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Minimality of the ball for a model of charged liquid droplets

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Abstract. We prove that charged liquid droplets minimizing *Debye–Hückel-type free energy* are spherical in the small charge regime. The variational model was proposed by Muratov and Novaga in 2016 to avoid the ill-posedness of the classical one. Combining a recent (partial) regularity result with the selection principle of Cicalese and Leonardi, we prove that the ball is the unique minimizer in the small charge regime.

1. Introduction

1.1. Background and description of the model

In this paper we deal with a variational model describing the shape of charged liquid droplets. We investigate the droplets minimizing a suitable free energy composed by an attractive term, coming from surface tension forces, and a repulsive one, due to the electric forces generated by the interaction between charged particles. Thanks to the particular structure of the energy, one may expect that for small values of the total charge the attractive part is predominant, in this way forcing the spherical shape.

The experiments agree with this guess – one observes the following phenomenon: the shape of the liquid droplet is spherical in a small charge regime. Then, as soon as the value of the total charge increases, the droplet gradually deforms into an ellipsoid, it develops conical singularities, the so-called Taylor cones [24], and finally the liquid starts emitting a thin jet [7, 8, 22, 25]. The first experiments were conducted by Zeleny in 1914 [26], but in a slightly different context.

Several mathematical models of charged liquid droplets have been studied over the years. A difficulty is that contrary to numerical and experimental observations, these models are in general mathematically ill posed; see [13]. For a more exhaustive discussion we refer the reader to [20].

The main issue with the variational model studied in [13] comes from the tendency of charges to concentrate at the interface of the liquid. To restore the well-posedness one should consider a physical regularizing mechanism in the functional. With this in mind,

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Muratov and Novaga [20] integrate the entropic effects associated with the presence of free ions in the liquid. The advantage of this model is that the charges are now distributed inside the droplet. More precisely, they suggest considering the following *Debye–Hückel-type free energy* (in every dimension):

$$\mathcal{F}_{\beta,K,Q}(E,u,\rho) := P(E) + Q^2\Big\{\int_{\mathbb{R}^n} a_E|\nabla u|^2\,dx + K\int_E \rho^2\,dx\Big\}.$$

Here $E \subset \mathbb{R}^n$ represents the droplet, $P(E)$ is the De Giorgi perimeter [17, Chapter 12], the constant $Q > 0$ is the total charge enclosed in E and

$$a_E(x) := \mathbf{1}_{E^c} + \beta\mathbf{1}_E,$$

where $\mathbf{1}_F$ is the characteristic function of a set F and $\beta \geq 1$[1] is the permittivity of the liquid.

The normalized density of charge $\rho \in L^2(\mathbb{R}^n)$ satisfies

$$\rho\mathbf{1}_{E^c} = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} \rho\,dx = 1, \tag{1.1}$$

and the electrostatic potential u is such that $\nabla u \in L^2(\mathbb{R}^n;\mathbb{R}^n)$ and

$$-\operatorname{div}(a_E\nabla u) = \rho \quad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{1.2}$$

For a fixed set E we define the set of admissible pairs of functions u and ρ:

$$\mathcal{A}(E) := \big\{(u,\rho) \in D^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n) : u \text{ and } \rho \text{ satisfy (1.2) and (1.1)}\big\},$$

where

$$D^1(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{\mathring{W}^{1,2}(\mathbb{R}^n)}, \qquad \|\varphi\|_{\mathring{W}^{1,2}(\mathbb{R}^n)} = \|\nabla\varphi\|_{L^2(\mathbb{R}^n)}.$$

Note that the class of admissible couples $\mathcal{A}(E)$ is non-empty only if $n \geq 3$ (see [21, Remark 2.2]). For this reason, the assumption $n \geq 3$ will be in force throughout the paper. The variational problem proposed in [20] is the following:

$$\min\big\{\mathcal{F}_{\beta,K,Q}(E,u,\rho) : |E| = V, E \subset B_R, (u,\rho) \in \mathcal{A}(E)\big\}.$$

The a priori boundedness assumption $E \subseteq B_R$ ensures the existence of a minimizer in the class of sets of finite perimeter with a prescribed volume [20, Theorem 3].

For convenience we introduce the following notation:

$$\mathcal{G}_{\beta,K}(E) := \inf_{(u,\rho)\in\mathcal{A}(E)}\Big\{\int_{\mathbb{R}^n} a_E|\nabla u|^2\,dx + K\int_E \rho^2\,dx\Big\}.$$

[1] Mathematically, considering $\beta < 1$ amounts to considering the complement of the set E in place of E. Our proof would work without change for the case $\beta < 1$ also. However, some changes would be needed in [21], so we would not be able to use their regularity results directly.

For $E \subset \mathbb{R}^n$ we set

$$\mathcal{F}_{\beta,K,Q}(E) := P(E) + Q^2 \mathcal{G}_{\beta,K}(E).$$

By scaling (see the introduction of [21]), we can reduce the problem to the case $|E| = |B_1|$ and so in the rest of the paper we will work with the following problem:

$$\min\{\mathcal{F}_{\beta,K,Q}(E) : |E| = |B_1|,\ E \subset B_R\}. \tag{$\mathcal{P}_{\beta,K,Q,R}$}$$

We will often omit the subscripts β and K as those are fixed physical parameters. We will also omit the subscript Q when it is clear from the context.

We note that the model we investigate can be seen as "interpolating" Gamow's model and the free interface problems arising in optimal design (see, for example, [10]). For the former, it has been recently shown [15, 16] that for small enough charges the unique minimizers are balls. However, in Gamow's model the non-local term is Lipschitz with respect to symmetric difference between sets, implying that on small scales the perimeter dominates the non-local part of the energy.

1.2. Main results

As we mentioned above, one can expect that the shape of the droplet in a small charge regime is spherical. We confirm this intuition by proving that the ball is the unique minimizer of the functional $\mathcal{F}$ for small values of the total charge Q. Precisely, we obtain the following result.

Theorem 1.1. *Fix $K > 0$, $\beta \geq 1$. Then there exists $Q_0 > 0$ such that for all $Q < Q_0$ and any $R \geq 1$, the only minimizers of ($\mathcal{P}_{\beta,K,Q,R}$) are balls of radius 1.*

The condition $E \subset B_R$ in the minimizing problem ($\mathcal{P}_{\beta,K,Q,R}$) is required to have existence of minimizers. However, thanks to Theorem 1.1 it can be dropped for small enough charges.

Corollary 1.2. *Fix $K > 0$, $\beta \geq 1$. Then there exists $Q_0 > 0$ such that for all $Q < Q_0$ the infimum in the problem*

$$\inf\{\mathcal{F}_{\beta,K,Q}(E) : |E| = |B_1|\} \tag{$\mathcal{P}_{\beta,K,Q}$}$$

is attained. Moreover, the only minimizers are balls of radius 1.

Remark 1.3. The constant Q_0 we obtain in the proof of Corollary 1.2 is the same as Q_0 in the statement of Theorem 1.1. However, we expect it to hold for bigger charges. Let Q_e be a maximal charge such that for any $Q < Q_e$ the minimizers of ($\mathcal{P}_{\beta,K,Q}$) exist and Q_b be a maximal charge such that for any $Q < Q_b$ the minimizers of ($\mathcal{P}_{\beta,K,Q}$) exist and are spherical. We conjecture that $Q_b < Q_e$.

For the proof of Theorem 1.1 we combine an improved version of (partial) regularity results for the minimizers of [21, Theorem 1.2] with second variation techniques. The first step is to obtain the partial $C^{2,\vartheta}$-regularity of minimizers. In fact, we are able to prove the following partial C^∞-regularity of minimizers, a result that is interesting in itself.

We refer the reader to Notation 2.1 for the definitions of $\mathbf{e}_E(x_0, r)$, $D_E(x_0, r)$ and $\mathbf{C}(x_0, r/2)$.

Theorem 1.4 (C^∞-regularity). *Given $n \geq 3$ and $A > 0$, there exists $\varepsilon_{\mathrm{reg}} = \varepsilon_{\mathrm{reg}}(n, A) > 0$ such that if E is a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$) with $Q + \beta + K + \frac{1}{K} \leq A$,*

$$x_0 \in \partial E \quad \text{and} \quad r + \mathbf{e}_E(x_0, r) + Q^2 D_E(x_0, r) \leq \varepsilon_{\mathrm{reg}},$$

then $E \cap \mathbf{C}(x_0, r/2)$ coincides with the epigraph of a C^∞-function f. In particular, we have that $\partial E \cap \mathbf{C}(x_0, r/2)$ is a C^∞ $(n-1)$-dimensional manifold. Moreover,

$$[f]_{C^k(\mathbf{D}(x_0', r/2))} \leq C(n, A, k, r)$$

for every $k \in \mathbb{N}$ with $k \geq 2$.

1.3. Strategy of the proof and structure of the paper

We use the selection principle, the technique introduced by Cicalese and Leonardi [5] for the proof of a quantitative isoperimetric inequality (see also [1], where the authors use a similar approach to investigate a non-local isoperimetric problem).

First, we enhance the regularity result obtained in [21]. Section 3 is dedicated to obtaining $C^{2,\vartheta}$ regularity of minimizers. Exploiting the Euler–Lagrange equation and the $C^{1,\eta}$-regularity of u up to the boundary ∂E, we deduce the partial $C^{2,\vartheta}$-regularity of minimizers (Theorem 3.9). This improvement of regularity is not trivial and requires delicate analysis of regularity for minimizers of $\mathcal{G}_{\beta,K}(E)$.

In Section 4, by a standard bootstrap argument, we obtain the partial smooth regularity of minimizers (Theorem 1.4).

To prove Theorem 1.1 we reduce our problem to the so-called *nearly spherical sets*. These are the sets which can be described as subgraphs of smooth functions defined over the boundary of the unitary ball. The advantage is that for this particular class of sets we are able to deduce a Taylor expansion for the energy near the ball B_1.

In Section 5 we show that a minimizer is nearly spherical whenever the total charge is small enough. First, we prove the L^1-convergence of the minimizers to the unitary ball and the convergence of the perimeters as the charge goes to zero. Thanks to *uniform* density estimates for the volume and the perimeter of a minimizer we obtain the Kuratowski convergence of sets as well as their boundaries. Finally, since for each Q small enough the corresponding minimal set E_Q has $C^{2,\vartheta}$-regular boundary (with uniform bounds), by Ascoli–Arzelà, up to extracting a subsequence, we easily get that E_Q converges to B_1 in a stronger $C^{2,\vartheta'}$-sense for every $\vartheta' < \vartheta$. This part is standard.

In Sections 6 and 7 we prove Theorem 1.1 for nearly spherical sets. To this end, we write the Taylor expansion of the energy $\mathcal{G}$ using shape derivatives and providing a bound for the "Hessian". A direct computation provides a similar (classical) bound for the perimeter and this allows us to conclude. Here some technical difficulties arise due to the non-local nature of the functional $\mathcal{G}_{\beta,K}$. Note also that the Euler–Lagrange equations we get have transmission boundary conditions while most of the existing literature only deals with Dirichlet or Neumann ones.

Remark 1.5. An extended version of the paper, containing some standard proofs and more detailed computations is available at [19].

2. Notation and preliminary results

In this section we fix the notation and collect some results obtained in [21] which will be useful in the proof of regularity.

Notation 2.1. Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$ and $r > 0$.

- We call $\mathbf{p}^\nu(x) := x - (x \cdot \nu)\nu$ and $\mathbf{q}^\nu(x) := (x \cdot \nu)\nu$, respectively, the *orthogonal projection* onto the plane $\nu^\perp$ and the *projection* on ν. For simplicity we write $\mathbf{p}(x) := \mathbf{p}^{e_n}(x)$ and $\mathbf{q}(x) := \mathbf{q}^{e_n}(x) = x_n$.
- We define the *cylinder* with center at $x_0 \in \mathbb{R}^n$ and radius $r > 0$ with respect to the direction $\nu \in \mathbb{S}^{n-1}$ as
$$\mathbf{C}(x_0, r, \nu) := \left\{x \in \mathbb{R}^n : |\mathbf{p}^\nu(x - x_0)| < r,\ |\mathbf{q}^\nu(x - x_0)| < r\right\},$$
and write $\mathbf{C}_r := \mathbf{C}(0, r, e_n)$, $\mathbf{C} := \mathbf{C}_1$.
- We denote the $(n-1)$-dimensional *disk* centered at $y_0 \in \mathbb{R}^{n-1}$ and of radius r by
$$\mathbf{D}(y_0, r) := \left\{y \in \mathbb{R}^{n-1} : |y - y_0| < r\right\}.$$
We let $\mathbf{D}_r := \mathbf{D}(0, r)$ and $\mathbf{D} := \mathbf{D}(0, 1)$.
- We define
$$\mathbf{e}_E(x, r) := \inf_{\nu \in \mathbb{S}^{n-1}} \frac{1}{r^{n-1}} \int_{\partial^* E \cap B_r(x)} \frac{|\nu_E(y) - \nu|^2}{2}\, d\mathcal{H}^{n-1}(y).$$
We call $\mathbf{e}_E(x, r)$ the *spherical excess*. Note that from the definition it follows that
$$\mathbf{e}_E(x, \lambda r) \le \frac{1}{\lambda^{n-1}} \mathbf{e}_E(x, r)$$
for any $\lambda \in (0, 1)$.
- Let $(u, \rho) \in \mathcal{A}(E)$ be the minimizer of $\mathcal{G}_{\beta,K}(E)$. We define the *normalized Dirichlet energy* at x as
$$D_E(x, r) := \frac{1}{r^{n-1}} \int_{B_r(x)} |\nabla u|^2\, dy.$$

Convention 2.2 (Universal constants). Let $A > 0$ be a positive constant. We say that

- the parameters β, K, Q with $\beta \ge 1$ are *controlled by* A if
$$\beta + K + \frac{1}{K} + Q \le A;$$
- a constant is *universal* if it depends only on the dimension n and on A.

Note that in particular universal constants *do not depend* on the size of the container where the minimization problem is set.

In the following theorem we collect some properties of minimizers. For the proofs we refer the reader to [21].

Theorem 2.3. *Let $E \subset \mathbb{R}^n$ be a set of finite measure. Then we have the following:*

(i) *There exists a unique pair $(u_E, \rho_E) \in \mathcal{A}(E)$ minimizing $\mathcal{G}_{\beta,K}(E)$. Moreover,*

$$u_E + K\rho_E = \mathcal{G}_{\beta,K}(E) \quad \text{in } E,$$

and

$$0 \leq u_E \leq \mathcal{G}_{\beta,K}(E), \quad 0 \leq K\rho_E \leq \mathcal{G}_{\beta,K}(E)\mathbf{1}_E.$$

In particular, $\rho_E \in L^\infty$ with

$$\|\rho_E\|_\infty \leq C(n, \beta, K, 1/|E|).$$

(ii) (Euler–Lagrange equation) *If E is a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$), then*

$$\begin{aligned}\int_{\partial^* E} \operatorname{div}_E \eta \, d\mathcal{H}^{n-1} - Q^2 \int_{\mathbb{R}^n} a_E \big(|\nabla u_E|^2 \operatorname{div} \eta - 2\nabla u_E \cdot (\nabla \eta \nabla u_E)\big) \, dx \\ - Q^2 K \int_{\mathbb{R}^n} \rho_E^2 \operatorname{div} \eta \, dx = 0\end{aligned}$$

for all $\eta \in C_c^1(B_R; \mathbb{R}^n)$ with $\int_E \operatorname{div} \eta \, dx = 0$.

(iii) (Compactness) *Let $K_h, Q_h \in \mathbb{R}$, $\beta_h \geq 1$ and $R_h \geq 1$ be such that*

$$K_h \to K > 0, \quad \beta_h \to \beta \geq 1, \quad R_h \to R \geq 1, \quad Q_h \to Q \geq 0,$$

when $h \to \infty$. For every $h \in \mathbb{N}$ let E_h be a minimizer of ($\mathcal{P}_{\beta_h,K_h,Q_h,R_h}$). Then, up to a non-relabeled subsequence, there exists a set of finite perimeter E such that

$$\lim_{h\to\infty} |E \Delta E_h| = 0.$$

Moreover, E is a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$) and

$$\mathcal{F}_{\beta,K,Q}(E) = \lim_{h\to\infty} \mathcal{F}_{\beta_h,K_h,Q_h}(E_h), \quad \lim_{h\to\infty} P(E_h) = P(E).$$

Let $A > 0$. For the following properties we require that β, K and Q are controlled by A.

(iv) (Boundedness of the normalized Dirichlet energy) *There exists a universal constant $C_e > 0$ such that, if E is a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$), then for all $x \in \overline{B_R}$,*

$$Q^2 D_E(x, r) = \frac{Q^2}{r^{n-1}} \int_{B_r(x)} |\nabla u|^2 \, dx \leq C_e.$$

(v) (Density estimates) *There exist universal constants* C_{o}, $C_{\mathrm{i}} > 0$ *and* $\bar{r} > 0$ *such that, if* E *is a minimizer of* $(\mathcal{P}_{\beta,K,Q,R})$, *then*[2]

$$\frac{1}{C_{\mathrm{i}}} r^{n-1} \le P(E, B_r(x)) \le C_{\mathrm{o}} r^{n-1} \quad \text{for all } x \in \partial E \text{ and } r \in (0, \bar{r})$$

and

$$\frac{1}{C_{\mathrm{i}}} \le \frac{|B_r(x) \cap E|}{|B_r(x)|} \le C_{\mathrm{o}} \quad \text{for all } x \in E \text{ and } r \in (0, \bar{r}).$$

(vi) (Excess improvement) *There exists a universal constant* $C_{\mathrm{dec}} > 0$ *such that for all* $\lambda \in (0, 1/4)$ *there exists* $\varepsilon_{\mathrm{dec}} = \varepsilon_{\mathrm{dec}}(n, A, \lambda) > 0$ *satisfying the following: if* E *is a minimizer of* $(\mathcal{P}_{\beta,K,Q,R})$ *and*

$$x \in \partial E, \quad r + Q^2 D_E(x, r) + \mathbf{e}_E(x, r) \le \varepsilon_{\mathrm{dec}},$$

then

$$Q^2 D_E(x, \lambda r) + \mathbf{e}_E(x, \lambda r) \le C_{\mathrm{dec}} \lambda (\mathbf{e}_E(x, r) + Q^2 D_E(x, r) + r).$$

(vii) (Decay of the Dirichlet energy) *There exists a universal constant* $C_{\mathrm{dir}} > 0$ *such that for all* $\lambda \in (0, 1/2)$ *there exists* $\varepsilon_{\mathrm{dir}} = \varepsilon_{\mathrm{dir}}(n, A, \lambda)$ *satisfying the following: if* E *is a minimizer of* $(\mathcal{P}_{\beta,K,Q,R})$, $x \in \partial E$ *and*

$$r + \mathbf{e}_E(x, r) \le \varepsilon_{\mathrm{dir}},$$

then

$$D_E(x, \lambda r) \le C_{\mathrm{dir}} \lambda (D_E(x, r) + r).$$

Proof. The proofs of (i), (iii), (iv), (v), (vi) and (vii) can be found respectively in [21, Proposition 2.3, Proposition 5.1, Lemma 6.5, Proposition 6.4, Proposition 6.6, Theorem 7.1, Proposition 7.6]. The Euler–Lagrange equation (ii) is given in [21, Corollary 3.3]. We believe it has a sign mistake; see the (classical) computation in [19]. ∎

We now state the ε-regularity theorem.

Theorem 2.4 ([21, Theorem 8.1]). *Let* $A > 0$, $\vartheta \in (0, 1)$, $R \ge 1$, *and let* β, K, Q *be controlled by* A. *There exist constants* $C_{\mathrm{reg}}(n, A, \theta) > 0$ *and* $\varepsilon_{\mathrm{reg}} = \varepsilon_{\mathrm{reg}}(n, A, \theta) > 0$ *such that if* E *is a minimizer of* $(\mathcal{P}_{\beta,K,Q,R})$, $x \in \partial E$, $r > 0$ *and* $\nu \in S^{n-1}$ *are such that*

$$r + Q^2 D_E(x, 2r) + \mathbf{e}_E(x, 2r, \nu) \le \varepsilon_{\mathrm{reg}},$$

[2] Here and in the sequel we will always work with the representative of E such that

$$\partial E = \left\{ x : \frac{|B_r(x) \setminus E|}{|B_r(x)|} \cdot \frac{|B_r(x) \cap E|}{|B_r(x)|} > 0 \text{ for all } r > 0 \right\};$$

see [17, Proposition 12.19].

then there exists a $C^{1,\vartheta}$ *function* $f:\mathbb{R}^{n-1}\to\mathbb{R}$ *with*[3]

$$f(0)=0,\quad |\nabla f(0)-\nu|^2+r^{\vartheta}[\nabla f]^2_{\vartheta/2}\le C_{\mathrm{reg}}(r+Q^2D_E(x,2r)+\mathbf{e}_E(x,2r,\nu)),$$

such that

$$E\cap B_r(x)=\big\{y\in B_r(x):\nu\cdot(y-x)\le f(\mathbf{p}^{\nu}(y-x))\big\}.$$

3. Higher regularity

In this section we improve Theorem 2.4. To be more precise, we deduce the partial $C^{2,\vartheta}$ regularity of minimizers.

The first step is to obtain better regularity for a couple $(u,\rho)\in\mathcal{A}(E)$, where $E\subset\mathbb{R}^n$ is a minimizer of problem ($\mathcal{P}_{\beta,K,Q,R}$): we prove that u is $C^{1,\eta}$-regular up to the boundary of E. We start with some preliminary results.

Notation 3.1. Let $E\subset\mathbb{R}^n$ be such that $\partial E\cap\mathbf{C}(x_0,r)$ is described by the graph of a regular function f.

- If $x\in\mathbb{R}^n$, we write $x=(x',x_n)$, where $x'\in\mathbb{R}^{n-1}$ and $x_n\in\mathbb{R}$.
- We denote by ν_E the outer unit normal to ∂E. Moreover, we extend ν_E at every point in the following way:

$$\nu_E(x',x_n)=\nu_E(x',f(x'))\quad\text{for all } x=(x',x_n)\in\mathbf{C}(x_0,r).$$

- Let u be a solution of

$$-\operatorname{div}(a_E\nabla u)=\rho_E\quad\text{in }\mathcal{D}'(B_r(x_0)),$$

  where

$$\rho_E\in L^{\infty}(B_r(x_0))\quad\text{and}\quad a_E=\beta\mathbf{1}_E+\mathbf{1}_{E^c}.$$

  We denote

$$T_Eu:=\partial_{\nu_E^{\perp}}u+(1+(\beta-1)\mathbf{1}_E)\partial_{\nu_E}u,$$

  where

$$\partial_{\nu_E^{\perp}}u:=\nabla u-(\nabla u\cdot\nu_E)\nu_E\quad\text{and}\quad\partial_{\nu_E}u:=(\nabla u\cdot\nu_E)\nu_E.$$

- We denote by

$$[g]_{x,r}:=\frac{1}{|B_r|}\int_{B_r(x)}g\,dy$$

  the *mean value* of $g\in L^1(B_r(x))$. We simply write $[g]_r:=[g]_{0,r}$.

