Benjamin–Feir Instability of Stokes Waves in Finite Depth

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Abstract

Whitham and Benjamin predicted in 1967 that small-amplitude periodic travel-ing Stokes waves of the 2d-gravity water waves equations are linearly unstable with respect to long-wave perturbations, if the depth h is larger than a critical threshold $h_{WB} \approx 1.363$. In this paper, we completely describe, for any *finite* value of $h > 0$, the four eigenvalues close to zero of the linearized equations at the Stokes wave, as the Floquet exponent μ is turned on. We prove, in particular, the existence of a unique depth h_{WB}, which coincides with the one predicted by Whitham and Ben-jamin, such that, for any $0 < h < h_{WB}$, the eigenvalues close to zero are purely imaginary and, for any $h > h_{WB}$, a pair of non-purely imaginary eigenvalues de-picts a closed figure "8", parameterized by the Floquet exponent. As $h \to h_{WB}^+$ the "8" collapses to the origin of the complex plane. The complete bifurcation diagram of the spectrum is not deduced as in deep water, since the limits $h \to +\infty$ (deep water) and $\mu \to 0$ (long waves) do not commute. In finite depth, the four eigen-values have all the same size $\mathcal{O}(\mu)$, unlike in deep water, and the analysis of their splitting is much more delicate, requiring, as a new ingredient, a non-perturbative step of block-diagonalization. Along the whole proof, the explicit dependence of the matrix entries with respect to the depth h is carefully tracked.

Contents

1. Introduction to Main Results

A classical problem in fluid dynamics, pioneered by the famous work of Stokes [36] in 1847, concerns the spectral stability/instability of periodic traveling waves—called Stokes waves– of the gravity water waves equations in any depth.

Benjamin and Feir [3], Lighthill [30] and Zakharov [40,42] discovered in the sixties, through experiments and formal arguments, that Stokes waves in deep water are unstable, proposing an heuristic mechanism which leads to the disintegration of wave trains. More precisely, these works predicted unstable eigenvalues of the linearized equations at the Stokes wave, near the origin of the complex plane, corresponding to small Floquet exponents μ or, equivalently, to long-wave perturbations. The same phenomenon was later predicted by Whitham [38] and Benjamin [2] for Stokes waves of wavelength $2\pi\kappa$, in finite depth h, provided that $\kappa h > 1.363$ approximately. This phenomenon is nowadays called "Benjamin–Feir"—or modulational– instability, and it is supported by an enormous amount of physical observations and numerical simulations, see e.g. [16,31]. We refer to [43] for an historical survey.

A serious difficulty for a rigorous mathematical proof of the Benjamin–Feir instability is that the perturbed eigenvalues bifurcate from the eigenvalue zero, which is *defective*, with multiplicity four. The first rigorous proof of a local branch of unstable eigenvalues close to zero for κh larger than the Whitham-Benjamin threshold $1.363\ldots$ was obtained by Bridges-Mielke [9] in finite depth (see also the preprint [23]). Their method, based on a spatial dynamics and a center manifold reduction, breaks down in deep water. For dealing with this case Nguyen-Strauss [33] have recently developed a new approach, based on a Lyapunov-Schmidt decomposition. Very recently Berti-Maspero-Ventura [6], in deep water, provided a detailed account of the splitting of the four eigenvalues close to zero, as the Floquet exponent is turned on (see also [7] for a review of this result).

The goal of this paper is to completely describe the Benjamin–Feir spectrum at any finite value of the depth h > 0. This analysis has fundamental physical importance, since real-life experiments are performed in water tanks (for example the original Benjamin and Feir experiments, in Feltham's National Physical Laboratory, had Stokes waves of wavelength 2.2 m and bottom's depth of 7.62 m, see [2]). The limits h $\rightarrow +\infty$ (infinite depth) and $\mu \rightarrow 0$ (long waves) do not commute and the emergence of Benjamin–Feir unstable eigenvalues in finite depth is *not* a direct followup of the infinite depth case.

Through out this paper, with no loss of generality, we consider 2π-periodic Stokes waves, i.e. with wave number $\kappa = 1$. In Theorems 2.5 and 1.1 we prove the existence of a unique depth h_{WB}, in perfect agreement with the Benjamin–Feir critical value 1.363..., such that

- *Shallow water case:* for any $0 < h < h_{WB}$ the eigenvalues close to zero are purely imaginary for Stokes waves of sufficiently small amplitude, see Fig. 2-left;
- *Sufficiently deep water case:* for any $h_{WB} < h < \infty$, there exists a pair of non-purely imaginary eigenvalues which traces a complete closed figure "8" (as shown in Fig. 2-right) parameterized by the Floquet exponent μ. By further

increasing μ, the eigenvalues recollide far from the origin on the imaginary axis where then they keep moving. As $\mathrm{h} \to \mathrm{h}_{\mathrm{WB}}^{+}$ the set of unstable Floquet exponents shrinks to zero and the Benjamin–Feir unstable eigenvalues collapse to the origin, see Fig. 3. This figure '8" was first numerically discovered by Deconink-Oliveras in [16].

We remark that the present approach provides a necessary and sufficient condition for the existence of unstable eigenvalues.

We encounter several differences between the current proof and the one of the infinite depth case in [6], the major of which we anticipate here. In the deep water ideal case it turns out that the "reduced" 4×4 matrix obtained by the Kato spectral procedure is a small perturbation of a block-diagonal matrix which shows up the Benjamin–Feir unstable eigenvalues. In finite depth this is not the case; the coupling between the 2×2 block-diagonal matrices and the out-diagonal ones is much stronger. The difference arises because, when $\mathrm{h} = +\infty$, the 4×4 reduced Kato matrix has two eigenvalues of size $\mathcal{O}(\mu)$ and the other two have the much bigger size $\mathcal{O}(\sqrt{\mu})$, whereas in finite depth all four eigenvalues are $\mathcal{O}(\mu)$. In turn, this is due to the different asymptotic expansions of the function

$$\sqrt{\mu \tanh(\mu \mathrm{h})} = \begin{cases} \sqrt{\mu} & \text{if } \mathrm{h} = +\infty, \\ \sqrt{\mathrm{h}}\mu + O(\mu^3) & \forall \mathrm{h} > 0 \text{ as } \mu \to 0, \end{cases}$$

appearing in the Floquet operator (see Sect. 2). This significantly increases the complexity of the spectral analysis. In order to rigorously compute the spectrum of the 4×4 reduced matrix in finite depth (not only providing a formal expansion) we introduce a novel *non-perturbative* step of block diagonalization, which considerably modifies the block-diagonal matrices (see comments below Theorem 2.5). Such procedure is uniform in h only on compact subsets of $(0, +\infty)$ and becomes singular in the deep water limit.

These differences indicate that the limits $\mathrm{h} \to +\infty$ (infinite depth) and $\mu \to 0$ (long wave) can not be simply interchanged, and the connection between the Benjamin–Feir instability in these two cases is far more complex: the modulational instability in infinite depth is not the limit of the finite depth one, nor the latter is a direct followup of the infinite depth case.

Let us now present, rigorously, our results.

Benjamin–Feir Instability in Finite Depth

We consider the pure gravity water waves equations for a bidimensional fluid occupying a region with finite depth h. With no loss of generality we set the gravity $g = 1$, see Remark 2.4. We consider a 2π-periodic Stokes wave with amplitude $0 < \epsilon \ll 1$ and speed

$$c_\epsilon = c_{\mathrm{h}} + \mathcal{O}(\epsilon^2), \quad c_{\mathrm{h}} := \sqrt{\tanh(\mathrm{h})}.$$

The linearized water waves equations at the Stokes wave are, in the inertial reference frame moving with speed c_ϵ, a linear time independent system of the form $h_t = \mathcal{L}_\epsilon h$ where $\mathcal{L}_\epsilon := \mathcal{L}_\epsilon(\mathrm{h})$ is a linear operator with 2π-periodic coefficients, see (2.17) (the

operator $\mathcal{L}_\epsilon$ in (2.17) is actually obtained conjugating the linearized water waves equations in the Zakharov formulation at the Stokes wave via the "good unknown of Alinhac" (2.11) and the Levi-Civita (2.16) invertible transformations). The operator $\mathcal{L}_\epsilon$ possesses the eigenvalue 0, which is defective, with multiplicity four, due to symmetries of the water waves equations. The problem is to prove that the linear system $h_t = \mathcal{L}_\epsilon h$ has solutions of the form $h(t, x) = \mathrm{Re}\left(e^{\lambda t} e^{\mathrm{i}\mu x} v(x)\right)$ where $v(x)$ is a 2π-periodic function, μ in $\mathbb{R}$ is the Floquet exponent and λ has positive real part, thus $h(t, x)$ grows exponentially in time. By Bloch-Floquet theory, such λ is an eigenvalue of the operator $\mathcal{L}_{\mu,\epsilon} := e^{-\mathrm{i}\mu x} \mathcal{L}_\epsilon\, e^{\mathrm{i}\mu x}$ acting on 2π-periodic functions.

The main result of this paper proves, for any finite value of the depth h, the full splitting of the four eigenvalues close to zero of the operator $\mathcal{L}_{\mu,\epsilon} := \mathcal{L}_{\mu,\epsilon}(\mathrm{h})$ when ϵ and μ are small enough, see Theorem 2.5. We first present Theorem 1.1 which focuses on the figure "8" formed by the Benjamin–Feir *unstable* eigenvalues.

We first need to introduce the "Whitham-Benjamin" function

$$e_{\mathrm{WB}} := e_{\mathrm{WB}}(\mathrm{h}) := \frac{1}{c_{\mathrm{h}}}\left[\frac{9c_{\mathrm{h}}^8 - 10c_{\mathrm{h}}^4 + 9}{8c_{\mathrm{h}}^6} \right.$$
$$\left. - \frac{1}{\mathrm{h} - \frac{1}{4}e_{12}^2}\left(1 + \frac{1 - c_{\mathrm{h}}^4}{2} + \frac{3}{4}\frac{(1 - c_{\mathrm{h}}^4)^2}{c_{\mathrm{h}}^2}\mathrm{h}\right)\right], \quad (1.1)$$

where $c_{\mathrm{h}} = \sqrt{\tanh(\mathrm{h})}$ is the speed of the linear Stokes wave, and

$$e_{12} := e_{12}(\mathrm{h}) := c_{\mathrm{h}} + c_{\mathrm{h}}^{-1}(1 - c_{\mathrm{h}}^4)\mathrm{h} > 0, \quad \forall \mathrm{h} > 0. \qquad (1.2)$$

The function $e_{\mathrm{WB}}(\mathrm{h})$ is well defined for any $\mathrm{h} > 0$ because the denominator $\mathrm{h} - \frac{1}{4}e_{12}^2 > 0$ in (1.1) is positive for any $\mathrm{h} > 0$, see Lemma 5.7. The function (1.1) coincides, up to a non zero factor, with the celebrated function obtained by Whitham [38], Benjamin [2] and Bridges-Mielke [9] which determines the "shallow/sufficiently deep" threshold regime. In particular the Whitham-Benjamin function $e_{\mathrm{WB}}(\mathrm{h})$ vanishes at $\mathrm{h}_{\mathrm{WB}} = 1.363...$, it is negative for $0 < \mathrm{h} < \mathrm{h}_{\mathrm{WB}}$, positive for $\mathrm{h} > \mathrm{h}_{\mathrm{WB}}$ and tends to 1 as $\mathrm{h} \to +\infty$, see Fig. 1. We also introduce the positive coefficient

$$e_{22} := e_{22}(\mathrm{h}) := \frac{(1 - c_{\mathrm{h}}^4)(1 + 3c_{\mathrm{h}}^4)\mathrm{h}^2 + 2c_{\mathrm{h}}^2(c_{\mathrm{h}}^4 - 1)\mathrm{h} + c_{\mathrm{h}}^4}{c_{\mathrm{h}}^3} > 0, \quad \forall \mathrm{h} > 0. \tag{1.3}$$

We remark that the functions $e_{12}(\mathrm{h}) > c_{\mathrm{h}}$ and $e_{22}(\mathrm{h}) > 0$ are positive for any $\mathrm{h} > 0$, tend to 0 as $\mathrm{h} \to 0^+$ and to 1 as $\mathrm{h} \to +\infty$, see Lemma 4.8.

Through out the paper we denote by $r(\epsilon^{m_1}\mu^{n_1}, \ldots, \epsilon^{m_p}\mu^{n_p})$ a real analytic function fulfilling for some $C > 0$ and ϵ, μ sufficiently small, the estimate $|r(\epsilon^{m_1}\mu^{n_1}, \ldots, \epsilon^{m_p}\mu^{n_p})| \leq C \sum_{j=1}^{p} |\epsilon|^{m_j}|\mu|^{n_j}$, where the constant $C := C(\mathrm{h})$ is uniform for h in any compact set of $(0, +\infty)$.
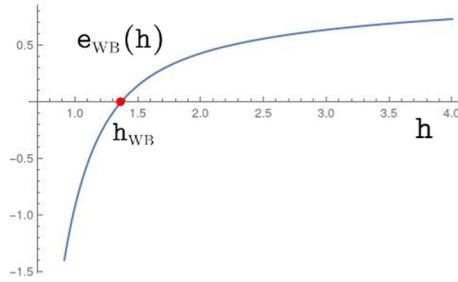
Fig. 1. Plot of the Whitham-Benjamin function $e_{WB}(h)$. The red dot shows its unique root $h_{WB} = 1.363\ldots$. which is the celebrated "shallow/sufficiently deep" water threshold predicted independently by Whitham (cfr. [38] p.49) and Benjamin (cfr. [2] p.68), and recovered in the rigorous proof of Bridges-Mielke [9, p. 183]

Theorem 1.1. (Benjamin–Feir unstable eigenvalues) *For any* $h > h_{WB}$, *there exist* $\epsilon_1, \mu_0 > 0$ *and an analytic function* $\underline{\mu} : [0, \epsilon_1) \to [0, \mu_0)$, *of the form*

$$\underline{\mu}(\epsilon) = e_h \epsilon (1 + r(\epsilon)), \quad e_h := \sqrt{\frac{8 e_{WB}(h)}{e_{22}(h)}}, \tag{1.4}$$

such that, for any $\epsilon \in [0, \epsilon_1)$, *the operator* $\mathcal{L}_{\mu,\epsilon}$ *has two eigenvalues* $\lambda_1^{\pm}(\mu, \epsilon)$ *of the form*

$$\begin{cases} i\frac{1}{2}\check{c}_h \mu + i\, r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \\ \quad \pm \frac{1}{8}\mu\sqrt{e_{22}(h)}(1 + r(\epsilon, \mu))\sqrt{\Delta_{BF}(h; \mu, \epsilon)}, & \forall \mu \in [0, \underline{\mu}(\epsilon)) \\ i\frac{1}{2}\check{c}_h \underline{\mu}(\epsilon) + i\, r(\epsilon^3), & \mu = \underline{\mu}(\epsilon) \\ i\frac{1}{2}\check{c}_h \mu + i\, r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \\ \quad \pm i\frac{1}{8}\mu\sqrt{e_{22}(h)}(1 + r(\epsilon, \mu))\sqrt{|\Delta_{BF}(h; \mu, \epsilon)|}, & \forall \mu \in (\underline{\mu}(\epsilon), \mu_0) \end{cases} \tag{1.5}$$

*where* $\check{c}_h := 2c_h - e_{12}(h) > 0$ *and* $\Delta_{BF}(h; \mu, \epsilon)$ *is the "Benjamin–Feir discriminant" function*

$$\Delta_{BF}(h; \mu, \epsilon) := 8 e_{WB}(h)\epsilon^2 + r_1(\epsilon^3, \mu\epsilon^2) - e_{22}(h)\mu^2 \big(1 + r_1''(\epsilon, \mu)\big). \tag{1.6}$$

Note that, for any $0 < \epsilon < \epsilon_1$ (*depending on* h) *the function* $\Delta_{BF}(h; \mu, \epsilon) > 0$ *is positive, respectively* < 0, *provided* $0 < \mu < \underline{\mu}(\epsilon)$, *respectively* $\mu > \underline{\mu}(\epsilon)$.

Let us make some comments.

1. BENJAMIN--FEIR UNSTABLE EIGENVALUES. For $h > h_{WB}$, according to (1.5), for values of the Floquet parameter $0 < \mu < \underline{\mu}(\epsilon)$, the eigenvalues $\lambda_1^{\pm}(\mu, \epsilon)$ have opposite non-zero real part. As μ tends to $\underline{\mu}(\epsilon)$, the two eigenvalues $\lambda_1^{\pm}(\mu, \epsilon)$ collide on the imaginary axis *far* from 0 (in the upper semiplane $\mathrm{Im}(\lambda) > 0$), along which they keep moving for $\mu > \underline{\mu}(\epsilon)$, see Figure 2. For $\mu < 0$ the operator $\mathcal{L}_{\mu,\epsilon}$ possesses the symmetric eigenvalues $\lambda_1^{\pm}(-\mu, \epsilon)$ in the semiplane $\mathrm{Im}(\lambda) < 0$. For
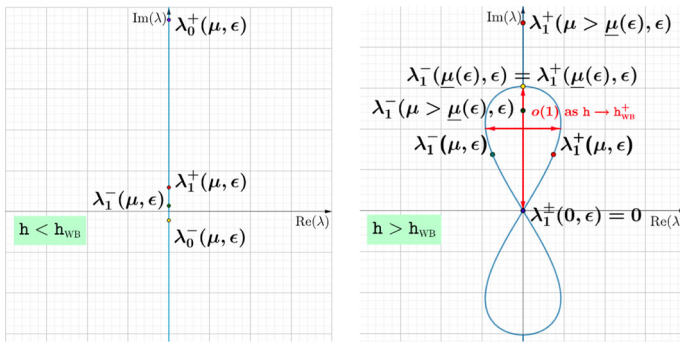
Fig. 2. The picture on the left shows, in the "shallow" water regime $h < h_{WB}$, the eigenvalues $\lambda_1^{\pm}(\mu, \epsilon)$ and $\lambda_0^{\pm}(\mu, \epsilon)$ which are purely imaginary. The picture on the right shows, in the "sufficiently deep" water regime $h > h_{WB}$, the eigenvalues $\lambda_1^{\pm}(\mu, \epsilon)$ in the complex λ-plane at fixed $|\epsilon| \ll 1$ as μ varies. This figure "8" depends on h and shrinks to 0 as $h \to h_{WB}^+$, see Fig. 3. As $h \to +\infty$ the spectrum resembles the one in deep water found in [6]

$\mu \in [0, \underline{\mu}(\epsilon)]$ we obtain the upper part of the figure "8", which is well approximated by the curves

$$\mu \mapsto \left(\pm \frac{\mu}{8} \sqrt{e_{22}} \sqrt{8 e_{WB} \epsilon^2 - e_{22} \mu^2}, \ \tfrac{1}{2} \check{c}_h \mu \right), \tag{1.7}$$

in accordance with the numerical simulations by Deconinck-Oliveras [16], and the formal expansions in [15]. Note that for $\mu > 0$ the imaginary part in (1.7) is positive because $\check{c}_h = c_h^{-1}(\tanh(h) - (1 - \tanh^2(h))h) > 0$ for any $h > 0$. The higher order "side-band" corrections of the eigenvalues $\lambda_1^{\pm}(\mu, \epsilon)$ in (1.5), provided by the analytic functions r, r_1, r_1'', r_2, are explicitly computable. We finally remark that the eigenvalues (1.5) are *not analytic* in (μ, ϵ) close to the value $(\underline{\mu}(\epsilon), \epsilon)$ where $\lambda_1^{\pm}(\mu, \epsilon)$ collide at the top of the figure "8" far from 0 (clearly they are continuous).

2. BEHAVIOUR NEAR THE WHITHAM-BENJAMIN DEPTH h_{WB}. As $h \to h_{WB}^+$ the constant $\epsilon_1 := \epsilon_1(h) > 0$ in Theorem 1.1 tends to zero, the set of unstable Floquet exponents $(0, \underline{\mu}(\epsilon))$ with $\underline{\mu}(\epsilon) = e_h \epsilon(1 + r(\epsilon))$ given in (1.4) shrinks to zero and the figure "8" of Benjamin–Feir unstable eigenvalues collapse to zero, see Fig. 3. In particular

$$\max_{\mu \in [0, \underline{\mu}(\epsilon)]} \operatorname{Re} \lambda_1^+(\mu, \epsilon) = \operatorname{Re} \lambda_1^+(\mu_{\max}, \epsilon) = \frac{1}{2} e_{WB}(h) \epsilon^2 + r(\epsilon^3) \ \text{ and } \tag{1.8}$$

tends to zero as $h \to h_{WB}^+$, since $0 < \epsilon < \epsilon_1(h)$ and $\epsilon_1(h) \to 0^+$.

3. RELATION WITH BRIDGES-MIELKE [9]. Bridges and Mielke describe the unstable eigenvalues very close to the origin, namely the cross amid the '8". In order to make a precise comparison with our result let us spell out the relation of the functions e_{WB}, e_{12} and e_{22} with the coefficients obtained in [9]. The Whitham-Benjamin function e_{WB} in (4.13) is $e_{WB} = (c_h h)^{-1} \nu(F)$, where $\nu(F)$ is defined in [9, formula (6.17)] and $F = c_h h^{-\frac{1}{2}}$ is the Froude number, cfr. [9, formula (3.4)]. Moreover the term
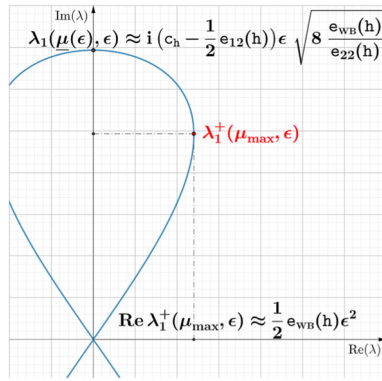
Fig. 3. The Benjamin–Feir eigenvalue $\lambda_1^+(\mu_{\max}, \epsilon)$ in (1.8) with maximal real part, as well as the whole figure "8" shrinks to zero as $h \to h_{\mathrm{WB}}^+$

e_{12} in (1.2) is $e_{12} = 2c_g$, where $c_g = \frac{1}{2}c_h\big(1 + F^{-2}\mathrm{sech}^2(h)\big)$ is the group velocity defined in Bridges-Mielke [9, formula (3.8)]. Finally $e_{22}(h) \propto \dot{c}_g$ where $\dot{c}_g$ is the derivative of the group velocity defined in [9, formula (6.15)], which for gravity waves is negative in any depth.

4. COMPLETE SPECTRUM NEAR 0. In Theorem 1.1 we have described just the two unstable eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ close to zero for $h > h_{\mathrm{WB}}$. There are also two larger purely imaginary eigenvalues of order $\mathcal{O}(\mu)$, see Theorem 2.5.

5. SHALLOW WATER REGIME. In the shallow water regime $0 < h < h_{\mathrm{WB}}$, we prove in Theorem 2.5 that all the four eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ close to zero remain purely imaginary for ϵ sufficiently small. The eigenvalue expansions of Theorem 2.5 become singular as $h \to 0^+$.

6. BEHAVIOR AT THE WHITHAM-BENJAMIN THRESHOLD h_{WB}. The analysis of Theorem 1.1 is not conclusive at the critical depth $h = h_{\mathrm{WB}}$. The reason is that $e_{\mathrm{WB}}(h_{\mathrm{WB}}) = 0$ and the Benjamin–Feir discriminant function (1.6) reduces to

$$\Delta_{\mathrm{BF}}(h_{\mathrm{WB}}; \mu, \epsilon) = r(\epsilon^3) + r(\mu\epsilon^2) - e_{22}(h_{\mathrm{WB}})\mu^2(1 + r_1''(\epsilon, \mu)). \qquad (1.9)$$

Thus its quadratic expansion is not sufficient anymore to determine the sign of $\Delta_{\mathrm{BF}}(h_{\mathrm{WB}}; \mu, \epsilon)$. Note that (1.9) could be positive due to the term $r(\epsilon^3)$ for ϵ and μ small enough. Actually the cubic term in $r(\epsilon^3) = \beta\epsilon^4 + \ldots$ vanishes and the coefficient β could be explicitly computed taking into account the fourth order expansion of the Stokes waves.

7. UNSTABLE FLOQUET EXPONENTS AND AMPLITUDES (μ, ϵ). In Theorem 2.5 we actually prove that the expansion (1.5) of the eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ holds for any value of (μ, ϵ) in a larger rectangle $[0, \mu_0) \times [0, \epsilon_0)$, and there exist Benjamin–Feir unstable eigenvalues if and only if the analytic function $\Delta_{\mathrm{BF}}(h; \mu, \epsilon)$ in (1.6) is positive. The zero set of $\Delta_{\mathrm{BF}}(h; \mu, \epsilon)$ is an analytic variety which, for $h > h_{\mathrm{WB}}$, is, restricted to the rectangle $[0, \mu_0) \times [0, \epsilon_1)$, the graph of the analytic function $\underline{\mu}(\epsilon) = e_h\epsilon(1 + r(\epsilon))$ in (1.4). This function is tangent at $\epsilon = 0$ to the straight
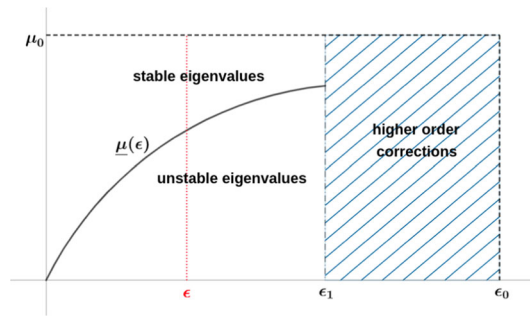
Fig. 4. The solid curve portrays the graph of the real analytic function $\underline{\mu}(\epsilon)$ in (1.4) as $h > h_{WB}$. For values of μ below this curve, the two eigenvalues $\lambda_1^\pm(\mu, \epsilon)$ have non zero real part. For μ above the curve, $\lambda_1^\pm(\mu, \epsilon)$ are purely imaginary. In the region $[\epsilon_1, \epsilon_0) \times [0, \mu_0)$ the eigenvalues are real/purely imaginary depending on the higher order corrections given by Theorem 2.5, which determine the sign of $\Delta_{BF}(h; \mu, \epsilon)$

line $\mu = e_h \epsilon$, and divides $[0, \mu_0) \times [0, \epsilon_1)$ in the region where $\Delta_{BF}(h; \mu, \epsilon) > 0$ –and thus the eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ have non-trivial real part–, from the "stable" one where all the eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ are purely imaginary, see Fig. 4. In the region $[0, \mu_0) \times [\epsilon_1, \epsilon_0)$ the higher order polynomial approximations of $\Delta_{BF}(h; \mu, \epsilon)$ (which are computable) will determine the sign of $\Delta_{BF}(h; \mu, \epsilon)$.

8. DEEP WATER LIMIT. Theorems 1.1 and 2.5 do not pass to the limit as $h \to +\infty$ since the remainders in the expansions of the eigenvalues are uniform only on any compact set of $h \in (0, +\infty)$. From a mathematical point of view, the difference is evident in the asymptotic behavior of $\tanh(h\mu)$ (and similar quantities) which, if $h = +\infty$, is identically equal to 1 for *any arbitrarily small* Floquet exponent μ, whereas $\tanh(h\mu) = O(\mu h)$ for any h finite, as $\mu \to 0$. Additional intermediate scaling regimes $h\mu \sim 1$, $h\mu \ll 1$, $h\mu \gg 1$ are possible. It is well-known (e.g. see [14]) that intermediate long-wave regimes of the water-waves equations formally lead to different physically-relevant limit equations as Boussinesq, KdV, NLS, Benjamin–Ono, etc...

We shall describe in detail the ideas of proof and the differences with the deep water case below the statement of Theorem 2.5.

Further literature. Modulational instability has been studied also for a variety of approximate water waves models, such as KdV, gKdV, NLS and the Whitham equation by, for instance, Whitham [39], Segur, Henderson, Carter and Hammack [35], Gallay and Haragus [18], Haragus and Kapitula [19], Bronski and Johnson [11], Johnson [25], Hur and Johnson [21], Bronski, Hur and Johnson [10], Hur and Pandey [22], Leisman, Bronski, Johnson and Marangell [28]. Also for these approximate models, numerical simulations predict a figure "8" similar to that in Fig. 2 for the bifurcation of the unstable eigenvalues close to zero.

Finally, we mention the nonlinear modulational instability result of Jin, Liao, and Lin [24] for several fluid model equations and the preprint by Chen-Su [12] for Stokes waves in deep water. Nonlinear transversal instability results of traveling

solitary water waves in finite depth decaying at infinity on $\mathbb{R}$ have been proved in [34] (in deep water no solitary wave exists [20,27]).

2. The Complete Benjamin–Feir Spectrum in Finite Depth

In this section we present in detail the complete spectral Theorem 2.5. We first introduce the pure gravity water waves equations and the Stokes waves solutions.
The water waves equations. We consider the Euler equations for a 2-dimensional incompressible, irrotational fluid under the action of gravity. The fluid fills the region

$$\mathcal{D}_\eta := \{(x, y) \in \mathbb{T} \times \mathbb{R} : -\mathrm{h} \leq y < \eta(t, x)\} , \quad \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z},$$

with finite depth and space periodic boundary conditions. The irrotational velocity field is the gradient of a harmonic scalar potential $\Phi = \Phi(t, x, y)$ determined by its trace $\psi(t, x) = \Phi(t, x, \eta(t, x))$ at the free surface $y = \eta(t, x)$. Actually Φ is the unique solution of the elliptic equation $\Delta\Phi = 0$ in $\mathcal{D}_\eta$ with Dirichlet datum $\Phi(t, x, \eta(t, x)) = \psi(t, x)$ and $\Phi_y(t, x, y) = 0$ at $y = -\mathrm{h}$.

