If we further assume the symmetry $\mathrm{d}\mu(z) = \mathrm{d}\mu(\overline{z})$, then $\Phi_m^*(z) = (-1)^m \Phi_m(z)$, and the orthonormal polynomial $\varphi_m(z)$ is not well-defined.

3.3.3 Quadrature formulas for Laurent polynomials

Consider the space Γ_n of Laurent polynomials W of the form

$$W(z) = \sum_{k=-n}^{n} a_k z^k,$$

where $a_k \in \mathbb{C}$. As a combined application of the two theories outlined above, we establish a quadrature formula for functions in Γ_n.

3.3.3.1 Auxiliary spaces

We consider the reproducing kernel Hilbert space of polynomials $\mathcal{H}_n(P)$ for particular choices of P. We set

$$P(z) = \varphi_{n+1}^*(z; \mathrm{d}\mu), \tag{3.15}$$

whenever the latter exists and is unique. If the measure μ is trivial and $\#\mathrm{supp}(\mu) = n+1$, then we define $P(z)$ by

$$P(z) = (1+i)\varphi_n^*(z;\mathrm{d}\mu) + iz\varphi_n(z;\mathrm{d}\mu). \tag{3.16}$$

Proposition 3.13. *Let P be given by (3.16). Then the following holds:*

(i) $\deg(P) = n+1$ *and* $P^*(z) = (1-i)z\varphi_n(z;\mathrm{d}\mu) - i\varphi_n^*(z;\mathrm{d}\mu)$*;*

(ii) $|P^*(z)| < |P(z)|$ *holds for all* $z \in \mathbb{D}$*;*

(iii) *P has no zeros on the unit circle $\partial\mathbb{D}$.*

Proof. (i). The leading coefficient of $P(z)$ is non-zero, hence $\deg(P) = n+1$. Thus, $P^*(z) = z^{n+1}\overline{P(1/\overline{z})} = (1-i)z\varphi_n(z;\mathrm{d}\mu) - i\varphi_n^*(z;\mathrm{d}\mu)$.

(ii). We compute

$$|P(z)|^2 - |P^*(z)|^2 = |\varphi_n^*(z)|^2 - |z\varphi_n(z)|^2.$$

By part (ii) of Lemma 3.12 we know

$$|z\varphi_n(z)| < |\varphi_n(z)| < |\varphi_n^*(z)|$$

holds for each $z \in \mathbb{D}$. Therefore the assertion follows.

(iii). On the unit circle, $|z\varphi_n(z)| = |\varphi_n(z)| = |\varphi_n^*(z)| > 0$. Therefore, for any $z \in \partial\mathbb{D}$,

$$|(1+i)\varphi_n^*(z;\mathrm{d}\mu)| = \sqrt{2}|z\varphi_n(z)| > |iz\varphi_n(z)|,$$

so $P(z) \neq 0$. □

In each case, the polynomial P has no zeros on $\partial\mathbb{D}$ and satisfies (3.8). Moreover, the reproducing kernel of $\mathcal{H}_n(P)$ is

$$K_n(w,z) = \frac{P(z)\overline{P(w)} - P^*(z)\overline{P^*(w)}}{1-\overline{w}z}, \quad \overline{w}z \neq 1. \tag{3.17}$$

Using these spaces, we obtain a family of quadrature formulas parametrized by $\theta \in \mathbb{T}$ for each probability measure μ. We prove the following generalization of the identity from [21, Corollary 26].

Lemma 3.14. *Let μ be a probability measure on $\partial\mathbb{D}$ and $W \in \Gamma_n$. Let P be given by (3.15) or by (3.16). For each fixed $\theta \in \mathbb{R}$ we have*

$$\int_{\partial\mathbb{D}} W(z)\,\mathrm{d}\mu(z) = \int_{\partial\mathbb{D}} \frac{W(z)}{|P(z)|^2}\,\mathrm{d}t = \sum_{T_\theta(\xi)=0} \frac{W(\xi)}{K_n(\xi,\xi)}, \tag{3.18}$$

where $z = e^{2\pi i t}$ and $T_\theta(z) = e^{i\theta}P(z) - e^{-i\theta}P^*(z)$.

Proof. Recall the choices of $P(z)$ from (3.15) and (3.16). Using (3.17) and the Christoffel–Darboux identity from Lemma 3.12 we obtain

$$K_n(w,z) = \sum_{j=0}^{n} \varphi_j(z;\mathrm{d}\mu)\,\overline{\varphi_j(w;\mathrm{d}\mu)}. \tag{3.19}$$

Fix $\theta \in \mathbb{R}$ and let $\xi_1, \xi_2, \ldots, \xi_{n+1}$ denote the zeros of $T_\theta(z)$. Set

$$q_j(z) = \frac{K_n(\xi_j, z)}{\sqrt{K_n(\xi_j,\xi_j)}}, \quad 1 \leqslant j \leqslant n+1.$$

Recalling that $\mathcal{H}_n(P) = \mathcal{H}_n(e^{i\theta}P)$ and the formula (3.10), the orthogonality of $K_n(\xi_j,\cdot)$ and $K_n(\xi_k,\cdot)$ in $\mathcal{H}_n(P)$ gives $K_n(\xi_j,\xi_k) = 0$ for $j \neq k$. Hence $q_1, q_2, \ldots, q_{n+1}$ is a basis of $\mathcal{P}_n(\mathbb{C})$. Using (3.19) we compute

$$\begin{aligned}
\langle q_j, q_k\rangle_{L^2(\partial\mathbb{D},\mu)} &= \frac{\langle K_n(\xi_j,\cdot), K_n(\xi_k,\cdot)\rangle_{L^2(\partial\mathbb{D},\mu)}}{\sqrt{K_n(\xi_j,\xi_j)}\sqrt{K_n(\xi_k,\xi_k)}} \\
&= \frac{1}{\sqrt{K_n(\xi_j,\xi_j)K_n(\xi_k,\xi_k)}} \sum_{0\leqslant r,s\leqslant n} \overline{\varphi_r(\xi_j)}\varphi_s(\xi_k)\,\langle \varphi_r, \varphi_s\rangle_{L^2(\partial\mathbb{D},\mu)} \\
&= \frac{K_n(\xi_j,\xi_k)}{\sqrt{K_n(\xi_j,\xi_j)K_n(\xi_k,\xi_k)}} = \langle q_j, q_k\rangle_{\mathcal{H}_n(P)}.
\end{aligned}$$

Since $q_1, \ldots, q_{n+1}$ is a basis of $\mathcal{P}_n(\mathbb{C})$, we conclude that the inner products $\langle\cdot,\cdot\rangle_{L^2(\partial\mathbb{D},\mu)}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}_n(P)}$ coincide on $\mathcal{P}_n(\mathbb{C})$.

We now turn to the statement of the lemma. By linearity, it suffices to prove (3.18) for $W_j(z) = z^j$ with $-n \leqslant j \leqslant n$. For $0 \leqslant j \leqslant n$ we have

$$\int_{\partial\mathbb{D}} z^j \,\mathrm{d}\mu(z) = \langle W_j, 1\rangle_{L^2(\partial\mathbb{D},\mu)} = \langle W_j, 1\rangle_{\mathcal{H}_n(P)} = \int_{\partial\mathbb{D}} \frac{z^j}{|P(z)|^2}\,\mathrm{d}t.$$

Moreover, using $\langle q_j, q_k\rangle = \delta_{j,k}$ to expand $1 = \sum_{k=1}^{n+1}\langle 1, q_k\rangle q_k(z)$, we obtain

$$\langle W_j, 1\rangle_{\mathcal{H}_n(P)} = \sum_{k=1}^{n+1} \overline{\langle 1, q_k\rangle}\langle W_j, q_k\rangle = \sum_{k=1}^{n+1} \frac{W_j(\xi_k)}{K_n(\xi_k,\xi_k)},$$

where the last identity uses $\langle Q, q_k\rangle = Q(\xi_k)/\sqrt{K_n(\xi_k,\xi_k)}$ for all $Q \in \mathcal{P}_n(\mathbb{C}) = \mathcal{H}_n(P)$.

For $-n \leqslant j \leqslant -1$, we write

$$\int_{\partial\mathbb{D}} z^j \,\mathrm{d}\mu(z) = \langle 1, W_{-j}\rangle = \int_{\partial\mathbb{D}} \frac{\overline{W_{-j}(z)}}{|P(z)|^2}\,\mathrm{d}t = \int_{\partial\mathbb{D}} \frac{z^j}{|P(z)|^2}\,\mathrm{d}t.$$

The final identity in (3.18) follows similarly, using $\overline{\xi_k^{-j}} = \xi_k^j$. □

Remark. The contents of Lemma 3.14 are also known as an $(n+1)$-point quadrature formula; a different proof can be found in [75, § 6 and § 7]. There, the polynomials T_θ are referred to as *para-orthogonal* polynomials. We chose to present the result through a general theory of Hilbert spaces of polynomials.

3.3.4 Proof of Theorem 3.1

We first prove part (ii) by an explicit construction. Let $\#\mathrm{supp}(\mu) = M \leqslant N < \infty$ and let $z_1, z_2, \ldots, z_M \in \partial\mathbb{D}$ be the points in $\mathrm{supp}(\mu)$. Since $\rho(\mu, N+1) \leqslant \rho(\mu, N)$, it suffices to show $\rho(\mu, M) = 0$. Consider the trigonometric polynomial

$$f(x) = |P(e^{2\pi i x})|^2,$$

where $P(z) = \prod_{j=1}^{M}(z - z_j)$. Clearly $\deg(f) = M$ and $\mathrm{r}(f) = 0$ since $f \geqslant 0$ everywhere on $\mathbb{T}$. Moreover,

$$\int_{\mathbb{T}} f(x)\,\mathrm{d}\mu(x) = \sum_{j=1}^{M} \mu(\{z_j\})|P(z_j)|^2 = 0.$$

Therefore $f \in \mathcal{T}(\mu, M)$ and $f \not\equiv 0$, hence $\rho(\mu, M) = 0$.

We now prove part (i). For the lower bound, we argue by contradiction. Suppose there exists $0 \not\equiv f \in \mathcal{T}(\mu, N)$ with $\mathrm{r}(f) < \theta_1$ and

$$f(x) = \sum_{n=-N}^{N} \widehat{f}(n)\, e^{2\pi i n x}.$$

Since f extends to a holomorphic function and $f \in \mathcal{T}(\mu, N)$, there exists $s \in (\mathrm{r}(f), \theta_1)$ such that $f(s) > 0$. Set $P(z)$ as in Section 3.3.3.1, i.e., $P(z) = \varphi^*_{N+1}(z; \mathrm{d}\mu)$ or $P(z) = (1+i)\varphi^*_N(z; \mathrm{d}\mu) + iz\varphi_N(z; \mathrm{d}\mu)$. Also set $A_{N+1}(z) = (P(z) + P^*(z))/2$ and consider the phase function φ defined in Section 3.3.1.2. Take any $\theta \in \mathbb{R}$ satisfying

$$\theta \equiv \varphi(s) \pmod{\pi}.$$

By Lemma 3.11, $T_\theta(z)$ has a simple zero at $e^{2\pi i s}$, so T_θ and $T_{\pi/2} = 2iA_{N+1}$ have distinct roots, giving $\theta \not\equiv \pi/2 \pmod{\pi}$. By the interlacing property of zeros of T_θ and $T_{\pi/2}$, the polynomial T_θ has no zeros in the arc $\{e^{2\pi i x} : -\mathrm{r}(f) \leqslant x \leqslant \mathrm{r}(f)\}$. Now let

$$W_f(z) = \sum_{n=-N}^{N} \widehat{f}(n)\, z^n$$

be the Laurent polynomial associated with f. Using $\int_{\mathbb{T}} f(x)\,\mathrm{d}\mu(x) \leqslant 0$ and Lemma 3.14 we reach the contradiction

$$0 \geqslant \int_{\mathbb{T}} f(x)\,\mathrm{d}\mu(x) = \int_{\partial\mathbb{D}} W_f(z)\,\mathrm{d}\mu(z) = \sum_{T_\theta(\xi)=0} \frac{W_f(\xi)}{K_N(\xi,\xi)} \geqslant \frac{f(s)}{K_N(e^{2\pi i s}, e^{2\pi i s})} > 0.$$

Therefore $\mathrm{r}(f) \geqslant \theta_1$.

To complete the proof, consider the function

$$f(x) = \frac{|A_{N+1}(e^{2\pi ix})|^2}{\cos(2\pi\theta_1) - \cos(2\pi x)}.$$

We claim that f is a real-valued trigonometric polynomial of degree exactly N. The numerator $|A_{N+1}(e^{2\pi ix})|^2$ is a non-negative trigonometric polynomial vanishing at $x = \theta_1$. In either case of the definition

$$A_{N+1}(z) = \frac{\varphi_{N+1}(z;\mathrm{d}\mu) + \varphi^*_{N+1}(z;\mathrm{d}\mu)}{2} \quad \text{or} \quad A_{N+1}(z) = \frac{z\varphi_N(z;\mathrm{d}\mu) + \varphi^*_N(z;\mathrm{d}\mu)}{2},$$

A_{N+1} has real coefficients by part (i) of Lemma 3.12, and hence the numerator also vanishes at $x = -\theta_1$. Since the denominator $\cos(2\pi\theta_1) - \cos(2\pi x)$ vanishes precisely at $x = \pm\theta_1$, the apparent singularities of f at these points are removable. Therefore f is a real-valued trigonometric polynomial of degree exactly N. By construction, f changes sign only at $x = \pm\theta_1$ and vanishes with multiplicity exactly two at every point $|x| \neq \theta_1$ for which $A_{N+1}(e^{2\pi ix}) = 0$. Applying (3.18) yields $\int_{\mathbb{T}} f(x)\,\mathrm{d}\mu(x) = 0$, so $f \in \mathcal{T}(\mu, N)$ with $\mathrm{r}(f) = \theta_1$. Combined with the lower bound, this gives $\rho(\mu, N) = \theta_1$.

It remains to establish the uniqueness of the extremizers. Let $g \in \mathcal{T}(\mu, N)$ satisfy $\mathrm{r}(g) = \theta_1$. Applying the quadrature formula to the Laurent polynomial W_g gives

$$0 \geqslant \int_{\mathbb{T}} g(x)\,\mathrm{d}\mu(x) = \sum_{A_{N+1}(\xi)=0} \frac{W_g(\xi)}{K_N(\xi,\xi)} \geqslant 0,$$

so equality holds throughout, forcing $W_g(\xi) = 0$ at every zero of A_{N+1}. Since $\mathrm{r}(g) = \theta_1$, we have $W_g(e^{2\pi ix}) \geqslant 0$ for all $\theta_1 \leqslant |x| \leqslant 1/2$. Consequently, g must vanish with even multiplicity at each point x with $|x| \neq \theta_1$ such that $A_{N+1}(e^{2\pi ix}) = 0$, which implies that g is a scalar multiple of f as given by (3.3). Since $\deg(g) \leqslant N = \deg(f)$, we conclude $g(x) = Cf(x)$ for some constant $C > 0$. This completes the proof.

3.4 Applications: proofs of Corollary 3.4 and Theorem 3.6

3.4.1 Relation between (EP1.2) and (EP3.2)

We start with an observation that establishes the connection between these two problems. Let the function $\mathrm{T} : [0,1] \to [0,1/2]$ be given by

$$\mathrm{T}(x) = \frac{1}{2\pi}\arccos(1-2x).$$

Proposition 3.15. *Let μ be a measure on $[0,1]$ and set its push-forward $\breve{\nu} := \mathrm{T}_{\#}\,\mu$. The extremal problems (EP1.2) and (EP3.2) are related by the identity*

$$\Omega(\mu, N) = \frac{1-\cos\big(2\pi\,\rho(\nu,N)\big)}{2},$$

where ν is the symmetric extension of $\breve{\nu}$ to $\mathbb{T}$.

Proof. The equivalence is established by two explicit constructions. If $p(x)$ is a polynomial in $\mathcal{P}(\mu,N)$, then the trigonometric polynomial $f(t) := p(\frac{1-\cos(2\pi t)}{2})$ belongs to $\mathcal{T}(\nu,N)$ and satisfies $\mathrm{r}(f) = \frac{1}{2\pi}\arccos(1-2\,\mathrm{r}(p))$. Indeed,

$$\int_{\mathbb{T}} f(t)\,\mathrm{d}\nu(t) = 2\int_0^{1/2} f(t)\,\mathrm{d}\breve{\nu}(t) = 2\int_0^1 p(x)\,\mathrm{d}\mu(x) \leqslant 0.$$

Conversely, given an even trigonometric polynomial $g \in \mathcal{T}(\nu,N)$, define

$$q(x) := (g\circ \mathrm{T})(x) = \sum_{k=0}^{N} a_k \cos\big(k\arccos(1-2x)\big) = \sum_{k=0}^{N} a_k\, T_k(1-2x),$$

where the last equality uses the Chebyshev polynomials T_k. Then q is a polynomial of degree at most N with $q \in \mathcal{P}(\mu,N)$. Together, these two constructions establish the equivalence between $\rho(\nu,N)$ and $\Omega(\mu,N)$ and show that the extremizers of both problems are in one-to-one correspondence. □

3.4.2 Proof of Corollary 3.4

Recall that the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ of degree n with parameters $\alpha,\beta > -1$ is defined by the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big[(1-x)^{n+\alpha}(1+x)^{n+\beta}\big], \quad x\in(-1,1).$$

When $\alpha=\beta=0$ the Jacobi polynomials reduce to the Legendre polynomials, and when $\alpha=\beta$ they become the Gegenbauer (ultraspherical) polynomials. Jacobi polynomials are the orthogonal polynomials associated with the measure

$$\mathrm{d}\omega_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}\mathrm{d}x, \quad x\in(-1,1).$$

Further background can be found in [114] and [72].

Let $\nu_{\alpha,\beta}$ be the measure on $[0,1]$ given by $\mathrm{d}\nu_{\alpha,\beta} = x^{\alpha}(1-x)^{\beta}\mathrm{d}x$. It is related to $\omega_{\alpha,\beta}$ via

$$\int_0^1 f(x)\,\mathrm{d}\nu_{\alpha,\beta}(x) = 2^{-(\alpha+\beta+1)}\int_{-1}^1 f((1-y)/2)\,\mathrm{d}\omega_{\alpha,\beta}(y).$$

From this identity, $\Psi_n(x;\mathrm{d}\nu_{\alpha,\beta}) = C_{\alpha,\beta}P_n^{(\alpha,\beta)}(1-2x)$ for $x\in[0,1]$. Moreover, $(1-x)\,\mathrm{d}\nu_{\alpha,\beta}(x) = \mathrm{d}\nu_{\alpha,\beta+1}(x)$. The polar part μ_{d-1} of the measure σ_d is given by (3.5) and is related to these measures with $\alpha = \beta = (d-2)/2$ via

$$\int_0^{1/2} f(\sin^2(\pi\theta)) \sin(2\pi\theta)^{d-1} \, \mathrm{d}\theta = \frac{2^{2\alpha}}{\pi} \int_0^1 f(x) \, \mathrm{d}\nu_{\alpha,\alpha}(x).$$

By Theorem 3.2, $\rho(d, \sigma_d, N) = 2\pi \, \rho(\mu_{d-1}, N)$. Applying Proposition 3.15 and Theorem 3.3 we conclude that

$$\rho(\mu_{d-1}, 2n-1) = \frac{1}{2\pi} \arccos(1 - 2\Omega(\nu_{\alpha,\alpha}, 2n-1)) = \frac{1}{2\pi} \arccos(1 - 2x_{n,1})$$

and

$$\rho(\mu_{d-1}, 2n) = \frac{1}{2\pi} \arccos(1 - 2\Omega(\nu_{\alpha,\alpha}, 2n)) = \frac{1}{2\pi} \arccos(1 - 2\xi_{n,1}),$$

where $x_{n,1}$ and $\xi_{n,1}$ are the smallest zeros of $\Psi_n(\cdot; \mathrm{d}\nu_{\alpha,\alpha})$ and $\Psi_n(\cdot; \mathrm{d}\nu_{\alpha,\alpha+1})$ on $[0,1]$, respectively. Using the identity $\Psi_n(x; \mathrm{d}\nu_{\alpha,\beta}) = C_{\alpha,\beta} P_n^{(\alpha,\beta)}(1-2x)$, we see that $1-2x_{n,1}$ is the largest zero of $P_n^{(\alpha,\alpha)}$ and $1-2\xi_{n,1}$ is the largest zero of $P_n^{(\alpha,\alpha+1)}$. This concludes the proof of Corollary 3.4. The result of Corollary 3.5 is an immediate consequence.

3.4.3 Proof of Theorem 3.6

First observe that if $N \geqslant m = \#\mathrm{supp}(\mu)$, then $\rho(\mu, N) = 0$. Suppose ξ is a set of nodes for a $(N+1)$-point quadrature formula, so there exist $\lambda_j > 0$ for $j = 1, 2, \ldots, N+1$. Taking the trigonometric polynomial $f(x) = |\Phi_m(e^{2\pi i x}; \mathrm{d}\mu)|^2$, we must have

$$0 = \int_{\mathbb{T}} f(x) \, \mathrm{d}\mu(x) = \sum_{j=1}^{N+1} \lambda_j \, f(\xi_j).$$

This forces f to vanish at each ξ_j, which is absurd since $\Phi_m(z; \mathrm{d}\mu)$ is a degree-m polynomial and cannot have $N+1 \geqslant m+1$ zeros. Thus no such quadrature formula exists, and $\Xi(\mu, N+1) = 0$.

Now assume $N < \#\mathrm{supp}(\mu)$, so $\rho(\mu, N) > 0$. By Lemma 3.14, several quadrature formulas exist, and we may assume $\Xi(\mu, N+1) > 0$. Fix $\varepsilon > 0$ sufficiently small and let ξ be a set of $N+1$ nodes with

$$\min\{|\xi_j| : \xi_j \in \xi\} > \Xi(\mu, N+1) - \varepsilon,$$

with associated weights $\lambda_j > 0$ for which (3.6) holds. Suppose for contradiction that there exists a non-trivial $f \in \mathcal{T}(\mu, N)$ with $\mathrm{r}(f) < \min\{|\xi_j| : \xi_j \in \xi\}$. Since f is non-negative at every node and has non-positive μ-integral, the quadrature identity (3.6) gives

$$0 \geqslant \int_{\mathbb{T}} f(x) \, \mathrm{d}\mu(x) = \sum_{j=1}^{N+1} \lambda_j \, f(\xi_j) \geqslant 0,$$

forcing f to vanish with even multiplicity at every point ξ_j with $|\xi_j| < 1/2$. However, by a classical result on the number of zeros of trigonometric polynomials [126, Chapter X, Theorem 1.7], a non-zero polynomial f of degree N has at most $2N$ zeros. Since the $N+1$ nodes of ξ contribute at least $2N+1 > 2N$ zeros, this is a contradiction. Therefore, for every $f \in \mathcal{T}(\mu, N)$,

$$\mathrm{r}(f) \geqslant \min\{|\xi_j| : \xi_j \in \xi\} > \Xi(\mu, N+1) - \varepsilon.$$

Taking the infimum over f and letting $\varepsilon \to 0$ gives $\rho(\mu, N) \geqslant \Xi(\mu, N+1)$.

For the converse, we construct a specific quadrature formula achieving equality. This was already done in the proof of Theorem 3.1: the zeros of $A_{N+1}(z; \mathrm{d}\mu)$ are the quadrature nodes for Laurent polynomials. Let $\theta_1, \theta_2, \ldots, \theta_{N+1}$ be such that $A_{N+1}(e^{2\pi i\theta_k}) = 0$ for each $1 \leqslant k \leqslant N+1$. Then $\rho(\mu, N) = \theta_1$ and by (3.18) the set $\{\theta_1, \ldots, \theta_{N+1}\}$ is a set of quadrature nodes, so $\Xi(\mu, N+1) \geqslant \theta_1 = \rho(\mu, N)$. This completes the proof.

3.5 Solution for polynomials: proof of Theorem 3.3

Proposition 3.15 yields the solution to (EP3.2) by lifting the measure to the circle $\mathbb{T}$ and applying Theorem 3.1. Here we present an alternative direct approach, inspired by the first proof of Theorem 2.2 in Section 2.3.2.1: we reduce (EP3.2) to an extremal problem involving the multiplication by x operator, then solve it via Gauss quadrature formulas on the real line.

For each finite Borel measure μ on $[0,1]$ we define its *degeneracy threshold*

$$d(\mu) = \inf \left\{ \deg(h) \; : \; \begin{array}{l} h \text{ is a polynomial, } h \not\equiv 0 \\ h \geqslant 0 \text{ on } [0,1], \text{ and } \displaystyle\int_0^1 h(x)\mathrm{d}\mu(x) = 0 \end{array} \right\}, \tag{3.20}$$

and set $d(\mu) = \infty$ if no such polynomial h exists.

Since $h \geqslant 0$ on $[0,1]$ forces every zero lying in the open interval $(0,1)$ to be a zero of even multiplicity, we deduce that

$$d(\mu) = 2\#\mathrm{supp}(\mu) - \gamma_0 - \gamma_1,$$

as stated in Section 3.1.3.

From this, the first part of Theorem 3.3 follows easily. If $N \geqslant d(\mu)$, consider the polynomial

$$p(x) = x^{\gamma_0}(1-x)^{\gamma_1} \prod_{\substack{t_k \in \mathrm{supp}(\mu) \\ 0 < t_k < 1}} (x - t_k)^2.$$

Clearly $\mathrm{r}(p) = 0$ and $\deg(p) = d(\mu)$. Hence $p \in \mathcal{P}(\mu, N)$, which implies $\Omega(\mu, N) = 0$.

We now turn to the case $1 \leqslant N < d(\mu)$. In this case, it follows from (3.20) that $\mathrm{r}(p) > 0$ for any $p \in \mathcal{P}(\mu, N)$.

3.5.1 Reduction to an equivalent problem

We begin by reducing the family $\mathcal{P}(\mu, N)$ to a more specific family of polynomials, in the spirit of a result from Section 2.3.1.3. For that, we use the following factorization lemma.

Lemma 3.16 (cf. Proposition 2.11)**.** *Let $n \geqslant 1$ and let $0 < t_1 < t_2 < \cdots < t_n < 1$ be distinct real numbers. Define the monic polynomial*

$$P(x) = \prod_{k=1}^{n} (x - t_k).$$

For any $r > t_n$, there exist polynomials Q and R, both non-negative on $\mathbb{R}$, such that:

(i) *$P(x) = (x - r)\, Q(x) + R(x)$ for all $x \in \mathbb{R}$;*

(ii) *the degrees of Q and R are given by*

$$\deg(Q) = \begin{cases} n-1, & \text{if } n \text{ is odd,} \\ n-2, & \text{if } n \text{ is even,} \end{cases} \qquad \deg(R) = \begin{cases} n-1, & \text{if } n \text{ is odd,} \\ n, & \text{if } n \text{ is even.} \end{cases}$$

The proof proceeds by induction on n and is a straightforward adaptation of the argument in Proposition 2.11; we therefore omit it.

Proposition 3.17. *Given μ and $1 \leqslant N < d(\mu)$. For any polynomial $p \in \mathcal{P}(\mu, N)$ there exists a polynomial $q \in \mathcal{P}(\mu, N)$ satisfying the following:*

(i) *$\mathrm{r}(q) = \mathrm{r}(p)$;*

(ii) *$q(x)$ changes sign only at $x = \mathrm{r}(q) = \mathrm{r}(p)$;*

(iii) *$\deg(q) = N$.*

Proof. Since $1 \leqslant N < d(\mu)$ we may assume there exists $p \in \mathcal{P}(\mu, N)$ with $0 < \mathrm{r}(p) < 1$. We first consider a simpler situation and reduce everything else to this simpler case. If the polynomial p changes its sign (in the interval $[0, 1]$) only at $\mathrm{r}(p)$ we can take

$$q(x) = \left(\frac{1-x}{1-\mathrm{r}(p)}\right)^{N-\deg(p)} p(x).$$

Note that $0 \leqslant q(x) \leqslant p(x)$ on $x \in [\mathrm{r}(p), 1]$, and $q(x) \leqslant p(x) \leqslant 0$ on $x \in [0, \mathrm{r}(p)]$. Hence,

$$\int_0^1 q(x)\, \mathrm{d}\mu(x) \leqslant \int_0^1 p(x)\, \mathrm{d}\mu(x) \leqslant 0$$

Now, it is not difficult to see that q satisfies all the necessary conditions. If $\deg(p) < N$, then using the fact that $p - q \geqslant 0$ on $[0, 1]$ and (3.20) we deduce the first inequality is strict. Therefore, in such a case we can produce a polynomial $q_1 = q - \int_0^1 q(x)\,\mathrm{d}\mu(x)$ so that $\mathrm{r}(q_1) < \mathrm{r}(q)$.

Now, suppose p changes sign at points $0 < t_1 < t_2 < \cdots < t_n < \mathrm{r}(p) < 1$, so that

$$p(x) = p_0(x)\,(x - \mathrm{r}(p))\prod_{k=1}^{n}(x - t_k),$$

where $p_0 \geqslant 0$ on $[0,1]$. We first produce a polynomial with a single sign change that lies in $\mathcal{P}(\mu, N)$ then apply the previous case.

Applying Lemma 3.16 with $r = \mathrm{r}(p)$, we obtain non-negative polynomials Q,R with $\deg(Q) \leqslant n-1$ and $\deg(R) \leqslant n$ satisfying

$$p(x) = p_0(x)\,(x - \mathrm{r}(p))^2\,Q(x) + p_0(x)\,(x - \mathrm{r}(p))\,R(x).$$

Denote by $p_1(x) = p_0(x)\,(x - \mathrm{r}(p))^2\,Q(x)$ and $p_2(x) = p_0(x)\,R(x)$. Since $p_0 \geqslant 0$ on $[0,1]$, the polynomials $p_1(x)$ and $p_2(x)$ are both non-negative on $[0,1]$. Moreover, we have a polynomial $\widetilde{p}(x) = (x - \mathrm{r}(p))p_2(x)$ which satisfies $\mathrm{r}(\widetilde{p}) = \mathrm{r}(p)$, and $p(x) - \widetilde{p}(x) = p_1(x) \geqslant 0$ on $[0,1]$. Hence,

$$\int_0^1 \widetilde{p}(x)\,\mathrm{d}\mu(x) < \int_0^1 p(x)\,\mathrm{d}\mu(x) \leqslant 0. \tag{3.21}$$

Thus $\widetilde{p} \in \mathcal{P}(\mu, N)$ and $\widetilde{p}$ changes sign only at $x = \mathrm{r}(p)$. Thus we are in the previous situation. Note that, the first inequality in (3.21) is strict, since $\deg(p - \widetilde{p}) < d(\mu)$ and $p(x) \geqslant \widetilde{p}(x)$ on $[0,1]$. Therefore, we are able to construct a polynomial with strictly smaller radius of last sign change in this scenario. □

Remark. As a consequence of the strict inequalities in the previous proof, any extremizer of the problem $\Omega(\mu, N)$ with $N < d(\mu)$ must be a polynomial of degree N with a single sign change at $x = \mathrm{r}(p)$ in the interval $[0,1]$.

The main result of this subsection reduces (EP3.2) to a simpler minimization problem over non-negative polynomials. The approach is inspired by Section 2.3.2.1. Let $1 \leqslant N < d(\mu)$ and define the space

$$\mathcal{Q}_N := \{q \in \mathbb{R}[x] : q \geqslant 0 \text{ on } [0,1],\ q \not\equiv 0,\ \deg(q) \leqslant N-1\}.$$

Over this space we define the functional $\mathrm{J} : \mathcal{Q}_N \to [0,1]$, given by

$$\mathrm{J}(q) := \frac{\int_0^1 x\,q(x)\,\mathrm{d}\mu(x)}{\int_0^1 q(x)\,\mathrm{d}\mu(x)}.$$

It is well-defined, as $\deg(q) < d(\mu)$ for each $q \in \mathcal{Q}_N$.

By Proposition 3.17, for computing $\Omega(\mu, N)$ it suffices to consider $p \in \mathcal{P}(\mu, N)$ of the form $p(x) = (x - r)\,q(x)$ with $q \in \mathcal{Q}_N$, for which the constraint $\int_0^1 p(x)\,\mathrm{d}\mu(x) \leqslant 0$

reads
$$r\int_0^1 q(x)\,\mathrm{d}\mu(x) \geqslant \int_0^1 x\,q(x)\,\mathrm{d}\mu(x), \quad \text{i.e., } r \geqslant \mathrm{J}(q).$$
As the task is to minimize r, we are led to the following problem — the polynomial analogue of (EP2.2).

Extremal problem 3.4 **(EP3.4)**. Let μ be a finite Borel measure on $[0,1]$ and $1 \leqslant N < d(\mu)$. Determine

$$\mathcal{M}(\mu,N) := \inf_{q\in\mathcal{Q}_N} \mathrm{J}(q) = \inf_{q\in\mathcal{Q}_N} \frac{\int_0^1 x\,q(x)\,\mathrm{d}\mu(x)}{\int_0^1 q(x)\,\mathrm{d}\mu(x)}.$$

The following proposition is the heart of the matter.