[3] Here,

$$[\nabla f]_{\vartheta/2}:=\sup_{x\neq y}\frac{|\nabla f(x)-\nabla f(y)|}{|x-y|^{\frac{\vartheta}{2}}}.$$

- We denote the restrictions of a function v to E and E^c by v^+ and v^- respectively:

$$v^+ := v\mathbf{1}_E, \quad v^- := v\mathbf{1}_{E^c}.$$

We are going to use the following lemma.

Lemma 3.2 ([3, Theorem 7.53]). *Let v be a solution of*

$$-\operatorname{div}(a_H \nabla v) = \rho_H \quad \text{in } \mathcal{D}'(B_1(x_0)),$$

where $\rho_H \in L^\infty(B_1(x_0))$ and

$$\begin{aligned} H &:= \{y \in \mathbb{R}^n : (y - x_0) \cdot e_n \le 0\}, \\ a_H &= \beta \mathbf{1}_H + \mathbf{1}_{H^c}. \end{aligned}$$

Then there exist $\gamma \in (0,1)$ and a constant $C_0 = C_0(n, \beta, \|\rho_H\|_\infty) > 0$ such that

$$\begin{aligned} &\int_{B_{\lambda r}(x_0)} |T_H v - [T_H v]_{x_0, \lambda r}|^2 \, dx \\ &\qquad \le C_0 \lambda^{n+2\gamma} \int_{B_r(x_0)} |T_H v - [T_H v]_{x_0, r}|^2 \, dx + C_0 r^{n+1} \end{aligned}$$

for all $\lambda \in (0,1)$ small enough. Note that $T_H v := (\partial_1 v, \ldots, \partial_{n-1} v, (1 + (\beta - 1)\mathbf{1}_H)\partial_n v)$.

We argue similarly to the proof of [3, Theorem 7.53] to show the following lemma.

Lemma 3.3. *Let $H \subset \mathbb{R}^n$ be the half-space $\{x_n > 0\}$. Let $v \in W^{1,2}(B_1)$ be a solution of*

$$-\operatorname{div}(A \nabla v) = \operatorname{div} G \quad \text{in } \mathcal{D}'(B_1). \tag{3.1}$$

Define

$$\begin{aligned} G^+ &:= G\mathbf{1}_H, \quad G^- := G\mathbf{1}_{H^c}, \\ A^+ &:= A\mathbf{1}_H, \quad A^- := A\mathbf{1}_{H^c}, \\ v^+ &:= v\mathbf{1}_H, \quad v^- := v\mathbf{1}_{H^c}. \end{aligned}$$

Suppose $G^+ \in C^{0,\alpha}(H)$, $G^- \in C^{0,\alpha}(H^c)$ and A is an elliptic matrix such that A^+ and A^- have coefficients respectively in $C^{0,\alpha}(B_1 \cap \bar{H})$ and $C^{0,\alpha}(B_1 \cap \overline{H^c})$. Then

$$\begin{aligned} v^+ &\in C^{1,\alpha}(B_{1/2} \cap \bar{H}), \\ v^- &\in C^{1,\alpha}(B_{1/2} \cap \overline{H^c}). \end{aligned}$$

Moreover, there exists a constant $C = C(\|G^+\|_{C^{0,\alpha}}, \|G^-\|_{C^{0,\alpha}}, \|A^+\|_{C^{0,\alpha}}, \|A^-\|_{C^{0,\alpha}}) > 0$ such that

$$[\nabla v^+]_{C^{0,\alpha}(\bar{H} \cap B_{1/2})} \le C \quad \text{and} \quad [\nabla v^-]_{C^{0,\alpha}(\overline{H^c} \cap B_{1/2})} \le C.$$

Proof. Fix $x_0 \in B_{1/2}$, and let r be such that $B_r(x_0) \subset B_1$. We denote by a^+ and a^- the averages of A in $B_r(x_0) \cap H$ and $B_r(x_0) \cap H^c$ respectively. In an analogous way we define g^+ and g^- as the averages of G in $B_r(x_0) \cap H$ and $B_r(x_0) \cap H^c$. For $x \in B_r(x_0)$ we set

$$\bar{A} := \begin{cases} a^+ & \text{if } x_n > 0, \\ a^- & \text{if } x_n < 0, \end{cases} \quad \text{and} \quad \bar{G} := \begin{cases} g^+ & \text{if } x_n > 0, \\ g^- & \text{if } x_n < 0. \end{cases}$$

By the assumptions of the lemma,

$$|A(x) - \bar{A}(x)| \le C r^\alpha \quad \text{and} \quad |G(x) - \bar{G}(x)| \le C r^\alpha. \tag{3.2}$$

Let w be the solution of

$$\begin{cases} -\operatorname{div}(\bar{A}\nabla w) = \operatorname{div} \bar{G} & \text{in } B_r, \\ w = v & \text{on } \partial B_r(x_0). \end{cases}$$

Note that the last equation can be rewritten as

$$\begin{cases} -\operatorname{div}(a^+\nabla w^+) = 0 & \text{in } H \cap B_r(x_0), \\ -\operatorname{div}(a^-\nabla w^-) = 0 & \text{in } H^c \cap B_r(x_0), \\ w^+ = w^- & \text{on } \partial H \cap B_r(x_0), \\ a^+\nabla w^+ \cdot e_n - a^-\nabla w^- \cdot e_n = -(g^+ \cdot e_n - g^- \cdot e_n) & \text{on } \partial H \cap B_r(x_0), \\ w = v & \text{on } \partial B_r(x_0), \end{cases} \tag{3.3}$$

where $w^+ := w\mathbf{1}_{H \cap B_r(x_0)}$, $w^- := w\mathbf{1}_{H^c \cap B_r(x_0)}$. For a function u set

$$\bar{D}_c u(x) = \sum_{i=1}^n \bar{A}_{i,n} \nabla_i u(x) + \bar{G} \cdot e_n, \tag{3.4}$$

$$D_c u(x) = \sum_{i=1}^n A_{i,n} \nabla_i u(x) + G \cdot e_n. \tag{3.5}$$

The reason for such a definition is that $D_c v$ and $\bar{D}_c w$ have no jumps on the boundary thanks to the transmission condition in (3.3).

Denote the tangential part of the gradient by D_τ, that is,

$$D_\tau u = \begin{pmatrix} \partial_{x_1} u \\ \partial_{x_2} u \\ \vdots \\ \partial_{x_{n-1}} u \end{pmatrix}.$$

We are going to estimate the decay of $D_\tau w$ and $\bar{D}_c w$, which will lead to Hölder continuity of $D_\tau v$ and $D_c v$, yielding the desired estimate on ∇v.

Step 1: Tangential derivatives of w. Since both $\bar{A}$ and $\bar{G}$ are constant along the tangential directions, the classical difference quotient method (see, for example, [12, Section 4.3]) gives us that $D_\tau w \in W^{1,2}_{\mathrm{loc}}(B_r(x_0))$ and $\operatorname{div}(\bar{A}\nabla(D_\tau w)) = 0$ in $B_r(x_0)$. Hence, Cacciop-poli's inequality holds:

$$\int_{B_\rho(x)} |\nabla(D_\tau w)|^2\,dy \le C\rho^{-2}\int_{B_{2\rho}(x)} |D_\tau w - (D_\tau w)_{x,2\rho}|^2\,dy \tag{3.6}$$

for all balls $B_{2\rho}(x) \subset B_r(x_0)$ and by De Giorgi's regularity theorem (see, for example, [3, Theorem 7.50]), $D_\tau w$ is Hölder continuous and thus, if $B_{\rho'}(x) \subset B_r(x_0)$,

$$\begin{aligned}\int_{B_\rho(x)} &|D_\tau w - (D_\tau w)_{x,\rho}|^2\,dy\\ &\le c\Big(\frac{\rho}{\rho'}\Big)^{n+2\gamma}\int_{B_{\rho'}(x)} |D_\tau w - (D_\tau w)_{x,\rho'}|^2\,dy \end{aligned}\tag{3.7}$$

for any $\rho \in (0, \rho'/2)$ and

$$\max_{B_{\rho'/2}(x)} |D_\tau w|^2 \le \frac{C}{(\rho')^n}\int_{B_{\rho'}(x)} |D_\tau w|^2\,dy. \tag{3.8}$$

Step 2: Regularity of $\bar{D}_c w$. First let us show that the distributional gradient of $\bar{D}_c w$ is given by the gradient of $\bar{D}_c w$ on the upper half-ball plus the one on the lower, i.e. that there is no contribution on the hyperplane. For that, we need to check that

$$-\int_{B_r(x_0)} \bar{D}_c w \operatorname{div}\varphi\,dx = \int_{B_r(x_0)^+} \nabla\bar{D}_c w\cdot\varphi\,dx + \int_{B_r(x_0)^-} \nabla\bar{D}_c w\cdot\varphi\,dx$$

for any $\varphi \in C^\infty_c(B_r(x_0);\mathbb{R}^n)$. Indeed, if we perform integration by parts on the left-hand side, we get

$$\begin{aligned}-\int_{B_r(x_0)} \bar{D}_c w \operatorname{div}\varphi\,dx &= \int_{B_r(x_0)^+} \nabla\bar{D}_c w\cdot\varphi\,dx + \int_{B_r(x_0)^-} \nabla\bar{D}_c w\cdot\varphi\,dx\\ &\quad - \int_{\partial H\cap B_r(x_0)}\Big(\sum_{i=1}^n a^+_{i,n}\nabla_i w(x) + g^+\cdot e_n\\ &\qquad\qquad - \sum_{i=1}^n a^-_{i,n}\nabla_i w(x) - g^-\cdot e_n\Big)(\varphi\cdot e_n)\,d\mathcal{H}^{n-1}\end{aligned}$$

for any $\varphi \in C^\infty_c(B_r(x_0);\mathbb{R}^n)$ and the last term vanishes thanks to the transmission condition in (3.3). Thus, the distributional gradient of $\bar{D}_c w$ coincides with the pointwise one.

Since $D_\tau(\bar{D}_c w) = \bar{D}_c(D_\tau w) - \bar{G}\cdot e_n$, the tangential derivatives of $\bar{D}_c w$ are in L^2_{loc}. As for the normal derivative, by equation (3.3) and definition (3.4), outside ∂H we have

$$\begin{aligned}\frac{\partial\bar{D}_c w}{\partial x_n}(x) = \nabla_n\Big(\sum_{i=1}^n \bar{A}_{i,n}\nabla_i w(x)\Big) &= \sum_{i=1}^n \nabla_i(\bar{A}_{n,i}\nabla_n w(x))\\ &= \sum_{i=1}^n \nabla_i\Big(-\sum_{j=1}^{n-1}\bar{A}_{j,i}\nabla_j w(x)\Big),\end{aligned}$$

yielding

$$\Big|\frac{\partial \bar{D}_c w}{\partial x_n}(x)\Big| \le C|\nabla D_\tau w(x)|.$$

It implies

$$|\nabla \bar{D}_c w(x)| \le C(|\nabla D_\tau w| + \|\bar{G}\|_{L^\infty}),$$

and thus $\bar{D}_c w$ is in $W^{1,2}_{\mathrm{loc}}$. Now, using Poincaré's inequality and (3.6), we have

$$\begin{aligned}\int_{B_\rho(x)} |\bar{D}_c w - (\bar{D}_c w)_{x,\rho}|^2\, dy &\le C\rho^2 \int_{B_\rho(x)} |\nabla(\bar{D}_c w)|^2\, dy \\ &\le C\rho^2 \int_{B_\rho(x)} |\nabla(D_\tau w)|^2\, dy + C\rho^{n+2} \\ &\le C \int_{B_{2\rho}(x)} |D_\tau w - (D_\tau w)_{x,2\rho}|^2\, dy + C\rho^{n+2}\end{aligned}$$

for any $B_{2\rho}(x) \subset B_r(x_0)$. Remembering (3.7), we obtain

$$\begin{aligned}&\int_{B_\rho(x)} |\bar{D}_c w - (\bar{D}_c w)_{x,\rho}|^2\, dy \\ &\quad\le C\Big(\frac{\rho}{r}\Big)^{n+2\gamma} \int_{B_{r/2}(x)} |D_\tau w - (D_\tau w)_{x,r/2}|^2\, dy + C\rho^{n+2} \\ &\quad\le C\Big(\frac{\rho}{r}\Big)^{n+2\gamma} \int_{B_r(x_0)} |D_\tau w|^2\, dy + C\rho^{n+2} \qquad (3.9)\end{aligned}$$

for any $x \in B_{r/4}(x_0)$, $\rho \le r/4$. Hence, by [3, Theorem 7.51], $\bar{D}_c w$ is Hölder continuous and

$$\max_{B_{r/4}(x_0)} |\bar{D}_c w|^2 \le \frac{C}{r^n} \int_{B_r(x_0)} |\nabla w|^2\, dy + C.$$

Note that from the definition of $\bar{D}_c w$ and using (3.8), we get the same bound holds for the full gradient ∇w:

$$\max_{B_{r/4}(x_0)} |\nabla w|^2 \le \frac{C}{r^n} \int_{B_r(x_0)} |\nabla w|^2\, dy + C. \qquad (3.10)$$

Step 3: Comparing v and w. Subtracting the equation for w from the equation for v we get

$$\begin{aligned}&\int_{B_r(x_0)} \sum_{i,j=1}^n \bar{A}_{i,j}(y)\Big(\frac{\partial v}{\partial y_i} - \frac{\partial w}{\partial y_i}\Big)\frac{\partial \varphi}{\partial y_j}\, dy \\ &\quad= \int_{B_r(x_0)} \sum_{i,j=1}^n (\bar{A}_{i,j}(y) - A_{i,j}(y))\frac{\partial v}{\partial y_i}\frac{\partial \varphi}{\partial y_j}\, dy \\ &\qquad+ \int_{B_r(x_0)} \sum_{i=1}^n (\bar{G}_i - G_i)\frac{\partial \varphi}{\partial y_i}\, dy \qquad (3.11)\end{aligned}$$

for any $\varphi \in W_0^{1,2}(B_r(x_0))$. We test (3.11) with $\varphi = v - w$ to get

$$\begin{aligned}
&\int_{B_r(x_0)} \sum_{i,j=1}^n \bar{A}_{i,j}(y) \frac{\partial(v-w)}{\partial y_i} \frac{\partial(v-w)}{\partial y_j}\, dy \\
&\quad = \int_{B_r(x_0)} \sum_{i,j=1}^n (\bar{A}_{i,j}(y) - A_{i,j}(y)) \frac{\partial v}{\partial y_i} \frac{\partial(v-w)}{\partial y_j}\, dy \\
&\qquad + \int_{B_r(x_0)} \sum_{i=1}^n (\bar{G}_i - G_i) \frac{\partial(v-w)}{\partial y_i}\, dy \\
&\quad \le C\left(\int_{B_r(x_0)} \sum_{i,j=1}^n r^\alpha \left|\frac{\partial v}{\partial y_i}\right| \left|\frac{\partial(v-w)}{\partial y_j}\right| dy + \int_{B_r(x_0)} \sum_{i=1}^n r^\alpha \left|\frac{\partial(v-w)}{\partial y_i}\right| dy \right) \\
&\quad \le C_\varepsilon r^\alpha \left(\int_{B_r(x_0)} r^{2\alpha} |\nabla v|^2\, dy + \int_{B_r(x_0)} r^{2\alpha}\, dy \right) \\
&\qquad + \varepsilon \int_{B_r(x_0)} |v-w|^2\, dy
\end{aligned} \tag{3.12}$$

for any $\varepsilon > 0$, where for the first inequality we used (3.2) and for the second one we used Young's inequality. Since A is elliptic, so is $\bar{A}$, and we can bound the left-hand side of (3.12) from below by Poincaré's inequality:

$$\int_{B_r(x_0)} \sum_{i,j=1}^n \bar{A}_{i,j}(y) \frac{\partial(v-w)}{\partial y_i} \frac{\partial(v-w)}{\partial y_j}\, dy \ge c \int_{B_r(x_0)} |v-w|^2\, dy.$$

Choosing $\varepsilon = \frac{c}{2}$ in (3.12), we get

$$\int_{B_r(x_0)} |\nabla v - \nabla w|^2\, dy \le C r^{2\alpha} \int_{B_r(x_0)} |\nabla v|^2\, dy + C r^{n+2\alpha}, \tag{3.13}$$

which in turn gives us

$$\begin{aligned}
\int_{B_\rho(x_0)} |\nabla v|^2\, dy &\le 2 \int_{B_\rho(x_0)} |\nabla w|^2\, dy + 2 \int_{B_\rho(x_0)} |\nabla v - \nabla w|^2\, dy \\
&\le 2\omega_n \rho^n \sup_{B_{r/4}(x_0)} |\nabla w|^2 + C r^{2\alpha} \int_{B_r(x_0)} |\nabla v|^2\, dy + C r^{n+2\alpha}
\end{aligned}$$

for $\rho \le r/4$. Recalling (3.10) we obtain

$$\begin{aligned}
\int_{B_\rho(x_0)} |\nabla v|^2\, dy &\le C\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |\nabla w|^2\, dy + C\rho^n \\
&\quad + C r^{2\alpha} \int_{B_r(x_0)} |\nabla v|^2\, dy + C r^{n+2\alpha} \\
&\le C\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |\nabla v|^2\, dy + C r^{2\alpha} \int_{B_r(x_0)} |\nabla v|^2\, dy + C r^n.
\end{aligned}$$

Now we can apply [3, Lemma 7.54] and get that there exists $r_0 > 0$ such that for $\rho < r/4 < r_0$,

$$\int_{B_\rho(x_0)} |\nabla v|^2 \, dy \le C\left(\frac{\rho}{r}\right)^{n-\alpha} \int_{B_r(x_0)} |\nabla v|^2 \, dy + C\rho^{n-\alpha}.$$

In particular, for $\rho < r_0$ we have

$$\int_{B_\rho(x_0)} |\nabla v|^2 \, dy \le C\rho^{n-\alpha}, \tag{3.14}$$

where $C = C(\|G^+\|_{C^{0,\alpha}}, \|G^-\|_{C^{0,\alpha}}, \|A^+\|_{C^{0,\alpha}}, \|A^-\|_{C^{0,\alpha}})$. Note that the L^2 norm of ∇v in B_1 is bounded by some constant depending only on L^∞ norms of A and G, as can be seen by testing equation (3.1) with v.

Step 4: Hölder continuity of ∇v. We show local Hölder continuity of $D_c v$ and $D_\tau v$, Hölder continuity of ∇v in $B_{1/2} \cap \bar{H}$ and in $B_{1/2} \cap \overline{H^c}$ follows immediately.

Take $\rho < r_0$, where r_0 is from the previous step. Let d be any real number. Using the definitions (3.4) and (3.5), we get

$$\begin{aligned}
&\int_{B_\rho(x_0)} |D_c v - d|^2 \, dy \\
&\quad = \int_{B_\rho(x_0)} \left| \bar{D}_c v - d + \sum_{i=1}^{n} (A_{i,n} - \bar{A}_{i,n}) \nabla_i v + (G - \bar{G}) \cdot e_n \right|^2 dy \\
&\quad \le 2 \int_{B_\rho(x_0)} |\bar{D}_c v - d|^2 \, dy + 4 \int_{B_\rho(x_0)} \left| \sum_{i=1}^{n} (A_{i,n} - \bar{A}_{i,n}) \nabla_i v \right|^2 dy \\
&\qquad + 4 \int_{B_\rho(x_0)} |(G - \bar{G}) \cdot e_n|^2 \, dy \\
&\quad \le 2 \int_{B_\rho(x_0)} \left| \bar{D}_c w - d + \sum_{i=1}^{n} \bar{A}_{i,n} (\nabla_i v - \nabla_i w) \right|^2 dy \\
&\qquad + C r^{2\alpha} \int_{B_\rho(x_0)} |\nabla v|^2 \, dy + C r^{n+2\alpha} \\
&\quad \le 4 \int_{B_\rho(x_0)} |\bar{D}_c w - d|^2 \, dy + C r^{n+2\alpha},
\end{aligned} \tag{3.15}$$

where we used inequalities (3.2) for the second-to-last inequality, and inequalities (3.14) and (3.13) for the last inequality. Thus, we have for $\rho < r/4$, $r < r_0$,

$$\begin{aligned}
\int_{B_\rho(x_0)} |D_c v - (D_c v)_{x_0,\rho}|^2 \, dy &\le \int_{B_\rho(x_0)} |D_c v - (\bar{D}_c w)_{x_0,\rho}|^2 \, dy \\
&\le 4 \int_{B_\rho(x_0)} |\bar{D}_c w - (\bar{D}_c w)_{x_0,\rho}|^2 \, dy + C r^{n+\alpha} \\
&\le C\left(\frac{\rho}{r}\right)^{n+2\gamma} \int_{B_r(x_0)} |D_\tau w|^2 \, dy + C r^{n+\alpha},
\end{aligned} \tag{3.16}$$

where we used the fact that $\int_\Omega |f(x)-t|^2\,dx$ is minimized by $t^* = \fint_\Omega f$ for the first inequality, the inequality (3.15) with $d = (\bar{D}_c w)_{x_0,\rho}$ for the second inequality, and (3.9) for the last inequality. Similarly, using (3.7) instead of (3.9) we get

$$\int_{B_\rho(x_0)} |D_\tau v - (D_\tau v)_{x_0,\rho}|^2\,dy \le C\left(\frac{\rho}{r}\right)^{n+2\gamma} \int_{B_r(x_0)} |D_\tau w|^2\,dy + Cr^{n+\alpha}. \tag{3.17}$$

Applying [3, Lemma 7.54] to (3.16) and (3.17), we deduce that $D_c v$ and $D_\tau v$ are Hölder continuous by [3, Theorem 7.51]. ■

Lemma 3.4. *Given a minimizer E of $(\mathcal{P}_{\beta,K,Q,R})$, let $(u,\rho) \in \mathcal{A}(E)$ be the minimizing pair of $\mathcal{G}_{\beta,K}(E)$. Let $A > 0$ and let β, K, Q be controlled by A. Let $f \in C^{1,\vartheta}(\mathbf{D}(x_0', r))$ and suppose that*

$$E \cap \mathbf{C}(x_0, r) = \left\{x = (x', x_n) \in \mathbf{D}(x_0', r) \times \mathbb{R} : x_n < f(x')\right\} \cap \mathbf{C}(x_0, r)$$

for some $r > 0$. Then for every $\gamma \in (0,1)$ there exist $C = C(n, A, \vartheta, \gamma, \|f\|_{C^{1,\vartheta}}) > 0$ such that the following inequality holds true:

$$Q^2 \int_{B_{\tilde{r}}(x_0)} |\nabla u|^2\,dx \le C\tilde{r}^{n-\gamma} \tag{3.18}$$

for every $\tilde{r} \le r$.