The time evolution of the fluid is determined by two boundary conditions at the free surface. The first is that the fluid particles remain, along the evolution, on the free surface (kinematic boundary condition), and the second one is that the pressure of the fluid is equal, at the free surface, to the constant atmospheric pressure (dynamic boundary condition). Then, as shown by Zakharov [41] and Craig-Sulem [13], the time evolution of the fluid is determined by the following equations for the unknowns $(\eta(t, x), \psi(t, x))$,

$$\eta_t = G(\eta)\psi , \quad \psi_t = -g\eta - \frac{\psi_x^2}{2} + \frac{1}{2(1 + \eta_x^2)}\big(G(\eta)\psi + \eta_x\psi_x\big)^2 , \quad (2.1)$$

where $g > 0$ is the gravity constant and $G(\eta) := G(\eta, \mathrm{h})$ denotes the Dirichlet-Neumann operator $[G(\eta)\psi](x) := \Phi_y(x, \eta(x)) - \Phi_x(x, \eta(x))\eta_x(x)$. In the sequel, with no loss of generality, we set the gravity constant $g = 1$, see Remark 2.4.

The equations (2.1) are the Hamiltonian system

$$\partial_t \begin{bmatrix} \eta \\ \psi \end{bmatrix} = \mathcal{J} \begin{bmatrix} \nabla_\eta \mathcal{H} \\ \nabla_\psi \mathcal{H} \end{bmatrix}, \quad \mathcal{J} := \begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}, \quad (2.2)$$

where ∇ denote the L^2-gradient, and the Hamiltonian $\mathcal{H}(\eta, \psi) := \frac{1}{2} \int_\mathbb{T} \big(\psi \, G(\eta)\psi + \eta^2\big)\mathrm{d}x$ is the sum of the kinetic and potential energy of the fluid. In addition of being Hamiltonian, the water waves system (2.1) possesses other important symmetries. First of all it is time reversible with respect to the involution

$$\rho \begin{bmatrix} \eta(x) \\ \psi(x) \end{bmatrix} := \begin{bmatrix} \eta(-x) \\ -\psi(-x) \end{bmatrix}, \quad \text{i.e. } \mathcal{H} \circ \rho = \mathcal{H} . \quad (2.3)$$

Moreover, the equation (2.1) is space invariant.
Stokes waves. The Stokes waves are traveling solutions of (2.1) of the form $\eta(t, x) = \check{\eta}(x - ct)$ and $\psi(t, x) = \check{\psi}(x - ct)$ for some real c and 2π-periodic

functions $(\check{\eta}(x), \check{\psi}(x))$. In a reference frame in translational motion with constant speed c, the water waves equations (2.1) become

$$\eta_t = c\eta_x + G(\eta)\psi \,, \quad \psi_t = c\psi_x - \eta - \frac{\psi_x^2}{2} + \frac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi + \eta_x\psi_x\big)^2 \tag{2.4}$$

and the Stokes waves $(\check{\eta}, \check{\psi})$ are equilibrium steady solutions of (2.4).

The bifurcation result of small amplitude of Stokes waves is due to Struik [37] in finite depth, and Levi-Civita [29], and Nekrasov [32] in infinite depth. We denote by $B(r) := \{x \in \mathbb{R}: |x| < r\}$ the real ball with center 0 and radius r.

Theorem 2.1. (Stokes waves) *For any* $h > 0$ *there exist* $\epsilon_* := \epsilon_*(h) > 0$ *and a unique family of real analytic solutions* $(\eta_\epsilon(x), \psi_\epsilon(x), c_\epsilon)$, *parameterized by the amplitude* $|\epsilon| \leq \epsilon_*$, *of*

$$c\,\eta_x + G(\eta)\psi = 0\,, \quad c\,\psi_x - \eta - \frac{\psi_x^2}{2} + \frac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi + \eta_x\psi_x\big)^2 = 0\,, \tag{2.5}$$

such that $\eta_\epsilon(x), \psi_\epsilon(x)$ *are* 2π-*periodic;* $\eta_\epsilon(x)$ *is even and* $\psi_\epsilon(x)$ *is odd, of the form*

$$\eta_\epsilon(x) = \epsilon \cos(x) + \epsilon^2(\eta_2^{[0]} + \eta_2^{[2]}\cos(2x)) + \mathcal{O}(\epsilon^3),$$
$$\psi_\epsilon(x) = \epsilon c_h^{-1}\sin(x) + \epsilon^2\psi_2^{[2]}\sin(2x) + \mathcal{O}(\epsilon^3)\,, \tag{2.6}$$
$$c_\epsilon = c_h + \epsilon^2 c_2 + \mathcal{O}(\epsilon^3) \quad \text{where} \quad c_h = \sqrt{\tanh(h)}\,,$$

and

$$\eta_2^{[0]} := \frac{c_h^4 - 1}{4c_h^2}\,, \quad \eta_2^{[2]} := \frac{3 - c_h^4}{4c_h^6}\,, \quad \psi_2^{[2]} := \frac{3 + c_h^8}{8c_h^7}\,, \tag{2.7}$$

$$c_2 := \frac{9 - 10c_h^4 + 9c_h^8}{16c_h^7} + \frac{(1-c_h^4)}{2c_h}\eta_2^{[0]} = \frac{-2c_h^{12} + 13c_h^8 - 12c_h^4 + 9}{16c_h^7}\,. \tag{2.8}$$

More precisely for any $\sigma \geq 0$ *and* $s > \frac{5}{2}$, *there exists* $\epsilon_* > 0$ *such that the map* $\epsilon \mapsto (\eta_\epsilon, \psi_\epsilon, c_\epsilon)$ *is analytic from* $B(\epsilon_*) \to H_{ev}^{\sigma,s}(\mathbb{T}) \times H_{odd}^{\sigma,s}(\mathbb{T}) \times \mathbb{R}$, *where* $H_{ev}^{\sigma,s}(\mathbb{T})$, *respectively* $H_{odd}^{\sigma,s}(\mathbb{T})$, *denote the space of even, respectively odd, real valued* 2π-*periodic analytic functions* $u(x) = \sum_{k \in \mathbb{Z}} u_k e^{ikx}$ *such that* $\|u\|_{\sigma,s}^2 := \sum_{k \in \mathbb{Z}} |u_k|^2 \langle k \rangle^{2s} e^{2\sigma|k|} < +\infty$.

The expansions (2.6)-(2.8) are derived in the Appendix B for completeness, although present in the literature (they coincide with [39, section 13, chapter 13] and [2, section 2]). Note that in the shallow water regime $h \to 0^+$ the expansions (2.6)-(2.8) become singular. For the analiticity properties of the maps stated in Theorem 2.1 we refer to [8].

We also mention that more general time quasi-periodic traveling Stokes waves—which are nonlinear superpositions of multiple Stokes waves traveling with rationally independent speeds—have been recently proved for (2.1) in [5] in finite depth, in [17] in infinite depth, and in [4] for capillary-gravity water waves in any depth.

Linearization at the Stokes waves. In order to determine the stability/instability of the Stokes waves given by Theorem 2.1, we linearize the water waves equations (2.4) with $c = c_\epsilon$ at $(\eta_\epsilon(x), \psi_\epsilon(x))$. In the sequel we closely follow [6] pointing out the differences of the finite depth case.

By using the shape derivative formula for the differential $d_\eta G(\eta)[\hat\eta]$ of the Dirichlet-Neumann operator one obtains the autonomous real linear system

$$\begin{bmatrix} \hat\eta_t \\ \hat\psi_t \end{bmatrix} = \begin{bmatrix} -G(\eta_\epsilon)B - \partial_x \circ (V - c_\epsilon) & G(\eta_\epsilon) \\ -1 + B(V - c_\epsilon)\partial_x - B\partial_x \circ (V - c_\epsilon) - BG(\eta_\epsilon) \circ B & -(V - c_\epsilon)\partial_x + BG(\eta_\epsilon) \end{bmatrix} \begin{bmatrix} \hat\eta \\ \hat\psi \end{bmatrix}, \tag{2.9}$$

where

$$V := V(x) := -B(\eta_\epsilon)_x + (\psi_\epsilon)_x \,,$$

$$B := B(x) := \frac{G(\eta_\epsilon)\psi_\epsilon + (\psi_\epsilon)_x(\eta_\epsilon)_x}{1 + (\eta_\epsilon)_x^2} = \frac{(\psi_\epsilon)_x - c_\epsilon}{1 + (\eta_\epsilon)_x^2}(\eta_\epsilon)_x \,. \tag{2.10}$$

The functions (V, B) are the horizontal and vertical components of the velocity field (Φ_x, Φ_y) at the free surface. Moreover $\epsilon \mapsto (V, B)$ is analytic as a map $B(\epsilon_0) \to H^{\sigma,s-1}(\mathbb{T}) \times H^{\sigma,s-1}(\mathbb{T})$. The real system (2.9) is Hamiltonian, i.e. of the form $\mathcal{J}\mathcal{A}$ with $\mathcal{A} = \mathcal{A}^\top$, where $\mathcal{A}^\top$ is the transposed operator with respect the scalar product of $L^2(\mathbb{T}, \mathbb{R}) \times L^2(\mathbb{T}, \mathbb{R})$. Moreover the linear operator in (2.9) is reversible, i.e. it anti-commutes with the involution ρ in (2.3).

Under the time-independent "good unknown of Alinhac" linear transformation

$$\begin{bmatrix} \hat\eta \\ \hat\psi \end{bmatrix} := Z \begin{bmatrix} u \\ v \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 \\ B & 1 \end{bmatrix}, \qquad Z^{-1} = \begin{bmatrix} 1 & 0 \\ -B & 1 \end{bmatrix}, \tag{2.11}$$

the system (2.9) assumes the simpler form

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \widetilde{\mathcal{L}}_\epsilon \begin{bmatrix} u \\ v \end{bmatrix}, \qquad \widetilde{\mathcal{L}}_\epsilon := \begin{bmatrix} -\partial_x \circ (V - c_\epsilon) & G(\eta_\epsilon) \\ -1 - (V - c_\epsilon)B_x & -(V - c_\epsilon)\partial_x \end{bmatrix}. \tag{2.12}$$

Next, we perform a conformal change of variables to flatten the water surface. Here the finite depth case induces a modification with respect to the deep water case. By [1, Appendix A], there exists a diffeomorphism of $\mathbb{T}$, $x \mapsto x + \mathfrak{p}(x)$, with a small 2π-periodic function $\mathfrak{p}(x)$, and a small constant $\mathfrak{f}_\epsilon$, such that, by defining the associated composition operator $(\mathfrak{P}u)(x) := u(x + \mathfrak{p}(x))$, the Dirichlet-Neumann operator can be written as [1, Lemma A.5]

$$G(\eta_\epsilon) = \partial_x \circ \mathfrak{P}^{-1} \circ \mathcal{H} \circ \tanh\big((\mathrm{h} + \mathfrak{f}_\epsilon)|D|\big) \circ \mathfrak{P}, \tag{2.13}$$

where $\mathcal{H}$ is the Hilbert transform, i.e. the Fourier multiplier operator

$$\mathcal{H}(e^{ijx}) := -i\,\mathrm{sign}(j)e^{ijx}, \quad \forall j \in \mathbb{Z} \setminus \{0\}, \quad \mathcal{H}(1) := 0.$$

The function $\mathfrak{p}(x)$ and the constant $\mathfrak{f}_\epsilon$ are determined as a fixed point of (see [1, formula (A.15)])

$$\mathfrak{p} = \frac{\mathcal{H}}{\tanh\big((\mathrm{h} + \mathfrak{f}_\epsilon)|D|\big)}[\eta_\epsilon(x + \mathfrak{p}(x))],$$

$$f_\epsilon := \frac{1}{2\pi} \int_{\mathbb{T}} \eta_\epsilon(x + \mathfrak{p}(x)) dx . \tag{2.14}$$

By the analyticity of the map $\epsilon \to \eta_\epsilon \in H^{\sigma,s}, \sigma > 0, s > 1/2$, the analytic implicit function theorem implies the existence of a solution $\epsilon \mapsto \mathfrak{p}(x) := \mathfrak{p}_\epsilon(x), \epsilon \mapsto f_\epsilon$, analytic as a map $B(\epsilon_0) \to H^s(\mathbb{T}) \times \mathbb{R}$. Moreover, since η_ϵ is even, the function $\mathfrak{p}(x)$ is odd. In Appendix B we prove the expansion

$$\mathfrak{p}(x) = \epsilon c_h^{-2} \sin(x) + \epsilon^2 \frac{(1 + c_h^4)(3 + c_h^4)}{8c_h^8} \sin(2x) + \mathcal{O}(\epsilon^3),$$

$$f_\epsilon = \epsilon^2 \frac{c_h^4 - 3}{4c_h^2} + \mathcal{O}(\epsilon^3) . \tag{2.15}$$

Under the symplectic and reversibility-preserving map

$$\mathcal{P} := \begin{bmatrix} (1 + \mathfrak{p}_x)\mathfrak{P} & 0 \\ 0 & \mathfrak{P} \end{bmatrix}, \tag{2.16}$$

the system (2.12) transforms, by (2.13), into the linear system $h_t = \mathcal{L}_\epsilon h$ where $\mathcal{L}_\epsilon$ is the Hamiltonian and reversible real operator

$$\mathcal{L}_\epsilon := \mathcal{P}\, \widetilde{\mathcal{L}}_\epsilon\, \mathcal{P}^{-1} = \begin{bmatrix} \partial_x \circ (c_h + p_\epsilon(x)) & |D| \tanh((h + f_\epsilon)|D|) \\ -(1 + a_\epsilon(x)) & (c_h + p_\epsilon(x))\partial_x \end{bmatrix}$$
$$= \mathcal{J} \begin{bmatrix} 1 + a_\epsilon(x) & -(c_h + p_\epsilon(x))\partial_x \\ \partial_x \circ (c_h + p_\epsilon(x)) & |D| \tanh((h + f_\epsilon)|D|) \end{bmatrix}, \tag{2.17}$$

where

$$c_h + p_\epsilon(x) := \frac{c_\epsilon - V(x + \mathfrak{p}(x))}{1 + \mathfrak{p}_x(x)},$$

$$1 + a_\epsilon(x) := \frac{1 + (V(x + \mathfrak{p}(x)) - c_\epsilon)B_x(x + \mathfrak{p}(x))}{1 + \mathfrak{p}_x(x)} . \tag{2.18}$$

By the analiticity results of the functions $V, B, \mathfrak{p}(x)$ given above, the functions p_ϵ and a_ϵ are analytic in ϵ as maps $B(\epsilon_0) \to H^s(\mathbb{T})$. In the Appendix B we prove the following expansions:

Lemma 2.2. *The analytic functions $p_\epsilon(x)$ and $a_\epsilon(x)$ in (2.18) are even in x, and*

$$p_\epsilon(x) = \epsilon p_1(x) + \epsilon^2 p_2(x) + \mathcal{O}(\epsilon^3), \qquad a_\epsilon(x) = \epsilon a_1(x) + \epsilon^2 a_2(x) + \mathcal{O}(\epsilon^3), \tag{2.19}$$

where

$$p_1(x) = p_1^{[1]} \cos(x), \qquad p_1^{[1]} := -2c_h^{-1}, \tag{2.20}$$

$$p_2(x) = p_2^{[0]} + p_2^{[2]} \cos(2x),$$

$$p_2^{[0]} := \frac{9 + 12c_h^4 + 5c_h^8 - 2c_h^{12}}{16c_h^7}, \qquad p_2^{[2]} := -\frac{3 + c_h^4}{2c_h^7}, \tag{2.21}$$

and

$$a_1(x) = a_1^{[1]} \cos(x), \qquad a_1^{[1]} := -(c_h^2 + c_h^{-2}), \qquad (2.22)$$

$$a_2(x) = a_2^{[0]} + a_2^{[2]} \cos(2x), \quad a_2^{[0]} := \frac{3}{2} + \frac{1}{2c_h^4}, \quad a_2^{[2]} := \frac{-14c_h^4 + 9c_h^8 - 3}{4c_h^8}. \qquad (2.23)$$

Bloch-Floquet expansion. Since the operator $\mathcal{L}_\epsilon$ in (2.17) has 2π-periodic coefficients, Bloch-Floquet theory guarantees that

$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_\epsilon) = \bigcup_{\mu \in [-\frac{1}{2}, \frac{1}{2})} \sigma_{L^2(\mathbb{T})}(\mathcal{L}_{\mu,\epsilon}) \quad \text{where} \quad \mathcal{L}_{\mu,\epsilon} := e^{-i\mu x} \mathcal{L}_\epsilon e^{i\mu x}.$$

The domain $[-\frac{1}{2}, \frac{1}{2})$ is called, in solid state physics, the "first zone of Brillouin". In particular, if λ is an eigenvalue of $\mathcal{L}_{\mu,\epsilon}$ on $L^2(\mathbb{T}, \mathbb{C}^2)$ with eigenvector $v(x)$, then $h(t, x) = e^{\lambda t} e^{i\mu x} v(x)$ solves $h_t = \mathcal{L}_\epsilon h$. We remark that: (i) if $A = \text{Op}(a)$ is a pseudo-differential operator with symbol $a(x, \xi)$, which is 2π periodic in x, then $A_\mu := e^{-i\mu x} A e^{i\mu x} = \text{Op}(a(x, \xi + \mu))$. (ii) If A is a real operator then $\overline{A_\mu} = A_{-\mu}$. As a consequence the spectrum $\sigma(A_{-\mu}) = \overline{\sigma(A_\mu)}$ and we can study $\sigma(A_\mu)$ just for $\mu > 0$. Furthermore $\sigma(A_\mu)$ is a 1-periodic set with respect to μ, so one can restrict to $\mu \in [0, \frac{1}{2})$.

By the previous remarks the Floquet operator associated with the real operator $\mathcal{L}_\epsilon$ in (2.17) is the complex *Hamiltonian* and *reversible* operator

$$
\mathcal{L}_{\mu,\epsilon} := \begin{bmatrix} (\partial_x + i\mu) \circ (c_h + p_\epsilon(x)) & |D + \mu| \tanh\left((h + f_\epsilon)|D + \mu|\right) \\ -(1 + a_\epsilon(x)) & (c_h + p_\epsilon(x))(\partial_x + i\mu) \end{bmatrix}
$$

$$
= \underbrace{\begin{bmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{bmatrix}}_{= \mathcal{J}} \underbrace{\begin{bmatrix} 1 + a_\epsilon(x) & -(c_h + p_\epsilon(x))(\partial_x + i\mu) \\ (\partial_x + i\mu) \circ (c_h + p_\epsilon(x)) & |D + \mu| \tanh\left((h + f_\epsilon)|D + \mu|\right) \end{bmatrix}}_{=: \mathcal{B}_{\mu,\epsilon}}.
$$

$$(2.24)$$

We regard $\mathcal{L}_{\mu,\epsilon}$ as an operator with domain $H^1(\mathbb{T}) := H^1(\mathbb{T}, \mathbb{C}^2)$ and range $L^2(\mathbb{T}) := L^2(\mathbb{T}, \mathbb{C}^2)$, equipped with the complex scalar product

$$(f, g) := \frac{1}{2\pi} \int_0^{2\pi} (f_1 \bar{g}_1 + f_2 \bar{g}_2) \, dx, \quad \forall f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \ g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in L^2(\mathbb{T}, \mathbb{C}^2). \qquad (2.25)$$

We also denote $\|f\|^2 = (f, f)$.

The complex operator $\mathcal{L}_{\mu,\epsilon}$ in (2.24) is *Hamiltonian* and *Reversible*.

Definition 2.3. (Complex Hamiltonian/Reversible operator) A complex operator $\mathcal{L} : H^1(\mathbb{T}, \mathbb{C}^2) \to L^2(\mathbb{T}, \mathbb{C}^2)$ is *Hamiltonian*, if $\mathcal{L} = \mathcal{J}\mathcal{B}$ where $\mathcal{B}$ is a self-adjoint operator, namely $\mathcal{B} = \mathcal{B}^*$, where $\mathcal{B}^*$ (with domain $H^1(\mathbb{T})$) is the adjoint with respect to the complex scalar product (2.25) of $L^2(\mathbb{T})$; it is *reversible* if

$$\mathcal{L} \circ \bar{\rho} = -\bar{\rho} \circ \mathcal{L}, \qquad (2.26)$$

where $\bar{\rho}$ is the complex involution (cfr. (2.3))

$$\bar{\rho} \begin{bmatrix} \eta(x) \\ \psi(x) \end{bmatrix} := \begin{bmatrix} \bar{\eta}(-x) \\ -\bar{\psi}(-x) \end{bmatrix}. \tag{2.27}$$

The property (2.26) for $\mathcal{L}_{\mu,\epsilon}$ follows because $\mathcal{L}_{\epsilon}$ is a real operator which is reversible with respect to the involution ρ in (2.3). Equivalently, since $\mathcal{J} \circ \bar{\rho} = -\bar{\rho} \circ \mathcal{J}$, the self-adjoint operator $\mathcal{B}_{\mu,\epsilon}$ is *reversibility-preserving*, i.e.

$$\mathcal{B}_{\mu,\epsilon} \circ \bar{\rho} = \bar{\rho} \circ \mathcal{B}_{\mu,\epsilon}. \tag{2.28}$$

In addition $(\mu, \epsilon) \to \mathcal{L}_{\mu,\epsilon} \in \mathcal{L}(H^1(\mathbb{T}), L^2(\mathbb{T}))$ is analytic, since the functions $\epsilon \mapsto a_\epsilon$, p_ϵ defined in (2.19) are analytic as maps $B(\epsilon_0) \to H^1(\mathbb{T})$ and $\mathcal{L}_{\mu,\epsilon}$ is analytic with respect to μ, since, for any $\mu \in [-\frac{1}{2}, \frac{1}{2})$,

$$|D + \mu| \tanh \big((\mathrm{h} + \mathrm{f}_\epsilon)|D + \mu|\big) = (D + \mu) \tanh \big((\mathrm{h} + \mathrm{f}_\epsilon)(D + \mu)\big). \tag{2.29}$$

We also note that (see [33, Section 5.1])

$$|D + \mu| = |D| + \mu(\mathrm{sgn}(D) + \Pi_0), \quad \forall \mu > 0, \tag{2.30}$$

where $\mathrm{sgn}(D)$ is the Fourier multiplier operator, acting on 2π-periodic functions, with symbol

$$\mathrm{sgn}(k) := 1 \,\forall k > 0, \quad \mathrm{sgn}(0) := 0, \quad \mathrm{sgn}(k) := -1 \,\forall k < 0, \tag{2.31}$$

and Π_0 is the projector operator on the zero mode, $\Pi_0 f(x) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) \mathrm{d}x$.

Remark 2.4. If $(\eta(x), \psi(x), c)$ solve the traveling wave equations (2.5) then the rescaled functions $(\tilde{\eta}(x), \tilde{\psi}(x), \tilde{c}) := (\eta(x), \sqrt{g}\psi(x), \sqrt{g}c)$ solve the same equations with gravity constant g instead of 1. The eigenvalues of the corresponding linearized operators (2.9) and (2.24) for a general gravity g are those of the $g = 1$ case multiplied by $\sqrt{g}$.

Our aim is to prove the existence of eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ in (2.24) with non zero real part. We remark that the Hamiltonian structure of $\mathcal{L}_{\mu,\epsilon}$ implies that eigenvalues with non zero real part may arise only from multiple eigenvalues of $\mathcal{L}_{\mu,0}$ ("Krein criterion"), because if λ is an eigenvalue of $\mathcal{L}_{\mu,\epsilon}$ then also $-\bar{\lambda}$ is, and the total algebraic multiplicity of the eigenvalues is conserved under small perturbation. We now describe the spectrum of $\mathcal{L}_{\mu,0}$.

The spectrum of $\mathcal{L}_{\mu,0}$. The spectrum of the Fourier multiplier matrix operator

$$\mathcal{L}_{\mu,0} = \begin{bmatrix} \mathrm{c_h}(\partial_x + \mathrm{i}\,\mu) & |D + \mu| \tanh \big(\mathrm{h}|D + \mu|\big) \\ -1 & \mathrm{c_h}(\partial_x + \mathrm{i}\,\mu) \end{bmatrix} \tag{2.32}$$

consists of the purely imaginary eigenvalues $\{\lambda_k^\pm(\mu), \ k \in \mathbb{Z}\}$, where

$$\lambda_k^\pm(\mu) := \mathrm{i} \left(\mathrm{c_h}(\pm k + \mu) \mp \sqrt{|k \pm \mu| \tanh(\mathrm{h}|k \pm \mu|)}\right). \tag{2.33}$$

For $\mu = 0$ the real operator $\mathcal{L}_{0,0}$ possesses the eigenvalue 0 with algebraic multiplicity 4,

$$\lambda_0^+(0) = \lambda_0^-(0) = \lambda_1^+(0) = \lambda_1^-(0) = 0,$$

and geometric multiplicity 3. A real basis of the Kernel of $\mathcal{L}_{0,0}$ is

$$f_1^+ := \begin{bmatrix} c_{\mathrm{h}}^{1/2}\cos(x) \\ c_{\mathrm{h}}^{-1/2}\sin(x) \end{bmatrix}, \quad f_1^- := \begin{bmatrix} -c_{\mathrm{h}}^{1/2}\sin(x) \\ c_{\mathrm{h}}^{-1/2}\cos(x) \end{bmatrix}, \quad f_0^- := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (2.34)$$

together with the generalized eigenvector

$$f_0^+ := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathcal{L}_{0,0} f_0^+ = -f_0^-. \tag{2.35}$$

Furthermore 0 is an isolated eigenvalue for $\mathcal{L}_{0,0}$, namely the spectrum $\sigma\left(\mathcal{L}_{0,0}\right)$ decomposes in two separated parts,

$$\sigma\left(\mathcal{L}_{0,0}\right) = \sigma'\left(\mathcal{L}_{0,0}\right) \cup \sigma''\left(\mathcal{L}_{0,0}\right), \quad \text{where} \quad \sigma'(\mathcal{L}_{0,0}) := \{0\}, \tag{2.36}$$

and $\sigma''(\mathcal{L}_{0,0}) := \{\lambda_k^\sigma(0), \ k \neq 0, 1, \sigma = \pm\}$.

We shall also use that, as proved in Theorem 4.1 in [33], the operator $\mathcal{L}_{0,\epsilon}$ possesses, for any sufficiently small $\epsilon \neq 0$, the eigenvalue 0 with a four dimensional generalized Kernel, spanned by ϵ-dependent vectors $U_1, \tilde{U}_2, U_3, U_4$ satisfying, for some real constant $\alpha_\epsilon, \beta_\epsilon$,

$$\mathcal{L}_{0,\epsilon} U_1 = 0, \ \mathcal{L}_{0,\epsilon} \tilde{U}_2 = 0, \ \mathcal{L}_{0,\epsilon} U_3 = \alpha_\epsilon \tilde{U}_2,$$

$$\mathcal{L}_{0,\epsilon} U_4 = -U_1 - \beta_\epsilon \tilde{U}_2, \quad U_1 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.37}$$

By Kato's perturbation theory (see Lemma 3.1 below) for any $\mu, \epsilon \neq 0$ sufficiently small, the perturbed spectrum $\sigma\left(\mathcal{L}_{\mu,\epsilon}\right)$ admits a disjoint decomposition as

$$\sigma\left(\mathcal{L}_{\mu,\epsilon}\right) = \sigma'\left(\mathcal{L}_{\mu,\epsilon}\right) \cup \sigma''\left(\mathcal{L}_{\mu,\epsilon}\right), \tag{2.38}$$

where $\sigma'\left(\mathcal{L}_{\mu,\epsilon}\right)$ consists of 4 eigenvalues close to 0. We denote by $\mathcal{V}_{\mu,\epsilon}$ the spectral subspace associated with $\sigma'\left(\mathcal{L}_{\mu,\epsilon}\right)$, which has dimension 4 and it is invariant by $\mathcal{L}_{\mu,\epsilon}$. Our goal is to prove that, for ϵ small, for values of the Floquet exponent μ in an interval of order ϵ, the 4×4 matrix which represents the operator $\mathcal{L}_{\mu,\epsilon}$: $\mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ possesses a pair of eigenvalues close to zero with opposite non zero real parts.

Before stating our main result, let us introduce a notation we shall use through all the paper.

- **Notation:** we denote by $\mathcal{O}(\mu^{m_1}\epsilon^{n_1}, \ldots, \mu^{m_p}\epsilon^{n_p})$, $m_j, n_j \in \mathbb{N}$ (for us $\mathbb{N} := \{1, 2, \ldots\}$), analytic functions of (μ, ϵ) with values in a Banach space X which satisfy, for some $C > 0$ uniform for h in any compact set of $(0, +\infty)$, the bound $\|\mathcal{O}(\mu^{m_j}\epsilon^{n_j})\|_X \leq C \sum_{j=1}^p |\mu|^{m_j}|\epsilon|^{n_j}$ for small values of (μ, ϵ). Similarly we denote $r_k(\mu^{m_1}\epsilon^{n_1}, \ldots, \mu^{m_p}\epsilon^{n_p})$ scalar functions $\mathcal{O}(\mu^{m_1}\epsilon^{n_1}, \ldots, \mu^{m_p}\epsilon^{n_p})$ which are also *real* analytic.

