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**Spin-cobordism and fermionic  $d = 2$   
anomalies**

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# Abstract

The aim of this work is to improve our description of global anomalies and the tools we have at our disposal for their computation. In particular, we focus on general fermionic quantum field theories with a global finite group symmetry  $G^f$  in 2-dimensions, with a special regard for the torus spacetime. The modular transformation properties of the family of partition functions with different backgrounds is determined by the 't Hooft anomaly of  $G^f$ . For a general  $G^f$ , possibly non-abelian or twisted, we provide a method to determine the modular transformations directly from the bulk 3d invertible topological quantum field theory (iTQFT) corresponding to the anomaly by inflow. We also describe a method of evaluating the character map from the real representation ring of  $G^f$  to the group which classifies anomalies. Physically the value of the map is given by the anomaly of free fermions in a given representation. We assume classification of the anomalies/iTQFTs by spin-cobordisms. As a byproduct, we provide explicit combinatorial expressions for corresponding spin-bordism invariants of abelian symmetry groups  $G^f$  in terms of surgery representation of arbitrary closed spin 3-manifolds. As an application, we compute the constraints that 't Hooft anomaly puts on the spectrum of infrared conformal field theories for various symmetry groups. In particular, we provide a first of such analysis for discrete non-abelians  $G^f$  or with a non-trivial twist of the  $\mathbb{Z}_2^f$  subgroup.

# Foreword

This thesis is divided into six chapters. In the first, we provide a general summary, background and motivation on the topic under study in this work. In Chapter 2 we revise how anomalies of global symmetries are described and their connection to higher dimensional invertible topological field theories via anomaly inflow. Chapter 3 is then devoted to the study of such topological theories and their aspects necessary for our analysis. In Chapter 4 we apply all the knowledge presented to the discussion of anomalies for  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2$  and develop techniques for their study. In Chapter 5 we apply further such techniques and generalize them so that can be efficiently applied for general symmetry groups  $G^f$ . Finally, in Chapter 6 we make use of the results following from these developments and apply them to modular bootstrap techniques to work out typical constraints on theories implied by the presence of anomalies.

The thesis is based on the following papers:

- [1] - A. Grigoletto and P. Putrov, “Spin-cobordisms, surgeries and fermionic modular bootstrap,” [arXiv:2106.16247 \[hep-th\]](#), in review process, *Comm. Math. Phys.*
- [2] - A. Grigoletto, “Anomalies of fermionic CFTs via cobordism and bootstrap,” [arXiv:2112.01485 \[hep-th\]](#)

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*Je n'ai pas le temps...*

*- Evariste Galois*

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# Chapter 1

## Introduction and summary

In the last few years a lot of new exciting developments in the study of symmetries in Quantum Field Theory (QFT) have started to emerge, opening the road to the development of many new powerful tools that are allowing us to efficiently probe new non-perturbative phenomena. Many new avenues are being opened, all of which are leading to uncover a new, deep and rich mathematical structure that enlarges the actual concept of symmetry. This thesis proposes to focus on a particular subset of these new developments, with the aim to organize the author's understanding and work on the topic. In particular, starting from their modern definition, the formalism required for the discussion of anomalies associated with invertible symmetries will be presented and expanded, as well as used to uncover (some of) the constraints that these imply on theories.

The concept of symmetry holds a fundamental role in the study of QFT since its first developments. With a special regard for the continuous case, symmetries have proved to be useful to get many sharp constraints in theories simply by looking at how matter has to organize itself in order to satisfy the rules they dictate. In particular, its concept has always been associated to the presence of an abstract group  $G$ , under which we expect the fields and states of the theory to transform. A fundamental property of these kinds of transformations is that they are invertible: under a transformation labelled by an element  $g \in G$ , we can always go back to the original setup by applying the transformation associated to another element, denoted  $g^{-1}$ . Nowadays, it starts to be clear that such requirement is not necessary to make sense of the notion of symmetry, which seems to be generalizable to a more ample, categorical sense. In particular, many different routes have been taken so far in order to generalize the classical concept of bosonic symmetries, which we can broadly summarize in a diagram like in Figure 1.1.

Several different and promising topics can be studied, many of which seem to be at first sight more interesting than focusing on the standard notion of invertible symmetry. Nevertheless, the analysis of this particular case is still of *fundamental* importance as it can show us the correct way to proceed in uncovering and formalizing these new, emerging branches of analysis of QFTs. In particular, the emerging

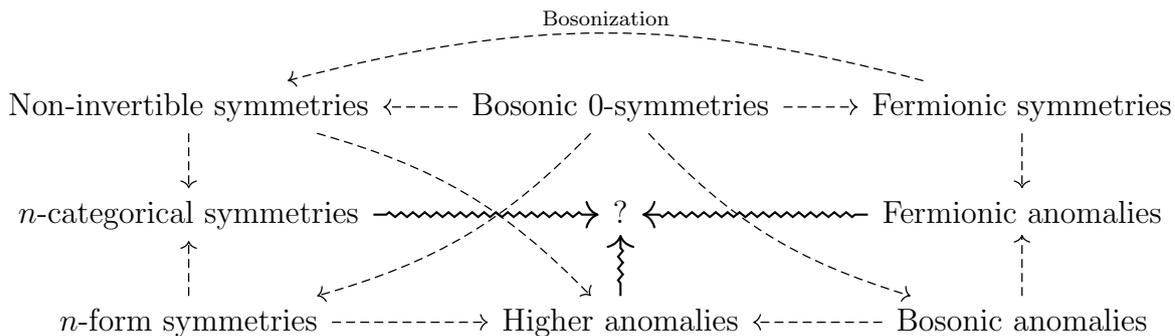


Figure 1.1: A schematic representation of the various developments that stems from the standard notion of symmetry.

idea that all symmetries, invertible or not, can be captured by a unique categorical framework, still lacks a proper formalization of the homotopical data that is necessary for the proper description of anomalies.

In fact, in the last decade there have been many new results in the description of 't Hooft anomalies and of their structure. Notably, this is thanks to the joint effort of the math, high-energy and condensed matter communities, which showed how anomalies are best described in a homotopy framework via the notion of *cobordism groups*, through the concepts of *anomaly inflow* and *relative field theories*. The further development of such concepts is thus crucial for the proper understanding of symmetries in QFTs. Moreover, the study of anomalies<sup>1</sup> is a very interesting argument *per se* and worth to be pursuit, as it still provides one the very few tools to analyze QFTs on non-perturbative level.

In particular, anomalies contain certain robust information about the dynamics of a theory. They normally take values in a discrete set which has a natural abelian group structure with respect to stacking of theories and which are invariant under all possible continuous deformations, including renormalization group flow. They are usually understood as non-invariance under diffeomorphisms or gauge transformations of background gauge fields on the quantum level. The global anomalies in particular exhibit themselves as non-invariance under “large” (meaning that they cannot be continuously deformed to identity) diffeomorphisms or gauge transformations. On the other hand, through the idea of anomaly inflow, it is well accepted that anomalies can be understood in terms of topological quantum field theories (TQFT) in one more dimension, see e.g. [3]. If the anomalous quantum field theory is put on the boundary of the spacetime of the corresponding TQFT, the combined system becomes invariant under diffeomorphisms and gauge transformations. Consider for example a partition function of the theory on a closed manifold with some particular background metric and global symmetry gauge field. A combination of a diffeomor-

<sup>1</sup>By default in this work, by “anomalies” we mean 't Hooft anomalies of gravitational and global symmetries. In particular the considered quantum field theories with anomalies are well defined, and the corresponding global symmetry is unbroken, when the background is trivial.

phism and a large gauge transformation that does not change the background (up to homotopy) will take the partition function to itself (possibly with a deformed background), up to an anomalous phase. The extra phase will then equal the partition function of the bulk TQFT on the corresponding mapping class torus [4–6]. More generally, one can consider a diffeomorphism that changes the homotopy class of the background gauge fields of global symmetries (note that all metrics are homotopic to each other). Such a diffeomorphism then takes the partition function to one with a different background, again, up to an anomalous phase. The phase is then determined by the linear evolution map that TQFT associates to the corresponding mapping cylinder.

The set of the phases that can appear forms an abelian group under stacking of decoupled theories together. The same can be said about the invertible bulk TQFTs, with a given global symmetry, that describe the anomalies by inflow. Therefore, schematically we will have the following homomorphism of abelian groups:

$$\{\text{invertible TQFTs}\} \longrightarrow \{\text{anomalous phases}\}. \quad (1.1)$$

One of the main goals of this work is to make this homomorphism and the group in the right hand side precise and explicit, using the classification of the invertible (unitary, or, equivalently, reflection-positive in the Euclidean setup) TQFTs by cobordisms [7–10]. A similar construction in the case of bosonic TQFTs classified by group cohomology [11, 12], when the 2d spacetime is two-torus was considered in [12–14].