Proposition 3.18. *Let μ be a finite Borel measure on $[0,1]$ and let N be an integer with $1 \leqslant N < d(\mu)$. Then:*

(i) $\Omega(\mu,N) = \mathcal{M}(\mu,N) > 0$*, and both infima are attained.*

(ii) *Every extremizer p (i.e. $0 \not\equiv p \in \mathcal{P}(\mu,N)$ with $\mathrm{r}(p) = \Omega(\mu,N)$) satisfies $\int_0^1 p\,\mathrm{d}\mu = 0$ and $\deg(p) = N$, and factors as*

$$p(x) = (x - \mathcal{M}(\mu,N))\,q(x), \qquad q \in \mathcal{Q}_N, \quad \mathrm{J}(q) = \mathcal{M}(\mu,N),$$

that is, q is a minimizer of J over $\mathcal{Q}_N$. Conversely, if $q \in \mathcal{Q}_N$ minimizes J, then $(x - \mathcal{M}(\mu,N))\,q \in \mathcal{P}(\mu,N)$ is an extremizer for $\Omega(\mu,N)$.

Consequently the extremizers of (EP3.2) are, up to the common factor $(x-\mathcal{M}(\mu,N))$, exactly the minimizers of J over $\mathcal{Q}_N$; in particular the extremizer is unique up to a positive constant if and only if the minimizer of J is.

Proof. For part (i) we first establish the identity by showing two inequalities.

For the upper bound let $q \in \mathcal{Q}_N$, set $p_q(x) := (x - \mathrm{J}(q))\,q(x)$. Then $\deg(p_q) \leqslant N$, $p_q \geqslant 0$ on $[\mathrm{J}(q),1]$, and by the definition of J we know $\int_0^1 p_q(x)\,\mathrm{d}\mu(x) = 0$, so $p_q \in \mathcal{P}(\mu,N)$ and $\Omega(\mu,N) \leqslant \mathrm{r}(p_q) = \mathrm{J}(q)$. Taking the infimum over $\mathcal{Q}_N$ yields $\Omega(\mu,N) \leqslant \mathcal{M}(\mu,N)$.

For the lower bound, take $0 \not\equiv p \in \mathcal{P}(\mu,N)$ and $r := \mathrm{r}(p)$. If $r = 0$, then $p \geqslant 0$ on $[0,1]$ and $\int_0^1 p(x)\,\mathrm{d}\mu(x) = 0$, so p contradicts (3.20) (as $\deg p \leqslant N < d(\mu)$); hence $r > 0$. Using Proposition 3.17 we may assume p satisfies the following:

(i) $\deg(p) = N$;

(ii) $p(x) = (x - \mathrm{r}(p))q(x)$, for some $q \in \mathcal{Q}_N$ and $\deg(q) = N-1$.

Since $\int_0^1 p(x)\,\mathrm{d}\mu(x) \leqslant 0$ we deduce,

$$\mathrm{r}(p) \geqslant \frac{\int_0^1 xq(x)\,\mathrm{d}\mu(x)}{\int_0^1 q(x)\,\mathrm{d}\mu(x)} = \mathrm{J}(q) \geqslant \mathcal{M}(\mu,N).$$

Taking infimum over all such p we conclude the lower bound. Combining these two bounds we get the identity $\Omega(\mu, N) = \mathcal{M}(\mu, N)$.

Now we show that in both cases the infimum is attained. Equip $\mathcal{P}_{N-1}(\mathbb{R})$ with the norm $\|\cdot\|_\infty$ and set

$$S_1 := \left\{ q \in \mathcal{Q}_N \ : \ \int_0^1 q(x)\,\mathrm{d}\mu(x) = 1 \right\} \subsetneq \mathcal{P}_{N-1}(\mathbb{R}).$$

The set S_1 is closed in $\mathcal{P}_{N-1}(\mathbb{R})$: non-negativity on $[0,1]$ is preserved under uniform limits, and the constraint $\int_0^1 q(x)\,\mathrm{d}\mu(x) = 1$ is the pre-image of $\{1\}$ under the continuous functional $p \mapsto \int_0^1 p(x)\,\mathrm{d}\mu(x)$; in particular no limit point is identically zero. We claim S_1 is bounded. If not, there is a sequence $q_k \in S_1$ with $\|q_k\|_\infty \geqslant k$. The normalized polynomials $\widetilde{q}_k := q_k/\|q_k\|_\infty$ lie on the unit sphere of the finite-dimensional space $(\mathcal{P}_{N-1}(\mathbb{R}), \|\cdot\|_\infty)$, which is compact; hence a subsequence converges uniformly to some $\widetilde{q}_0 \in \mathcal{P}_{N-1}(\mathbb{R})$ with $\|\widetilde{q}_0\|_\infty = 1$. By continuity of the integral,

$$\int_0^1 \widetilde{q}_0(x)\,\mathrm{d}\mu(x) = \lim_{k\to\infty} \frac{1}{\|q_{n_k}\|_\infty} = 0.$$

Since $\deg(\widetilde{q}_0) \leqslant N-1 < d(\mu)$ and $\widetilde{q}_0 \geqslant 0$ on $[0,1]$, this forces $\widetilde{q}_0 \equiv 0$, contradicting $\|\widetilde{q}_0\|_\infty = 1$. Hence S_1 is compact. Since J is scale-invariant, $\mathcal{M}(\mu, N)$ equals the minimum of $q \mapsto \int_0^1 x\,q(x)\,\mathrm{d}\mu(x)$ over the compact set S_1, so the minimum is attained at some q^*; the construction used for the upper bound then yields an extremizer for $\Omega(\mu, N)$.

For the strict positivity of the values we argue as follows. Suppose that $\mathcal{M}(\mu, N) = 0$, then there must exist a polynomial $q^* \in \mathcal{Q}_N$ such that $\mathrm{J}(q^*) = \mathcal{M}(\mu, N) = 0$. Hence, xq^* would be non-zero, non-negative polynomial of degree $\leqslant N < d(\mu)$, with $\int_0^1 x\,q^*(x)\,\mathrm{d}\mu(x) = 0$, again contradicting (3.20). Hence $\mathcal{M}(\mu, N) > 0$.

Now, we establish part (ii). If $r = \mathrm{r}(p) = \Omega(\mu, N) = \mathcal{M}(\mu, N)$, then from the proof of Proposition 3.17 we conclude that

$$p(x) = (x - \mathrm{r}(p))q(x), \ \text{ for some } q \in \mathcal{Q}_N \text{ with } \deg(q) = N-1.$$

Moreover, $\int_0^1 p(x)\,\mathrm{d}\mu(x) = 0$. If it was strictly negative, then the polynomial $\widetilde{p} = p - \int_0^1 p(x)\,\mathrm{d}\mu(x)$ would satisfy $\mathrm{r}(\widetilde{p}) < \mathrm{r}(p)$, contradicting the minimality of $\mathrm{r}(p)$. As a consequence of this integral identity we deduce, $\mathrm{J}(q) = \mathrm{r}(p)$ concluding q is an extremizer of the problem (EP3.4).

For the converse we take the polynomial $q \in \mathcal{Q}_N$ with $\mathrm{J}(q) = \mathcal{M}(\mu, N)$. Consider the polynomial

$$p(x) = (x - \mathcal{M}(\mu, N))q(x).$$

It is immediate that $p \in \mathcal{P}(\mu, N)$ and $\mathrm{r}(p) = \mathcal{M}(\mu, N) = \Omega(\mu, N)$ thus it is an extremizer for the problem (EP3.2). $\square$

Now we have reduced the sign uncertainty problem into an explicit minimization problem involving multiplication by x operator. From now on we focus on the problem (EP3.4).

3.5.2 Solution to the problem (EP3.4)

3.5.2.1 *Characterization of non-negative polynomials on* $[0,1]$

The following is a classical result in the theory of orthogonal polynomials, provable via the Fejér–Riesz theorem; see [114, § 1.21].

Lemma 3.19 (Theorem of Lukács)**.** *Let $q(x)$ be a polynomial of degree k with $q(x) \geqslant 0$ for all $x \in [0,1]$. Then*

$$q(x) = \begin{cases} x\, a(x)^2 + (1-x)\, b(x)^2, & \text{if } k \text{ is odd;} \\ c(x)^2 + x(1-x)\, d(x)^2, & \text{if } k \text{ is even,} \end{cases}$$

where a, b, c, d are polynomials with real coefficients and each summand on the right has degree at most k.

3.5.2.2 *Orthogonal polynomials on the real line*

Given a finite Borel measure μ on $[0,1]$, let $1 \equiv \Psi_0, \Psi_1, \dots, \Psi_n(x; \mathrm{d}\mu), \dots$ be the unique monic orthogonal polynomials in $L^2([0,1], \mu)$. For each $\Psi_n(x; \mathrm{d}\mu)$ we denote its zeros by $x_{n,k}$ for $1 \leqslant k \leqslant n$. The following lemma collects the necessary facts about their zeros.

Lemma 3.20 (cf. [72, § 3.2–3.4])**.** *Let μ be a probability measure on $[0,1]$ and let $n \geqslant 1$ be such that $\Psi_n(x; \mathrm{d}\mu)$ exists, and $x_{n,k}$ denote its zeros.*

(i) *the polynomials $\Psi_0, \Psi_1, \dots, \Psi_n$ satisfy the Christoffel–Darboux identities:*

$$\sum_{k=0}^{n-1} \frac{\Psi_k(x)\Psi_k(y)}{\|\Psi_k\|_2^2} = \frac{1}{\|\Psi_{n-1}\|_2^2} \frac{\Psi_n(x)\Psi_{n-1}(y) - \Psi_n(y)\Psi_{n-1}(x)}{x-y}, \qquad x \neq y,$$

and

$$\sum_{k=0}^{n-1} \frac{\Psi_k^2(x)}{\|\Psi_k\|_2^2} = \frac{\Psi_n'(x)\Psi_{n-1}(x) - \Psi_n(x)\Psi_{n-1}'(x)}{\|\Psi_{n-1}\|_2^2}, \qquad x \in \mathbb{R};$$

(ii) *if $n < \#\mathrm{supp}(\mu)$, then $\Psi_n(x; \mathrm{d}\mu)$ has exactly n distinct zeros in $(0,1)$;*

(iii) *between any two consecutive zeros of $\Psi_n(x; \mathrm{d}\mu)$ there is exactly one zero of $\Psi_{n-1}(x; \mathrm{d}\mu)$:*

$$x_{n,k} < x_{n-1,k} < x_{n,k+1}, \qquad k = 1, \dots, n-1;$$

(iv) *the zeros* $x_{n,1}, \dots, x_{n,n}$ *are nodes of a Gauss quadrature formula: there exist weights* $\lambda^{\mu}_{n,k} > 0$, $1 \leqslant k \leqslant n$, *such that*

$$\int_0^1 q(x)\,\mathrm{d}\mu(x) = \sum_{k=1}^{n} \lambda^{\mu}_{n,k}\, q(x_{n,k})$$

for every polynomial q *of degree at most* $2n-1$.

3.5.2.3 Comparison of zeros

Let μ be a probability measure and let $n \geqslant 1$ be such that $\Psi_n(x;\mathrm{d}\mu)$ exists and $x_{n,k}$ denote its zeros. Define two measures μ_0 and μ_1 by

$$\mathrm{d}\mu_0(x) \coloneqq x\,\mathrm{d}\mu(x), \qquad \mathrm{d}\mu_1(x) \coloneqq (1-x)\,\mathrm{d}\mu(x).$$

Denote the zeros of the monic orthogonal polynomials (whenever they exist) $\Psi_n(x;\mathrm{d}\mu_0)$ and $\Psi_n(x;\mathrm{d}\mu_1)$ by $\zeta_{n,k}$ and $\xi_{n,k}$, respectively. We have the following relation for these zeros, provided that all orthogonal polynomials exist.

Proposition 3.21. *Let* μ *be a probability measure and let* $n \geqslant 1$ *be such that* $2n < d(\mu)$. *In this case, all the polynomials* $\Psi_n(x;\mathrm{d}\mu)$, $\Psi_n(x;\mathrm{d}\mu_0)$ *and* $\Psi_n(x;\mathrm{d}\mu_1)$ *exist and for each* $1 \leqslant k \leqslant n$, *we have the following*

$$\xi_{n,k} < x_{n,k} < \zeta_{n,k}.$$

Proof. First of all, note that the condition $2n < d(\mu)$ implies $n \leqslant \#\mathrm{supp}(\mu) - 1$, where the equality may occur if and only if at most one of the points $x = 0$ and $x = 1$ belong to the support of μ. From this it is immediate that polynomials $\Psi_n(x;\mathrm{d}\mu)$, $\Psi_{n+1}(x;\mathrm{d}\mu)$, $\Psi_n(x;\mathrm{d}\mu_0)$, and $\Psi_n(x;\mathrm{d}\mu_1)$ exist.

Define the *ratio function* R_n by

$$R_n(x) \coloneqq \frac{\Psi_{n+1}(x;\mathrm{d}\mu)}{\Psi_n(x;\mathrm{d}\mu)}.$$

By part (ii) of Lemma 3.20, $R_n(0)$ and $R_n(1)$ are well-defined and finite. The monic orthogonal polynomials associated with μ_0 and μ_1 can be expressed as

$$\Psi_n(x;\mathrm{d}\mu_1) = \frac{R_n(1)\Psi_n(x;\mathrm{d}\mu) - \Psi_{n+1}(x;\mathrm{d}\mu)}{1-x},$$

and

$$\Psi_n(x;\mathrm{d}\mu_0) = \frac{\Psi_{n+1}(x;\mathrm{d}\mu) - R_n(0)\Psi_n(x;\mathrm{d}\mu)}{x}.$$

Indeed, these are monic polynomials orthogonal to all monomials z^k for $0 \leqslant k \leqslant n-1$ with respect to their respective measures. By part (iii) of Lemma 3.20, $\Psi_{n+1}(x;\mathrm{d}\mu)$ and $\Psi_n(x;\mathrm{d}\mu)$ have no common zeros, so R_n has poles exactly at the zeros of $\Psi_n(x;\mathrm{d}\mu)$.

Suppose by contradiction that $\xi_{n,1} = 0$. By part (ii) of Lemma 3.20 we deduce that $n = \#\mathrm{supp}(\mu_1) \geqslant \#\mathrm{supp}(\mu) - 1$. Which implies that $\{0, 1\} \subseteq \mathrm{supp}(\mu)$ contradicting the inequality $2n < d(\mu)$. Similarly, we can exclude the possibilities of $\zeta_{n,n} = 1$. It follows from the definition that $\xi_{n,n} < 1$ and $\zeta_{n,1} > 0$ as these endpoints cannot be in their support as a separate point masses. Therefore, we conclude that $\zeta_{n,k}, \xi_{n,k} \in (0, 1)$ for each $1 \leqslant k \leqslant n$. Thus

$$\Psi_n(x; \mathrm{d}\mu_1) = 0 \iff x \neq 1 \text{ and } R_n(x) = R_n(1),$$

and

$$\Psi_n(x; \mathrm{d}\mu_0) = 0 \iff x \neq 0 \text{ and } R_n(x) = R_n(0).$$

Using the Christoffel–Darboux identity of Lemma 3.20 we compute

$$\begin{aligned} R_n'(x) &= \frac{\Psi_{n+1}'(x; \mathrm{d}\mu)\Psi_n(x; \mathrm{d}\mu) - \Psi_{n+1}(x; \mathrm{d}\mu)\Psi_n'(x; \mathrm{d}\mu)}{\Psi_n^2(x; \mathrm{d}\mu)} \\ &= \sum_{k=0}^{n} \frac{\|\Psi_n\|_2^2}{\|\Psi_k\|_2^2} \frac{\Psi_k^2(x; \mathrm{d}\mu)}{\Psi_n^2(x; \mathrm{d}\mu)} > 0 \end{aligned}$$

for all $x \neq x_{n,k}$. By part (iii) of Lemma 3.20, $x_{n+1,k} < x_{n,k}$ for each $k = 1, \dots, n$, and R_n maps each of the intervals

$$I_{n,0} := (-\infty, x_{n,1}), \quad I_{n,j} := (x_{n,j}, x_{n,j+1}) \text{ for } 1 \leqslant j \leqslant n-1, \quad I_{n,n} := (x_{n,n}, +\infty)$$

bijectively onto $(-\infty, +\infty)$, with $x_{n+1,k} \in I_{n,k-1}$ for each $k = 1, \dots, n+1$.

For $\Psi_n(x; \mathrm{d}\mu_1)$: its zeros are the solutions $x \neq 1$ of $R_n(x) = R_n(1)$. Note that $R_n(x_{n+1,k}) = 0$ for all $1 \leqslant k \leqslant n+1$.

If $x_{n+1,n+1} = 1$, that means $n + 1 = \#\mathrm{supp}(\mu)$ and $\gamma_1 = 1$. In this case, we deduce that $\xi_{n,k} = x_{n+1,k}$ for all $1 \leqslant k \leqslant n$. Using part (iii) of Lemma 3.20 we know $\xi_{n,k} < x_{n,k}$. If $x_{n,n} < x_{n+1,n+1} < 1$, by the fact that $R_n' > 0$ on $I_{n,n}$, we deduce $R_n(1) > R_n(x_{n+1,n+1}) = 0$. Thus, we have a unique solution of $R_n(x) = R_n(1)$ on $I_{n,n}$, that is $x = 1$ itself, which is excluded. Hence no zero $\xi_{n,j}$ lies in $I_{n,n}$. On each $I_{n,k-1}$ for $k = 1, \dots, n$, bijectivity gives exactly one solution; since $R_n(x_{n+1,k}) = 0 < R_n(1)$, strict monotonicity forces a zero of $\Psi_n(x; \mathrm{d}\mu_1)$ belong to $(x_{n+1,k}, x_{n,k}) \subset I_{n,k-1}$. Since $\Psi_n(x; \mathrm{d}\mu_1)$ has exactly n zeros by part (ii) of Lemma 3.20, we conclude $x_{n+1,k} < \xi_{n,k} < x_{n,k}$ for each $k = 1, \dots, n$. In either case, we deduce $\xi_{n,k} < x_{n,k}$.

The argument for $\Psi_n(x; \mathrm{d}\mu_0)$ is analogous: its zeros are the solutions $x \neq 0$ of $R_n(x) = R_n(0)$. If $x_{n+1,1} = 0$, then we would have $n + 1 = \#\mathrm{supp}(\mu)$ and $\gamma_0 = 1$. Similarly, we would have $\zeta_{n,k} = x_{n+1,k+1}$ for each $1 \leqslant k \leqslant n$ and the inequality $\zeta_{n,k} > x_{n,k}$ follows from Lemma 3.20 part (iii).

If $x_{n+1,1} > 0$ then, by the fact that $R_n' > 0$ on $I_{n,0}$, we have that $R_n(x_{n+1,1}) =$

$0 > R_n(0)$ and there is a unique solution to the equation $R_n(x) = R_n(0)$ on $I_{n,0}$ that is $x = 0$, which is not a zero of $\Psi_n(x; \mathrm{d}\mu_0)$. On each $I_{n,k}$ for $k = 1, \dots, n$, since $R_n(x_{n+1,k+1}) = 0 > R_n(0)$, the unique solution lies in $(x_{n,k}, x_{n+1,k+1}) \subset I_{n,k}$, giving $x_{n,k} < \zeta_{n,k}$ for each $k = 1, \dots, n$. □

3.5.2.4 Solution to the problem (EP3.4)

With all the necessary results established we proceed with the complete solution to the problem (EP3.4), which is the content of the next proposition.

Proposition 3.22. *Let μ be a finite Borel measure on $[0, 1]$ and let $1 \leqslant N < d(\mu)$. Let μ_1 be defined by $\mathrm{d}\mu_1 := (1 - x)\,\mathrm{d}\mu$. Then we have the following*

(i) *If $N = 2n - 1$, then $\mathcal{M}(\mu, 2n - 1) = x_{n,1}$, the smallest zero of $\Psi_n(x; \mathrm{d}\mu)$. Moreover, every extremizer has the following form*

$$C \left(\frac{\Psi_n(x; \mathrm{d}\mu)}{x - x_{n,1}} \right)^2, \text{ for some } C > 0.$$

(ii) *If $N = 2n$, then $\mathcal{M}(\mu, 2n) = \xi_{n,1}$, the smallest zero of $\Psi_n(x; \mathrm{d}\mu_1)$. Moreover, every extremizer has the following form*

$$C(1 - x) \left(\frac{\Psi_n(x; \mathrm{d}\mu_1)}{x - \xi_{n,1}} \right)^2, \text{ for some } C > 0.$$

Proof. Part (i). Assume $N = 2n - 1 < d(\mu)$. Thus, we have $n \leqslant \#\mathrm{supp}(\mu)$, where the equality is possible if and only if $\gamma_0 = \gamma_1 = 0$. Therefore, $\Psi_n(x; \mathrm{d}\mu)$ is well-defined and all its zeros lie in $(0, 1)$. Using the quadrature formula from part (iv) of Lemma 3.20 for polynomials $xq(x)$ and $q(x)$, where $q \in \mathcal{Q}_{2n-1}$, we get

$$\int_0^1 xq(x)\,\mathrm{d}\mu(x) = \sum_{k=1}^{n} \lambda^{\mu}_{n,k}\, x_{n,k}\, q(x_{n,k}) \geqslant x_{n,1} \sum_{k=1}^{n} \lambda^{\mu}_{n,k}\, q(x_{n,k}) = x_{n,1} \int_0^1 q(x)\,\mathrm{d}\mu(x).$$

Therefore, we conclude $\mathrm{J}(q) \geqslant x_{n,1}$ for each $q \in \mathcal{Q}_{2n-1}$. Note that, the equality in the above estimate can occur if and only if $q(x_{n,k}) = 0$ for all $2 \leqslant k \leqslant n$. Since, $q \in \mathcal{Q}_{2n-1}$ and $0 < x_{n,k} < 1$ for all k we conclude that

$$q(x) = C \left(\frac{\Psi_n(x; \mathrm{d}\mu)}{x - x_{n,1}} \right)^2, \text{ for some } C > 0.$$

Part (ii). Assume $N = 2n < d(\mu)$. Using Proposition 3.18 we know the extremizers of (EP3.4) must have highest possible degree. Hence we are dealing with polynomials $q \in \mathcal{Q}_{2n}$ such that $\deg(q) = 2n - 1$. Using the characterization of non-negative polynomials from Lemma 3.19 we can write

$$q(x) = xa(x)^2 + (1 - x)b(x)^2, \text{ for some } a, b \in \mathcal{P}_{n-1}(\mathbb{R}).$$

Next, we apply Gauss quadrature formula for measures μ_0 and μ_1 in the following way

$$\begin{aligned}\int_0^1 xq(x)\,\mathrm{d}\mu(x) &= \int_0^1 xa(x)^2\,\mathrm{d}\mu_0(x) + \int_0^1 xb(x)^2\,\mathrm{d}\mu_1(x)\\ &= \sum_{k=1}^n \lambda_{n,k}^{\mu_0}\,\zeta_{n,k}\,a(\zeta_{n,k})^2 + \lambda_{n,k}^{\mu_1}\,\xi_{n,k}\,b(\xi_{n,k})^2\\ &\geqslant \xi_{n,1}\left(\sum_{k=1}^n \lambda_{n,k}^{\mu_0}\,a(\zeta_{n,k})^2 + \lambda_{n,k}^{\mu_1}\,b(\xi_{n,k})^2\right)\\ &= \xi_{n,1}\int_0^1 xa(x)^2 + (1-x)b(x)^2\,\mathrm{d}\mu(x) = \xi_{n,1}\int_0^1 q(x)\,\mathrm{d}\mu(x).\end{aligned}$$

Here we have used the estimate $\zeta_{n,k} > \xi_{n,k} \geqslant \xi_{n,1}$ which we established in Proposition 3.21. Note that the equality in the above estimate can only occur if and only if $a(\zeta_{n,k}) = 0$ for all $1 \leqslant k \leqslant n$ and $b(\xi_{n,k}) = 0$ for all $2 \leqslant k \leqslant n$. It means equality can occur when $q(x) = C(1-x)\prod_{k=2}^n (x-\xi_{n,k})^2$. As a consequence of this estimate we know $\mathrm{J}(q) \geqslant \xi_{n,1}$ for all $q \in \mathcal{Q}_{2n}$ and the equality is attained. Therefore, we conclude the assertion of part (ii). □

Now, putting together Proposition 3.18 and Proposition 3.22 we obtain Theorem 3.3.

Chapter 4

Sharp Fourier extension inequalities for cones over finite fields

4.1 Introduction

The Fourier restriction problem over finite fields, its general setup, and the state of the art regarding sharp extension constants were surveyed in Section 1.2. The present chapter, based on [A3], establishes the sharp $L^2 \to L^4$ extension constant for the punctured $(2,2)$-cone in $\mathbb{F}_q^4$ (for all odd prime powers q) and provides a complete classification of its extremizers.

4.1.1 Setup and main results

We work over a finite field $\mathbb{F}_q$ with $q = p^n$ elements, where p is an odd prime and $n \geqslant 1$; this assumption is in force throughout the chapter. A *non-principal additive character* is a non-constant group homomorphism $e : (\mathbb{F}_q, +) \to (\mathbb{C}^\times, \cdot)$; these take the form

$$e_a(x) = \exp\left(\frac{2\pi i}{p}\,\mathrm{Tr}_n(ax)\right), \qquad a \in \mathbb{F}_q^\times,$$

where $\mathrm{Tr}_n : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map given by

$$\mathrm{Tr}_n(x) = \sum_{k=0}^{n-1} x^{p^k}.$$

We fix once and for all a non-principal character $e = e_1$. Given a non-empty set $\mathcal{S} \subset \mathbb{F}_q^{d+1}$ with normalized counting measure σ (mass $1/\#\mathcal{S}$ per point), the *extension operator* is

$$(f\sigma)^\vee(\boldsymbol{x}) = \frac{1}{\#\mathcal{S}} \sum_{\boldsymbol{\eta} \in \mathcal{S}} f(\boldsymbol{\eta})\, e(\boldsymbol{\eta} \cdot \boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{F}_q^{d+1},$$

and the *sharp extension constant* is

$$\mathfrak{R}^*_{\mathcal{S}}(r \to s) := \sup_{f \not\equiv 0} \frac{\|(f\sigma)^\vee\|_{L^s(\mathbb{F}_q^{d+1}, \mathrm{d}\boldsymbol{x})}}{\|f\|_{L^r(\mathcal{S}, \sigma)}}, \tag{4.1}$$

where $\mathrm{d}\boldsymbol{x}$ is the counting measure on $\mathbb{F}_q^{d+1}$. The main object of study is the *punctured cone with signature* $(2,2)$ in $\mathbb{F}_q^4$:

$$\Gamma^3_{(2,2)} := \{(\eta_1, \eta_2, \eta_3, \eta_4) \in \mathbb{F}_q^4 \setminus \{\mathbf{0}\} : \eta_1^2 + \eta_2^2 = \eta_3^2 + \eta_4^2\}.$$

Remark. The removal of the origin is essential. By [65, Theorem 1.6], constant functions are *not* extremizers for the extension inequality on the full cone $\Upsilon^3_{(a,b)}$, whereas Theorem 4.2 shows they are extremizers (up to character modulation) on the punctured cone $\Gamma^3_{(2,2)}$. This illustrates how sharp extension inequalities are sensitive to small geometric perturbations of the underlying surface.

The main results of this chapter are the following.

Theorem 4.1 (Optimal constant)**.** *For all odd prime powers q,*

$$\mathfrak{R}^*_{\Gamma^3_{(2,2)}}(2 \to 4) = q \left(\frac{q^5 + 4q^4 - 4q^3 - 6q^2 + 3q + 3}{(q+1)^6 (q-1)^3} \right)^{1/4}. \tag{4.2}$$

Theorem 4.2 (Classification of extremizers)**.** *A function $f : \Gamma^3_{(2,2)} \to \mathbb{C}$ is an extremizer of (4.1) if and only if*

$$f(\eta_1, \eta_2, \eta_3, \eta_4) = \lambda \cdot \exp\Big(\frac{2\pi i}{p} \operatorname{Tr}_n(a_1\eta_1 + a_2\eta_2 + a_3\eta_3 + a_4\eta_4) \Big) \tag{4.3}$$

for some $\lambda \in \mathbb{C}^\times$ and $(a_1, a_2, a_3, a_4) \in \mathbb{F}_q^4$.

When $q \equiv 1 \pmod 4$, there exists $\omega \in \mathbb{F}_q^\times$ with $\omega^2 = -1$, and the substitution $\eta_4 \mapsto \omega\eta_4$ identifies $\Gamma^3_{(2,2)}$ with $\Gamma^3_{(3,1)}$. Hence:

Corollary 4.3. *When $q \equiv 1 \pmod 4$, the sharp constant $\mathfrak{R}^*_{\Gamma^3_{(3,1)}}(2 \to 4)$ is given by (4.2), and the extremizers are precisely the functions of the form (4.3).*

Together with [65], which establishes $\mathfrak{R}^*_{\Gamma^3_{(3,1)}}(2 \to 4)$ when $q \equiv 3 \pmod 4$, Theorem 4.1 and Corollary 4.3 complete the determination of all sharp $L^2 \to L^4$ extension constants for punctured cones in $\mathbb{F}_q^4$.

Remark. The Euclidean analogue of Theorem 4.1 for the $(2,2)$-cone $\{\boldsymbol{x} \in \mathbb{R}^4 : x_1^2 + x_2^2 = x_3^2 + x_4^2\}$ remains open, in contrast to the $(3,1)$-cone in $\mathbb{R}^4$, whose sharp $L^2 \to L^4$ constant was determined by Foschi [49] with Gaussians as the unique extremizers (see also [9, 18, 60, 71]).

By [65, Proposition 2.1], Theorems 4.1 and 4.2 are equivalent to the following. Let $\mathbf{C}_{\Gamma^3_{(2,2)}}(2\to 4)$ be the smallest constant such that

$$\sum_{\boldsymbol{\xi}\in\mathbb{F}_q^4}\left|\sum_{\substack{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\in\Gamma^3_{(2,2)}\\ \boldsymbol{\eta}_1+\boldsymbol{\eta}_2=\boldsymbol{\xi}}} f(\boldsymbol{\eta}_1)\, f(\boldsymbol{\eta}_2)\right|^2 \leqslant \mathbf{C}_{\Gamma^3_{(2,2)}}(2\to 4)\left(\sum_{\boldsymbol{\xi}\in\Gamma^3_{(2,2)}}|f(\boldsymbol{\xi})|^2\right)^2 \tag{4.4}$$

holds for all $f:\Gamma^3_{(2,2)}\to\mathbb{C}$. The two claims to be established are:

(i) the sharp constant in (4.4) is

$$\mathbf{C}_{\Gamma^3_{(2,2)}}(2\to 4)=\frac{q^5+4q^4-4q^3-6q^2+3q+3}{(q+1)^2(q-1)};$$

(ii) the extremizers of (4.4) are precisely the functions of the form (4.3).

The proofs of (i) and (ii) constitute the remainder of this chapter.

4.1.2 Outline of the approach

4.1.2.1 Comparison with prior work

González-Riquelme and Oliveira e Silva [65] established $\mathfrak{R}^*_{\Gamma^3_{(3,1)}}(2\to 4)$ when $q\equiv 3 \pmod 4$ by exploiting the following rigidity: when -1 is not a square in $\mathbb{F}_q$, every decomposition $\boldsymbol{\xi}=\boldsymbol{\eta}_1+\boldsymbol{\eta}_2$ with $\boldsymbol{\xi},\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\in\Gamma^3_{(3,1)}$ forces $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ to lie on the projective line $\{\lambda\boldsymbol{\xi}:\lambda\in\mathbb{F}_q^\times\}$ through $\boldsymbol{\xi}$. These lines partition $\Gamma^3_{(3,1)}$ and render the estimates tractable. When $q\equiv 1 \pmod 4$ (or when the cone has signature $(2,2)$), this line structure breaks down: a point $\boldsymbol{\xi}\in\Gamma^3_{(2,2)}$ admits decompositions $\boldsymbol{\eta}_1+\boldsymbol{\eta}_2=\boldsymbol{\xi}$ with $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2$ not proportional to $\boldsymbol{\xi}$. A more refined analysis is therefore required.

4.1.2.2 The decomposition sets and their structure

For each $\boldsymbol{\xi}\in\mathbb{F}_q^4$, define the *cone decomposition set*

$$\Sigma_{\boldsymbol{\xi}} \coloneqq \{(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)\in(\Gamma^3_{(2,2)})^2:\boldsymbol{\eta}_1+\boldsymbol{\eta}_2=\boldsymbol{\xi}\}.$$

The inner sum in (4.4) is supported on $\Sigma_{\boldsymbol{\xi}}$, so the proof of (4.4) requires a precise understanding of these sets.