Our complete spectral result is the following:

Theorem 2.5. (Complete Benjamin–Feir spectrum) *There exist $\epsilon_0, \mu_0 > 0$, uniformly for the depth $\mathtt{h}$ in any compact set of $(0, +\infty)$, such that, for any $0 < \mu < \mu_0$ and $0 \leq \epsilon < \epsilon_0$, the operator $\mathcal{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ can be represented by a 4×4 matrix of the form*

$$\left(\begin{array}{c|c} \mathtt{U} & 0 \\ \hline 0 & \mathtt{S} \end{array}\right), \tag{2.39}$$

where $\mathtt{U}$ and $\mathtt{S}$ are 2×2 matrices, with identical diagonal entries each, of the form

$$\mathtt{U} = \begin{pmatrix} \mathrm{i}\left((c_{\mathtt{h}} - \frac{1}{2}e_{12})\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) & -e_{22}\frac{\mu}{8}(1 + r_5(\epsilon, \mu)) \\ -\mu\epsilon^2 e_{\mathrm{WB}} + r_1'(\mu\epsilon^3, \mu^2\epsilon^2) + e_{22}\frac{\mu^3}{8}(1 + r_1''(\epsilon, \mu)) & \mathrm{i}\left((c_{\mathtt{h}} - \frac{1}{2}e_{12})\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) \end{pmatrix},$$

$$\mathtt{S} = \begin{pmatrix} \mathrm{i}\,c_{\mathtt{h}}\mu + \mathrm{i}\,r_9(\mu\epsilon^2, \mu^2\epsilon) & \tanh(\mathtt{h}\mu) + r_{10}(\mu\epsilon) \\ -\mu + r_8(\mu\epsilon^2, \mu^3\epsilon) & \mathrm{i}\,c_{\mathtt{h}}\mu + \mathrm{i}\,r_9(\mu\epsilon^2, \mu^2\epsilon) \end{pmatrix}, \tag{2.40}$$

where $e_{\mathrm{WB}}, e_{12}, e_{22}$ are defined in (1.1)*,* (1.2)*,* (1.3)*. The eigenvalues of $\mathtt{U}$ have the form*

$$\begin{aligned} \lambda_1^{\pm}(\mu, \epsilon) = &\,\mathrm{i}\,\frac{1}{2}\check{c}_{\mathtt{h}}\mu + \mathrm{i}\,r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \\ &\pm \frac{1}{8}\mu\sqrt{e_{22}(\mathtt{h})(1 + r_5(\epsilon, \mu))}\sqrt{\Delta_{\mathrm{BF}}(\mathtt{h}; \mu, \epsilon)}, \end{aligned} \tag{2.41}$$

where $\check{c}_{\mathtt{h}} := 2c_{\mathtt{h}} - e_{12}(\mathtt{h})$ and $\Delta_{\mathrm{BF}}(\mathtt{h}; \mu, \epsilon)$ is the Benjamin–Feir discriminant function (1.6) *(with $r_1(\epsilon^3, \mu\epsilon^2) := -8r_1'(\epsilon^3, \mu\epsilon^2)$). As $e_{22}(\mathtt{h}) > 0$, they have non-zero real part if and only if $\Delta_{\mathrm{BF}}(\mathtt{h}; \mu, \epsilon) > 0$.*

The eigenvalues of the matrix $\mathtt{S}$ are a pair of purely imaginary eigenvalues of the form

$$\lambda_0^{\pm}(\mu, \epsilon) = \mathrm{i}\,c_{\mathtt{h}}\mu\big(1 + r_9(\epsilon^2, \mu\epsilon)\big) \mp \mathrm{i}\sqrt{\mu\tanh(\mathtt{h}\mu)}\big(1 + r(\epsilon)\big). \tag{2.42}$$

For $\epsilon = 0$ the eigenvalues $\lambda_1^{\pm}(\mu, 0), \lambda_0^{\pm}(\mu, 0)$ coincide with those in (2.33)*.*

Remark 2.6. At $\epsilon = 0$, the eigenvalues in (2.41) have the Taylor expansion

$$\lambda_1^{\pm}(\mu, 0) = \mathrm{i}\left(c_{\mathtt{h}} - \frac{1}{2}e_{12}(\mathtt{h})\right)\mu \pm \mathrm{i}\,\frac{e_{22}(\mathtt{h})}{8}\mu^2 + \mathcal{O}(\mu^3),$$

which coincides with the one of $\lambda_1^{\pm}(\mu)$ in (2.33), in view of the coefficients $e_{12}(\mathtt{h})$ and $e_{22}(\mathtt{h})$ defined in (1.2), (1.3).

We conclude this section by describing our approach in detail.

Ideas and scheme of proof. The first step is to exploit as in [6] Kato's theory to prolong the unperturbed symplectic basis $\{f_1^{\pm}, f_0^{\pm}\}$ of $\mathcal{V}_{0,0}$ in (2.34)-(2.35) into a symplectic basis $\{f_k^{\sigma}(\mu, \epsilon), k = 0, 1, \sigma = \pm\}$ of the spectral subspace $\mathcal{V}_{\mu,\epsilon}$ associated with $\sigma'(\mathcal{L}_{\mu,\epsilon})$ in (2.38), depending analytically on μ, ϵ.

Its expansion in μ, ϵ is provided in Lemma 4.2. This procedure reduces our spectral problem to determine the eigenvalues of the 4×4 Hamiltonian and reversible

matrix $\mathtt{L}_{\mu,\epsilon}$ (Lemma 3.4), representing the action of the operator $\mathcal{L}_{\mu,\epsilon} - \mathrm{i}\,\mathtt{c}_\mathrm{h}\mu$ on $\{f_k^\sigma(\mu,\epsilon)\}$. In Proposition 4.3 we prove that

$$\mathtt{L}_{\mu,\epsilon} = \mathtt{J}_4 \begin{pmatrix} E & F \\ F^* & G \end{pmatrix} = \begin{pmatrix} \mathtt{J}_2 E & \mathtt{J}_2 F \\ \mathtt{J}_2 F^* & \mathtt{J}_2 G \end{pmatrix} \qquad \text{where}$$

$$\mathtt{J}_4 = \begin{pmatrix} \mathtt{J}_2 & 0 \\ 0 & \mathtt{J}_2 \end{pmatrix}, \quad \mathtt{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.43}$$

and the 2×2 matrices E, G, F have the expansions (4.10)-(4.12). In finite depth this computation is much more involved than in deep water, as we need to track the exact dependence of the matrix entries with respect to h. In particular the matrix E is

$$E = \begin{pmatrix} \mathtt{e}_{11}\epsilon^2(1 + r_1'(\epsilon, \mu\epsilon)) - \mathtt{e}_{22}\frac{\mu^2}{8}(1 + r_1''(\epsilon, \mu)) & \mathrm{i}\left(\frac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) \\ -\mathrm{i}\left(\frac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) & -\mathtt{e}_{22}\frac{\mu^2}{8}(1 + r_5(\epsilon, \mu)) \end{pmatrix} \tag{2.44}$$

where the coefficients $\mathtt{e}_{11}$ and $\mathtt{e}_{22}$, defined in (4.13) and (1.3), are strictly positive *for any* value of $\mathrm{h} > 0$. Thus the submatrix $\mathtt{J}_2 E$ has a pair of eigenvalues with nonzero real part, for any value of $\mathrm{h} > 0$, provided $0 < \mu < \overline{\mu}(\epsilon) \sim \epsilon$. On the other hand, it has to come out that the complete 4×4 matrix $\mathtt{L}_{\mu,\epsilon}$ possesses unstable eigenvalues *if and only if* the depth exceeds the celebrated Whitham-Benjamin threshold $\mathrm{h}_{\mathrm{WB}} \sim 1.363\ldots$. Indeed the correct eigenvalues of $\mathtt{L}_{\mu,\epsilon}$ are not a small perturbation of those of $\begin{pmatrix} \mathtt{J}_2 E & 0 \\ 0 & \mathtt{J}_2 G \end{pmatrix}$ and will emerge only after one non-perturbative step of block diagonalization. This was not the case in the infinitely deep water case [6], where the corresponding submatrix $\mathtt{J}_2 E$ showed up the Benjamin–Feir eigenvalues, and we only had to check their stability under perturbation.

Remark 2.7. We underline that (2.44) is not a simple Taylor expansion in μ, ϵ: note that the $(2, 2)$-entry in (2.44) does not have any term $\mathcal{O}(\epsilon^m)$ nor $\mathcal{O}(\mu\epsilon^m)$ *for any* $m \in \mathbb{N}$. These terms could change the sign of the entry $(2, 2)$ which instead, in (2.44), is always negative (recall that $\mathtt{e}_{22}(\mathrm{h}) > 0$). We prove the absence of terms ϵ^m exploiting the structural information (2.37) concerning the four dimensional generalized Kernel of the operator $\mathcal{L}_{0,\epsilon}$ for any $\epsilon > 0$, see Lemma 4.4. We also note that the 2×2 matrices $\mathtt{J}_2 E$ and $\mathtt{J}_2 G$ in (2.43) have both eigenvalues of size $\mathcal{O}(\mu)$. As already mentioned in the introduction, this is a crucial difference with the deep water case, where the eigenvalues of $\mathtt{J}_2 G$ are $\mathcal{O}(\sqrt{\mu})$.

In order to determine the spectrum of the matrix $\mathtt{L}_{\mu,\epsilon}$ in (2.43), we perform a block diagonalization of $\mathtt{L}_{\mu,\epsilon}$ to eliminate the coupling term $\mathtt{J}_2 F$ (which has size ϵ, see (4.12)). We proceed, in Sect. 5, in three steps.

1. *Symplectic rescaling*. We first perform a symplectic rescaling which is singular at $\mu = 0$, see Lemma 5.1, obtaining the matrix $\mathtt{L}_{\mu,\epsilon}^{(1)}$. The effects are twofold: (i) the diagonal elements of

$$E^{(1)} = \begin{pmatrix} \mathsf{e}_{11}\mu\epsilon^2(1 + r_1'(\epsilon, \mu\epsilon)) - \mathsf{e}_{22}\frac{\mu^3}{8}(1 + r_1''(\epsilon, \mu)) & \mathrm{i}\left(\frac{1}{2}\mathsf{e}_{12}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) \\ -\mathrm{i}\left(\frac{1}{2}\mathsf{e}_{12}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) & -\mathsf{e}_{22}\frac{\mu}{8}(1 + r_5(\epsilon, \mu)) \end{pmatrix}$$

(2.45)

have size $\mathcal{O}(\mu)$, as well as those of $G^{(1)}$, and (ii) the matrix $F^{(1)}$ has the smaller size $\mathcal{O}(\mu\epsilon)$.

2. *Non-perturbative step of block-diagonalization (Section 5.1)*. Inspired by KAM theory, we perform one step of block decoupling to decrease further the size of the off-diagonal blocks. This step *modifies* the matrix $\mathtt{J}_2 E^{(1)}$ in a substantial way, by a term $\mathcal{O}(\mu\epsilon^2)$. Let us explain better this step. In order to reduce the size of $\mathtt{J}_2 F^{(1)}$, we conjugate $\mathtt{L}_{\mu,\epsilon}^{(1)}$ by the symplectic matrix $\exp(S^{(1)})$, where $S^{(1)}$ is a Hamiltonian matrix with the same form of $\mathtt{J}_2 F^{(1)}$, see (5.9). The transformed matrix $\mathtt{L}_{\mu,\epsilon}^{(2)} = \exp(S^{(1)})\mathtt{L}_{\mu,\epsilon}^{(1)}\exp(-S^{(1)})$ has the Lie expansion[1]

$$\begin{aligned} \mathtt{L}_{\mu,\epsilon}^{(2)} = &\begin{pmatrix} \mathtt{J}_2 E^{(1)} & 0 \\ 0 & \mathtt{J}_2 G^{(1)} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & \mathtt{J}_2 F^{(1)} \\ \mathtt{J}_2[F^{(1)}]^* & 0 \end{pmatrix} + \left[S^{(1)}, \begin{pmatrix} \mathtt{J}_2 E^{(1)} & 0 \\ 0 & \mathtt{J}_2 G^{(1)} \end{pmatrix}\right] \\ &+ \frac{1}{2}\left[S^{(1)}, \left[S^{(1)}, \begin{pmatrix} \mathtt{J}_2 E^{(1)} & 0 \\ 0 & \mathtt{J}_2 G^{(1)} \end{pmatrix}\right]\right] + \left[S^{(1)}, \begin{pmatrix} 0 & \mathtt{J}_2 F^{(1)} \\ \mathtt{J}_2[F^{(1)}]^* & 0 \end{pmatrix}\right] + \text{h.o.t..} \end{aligned}$$

(2.46)

The first line in the right hand side of (2.46) is the previous block-diagonal matrix, the second line of (2.46) is a purely off-diagonal matrix and the third line is the sum of two block-diagonal matrices and "h.o.t." collects terms of much smaller size. $S^{(1)}$ is determined in such a way that the second line of (2.46) vanishes, and therefore the remaining off-diagonal matrices (appearing in the h.o.t. remainder) are smaller in size. Unlike the infinitely deep water case [6], the block-diagonal corrections in the third line of (2.46) are *not* perturbative, modifying substantially the block-diagonal part. More precisely we obtain that $\mathtt{L}_{\mu,\epsilon}^{(2)}$ has the form (5.10) with

$$E^{(2)} := \begin{pmatrix} \mu\epsilon^2 \mathsf{e}_{\text{WB}} + r_1'(\mu\epsilon^3, \mu^2\epsilon^2) - \mathsf{e}_{22}\frac{\mu^3}{8}(1 + r_1''(\epsilon, \mu)) & \mathrm{i}\left(\frac{1}{2}\mathsf{e}_{12}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) \\ -\mathrm{i}\left(\frac{1}{2}\mathsf{e}_{12}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) & -\mathsf{e}_{22}\frac{\mu}{8}(1 + r_5(\epsilon, \mu)) \end{pmatrix}.$$

Note the appearance of the Whitham-Benjamin function $\mathsf{e}_{\text{WB}}(\mathsf{h})$ in the (1,1)-entry of $E^{(2)}$, which changes sign at the critical depth h_{WB}, see Fig. 1, unlike the coefficient $\mathsf{e}_{11}(\mathsf{h}) > 0$ in (2.45). If $\mathsf{e}_{\text{WB}}(\mathsf{h}) > 0$ and ϵ and μ are sufficiently small, the matrix $\mathtt{J}_2 E^{(2)}$ has eigenvalues with non-zero real part (recall that $\mathsf{e}_{22}(\mathsf{h}) > 0$ for any h). On the contrary, if $\mathsf{e}_{\text{WB}}(\mathsf{h}) < 0$, then the eigenvalues of $\mathtt{J}_2 E^{(2)}$ lay on the imaginary axis.

3. *Complete block-diagonalization (Section 5.2)*. In Lemma 5.9 we completely block-diagonalize $\mathtt{L}_{\mu,\epsilon}^{(2)}$ by means of a standard implicit function theorem, finally proving that $\mathtt{L}_{\mu,\epsilon}$ is conjugated to the matrix (2.39).

---

[1] Recall that $\exp(S)L\exp(-S) = \sum_{n\geq 0}\frac{1}{n!}\mathrm{ad}_S^n(L)$, where $\mathrm{ad}_S^0(L) := L$, $\mathrm{ad}_S^n(L) = [S, \mathrm{ad}_S^{n-1}(L)]$ for $n \geq 1$.

3. Perturbative Approach to the Separated Eigenvalues

We apply Kato's similarity transformation theory [26, I-§4-6, II-§4] to study the splitting of the eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ close to 0 for small values of μ and ϵ, following [6]. First of all, it is convenient to decompose the operator $\mathcal{L}_{\mu,\epsilon}$ in (2.24) as

$$\mathcal{L}_{\mu,\epsilon} = i \, c_h \mu + \mathscr{L}_{\mu,\epsilon} \,, \qquad \mu > 0 \,, \tag{3.1}$$

where, using also (2.30), $\mathscr{L}_{\mu,\epsilon}$ is the Hamiltonian operator

$$
\begin{aligned}
\mathscr{L}_{\mu,\epsilon} &= \mathcal{J} \, \mathcal{B}_{\mu,\epsilon} \,, \\
\mathcal{B}_{\mu,\epsilon} &:= \begin{bmatrix} 1 + a_\epsilon(x) & -(c_h + p_\epsilon(x))\partial_x - i \, \mu \, p_\epsilon(x) \\ \partial_x \circ (c_h + p_\epsilon(x)) + i \, \mu \, p_\epsilon(x) & |D + \mu| \, \tanh\big((h + f_\epsilon)|D + \mu|\big) \end{bmatrix}
\end{aligned}
\tag{3.2}
$$

with $\mathcal{B}_{\mu,\epsilon}$ selfadjoint, and it is also reversible, namely it satisfies, by (2.26),

$$\mathscr{L}_{\mu,\epsilon} \circ \bar{\rho} = -\bar{\rho} \circ \mathscr{L}_{\mu,\epsilon} \,, \qquad \bar{\rho} \text{ defined in (2.27)} \,, \tag{3.3}$$

whereas $\mathcal{B}_{\mu,\epsilon}$ is reversibility-preserving, i.e. fulfills (2.28). Note also that $\mathcal{B}_{0,\epsilon}$ is a real operator.

The scalar operator $i \, c_h \mu \equiv i \, c_h \mu \, \text{Id}$ just translates the spectrum of $\mathcal{L}_{\mu,\epsilon}$ along the imaginary axis of the quantity $i \, c_h \mu$, that is, in view of (3.1), $\sigma(\mathcal{L}_{\mu,\epsilon}) = i \, c_h \mu + \sigma(\mathscr{L}_{\mu,\epsilon})$. Thus in the sequel we focus on studying the spectrum of $\mathscr{L}_{\mu,\epsilon}$.

Note also that $\mathscr{L}_{0,\epsilon} = \mathcal{L}_{0,\epsilon}$ for any $\epsilon \geq 0$. In particular $\mathscr{L}_{0,0}$ has zero as isolated eigenvalue with algebraic multiplicity 4, geometric multiplicity 3 and generalized kernel spanned by the vectors $\{f_1^+, f_1^-, f_0^+, f_0^-\}$ in (2.34), (2.35); furthermore, its spectrum is separated as in (2.36). For any $\epsilon \neq 0$ small, $\mathscr{L}_{0,\epsilon}$ has zero as isolated eigenvalue with geometric multiplicity 2, and two generalized eigenvectors satisfying (2.37).

We remark that, in view of (2.30), the operator $\mathscr{L}_{\mu,\epsilon}$ is analytic with respect to μ. The operator $\mathscr{L}_{\mu,\epsilon} : Y \subset X \to X$ has domain $Y := H^1(\mathbb{T}) := H^1(\mathbb{T}, \mathbb{C}^2)$ and range $X := L^2(\mathbb{T}) := L^2(\mathbb{T}, \mathbb{C}^2)$.

Lemma 3.1. (Kato theory for separated eigenvalues) *Let* Γ *be a closed, counterclockwise-oriented curve around* 0 *in the complex plane separating* $\sigma'(\mathscr{L}_{0,0}) = \{0\}$ *and the other part of the spectrum* $\sigma''(\mathscr{L}_{0,0})$ *in* (2.36). *There exist* $\epsilon_0, \mu_0 > 0$ *such that for any* $(\mu, \epsilon) \in B(\mu_0) \times B(\epsilon_0)$ *the following statements hold:*

1. *The curve* Γ *belongs to the resolvent set of the operator* $\mathscr{L}_{\mu,\epsilon} : Y \subset X \to X$ *defined in* (3.2).
2. *The operators*

$$P_{\mu,\epsilon} := -\frac{1}{2\pi i} \oint_\Gamma (\mathscr{L}_{\mu,\epsilon} - \lambda)^{-1} d\lambda : X \to Y \tag{3.4}$$

are well defined projectors commuting with $\mathscr{L}_{\mu,\epsilon}$, *i.e.* $P_{\mu,\epsilon}^2 = P_{\mu,\epsilon}$ *and* $P_{\mu,\epsilon}\mathscr{L}_{\mu,\epsilon} = \mathscr{L}_{\mu,\epsilon}P_{\mu,\epsilon}$. *The map* $(\mu, \epsilon) \mapsto P_{\mu,\epsilon}$ *is analytic from* $B(\mu_0) \times B(\epsilon_0)$ *to* $\mathcal{L}(X, Y)$.

3. *The domain Y of the operator $\mathscr{L}_{\mu,\epsilon}$ decomposes as the direct sum*

$$Y = \mathcal{V}_{\mu,\epsilon} \oplus Ker(P_{\mu,\epsilon}), \quad \mathcal{V}_{\mu,\epsilon} := Rg(P_{\mu,\epsilon}) = Ker(\mathrm{Id} - P_{\mu,\epsilon}),$$

of closed invariant subspaces, namely $\mathscr{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}, \mathscr{L}_{\mu,\epsilon} : Ker(P_{\mu,\epsilon}) \to Ker(P_{\mu,\epsilon})$. Moreover

$$\sigma(\mathscr{L}_{\mu,\epsilon}) \cap \{z \in \mathbb{C} \text{ inside } \Gamma\} = \sigma(\mathscr{L}_{\mu,\epsilon}|_{\mathcal{V}_{\mu,\epsilon}}) = \sigma'(\mathscr{L}_{\mu,\epsilon}),$$
$$\sigma(\mathscr{L}_{\mu,\epsilon}) \cap \{z \in \mathbb{C} \text{ outside } \Gamma\} = \sigma(\mathscr{L}_{\mu,\epsilon}|_{Ker(P_{\mu,\epsilon})}) = \sigma''(\mathscr{L}_{\mu,\epsilon}).$$

4. *The projectors $P_{\mu,\epsilon}$ are similar one to each other; the transformation operators*

$$U_{\mu,\epsilon} := \left(\mathrm{Id} - (P_{\mu,\epsilon} - P_{0,0})^2\right)^{-1/2}\left[P_{\mu,\epsilon}P_{0,0} + (\mathrm{Id} - P_{\mu,\epsilon})(\mathrm{Id} - P_{0,0})\right] \tag{3.5}$$

are bounded and invertible in Y and in X, with inverse

$$U_{\mu,\epsilon}^{-1} = \left[P_{0,0}P_{\mu,\epsilon} + (\mathrm{Id} - P_{0,0})(\mathrm{Id} - P_{\mu,\epsilon})\right]\left(\mathrm{Id} - (P_{\mu,\epsilon} - P_{0,0})^2\right)^{-1/2},$$

and $U_{\mu,\epsilon}P_{0,0}U_{\mu,\epsilon}^{-1} = P_{\mu,\epsilon}$ as well as $U_{\mu,\epsilon}^{-1}P_{\mu,\epsilon}U_{\mu,\epsilon} = P_{0,0}$.[2] The map $(\mu, \epsilon) \mapsto U_{\mu,\epsilon}$ is analytic from $B(\mu_0) \times B(\epsilon_0)$ to $\mathcal{L}(Y)$.

5. *The subspaces $\mathcal{V}_{\mu,\epsilon} = Rg(P_{\mu,\epsilon})$ are isomorphic one to each other: $\mathcal{V}_{\mu,\epsilon} = U_{\mu,\epsilon}\mathcal{V}_{0,0}$. In particular $\dim \mathcal{V}_{\mu,\epsilon} = \dim \mathcal{V}_{0,0} = 4$, for any $(\mu, \epsilon) \in B(\mu_0) \times B(\epsilon_0)$.*

Proof. For any $\lambda \in \mathbb{C}$ we decompose $\mathscr{L}_{\mu,\epsilon} - \lambda = \mathscr{L}_{0,0} - \lambda + \mathcal{R}_{\mu,\epsilon}$ where $\mathscr{L}_{0,0} = \begin{bmatrix} c_h \partial_x & |D| \tanh(h|D|) \\ -1 & c_h \partial_x \end{bmatrix}$ and

$$\mathcal{R}_{\mu,\epsilon} := \mathscr{L}_{\mu,\epsilon} - \mathscr{L}_{0,0} = \begin{bmatrix} (\partial_x + i\mu)p_\epsilon(x) & f_{\mu,\epsilon}(D) \\ -a_\epsilon(x) & p_\epsilon(x)(\partial_x + i\mu) \end{bmatrix} : Y \to X,$$

having used also (2.30) and setting

$$f_{\mu,\epsilon}(D) := |D + \mu| \tanh\left((h + f_\epsilon)|D + \mu|\right) - |D| \tanh(h|D|) \in \mathcal{L}(Y),$$
$$\|f_{\mu,\epsilon}(D)\|_{\mathcal{L}(Y)} = \mathcal{O}(\mu, \epsilon).$$

For any $\lambda \in \Gamma$, the operator $\mathscr{L}_{0,0} - \lambda$ is invertible with inverse

$$(\mathscr{L}_{0,0} - \lambda)^{-1}$$
$$= \mathrm{Op}\left(\frac{1}{(i c_h k - \lambda)^2 + |k| \tanh(h|k|)}\begin{bmatrix} i c_h k - \lambda & -|k| \tanh(h|k|) \\ 1 & i c_h k - \lambda \end{bmatrix}\right) : X \to Y.$$

---

[2] The operator $(\mathrm{Id} - R)^{-\frac{1}{2}}$ is defined, for any operator R satisfying $\|R\|_{\mathcal{L}(Y)} < 1$, by the power series

$$(\mathrm{Id} - R)^{-\frac{1}{2}} := \sum_{k=0}^{\infty} \binom{-1/2}{k}(-R)^k = \mathrm{Id} + \frac{1}{2}R + \frac{3}{8}R^2 + \mathcal{O}(R^3). \tag{3.6}$$

Hence, for $|\epsilon| < \epsilon_0$ and $|\mu| < \mu_0$ small enough, uniformly on the compact set Γ, the operator $(\mathcal{L}_{0,0} - \lambda)^{-1}\mathcal{R}_{\mu,\epsilon} : Y \to Y$ is bounded, with small operatorial norm. Then $\mathcal{L}_{\mu,\epsilon} - \lambda$ is invertible by Neumann series and Γ belongs to the resolvent set of $\mathcal{L}_{\mu,\epsilon}$. The remaining part of the proof follows exactly as in Lemma 3.1 in [6].
□

The Hamiltonian and reversible nature of the operator $\mathcal{L}_{\mu,\epsilon}$, see (3.2) and (3.3), imply additional algebraic properties for spectral projectors $P_{\mu,\epsilon}$ and the transformation operators $U_{\mu,\epsilon}$. By Lemma 3.2 in [6] we have that:

Lemma 3.2. *For any* $(\mu, \epsilon) \in B(\mu_0) \times B(\epsilon_0)$, *the following holds true:*

(i) *The projectors* $P_{\mu,\epsilon}$ *defined in* (3.4) *are skew-Hamiltonian, namely* $\mathcal{J}P_{\mu,\epsilon} = P^*_{\mu,\epsilon}\mathcal{J}$, *and reversibility preserving, i.e.* $\bar{\rho}P_{\mu,\epsilon} = P_{\mu,\epsilon}\bar{\rho}$.
(ii) *The transformation operators* $U_{\mu,\epsilon}$ *in* (3.5) *are symplectic, namely* $U^*_{\mu,\epsilon}\mathcal{J}U_{\mu,\epsilon} = \mathcal{J}$, *and reversibility preserving.*
(iii) $P_{0,\epsilon}$ *and* $U_{0,\epsilon}$ *are real operators, i.e.* $\bar{P}_{0,\epsilon} = P_{0,\epsilon}$ *and* $\bar{U}_{0,\epsilon} = U_{0,\epsilon}$.

By the previous lemma, the linear involution $\bar{\rho}$ commutes with the spectral projectors $P_{\mu,\epsilon}$ and then $\bar{\rho}$ leaves invariant the subspace $\mathcal{V}_{\mu,\epsilon} = \mathrm{Rg}(P_{\mu,\epsilon})$.
Symplectic and reversible basis of $\mathcal{V}_{\mu,\epsilon}$. It is convenient to represent the Hamiltonian and reversible operator $\mathcal{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ in a basis which is symplectic and reversible, according to the following definition:

Definition 3.3. (Symplectic and reversible basis) A basis $\mathtt{F} := \{\mathtt{f}^+_1, \mathtt{f}^-_1, \mathtt{f}^+_0, \mathtt{f}^-_0\}$ of $\mathcal{V}_{\mu,\epsilon}$ is *symplectic* if, for any $k, k' = 0, 1$,

$$\left(\mathcal{J}\mathtt{f}^-_k, \mathtt{f}^+_k\right) = 1, \quad \left(\mathcal{J}\mathtt{f}^\sigma_k, \mathtt{f}^\sigma_k\right) = 0, \quad \forall\sigma = \pm;$$
$$\text{if } k \neq k' \text{ then } \left(\mathcal{J}\mathtt{f}^\sigma_k, \mathtt{f}^{\sigma'}_{k'}\right) = 0, \quad \forall\sigma, \sigma' = \pm. \tag{3.7}$$

This is *reversible* if

$$\bar{\rho}\mathtt{f}^+_1 = \mathtt{f}^+_1, \quad \bar{\rho}\mathtt{f}^-_1 = -\mathtt{f}^-_1, \quad \bar{\rho}\mathtt{f}^+_0 = \mathtt{f}^+_0, \quad \bar{\rho}\mathtt{f}^-_0 = -\mathtt{f}^-_0,$$
$$\text{i.e. } \bar{\rho}\mathtt{f}^\sigma_k = \sigma\mathtt{f}^\sigma_k, \quad \forall\sigma = \pm, k = 0, 1. \tag{3.8}$$

We use the following notation along the paper: we denote by $even(x)$ a real 2π-periodic function which is even in x, and by $odd(x)$ a real 2π-periodic function which is odd in x.