One can use this map in two ways. First, it can be used to determine the anomaly of a given theory, if one knows how to calculate a partition function and determine how it transforms under large diffeomorphisms. In this case the anomaly will be computed starting from the details of the theory, which are known. The other way around, one can use this map to determine what general constraints a given anomaly imposes on the dynamics of the class of theories which exhibit it, for example using modular bootstrap. Therefore, in this case one discuss properties that must hold in full generality, independently from the details of how the anomaly is realized case by case.

Note that one can often determine the anomalous phases indirectly, by considering a particular theory that realizes a given anomaly. One of the most common ways to realize anomalies is to consider free fermions transforming in a non-trivial way under the considered symmetry. However it may happen that partition functions in certain background vanish (unless appropriate observables are inserted, as in the case of free fermions), which can lead to ambiguities in determining the phases. Calculating anomalies using directly invertible TQFTs leaves no ambiguity about which features are general and which are specific for a particular theory.

Nevertheless, considering free fermion theories provides us with a natural homomorphism from the free abelian group generated by representations of the symmetry (also known as representation ring, if one also considers the tensor product operation) to the group of anomalies. This map takes a representation to the anomaly of the theory of free fermions in this representation. So altogether we have a sequence

of two homomorphisms, which schematically reads as follows:

$$\{\text{representations}\} \longrightarrow \{\text{invertible TQFTs}\} \longrightarrow \{\text{anomalous phases}\}. \quad (1.2)$$

The maps in the sequence are natural in the sense that, if one considers a homomorphism between two different symmetry groups (e.g. corresponding to breaking of a symmetry to a subgroup), the induced maps between the three groups in the sequence (5.2) commute with the maps in the sequence. In mathematical terminology this means that we have a sequence of natural transformations between three (contravariant) functors from the category of groups to the category of abelian groups.

As a main example of this, after a review of the necessary notions in Chapters 2 and 3, in Chapter 4 we study in detail the case of  $\mathbb{Z}_2^f \times \mathbb{Z}_2$  global symmetry in two spacetime dimensions. Here  $\mathbb{Z}_2^f$  is fermionic parity and  $\mathbb{Z}_2$  a unitary global symmetry. Assuming absence of perturbative gravitational anomaly, the anomalies in this case are known to be classified by a  $\mathbb{Z}_8$  group [15, 16] and the map (1.1) turns out to be injective. In particular, we compute it via three different (but closely related) approaches: 1) explicit calculation, using geometric definition of the corresponding spin-bordism invariant, 2) using operations on symmetry defects in 2d, 3) using surgery representation of closed 3-manifolds. We also show how the anomalous action of the symmetry generators on the Hilbert space analyzed in [17] can be geometrically interpreted in terms of the extension of the corresponding charge operators to the bulk TQFT.

After this detailed analysis, in Chapter 5 we further apply the techniques developed for such starting case to the computation of anomalies for general symmetry groups  $G^f$ . In this way, we provide a method of explicit evaluation of the maps in the sequence (1.2) for 2d theories on  $\mathbb{T}^2$  with general, possibly non-abelian or twisted, symmetry groups.

Finally, as an application, following the techniques of [18–21], in Chapter 6 we perform the modular bootstrap of the spectrum of conformal field theories with given anomaly and various symmetry groups  $G^f$ . Specifically, we devise how to compute the spin-selection rules for generic theories in terms of anomalies via the techniques developed in the previous chapters, in order to apply such knowledge to compute bounds on the existence of symmetry-preserving scalar operators possibly associated to relevant/marginal deformations.

## Conventions

For ease of the reader, here we summarize few notation conventions that we are going to adopt in the rest of the work.

- We will work in Euclidean spacetime. All the manifolds considered will be Riemannian, with positive metric signature, i.e. locally  $g_{\mu\nu} = \delta_{\mu\nu}$ .

- We will denote with  $X$  manifolds of generic dimension  $d$ . Moreover, whenever we start from a fixed dimension  $d$ , manifolds of dimension  $d + 1$  and  $d + 2$  will be denoted respectively as  $Y$  and  $W$ .
- Unless specified, from the second half of Chapter 2 we will specialize to the fixed spacetime dimension  $d = 2$ . Therefore,  $X$  will be mostly reserved for 2-dimensional manifolds,  $Y$  for 3-manifolds and  $W$  for 4-manifolds.
- We use the same notation to denote a manifold and its  $H$ -structure. Moreover, the opposite  $H$ -structure on a manifold  $X$  is denoted by  $\overline{X}$ .
- Our discussion will be based on unitary symmetry groups  $G$ . This means that the analysis requires minor modifications if one is interested in the treatment of antiunitary symmetries, like time-reversal.
- Topological cochains, cocycles, coboundary and cohomology groups will be denoted by capital letters, e.g. in the  $\mathbb{Z}$ -valued case we write  $C^\bullet(-; \mathbb{Z})$ ,  $Z^\bullet(-; \mathbb{Z})$ ,  $B^\bullet(-; \mathbb{Z})$  and  $H^\bullet(-; \mathbb{Z})$ . Instead, group cochain, cocycles and cohomologies will be denoted by capital cursive letters, e.g.  $\mathcal{H}^\bullet(-; \mathbb{Z})$ .
- The symbol  $\cup$  is reserved to denote the standard cup product in cohomology and  $\cup_i$  the higher cup product. Instead,  $\sqcup$  will denote the disjoint union of manifolds.
- The trivial element for generic groups and ring is identified with  $\mathbb{1}$ .
- The trivial map, both between groups and rings, is denoted as  $\tau_\emptyset$ .
- The cyclic groups are denoted by  $\mathbb{Z}_n$  and, unless specified, the additive notation is used. Instead,  $\mathbb{Z}_+$  denotes the monoid of integers  $n \geq 0$ .
- A group with the additional superscript  $f$ , e.g.  $G^f$ , is intended to have a subgroup  $\mathbb{Z}_2^f \subseteq G^f$  identified with the fermionic parity.
- For a given group homomorphism

$$f : G \rightarrow G'$$

we will denote by  $f^*$ ,  $\hat{f}_*$  and  $\hat{f}^*$  the induced pullbacks and push-forward

$$f^* : RO(G') \rightarrow RO(G),$$

$$\hat{f}_* : \Omega_d^{\text{Spin}}(BG) \rightarrow \Omega_d^{\text{Spin}}(BG'),$$

$$\hat{f}^* : \text{Hom}(\Omega_d^{\text{Spin}}(BG'), U(1)) \rightarrow \text{Hom}(\Omega_d^{\text{Spin}}(BG), U(1)).$$

# Chapter 2

## Global anomaly inflow

Symmetries detain a fundamental role in our comprehension of quantum systems: they provide an organizational principle under which it is possible to infer and explain various phenomena. Therefore, understanding all the possible information that they yield is an important task. One of their particular features, which will be the object of this thesis, is that symmetries can be *anomalous*.

By definition, we consider a group  $G$  to be a global symmetry of a Quantum Field Theory (QFT) if its operators transform via a (anti)unitary representation  $U_g$ , such that any correlation function is kept invariant

$$\langle \mathcal{O}(x) \cdots \rangle = \langle U_g[\mathcal{O}(x)] \cdots \rangle, \quad \forall g \in G. \quad (2.1)$$

For a Lagrangian theory with a field content  $\{\Phi_i\}_{i \in \mathcal{I}}$ , this would correspond to the invariance of the action under  $G$

$$S = \int d^d x \mathcal{L}(\Phi_i) \stackrel{!}{=} \int d^d x \mathcal{L}(U_g[\Phi_i]), \quad \forall g \in G. \quad (2.2)$$

Thanks to Noether's theorem, at the classical level any such continuous symmetry group provides also a conserved current<sup>1</sup>  $j = j_\mu^a T^a dx^\mu$  that must satisfy

$$D \star j = 0. \quad (2.4)$$

Naively, one would guess that (2.4) should hold also at the quantum level and be in correspondence to (2.1), as they both arise from the same symmetry invariance. However, it turns out that this is not always the case, as (2.4) (and with it, also the associated Ward identities) can be broken at the quantum level if the gauge field

---

<sup>1</sup>Here  $T^a$  denote the generators of the Lie algebra  $\mathfrak{g}$  of  $G$ , while  $D := d + [A, \cdot]$ . At this point the gauge field  $A$  is treated as a *background* gauge field and, if not present, the equation would simply be

$$d \star j = 0. \quad (2.3)$$

Note that having the covariant conservation of the current does *not* rely on  $A$  being dynamical.

corresponding to the symmetry is turned on. In this case the expression will be of the more general form

$$\langle D \star j \rangle = \mathcal{I}(A). \quad (2.5)$$

Such equation signal a particular property of the theory that is being analyzed, namely it cannot be invariant under gauge transformations of the background field  $A$ . Therefore, if there is no renormalization scheme under which one is able to cure the expression and reabsorb  $\mathcal{I}(A)$  by adding some proper counterterms, the theory is said to exhibit a perturbative 't Hooft anomaly. Although this may seem problematic, note that even in this case  $G$  is a genuine symmetry and the original theory is not pathological *per se*, since its correlation functions are still symmetry-preserving. Indeed, the conservation of the current is modified only when turning on a background gauge field  $A$ . The only novelty that (2.5) implies is that in such instance  $G$  cannot be consistently gauged, as the gauge invariance of the - potentially dynamical - gauge field  $A$  would be lost.