The cardinality $\#\Sigma_{\boldsymbol{\xi}}$ depends only on which of three cases $\boldsymbol{\xi}$ falls into: $\boldsymbol{\xi}\in\Gamma^3_{(2,2)}$, $\boldsymbol{\xi}=\mathbf{0}$, or $\boldsymbol{\xi}$ a generic point in $\mathbb{F}_q^4\setminus(\Gamma^3_{(2,2)}\cup\{\mathbf{0}\})$. The dominant contributions to (4.4) come from $\boldsymbol{\xi}\in\Gamma^3_{(2,2)}\cup\{\mathbf{0}\}$; the generic contribution is controlled directly by Cauchy–Schwarz. For $\boldsymbol{\xi}\in\Gamma^3_{(2,2)}$ we show that the first components of the pairs in $\Sigma_{\boldsymbol{\xi}}$, together with $\boldsymbol{\xi}$ itself, form the union of two *punctured affine planes* sharing a common *punctured line*, all contained in $\Gamma^3_{(2,2)}$; in particular $\Sigma_{\boldsymbol{\xi}}$ is in bijection with this union minus the point $\boldsymbol{\xi}$, via $\boldsymbol{\eta} \mapsto (\boldsymbol{\eta}, \boldsymbol{\xi} - \boldsymbol{\eta})$. The resulting planes foliate $\Gamma^3_{(2,2)}$. This decomposition is described in the following section.

4.2 Geometric structure of $\Sigma_{\boldsymbol{\xi}}$

With a slight abuse of notation throughout the remainder of this chapter we write

$$\Gamma^3 := \Gamma^3_{(2,2)}.$$

First, let us observe that by the change of variables $(\eta_1, \eta_2, \eta_3, \eta_4) \mapsto (\eta_1{-}\eta_3,\ \eta_1{+}\eta_3,\ \eta_4{-}\eta_2,\ \eta_4{+}\eta_2)$, we can use the definition

$$\Gamma^3 = \{(\eta_1, \eta_2, \eta_3, \eta_4) \in \mathbb{F}_q^4 \setminus \{\mathbf{0}\} : \eta_1\eta_2 = \eta_3\eta_4\}.$$

The first thing we observe is that $\#\Gamma^3 = (q-1)(q+1)^2$. This follows immediately by considering two cases $\eta_1 = 0$ and $\eta_1 \neq 0$, and then counting all possible values for the remaining coordinates.

As mentioned in Section 4.1, one of the key points of our argument is to provide a clear description of the sets $\Sigma_{\boldsymbol{\xi}}$. There is a natural parametrization of the cone Γ^3 that makes this possible.

Lemma 4.4. *There is a parametrization of* Γ^3 *with the Cartesian product* $\mathbb{F}_q^\times \times \mathbb{P}^1 \times \mathbb{P}^1$*, where*

$$\mathbb{P}^1 := \{(1, y) \in \mathbb{F}_q^2 : y \in \mathbb{F}_q\} \cup \{(0, 1)\}$$

denotes one-dimensional projective space over $\mathbb{F}_q$*.*

Proof. Consider the map $\mathfrak{s} : \mathbb{F}_q^\times \times \mathbb{P}^1 \times \mathbb{P}^1 \to \Gamma^3$ given by

$$\mathfrak{s}(\lambda, \alpha, \beta) := (\lambda\alpha_1\beta_1,\ \lambda\alpha_2\beta_2,\ \lambda\alpha_1\beta_2,\ \lambda\alpha_2\beta_1), \tag{4.5}$$

for every $(\lambda, \alpha, \beta) \in \mathbb{F}_q^\times \times \mathbb{P}^1 \times \mathbb{P}^1$, where $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$. We claim $\mathfrak{s}$ is a bijection. Note the identities:

$$\begin{aligned}
\mathfrak{s}(\lambda, (1, \alpha_2), (1, \beta_2)) &= (\lambda,\ \lambda\alpha_2\beta_2,\ \lambda\beta_2,\ \lambda\alpha_2);\\
\mathfrak{s}(\lambda, (1, \alpha_2), (0, 1)) &= (0,\ \lambda\alpha_2,\ \lambda,\ 0);\\
\mathfrak{s}(\lambda, (0, 1), (1, \beta_2)) &= (0,\ \lambda\beta_2,\ 0,\ \lambda);\\
\mathfrak{s}(\lambda, (0, 1), (0, 1)) &= (0,\ \lambda,\ 0,\ 0).
\end{aligned}$$

Using these identities, if $\mathfrak{s}(\lambda, \alpha, \beta) = \mathfrak{s}(\mu, \gamma, \delta)$ for some (λ, α, β) and (μ, γ, δ) in $\mathbb{F}_q^\times \times \mathbb{P}^1 \times \mathbb{P}^1$, then $\alpha = \gamma$, $\beta = \delta$, and $\lambda = \mu$, so $\mathfrak{s}$ is injective. Since $\#\Gamma^3 = (q-1)(q+1)^2 = \#(\mathbb{F}_q^\times \times \mathbb{P}^1 \times \mathbb{P}^1)$, the map $\mathfrak{s}$ is bijective. □

Remark. This parametrization is a special case of a general family of maps known as Segre embeddings of projective spaces [106].

We now turn to the study of the sets $\Sigma_{\boldsymbol{\xi}}$.

Proposition 4.5. *For each $\boldsymbol{\xi} \in \mathbb{F}_q^4$, the cardinality of $\Sigma_{\boldsymbol{\xi}}$ is*

$$\#\Sigma_{\boldsymbol{\xi}} = \begin{cases} (q+1)^2(q-1), & \text{if } \boldsymbol{\xi} = \mathbf{0}, \\ 2q^2 - q - 2, & \text{if } \boldsymbol{\xi} \in \Gamma^3, \\ q^2 + q, & \text{otherwise.} \end{cases}$$

Proof. When $\boldsymbol{\xi} = \mathbf{0}$, we have $\Sigma_{\mathbf{0}} = \{(\boldsymbol{\rho}, -\boldsymbol{\rho}) : \boldsymbol{\rho} \in \Gamma^3\}$, which has the same cardinality as Γ^3, so $\#\Sigma_{\mathbf{0}} = (q-1)(q+1)^2$. Now suppose $\boldsymbol{\xi} \neq \mathbf{0}$. Consider the set

$$H_{\boldsymbol{\xi}} \coloneqq \{\boldsymbol{\eta} \in \Gamma^3 : (\xi_1 - \eta_1)(\xi_2 - \eta_2) = (\xi_3 - \eta_3)(\xi_4 - \eta_4)\}. \tag{4.6}$$

If $\boldsymbol{\xi} \in \Gamma^3$, then $\boldsymbol{\xi} \in H_{\boldsymbol{\xi}}$, yet the pair $(\boldsymbol{\xi}, \mathbf{0})$ is absent from $\Sigma_{\boldsymbol{\xi}}$, so $\#\Sigma_{\boldsymbol{\xi}} = \#H_{\boldsymbol{\xi}} - 1$. This is where the parametrization (4.5) has a clear advantage over Cartesian coordinates. For $\boldsymbol{\xi} = \mathfrak{s}(\lambda, \alpha, \beta)$ and $\boldsymbol{\eta} = \mathfrak{s}(\mu, \gamma, \delta)$ in Γ^3, the condition (4.6) is equivalent to

$$\alpha_1\beta_1\gamma_2\delta_2 + \alpha_2\beta_2\gamma_1\delta_1 = \alpha_1\beta_2\gamma_2\delta_1 + \alpha_2\beta_1\gamma_1\delta_2.$$

Factoring this equation shows it holds if and only if $\alpha = \gamma$ or $\beta = \delta$. Since $\#\mathbb{P}^1 = q+1$, by inclusion-exclusion we get $\#H_{\boldsymbol{\xi}} = (q-1)(2q+1)$ for each $\boldsymbol{\xi} \in \Gamma^3$, so $\#\Sigma_{\boldsymbol{\xi}} = 2q^2 - q - 2$.

Now suppose $\boldsymbol{\xi} \notin \Gamma^3$ and $\boldsymbol{\xi} \neq \mathbf{0}$. In this case $\boldsymbol{\xi} \notin H_{\boldsymbol{\xi}}$, so $\#\Sigma_{\boldsymbol{\xi}} = \#H_{\boldsymbol{\xi}}$. By symmetry of the equation in (4.6) it is not restrictive to assume $\xi_1 = 1$. We can then isolate η_2:

$$\eta_2 = \xi_4\eta_3 + \xi_3\eta_4 - \xi_2\eta_1 + \xi_2 - \xi_3\xi_4.$$

Substituting into the equation $\eta_1\eta_2 = \eta_3\eta_4$ of the cone gives

$$(\xi_3\eta_1 - \eta_3)\eta_4 = \eta_1(\xi_2\eta_1 - \xi_4\eta_3 + \xi_3\xi_4 - \xi_2).$$

If $\eta_3 = \xi_3\eta_1$, this simplifies to $\eta_1(\eta_1 - 1)(\xi_2 - \xi_3\xi_4) = 0$. Since $\boldsymbol{\xi} \notin \Gamma^3$, we have $\eta_1 \in \{0, 1\}$, and for each choice η_4 is free, giving $2q$ solutions. If $\eta_3 \neq \xi_3\eta_1$, for each of the q values of η_1 the coordinate η_3 takes $q-1$ values, giving $q^2 - q$ further solutions. Hence $\#H_{\boldsymbol{\xi}} = 2q + q^2 - q = q^2 + q$. □

Remark. Note that the set $H_{\boldsymbol{\xi}}$ has a structure that is easier to handle when $\boldsymbol{\xi} \in \Gamma^3$ than in the other cases. The following properties of $H_{\boldsymbol{\xi}}$ will be used throughout.

(i) If $\boldsymbol{\eta} \in H_{\boldsymbol{\xi}}$ then $\lambda\boldsymbol{\eta} \in H_{\boldsymbol{\xi}}$ for each $\lambda \in \mathbb{F}_q^\times$; also, $H_{\boldsymbol{\xi}} = H_{\lambda\boldsymbol{\xi}}$ for any $\lambda \in \mathbb{F}_q^\times$.

(ii) $\boldsymbol{\eta} \in H_{\boldsymbol{\xi}} \setminus \{\boldsymbol{\xi}\} \iff \boldsymbol{\xi} - \boldsymbol{\eta} \in H_{\boldsymbol{\xi}} \setminus \{\boldsymbol{\xi}\}$.

(iii) $\boldsymbol{\eta} \in H_{\boldsymbol{\xi}} \iff \boldsymbol{\xi} \in H_{\boldsymbol{\eta}}$.

Now, we look more closely at the structure of $H_{\boldsymbol{\xi}}$ for $\boldsymbol{\xi} \in \Gamma^3$. Recall that in the proof of Proposition 4.5 we established that, for $\boldsymbol{\xi} = \mathfrak{s}(\lambda, \alpha, \beta)$,

$$H_{\boldsymbol{\xi}} = \{\mathfrak{s}(\mu, \alpha, y) : (\mu, y) \in \mathbb{F}_q^\times \times \mathbb{P}^1\} \cup \{\mathfrak{s}(\mu, x, \beta) : (\mu, x) \in \mathbb{F}_q^\times \times \mathbb{P}^1\}.$$

Since these sets have geometric significance and play a crucial role in our estimates, we introduce the following disjoint subsets:

$$\begin{aligned} H_{\boldsymbol{\xi}}^+ &:= \{\mathfrak{s}(\mu, \alpha, y) : \mu \in \mathbb{F}_q^\times,\ y \in \mathbb{P}^1 \setminus \{\beta\}\}, \\ H_{\boldsymbol{\xi}}^- &:= \{\mathfrak{s}(\mu, x, \beta) : \mu \in \mathbb{F}_q^\times,\ x \in \mathbb{P}^1 \setminus \{\alpha\}\}, \end{aligned}$$

and the line

$$\mathcal{L}_{\boldsymbol{\xi}} := \{\mu\boldsymbol{\xi} : \mu \in \mathbb{F}_q^\times\} = \{\mathfrak{s}(\mu, \alpha, \beta) : \mu \in \mathbb{F}_q^\times\}.$$

Thus $H_{\boldsymbol{\xi}} = H_{\boldsymbol{\xi}}^+ \cup H_{\boldsymbol{\xi}}^- \cup \mathcal{L}_{\boldsymbol{\xi}}$, with $\#H_{\boldsymbol{\xi}}^\pm = q(q-1)$ and $\#\mathcal{L}_{\boldsymbol{\xi}} = q-1$.

For the case $\xi_i \neq 0$ for $i = 1,2,3,4$, the sets $H_{\boldsymbol{\xi}}^+ \cup \mathcal{L}_{\boldsymbol{\xi}}$ and $H_{\boldsymbol{\xi}}^- \cup \mathcal{L}_{\boldsymbol{\xi}}$ admit the following Cartesian descriptions:

$$H_{\boldsymbol{\xi}}^+ \cup \mathcal{L}_{\boldsymbol{\xi}} = \{(t\xi_1,\ s\xi_2,\ s\xi_3,\ t\xi_4) : t, s \in \mathbb{F}_q,\ (t,s) \neq (0,0)\},$$

and similarly

$$H_{\boldsymbol{\xi}}^- \cup \mathcal{L}_{\boldsymbol{\xi}} = \{(t\xi_1,\ s\xi_2,\ t\xi_3,\ s\xi_4) : t, s \in \mathbb{F}_q,\ (t,s) \neq (0,0)\}.$$

The parametrization is slightly different when some coordinates of $\boldsymbol{\xi}$ vanish, but in all cases the sets $H_{\boldsymbol{\xi}}^+ \cup \mathcal{L}_{\boldsymbol{\xi}}$ and $H_{\boldsymbol{\xi}}^- \cup \mathcal{L}_{\boldsymbol{\xi}}$ are closed under addition after including $\mathbf{0}$, and thus represent *punctured planes* in $\mathbb{F}_q^4$. We denote them respectively by $A_{\boldsymbol{\xi}}^+$ and $A_{\boldsymbol{\xi}}^-$.

Proposition 4.6. *For points $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in Γ^3 the following hold.*

(i) $H_{\boldsymbol{\xi}}^\pm = H_{\lambda\boldsymbol{\xi}}^\pm$ *and* $A_{\boldsymbol{\xi}}^\pm = A_{\lambda\boldsymbol{\xi}}^\pm$ *for all* $\lambda \in \mathbb{F}_q^\times$.

(ii) $\boldsymbol{\eta} \in H_{\boldsymbol{\xi}}^\pm \iff \boldsymbol{\xi} \in H_{\boldsymbol{\eta}}^\pm$.

(iii) $\boldsymbol{\eta} \in H_{\boldsymbol{\xi}}^\pm \iff \boldsymbol{\xi} + \boldsymbol{\eta} \in H_{\boldsymbol{\xi}}^\pm$.

(iv) *The sets $A_{\boldsymbol{\xi}}^\pm$ and $A_{\boldsymbol{\eta}}^\pm$ are either disjoint or equal.*

(v) *The sets $A_{\boldsymbol{\xi}}^\pm$ and $A_{\boldsymbol{\eta}}^\mp$ always intersect along a punctured line (punctured at $\mathbf{0}$).*

Proof. Part (i) follows immediately from the definitions of $H_{\boldsymbol{\xi}}^\pm$. In the rest of the proof we focus on $H_{\boldsymbol{\xi}}^+$ and $A_{\boldsymbol{\xi}}^+$; the other cases are analogous. Let $\boldsymbol{\xi} = \mathfrak{s}(\lambda, \alpha, \beta)$ and assume $\boldsymbol{\eta} \in H_{\boldsymbol{\xi}}^+$, so $\boldsymbol{\eta} = \mathfrak{s}(\mu, \alpha, \gamma)$ for some $(\mu, \gamma) \in \mathbb{F}_q^\times \times (\mathbb{P}^1 \setminus \{\beta\})$. Since $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ share the same second coordinate α in the parametrization, part (ii) follows. For part (iii), compute

$$\begin{aligned} \boldsymbol{\xi} + \boldsymbol{\eta} &= \mathfrak{s}(\lambda, \alpha, \beta) + \mathfrak{s}(\mu, \alpha, \gamma) \\ &= (\alpha_1(\lambda\beta_1 + \mu\gamma_1),\ \alpha_2(\lambda\beta_2 + \mu\gamma_2),\ \alpha_1(\lambda\beta_2 + \mu\gamma_2),\ \alpha_2(\lambda\beta_1 + \mu\gamma_1)) \end{aligned}$$

$$= \begin{cases} \mathfrak{s}\Big(\lambda\beta_1+\mu\gamma_1,\ \alpha,\ \Big(1, \dfrac{\lambda\beta_2+\mu\gamma_2}{\lambda\beta_1+\mu\gamma_1}\Big)\Big), & \text{if } \lambda\beta_1+\mu\gamma_1 \neq 0, \\ \mathfrak{s}\big(\lambda\beta_2+\mu\gamma_2,\ \alpha,\ (0,1)\big), & \text{if } \lambda\beta_1+\mu\gamma_1 = 0. \end{cases}$$

In both cases $\boldsymbol{\xi}+\boldsymbol{\eta} \in H^+_{\boldsymbol{\xi}}$. For the converse, if $\boldsymbol{\xi}+\boldsymbol{\eta} \in H^+_{\boldsymbol{\xi}}$ for some $\boldsymbol{\eta} \in \Gamma^3$, then $-\boldsymbol{\xi}-\boldsymbol{\eta}$ also belongs to $H^+_{\boldsymbol{\xi}}$, and since $-\boldsymbol{\eta} = \boldsymbol{\xi} + (-\boldsymbol{\xi}-\boldsymbol{\eta}) \in H^+_{\boldsymbol{\xi}}$, we get $\boldsymbol{\eta} \in H^+_{\boldsymbol{\xi}}$.

For part (iv), all elements of $A^+_{\boldsymbol{\xi}}$ share the same second coordinate α in the parametrization. Hence if $A^+_{\boldsymbol{\xi}} \cap A^+_{\boldsymbol{\eta}}$ is non-empty, both sets have the same second coordinate and thus coincide.

For part (v), given $\boldsymbol{\xi} = \mathfrak{s}(\lambda,\alpha,\beta)$ and $\boldsymbol{\eta} = \mathfrak{s}(\mu,\gamma,\delta)$, a point $\boldsymbol{\rho} \in \Gamma^3$ lies in $A^+_{\boldsymbol{\xi}} \cap A^-_{\boldsymbol{\eta}}$ if and only if $\boldsymbol{\rho} = \mathfrak{s}(\sigma,\alpha,\delta)$ for some $\sigma \in \mathbb{F}_q^\times$, which is precisely a punctured line. $\square$

Proposition 4.7. *The following identity holds:*

$$\sum_{\boldsymbol{\xi}\in\Gamma^3}\Bigg(\sum_{\boldsymbol{\eta}_1\in A^+_{\boldsymbol{\xi}}} f(\boldsymbol{\eta}_1)^2\Bigg)\Bigg(\sum_{\boldsymbol{\eta}_2\in A^-_{\boldsymbol{\xi}}} f(\boldsymbol{\eta}_2)^2\Bigg) = (q-1)\Bigg(\sum_{\boldsymbol{\xi}\in\Gamma^3} f(\boldsymbol{\xi})^2\Bigg)^2. \tag{4.7}$$

Proof. Rewrite the inner sums using the parametrization $\mathfrak{s}$. For $\boldsymbol{\xi} = \mathfrak{s}(\lambda,\alpha,\beta)$,

$$\sum_{\boldsymbol{\eta}_1\in A^+_{\boldsymbol{\xi}}} f(\boldsymbol{\eta}_1)^2 = \sum_{\substack{\mu\in\mathbb{F}_q^\times \\ y\in\mathbb{P}^1}} f(\mathfrak{s}(\mu,\alpha,y))^2,$$

and

$$\sum_{\boldsymbol{\eta}_2\in A^-_{\boldsymbol{\xi}}} f(\boldsymbol{\eta}_2)^2 = \sum_{\substack{\mu\in\mathbb{F}_q^\times \\ x\in\mathbb{P}^1}} f(\mathfrak{s}(\mu,x,\beta))^2.$$

Thus the left-hand side of (4.7) equals

$$\begin{aligned} &\sum_{\substack{\lambda\in\mathbb{F}_q^\times \\ \alpha,\beta\in\mathbb{P}^1}}\Bigg(\sum_{\substack{\mu_1\in\mathbb{F}_q^\times \\ y\in\mathbb{P}^1}} f(\mathfrak{s}(\mu_1,\alpha,y))^2\Bigg)\Bigg(\sum_{\substack{\mu_2\in\mathbb{F}_q^\times \\ x\in\mathbb{P}^1}} f(\mathfrak{s}(\mu_2,x,\beta))^2\Bigg) \\ &\qquad = \sum_{\lambda\in\mathbb{F}_q^\times}\Bigg(\sum_{\substack{\mu_1\in\mathbb{F}_q^\times \\ \alpha,y\in\mathbb{P}^1}} f(\mathfrak{s}(\mu_1,\alpha,y))^2\Bigg)\Bigg(\sum_{\substack{\mu_2\in\mathbb{F}_q^\times \\ x,\beta\in\mathbb{P}^1}} f(\mathfrak{s}(\mu_2,x,\beta))^2\Bigg). \end{aligned}$$

The two inner sums are equal and each equals $\sum_{\boldsymbol{\xi}\in\Gamma^3} f(\boldsymbol{\xi})^2$. Summing over $\lambda \in \mathbb{F}_q^\times$ (contributing a factor of $q-1$) gives the result. $\square$

4.3 Proof of Theorem 4.1

In this section, we provide the argument that allows us to conclude the sharp constant in (4.4).

Step 1: Symmetrization. We employ the method of antipodal symmetrization from [37, 50]. The right-hand side of (4.4) does not change if we replace $f(\boldsymbol{\xi})$ by

$$f_\sharp(\boldsymbol{\xi}) := \sqrt{\frac{|f(\boldsymbol{\xi})|^2 + |f(-\boldsymbol{\xi})|^2}{2}}, \qquad \boldsymbol{\xi} \in \Gamma^3.$$

Moreover, this replacement does not reduce the left-hand side. To see this, set

$$\mathrm{d}\Sigma := \boldsymbol{\delta}(\boldsymbol{\eta}_1 + \boldsymbol{\eta}_2 + \boldsymbol{\eta}_3 + \boldsymbol{\eta}_4)\, \mathrm{d}\boldsymbol{\eta}_1\, \mathrm{d}\boldsymbol{\eta}_2\, \mathrm{d}\boldsymbol{\eta}_3\, \mathrm{d}\boldsymbol{\eta}_4,$$

where $\boldsymbol{\delta}$ is the Dirac mass at $\mathbf{0}$, so that

$$\begin{aligned}\sum_{\boldsymbol{\xi} \in \mathbb{F}_q^4} \Bigg| \sum_{\substack{\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in \Gamma^3 \\ \boldsymbol{\eta}_1 + \boldsymbol{\eta}_2 = \boldsymbol{\xi}}} f(\boldsymbol{\eta}_1) f(\boldsymbol{\eta}_2) \Bigg|^2 &= \int_{(\Gamma^3)^4} f(\boldsymbol{\eta}_1) f(\boldsymbol{\eta}_2) \overline{f(-\boldsymbol{\eta}_3)}\, \overline{f(-\boldsymbol{\eta}_4)}\, \mathrm{d}\Sigma \\ &= \int_{(\Gamma^3)^4} f(\boldsymbol{\eta}_1)\, \overline{f(-\boldsymbol{\eta}_2)}\, f(\boldsymbol{\eta}_3)\, \overline{f(-\boldsymbol{\eta}_4)}\, \mathrm{d}\Sigma = Q(f, f, f, f),\end{aligned}$$

where

$$Q(f_1, f_2, f_3, f_4) := \int_{(\Gamma^3)^4} f_1(\boldsymbol{\eta}_1)\, \overline{f_2(-\boldsymbol{\eta}_2)}\, f_3(\boldsymbol{\eta}_3)\, \overline{f_4(-\boldsymbol{\eta}_4)}\, \mathrm{d}\Sigma.$$

Symmetry of the measure $\mathrm{d}\Sigma$ allows us to write

$$Q(f, f, f, f) = Q(f^*, f^*, f, f),$$

where $f^*(\boldsymbol{\xi}) := \overline{f(-\boldsymbol{\xi})}$. Hence, we take the average of these two expressions

$$\begin{aligned} Q(f, f, f, f) &= \int_{(\Gamma^3)^4} \left(\frac{f(\boldsymbol{\eta}_1)\overline{f(-\boldsymbol{\eta}_2)} + \overline{f(-\boldsymbol{\eta}_1)} f(\boldsymbol{\eta}_2)}{2} \right) f(\boldsymbol{\eta}_3)\, \overline{f(-\boldsymbol{\eta}_4)}\, \mathrm{d}\Sigma \\ &= \operatorname{Re} \int_{(\Gamma^3)^4} \left(\frac{f(\boldsymbol{\eta}_1)\overline{f(-\boldsymbol{\eta}_2)} + \overline{f(-\boldsymbol{\eta}_1)} f(\boldsymbol{\eta}_2)}{2} \right) f(\boldsymbol{\eta}_3)\, \overline{f(-\boldsymbol{\eta}_4)}\, \mathrm{d}\Sigma. \end{aligned}$$

Using the inequality $\operatorname{Re}(z) \leqslant |z|$ and then Cauchy–Schwarz we get

$$\begin{aligned} Q(f, f, f, f) &\leqslant \int_{(\Gamma^3)^4} \left| \frac{f(\boldsymbol{\eta}_1)\overline{f(-\boldsymbol{\eta}_2)} + \overline{f(-\boldsymbol{\eta}_1)} f(\boldsymbol{\eta}_2)}{2} \right| \left| f(\boldsymbol{\eta}_3)\, \overline{f(-\boldsymbol{\eta}_4)} \right| \mathrm{d}\Sigma \\ &\overset{\text{C-S}}{\leqslant} \int_{(\Gamma^3)^4} f_\sharp(\boldsymbol{\eta}_1)\, f_\sharp(\boldsymbol{\eta}_2)\, |f(\boldsymbol{\eta}_3)||f(-\boldsymbol{\eta}_4)|\, \mathrm{d}\Sigma = Q(f_\sharp, f_\sharp, |f|, |f|). \end{aligned}$$

Note that the equality is attained if and only if the following conditions are satisfied

- $\left(f(\boldsymbol{\eta}_1)\overline{f(-\boldsymbol{\eta}_2)} + \overline{f(-\boldsymbol{\eta}_1)} f(\boldsymbol{\eta}_2)\right) f(\boldsymbol{\eta}_3)\, \overline{f(-\boldsymbol{\eta}_4)} \geqslant 0$, for all $\boldsymbol{\eta}_i \in \Gamma^3$ such that $\boldsymbol{\eta}_1 + \boldsymbol{\eta}_2 + \boldsymbol{\eta}_3 + \boldsymbol{\eta}_4 = \mathbf{0}$;

- $f(\boldsymbol{\eta}_1) = C(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2) f(-\boldsymbol{\eta}_2)$ and $\overline{f(-\boldsymbol{\eta}_1)} = C(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)\overline{f(\boldsymbol{\eta}_2)}$, for some $C(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in \mathbb{C}$, for all $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in \Gamma^3$.

Also notice that $Q(f, f, f, f) = Q(f, f, f^*, f^*)$, therefore, the same argument gives

$$Q(f, f, f, f) \leqslant Q(f_\sharp, f_\sharp, |f|, |f|) \leqslant Q(f_\sharp, f_\sharp, f_\sharp, f_\sharp), \tag{4.8}$$

with equality subject to the same conditions. Henceforth, we may assume that the function f is even and takes only non-negative values.

For simplicity, we introduce the following notation for expressions that appear frequently throughout the proof. Let g be a function on Γ^3; for any $A \subseteq \Gamma^3$ and $\boldsymbol{\xi} \in \mathbb{F}_q^4$ we set

$$\sum_A g \coloneqq \sum_{\boldsymbol{\eta}\in A} g(\boldsymbol{\eta}),$$
$$\sum_{A,\, \boldsymbol{\xi}} g \cdot g \coloneqq \sum_{\substack{\boldsymbol{\eta}\in A \\ (\boldsymbol{\xi}-\boldsymbol{\eta})\in\Gamma^3}} g(\boldsymbol{\eta})\, g(\boldsymbol{\xi}-\boldsymbol{\eta}).$$

Step 2: Mass transport. We split the sum on the left side of (4.4) into three parts: Γ^3 and $\{\mathbf{0}\}$, and the rest of the points in $\mathbb{F}_q^4$,

$$\sum_{\boldsymbol{\xi}\in\mathbb{F}_q^4}\Big(\sum_{\Gamma^3,\,\boldsymbol{\xi}} f\cdot f\Big)^2 = \sum_{\boldsymbol{\xi}\in\Gamma^3}\Big(\sum_{\Gamma^3,\,\boldsymbol{\xi}} f\cdot f\Big)^2 + \Big(\sum_{\boldsymbol{\xi}\in\Gamma^3} f(\boldsymbol{\xi})f(-\boldsymbol{\xi})\Big)^2 + \sum_{\boldsymbol{\xi}\notin\Gamma_0^3}\Big(\sum_{\Gamma^3,\,\boldsymbol{\xi}} f\cdot f\Big)^2,$$

where $\Gamma_0^3 = \Gamma^3 \cup \{\mathbf{0}\}$.

To begin with, we estimate the terms outside the set Γ_0^3 by employing the Cauchy–Schwarz inequality alongside the precise formula laid out in Proposition 4.5. Subsequently, we complete the expression by strategically adding and subtracting suitable terms as follows

$$\begin{aligned}
\sum_{\boldsymbol{\xi}\notin\Gamma_0^3}\Big(\sum_{\Gamma^3,\,\boldsymbol{\xi}} f\cdot f\Big)^2 &\leqslant \sum_{\boldsymbol{\xi}\notin\Gamma_0^3}\Big(\sum_{\Gamma^3,\,\boldsymbol{\xi}} 1\cdot 1\Big)\Big(\sum_{\Gamma^3,\,\boldsymbol{\xi}} f^2\cdot f^2\Big) = (q^2+q)\sum_{\boldsymbol{\xi}\notin\Gamma_0^3}\Big(\sum_{\Gamma^3,\,\boldsymbol{\xi}} f^2\cdot f^2\Big) \\
&= (q^2+q)\sum_{\boldsymbol{\xi}\in\mathbb{F}_q^4}\Big(\sum_{\Gamma^3,\,\boldsymbol{\xi}} f^2\cdot f^2\Big) - (q^2+q)\sum_{\boldsymbol{\xi}\in\Gamma_0^3}\Big(\sum_{\Gamma^3,\,\boldsymbol{\xi}} f^2\cdot f^2\Big) \\
&= (q^2+q)\Big(\sum_{\Gamma^3} f^2\Big)^2 - (q^2+q)\sum_{\boldsymbol{\xi}\in\Gamma^3}\Big(\sum_{\Gamma^3,\,\boldsymbol{\xi}} f^2\cdot f^2\Big) - (q^2+q)\sum_{\Gamma^3} f^4,
\end{aligned}$$

where the first term in the last line comes from the change of the order of summation in the corresponding sum above. Observe that equality is achieved when, for every fixed $\boldsymbol{\xi} \notin \Gamma_0^3$, the identity $f(\boldsymbol{\xi}-\boldsymbol{\eta}) = C(\boldsymbol{\xi})f(\boldsymbol{\eta})$ holds for every $\boldsymbol{\eta} \in \Gamma^3$ with $\boldsymbol{\xi}-\boldsymbol{\eta} \in \Gamma^3$, where $C(\boldsymbol{\xi})$ is a constant depending solely on $\boldsymbol{\xi}$.