By the definition of the involution $\bar{\rho}$ in (2.27), the real and imaginary parts of a reversible basis $\mathtt{F} = \{\mathtt{f}^\pm_k\}$, $k = 0, 1$, enjoy the following parity properties (cfr. Lemma 3.4 in [6])

$$\mathtt{f}^+_k(x) = \begin{bmatrix} even(x) + \mathrm{i}\,odd(x) \\ odd(x) + \mathrm{i}\,even(x) \end{bmatrix}, \quad \mathtt{f}^-_k(x) = \begin{bmatrix} odd(x) + \mathrm{i}\,even(x) \\ even(x) + \mathrm{i}\,odd(x) \end{bmatrix}. \tag{3.9}$$

By Lemmata 3.5 and 3.6 in [6] we have

Lemma 3.4. *The* 4×4 *matrix that represents the Hamiltonian and reversible operator* $\mathscr{L}_{\mu,\epsilon} = \mathcal{J}\mathcal{B}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ *with respect to a symplectic and reversible basis* $\mathtt{F} = \{\mathtt{f}_1^+, \mathtt{f}_1^-, \mathtt{f}_0^+, \mathtt{f}_0^-\}$ *of* $\mathcal{V}_{\mu,\epsilon}$ *is*

$$\mathtt{J}_4\mathtt{B}_{\mu,\epsilon}, \quad \mathtt{J}_4 := \left(\begin{array}{c|c} \mathtt{J}_2 & 0 \\ \hline 0 & \mathtt{J}_2 \end{array}\right), \quad \mathtt{J}_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{where} \quad \mathtt{B}_{\mu,\epsilon} = \mathtt{B}_{\mu,\epsilon}^* \quad (3.10)$$

is the self-adjoint matrix

$$\mathtt{B}_{\mu,\epsilon} = \begin{pmatrix} \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_1^+\, , \mathtt{f}_1^+\right) & \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_1^-\, , \mathtt{f}_1^+\right) & \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_0^+\, , \mathtt{f}_1^+\right) & \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_0^-\, , \mathtt{f}_1^+\right) \\ \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_1^+\, , \mathtt{f}_1^-\right) & \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_1^-\, , \mathtt{f}_1^-\right) & \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_0^+\, , \mathtt{f}_1^-\right) & \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_0^-\, , \mathtt{f}_1^-\right) \\ \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_1^+\, , \mathtt{f}_0^+\right) & \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_1^-\, , \mathtt{f}_0^+\right) & \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_0^+\, , \mathtt{f}_0^+\right) & \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_0^-\, , \mathtt{f}_0^+\right) \\ \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_1^+\, , \mathtt{f}_0^-\right) & \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_1^-\, , \mathtt{f}_0^-\right) & \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_0^+\, , \mathtt{f}_0^-\right) & \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_0^-\, , \mathtt{f}_0^-\right) \end{pmatrix}.$$
$$(3.11)$$

The entries of the matrix $\mathtt{B}_{\mu,\epsilon}$ *are alternatively real or purely imaginary: for any* $\sigma = \pm$, $k = 0, 1$,

$$\left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_k^\sigma\, , \mathtt{f}_{k'}^\sigma\right) \text{ is real,} \qquad \left(\mathcal{B}_{\mu,\epsilon}\, \mathtt{f}_k^\sigma\, , \mathtt{f}_{k'}^{-\sigma}\right) \text{ is purely imaginary.} \quad (3.12)$$

It is convenient to give a name to the matrices of the form obtained in Lemma 3.4.

Definition 3.5. A $2n \times 2n$, $n = 1, 2$, matrix of the form $\mathtt{L} = \mathtt{J}_{2n}\mathtt{B}$ is *Hamiltonian* if $\mathtt{B}$ is a self-adjoint matrix, i.e. $\mathtt{B} = \mathtt{B}^*$. It is *reversible* if $\mathtt{B}$ is reversibility-preserving, i.e. $\rho_{2n} \circ \mathtt{B} = \mathtt{B} \circ \rho_{2n}$, where

$$\rho_4 := \begin{pmatrix} \rho_2 & 0 \\ 0 & \rho_2 \end{pmatrix}, \qquad \rho_2 := \begin{pmatrix} \mathfrak{c} & 0 \\ 0 & -\mathfrak{c} \end{pmatrix},$$

and $\mathfrak{c} : z \mapsto \bar{z}$ is the conjugation of the complex plane. Equivalently, $\rho_{2n} \circ \mathtt{L} = -\mathtt{L} \circ \rho_{2n}$.

The transformations preserving the Hamiltonian structure are called *symplectic*, and satisfy

$$Y^*\mathtt{J}_4 Y = \mathtt{J}_4. \quad (3.13)$$

If Y is symplectic then Y^* and Y^{-1} are symplectic as well. A Hamiltonian matrix $\mathtt{L} = \mathtt{J}_4\mathtt{B}$, with $\mathtt{B} = \mathtt{B}^*$, is conjugated through Y in the new Hamiltonian matrix

$$\mathtt{L}_1 = Y^{-1}\mathtt{L}Y = Y^{-1}\mathtt{J}_4 Y^{-*}Y^*\mathtt{B}Y = \mathtt{J}_4\mathtt{B}_1 \quad \text{where} \quad \mathtt{B}_1 := Y^*\mathtt{B}Y = \mathtt{B}_1^*. \quad (3.14)$$

A 4×4 matrix $\mathtt{B} = (\mathtt{B}_{ij})_{i,j=1,\dots,4}$ is reversibility-preserving if and only if its entries are alternatively real and purely imaginary, namely $\mathtt{B}_{ij}$ is real when $i + j$ is even and purely imaginary otherwise, as in (3.12). A 4×4 complex matrix $\mathtt{L} = (\mathtt{L}_{ij})_{i,j=1,\dots,4}$ is reversible if and only if $\mathtt{L}_{ij}$ is purely imaginary when $i + j$ is even and real otherwise.

Finally, we mention that the flow of a Hamiltonian reversibility-preserving matrix is symplectic and reversibility-preserving (see Lemma 3.8 in [6]).

4. Matrix Representation of $\mathscr{L}_{\mu,\epsilon}$ on $\mathcal{V}_{\mu,\epsilon}$

Using the transformation operators $U_{\mu,\epsilon}$ in (3.5), we construct the basis of $\mathcal{V}_{\mu,\epsilon}$

$$\mathcal{F} := \left\{ f_1^+(\mu, \epsilon),\ f_1^-(\mu, \epsilon),\ f_0^+(\mu, \epsilon),\ f_0^-(\mu, \epsilon) \right\},$$
$$f_k^\sigma(\mu, \epsilon) := U_{\mu,\epsilon} f_k^\sigma,\quad \sigma = \pm,\ k = 0, 1, \tag{4.1}$$

where

$$f_1^+ = \begin{bmatrix} c_{\mathrm{h}}^{1/2} \cos(x) \\ c_{\mathrm{h}}^{-1/2} \sin(x) \end{bmatrix}, \quad f_1^- = \begin{bmatrix} -c_{\mathrm{h}}^{1/2} \sin(x) \\ c_{\mathrm{h}}^{-1/2} \cos(x) \end{bmatrix}, \quad f_0^+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f_0^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{4.2}$$

form a basis of $\mathcal{V}_{0,0} = \mathrm{Rg}(P_{0,0})$, cfr. (2.34)-(2.35). Note that the real valued vectors $\{f_1^\pm, f_0^\pm\}$ form a symplectic and reversible basis for $\mathcal{V}_{0,0}$, according to Definition 3.3. Then, by Lemma 3.2 and 3.1 we deduce that (cfr. Lemma 4.1 in [6]):

Lemma 4.1. *The basis $\mathcal{F}$ of $\mathcal{V}_{\mu,\epsilon}$ defined in (4.1), is symplectic and reversible, i.e. satisfies (3.7) and (3.8). Each map $(\mu, \epsilon) \mapsto f_k^\sigma(\mu, \epsilon)$ is analytic as a map $B(\mu_0) \times B(\epsilon_0) \to H^1(\mathbb{T})$.*

In the next lemma we expand the vectors $f_k^\sigma(\mu, \epsilon)$ in (μ, ϵ). We denote by $even_0(x)$ a real, even, 2π-periodic function with zero space average. In the sequel $\mathcal{O}(\mu^m \epsilon^n) \begin{bmatrix} even(x) \\ odd(x) \end{bmatrix}$ denotes an analytic map in (μ, ϵ) with values in $H^1(\mathbb{T}, \mathbb{C}^2)$, whose first component is $even(x)$ and the second one $odd(x)$; we have a similar meaning for $\mathcal{O}(\mu^m \epsilon^n) \begin{bmatrix} odd(x) \\ even(x) \end{bmatrix}$, etc....

Lemma 4.2. (Expansion of the basis $\mathcal{F}$) *For small values of (μ, ϵ) the basis $\mathcal{F}$ in (4.1) has the expansion*

$$f_1^+(\mu, \epsilon) = \begin{bmatrix} c_{\mathrm{h}}^{\frac{1}{2}} \cos(x) \\ c_{\mathrm{h}}^{-\frac{1}{2}} \sin(x) \end{bmatrix} + \mathrm{i}\, \frac{\mu}{4} \gamma_{\mathrm{h}} \begin{bmatrix} c_{\mathrm{h}}^{\frac{1}{2}} \sin(x) \\ c_{\mathrm{h}}^{-\frac{1}{2}} \cos(x) \end{bmatrix} + \epsilon \begin{bmatrix} \alpha_{\mathrm{h}} \cos(2x) \\ \beta_{\mathrm{h}} \sin(2x) \end{bmatrix}$$
$$+ \mathcal{O}(\mu^2) \begin{bmatrix} even_0(x) + \mathrm{i}\, odd(x) \\ odd(x) + \mathrm{i}\, even_0(x) \end{bmatrix} + \mathcal{O}(\epsilon^2) \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix}$$
$$+ \mathrm{i}\, \mu\epsilon \begin{bmatrix} odd(x) \\ even(x) \end{bmatrix} + \mathcal{O}(\mu^2 \epsilon, \mu\epsilon^2), \tag{4.3}$$

$$f_1^-(\mu, \epsilon) = \begin{bmatrix} -c_{\mathrm{h}}^{\frac{1}{2}} \sin(x) \\ c_{\mathrm{h}}^{-\frac{1}{2}} \cos(x) \end{bmatrix} + \mathrm{i}\, \frac{\mu}{4} \gamma_{\mathrm{h}} \begin{bmatrix} c_{\mathrm{h}}^{\frac{1}{2}} \cos(x) \\ -c_{\mathrm{h}}^{-\frac{1}{2}} \sin(x) \end{bmatrix} + \epsilon \begin{bmatrix} -\alpha_{\mathrm{h}} \sin(2x) \\ \beta_{\mathrm{h}} \cos(2x) \end{bmatrix}$$
$$+ \mathcal{O}(\mu^2) \begin{bmatrix} odd(x) + \mathrm{i}\, even_0(x) \\ even_0(x) + \mathrm{i}\, odd(x) \end{bmatrix} + \mathcal{O}(\epsilon^2) \begin{bmatrix} odd(x) \\ even(x) \end{bmatrix}$$
$$+ \mathrm{i}\, \mu\epsilon \begin{bmatrix} even(x) \\ odd(x) \end{bmatrix} + \mathcal{O}(\mu^2 \epsilon, \mu\epsilon^2), \tag{4.4}$$

$$f_0^+(\mu, \epsilon) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \epsilon \delta_h \begin{bmatrix} c_h^{\frac{1}{2}} \cos(x) \\ -c_h^{-\frac{1}{2}} \sin(x) \end{bmatrix} + \mathcal{O}(\epsilon^2) \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix}$$

$$+ \mathrm{i}\,\mu\epsilon \begin{bmatrix} odd(x) \\ even_0(x) \end{bmatrix} + \mathcal{O}(\mu^2\epsilon, \mu\epsilon^2), \tag{4.5}$$

$$f_0^-(\mu, \epsilon) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \mathrm{i}\,\mu\epsilon \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix} + \mathcal{O}(\mu^2\epsilon, \mu\epsilon^2), \tag{4.6}$$

where the remainders $\mathcal{O}()$ are vectors in $H^1(\mathbb{T})$ and

$$\alpha_h := \frac{1}{2} c_h^{-\frac{11}{2}} (3 + c_h^4), \quad \beta_h := \frac{1}{4} c_h^{-\frac{13}{2}} (1 + c_h^4)(3 - c_h^4),$$

$$\gamma_h := 1 + \frac{h(1 - c_h^4)}{c_h^2}, \quad \delta_h := \frac{3 + c_h^4}{4 c_h^{\frac{5}{2}}}. \tag{4.7}$$

For $\mu = 0$ the basis $\{f_k^\pm(0, \epsilon), k = 0, 1\}$ is real and

$$f_1^+(0, \epsilon) = \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix}, \quad f_1^-(0, \epsilon) = \begin{bmatrix} odd(x) \\ even(x) \end{bmatrix},$$

$$f_0^+(0, \epsilon) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix}, \quad f_0^-(0, \epsilon) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4.8}$$

Proof. The long calculations are given in Appendix A. □

We now state the main result of this section.

Proposition 4.3. *The matrix that represents the Hamiltonian and reversible operator $\mathscr{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ in the symplectic and reversible basis $\mathcal{F}$ of $\mathcal{V}_{\mu,\epsilon}$ defined in (4.1), is a Hamiltonian matrix $\mathsf{L}_{\mu,\epsilon} = \mathsf{J}_4 \mathsf{B}_{\mu,\epsilon}$, where $\mathsf{B}_{\mu,\epsilon}$ is a self-adjoint and reversibility preserving (i.e. satisfying (3.12)) 4×4 matrix of the form*

$$\mathsf{B}_{\mu,\epsilon} = \begin{pmatrix} E & F \\ F^* & G \end{pmatrix}, \quad E = E^*, \quad G = G^*, \tag{4.9}$$

where E, F, G are the 2×2 matrices

$$E := \begin{pmatrix} \mathsf{e}_{11}\epsilon^2(1 + r_1'(\epsilon, \mu\epsilon)) - \mathsf{e}_{22}\frac{\mu^2}{8}(1 + r_1''(\epsilon, \mu)) & \mathrm{i}\left(\frac{1}{2}\mathsf{e}_{12}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) \\ -\mathrm{i}\left(\frac{1}{2}\mathsf{e}_{12}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) & -\mathsf{e}_{22}\frac{\mu^2}{8}(1 + r_5(\epsilon, \mu)) \end{pmatrix} \tag{4.10}$$

$$G := \begin{pmatrix} 1 + r_8(\epsilon^2, \mu^2\epsilon) & -\mathrm{i}\,r_9(\mu\epsilon^2, \mu^2\epsilon) \\ \mathrm{i}\,r_9(\mu\epsilon^2, \mu^2\epsilon) & \mu \tanh(h\mu) + r_{10}(\mu^2\epsilon) \end{pmatrix} \tag{4.11}$$

$$F := \begin{pmatrix} \mathsf{f}_{11}\epsilon + r_3(\epsilon^3, \mu\epsilon^2, \mu^2\epsilon) & \mathrm{i}\,\mu\epsilon\, c_h^{-\frac{1}{2}} + \mathrm{i}\,r_4(\mu\epsilon^2, \mu^2\epsilon) \\ \mathrm{i}\,r_6(\mu\epsilon) & r_7(\mu^2\epsilon) \end{pmatrix}, \tag{4.12}$$

with e_{12} and e_{22} given in (1.2) and (1.3) respectively, and

$$\mathsf{e}_{11} := \frac{9c_h^8 - 10c_h^4 + 9}{8c_h^7} = \frac{9(1 - c_h^4)^2 + 8c_h^4}{8c_h^7} > 0, \quad \mathsf{f}_{11} := \frac{1}{2}c_h^{-\frac{3}{2}}(1 - c_h^4). \tag{4.13}$$

The rest of this section is devoted to the proof of Proposition 4.3.
We decompose $\mathcal{B}_{\mu,\epsilon}$ in (3.2) as

$$\mathcal{B}_{\mu,\epsilon} = \mathcal{B}_\epsilon + \mathcal{B}^\flat + \mathcal{B}^\sharp,$$

where $\mathcal{B}_\epsilon$, $\mathcal{B}^\flat$, $\mathcal{B}^\sharp$ are the self-adjoint and reversibility preserving operators

$$\mathcal{B}_\epsilon := \mathcal{B}_{0,\epsilon} := \begin{bmatrix} 1 + a_\epsilon(x) & -(c_h + p_\epsilon(x))\partial_x \\ \partial_x \circ (c_h + p_\epsilon(x)) & |D|\tanh((h + f_\epsilon)|D|) \end{bmatrix}, \qquad (4.14)$$

$$\mathcal{B}^\flat := \begin{bmatrix} 0 & 0 \\ 0 & |D + \mu|\tanh((h + f_\epsilon)|D + \mu|) - |D|\tanh((h + f_\epsilon)|D|) \end{bmatrix}, \qquad (4.15)$$

$$\mathcal{B}^\sharp := \mu \begin{bmatrix} 0 & -i\, p_\epsilon \\ i\, p_\epsilon & 0 \end{bmatrix}. \qquad (4.16)$$

In view of (2.29), the operator $\mathcal{B}^\flat$ is analytic in μ.

Lemma 4.4. (Expansion of B_ϵ) *The self-adjoint and reversibility preserving matrix $B_\epsilon := B_\epsilon(\mu)$ associated, as in (3.11), with the self-adjoint and reversibility preserving operator $\mathcal{B}_\epsilon$ defined in (4.14), with respect to the basis $\mathcal{F}$ of $\mathcal{V}_{\mu,\epsilon}$ in (4.1), expands as*

$$B_\epsilon = \left(\begin{array}{cc|cc} e_{11}\epsilon^2 + \zeta_h\mu^2 + r_1(\epsilon^3, \mu\epsilon^3) & i\, r_2(\mu\epsilon^2) & f_{11}\epsilon + r_3(\epsilon^3, \mu\epsilon^2) & i\, r_4(\mu\epsilon^3) \\ -i\, r_2(\mu\epsilon^2) & \zeta_h\mu^2 & i\, r_6(\mu\epsilon) & 0 \\ \hline f_{11}\epsilon + r_3(\epsilon^3, \mu\epsilon^2) & -i\, r_6(\mu\epsilon) & 1 + r_8(\epsilon^2, \mu\epsilon^2) & i\, r_9(\mu\epsilon^2) \\ -i\, r_4(\mu\epsilon^3) & 0 & -i\, r_9(\mu\epsilon^2) & 0 \end{array} \right)$$
$$+ \mathcal{O}(\mu^2\epsilon, \mu^3), \qquad (4.17)$$

where e_{11}, f_{11} are defined respectively in (4.13), and

$$\zeta_h := \tfrac{1}{8}c_h\gamma_h^2. \qquad (4.18)$$

Proof. We expand the matrix $B_\epsilon(\mu)$ as

$$B_\epsilon(\mu) = B_\epsilon(0) + \mu(\partial_\mu B_\epsilon)(0) + \frac{\mu^2}{2}(\partial_\mu^2 B_0)(0) + \mathcal{O}(\mu^2\epsilon, \mu^3). \qquad (4.19)$$

The matrix $B_\epsilon(0)$. The main result of this long paragraph is to prove that the matrix $B_\epsilon(0)$ has the expansion (4.23). The matrix $B_\epsilon(0)$ is real, because the operator $\mathcal{B}_\epsilon$ is real and the basis $\{f_k^\pm(0, \epsilon)\}_{k=0,1}$ is real. Consequently, by (3.12), its matrix elements $(B_\epsilon(0))_{i,j}$ are real whenever $i + j$ is even and vanish for $i + j$ odd. In addition $f_0^-(0, \epsilon) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by (4.8), and, by (4.14), we get $\mathcal{B}_\epsilon f_0^-(0, \epsilon) = 0$, for any ϵ. We deduce that the self-adjoint matrix $B_\epsilon(0)$ has the form

$$B_\epsilon(0) = \left(\mathcal{B}_\epsilon\, f_k^\sigma(0, \epsilon),\, f_{k'}^{\sigma'}(0, \epsilon) \right)_{k,k'=0,1,\sigma,\sigma'=\pm}$$

$$
= \begin{pmatrix}
E_{11}(0,\epsilon) & 0 & F_{11}(0,\epsilon) & 0 \\
0 & E_{22}(0,\epsilon) & 0 & 0 \\
\hline
F_{11}(0,\epsilon) & 0 & G_{11}(0,\epsilon) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \tag{4.20}
$$

where $E_{11}(0,\epsilon), E_{22}(0,\epsilon), G_{11}(0,\epsilon), F_{11}(0,\epsilon)$ are real. We claim that $E_{22}(0,\epsilon) = 0$ for any ϵ. As a first step, following [6], we prove that

$$
\text{either} \quad E_{22}(0,\epsilon) \equiv 0, \qquad \text{or} \quad E_{11}(0,\epsilon) \equiv 0 \equiv F_{11}(0,\epsilon). \tag{4.21}
$$

Indeed, by (2.37), the operator $\mathscr{L}_{0,\epsilon} \equiv \mathcal{L}_{0,\epsilon}$ possesses, for any sufficiently small $\epsilon \neq 0$, the eigenvalue 0 with a four dimensional generalized Kernel $\mathcal{W}_\epsilon :=$ span$\{U_1, \tilde{U}_2, U_3, U_4\}$, spanned by ϵ-dependent vectors $U_1, \tilde{U}_2, U_3, U_4$. By Lemma 3.1 it results that $\mathcal{W}_\epsilon = \mathcal{V}_{0,\epsilon} = \text{Rg}(P_{0,\epsilon})$ and by (2.37) we have $\mathscr{L}_{0,\epsilon}^2 = 0$ on $\mathcal{V}_{0,\epsilon}$. Thus the matrix

$$
\mathsf{L}_\epsilon(0) := \mathsf{J}_4 \mathsf{B}_\epsilon(0) = \begin{pmatrix}
0 & E_{22}(0,\epsilon) & 0 & 0 \\
-E_{11}(0,\epsilon) & 0 & -F_{11}(0,\epsilon) & 0 \\
\hline
0 & 0 & 0 & 0 \\
-F_{11}(0,\epsilon) & 0 & -G_{11}(0,\epsilon) & 0
\end{pmatrix}, \tag{4.22}
$$

which represents $\mathscr{L}_{0,\epsilon} : \mathcal{V}_{0,\epsilon} \to \mathcal{V}_{0,\epsilon}$, satisfies $\mathsf{L}_\epsilon^2(0) = 0$, namely

$$
\mathsf{L}_\epsilon^2(0) = - \begin{pmatrix}
(E_{11}E_{22})(0,\epsilon) & 0 & (F_{11}E_{22})(0,\epsilon) & 0 \\
0 & (E_{11}E_{22})(0,\epsilon) & 0 & 0 \\
\hline
0 & 0 & 0 & 0 \\
0 & (F_{11}E_{22})(0,\epsilon) & 0 & 0
\end{pmatrix} = 0,
$$

which implies (4.21). We now prove that the matrix $\mathsf{B}_\epsilon(0)$ defined in (4.20) expands as

$$
\mathsf{B}_\epsilon(0) = \begin{pmatrix}
\mathsf{e}_{11}\epsilon^2 + r(\epsilon^3) & 0 & \mathsf{f}_{11}\epsilon + r(\epsilon^3) & 0 \\
0 & 0 & 0 & 0 \\
\hline
\mathsf{f}_{11}\epsilon + r(\epsilon^3) & 0 & 1 + r(\epsilon^2) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \tag{4.23}
$$

where e_{11} and f_{11} are in (4.29) and (4.32). We expand the operator $\mathcal{B}_\epsilon$ in (4.14) as

$$
\mathcal{B}_\epsilon = \mathcal{B}_0 + \epsilon \mathcal{B}_1 + \epsilon^2 \mathcal{B}_2 + \mathcal{O}(\epsilon^3), \quad \mathcal{B}_0 := \begin{bmatrix} 1 & -\mathsf{c_h}\partial_x \\ \mathsf{c_h}\partial_x & |D|\tanh(\mathsf{h}|D|) \end{bmatrix},
$$
$$
\mathcal{B}_1 := \begin{bmatrix} a_1(x) & -p_1(x)\partial_x \\ \partial_x \circ p_1(x) & 0 \end{bmatrix}, \quad \mathcal{B}_2 := \begin{bmatrix} a_2(x) & -p_2(x)\partial_x \\ \partial_x \circ p_2(x) & -\mathsf{f}_2\partial_x^2(1 - \tanh^2(\mathsf{h}|D|)) \end{bmatrix}, \tag{4.24}
$$

where the remainder term $\mathcal{O}(\epsilon^3) \in \mathcal{L}(Y, X)$, the functions a_1, p_1, a_2, p_2 are given in (2.20)-(2.23) and, in view of (2.15), $\mathsf{f}_2 := \frac{1}{4}\mathsf{c_h}^{-2}(\mathsf{c_h}^4 - 3)$.

• *Expansion of* $E_{11}(0,\epsilon) = \mathsf{e}_{11}\epsilon^2 + r(\epsilon^3)$. By (4.3) we split the real function $f_1^+(0,\epsilon)$ as

$$
f_1^+(0,\epsilon) = f_1^+ + \epsilon f_{1_1}^+ + \epsilon^2 f_{1_2}^+ + \mathcal{O}(\epsilon^3),
$$
$$
f_1^+ = \begin{bmatrix} \mathsf{c_h}^{\frac{1}{2}}\cos(x) \\ \mathsf{c_h}^{-\frac{1}{2}}\sin(x) \end{bmatrix}, \quad f_{1_1}^+ := \begin{bmatrix} \alpha_\mathsf{h}\cos(2x) \\ \beta_\mathsf{h}\sin(2x) \end{bmatrix}, \quad f_{1_2}^+ := \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix}, \tag{4.25}
$$

where both $f_{1_2}^+$ and $\mathcal{O}(\epsilon^3)$ are vectors in $H^1(\mathbb{T})$. Since $\mathcal{B}_0 f_1^+ = \mathcal{J}^{-1}\mathcal{L}_{0,0} f_1^+ = 0$, and both $\mathcal{B}_0$, $\mathcal{B}_1$ are self-adjoint real operators, it results

$$
\begin{aligned}
E_{11}(0,\epsilon) &= \left(\mathcal{B}_\epsilon f_1^+(0,\epsilon),\, f_1^+(0,\epsilon)\right) \\
&= \epsilon\left(\mathcal{B}_1 f_1^+,\, f_1^+\right) + \epsilon^2\left[\left(\mathcal{B}_2 f_1^+,\, f_1^+\right) + 2\left(\mathcal{B}_1 f_1^+,\, f_{1_1}^+\right) + \left(\mathcal{B}_0 f_{1_1}^+,\, f_{1_1}^+\right)\right] \\
&\quad + \mathcal{O}(\epsilon^3).
\end{aligned}
\tag{4.26}
$$

By (4.24) one has

$$
\mathcal{B}_1 f_1^+ = \begin{bmatrix} A_1(1+\cos(2x)) \\ B_1 \sin(2x) \end{bmatrix}, \quad
\mathcal{B}_2 f_1^+ = \begin{bmatrix} A_2\cos(x) + A_3\cos(3x) \\ B_2\sin(x) + B_3\sin(3x) \end{bmatrix},
$$

$$
\mathcal{B}_0 f_{1_1}^+ = \begin{bmatrix} A_4\cos(2x) \\ B_4\sin(2x) \end{bmatrix},
\tag{4.27}
$$

with

$$
\begin{aligned}
A_1 &:= \tfrac{1}{2}(a_1^{[1]} c_{\mathrm{h}}^{\frac{1}{2}} - p_1^{[1]} c_{\mathrm{h}}^{-\frac{1}{2}}), \qquad B_1 := -p_1^{[1]} c_{\mathrm{h}}^{\frac{1}{2}}, \\
A_2 &:= c_{\mathrm{h}}^{\frac{1}{2}} a_2^{[0]} - c_{\mathrm{h}}^{-\frac{1}{2}} p_2^{[0]} + \tfrac{1}{2} c_{\mathrm{h}}^{\frac{1}{2}} a_2^{[2]} - \tfrac{1}{2} c_{\mathrm{h}}^{-\frac{1}{2}} p_2^{[2]}, \qquad A_4 := \alpha_{\mathrm{h}} - 2\beta_{\mathrm{h}} c_{\mathrm{h}}, \\
B_2 &:= -c_{\mathrm{h}}^{\frac{1}{2}} p_2^{[0]} - \tfrac{1}{2} c_{\mathrm{h}}^{\frac{1}{2}} p_2^{[2]} + c_{\mathrm{h}}^{-\frac{1}{2}} f_2(1 - c_{\mathrm{h}}^4), \qquad B_4 := -2\alpha_{\mathrm{h}} c_{\mathrm{h}} + \frac{4 c_{\mathrm{h}}^2}{1 + c_{\mathrm{h}}^4}\beta_{\mathrm{h}}.
\end{aligned}
\tag{4.28}
$$

By (4.27) and (4.25), we deduce

$$
\begin{aligned}
E_{11}(0,\epsilon) &= \mathrm{e}_{11}\epsilon^2 + r(\epsilon^3), \\
\mathrm{e}_{11} &:= \frac{1}{2}\left(A_2 c_{\mathrm{h}}^{\frac{1}{2}} + B_2 c_{\mathrm{h}}^{-\frac{1}{2}} + 2\alpha_{\mathrm{h}} A_1 + 2 B_1 \beta_{\mathrm{h}} + \alpha_{\mathrm{h}} A_4 + \beta_{\mathrm{h}} B_4\right).
\end{aligned}
\tag{4.29}
$$

By (4.29), (4.28), (4.7), (2.20)-(2.23) we obtain (4.13). Since $\mathrm{e}_{11} > 0$ the second alternative in (4.21) is ruled out, implying $E_{22}(0,\epsilon) \equiv 0$.