The (non-)conservation of currents we just presented is related to *perturbative* anomalies, first discovered by the computations of Adler, Bell, and Jackiw (ABJ) [22, 23]. Since then, the study of anomalies has been proven of fundamental importance to study non-perturbative effects in QFTs, see as a partial list of examples [24–36] and references therein. Indeed, one of the main features of anomalies is that they are robust under symmetry-preserving deformations of the theory and thus so must be all the statements that follow by their existence. In particular, they must be preserved under the renormalization group (RG) flow, offering one of the very few exact tools we have at our disposal to probe the connection between UV and IR physics.

The focus of our work will be mainly on *global* anomalies, i.e. anomalies that are connected to the global structure of the symmetry group  $G$ . In this case there is no Noether current of which the conservation can be used to probe the presence of the anomaly itself. Instead, its presence must be signaled by a more comprehensive analysis of the problem from the mathematical point of view. Summarizing this view for the reader will be the scope of this first Chapter.

## 2.1 General anomalies framework

In order to better introduce anomalies, it is best to start by considering the behaviour of fermions, which we know admit a Lagrangian description, under symmetry transformations. This will provide a solid framework to describe all kinds of anomalies and later introduce the concept of *anomaly inflow*.

Therefore, suppose we consider a set of complex massless fermions  $\psi_i$  that transform under a representation  $\mathbf{R}$  of a symmetry group  $G$  and let us turn on the background gauge field  $A$  associated to it. Then, if it is possible to decouple their action

from that of the other fields  $\Phi_i$ , we can write

$$S = S_\psi + S_\Phi, \quad S_\psi = \sum_i i \int_X \bar{\psi}_i \not{D} \psi_i, \quad (2.6)$$

where  $\not{D} = \gamma^\mu D_\mu$ , allowing to rewrite their contribution to the (Euclidean) partition function on a manifold  $X$  as

$$Z[X, A] = e^{-W[X, A]} = \int \mathcal{D}\bar{\psi}_i \mathcal{D}\psi_i e^{-S_\psi(X, \psi_i, A)}. \quad (2.7)$$

Therefore, being this action quadratic, its evaluation is straightforward and for Dirac fermions leads to the formal expression

$$Z[X, A] = \det(i\not{D}) = \prod_k \lambda_k. \quad (2.8)$$

Here  $\lambda_k$  are the eigenvalues of  $i\not{D}$ , which are real since this operator is self-adjoint. However, it is obvious that such expression needs to be regularized, so one must proceed to handle it with care. Not only one has to find a suitable regularization to make sense of (2.8), but worse, if we do not limit ourselves to work with Dirac fermions, then the operator  $i\not{D}$  is not always an endomorphism, as it happens when fermions with different chiralities are organized into different representations of  $G$ . The standard example is by considering Weyl fermions in  $d = 2n$ , where  $i\not{D}$  splits into the two operators

$$i\not{D}^{+-} : H_+(X) \rightarrow H_-(X), \quad i\not{D}^{-+} : H_-(X) \rightarrow H_+(X), \quad (2.9)$$

adjoint to each other. Here  $H_\pm(X)$  denote the (infinite dimensional) functional space of sections  $H_\pm(X) := \Gamma(S_\pm \otimes E)$ , where  $S_\pm$  denotes the chirality split of the spinor bundle  $S = S_+ \oplus S_-$  on  $X$  and  $E$  is the vector bundle associated to the principal  $G$ -bundle of the symmetry via representation  $\mathbf{R} = \mathbf{R}_+ \oplus \mathbf{R}_-$ . In other words,  $H_\pm(X)$  is the space of Weyl fermions that transform under  $\mathbf{R}$  with positive (negative) chirality, i.e. such that  $\gamma_{2n+1}\psi = \pm\psi$ , and the operators (2.9) change such chirality.

From this point of view, one sees that in general it may be problematic to define an analogue expression to (2.8) and it may not be possible to define the partition function as a well behaved  $\mathbb{C}$ -valued function of the background gauge field. For example, it is tempting to guess that the partition function for a single Weyl fermion of positive chirality is formally

$$Z[X, A] = \det(i\not{D}^{+-}). \quad (2.10)$$

However, this clearly does not make sense per se and one should find a correct way to interpret it. If we happen to be in such an instance, it is possible that in the process of regularizing and making sense of the expression we are dealing with, one is not able to conserve also its original invariance under gauge transformations of  $A$ .

This in return implies that, as we already anticipated, even in the case we are able to properly regularize it, the symmetry  $G$  cannot be consistently gauged. For instance, one may try to make sense of (2.10) by making use of the fact that  $H_{\pm}(X)$  have a natural hermitian inner product<sup>2</sup> [37], which allows to have a well-defined absolute value

$$\left| \det \left( i\mathcal{D}^{+-} \right) \right| = \sqrt{\det \left[ \left( i\mathcal{D}^{-+} \right) \left( i\mathcal{D}^{+-} \right) \right]} = \sqrt{\det \left( i\mathcal{D} \right)}. \quad (2.11)$$

The next step would be to identify (2.10) with (2.11), which indeed is gauge invariant. However, this does not produce a smooth function in the space of configurations of the background gauge field  $A$ , as it behaves like a square root function  $|w| = \sqrt{w^2}$  around the field configurations  $A_0$  for which there is a zero eigenvalue [6, 37]. Therefore, as we will see in the next sections, one needs more effort to regularize (2.10) and the final price needed to pay to achieve it is indeed a potential loss of gauge invariance. Let us mention that also the more innocent expression (2.8) may be affected in principle by similar problems, though they are in general more subtle, as they are purely associated to the regularization of the product of eigenvalues and to global anomalies.

The phenomena we just explained is precisely what always happens in the presence of an anomaly and therefore it is taken as its defining feature [37–40].

**Definition 2.12.** *We say that a theory is anomalous if, given the collection of its background field  $\mathcal{F}(X)$  over  $X$ , its partition function defines a section of a non-trivial complex line bundle  $\mathfrak{D}(X)$ , usually called determinant line bundle,*

$$Z[X, \mathcal{F}] \in \Gamma[\mathfrak{D}(X)], \quad \mathbb{C} \rightarrow \mathfrak{D}(X) \rightarrow \mathcal{F}(X). \quad (2.13)$$

*Analogously,  $Z$  will be physically well-defined and gauge-invariant if such bundle is trivial, in which case the theory is said to be anomaly-free.*

Back to the free fermions example, we have  $\mathcal{F}(X) = \{A\}$  and the non-trivial bundle condition is reflected by the fact that for some gauge transformation  $A \rightarrow A^g$ , we have

$$Z[X, A] \neq Z[X, A^g]. \quad (2.14)$$

Along our effort to try to define a proper way to regularize anomalous partition functions, let us note that, if a set of fermions admits a  $G$ -preserving mass deformation to the theory, i.e. a nondegenerate skew-symmetric bilinear form  $m : S \otimes S \rightarrow \mathbb{R}$  such that the action term is  $G$ -invariant

$$S_{\text{mass}} \equiv \int_X m(\psi_1, \psi_2) = \int_X m(g \cdot \psi_1, g \cdot \psi_2), \quad \forall g \in G, \quad (2.15)$$

then their theory is always anomaly-free. Indeed, in this case it is possible to prove that the theory can be regularized in a gauge invariant way via Pauli-Villars regu-