Based on this estimate, to prove (4.4) it suffices to show

$$\begin{aligned}\sum_{\boldsymbol{\xi}\in\Gamma^3}\Big(\sum_{H_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big)^2 - q(q+1)\sum_{\boldsymbol{\xi}\in\Gamma^3}\sum_{H_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f^2\cdot f^2 \\ - q(q+1)\sum_{\Gamma^3} f^4 - \frac{2q^4-5q^3-5q^2+5q+4}{(q-1)(q+1)^2}\Big(\sum_{\Gamma^3} f^2\Big)^2 \leqslant 0.\end{aligned} \tag{4.9}$$

Now we utilize the geometric information about the sets $H_{\boldsymbol{\xi}}$. Rewrite the first term by separating the following sets in the inner sum $\mathcal{L}^\circ_{\boldsymbol{\xi}} := \mathcal{L}_{\boldsymbol{\xi}}\setminus\{\boldsymbol{\xi}\}$, $H^+_{\boldsymbol{\xi}}$, and $H^-_{\boldsymbol{\xi}}$.

$$\begin{aligned}\sum_{\boldsymbol{\xi}\in\Gamma^3}\Big(\sum_{H_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big)^2 &= \sum_{\boldsymbol{\xi}\in\Gamma^3}\Big\{\Big(\sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big)^2 + \Big(\sum_{H^+_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big)^2 + \Big(\sum_{H^-_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big)^2\Big\} \\ &+ 2\sum_{\boldsymbol{\xi}\in\Gamma^3}\Big\{\Big(\sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big)\Big(\sum_{H^+_{\boldsymbol{\xi}}\cup H^-_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big) + \Big(\sum_{H^+_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big)\Big(\sum_{H^-_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big)\Big\}.\end{aligned}$$

Similarly

$$\sum_{\boldsymbol{\xi}\in\Gamma^3}\sum_{H_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f^2\cdot f^2 = \sum_{\boldsymbol{\xi}\in\Gamma^3}\sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f^2\cdot f^2 + \sum_{\boldsymbol{\xi}\in\Gamma^3}\sum_{H^+_{\boldsymbol{\xi}}\cup H^-_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f^2\cdot f^2.$$

Step 3: Application of Proposition 4.7. This is an important part of our argument. We can apply Proposition 4.7 for the last term on the left-hand side of the inequality (4.9). For simplicity, we denote by

$$M_q = \frac{2q^4-5q^3-5q^2+5q+4}{(q-1)(q+1)^2}$$

and the last term of (4.9) can be replaced by

$$\frac{M_q}{q-1}\sum_{\boldsymbol{\xi}\in\Gamma^3}\Big(\sum_{A^+_{\boldsymbol{\xi}}} f^2\Big)\Big(\sum_{A^-_{\boldsymbol{\xi}}} f^2\Big),$$

which we can write as

$$\frac{M_q}{q-1}\sum_{\boldsymbol{\xi}\in\Gamma^3}\Big\{\Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2\Big)^2 + \Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2\Big)\Big(\sum_{H^+_{\boldsymbol{\xi}}\cup H^-_{\boldsymbol{\xi}}} f^2\Big) + \Big(\sum_{H^+_{\boldsymbol{\xi}}} f^2\Big)\Big(\sum_{H^-_{\boldsymbol{\xi}}} f^2\Big)\Big\}.$$

Step 4: Separation of the sums over $H^+_{\boldsymbol{\xi}}$ and $H^-_{\boldsymbol{\xi}}$. We have rewritten the expression on the left of (4.9). In this rewritten version, there are only two expressions where sums over $H^+_{\boldsymbol{\xi}}$ and $H^-_{\boldsymbol{\xi}}$ are multiplied by each other. We want to replace such terms with terms where there are no mixed products, using the AM–GM inequality. Grouping them, we have

$$
\begin{aligned}
2\sum_{\boldsymbol{\xi}\in\Gamma^3}\Big(\sum_{H_{\boldsymbol{\xi}}^+,\,\boldsymbol{\xi}} f\cdot f\Big)\Big(\sum_{H_{\boldsymbol{\xi}}^-,\,\boldsymbol{\xi}} f\cdot f\Big) &- \frac{M_q}{q-1}\sum_{\boldsymbol{\xi}\in\Gamma^3}\Big(\sum_{H_{\boldsymbol{\xi}}^+} f^2\Big)\Big(\sum_{H_{\boldsymbol{\xi}}^-} f^2\Big) \\
&\overset{\text{C-S}}{\leqslant} \Big(2-\frac{M_q}{q-1}\Big)\sum_{\boldsymbol{\xi}\in\Gamma^3}\Big(\sum_{H_{\boldsymbol{\xi}}^+,\,\boldsymbol{\xi}} f\cdot f\Big)\Big(\sum_{H_{\boldsymbol{\xi}}^-,\,\boldsymbol{\xi}} f\cdot f\Big) \\
&\overset{\text{AM-GM}}{\leqslant} \Big(1-\frac{M_q}{2(q-1)}\Big)\sum_{\boldsymbol{\xi}\in\Gamma^3}\Big\{\Big(\sum_{H_{\boldsymbol{\xi}}^+,\,\boldsymbol{\xi}} f\cdot f\Big)^2+\Big(\sum_{H_{\boldsymbol{\xi}}^-,\,\boldsymbol{\xi}} f\cdot f\Big)^2\Big\}.
\end{aligned}
$$

It is crucial to note that $2-\frac{M_q}{q-1}=\frac{5q^3+q^2-5q-2}{(q^2-1)^2}>0$ for every odd prime power $q\geqslant 3$. In the first inequality above, equality occurs if and only if $f(\boldsymbol{\xi}-\boldsymbol{\eta})=f(\boldsymbol{\eta})\,C^{\pm}(\boldsymbol{\xi})$ for all $\boldsymbol{\eta}\in H_{\boldsymbol{\xi}}^{\pm}$ and all $\boldsymbol{\xi}\in\Gamma^3$. In the second inequality, equality occurs if and only if the factors

$$
\sum_{H_{\boldsymbol{\xi}}^+,\,\boldsymbol{\xi}} f\cdot f \quad\text{and}\quad \sum_{H_{\boldsymbol{\xi}}^-,\,\boldsymbol{\xi}} f\cdot f
$$

are equal.

Combining the estimates from Steps 2–4, we obtain

$$
\begin{aligned}
\text{LHS of (4.9)} \leqslant{} & \sum_{\boldsymbol{\xi}\in\Gamma^3}\Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}^\circ,\,\boldsymbol{\xi}} f\cdot f\Big)^2 \\
&+\Big(2-\frac{M_q}{2(q-1)}\Big)\sum_{\boldsymbol{\xi}\in\Gamma^3}\Big\{\Big(\sum_{H_{\boldsymbol{\xi}}^+,\,\boldsymbol{\xi}} f\cdot f\Big)^2+\Big(\sum_{H_{\boldsymbol{\xi}}^-,\,\boldsymbol{\xi}} f\cdot f\Big)^2\Big\} \\
&+2\sum_{\boldsymbol{\xi}\in\Gamma^3}\Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}^\circ,\,\boldsymbol{\xi}} f\cdot f\Big)\Big(\sum_{H_{\boldsymbol{\xi}}^+\cup H_{\boldsymbol{\xi}}^-,\,\boldsymbol{\xi}} f\cdot f\Big) \\
&-q(q+1)\sum_{\Gamma^3} f^4 - q(q+1)\sum_{\boldsymbol{\xi}\in\Gamma^3}\Big\{\sum_{\mathcal{L}_{\boldsymbol{\xi}}^\circ,\,\boldsymbol{\xi}} f^2\cdot f^2+\sum_{H_{\boldsymbol{\xi}}^+\cup H_{\boldsymbol{\xi}}^-,\,\boldsymbol{\xi}} f^2\cdot f^2\Big\} \\
&-\frac{M_q}{q-1}\sum_{\boldsymbol{\xi}\in\Gamma^3}\Big\{\Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2\Big)^2+\Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2\Big)\Big(\sum_{H_{\boldsymbol{\xi}}^+\cup H_{\boldsymbol{\xi}}^-} f^2\Big)\Big\}.
\end{aligned}
\tag{4.10}
$$

It suffices to show that the right-hand side of (4.10) is non-positive. Now, we will divide the expression into several parts in a meaningful way and show that each part we consider is non-positive.

Step 5: Regroup terms of the same plane. We examine each plane $A_{\boldsymbol{\xi}}^{\pm}$ separately. It follows from Proposition 4.6 that there are exactly $q+1$ pairwise disjoint planes of the form $A_{\boldsymbol{\xi}}^+$; the same conclusion is true for $A_{\boldsymbol{\xi}}^-$. We enumerate them as $A_i^{\pm}$, for $1\leqslant i\leqslant q+1$. Thus, we have the following identity

$$
\sum_{\boldsymbol{\xi}\in\Gamma^3} g(\boldsymbol{\xi})=\sum_{i=1}^{q+1}\sum_{\boldsymbol{\xi}\in A_i^+} g(\boldsymbol{\xi})=\sum_{i=1}^{q+1}\sum_{\boldsymbol{\xi}\in A_i^-} g(\boldsymbol{\xi}).
$$

Now, we pick a plane A among the planes A_i^+, and analyze the following expression

$$\begin{aligned}
S(A) \coloneqq \sum_{\boldsymbol{\xi}\in A} \frac{1}{2}\Big(\sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big)^2 &+ \Big(2-\frac{M_q}{2(q-1)}\Big)\sum_{\boldsymbol{\xi}\in A}\Big(\sum_{H^+_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big)^2 \\
&+ 2\sum_{\boldsymbol{\xi}\in A}\Big(\sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big)\Big(\sum_{H^+_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big) - \frac{q(q+1)}{2}\sum_{\boldsymbol{\xi}\in A}\sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f^2\cdot f^2 \\
&- q(q+1)\sum_{\boldsymbol{\xi}\in A}\sum_{H^+_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f^2\cdot f^2 - \frac{q(q+1)}{2}\sum_A f^4 \\
&- \frac{M_q}{2(q-1)}\sum_{\boldsymbol{\xi}\in A}\Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2\Big)^2 - \frac{M_q}{q-1}\sum_{\boldsymbol{\xi}\in A}\Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2\Big)\Big(\sum_{H^+_{\boldsymbol{\xi}}} f^2\Big).
\end{aligned}$$

Note that some of the terms in (4.10) appear in two different places, which are $S(A_i^+)$ and $S(A_j^-)$ for appropriate i, j. Hence, we are taking such terms with the coefficient $1/2$ in the definition of $S(A)$.

To estimate the quantity $S(A)$, we use the following inequalities

$$\begin{aligned}
\Big(\sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big)^2 &\leqslant (q-2)\sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f^2\cdot f^2, \\
\Big(\sum_{H^+_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big)^2 &\leqslant q(q-1)\sum_{H^+_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f^2\cdot f^2, \\
\Big(\sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big)\Big(\sum_{H^+_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f\cdot f\Big) &\leqslant \Big(\sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}}} f^2\Big)\Big(\sum_{H^+_{\boldsymbol{\xi}}} f^2\Big),
\end{aligned}$$

where the first and the second follow from the Cauchy–Schwarz inequality, and the last one can be deduced from the AM–GM inequality. Therefore, the equality in these inequalities occurs if and only if the following conditions are satisfied, respectively:

(i) $f(\boldsymbol{\eta}) = C_1(\boldsymbol{\xi})$ for all $\boldsymbol{\eta}\in\mathcal{L}^\circ_{\boldsymbol{\xi}}$;
(ii) $f(\boldsymbol{\eta}) = C_2(\boldsymbol{\xi})$ for all $\boldsymbol{\eta}\in H^+_{\boldsymbol{\xi}}$;
(iii) $f(\boldsymbol{\eta}) = f(\boldsymbol{\xi}-\boldsymbol{\eta})$ for all $\boldsymbol{\eta}\in\mathcal{L}^\circ_{\boldsymbol{\xi}}\cup H^+_{\boldsymbol{\xi}}$.

Thus, we have the following

$$\begin{aligned}
S(A) \leqslant \frac{q-2}{2}\sum_{\boldsymbol{\xi}\in A}\sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f^2\cdot f^2 &+ \Big(2-\frac{M_q}{2(q-1)}\Big)q(q-1)\sum_{\boldsymbol{\xi}\in A}\sum_{H^+_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f^2\cdot f^2 \\
&+ 2\sum_{\boldsymbol{\xi}\in A}\Big(\sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}}} f^2\Big)\Big(\sum_{H^+_{\boldsymbol{\xi}}} f^2\Big) - \frac{q(q+1)}{2}\sum_{\boldsymbol{\xi}\in A}\sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f^2\cdot f^2 \\
&- q(q+1)\sum_{\boldsymbol{\xi}\in A}\sum_{H^+_{\boldsymbol{\xi}},\,\boldsymbol{\xi}} f^2\cdot f^2 - \frac{q(q+1)}{2}\sum_A f^4
\end{aligned}$$

$$- \frac{M_q}{2(q-1)} \sum_{\boldsymbol{\xi}\in A} \Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2 \Big)^2 - \frac{M_q}{q-1} \sum_{\boldsymbol{\xi}\in A} \Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2 \Big) \Big(\sum_{H^+_{\boldsymbol{\xi}}} f^2 \Big).$$

At this moment, we are able to apply Fubini's theorem for the iterated sums and simplify the expression on the right. Using the facts from Proposition 4.6, one can see that the following identities hold:

$$\sum_{\boldsymbol{\xi}\in A} \sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}},\, \boldsymbol{\xi}} f^2 \cdot f^2 = \sum_{\boldsymbol{\xi}\in A} f(\boldsymbol{\xi})^2 \sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}}} f^2 = \frac{1}{q-1} \sum_{\boldsymbol{\xi}\in A} \Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2 \Big)^2 - \sum_A f^4,$$

$$\sum_{\boldsymbol{\xi}\in A} \sum_{H^+_{\boldsymbol{\xi}},\, \boldsymbol{\xi}} f^2 \cdot f^2 = \sum_{\boldsymbol{\xi}\in A} f(\boldsymbol{\xi})^2 \sum_{H^+_{\boldsymbol{\xi}}} f^2 = \frac{1}{q-1} \sum_{\boldsymbol{\xi}\in A} \Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2 \Big) \Big(\sum_{H^+_{\boldsymbol{\xi}}} f^2 \Big),$$

$$\begin{aligned} \sum_{\boldsymbol{\xi}\in A} \Big(\sum_{\mathcal{L}^\circ_{\boldsymbol{\xi}}} f^2 \Big) \Big(\sum_{H^+_{\boldsymbol{\xi}}} f^2 \Big) &= \sum_{\boldsymbol{\xi}\in A} \Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2 \Big) \Big(\sum_{H^+_{\boldsymbol{\xi}}} f^2 \Big) - \sum_{\boldsymbol{\xi}\in A} f(\boldsymbol{\xi})^2 \Big(\sum_{H^+_{\boldsymbol{\xi}}} f^2 \Big) \\ &= \frac{q-2}{q-1} \sum_{\boldsymbol{\xi}\in A} \Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2 \Big) \Big(\sum_{H^+_{\boldsymbol{\xi}}} f^2 \Big). \end{aligned}$$

From these identities, it follows that

$$\begin{aligned} S(A) \leqslant &- \frac{q^5 + 3q^4 - 4q^3 - 4q^2 + 3q + 2}{2(q^2-1)^2} \sum_{\boldsymbol{\xi}\in A} \Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2 \Big)^2 \\ &+ \frac{q\left(q^3 + 3q^2 - 3q - 4\right)}{2\left(q^2-1\right)^2} \sum_{\boldsymbol{\xi}\in A} \Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2 \Big) \Big(\sum_{H^+_{\boldsymbol{\xi}}} f^2 \Big) - \frac{q-2}{2} \sum_A f^4. \end{aligned}$$

Finally, we need one more identity to make the expression simpler

$$\begin{aligned} \sum_{\boldsymbol{\xi}\in A} \Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2 \Big) \Big(\sum_{H^+_{\boldsymbol{\xi}}} f^2 \Big) &= \sum_{\boldsymbol{\xi}\in A} \Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2 \Big) \Big(\sum_A f^2 \Big) - \sum_{\boldsymbol{\xi}\in A} \Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2 \Big)^2 \\ &= (q-1) \Big(\sum_A f^2 \Big)^2 - \sum_{\boldsymbol{\xi}\in A} \Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2 \Big)^2. \end{aligned}$$

Hence, the above estimate simplifies to the following

$$\begin{aligned} S(A) \leqslant &- \frac{q^4 + 3q^3 - 4q^2 - 3q + 2}{2(q-1)^2(q+1)} \sum_{\boldsymbol{\xi}\in A} \Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2 \Big)^2 - \frac{q-2}{2} \sum_A f^4 \\ &+ \frac{q\left(q^3 + 3q^2 - 3q - 4\right)}{2(q-1)\left(q+1\right)^2} \Big(\sum_A f^2 \Big)^2. \end{aligned}$$

Note the following implication of the Cauchy–Schwarz inequality

$$\sum_A f^4 \geqslant \frac{1}{q^2-1} \Big(\sum_A f^2 \Big)^2,$$

where the equality happens if and only if f is a constant function on A.

Thus, it follows that

$$S(A) \leqslant \frac{q^4+3q^3-4q^2-3q+2}{2(q+1)^2(q-1)}\Big(\sum_A f^2\Big)^2 - \frac{q^4+3q^3-4q^2-3q+2}{2(q-1)^2(q+1)}\sum_{\boldsymbol{\xi}\in A}\Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2\Big)^2$$
$$= \frac{q^4+3q^3-4q^2-3q+2}{2(q+1)^2(q-1)}\left(\Big(\sum_A f^2\Big)^2 - \frac{q+1}{q-1}\sum_{\boldsymbol{\xi}\in A}\Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2\Big)^2\right).$$

We know that the first factor is a non-negative number for all $q > 2$. Moreover, notice that

$$\Big(\sum_A f^2\Big)^2 = \left(\frac{1}{q-1}\sum_{\boldsymbol{\xi}\in A}\Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2\Big)\right)^2 \overset{\text{C-S}}{\leqslant} \frac{q^2-1}{(q-1)^2}\sum_{\boldsymbol{\xi}\in A}\Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2\Big)^2 = \frac{q+1}{q-1}\sum_{\boldsymbol{\xi}\in A}\Big(\sum_{\mathcal{L}_{\boldsymbol{\xi}}} f^2\Big)^2.$$

Therefore, we conclude that $S(A) \leqslant 0$. The equality happens if and only if all the inequalities above are equalities, which occurs if and only if f is a constant function on A. To complete the proof, we note that the choice of A was arbitrary; we can argue for any plane $A_i^{\pm}$ in a similar way. Combining all such inequalities, we deduce that (4.9) holds. Equality in (4.9) is attained if and only if f is constant on each of the planes $A_i^{\pm}$. By Proposition 4.6, we know A_i^{+} and A_j^{-} have a non-trivial intersection; therefore, the constants are the same. Varying i and j, we see that f has to be a constant function on Γ^3. □

4.4 Classification of extremizers: Proof of Theorem 4.2

In this section, we will give a complete classification of the functions f on Γ^3 for which the inequality (4.4) becomes equality. We assume henceforth that the extremizers are not constantly zero. We begin with the following observation.

Lemma 4.8. *Let $f \not\equiv 0$ be a function on Γ^3 for which (4.4) becomes equality. Then the following must hold:*

(i) *$|f|$ is a constant function;*

(ii) *If $\varphi(\boldsymbol{\xi}) := f(\boldsymbol{\xi})/|f(\boldsymbol{\xi})|$, then*

$$\varphi(\boldsymbol{x})\varphi(\boldsymbol{y}) = \varphi(\boldsymbol{z})\varphi(\boldsymbol{w}), \tag{4.11}$$

for all $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w} \in \Gamma^3$ such that $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{z}+\boldsymbol{w}$.

Proof. Since the estimate (4.4) becomes an equality, it must be true that the inequality (4.8) also becomes an equality, and the symmetrization $f_\sharp$ of the function f has to be a constant function on Γ^3. Thus, we got the necessary condition on the magnitude of the extremizers, i.e. $|f(\boldsymbol{\eta})|^2 + |f(-\boldsymbol{\eta})|^2 = \text{const} \neq 0$ for all $\boldsymbol{\eta} \in \Gamma^3$. Combining the conditions for the equality of (4.8) and the fact that $f_\sharp = \text{const}$, we deduce that we must have the following identities,

$$\begin{aligned} f(\boldsymbol{\eta}_1) &= C(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2) f(-\boldsymbol{\eta}_2);\\ \overline{f(-\boldsymbol{\eta}_1)} &= C(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)\overline{f(\boldsymbol{\eta}_2)}, \end{aligned}$$

for all $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2 \in \Gamma^3$ and for some number $C(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2) \in \mathbb{C}$. Looking at the sum of the squares of these two equations, we deduce that $|C(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)| = 1$, therefore, $|f|$ is constant. Using this fact, we conclude that $f(\boldsymbol{\rho}) \neq 0$ for all $\boldsymbol{\rho} \in \Gamma^3$; otherwise we see $f \equiv 0$. Therefore, extremizers never vanish and we define the phase function $\varphi : \Gamma^3 \to \{z \in \mathbb{C} : |z| = 1\}$ as $\varphi(\boldsymbol{\rho}) \coloneqq f(\boldsymbol{\rho})/|f(\boldsymbol{\rho})|$.

The other condition for the equality in (4.8) is the following

$$C(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2) f_\sharp(-\boldsymbol{\eta}_2) f(\boldsymbol{\eta}_3)\overline{f(-\boldsymbol{\eta}_4)} \geqslant 0, \tag{4.12}$$

for all $\boldsymbol{\eta}_1+\boldsymbol{\eta}_2+\boldsymbol{\eta}_3+\boldsymbol{\eta}_4 = \mathbf{0}$, $\boldsymbol{\eta}_i \in \Gamma^3$. First of all, we observe that each of the factors in the expression above is non-zero by the previous deductions; therefore, the inequality must be strict. Hence, the condition in the equation (4.12) is equivalent to

$$\operatorname{Arg}\Big(C(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2) f(\boldsymbol{\eta}_3)\overline{f(-\boldsymbol{\eta}_4)}\Big) = 0 \pmod{2\pi},$$

for all $\boldsymbol{\eta}_1+\boldsymbol{\eta}_2+\boldsymbol{\eta}_3+\boldsymbol{\eta}_4 = \mathbf{0}$, $\boldsymbol{\eta}_i \in \Gamma^3$. Clearly, by varying the elements $\boldsymbol{\eta}_3$ and $\boldsymbol{\eta}_4$ in this condition we conclude that

$$\operatorname{Arg}\Big(f(\boldsymbol{\eta}_3)\overline{f(-\boldsymbol{\eta}_4)}\Big) = \operatorname{Arg}\Big(f(\boldsymbol{\eta}_3')\overline{f(-\boldsymbol{\eta}_4')}\Big) \pmod{2\pi},$$

for all $\boldsymbol{\eta}_3+\boldsymbol{\eta}_4 = \boldsymbol{\eta}_3'+\boldsymbol{\eta}_4' \neq \mathbf{0}$. If we write this equation in terms of the phase function, we get

$$\frac{\varphi(\boldsymbol{\eta}_3)}{\varphi(-\boldsymbol{\eta}_4)} = \frac{\varphi(\boldsymbol{\eta}_3')}{\varphi(-\boldsymbol{\eta}_4')}.$$

After we shuffle the terms and rename the variables, we arrive at the functional equation (4.11) for φ. □

In the rest of this section, we study the solutions to the functional equation (4.11).

Proposition 4.9. *Let p be an odd prime and $q = p^n$. If a map $\psi : \mathbb{F}_q^2 \setminus \{\mathbf{0}\} \to \mathbb{C}^\times$ satisfies*

$$\psi(\boldsymbol{x})\psi(\boldsymbol{y}) = \psi(\boldsymbol{z})\psi(\boldsymbol{w}), \tag{4.13}$$

for all non-zero $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}$, and $\boldsymbol{w}$ such that $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{z}+\boldsymbol{w}$, then it must be a multiple of a character. More precisely, there exists $\lambda \in \mathbb{C}^\times$ and $(a_1,a_2) \in \mathbb{F}_q^2$ such that

$$\psi(x_1,x_2) = \lambda \cdot \exp\left(\frac{2\pi i}{p}\operatorname{Tr}_n(a_1x_1+a_2x_2)\right), \tag{4.14}$$

for all $(x_1,x_2) \in \mathbb{F}_q^2 \setminus \{\mathbf{0}\}$.

Proof. From (4.13) it follows that $\psi(\boldsymbol{x})\psi(-\boldsymbol{x}) = \text{const}$, for all $\boldsymbol{x} \neq \mathbf{0}$. Since every non-zero scalar multiple of ψ satisfies the functional equation (4.13), we may assume (up to a constant factor) that

$$\psi(\boldsymbol{x})\psi(-\boldsymbol{x}) = 1, \qquad \text{for all } \boldsymbol{x} \neq \mathbf{0}. \tag{4.15}$$

Under these conditions, we still have two choices for the sign of ψ; thus, we can try to identify the solutions up to a factor of ± 1.

Step 1: $\psi(\boldsymbol{x})^{2p} = 1$, for all $\boldsymbol{x} \neq \mathbf{0}$.

Fix $\boldsymbol{x} \neq \mathbf{0}$. Since there are at least 3 elements in $\mathbb{F}_q^2 \setminus \{\mathbf{0}\}$ one finds $\boldsymbol{y} \in \mathbb{F}_q^2 \setminus \{\mathbf{0}\}$ such that $\boldsymbol{y} \neq \pm\boldsymbol{x}$. Therefore, using (4.15) and (4.13) consecutively, we deduce

$$\psi(\boldsymbol{x})^2 = \psi(\boldsymbol{x})\psi(\boldsymbol{y})\psi(-\boldsymbol{y})\psi(\boldsymbol{x}) = \psi\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}\right)^2 \psi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{2}\right)^2 = \psi\left(\frac{\boldsymbol{x}}{2}\right)^4. \tag{4.16}$$

If we assume $p \neq 3$, again using (4.15) we can obtain the relation

$$\psi(3\boldsymbol{x}) = \psi(3\boldsymbol{x})\psi(-\boldsymbol{x})\psi(\boldsymbol{x}) = \psi(\boldsymbol{x})^3.$$

Repeating similar argument several times we show that $\psi((p-2)\boldsymbol{x}) = \psi(\boldsymbol{x})^{p-2}$, which is also equivalent to the identity $\psi(2\boldsymbol{x}) = \psi(\boldsymbol{x})^{2-p}$. Combining the last identity with (4.16) we get

$$\psi(\boldsymbol{x})^4 = \psi(2\boldsymbol{x})^2 = \psi(\boldsymbol{x})^{4-2p},$$

which implies that $\psi(\boldsymbol{x})^{2p} = 1$ for the case $p \neq 3$.

Now suppose $p = 3$. In this case, we cannot use the previous argument simply because $3\boldsymbol{x} = \mathbf{0}$. What we can use is the fact that $2^{-1} = 2$ and (4.16). Employing these facts, one can show

$$\psi(\boldsymbol{x})^2 = \psi(2\boldsymbol{x})^4 = \psi(4\boldsymbol{x})^8 = \psi(\boldsymbol{x})^8.$$

Therefore, $\psi(\boldsymbol{x})^6 = 1$. It completes the proof of step 1.

Step 2: $\psi(2\boldsymbol{x}) = \pm\psi(\boldsymbol{x})^2$, where the choice of $\pm$ is independent of $\boldsymbol{x}$.

From (4.16) it is immediate that $\psi(2\boldsymbol{x}) \in \{\pm\psi(\boldsymbol{x})^2\}$. We argue by contradiction, suppose there are two elements $\boldsymbol{x}, \boldsymbol{y}$ such that $\psi(2\boldsymbol{x}) = \psi(\boldsymbol{x})^2$ and $\psi(2\boldsymbol{y}) = -\psi(\boldsymbol{y})^2$. In such a case, using the conclusion of step 1, we are led to the following identity

$$-1 = (-1)^p = \left(-\psi(\boldsymbol{x})^2\psi(\boldsymbol{y})^2\right)^p = \psi(2\boldsymbol{x})^p\psi(2\boldsymbol{y})^p = \psi(\boldsymbol{x}+\boldsymbol{y})^{2p} = 1,$$

which is absurd. Thus, the choice of the sign in the identity is uniform.

Using step 2, we can assume that $\psi(2\boldsymbol{x}) = \psi(\boldsymbol{x})^2$; otherwise, we work with $-\psi$, which is also a solution to the equation (4.13) with (4.15). Define an extension Ψ of the function ψ given by

$$\Psi(\boldsymbol{x}) := \begin{cases} \psi(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{F}_q^2 \setminus \{\mathbf{0}\}; \\ 1, & \boldsymbol{x} = \mathbf{0}. \end{cases}$$

From this definition, it is immediate that $\Psi(\boldsymbol{x})\Psi(\boldsymbol{y}) = \Psi(\boldsymbol{z})\Psi(\boldsymbol{w})$ holds for all $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{z} + \boldsymbol{w}$. In particular, $\boldsymbol{x} + \boldsymbol{y} = (\boldsymbol{x} + \boldsymbol{y}) + \mathbf{0}$, implies

$$\Psi(\boldsymbol{x})\Psi(\boldsymbol{y}) = \Psi(\boldsymbol{x} + \boldsymbol{y}),$$

for all $\boldsymbol{x}, \boldsymbol{y}$. Thus Ψ is a character on the additive group $(\mathbb{F}_q^2, +)$. Now it is clear that there exists $(a_1, a_2) \in \mathbb{F}_q^2$ such that

$$\Psi(\boldsymbol{x}) = \exp\left(\frac{2\pi i}{p}\operatorname{Tr}_n(a_1 x_1 + a_2 x_2)\right),$$

for all $\boldsymbol{x} = (x_1, x_2) \in \mathbb{F}_q^2$. In particular, $\psi(\boldsymbol{x})$ has the same representation for all $\boldsymbol{x} \neq \mathbf{0}$. Since we have done several reductions on the scalar factor, the general solution to the functional equation (4.11) is given by (4.14). $\square$

Now, using this result, we can classify all the extremizers.

Proof of Theorem 4.2. We know that the extremizers have constant absolute value; therefore, we can assume $f(\boldsymbol{x}) = \lambda\,\varphi(\boldsymbol{x})$, for some $\lambda \in \mathbb{C}^\times$, and here φ is the phase function of f. We first observe that any map $f : \Gamma^3 \to \mathbb{C}$ which acts as follows

$$(\eta_1, \eta_2, \eta_3, \eta_4) \mapsto \lambda \cdot \exp\left(\frac{2\pi i}{p}\operatorname{Tr}_n(a_1\eta_1 + a_2\eta_2 + a_3\eta_3 + a_4\eta_4)\right)$$

for given $a_1, a_2, a_3, a_4 \in \mathbb{F}_q$ and $\lambda \in \mathbb{C}^\times$ is an extremizer for the inequality (4.4). Indeed,

$$\begin{aligned} \sum_{\boldsymbol{\xi}\in\mathbb{F}_q^4}\Bigg|\sum_{\substack{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\in\Gamma^3\\ \boldsymbol{\eta}_1+\boldsymbol{\eta}_2=\boldsymbol{\xi}}} f(\boldsymbol{\eta}_1)f(\boldsymbol{\eta}_2)\Bigg|^2 &= |\lambda|^4\sum_{\boldsymbol{\xi}\in\mathbb{F}_q^4}\Bigg|\sum_{\substack{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\in\Gamma^3\\ \boldsymbol{\eta}_1+\boldsymbol{\eta}_2=\boldsymbol{\xi}}}\exp\left(\frac{2\pi i}{p}\operatorname{Tr}_n\Big((\boldsymbol{\eta}_1+\boldsymbol{\eta}_2)\cdot\boldsymbol{a}\Big)\right)\Bigg|^2 \\ &= |\lambda|^4\sum_{\boldsymbol{\xi}\in\mathbb{F}_q^4}(\#\Sigma_{\boldsymbol{\xi}})^2 = \mathbf{C}_{\Gamma^3_{(2,2)}}(2\to 4)\Bigg(\sum_{\boldsymbol{\xi}\in\Gamma^3_{(2,2)}}|f(\boldsymbol{\xi})|^2\Bigg)^2. \end{aligned}$$

Now, we prove that these are the only extremizers with $|f| = |\lambda|$. Since the phase function φ has modulus 1 by (4.11) and the previous proposition, we know that $\varphi|_A = c\,\chi$, where $c \in \{z \in \mathbb{C} : |z| = 1\}$ and χ is a character on the complete plane $A \cup \{\mathbf{0}\}$, for any punctured plane $A \subset \Gamma^3$. Up to a constant factor, we may assume $c = 1$. Hence, we have that φ acts as a character in any punctured plane contained in the cone. In particular, for

$$A_1 \coloneqq \left\{(0,\eta_2,0,\eta_4) : (\eta_2,\eta_4) \in \mathbb{F}_q^2 \setminus \{\mathbf{0}\}\right\},$$
$$A_2 \coloneqq \left\{(\eta_1,0,\eta_3,0) : (\eta_1,\eta_3) \in \mathbb{F}_q^2 \setminus \{\mathbf{0}\}\right\},$$

we have: $\varphi(0,\eta_2,0,\eta_4) = \exp\left(\frac{2\pi i}{p}\operatorname{Tr}_n(a_2\eta_2+a_4\eta_4)\right)$ and $\varphi(\eta_1,0,\eta_3,0) = \exp\left(\frac{2\pi i}{p}\operatorname{Tr}_n(a_1\eta_1+a_3\eta_3)\right)$, for some $a_1,a_2,a_3,a_4 \in \mathbb{F}_q$. Consider the function $\psi : \Gamma^3 \to \{z\in\mathbb{C} : |z|=1\}$ given by

$$\psi(\eta_1,\eta_2,\eta_3,\eta_4) = \frac{1}{\varphi(\eta_1,\eta_2,\eta_3,\eta_4)}\exp\left(\frac{2\pi i}{p}\operatorname{Tr}_n(a_1\eta_1+a_2\eta_2+a_3\eta_3+a_4\eta_4)\right).$$

Note that also ψ must satisfy (4.11). Observe that $\psi(0,\eta_2,0,\eta_4) = 1$ and $\psi(\eta_1,0,\eta_3,0)=1$ for any $(\eta_1,\eta_3),(\eta_2,\eta_4)\in\mathbb{F}_q^2\setminus\{\mathbf{0}\}$. Now, take any $(\eta_1,\eta_2,\eta_3,\eta_4)\in\Gamma^3\setminus(A_1\cup A_2)$ and take a nonzero $(\rho_2,\rho_4)\neq(-\eta_2,-\eta_4)$; then we have that

$$\begin{aligned}\psi(\eta_1,\eta_2,\eta_3,\eta_4) =& \psi(\eta_1,\eta_2,\eta_3,\eta_4)\psi(0,\rho_2,0,\rho_4)\\ &= \psi(\eta_1,0,\eta_3,0)\psi(0,\eta_2+\rho_2,0,\eta_4+\rho_4) = 1.\end{aligned}$$

From this, we conclude that $\varphi(\boldsymbol{\eta})$ is exactly given by

$$\exp\left(\frac{2\pi i}{p}\operatorname{Tr}_n(a_1\eta_1+a_2\eta_2+a_3\eta_3+a_4\eta_4)\right).$$

□

Chapter 5

Fourier optimization and pair correlation problems

5.1 Introduction

This chapter is based on [A4]. An overview of the main results and their context was presented in Section 1.3.1; here we develop the full details.