- *Expansion of* $G_{11}(0,\epsilon) = 1 + r(\epsilon^2)$. By (4.5) we split the real-valued function $f_0^+(0,\epsilon)$ as

$$
f_0^+(0,\epsilon) = f_0^+ + \epsilon f_{0_1}^+ + \epsilon^2 f_{0_2}^+ + \mathcal{O}(\epsilon^3), \quad f_0^+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
$$

$$
f_{0_1}^+ := \delta_{\mathrm{h}} \begin{bmatrix} c_{\mathrm{h}}^{\frac{1}{2}}\cos(x) \\ -c_{\mathrm{h}}^{-\frac{1}{2}}\sin(x) \end{bmatrix}, \quad f_{0_2}^+ := \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix}.
\tag{4.30}
$$

Since, by (2.34) and (4.24), $\mathcal{B}_0 f_0^+ = f_0^+$, using that $\mathcal{B}_0$, $\mathcal{B}_1$ are self-adjoint real operators, and $\|f_0^+\| = 1$, $(f_0^+, f_{0_1}^+)$, we have $G_{11}(0,\epsilon) = \left(\mathcal{B}_\epsilon f_0^+(0,\epsilon),\, f_0^+(0,\epsilon)\right) = 1 + \epsilon\left(\mathcal{B}_1 f_0^+,\, f_0^+\right) + r(\epsilon^2)$. By (4.24) and (2.20)-(2.23) one has

$$
\mathcal{B}_1 f_0^+ = \begin{bmatrix} a_1^{[1]}\cos(x) \\ -p_1^{[1]}\sin(x) \end{bmatrix}
\tag{4.31}
$$

and, by (4.30), we deduce $G_{11}(0,\epsilon) = 1 + r(\epsilon^2)$.

• *Expansion of $F_{11}(0, \epsilon) = \mathtt{f}_{11}\epsilon + r(\epsilon^3)$.* By (4.24), (4.25), (4.30), using that $\mathcal{B}_0, \mathcal{B}_1$ are self-adjoint and real, and $\mathcal{B}_0 f_1^+ = 0$, $\mathcal{B}_0 f_0^+ = f_0^+$, we obtain

$$
\begin{aligned}
F_{11}(0, \epsilon) = \; & \epsilon \left[\left(\mathcal{B}_1 f_1^+, \, f_0^+ \right) + \left(f_{1_1}^+, \, f_0^+ \right) \right] \\
& + \epsilon^2 \left[\left(\mathcal{B}_2 f_1^+, \, f_0^+ \right) + \left(\mathcal{B}_1 f_1^+, \, f_{0_1}^+ \right) + \left(\mathcal{B}_1 f_0^+, \, f_{1_1}^+ \right) \right. \\
& \left. + \left(f_{1_2}^+, \, f_0^+ \right) + \left(\mathcal{B}_0 f_{1_1}^+, \, f_{0_1}^+ \right) \right] + r(\epsilon^3) .
\end{aligned}
$$

By (4.25), (4.27), (4.28), (4.30), (4.31), all these scalar products vanish but the first one, and then

$$
F_{11}(0, \epsilon) = \mathtt{f}_{11}\epsilon + r(\epsilon^3) , \quad \mathtt{f}_{11} := A_1 = \tfrac{1}{2} \left(a_1^{[1]} \mathtt{c}_{\mathtt{h}}^{\frac{1}{2}} - p_1^{[1]} \mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}} \right), \quad (4.32)
$$

which, by substituting the expressions of $a_1^{[1]}$, $p_1^{[1]}$ in Lemma 2.2, gives the expression in (4.13).

The expansion (4.23) in proved.

Linear terms in μ. We now compute the terms of $\mathcal{B}_\epsilon(\mu)$ that are linear in μ. It results

$$
\partial_\mu \mathcal{B}_\epsilon(0) = X + X^* \quad \text{where} \quad X := \left(\mathcal{B}_\epsilon f_k^\sigma(0, \epsilon), (\partial_\mu f_{k'}^{\sigma'})(0, \epsilon) \right)_{k,k'=0,1,\sigma,\sigma'=\pm} . \tag{4.33}
$$

We now prove that

$$
X = \begin{pmatrix}
\mathcal{O}(\epsilon^3) & 0 & \mathcal{O}(\epsilon^2) & 0 \\
\mathcal{O}(\epsilon^2) & 0 & \mathcal{O}(\epsilon) & 0 \\
\mathcal{O}(\epsilon^3) & 0 & \mathcal{O}(\epsilon^2) & 0 \\
\mathcal{O}(\epsilon^3) & 0 & \mathcal{O}(\epsilon^2) & 0
\end{pmatrix} . \tag{4.34}
$$

The matrix $\mathtt{L}_\epsilon(0)$ in (4.22) where $E_{22}(0, \epsilon) = 0$, represents the action of the operator $\mathscr{L}_{0,\epsilon} : \mathcal{V}_{0,\epsilon} \to \mathcal{V}_{0,\epsilon}$ in the basis $\{ f_k^\sigma(0, \epsilon) \}$ and then we deduce that $\mathscr{L}_{0,\epsilon} f_1^-(0, \epsilon) = 0$, $\mathscr{L}_{0,\epsilon} f_0^-(0, \epsilon) = 0$. Thus also $\mathcal{B}_\epsilon f_1^-(0, \epsilon) = 0$, $\mathcal{B}_\epsilon f_0^-(0, \epsilon) = 0$, and the second and the fourth column of the matrix X in (4.34) are zero. To compute the other two columns we use the expansion of the derivatives. In view of (4.3)–(4.6) and by denoting with a dot the derivative w.r.t. μ, one has

$$
\begin{aligned}
\dot{f}_1^+(0, \epsilon) &= \frac{\mathrm{i}}{4} \gamma_{\mathtt{h}} \begin{bmatrix} \mathtt{c}_{\mathtt{h}}^{\frac{1}{2}} \sin(x) \\ \mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}} \cos(x) \end{bmatrix} + \mathrm{i}\epsilon \begin{bmatrix} odd(x) \\ even(x) \end{bmatrix} + \mathcal{O}(\epsilon^2) , \\
\dot{f}_0^+(0, \epsilon) &= \mathrm{i}\epsilon \begin{bmatrix} odd(x) \\ even_0(x) \end{bmatrix} + \mathcal{O}(\epsilon^2) , \\
\dot{f}_1^-(0, \epsilon) &= \frac{\mathrm{i}}{4} \gamma_{\mathtt{h}} \begin{bmatrix} \mathtt{c}_{\mathtt{h}}^{\frac{1}{2}} \cos(x) \\ -\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}} \sin(x) \end{bmatrix} + \mathrm{i}\epsilon \begin{bmatrix} even(x) \\ odd(x) \end{bmatrix} + \mathcal{O}(\epsilon^2) , \\
\dot{f}_0^-(0, \epsilon) &= \mathrm{i}\epsilon \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix} + \mathcal{O}(\epsilon^2) .
\end{aligned} \tag{4.35}
$$

In view of (2.2), (4.3)–(4.6), (4.22), (4.8), (4.29),(4.32), and since $\mathcal{B}_\epsilon f_k^\sigma(0,\epsilon) = -\mathcal{J}\mathcal{L}_\epsilon f_k^\sigma(0,\epsilon)$, we have

$$
\begin{aligned}
\mathcal{B}_\epsilon f_1^+(0,\epsilon) &= E_{11}(0,\epsilon)\,\mathcal{J}f_1^-(0,\epsilon) + F_{11}(0,\epsilon)\,\mathcal{J}f_0^- \\
&= \epsilon\begin{bmatrix} \mathsf{f}_{11} \\ 0 \end{bmatrix} + \epsilon^2 \mathsf{e}_{11}\begin{bmatrix} \mathsf{c}_{\mathsf{h}}^{-\frac{1}{2}}\cos(x) \\ \mathsf{c}_{\mathsf{h}}^{\frac{1}{2}}\sin(x) \end{bmatrix} + \mathcal{O}(\epsilon^3), \\
\mathcal{B}_\epsilon f_0^+(0,\epsilon) &= F_{11}(0,\epsilon)\,\mathcal{J}f_1^-(0,\epsilon) + G_{11}(0,\epsilon)\,\mathcal{J}f_0^- \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \epsilon\,\mathsf{f}_{11}\begin{bmatrix} \mathsf{c}_{\mathsf{h}}^{-\frac{1}{2}}\cos(2x) \\ \mathsf{c}_{\mathsf{h}}^{\frac{1}{2}}\sin(2x) \end{bmatrix} + \mathcal{O}(\epsilon^2).
\end{aligned}
\tag{4.36}
$$

We deduce (4.34) by (4.35) and (4.36).

Quadratic terms in μ. By denoting with a double dot the double derivative w.r.t. μ, we have

$$
\begin{aligned}
\partial_\mu^2 \mathsf{B}_0(0) &= \left(\mathcal{B}_0 f_k^\sigma\,,\,\ddot{f}_{k'}^{\sigma'}(0,0)\right) + \left(\ddot{f}_k^\sigma(0,0)\,,\,\mathcal{B}_0 f_k^{\sigma'}\right) \\
&\quad + 2\left(\mathcal{B}_0 \dot{f}_k^\sigma(0,0)\,,\,\dot{f}_{k'}^{\sigma'}(0,0)\right) =: Y + Y^* + 2Z.
\end{aligned}
\tag{4.37}
$$

We claim that $Y = 0$. Indeed, its first, second and fourth column are zero, since $\mathcal{B}_0 f_k^\sigma = 0$ for $f_k^\sigma \in \{f_1^+, f_1^-, f_0^-\}$. The third column is also zero by noting that $\mathcal{B}_0 f_0^+ = f_0^+$ and

$$
\ddot{f}_1^+(0,0) = \begin{bmatrix} even_0(x) + \mathrm{i}\,odd(x) \\ odd(x) + \mathrm{i}\,even_0(x) \end{bmatrix}, \quad \ddot{f}_1^-(0,0) = \begin{bmatrix} odd(x) + \mathrm{i}\,even_0(x) \\ even_0(x) + \mathrm{i}\,odd(x) \end{bmatrix},
$$

$$
\ddot{f}_0^+(0,0) = \ddot{f}_0^-(0,0) = 0.
$$

We claim that

$$
Z = \left(\mathcal{B}_0 \dot{f}_k^\sigma(0,0)\,,\,\dot{f}_{k'}^{\sigma'}(0,0)\right)_{\substack{k,k'=0,1,\\ \sigma,\sigma'=\pm}} = \begin{pmatrix} \zeta_{\mathsf{h}} & 0 & 0 & 0 \\ 0 & \zeta_{\mathsf{h}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\tag{4.38}
$$

with ζ_{h} as in (4.18). Indeed, by (4.35), we have $\dot{f}_0^+(0,0) = \dot{f}_0^-(0,0) = 0$. Therefore the last two columns of Z, and by self-adjointness the last two rows, are zero. By (4.24), (4.35) we obtain the matrix (4.38) with

$$
\zeta_{\mathsf{h}} := \left(\mathcal{B}_0 \dot{f}_1^+(0,0)\,,\,\dot{f}_1^+(0,0)\right) = \left(\mathcal{B}_0 \dot{f}_1^-(0,0)\,,\,\dot{f}_1^-(0,0)\right) = \tfrac{1}{8}\mathsf{c}_{\mathsf{h}}\gamma_{\mathsf{h}}^2.
$$

In conclusion (4.19), (4.33), (4.34), (4.37), the fact that $Y = 0$ and (4.38) imply (4.17), using also the selfadjointness of B_ϵ and (3.12). $\quad\square$

We now consider $\mathcal{B}^\flat$.

Lemma 4.5. (Expansion of $\mathrm{B}^\flat$) *The self-adjoint and reversibility-preserving matrix* $\mathrm{B}^\flat$ *associated, as in* (3.11)*, to the self-adjoint and reversibility-preserving operator* $\mathcal{B}^\flat$*, defined in* (4.15)*, with respect to the basis* $\mathcal{F}$ *of* $\mathcal{V}_{\mu,\epsilon}$ *in* (4.1)*, admits the expansion*

$$
\mathrm{B}^\flat = \left(
\begin{array}{cc|cc}
-\frac{\mu^2}{4}\boldsymbol{b}_\mathrm{h} & \mathrm{i}\left(\frac{\mu}{2}\mathrm{e}_{12} + r_2(\mu\epsilon^2)\right) & 0 & 0 \\
-\mathrm{i}\left(\frac{\mu}{2}\mathrm{e}_{12} + r_2(\mu\epsilon^2)\right) & -\frac{\mu^2}{4}\boldsymbol{b}_\mathrm{h} & \mathrm{i}\,r_6(\mu\epsilon) & 0 \\
\hline
0 & -\mathrm{i}\,r_6(\mu\epsilon) & 0 & 0 \\
0 & 0 & 0 & \mu\tanh(\mathrm{h}\mu)
\end{array}
\right)
$$
$$+\,\mathcal{O}(\mu^2\epsilon, \mu^3),\tag{4.39}$$

where e_{12} *is defined in* (1.2) *and*

$$\boldsymbol{b}_\mathrm{h} := \gamma_\mathrm{h}\mathrm{c}_\mathrm{h} + \mathrm{c}_\mathrm{h}^{-1}\mathrm{h}(1 - \mathrm{c}_\mathrm{h}^4)(\gamma_\mathrm{h} - 2(1 - \mathrm{c}_\mathrm{h}^2\mathrm{h})) \ .\tag{4.40}$$

Proof. We have to compute the expansion of the matrix entries $(\mathcal{B}^\flat f_k^\sigma(\mu, \epsilon), f_{k'}^{\sigma'}(\mu, \epsilon))$. First, by (4.6), (4.15) and since $\mathrm{f}_\epsilon = O(\epsilon^2)$ (cfr. (2.15)) we have

$$\mathcal{B}^\flat f_0^-(\mu, \epsilon) = \begin{bmatrix} 0 \\ \mu\tanh\left(\mathrm{h}\mu\right) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{O}(\mu^2\epsilon) \end{bmatrix}.$$

Hence, by (4.3)–(4.6), the entries of the last column (and row) of $\mathrm{B}^\flat$ are

$$
\begin{aligned}
&\left(\mathcal{B}^\flat f_0^-(\mu, \epsilon), f_1^+(\mu, \epsilon)\right) = \mathcal{O}(\mu^2\epsilon)\,, \\
&\left(\mathcal{B}^\flat f_0^-(\mu, \epsilon), f_1^-(\mu, \epsilon)\right) = \mu\tanh(\mathrm{h}\mu)\mathcal{O}(\epsilon^2) + \mathcal{O}(\mu^2\epsilon^2) = \mathcal{O}(\mu^2\epsilon^2) \\
&\left(\mathcal{B}^\flat f_0^-(\mu, \epsilon), f_0^+(\mu, \epsilon)\right) = \mathcal{O}(\mu^2\epsilon, \mu^3)\,, \\
&\left(\mathcal{B}^\flat f_0^-(\mu, \epsilon), f_0^-(\mu, \epsilon)\right) = \mu\tanh(\mathrm{h}\mu) + \mathcal{O}(\mu^2\epsilon)\,,
\end{aligned}
$$

in agreement with (4.39).

In order to compute the other matrix entries we expand $\mathcal{B}^\flat$ in (4.15) at $\mu = 0$, obtaining

$$\mathcal{B}^\flat = \mu\mathcal{B}_1^\flat(0) + \mu\mathcal{R}^\flat(\epsilon) + \mu^2\mathcal{B}_2^\flat + \mathcal{O}(\mu^2\epsilon, \mu^3)\,, \quad \text{where}$$
$$\mathcal{B}_1^\flat(0) := \left[\mathrm{h}D\left(1 - \tanh^2(\mathrm{h}|D|)\right) + \mathrm{sgn}(D)\tanh(\mathrm{h}|D|)\right]\Pi_{\mathbb{I}}\,, \quad \Pi_{\mathbb{I}} := \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{bmatrix},\tag{4.41}$$
$$\mathcal{R}^\flat(\epsilon) := \mathcal{O}(\epsilon^2)\Pi_{\mathbb{I}}\,, \qquad \mathcal{B}_2^\flat := \left[\mathrm{h}\left(1 - \tanh^2(\mathrm{h}|D|)\right)\left(1 - \mathrm{h}\tanh(\mathrm{h}|D|)|D|\right)\right]\Pi_{\mathbb{I}}\,.$$

We note that

$$
\begin{aligned}
\mu\left(\mathcal{R}^\flat(\epsilon)f_k^\sigma(\mu, \epsilon), f_{k'}^{\sigma'}(\mu, \epsilon)\right) &= \mu\left(\mathcal{R}^\flat f_k^\sigma(0, \epsilon), f_{k'}^{\sigma'}(0, \epsilon)\right) + \mathcal{O}(\mu^2\epsilon^2) \\
&= \begin{cases} \mathcal{O}(\mu^2\epsilon^2) & \text{if } \sigma = \sigma'\,, \\ \mathcal{O}(\mu\epsilon^2) & \text{if } \sigma \neq \sigma'\,. \end{cases}
\end{aligned}\tag{4.42}
$$

Indeed, if $\sigma = \sigma'$, $\left(\mathcal{R}^\flat f_k^\sigma(0, \epsilon), f_{k'}^{\sigma'}(0, \epsilon)\right)$ is real by (3.12), but purely imag-
inary[3] too, since the operator $\mathcal{R}^\flat$ is purely imaginary (as $\mathcal{B}^\flat$ is) and the basis
$\{f_k^{\pm}(0, \epsilon)\}_{k=0,1}$ is real. The terms (4.42) contribute to $r_2(\mu\epsilon^2)$ and $r_6(\epsilon\mu)$ in (4.39).

Next we compute the other scalar products. By (4.3), (4.41), and the identities
$\mathrm{sgn}(D)\sin(kx) = -\mathrm{i}\cos(kx)$ and $\mathrm{sgn}(D)\cos(kx) = \mathrm{i}\sin(kx)$ for any $k \in \mathbb{N}$, we
have

$$\mu\mathcal{B}_1^\flat(0)f_1^+(\mu, \epsilon) = -\mathrm{i}\,\mu\flat_1\begin{bmatrix} 0 \\ \cos(x) \end{bmatrix} - \frac{\mu^2}{4}\gamma_\mathrm{h}\flat_1\begin{bmatrix} 0 \\ \sin(x) \end{bmatrix}$$
$$-\mathrm{i}\,\mu\epsilon\flat_2\begin{bmatrix} 0 \\ \cos(2x) \end{bmatrix} + \mathrm{i}\,\mathcal{O}(\mu\epsilon^2)\begin{bmatrix} 0 \\ even_0(x) \end{bmatrix} + \mathcal{O}(\mu^2\epsilon, \mu^3)$$

where

$$\begin{aligned}
\flat_1 &:= c_\mathrm{h}^{-\frac{1}{2}}(c_\mathrm{h}^2 + (1 - c_\mathrm{h}^4)h) \\
\flat_2 &:= \beta_\mathrm{h}\Big(\tanh(2h) + 2h(1 - \tanh^2(2h))\Big) \\
&= \beta_\mathrm{h}\Big(\frac{2c_\mathrm{h}^2}{1 + c_\mathrm{h}^4} + 2h\big(1 - \frac{4c_\mathrm{h}^4}{(1 + c_\mathrm{h}^4)^2}\big)\Big).
\end{aligned} \qquad (4.43)$$

Similarly $\mu^2\mathcal{B}_2^\flat f_1^+(\mu, \epsilon) = \mu^2\flat_3\begin{bmatrix} 0 \\ \sin(x) \end{bmatrix} + \mathcal{O}(\mu^2\epsilon, \mu^3)$, where

$$\flat_3 := h\big(1 - \tanh^2(h)\big)\big(1 - \tanh(h)h\big)c_\mathrm{h}^{-\frac{1}{2}} = h(1 - c_\mathrm{h}^4)(1 - c_\mathrm{h}^2 h)c_\mathrm{h}^{-\frac{1}{2}}. \quad (4.44)$$

Analogously, using (4.4),

$$\mu\mathcal{B}_1^\flat(0)f_1^-(\mu, \epsilon) = \mathrm{i}\,\mu\flat_1\begin{bmatrix} 0 \\ \sin(x) \end{bmatrix} - \frac{\mu^2}{4}\gamma_\mathrm{h}\flat_1\begin{bmatrix} 0 \\ \cos(x) \end{bmatrix}$$
$$+\mathrm{i}\,\mu\epsilon\flat_3\begin{bmatrix} 0 \\ \sin(2x) \end{bmatrix} + \mathrm{i}\,\mathcal{O}(\mu\epsilon^2)\begin{bmatrix} 0 \\ odd(x) \end{bmatrix} + \mathcal{O}(\mu^2\epsilon, \mu^3),$$

and $\mu^2\mathcal{B}_2^\flat f_1^-(\mu, \epsilon) = \mu^2\flat_3\begin{bmatrix} 0 \\ \cos(x) \end{bmatrix} + \mathcal{O}(\mu^2\epsilon, \mu^3)$, with $\flat_j$, $j = 1, 2, 3$, defined
in (4.43) and (4.44). In addition, by (4.5)–(4.6), we get that

$$\mu\mathcal{B}_1^\flat(0)f_0^+(\mu, \epsilon) = \mathrm{i}\,\mu\epsilon\delta_\mathrm{h}\flat_1\begin{bmatrix} 0 \\ \cos(x) \end{bmatrix} + \mathrm{i}\,\mathcal{O}(\mu\epsilon^2)\begin{bmatrix} 0 \\ even_0(x) \end{bmatrix} + \mathcal{O}(\mu^2\epsilon),$$
$$\mu^2\mathcal{B}_2^\flat f_0^+(\mu, \epsilon) = \begin{bmatrix} 0 \\ \mathcal{O}(\mu^2\epsilon) \end{bmatrix}$$

---

[3] An operator $\mathcal{A}$ is *purely imaginary* if $\bar{\mathcal{A}} = -\mathcal{A}$. A purely imaginary operator sends real
functions into purely imaginary ones.

with b_1 in (4.43). By taking the scalar products of the above expansions of $\mathcal{B}^\flat f_k^\sigma(\mu, \epsilon)$ with the functions $f_{k'}^{\sigma'}(\mu, \epsilon)$ expanded as in (4.3)-(4.6) we obtain that (recall that the scalar product is conjugate-linear in the second component)

$$\left(\mu\mathcal{B}_1^\flat(0)f_1^+(\mu, \epsilon), f_1^+(\mu, \epsilon)\right), \ \left(\mu\mathcal{B}_1^\flat(0)f_1^-(\mu, \epsilon), f_1^-(\mu, \epsilon)\right)$$
$$= -\frac{\mu^2}{4}\gamma_h b_1 c_h^{-\frac{1}{2}} + \mathcal{O}(\mu^2\epsilon, \mu^3)$$
$$\left(\mu^2\mathcal{B}_2^\flat f_1^+(\mu, \epsilon), f_1^+(\mu, \epsilon)\right), \ \left(\mu^2\mathcal{B}_2^\flat f_1^-(\mu, \epsilon), f_1^-(\mu, \epsilon)\right)$$
$$= \frac{\mu^2}{2}b_3 c_h^{-\frac{1}{2}} + \mathcal{O}(\mu^2\epsilon, \mu^3)$$

and, recalling (4.41), (4.43), (4.44), we deduce the expansion of the entries $(1, 1)$ and $(2, 2)$ of the matrix $\mathrm{B}^\flat$ in (4.39) with $\mathbf{b}_h = c_h^{-\frac{1}{2}}(\gamma_h b_1 - 2b_3)$ in (4.40). Moreover

$$\left(\mu\mathcal{B}_1^\flat(0)f_1^-(\mu, \epsilon), f_1^+(\mu, \epsilon)\right) = i\frac{\mu}{2}e_{12} + \mathcal{O}(\mu\epsilon^2, \mu^2\epsilon, \mu^3),$$
$$\left(\mu^2\mathcal{B}_2^\flat f_1^-(\mu, \epsilon), f_1^+(\mu, \epsilon)\right) = \mathcal{O}(\mu^3, \mu^2\epsilon),$$

where $e_{12} := b_1 c_h^{-\frac{1}{2}}$ is equal to (1.2). Finally we obtain

$$\left(\mu(\mathcal{B}_1^\flat(0) + \mu\mathcal{B}_2^\flat)f_1^-(\mu, \epsilon), f_0^+(\mu, \epsilon)\right) = \mathcal{O}(\mu\epsilon, \mu^3)$$
$$\left(\mu(\mathcal{B}_1^\flat(0) + \mu\mathcal{B}_2^\flat)f_1^+(\mu, \epsilon), f_0^+(\mu, \epsilon)\right) = \mathcal{O}(\mu^3, \mu^2\epsilon),$$
$$\left(\mu(\mathcal{B}_1^\flat(0) + \mu\mathcal{B}_2^\flat)f_0^+(\mu, \epsilon), f_0^+(\mu, \epsilon)\right) = \mathcal{O}(\mu^2\epsilon^2).$$

The expansion (4.39) is proved. $\quad\square$

Finally, we consider $\mathcal{B}^\sharp$.

Lemma 4.6. (Expansion of $\mathrm{B}^\sharp$) *The self-adjoint and reversibility-preserving matrix* $\mathrm{B}^\sharp$ *associated, as in* (3.11)*, to the self-adjoint and reversibility-preserving operators* $\mathcal{B}^\sharp$*, defined in* (4.16)*, with respect to the basis* $\mathcal{F}$ *of* $\mathcal{V}_{\mu,\epsilon}$ *in* (4.1)*, admits the expansion*

$$\mathrm{B}^\sharp = \left(\begin{array}{cc|cc} 0 & i\,r_2(\mu\epsilon^2) & 0 & i\,\mu\epsilon c_h^{-\frac{1}{2}} + i\,r_4(\mu\epsilon^2) \\ -i\,r_2(\mu\epsilon^2) & 0 & -i\,r_6(\mu\epsilon) & 0 \\ \hline 0 & i\,r_6(\mu\epsilon) & 0 & -i\,r_9(\mu\epsilon^2) \\ -i\,\mu\epsilon c_h^{-\frac{1}{2}} - i\,r_4(\mu\epsilon^2) & 0 & i\,r_9(\mu\epsilon^2) & 0 \end{array}\right)$$
$$+ \mathcal{O}(\mu^2\epsilon). \tag{4.45}$$

Proof. Since $\mathcal{B}^\sharp = -i\,\mu p_\epsilon \mathcal{J}$ and $p_\epsilon = \mathcal{O}(\epsilon)$ by (2.19), we have the expansion

$$\left(\mathcal{B}^\sharp f_k^\sigma(\mu, \epsilon), f_{k'}^{\sigma'}(\mu, \epsilon)\right) = \left(\mathcal{B}^\sharp f_k^\sigma(0, \epsilon), f_{k'}^{\sigma'}(0, \epsilon)\right) + \mathcal{O}(\mu^2\epsilon). \tag{4.46}$$

The matrix entries $(\mathcal{B}^\sharp f_k^\sigma(0, \epsilon), f_{k'}^\sigma(0, \epsilon)), k, k' = 0, 1, \sigma = \{\pm\}$ are zero, because they are simultaneously real by (3.12), and purely imaginary, being the operator $\mathcal{B}^\sharp$ purely imaginary and the basis $\{f_k^\pm(0, \epsilon)\}_{k=0,1}$ real. Hence $\mathcal{B}^\sharp$ has the form

$$
\mathcal{B}^\sharp = \begin{pmatrix} 0 & i\beta & 0 & i\delta \\ -i\beta & 0 & -i\gamma & 0 \\ 0 & i\gamma & 0 & i\eta \\ -i\delta & 0 & -i\eta & 0 \end{pmatrix} + \mathcal{O}(\mu^2\epsilon) \quad \text{where} \quad \begin{cases} \left(\mathcal{B}^\sharp f_1^-(0, \epsilon), f_1^+(0, \epsilon)\right) =: i\beta, \\ \left(\mathcal{B}^\sharp f_1^-(0, \epsilon), f_0^+(0, \epsilon)\right) =: i\gamma, \\ \left(\mathcal{B}^\sharp f_0^-(0, \epsilon), f_1^+(0, \epsilon)\right) =: i\delta, \\ \left(\mathcal{B}^\sharp f_0^-(0, \epsilon), f_0^+(0, \epsilon)\right) =: i\eta, \end{cases}
$$
(4.47)

and $\alpha, \beta, \gamma, \delta$ are real numbers. As $\mathcal{B}^\sharp = \mathcal{O}(\mu\epsilon)$ in $\mathcal{L}(Y)$, we deduce that $\gamma = r(\mu\epsilon)$. Let us compute the expansion of β, δ and η. By (2.20) and (2.2) we write the operator $\mathcal{B}^\sharp$ in (4.16) as

$$
\mathcal{B}^\sharp = i\mu\epsilon\mathcal{B}_1^\sharp + \mathcal{O}(\mu\epsilon^2), \quad \mathcal{B}_1^\sharp := 2c_h^{-1}\cos(x)\begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix},
$$
(4.48)

with $\mathcal{O}(\mu\epsilon^2) \in \mathcal{L}(Y)$. In view of (4.3)–(4.6), $f_1^\pm(0, \epsilon) = f_1^\pm + \mathcal{O}(\epsilon)$, $f_0^+(0, \epsilon) = f_0^+ + \mathcal{O}(\epsilon)$, $f_0^-(0, \epsilon) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where f_k^σ are in (4.2). By (4.48) we have $\mathcal{B}_1^\sharp f_1^- = \begin{bmatrix} c_h^{-\frac{3}{2}}(1 + \cos(2x)) \\ c_h^{-\frac{1}{2}}\sin(2x) \end{bmatrix}$, $\mathcal{B}_1^\sharp f_0^- = \begin{bmatrix} 2c_h^{-1}\cos(x) \\ 0 \end{bmatrix}$ and then

$$
\beta = \mu\epsilon\left(\mathcal{B}_1^\sharp f_1^-, f_1^+\right) + r(\mu\epsilon^2) = r(\mu\epsilon^2),
$$

$$
\delta = \mu\epsilon\left(\mathcal{B}_1^\sharp f_0^-, f_1^+\right) + r(\mu\epsilon^2) = \mu\epsilon c_h^{-\frac{1}{2}} + r(\mu\epsilon^2),
$$

$$
\eta = \mu\epsilon\left(\mathcal{B}_1^\sharp f_0^-, f_0^+\right) + r(\mu\epsilon^2) = r(\mu\epsilon^2).
$$

This proves (4.45). $\quad\square$

Lemmata 4.4, 4.5, 4.6 imply (4.9) where the matrix E has the form (4.10) and

$$
e_{22} := 2(\mathbf{b}_h - 4\zeta_h) = 2\gamma_h c_h + 2c_h^{-1}h(1 - c_h^4)(\gamma_h - 2(1 - c_h^2h)) - c_h\gamma_h^2,
$$

with $\mathbf{b}_h$ in (4.40) and ζ_h in (4.18). The term e_{22} has the expansion in (1.3). Moreover

$$
G := G(\mu, \epsilon) = \begin{pmatrix} 1 + r_8(\epsilon^2, \mu^2\epsilon, \mu^3) & -i r_9(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \\ i r_9(\mu\epsilon^2, \mu^2\epsilon, \mu^3) & \mu\tanh(h\mu) + r_{10}(\mu^2\epsilon, \mu^3) \end{pmatrix}
$$
(4.49)

$$
F := F(\mu, \epsilon) = \begin{pmatrix} f_{11}\epsilon + r_3(\epsilon^3, \mu\epsilon^2, \mu^2\epsilon, \mu^3) & i\mu\epsilon c_h^{-\frac{1}{2}} + i r_4(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \\ i r_6(\mu\epsilon, \mu^3) & r_7(\mu^2\epsilon, \mu^3) \end{pmatrix}.
$$
(4.50)

In order to deduce the expansion (4.11)–(4.12) of the matrices F, G we exploit further information for

$$
\mathscr{L}_{\mu,0} := \mathcal{J}\mathcal{B}_{\mu,0}, \quad \mathcal{B}_{\mu,0} := \begin{bmatrix} 1 & -c_h\partial_x \\ c_h\partial_x & |D + \mu|\tanh\left(h|D + \mu|\right) \end{bmatrix}.
$$
(4.51)

We have

Lemma 4.7. *At* $\epsilon = 0$ *the matrices are* $F(\mu, 0) = 0$ *and* $G(\mu, 0) = \begin{pmatrix} 1 & 0 \\ 0 & \mu \tanh(h\mu) \end{pmatrix}$.