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<sup>2</sup>The inner product between two fermions  $\psi_{1,2} \in H(X)$  is defined by  $(\psi_1, \psi_2)_X := \int_X \bar{\psi}_1 \psi_2$ .

larization<sup>3</sup>, for details see [6]. In other words, such regularization guarantees that in that case the bundle  $\mathfrak{D}(X)$  is trivial. Moreover, this also means that the modulus of  $Z$  is always gauge-invariant. Consider for example a set of fermions  $\psi$  and take a conjugate set  $\chi$ , transforming under any symmetries as the complex conjugate of  $\psi$ . The set  $\psi \oplus \chi$  will then admit such a  $G$ -equivariant mass deformation (2.15) and its partition function, which will not be anomalous, will be

$$Z_\psi[X, A]Z_\chi[X, A] = Z_\psi[X, A]\bar{Z}_\psi[X, A] = |Z_\psi[X, A]|^2. \quad (2.16)$$

From this, the same anomaly-free conclusion must apply also for  $|Z_\psi[X, A]|$ . Indeed, this modulus corresponds exactly to our first guess of the total partition function (2.11) in the case of a single Weyl fermion  $\psi_+$ : its gauge invariance is precisely why it was so appealing in the first place. Note also that its problem of not being smooth is not present for the squared modulus, as one should correctly expect by describing the partition function of a single Dirac fermion  $\psi = \psi_+ \oplus \psi_-$ .

Therefore, the general lesson one may understand from this observation is that the source of the anomaly is deeply connected with the phase that partition functions exhibit after regularization. As we will see, both perturbative and global anomalies can be traced back to this ambiguity, which is expected to hold in complete generality for any kind of anomaly. A very loose explanation of this can be found by noticing that turning on a background gauge field is equivalent to coupling the theory to a principal  $G$ -bundle with a non-trivial connection  $A \neq 0$ . As a consequence, 't Hooft anomalies should depend on the topology of the system, as they are not visible in presence of a trivial bundle. Moreover, terms in the action which are sensible to the topology of the manifold  $X$  are mostly imaginary, as they arise as pullbacks of  $d$ -forms from some appropriate target spaces into the spacetime  $X$ . Therefore, they will be generally proportional to the Levi-Civita tensor  $\epsilon^{\mu_1 \dots \mu_d}$ . Because of its transformation properties, this ultimately means that under Wick rotation from Minkowski to Euclidean signature their total contribution to the effective action will be indeed purely imaginary and affect only the complex phase of  $Z$ . Thus, it is no surprise that 't Hooft anomalies should make their appearance only in the definition of the phase of  $Z$  too.

### 2.1.1 Perturbative case

Based on our discussion, at the perturbative level the presence of a 't Hooft anomaly must be signaled by the change of the partition function under symmetry transformation  $A \rightarrow A^\varrho = A + \delta_\lambda A$  such that

$$Z[X, A + \delta_\lambda A] = Z[X, A] \exp \left( -2\pi i \int_X \mathcal{I}_d(A, \lambda) \right). \quad (2.17)$$

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<sup>3</sup>Note that Pauli-Villars regularization is always available, but it is only in the case when fermions have a bare mass that it is able to provide a good definition of the partition function. If this is not possible, such regularization can be shown to well-define only the absolute value of it.

The source of the additional term  $\mathcal{I}_d(A, \lambda)$  is related to the non-invariance of the path integral measure under symmetry transformation, as it is possible to see via Fujikawa's method for fermionic fields [41]. Notably, the polynomial  $\mathcal{I}_d(A, \lambda)$  turns out to be always real, in agreement with what we already said. Note also how this term is what defines the non-conservation of the Noether current (2.5) as well, as it can be immediately seen by recalling that for  $\delta_\lambda A = D\lambda$

$$\delta_\lambda W[A] = \int_X \delta_\lambda A \wedge \frac{\delta W[A]}{\delta A} = - \int_X \lambda \langle D \star j \rangle \quad (2.18)$$

and, at the same time, via (2.17)

$$\delta_\lambda W[A] = 2\pi i \int_X \mathcal{I}_d(A, \lambda). \quad (2.19)$$

Perturbative 't Hooft anomalies can be analyzed in different ways. The first more traditional approach is to figure out their expression by computing  $(d/2 + 1)$ -polygon one loop diagrams, which indeed contain all the possible information about them [42, 43]. Alternatively, it is possible to uncover them via the Wess-Zumino descent equation [44, 45], which relates  $\mathcal{I}_d(A, \lambda)$  to a  $d + 2$  dimensional polynomial  $\Phi_{d+2}$ , function of the background fields of the theory. The relation found through this procedure is

$$d\mathcal{I}_d(A, \lambda) = I_{d+1}(A, \lambda), \quad I_{d+1}(A, \lambda) = \delta_\lambda \mathcal{I}_{d+1}(A), \quad d\mathcal{I}_{d+1}(A) = \Phi_{d+2}. \quad (2.20)$$

Notably, the  $\mathcal{I}_{d+1}(A)$  polynomial that appears in this cascade of polynomials is the Chern-Simons action of  $A$  in  $d + 1$  dimensions, while  $\Phi_{d+2}$  is the index density in the Atiyah-Singer index theorem [46–48]. Another important detail is that  $\mathcal{I}_{d+1}$  and  $\Phi_{d+2}$  are defined over manifolds  $Y$  and  $W$  of higher dimension, equal to their degree. The relation between these manifolds with our initial  $X$  will be elucidated in the next sections.

For ease of exposition, the examples we focused on so far regard perturbative anomalies of gauge fields. However, one may consider also *gravitational* anomalies, which are related to the behaviour of the theory under change of metric. Indeed, also such transformations (or even their mixing with gauge transformations) may be a priori anomalous, so one should account them. Remarkably, the descent equation takes care of these as well and the most general expression of  $\Phi_{d+2}$  will have a dependence on the metric of the theory  $g$ , expressed in terms of the Riemann tensor  $R$ . More precisely,  $\Phi_{d+2}$  identifies a cohomology element in  $H^{d+2}(W; \mathbb{Q})$  written as the truncation of a closed, gauge-invariant polynomial of the background gauge fields  $(A, g)$ . Therefore, it is a linear combination of characteristic classes associated to  $TW$  and the principal bundle  $P \rightarrow W$  described by  $A$ . All in all, the full expression of this polynomial is

$$\Phi_{d+2} = \hat{A}(R) \operatorname{tr}_{\mathbf{R}} \exp \left( \frac{F}{2\pi} \right) \Big|_{d+2}, \quad (2.21)$$

where  $\hat{A}(R)$  denotes the Dirac genus that describes the aforementioned gravitational contribution, written in terms of Potryagin classes  $p_i(TW) \in H^{4i}(W; \mathbb{Z})$ , and the second term is the Chern character, written in term of Chern classes  $c_i(P) \in H^{2i}(W; \mathbb{Z})$ .

At this point, given the relations (2.20), in the presence of solely perturbative anomalies one may be able to find a proper definition of anomalous partition functions. In the light of the fact that their modulus is always anomaly-free, it is tempting to define

$$Z[X, A] = |Z[X, A]| \exp \left( -2\pi i \int_Y \mathcal{I}_{d+1}(A) \right), \quad (2.22)$$

where the fields of the theory in  $X$  have been properly extended into<sup>4</sup> the  $d + 1$  manifold  $Y$ , such that  $X = \partial Y$ .

**Example 2.23.** A standard example of perturbative anomalies is given by a set of  $n$  Weyl fermion  $\psi_+^i$  with  $G = U(1)$  symmetry in  $d = 4$ . In this case the non-conservation of the  $U(1)$  current signaling a 't Hooft anomaly is given by

$$d \star j = \frac{1}{4\pi} F \wedge F. \quad (2.24)$$

Moreover, following (2.20) one uncovers the presence of a mixed gravitational and gauge anomaly as well. Indeed

$$\Phi_6 = \frac{1}{6(2\pi)^3} \text{tr} (F \wedge F \wedge F) - \frac{1}{6(4\pi)^3} \text{tr} (F) \text{tr} (R \wedge R). \quad (2.25)$$

Therefore the general 5-dimensional polynomial is

$$\mathcal{I}_5(A) = \frac{1}{8\pi^3} \text{tr} (A \wedge F \wedge F) - \frac{1}{6(4\pi)^3} \text{tr} (F) \text{tr} \left( \omega d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right), \quad (2.26)$$

where  $\omega$  is the standard spin-connection associated with the 5-dimensional metric. By looking only at gauge transformations, (2.24) follows immediately. Finally, *if* the theory does not have non-perturbative anomalies, one could assume as a regularized version of the partition function the expression (2.22) with the proper term (2.26), where  $\partial Y = X$  as discussed. Indeed, we anticipate here that for  $G = U(1)$  there are no additional anomalies<sup>5</sup> and this example provides the first important expression of the idea of anomaly inflow, first expressed for  $d = 2$  in [49], that is that the presence of an anomaly captures the dependence of the theory on an higher dimensional extension, from which such anomaly “flows”.