Proof. Let us first show that it is sufficient to prove (3.18) for $\tilde{r} \le \bar{r}$ for some small $\bar{r} = \bar{r}(n, A, \vartheta, \gamma, \|f\|_{C^{1,\vartheta}}) > 0$. Indeed, suppose $\tilde{r} \in (\bar{r}, r)$. Then by Theorem 2.3 (iv) there exists a universal constant C such that

$$Q^2 \int_{B_{\tilde{r}}(x_0)} |\nabla u|^2\,dx \le C\tilde{r}^{n-1} = C\tilde{r}^{n-\gamma}\tilde{r}^{\gamma-1} \le (C\bar{r}^{\gamma-1})\tilde{r}^{n-\gamma}.$$

Thus we only need to prove (3.18) for small $\tilde{r}$.

Fix $\gamma \in (0,1)$. Choose $\lambda = \lambda(\gamma, n, A) \in (0, 1/4)$ such that

$$(1 + C_{\mathrm{dec}})\lambda \le \lambda^{1-\gamma},$$

where $C_{\mathrm{dec}} = C_{\mathrm{dec}}(n, A)$ is as in Theorem 2.3 (vi). Let $s = s(\lambda, n, A) < \frac{1}{2}$ be such that

$$C_{\mathrm{dir}}(C_{\mathrm{e}} + 1)s \le \frac{\varepsilon_{\mathrm{dec}}(\lambda)}{2},$$

where $\varepsilon_{\mathrm{dec}} = \varepsilon_{\mathrm{dec}}(n, A, \lambda)$, $C_{\mathrm{dir}} = C_{\mathrm{dir}}(n, A)$ and $C_{\mathrm{e}} = C_{\mathrm{e}}(n, A)$ are as in Theorem 2.4 and Theorem 2.3 (vii), (iv). Define

$$\varepsilon(\lambda, n, A) := \min\left\{s^{n-1}\frac{\varepsilon_{\mathrm{dec}}(\lambda)}{2}, \varepsilon_{\mathrm{dir}}(s)\right\}.$$

Since $\partial E \cap \mathbf{C}(x_0, r)$ is regular, we can take a radius $0 < \bar{r} = \bar{r}(\lambda, n, A, \|f\|_{C^{1,\vartheta}}) < \min(r, 1, \frac{1}{Q^2})$ such that

$$\bar{r} + \mathbf{e}_E(x_0, \bar{r}) \le \varepsilon.$$

Now a straightforward application of Theorem 2.3 (iv), (vi), (vii) gives us (3.18). ■

Proposition 3.5. *Let E be a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$), let $(u,\rho) \in \mathcal{A}(E)$ be the minimizing pair of $\mathcal{G}_{\beta,K}(E)$, $x_0 \in \partial E$, $\nu_E(x_0) = e_n$ and $f \in C^{1,\vartheta}(\mathbf{D}(x_0', r))$. Let $A > 0$ and let β, K, Q be controlled by A. Suppose that*

$$E \cap \mathbf{C}(x_0, r) = \{x = (x', x_n) \in \mathbf{D}(x_0', r) \times \mathbb{R} : x_n < f(x')\} \cap \mathbf{C}(x_0, r)$$

for some $0 < r \leq 1$. Then there exist $\alpha = \alpha(\vartheta) \in (0,1)$ and a constant $C = C(n, A, \vartheta, \|\rho\|_\infty, \|f\|_{C^{1,\vartheta}}) > 0$ such that

$$\begin{aligned} Q^2 \int_{B_{\lambda r}(x_0)} &|T_E u - [T_E u]_{x_0,\lambda r}|^2 \, dx \\ &\leq C Q^2 \lambda^{n+2\alpha} \int_{B_r(x_0)} |T_E u - [T_E u]_{x_0,r}|^2 \, dx + C r^{n+\alpha} \end{aligned} \tag{3.19}$$

for all $\lambda > 0$ small enough.

Proof. Without loss of generality assume $x_0 = 0$. Let $\lambda \in (0, 1/2)$ be such that Lemma 3.2 holds and let v be the solution of

$$\begin{cases} -\operatorname{div}(a_H \nabla v) = \rho & \text{in } B_{r/2}, \\ v = u & \text{on } \partial B_{r/2}, \end{cases}$$

where H is the half-space $\{x = (x', x_n) : x_n < 0\}$. In particular, $w = v - u \in W_0^{1,2}(B_{r/2})$ and

$$-\operatorname{div}(a_H \nabla w) = -\operatorname{div}((a_E - a_H)\nabla u). \tag{3.20}$$

Since $[T_E g]_s$ minimizes the functional $m \mapsto \int_{B_s} |T_E g - m|^2 \, dx$, we have

$$\begin{aligned} \int_{B_{\lambda r}} |T_E u - [T_E u]_{\lambda r}|^2 \, dx &\leq \int_{B_{\lambda r}} |T_E u - [T_H u]_{\lambda r}|^2 \, dx \\ &\leq 2\Big(\int_{B_{\lambda r}} |T_H u - [T_H u]_{\lambda r}|^2 \, dx \\ &\qquad + \int_{B_{\lambda r}} |T_E u - T_H u|^2 \, dx \Big). \end{aligned} \tag{3.21}$$

Now we want to estimate the first term on the right-hand side of (3.21). Notice that, since $u = v - w$, by linearity of T_H we have

$$|T_H u - [T_H u]_{\lambda r}|^2 \leq 2\big(|T_H v - [T_H v]_{\lambda r}|^2 + |T_H w - [T_H w]_{\lambda r}|^2\big).$$

Hence, integrating the above inequality on $B_{\lambda r}$ we obtain

$$\begin{aligned} \int_{B_{\lambda r}} &|T_H u - [T_H u]_{\lambda r}|^2 \, dx \\ &\leq 2\Big(\int_{B_{\lambda r}} |T_H v - [T_H v]_{\lambda r}|^2 \, dx + \int_{B_{\lambda r}} |T_H w - [T_H w]_{\lambda r}|^2 \, dx \Big) \end{aligned}$$

$$\le 2\Big(\int_{B_{\lambda r}} |T_H v - [T_H v]_{\lambda r}|^2\,dx + \int_{B_{\lambda r}} |T_H w|^2\,dx\Big)$$
$$\le C\Big(\int_{B_{\lambda r}} |\nabla w|^2\,dx + \int_{B_{\lambda r}} |T_H v - [T_H v]_{\lambda r}|^2\,dx\Big). \tag{3.22}$$

Now we estimate the second term on the right-hand side of (3.21):

$$\begin{aligned}|T_E u - T_H u| &= \big|\nabla u - (\nabla u\cdot\nu_E)\nu_E + (1+(\beta-1)\mathbf{1}_E)(\nabla u\cdot\nu_E)\nu_E \\ &\qquad - \big(\nabla u - (\nabla u\cdot e_n)e_n + (1+(\beta-1)\mathbf{1}_H)(\nabla u\cdot e_n)e_n\big)\big| \\ &= \big|(\nabla u\cdot e_n)e_n - (\nabla u\cdot\nu_E)\nu_E + (1+(\beta-1)\mathbf{1}_E)(\nabla u\cdot\nu_E)\nu_E \\ &\qquad - \big((1+(\beta-1)\mathbf{1}_H)(\nabla u\cdot e_n)e_n\big)\big| \\ &\le (1+\beta)|(\nabla u\cdot e_n)e_n - (\nabla u\cdot\nu_E)\nu_E| \\ &\quad + \big|\big((1+(\beta-1)\mathbf{1}_E) - (1+(\beta-1)\mathbf{1}_H)\big)(\nabla u\cdot e_n)e_n\big| \\ &= (1+\beta)\big|\big((\nabla u\cdot e_n) - (\nabla u\cdot\nu_E)\big)e_n + (\nabla u\cdot\nu_E)(e_n-\nu_E)\big| \\ &\quad + (\beta-1)\mathbf{1}_{E\Delta H}|\nabla u\cdot e_n| \\ &\le \big(2(1+\beta)|\nu_E - e_n| + (\beta-1)\mathbf{1}_{E\Delta H}\big)|\nabla u|.\end{aligned}$$

Therefore,

$$\int_{B_{\lambda r}} |T_E u - T_H u|^2\,dx$$
$$\le C\Big(\int_{B_{\lambda r}} |\nabla u|^2|\nu_E - e_n|^2\,dx + \int_{B_{\lambda r}} |\nabla u|^2\mathbf{1}_{E\Delta H}\,dx\Big). \tag{3.23}$$

Combining (3.21), (3.22) and (3.23) we obtain

$$\begin{aligned}\int_{B_{\lambda r}} |T_E u - [T_E u]_{\lambda r}|^2\,dx &\le C\int_{B_{\lambda r}} |T_H v - [T_H v]_{\lambda r}|^2\,dx + C\int_{B_{r/2}} |\nabla w|^2\,dx \\ &\quad + C\Big(\int_{B_{\lambda r}} |\nabla u|^2|\nu_E - e_n|^2\,dx + \int_{B_{\lambda r}} |\nabla u|^2\mathbf{1}_{E\Delta H}\,dx\Big).\end{aligned}$$

By Lemma 3.2 we have

$$\int_{B_{\lambda r}} |T_H v - [T_H v]_{\lambda r}|^2\,dx \le C\lambda^{n+2\gamma}\int_{B_{r/2}} |T_H v - [T_H v]_{r/2}|^2\,dx + Cr^{n+1}.$$

Arguing as above, one can easily see that

$$\begin{aligned}\int_{B_{r/2}} |T_H v - [T_H v]_{r/2}|^2\,dx &\le C\int_{B_{r/2}} |T_E u - [T_E u]_{r/2}|^2\,dx + C\int_{B_{r/2}} |\nabla w|^2\,dx \\ &\quad + C\Big(\int_{B_{r/2}} |\nabla v|^2|\nu_E - e_n|^2\,dx + \int_{B_{r/2}} |\nabla v|^2\mathbf{1}_{E\Delta H}\,dx\Big).\end{aligned}$$

We note that

$$\begin{aligned}\int_{B_{r/2}} &|\nabla v|^2|\nu_E - e_n|^2\,dx + \int_{B_{r/2}} |\nabla v|^2 \mathbf{1}_{E\Delta H}\,dx\\ &\le 2\int_{B_{r/2}} |\nabla w|^2|\nu_E - e_n|^2\,dx + \int_{B_{r/2}} |\nabla u|^2|\nu_E - e_n|^2\,dx\\ &\quad + 2\int_{B_{r/2}} |\nabla w|^2\mathbf{1}_{E\Delta H}\,dx + 2\int_{B_{r/2}} |\nabla u|^2\mathbf{1}_{E\Delta H}\,dx\\ &\le C\int_{B_{r/2}} |\nabla w|^2\,dx + \int_{B_{r/2}} |\nabla u|^2|\nu_E - e_n|^2\,dx\\ &\quad + 2\int_{B_{r/2}} |\nabla u|^2\mathbf{1}_{E\Delta H}\,dx.\end{aligned}$$

Bringing it all together, we get

$$\begin{aligned}\int_{B_{\lambda r}} |T_E u - [T_E u]_{\lambda r}|^2\,dx \le C\lambda^{n+2\gamma}&\int_{B_{r/2}} |T_E u - [T_E u]_{r/2}|^2\,dx + Cr^{n+1}\\ &+ C\int_{B_{r/2}} |\nabla u|^2|\nu_E - e_n|^2\,dx\\ &+ C\int_{B_{r/2}} |\nabla u|^2\mathbf{1}_{E\Delta H}\,dx + C\int_{B_{r/2}} |\nabla w|^2\,dx. \qquad (3.24)\end{aligned}$$

We need to estimate the last three terms on the right-hand side of the above inequality. Since E is parametrized by $f \in C^{1,\vartheta}(\mathbf{D}_r)$ in the cylinder $\mathbf{C}(x_0, r)$, there exists a constant $C > 0$ such that

$$\frac{|(E\Delta H)\cap B_r|}{|B_r|} \le Cr^{\vartheta}. \qquad (3.25)$$

We will estimate the last two terms together. Testing (3.20) with w, we deduce

$$\int_{B_{r/2}} |\nabla w|^2\,dx \le \int_{B_{r/2}} a_H|\nabla w|^2\,dx = \int_{B_{r/2}} (a_E - a_H)\nabla u\cdot\nabla w\,dx. \qquad (3.26)$$

Applying the inequality of arithmetic and geometric means in (3.26) we obtain

$$\int_{B_{r/2}} |\nabla w|^2\,dx \le \frac{1}{2}\int_{B_{r/2}} |\nabla w|^2\,dx + \frac{1}{2}\int_{B_{r/2}} (a_E - a_H)^2|\nabla u|^2\,dx,$$

which yields

$$\int_{B_{r/2}} |\nabla w|^2\,dx \le C\int_{(E\Delta H)\cap B_{r/2}} |\nabla u|^2\,dx. \qquad (3.27)$$

By higher integrability [21, Lemma 6.1], there exists $p > 1$ such that

$$\left(\frac{1}{|B_{r/2}|}\int_{B_{r/2}} |\nabla u|^{2p}\,dx\right)^{\frac{1}{p}} \le C\frac{1}{|B_r|}\int_{B_r} |\nabla u|^2\,dx + Cr^{n+2}\|\rho\|_\infty^2. \qquad (3.28)$$

Hence, by exploiting Hölder inequality, (3.25) and (3.28) we have

$$\begin{aligned}\int_{(E\Delta H)\cap B_{r/2}} |\nabla u|^2\,dx &\le |(E\Delta H)\cap B_{r/2}|^{1-\frac{1}{p}}\Big(\int_{B_{r/2}} |\nabla u|^{2p}\,dx\Big)^{\frac{1}{p}}\\ &\le C|B_r|\Big(\frac{|(E\Delta H)\cap B_r|}{|B_r|}\Big)^{1-\frac{1}{p}}\Big(\frac{1}{|B_{r/2}|}\int_{B_{r/2}} |\nabla u|^{2p}\,dx\Big)^{\frac{1}{p}}\\ &\le C r^{\vartheta(1-\frac{1}{p})}\Big\{\int_{B_r} |\nabla u|^2\,dx + r^{n+2}\|\rho\|_\infty^2\Big\}. \end{aligned} \tag{3.29}$$

Therefore, (3.27) together with (3.29) (recall $r < 1$) yields

$$\begin{aligned}&\int_{B_{r/2}} |\nabla u|^2 \mathbf{1}_{E\Delta H}\,dx + \int_{B_{r/2}} |\nabla w|^2\,dx\\ &\quad\le C\Big\{r^{\vartheta(1-\frac{1}{p})}\int_{B_r} |\nabla u|^2\,dx + r^{n+2}\|\rho\|_\infty^2\Big\}. \end{aligned} \tag{3.30}$$

On the other hand, by Lemma 3.4 we have

$$Q^2\int_{B_{r/2}} |\nabla u|^2\,dx \le C\Big(\frac{r}{2}\Big)^{n-\gamma}. \tag{3.31}$$

Hence, combining (3.30) and (3.31), we obtain

$$Q^2\Big(\int_{B_{r/2}} |\nabla u|^2\mathbf{1}_{E\Delta H}\,dx + \int_{B_{r/2}} |\nabla w|^2\Big) \le C\{r^{\vartheta(1-\frac{1}{p})+n-\gamma} + r^{n+2}\|\rho\|_\infty^2\}.$$

Finally, we estimate the second term in (3.24). Notice that

$$\begin{aligned}\int_{B_{r/2}} |\nabla u|^2|\nu_E - e_n|^2\,dx &= \int_{B_{r/2}} |\nabla u(x',x_n)|^2|\nu_E(x',x_n) - e_n|^2\,dx\\ &= \int_{B_{r/2}} |\nabla u|^2|\nu_E(x',f(x')) - e_n|^2\,dx.\end{aligned}$$

Since $\sqrt{1+t} \le 1 + \frac{t}{2}$ for every $t > 0$,

$$\begin{aligned}|\nu_E(x',f(x')) - e_n|^2 &= 2 - \frac{2}{\sqrt{1+|\nabla f(x')|^2}}\\ &\le 2\Big(\frac{\sqrt{1+|\nabla f(x')|^2}-1}{\sqrt{1+|\nabla f(x')|^2}}\Big) \le |\nabla f(x')|^2. \end{aligned} \tag{3.32}$$

Thanks to (3.31) and (3.32), and using that ∇f is ϑ-Hölder, we deduce

$$Q^2\int_{B_{r/2}} |\nabla u|^2|\nu_E - e_n|^2\,dx \le C r^{n+2\vartheta-\gamma}. \tag{3.33}$$

Let

$$\alpha := \min\{\gamma, \vartheta(1-1/p)-\gamma, 2\vartheta-\gamma\}.$$

Therefore, by multiplying (3.24) and (3.30) with Q^2 we have that (3.33) implies (3.19). ■

We are now ready to prove that u is regular up to the boundary. Recall that $u^+ = u\mathbf{1}_E$ and $u^- = u\mathbf{1}_{E^c}$.

Theorem 3.6. *Let E be a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, let $(u,\rho) \in \mathcal{A}(E)$ be the minimizing pair of $\mathcal{G}_{\beta,K}(E)$, $x_0 \in \partial E$, $\nu_E(x_0) = e_n$ and $f \in C^{1,\vartheta}(\mathbf{D}(x_0', r))$. Let $A > 0$ and let β, K, Q be controlled by A. Suppose*

$$E \cap \mathbf{C}(x_0, r) = \{x = (x', x_n) \in \mathbf{D}(x_0', r) \times \mathbb{R} : x_n < f(x')\} \cap \mathbf{C}(x_0, r)$$

for some $0 < r \le 1$. Then there exists $\eta = \eta(\vartheta) \in (0,1)$ such that $u^+ \in C^{1,\eta}(\bar{E} \cap \mathbf{C}_{r/2}(x_0))$ and $u^- \in C^{1,\eta}(\bar{E}^c \cap \mathbf{C}_{r/2}(x_0))$. Furthermore, let $A > 0$ and let β, K, Q be controlled by A. Then there exists a constant $C = C(n, A, \vartheta, \|\rho\|_\infty, \|f\|_{C^{1,\vartheta}}) > 0$ such that

$$\|Qu^+\|_{C^{1,\eta}(\bar{E}\cap \mathbf{C}_{r/2}(x_0))} \le C \quad \textit{and} \quad \|Qu^-\|_{C^{1,\eta}(\bar{E}^c\cap \mathbf{C}_{r/2}(x_0))} \le C. \tag{3.34}$$

Proof. Let $u_Q := Qu$. By Proposition 3.5 there exists $C = C(n, A, \vartheta, \|\rho\|_\infty, \|f\|_{C^{1,\vartheta}}) > 0$ such that

$$\int_{B_{\lambda r}(x_0)} |T_E u_Q - [T_E u_Q]_{x_0,\lambda r}|^2 \, dx \le C\lambda^{n+2\alpha} \int_{B_r(x_0)} |T_E u_Q - [T_E u_Q]_{x_0,r}|^2 \, dx + C r^{n+\alpha},$$

where $\alpha \in (0,1)$ is as in Proposition 3.5. Therefore, [3, Lemma 7.54] implies that there exists a constant $C = C(n, A, \vartheta, \|\rho\|_\infty, \|f\|_{C^{1,\vartheta}}) > 0$ such that

$$\frac{1}{|B_r|} \int_{B_r(x_0)} |T_E u_Q - [T_E u_Q]_{x,r}|^2 \, dy \le C\left(\frac{r}{R}\right)^{2\eta}$$

for some $\eta = \eta(\vartheta) \in (0,1)$. Hence, by [3, Theorem 7.51], recalling the definition of T_E, we get $u_Q\mathbf{1}_E \in C^{1,\eta}(\bar{E} \cap \mathbf{C}_{r/2}(x_0))$ and $u_Q\mathbf{1}_{E^c} \in C^{1,\eta}(\bar{E}^c \cap \mathbf{C}_{r/2}(x_0))$ and (3.34). ■

In the next proposition we rewrite the Euler–Lagrange equation (see Theorem 2.3 (ii)) in a more convenient form by exploiting the regularity of ∂E.

Proposition 3.7 (Euler–Lagrange equation). *Let E be a minimizer for $(\mathcal{P}_{\beta,K,Q,R})$ and $(u,\rho) \in \mathcal{A}(E)$, $x_0 \in \partial E$, $\nu_E(x_0) = e_n$. Let $A > 0$ and let β, K, Q be controlled by A. Assume that $f \in C^{1,\vartheta}(\mathbf{D}(x_0', r))$ and*

$$E \cap \mathbf{C}(x_0, r) = \{x = (x', x_n) \in \mathbf{D}(x_0', r) \times \mathbb{R} : x_n < f(x')\} \cap \mathbf{C}(x_0, r)$$

for some $0 < r \le 1$. *Then there exists a constant* $C = C(n, A, \vartheta, \|\rho\|_\infty, \|f\|_{C^{1,\vartheta}}) > 0$ *such that*

$$
\begin{aligned}
-\operatorname{div}&\Big(\frac{\nabla f(x')}{\sqrt{1+|\nabla f(x')|^2}}\Big)\\
&= Q^2(\beta|\nabla u^+|^2 - |\nabla u^-|^2 + K\rho^2)(x', f(x'))\\
&\quad - 2Q^2(\beta\partial_n u^+\nabla u^+ - \partial_n u^-\nabla u^-)(x', f(x'))\cdot(-\nabla f(x'), 1) + C
\end{aligned}
\tag{3.35}
$$

weakly in $\mathbf{D}(x_0', r)$.

Proof. Let $E \subset \mathbb{R}^n$ be a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$ and let $(u,\rho) \in \mathcal{A}(E)$.