The seminal work of Montgomery [94] inaugurated the study of the distribution of non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of the Riemann zeta function $\zeta(s)$ along the critical line via their pair correlation. In this framework, Montgomery introduced the form factor $F(\alpha, T)$, a weighted exponential sum over pairs of zeros, and stated a conjectural asymptotic formula for its behaviour as $T \to \infty$, now known as the pair correlation conjecture (PCC). Subsequently, Goldston [57] established the equivalence of PCC with certain averaged asymptotic estimates of $F(\alpha, T)$. This result provided the principal motivation for the study of such average estimates, an investigation subsequently pursued in [30, 32, 58].

The approach of [94] has been extended to many other settings, including the pair correlation of zeros of L-functions, Dirichlet series, and similar objects; see [8, 36, 46, 64, 76, 96] for examples. In this chapter we look at the problem of average estimates from a much broader perspective, introducing an axiomatic approach which allows us to bound the long-term averages of analogues of Montgomery's function in a wide variety of settings. We connect these average bounds to two Fourier optimization problems and analyze them using techniques from Fourier analysis and the theory of reproducing kernel Hilbert spaces.

As a consequence of the bounds we obtain, we address a conjecture of Gonek and Ki [64, Conjecture 2], related to the pair correlation of zeros of $\operatorname{Re}\zeta$ and $\operatorname{Im}\zeta$ along vertical lines in the critical strip, and show that the relevant form factor cannot have the conjectured expression in certain regimes.

5.1.1 Montgomery's pair correlation conjecture

Assuming the Riemann hypothesis (RH), let $\rho = \frac{1}{2} + i\gamma$ denote a non-trivial zero of $\zeta(s)$. Montgomery [94] conjectured that, assuming RH, the pair correlation of zeros of ζ has the following limiting distribution: for any fixed $\beta > 0$,

$$\left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\substack{0<\gamma,\gamma'\leqslant T \\ 0<\gamma-\gamma'\leqslant \frac{2\pi\beta}{\log T}}} 1 \sim \int_0^\beta \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) \mathrm{d}u, \quad \text{as } T \to \infty. \tag{5.1}$$

This is Montgomery's pair correlation conjecture (PCC). To study (5.1), Montgomery introduced the form factor

$$F(\alpha, T) := \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma'\leqslant T} T^{i\alpha(\gamma-\gamma')} w(\gamma - \gamma'), \tag{5.2}$$

where $\alpha \in \mathbb{R}$, $T \geqslant 2$, $w(u) = \frac{4}{4+u^2}$, and the sums are taken over ordinates of non-trivial zeros of $\zeta(s)$, counting multiplicity.

For any $r \in L^1(\mathbb{R})$ with $\widehat{r} \in L^1(\mathbb{R})$, the Fourier inversion formula gives

$$\left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma'\leqslant T} r\left((\gamma-\gamma')\frac{\log T}{2\pi}\right) w(\gamma-\gamma') = \int_{\mathbb{R}} \widehat{r}(\alpha)\, F(\alpha, T)\, \mathrm{d}\alpha.$$

The connection between (5.1) and (5.2) is visible from this identity: by testing against an appropriate sequence of functions r approximating the characteristic function of $(0, \beta]$, the pair correlation asymptotic (5.1) follows from knowledge of $F(\alpha, T)$ as $T \to \infty$.

Montgomery [94] and later Goldston and Montgomery [59, Lemma 8] proved, conditionally on RH, that

$$F(\alpha, T) = \left(T^{-2|\alpha|}\log T + |\alpha|\right)\left(1 + O\left(\sqrt{\frac{\log\log T}{\log T}}\right)\right), \quad \text{as } T \to \infty, \tag{5.3}$$

uniformly for $0 \leqslant |\alpha| \leqslant 1$. Montgomery also conjectured that for $|\alpha| \geqslant 1$, uniformly on bounded intervals,

$$F(\alpha, T) \sim 1, \quad \text{as } T \to \infty, \tag{5.4}$$

which together with (5.3) would give a complete description of the asymptotic behavior of $F(\alpha, T)$. The conjunction of (5.3) and (5.4) is called the strong pair correlation conjecture and implies (5.1).

In 1988, Goldston [57] proved that PCC is equivalent to the asymptotic

$$\frac{1}{\ell}\int_b^{b+\ell} F(\alpha, T)\, \mathrm{d}\alpha \sim 1, \quad \text{as } T \to \infty,$$

for any $b \geqslant 1$ and $\ell > 0$. More recently, Carneiro, Milinovich, and Ramos [32] proved, assuming RH, that

$$0.9303 + o(1) < \frac{1}{\ell}\int_b^{b+\ell} F(\alpha, T)\,\mathrm{d}\alpha < 1.3208 + o(1), \quad \text{as } T \to \infty,$$

uniformly in $b \geqslant 1$ for the upper bound and for $\ell \geqslant \ell_0(b)$ for the lower bound, improving previous results of [30]. They obtained this result by introducing new averaging mechanisms and making use of the class of test functions introduced by Cohn and Elkies [38] for the sphere packing problem. The present chapter generalizes and extends this framework.

5.1.2 The axiomatic formulation

Our main purpose is to obtain a unified approach for various pair correlation functions, capturing the behavior of a wide variety of examples. Our objects of interest are sequences $\Gamma(T) = (\gamma_n(T))$ of real numbers depending on a parameter $T \geqslant T_0$; the elements of the sequence are not necessarily distinct. For most number-theoretic instances, such as when Γ consists of the ordinates of non-trivial zeros of a typical L-function, the sequence is independent of T. The dependence on T is included because we are also interested in cases such as those studied by Gonek and Ki [64], where the pair correlation of zeros of $\operatorname{Re}\zeta(s)$ along the lines $\operatorname{Re}(s) = \frac{1}{2} - \frac{c}{\log T}$ is considered; these naturally vary with T.

Sequences of non-trivial zeros of a typical L-function share the behavior that the number of elements with height at most $T > 0$ is asymptotically $\frac{\lambda T}{2\pi}\log T$ and the average spacing is $\frac{\lambda}{2\pi}\log T$ for some $\lambda > 0$. Motivated by this, we introduce our first assumption.

(H1) There is a $\lambda > 0$ such that

$$\liminf_{T\to\infty} \frac{\#\{n : \gamma_n(T) \in (0, T]\}}{\frac{\lambda T}{2\pi}\log T} = 1. \tag{5.5}$$

In view of this, we associate a form factor to Γ in analogy with Montgomery's (5.2):

$$\mathfrak{F}_\Gamma(\alpha, T) := \left(\frac{\lambda T}{2\pi}\log T\right)^{-1} \sum_{\gamma,\gamma' \in (0,T]\cap\Gamma(T)} T^{i\lambda\alpha(\gamma-\gamma')}\, w(\gamma - \gamma'), \tag{5.6}$$

where $\alpha \in \mathbb{R}$, $T \geqslant T_0$, and $w(u) = \frac{4}{4+u^2}$; elements in the sum are counted with multiplicity. The choice that the γ's lie in $(0, T]$ in both (5.5) and (5.6) is made for simplicity and is not essential; the interval $(0, T]$ can be replaced by $(T, 2T]$ or $[-T/2, T/2]$ with no change in the arguments.

From the definition, $\mathfrak{F}_\Gamma(\alpha,T)$ is real-valued and even. It is also non-negative, as follows from the identity

$$\sum_{\gamma,\gamma'\in(0,T]\cap\Gamma(T)} T^{i\lambda\alpha(\gamma'-\gamma)} w(\gamma'-\gamma) = 2\pi\int_{-\infty}^{\infty} e^{-4\pi|u|}\left|\sum_{\gamma\in(0,T]\cap\Gamma(T)} T^{i\lambda\alpha\gamma} e^{2\pi i\gamma u}\right|^2 \mathrm{d}u,$$

valid for all $\alpha\in\mathbb{R}$.

Our goal is to obtain effective bounds for the long-term averages $\frac{1}{\ell}\int_b^{b+\ell}\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha$ as $T\to\infty$, assuming some information about the asymptotic behavior of $\mathfrak{F}_\Gamma(\alpha,T)$. This information will come in the form of a convergence of measures. We say that a Borel measure ν is a *limiting measure* for $\mathfrak{F}_\Gamma(\alpha,T)$ in $(-\Delta,\Delta]$ if for every $\varphi\in C_c((-\Delta,\Delta))$,

$$\lim_{T\to\infty}\int_{\mathbb{R}}\varphi(\alpha)\,\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha = \int_{\mathbb{R}}\varphi(\alpha)\,\mathrm{d}\nu(\alpha). \tag{5.7}$$

In other words, the measures $\mathfrak{F}_\Gamma(\alpha,T)\mathrm{d}\alpha$ converge in the weak* sense to ν as measures on $(-\Delta,\Delta)$ as $T\to\infty$. Such a convergence assumption constitutes our second axiom.

(H2) There is a $\Delta>0$ and a finite Borel measure ν such that ν is a limiting measure for $\mathfrak{F}_\Gamma(\alpha,T)$ in $[-\Delta,\Delta]$.

Remark. Condition (5.7) involves only test functions supported in compact subsets of the open interval $(-\Delta,\Delta)$. It therefore determines ν on $(-\Delta,\Delta)$, but carries no information about possible point masses at $\pm\Delta$.

As a canonical example, when Γ is the set of ordinates of non-trivial zeros of $\zeta(s)$, the form factor $\mathfrak{F}_\Gamma(\alpha,T)$ is Montgomery's $F(\alpha,T)$ and (5.3) implies that, under RH, the measure ν given by

$$\mathrm{d}\nu(\alpha) = \boldsymbol{\delta}(\alpha) + |\alpha|\,\mathrm{d}\alpha \tag{5.8}$$

is the limiting measure of $F(\alpha,T)$ in $[-1,1]$, where $\boldsymbol{\delta}$ denotes the Dirac mass at the origin. Further examples of form factors and their associated limiting measures are discussed in Section 5.1.6.

5.1.3 Bounds via Fourier optimization

We obtain lower and upper bounds for the long-term averages of form factors through a Fourier optimization approach. There are two relevant extremal problems; we study both in subsequent sections.

As discussed previously, the first extremal problem makes use of the class $\mathcal{A}_\Delta$. Recall that, for $\Delta>0$, the class $\mathcal{A}_\Delta$ consists of continuous even functions $g:\mathbb{R}\to\mathbb{R}$ satisfying:

(i) g and $\widehat{g}\in L^1(\mathbb{R})$;

(ii) $\widehat{g}(\alpha)\leqslant 0$ for $\alpha\in\mathbb{R}\setminus[-\Delta,\Delta]$;

(iii) $g(\alpha)\geqslant 0$ for all $\alpha\in\mathbb{R}$.

Given a finite Borel measure ν supported in $[-\Delta, \Delta]$, define the functional

$$\Phi_\nu(g) := \int_{(-\Delta,\Delta)} \widehat{g}(\alpha)\, \mathrm{d}\nu(\alpha).$$

Extremal problem 5.1 **(EP5.1)**. Given $\Delta > 0$ and a finite Borel measure ν on $[-\Delta, \Delta]$, find the quantity

$$\mathcal{C}_\nu := \inf_{\substack{g\in\mathcal{A}_\Delta \\ g(0)>0}} \frac{\Phi_\nu(g)}{g(0)}.$$

If a sequence Γ satisfies (H1) and (H2), we can provide upper and lower bounds for averages of $\mathfrak{F}_\Gamma(\alpha, T)$ in terms of $\mathcal{C}_\nu$ by applying the averaging mechanism of Carneiro, Milinovich, and Ramos [32] in our generic setting. This leads to our first main result.

Theorem 5.1. *Let Γ be a sequence of real numbers satisfying* (H1) *and* (H2)*, and let ν be the limiting measure for $\mathfrak{F}_\Gamma(\alpha, T)$ in $[-\Delta, \Delta]$. Given $\varepsilon > 0$, for any $b \geqslant 1$ and sufficiently large ℓ, one has*

$$1 + s_0(\mathcal{C}_\nu - 1) - \varepsilon + o(1) < \frac{1}{\ell}\int_b^{b+\ell} \mathfrak{F}_\Gamma(\alpha, T)\, \mathrm{d}\alpha < \mathcal{C}_\nu + \varepsilon + o(1),$$

as $T \to \infty$, with $\ell \geqslant \ell_0(\varepsilon)$ for the upper bound and $\ell \geqslant \ell_0(b, \varepsilon)$ for the lower bound.

Remark. By the inequalities above, the solution to (EP5.1) must satisfy $\mathcal{C}_\nu \geqslant 1$; this is a necessary condition on the limiting measure ν of $\mathfrak{F}_\Gamma(\alpha, T)$.

Remark. Since s_0 is a negative quantity, the lower bound in Theorem 5.1 can actually be negative if $\mathcal{C}_\nu$ is larger than a certain number. This motivates our next theorem.

We can strengthen the lower bound with the aid of a second extremal problem. Given $\beta > 0$, introduce the class $\mathcal{R}_\beta$ of continuous functions $g : \mathbb{R} \to \mathbb{R}$ satisfying:

(i) g and $\widehat{g} \in L^1(\mathbb{R})$;

(ii) $\widehat{g}(\alpha) \leqslant \mathbf{1}_{[-\beta,\beta]}(\alpha)$ for all $\alpha \in \mathbb{R}$;

(iii) $g(\alpha) \geqslant 0$ for all $\alpha \in \mathbb{R}$.

Extremal problem 5.2 **(EP5.2)**. Given $\beta > 0$, determine the value of

$$\mathcal{D}_\beta := \sup_{g\in\mathcal{R}_\beta} g(0) = \sup_{g\in\mathcal{R}_\beta} \int_{\mathbb{R}} \widehat{g}(\alpha)\, \mathrm{d}\alpha.$$

We solve (EP5.2) completely in Section 5.2.2, showing that $\mathcal{D}_\beta = \beta$ (see Proposition 5.14). Assuming (H1) and (H2), we obtain the following lower bound for symmetric averages of $\mathfrak{F}_\Gamma$, which is independent of the size of the interval $[-\Delta, \Delta]$.

Theorem 5.2. *Let Γ be a sequence of real numbers satisfying* (H1) *and* (H2). *Given $\varepsilon > 0$, for any $b \geqslant 1$ and $\ell \geqslant \ell_0(b, \varepsilon)$, one has*

$$\frac{1}{\ell}\int_b^{b+\ell} \mathfrak{F}_\Gamma(\alpha, T)\,\mathrm{d}\alpha > \frac{1}{2} - \varepsilon + o(1), \quad \text{as } T \to \infty.$$

Remark. The proof of Theorem 5.2 shows that assumption (H2) is not needed to obtain an analogous result for averages over symmetric intervals (see Lemma 5.13). One can also verify that Theorem 5.2 holds under the weaker assumption that there exists $\Delta > 0$ for which $\limsup_{T\to\infty} \int_{-\Delta}^{\Delta} \mathfrak{F}_\Gamma(\alpha, T)\,\mathrm{d}\alpha$ is finite.

5.1.4 Reproducing kernel Hilbert spaces

For a given measure ν, finding the exact value of $\mathcal{C}_\nu$ is in general difficult. We use reproducing kernel Hilbert space theory to obtain an upper bound for $\mathcal{C}_\nu$ by working over a subclass of $\mathcal{A}_\Delta$.

A Hilbert space $\mathcal{H}$ of entire functions in which the evaluation functionals $f \mapsto f(w)$ are continuous for all $w \in \mathbb{C}$ is called a *reproducing kernel Hilbert space* because, by the Riesz representation theorem, there exists $K : \mathbb{C}^2 \to \mathbb{C}$ such that $K(w, \cdot) \in \mathcal{H}$ and

$$f(w) = \langle f, K(w, \cdot)\rangle_{\mathcal{H}}$$

for all $f \in \mathcal{H}$ and $w \in \mathbb{C}$. This K is called the *reproducing kernel* of $\mathcal{H}$.

The classical example is the *Paley–Wiener space* $PW(\pi\Delta)$ for $\Delta > 0$, consisting of all entire functions $f : \mathbb{C} \to \mathbb{C}$ of exponential type at most $\pi\Delta$ whose restriction to $\mathbb{R}$ lies in $L^2(\mathbb{R})$. This forms a reproducing kernel Hilbert space under the $L^2(\mathbb{R})$-inner product. By the Paley–Wiener theorem [84, §10], $f \in L^2(\mathbb{R})$ extends to a function in $PW(\pi\Delta)$ if and only if $\widehat{f}$ is supported in $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$.

We are interested in weighted versions of $PW(\pi\Delta)$. Given a finite Borel measure ν supported in $[-\Delta, \Delta]$, let $\mathcal{H}_\nu$ be the set of entire functions of exponential type at most $\pi\Delta$ satisfying

$$\|f\|_\nu^2 := \int_{\mathbb{R}} |f(x)|^2\,\widehat{\nu}(x)\,\mathrm{d}x < \infty,$$

where $\widehat{\nu}$ is the Fourier transform of ν, which is defined as

$$\widehat{\nu}(x) = \int_{-\infty}^{\infty} e^{-2\pi i x t}\,\mathrm{d}\nu(t).$$

We restrict to the class of measures ν supported in $[-\Delta, \Delta]$ of the form

$$\mathrm{d}\nu(\alpha) = c_1\boldsymbol{\delta}(\alpha) + c_2|\alpha|e^{-c_3|\alpha|}\mathrm{d}\alpha, \tag{5.9}$$

for $\alpha \in [-\Delta, \Delta]$, where $c_1 > 0$ and $c_2, c_3 \geqslant 0$. This class covers many examples of interest from analytic number theory (see Section 5.1.6). For such measures, we establish the following.

Theorem 5.3. *Let ν be a measure on $[-\Delta, \Delta]$ given by (5.9). If $\frac{c_2}{c_1}\Delta^2 \leqslant \frac{5}{3}$ and $c_3 \geqslant 0$, then $\|\cdot\|_\nu$ is a norm making $\mathcal{H}_\nu$ a reproducing kernel Hilbert space. Moreover, $\mathcal{H}_\nu = PW(\pi\Delta)$ as sets, and there exist constants $a, b > 0$ depending only on c_1, c_2, and Δ such that*

$$a\|f\|_{L^2(\mathbb{R})} \leqslant \|f\|_\nu \leqslant b\|f\|_{L^2(\mathbb{R})}. \tag{5.10}$$

To establish Theorem 5.3, one finds upper and lower bounds for $\widehat{\nu}(x)$, which introduces the constraint $\frac{c_2}{c_1}\Delta^2 \leqslant \frac{5}{3}$. This is a sufficient condition for the method used here; the value $\frac{5}{3}$ can be slightly improved at the cost of increased technicality, but is retained for clarity.

By restricting to the subclass of $\mathcal{A}_\Delta$ consisting of functions f with $\operatorname{supp}\widehat{f} \subseteq [-\Delta, \Delta]$ and exploiting the reproducing kernel structure of $\mathcal{H}_\nu$, we obtain an upper bound for $\mathcal{C}_\nu$. This restricted version of (EP5.1) was solved by Carneiro, Chandee, Littmann, and Milinovich [28] in the case of the measure (5.8) via reproducing kernel theory. We adapt their approach to establish the following.

Lemma 5.4. *Let ν be a finite Borel measure supported in $[-\Delta, \Delta]$ such that the norm equivalence (5.10) holds. Then $(\mathcal{H}_\nu, \|\cdot\|_\nu)$ is a reproducing kernel Hilbert space with $\mathcal{H}_\nu = PW(\pi\Delta)$ as sets, and*

$$\mathcal{C}_\nu \leqslant \frac{1}{K_\nu(0,0)},$$

where K_ν is its reproducing kernel.

This is the link between explicit bounds and reproducing kernels which underpins our strategy. A short proof of Lemma 5.4 is included in Section 5.3.

5.1.5 Explicit bounds for averages

By Lemma 5.4, an effective upper bound for $\mathcal{C}_\nu$ follows once the value $K_\nu(0,0)$ of the reproducing kernel at the origin is known. The next results compute $K_\nu(0,0)$ for the class of measures (5.9). The two cases $c_3 = 0$ and $c_3 > 0$ are treated separately because of the significant increase in computational difficulty in the latter.

Theorem 5.5. *Let $\Delta > 0$, $c_1 > 0$, and $c_2 \geqslant 0$ satisfy $\frac{c_2}{c_1}\Delta^2 \leqslant \frac{5}{3}$. For the measure ν on $[-\Delta, \Delta]$ given by $\mathrm{d}\nu(\alpha) = c_1\boldsymbol{\delta}(\alpha) + c_2|\alpha|\,\mathrm{d}\alpha$, we have*

$$K_\nu(0,0) = \sqrt{\frac{2}{c_1c_2}}\,\frac{\sin\Big(\sqrt{\frac{c_2}{2c_1}}\,\Delta\Big)}{\cos\Big(\sqrt{\frac{c_2}{2c_1}}\,\Delta\Big) + \sqrt{\frac{c_2}{2c_1}}\,\Delta\,\sin\Big(\sqrt{\frac{c_2}{2c_1}}\,\Delta\Big)}.$$

To state the result when $c_3 \neq 0$, define the auxiliary functions of $\eta \in \mathbb{C}$:

$$A(\eta) \coloneqq 1 + \frac{c_2}{c_1}\int_{-\Delta/2}^{\Delta/2} \cosh(\eta\alpha)\,|\alpha|\,e^{-c_3|\alpha|}\,\mathrm{d}\alpha, \tag{5.11}$$

$$B(\eta) \coloneqq \eta^2 + 2\frac{c_2}{c_1} - c_3^2 + 2\frac{c_2c_3}{c_1}\int_{-\Delta/2}^{\Delta/2} \cosh(\eta\alpha)\,e^{-c_3|\alpha|}\,\mathrm{d}\alpha. \tag{5.12}$$

We also set $D(\eta_1,\eta_2) := A(\eta_1)B(\eta_2) - B(\eta_1)A(\eta_2)$.

Theorem 5.6. *Let $\Delta, c_1, c_2, c_3 > 0$ satisfy $\frac{c_2}{c_1}\Delta^2 \leqslant \frac{5}{3}$. For the measure ν on $[-\Delta,\Delta]$ given by $\mathrm{d}\nu(\alpha) = c_1\boldsymbol{\delta}(\alpha) + c_2|\alpha|e^{-c_3|\alpha|}\mathrm{d}\alpha$, we have*

$$\begin{aligned} K_\nu(0,0) = {} & \frac{2}{D(\eta_1,\eta_2)}\left(\frac{1}{c_1} - \frac{A(0)c_3^2}{2c_2 + c_3^2 c_1}\right)\left(\frac{\sinh\frac{\Delta\eta_1}{2}}{\eta_1}B(\eta_2) - \frac{\sinh\frac{\Delta\eta_2}{2}}{\eta_2}B(\eta_1)\right) \\ & + \frac{2}{D(\eta_1,\eta_2)}\left(\frac{c_3^2}{c_1} + \frac{B(0)c_3^2}{2c_2 + c_3^2 c_1}\right)\left(\frac{\sinh\frac{\Delta\eta_1}{2}}{\eta_1}A(\eta_2) - \frac{\sinh\frac{\Delta\eta_2}{2}}{\eta_2}A(\eta_1)\right) \\ & + \frac{\Delta c_3^2}{2c_2 + c_3^2 c_1}, \end{aligned} \tag{5.13}$$

where η_1 and η_2 are roots of

$$c_1\eta^4 + 2(c_2 - c_1c_3^2)\eta^2 + c_3^2(2c_2 + c_1c_3^2) = 0$$

with $\eta_1 + \eta_2 \neq 0$. The expression (5.13) is interpreted as a limit when $\eta_1 = \eta_2$.

Remark. When c_3 is sufficiently large, one has the asymptotic

$$\frac{1}{K_\nu(0,0)} = \frac{c_1}{\Delta} + O\left(\frac{c_2\Delta}{c_3}\right).$$

In Section 5.4 we compute explicit expressions for the full reproducing kernel $K_\nu(w,z)$ when $c_3 = 0$, and for $K_\nu(0,z)$ when $c_3 > 0$ (see Theorems 5.15 and 5.17, respectively). The reproducing kernel satisfies a certain integral equation depending on ν; we construct solutions for the class (5.9) by deriving and solving suitable ordinary differential equations (Lemmas 5.16 and 5.18).

Combining Theorems 5.1, 5.2, 5.5 and 5.6 and Lemma 5.4 we obtain the following explicit estimate for the long-term averages of $\mathfrak{F}_\Gamma(\alpha,T)$.

Corollary 5.7. *Let Γ be a sequence of real numbers satisfying* (H1) *and* (H2)*, and assume that the limiting measure ν for $\mathfrak{F}_\Gamma(\alpha,T)$ in $[-\Delta,\Delta]$ is given by* (5.9)*. If $\frac{c_2}{c_1}\Delta^2 \leqslant \frac{5}{3}$ and $c_3 \geqslant 0$, then for all $\varepsilon > 0$, $b \geqslant 1$, and sufficiently large ℓ, one has*

$$\left(\frac{1}{2} + s_0\left(\frac{1}{K_\nu(0,0)} - 1\right)\right)_+ + \frac{1}{2} - \varepsilon + o(1) < \frac{1}{\ell}\int_b^{b+\ell} \mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha < \frac{1}{K_\nu(0,0)} + \varepsilon + o(1),$$

as $T \to \infty$, with $\ell \geqslant \ell_0(\varepsilon)$ for the upper bound and $\ell \geqslant \ell_0(b,\varepsilon)$ for the lower bound, where $K_\nu(0,0)$ is given by Theorem 5.5 and Theorem 5.6.

5.1.6 Applications

We apply the results above to three main classes of sequences arising from zeros of L-functions or similar objects:

(i) primitive elements of the Selberg class;

(ii) Dedekind zeta functions of number fields;

(iii) real and imaginary parts of the Riemann zeta function (and suitable generalizations).

5.1.6.1 Primitive L-functions in the Selberg class

The Selberg class $\mathcal{S}$, introduced by Selberg [107], is the set of meromorphic functions $L(s)$ satisfying five fundamental axioms: Dirichlet series representation, analytic continuation, functional equation, Ramanujan hypothesis, and Euler product. Associated with each $L \in \mathcal{S}$ is its degree d_L, determined by the functional equation. Although not assumed in the axioms, it is conjectured that d_L is always a non-negative integer, which is supported by all known examples.

Murty and Perelli [96] studied the pair correlation of zeros of primitive elements of $\mathcal{S}$ (those that cannot be factored into non-trivial elements). Here Γ consists of the ordinates of non-trivial zeros of a primitive $L \in \mathcal{S}$, i.e., $\Gamma = \{\gamma : \rho = \beta + i\gamma,\ L(\rho) = 0,\ 0 \leqslant \beta \leqslant 1\}$. The argument principle gives the zero counting function

$$N_L(T) = \#\{\rho = \beta + i\gamma : L(\rho) = 0,\ 0 \leqslant \beta \leqslant 1,\ -T/2 \leqslant \gamma \leqslant T/2\} = \frac{d_L T}{2\pi} \log T + O(T),$$

so Γ satisfies (H1) with $\lambda = d_L$, after replacing $(0, T]$ by $[-T/2, T/2]$ in (5.5). This change does not affect any of the arguments.

Recall the Euler product axiom: for $L \in \mathcal{S}$, $\log L(s) = \sum_{n=1}^{\infty} b_L(n) n^{-s}$, where $b_L(n) = 0$ unless $n = p^k$ and $b_L(n) \leqslant n^{\theta}$ for some $\theta < 1/2$. Murty and Perelli proposed the following hypothesis on the coefficients $b_L(n)$.

Let $v_L(n) := b_L(n) \log n$. Then $\sum_{n \leqslant x} v_L(n) \overline{v_L(n)} = (1 + o(1)) x \log x$ as $x \to \infty$.

Under GRH and Hypothesis MP, Murty and Perelli [96] proved that

$$\mathfrak{F}_{\Gamma}(\alpha, T) \sim d_L T^{-2|\alpha| d_L} \log T + |\alpha|, \quad \text{as } T \to \infty,$$

uniformly for $0 \leqslant |\alpha| \leqslant \frac{1}{d_L}(1 - \delta)$, for any $\delta > 0$. Therefore Γ satisfies (H2) with limiting measure ν in $\left[-\frac{1}{d_L}, \frac{1}{d_L}\right]$ given by

$$\mathrm{d}\nu(\alpha) = \boldsymbol{\delta}(\alpha) + |\alpha| \, \mathrm{d}\alpha.$$

Corollary 5.7 then yields the following.

Corollary 5.8. *Assume GRH and Hypothesis MP. If Γ is the sequence of ordinates of the non-trivial zeros of a primitive element of $\mathcal{S}$ of degree m, then for all $\varepsilon > 0$ and*

$b \geqslant 1$,

$$\begin{aligned}\Big(\frac{1}{2} + s_0\Big(\frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2m}} - \frac{2m-1}{2m}\Big)\Big)_+ + \frac{1}{2} - \varepsilon + o(1)\\ < \frac{1}{\ell}\int_b^{b+\ell} \mathfrak{F}_\Gamma(\alpha, T)\,\mathrm{d}\alpha < \frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2m}} + \frac{1}{2m} + \varepsilon + o(1),\end{aligned}$$

as $T \to \infty$, with $\ell \geqslant \ell_0(\varepsilon)$ for the upper bound and $\ell \geqslant \ell_0(b, \varepsilon)$ for the lower bound.

5.1.6.2 Dedekind zeta functions of number fields

The Dedekind zeta function $\zeta_K(s)$ of a number field $K/\mathbb{Q}$ of degree n with ring of integers $\mathcal{O}_K$ is defined by $\zeta_K(s) = \sum_{I \subseteq \mathcal{O}_K} N(I)^{-s}$, where the sum runs over ideals I in $\mathcal{O}_K$ and $N(I)$ is the norm. For an abelian extension $K/\mathbb{Q}$, one has $\zeta_K(s) = \zeta(s)\prod_{j=1}^{n-1} L(s, \chi_j)$ for certain Dirichlet characters χ_j; this is a non-primitive element of $\mathcal{S}$ of degree n.

In a recent preprint, de Laat, Rolen, Tripp, and Wagner [42] studied the pair correlation of zeros of $\zeta_K(s)$ under GRH. If Γ consists of the ordinates of zeros of ζ_K in the critical strip, standard results confirm (H1). Furthermore, they established that

$$\mathfrak{F}_\Gamma(\alpha, T) \sim nT^{-2n|\alpha|}\log T + n|\alpha|, \quad \text{as } T \to \infty,$$

uniformly for $|\alpha| \leqslant \frac{1}{n} - \delta$ with $\delta > 0$. Thus (H2) holds with limiting measure ν in $[-\frac{1}{n}, \frac{1}{n}]$ given by

$$\mathrm{d}\nu(\alpha) = \boldsymbol{\delta}(\alpha) + n|\alpha|\,\mathrm{d}\alpha.$$

Applying Corollary 5.7 gives the following.

Corollary 5.9. *Assuming GRH, if Γ is the sequence of zeros of the Dedekind zeta function $\zeta_K(s)$ of a degree n abelian extension $K/\mathbb{Q}$, then for all $\varepsilon > 0$ and $b \geqslant 1$,*

$$\begin{aligned}\Big(\frac{1}{2} + s_0\Big(\sqrt{\frac{n}{2}}\cot\frac{1}{\sqrt{2n}} - \frac{1}{2}\Big)\Big)_+ + \frac{1}{2} - \varepsilon + o(1)\\ < \frac{1}{\ell}\int_b^{b+\ell} \mathfrak{F}_\Gamma(\alpha, T)\,\mathrm{d}\alpha < \sqrt{\frac{n}{2}}\cot\frac{1}{\sqrt{2n}} + \frac{1}{2} + \varepsilon + o(1),\end{aligned}$$

as $T \to \infty$, with $\ell \geqslant \ell_0(\varepsilon)$ for the upper bound and $\ell \geqslant \ell_0(b, \varepsilon)$ for the lower bound.