<sup>4</sup>It is not always guaranteed that such a manifold exists. Indeed, at this level (2.22) is still a proposal. Such issue will be addressed later on in Section 2.2.

<sup>5</sup>As an anticipation for the next sections, this can be seen by the fact that  $\text{Hom}(\Omega_5^{\text{Spin}}(BU(1)), U(1)) = 0$ .

### 2.1.2 Non-perturbative case

Non-perturbative anomalies are associated to transformations of the background field not connected to the identity. Therefore, they are heavily dependent on the topology of the symmetry group  $G$  that pertains the theory under study. For this reason, we will also call these *global* anomalies.

For this kind of anomalies one cannot rely on perturbative expansions and the more classical tools fail to provide an answer. Therefore, to access them one needs to develop other, new methods, which are sensible to the global structure of  $G$  as well. The traditional method to deal with non-perturbative anomalies is based on looking at particular  $d + 1$  closed manifolds over which we extend our theory; we recall the basic idea in the following. In particular, the original study of global anomalies started by focusing on the presence of non-trivial homotopy groups  $\pi_n(G)$ . Albeit it is by now clear that this is not the fully correct way to proceed, we will follow such a historical approach to better introduce the overall picture, explained in the next section and which relies on the concept of anomaly inflow and cobordism groups.

Suppose now that we want to analyze the behaviour of a theory which is perturbatively anomaly-free, so  $\Phi_{d+2} = 0$ . The next step is to try to figure out whether gauge transformations not connected to the identity exist in the first place. If not, obviously the theory will not be anomalous. Instead, if these are present, one must make sure that they are free of anomalies as well. To do so, we consider the theory to be defined on the 1-point compactification  $S^d$  of the  $\mathbb{R}^d$  spacetime, by the standard argument that the fields have to vanish sufficiently fast at infinity,  $|x| \rightarrow \infty$ . The possibly non-trivial configurations of the connection  $A$  associated to the symmetry group  $G$  are found by checking whether  $\pi_d(G) \neq 0$ . Indeed, if this is the case, there can indeed be a gauge transformations  $A \rightarrow A^e = \varrho(A + id)\varrho^{-1}$  such that  $[\varrho] \neq [\mathbb{I}]$  in  $\pi_d(G)$  and that cannot be perturbatively reproduced on the spacetime of our choice. For these sets of transformations in principle we may have

$$Z[S^d, A^e] = Z[S^d, A]e^{2\pi i\alpha_e[S^d, A]}. \quad (2.27)$$

One of the first ideas to figure out the potential existence of non-perturbative anomalies is to try make use of our knowledge of the perturbative ones for another  $\hat{G}$ . In particular, we could suppose that the symmetry group  $G$  we are interested in can be embedded into a bigger  $\hat{G}$ ,  $G \subset \hat{G}$ , with  $\pi_d(\hat{G}) = 0$ . At this point it is clear that any potential global anomaly from  $G$  must come from a perturbative one of  $\hat{G}$ . To implement this idea, we recall the original procedure described by [50]; for a recent review see also [51, 52].

For any  $G \subset \hat{G}$  we have the short exact sequence

$$1 \longrightarrow G \xrightarrow{\iota} \hat{G} \xrightarrow{p} \hat{G}/G \longrightarrow 1, \quad (2.28)$$

which induces the long exact sequence

$$\dots \xrightarrow{\iota_*} \pi_{d+1}(\hat{G}) \xrightarrow{p_*} \pi_{d+1}(\hat{G}/G) \xrightarrow{\partial} \pi_d(G) \xrightarrow{\iota_*} \underbrace{\pi_d(\hat{G})}_{=0} \longrightarrow \dots \quad (2.29)$$

For simplicity, let's describe the case when  $\pi_d(G) = \mathbb{Z}_k$  and  $\pi_{d+1}(\hat{G}) = \pi_{d+1}(\hat{G}/G) = \mathbb{Z}$ . For instance, we happen to be in such situation when  $d = 2n$  and  $G = SU(n)$ ,  $\hat{G} = SU(n+1)$ , like for the famous  $SU(2)$  Witten's anomaly [4].

At this point, one can work out the potential anomaly by building the *mapping sphere*  $S_\varrho^{d+1}$  corresponding to a non-trivial  $[\varrho] \in \pi_d(G) = \mathbb{Z}_k$ . Suppose we consider  $A$  defining a trivial  $G$  bundle on  $S^d$ . Since the bundle is trivial, we can extend our theory to a disk  $D^{d+1}$  with no problem. Moreover, the same can be done if one were to start with  $A^e$ , getting a theory with a possibly different extension on a disk, labelled as  $D_\varrho^{d+1}$ . At this point we can join the two extended theory to get a theory defined on<sup>6</sup>  $S_\varrho^{d+1} := D^{d+1} \sqcup \bar{D}_\varrho^{d+1}$ , where our original theory will lie in the equator  $S^d$ . We call the associated  $G$  bundle  $P_{[\varrho]}$ , since its equivalence class is classified by  $[\varrho]$ . Indeed, recall that for any compact Lie group we have the natural isomorphism

$$\pi_d(G) \cong \pi_{d+1}(BG). \quad (2.30)$$

Here  $BG$  is the classifying space associated to  $G$ , which has the property that maps  $f : X \rightarrow BG$  define up to homotopy the equivalence classes of principal  $G$  bundles over  $X$ . Therefore, the various equivalence classes of bundles in  $S^{d+1}$  are properly captured by  $[\varrho]$ . Then, in virtue of the hypothesis that  $\pi_d(\hat{G}) = \pi_{d+1}(B\hat{G}) = 0$ , we can also view  $P_{[\varrho]}$  as a  $\hat{G}$  bundle and then extend it to a  $\hat{G}$ -bundle on  $D^{d+2}$ . Such extension is now classified by the elements  $[\xi] \in \pi_{d+1}(\hat{G}/G)$  such that  $\partial[\xi] = [\varrho]$ , so we denote the new bundle as  $\hat{P}_{[\xi]}$ . The anomaly of our original theory can then be seen as an anomaly of  $\hat{G}$  and thus will arise from its  $d+2$  anomaly polynomial  $\hat{\Phi}_{d+2}$  for  $\hat{G}$  evaluated on  $D^{d+2}$ . Moreover, in virtue of the setup we are working with, this can be seen as a map

$$\left( \int_{D^{d+2}} \hat{\Phi}_{d+2}(\hat{P}_{[\xi]}) \right) : \pi_{d+1}(\hat{G}/G) \rightarrow \mathbb{R}, \quad (2.31)$$

since  $\hat{\Phi}_{d+2}$  is a closed form that by hypothesis must vanish once restricted to  $G$ . Here we made explicit that the integration is carried out when the field strength appearing in  $\hat{\Phi}_{d+2}$  describes  $\hat{P}_{[\xi]}$ . From this, the final anomaly  $G$  is simply given by the evaluation of (2.31) for the element  $[\xi] \in \pi_{d+1}(\hat{G}/G)$ . One can do more: indeed, notice that  $\partial(k[\xi]) = 0$ , so the  $k$ -th multiple of  $[\xi]$ , i.e.  $k[\xi]$ , must be the image of some element, namely  $k[\xi] = p_*([\hat{\varrho}])$ , with  $[\hat{\varrho}] \in \pi_{d+1}(\hat{G})$ . This means that a trivial  $\hat{G}$  bundle on  $D^{d+2}$  can be analogously merged with the bundle associated to  $[\hat{\varrho}]$  to get a  $\hat{G}$  theory defined over the mapping sphere  $S_\varrho^{d+2}$ , which will produce an anomaly equal to

$$\int_{S_\varrho^{d+2}} \hat{\Phi}_{d+2} = \int_{D^{d+2}} \hat{\Phi}_{d+2}(\hat{P}_{k[\xi]}). \quad (2.32)$$

This quantity is often easily computable and much easier to deal with than (2.31). Not only that, but (2.32) is also connected to the original global anomaly for  $G$ , as

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<sup>6</sup>Here  $\bar{D}_\varrho^{d+1}$  denotes a generalization of the orientation reversal of the manifold  $D_\varrho^{d+1}$ , which we will explain in Section 3.4.

this must be its  $k$ -root and thus is

$$\exp(2\pi i \alpha_\varrho[A, S^d]) = \exp\left(\frac{2\pi i}{k} \int_{S^{d+2}} \hat{\Phi}_{d+2}(\hat{P}_{k[\xi]})\right). \quad (2.33)$$

The method of evaluation of the global anomalies we just described has some drawbacks, but it is very useful to illustrate one concept, namely that it is possible to have an *anomaly interplay* for various symmetry groups, which may mix perturbative and non-perturbative anomalies. This will be an important consideration to take into account, as it will be useful for our analysis in the later chapters.