Notice that $E \cap \mathbf{C}(x_0, r)$ is an open set of $\mathbb{R}^n$. Moreover, by an approximation argument, we can integrate the following identity over $E \cap \mathbf{C}(x_0, r)$:

$$
\begin{aligned}
|\nabla u^+|^2 \operatorname{div}\eta &= \operatorname{div}(|\nabla u^+|^2\eta) - \nabla|\nabla u^+|^2\cdot\eta\\
&= \operatorname{div}(|\nabla u^+|^2\eta) - 2\operatorname{div}(\nabla u^+(\nabla u^+\cdot\eta)) + 2\Delta u^+\nabla u^+\cdot\eta\\
&\quad + 2\nabla u^+\cdot(\nabla\eta\nabla u^+)
\end{aligned}
$$

for every $\eta \in C_c^\infty(\mathbf{C}(x_0, r), \mathbb{R}^n)$. Therefore,

$$
\begin{aligned}
\int_{E\cap\mathbf{C}(x_0,r)} &\big(|\nabla u^+|^2\operatorname{div}\eta - 2\nabla u^+\cdot(\nabla\eta\nabla u^+)\big)\,dx\\
&= \int_{E\cap\mathbf{C}(x_0,r)} \operatorname{div}(|\nabla u^+|^2\eta)\,dx - \int_{E\cap\mathbf{C}(x_0,r)} 2\operatorname{div}(\nabla u^+(\nabla u^+\cdot\eta))\,dx\\
&\quad + \int_{E\cap\mathbf{C}(x_0,r)} 2\Delta u^+\nabla u^+\cdot\eta\,dx.
\end{aligned}
\tag{3.36}
$$

On the other hand, since $(u,\rho) \in \mathcal{A}(E)$, we have

$$-\beta\Delta u^+ = \rho \quad \text{in } \mathcal{D}'(E\cap\mathbf{C}(x_0,r)).$$

Moreover, by Theorem 2.3 (i) we deduce

$$\nabla u^+ = -K\nabla\rho \quad \text{in } E\cap\mathbf{C}(x_0,r).$$

Then, by multiplying equation (3.36) by β, we have

$$
\begin{aligned}
\int_{E\cap\mathbf{C}(x_0,r)} &\beta\big(|\nabla u^+|^2\operatorname{div}\eta - 2\nabla u^+\cdot(\nabla\eta\nabla u^+)\big)\,dx\\
&= \int_{E\cap\mathbf{C}(x_0,r)} \beta\operatorname{div}(|\nabla u^+|^2\eta)\,dx - \int_{E\cap\mathbf{C}(x_0,r)} 2\beta\operatorname{div}(\nabla u^+(\nabla u^+\cdot\eta))\,dx\\
&\quad + K\int_{E\cap\mathbf{C}(x_0,r)} 2\rho\nabla\rho\cdot\eta\,dx.
\end{aligned}
\tag{3.37}
$$

Integrating by parts the first and the second term on the right-hand side of (3.37), we can write

$$\begin{aligned}\int_{E\cap \mathbf{C}(x_0,r)} &\beta\big(|\nabla u^+|^2 \operatorname{div}\eta - 2\nabla u^+\cdot(\nabla\eta\nabla u^+)\big)\,dx\\ &= \int_{\partial E\cap \mathbf{C}(x_0,r)} \beta|\nabla u^+|^2\eta\cdot\nu_E\,d\mathcal{H}^{n-1}\\ &\quad - \int_{\partial E\cap \mathbf{C}(x_0,r)} 2\beta(\nabla u^+\cdot\eta)(\nabla u^+\cdot\nu_E)\,d\mathcal{H}^{n-1}\\ &\quad + K\int_{E\cap \mathbf{C}(x_0,r)} 2\rho\nabla\rho\cdot\eta\,dx. \end{aligned}\tag{3.38}$$

Arguing as above, one can also prove

$$\begin{aligned}\int_{E^c\cap \mathbf{C}(x_0,r)} &\big(|\nabla u^-|^2 \operatorname{div}\eta - 2\nabla u^-\cdot(\nabla\eta\nabla u^-)\big)\,dx\\ &= \int_{E^c\cap \mathbf{C}(x_0,r)} \operatorname{div}(|\nabla u^-|^2\eta)\,dx - \int_{E^c\cap \mathbf{C}(x_0,r)} 2\operatorname{div}(\nabla u^-(\nabla u^-\cdot\eta))\,dx. \end{aligned}\tag{3.39}$$

Integrating by parts the right-hand side of (3.39), we can write

$$\begin{aligned}\int_{E^c\cap \mathbf{C}(x_0,r)} &\big(|\nabla u^-|^2 \operatorname{div}\eta - 2\nabla u^-\cdot(\nabla\eta\nabla u^-)\big)\,dx\\ &= -\int_{\partial E\cap \mathbf{C}(x_0,r)} |\nabla u^-|^2\eta\cdot\nu_E\,d\mathcal{H}^{n-1}\\ &\quad + \int_{\partial E\cap \mathbf{C}(x_0,r)} 2(\nabla u^-\cdot\eta)(\nabla u^-\cdot\nu_E)\,d\mathcal{H}^{n-1}. \end{aligned}\tag{3.40}$$

Therefore, combining (3.38) and (3.40), we get

$$\begin{aligned}\int_{\mathbb{R}^n} &a_E\big(\operatorname{div}\eta|\nabla u|^2 - 2\nabla u\cdot(\nabla\eta\nabla u)\big)\,dx\\ &= \int_{\partial E}(\beta|\nabla u^+|^2 - |\nabla u^-|^2)\eta\cdot\nu_E\,d\mathcal{H}^{n-1}\\ &\quad - \int_{\partial E\cap \mathbf{C}(x_0,r)} 2\big(\beta(\nabla u^+\cdot\eta)(\nabla u^+\cdot\nu_E) - (\nabla u^-\cdot\eta)(\nabla u^-\cdot\nu_E)\big)\,d\mathcal{H}^{n-1}\\ &\quad + K\int_{E\cap \mathbf{C}(x_0,r)} 2\rho\nabla\rho\cdot\eta\,dx. \end{aligned}\tag{3.41}$$

Notice that the following identity holds true:

$$\begin{aligned}K\int_{\mathbb{R}^n}\rho^2\operatorname{div}\eta\,dx &= K\int_{E\cap \mathbf{C}(x_0,r)}\operatorname{div}(\rho^2\eta)\,dx - K\int_{E\cap \mathbf{C}(x_0,r)} 2\rho\nabla\rho\cdot\eta\,dx\\ &= K\int_{\partial E\cap \mathbf{C}(x_0,r)}\rho^2\eta\cdot\nu_E\,d\mathcal{H}^{n-1}\\ &\quad - K\int_{E\cap \mathbf{C}(x_0,r)} 2\rho\nabla\rho\cdot\eta\,dx. \end{aligned}\tag{3.42}$$

Combining the Euler–Lagrange equation of Theorem 2.3 (ii), (3.41) and (3.42), we find

$$\begin{aligned}\int_{\partial E} \operatorname{div}_E \eta \, d\mathcal{H}^{n-1} &= Q^2 \int_{\partial E} (\beta|\nabla u^+|^2 - |\nabla u^-|^2 + K\rho^2)\eta \cdot \nu_E \, d\mathcal{H}^{n-1} \\ &\quad - 2Q^2 \int_{\partial E} \beta(\eta \cdot \nabla u^+)(\nabla u^+ \cdot \nu_E) - (\eta \cdot \nabla u^-)(\nabla u^- \cdot \nu_E) \, d\mathcal{H}^{n-1} \end{aligned} \tag{3.43}$$

for every $\eta \in C_c^1(B_r(x_0), \mathbb{R}^n)$ with $\int_E \operatorname{div} \eta \, dx = 0$.

Now we are ready to prove (3.35). The tangential divergence of η on ∂E is

$$\operatorname{div}_E \eta := \operatorname{div} \eta - \sum_{i,j=1}^{n} (\nu_E)_i (\nu_E)_j \partial_j \eta_i \quad \text{on } \partial E, \tag{3.44}$$

where $\nu_E \colon \partial E \to \mathbb{S}^{n-1}$ is the normal vector to ∂E:

$$\nu_E := \frac{1}{\sqrt{1 + |\nabla f|^2}}(-\nabla f, 1).$$

Let $\eta := (0, \dots, 0, \eta_n)$; then by (3.44) we have

$$\operatorname{div}_E \eta := \partial_n \eta_n + \frac{1}{1 + |\nabla f|^2}\left\{\sum_{j=1}^{n-1} \partial_j \eta_n \partial_j f - \partial_n \eta_n\right\} \quad \text{on } \partial E. \tag{3.45}$$

Choose $\eta_n(x) := \varphi(\mathbf{p}x)s(x_n)$, where $\varphi \in C_c^1(\mathbf{D}(x_0', r))$ is such that $\int_{\mathbf{D}(x_0',r)} \varphi = 0$ and $s \colon (-1, 1) \to \mathbb{R}^n$ is such that $s(t) = 1$ for every $|t| \le \|f\|_\infty$. Since now η_n does not depend on the nth component on ∂E, we have

$$\eta \cdot \nu_E = \frac{\varphi(\mathbf{p}x)}{\sqrt{1 + |\nabla f|^2}} \quad \text{on } \partial E \cap \mathbf{C}(x_0, r), \tag{3.46}$$

and the above equation (3.45) reads

$$\operatorname{div}_E \eta = \frac{1}{1 + |\nabla f|^2} \nabla \varphi \cdot \nabla f \quad \text{on } \partial E \cap \mathbf{C}(x_0, r). \tag{3.47}$$

Moreover,

$$\begin{aligned}\int_E \operatorname{div} \eta \, dx &= \int_{\partial E} (\eta \cdot \nu_E) \, d\mathcal{H}^{n-1} = \int_{\partial E \cap \mathbf{C}(x_0,r)} \eta_n (\nu_E \cdot e_n) \, d\mathcal{H}^{n-1} \\ &= \int_{\partial E \cap \mathbf{C}(x_0,r)} \varphi(\mathbf{p}x) s(f(x)) (\nu_E \cdot e_n) \, d\mathcal{H}^{n-1} \\ &= \int_{\partial E \cap \mathbf{C}(x_0,r)} \frac{\varphi(\mathbf{p}x)}{\sqrt{1 + |\nabla f(\mathbf{p}x)|}} \, d\mathcal{H}^{n-1} = \int_{\mathbf{p}(\partial E \cap \mathbf{C}(x_0,r))} \varphi \, dx = 0.\end{aligned}$$

This implies that η is admissible in (3.43). Hence, by using η as a test function in (3.43), by combining (3.46) and (3.47) and using a change of variables, we have

$$\begin{aligned}\int_{\mathbf{D}(x_0',r)} &\frac{\nabla f}{1+|\nabla f|^2}\cdot\nabla\varphi\sqrt{1+|\nabla f|^2}\,dx' \\ &= Q^2\int_{\mathbf{D}(x_0',r)}\big(\beta|\nabla u^+|^2-|\nabla u^-|^2+K\rho^2\big)(x',f(x'))\varphi(x')\,dx' \\ &\quad -2Q^2\int_{\mathbf{D}(x_0',r)}\big(\beta\partial_n u^+\nabla u^+-\partial_n u^-\nabla u^-\big)(x',f(x'))\cdot(-\nabla f,1)\varphi(x')\,dx'\end{aligned}$$

for any $\varphi\in C_c^1(\mathbf{D}(x_0',r))$ with $\int_{\mathbf{D}(x_0',r)}\varphi=0$. ■

Corollary 3.8. *Let E be a minimizer for $(\mathcal{P}_{\beta,K,Q,R})$ and $(u,\rho)\in\mathcal{A}(E)$, $x_0\in\partial E$, $\nu_E(x_0)=e_n$. Assume that $f\in C^{1,\vartheta}(\mathbf{D}(x_0',r))$ and*

$$E\cap\mathbf{C}(x_0,r)=\big\{x=(x',x_n)\in\mathbf{D}(x_0',r)\times\mathbb{R}:x_n<f(x')\big\}\cap\mathbf{C}(x_0,r)$$

for some $0<r\le 1$. Then there exists a vector field $M:\mathbb{R}^n\to\mathbb{R}^n$ such that the matrix $\nabla M(\nabla f)$ is uniformly elliptic and Hölder continuous and a Hölder-continuous function G such that

$$-\operatorname{div}(\nabla M(\nabla f)\nabla\partial_i f)=\partial_i G\quad\textit{weakly on } \partial E\cap\mathbf{C}(x_0,r/2)$$

for every $i=1,\dots,n$.

Proof. Exploiting Proposition 3.7 we have

$$-\operatorname{div}\Big(\frac{\nabla f(x')}{\sqrt{1+|\nabla f(x')|^2}}\Big)=G(x',f(x'))\quad\text{for a.e. } x'\in\mathbf{D}(x_0',r/2),\tag{3.48}$$

where, for $x'\in\mathbf{D}(x_0',r/2)$,

$$\begin{aligned}G(x',f(x'))&=Q^2(\beta|\nabla u^+|^2-|\nabla u^-|^2+K\rho^2)(x',f(x'))\\ &\quad-2Q^2(\beta\partial_n u^+\nabla u^+-\partial_n u^-\nabla u^-)(x',f(x'))\cdot(-\nabla f(x'),1)+C.\end{aligned}$$

Hence, (3.48) is equivalent to

$$-\operatorname{div}(M(\nabla f))=G\quad\text{a.e. on } \partial E\cap\mathbf{C}(x_0,r/2),\tag{3.49}$$

where

$$M(\xi):=\frac{\xi}{\sqrt{1+|\xi|^2}}\quad\text{for all } \xi\in\mathbb{R}^n.$$

By [17, Theorem 27.1] we can take the derivatives of (3.49). Then

$$-\operatorname{div}(\nabla M(\nabla f)\nabla\partial_i f)=\partial_i G\quad\text{a.e. on } \partial E\cap\mathbf{C}(x_0,r/2)$$

for every $i = 1, \dots, n$. Notice that

$$\nabla M(\xi) = \frac{1}{\sqrt{1+|\xi|^2}}\Big(\mathrm{Id} - \frac{\xi \otimes \xi}{1+|\xi|^2}\Big) \quad \text{for all } \xi \in \mathbb{R}^n,$$

meaning that the matrix $\nabla M(\nabla f)$ is uniformly elliptic; more precisely,

$$|\eta|^2 \geq \nabla M(\nabla f)\eta \cdot \eta \geq (1 + \|\nabla f\|_\infty)^{-3/2}|\eta|^2 \quad \text{for all } \eta \in \mathbb{R}^n.$$

Theorem 3.6 gives us Hölder bounds for $Q\nabla u$, while Theorem 2.3 (i) ensures that ρ is Hölder continuous. Thus, G is Hölder continuous. By the definition of M and by the regularity of f we also have that $\nabla M(\nabla f)$ is Hölder continuous. ∎

We prove now the partial $C^{2,\vartheta}$-regularity of minimizers.

Theorem 3.9 ($C^{2,\vartheta}$-regularity). *Given $n \geq 3$, $A > 0$ and $\vartheta \in (0, 1/2)$, there exists $\varepsilon_{\mathrm{reg}} = \varepsilon_{\mathrm{reg}}(n, A, \vartheta) > 0$ such that if E is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, β, K, Q are controlled by A, $x_0 \in \partial E$ and*

$$r + \mathbf{e}_E(x_0, r) + Q^2 D_E(x_0, r) \leq \varepsilon_{\mathrm{reg}},$$

then $E \cap \mathbf{C}(x_0, r/2)$ coincides with the epigraph of a $C^{2,\vartheta}$-function f. In particular, we have that $\partial E \cap \mathbf{C}(x_0, r/2)$ is a $C^{2,\vartheta}$ $(n-1)$-dimensional manifold and

$$[f]_{C^{2,\vartheta}(\mathbf{D}(x_0', r/2))} \leq C(n, A, \vartheta, r). \tag{3.50}$$

Proof. Choose $\varepsilon_{\mathrm{reg}}$ as the minimum between the one in Theorem 2.4 and 1. Then there exists $f \in C^{1,\vartheta}(\mathbf{D}(x_0', r/2))$ such that

$$E \cap \mathbf{C}(x_0, r/2) = \big\{x = (x', x_n) \in \mathbf{D}(x_0', r/2) \times \mathbb{R} : x_n < f(x')\big\}.$$

By Corollary 3.8 we have

$$-\operatorname{div}(\nabla M(\nabla f)\nabla \partial_i f) = \partial_i G \quad \text{a.e. on } \partial E \cap \mathbf{C}(x_0, r/2).$$

Hence the following Schauder estimates hold:

$$[f]_{C^{2,\vartheta}(\mathbf{D}(x_0', r/2))} \leq C r^{-2-\vartheta}\{\|\nabla f\|_{L^2(\mathbf{D}(x_0', 3r/4))} + [G]_{C^{0,\eta}(\mathbf{C}(x_0, 3r/4))}\}$$

for some universal constant C. By the definition of G, recalling (3.34), Theorem 2.3 (i), Theorem 2.4, and using the Poincaré inequality, one can easily see that there exists $C = C(n, A, \vartheta, r) > 0$ such that

$$[G]_{C^{0,\vartheta}(\mathbf{C}(x_0, r/2))} \leq C(n, A, \vartheta, r).$$

By Theorem 2.4 it follows that

$$\frac{1}{r^{n-1}} \int_{\mathbf{D}(x_0', 3r/4)} |\nabla f|^2 \, dz \leq C \mathbf{e}_E(x_0, r) \leq C \varepsilon_{\mathrm{reg}},$$

which implies (3.50). ∎

Remark 3.10. By scaling one can get that the constant on the right-hand side of (3.50) grows as $r^{-1-\vartheta}$.

Remark 3.11. A minimizer E_Q of problem ($\mathcal{P}_{\beta,K,Q,R}$) satisfies the hypothesis of Theorems 3.9 and 1.4 whenever $Q > 0$ is small enough. Indeed, assume $x_0 \in \partial B_1$. Then, by the regularity of ∂B_1, there exists a radius $r = r(n) > 0$ such that

$$r + \mathbf{e}_{B_1}(x_0, 2r) \le \frac{\varepsilon_{\mathrm{reg}}}{2},$$

where $\varepsilon_{\mathrm{reg}}$ is as in Theorem 1.4. On the other hand, by Proposition 5.3 we have that E_Q converges to B_1 in the Kuratowski sense when $Q \to 0$. Hence, by properties of the excess function, $\mathbf{e}_{E_Q}(x_0, 2r) \to \mathbf{e}_{B_1}(x_0, 2r)$ when $Q \to 0$.

As for the Dirichlet energy, we recall that by Theorem 2.3 (iii) we have

$$\mathcal{F}_{\beta,K,Q}(E_Q) \to \mathcal{F}_{\beta,K,0}(B_1) = P(B_1), \quad P(E_Q) \to P(B_1).$$

On the other hand,

$$\begin{aligned}\mathcal{F}_{\beta,K,Q}(E_Q) = P(E_Q) + Q^2 \mathcal{G}_{\beta,K}(E_Q) &\ge P(E_Q) + Q^2 \int_{\mathbb{R}^n} a_{E_Q} |\nabla u|^2 \, dx \\ &\ge P(E_Q) + Q^2 (2r)^{n-1} D_{E_Q}(x_0, 2r).\end{aligned}$$

Thus, $Q^2 D_{E_Q}(x_0, 2r) \to 0$ when $Q \to 0$. Therefore,

$$r + \mathbf{e}_{E_Q}(x_0, 2r) + Q^2 D_{E_Q}(x_0, 2r) \le \varepsilon_{\mathrm{reg}}$$

when $Q > 0$ is small enough.

4. C^∞ regularity

In this section, by a bootstrap argument, we obtain the C^∞ partial regularity of minimizers. Since this result is not necessary for the proof of the main theorem, the reader may skip it unless interested.

Improving the regularity from $C^{2,\eta}$ to C^∞ is easier than from $C^{1,\eta}$ to $C^{2,\eta}$, because we can straighten the boundary in a nice way once it is C^2. More precisely, we have the following lemma.

Lemma 4.1. *Let $k \in \mathbb{N}$, $k \ge 2$ and f is $C^{k,\vartheta}(\mathbf{D})$. There exists $\varepsilon > 0$ such that if*

$$\|f\|_{C^{2,\vartheta}(\mathbf{D})} \le \varepsilon \quad \text{and} \quad f(0) = 0,$$

then there exists a diffeomorphism $\Phi \in C^{k-1,\vartheta}$, $\Phi\colon \mathbf{C}_{1-\varepsilon} \to \mathbf{C}_{1-\varepsilon}$, such that

$$\Phi(\Gamma_f \cap \mathbf{C}_{1-\varepsilon}) = \big\{x = (x', x_n) \in \mathbf{D}_{1-\varepsilon} \times \mathbb{R} : x_n = 0\big\},$$

where Γ_f is the graph of f. Moreover,

$$\begin{aligned} &\big(\nabla\Phi(\Phi^{-1}(x))\big(\nabla\Phi(\Phi^{-1}(x))\big)^T\big)_{jn} = 0 \quad \textit{for all } j \neq n, \\ &\big(\nabla\Phi(\Phi^{-1}(x))\big(\nabla\Phi(\Phi^{-1}(x))\big)^T\big)_{nn} \neq 0. \end{aligned} \tag{4.1}$$

Proof. Define

$$\Psi(x', x_n) := (x', f(x')) + x_n \frac{(-\nabla f(x'), 1)}{\sqrt{1 + |\nabla f(x')|^2}} \quad \text{for all } x = (x', x_n) \in \mathbf{C}_{1-\varepsilon};$$

then $\Phi := \Psi^{-1}$ is the desired diffeomorphism. ∎

Lemma 4.2. *Let k be a positive integer and let f be a $C^{k+1,\vartheta}$-Hölder-continuous function defined on $\mathbf{D}(x_0, r)$ such that $\|f\|_{C^{k+1,\vartheta}} \leq \varepsilon$ for some $\varepsilon > 0$ and*

$$E \cap \mathbf{C}(x_0, r) = \big\{x = (x', x_n) \in \mathbf{D}(x_0', r) \times \mathbb{R} : x_n < f(x')\big\} \cap \mathbf{C}(x_0, r).$$

Suppose v is a solution of

$$-\operatorname{div}(a_E \nabla v) = h \quad \textit{in } \mathcal{D}'(B_r(x_0)), \quad a_E := \mathbf{1}_{E^c} + \beta \mathbf{1}_E,$$

with h^+ and h^- $C^{k,\eta}$-Hölder continuous respectively on $\bar{E} \cap \mathbf{C}(x_0, r)$ and $\overline{E^c} \cap \mathbf{C}(x_0, r)$, where $h^+ = h\mathbf{1}_E$, $h^- = h\mathbf{1}_{E^c}$. Then v^+, v^- are $C^{k+1,\eta}$-Hölder continuous respectively on $\bar{E} \cap \mathbf{C}(x_0, r)$ and $\overline{E^c} \cap \mathbf{C}(x_0, r)$.

Moreover,

$$\|v^-\|_{C^{k+1,\eta}(\bar{E}^c \cap \mathbf{C}(x_0,r))} \leq C \quad \textit{and} \quad \|v^+\|_{C^{k+1,\eta}(\bar{E} \cap \mathbf{C}(x_0,r))} \leq C$$

for some constant $C \geq 0$ which depends on the $C^{k,\eta}$- Hölder norms of h^+ and h^- and on the $C^{k+1,\vartheta}$ norm of f.