5.1.6.3 The real and imaginary parts of the Riemann zeta function

Our third application concerns the pair correlation of zeros of $\operatorname{Re}\zeta$ and $\operatorname{Im}\zeta$ and their generalizations. Following Gonek–Ki [64] and the works of Garaev [56] and Ki [80], for $0 < a \leqslant \frac{1}{2}$ and $\theta \in [0, 2\pi)$, consider

$$f_a(w, \theta) := \zeta(a + w) + e^{i\theta}\zeta(a - w).$$

The zeros of $f_a(w,0)$ and $f_a(w,\pi)$ coincide with those of $\operatorname{Re}\zeta(a+w)$ and $\operatorname{Im}\zeta(a+w)$ on the line $\operatorname{Re}(w)=0$, respectively. Under RH, Ki [80] showed that all but finitely many non-real zeros of $f_a(w,\theta)$ lie on $\operatorname{Re}(w)=0$ for any $0<a\leqslant\frac{1}{2}$, with a lower bound for the exceptional height that can be taken uniform in $a\in(0,\frac{1}{2}]$ for fixed θ.

Assuming RH, Gonek and Ki [64] proved that the zeros of $f_a(w,\theta)$ are simple after a sufficiently large height for fixed $0<a<\frac{1}{2}$ and $\theta\in[0,2\pi)$. We focus on the case $a=\frac{1}{2}-\frac{c}{\log T}$ with $c>0$ fixed (and θ fixed), letting $\Gamma(T)$ denote the sequence of ordinates (γ_a) for $T>T_0$. We work with elements $\gamma,\gamma'\in\Gamma(T)$ from $[T,2T]$ to match the setup of [64]; this change has no effect on the arguments. Garaev [56] and Ki [80] established that

$$N_a(T) \coloneqq \#\{\rho_a : T\leqslant\gamma_a\leqslant 2T\} = \frac{T}{\pi}\log T + O(T),$$

so $\Gamma(T)$ satisfies (H1) with $\lambda=2$.

For the associated form factor, Gonek and Ki [64, Theorem 3] proved the following: let $0<\delta<\frac{1}{2}$; then, under RH, for $T\geqslant T_0$ and $\delta\leqslant a\leqslant\frac{1}{2}$,

$$\mathfrak{F}_\Gamma(\alpha,T)\sim 2T^{-4|\alpha|}\log T + \frac{2|\alpha|}{(a+\frac{1}{2})(3-2a)}\,T^{(4a-2)|\alpha|}, \quad \text{as } T\to\infty,$$

uniformly for $0\leqslant|\alpha|\leqslant\frac{1}{2}-\delta$. As $T\to\infty$ with $a=\frac{1}{2}-\frac{c}{\log T}$, the second term satisfies $T^{(4a-2)|\alpha|}=T^{-4c|\alpha|/\log T}\to e^{-4c|\alpha|}$, so $\Gamma(T)$ satisfies (H2) with limiting measure ν in $[-\frac{1}{2},\frac{1}{2}]$ given by

$$\mathrm{d}\nu(\alpha)=\boldsymbol{\delta}(\alpha)+|\alpha|e^{-4c|\alpha|}\mathrm{d}\alpha.$$

This is a measure of the form (5.9) with $\Delta=\frac{1}{2}$, $c_1=1$, $c_2=1$, and $c_3=4c$. Applying Corollary 5.7 gives the following result.

Corollary 5.10. *Assume RH. Let $\theta\in[0,2\pi)$, $a=\frac{1}{2}-\frac{c}{\log T}$ with $c\geqslant 0$, and let $\Gamma(T)$ consist of the zeros of $f_a(w,\theta)$ along $\operatorname{Re}(w)=0$. For all $\varepsilon>0$ and $b\geqslant 1$,*

$$1+s_0\left(\frac{1}{K_\nu(0,0)}-1\right)-\varepsilon+o(1)<\frac{1}{\ell}\int_b^{b+\ell}\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha<\frac{1}{K_\nu(0,0)}+\varepsilon+o(1),$$

as $T\to\infty$, with $\ell\geqslant\ell_0(\varepsilon)$ for the upper bound and $\ell\geqslant\ell_0(b,\varepsilon)$ for the lower bound, where $K_\nu(0,0)$ is given by Theorem 5.5 and Theorem 5.6 with $\Delta=\frac{1}{2}$, $c_1=1$, $c_2=1$, and $c_3=4c$.

In particular, if Γ is either the sequence of zeros of $\operatorname{Re}\zeta$ or that of $\operatorname{Im}\zeta$ along the critical line $\operatorname{Re}(s)=\frac{1}{2}$ (corresponding to $c=0$), then for all $\varepsilon>0$ and $b\geqslant 1$,

$$0.7467\cdots-\varepsilon+o(1)<\frac{1}{\ell}\int_b^{b+\ell}\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha<2.1659\cdots+\varepsilon+o(1),$$

as $T\to\infty$, with $\ell\geqslant\ell_0(\varepsilon)$ for the upper bound and $\ell\geqslant\ell_0(b,\varepsilon)$ for the lower bound.

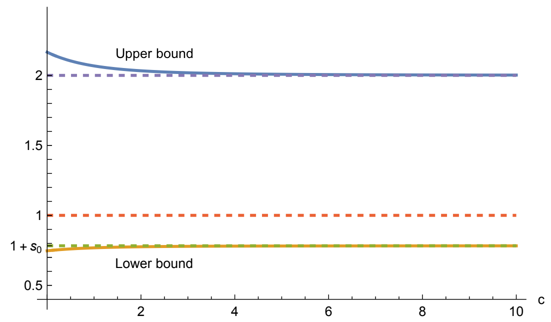

FIGURE 5.1: Upper and lower bounds for $\frac{1}{\ell}\int_b^{b+\ell}\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha$ as a function of c, where $c_1 = 1$, $c_2 = 1$, $c_3 = 4c$, and $\Delta = 0.5$.

Figure 5.1 illustrates the upper and lower bounds given in Corollary 5.10.

One implication of Corollary 5.10 concerns Conjecture 2 of Gonek and Ki [64], which we now recall.

Conjecture 5.11 (Gonek–Ki)**.** *Suppose $\theta \in [0, 2\pi)$ is fixed, $a = \frac{1}{2} - \frac{c}{\log T}$ with $0 < c = o(\log T)$, and $A > 0$ is arbitrarily large but fixed. Then, if $\Gamma(T)$ consists of the zeros of $f_a(w,\theta)$ along* $\mathrm{Re}(w) = 0$,

$$\mathfrak{F}_\Gamma(\alpha,T) = (2+o(1))T^{-4|\alpha|}\log T + \frac{\min(2|\alpha|,1)}{(a+\frac{1}{2})(3-2a)}\,T^{(4a-2)|\alpha|} + o(1),$$

as $T\to\infty$, uniformly for $0 \leqslant |\alpha| \leqslant A$.

If Conjecture 5.11 were true, then for $c > 0$ fixed and any $b > \frac{1}{2}$, $\ell > 0$, integrating and passing to the limit $T\to\infty$ would give

$$\lim_{T\to\infty}\frac{1}{\ell}\int_b^{b+\ell}\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha = \frac{e^{-4cb}(1-e^{-4c\ell})}{8c\ell}.$$

As either $b\to\infty$ or $\ell\to\infty$, this right-hand side tends to 0, so the conjecture would predict that for sufficiently large b or ℓ the averages of $\mathfrak{F}_\Gamma(\alpha,T)$ converge to 0. The lower bounds of Corollary 5.10 show this is impossible, giving the following result.

Corollary 5.12. Conjecture 5.11 *cannot hold when $c > 0$ is any fixed number.*

It remains an interesting open question whether Conjecture 5.11 holds when $c = o(1)$.

5.2 Fourier optimization bounds

5.2.1 Proof of Theorem 5.1

Fix $g \in \mathcal{A}_\Delta$ with $g(0) > 0$ and let ϕ be a Schwartz function satisfying $\phi, \widehat{\phi} \geqslant 0$ and $\|\phi\|_1 = 1$ with $\mathrm{supp}(\widehat{\phi}) \subset [-1,1]$. Define $\phi^\varepsilon(x) = \frac{1}{\varepsilon}\phi(x/\varepsilon)$ and set $g_\varepsilon := g * \phi^\varepsilon$. We find that g_ε occurs in $\mathcal{A}_\Delta^{BL}$, the collection of functions $f \in \mathcal{A}_\Delta$ such that $\mathrm{supp}\,(\widehat{f})$ is compact, and $\lim_{\varepsilon\to 0}\frac{\Phi_\nu(g_\varepsilon)}{g_\varepsilon(0)} = \frac{\Phi_\nu(g)}{g(0)}$. Thus, given a $\delta > 0$, there is a $g \in \mathcal{A}_\Delta$ and a suitably small ε for which $\frac{\Phi_\nu(g_\varepsilon)}{g_\varepsilon(0)} \leqslant \mathcal{C}_\nu + \delta$.

Consequently, for a given $\varepsilon > 0$ there is a function $h \in \mathcal{A}_\Delta^{BL} \subset \mathcal{A}_\Delta$, that satisfies the inequality

$$\Phi_\nu(h) \leqslant \mathcal{C}_\nu + \frac{\varepsilon}{3}.$$

Without loss of generality, we normalize h to have $h(0) = 1$. Now, pick $g \in \mathcal{A}^{BL}_{(1-\delta)\Delta}$ defined on the Fourier transform side by $\widehat{g}(\alpha) = \frac{1}{1-\delta}\widehat{h}\left(\frac{1}{1-\delta}\alpha\right)$ for a $0 < \delta < \frac{1}{2}$ small enough such that

$$\int_{-(1-\delta)\Delta}^{(1-\delta)\Delta} \widehat{g}(\alpha)\,\mathrm{d}\nu(\alpha) \leqslant \Phi_\nu(h) + \frac{\varepsilon}{3}.$$

and $\operatorname{supp}(\widehat{g}) \subset [-M, M]$, with M a parameter uniform in δ.

The core of the proof is an inequality regarding a convenient approximation of the characteristic functions of intervals (established by Carneiro, Milinovich, and Ramos in [32, eq: (2.6)] and reproduced here as (5.14)). For $L > M$, define the function

$$\mathcal{G}_L(x) = (\widehat{g} * \mathbf{1}_{[-L,L]})(x) = \int_{x-L}^{x+L} \widehat{g}(t)\mathrm{d}t.$$

5.2.1.1 Upper bound

Setting $L = M + \frac{\ell}{2}$, it follows that

$$\mathcal{G}_L\left(x - b - \frac{\ell}{2}\right) \geqslant \mathbf{1}_{[b,b+\ell]}(x) - \|\widehat{h}\|_1 \cdot \mathbf{1}_{b+\frac{\ell}{2}+I_L}(x), \tag{5.14}$$

where $I_L \coloneqq [-L - M, -L + M] \bigcup [L - M, L + M]$. The above inequality implies that

$$\int_b^{b+\ell} \mathfrak{F}_\Gamma(\alpha, T)\,\mathrm{d}\alpha \leqslant \int_{-\infty}^{\infty} \mathcal{G}_L\left(\alpha - b - \frac{\ell}{2}\right) \mathfrak{F}_\Gamma(\alpha, T)\,\mathrm{d}\alpha + \|\widehat{h}\|_1 \int_{b+\frac{\ell}{2}+I_L} \mathfrak{F}_\Gamma(\alpha, T)\,\mathrm{d}\alpha. \tag{5.15}$$

By the definition of $\mathcal{A}_\Delta$ and assumption (H2), we also get

$$\begin{aligned}\int_{-\infty}^{\infty} \widehat{g}(\alpha)\,\mathfrak{F}_\Gamma(\alpha, T)\,\mathrm{d}\alpha &\leqslant \int_{-(1-\delta)\Delta}^{(1-\delta)\Delta} \widehat{g}(\alpha)\,\mathfrak{F}_\Gamma(\alpha, T)\,\mathrm{d}\alpha \\ &= \int_{-(1-\delta)\Delta}^{(1-\delta)\Delta} \widehat{g}(\alpha)\,\mathrm{d}\nu(\alpha) + o(1) \leqslant \Phi_\nu(h) + \frac{\varepsilon}{3} + o(1),\end{aligned} \tag{5.16}$$

as $T \to \infty$. So that, by Fourier inversion,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathcal{G}_L\left(\alpha - b - \frac{\ell}{2}\right) \mathfrak{F}_\Gamma(\alpha, T)\, \mathrm{d}\alpha \\
&\quad = \frac{\operatorname{Re} \sum\limits_{0<\gamma,\gamma'\leqslant T} T^{i\lambda(b+\frac{\ell}{2})(\gamma-\gamma')} (\mathscr{F}^{-1}[\mathbf{1}_{[-L,L]}] \cdot g)\left((\gamma-\gamma')\frac{\lambda \log T}{2\pi}\right) w(\gamma-\gamma')}{\frac{\lambda T}{2\pi} \log T} \\
&\quad \leqslant 2L \left(\frac{\lambda T}{2\pi} \log T\right)^{-1} \sum_{0<\gamma,\gamma'\leqslant T} g\left((\gamma-\gamma')\frac{\lambda \log T}{2\pi}\right) w(\gamma-\gamma') \\
&\quad = 2L \int_{-\infty}^{\infty} \widehat{g}(\alpha)\, \mathfrak{F}_\Gamma(\alpha, T)\, \mathrm{d}\alpha \\
&\quad \leqslant 2L \left\{\Phi_\nu(h) + \frac{\varepsilon}{3} + o(1)\right\},
\end{aligned}$$

which is almost enough to establish the upper bound. It remains to show that the second integral in the right-hand side of (5.15) is bounded. But this follows since $|I_L| = 4M$, which is a finite length, and hence, $\int_{b+\frac{\ell}{2}+I_L} \mathfrak{F}_\Gamma(\alpha, T)\, \mathrm{d}\alpha \ll 4M$, by the argument of Goldston [57, eq: (2.6)]. Thus, for $\ell \geqslant \ell_0(\varepsilon)$, we get

$$\frac{1}{\ell} \int_b^{b+\ell} \mathfrak{F}_\Gamma(\alpha, T)\, \mathrm{d}\alpha \leqslant \frac{2L}{\ell} \left\{\Phi_\nu(h) + \frac{\varepsilon}{3} + o(1)\right\} \leqslant \mathcal{C}_\nu + \varepsilon + o(1).$$

5.2.1.2 Lower bound

For the lower bound set $L = M + \beta$ and observe that by appropriately shifting in (5.14)

$$\mathbf{1}_{[-\beta,\beta]}(x) + \|\widehat{h}\|_1 \cdot \mathbf{1}_{I_L}(x) \geqslant \mathcal{G}_L(x),$$

whence we get the inequality

$$\int_{-\beta}^{\beta} \mathfrak{F}_\Gamma(\alpha, T)\mathrm{d}\alpha + \|\widehat{h}\|_1 \int_{I_L} \mathfrak{F}_\Gamma(\alpha, T)\, \mathrm{d}\alpha \geqslant \int_{\mathbb{R}} \mathcal{G}_L(\alpha)\, \mathfrak{F}_\Gamma(\alpha, T)\, \mathrm{d}\alpha.$$

Now, letting m_γ be the multiplicity of the element γ of the sequence (that is, $m_\gamma := \#\{n : \gamma_n = \gamma\}$), we deduce that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathcal{G}_L(\alpha)\mathfrak{F}_\Gamma(\alpha, T)\mathrm{d}\alpha \\
&\quad = \frac{2\pi}{\lambda T \log T} \sum_{0<\gamma,\gamma'\leqslant T} g\left((\gamma-\gamma')\frac{\lambda \log T}{2\pi}\right)\left(\frac{2L \sin\left(L(\gamma-\gamma')\lambda \log T\right)}{L(\gamma-\gamma')\lambda \log T}\right) w(\gamma-\gamma') \\
&\quad \geqslant 2L \frac{2\pi}{\lambda T \log T} \left\{ \sum_{0<\gamma\leqslant T} m_\gamma + \min_{x\in\mathbb{R}} \frac{\sin x}{x} \sum_{\substack{0<\gamma,\gamma'\leqslant T \\ \gamma\neq\gamma'}} g\left((\gamma-\gamma')\frac{\lambda \log T}{2\pi}\right) w(\gamma-\gamma') \right\} \\
&\quad = 2L \frac{2\pi}{\lambda T \log T} \left\{ (1-s_0) \sum_{0<\gamma\leqslant T} m_\gamma + s_0 \sum_{0<\gamma,\gamma'\leqslant T} g\left((\gamma-\gamma')\frac{\lambda \log T}{2\pi}\right) w(\gamma-\gamma') \right\}
\end{aligned}$$

$$\geqslant 2L\left\{\frac{2\pi}{\lambda T\log T}(1-s_0)\sum_{0<\gamma\leqslant T}1+s_0\int_{-\infty}^{\infty}\widehat{g}(\alpha)\,\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha\right\}.$$

Thus, recalling s_0 is negative, we apply (5.16) and assumption (H1) to get

$$\begin{aligned}\int_{-\infty}^{\infty}\mathcal{G}_L(\alpha)\,\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha &\geqslant 2L\left[1-s_0+s_0\Phi_\nu(h)+s_0\frac{\varepsilon}{3}+o(1)\right]\\ &\geqslant 2\beta\left[1-s_0+s_0\Phi_\nu(h)+s_0\frac{\varepsilon}{3}+o(1)\right],\end{aligned}$$

as $T\to\infty$. So that we have

$$\begin{aligned}&\liminf_{T\to\infty}\frac{1}{2\beta}\int_{-\beta}^{\beta}\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha\\ &\quad\geqslant\left[1-s_0+s_0\Phi_\nu(h)+s_0\frac{\varepsilon}{3}+o(1)\right]-\frac{1}{2\beta}\limsup_{T\to\infty}\|\widehat{h}\|_1\int_{I_L}\mathfrak{F}_\Gamma(\alpha,T)\mathrm{d}\alpha.\end{aligned}\tag{5.17}$$

Note that $|I_L|=4M$, a finite length, and thus the integral on the right-hand side of the inequality above is bounded. Therefore, by (5.17) and our choice of h, for $\beta\geqslant\beta_0(\varepsilon)$ we have

$$\liminf_{T\to\infty}\frac{1}{2\beta}\int_{-\beta}^{\beta}\mathfrak{F}_\Gamma(\alpha,T)\mathrm{d}\alpha\geqslant 1-s_0+s_0(\mathcal{C}_\nu+\varepsilon).$$

Also, when $\beta=b+\ell$, the evenness of $\mathfrak{F}_\Gamma(\alpha)$ yields

$$\begin{aligned}\frac{1}{\ell}\int_b^{b+\ell}\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha &=\frac{2\beta}{2\ell}\frac{1}{2\beta}\int_{-\beta}^{\beta}\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha-\frac{1}{2\ell}\int_{-b}^{b}\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha\\ &=\frac{1}{2\beta}\int_{-\beta}^{\beta}\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha\\ &\quad+\frac{b}{\ell}\left(\frac{1}{2\beta}\int_{-\beta}^{\beta}\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha-\frac{1}{2b}\int_{-b}^{b}\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha\right).\end{aligned}\tag{5.18}$$

Note that $\int_{-\beta}^{\beta}\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha\ll\beta$, which makes the averages appearing in the equation above uniformly bounded. Hence, by using (5.17) and (5.18), we have that for any $\varepsilon>0$ and given $b\geqslant 1$ there exists $\ell_0(b,\varepsilon)$ such that for all $\ell\geqslant\ell_0(b,\varepsilon)$

$$\frac{1}{\ell}\int_b^{b+\ell}\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha\geqslant 1-s_0+s_0\mathcal{C}_\nu-\varepsilon+o(1),$$

as $T\to\infty$, finishing the proof.

5.2.2 Proof of Theorem 5.2

For any non-negative function $g\in L^1(\mathbb{R})$ such that $\widehat{g}\in L^1(\mathbb{R})$, we can write by the Fourier inversion formula

$$\int_{-\infty}^{\infty}\widehat{g}(\alpha)\,\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha=\frac{2\pi}{\lambda T\log T}\sum_{0<\gamma,\gamma'\leqslant T}g\left((\gamma-\gamma')\frac{\lambda\log T}{2\pi}\right)w(\gamma-\gamma')$$

$$= \frac{2\pi}{\lambda T \log T} g(0) \sum_{0<\gamma\leqslant T} m_\gamma + \frac{2\pi}{\lambda T \log T} \sum_{\substack{0<\gamma,\gamma'\leqslant T\\ \gamma\neq\gamma'}} g\Big((\gamma-\gamma')\frac{\lambda \log T}{2\pi}\Big) w(\gamma-\gamma').$$

Since the sum of off-diagonal terms is non-negative, we have by (H1) that, as $T\to\infty$,

$$\int_{-\infty}^{\infty} \widehat{g}(\alpha)\,\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha \geqslant g(0)+o(1).$$

Now if g is such that $\widehat{g}(\alpha)\leqslant \mathbf{1}_{[-\beta,\beta]}(\alpha)$, we have

$$\int_{-\beta}^{\beta} \mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha \geqslant \int_{-\infty}^{\infty} \widehat{g}(\alpha)\,\mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha \geqslant g(0)+o(1),$$

as $T\to\infty$, leading naturally to the extremal problem (EP5.2) defined in Section 5.1.3.

We have thus just proven a lower bound depending on the Fourier optimization constant $\mathcal{D}_\beta$.

Lemma 5.13. *Let Γ be a sequence of real numbers satisfying* (H1)*. The following bound holds as $T\to\infty$:*

$$\int_{-\beta}^{\beta} \mathfrak{F}_\Gamma(\alpha,T)\,\mathrm{d}\alpha \geqslant \mathcal{D}_\beta+o(1).$$

We know how to solve problem (EP5.2) exactly. This is the next step towards the lower bound independent of Δ in Theorem 5.2.

Proposition 5.14. *For all $\beta>0$, we have the equality $\mathcal{D}_\beta=\beta$.*

Proof. With no loss of generality, we can assume $\beta=1$. This is because for any $g\in\mathcal{R}_\beta$, one can rescale g to get an $h\in\mathcal{R}_1$ by setting $h(x)=\beta\, g(\beta x)$. This goes in both directions, plainly yielding $\mathcal{D}_\beta=\beta\,\mathcal{D}_1$.

If we pick $g_0(x)=\left(\frac{\sin \pi x}{\pi x}\right)^2$, we have $\widehat{g_0}(\alpha)=(1-|\alpha|)_+$, it is clear that $g_0\in\mathcal{R}_1$, so that $\mathcal{D}_1\geqslant 1$. Now we prove that we can do no better than that. Let $g\in\mathcal{R}_1$ and ϕ be a non-negative smooth function of compact support such that $\widehat{\phi}\geqslant 0$ and $\int\phi=1$. If $\phi^\varepsilon(x):=\frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})$, the function $g_\varepsilon=g*\phi^\varepsilon$ is integrable by Young's convolution inequality, it is non-negative and, since $\widehat{g_\varepsilon}(\alpha)=\widehat{g}(\alpha)\,\widehat{\phi}(\varepsilon\alpha)$, it also satisfies $\widehat{g_\varepsilon}(\alpha)\leqslant\mathbf{1}_{[-1,1]}(\alpha)$. Note that g_ε has bounded variation, which can be seen through another application of Young's convolution inequality. We can therefore apply the Poisson summation formula to obtain

$$\sum_{n\in\mathbb{Z}} g_\varepsilon(n)=\sum_{k\in\mathbb{Z}}\widehat{g_\varepsilon}(k),$$

but since $g_\varepsilon\geqslant 0$ and $\widehat{g_\varepsilon}\leqslant 0$ outside of $[-1,1]$, we get

$$g_\varepsilon(0)\leqslant\sum_{n\in\mathbb{Z}} g_\varepsilon(n)=\sum_{k\in\mathbb{Z}}\widehat{g_\varepsilon}(k)\leqslant\widehat{g_\varepsilon}(0)=\widehat{g}(0)\,\widehat{\phi}(0)=\widehat{g}(0).$$

Since g is a continuous function, we have that $g_\varepsilon(x) \to g(x)$ converges uniformly in compact sets as $\varepsilon \to 0$, so that $g(0) \leqslant \widehat{g}(0) \leqslant 1$, implying $\mathcal{D}_1 \leqslant 1$ because g was arbitrary. We thus have $\mathcal{D}_1 = 1$, concluding the proof. $\square$

Putting together Lemma 5.13, Proposition 5.14 and the identity (5.18), we get Theorem 5.2.

5.3 The reproducing kernel Hilbert space $\mathcal{H}_\nu$

5.3.1 Proof of Theorem 5.3

Taking the Fourier transform of the measure ν which is compactly supported in $[-\Delta, \Delta]$, we obtain

$$\begin{aligned} \widehat{\nu}(x) = c_1 + \frac{c_2\, e^{-c_3\Delta}}{(c_3^2 + 4\pi^2x^2)^2}\Big\{ & 2e^{c_3\Delta}(c_3^2 - 4\pi^2x^2) \\ & - 2(c_3^2 - 4\pi^2x^2 + c_3^3\Delta + 4c_3\pi^2x^2\Delta)\cos(2\pi\Delta x) \\ & + 4\pi x(2c_3 + c_3^2\Delta + 4\pi^2x^2\Delta)\sin(2\pi\Delta x)\Big\}. \end{aligned} \tag{5.19}$$

If we prove that this function is bounded above and below away from zero, i.e., that there exists $a, b > 0$ such that $a^2 \leqslant \widehat{\nu}(x) \leqslant b^2$, we are done, because then $\widehat{\nu}(x)\,\mathrm{d}x$ is a positive measure equivalent to Lebesgue, and we have the inequalities

$$a^2 \int_{\mathbb{R}} g(x)\,\mathrm{d}x \leqslant \int_{\mathbb{R}} g(x)\,\widehat{\nu}(x)\,\mathrm{d}x \leqslant b^2 \int_{\mathbb{R}} g(x)\,\mathrm{d}x$$

for all Lebesgue measurable $g \geqslant 0$, whence we also have (5.10) by taking $g = |f|^2$ for $f \in L^2(\mathbb{R})$. In particular, this means $\mathcal{H}_\nu$ is a normed linear space, and its completeness follows from the completeness of $PW(\pi\Delta)$. This likewise applies to the reproducing kernel property.

Now, we can trivially estimate it from above

$$|\widehat{\nu}(x)| \leqslant c_1 + c_2 \int_{-\Delta}^{\Delta} |\alpha|\; e^{-c_3|\alpha|}\mathrm{d}\alpha \leqslant c_1 + c_2\Delta^2,$$

and to find a lower bound, set $t \coloneqq 2\pi\Delta x$ and $\sigma \coloneqq c_3\Delta$ and write

$$\widehat{\nu}(x) = c_2\Delta^2\left\{\frac{c_1}{\Delta^2 c_2} - G(\sigma, t)\right\},$$

where

$$\begin{aligned} G(\sigma, t) = -\frac{e^{-\sigma}}{(\sigma^2 + t^2)^2}\Big\{ & 2e^{\sigma}(\sigma^2 - t^2) - 2(\sigma^2 - t^2 + \sigma^3 + t^2\sigma)\cos(t) \\ & + 2t(2\sigma + \sigma^2 + t^2)\sin(t)\Big\}. \end{aligned}$$

We will now show that $G(\sigma, t)$ cannot be too large.

Observe first that this is a harmonic function. In fact, it is the real part of a holomorphic function of the variable $z = -\sigma + it$. Explicitly,

$$G(\sigma, t) = \mathrm{Re}\left\{\frac{e^z(1-z)}{z^2} - \frac{1}{z^2}\right\}.$$

Hence, if we let Ω_R be the region $\{-\sigma + it : \sigma > 0, t > 0, |z| < R\}$, we have by the maximum principle that

$$\max_{z\in\overline{\Omega}_R} G(\sigma, t) \leqslant \max_{z\in\partial\Omega_R} \mathrm{Re}\left\{\frac{e^z(1-z)}{z^2} - \frac{1}{z^2}\right\},$$

but, when $|z| = R$, we have that

$$\left|\mathrm{Re}\left\{\frac{e^z(1-z)}{z^2} - \frac{1}{z^2}\right\}\right| \leqslant \frac{e^{-\sigma}(1+R)+1}{R^2},$$

which becomes very small when $R \to \infty$. Moreover, when $t = 0$, we have

$$G(\sigma, 0) = -\frac{e^{-\sigma}}{\sigma^4}\left\{2e^{\sigma}(\sigma^2) - 2(\sigma^2+\sigma^3)\right\} = \frac{2}{\sigma^2}\left\{-1 + e^{-\sigma}(1+\sigma)\right\} \leqslant 0$$

by the elementary inequality $1 + \sigma \leqslant e^{\sigma}$ for $\sigma \geqslant 0$. So that to bound G in the first quadrant, it is enough to consider it on the half-line $\sigma = 0$, $t \geqslant 0$. That is,

$$\sup_{\sigma\geqslant 0, t\geqslant 0} G(\sigma, t) = \sup_{t>0} \frac{2 - 2\cos(t) - 2t\sin(t)}{t^2} = 0.586\ldots.$$

Thus, choosing $\frac{c_1}{\Delta^2 c_2} \geqslant \frac{3}{5} > 0.586\ldots$, we obtain that $\widehat{\nu}$ is bounded below uniformly away from zero, concluding the proof.

Remark. By a standard procedure involving a Fourier uncertainty principle (as used in [28, proof of Lemma 12]), we can extend the range of admissible values of c_1, c_2, and Δ all the way up to $\frac{c_2}{c_1}\Delta^2 \leqslant \frac{1}{0.586\ldots}$ to guarantee that $(\mathcal{H}_\nu, \|\cdot\|_\nu)$ is a reproducing kernel Hilbert space.

5.3.2 Proof of Lemma 5.4

It is immediate from the inequalities in (5.10) that $\mathcal{H}_\nu$ is a reproducing kernel Hilbert space which is equal to $PW(\pi\Delta)$ as a set.

Returning to (EP5.1), we recall that, to find $\mathcal{C}_\nu$, we want to minimize

$$\frac{\Phi_\nu(g)}{g(0)} = \frac{1}{g(0)}\int_{(-\Delta,\Delta)} \widehat{g}(\alpha)\,\mathrm{d}\nu(\alpha),$$

over g in the class $\mathcal{A}_\Delta$. We restrict ourselves to a subclass of bandlimited functions to obtain an upper bound for $\mathcal{C}_\nu$. This is an adaptation of the proof of [28, Corollary 14] to our setting.

Denote by $\mathcal{B}_\Delta \coloneqq \{g \in \mathcal{A}_\Delta : \mathrm{supp}(\widehat{g}) \subseteq [-\Delta, \Delta]\}$ and let $g \in \mathcal{B}_\Delta \subset \mathcal{A}_\Delta$. Since $\mathrm{supp}(\widehat{g}) \subseteq [-\Delta, \Delta]$, we know that g has a unique extension to an entire function of exponential type at most $2\pi\Delta$, which we also denote by g. Since $g(x) \geqslant 0$ and $g \in L^1(\mathbb{R})$ by Krein's decomposition theorem [1, p. 154] there exists an entire function $f \in PW(\pi\Delta)$ such that

$$g(z) = f(z)\overline{f(\bar{z})}$$

for all $z \in \mathbb{C}$. By (5.10), f is also a member of $\mathcal{H}_\nu$. Applying Plancherel's formula,

$$\int_{(-\Delta,\Delta)} \widehat{g}(\alpha)\,\mathrm{d}\nu(\alpha) = \int_{\mathbb{R}} g(x)\,\widehat{\nu}(x)\,\mathrm{d}x = \int_{\mathbb{R}} |f(x)|^2\,\widehat{\nu}(x)\,\mathrm{d}x.$$

On the other hand, by the reproducing kernel property and the Cauchy–Schwarz inequality,

$$g(0) = |f(0)|^2 = |\langle f, K_\nu(0,\cdot)\rangle|^2 \leqslant K_\nu(0,0) \int_{\mathbb{R}} |f(x)|^2\,\widehat{\nu}(x)\,\mathrm{d}x,$$

so that

$$\frac{1}{g(0)} \int_{(-\Delta,\Delta)} \widehat{g}(\alpha)\,\mathrm{d}\nu(\alpha) \geqslant \frac{1}{K_\nu(0,0)}.$$

Now, equality can be attained in the above inequality if and only if g is a multiple of $K_\nu(0,z)\overline{K_\nu(0,\bar{z})}$, which occurs in $\mathcal{B}_\Delta$ since $K_\nu(0,\cdot) \in L^2(\mathbb{R})$ by (5.10). This is the best bound we can obtain for $\mathcal{C}_\nu$ working within the class $\mathcal{B}_\Delta$, and it gives us what we had set out to prove.