At this point, one starts to understand that in order to access global anomalies, it is necessary to associate some non-trivial topological setup to the potentially interesting gauge transformations disconnected from identity. This will be the route that we try to push further. In particular, the immediate extension of the idea explained above should eventually be able to provide a recipe that it is possible to use for any non-trivial gauge transformation and symmetry group  $G$ , without being reliant on finding a proper embedding of this in some bigger group  $\hat{G}$  with particular topology.

The fundamental observation necessary to understand how to provide such a generalized description of global anomalies is that, in principle, these should be manifest by checking the behaviour of the partition function along an interpolating (connected) path in the configuration space of the fields between two gauge equivalent field configurations, e.g.  $A$  and  $A^\varrho$  [5]. Note that, since  $A$  and  $A^\varrho$  cannot be continuously connected only through gauge equivalent configurations, one has to necessarily cross other inequivalent field configurations to do so. However, if we are not in the presence of an anomaly, since the initial and final configurations are gauge equivalent, the expectation is that the partition function should be identical in the two instances. If this is the case, we can infer that the theory is anomaly-free, at least for what regards the non-connected gauge transformation considered. To label the theory completely anomaly-free, such check should be done for all the different non-trivial equivalence classes  $[\varrho] \in \pi_d(G)$ .

The standard technique to achieve what we have described is to consider the interpolation

$$A^t := (1-t)A + tA^\varrho, \quad t \in I := [0, 1]. \quad (2.34)$$

With a similar philosophy used for the mapping spheres, one may consider the field (2.34) as the extension of  $A$  in the  $d+1$ -manifold  $(S^d \times I)_\varrho$ . Manifolds of the generic expression  $(X \times I)_\varrho$ , where the two boundaries describe the same theory up to a gauge transformation and/or diffeomorphism  $\varrho$  are called *mapping cylinders* and will be useful also later on. These manifolds are topologically equivalent to the product  $X \times I$ , with the only difference that at  $t=1$  all the relevant fields  $\{\mathcal{F}\}$  are transformed by the action of  $\varrho$ . Since the boundaries of  $(X \times I)_\varrho$  are physically equivalent, one can also close the mapping cylinder and get the manifold  $(X \times S^1)_\varrho$ , which is formally defined as

$$(X \times S^1)_\varrho := (X \times I) / \sim_\varrho, \quad \sim_\varrho: \begin{cases} (x, 0) \sim (\varrho(x), 1), \\ \mathcal{F}(x, 0) \sim \mathcal{F}^\varrho(x, 1). \end{cases} \quad (2.35)$$

This kind of manifold is instead known as the *mapping torus* associated to the transformation<sup>7</sup>  $\varrho$ .

With this construction at hand, it is possible to show that the anomaly must describe a topological invariant of the associated mapping torus [5, 6]. Note also that due to the generality of the construction, it is not necessary to restrict ourselves to the description of theories on  $S^d$  anymore, but we can consider them on a general  $X$ . In this case the caveat is that non-perturbative gauge transformations will not be labeled by  $\pi_d(G)$  anymore, but rather by the homotopy classes of maps  $\varrho : X \rightarrow G$ . Moreover, notice that now  $\varrho$  can also incorporate a large diffeomorphism of  $X$ . With this premise, the general expression for the anomaly associated to such a  $\varrho$  should be equal to

$$\exp(2\pi i \alpha_\varrho[X, A]) = \frac{Z[X, A^\varrho]}{Z[X, A]} = \exp\left(-\int_0^1 dt \frac{d}{dt} W[X, A^t]\right), \quad (2.36)$$

where  $W[X, A]$  is the standard effective action of the theory on  $X$ , already defined in (2.7). Breaking down (2.36) is complicated in general, but it is possible to show that, in the usual case of massless chiral fermions where one can reduce themself to the case where the metric of  $(X \times S^1)_\varrho$  has an adiabatic dependence on  $t$  and thus allow to apply the adiabatic approximation in quantum mechanics [5], this expression turns out to be equal to

$$2\pi i \alpha_\varrho[A, X] = i\pi \eta[(X \times S^1)_\varrho]. \quad (2.37)$$

Here  $\eta$  is the Atiyah-Patodi-Singer (APS)  $\eta$ -invariant [46, 53] and is the topological invariant we were talking about. This is defined as the regularized sum of the signs of the eigenvalues  $\lambda_k$  of the Dirac operator  $i\mathcal{D}$  on  $(X \times S^1)_\varrho$ . One possible choice of regularization is given by

$$\eta := \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2} \sum_k e^{-\varepsilon |\lambda_k|} \text{sign } \lambda_k, \quad (2.38)$$

where the zero modes  $\lambda_k = 0$  are treated conventionally by  $\text{sign}(0) = 1$ .

To understand when (2.37) may be non-zero, even in the absence of zero-modes, recall that generally partition functions of massless fermions will be a regularized product of eigenvalues  $\lambda_k$ . For example, for a single Weyl fermion in  $d = 2n$  the partition function is of the form

$$Z[X, A] = \sqrt{\det(i\mathcal{D})} = \prod_k \widetilde{\lambda}_k, \quad (2.39)$$

where  $\widetilde{\prod}$  means that we take only half of the original eigenvalues  $\lambda_k$  of the Dirac operator. However, being this adjoint, the  $\lambda_k$  are real and always present in couples  $(\lambda_k, -\lambda_k)$ , as one can see by the fact that if  $i\mathcal{D}\psi_k = \lambda_k\psi_k$ , then  $\gamma_{2n+1}\psi_k = \psi_{-k}$ .

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<sup>7</sup>Note that these kinds of constructions can be defined equally well also for perturbative transformations.

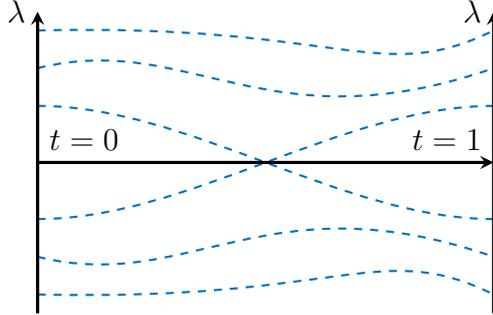


Figure 2.1: Example of a spectral flow of the eigenvalues of  $i\mathcal{D}$  along the path (2.34). The gauge equivalent field configurations  $A$  and  $A^e$  yield the same eigenvalues at  $t = 0, 1$ . However, there may be a net flow along the path on inequivalent configurations corresponding to  $A^t$  for  $0 < t < 1$ .

Therefore, the definition of (2.39) for a particular background  $A$  may be set by choosing a particular representative  $\lambda_k$  out of each couple  $(\lambda_k, -\lambda_k)$ . Suppose for instance that there are no zero modes and that we choose always the eigenvalues  $\lambda_k > 0$ . The partition function is at this point perfectly well defined for  $A$ . However, the ambiguity associated to a potential presence of an anomaly is now translated as the possibility that the partition function changes sign for gauge equivalent configurations  $A^e$ . As a confirmation, let's now look at the interpolating paths (2.34). Albeit the eigenvalues of  $i\mathcal{D}$  are unchanged at  $t = 0, 1$ , it may indeed happen that along the path there is a non-zero flow of eigenvalues which will change the total phase of  $Z$ , see Figure 2.1. If this is the case, the choice of regularization we have chosen is not consistent if one wants it to be applied to *all* the gauge-equivalent configurations  $A^e$  while trying to get at the same time a smooth function. Therefore, the gauge invariance must be dropped to account for these changes of signs in order to get a proper smooth regularization and the theory is anomalous.