Proof. Assume $x_0 = 0$. Let $H := \{x \in \mathbb{R}^n : x_n = x \cdot e_n \leq 0\}$ be the half-space in $\mathbb{R}^n$. By Lemma 4.1, we can assume that

$$\Gamma_f \cap \mathbf{C}_r = \partial H \cap \mathbf{C}_r,$$

where $\Gamma_f \cap \mathbf{C}_{r/2} := \{(x', f(x')) : x' \in \mathbf{D}_r\}$, $f(0) = 0$, and that v solves the equation

$$-\operatorname{div}(a_H A \nabla v) = h, \tag{4.2}$$

where by (4.1), A is a $C^{k-1,\vartheta}$-continuous elliptic matrix such that $A_{jn} = 0$ for every $j \neq n$, $A_{nn} \neq 0$.

We continue the proof by induction on k. For clarity, we do the detailed computations for the case $k = 1$, the case of general k is analogous.

Case $k = 1$. Taking the derivatives with respect to the tangential coordinates $j \neq n$ of (4.2), we deduce

$$\begin{aligned} -\operatorname{div}(a_H A \nabla \partial_j v) &= \partial_j h + \operatorname{div}(\partial_j (a_H A) \nabla v) \\ &= \operatorname{div}(h e_j + \partial_j (a_H A) \nabla v) \quad \text{in } \mathcal{D}'(\mathbb{R}^n). \end{aligned}$$

Notice that a_H is constant along tangential directions and that $(a_H A)^+$, $(a_H A)^-$ have coefficients respectively in $C^{0,\eta}(\bar{H}^c \cap \mathbf{C}_r)$ and $C^{0,\eta}(\bar{H} \cap C_r)$. Furthermore,

$$(h e_j + \partial_j (a_H A) \nabla v)^+ \in C^{0,\eta}(\bar{H}^c \cap \mathbf{C}_r)$$

and

$$(h e_j + \partial_j (a_H A) \nabla v)^- \in C^{0,\eta}(\bar{H} \cap \mathbf{C}_r).$$

Hence, exploiting Lemma 3.3 we deduce

$$\partial_j v^+ \in C^{1,\eta}(\bar{H} \cap \mathbf{C}_r) \quad \text{and} \quad \partial_j v^- \in C^{1,\eta}(\overline{H^c} \cap \mathbf{C}_r) \quad \text{for all } j \neq n. \tag{4.3}$$

Furthermore, by (4.2) we have

$$-\sum_{i,j=1}^{n} \{a_H A_{ij} \partial_{ij} v + \partial_i (a_H A_{ij}) \partial_j v\} = h.$$

Thanks to the form of the matrix A we obtain

$$-a_H A_{nn} \partial_{nn} v = \sum_{i,j \neq n} \{a_H A_{ij} \partial_{ij} v + \partial_i (a_H A_{ij}) \partial_j v\} + h.$$

Since the right-hand side of the previous equation is Hölder continuous, we have

$$\partial_{nn} v^+ \in C^{0,\eta}(\bar{H}^c \cap \mathbf{C}_r) \quad \text{and} \quad \partial_{nn} v^- \in C^{0,\eta}(\bar{H} \cap \mathbf{C}_r).$$

Moreover, (4.3) implies

$$\partial_{nj} v^+ \in C^{0,\eta}(\bar{H}^c \cap \mathbf{C}_r) \quad \text{and} \quad \partial_{nj} v^- \in C^{0,\eta}(\bar{H} \cap \mathbf{C}_r)$$

for every $j \neq n$. Therefore,

$$v^+ \in C^{2,\eta}(\bar{H}^c \cap \mathbf{C}_r) \quad \text{and} \quad v^- \in C^{2,\eta}(\bar{H}^c \cap \mathbf{C}_r).$$

By Lemma 3.3 we deduce also that

$$\|\nabla v^+\|_{C^{1,\eta}(\bar{H} \cap \mathbf{C}_r)} \quad \text{and} \quad \|\nabla v^-\|_{C^{1,\eta}(\overline{H^c} \cap \mathbf{C}_r)}$$

are bounded by a constant which depends on the Hölder norms of ∇h^+, ∇h^-, the coefficients of $(a_H A)^+$ and $(a_H A)^-$.

General k. Analogous to the case $k = 1$. ∎

Proof of Theorem 1.4. Fix $\vartheta = \frac{1}{4}$. If we prove

$$[f]_{C^{k,\vartheta}(\mathbf{D}(x_0',r/2))} \leq C(n, A, k, r, \vartheta),$$

we prove the theorem since $[f]_{C^k}$ is bounded by $[f]_{C^{k,\vartheta}}$.

If we choose $\varepsilon_{\mathrm{reg}}$ as in Theorem 3.9, then there exists $f \in C^{2,\vartheta}(\mathbf{D}(x_0', r/2))$ such that

$$E \cap \mathbf{C}(x_0, r/2) = \big\{x = (x', x_n) \in \mathbf{D}(x_0', r/2) \times \mathbb{R} : x_n < f(x')\big\}.$$

By Corollary 3.8 we have

$$-\operatorname{div}(\nabla M(\nabla f)\nabla \partial_i f) = \partial_i G \quad \text{a.e. on } \partial E \cap \mathbf{C}(x_0, r/2) \tag{4.4}$$

for every $i = 1, \dots, n$, with $\nabla M(\nabla f)$ uniformly elliptic and Hölder continuous and G-Hölder continuous.

Now we argue by induction on k. The induction step is divided into two parts:

Claim 1: f is C^k-Hölder continuous $\Rightarrow u^+, u^-$ are C^k-Hölder continuous respectively on $\bar{E} \cap \mathbf{C}(x_0, r/2)$ and $\overline{E^c} \cap \mathbf{C}(x_0, r/2)$.

Moreover, there exists a universal constant $C = C(n, A) > 0$ and $\eta \in (0, \frac{1}{2})$ such that

$$\|Qu^+\|_{C^{k,\eta}(\bar{E}\cap \mathbf{C}(x_0,r/2))} \leq C \quad \text{and} \quad \|Qu^-\|_{C^{k,\eta}(\overline{E^c}\cap \mathbf{C}(x_0,r/2))} \leq C. \tag{4.5}$$

Claim 2: f is C^k-Hölder continuous $\Rightarrow$ f is C^{k+1}-Hölder continuous.

To prove Claim 1, we apply Lemma 4.2 to $v = Qu$ and $h = Q\rho$. By (3.34) the norms

$$\|Q\nabla u^+\|_{C^{0,\eta}(\bar{H}\cap \mathbf{C}_{r/2})} \quad \text{and} \quad \|Q\nabla u^-\|_{C^{0,\eta}(\overline{H^c}\cap \mathbf{C}_{r/2})}$$

are bounded by a universal constant. That gives us (4.5).

As for Claim 2, notice that by the definition of M, since f is C^k-Hölder continuous, we have that $\nabla M(\nabla f)$ in (4.4) is C^{k-1}-Hölder continuous. By Claim 1 we deduce that G is C^{k-1}-Hölder continuous with its norm uniformly bounded. Then, using Schauder estimates for (4.4), we get that f is C^{k+1}-Hölder continuous. ∎

5. Reduction to nearly spherical sets

In this section, by combining Proposition 5.3 with the higher regularity (Theorem 3.9), we prove that for small enough values of the total charge the minimizers are *nearly spherical sets*. Recall the following definition.

Definition 5.1 ($C^{2,\gamma}$-nearly spherical set). An open bounded set $\Omega \subset \mathbb{R}^n$ is called *nearly spherical* of class $C^{2,\gamma}$ parametrized by φ, if there exists $\varphi \in C^{2,\gamma}$ with $\|\varphi\|_{L^\infty} < \frac{1}{2}$ such that

$$\partial\Omega = \big\{(1+\varphi(x))x : x \in \partial B_1\big\}.$$

We first show the L^∞-closeness of minimizers to the unitary ball in the small charge regime. Let us start with the following proposition.

Proposition 5.2 (L^1-closeness to the ball). *Let $E \subset \mathbb{R}^n$ be a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$). Then there exists a point $x_0 \in \mathbb{R}^n$ such that*

$$|E\Delta B_1(x_0)|^2 \le C(P(E) - P(B_1)) \le CQ^2\mathcal{G}_{\beta,K}(B_1).$$

Proof. By the quantitative isoperimetric inequality [11, Theorem 1.1], there exists a point $x_0 \in \mathbb{R}^n$ such that

$$|E\Delta B_1(x_0)|^2 \le C(P(E) - P(B_1))$$

for some constant $C = C(n) > 0$. By the minimality of E we have

$$\mathcal{F}(E) = P(E) + Q^2\mathcal{G}_{\beta,K}(E) \le P(B_1) + Q^2\mathcal{G}_{\beta,K}(B_1) = \mathcal{F}(B_1).$$

Hence,

$$|E\Delta B_1(x_0)|^2 \le C(P(E) - P(B_1)) \le CQ^2\mathcal{G}_{\beta,K}(B_1)$$

for some constant $C = C(n) > 0$. ■

Thanks to the density estimates (see Theorem 2.3 (v)), we can now get the L^∞ convergence.

Proposition 5.3 (L^∞-closeness to the ball). *Let $\{Q_h\}_{h\in\mathbb{N}}$ be a sequence such that $Q_h > 0$ and $Q_h \to 0$ when $h \to \infty$. Let $\{E_h\}_{h\in\mathbb{N}}$ be a sequence of minimizers of ($\mathcal{P}_{\beta,K,Q_h,R}$). Then, up to translations, $E_h \to \overline{B}_1$ and $\partial E_h \to \partial B_1$ in the Kuratowski sense.*

Proof. First, by Proposition 5.2 we can easily get that up to translations $E_h \to B_1$ in L^1. Indeed, Proposition 5.2 gives us that for every h there exists a point x_h such that

$$|E_h\Delta B_1(x_h)|^2 \le C(P(E_h) - P(B_1)) \le CQ^2\mathcal{G}_{\beta,K}(B_1).$$

Since $\mathcal{G}_{\beta,K}$ is invariant under translations, we can assume that $x_h = 0$ for every h, so that

$$|E_h\Delta B_1|^2 \le C(P(E_h) - P(B_1)) \le CQ^2\mathcal{G}_{\beta,K}(B_1).$$

Then $Q_h \to 0$ implies $E_h \to B_1$ in L^1 and $P(E_h) \to P(B_1)$ when $h \to \infty$.

The rest of the argument is classical and follows from the density estimates (see Theorem 2.3 (v)). We do not include it here for brevity; see the proof of [19, Proposition 3.2]. ■

Remark 5.4. Note that a priori the translated sets are not necessarily inside the ball B_R, so they might not be minimizers for ($\mathcal{P}_{\beta,K,Q,R}$). However, in $\mathbb{R}^n$ Kuratowski convergence is equivalent to Hausdorff convergence, so for h big enough the translated sets will be in the ball $B_{1+\varepsilon}$ for any $\varepsilon > 0$ (and hence in B_R since $R > 1$).

Theorem 5.5. *Let $\{Q_h\}_{h\in\mathbb{N}}$ be a sequence such that $Q_h > 0$ and $Q_h \to 0$ when $h \to \infty$, let β_h and K_h be controlled by A and let $R_h > 1$. Let $\{E_h\}_{h\in\mathbb{N}}$ be a sequence of minimizers of ($\mathcal{P}_{\beta_h,K_h,Q_h,R_h}$). Then for h big enough E_h is nearly spherical of class C^∞, i.e. there exists $\varphi_h \in C^\infty$ with uniform bounds and $\|\varphi_h\|_{L^\infty} < \frac{1}{2}$ such that*

$$\partial E_h = \big\{(1 + \varphi_h(x))x : x \in \partial B_1\big\}.$$

Moreover, $\|\varphi_h\|_{C^k} \to 0$ when $h \to \infty$ for every $k \in \mathbb{N}$.

Proof. We do not provide a proof since this result is classical. See, for example, [4, Proof of Proposition 4.4]. Note that all the regularity estimates we obtain are universal, so they do not depend on R and depend on A rather than on specific values of β and K. ∎

6. Theorem 1.1 for nearly spherical sets

To prove Theorem 1.1 for nearly spherical sets we are going to write the Taylor expansion for the energy. We only need to deal with the repulsive term $\mathcal{G}$, as the expansion for the perimeter is well known. To this end, we need to compute shape derivatives of the energy $\mathcal{G}$ near the ball and get a bound on the second derivative. For the convenience of the reader we show these calculations later in Section 7 as they are rather technical.

In this section we first replace our problem with an equivalent one and write the Euler–Lagrange equations for it. We do so to facilitate the computations of Section 7. We then conclude the proof of Theorem 1.1 for nearly spherical sets given the Taylor expansion. Thanks to the quantitative isoperimetric inequality for nearly spherical sets, we see that we can be crude in the bounds of Section 7 as we have a small parameter in front of the disaggregating term.

6.1. Changing minimization problem

For a fixed domain E we are solving the following minimization problem:

$$\mathcal{G}(E) = \inf_{\substack{u\in H^1(\mathbb{R}^n) \\ \rho\mathbf{1}_{E^c}=0}} \Big\{\tfrac{1}{2}\int_{\mathbb{R}^n}(a_E|\nabla u|^2 + K\rho^2)\,dx : -\operatorname{div}(a_E\nabla u) = \rho,\ \int_{\mathbb{R}^n}\rho\,dx = 1\Big\}.$$

We want to get rid of the constraints and make it a minimization problem over single functions rather than over pairs. More precisely, we prove the following lemma.

Lemma 6.1. *For any $E \subset \mathbb{R}^n$ the energy $\mathcal{G}$ can be represented in the following way:*

$$\mathcal{G}(E) = \frac{K}{2|E|} - \inf_{\psi \in H^1(\mathbb{R}^n)} \Big(\frac{1}{2} \int_{\mathbb{R}^n} a_E |\nabla \psi|^2 \, dx + \frac{1}{|E|} \int_E \psi \, dx - \frac{1}{2|E|K} \Big(\int_E \psi \, dx \Big)^2 + \frac{1}{2K} \int_E \psi^2 \, dx \Big).$$

Proof. We use an "infinite-dimensional Lagrange multiplier":

$$\begin{aligned}
\mathcal{G}(E) &= \inf_{\substack{u \in H^1(\mathbb{R}^n) \\ \rho \mathbf{1}_{E^c} = 0}} \Big\{ \frac{1}{2} \int_{\mathbb{R}^n} a_E |\nabla u|^2 \, dx + \frac{1}{2} \int_E K \rho^2 \, dx \\
&\qquad + \sup_{\psi \in H^1(\mathbb{R}^n)} \Big[\int_{\mathbb{R}^n} (a_E \nabla u \cdot \nabla \psi - \rho \psi) \, dx \Big] : \int_E \rho \, dx = 1 \Big\} \\
&= \inf_{\substack{u \in H^1(\mathbb{R}^n) \\ \rho \mathbf{1}_{E^c} = 0}} \sup_{\psi \in H^1(\mathbb{R}^n)} \Big\{ \frac{1}{2} \int_{\mathbb{R}^n} a_E (|\nabla u|^2 + 2 \nabla u \cdot \nabla \psi) \, dx \\
&\qquad + \frac{1}{2} \int_E (K \rho^2 - 2 \rho \psi) \, dx : \int_E \rho \, dx = 1 \Big\}.
\end{aligned}$$

The convexity of the problem allows us to use the Sion minimax theorem ([23, Corollary 3.3]) and interchange the infimum and the supremum:

$$\begin{aligned}
\mathcal{G}(E) &= \sup_{\psi \in H^1(\mathbb{R}^n)} \inf_{\substack{u \in H^1(\mathbb{R}^n) \\ \rho \mathbf{1}_{E^c} = 0}} \Big\{ \frac{1}{2} \int_{\mathbb{R}^n} a_E (|\nabla u|^2 + 2 \nabla u \cdot \nabla \psi) \, dx \\
&\qquad + \frac{1}{2} \int_E (K \rho^2 - 2 \rho \psi) \, dx : \int_E \rho \, dx = 1 \Big\} \\
&= \sup_{\psi \in H^1(\mathbb{R}^n)} \Big\{ \inf_{u \in H^1(\mathbb{R}^n)} \frac{1}{2} \int_{\mathbb{R}^n} a_E (|\nabla u|^2 + 2 \nabla u \cdot \nabla \psi) \, dx \\
&\qquad + \inf_{\substack{\rho \mathbf{1}_{E^c} = 0 \\ \int_E \rho \, dx = 1}} \frac{1}{2} \int_E (K \rho^2 - 2 \rho \psi) \, dx \Big\}.
\end{aligned}$$

We denote the infimums inside by I and II, that is,

$$\mathrm{I} := \inf_{u \in H^1(\mathbb{R}^n)} \Big\{ \frac{1}{2} \int_{\mathbb{R}^n} a_E (|\nabla u|^2 + 2 \nabla u \cdot \nabla \psi) \, dx \Big\},$$

$$\mathrm{II} := \inf_{\rho} \Big\{ \frac{1}{2} \int_E (K \rho^2 - 2 \rho \psi) \, dx : \int_E \rho \, dx = 1 \Big\}.$$

We want to compute both I and II in terms of ψ.

For I it is immediate. Since a_E is positive we get

$$\begin{aligned}
\mathrm{I} &= \inf_{u\in H^1(\mathbb{R}^n)}\left\{\frac{1}{2}\int_{\mathbb{R}^n} a_E(|\nabla u|^2 + 2\nabla u\cdot\nabla\psi)\,dx\right\}\\
&= \inf_{u\in H^1(\mathbb{R}^n)}\left\{\frac{1}{2}\int_{\mathbb{R}^n} a_E(|\nabla u+\nabla\psi|^2 - |\nabla\psi|^2)\,dx\right\}\\
&= -\frac{1}{2}\int_{\mathbb{R}^n} a_E|\nabla\psi|^2\,dx.
\end{aligned}$$

We note that the corresponding minimizing u equals $-\psi$.

To compute II, note that

$$\begin{aligned}
\mathrm{II} &= \inf_{\rho}\left\{\frac{1}{2}\int_E (K\rho^2 - 2\rho\psi)\,dx : \int_E \rho\,dx = 1\right\}\\
&= \inf_{\rho}\left\{\frac{1}{2}\int_E\Big(\sqrt{K}\rho - \frac{\psi}{\sqrt{K}}\Big)^2 dx : \int_E \rho\,dx = 1\right\} - \frac{1}{2K}\int_E \psi^2\,dx\\
&= \frac{K}{2}\inf_{f}\left\{\int_E\Big(f - \Big(\frac{\psi}{K} - \frac{1}{|E|}\Big)\Big)^2 dx : \int_E f\,dx = 0\right\} - \frac{1}{2K}\int_E \psi^2\,dx.
\end{aligned}$$

Then the minimizing function f^* is the projection in $L^2(E)$ of a function $(\frac{\psi}{K} - \frac{1}{|E|})$ onto the linear space $\{f : \int_E f\,dx = 0\}$. Thus, $f^* = (\frac{\psi}{K} - \frac{1}{|E|}) - c$, where c is the constant such that $\int_E f^* = 0$, i.e. $c = \frac{1}{|E|}\int_E(\frac{\psi}{K} - \frac{1}{|E|})$. The corresponding minimizing ρ equals $\mathbf{1}_E\frac{1}{K}(\psi + \frac{1}{|E|}(1 - \frac{1}{K}\int_E \psi\,dx)K)$.

Bringing it all together,

$$\begin{aligned}
\mathcal{G}(E) &= \frac{K}{2|E|} + \sup_{\psi\in H^1(\mathbb{R}^n)}\Big(-\frac{1}{2}\int_{\mathbb{R}^n} a_E|\nabla\psi|^2\,dx - \frac{1}{|E|}\int_E \psi\,dx\\
&\qquad + \frac{1}{2|E|K}\Big(\int_E \psi\,dx\Big)^2 - \frac{1}{2K}\int_E \psi^2\,dx\Big)\\
&= \frac{K}{2|E|} - \inf_{\psi\in H^1(\mathbb{R}^n)}\Big(\frac{1}{2}\int_{\mathbb{R}^n} a_E|\nabla\psi|^2\,dx + \frac{1}{|E|}\int_E \psi\,dx\\
&\qquad - \frac{1}{2|E|K}\Big(\int_E \psi\,dx\Big)^2 + \frac{1}{2K}\int_E \psi^2\,dx\Big). \qquad \blacksquare
\end{aligned}$$

6.2. Euler–Lagrange

We now consider the following minimization problem:

$$\begin{aligned}
\mathcal{J}(E) = \inf_{\psi\in H^1(\mathbb{R}^N)}\Big(&\frac{1}{2}\int_{\mathbb{R}^n} a_E|\nabla\psi|^2\,dx + \frac{1}{|E|}\int_E \psi\,dx - \frac{1}{2|E|K}\Big(\int_E \psi\,dx\Big)^2\\
&+ \frac{1}{2K}\int_E \psi^2\,dx\Big).
\end{aligned} \tag{6.1}$$

Remark 6.2. Note that $\mathcal{J}(E) \le 0$. By Lemma 6.1,

$$\mathcal{G}(E) = \frac{K}{2|E|} - \mathcal{J}(E).$$

By [21, inequality (2.1)], $\mathcal{G}(E) \le C(n, K, \beta, |E|)$. This implies that

$$|\mathcal{J}(E)| \le C(n, K, \beta, |E|). \tag{6.2}$$

A minimizer for this problem exists, and it is unique by convexity. Indeed, to see the coercivity of the functional note that

$$-\frac{1}{2|E|K}\left(\int_E \psi\, dx\right)^2 + \frac{1}{2K}\int_E \psi^2\, dx \ge 0$$

by the Jensen inequality. As for convexity, we use that

$$-\frac{1}{2|E|K}\left(\int_E \psi\, dx\right)^2 + \frac{1}{2K}\int_E \psi^2\, dx = \frac{1}{2K}\int_E \left(\psi - \fint_E \psi\, dy\right)^2 dx.$$

Note that the minimizers in the definitions of $\mathcal{J}$ and $\mathcal{G}$ coincide since the set is fixed. We denote the minimizer by ψ_E. We would also need the interior and exterior restrictions of the function ψ_E, i.e.

$$\psi_E^+ := \psi_E \mathbf{1}_E, \quad \psi_E^- := \psi_E \mathbf{1}_{E^c}.$$

Proposition 6.3. *The following identities hold for* ψ_E*:*

(i) (Euler–Lagrange equation, integral form) *For any* $\Psi \in D^1(\mathbb{R}^n)$,

$$\begin{aligned}\int_{\mathbb{R}^n} a_E \nabla\psi_E \cdot \nabla\Psi\, dx &+ \frac{1}{K}\int_E \psi_E \Psi\, dx \\ &+ \frac{1}{|E|}\left(\int_E \Psi\, dx\right)\left(1 - \frac{1}{K}\int_E \psi_E\, dx\right) \\ &= \int_{\mathbb{R}^n} \Psi\Big(\frac{\mathbf{1}_E \psi_E}{K} - \operatorname{div}(a_E \nabla\psi_E)\Big)\, dx \\ &+ \int_{\partial E} (\beta\nabla\psi_E^+ - \nabla\psi_E^-) \cdot \nu \Psi\, d\mathcal{H}^{n-1} \\ &+ \frac{1}{|E|}\left(\int_E \Psi\, dx\right)\left(1 - \frac{1}{K}\int_E \psi_E\, dx\right) = 0.\end{aligned} \tag{6.3}$$

(ii) (Euler–Lagrange equation)

$$\begin{cases} -\beta\Delta\psi_E = -\frac{1}{K}\psi_E + \frac{2}{K}\mathcal{J}(E) - \frac{1}{|E|} & \text{in } E, \\ \Delta\psi_E = 0 & \text{in } E^c, \\ \psi_E^+ = \psi_E^- & \text{on } \partial E, \\ \beta\nabla\psi_E^+ \cdot \nu = \nabla\psi_E^- \cdot \nu & \text{on } \partial E. \end{cases} \tag{6.4}$$

(iii)
$$\mathcal{J}(E) = \frac{1}{2|E|}\int_E \psi_E \, dx. \tag{6.5}$$

(iv) *There exists a constant* $C = C(n, K, \beta, |E|)$ *such that*
$$\int_{\mathbb{R}^n} a_E |\nabla \psi_E|^2 \, dx \le C. \tag{6.6}$$

Proof. Equations (6.3) and (6.4) are the standard Euler–Lagrange equations for problem (6.1).