5.4 Computing reproducing kernels

In this section, we will study the reproducing kernel $K_\nu : \mathbb{C}^2 \to \mathbb{C}$ of the space $\mathcal{H}_\nu$, providing proofs for Theorems 5.5 and 5.6. The starting point for both proofs is the same. For all $f \in \mathcal{H}_\nu$ and $w \in \mathbb{C}$, we must have by the reproducing kernel property and Plancherel's formula that

$$f(w) = \int_{\mathbb{R}} f(x)\overline{K_\nu(w,x)}\,\widehat{\nu}(x)\,\mathrm{d}x = \int_{-\Delta/2}^{\Delta/2} \widehat{f}(\xi)\mathscr{F}^{-1}\left[\overline{K_\nu(w,\cdot)}\,\widehat{\nu}\right](\xi)\,\mathrm{d}\xi.$$

But by Fourier inversion

$$f(w) = \int_{-\Delta/2}^{\Delta/2} \widehat{f}(\xi)e^{2\pi i w\xi}\,\mathrm{d}\xi$$

as well. Since $\mathscr{F}(\mathcal{H}_\nu) = \mathscr{F}(PW(\pi\Delta)) = L^2([-\frac{\Delta}{2}, \frac{\Delta}{2}])$ by the Paley–Wiener theorem, we must therefore have

$$\int_{\mathbb{R}} \overline{K_\nu(w,x)}\,\widehat{\nu}(x)e^{2\pi i\xi x}\,\mathrm{d}x = \mathscr{F}^{-1}\left[\overline{K_\nu(w,\cdot)}\,\widehat{\nu}\right](\xi) = e^{2\pi i w\xi}$$

for almost all $\xi \in [-\frac{\Delta}{2}, \frac{\Delta}{2}]$. Denoting by $u_w = \mathscr{F}\left[\overline{K_\nu(w,\cdot)}\right]$, we will thus seek to solve for u_w in the $L^2([-\frac{\Delta}{2}, \frac{\Delta}{2}])$ equation given by

$$(u_w * \nu)(\xi) = \int_{\mathbb{R}} u_w(\xi - \alpha)\, \mathrm{d}\nu(\alpha) = e^{-2\pi i w \xi} \tag{5.20}$$

for almost all $\xi \in [-\frac{\Delta}{2}, \frac{\Delta}{2}]$.

5.4.1 Proof of Theorem 5.5

We obtain the expression for the reproducing kernel $K_\nu(w,z)$ of the space $\mathcal{H}_\nu$ with weight given by (5.19), which in this case simplifies to

$$\widehat{\nu}(x) = c_1 + c_2 \Delta \frac{\sin 2\pi\Delta x}{\pi x} - c_2 \left(\frac{\sin \pi\Delta x}{\pi x}\right)^2.$$

To state our result, we define the functions

$$\begin{aligned}
a(w) &:= \frac{-2c_2\left(\cos(\pi\Delta w) + \pi\Delta w \sin(\pi\Delta w)\right)}{c_1(2c_1\pi^2 w^2 - c_2)\left(2\cos\left(\sqrt{\frac{c_2}{2c_1}}\Delta\right) + \sqrt{\frac{2c_2}{c_1}}\Delta \sin\left(\sqrt{\frac{c_2}{2c_1}}\Delta\right)\right)};\\
b(w) &:= \sqrt{\frac{2c_2}{c_1}} \cdot \frac{i\pi w \cos(\pi\Delta w)}{(2c_1\pi^2 w^2 - c_2)\cos\left(\sqrt{\frac{c_2}{2c_1}}\Delta\right)};\\
c(w) &:= \frac{2\pi^2 w^2}{2c_1\pi^2 w^2 - c_2},
\end{aligned}$$

and

$$\begin{aligned}
q(z) &:= \frac{\sqrt{2c_1c_2}\sin\left(\sqrt{\frac{c_2}{2c_1}}\Delta\right)\cos(\pi\Delta z) - 2\pi c_1 z \cos\left(\sqrt{\frac{c_2}{2c_1}}\Delta\right)\sin(\pi\Delta z)}{c_2 - 2c_1\pi^2 z^2};\\
r(z) &:= \frac{i\sqrt{2c_1c_2}\cos\left(\sqrt{\frac{c_2}{2c_1}}\Delta\right)\sin(\pi\Delta z) + i2\pi c_1 z \sin\left(\sqrt{\frac{c_2}{2c_1}}\Delta\right)\cos(\pi\Delta z)}{c_2 - 2c_1\pi^2 z^2}.
\end{aligned}$$

We will show a stronger result than Theorem 5.5 in the following theorem.

Theorem 5.15. *Under the hypotheses of* Theorem 5.5 *we have*

$$K_\nu(w,z) = a(\bar{w})q(z) + b(\bar{w})r(z) + c(\bar{w})\frac{\sin \pi\Delta(z - \bar{w})}{\pi(z - \bar{w})},$$

where the above expression is understood in the limit sense when either z or w equal $\pm\frac{1}{\pi}\sqrt{\frac{c_2}{2c_1}}$.

In particular,

$$K_\nu(0,0) = a(0)\, q(0) = \sqrt{\frac{2}{c_1c_2}} \frac{\sin\left(\sqrt{\frac{c_2}{2c_1}}\Delta\right)}{\cos\left(\sqrt{\frac{c_2}{2c_1}}\Delta\right) + \sqrt{\frac{c_2}{2c_1}}\Delta \sin\left(\sqrt{\frac{c_2}{2c_1}}\Delta\right)}.$$

Thus, Theorem 5.5 follows as a consequence of this result.

The proof of Theorem 5.15 is obtained by solving the functional equation (5.20). We collect the results about this equation in the following lemma.

Lemma 5.16. *Let $c_1, c_2 > 0$, $w \in \mathbb{C}$, $\Delta > 0$. When $\frac{c_2}{c_1}\Delta^2 < 2$, the following statements hold.*

(i) *There exists a unique function $u_w \in L^2(\mathbb{R})$ with* $\operatorname{supp} u_w \subseteq [-\frac{\Delta}{2}, \frac{\Delta}{2}]$ *such that*

$$c_1 u_w(\xi) + c_2 \int_{-\Delta}^{\Delta} u_w(\xi - \alpha)|\alpha| \, \mathrm{d}\alpha = e^{-2\pi i \xi w}$$

for almost all $\xi \in [-\frac{\Delta}{2}, \frac{\Delta}{2}]$.

(ii) *When $w \neq \pm\frac{1}{\pi}\sqrt{\frac{c_2}{2c_1}}$, the function u_w is given by the expression*

$$u_w(\xi) = a(w)\cos\left(\sqrt{\frac{2c_2}{c_1}}\xi\right) + b(w)\sin\left(\sqrt{\frac{2c_2}{c_1}}\xi\right) + c(w)e^{-2\pi i w \xi} \tag{5.21}$$

in the interval $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$ and is identically zero outside it.

(iii) *If we define the entire function*

$$k_w(z) := \int_{-\Delta/2}^{\Delta/2} u_w(\alpha) e^{2\pi i \alpha z} \, \mathrm{d}\alpha,$$

then $k_w \in PW(\pi\Delta)$ and

$$f(w) = \int_{\mathbb{R}} f(x)\, k_w(x)\, \widehat{\nu}(x) \, \mathrm{d}x$$

for all $f \in PW(\pi\Delta)$.

Proof. Henceforth, we will use the abbreviation $I = [-\frac{\Delta}{2}, \frac{\Delta}{2}]$. In the first part, we are asserting the existence and uniqueness of solutions to the functional equation

$$u + Tu = \frac{1}{c_1}e^{-2\pi i w \xi} \tag{5.22}$$

in $L^2(I)$, where $T : L^2(I) \to L^2(I)$ is the operator given by

$$(Tu)(\xi) := \frac{c_2}{c_1}\int_{-\Delta/2}^{\Delta/2} u(\alpha)|\xi - \alpha| \, \mathrm{d}\alpha.$$

Now, T is a Hilbert–Schmidt operator, since the kernel $S(x, y) = |x - y|$ is in $L^2(I \times I)$. The operator T is therefore compact so, by Fredholm's alternative, there exists a unique solution to (5.22) if and only if the equation

$$u + Tu = 0$$

only admits the trivial solution. Suppose $u \in L^2(I)$ is a solution to the above equation, i.e.,

$$u(\xi) = -\frac{c_2}{c_1}\int_{-\Delta/2}^{\Delta/2} u(\alpha)|\xi-\alpha|\,\mathrm{d}\alpha$$

for a.e. $\xi \in I$. It follows that $u \in L^\infty(I)$ by, say, the Cauchy–Schwarz inequality, and that

$$|u(\xi)| \leqslant \|u\|_{L^\infty(I)}\frac{c_2}{c_1}\int_{-\Delta/2}^{\Delta/2}|\xi-\alpha|\,\mathrm{d}\alpha$$

for a.e. ξ, yielding

$$\|u\|_{L^\infty(I)} \leqslant \|u\|_{L^\infty(I)}\,\frac{c_2}{c_1}\sup_{\xi\in I}\int_{-\Delta/2}^{\Delta/2}|\xi-\alpha|\,\mathrm{d}\alpha = \frac{c_2}{c_1}\frac{\Delta^2}{2}\|u\|_{L^\infty(I)},$$

which, for the admissible values of $c_1, c_2,$ and Δ, is only possible when u is trivial, proving (i).

Now, formally differentiating equation (5.20), we get

$$c_1u'(\xi) + c_2\int_{-\Delta}^{\Delta} u'(\xi-\alpha)|\alpha|\,\mathrm{d}\alpha = -2\pi i w e^{-2\pi i w\xi}. \tag{5.23}$$

Doing it once more yields the ordinary differential equation

$$c_1u''(\xi) + 2c_2u(\xi) = -4\pi^2w^2e^{-2\pi i w\xi}. \tag{5.24}$$

In turn, a solution to the above ODE for $\xi \in I$, subject to the boundary conditions (obtained by plugging in $\xi = 0$ in the expressions (5.20) and (5.23))

$$\begin{cases} c_1u(0) + c_2\int_{-\Delta/2}^{\Delta/2} u(\alpha)|\alpha|\,\mathrm{d}\alpha = 1; \\ c_1u'(0) - c_2\int_{-\Delta/2}^{\Delta/2} u(\alpha)\,\mathrm{sgn}\,(\alpha)\,\mathrm{d}\alpha = -2\pi i w, \end{cases} \tag{5.25}$$

is a solution to the functional equation and, therefore, its only solution. Indeed, integrating the ODE (5.24) from 0 to a $\xi \in (0, \frac{\Delta}{2})$,

$$c_1u'(\xi) + 2c_2\int_0^\xi u(\alpha)\,\mathrm{d}\alpha = -2\pi i w e^{-2\pi i w\xi} + 2\pi i w + c_1u'(0)$$

and, integrating again,

$$c_1u(\xi) + 2c_2\int_0^\xi\int_0^\alpha u(\beta)\,\mathrm{d}\beta\mathrm{d}\alpha = e^{-2\pi i w\xi} - 1 + 2\pi i w\xi + c_1u(0) + c_1u'(0)\xi$$

which, by Fubini's theorem, is

$$c_1u(\xi) + 2c_2\int_0^\xi(\xi-\beta)u(\beta)\,\mathrm{d}\beta = e^{-2\pi i w\xi} - 1 + 2\pi i w\xi + c_1u(0) + c_1u'(0)\xi.$$

Applying the boundary conditions, we get

$$c_1 u(\xi) + 2c_2 \int_0^\xi (\xi - \beta) u(\beta)\, \mathrm{d}\beta = e^{-2\pi i w \xi} - c_2 \int_{-\Delta/2}^{\Delta/2} u(\alpha)|\alpha|\, \mathrm{d}\alpha + c_2 \int_{-\Delta/2}^{\Delta/2} \xi u(\alpha)\, \mathrm{sgn}\,(\alpha) \mathrm{d}\alpha,$$

which, after rearranging, is exactly

$$c_1 u(\xi) + c_2 \int_{-\Delta/2}^{\Delta/2} u(\alpha)|\xi - \alpha|\, \mathrm{d}\alpha = e^{-2\pi i w \xi},$$

and the same can be done for negative ξ, proving the assertion that the solution of the ODE is a solution to the functional equation.

Now it remains to be shown that the expression in (5.21) solves the ODE with the boundary conditions (5.25) to conclude the proof of (ii). But a solution to (5.24) must be of the form

$$u(\xi) = A \cos\left(\sqrt{\frac{2c_2}{c_1}}\xi\right) + B \sin\left(\sqrt{\frac{2c_2}{c_1}}\xi\right) + \frac{2\pi^2 w^2}{2c_1\pi^2 w^2 - c_2} e^{-2\pi i w \xi}$$

for all $\xi \in I$ for some coefficients A and B. We use (5.25) to identify the coefficients, whence it follows $A = a(w)$ and $B = b(w)$ exactly.

Now suppose $f \in PW(\pi\Delta)$ and let k_w be defined as before. Since u_w is in $L^2(I)$, it is clear that $k_w \in PW(\pi\Delta)$. By applying Plancherel's formula and Fourier inversion,

$$\begin{aligned}
\int_{\mathbb{R}} f(x) k_w(x)\, \widehat{\nu}(x)\, \mathrm{d}x &= \int_{-\Delta/2}^{\Delta/2} \widehat{f}(\xi)\, \mathscr{F}^{-1}[k_w \cdot \widehat{\nu}\,](\xi)\, \mathrm{d}\xi \\
&= \int_{-\Delta/2}^{\Delta/2} \widehat{f}(\xi) \left\{ c_1 u_w(-\xi) + c_2 \int_{-\Delta}^{\Delta} u_w(-\xi - \alpha)|\alpha|\, \mathrm{d}\alpha \right\} \mathrm{d}\xi \\
&= \int_{-\Delta/2}^{\Delta/2} \widehat{f}(\xi)\, e^{2\pi i \xi w}\, \mathrm{d}\xi \\
&= f(w),
\end{aligned}$$

proving item (iii) of the lemma. □

As per the lemma, Theorem 5.15 will now follow by taking the inverse Fourier transform of (5.21) to obtain

$$k_w(z) = a(w) q(z) + b(w) r(z) + c(w) \frac{\sin \pi\Delta(z - w)}{\pi(z - w)},$$

and taking $K_\nu(w, z) = \overline{k_w(\bar{z})}$, which gives the desired expression whenever $w \neq \pm\frac{1}{\pi}\sqrt{\frac{c_2}{2c_1}}$. However, we know that $K_\nu(w, z)$ always exists when $\frac{c_2}{c_1}\Delta^2 \leqslant 5/3$ and that

$$K_\nu(w, z) = \langle K_\nu(w, \cdot), K_\nu(z, \cdot)\rangle = \overline{\langle K_\nu(z, \cdot), K_\nu(w, \cdot)\rangle} = \overline{K_\nu(z, w)},$$

which means that, since K_ν is holomorphic in the first variable, it must be anti-holomorphic in the second variable. In particular, the limit as $w \to \pm\frac{1}{\pi}\sqrt{\frac{c_2}{2c_1}}$ of

$$a(\bar{w})q(z) + b(\bar{w})r(z) + c(\bar{w})\frac{\sin \pi\Delta(z-\bar{w})}{\pi(z-\bar{w})}$$

must always exist, which yields the full theorem.

5.4.2 Proof of Theorem 5.6

It follows from the hypothesis of Theorem 5.6 that we can apply Theorem 5.3. Hence, $\mathcal{H}_\nu$ is a reproducing kernel Hilbert space.

Let η_1 and η_2 be the solutions of the equation

$$\eta^4 + 2\left(\frac{c_2}{c_1} - c_3^2\right)\eta^2 + c_3^2\left(2\frac{c_2}{c_1} + c_3^2\right) = 0, \tag{5.26}$$

such that $\eta_1+\eta_2 \neq 0$. Without loss of generality we assume $\mathrm{Re}(\eta_1) \geqslant 0$ and $\mathrm{Re}(\eta_2) \geqslant 0$. Also, define

$$\mu := \frac{c_3^2}{2c_2 + c_3^2 c_1}$$

and the following auxiliary function

$$C(\eta, z) := \frac{4\pi z \sin(\pi\Delta z)\cosh\frac{\Delta\eta}{2} + 2\eta\cos(\pi\Delta z)\sinh\frac{\Delta\eta}{2}}{4\pi^2 z^2 + \eta^2}.$$

Note that, for any fixed η, the values $C(\eta, \pm i\frac{\eta}{2\pi})$ are understood as the limits $\lim\limits_{z\to\pm i\frac{\eta}{2\pi}} C(\eta, z)$.

Theorem 5.17. *Under the hypotheses of Theorem 5.6 we have the reproducing kernel*

$$\begin{aligned} K_\nu(0,z) = &\left(\frac{1}{c_1} - A(0)\mu\right)\frac{C(\eta_1,z)B(\eta_2) - C(\eta_2,z)B(\eta_1)}{A(\eta_1)B(\eta_2) - A(\eta_2)B(\eta_1)} \\ &+ \left(\frac{c_3^2}{c_1} + B(0)\mu\right)\frac{C(\eta_1,z)A(\eta_2) - C(\eta_2,z)A(\eta_1)}{A(\eta_1)B(\eta_2) - A(\eta_2)B(\eta_1)} + \mu\frac{\sin(\pi\Delta z)}{\pi z} \end{aligned} \tag{5.27}$$

of the space $\mathcal{H}_\nu$, where the expression is interpreted as a limit whenever $\eta_1 = \eta_2$.

In particular, when $z = 0$ we have (5.13), concluding the proof of Theorem 5.6. The proof of Theorem 5.17 is analogous to the proof of Theorem 5.15 and it follows from the lemma below.

Lemma 5.18. *Let $c_1, c_2, c_3 > 0$, $w \in \mathbb{C}$, and $\Delta > 0$. When $\frac{c_2}{c_1}\Delta^2 < 2$ the following statements hold.*

(i) *There exists a unique function $u_w \in L^2(\mathbb{R})$ with $\mathrm{supp}\, u_w \subseteq [-\frac{\Delta}{2}, \frac{\Delta}{2}]$ such that*

$$c_1 u_w(\xi) + c_2\int_{-\Delta}^{\Delta} u_w(\xi - \alpha)|\alpha|\, e^{-c_3|\alpha|}\mathrm{d}\alpha = e^{-2\pi i w\xi} \tag{5.28}$$

for almost all $\xi \in [-\frac{\Delta}{2}, \frac{\Delta}{2}]$.

(ii) *The function* u_w *is a solution of the ODE*

$$c_1 u^{(4)}(\xi) + 2(c_2 - c_1 c_3^2)u''(\xi) + (2c_2c_3^2 + c_1c_3^4)u(\xi) = (4\pi^2 w^2 + c_3^2)^2 e^{-2\pi i w\xi},$$

subject to the conditions

$$\begin{cases} c_1 u(0) + c_2 \int_{-\Delta/2}^{\Delta/2} u(\alpha)|\alpha| e^{-c_3|\alpha|}\,\mathrm{d}\alpha = 1; \\ c_1 u'(0) - c_2 \displaystyle\int_{-\Delta/2}^{\Delta/2} u(\alpha)\operatorname{sgn}\alpha\, e^{-c_3|\alpha|}\,\mathrm{d}\alpha \\ \qquad\qquad\qquad + c_2 c_3 \displaystyle\int_{-\Delta/2}^{\Delta/2} u(\alpha)\alpha\, e^{-c_3|\alpha|}\,\mathrm{d}\alpha = -2\pi i w; \\ c_1 u''(0) + (2c_2 - c_1c_3^2)u(0) - 2c_2c_3 \int_{-\Delta/2}^{\Delta/2} u(\alpha) e^{-c_3|\alpha|}\,\mathrm{d}\alpha = -4\pi^2 w^2 - c_3^2; \\ c_1 u'''(0) - (c_1c_3^2 - 2c_2)u'(0) \\ \qquad\qquad - 2c_2c_3^2 \displaystyle\int_{-\Delta/2}^{\Delta/2} u(\alpha)\operatorname{sgn}\alpha\, e^{-c_3|\alpha|}\,\mathrm{d}\alpha = (2\pi i w)(4\pi^2 w^2 + c_3^2), \end{cases}$$

in the interval $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$ *and identically zero outside it.*

(iii) *If we define the entire function*

$$k_w(z) \coloneqq \int_{-\Delta/2}^{\Delta/2} u_w(\alpha) e^{2\pi i \alpha z}\,\mathrm{d}\alpha,$$

then $k_w \in PW(\pi\Delta)$ *and*

$$f(w) = \int_{\mathbb{R}} f(x)\, k_w(x)\, \widehat{\nu}(x)\,\mathrm{d}x$$

for all $f \in PW(\pi\Delta)$.

Proof. Throughout we will use the abbreviation $I = [-\frac{\Delta}{2}, \frac{\Delta}{2}]$. The proof of items (i) and (iii) are analogous to that of Lemma 5.16. Now knowing that there is a solution u_w to (5.28) in $L^2(I)$, we can quickly see $u_w \in C^\infty(I)$, since (5.28) is equivalent to

$$\begin{aligned} c_1 u_w(\xi) &= e^{-2\pi i w\xi} - c_2 \int_{-\Delta}^{\Delta} u_w(\xi - \alpha)|\alpha|\ e^{-c_3|\alpha|}\mathrm{d}\alpha \\ &= e^{-2\pi i w\xi} - c_2 \int_{-\Delta/2}^{\Delta/2} u_w(\alpha)|\xi - \alpha|\ e^{-c_3|\xi-\alpha|}\mathrm{d}\alpha \\ &= e^{-2\pi i w\xi} - c_2 \int_{-\Delta/2}^{\xi} u_w(\alpha)(\xi - \alpha)e^{-c_3(\xi-\alpha)}\mathrm{d}\alpha \\ &\qquad + c_2 \int_{\xi}^{\Delta/2} u_w(\alpha)(\xi - \alpha)e^{c_3(\xi-\alpha)}\mathrm{d}\alpha, \end{aligned} \tag{5.29}$$

so that it is immediate that the integrability of u_w implies that it is absolutely continuous in I. But then if u_w is absolutely continuous, the same equation tells us it is C^1 with an absolutely continuous derivative by the fundamental theorem of calculus. Proceeding inductively, it follows that u_w is smooth in the interval I.

To obtain the ODE, it is enough to take derivatives of the expression (5.29). By differentiating and collecting the appropriate terms, we get

$$\begin{aligned} c_1 u_w'(\xi) = -2\pi i w e^{-2\pi i w\xi} - c_2 \int_{-\Delta/2}^{\Delta/2} u_w(\alpha)\,\mathrm{sgn}(\xi-\alpha) e^{-c_3|\xi-\alpha|} \mathrm{d}\alpha \\ + c_2 c_3 \int_{-\Delta/2}^{\Delta/2} u_w(\alpha)(\xi-\alpha) e^{-c_3|\xi-\alpha|} \mathrm{d}\alpha. \end{aligned} \tag{5.30}$$

Doing it again,

$$\begin{aligned} c_1 u_w''(\xi) = (-4\pi^2 w^2 - c_3^2) e^{-2\pi i w\xi} + (c_1 c_3^2 - 2c_2) u_w(\xi) \\ + 2c_2 c_3 \int_{-\Delta/2}^{\Delta/2} u_w(\alpha) e^{-c_3|\xi-\alpha|} \mathrm{d}\alpha, \end{aligned} \tag{5.31}$$

a third time,

$$\begin{aligned} c_1 u_w'''(\xi) = (2\pi i w)(4\pi^2 w^2 + c_3^2) e^{-2\pi i w\xi} + (c_1 c_3^2 - 2c_2) u_w'(\xi) \\ - 2c_2 c_3^2 \int_{-\Delta/2}^{\Delta/2} u_w(\alpha)\,\mathrm{sgn}(\xi-\alpha)\, e^{-c_3|\xi-\alpha|} \mathrm{d}\alpha, \end{aligned} \tag{5.32}$$

and a last time

$$c_1 u_w^{(4)}(\xi) = (4\pi^2 w^2 + c_3^2)^2 e^{-2\pi i w\xi} + 2(c_1 c_3^2 - c_2) u_w''(\xi) - (c_1 c_3^2 + 2c_2) c_3^2 u_w(\xi),$$

which is the desired equation. The boundary conditions are recovered by plugging in $\xi = 0$ in the integral equations (5.29), (5.30), (5.31), and (5.32), which u_w and its derivatives must satisfy. $\square$

Proof of Theorem 5.17. Applying item (ii) of Lemma 5.18 to the case $w = 0$, we obtain that, in this case, the function $u = u_0$ is the solution to the ODE

$$c_1 u^{(4)}(\xi) + 2(c_2 - c_1 c_3^2) u''(\xi) + (2c_2 c_3^2 + c_1 c_3^4) u(\xi) = c_3^4, \tag{5.33}$$

subject to the conditions

$$\begin{cases} c_1 u(0) + c_2 \int_{-\Delta/2}^{\Delta/2} u(\alpha)|\alpha| e^{-c_3|\alpha|}\, \mathrm{d}\alpha = 1; \\ c_1 u'(0) - c_2 \int_{-\Delta/2}^{\Delta/2} u(\alpha)\,\mathrm{sgn}\,\alpha\, e^{-c_3|\alpha|}\, \mathrm{d}\alpha + c_2 c_3 \int_{-\Delta/2}^{\Delta/2} u(\alpha)\alpha\, e^{-c_3|\alpha|}\, \mathrm{d}\alpha = 0; \\ c_1 u''(0) + (2c_2 - c_1 c_3^2) u(0) - 2c_2 c_3 \int_{-\Delta/2}^{\Delta/2} u(\alpha) e^{-c_3|\alpha|}\, \mathrm{d}\alpha = -c_3^2; \\ c_1 u'''(0) - (c_1 c_3^2 - 2c_2) u'(0) - 2c_2 c_3^2 \int_{-\Delta/2}^{\Delta/2} u(\alpha)\,\mathrm{sgn}\,\alpha\, e^{-c_3|\alpha|}\, \mathrm{d}\alpha = 0 \end{cases}$$

in the interval $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$ and identically zero outside it.

Knowing that the kernel $K_\nu(0,z)$ must be even allows us to conclude that u must likewise be even, which further simplifies the boundary conditions to

$$\begin{cases} c_1 u(0) + c_2 \int_{-\Delta/2}^{\Delta/2} u(\alpha)|\alpha| e^{-c_3|\alpha|}\,\mathrm{d}\alpha = 1; \\ u'(0) = 0; \\ c_1 u''(0) + (2c_2 - c_1 c_3^2)u(0) - 2c_2c_3 \int_{-\Delta/2}^{\Delta/2} u(\alpha) e^{-c_3|\alpha|}\,\mathrm{d}\alpha = -c_3^2; \\ u'''(0) = 0. \end{cases} \tag{5.34}$$

To emphasize what are the crucial parameters for the solution to the ODE (5.33) we denote by $\lambda = c_2/c_1$. By rewriting the equation, we get that u is a solution to the following differential equation:

$$u^{(4)}(\xi) + 2(\lambda - c_3^2)u''(\xi) + (2\lambda c_3^2 + c_3^4)u(\xi) = \frac{c_3^4}{c_1}.$$

Thus $u(\xi) = v(\xi) + \mu$, where $v(\xi)$ is a solution to the homogeneous ODE with constant coefficients. To obtain a representation of the function $v(\xi)$ we have to solve the corresponding characteristic equation (5.26). Since we are assuming $\lambda, c_3 > 0$, all solutions to (5.26) are different from 0, and it follows that

$$\eta^2 = \begin{cases} c_3^2 - \lambda \pm \sqrt{\lambda}\sqrt{\lambda - 4c_3^2}, & \text{if } \lambda \geqslant 4c_3^2; \\ c_3^2 - \lambda \pm i\sqrt{\lambda}\sqrt{4c_3^2 - \lambda}, & \text{if } \lambda < 4c_3^2 \end{cases}$$

for the solutions η. Since the expression on the right-hand side is never be non-negative, we conclude that there are no real solutions to equation (5.26). Suppose the numbers $\eta_1, \eta_2, -\eta_1, -\eta_2$ are all the solutions listed according to the multiplicity (i.e., η_2 may be equal to η_1). We assume $\operatorname{Re}\eta_1, \operatorname{Re}\eta_2 \geqslant 0$. The following situations may occur:
Case 1: $\eta_2 \neq \eta_1 \iff \lambda \neq 4c_3^2$.
Case 2: $\eta_2 = \eta_1 \iff \lambda = 4c_3^2$.
Assuming $\eta_2 \neq \eta_1$ we can write the general solution to the ODE (5.33) by

$$u(\xi) = \widetilde{T}_1 e^{\eta_1\xi} + \widetilde{T}_2 e^{\eta_2\xi} + \widetilde{T}_3 e^{-\eta_1\xi} + \widetilde{T}_4 e^{-\eta_2\xi} + \mu$$

for all $\xi \in [-\frac{\Delta}{2}, \frac{\Delta}{2}]$. Otherwise, in the case $\eta_2 = \eta_1$, the solution can be written as

$$u(\xi) = (\widetilde{T}_1 + \widetilde{T}_2\xi)e^{\eta_1\xi} + (\widetilde{T}_3 + \widetilde{T}_4\xi)e^{-\eta_1\xi} + \mu$$

for all $\xi \in [-\frac{\Delta}{2}, \frac{\Delta}{2}]$. Note that the function $K_\nu(0,\cdot)$ is a real entire[1] and even function. Thus its Fourier transform, $u(\xi)$, is a real-valued, even function.

[1]We call a function $f : \mathbb{C} \to \mathbb{C}$ *real entire* if f is analytic in $\mathbb{C}$ and $f(x)$ is real for all x real.

Suppose we are in the case where $\lambda \neq 4c_3^2$. The above-mentioned facts about $u(\xi)$ lead us to the following reduction:

$$u(\xi) = T_1 \cosh \eta_1 \xi + T_2 \cosh \eta_2 \xi + \mu$$

for all $\xi \in [-\frac{\Delta}{2}, \frac{\Delta}{2}]$, where T_1 and T_2 are appropriate complex numbers. We will identify these coefficients using the conditions in (5.34), which become

$$\frac{1}{c_1} = u(0) + \lambda \int_{-\Delta/2}^{\Delta/2} u(\alpha)|\alpha|\ e^{-c_3|\alpha|} \mathrm{d}\alpha, \tag{5.35}$$

$$-\frac{c_3^2}{c_1} = u''(0) + (2\lambda - c_3^2)u(0) - 2\lambda c_3 \int_{-\Delta/2}^{\Delta/2} u(\alpha) e^{-c_3|\alpha|}\, \mathrm{d}\alpha. \tag{5.36}$$

Using the notation from (5.11) and (5.12) in the equations (5.35) and (5.36) we get

$$\begin{cases} \dfrac{1}{c_1} - A(0)\mu = A(\eta_1)T_1 + A(\eta_2)T_2 \\ -\dfrac{c_3^2}{c_1} - B(0)\mu = B(\eta_1)T_1 + B(\eta_2)T_2. \end{cases}$$

In order to have a unique solution to this system of equations, we need to check whether the quantity

$$D(\eta_1, \eta_2) = A(\eta_1)B(\eta_2) - B(\eta_1)A(\eta_2)$$

is non-zero or not. Assuming $\lambda\Delta^2 \leqslant 5/3$ we can ensure that the quantity is not zero. The details for this can be found in the appendix of the article [A4].

It follows from these equations that

$$T_1 = \frac{1}{D(\eta_1, \eta_2)} \left\{ \left(\frac{1}{c_1} - A(0)\mu \right) B(\eta_2) + \left(\frac{c_3^2}{c_1} + B(0)\mu \right) A(\eta_2) \right\}$$

and

$$T_2 = -\frac{1}{D(\eta_1, \eta_2)} \left\{ \left(\frac{1}{c_1} - A(0)\mu \right) B(\eta_1) + \left(\frac{c_3^2}{c_1} + B(0)\mu \right) A(\eta_1) \right\}.$$

Thus,

$$\begin{aligned} u(\xi) = \frac{1}{D(\eta_1, \eta_2)} \Bigg[& \left\{ B(\eta_2) \left(\frac{1}{c_1} - A(0)\mu \right) + A(\eta_2) \left(\frac{c_3^2}{c_1} + B(0)\mu \right) \right\} \cosh \eta_1 \xi \\ & - \left\{ A(\eta_1) \left(\frac{c_3^2}{c_1} + B(0)\mu \right) + B(\eta_1) \left(\frac{1}{c_1} - A(0)\mu \right) \right\} \cosh \eta_2 \xi \Bigg] + \mu. \end{aligned}$$

In the second case, one can repeat this argument with a slight modification and get the following expression

$$\begin{aligned} u(\xi) = \mu &+ \frac{\left\{B'(\eta_1)\left(\frac{1}{c_1} - A(0)\mu\right) + A'(\eta_1)\left(\frac{c_3^2}{c_1} + B(0)\mu\right)\right\}\cosh \eta_1 \xi}{A(\eta_1)B'(\eta_1) - A'(\eta_1)B(\eta_1)} \\ &- \frac{\left\{A(\eta_1)\left(\frac{c_3^2}{c_1} + B(0)\mu\right) + B(\eta_1)\left(\frac{1}{c_1} - A(0)\mu\right)\right\} \cdot \xi \cdot \sinh \eta_1 \xi}{A(\eta_1)B'(\eta_1) - A'(\eta_1)B(\eta_1)}, \end{aligned}$$

where $A' = \frac{\mathrm{d}A}{\mathrm{d}\eta}$ and $B' = \frac{\mathrm{d}B}{\mathrm{d}\eta}$.