Note that formula (2.37) is quite general and in principle our analysis suggests that it is sufficient to compute it for any mapping tori  $(X \times S^1)_e$  to label the theory anomalous or not. However, there are still some shortcomings which one would like to be able to address with a more comprehensive analysis. For example, in the following we have always supposed to deal with continuous Lie groups, but global anomalies may be present also for finite symmetries. In this case one is not able to continuously interpolate between various field configurations and standard homotopy groups are of no help anymore, since  $\pi_{d>1}(G) = 0$ . Moreover, for simplicity we always discussed global anomalies in absence of perturbative ones. However, it is legitimate to ask if, by any chance, there is some particular splitting between the two such that even asking about global anomalies in presence of perturbative ones makes sense in the first place. Another important problem is that, while (2.37) is very useful in the case of massless fermions, we still have no idea about the nature of the topological invariants that should describe anomalies for more generic theories. In the case of perturbative anomalies, instead, we have been able to extract the general anomaly

equations (2.20). One may wonder if something similar can be done for the global case. We will answer these questions in the following section.

## 2.2 Anomaly inflow

So far by presenting the standard techniques used to understand anomalies we noticed a striking connection between them and extensions of the interesting theory in one (or even two) dimension higher. This is by no means a coincidence and, in order to uncover the full picture, we will once again make use of the description of massless fermions as a way to extract general informations on the anomaly. As a result, we learn that anomalous theories can be described in terms of *anomaly inflow*, namely they describe a consistent theory once they are seen as the description of boundary degrees of freedom of some higher dimensional theory. This profound statement will be crucial in our work, as it will help tremendously in the classification and analysis of anomalies, by translating such problems into the classification of the correspondent higher dimensional bulk descriptions.

### 2.2.1 The $\eta$ invariant

Let's try to find a correct regularization of the partition function for sets of massless chiral fermions in  $d$  dimensions by proceeding in a sort of counterintuitive way: we look at the behaviour of a massive Dirac fermion  $\Psi$  in a  $d + 1$  manifold  $Y$ . Per usual, its action is of the form

$$S = \int_Y \bar{\Psi} i(\not{D} + m)\Psi. \quad (2.40)$$

Now, let's try to make a connection between this and a description of some  $d$  dimensional degrees of freedom. To do so, we suppose that  $Y$  has a non-trivial boundary,  $\partial Y = X$ . Then, in order to get a partition function and sensible correlation functions out of it, boundary conditions must be specified. In an open set around the boundary, the manifold  $Y$  will be of the general form  $Y \approx (-\epsilon, 0] \times X$ , labeled by a set of local coordinates  $(\tau, x^i)$ . From this we choose as appropriate boundary condition for the fermion

$$(1 - \gamma^\tau)\Psi|_{\tau=0} = 0, \quad (2.41)$$

where  $\gamma^\tau$  is the gamma matrix corresponding to the normal direction to the boundary. Therefore, near it the action will be of the form

$$S \sim \int_{(-\epsilon, 0] \times X} \bar{\Psi} i\gamma^\tau \left( \frac{\partial}{\partial \tau} + \not{D}_d + \gamma^\tau m \right) \Psi, \quad (2.42)$$

where  $\not{D}_d = \gamma^\tau \gamma^i D_i$ . To fix our ideas now suppose that  $m < 0$ . By admitting the presence of a non-trivial boundary, what we find is that the  $d + 1$  fermion exhibits a mode  $\psi$  localized along it, which satisfies

$$\Psi = e^{|m|\tau} \psi_+, \quad (1 - \gamma^\tau)\psi_+ = 0, \quad i\not{D}_d \psi_+ = 0. \quad (2.43)$$

Two important properties characterize it: first, it is chiral<sup>8</sup> along  $X$ , and second, it is massless. We will skip the details of the proof, which can be found in [3], but for us it is enough to advocate that, based on these properties, in the limit of large mass  $|m| \gg 0$  the partition function of  $\Psi$  can equivalently be seen as the partition function that one would associate to the massless chiral mode  $\psi_+$ , once properly regularized through Pauli-Villars. This is the anomaly inflow statement for massless fermion theories: a partition function of a potentially anomalous set of fermions does not make sense *per se*, but rather as seen as the edge description of a higher dimensional theory with a non-trivial bulk description. Indeed, in this case the final result of the regularization procedure gives us as the final partition function

$$Z[X, A] = \left| \det \left( i\mathcal{D}_d^{+-} \right) \right| \exp(-2\pi i \eta[Y]), \quad (2.44)$$

which we notice to have a striking resemblance to the possible hypothesis that were discussed in the previous sections. Here  $\eta[Y]$  is the  $\eta$  invariant of the initial  $d + 1$  dimensional theory, already encountered in (2.37). However, note that in (2.44) is defined for a manifold with boundary. In this case, such invariant is intended to be defined by imposing APS boundary conditions [37, 46], given by<sup>9</sup>

$$\Psi_- = T\Psi_+, \quad (2.45)$$

where  $T$  is a unitary linear map

$$T : H_+(X) \rightarrow H_-(X). \quad (2.46)$$

Note also that (2.44) is smooth, as the discontinuity of the modulus in proximity of field configurations with a zero eigenvalues is compensated by the jump that  $\eta$  has as well. Let us finally mention that the procedure presented and formula (2.44) hold for any dimension  $d$ , be it even or odd. While for  $X$  even-dimensional the notion of chirality is well-known, for  $d = 2n + 1$  there is only one spinor representation of  $\text{Spin}(n)$  and an associated spin bundle  $\mathcal{S}$ . In this case, as already mentioned, the chirality notion needs to be understood in the sense that the spinor bundle  $S$  defined on  $Y$  splits on the boundary as  $S|_X = \mathcal{S} \oplus \mathcal{S}$ , i.e.  $S_\pm = \mathcal{S}$ , and one is able to reduce (2.44) to the standard determinant formula (2.8), since  $H_+(X) \cong H_-(X)$ . A different matter is when the  $\text{pin}^\pm$  case is discussed, where there are two inequivalent spin bundles, be them  $\mathcal{S}$  and  $\tilde{\mathcal{S}}$ , in which case the procedure gives  $S|_X = \mathcal{S} \oplus \tilde{\mathcal{S}}$  and one can discuss a more proper chirality notion as in the even dimensional case. For more details on this matter, see again [3, 6, 37].

By looking at (2.44) as the partition function associated to a chiral massless boundary fermion, notice that the presence of an anomaly acquire an important

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<sup>8</sup>Here with chirality we mean the standard notion in even dimension. In odd dimension this is not standard terminology. However, since  $\{\gamma^\tau, i\mathcal{D}_d\} = 0$ ,  $\gamma^\tau$  still defines a split of the spinorial bundle on the boundary,  $S|_X = S_+ \oplus S_-$ .

<sup>9</sup>Here  $\Psi_\pm$  are the chirality sectors of  $\Psi$  defined on the boundary, i.e. sections of  $S_\pm$ .

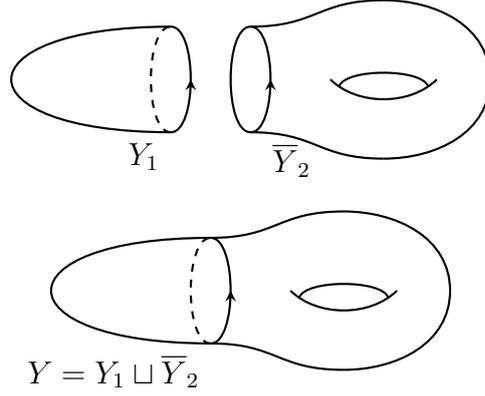


Figure 2.2: Once we have two manifolds  $Y_{1,2}$  with a common boundary we can glue them to get a single new manifold. In principle, the relation holds also if  $Y_{1,2}$  have more than one boundary term each,  $\partial Y_i = \bigsqcup_j X_j^i$ , and we choose to glue only a subset of those they have in common, e.g.  $X_1^1 = X_1^2$ .

meaning: it signals a hidden dependence (at least from the massless chiral fermion point of view) of the regularization on the manifold  $Y$  over which we decided to define the correspondent bulk. This can be immediately seen by means of the gluing property of the  $\eta$  invariant [53], which states that given two different manifolds  $Y_1$  and  $Y_2$  such that  $\partial Y_1 = \partial Y_2$ , it holds

$$\exp(-2\pi i\eta[Y_1]) \exp(-2\pi i\eta[\bar{Y}_2]) = \exp(-2\pi i\eta[Y_1 \sqcup \bar{Y}_2]). \quad (2.47)$$