To prove (6.5) we use ψ_E as a test function in (6.3).

To see (6.6), we use ψ_E as a test function in (6.3) and the Cauchy–Schwarz inequality to get
$$\int_{\mathbb{R}^n} a_E |\nabla \psi_E|^2 \, dx \le -\frac{1}{|E|}\left(\int_E \psi_E \, dx\right).$$
Now we apply (6.5) and (6.2) to obtain
$$\int_{\mathbb{R}^n} a_E |\nabla \psi_E|^2 \, dx \le -2\mathcal{J}(E) \le 2C(n, K, \beta, |E|). \qquad \blacksquare$$

Proposition 6.4. *Let* ψ_0 *be the minimizer for* $\mathcal{J}(B_1)$. *Then* ψ_0 *is radial.*

Proof. Let $R\colon \mathbb{R}^n \to \mathbb{R}^n$ be any rotation. Since $R(B_1) = B_1$, $\psi_0 \circ R$ is also a minimizer for $\mathcal{J}(B_1)$. But the minimizer is unique, so we obtain that $\psi_0 \circ R = \psi_0$ for any rotation R. This implies that ψ_0 is radial. $\blacksquare$

6.3. Proof of Theorem 1.1

We will use the following notation.

Definition 6.5. For an open set Ω, x_Ω denotes the barycenter of Ω, namely
$$x_\Omega = \frac{1}{|\Omega|}\int_\Omega x \, dx.$$

We want to prove that for Q small enough the only minimizer of $\mathcal{F}(\Omega) = P(\Omega) + Q^2\mathcal{G}(\Omega)$ for Ω nearly spherical is a ball.

We will use the following theorem proved by Fuglede.

Theorem 6.6 ([9, Theorem 1.2]). *There exists a constant* $c = c(N)$ *such that for any* Ω *a nearly spherical set parametrized by* φ *with* $|\Omega| = |B_1|$, $x_\Omega = 0$, *the following inequality holds:*
$$P(\Omega) - P(B_1) \ge c\|\varphi\|^2_{H^1(\partial B_1)}.$$

We will also need the following bound on the energy $\mathcal{J}$; see Section 7 for the proof.

Lemma 6.7. *Given* $\vartheta \in (0, 1]$, *there exists* $\delta = \delta(N, \vartheta) > 0$ *and a constant* $C = C(\delta)$ *such that for every nearly spherical set* E *parametrized by* φ *with* $\|\varphi\|_{C^{2,\vartheta}(\partial B_1)} < \delta$ *and* $|E| = |B_1|$, *we have*
$$-\mathcal{J}(E) \ge -\mathcal{J}(B_1) - C\|\varphi\|^2_{H^1(\partial B_1)}.$$

Remark 6.8. One can show that at the ball a stronger estimate holds. More precisely, we have

$$\partial^2 \mathcal{G}(B_1)[\varphi,\varphi] \geq -c\|\varphi\|^2_{H^{\frac{1}{2}}(\partial B_1)}.$$

For the detailed computations see [19, Appendix A].

Finally, we are ready to prove the main result of the paper.

Proof of Theorem 1.1. Argue by contradiction. Suppose there exists a sequence of minimizers E_h corresponding to $Q_h \to 0$ such that E_h are not balls. By Theorem 5.5 we have that starting from a certain h the sets (possibly, translated) are nearly spherical parametrized by φ_h with $\|\varphi_h\|_{C^{2,\gamma}(\partial B_1)} < \delta$, where δ is the one of Lemma 6.7.

To apply Theorem 6.6 and Lemma 6.7 we need the sets to have barycenters at the origin. It is not necessarily true for the sequence E_h; however, we can exploit the fact that nearly spherical sets have barycenters close to the origin. We choose the sequence E_h so that $E_h \to B_1$ in L^∞. One can easily show that it implies $x_{E_h} \to 0$. So if we now look at the sequence of sets $\tilde{E}_h = \{x - x_{E_h} : x \in E_h\}$, we see that $\tilde{E}_h \to B_1$ in L^∞ and $x_{\tilde{E}_h} = 0$. It remains to apply Theorem 5.5 to the sequence $\{E_h\}$ to see that these new translated sets are still nearly spherical. For the sake of simplicity let us not rename the sequence and assume that the sequence $\{E_h\}$ is such that $x_{E_h} = 0$.

Now we can apply Theorem 6.6 and Lemma 6.7. We want to show that $\mathcal{F}(E_h) > F(B_1)$ for h big enough. Indeed, if Q_h is small enough, we have

$$\begin{aligned}
\mathcal{F}(E_h) = P(E_h) + Q_h^2\mathcal{G}(E_h) &\geq P(B_1) + c\|\varphi_h\|^2_{H^1(\partial B_1)} + Q_h^2\Big(\frac{K}{2|B_1|} - \mathcal{I}(E_h)\Big) \\
&\geq P(B_1) + c\|\varphi_h\|^2_{H^1(\partial B_1)} + Q_h^2\Big(\frac{K}{2|B_1|} - \mathcal{I}(B_1) - c'\|\varphi_h\|^2_{H^1(\partial B_1)}\Big) \\
&> P(B_1) + Q_h^2\Big(\frac{K}{2|B_1|} - \mathcal{I}(B_1)\Big) = \mathcal{F}(B_1). \qquad \blacksquare
\end{aligned}$$

We can now prove Corollary 1.2, which follows from Theorem 1.1 and properties of minimizers established in [21].

Proof of Corollary 1.2. Let Q_0 be the one of Theorem 1.1. Let E be an open set such that $|E| = |B_1|$. Let us show that $\mathcal{F}(E) \geq \mathcal{F}(B_1)$. If E is bounded, then $\mathcal{F}(E) \geq F(B_1)$ by Theorem 1.1. Assume now that E is unbounded.

We can assume that E is of finite perimeter, since otherwise $\mathcal{F}(E) = \infty$. Then, by [17, Remark 13.12], there exists a sequence $R_h \to \infty$ such that $E \cap B_{R_h} \to E$ in L^1, $P(E \cap B_{R_h}) \to P(E)$. Rescale the sets so that their volumes are the same as that of the unit ball, i.e.

$$\Omega_h = \alpha_h(E \cap B_{R_h}) \quad \text{with } \alpha_h = \Big(\frac{|B_1|}{|E \cap B_{R_h}|}\Big)^{1/n}.$$

Note that since $|E| = |B_1|$, $\alpha_h \to 1$, so also for Ω_h we have $|\Omega_h \Delta E| \to 0$, $P(\Omega_h) \to P(E)$. Now, by the continuity of the functional $\mathscr{G}$ in L^1 (see [21, Proposition 2.6]), we get

$$\mathscr{F}(\Omega_h) = P(\Omega_h) + \mathscr{G}(\Omega_h) \to P(E) + \mathscr{G}(E) = \mathscr{F}(E). \tag{6.7}$$

On the other hand, $\Omega_h \subset \alpha_h B_{R_h}$, so it is bounded and hence, by Theorem 1.1,

$$\mathscr{F}(\Omega_h) \geq \mathscr{F}(B_1) \quad \text{for every } h.$$

Combining the last inequality with (6.7), we get $\mathscr{F}(E) \geq \mathscr{F}(B_1)$. Thus, the infimum in problem ($\mathscr{P}_{\beta,K,Q}$) is achieved on balls.

Let us show that the only minimizers are the unit balls. Let E be a minimizer for ($\mathscr{P}_{\beta,K,Q}$). If E is bounded, then by Theorem 1.1 it should be a ball of radius 1. We now explain why E cannot be unbounded. Indeed, suppose the contrary holds. Then we can find a sequence of points x_k such that $x_k \in E$, $|x_k - x_j| \geq 1$ for $k \neq j$ (for example, we can define $x_k := E \backslash B_{\max\{|x_1|,|x_2|,\ldots,|x_{k-1}|\}+1}$). Now, by density estimates for minimizers (Theorem 2.3 (v)), we have

$$\frac{|B_r(x) \cap E|}{|B_r|} \geq \frac{1}{C} \quad \text{for } x \in E,\ r \in (0, \bar{r}). \tag{6.8}$$

Note that even though Theorem 2.3 (v) deals with minimizers of ($\mathscr{P}_{\beta,K,Q,R}$), the constants C and $\bar{r}$ do not depend on R, so it applies in our case. It remains to use (6.8) for $x = x_k$ and $r = \min(1/2\bar{r}, 1/2)$ to see that

$$|E| \geq \sum_{k=1}^{\infty} |B_r(x_k) \cap E| \geq \sum_{k=1}^{\infty} \frac{|B_r|}{C} = \infty,$$

which contradicts the fact that $|E| = |B_1|$. Thus, E is bounded and it is a ball of radius 1. ■

7. Proof of Lemma 6.7

We will need the following technical lemma, which is almost identical to [4, Lemma A.1]. Since we need a slightly different conclusion than in [4], we repeat the proof here. Throughout this section we will be using the following notation.

Notation 7.1. We denote by $J_{\Phi_t}(x)$ the Jacobian of Φ_t at x:

$$J_{\Phi_t}(x) = \det \nabla \Phi_t(x).$$

Lemma 7.2. *Given $\vartheta \in (0, 1]$ there exists $\delta = \delta(n, \vartheta) > 0$, a modulus of continuity ω and a constant $C = C(n, \theta)$ such that for every nearly spherical set E parametrized by φ with $\|\varphi\|_{C^{2,\vartheta}(\partial B_1)} < \delta$ and $|\Omega| = |B_1|$, we can find an autonomous vector field X_φ for which the following holds true:*

(i) div $X_\varphi = 0$ *in a* δ*-neighborhood of* ∂B_1*;*

(ii) *if* $\Phi_t := \Phi(t, x)$ *is the flow of* X_φ*, i.e.*

$$\partial_t \Phi_t = X_\varphi(\Phi_t), \quad \Phi_0(x) = x,$$

then $\Phi_1(\partial B_1) = \partial E$ *and* $|\Phi_t(B_1)| = |B_1|$ *for all* $t \in [0, 1]$*;*

(iii) *denote* $E_t := \Phi_t(B_1)$*; then*

$$\|\Phi_t - Id\|_{C^{2,\vartheta}} \le \omega(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}) \quad \textit{for every } t \in [0, 1], \tag{7.1}$$

$$|J_\Phi| \le C \quad \textit{in a neighborhood of } B_1, \tag{7.2}$$

$$\|X \cdot \nu\|_{H^1(\partial E_t)} \le C\|\varphi\|_{H^1(\partial B_1)}, \tag{7.3}$$

and for the tangential part of X*, defined as* $X = X - (X \cdot \nu)\nu$*, there holds*

$$|X^\tau| \le \omega(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)})|X \cdot \nu| \quad \textit{on } \partial E_t. \tag{7.4}$$

Proof. Such a vector field can be constructed for any smooth set, see for example [6]. However, for the ball one can write an explicit expression in a neighborhood of ∂B_1. The proof for the case of the ball can be found in [4, Lemma A.1]. For the convenience of the reader we provide the expression here, as well as a brief explanation of how to get the needed bounds. In polar coordinates, $\rho = |x|$, $\theta = x/|x|$ the field looks like

$$X_\varphi(\rho, \theta) = \frac{(1 + \varphi(\theta))^n - 1}{n\rho^{n-1}}\theta,$$

$$\Phi_t(\rho, \theta) = \big(\rho^n + t\big((1 + \varphi(\theta))^n - 1\big)\big)^{\frac{1}{N}}\theta$$

for $|\rho - 1| \ll 1$. Then we extend this vector field globally in order to satisfy (7.1). Notice that (7.2) is a direct consequence of (7.1).

By direct computation we get

$$(X \cdot \theta) \circ \Phi_t - X \cdot \nu_{\partial B_1} = (X \cdot \nu_{\partial B_1}) f \text{ on } \partial B_1, \tag{7.5}$$

with $\|f\|_{C^{2,\vartheta}(\partial B_1)} \le \omega(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)})$. Now we can get bound (7.3). Indeed, (7.5) together with (7.2) gives us

$$\|X \cdot \nu\|_{H^1(\partial E_t)} \le C\|X \cdot \nu\|_{H^1(\partial B_1)}.$$

From the definition of X, on ∂B_1 we have

$$\varphi - X \cdot \nu = \frac{1}{n}\sum_{i=2}^{n}\binom{n}{i}\varphi^i,$$

and thus

$$\|\varphi - X \cdot \nu\|_{H^1(\partial B_1)} \le \omega(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)})\|X \cdot \nu\|_{H^1(\partial B_1)},$$

yielding inequality (7.3).

To see (7.4) we use that by definition X is parallel to θ close to ∂B_1. Thus,

$$\begin{aligned}|X^\tau \circ \Phi_t| &= |((X\cdot\theta)\theta)\circ\Phi_t - ((X\cdot\nu)\nu)\circ\Phi_t| \\ &= \big|(X\cdot\nu_{\partial B_1})(1+\omega(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}))\nu_{\partial B_1}(1+\omega(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)})) \\ &\qquad - (X\cdot\nu_{\partial B_1})(1+\omega(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}))\nu_{\partial B_1}(1+\omega(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}))\big| \\ &= \omega(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)})|(X\cdot\nu)\circ\Phi_t|. \end{aligned}$$

■

Let E be the nearly spherical set from Lemma 6.7 and let E_t be the sequence provided by Lemma 7.2. In what follows we omit the subscript φ for brevity.

7.1. First derivative

We want to compute $\frac{d}{dt}\mathcal{J}(E_t)$. Let ψ_t be the minimizer in the minimization problem (6.1) for E_t. Recall that by (6.4) it means that ψ_t satisfies

$$\begin{cases} -\beta\Delta\psi_t = -\dfrac{1}{K}\psi_t + \dfrac{2}{K}\mathcal{J}(E_t) - \dfrac{1}{|B_1|} & \text{in } E_t, \\ \Delta\psi_t = 0 & \text{in } E_t^c, \\ \psi_t^+ = \psi_t^- & \text{on } \partial E_t, \\ \beta\nabla\psi_t^+\cdot\nu = \nabla\psi_t^-\cdot\nu & \text{on } \partial E_t. \end{cases} \tag{7.6}$$

First we notice that ψ_t is regular since it is a solution to a transmission problem. More precisely, by Lemma 4.2, the following holds.

Proposition 7.3. *There exists $\delta > 0$ such that if $\|\varphi\|_{C^{2,\vartheta}(\partial B_1)} < \delta$, then*

$$\|\psi_t\|_{C^2(\overline{E_t})} \le C \quad \textit{for every } t \in [0,1].$$

To compute the derivative of $\mathcal{J}(E_t)$ we would like to use the Hadamard formula (see [14, Chapter 5]). For that, we first need to prove the following proposition.

Proposition 7.4. *The function $t \mapsto \psi_t$ is differentiable in t and its derivative $\dot\psi_t$ satisfies*

$$\begin{cases} -\beta\Delta\dot\psi_t = -\dfrac{1}{K}\dot\psi_t + \dfrac{2}{K}\dot{\mathcal{J}}(E_t) & \text{in } E_t, \\ \Delta\dot\psi_t = 0 & \text{in } E_t^c, \\ \dot\psi_t^+ - \dot\psi_t^- = -(\nabla\psi_t^+ - \nabla\psi_t^-)\cdot\nu(X\cdot\nu) & \text{on } \partial E_t, \\ \beta\nabla\dot\psi_t^+\cdot\nu - \nabla\dot\psi_t^-\cdot\nu = -((\beta\nabla[\nabla\psi_t^+] - \nabla[\nabla\psi_t^-])X)\cdot\nu & \text{on } \partial E_t, \end{cases} \tag{7.7}$$

where $\dot{\mathcal{J}}(E_t) := \frac{d}{dt}\mathcal{J}(E_t)$.

Proof. The proof is standard; see [14, Chapter 5] for the general strategy and [2, Theorem 3.1] for a different kind of a transmission problem. We were unable to find a result covering our particular case in the literature, so we provide a proof here.

We first deal with the material derivative of the function ψ, i.e. we shall look at the function $t \mapsto \widetilde{\psi}_t := \psi_t(\Phi_t(x))$. The advantage is that its derivative in time is in H^1 as we will see. Note that the time derivative of ψ_t itself is not in H^1 as it has a jump on ∂E_t.

Step 1: Moving everything to a fixed domain. We introduce the following notation:

$$A_t(x) := D\Phi_t^{-1}(x)(D\Phi_t^{-1})^t(x)J_{\Phi_t}(x).$$

Note that A_t is symmetric and positive definite and for t small enough it is elliptic with a constant independent of t.

Now we perform a change of variables in Euler–Lagrange equation for ψ_t (6.3) to get the Euler–Lagrange equation for $\widetilde{\psi}_t$:

$$\int_{\mathbb{R}^n} \nabla\Psi(a_B A_t \nabla\widetilde{\psi}_t)\,dx + \frac{1}{K}\int_{B_1} \Psi\widetilde{\psi}_t J_{\Phi_t}(x)\,dx + \frac{1}{|B_1|}\Big(\int_{B_1} \Psi J_{\Phi_t}(x)\,dx\Big)\Big(1 - \frac{1}{K}\int_{B_1} \widetilde{\psi}_t J_{\Phi_t}(x)\,dx\Big) = 0 \tag{7.8}$$

for any $\Psi \in D^1(\mathbb{R}^n)$.

Step 2: Convergence of the material derivative. Let us for convenience denote

$$f(t) := \frac{1}{|B_1|}\Big(1 - \frac{1}{K}\int_{B_1} \widetilde{\psi}_t J_{\Phi_t}(x)\,dx\Big).$$

We write the difference of equations (7.8) for $\widetilde{\psi}_{t+h}$ and $\widetilde{\psi}_t$ and divide it by h to get

$$\begin{aligned}\int_{\mathbb{R}^n} \nabla\Psi\Big(a_{B_1}\frac{A_{t+h}\nabla\widetilde{\psi}_{t+h} - A_t\nabla\widetilde{\psi}_t}{h}\Big)\,dx &+ \frac{1}{K}\int_{B_1} \Psi\Big(\frac{\widetilde{\psi}_{t+h} - \widetilde{\psi}_t}{h}\Big)J_{\Phi_t}(x)\,dx \\ &+ \frac{1}{K}\int_{B_1} \Psi\widetilde{\psi}_{t+h}\frac{J_{\Phi_{t+h}} - J_{\Phi_t}}{h}\,dx + \Big(\int_{B_1} \Psi J_{\Phi_t}(x)\,dx\Big)\frac{f(t+h) - f(t)}{h} \\ &+ \Big(\int_{B_1} \Psi\frac{J_{\Phi_{t+h}}(x) - J_{\Phi_t}(x)}{h}\,dx\Big)f(t+h) = 0\end{aligned}$$

for any $\Psi \in D^1(\mathbb{R}^n)$.

Now, introducing $g_h(x) := \frac{\widetilde{\psi}_{t+h} - \widetilde{\psi}_t}{h}$ also for convenience, we get

$$\begin{aligned}\int_{\mathbb{R}^n} \nabla\Psi(a_{B_1}A_{t+h}\nabla g_h)\,dx &+ \frac{1}{K}\int_{B_1} \Psi g_h J_{\Phi_t}(x)\,dx \\ &+ \int_{\mathbb{R}^n} \nabla\Psi\Big(a_{B_1}\frac{A_{t+h} - A_t}{h}\nabla\widetilde{\psi}_t\Big)\,dx + \frac{1}{K}\int_{B_1} \Psi\widetilde{\psi}_{t+h}\frac{J_{\Phi_{t+h}} - J_{\Phi_t}}{h}\,dx \\ &+ \Big(\int_{B_1} \Psi J_{\Phi_t}(x)\,dx\Big)\frac{f(t+h) - f(t)}{h} \\ &+ \Big(\int_{B_1} \Psi\frac{J_{\Phi_{t+h}}(x) - J_{\Phi_t}(x)}{h}\,dx\Big)f(t+h) = 0\end{aligned} \tag{7.9}$$

for any $\Psi \in D^1(\mathbb{R}^n)$.