By taking the inverse Fourier transform of u, we get the expression for k_0 of item (iii) in Lemma 5.18, which is real entire, and so it also yields the formula for $K_\nu(0, z)$ in (5.27). $\square$

Chapter 6

Littlewood's estimates for L-functions in the hyperelliptic ensemble

6.1 Introduction

This chapter is based on [A5]. An overview of the main results and their context was presented in Section 1.3.2; here we develop the full details.

6.1.1 Background

Recall from Section 1.3.2 the classical argument function $S(t) := \frac{1}{\pi}\arg\zeta(\frac{1}{2}+it)$ and its antiderivatives $S_n(t)$, defined by $S_0(t) = S(t)$ and

$$S_n(t) := \int_0^t S_{n-1}(\tau)\,\mathrm{d}\tau + \delta_n\,, \qquad n \geqslant 1\,,$$

where the constants δ_n are the unique choices ensuring mean value zero; see [19, 54, 88] for explicit formulae. Under the Riemann hypothesis (RH), Littlewood proved in [88] that

$$\log\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \ll \frac{\log t}{\log\log t} \qquad \text{and} \qquad |S_n(t)| \ll \frac{\log t}{(\log\log t)^{n+1}}\,.$$

The current best estimates are discussed in Section 1.3.2, see (1.8)– (1.10). For the convenience of the reader we recall them here.

$$\log\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \leqslant \left(\frac{\log 2}{2}+o(1)\right)\frac{\log t}{\log\log t}\,, \tag{6.1}$$

$$|S(t)| \leqslant \left(\frac{1}{4}+o(1)\right)\frac{\log t}{\log\log t}\,, \tag{6.2}$$

and, more generally, for $n \geqslant 1$,

$$-\left(C_n^- + o(1)\right)\frac{\log t}{(\log\log t)^{n+1}} \leqslant S_n(t) \leqslant \left(C_n^+ + o(1)\right)\frac{\log t}{(\log\log t)^{n+1}}\,, \tag{6.3}$$

where, for $n = 4k \pm 1$,

$$C_n^{\mp} = \frac{\zeta(n+1)}{\pi \cdot 2^{n+1}} \qquad \text{and} \qquad C_n^{\pm} = \frac{(1-2^{-n})\,\zeta(n+1)}{\pi \cdot 2^{n+1}}\,, \tag{6.4}$$

and, for $n \geqslant 2$ even,

$$\begin{aligned} C_n^{+} = C_n^{-} &= \left[\frac{2(C_{n+1}^{+} + C_{n+1}^{-})C_{n-1}^{+}C_{n-1}^{-}}{C_{n-1}^{+} + C_{n-1}^{-}}\right]^{1/2} \\ &= \frac{\sqrt{2}}{\pi\, 2^{n+1}} \left[\frac{(1-2^{-n-2})(1-2^{-n+1})\,\zeta(n)\,\zeta(n+2)}{1-2^{-n}}\right]^{1/2}. \end{aligned} \tag{6.5}$$

The estimates (6.1)–(6.3) are what we wish to discuss in a function field setup.

6.1.2 Analogues over function fields

6.1.2.1 Setup

We briefly recall the setup from Section 1.3.2. Fix a finite field $\mathbb{F}_q$ of odd cardinality; a general reference for the background is [104]. For $f \in \mathbb{F}_q[x]$ its degree is denoted by $\deg(f)$. If f is a non-zero polynomial in $\mathbb{F}_q[x]$ then the norm of f is defined as $|f| := q^{\deg(f)}$. If $f = 0$, we set $|f| = 0$. A monic irreducible polynomial P, of degree at least 1, is called a *prime polynomial* or simply a *prime*. If P is a prime polynomial, the quadratic residue symbol $\left(\frac{f}{P}\right)$ is defined by

$$\left(\frac{f}{P}\right) = \begin{cases} 0, & \text{if } P \mid f; \\ 1, & \text{if } P \nmid f \text{ and } f \text{ is a square modulo } P; \\ -1, & \text{if } P \nmid f \text{ and } f \text{ is not a square modulo } P. \end{cases}$$

If $D \in \mathbb{F}_q[x]$ has prime factorization $D = P_1^{a_1} \cdots P_k^{a_k}$, we set

$$\left(\frac{f}{D}\right) := \prod_{i=1}^{k} \left(\frac{f}{P_i}\right)^{a_i}.$$

The quadratic reciprocity law states that for A, B non-zero, coprime monic polynomials,

$$\left(\frac{A}{B}\right) = \left(\frac{B}{A}\right)(-1)^{\frac{(q-1)}{2}\deg(A)\deg(B)}.$$

For $d = 2g+1$ odd, the hyperelliptic ensemble $\mathcal{H}_d$ consists of all monic, square-free polynomials in $\mathbb{F}_q[x]$ of degree d, and each $D \in \mathcal{H}_d$ gives rise to a quadratic Dirichlet character $\chi_D(f) = \left(\frac{D}{f}\right)$ and to an L-function, written in terms of $u = q^{-s}$ as

$$\mathcal{L}(u, \chi_D) = \sum_{f \text{ monic}} \chi_D(f)\, u^{\deg f} = \prod_{P \text{ prime}} (1 - \chi_D(P)\, u^{\deg P})^{-1}, \qquad |u| < q^{-1}. \tag{6.6}$$

By [104, Proposition 4.3], $\mathcal{L}(u,\chi_D)$ is a polynomial of degree $2g$. The classical work of Weil [121] on the Riemann hypothesis for curves over finite fields (the curve here being $C_D : y^2 = D(x)$, which is non-singular of genus g) yields the functional equation

$$\mathcal{L}(u,\chi_D) = (qu^2)^g\,\mathcal{L}\Big(\tfrac{1}{qu},\chi_D\Big) \tag{6.7}$$

and the Riemann hypothesis: all zeros of $\mathcal{L}(u,\chi_D)$ lie on the circle $|u| = q^{-1/2}$. Setting $\mathsf{e}(x) := e^{2\pi i x}$, we write the $2g$ zeros as $u_j = q^{-1/2}\mathsf{e}(\theta_j)$, so that

$$\mathcal{L}(u,\chi_D) = \prod_{j=1}^{2g}\left(1 - \frac{u}{u_j}\right). \tag{6.8}$$

From (6.7), the complex zeros come in conjugate pairs and $\mathcal{L}(u,\chi_D)$ is real-valued for $u \in \mathbb{R}$. The critical line $s = \frac{1}{2} + it$ becomes the critical circle $u = q^{-1/2}\mathsf{e}(\theta)$, $\theta \in \mathbb{T}$. Since the Riemann hypothesis holds unconditionally, all estimates below are unconditional. We refer to [5, 15, 16, 45, 47, 105] for further background on L-functions in this setting.

6.1.2.2 The modulus on the critical circle

Theorem 6.1. *For $d = 2g+1$, $D \in \mathcal{H}_d$, and $\theta \in \mathbb{T}$, as $d \to \infty$,*

$$\log\big|\mathcal{L}(q^{-1/2}\mathsf{e}(\theta),\chi_D)\big| \leqslant \left(\frac{\log 2}{2} + o(1)\right)\frac{d}{\log_q d}. \tag{6.9}$$

This result is not new. It was independently obtained by Florea [47, Corollary 8.2] and by Bucur, Costa, David, Guerreiro, and Lowry-Duda [14, Theorem 20], with different proofs. We provide a short proof within the framework developed here, which is similar in spirit to [14, Theorem 20]. A previous result of Altuğ and Tsimerman [2, Theorem 3.4] gives a bound of the same type at the central point $\theta = 0$, with an explicit error term, but with the constant 1 instead of $(\log 2)/2$.

6.1.2.3 The argument on the critical circle

Following the notation of [5, §4], when $q^{-1/2}\mathsf{e}(\theta)$ is not a zero of $\mathcal{L}(u,\chi_D)$, define

$$S(\theta,\chi_D) := -\frac{1}{\pi}\,\Delta_{\Gamma_\theta}\arg\mathcal{L}(u,\chi_D)\,, \tag{6.10}$$

where Γ_θ is the path consisting of the positively oriented circular arc from q^{-2} to $q^{-2}\mathsf{e}(\theta)$ followed by the radial segment from $q^{-2}\mathsf{e}(\theta)$ to $q^{-1/2}\mathsf{e}(\theta)$, and $\Delta_{\Gamma_\theta}\arg$ denotes the variation of the argument along this path. If $q^{-1/2}\mathsf{e}(\theta)$ is a zero of $\mathcal{L}(u,\chi_D)$, set $S(\theta,\chi_D) := \frac{1}{2}\lim_{\varepsilon\to 0}(S(\theta-\varepsilon,\chi_D) + S(\theta+\varepsilon,\chi_D))$. By the argument principle, this is a 1-periodic function on θ.

The bound $S(\theta, \chi_D) \ll d/\log_q d$ was established in [5, Theorem 9], and a careful reading of their proof yields $S(\theta, \chi_D) \leqslant (\frac{1}{2} + o(1))d/\log_q d$. We refine this to obtain the analogue of (6.2).

Theorem 6.2. *For $d = 2g+1$, $D \in \mathcal{H}_d$, and $\theta \in \mathbb{T}$, as $d \to \infty$,*

$$S(\theta, \chi_D) \leqslant \left(\frac{1}{4} + o(1)\right) \frac{d}{\log_q d}.$$

6.1.2.4 The antiderivatives of the argument

The function $\theta \mapsto S(\theta, \chi_D)$ has mean value zero in $\mathbb{T}$ (see Lemma 6.5 below). We define a sequence of antiderivatives by setting $S_0(\theta, \chi_D) = S(\theta, \chi_D)$ and, for $n \geqslant 1$ and $\theta \in \mathbb{T}$,

$$S_n(\theta, \chi_D) := \int_0^\theta S_{n-1}(\alpha, \chi_D)\, \mathrm{d}\alpha + c_n\,, \tag{6.11}$$

where the constant $c_n = c_n(\chi_D)$ is chosen so that $\int_{\mathbb{T}} S_n(\theta, \chi_D)\, \mathrm{d}\theta = 0$. The function S_0 has jump discontinuities at the θ_j, but all S_n with $n \geqslant 1$ are continuous, and the mean-value zero property of S_{n-1} ensures that each S_n is 1-periodic.

Before stating the bounds on $S_n(\theta, \chi_D)$, we recall the Bernoulli polynomials $B_n(x)$, defined by the generating function

$$\frac{t\, e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!}\, t^n\,, \qquad |t| < 2\pi\,,$$

and the associated Bernoulli periodic functions

$$\mathcal{B}_n(x) := B_n(x - \lfloor x \rfloor)\,, \qquad n \geqslant 0,\ n \neq 1\,,$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. For $n = 1$ we make a minor adjustment $\mathcal{B}_1(x) := x - \lfloor x \rfloor - \frac{1}{2}$ for $x \notin \mathbb{Z}$ and $\mathcal{B}_1(x) := 0$ for $x \in \mathbb{Z}$. Throughout this chapter we set

$$M_n := \max_{x \in [0,1]} B_n(x) \qquad \text{and} \qquad m_n := \min_{x \in [0,1]} B_n(x)\,. \tag{6.12}$$

Theorem 6.3. *For $d = 2g+1$, $D \in \mathcal{H}_d$, $n \geqslant 1$, and $\theta \in \mathbb{T}$, as $d \to \infty$,*

$$-\left(\frac{A_n^-}{(2\pi)^n} + o(1)\right) \frac{d}{(\log_q d)^{n+1}} \leqslant S_n(\theta, \chi_D) \leqslant \left(\frac{A_n^+}{(2\pi)^n} + o(1)\right) \frac{d}{(\log_q d)^{n+1}}\,, \tag{6.13}$$

where

$$A_n^- := \frac{\pi^n}{2} \cdot \frac{M_{n+1}}{(n+1)!} \qquad \textit{and} \qquad A_n^+ := -\frac{\pi^n}{2} \cdot \frac{m_{n+1}}{(n+1)!}\,. \tag{6.14}$$

Remark. (i) The factor $(2\pi)^n$ dividing $A_n^{\pm}$ in (6.13) is a normalization due to working with 1-periodic functions rather than 2π-periodic ones. If instead one sets $\widetilde{S}_0(\tau,\chi_D) := S_0(\frac{\tau}{2\pi},\chi_D)$ for $\tau \in \mathbb{R}/2\pi\mathbb{Z}$, its antiderivatives satisfy $\widetilde{S}_n(\tau,\chi_D) = (2\pi)^n S_n(\frac{\tau}{2\pi},\chi_D)$, and the factor $(2\pi)^n$ disappears from (6.13).

(ii) An explicit formula for the constants c_n follows from Lemma 6.5 and the identity $\mathcal{B}'_{n+1} = (n+1)\mathcal{B}_n$ for $n \geqslant 1$:

$$c_n = -\frac{1}{(n+1)!}\sum_{j=1}^{2g}\mathcal{B}_{n+1}(-\theta_j)\,.$$

For even n, one has $c_n = 0$, since $\mathcal{B}_{n+1}(-\theta) = (-1)^{n+1}\mathcal{B}_{n+1}(\theta)$ and in particular $\mathcal{B}_{n+1}(0) = \mathcal{B}_{n+1}(\frac{1}{2}) = 0$ when n is even.

(iii) In each of Theorems 6.1, 6.2 and 6.3 the term $o(1)$ is in fact $O\Big(\log_q \log_q d\,/\,\log_q d\Big)$, with the implicit constant depending on n in Theorem 6.3.

(iv) One can also treat even degrees $d = 2g+2$ in the definition of $\mathcal{H}_d$. In this case $\mathcal{L}(u,\chi_D)$ is a polynomial of degree $2g+1$ with one trivial zero at $u=1$ and the remaining $2g$ zeros on $|u| = q^{-1/2}$. The results of Theorems 6.1, 6.2 and 6.3 continue to hold with minor modifications to accommodate the $O(1)$ contribution from the trivial zero. We restrict to odd d throughout, as is standard in this model; see, e.g., [5, 15, 16, 47].

6.1.3 Lessons from the function field setup

6.1.3.1 Matching constants

We elaborate on the comparison between Theorem 6.3 and the classical bound (6.3). It is well known (see [83, Theorem 1]) that for $n = 4k+1$:

$$M_{n+1} = B_{n+1}(0) = \frac{2\,(n+1)!\,\zeta(n+1)}{(2\pi)^{n+1}};$$
$$m_{n+1} = B_{n+1}\Big(\frac{1}{2}\Big) = -\frac{2\,(n+1)!\,(1-2^{-n})\,\zeta(n+1)}{(2\pi)^{n+1}},$$

and for $n = 4k-1$ the roles of 0 and $\frac{1}{2}$ are reversed and the values are given by

$$M_{n+1} = B_{n+1}\Big(\frac{1}{2}\Big) = \frac{2\,(n+1)!\,(1-2^{-n})\,\zeta(n+1)}{(2\pi)^{n+1}};$$
$$m_{n+1} = B_{n+1}(0) = -\frac{2\,(n+1)!\,\zeta(n+1)}{(2\pi)^{n+1}}.$$

This plainly implies that, for n odd, we have the exact match

$$A_n^- = C_n^- \qquad \text{and} \qquad A_n^+ = C_n^+$$

with the constants in (6.4). The constant $(\log 2)/2$ in (6.9) likewise matches the classical bound (6.1).

For $n \geqslant 2$ even, Lehmer [83, Eqs. (15)–(19)] showed that $M_{n+1} = -m_{n+1}$ and

$$\frac{2\,(n+1)!\,(1-3^{-n})}{(2\pi)^{n+1}} < M_{n+1} < \frac{2\,(n+1)!}{(2\pi)^{n+1}}\,,$$

which implies

$$\frac{1-3^{-n}}{\pi \cdot 2^{n+1}} < A_n^- = A_n^+ < \frac{1}{\pi \cdot 2^{n+1}}. \tag{6.15}$$

Inequality (6.15) in turn implies that, for $n \geqslant 2$ even,

$$A_n^{\pm} < C_n^{\pm},$$

with $C_n^{\pm}$ defined in (6.5). In particular $A_n^{\pm} \sim \frac{1}{\pi \cdot 2^{n+1}}$ while $C_n^{\pm} \sim \frac{\sqrt{2}}{\pi \cdot 2^{n+1}}$ as $n \to \infty$, so the function field bounds are strictly sharper by a factor of $\sqrt{2}$ asymptotically.

6.1.3.2 Strategy and mysterious connections

A classical problem in approximation theory is that of finding trigonometric polynomials of a given degree that majorize or minorize a given periodic function while minimizing the $L^1(\mathbb{T})$ error; see, e.g., [17, 24, 95, 118]. This can be seen as the periodic analogue of the classical Beurling–Selberg extremal problem of finding one-sided bandlimited approximations to a given function on $\mathbb{R}$ while minimizing the $L^1(\mathbb{R})$ error; see, e.g., [20, 24, 26, 118]. These two worlds meet in this chapter in a somewhat mysterious way.

The proofs of Theorems 6.1, 6.2 and 6.3 share a unified strategy: (i) represent the quantity of interest as a sum of translates of a special periodic function over the angles θ_j; (ii) majorize or minorize that function by a trigonometric polynomial of degree N, using the extremal constructions of [17, 24]; (iii) bound the sum over zeros via the prime polynomial theorem and choose N optimally. The periodic function arising for Theorem 6.1 is $\varphi(\theta) = \log(2|\sin(\pi\theta)|)$, while that for $S_n(\theta, \chi_D)$ in Theorems 6.2 and 6.3 is the Bernoulli periodic function $\mathcal{B}_{n+1}(\theta)$. The extremal trigonometric polynomials for these functions were characterized in [17, 24].

The proofs of the classical estimates (6.1)–(6.3) for odd n follow a parallel strategy via the Guinand–Weil explicit formula, using Beurling–Selberg extremal majorants and minorants of certain special real-valued functions; see [19, 25, 35]. The functions appearing in the two settings are paired in the following table.

	periodic setting	real line setting
modulus	$\log(2\lvert\sin(\pi\theta)\rvert)$	$\log\left(\dfrac{4+x^2}{x^2}\right)$
S_0	$\mathcal{B}_1(\theta)$	$\arctan\left(\dfrac{1}{x}\right) - \dfrac{x}{1+x^2}$
S_{2k+1}	$\mathcal{B}_{2k+2}(\theta)$	$\dfrac{1}{2k+1}\left[(-1)^{k+1}x^{2k+1}\arctan\left(\dfrac{1}{x}\right) + \displaystyle\sum_{\ell=0}^{k}\dfrac{(-1)^{k-\ell}}{2\ell+1}x^{2k-2\ell}\right]$

For each of the functions in the right column, the Beurling–Selberg extremal problem has been solved [20, 24, 26]. Although the functions in the two columns bear no obvious relation to each other, when the solutions of both extremal problems are placed in their respective explicit formulas, the matching constants $A_n^{\pm} = C_n^{\pm}$ emerge for $n = 0$ and n odd (and the factor $(\log 2)/2$ for the modulus).

For even $n = 2k$, $k \geqslant 1$, the natural real-line function associated to bounding $S_{2k}(t)$ is

$$f_{2k}(x) := (-1)^k x^{2k}\arctan\left(\frac{1}{x}\right) + \sum_{\ell=0}^{k-1}\frac{(-1)^{k-\ell+1}}{2\ell+1}x^{2k-2\ell-1} - \frac{x}{(2k+1)(1+x^2)}\,.$$

The Beurling–Selberg extremal problem for f_{2k} remains open; these are odd, continuous functions, a class known to be particularly difficult. The authors of [19] develop an alternative (non-optimal) approach to arrive at (6.3)–(6.5) in this case. We expect that solving the Beurling–Selberg problem for f_{2k} and running the complete strategy would yield the constants matching the ones we obtain in Theorem 6.3.

6.2 Auxiliary results

6.2.1 The prime polynomial theorem

The zeta function over $\mathbb{A} = \mathbb{F}_q[x]$ is defined by

$$\zeta_{\mathbb{A}}(s) := \sum_{f \text{ monic}}\frac{1}{|f|^s} = \prod_{P \text{ prime}}\left(1-|P|^{-s}\right)^{-1}, \qquad \mathrm{Re}(s) > 1.$$

With the usual change of variables $u = q^{-s}$, and observing that there are q^k monic polynomials of degree k, one arrives at

$$\mathcal{Z}_{\mathbb{A}}(u) = \sum_{f \text{ monic}} u^{\deg(f)} = \prod_{P \text{ prime}}\left(1-u^{\deg(P)}\right)^{-1} = \frac{1}{1-qu}. \tag{6.16}$$

Logarithmic differentiation of (6.16) then yields the prime polynomial theorem

$$\sum_{\substack{f \text{ monic} \\ \deg(f)=k}} \Lambda(f) = q^k, \tag{6.17}$$

where, for a monic polynomial f, we write $\Lambda(f) \coloneqq \deg(P)$ if $f = P^k$ for some prime polynomial P and positive integer k, and $\Lambda(f) \coloneqq 0$ otherwise.

In a similar way, logarithmic differentiation of (6.6) and (6.8) yields

$$\sum_{k=1}^{\infty} \left(\sum_{\substack{f \text{ monic} \\ \deg(f)=k}} \chi_D(f)\,\Lambda(f) \right) u^{k-1} = \frac{\mathcal{L}'}{\mathcal{L}}(u, \chi_D) = -\sum_{k=1}^{\infty} \left(\sum_{j=1}^{2g} u_j^{-k} \right) u^{k-1}. \tag{6.18}$$

Recalling that $u_j = q^{-1/2}\mathsf{e}(\theta_j)$, expression (6.18) leads to the following identity, for each $k \geqslant 1$,

$$-\sum_{j=1}^{2g} \mathsf{e}(-k\theta_j) = \frac{1}{q^{k/2}} \sum_{\substack{f \text{ monic} \\ \deg(f)=k}} \chi_D(f)\,\Lambda(f). \tag{6.19}$$

This is the standard explicit formula for the L-functions in our study. It has been previously used, for instance, in [5, 16, 105].

By combining (6.17) and (6.19) we get, for each $k \neq 0$,

$$\left| \sum_{j=1}^{2g} \mathsf{e}(k\theta_j) \right| = \left| \frac{1}{q^{|k|/2}} \sum_{\substack{f \text{ monic} \\ \deg(f)=|k|}} \chi_D(f)\,\Lambda(f) \right| \leqslant \frac{1}{q^{|k|/2}} \sum_{\substack{f \text{ monic} \\ \deg(f)=|k|}} \Lambda(f) = q^{|k|/2}. \tag{6.20}$$

Estimate (6.20) will be later used in the proofs of Theorems 6.1, 6.2 and 6.3.

6.2.2 Representation formulas

We start here with a simple observation.

Lemma 6.4. *For each $\theta \in \mathbb{T}$, we have*

$$\log\left|\mathcal{L}(q^{-1/2}\mathsf{e}(\theta), \chi_D)\right| = \sum_{j=1}^{2g} \log\left(2|\sin \pi(\theta - \theta_j)|\right).$$

Proof. This plainly follows from (6.8). □

Recall that, for $n \geqslant 1$, the Bernoulli periodic functions have the Fourier series expansion

$$\mathcal{B}_n(\theta) = -\frac{n!}{(2\pi i)^n} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k^n}\,\mathsf{e}(k\theta) \tag{6.21}$$

(for $n = 1$ the equality above is understood as the limit of the symmetric partial sums). In particular, for $n \geqslant 1$, note that each $\mathcal{B}_n$ has mean value zero in $\mathbb{T}$, i.e., $\widehat{\mathcal{B}_n}(0) = 0$.

We proceed with a formula that connects the functions $S_n(\theta, \chi_D)$, for $n \geqslant 0$, to the Bernoulli periodic functions. The case $n = 0$ of the following lemma already appears in [5, Theorem 4], and we include a brief proof for the convenience of the reader.

Lemma 6.5. *Let n be a non-negative integer. For each $\theta \in \mathbb{T}$, we have*

$$S_n(\theta, \chi_D) = -\frac{1}{(n+1)!}\sum_{j=1}^{2g} \mathcal{B}_{n+1}(\theta - \theta_j). \tag{6.22}$$

Proof. We first prove the case $n = 0$. Assume first that $\theta \neq \theta_j$. Considering the principal branch of the logarithm, from (6.8) and definition (6.10) we get

$$\begin{aligned} S(\theta, \chi_D) &= -\frac{1}{\pi}\Delta_{\Gamma_\theta} \arg \mathcal{L}(u, \chi_D) = -\frac{1}{\pi}\sum_{j=1}^{2g}\Delta_{\Gamma_\theta}\arg\left(1 - \frac{u}{u_j}\right) \\ &= -\frac{1}{\pi}\sum_{j=1}^{2g}\mathrm{Im}\left(\log\left(1 - \mathsf{e}(\theta - \theta_j)\right) - \log\left(1 - q^{-3/2}\mathsf{e}(-\theta_j)\right)\right). \end{aligned}$$

Since the zeros $u_j = q^{-1/2}\mathsf{e}(\theta_j)$ appear in conjugate pairs, we have

$$\sum_{j=1}^{2g}\mathrm{Im}\log\left(1 - q^{-3/2}\mathsf{e}(-\theta_j)\right) = 0.$$

Noting that

$$\mathrm{Im}\log\left(1 - \mathsf{e}(\theta)\right) = \pi\,\mathcal{B}_1(\theta)$$

for any $\theta \in \mathbb{R}/\mathbb{Z} \setminus \{0\}$, we arrive at the desired identity (6.22) in the case $n = 0$. The identity continues to hold when $\theta = \theta_j$ since both sides of (6.22) are normalized functions (i.e., the value at a discontinuity point is the average of the side limits).

The general case plainly follows by induction from the way we defined $S_n(\theta, \chi_D)$ in (6.11), using the fact that the right-hand side of (6.22) has mean value zero in $\mathbb{T}$, and $\mathcal{B}'_{n+1}(x) = (n+1)\mathcal{B}_n(x)$ for $n \geqslant 1$ (for $n = 1$ it suffices that this holds on $\mathbb{T} \setminus \{0\}$). ∎

6.2.3 Extremal trigonometric polynomials

We collect here a few results on one-sided approximations of periodic functions by trigonometric polynomials that will be suitable for our purposes. The first one is a result from Carneiro and Vaaler [24, Theorem 1.5].

Lemma 6.6. *Let $\varphi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be given by $\varphi(\theta) = \log\left(2|\sin(\pi\theta)|\right)$, and let N be a non-negative integer. Then there exists a unique real-valued trigonometric polynomial*

$U_N(\theta) = \sum_{k=-N}^{N} \widehat{U_N}(k)\,\mathsf{e}(k\theta)$ *such that* $\varphi(\theta) \leqslant U_N(\theta)$ *for each* $\theta \in \mathbb{T}$ *and such that*

$$\int_{\mathbb{T}} U_N(\theta)\,\mathrm{d}\theta = \frac{\log 2}{N+1}.$$

Moreover, for each integer k *with* $1 \leqslant |k| \leqslant N$*, one has*

$$-\frac{1}{2|k|} \leqslant \widehat{U_N}(k) \leqslant 0.$$

The next lemma is a result from [17, Theorem 1], characterizing the extremal approximations to the Bernoulli periodic functions. It is worth mentioning that these, in turn, rely on the solution of the Beurling–Selberg extremal problem for the functions $x^k \operatorname{sgn}(x)$, for $k \geqslant 1$, due to Littmann [89] (the case $k = 0$ is originally due to Beurling, as remarked in Vaaler's classical survey [118]). For $n \geqslant 0$, recall the definition of M_n and m_n in (6.12).

Lemma 6.7. *Let* n *and* N *be non-negative integers. There exist unique trigonometric polynomials* $P^{\pm}_{n+1,N}(\theta) = \sum_{k=-N}^{N} \widehat{P^{\pm}_{n+1,N}}(k)\,\mathsf{e}(k\theta)$ *such that*

$$P^{-}_{n+1,N}(\theta) \leqslant \mathcal{B}_{n+1}(\theta) \leqslant P^{+}_{n+1,N}(\theta)$$

for each $\theta \in \mathbb{R}/\mathbb{Z}$ *and*

$$\int_{\mathbb{T}} P^{-}_{n+1,N}(\theta)\,\mathrm{d}\theta = \frac{m_{n+1}}{(N+1)^{n+1}} \qquad \text{and} \qquad \int_{\mathbb{T}} P^{+}_{n+1,N}(\theta)\,\mathrm{d}\theta = \frac{M_{n+1}}{(N+1)^{n+1}}.$$

Moreover, for each integer k *with* $1 \leqslant |k| \leqslant N$*, one has*

$$\left|\widehat{P^{\pm}_{n+1,N}}(k)\right| \ll_n \frac{1}{k^{n+1}}. \tag{6.23}$$

Proof of (6.23)*.* The estimate (6.23) is not explicitly mentioned in [17, Theorem 1] and we give a brief proof. We deal with the case of $P^{+}_{n+1,N}$, and the estimate for $P^{-}_{n+1,N}$ is analogous. Let $D^{+}_{n+1,N}(\theta) \coloneqq P^{+}_{n+1,N}(\theta) - \mathcal{B}_{n+1}(\theta)$. Since $D^{+}_{n+1,N}(\theta) \geqslant 0$, we have

$$\left|\widehat{D^{+}_{n+1,N}}(k)\right| \leqslant \widehat{D^{+}_{n+1,N}}(0) = \widehat{P^{+}_{n+1,N}}(0) = \frac{M_{n+1}}{(N+1)^{n+1}}. \tag{6.24}$$

By the triangle inequality,

$$\left|\widehat{P^{+}_{n+1,N}}(k)\right| \leqslant \left|\widehat{D^{+}_{n+1,N}}(k)\right| + \left|\widehat{\mathcal{B}_{n+1}}(k)\right|,$$

and estimate (6.23) plainly follows from (6.24) and (6.21), since $\widehat{\mathcal{B}_{n+1}}(k) = -\frac{(n+1)!}{(2\pi i k)^{n+1}}$. $\square$

6.3 Proofs of Theorems 6.1, 6.2 and 6.3

We give a unified proof of Theorems 6.1, 6.2 and 6.3. In each situation we want to bound a certain periodic function $F(\cdot, \chi_D) : \mathbb{T} \to \mathbb{R}$ that can be written as

$$F(\theta, \chi_D) = \sum_{j=1}^{2g} G(\theta - \theta_j)\,,$$

with $G : \mathbb{T} \to \mathbb{R}$ being a suitable periodic function of mean value zero (these are Lemma 6.4 and Lemma 6.5). Suppose we can solve the problem of majorizing G by a trigonometric polynomial of a fixed degree, optimizing the $L^1(\mathbb{T})$-error. In our framework (Lemma 6.6 and Lemma 6.7), this means that, for each non-negative N, there exists a trigonometric polynomial $V_N(\theta) = \sum_{k=-N}^{N} \widehat{V_N}(k)\,\mathsf{e}(k\theta)$ such that:

(i) $G(\theta) \leqslant V_N(\theta)$ for all $\theta \in \mathbb{T}$.

(ii) For some constants κ and n (independent of N), we have

$$\int_{\mathbb{T}} V_N(\theta)\,\mathrm{d}\theta = \frac{\kappa}{(N+1)^{n+1}}.$$

(iii) $|\widehat{V_N}(k)| \ll 1$ for integers k with $1 \leqslant |k| \leqslant N$, with the implicit constant independent of N.

Under such conditions, we have

$$\begin{aligned} F(\theta, \chi_D) &= \sum_{j=1}^{2g} G(\theta - \theta_j) \leqslant \sum_{j=1}^{2g} V_N(\theta - \theta_j) = \sum_{j=1}^{2g} \sum_{k=-N}^{N} \widehat{V_N}(k)\,\mathsf{e}(k(\theta - \theta_j)) \\ &= 2g\,\widehat{V_N}(0) + \sum_{j=1}^{2g} \sum_{\substack{k=-N \\ k\neq 0}}^{N} \widehat{V_N}(k)\,\mathsf{e}(k(\theta - \theta_j)) \\ &\leqslant 2g\,\widehat{V_N}(0) + \sum_{\substack{k=-N \\ k\neq 0}}^{N} \left|\widehat{V_N}(k)\right| \left|\sum_{j=1}^{2g} \mathsf{e}(k\theta_j)\right| \\ &\leqslant 2g\,\widehat{V_N}(0) + \sum_{\substack{k=-N \\ k\neq 0}}^{N} \left|\widehat{V_N}(k)\right| q^{|k|/2} = \frac{2g\,\kappa}{(N+1)^{n+1}} + O(Nq^{N/2})\,, \end{aligned}$$

where we have used (6.20). Choosing $N \sim 2\log_q d - (2n+6)\log_q \log_q d$, we get

$$F(\theta, \chi_D) \leqslant \left(\frac{\kappa}{2^{n+1}} + o(1)\right) \frac{d}{(\log_q d)^{n+1}}.$$

The term $o(1)$ above is $O(\log_q \log_q d \,/\, \log_q d)$ (the implicit constant might depend on n here).

The proof of the lower bound is analogous, using the trigonometric polynomials that minorize G.

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