Proof. By Lemma A.5 and (4.51) we have $\mathcal{B}_{\mu,0} f_0^+(\mu, 0) = f_0^+$ and $\mathcal{B}_{\mu,0} f_0^-(\mu, 0) = \mu \tanh(h\mu) f_0^-$, for any μ. Then the lemma follows recalling (3.11) and the fact that $f_1^+(\mu, 0)$ and $f_1^-(\mu, 0)$ have zero space average by Lemma A.5. □

In view of Lemma 4.7 we deduce that the matrices (4.49) and (4.50) have the form (4.11) and (4.12). This completes the proof of Proposition 4.3.

We now show that the constant e_{22} in (1.3) is positive for any depth $h > 0$.

Lemma 4.8. *For any* $h > 0$ *the term* e_{22} *in* (1.3) *is positive,* $e_{22} \to 0$ *as* $h \to 0^+$ *and* $e_{22} \to 1$ *as* $h \to +\infty$. *As a consequence for any* $h_0 > 0$ *the term* e_{22} *is bounded from below uniformly in* $h > h_0$.

Proof. The quantity $z := c_h^2 = \tanh(h)$ is in $(0, 1)$ for any $h > 0$. Then the quadratic polynomial $(0, +\infty) \ni h \mapsto (1 - z^2)(1 + 3z^2)h^2 + 2z(z^2 - 1)h + z^2$ is positive because its discriminant $-4z^4(1 - z^2)$ is negative as $0 < z^2 < 1$. The limits for $h \to 0^+$ and $h \to +\infty$ follow by inspection. □

5. Block-Decoupling and Emergence of the Whitham–Benjamin Function

In this section we block-decouple the 4×4 Hamiltonian matrix $L_{\mu,\epsilon} = J_4 B_{\mu,\epsilon}$ obtained in Proposition 4.3.

We first perform a singular symplectic and reversibility-preserving change of coordinates.

Lemma 5.1. (Singular symplectic rescaling) *The conjugation of the Hamiltonian and reversible matrix* $L_{\mu,\epsilon} = J_4 B_{\mu,\epsilon}$ *obtained in Proposition 4.3 through the symplectic and reversibility-preserving* 4×4-*matrix*

$$Y := \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \quad \text{with} \quad Q := \begin{pmatrix} \mu^{\frac{1}{2}} & 0 \\ 0 & \mu^{-\frac{1}{2}} \end{pmatrix}, \quad \mu > 0, \tag{5.1}$$

yields the Hamiltonian and reversible matrix

$$L_{\mu,\epsilon}^{(1)} := Y^{-1} L_{\mu,\epsilon} Y = J_4 B_{\mu,\epsilon}^{(1)} = \begin{pmatrix} J_2 E^{(1)} & J_2 F^{(1)} \\ J_2 [F^{(1)}]^* & J_2 G^{(1)} \end{pmatrix} \tag{5.2}$$

where $B_{\mu,\epsilon}^{(1)}$ *is a self-adjoint and reversibility-preserving* 4×4 *matrix*

$$B_{\mu,\epsilon}^{(1)} = \begin{pmatrix} E^{(1)} & F^{(1)} \\ [F^{(1)}]^* & G^{(1)} \end{pmatrix}, \quad E^{(1)} = [E^{(1)}]^*, \quad G^{(1)} = [G^{(1)}]^*, \tag{5.3}$$

where the 2×2 reversibility-preserving matrices $E^{(1)}$, $G^{(1)}$ and $F^{(1)}$ extend analytically at $\mu = 0$ with the following expansion

$$E^{(1)} = \begin{pmatrix} e_{11}\mu\epsilon^2(1 + r'_1(\epsilon, \mu\epsilon)) - e_{22}\frac{\mu^3}{8}(1 + r''_1(\epsilon, \mu)) & i\left(\frac{1}{2}e_{12}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) \\ -i\left(\frac{1}{2}e_{12}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) & -e_{22}\frac{\mu}{8}(1 + r_5(\epsilon, \mu)) \end{pmatrix},$$

(5.4)

$$G^{(1)} = \begin{pmatrix} \mu + r_8(\mu\epsilon^2, \mu^3\epsilon) & -ir_9(\mu\epsilon^2, \mu^2\epsilon) \\ ir_9(\mu\epsilon^2, \mu^2\epsilon) & \tanh(h\mu) + r_{10}(\mu\epsilon) \end{pmatrix},$$

(5.5)

$$F^{(1)} = \begin{pmatrix} f_{11}\mu\epsilon + r_3(\mu\epsilon^3, \mu^2\epsilon^2, \mu^3\epsilon) & i\mu\epsilon c_h^{-\frac{1}{2}} + ir_4(\mu\epsilon^2, \mu^2\epsilon) \\ ir_6(\mu\epsilon) & r_7(\mu\epsilon) \end{pmatrix}$$

(5.6)

where e_{11}, e_{12}, e_{22}, f_{11} *are defined in* (4.13), (1.2), (1.3).

Remark 5.2. The matrix $L^{(1)}_{\mu,\epsilon}$, a priori defined only for $\mu \neq 0$, extends analytically to the zero matrix at $\mu = 0$. For $\mu \neq 0$ the spectrum of $L^{(1)}_{\mu,\epsilon}$ coincides with the spectrum of $L_{\mu,\epsilon}$.

Proof. The matrix Y is symplectic, i.e. (3.13) holds, and since μ is real, it is reversibility preserving, i.e. satisfies (3.12). By (3.14),

$$B^{(1)}_{\mu,\epsilon} = Y^* B_{\mu,\epsilon} Y = \begin{pmatrix} E^{(1)} & F^{(1)} \\ [F^{(1)}]^* & G^{(1)} \end{pmatrix},$$

with, Q being self-adjoint, $E^{(1)} = QEQ = [E^{(1)}]^*$, $G^{(1)} = QGQ = [G^{(1)}]^*$ and $F^{(1)} = QFQ$. In view of (4.10)–(4.12), we obtain (5.4)–(5.6). □

5.1. Non-perturbative Step of Block-Decoupling

We first verify that the quantity $D_h := h - \frac{1}{4}e_{12}^2$ is nonzero for any $h > 0$. In view of the comment 3 after Theorem 1.1, we have that $D_h = h - c_g^2$. The non-degeneracy property $D_h \neq 0$ corresponds to that in Bridges-Mielke [9, p.183] and [38, p.409].

Lemma 5.3. *For any* $h > 0$ *it results*

$$D_h := h - \frac{1}{4}e_{12}^2 > 0, \quad and \quad \lim_{h \to 0^+} D_h = 0.$$

(5.7)

Proof. We write $D_h = (\sqrt{h} + \frac{1}{2}e_{12})(\sqrt{h} - \frac{1}{2}e_{12})$ whose first factor is positive for $h > 0$. We claim that also the second factor is positive. In view of (1.2) it is equal to $\frac{1}{2}c_h^{-1} f(h)$ with

$$f(h) := \left(\sqrt{h}\tanh(h) - \sqrt{h} + \sqrt{\tanh(h)}\right)\left(\sqrt{h}\tanh(h) + \sqrt{h} - \sqrt{\tanh(h)}\right)$$
$$=: q(h)p(h).$$

The function $p(h)$ is positive since $h > \tanh(h)$ for any $h > 0$. We claim that also the function $q(h)$ is positive. Indeed its derivative

$$q'(h) = \frac{1 - \tanh(h)}{2\sqrt{h}\sqrt{\tanh(h)}}\left(-\sqrt{\tanh(h)} + \sqrt{h} + \sqrt{h}\tanh(h)\right) + \sqrt{h}\left(1 - \tanh^2(h)\right) > 0$$

for any $h > 0$. Since $q(0) = 0$ we deduce that $q(h) > 0$ for any $h > 0$. This proves the lemma. □

We now state the main result of this section.

Lemma 5.4. (Step of block-decoupling) *There exists a 2×2 reversibility-preserving matrix X, analytic in (μ, ϵ), of the form*

$$X := \begin{pmatrix} x_{11} & i\,x_{12} \\ i\,x_{21} & x_{22} \end{pmatrix} \qquad \text{with } x_{ij} \in \mathbb{R}, \ i, j = 1, 2,$$

$$= \begin{pmatrix} r_{11}(\epsilon) & i\,r_{12}(\epsilon) \\ -i\,\tfrac{1}{2}\mathtt{D}_{\mathtt{h}}^{-1}(\mathtt{e}_{12}\mathtt{f}_{11} + 2\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}})\epsilon + i\,r_{21}(\epsilon^2, \mu\epsilon) & \tfrac{1}{2}\mathtt{D}_{\mathtt{h}}^{-1}(\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}}\mathtt{e}_{12} + 2\mathtt{h}\mathtt{f}_{11})\epsilon + r_{22}(\epsilon^2, \mu\epsilon) \end{pmatrix},$$

(5.8)

where $\mathtt{e}_{12}$, $\mathtt{f}_{11}$ are defined in (1.2), (4.13) and $\mathtt{D}_{\mathtt{h}}$ is the positive constant in (5.7), such that the following holds true. By conjugating the Hamiltonian and reversible matrix $\mathtt{L}_{\mu,\epsilon}^{(1)}$, defined in (5.2), with the symplectic and reversibility-preserving 4×4 matrix

$$\exp\left(S^{(1)}\right), \quad \text{where} \quad S^{(1)} := \mathtt{J}_4 \begin{pmatrix} 0 & \Sigma \\ \Sigma^* & 0 \end{pmatrix}, \qquad \Sigma := \mathtt{J}_2 X, \qquad (5.9)$$

we get the Hamiltonian and reversible matrix

$$\mathtt{L}_{\mu,\epsilon}^{(2)} := \exp\left(S^{(1)}\right) \mathtt{L}_{\mu,\epsilon}^{(1)} \exp\left(-S^{(1)}\right) = \mathtt{J}_4 \mathtt{B}_{\mu,\epsilon}^{(2)} = \begin{pmatrix} \mathtt{J}_2 E^{(2)} & \mathtt{J}_2 F^{(2)} \\ \mathtt{J}_2 [F^{(2)}]^* & \mathtt{J}_2 G^{(2)} \end{pmatrix},$$

(5.10)

where the reversibility-preserving 2×2 self-adjoint matrix $[E^{(2)}]^* = E^{(2)}$ has the form

$$E^{(2)} = \begin{pmatrix} \mu\epsilon^2 \mathtt{e}_{\mathrm{WB}} + r_1'(\mu\epsilon^3, \mu^2\epsilon^2) - \mathtt{e}_{22}\tfrac{\mu^3}{8}(1 + r_1''(\epsilon, \mu)) & i\left(\tfrac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) \\ -i\left(\tfrac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) & -\mathtt{e}_{22}\tfrac{\mu}{8}(1 + r_5(\epsilon, \mu)) \end{pmatrix},$$

(5.11)

where

$$\mathtt{e}_{\mathrm{WB}} = \mathtt{e}_{11} - \mathtt{D}_{\mathtt{h}}^{-1}\left(\mathtt{c}_{\mathtt{h}}^{-1} + \mathtt{h}\mathtt{f}_{11}^2 + \mathtt{e}_{12}\mathtt{f}_{11}\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}}\right) \qquad (5.12)$$

(with constants $\mathtt{e}_{11}$, $\mathtt{D}_{\mathtt{h}}$, $\mathtt{f}_{11}$, $\mathtt{e}_{12}$, defined in (4.13), (5.7), (1.2)), is the Whitham-Benjamin function defined in (1.1), the reversibility-preserving 2×2 self-adjoint matrix $[G^{(2)}]^* = G^{(2)}$ has the form

$$G^{(2)} = \begin{pmatrix} \mu + r_8(\mu\epsilon^2, \mu^3\epsilon) & -i\,r_9(\mu\epsilon^2, \mu^2\epsilon) \\ i\,r_9(\mu\epsilon^2, \mu^2\epsilon) & \tanh(\mathtt{h}\mu) + r_{10}(\mu\epsilon) \end{pmatrix}, \qquad (5.13)$$

and

$$F^{(2)} = \begin{pmatrix} r_3(\mu\epsilon^3) & i\,r_4(\mu\epsilon^3) \\ i\,r_6(\mu\epsilon^3) & r_7(\mu\epsilon^3) \end{pmatrix}. \qquad (5.14)$$

The rest of the section is devoted to the proof of Lemma 5.4. For simplicity let $S = S^{(1)}$.

The matrix $\exp(S)$ is symplectic and reversibility-preserving because the matrix S in (5.9) is Hamiltonian and reversibility-preserving, cfr. Lemma 3.8 in [6]. Note that S is reversibility preserving, since X has the form (5.8).

We now expand in Lie series the Hamiltonian and reversible matrix $L_{\mu,\epsilon}^{(2)} = \exp(S)L_{\mu,\epsilon}^{(1)}\exp(-S)$.

We split $L_{\mu,\epsilon}^{(1)}$ into its 2×2-diagonal and off-diagonal Hamiltonian and reversible matrices

$$L_{\mu,\epsilon}^{(1)} = D^{(1)} + R^{(1)},$$

$$D^{(1)} := \begin{pmatrix} D_1 & 0 \\ 0 & D_0 \end{pmatrix} := \begin{pmatrix} J_2 E^{(1)} & 0 \\ 0 & J_2 G^{(1)} \end{pmatrix}, \quad R^{(1)} := \begin{pmatrix} 0 & J_2 F^{(1)} \\ J_2 [F^{(1)}]^* & 0 \end{pmatrix},$$

$$(5.15)$$

and we perform the Lie expansion

$$
\begin{aligned}
L_{\mu,\epsilon}^{(2)} = \exp(S)L_{\mu,\epsilon}^{(1)}\exp(-S) &= D^{(1)} + \left[S, \, D^{(1)}\right] \\
&+ \frac{1}{2}[S, [S, D^{(1)}]] + R^{(1)} + [S, R^{(1)}] \\
&+ \frac{1}{2}\int_0^1 (1-\tau)^2 \exp(\tau S)\mathrm{ad}_S^3(D^{(1)})\exp(-\tau S)\,\mathrm{d}\tau \\
&+ \int_0^1 (1-\tau)\exp(\tau S)\,\mathrm{ad}_S^2(R^{(1)})\exp(-\tau S)\,\mathrm{d}\tau
\end{aligned}
\tag{5.16}
$$

where $\mathrm{ad}_A(B) := [A, B] := AB - BA$ denotes the commutator between the linear operators A, B.

We look for a 4×4 matrix S as in (5.9) (which is Hamiltonian, reversibility-preserving and off-diagonal as the term $R^{(1)}$ we wish to eliminate) that solves the homological equation $R^{(1)} + \left[S, \, D^{(1)}\right] = 0$, which, recalling (5.15), reads

$$\begin{pmatrix} 0 & J_2 F^{(1)} + J_2 \Sigma D_0 - D_1 J_2 \Sigma \\ J_2 [F^{(1)}]^* + J_2 \Sigma^* D_1 - D_0 J_2 \Sigma^* & 0 \end{pmatrix} = 0. \tag{5.17}$$

Note that the equation $J_2 F^{(1)} + J_2 \Sigma D_0 - D_1 J_2 \Sigma = 0$ implies also $J_2 [F^{(1)}]^* + J_2 \Sigma^* D_1 - D_0 J_2 \Sigma^* = 0$ and viceversa. Thus, writing $\Sigma = J_2 X$, namely $X = -J_2 \Sigma$, the equation (5.17) amounts to solve the "Sylvester" equation

$$D_1 X - X D_0 = -J_2 F^{(1)}. \tag{5.18}$$

We write the matrices $E^{(1)}$, $F^{(1)}$, $G^{(1)}$ in (5.2) as

$$E^{(1)} = \begin{pmatrix} E_{11}^{(1)} & i\,E_{12}^{(1)} \\ -i\,E_{12}^{(1)} & E_{22}^{(1)} \end{pmatrix}, \quad F^{(1)} = \begin{pmatrix} F_{11}^{(1)} & i\,F_{12}^{(1)} \\ i\,F_{21}^{(1)} & F_{22}^{(1)} \end{pmatrix},$$

$$G^{(1)} = \begin{pmatrix} G_{11}^{(1)} & i\,G_{12}^{(1)} \\ -i\,G_{12}^{(1)} & G_{22}^{(1)} \end{pmatrix} \tag{5.19}$$

where the real numbers $E_{ij}^{(1)}$, $F_{ij}^{(1)}$, $G_{ij}^{(1)}$, $i, j = 1, 2$, have the expansion in (5.4), (5.5), (5.6). Thus, by (5.15), (5.8) and (5.19), the equation (5.18) amounts to solve the 4×4 real linear system

$$
\underbrace{\begin{pmatrix}
G_{12}^{(1)} - E_{12}^{(1)} & G_{11}^{(1)} & E_{22}^{(1)} & 0 \\
G_{22}^{(1)} & G_{12}^{(1)} - E_{12}^{(1)} & 0 & -E_{22}^{(1)} \\
E_{11}^{(1)} & 0 & G_{12}^{(1)} - E_{12}^{(1)} & -G_{11}^{(1)} \\
0 & -E_{11}^{(1)} & -G_{22}^{(1)} & G_{12}^{(1)} - E_{12}^{(1)}
\end{pmatrix}}_{=:\mathcal{A}}
\underbrace{\begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix}}_{=:\vec{x}}
=
\underbrace{\begin{pmatrix} -F_{21}^{(1)} \\ F_{22}^{(1)} \\ -F_{11}^{(1)} \\ F_{12}^{(1)} \end{pmatrix}}_{=:\vec{f}}.
$$

(5.20)

We solve this system using the following result, verified by a direct calculus:

Lemma 5.5. *The determinant of the matrix*

$$
A := \begin{pmatrix}
a & b & c & 0 \\
d & a & 0 & -c \\
e & 0 & a & -b \\
0 & -e & -d & a
\end{pmatrix}
$$

(5.21)

where a, b, c, d, e are real numbers, is

$$
\det A = a^4 - 2a^2(bd + ce) + (bd - ce)^2
$$
$$
= (bd - a^2)^2 - 2ce\left(a^2 + bd - \frac{1}{2}ce\right).
$$

(5.22)

If $\det A \neq 0$ then A is invertible and

$$
A^{-1} = \frac{1}{\det A}
\begin{pmatrix}
a\left(a^2 - bd - ce\right) & b\left(-a^2 + bd - ce\right) & -c\left(a^2 + bd - ce\right) & -2abc \\
d\left(-a^2 + bd - ce\right) & a\left(a^2 - bd - ce\right) & 2acd & -c\left(-a^2 - bd + ce\right) \\
-e\left(a^2 + bd - ce\right) & 2abe & a\left(a^2 - bd - ce\right) & b\left(a^2 - bd + ce\right) \\
-2ade & -e\left(-a^2 - bd + ce\right) & d\left(a^2 - bd + ce\right) & a\left(a^2 - bd - ce\right)
\end{pmatrix}.
$$

(5.23)

The Sylvester matrix $\mathcal{A}$ in (5.20) has the form (5.21) where, by (5.4)-(5.6) and since $\tanh(h\mu) = h\mu + r(\mu^3)$,

$$
a = G_{12}^{(1)} - E_{12}^{(1)} = -e_{12}\frac{\mu}{2}\left(1 + r(\epsilon^2, \mu\epsilon, \mu^2)\right), \quad b = G_{11}^{(1)} = \mu + r_8(\mu\epsilon^2, \mu^3\epsilon),
$$
$$
c = E_{22}^{(1)} = -e_{22}\frac{\mu}{8}(1 + r_5(\epsilon, \mu)), \quad d = G_{22}^{(1)} = \mu h + r(\mu\epsilon, \mu^3),
$$
$$
e = E_{11}^{(1)} = r(\mu\epsilon^2, \mu^3),
$$

(5.24)

where e_{12} and e_{22}, defined respectively in (1.2), (1.3), are positive for any $h > 0$.

By (5.22), the determinant of the matrix $\mathcal{A}$ is

$$
\det \mathcal{A} = (bd - a^2)^2 + r(\mu^4\epsilon^2, \mu^6) = \mu^4 D_h^2(1 + r(\epsilon, \mu^2))
$$

(5.25)

where D_h is defined in (5.7). By (5.23), (5.24), (5.25) and, since $D_h = h - \frac{1}{4}e_{12}^2$, we obtain

$$
\mathcal{A}^{-1} = (1 + r(\epsilon, \mu))\frac{1}{\mu D_h^2}
$$

$$\begin{pmatrix} \frac{1}{2}e_{12}D_h & D_h & \frac{1}{32}e_{22}(e_{12}^2+4h) & -\frac{1}{8}e_{12}\,e_{22} \\ hD_h & \frac{1}{2}e_{12}D_h & \frac{1}{8}e_{12}e_{22}h & -\frac{1}{32}e_{22}\,(e_{12}^2+4h) \\ r(\epsilon^2,\mu^2) & r(\epsilon^2,\mu^2) & \frac{1}{2}e_{12}D_h & -D_h \\ r(\epsilon^2,\mu^2) & r(\epsilon^2,\mu^2) & -hD_h & \frac{1}{2}e_{12}D_h \end{pmatrix}.$$

$$(5.26)$$

Therefore, for any $\mu \neq 0$, there exists a unique solution $\vec{x} = A^{-1}\vec{f}$ of the linear system (5.20), namely a unique matrix X which solves the Sylvester equation (5.18).

Lemma 5.6. *The matrix solution X of the Sylvester equation* (5.18) *is analytic in* (μ, ϵ), *and admits an expansion as in* (5.8).

Proof. By (5.20), (5.26), (5.19), (5.6) we obtain, for any $\mu \neq 0$

$$\begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} = \frac{1}{D_h^2} \begin{pmatrix} \frac{1}{2}e_{12}D_h & D_h & \frac{1}{32}e_{22}(e_{12}^2+4h) & -\frac{1}{8}e_{12}\,e_{22} \\ hD_h & \frac{1}{2}e_{12}D_h & \frac{1}{8}e_{12}e_{22}h & -\frac{1}{32}e_{22}\,(e_{12}^2+4h) \\ r(\epsilon^2,\mu^2) & r(\epsilon^2,\mu^2) & \frac{1}{2}e_{12}D_h & -D_h \\ r(\epsilon^2,\mu^2) & r(\epsilon^2,\mu^2) & -hD_h & \frac{1}{2}e_{12}D_h \end{pmatrix}$$

$$\begin{pmatrix} r(\epsilon) \\ r(\epsilon) \\ -f_{11}\epsilon + r(\epsilon^3, \mu\epsilon^2, \mu^2\epsilon) \\ c_h^{-\frac{1}{2}}\epsilon + r(\epsilon^2, \mu\epsilon) \end{pmatrix} (1 + r(\epsilon, \mu)),$$

which proves (5.8). In particular each x_{ij} admits an analytic extension at $\mu = 0$. Note that, for $\mu = 0$, one has $E^{(2)} = G^{(2)} = F^{(2)} = 0$ and the Sylvester equation reduces to tautology. □

Since the matrix S solves the homological equation $[S, D^{(1)}] + R^{(1)} = 0$, identity (5.16) simplifies to

$$L_{\mu,\epsilon}^{(2)} = D^{(1)} + \frac{1}{2}[S, R^{(1)}] + \frac{1}{2}\int_0^1 (1-\tau^2)\exp(\tau S)\operatorname{ad}_S^2(R^{(1)})\exp(-\tau S)d\tau.$$

$$(5.27)$$

The matrix $\frac{1}{2}[S, R^{(1)}]$ is, by (5.9), (5.15), the block-diagonal Hamiltonian and reversible matrix

$$\frac{1}{2}[S, R^{(1)}]$$
$$= \begin{pmatrix} \frac{1}{2}J_2(\Sigma J_2[F^{(1)}]^* - F^{(1)}J_2\Sigma^*) & 0 \\ 0 & \frac{1}{2}J_2(\Sigma^*J_2F^{(1)} - [F^{(1)}]^*J_2\Sigma) \end{pmatrix} \quad (5.28)$$
$$= \begin{pmatrix} J_2\tilde{E} & 0 \\ 0 & J_2\tilde{G} \end{pmatrix},$$

where, since $\Sigma = J_2 X$,

$$\tilde{E} := \mathbf{Sym}(J_2 X J_2[F^{(1)}]^*), \qquad \tilde{G} := \mathbf{Sym}(X^* F^{(1)}), \qquad (5.29)$$

denoting $\mathbf{Sym}(A) := \frac{1}{2}(A + A^*)$.

Lemma 5.7. *The self-adjoint and reversibility-preserving matrices $\tilde{E}$, $\tilde{G}$ in* (5.29) *have the form*

$$\tilde{E} = \begin{pmatrix} \tilde{\mathsf{e}}_{11}\mu\epsilon^2 + \tilde{r}_1(\mu\epsilon^3, \mu^2\epsilon^2) & \mathrm{i}\,\tilde{r}_2(\mu\epsilon^2) \\ -\mathrm{i}\,\tilde{r}_2(\mu\epsilon^2) & \tilde{r}_5(\mu\epsilon^2) \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} \tilde{r}_8(\mu\epsilon^2) & \mathrm{i}\,\tilde{r}_9(\mu\epsilon^2) \\ -\mathrm{i}\,\tilde{r}_9(\mu\epsilon^2) & \tilde{r}_{10}(\mu\epsilon^2) \end{pmatrix},$$

$$\tilde{\mathsf{e}}_{11} := -\mathsf{D}_{\mathsf{h}}^{-1}\big(\mathsf{c}_{\mathsf{h}}^{-1} + \mathsf{h}\mathsf{f}_{11}^2 + \mathsf{e}_{12}\mathsf{f}_{11}\mathsf{c}_{\mathsf{h}}^{-\frac{1}{2}}\big).$$

$$(5.30)$$

Proof. For simplicity we set $F = F^{(1)}$. By (5.8), (5.6), one has

$$\mathsf{J}_2 X \mathsf{J}_2 F^* = \begin{pmatrix} x_{21}F_{12} - x_{22}F_{11} & \mathrm{i}\,(x_{21}F_{22} + x_{22}F_{21}) \\ \mathrm{i}\,(x_{11}F_{12} + x_{12}F_{11}) & -x_{11}F_{22} + x_{12}F_{21} \end{pmatrix}$$
$$= \begin{pmatrix} \tilde{\mathsf{e}}_{11}\mu\epsilon^2 + r(\mu\epsilon^3, \mu^2\epsilon^2) & \mathrm{i}\,r(\mu\epsilon^2) \\ \mathrm{i}\,r(\mu\epsilon^2) & r(\mu\epsilon^2) \end{pmatrix},$$

with $\tilde{\mathsf{e}}_{11}$ being defined as in (5.30). The expansion of $\tilde{E}$ in (5.30) follows in view of (5.29). Since $X = \mathcal{O}(\epsilon)$ by (5.8) and $F = O(\mu\epsilon)$ by (5.6) we deduce that $X^*F = \mathcal{O}(\mu\epsilon^2)$ and the expansion of $\tilde{G}$ in (5.30) follows. $\quad\square$

Note that the term $\tilde{\mathsf{e}}_{11}\mu\epsilon^2$ in the matrix $\tilde{E}$ in (5.29)–(5.30), has the same order of the (1, 1)-entry of $E^{(1)}$ in (5.4), thus will contribute to the Whitham-Benjamin function e_{WB} in the (1, 1)-entry of $E^{(2)}$ in (5.11). Finally we show that the last term in (5.27) is small.