Now we want to get a uniform bound on g_h in $D^1(\mathbb{R}^n)$. To do that we argue in a way similar to the proof of (6.6). We use g_h as a test function in (7.9) and get

$$\begin{aligned}
&\int_{\mathbb{R}^n} a_{B_1} \nabla g_h \cdot (A_{t+h} \nabla g_h)\,dx + \frac{1}{K}\int_{B_1} g_h^2 J_{\Phi_t}(x)\,dx \\
&\quad + \int_{\mathbb{R}^n} a_{B_1} \nabla g_h \cdot \Big(\frac{A_{t+h}-A_t}{h}\nabla \widetilde{\psi}_t\Big)\,dx + \frac{1}{K}\int_{B_1} g_h \widetilde{\psi}_{t+h} \frac{J_{\Phi_{t+h}} - J_{\Phi_t}}{h}\,dx \\
&\quad + \Big(\int_{B_1} g_h J_{\Phi_t}(x)\,dx\Big)\frac{f(t+h)-f(t)}{h} \\
&\quad + \Big(\int_{B_1} g_h \frac{J_{\Phi_{t+h}}(x) - J_{\Phi_t}(x)}{h}\,dx\Big) f(t+h) = 0.
\end{aligned}$$

Since $\frac{A(t+h,x)-A(t,x)}{h}$ is bounded in L^∞ and A_t is uniformly elliptic we know that there exists some positive constant c independent of h such that

$$\begin{aligned}
&\int_{\mathbb{R}^n} a_{B_1} \nabla g_h \cdot (A_{t+h} \nabla g_h)\,dx + \int_{\mathbb{R}^n} a_{B_1} \nabla g_h \cdot \Big(\frac{A_{t+h}-A_t}{h}\nabla \widetilde{\psi}_t\Big)\,dx \\
&\quad \geq c\int_{\mathbb{R}^n} |\nabla g_h|^2\,dx - C\int_{\mathbb{R}^n} |\nabla \psi_t|^2\,dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
&c\int_{\mathbb{R}^n} |\nabla g_h|^2\,dx + \frac{1}{K}\int_{B_1} g_h^2 J_{\Phi_t}(x)\,dx \\
&\quad \leq C\int_{\mathbb{R}^n} |\nabla \psi_t|^2\,dx + \frac{1}{K}\int_{B_1} \Big|g_h \widetilde{\psi}_{t+h}\frac{J_{\Phi_{t+h}} - J_{\Phi_t}}{h}\Big|\,dx \\
&\qquad + \Big|\frac{f(t+h)-f(t)}{h}\Big| \int_{B_1} |g_h J_{\Phi_t}(x)|\,dx \\
&\qquad + |f(t+h)| \int_{B_1} \Big|g_h \frac{J_{\Phi_{t+h}}(x) - J_{\Phi_t}(x)}{h}\Big|\,dx \\
&\quad \leq C + C\int_{B_1} |g_h|\,dx + \Big|\frac{f(t+h)-f(t)}{h}\Big| \int_{B_1} |g_h|\,dx \\
&\qquad + |f(t+h)| \int_{B_1} |g_h|\,dx,
\end{aligned} \tag{7.10}$$

where in the last inequality we used inequality (6.6), Proposition 7.3 and (7.1). We want to show now that f is bounded and Lipschitz. Indeed, we recall the definition of f and use the definition of $\widetilde{\psi}_t$ and (6.5):

$$f(t) = \frac{1}{|B_1|}\Big(1 - \frac{1}{K}\int_{B_1} \widetilde{\psi}_t J_{\Phi_t}(x)\,dx\Big) = \frac{1}{|B_1|} - \frac{2}{K}\mathcal{I}(E_t).$$

We get that f is bounded by (6.2). To get Lipschitz continuity, we notice that by direct computation in Lagrangian coordinates one can get that $\frac{\mathcal{I}(E_{t+h})-\mathcal{I}(E_t)}{h}$ is uniformly bounded; see [21, Lemma 3.2]. Plugging this information into (7.10), we get

$$c\int_{\mathbb{R}^n}|\nabla g_h|^2\,dx+\frac{1}{K}\int_{B_1}g_h^2J_{\Phi_t}(x)\,dx\le C+C\int_{B_1}|g_h|\,dx.$$

Finally, we use Young's inequality and (7.1) to obtain

$$c\int_{\mathbb{R}^n}|\nabla g_h|^2\,dx+\frac{1}{2K}\int_{B_1}g_h^2J_{\Phi_t}(x)\,dx\le C.$$

Thus, g_h is uniformly bounded in $D^1(\mathbb{R}^n)$ and up to a subsequence, there exists a weak limit g_0 as h goes to zero. Note that g_0 satisfies

$$\begin{aligned}\int_{\mathbb{R}^n}&\nabla\Psi(a_{B_1}A_t\nabla g_0)\,dx+\int_{\mathbb{R}^n}\nabla\Psi\Big(a_{B_1}\frac{d}{dt}A_t\nabla\widetilde{\psi}_t\Big)\,dx\\&+\frac{1}{K}\int_{B_1}\Psi g_0J_{\Phi_t}(x)\,dx+\frac{1}{K}\int_{B_1}\Psi\widetilde{\psi}_t\dot{J}_{\Phi_t}\,dx\\&-\frac{1}{|B_1|K}\Big(\int_{B_1}\Psi J_{\Phi_t}(x)\,dx\Big)\Big(\int_{B_1}g_0J_{\Phi_t}(x)\,dx-\int_{B_1}\widetilde{\psi}_t\dot{J}_{\Phi_t}\,dx\Big)\\&+\frac{1}{|B_1|}\Big(\int_{B_1}\Psi\dot{J}_{\Phi_t}(x)\,dx\Big)\Big(1-\frac{1}{K}\int_{B_1}\widetilde{\psi}_tJ_{\Phi_t}(x)\,dx\Big)=0\end{aligned}\tag{7.11}$$

for any $\Psi\in D^1(\mathbb{R}^n)$. Let us show that equation (7.11) has a unique solution. To that end, assume that both g_0 and g_0' are solutions of (7.11). Then their difference $w=g_0-g_0'$ satisfies

$$\begin{aligned}\int_{\mathbb{R}^n}&\nabla\Psi(a_{B_1}A_t\nabla w)\,dx+\frac{1}{K}\int_{B_1}\Psi wJ_{\Phi_t}(x)\,dx\\&-\frac{1}{|B_1|K}\int_{B_1}\Psi J_{\Phi_t}(x)\,dx\int_{B_1}wJ_{\Phi_t}(x)\,dx=0\end{aligned}\tag{7.12}$$

for any $\Psi\in D^1(\mathbb{R}^n)$. Since $w\in D^1(\mathbb{R}^n)$, we can test (7.12) with w and get

$$\int_{\mathbb{R}^n}\nabla w(a_{B_1}A_t\nabla w)\,dx+\frac{1}{K}\int_{B_1}w^2J_{\Phi_t}(x)\,dx-\frac{1}{|B_1|K}\Big(\int_{B_1}wJ_{\Phi_t}(x)\,dx\Big)^2=0.$$

By the Cauchy–Schwarz inequality, it yields

$$\int_{\mathbb{R}^n}\nabla w(a_{B_1}A_t\nabla w)\,dx\le 0,$$

which in turn gives us $w=0$ by ellipticity of A_t. Thus, the solution of (7.11) is unique and thus the whole sequence g_h converges to g_0.

To get the strong convergence of the material derivative, we observe that using g_h as a test function in its Euler–Lagrange equation, we get the convergence of the norm in H^1 to the norm of g_0. That, together with weak convergence, gives us strong convergence of g_h.

Step 3: Existence of the shape derivative. We want to show that

$$\dot{\psi}_t = \frac{d}{dt}\widetilde{\psi}_t - X \cdot \nabla \psi_t$$

in $D^1(E_t) \cap D^1(E_t^c)$. Indeed, since $\psi_t(x) = \dot{\psi}_t(\Phi_t^{-1}(x))$, we have

$$\frac{\psi_{t+h}(x) - \psi_{t+h}(x)}{h} = \frac{\psi_{t+h}(\Phi_{t+h}^{-1}(x)) - \psi_t(\Phi_{t+h}^{-1}(x))}{h} + \frac{\psi_t(\Phi_{t+h}^{-1}(x)) - \psi_t(\Phi_t^{-1}(x))}{h}.$$

The first term on the right-hand side converges strongly to $\frac{d}{dt}\psi_t(\Phi_t^{-1}(x))$ as h goes to 0 by Step 2 and continuity of Φ_t. As for the second term, by Proposition 7.3 and the definition of Φ, it converges to $-\nabla\psi_t(\Phi_t^{-1}(x)) \cdot X$ strongly in $D^1(E_t) \cap D^1(E_t^c)$.

Step 4: Equation for the shape derivative. Now that we know that $t \mapsto \psi_t$ is differentiable, we can differentiate the Euler–Lagrange equation for ψ_t given by (7.6) and we get

$$\begin{cases} -\beta\Delta\dot{\psi}_t = -\dfrac{1}{K}\dot{\psi}_t + \dfrac{2}{K}\dot{\mathcal{J}}(E_t) & \text{in } E_t, \\ \Delta\dot{\psi}_t = 0 & \text{in } E_t^c, \\ \dot{\psi}_t^+ - \dot{\psi}_t^- = -(\nabla\psi_t^+ - \nabla\psi_t^-) \cdot X & \text{on } \partial E_t, \\ \beta\nabla\dot{\psi}_t^+ \cdot \nu - \nabla\dot{\psi}_t^1 \cdot \nu = -((\beta\nabla[\nabla\psi_t^+] - \nabla[\nabla\psi_t^-])X) \cdot \nu & \text{on } \partial E_t. \end{cases}$$

Now we can use the boundary conditions in (7.6) to get rid of the tangential part in the right-hand side. Indeed,

$$-(\nabla\psi_t^+ - \nabla\psi_t^-) \cdot X = -(\nabla^\tau\psi_t^+ - \nabla^\tau\psi_t^-) \cdot X^\tau - (\nabla\psi_t^+ - \nabla\psi_t^-) \cdot \nu(X \cdot \nu)$$

and $\nabla^\tau\psi_t^+ = \nabla^\tau\psi_t^-$ by differentiating the equality $\psi_t^+ = \psi_t^-$ on the boundary of E_t. ∎

The following observation, which is a consequence of equality for $\dot{\psi}_t$ will be useful for us.

Lemma 7.5. *There exists $f \in H^{3/2}(E_t) \cap H^{3/2}(E_t^c)$ such that*

$$f^\pm = \nabla\psi_t^\pm \cdot X \text{ on } \partial E_t, \quad \|f^\pm\|_{H^{3/2}} \le C\|\nabla\psi_t^\pm \cdot X\|_{H^1(\partial E_t)}. \tag{7.13}$$

Consider the function $v := \dot{\psi}_t + f$. Then v satisfies the equations

$$\begin{cases} -\beta\Delta v = -\dfrac{1}{K}v + \dfrac{2}{K}\dot{\mathcal{J}}(E_t) - \beta\Delta f + \dfrac{1}{K}f & \text{in } E_t, \\ \Delta v = \Delta f & \text{in } E_t^c, \\ v^+ - v^- = 0 & \text{on } \partial E_t, \\ \beta\nabla v^+ \cdot \nu - \nabla v^- \cdot \nu = (-(\beta\nabla[\nabla\psi_t^+] - \nabla[\nabla\psi_t^-])X + \beta\nabla f^+ - \nabla f^-) \cdot \nu & \text{on } \partial E_t. \end{cases}$$

$$v = \dot{\psi}_t^\pm + \nabla\psi_t^\pm \cdot X \quad \text{on } \partial E_t.$$

Moreover, the following bounds hold:

$$\|v\|_{W^{1,2}(E_t)} + \|v\|_{D^{1,2}(E_t^c)} \leq C(|\dot{\mathcal{J}}(E_t)| + \|X \cdot \nu\|_{H^1(\partial E_t)}), \tag{7.14}$$

$$\|v\|_{L^{2^*}(\mathbb{R}^n)} \leq C(|\dot{\mathcal{J}}(E_t)| + \|X \cdot \nu\|_{H^1(\partial E_t)}). \tag{7.15}$$

Proof. The function f exists since $\nabla\psi_t^\pm \cdot X \in H^1(\partial E_t)$. The equation for v follows from the equation for $\dot{\psi}_t$ and the definition of f. Using the divergence theorem, we get

$$\begin{aligned}
&\int_{E_t} \frac{1}{K} v^2\, dx + \int_{E_t} \beta |\nabla v|^2\, dx + \int_{E_t^c} |\nabla v|^2\, dx \\
&\quad = \int_{E_t} \Big(\frac{2}{K}\dot{\mathcal{J}}(E_t) - \beta\Delta f + \frac{1}{K} f\Big) v\, dx - \int_{E_t^c} \Delta f v\, dx \\
&\qquad + \int_{\partial E_t} \big(\big(-(\beta\nabla[\nabla\psi_t^+] - \nabla[\nabla\psi_t^-])X + \beta\nabla f^+ - \nabla f^-\big)\cdot\nu\big) v\, dx,
\end{aligned}$$

which by the Young, Cauchy–Schwarz and trace inequalities, recalling (7.13), implies that

$$\|v\|_{W^{1,2}(E_t)} + \|v\|_{D^1(E_t^c)} \leq C(|\dot{\mathcal{J}}(E_t)| + \|\nabla\psi_t \cdot X\|_{H^1(\partial E_t)}),$$

which in turn implies by Proposition 7.3 and (7.4),

$$\|v\|_{W^{1,2}(E_t)} + \|v\|_{D^1(E_t^c)} \leq C(|\dot{\mathcal{J}}(E_t)| + \|X \cdot \nu\|_{H^1(\partial E_t)}).$$

Moreover, we can also bound the L^{2^*} norm of v. Indeed, since v does not have a jump on the boundary of E_t, we know by (7.14) that it belongs to the space $D^1(\mathbb{R}^n)$. Thus, employing the Gagliardo–Nirenberg–Sobolev inequality we get (7.15). ■

Proposition 7.6. *For any $t \in [0, 1]$,*

$$\begin{aligned}
\dot{\mathcal{J}}(E_t) &= \Big(1 - \frac{1}{K}\int_{E_t} \psi_t\, dx\Big)\frac{1}{|E_t|}\int_{\partial E_t} \psi_{E_t}(X\cdot\nu)\, d\mathcal{H}^{n-1} \\
&\quad + \frac{1}{2}\int_{\partial E_t} (\beta|\nabla\psi_t^+|^2 - |\nabla\psi_t^-|^2)(X\cdot\nu)\, d\mathcal{H}^{n-1} \\
&\quad + \frac{1}{2K}\int_{\partial E_t} \psi_t^2 (X\cdot\nu)\, d\mathcal{H}^{n-1} \\
&\quad - \int_{\partial E_t} (\nabla\psi_t^-\cdot\nu)((\nabla\psi_t^+ - \nabla\psi_t^-)\cdot\nu)(X\cdot\nu)\, d\mathcal{H}^{n-1} \\
&= \Big(1 - \frac{1}{K}\int_{E_t} \psi_t\, dx\Big)\frac{1}{|E_t|}\int_{E_t} \operatorname{div}(\psi_t X)\, dx + \frac{1}{2}\int_{\mathbb{R}^n} \operatorname{div}(a_{E_t}|\nabla\psi_t|^2 X)\, dx \\
&\quad + \frac{1}{2K}\int_{E_t} \operatorname{div}(\psi_t^2 X)\, dx - \int_{\mathbb{R}^n} \operatorname{div}(a_{E_t}(\nabla\psi_t\cdot\nu)^2 X)\, dx.
\end{aligned}$$

In particular,

$$\dot{\mathcal{J}}(B_1) = 0.$$

Proof. Since ψ_t is the minimizer for $\mathcal{J}$, we have by (6.1),

$$\begin{aligned}\mathcal{J}(E_t) &= \frac{1}{2}\int_{\mathbb{R}^n} a_{E_t}|\nabla\psi_t|^2\,dx + \frac{1}{|E_t|}\int_{E_t}\psi_t\,dx - \frac{1}{2|E_t|K}\Big(\int_{E_t}\psi_t\,dx\Big)^2 \\ &\quad + \frac{1}{2K}\int_{E_t}\psi_t^2\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}^n} a_{E_t}|\nabla\psi_t|^2\,dx + 2\mathcal{J}(E_t) - \frac{2|E_t|}{K}\mathcal{J}(E_t)^2 + \frac{1}{2K}\int_{E_t}\psi_t^2\,dx, \qquad (7.16)\end{aligned}$$

where we used (6.5) for the last equality.

Now we differentiate (7.16) to get

$$\begin{aligned}\dot{\mathcal{J}}(E_t) &= \int_{\mathbb{R}^n} a_{E_t}\nabla\psi_t\cdot\nabla\dot\psi_t\,dx + \frac{1}{2}\int_{\partial E_t}(\beta|\nabla\psi_t^+|^2 - |\nabla\psi_t^-|^2)(X\cdot\nu)\,d\mathcal{H}^{n-1} \\ &\quad + 2\dot{\mathcal{J}}(E_t) - \frac{2}{K}|E_t|2\dot{\mathcal{J}}(E_t)\mathcal{J}(E_t) + \frac{1}{K}\int_{E_t}\dot\psi_t\psi_t\,dx \\ &\quad + \frac{1}{2K}\int_{\partial E_t}\psi_t^2(X\cdot\nu)\,dx.\end{aligned}$$

We note that by (6.5),

$$\dot{\mathcal{J}}(E_t) = \frac{1}{2|E_t|}\int_{E_t}\dot\psi_t\,dx + \frac{1}{2|E_t|}\int_{\partial E_t}\psi_t(X\cdot\nu)\,d\mathcal{H}^{n-1}. \qquad (7.17)$$

Using (7.17) and (7.7), we obtain

$$\begin{aligned}\dot{\mathcal{J}}(E_t) &= -\frac{1}{|E_t|}\Big(\int_{E_t}\dot\psi_t\,dx\Big)\Big(1 - \frac{1}{K}\int_{E_t}\psi_t\,dx\Big) + 2\dot{\mathcal{J}}(E_t)\Big(1 - \frac{1}{K}\int_{E_t}\psi_t\,dx\Big) \\ &\quad + \frac{1}{2}\int_{\partial E_t}(\beta|\nabla\psi_t^+|^2 - |\nabla\psi_t^-|^2)(X\cdot\nu)\,d\mathcal{H}^{n-1} + \frac{1}{2K}\int_{\partial E_t}\psi_t^2(X\cdot\nu)\,dx \\ &\quad + \int_{\partial E_t}(\beta\dot\psi^+\nabla\psi_t^+\cdot\nu - \dot\psi^-\nabla\psi_t^-\cdot\nu)\,d\mathcal{H}^{n-1} \\ &= \Big(1 - \frac{1}{K}\int_{E_t}\psi_t\,dx\Big)\frac{1}{|E_t|}\int_{\partial E_t}\psi_{E_t}(X\cdot\nu)\,d\mathcal{H}^{n-1} \\ &\quad + \frac{1}{2}\int_{\partial E_t}(\beta|\nabla\psi_t^+|^2 - |\nabla\psi_t^-|^2)(X\cdot\nu)\,d\mathcal{H}^{n-1} + \frac{1}{2K}\int_{\partial E_t}\psi_t^2(X\cdot\nu)\,d\mathcal{H}^{n-1} \\ &\quad - \int_{\partial E_t}(\nabla\psi_t^-\cdot\nu)((\nabla\psi_t^+ - \nabla\psi_t^-)\cdot\nu)(X\cdot\nu)\,d\mathcal{H}^{n-1} \\ &= \Big(1 - \frac{1}{K}\int_{E_t}\psi_t\,dx\Big)\frac{1}{|E_t|}\int_{E_t}\operatorname{div}(\psi_t X)\,dx + \frac{1}{2}\int_{\mathbb{R}^n}\operatorname{div}(a_{E_t}|\nabla\psi_t|^2 X)\,dx \\ &\quad + \frac{1}{2K}\int_{E_t}\operatorname{div}(\psi_t^2 X)\,dx - \int_{\mathbb{R}^n}\operatorname{div}(a_{E_t}(\nabla\psi_t\cdot\nu)^2 X)\,dx.\end{aligned}$$

Note that from the second-to-last expression it is easy to see that $\dot{\mathcal{J}}(B_1) = 0$ as ψ_0 is radial by Proposition 6.4 and the volume of E_t is constant (hence $\int_{\partial B_1}(X\cdot\nu)\,d\mathcal{H}^{n-1} = 0$). ■

7.2. Second derivative

Now we differentiate again to get

$$
\begin{aligned}
\ddot{\mathcal{J}}(E_t) = &-\frac{2}{K}\dot{\mathcal{J}}(E_t)\int_{E_t}\operatorname{div}(\psi_t X)\,dx \\
&+\frac{1-\frac{2}{K}|E_t|\mathcal{J}(E_t)}{|E_t|}\left(\int_{E_t}\operatorname{div}(\dot{\psi}_t X)\,dx+\int_{\partial E_t}\operatorname{div}(\psi_t X)(X\cdot\nu)\,d\mathcal{H}^{n-1}\right) \\
&+\int_{\partial E_t}(\beta\nabla\psi_t^+\cdot\nabla\dot{\psi}_t^+-\nabla\psi_t^-\cdot\nabla\dot{\psi}_t^-)(X\cdot\nu)\,d\mathcal{H}^{n-1} \\
&+\frac{1}{2}\int_{\partial E_t}\nabla\big[\beta|\nabla\psi_t^+|^2-|\nabla\psi_t^-|^2\big]\cdot X(X\cdot\nu)\,d\mathcal{H}^{n-1} \\
&+\frac{1}{K}\int_{\partial E_t}\psi_t\dot{\psi}_t(X\cdot\nu)\,d\mathcal{H}^{n-1}+\frac{1}{K}\int_{\partial E_t}\psi_t\nabla\psi_t\cdot X(X\cdot\nu)\,d\mathcal{H}^{n-1} \\
&-2\int_{\partial E_t}\big(\beta(\nabla\dot{\psi}_t^+\cdot\nu)(\nabla\psi_t^+\cdot\nu)-(\nabla\dot{\psi}_t^-\cdot\nu)(\nabla\psi_t^-\cdot\nu)\big)(X\cdot\nu)\,d\mathcal{H}^{n-1} \\
&-\int_{\partial E_t}\nabla\big[\beta(\nabla\psi_t^+\cdot\nu)^2-(\nabla\psi_t^-\cdot\nu)^2\big]\cdot X(X\cdot\nu)\,d\mathcal{H}^{n-1}.
\end{aligned}
$$

Using that the vector field X is divergence-free in the neighborhood of ∂B_1 we get for t small enough,

$$
\begin{aligned}
\ddot{\mathcal{J}}(E_t) = &-\frac{2}{K}\dot{\mathcal{J}}(E_t)\int_{\partial E_t}\psi_t(X\cdot\nu)\,d\mathcal{H}^{n-1} \\
&+\frac{1-\frac{2}{K}|E_t|\mathcal{J}(E_t)}{|E_t|}\left(\int_{\partial E_t}\dot{\psi}_t(X\cdot\nu)\,d\mathcal{H}^{n-1}+\int_{\partial E_t}(\nabla\psi_t^+\cdot X)(X\cdot\nu)\,d\mathcal{H}^{n-1}\right) \\
&+\int_{\partial E_t}(\beta\nabla\psi_t^+\cdot\nabla\dot{\psi}_t^+-\nabla\psi_t^-\cdot\nabla\dot{\psi}_t^-)(X\cdot\nu)\,d\mathcal{H}^{n-1} \\
&+\frac{1}{2}\int_{\partial E_t}\nabla\big[\beta|\nabla\psi_t^+|^2-|\nabla\psi_t^-|^2\big]\cdot X(X\cdot\nu)\,d\mathcal{H}^{n-1} \\
&+\frac{1}{K}\int_{\partial E_t}\psi_t\dot{\psi}_t^+(X\cdot\nu)\,d\mathcal{H}^{n-1}+\frac{1}{K}\int_{\partial E_t}\psi_t\nabla\psi_t^+\cdot X(X\cdot\nu)\,d\mathcal{H}^{n-1} \\
&-2\int_{\partial E_t}\big(\beta(\nabla\dot{\psi}_t^+\cdot\nu)(\nabla\psi_t^+\cdot\nu)-(\nabla\dot{\psi}_t^-\cdot\nu)(\nabla\psi_t^-\cdot\nu)\big)(X\cdot\nu)\,d\mathcal{H}^{n-1} \\
&-\int_{\partial E_t}\nabla\big[\beta(\nabla\psi_t^+\cdot\nu)^2-(\nabla\psi_t^-\cdot\nu)^2\big]\cdot X(X\cdot\nu)\,d\mathcal{H}^{n-1}.
\end{aligned}
\tag{7.18}
$$

Now, to prove Lemma 6.7 we only need the following bound on the second derivative.