Lemma 5.8. *The 4×4 Hamiltonian and reversibility matrix*

$$\frac{1}{2}\int_0^1 (1 - \tau^2)\,\exp(\tau S)\,\mathrm{ad}_S^2(R^{(1)})\,\exp(-\tau S)\,\mathrm{d}\tau = \begin{pmatrix} \mathsf{J}_2\widehat{E} & \mathsf{J}_2 F^{(2)} \\ \mathsf{J}_2[F^{(2)}]^* & \mathsf{J}_2\widehat{G} \end{pmatrix} \quad (5.31)$$

where the 2×2 self-adjoint and reversible matrices $\widehat{E}$, $\widehat{G}$ have entries

$$\widehat{E}_{ij}\,\widehat{G}_{ij} = r(\mu\epsilon^3), \quad i, j = 1, 2, \qquad (5.32)$$

and the 2×2 reversible matrix $F^{(2)}$ admits an expansion as in (5.14).

Proof. Since S and $R^{(1)}$ are Hamiltonian and reversibility-preserving then $\mathrm{ad}_S R^{(1)} = [S, R^{(1)}]$ is Hamiltonian and reversibility-preserving as well. Thus each $\exp(\tau S)\,\mathrm{ad}_S^2(R^{(1)})\,\exp(-\tau S)$ is Hamiltonian and reversibility-preserving, and formula (5.31) holds. In order to estimate its entries we first compute $\mathrm{ad}_S^2(R^{(1)})$. Using the form of S in (5.9) and $[S, R^{(1)}]$ in (5.28) one gets

$$\mathrm{ad}_S^2(R^{(1)}) = \begin{pmatrix} 0 & \mathsf{J}_2\tilde{F} \\ \mathsf{J}_2\tilde{F}^* & 0 \end{pmatrix} \quad \text{where} \quad \tilde{F} := 2\left(\Sigma\mathsf{J}_2\tilde{G} - \tilde{E}\mathsf{J}_2\Sigma\right) \quad (5.33)$$

and $\tilde{E}$, $\tilde{G}$ are defined in (5.29). Since $\tilde{E}$, $\tilde{G} = \mathcal{O}(\mu\epsilon^2)$ by (5.30), and $\Sigma = \mathsf{J}_2 X = \mathcal{O}(\epsilon)$ by (5.8), we deduce that $\tilde{F} = \mathcal{O}(\mu\epsilon^3)$. Then, for any $\tau \in [0, 1]$, the matrix $\exp(\tau S)\,\mathrm{ad}_S^2(R^{(1)})\,\exp(-\tau S) = \mathrm{ad}_S^2(R^{(1)})(1 + \mathcal{O}(\mu, \epsilon))$. In particular the matrix $F^{(2)}$ in (5.31) has the same expansion of $\tilde{F}$, namely $F^{(2)} = \mathcal{O}(\mu\epsilon^3)$, and the matrices $\widehat{E}$, $\widehat{G}$ have entries as in (5.32). $\quad\square$

Proof of Lemma 5.4. It follows by (5.27)–(5.28), (5.15) and Lemmata 5.7 and 5.8. The matrix $E^{(2)} := E^{(1)} + \tilde{E} + \widehat{E}$ has the expansion in (5.11), with $e_{wB} = e_{11} + \tilde{e}_{11}$ as in (5.12). Similarly $G^{(2)} := G^{(1)} + \tilde{G} + \widehat{G}$ has the expansion in (5.13).   □

5.2. Complete Block-Decoupling and Proof of the Main Results

We now block-diagonalize the 4×4 Hamiltonian and reversible matrix $\mathtt{L}^{(2)}_{\mu,\epsilon}$ in (5.10). First we split it into its 2×2-diagonal and off-diagonal Hamiltonian and reversible matrices

$$\mathtt{L}^{(2)}_{\mu,\epsilon} = D^{(2)} + R^{(2)} \,,$$
$$D^{(2)} := \begin{pmatrix} \mathtt{J}_2 E^{(2)} & 0 \\ 0 & \mathtt{J}_2 G^{(2)} \end{pmatrix}, \quad R^{(2)} := \begin{pmatrix} 0 & \mathtt{J}_2 F^{(2)} \\ \mathtt{J}_2 [F^{(2)}]^* & 0 \end{pmatrix}. \tag{5.34}$$

Lemma 5.9. *There exist a 4×4 reversibility-preserving Hamiltonian matrix $S^{(2)} := S^{(2)}(\mu, \epsilon)$ of the form (5.9), analytic in (μ, ϵ), of size $\mathcal{O}(\epsilon^3)$, and a 4×4 block-diagonal reversible Hamiltonian matrix $P := P(\mu, \epsilon)$, analytic in (μ, ϵ), of size $\mathcal{O}(\mu\epsilon^6)$ such that*

$$\exp(S^{(2)})(D^{(2)} + R^{(2)}) \exp(-S^{(2)}) = D^{(2)} + P \,. \tag{5.35}$$

Proof. We set for brevity $S = S^{(2)}$. The equation (5.35) is equivalent to the system

$$\begin{cases} \Pi_D \big(e^S (D^{(2)} + R^{(2)}) e^{-S} \big) - D^{(2)} = P \\ \Pi_\varnothing \big(e^S (D^{(2)} + R^{(2)}) e^{-S} \big) = 0 \,, \end{cases} \tag{5.36}$$

where Π_D is the projector onto the block-diagonal matrices and $\Pi_\varnothing$ onto the block-off-diagonal ones. The second equation in (5.36) is equivalent, by a Lie expansion, and since $[S, R^{(2)}]$ is block-diagonal, to

$$R^{(2)} + \big[S, D^{(2)} \big] + \underbrace{\Pi_\varnothing \int_0^1 (1 - \tau) e^{\tau S} \mathrm{ad}_S^2 (D^{(2)} + R^{(2)}) e^{-\tau S} \mathrm{d}\tau}_{=: \mathcal{R}(S)} = 0 \,. \tag{5.37}$$

The "nonlinear homological equation" (5.37),

$$[S, D^{(2)}] = -R^{(2)} - \mathcal{R}(S) \,, \tag{5.38}$$

is equivalent to solve the 4×4 real linear system

$$\mathcal{A}\vec{x} = \vec{f}(\mu, \epsilon, \vec{x}), \quad \vec{f}(\mu, \epsilon, \vec{x}) = \mu\vec{v}(\mu, \epsilon) + \mu\vec{g}(\mu, \epsilon, \vec{x}) \tag{5.39}$$

associated, as in (5.20), to (5.38). The vector $\mu\vec{v}(\mu, \epsilon)$ is associated with $-R^{(2)}$ where $R^{(2)}$ is in (5.34). The vector $\mu\vec{g}(\mu, \epsilon, \vec{x})$ is associated with the matrix $-\mathcal{R}(S)$, which is a Hamiltonian and reversible block-off-diagonal matrix (i.e of the form (5.15)). The factor μ is present in $D^{(2)}$ and $R^{(2)}$, see (5.11), (5.13), (5.14) and the

analytic function $\vec{g}(\mu, \epsilon, \vec{x})$ is quadratic in $\vec{x}$ (for the presence of ad_S^2 in $\mathcal{R}(S)$). In view of (5.14) one has

$$\mu\vec{v}(\mu, \epsilon) := (-F_{21}^{(2)}, F_{22}^{(2)}, -F_{11}^{(2)}, F_{12}^{(2)})^\top, \quad F_{ij}^{(2)} = r(\mu\epsilon^3). \qquad (5.40)$$

System (5.39) is equivalent to $\vec{x} = \mathcal{A}^{-1}\vec{f}(\mu, \epsilon, \vec{x})$ and, writing $\mathcal{A}^{-1} = \frac{1}{\mu}\mathcal{B}(\mu, \epsilon)$ (cfr. (5.26)), to

$$\vec{x} = \mathcal{B}(\mu, \epsilon)\vec{v}(\mu, \epsilon) + \mathcal{B}(\mu, \epsilon)\vec{g}(\mu, \epsilon, \vec{x}).$$

By the implicit function theorem this equation admits a unique small solution $\vec{x} = \vec{x}(\mu, \epsilon)$, analytic in (μ, ϵ), with size $\mathcal{O}(\epsilon^3)$ as $\vec{v}$ in (5.40). Then the first equation of (5.36) gives $P = [S, R^{(2)}] + \Pi_D \int_0^1 (1-\tau)e^{\tau S}\mathrm{ad}_S^2(D^{(2)} + R^{(2)})e^{-\tau S}\mathrm{d}\tau$, and its estimate follows from those of S and $R^{(2)}$ (see (5.14)). □

Proof of Theorems 2.5 and 1.1. By Lemma 5.9 and recalling (3.1) the operator $\mathcal{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ is represented by the 4×4 Hamiltonian and reversible matrix

$$\mathrm{i}\, c_\mathrm{h}\mu + \exp(S^{(2)})\mathrm{L}_{\mu,\epsilon}^{(2)}\exp(-S^{(2)}) = \mathrm{i}\, c_\mathrm{h}\mu + \begin{pmatrix} \mathsf{J}_2 E^{(3)} & 0 \\ 0 & \mathsf{J}_2 G^{(3)} \end{pmatrix} =: \begin{pmatrix} \mathsf{U} & 0 \\ 0 & \mathsf{S} \end{pmatrix},$$

where the matrices $E^{(3)}$ and $G^{(3)}$ expand as in (5.11), (5.13). Consequently the matrices U and S expand as in (2.40). Theorem 2.5 is proved. Theorem 1.1 is a straightforward corollary. The function $\mu(\epsilon)$ in (1.4) is defined as the implicit solution of the function $\Delta_{\mathrm{BF}}(\mathsf{h}; \mu, \epsilon)$ in (1.6) for ϵ small enough, depending on h. □

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Declarations

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A. Expansion of the Kato Basis

In this appendix we prove Lemma 4.2. We provide the expansion of the basis $f_k^\pm(\mu, \epsilon) = U_{\mu,\epsilon} f_k^\pm$, $k = 0, 1$, in (4.1), where $f_k^\pm$ defined in (4.2) belong to the subspace $\mathcal{V}_{0,0} := \mathrm{Rg}(P_{0,0})$. We first Taylor-expand the transformation operators $U_{\mu,\epsilon}$ defined in (3.5). We denote ∂_ϵ with a prime and ∂_μ with a dot.

Lemma A.1. *The first jets of $U_{\mu,\epsilon} P_{0,0}$ are*

$$U_{0,0} P_{0,0} = P_{0,0}, \quad U_{0,0}' P_{0,0} = P_{0,0}' P_{0,0}, \quad \dot{U}_{0,0} P_{0,0} = \dot{P}_{0,0} P_{0,0}, \tag{A.1}$$

$$\dot{U}_{0,0}' P_{0,0} = \left(\dot{P}_{0,0}' - \tfrac{1}{2} P_{0,0} \dot{P}_{0,0}' \right) P_{0,0}, \tag{A.2}$$

where

$$P_{0,0}' = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\mathcal{L}_{0,0} - \lambda)^{-1} \mathcal{L}_{0,0}' (\mathcal{L}_{0,0} - \lambda)^{-1} \mathrm{d}\lambda, \tag{A.3}$$

$$\dot{P}_{0,0} = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\mathcal{L}_{0,0} - \lambda)^{-1} \dot{\mathcal{L}}_{0,0} (\mathcal{L}_{0,0} - \lambda)^{-1} \mathrm{d}\lambda, \tag{A.4}$$

and

$$\dot{P}_{0,0}' = -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\mathcal{L}_{0,0} - \lambda)^{-1} \dot{\mathcal{L}}_{0,0} (\mathcal{L}_{0,0} - \lambda)^{-1} \mathcal{L}_{0,0}' (\mathcal{L}_{0,0} - \lambda)^{-1} \mathrm{d}\lambda \tag{A.5a}$$

$$- \frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\mathcal{L}_{0,0} - \lambda)^{-1} \mathcal{L}_{0,0}' (\mathcal{L}_{0,0} - \lambda)^{-1} \dot{\mathcal{L}}_{0,0} (\mathcal{L}_{0,0} - \lambda)^{-1} \mathrm{d}\lambda \tag{A.5b}$$

$$+ \frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\mathcal{L}_{0,0} - \lambda)^{-1} \dot{\mathcal{L}}_{0,0}' (\mathcal{L}_{0,0} - \lambda)^{-1} \mathrm{d}\lambda. \tag{A.5c}$$

The operators $\mathcal{L}_{0,0}'$ and $\dot{\mathcal{L}}_{0,0}$ are

$$\mathcal{L}_{0,0}' = \begin{bmatrix} \partial_x \circ p_1(x) & 0 \\ -a_1(x) & p_1(x) \circ \partial_x \end{bmatrix}, \quad \dot{\mathcal{L}}_{0,0} = \begin{bmatrix} 0 & \mathrm{sgn}(D) m(D) \\ 0 & 0 \end{bmatrix}, \tag{A.6}$$

where $\mathrm{sgn}(D)$ is defined in (2.31) and $m(D)$ is the real, even operator

$$m(D) := \tanh(\mathrm{h}|D|) + \mathrm{h}|D|(1 - \tanh^2(\mathrm{h}|D|)) \tag{A.7}$$

and $a_1(x)$ and $p_1(x)$ are given in Lemma 2.2.
　　The operator $\dot{\mathcal{L}}_{0,0}'$ is

$$\dot{\mathcal{L}}_{0,0}' = \begin{bmatrix} \mathrm{i}\, p_1(x) & 0 \\ 0 & \mathrm{i}\, p_1(x) \end{bmatrix}. \tag{A.8}$$

Proof. By (3.5) and (3.6) one has the Taylor expansion in $\mathcal{L}(Y)$

$$U_{\mu,\epsilon} P_{0,0} = P_{\mu,\epsilon} P_{0,0} + \frac{1}{2}(P_{\mu,\epsilon} - P_{0,0})^2 P_{\mu,\epsilon} P_{0,0} + \mathcal{O}(P_{\mu,\epsilon} - P_{0,0})^4,$$

where $\mathcal{O}(P_{\mu,\epsilon} - P_{0,0})^4 = \mathcal{O}(\epsilon^4, \epsilon^3\mu, \epsilon^2\mu^2, \epsilon\mu^3, \mu^4) \in \mathcal{L}(Y)$. Consequently one derives (A.1), (A.2), using also the identity $\dot{P}_{0,0} P'_{0,0} P_{0,0} + P'_{0,0} \dot{P}_{0,0} P_{0,0} = -P_{0,0} \dot{P}'_{0,0} P_{0,0}$, which follows differentiating $P^2_{\mu,\epsilon} = P_{\mu,\epsilon}$. Differentiating (3.4) one gets (A.3)-(A.5c). Formulas (A.6)-(A.8) follow by (3.2) using also that the Fourier multiplier $\Pi_0\big(\tanh(\mathrm{h}|D|) + \mathrm{h}|D|(1 - \tanh^2(\mathrm{h}|D|))\big) = 0$. $\quad\square$

By the previous lemma we have the Taylor expansion

$$f_k^\sigma(\mu, \epsilon) = f_k^\sigma + \epsilon P'_{0,0} f_k^\sigma + \mu \dot{P}_{0,0} f_k^\sigma$$
$$+ \mu\epsilon\big(\dot{P}'_{0,0} - \frac{1}{2}P_{0,0}\dot{P}'_{0,0}\big)f_k^\sigma + \mathcal{O}(\mu^2, \epsilon^2). \tag{A.9}$$

In order to compute the vectors $P'_{0,0} f_k^\sigma$ and $\dot{P}_{0,0} f_k^\sigma$ using (A.3) and (A.4), it is useful to know the action of $(\mathcal{L}_{0,0} - \lambda)^{-1}$ on the vectors

$$f_k^+ := \begin{bmatrix} c_\mathrm{h}^{1/2} \cos(kx) \\ c_\mathrm{h}^{-1/2} \sin(kx) \end{bmatrix},$$

$$f_k^- := \begin{bmatrix} -c_\mathrm{h}^{1/2} \sin(kx) \\ c_\mathrm{h}^{-1/2} \cos(kx) \end{bmatrix}, \tag{A.10}$$

$$f_{-k}^+ := \begin{bmatrix} c_\mathrm{h}^{1/2} \cos(kx) \\ -c_\mathrm{h}^{-1/2} \sin(kx) \end{bmatrix}, \quad f_{-k}^- := \begin{bmatrix} c_\mathrm{h}^{1/2} \sin(kx) \\ c_\mathrm{h}^{-1/2} \cos(kx) \end{bmatrix}, \quad k \in \mathbb{N}.$$

Lemma A.2. *The space $H^1(\mathbb{T})$ decomposes as $H^1(\mathbb{T}) = \mathcal{V}_{0,0} \oplus \mathcal{U} \oplus \mathcal{W}_{H^1}$, with $\mathcal{W}_{H^1} = \overline{\bigoplus_{k=2}^{\infty} \mathcal{W}_k}^{H^1}$ where the subspaces $\mathcal{V}_{0,0}, \mathcal{U}$ and $\mathcal{W}_k$, defined below, are invariant under $\mathcal{L}_{0,0}$ and the following properties hold:*

(i) *$\mathcal{V}_{0,0} = span\{f_1^+, f_1^-, f_0^+, f_0^-\}$ is the generalized kernel of $\mathcal{L}_{0,0}$. For any $\lambda \neq 0$ the operator $\mathcal{L}_{0,0} - \lambda : \mathcal{V}_{0,0} \to \mathcal{V}_{0,0}$ is invertible and*

$$(\mathcal{L}_{0,0} - \lambda)^{-1} f_1^+ = -\frac{1}{\lambda}f_1^+, \quad (\mathcal{L}_{0,0} - \lambda)^{-1} f_1^- = -\frac{1}{\lambda}f_1^-,$$

$$(\mathcal{L}_{0,0} - \lambda)^{-1} f_0^- = -\frac{1}{\lambda}f_0^-, \tag{A.11}$$

$$(\mathcal{L}_{0,0} - \lambda)^{-1} f_0^+ = -\frac{1}{\lambda}f_0^+ + \frac{1}{\lambda^2}f_0^-. \tag{A.12}$$

(ii) $\mathcal{U} := span\left\{f_{-1}^+, f_{-1}^-\right\}$. For any $\lambda \neq \pm 2\mathrm{i}$ the operator $\mathcal{L}_{0,0} - \lambda : \mathcal{U} \to \mathcal{U}$ is invertible and

$$
\begin{aligned}
(\mathcal{L}_{0,0} - \lambda)^{-1} f_{-1}^+ &= \frac{1}{\lambda^2 + 4c_\mathrm{h}^2} \left(-\lambda f_{-1}^+ + 2c_\mathrm{h} f_{-1}^-\right), \\
(\mathcal{L}_{0,0} - \lambda)^{-1} f_{-1}^- &= \frac{1}{\lambda^2 + 4c_\mathrm{h}^2} \left(-2c_\mathrm{h} f_{-1}^+ - \lambda f_{-1}^-\right).
\end{aligned}
\tag{A.13}
$$

(iii) Each subspace $\mathcal{W}_k := span\left\{f_k^+,\ f_k^-, f_{-k}^+,\ f_{-k}^-\right\}$ is invariant under $\mathcal{L}_{0,0}$.
Let $\mathcal{W}_{L^2} = \overline{\bigoplus_{k=2}^{\infty} \mathcal{W}_k}^{L^2}$. For any $|\lambda| < \delta(\mathrm{h})$ small enough, the operator $\mathcal{L}_{0,0} - \lambda : \mathcal{W}_{H^1} \to \mathcal{W}_{L^2}$ is invertible and for any $f \in \mathcal{W}_{L^2}$

$$
\begin{aligned}
(\mathcal{L}_{0,0} - \lambda)^{-1} f &= \left(c_\mathrm{h}^2 \partial_x^2 + |D| \tanh(\mathrm{h}|D|)\right)^{-1} \begin{bmatrix} c_\mathrm{h}\partial_x & -|D|\tanh(\mathrm{h}|D|) \\ 1 & c_\mathrm{h}\partial_x \end{bmatrix} f \\
&\quad + \lambda \varphi_f(\lambda, x),
\end{aligned}
\tag{A.14}
$$

for some analytic function $\lambda \mapsto \varphi_f(\lambda, \cdot) \in H^1(\mathbb{T}, \mathbb{C}^2)$.

Proof. By inspection the spaces $\mathcal{V}_{0,0}, \mathcal{U}$ and $\mathcal{W}_k$ are invariant under $\mathcal{L}_{0,0}$ and, by Fourier series, they decompose $H^1(\mathbb{T}, \mathbb{C}^2)$. Formulas (A.11)–(A.12) follow using that f_1^+, f_1^-, f_0^- are in the kernel of $\mathcal{L}_{0,0}$, and $\mathcal{L}_{0,0} f_0^+ = -f_0^-$. Formula (A.13) follows using that $\mathcal{L}_{0,0} f_{-1}^+ = -2c_\mathrm{h} f_{-1}^-$ and $\mathcal{L}_{0,0} f_{-1}^- = 2c_\mathrm{h} f_{-1}^+$. Let us prove item (iii). Let $\mathcal{W} := \mathcal{W}_{H^1}$. The operator $(\mathcal{L}_{0,0} - \lambda \mathrm{Id})\big|_{\mathcal{W}}$ is invertible for any $\lambda \notin \{\pm\mathrm{i}\sqrt{|k|\tanh(\mathrm{h}|k|)} \pm \mathrm{i}k c_\mathrm{h}, k \geq 2, k \in \mathbb{N}\}$ and

$$
(\mathcal{L}_{0,0}\big|_{\mathcal{W}})^{-1} = \left(c_\mathrm{h}^2 \partial_x^2 + |D| \tanh(\mathrm{h}|D|)\right)^{-1} \begin{bmatrix} c_\mathrm{h}\partial_x & -|D|\tanh(\mathrm{h}|D|)) \\ 1 & c_\mathrm{h}\partial_x \end{bmatrix}_{|\mathcal{W}}.
$$

By Neumann series, for any λ such that $|\lambda| \|(\mathcal{L}_{0,0}\big|_{\mathcal{W}})^{-1}\|_{\mathcal{L}(\mathcal{W}, H^1(\mathbb{T}))} < 1$ we have

$$
\begin{aligned}
(\mathcal{L}_{0,0}\big|_{\mathcal{W}} - \lambda)^{-1} &= (\mathcal{L}_{0,0}\big|_{\mathcal{W}})^{-1}\left(\mathrm{Id} - \lambda(\mathcal{L}_{0,0}\big|_{\mathcal{W}})^{-1}\right)^{-1} \\
&= (\mathcal{L}_{0,0}\big|_{\mathcal{W}})^{-1} \sum_{k \geq 0}((\mathcal{L}_{0,0}\big|_{\mathcal{W}})^{-1}\lambda)^k.
\end{aligned}
$$

Formula (A.14) follows with $\varphi_f(\lambda, x) := (\mathcal{L}_{0,0}\big|_{\mathcal{W}})^{-1} \sum_{k \geq 1} \lambda^{k-1}[(\mathcal{L}_{0,0}\big|_{\mathcal{W}})^{-1}]^k f$. $\square$

We shall also use the following formulas obtained by (A.6), (A.7) and (4.2):

$$\mathscr{L}'_{0,0}f_1^+ = \begin{bmatrix} 2c_h^{-1/2}\,\sin(2x) \\ \frac{1}{2}c_h^{5/2}(1-c_h^{-4})(1+\cos(2x)) \end{bmatrix},$$

$$\mathscr{L}'_{0,0}f_1^- = \begin{bmatrix} 2\,c_h^{-1/2}\,\cos(2x) \\ -\frac{1}{2}c_h^{5/2}(1-c_h^{-4})\sin(2x) \end{bmatrix},$$

$$\mathscr{L}'_{0,0}f_0^+ = \begin{bmatrix} 2c_h^{-1}\sin(x) \\ \left(c_h^2+c_h^{-2}\right)\cos(x) \end{bmatrix}, \qquad \mathscr{L}'_{0,0}f_0^- = 0, \tag{A.15}$$

$$\dot{\mathscr{L}}_{0,0}f_1^+ = -i\,b(h)\begin{bmatrix}\cos(x)\\0\end{bmatrix}, \qquad \dot{\mathscr{L}}_{0,0}f_1^- = i\,b(h)\begin{bmatrix}\sin(x)\\0\end{bmatrix},$$

$$b(h) := c_h^{-1/2}\big(c_h^2+h(1-c_h^4)\big),$$

$$\dot{\mathscr{L}}_{0,0}f_0^+ = 0, \qquad \dot{\mathscr{L}}_{0,0}f_0^- = 0.$$

Remark. In deep water we have $\dot{\mathscr{L}}_{0,0}f_0^- = f_0^+$ (cfr. formula (A.14) in [6]). In finite depth instead $\dot{\mathscr{L}}_{0,0}f_0^- = 0$ because the Fourier multiplier $\mathrm{sgn}(D)m(D)$ in (A.7) vanishes on the constants.

We finally compute $P'_{0,0}f_k^\sigma$ and $\dot{P}_{0,0}f_k^\sigma$.

Lemma A.3. *One has*

$$P'_{0,0}f_1^+ = \begin{bmatrix} \frac{1}{2}c_h^{-\frac{11}{2}}(3+c_h^4)\,\cos(2x) \\ \frac{1}{4}c_h^{-\frac{13}{2}}(1+c_h^4)(3-c_h^4)\sin(2x) \end{bmatrix},$$

$$P'_{0,0}f_1^- = \begin{bmatrix} -\frac{1}{2}c_h^{-\frac{11}{2}}(3+c_h^4)\,\sin(2x) \\ \frac{1}{4}c_h^{-\frac{13}{2}}(1+c_h^4)(3-c_h^4)\cos(2x) \end{bmatrix}, \tag{A.16}$$

$$P'_{0,0}f_0^+ = \frac{1}{4}c_h^{-\frac{5}{2}}(3+c_h^4)f_{-1}^+, \quad P'_{0,0}f_0^- = 0, \quad \dot{P}_{0,0}f_0^+ = 0, \quad \dot{P}_{0,0}f_0^- = 0,$$

$$\dot{P}_{0,0}f_1^+ = \frac{i}{4}\big(1+c_h^{-2}h(1-c_h^4)\big)f_{-1}^-, \quad \dot{P}_{0,0}f_1^- = \frac{i}{4}\big(1+c_h^{-2}h(1-c_h^4)\big)f_{-1}^+.$$

Proof. We first compute $P'_{0,0}f_1^+$. By (A.3), (A.11) and (A.15) we deduce

$$P'_{0,0}f_1^+ = -\frac{1}{2\pi i}\oint_\Gamma \frac{1}{\lambda}(\mathscr{L}_{0,0}-\lambda)^{-1}\begin{bmatrix} 2c_h^{-1/2}\,\sin(2x) \\ \frac{1}{2}c_h^{5/2}(1-c_h^{-4})(1+\cos(2x)) \end{bmatrix}d\lambda.$$

We note that $\begin{bmatrix} 2c_h^{-1/2}\,\sin(2x) \\ \frac{1}{2}c_h^{5/2}(1-c_h^{-4})(1+\cos(2x)) \end{bmatrix} = \frac{1}{2}c_h^{5/2}(1-c_h^{-4})f_0^- +\mathcal{W}$. Therefore by (A.11) and (A.14) there is an analytic function $\lambda \mapsto \varphi(\lambda,\cdot) \in H^1(\mathbb{T},\mathbb{C}^2)$ so that

$$P'_{0,0}f_1^+ = -\frac{1}{2\pi i}\oint_\Gamma \frac{1}{\lambda}\Big(-\frac{c_h^{5/2}(1-c_h^{-4})}{2\lambda}f_0^- -\frac{1+c_h^4}{4c_h^6}\begin{bmatrix} 2c_h\frac{c_h^{-\frac{1}{2}}(3+c_h^4)}{1+c_h^4}\cos(2x) \\ c_h^{-\frac{1}{2}}(3-c_h^4)\sin(2x) \end{bmatrix}$$

$$+\lambda\varphi(\lambda)\Big)\,d\lambda\,,$$

where we exploited the identity $\tanh(2h) = \frac{2c_h^2}{1+c_h^4}$ in applying (A.14). Thus, by means of residue Theorem we obtain the first identity in (A.16). Similarly one computes $P_{0,0}'f_1^-$. By (A.3), (A.11) and (A.15), one has $P_{0,0}'f_0^- = 0$. Next we compute $P_{0,0}'f_0^+$. By (A.3), (A.11), (A.12) and (A.15) we get

$$P_{0,0}'f_0^+ = -\frac{1}{2\pi i}\oint_\Gamma \frac{1}{\lambda}(\mathscr{L}_{0,0}-\lambda)^{-1}\begin{bmatrix} 2c_h^{-1}\sin(x) \\ (c_h^2 + c_h^{-2})\cos(x) \end{bmatrix}d\lambda\,.$$

Next we decompose $\begin{bmatrix} 2c_h^{-1}\sin(x) \\ (c_h^2 + c_h^{-2})\cos(x) \end{bmatrix} = \underbrace{\frac{1}{2}c_h^{-\frac{3}{2}}(c_h^4+3)}_{=:\alpha}\,f_{-1}^- + \underbrace{\frac{1}{2}c_h^{-\frac{3}{2}}(c_h^4-1)}_{=:\beta}\,f_1^-\,.$

By (A.15) and (A.13) we get

$$P_{0,0}'f_0^+ = -\frac{1}{2\pi i}\oint_\Gamma\Big(-\frac{2\alpha c_h}{\lambda(\lambda^2+4c_h^2)}f_1^+ - \frac{\alpha}{\lambda^2+4c_h^2}f_{-1}^- + \frac{\beta}{\lambda^2}f_1^-\Big)d\lambda = \frac{\alpha}{2c_h}f_{-1}^+\,,$$

where in the last step we used the residue theorem. We compute now $\dot{P}_{0,0}f_1^+$. First we have $\dot{P}_{0,0}f_1^+ = \frac{i}{2\pi i}b(h)\oint_\Gamma\frac{1}{\lambda}(\mathscr{L}_{0,0}-\lambda)^{-1}\begin{bmatrix}\cos(x)\\0\end{bmatrix}d\lambda$, where $b(h)$ is in (A.15), and then, writing $\begin{bmatrix}\cos(x)\\0\end{bmatrix} = \frac{1}{2}c_h^{-\frac{1}{2}}(f_1^+ + f_{-1}^+)$ and using (A.13), we conclude using again the residue theorem $\dot{P}_{0,0}f_1^+ = \frac{i}{4}\big(1 + h(1-c_h^4)c_h^{-2}\big)f_{-1}^-$. The computation of $\dot{P}_{0,0}f_1^-$ is analogous. Finally, in view of (A.15), we have

$$\dot{P}_{0,0}f_0^+ = \frac{1}{2\pi i}\oint_\Gamma(\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathcal{L}}_{0,0}\Big(\frac{1}{\lambda^2}f_0^- - \frac{1}{\lambda}f_0^+\Big)d\lambda = 0\,,$$

$$\dot{P}_{0,0}f_0^- = -\frac{1}{2\pi i}\oint_\Gamma\frac{1}{\lambda}(\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathcal{L}}_{0,0}f_0^-\,d\lambda = 0\,.$$

In conclusion all the formulas in (A.16) are proved.   □

So far we have obtained the linear terms of the expansions (4.3), (4.4), (4.5), (4.6). We now provide further information about the expansion of the basis at $\mu = 0$. The proof of the next lemma follows as that of Lemma A.4 in [6].