Lemma 7.7. *There exist $\delta > 0$ and a constant C such that if $\|\varphi\|_{C^{2,\vartheta}} < \delta$, then*

$$|\ddot{\mathcal{J}}(E_t)| \le C\|X\cdot\nu\|^2_{H^1(\partial B_1)}.$$

We will need the following proposition.

Proposition 7.8.

$$\|\dot{\psi}_t^+\|_{H^1(\partial E_t)} + \|\dot{\psi}_t^-\|_{H^1(\partial E_t)} \le C(\|X\cdot\nu\|_{H^1(\partial E_t)} + |\dot{\mathcal{J}}(E_t)|).$$

To prove the proposition we will use the following theorem concerning Sobolev bounds.

Theorem 7.9 ([18, Theorem 4.20]). *Let G_1 and G_2 be bounded open subsets of $\mathbb{R}^n$ such that $\overline{G}_1 \Subset G_2$ and G_1 intersects an $(n-1)$-dimensional manifold Γ, and put*

$$\Omega_j^{\pm} = G_j \cap \Omega^{\pm} \quad \text{and} \quad \Gamma_j = G_j \cap \Gamma \quad \text{for } j = 1, 2.$$

Suppose, for an integer $r \ge 0$, that Γ_2 is $C^{r+1,1}$, and consider two equations

$$\mathcal{P}u^{\pm} = f^{\pm} \quad \text{on } \Omega_2^{\pm},$$

where $\mathcal{P}$ is strongly elliptic on G_2 with coefficients in $C^{r,1}(\overline{\Omega_2^{\pm}})$. If $u \in L^2(G_2)$ satisfies

$$u^{\pm} \in H^1(\Omega_2^{\pm}), \quad [u]_\Gamma \in H^{r+\frac{3}{2}}(\Gamma_2), \quad [\mathcal{B}_\nu u]_\Gamma \in H^{r+\frac{1}{2}}(\Gamma_2),^{4}$$

and if $f^{\pm} \in H^r(\Omega_2^{\pm})$, then $u^{\pm} \in H^{r+2}(\Omega_1^{\pm})$ and

$$\begin{aligned}\|u^+\|_{H^{r+2}(\Omega_1^+)} + \|u^+\|_{H^{r+2}(\Omega_1^-)} \le{}& C(\|u^+\|_{H^1(\Omega_2^+)} + \|u^-\|_{H^1(\Omega_2^-)}) \\ &+ C\big(\|[u]_{\Gamma_2}\|_{H^{r+\frac{3}{2}}(\Gamma_2)} + \|[\mathcal{B}_\nu u]_{\Gamma_2}\|_{H^{r+\frac{1}{2}}(\Gamma_2)}\big) \\ &+ C(\|f^+\|_{H^r(\Omega_2^+)} + \|f^-\|_{H^r(\Omega_2^-)}).\end{aligned}$$

We need an analogue of the above theorem for $r = -\frac{1}{2}$. To get it, we are going to interpolate between $r = 0$ and $r = -1$. We first prove the following lemma.

Lemma 7.10. *Let E be a set with the boundary in $C^{1,1}$ and let $R > 0$ be such that $B_R \supset \overline{E}$. Consider the equations*

$$\begin{cases} \beta\Delta u^+ = f^+ & \text{in } E, \\ \Delta u^- = f^- & \text{in } B_R\setminus E, \\ u^+ = u^- & \text{on } \partial E, \\ \beta\nabla u^+\cdot\nu - \nabla u^-\cdot\nu = g & \text{on } \partial E, \\ u^- = 0 & \text{on } \partial B_R, \end{cases} \tag{7.19}$$

where $f^+ \in H^{-1}(E)$, $f^- \in H^{-1}(B_R\setminus E)$ and $g \in H^{-1/2}(\partial E)$ are given. Then there exists u, the solution of (7.19) in $W_0^{1,2}(B_R)$, and it satisfies

$$\|u\|^2_{H^1(B_R)} \le C(\|f^+\|^2_{H^{-1}(E)} + \|f^-\|^2_{H^{-1}(B_R\setminus E)} + \|g\|^2_{H^{-1/2}(\partial E)}) \tag{7.20}$$

[4] Here, $\mathcal{B}_\nu$ denotes the conormal derivative. In our case it reduces to $a_E\partial_\nu$ since we deal with the Laplacian.

with $C = C(n, R) > 0$. *Moreover, if* $f^+ \in H^{-1/2}(E)$, $f^- \in H^{-1/2}(B_R \setminus E)$ *and* $g \in L^2(\partial E)$, *then*

$$\|u\|^2_{H^{3/2}(B_R)} \le C(\|f^+\|^2_{H^{-1/2}(E)} + \|f^-\|^2_{H^{-1/2}(B_R\setminus E)} + \|g\|^2_{L^2(\partial E)}) \tag{7.21}$$

with $C = C(n, R) > 0$.

Proof. First we observe that the solution in H^1 exists since it is a minimizer of the following convex functional:

$$\int_{E_t} \Big(\frac{1}{2}\beta|\nabla u^+|^2 - f^+u^+\Big)\,dx + \int_{E_t^c} \Big(\frac{1}{2}|\nabla u^-|^2 - f^-u^-\Big)\,dx + \int_{\partial E_t} g(u^+ - u^-)\,d\mathcal{H}^{n-1}.$$

Note that if we test the equation with the solution itself, we get

$$\int_{E_t} \frac{1}{2}\beta|\nabla u^+|^2\,dx + \int_{E_t^c} \frac{1}{2}|\nabla u^-|^2\,dx = -\int_{E_t} f^+u^+\,dx - \int_{E_t} f^-u^-\,dx + \int_{\partial E_t} u^+g\,d\mathcal{H}^{n-1}.$$

By Poincaré, Cauchy–Schwarz, Young and the trace inequality we obtain (7.20).

Now we consider an operator that takes the functions of the right-hand side and returns the solution of the corresponding transmission problem, i.e. we define $T(f_1, f_2, g)$ for $f_1 \in H^r(E_t)$, $f_2 \in H^r(E_t^c)$, $g \in H^{r+\frac{1}{2}}(\partial E_t)$ as the only H^1 solution of (7.19).

By (7.20), $T\colon H^r \times H^r \times H^{r+\frac{1}{2}} \to H^{r+2}$ for $r = -1$. Moreover, (7.20) together with Theorem 7.9 yields $T\colon H^r \times H^r \times H^{r+\frac{1}{2}} \to H^{r+2}$ for $r \ge 0$, an integer. Thus, interpolating between $r = 0$ and $r = -1$ we get that

$$T\colon H^{-\frac{1}{2}} \times H^{-\frac{1}{2}} \times L^2 \to H^{\frac{3}{2}},$$

so (7.21) holds for an appropriately regular right-hand side. ∎

Proof of Proposition 7.8. Since we are interested only in the value of $\dot{\psi}_t$ on ∂E_t, we multiply it by a cut-off function η. The function $\eta \in C_c^\infty(\mathbb{R}^n)$ is such that

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ in } B_2, \quad \eta \equiv 0 \text{ outside } B_3, \quad |\nabla\eta| \le 2, \quad |\Delta\eta| \le C(n).$$

We would also like to eliminate the jump on the boundary in order to use Lemma 7.10, so we consider a function $u := v\eta$, where v is as in Lemma 7.5 (we recall that $v = \dot{\psi}_t + f$, where f is an $H^{3/2}$ continuation of $\nabla\psi_t \cdot X$ from ∂E_t inside and outside). For δ small enough, all sets E_t lie inside B_2, so

$$u = \dot{\psi}_t + \nabla\psi_t \cdot X \quad \text{on } \partial E_t. \tag{7.22}$$

Note that u satisfies

$$\begin{cases} -\beta\Delta u = -\frac{1}{K}v + \frac{2}{K}\dot{\mathcal{J}}(E_t) + \Delta f & \text{in } E_t, \\ \Delta u = \nabla v \cdot \nabla \eta + (\dot{\psi}_t + f)\Delta\eta & \text{in } E_t^c, \\ u^+ - u^- = 0 & \text{on } \partial E_t, \\ \beta\nabla u^+ \cdot \nu - \nabla u^- \cdot \nu = \big(-(\beta\nabla[\nabla\psi_t^+] - \nabla[\nabla\psi_t^-])X + \beta\nabla f^+ - \nabla f^-\big)\cdot\nu & \text{on } \partial E_t, \\ u = 0 & \text{on } \partial B_3. \end{cases}$$

By Lemma 7.10,

$$\begin{aligned} &\|u^+\|_{H^{\frac{3}{2}}(E_t)} + \|u^-\|_{H^{\frac{3}{2}}(E_t^c)} \\ &\quad\le C\big(\|(\beta\nabla[\nabla\psi_t^+\cdot X])\cdot\nu\|_{L^2(\Gamma_2)} + \|(\nabla[\nabla\psi_t^-\cdot X])\cdot\nu\|_{L^2(\Gamma_2)}\big) \\ &\qquad + C\big(\|(\beta\nabla[\nabla\psi_t^+]\cdot X)\cdot\nu\|_{L^2(\Gamma_2)} + \|(\nabla[\nabla\psi_t^-]\cdot X)\cdot\nu\|_{L^2(\Gamma_2)}\big) \\ &\qquad + C\Big(\Big\|\frac{1}{K}v\Big\|_{H^{-\frac{1}{2}}(E_t)} + \Big\|\frac{2}{K}\dot{\mathcal{J}}(E_t)\Big\|_{H^{-\frac{1}{2}}(E_t)} + \|\Delta f\|_{H^{-\frac{1}{2}}(E_t)}\Big) \\ &\qquad + C\big(\|\nabla v\cdot\nabla\eta\|_{H^{-\frac{1}{2}}(E_t^c)} + \|v\Delta\eta\|_{H^{-\frac{1}{2}}(E_t^c)}\big). \end{aligned}$$

Now we employ Proposition 7.3, inequality (7.4) and the definition of f to get

$$\begin{aligned} \|u^+\|_{H^{\frac{3}{2}}(E_t)} + \|u^-\|_{H^{\frac{3}{2}}(E_t^c)} &\le C(\|X\cdot\nu\|_{H^1(\partial E_t)} + |\dot{\mathcal{J}}(E_t)|) \\ &\quad + C(\|\nabla v\cdot\nabla\eta\|_{H^{-\frac{1}{2}}(E_t^c)} + \|v\Delta\eta\|_{H^{-\frac{1}{2}}(E_t^c)}). \end{aligned}$$

Remembering (7.22), using the trace inequality and properties of η, we have

$$\begin{aligned} &\|\dot{\psi}_t^+\|_{H^1}(\partial E_t) + \|\dot{\psi}_t^-\|_{H^1(\partial E_t)} \\ &\quad\le C(\|X\cdot\nu\|_{H^1(\partial E_t)} + |\dot{\mathcal{J}}(E_t)|) + C(\|\nabla v\cdot\nabla\eta\|_{H^{-\frac{1}{2}}(E_t^c)} + \|v\Delta\eta\|_{H^{-\frac{1}{2}}(E_t^c)}) \\ &\quad\le C(\|X\cdot\nu\|_{H^1(\partial E_t)} + |\dot{\mathcal{J}}(E_t)|) + C(\|\nabla v\cdot\nabla\eta\|_{L^2(E_t^c)} + \|v\Delta\eta\|_{L^2(E_t^c)}) \\ &\quad\le C(\|X\cdot\nu\|_{H^1(\partial E_t)} + |\dot{\mathcal{J}}(E_t)|) + C(\|\nabla v\|_{L^2(E_t^c)} + \|v\|_{L^2(B_3\setminus B_2)}). \end{aligned}$$

Now it remains to recall bounds (7.14) and (7.15) and notice that $\|\cdot\|_{L^2(B_3\setminus B_2)} \le C\|\cdot\|_{L^{2^*}(B_3\setminus B_2)}$. ■

Proof of Lemma 7.7. Let us first show that the lemma is implied by the following claim.

Claim: $|\ddot{\mathcal{J}}(E_t)| \le C\big(\|X\cdot\nu\|^2_{H^1(\partial B_1)} + \dot{\mathcal{J}}(E_t)\|X\cdot\nu\|_{H^1(\partial B_1)}\big)$.

Indeed, suppose we proved the claim. Denote $\dot{\mathcal{J}}(E_t)$ by $h(t)$. Then we know the following:

$$\begin{cases} |h'(t)| \le C\big(\|X\cdot\nu\|^2_{H^1(\partial B_1)} + h(t)\|X\cdot\nu\|_{H^1(\partial B_1)}\big), \\ h(0) = 0. \end{cases}$$

Let us show that

$$|h(t)| \leq \|X \cdot \nu\|_{H^1(\partial B_1)}; \tag{7.23}$$

then the lemma will follow immediately. Suppose that there exists a time $t \in (0, 1]$ such that the inequality (7.23) fails. We denote by t^* the first time when it happens, i.e.

$$t^* := \inf_{t \in [0,1]} \big\{t : (7.23) \text{ fails}\big\}.$$

Since inequality (7.23) is true for $t = 0$, the following holds:

$$|h(t^*)| = \|X \cdot \nu\|_{H^1(\partial B_1)}, \quad |h(t)| \leq \|X \cdot \nu\|_{H^1(\partial B_1)} \text{ for } t \in [0, t^*].$$

Now, as $h(0) = 0$, we can write

$$h(t^*) = \int_0^{t^*} h'(t)\,dt$$

and thus

$$\begin{aligned}
\|X \cdot \nu\|_{H^1(\partial B_1)} = |h(t^*)| &\leq \int_0^{t^*} |h'(t)|, dt \\
&\leq \int_0^{t^*} C(\|X \cdot \nu\|^2_{H^1(\partial B_1)} + h(t)\|X \cdot \nu\|_{H^1(\partial B_1)})dt \\
&\leq 2C\|X \cdot \nu\|^2_{H^1(\partial B_1)}.
\end{aligned}$$

However, that cannot hold for $\|X \cdot \nu\|_{H^1(\partial B_1)}$ small enough. That means that (7.23) holds for all times t.

Proof of the claim. By (7.18) we have

$$\begin{aligned}
\ddot{\mathcal{J}}(E_t) = &-\frac{2}{K}\dot{\mathcal{J}}(E_t)\int_{\partial E_t} \psi_t(X \cdot \nu)\,d\mathcal{H}^{n-1} \\
&+ \frac{1}{2}\int_{\partial E_t} \nabla\big[\beta|\nabla\psi_t^+|^2 - |\nabla\psi_t^-|^2\big] \cdot X(X \cdot \nu)\,d\mathcal{H}^{n-1} \\
&+ \int_{\partial E_t} \Big(\frac{1 - \frac{2}{K}|B_1|\mathcal{J}(E_t)}{|B_1|} + \frac{1}{K}\psi_t\Big)(\nabla\psi_t^+ \cdot X)(X \cdot \nu)\,d\mathcal{H}^{n-1} \\
&+ \int_{\partial E_t} \Big(\Big(\frac{1 - \frac{2}{K}|B_1|\mathcal{J}(E_t)}{|B_1|}\Big) + \frac{1}{K}\psi_t\Big)\dot{\psi}_t^+(X \cdot \nu)\,d\mathcal{H}^{n-1} \\
&+ \int_{\partial E_t} (\beta\nabla\psi_t^+ \cdot \nabla\dot{\psi}_t^+ - \nabla\psi_t^- \cdot \nabla\dot{\psi}_t^-)(X \cdot \nu)\,d\mathcal{H}^{n-1} \\
&- 2\int_{\partial E_t} \big(\beta(\nabla\dot{\psi}_t^+ \cdot \nu)(\nabla\psi_t^+ \cdot \nu) - (\nabla\dot{\psi}_t^- \cdot \nu)(\nabla\psi_t^- \cdot \nu)\big)(X \cdot \nu)\,d\mathcal{H}^{n-1} \\
&- \int_{\partial E_t} \nabla\big[\beta(\nabla\psi_t^+ \cdot \nu)^2 - (\nabla\psi_t^- \cdot \nu)^2\big] \cdot X(X \cdot \nu)\,d\mathcal{H}^{n-1} \\
=: &\, I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t) + I_7(t).
\end{aligned}$$

We start with I_1. Using the expression for $\dot{\mathcal{J}}(E_t)$ obtained in Proposition 7.6, we get

$$\begin{aligned}
-\frac{K}{2} I_1(t) &= \dot{\mathcal{J}}(E_t) \int_{\partial E_t} \psi_t (X \cdot \nu)\, d\mathcal{H}^{n-1} \\
&= \Big(1 - \frac{1}{K} \int_{E_t} \psi_t \, dx\Big) \frac{1}{|B_1|} \Big(\int_{\partial E_t} \psi_t (X \cdot \nu)\, d\mathcal{H}^{n-1}\Big)^2 \\
&\quad + \frac{1}{2} \int_{\partial E_t} (\beta |\nabla \psi_t^+|^2 - |\nabla \psi_t^-|^2)(X \cdot \nu)\, d\mathcal{H}^{n-1} \int_{\partial E_t} \psi_t (X \cdot \nu)\, d\mathcal{H}^{n-1} \\
&\quad + \frac{1}{2K} \int_{\partial E_t} \psi_t^2 (X \cdot \nu)\, d\mathcal{H}^{n-1} \int_{\partial E_t} \psi_t (X \cdot \nu)\, d\mathcal{H}^{n-1}.
\end{aligned}$$

Thus,

$$|I_1(t)| \le g(\|\psi_t\|_{C^1(\overline{E_t})}) \|X \cdot \nu\|^2_{L^1(\partial E_t)}$$

for some bounded function g.

To prove the bounds for I_2, I_3 and I_7, we rewrite X as $(X \cdot \nu)\nu + X^\tau$ and use that

$$|X^\tau \circ \Phi_t| \le \omega(\|\varphi\|_{C^{2,\vartheta}}) |X \cdot \nu_{B_1}|.$$

Indeed,

$$\begin{aligned}
I_2(t) &= \frac{1}{2} \int_{\partial E_t} \nabla\big[\beta |\nabla \psi_t^+|^2 - |\nabla \psi_t^-|^2\big] \cdot X (X \cdot \nu)\, d\mathcal{H}^{n-1} \\
&= \frac{1}{2} \int_{\partial E_t} \nabla\big[\beta |\nabla \psi_t^+|^2 - |\nabla \psi_t^-|^2\big] \cdot \nu (X \cdot \nu)^2\, d\mathcal{H}^{n-1} \\
&\quad + \frac{1}{2} \int_{\partial E_t} \nabla\big[\beta |\nabla \psi_t^+|^2 - |\nabla \psi_t^-|^2\big] \cdot X^\tau (X \cdot \nu)\, d\mathcal{H}^{n-1}
\end{aligned}$$

and thus

$$|I_2(t)| \le g(\|\psi_t\|_{C^2(\overline{E_t})}) \|X \cdot \nu\|^2_{L^2(\partial E_t)}$$

for some bounded function g. The terms I_3 and I_7 are treated in the same way.

To bound I_4, I_5 and I_6 we use Propositions 7.8 and 7.3. Let us show that the inequality for I_5, I_4 and I_6 can be treated in a similar way:

$$\begin{aligned}
&\Big| \int_{\partial E_t} (\beta \nabla \psi_t^+ \cdot \nabla \dot{\psi}_t^+ - \nabla \psi_t^- \cdot \nabla \dot{\psi}_t^-)(X \cdot \nu)\, d\mathcal{H}^{n-1} \Big| \\
&\quad \le \int_{\partial E_t} (|\beta \nabla \psi_t^+ \cdot \nabla \dot{\psi}_t^+| + |\nabla \psi_t^- \cdot \nabla \dot{\psi}_t^-|) |X \cdot \nu|\, d\mathcal{H}^{n-1} \\
&\quad \le \Big(\Big(\int_{\partial E_t} |\beta \nabla \psi_t^+ \cdot \nabla \dot{\psi}_t^+|^2\, d\mathcal{H}^{n-1}\Big)^{\frac{1}{2}} \\
&\qquad + \Big(\int_{\partial E_t} |\nabla \psi_t^- \cdot \nabla \dot{\psi}_t^-|^2\, d\mathcal{H}^{n-1}\Big)^{\frac{1}{2}}\Big) \|X \cdot \nu\|_{L^2(\partial E_t)}
\end{aligned}$$

$$\leq g(\|\psi_t\|_{C^2(\overline{E_t})})\Big(\Big(\int_{\partial E_t}|\nabla\dot{\psi}_t^+|^2\,d\mathcal{H}^{n-1}\Big)^{\frac{1}{2}} + \Big(\int_{\partial E_t}|\nabla\dot{\psi}_t^-|^2\,d\mathcal{H}^{n-1}\Big)^{\frac{1}{2}}\Big)\|X\cdot\nu\|_{L^2(\partial E_t)}$$

$$\leq g(\|\psi_t\|_{C^2(\overline{E_t})})(\|X\cdot\nu\|_{H^1(\partial E_t)} + |\dot{\mathcal{J}}(E_t)|)\|X\cdot\nu\|_{L^2(\partial E_t)} \qquad \blacksquare$$

■

Now we are ready to prove Lemma 6.7.

Proof of Lemma 6.7.

$$\mathcal{J}(E) = \mathcal{J}(B_1) + \dot{\mathcal{J}}(B_1) + \int_0^1 (1-s)\ddot{\mathcal{J}}(E_s)\,ds.$$

By Proposition 7.6 we know that $\dot{\mathcal{J}}(B_1) = 0$. Now use Lemma 7.7 to bound the integral. ■

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