Lemma A.4. *The basis $\{f_k^\sigma(0,\epsilon),\ k = 0, 1, \sigma = \pm\}$ is real. For any ϵ it results $f_0^-(0,\epsilon) \equiv f_0^-$. The property (4.8) holds.*

We now provide further information about the expansion of the basis at $\epsilon = 0$. The following lemma follows as Lemma A.5 in [6]. The key observation is that the operator $\mathscr{L}_{\mu,0}\big|_{\mathcal{Z}}$, where $\mathcal{Z}$ is the invariant subspace $\mathcal{Z} := \mathrm{span}\{f_0^+,\ f_0^-\}$, has the two eigenvalues $\pm i\sqrt{\mu\tanh(h\mu)}$, which, for small μ, lie inside the loop Γ around 0 in (3.4).

Lemma A.5. *For any small μ, we have $f_0^+(\mu,0) \equiv f_0^+$ and $f_0^-(\mu,0) \equiv f_0^-$. Moreover the vectors $f_1^+(\mu,0)$ and $f_1^-(\mu,0)$ have both components with zero space average.*

We finally consider the $\mu\epsilon$ term in the expansion (A.9).

Lemma A.6. *The derivatives* $(\partial_\mu \partial_\epsilon f_k^\sigma)(0, 0) = \left(\dot{P}'_{0,0} - \frac{1}{2} P_{0,0} \dot{P}'_{0,0} \right) f_k^\sigma$ *satisfy*

$$
(\partial_\mu \partial_\epsilon f_1^+)(0, 0) = \mathrm{i} \begin{bmatrix} odd(x) \\ even(x) \end{bmatrix}, \qquad (\partial_\mu \partial_\epsilon f_1^-)(0, 0){-} = \mathrm{i} \begin{bmatrix} even(x) \\ odd(x) \end{bmatrix},
$$

$$
(\partial_\mu \partial_\epsilon f_0^+)(0, 0) = \mathrm{i} \begin{bmatrix} odd(x) \\ even_0(x) \end{bmatrix}, \qquad (\partial_\mu \partial_\epsilon f_0^-)(0, 0) = \mathrm{i} \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix}.
$$

$$(A.17)$$

Proof. We prove that $\dot{P}'_{0,0}$= (A.5a) + (A.5b) + (A.5c) is purely imaginary, see footnote 3. This follows since the operators in (A.5a), (A.5b) and (A.5c) are purely imaginary because $\dot{\mathscr{L}}_{0,0}$ is purely imaginary, $\mathscr{L}'_{0,0}$ in (A.6) is real and $\dot{\mathscr{L}}'_{0,0}$ in (A.8) is purely imaginary (argue as in Lemma 3.2-(iii) of [6]). Then, applied to the real vectors f_k^σ, $k = 0, 1$, $\sigma = \pm$, give purely imaginary vectors.

The property (3.9) implies that $(\partial_\mu \partial_\epsilon f_k^\sigma)(0, 0)$ have the claimed parity structure in (A.17). We shall now prove that $(\partial_\mu \partial_\epsilon f_0^\pm)(0, 0)$ have zero average. We have, by (A.12) and (A.15)

$$
(A.5a) f_0^+ := \frac{1}{2\pi\mathrm{i}} \oint_\Gamma (\mathscr{L}_{0,0} - \lambda)^{-1} \dot{\mathscr{L}}_{0,0} (\mathscr{L}_{0,0} - \lambda)^{-1} \frac{1}{\lambda} \begin{bmatrix} 2c_\mathrm{h}^{-1} \sin(x) \\ \left(c_\mathrm{h}^2 + c_\mathrm{h}^{-2} \right) \cos(x) \end{bmatrix} \mathrm{d}\lambda
$$

and since the operators $(\mathscr{L}_{0,0} - \lambda)^{-1}$ and $\dot{\mathscr{L}}_{0,0}$ are both Fourier multipliers, hence they preserve the absence of average of the vectors, then (A.5a) f_0^+ has zero average. Next (A.5b) $f_0^+ = 0$ since $\dot{\mathscr{L}}_{0,0} f_0^\pm = 0$, cfr. (2.31). Finally, by (A.12) and (A.8), where $p_1(x) = p_1^{[1]} \cos(x)$,

$$
(A.5c) f_0^+ = \frac{\mathrm{i}\, p_1^{[1]}}{2\pi\mathrm{i}} \oint_\Gamma (\mathscr{L}_{0,0} - \lambda)^{-1} \left(-\frac{1}{\lambda} \begin{bmatrix} \cos(x) \\ 0 \end{bmatrix} + \frac{1}{\lambda^2} \begin{bmatrix} 0 \\ \cos(x) \end{bmatrix} \right) \mathrm{d}\lambda
$$

is a vector with zero average. We conclude that $\dot{P}'_{0,0} f_0^+$ is an imaginary vector with zero average, as well as $(\partial_\mu \partial_\epsilon f_0^+)(0, 0)$ since $P_{0,0}$ sends zero average functions in zero average functions. Finally, by (3.9), $(\partial_\mu \partial_\epsilon f_0^+)(0, 0)$ has the claimed structure in (A.17).

We finally consider $(\partial_\mu \partial_\epsilon f_0^-)(0, 0)$. By (A.11) and $\mathscr{L}'_{0,0} f_0^- = 0$ (cfr. (A.15)), it results

$$
(A.5a) f_0^- = -\frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{(\mathscr{L}_{0,0} - \lambda)^{-1}}{\lambda} \dot{\mathscr{L}}_{0,0} (\mathscr{L}_{0,0} - \lambda)^{-1} \mathscr{L}'_{0,0} f_0^- \mathrm{d}\lambda = 0 \,.
$$

Next by (A.11) and $\dot{\mathscr{L}}_{0,0} f_0^- = 0$ we get (A.5b) $f_0^- = 0$. Finally by (A.11) and (A.8)

$$
(A.5c) f_0^- = -\frac{1}{2\pi\mathrm{i}} \oint_\Gamma (\mathscr{L}_{0,0} - \lambda)^{-1} \frac{1}{\lambda} \begin{bmatrix} 0 \\ \mathrm{i}\, p_1^{[1]} \cos(x) \end{bmatrix} \mathrm{d}\lambda
$$

has zero average since $(\mathscr{L}_{0,0} - \lambda)^{-1}$ is a Fourier multiplier (and thus preserves average absence). □

This completes the proof of Lemma 4.2.

B. Expansion of the Stokes Waves in Finite Depth

In this Appendix we provide the expansions (2.6)–(2.7), (2.15), (2.20)–(2.23).

Proof of (2.6)-(2.7). Writing

$$\eta_\epsilon(x) = \epsilon \eta_1(x) + \epsilon^2 \eta_2(x) + \mathcal{O}(\epsilon^3),$$
$$\psi_\epsilon(x) = \epsilon \psi_1(x) + \epsilon^2 \psi_2(x) + \mathcal{O}(\epsilon^3),$$
$$c_\epsilon = c_h + \epsilon c_1 + \epsilon^2 c_2 + \mathcal{O}(\epsilon^3),$$

(B.1)

where η_i is $even(x)$ and ψ_i is $odd(x)$ for $i = 1, 2$, we solve order by order in ϵ the equations (2.5), that we rewrite as

$$\begin{cases} -c\,\psi_x + \eta + \dfrac{\psi_x^2}{2} - \dfrac{\eta_x^2}{2(1+\eta_x^2)}(c - \psi_x)^2 = 0 \\ c\,\eta_x + G(\eta)\psi = 0, \end{cases}$$

(B.2)

having substituted $G(\eta)\psi$ with $-c\,\eta_x$ in the first equation. We expand the Dirichlet-Neumann operator $G(\eta) = G_0 + G_1(\eta) + G_2(\eta) + \mathcal{O}(\eta^3)$ where, according to [13, formula (2.14)],

$$\begin{aligned} G_0 &:= D \tanh(hD) = |D| \tanh(h|D|), \\ G_1(\eta) &:= D\big(\eta - \tanh(hD)\eta \tanh(hD)\big)D \\ &= -\partial_x \eta \partial_x - |D| \tanh(h|D|)\eta|D| \tanh(h|D|), \\ G_2(\eta) &:= -\frac{1}{2}D\Big(D\eta^2 \tanh(hD) + \tanh(hD)\eta^2 D \\ &\quad - 2\tanh(hD)\eta D \tanh(hD)\eta \tanh(hD)\Big)D. \end{aligned}$$

(B.3)

First order in ϵ. Substituting in (B.2) the expansions in (B.1), we get the linear system

$$\begin{cases} -c_h(\psi_1)_x + \eta_1 = 0 \\ c_h(\eta_1)_x + G_0\psi_1 = 0, \end{cases} \quad \text{i.e.} \quad \begin{bmatrix} \eta_1 \\ \psi_1 \end{bmatrix} \in \operatorname{Ker} \mathcal{B}_0 \text{ with } \mathcal{B}_0 := \begin{bmatrix} 1 & -c_h\partial_x \\ c_h\partial_x & G_0 \end{bmatrix},$$

(B.4)

where η_1 is $even(x)$ and ψ_1 is $odd(x)$. $\quad\square$

Lemma B.1. *The kernel of the linear operator $\mathcal{B}_0$ in (B.4) is*

$$\operatorname{Ker} \mathcal{B}_0 = span\left\{ \begin{bmatrix} \cos(x) \\ c_h^{-1}\sin(x) \end{bmatrix} \right\}.$$

(B.5)

Proof. The action of $\mathcal{B}_0$ on each subspace $span\left\{ \begin{bmatrix} \cos(kx) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \sin(kx) \end{bmatrix} \right\}, k \in \mathbb{N}$, is represented by the 2×2 matrix $\begin{bmatrix} 1 & -c_h k \\ -c_h k & k \tanh(hk) \end{bmatrix}$. Its determinant $k \tanh(hk) -$

$c_h^2 k^2 = k^2 \left(\frac{\tanh(hk)}{k} - \tanh(h) \right)$ vanishes if and only if $k = 1$. Indeed the function $x \mapsto \frac{\tanh(hx)}{x}$ is monotonically decreasing for $x > 0$, since its derivative $\frac{2xh - \sinh(2hx)}{2\cosh^2(hx)x^2}$ is negative for $x > 0$. For $k = 1$ we obtain the kernel of $\mathcal{B}_0$ given in (B.5). For $k = 0$ it has no kernel since $\psi_1(x)$ is odd. $\quad\square$

We set $\eta_1(x) := \cos(x)$, $\psi_1(x) := c_h^{-1} \sin(x)$ in agreement with (2.6).

Second order in ϵ. By (B.2), and since $c_h^2(\eta_1)_x^2 = (G_0\psi_1)^2$, we get the linear system

$$\mathcal{B}_0 \begin{bmatrix} \eta_2 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} c_1(\psi_1)_x - \frac{1}{2}(\psi_1)_x^2 + \frac{1}{2}(G_0\psi_1)^2 \\ -c_1(\eta_1)_x - G_1(\eta_1)\psi_1 \end{bmatrix}, \tag{B.6}$$

where $\mathcal{B}_0$ is the self-adjoint operator in (B.4). System (B.6) admits a solution if and only if its right-hand term is orthogonal to the Kernel of $\mathcal{B}_0$ in (B.5), namely

$$\left(\begin{bmatrix} c_1(\psi_1)_x - \frac{1}{2}(\psi_1)_x^2 + \frac{1}{2}(G_0\psi_1)^2 \\ -c_1(\eta_1)_x - G_1(\eta_1)\psi_1 \end{bmatrix}, \begin{bmatrix} \cos(x) \\ c_h^{-1}\sin(x) \end{bmatrix} \right) = 0. \tag{B.7}$$

In view of the first order expansion (2.6), (B.3) and the identity $\tanh(2h) = \dfrac{2c_h^2}{1 + c_h^4}$, it results $[G_0\psi_1](x) = c_h \sin(x)$, $[G_1(\eta_1)\psi_1](x) = \frac{1 - c_h^4}{c_h(1 + c_h^4)} \sin(2x)$ so that (B.7) implies $c_1 = 0$, in agrement with (2.6). Equation (B.6) reduces to

$$\begin{bmatrix} 1 & -c_h\partial_x \\ c_h\partial_x & G_0 \end{bmatrix} \begin{bmatrix} \eta_2 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}(c_h^{-2} - c_h^2) - \frac{1}{4}(c_h^{-2} + c_h^2)\cos(2x) \\ -\frac{1 - c_h^4}{c_h(1 + c_h^4)}\sin(2x) \end{bmatrix}. \tag{B.8}$$

Setting $\eta_2 = \eta_2^{[0]} + \eta_2^{[2]}\cos(2x)$ and $\psi_2 = \psi_2^{[2]}\sin(2x)$, system (B.8) amounts to

$$\begin{cases} \eta_2^{[0]} + \left(\eta_2^{[2]} - 2c_h\psi_2^{[2]}\right)\cos(2x) = -\frac{1}{4}\left(c_h^{-2} - c_h^2\right) - \frac{1}{4}\left(c_h^{-2} + c_h^2\right)\cos(2x) \\ \left(-2c_h\eta_2^{[2]} + 2\psi_2^{[2]}\tanh(2h)\right)\sin(2x) = -\frac{1 - c_h^4}{c_h(1 + c_h^4)}\sin(2x), \end{cases}$$

which leads to the expansions of $\eta_2^{[0]}, \eta_2^{[2]}, \psi_2^{[2]}$ given in (2.6)-(2.7).

Third order in ϵ. It remains to determine c_2 in (2.8). We get the linear system

$$\mathcal{B}_0 \begin{bmatrix} \eta_3 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} c_2(\psi_1)_x - (\psi_1)_x(\psi_2)_x - (\eta_1)_x^2(\psi_1)_x c_h + (\eta_1)_x(\eta_2)_x c_h^2 \\ -c_2(\eta_1)_x - G_1(\eta_1)\psi_2 - G_1(\eta_2)\psi_1 - G_2(\eta_1)\psi_1 \end{bmatrix}. \tag{B.9}$$

System (B.9) has a solution if and only if the right hand side is orthogonal to the Kernel of $\mathcal{B}_0$ given in (B.5). This condition determines uniquely c_2. Denoting Π_1 the L^2-orthogonal projector on span $\{\cos(x), \sin(x)\}$, we get that

$$c_2(\psi_1)_x = c_2 c_h^{-1}\cos(x), \quad c_2(\eta_1)_x = -c_2\sin(x),$$

$$\Pi_1[(\psi_1)_x(\psi_2)_x] = \psi_2^{[2]}c_h^{-1}\cos(x),$$

$$\Pi_1[c_h(\eta_1)_x^2(\psi_1)_x] = \frac{1}{4}\cos(x), \quad \Pi_1[c_h^2(\eta_1)_x(\eta_2)_x] = \eta_2^{[2]}c_h^2\cos(x),$$

and, in view of (B.3), and (2.6), (2.7),

$$\Pi_1[G_1(\eta_1)\psi_2] = \psi_2^{[2]}\frac{1 - c_h^4}{1 + c_h^4}\sin(x)\,, \quad \Pi_1[G_2(\eta_1)\psi_1] = c_h\frac{3c_h^4 - 1}{4(1 + c_h^4)}\sin(x)\,,$$

$$\Pi_1[G_1(\eta_2)\psi_1] = c_h^{-1}\Big(\eta_2^{[0]}(1 - c_h^4) + \tfrac{1}{2}\eta_2^{[2]}(1 + c_h^4)\Big)\sin(x)\,.$$

Therefore the orthogonality condition proves (2.8).

Proof of (2.15). We expand the function $\mathfrak{p}(x) = \epsilon\mathfrak{p}_1(x) + \epsilon^2\mathfrak{p}_2(x) + \mathcal{O}(\epsilon^3)$ defined by the fixed point equation (2.14). We first note that the constant $\mathfrak{f}_\epsilon = \mathcal{O}(\epsilon^2)$ because $\eta_1(x) = \cos(x)$ has zero average. Then $\mathfrak{p}(x) = \frac{\mathcal{H}}{\tanh(\mathsf{h}|D|)}\big[\epsilon\eta_1 + \epsilon^2(\eta_2 + (\eta_1)_x\mathfrak{p}_1) + \mathcal{O}(\epsilon^3)\big]$, and, using that $\mathcal{H}\cos(kx) = \sin(kx)$, for any $k \in \mathbb{N}$, we get

$$\mathfrak{p}_1(x) = \frac{\mathcal{H}}{\tanh(\mathsf{h}|D|)}\cos(x) = c_h^{-2}\sin(x)\,, \tag{B.10}$$

$$\mathfrak{p}_2(x) = \frac{\mathcal{H}}{\tanh(\mathsf{h}|D|)}((\eta_1)_x\mathfrak{p}_1 + \eta_2) = \frac{(1 + c_h^4)(c_h^4 + 3)}{8c_h^8}\sin(2x)\,. \tag{B.11}$$

Finally,

$$\mathfrak{f}_\epsilon = \frac{\epsilon^2}{2\pi}\int_{\mathbb{T}}\big(\eta_2 + (\eta_1)_x\mathfrak{p}_1\big)\mathrm{d}x + \mathcal{O}(\epsilon^3)$$

$$= \epsilon^2\big(\eta_2^{[0]} - \tfrac{1}{2}c_h^{-2}\big) + \mathcal{O}(\epsilon^3) \stackrel{(2.7)}{=} \epsilon^2\frac{c_h^4 - 3}{4c_h^2} + \mathcal{O}(\epsilon^3)\,.$$

The expansion (2.15) is proved.   □

Proof of Lemma 2.2. In view of (2.6)–(2.7), the expansions of the functions B, V in (2.10) are

$$B =: \epsilon B_1(x) + \epsilon^2 B_2(x) + \mathcal{O}(\epsilon^3) = \epsilon c_h\sin(x) + \epsilon^2\frac{3 - 2c_h^4}{2c_h^5}\sin(2x) + \mathcal{O}(\epsilon^3) \tag{B.12}$$

and

$$V =: \epsilon V_1(x) + \epsilon^2 V_2(x) + \mathcal{O}(\epsilon^3) = \epsilon c_h^{-1}\cos(x)$$

$$+ \epsilon^2\Big[\frac{c_h}{2} + \frac{3 - c_h^8}{4c_h^7}\cos(2x)\Big] + \mathcal{O}(\epsilon^3)\,. \tag{B.13}$$

In view of (2.18), denoting derivatives w.r.t x with a prime and suppressing dependence on x when trivial, we have

$$c_h + p_\epsilon(x) = (c_h + \epsilon^2 c_2 - V(x) - V'(x)\mathfrak{p}(x)$$

$$+ \mathcal{O}(\epsilon^3))(1 - \mathfrak{p}'(x) + (\mathfrak{p}'(x))^2 + \mathcal{O}(\epsilon^3))$$

$$= c_h + \epsilon\underbrace{(-V_1 - c_h\mathfrak{p}_1')}_{=:p_1}$$

$$+ \epsilon^2 \underbrace{\left(c_2 + V_1 \mathfrak{p}'_1 - V_2 - V'_1 \mathfrak{p}_1 - c_{\mathrm{h}} \mathfrak{p}'_2 + c_{\mathrm{h}} (\mathfrak{p}'_1)^2\right)}_{=:p_2} + \mathcal{O}(\epsilon^3) .$$

(B.14)

Similarly, by (2.18),

$$1 + a_\epsilon(x) := \frac{1}{1 + \mathfrak{p}_x(x)} - (c_{\mathrm{h}} + p_\epsilon(x)) B_x(x + \mathfrak{p}(x))$$

$$= 1 + \epsilon \underbrace{\left(- \mathfrak{p}'_1 - c_{\mathrm{h}} B'_1\right)}_{=:a_1}$$

$$+ \epsilon^2 \underbrace{\left((\mathfrak{p}'_1)^2 - \mathfrak{p}'_2 - c_{\mathrm{h}} B'_2 - c_{\mathrm{h}} B''_1 \mathfrak{p}_1(x) + B'_1 V_1 + c_{\mathrm{h}} B'_1 \mathfrak{p}'_1\right)}_{=:a_2} + \mathcal{O}(\epsilon^3) .$$

(B.15)

By (B.13), (B.10), (2.6), (B.11), (B.12) we deduce that the functions p_1, p_2, a_1, a_2 in (B.14) and (B.15) have an expansion as in (2.20)–(2.23). □

References

1. BALDI, P., BERTI, M., HAUS, E., MONTALTO, R.: Time quasi-periodic gravity water waves in finite depth. *Inventiones Math.* **214**(2), 739–911, 2018
2. BENJAMIN, T.: Instability of periodic wavetrains in nonlinear dispersive systems. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **299**456, 1967
3. BENJAMIN, T., FEIR, J.: The disintegration of wave trains on deep water, Part 1. *Theory. J. Fluid Mech.* **27**(3), 417–430, 1967
4. BERTI, M., FRANZOI, L., MASPERO, A.: Traveling quasi-periodic water waves with constant vorticity. *Arch. Ration. Mech.* **240**, 99–202, 2021
5. BERTI, M., FRANZOI, L., MASPERO, A.: Pure gravity traveling quasi-periodic water waves with constant vorticity. arXiv:2101.12006, 2021, to appear in Comm. Pure Appl. Math.
6. BERTI, M., MASPERO, A., VENTURA, P.: Full description of Benjamin-Feir instability of Stokes waves in deep water. *Inventiones Math.* **230**, 651–711, 2022
7. BERTI, M., MASPERO, A., VENTURA, P.: Benjamin–Feir instability of Stokes waves. *Rend. Lincei Mat. Appl.* **33**, 399–412, 2022
8. BERTI, M., MASPERO, A., VENTURA, P.: On the analyticity of the Dirichlet-Neumann operator and Stokes waves. *Rend. Lincei Mat. Appl.* **33**, 611–650, 2022. https://doi.org/10.4171/RLM/983
9. BRIDGES, T., MIELKE, A.: A proof of the Benjamin–Feir instability. *Arch. Ration. Mech. Anal.* **133**(2), 145–198, 1995
10. BRONSKI, J., HUR, V., JOHNSON, M.: Modulational Instability in Equations of KdV Type. In: Tobisch E. (eds) *New Approaches to Nonlinear Waves*. Lecture Notes in Physics, vol 908. Springer, 2016
11. BRONSKI, J., JOHNSON, M.: The modulational instability for a generalized Korteweg-de Vries equation. *Arch. Ration. Mech. Anal.* **197**(2), 357–400, 2010
12. CHEN, G., SU, Q.: Nonlinear modulational instabililty of the Stokes waves in 2d full water waves. 2023. https://doi.org/10.1007/s00220-023-04747-0
13. CRAIG, W., SULEM, C.: Numerical simulation of gravity waves. *J. Comput. Phys.* **108**(1), 73–83, 1993
14. CRAIG, W., GUYENNE, P., KALISCH, H.: Hamiltonian long-wave expansions for free surfaces and interfaces. *Comm. Pure Appl. Math.* **58**(12), 1587–1641, 2005

15. CREEDON, R., DECONINCK, B.: A High-Order Asymptotic Analysis of the Benjamin-Feir Instability Spectrum in Arbitrary Depth. *J. Fluid Mech.* **956**, A29, 2023
16. DECONINCK, B., OLIVERAS, K.: The instability of periodic surface gravity waves. *J. Fluid Mech.* **675**, 141–167, 2011
17. FEOLA, R., GIULIANI, F.: Quasi-periodic traveling waves on an infinitely deep fluid under gravity. arXiv:2005.08280, to appear on Memoires American Mathematical Society
18. GALLAY, T., HARAGUS, M.: Stability of small periodic waves for the nonlinear Schrödinger equation. *J. Differ. Equ.* **234**, 544–581, 2007
19. HARAGUS, M., KAPITULA, T.: On the spectra of periodic waves for infinite-dimensional Hamiltonian systems. *Phys. D* **237**, 2649–2671, 2008
20. HUR, V.: No solitary waves exist on 2D deep water. *Nonlinearity* **25**(12), 3301–3312, 2012
21. HUR, V., JOHNSON, M.: Modulational instability in the Whitham equation for water waves. *Stud. Appl. Math.* **134**(1), 120–143, 2015
22. HUR, V., PANDEY, A.: Modulational instability in nonlinear nonlocal equations of regularized long wave type. *Phys. D* **325**, 98–112, 2016
23. HUR, V., YANG, Z.: Unstable Stokes waves. *Arch. Rational Mech. Anal.* **247**, 62, 2023
24. JIN, J., LIAO, S., LIN, Z.: Nonlinear modulational instability of dispersive PDE models. *Arch. Ration. Mech. Anal.* **231**(3), 1487–1530, 2019
25. JOHNSON, M.: Stability of small periodic waves in fractional KdV type equations. *SIAM J. Math. Anal.* **45**, 2529–3228, 2013
26. KATO, T.: Perturbation theory for linear operators. Die Grundlehren der mathematischen Wissenschaften, Band 132 Springer-Verlag New York, Inc., New York, 1966
27. IFRIM, M., TATARU, D.: No solitary waves in 2D gravity and capillary waves in deep water. *Nonlinearity* **33**, 5457, 2020
28. LEISMAN, K., BRONSKI, J., JOHNSON, M., MARANGELL, R.: Stability of traveling wave solutions of nonlinear dispersive equations of NLS type. *Arch. Ration. Mech. Anal.* **240**, 927–969, 2021
29. LEVI-CIVITA, T.: Détermination rigoureuse des ondes permanentes d' ampleur finie. *Math. Ann.* **93**, 264–314, 1925
30. LIGHTHILL, M.J.: Contribution to the theory of waves in nonlinear dispersive systems. *IMA J. Appl. Math.* **1**(3), 269–306, 1965
31. KOROTKEVICH, A.O., DYACHENKO, A.I., ZAKHAROV, V.E.: Numerical simulation of surface waves instability on a homogeneous grid. *Phys. D Nonlinear Phenom. Volumes* **321–322**, 51–66, 2016
32. NEKRASOV, A.: On steady waves. Izv. Ivanovo–Voznesenk. Politekhn. 3, 1921
33. NGUYEN, H., STRAUSS, W.: Proof of modulational instability of Stokes waves in deep water. *Commun. Pure Appl. Math.* **76**(5), 899–1136, 2023
34. ROUSSET, F., TZVETKOV, N.: Transverse instability of the line solitary water-waves. *Inventiones Math.* **184**, 257–388, 2011
35. SEGUR, H., HENDERSON, D., CARTER, J., HAMMACK, J.: Stabilizing the Benjamin–Feir instability. *J. Fluid Mech.* **539**, 229–271, 2005
36. STOKES, G.: On the theory of oscillatory waves. *Trans. Cambridge Phil. Soc.* **8**, 441–455, 1847
37. STRUIK, D.: Détermination rigoureuse des ondes irrotationelles périodiques dans un canal á profondeur finie. *Math. Ann.* **95**, 595–634, 1926
38. WHITHAM, G.B.: Non-linear dispersion of water waves. J. Fluid Mech, volume 26 part 2 pp. 399–412, 1967
39. WHITHAM, G.B.: Linear and Nonlinear Waves. John Wiley-Sons, New York (1974)
40. ZAKHAROV, V.: The instability of waves in nonlinear dispersive media. *J. Exp. Theor. Phys.* **24**(4), 740–744, 1967
41. ZAKHAROV, V.: Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Zhurnal Prikladnoi Mekhaniki i Teckhnicheskoi Fiziki* **9**(2), 86–94, 1969
42. ZAKHAROV, V., KHARITONOV, V.: Instability of monochromatic waves on the surface of a liquid of arbitrary depth. *J. Appl. Mech. Tech. Phys.* **11**, 747–751, 1970

43. Zakharov, V., Ostrovsky, L.: Modulation instability: the beginning. *Phys. D* **238**(5), 540–548, 2009

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