Here  $\bar{Y}_2$  denotes the manifold  $Y_2$  equipped with the opposite spin structure than the one it is originally defined with. We will make this notion more precise in Section 3.4. For the time being, it is sufficient to note that this is the condition that allows us to glue  $Y_1$  and  $Y_2$  along their boundary to get the closed manifold  $Y := Y_1 \sqcup \bar{Y}_2$ , see Figure 2.2, and that the following identity holds

$$\exp(-2\pi i\eta[\bar{Y}_i]) = \exp(2\pi i\eta[Y_i]). \quad (2.48)$$

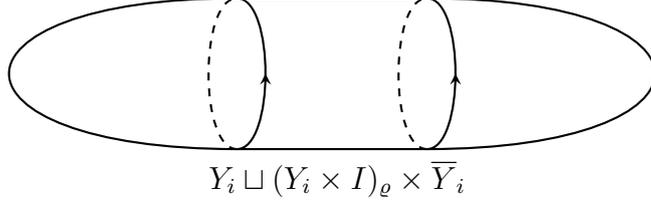
With (2.44) at hand we can now reinterpret the condition of a theory to be anomaly-free. Indeed, as we already suggested, the requirement is that (2.44) does not depend on the choice of extension  $Y_i$ , so that it can correctly be interpreted as the partition function of a theory on  $X$ . In other words

$$\exp(-2\pi i\eta[Y]) = 1 \quad (2.49)$$

for any closed manifold  $Y$ . If it were otherwise, we would have that

$$\frac{|Z[X, A]| \exp(2\pi i\eta[Y_1])}{|Z[X, A]| \exp(2\pi i\eta[Y_2])} = \exp(-2\pi i\eta[Y_1 \sqcup \bar{Y}_2]) \neq 1 \quad (2.50)$$

for  $\partial Y_1 = \partial Y_2 = X$ , expressing that the theory cannot be local and well-defined simply in terms of  $X$ , as its regularization would depend on some unphysical detail, i.e. the choice of  $Y_i$ .


 Figure 2.3: The closed manifold  $Y$  associated to perturbative anomalies.

We can also re-interpret what means for a theory to have perturbative and global anomalies. Let's focus on the first kind for example. Consider the case when a  $(d+1)$  manifold  $Y$  is a boundary  $Y = \partial W$ . In this case the APS index theorem tells us that the index of the Dirac operator  $i\mathcal{D}_W$  on  $W$  is given by

$$\text{Index}(i\mathcal{D}_W) = \int_Y \mathcal{I}_{d+1} - \eta[Y], \quad (2.51)$$

where we recall that  $\text{Index}(i\mathcal{D}_W) := \dim \ker i\mathcal{D}_W^{+-} - \dim \ker i\mathcal{D}_W^{-+}$  is always an integer. It follows that a perturbative gauge variation of the exponentiated  $\eta$  invariant is equivalent to the variation of the Chern-Simons term, so that one can substitute in this case

$$Z[X, A] = \left| \det \left( i\mathcal{D}_d^{+-} \right) \right| \exp \left( -2\pi i \int_{Y_i} \mathcal{I}_{d+1} \right) \quad (2.52)$$

for some  $Y_i$  and get the same expression for the regularized partition function (2.22) we tried to guess when dealing only with perturbative anomalies. Indeed, notice that a perturbative gauge transformation  $A \rightarrow A^\varrho$  is associated to the mapping cylinder  $(X \times I)_\varrho$ . Therefore the variation of the partition function can be captured by evaluating the  $\eta$  invariant on the closed manifold  $Y = Y_i \sqcup (Y_i \times I)_\varrho \sqcup \bar{Y}_i$ , see Figure 2.3, since

$$\frac{Z[X, A^\varrho]}{Z[X, A]} = \frac{\exp(-2\pi i \eta[Y_i \sqcup (Y_i \times I)_\varrho])}{\exp(2\pi i \eta[Y_i])} = \exp(-2\pi i \eta[Y]). \quad (2.53)$$

Equivalently, the same variation will be given by the evaluation of the Chern-Simons term, as by (2.17) we have

$$\exp \left( -2\pi i \int_X \mathcal{I}_d \right) = \exp \left( -2\pi i \int_{Y_i} \delta_\varrho \mathcal{I}_{d+1} \right) = \exp \left( -2\pi i \int_Y \mathcal{I}_{d+1} \right). \quad (2.54)$$

Therefore, one can reinterpret the condition of absence of perturbative anomalies as the requirement that for closed  $(d+1)$ -manifolds  $Y$  that can be seen as a boundary  $Y = \partial W$ , the  $\eta$  invariant must vanish. In this case only global anomalies are left and the exponentiated  $\eta$  invariant becomes a topological invariant. More precisely it turns to be a cobordism invariant, i.e.

$$\exp(2\pi i \eta) \in \text{Hom} \left( \Omega_{d+1}^{\text{Spin}}(BG), U(1) \right). \quad (2.55)$$

Here  $\Omega_n^{\text{Spin}}(BG)$  is a bordism group, defined by the following:

**Definition 2.56.** The bordism group  $\Omega_n^{\text{Spin}}(BG)$  is defined as the group of equivalence classes  $[(X, f)]$  of closed  $n$ -dimensional manifolds equipped with a spin structure and a principal  $G$ -bundle, identified by a map  $f : X \rightarrow BG$ , where:

- The equivalence relation is defined by

$$(X, f) \sim (X', f') \iff \exists (Y, F : Y \rightarrow BG) \mid \partial Y = X \sqcup X', F|_X = f, F|_{X'} = f'. \quad (2.57)$$

Here the manifold  $Y$  is called bordism between  $X$  and  $X'$  and is equipped with a spin structure, which restriction on the boundary equals the spin structure on  $X$  and  $X'$ .

- The group operation is instead given by the disjoint union

$$[(X, f)] \cdot [(X', f')] := [(X \sqcup X', f \sqcup f')], \quad (2.58)$$

- The trivial element is equal to the equivalence class of the empty manifold  $[\emptyset]$ .

Note that stating  $[(X, f)] = [\emptyset]$  is equivalent to saying that  $(X, f)$  can be extended to one dimension higher. We will often identify the map  $f$  with  $A$ , which stands also for the connection of the corresponding bundle, as the two are in correspondence.

At this point the existence or not of global anomalies can be inferred by checking whether (2.49) holds for any closed  $Y \in \Omega_{d+1}^{\text{Spin}}(BG)$ , which includes those that lie in the non-trivial equivalence classes as well. Note that part of this statement was already described by the previous knowledge about global anomalies, which stated that a theory can be considered anomaly-free also at the non-perturbative level if the exponentiated  $\eta$  invariant computed for mapping tori of large transformations is trivial. The novelty that anomaly inflow teaches us is that this must hold for *any* closed  $(d + 1)$ -manifold and not only those that can be described in this way. This in principle is much more constraining! However, in return this requirements guarantess that an anomaly-free theory can be correctly defined over *any* manifold, without having to worry case by case about possible higher dimensional regularizing extensions.

An important hypothesis that we used so far in the discussion, especially in the form (2.44), is that starting from a generic manifold  $X$  with some  $G$ -bundle and spin structure, it is always possible to find a manifold  $Y$  (with a  $G$ -bundle and spin structure) that bounds it. However, this is not at all trivial, as we may have a nontrivial bordism group  $\Omega_d^{\text{Spin}}(BG)$ . If so, what about regularizing the partition function of massless fermions on  $(X, f)$  such that  $[(X, f)] \neq [\emptyset]$ ? In this case, the theory does not admit a unique regularization, but rather it is defined up to a choice that we now explain. To be concrete, suppose that  $\Omega_d^{\text{Spin}}(BG) = \mathbb{Z}_k$  and consider a manifold  $X$  that lies in the equivalence class generating such non-trivial group. Then  $k$  disjoint unions of  $X$ , which we denote by  $X^k$ , can be bounded by a proper

manifold  $Y$ . Therefore, we are able to define the regularized theory on  $X^k$  via (2.44) and, after a choice of a  $k^{\text{th}}$  root, we can define

$$Z[X, A] := Z[X^k, A]^{1/k} = \left| \det \left( i\mathcal{D}_d^{+-} \right) \right|^{1/k} \exp \left( -\frac{2\pi i}{k} \eta[Y] \right). \quad (2.59)$$

From this, the definition of the partition function over any other manifold is straightforward, with the premise that one needs to identify first its bordism class. Note that this expression can change by taking a different choice of the root, which is equivalent to add to our regularization a map  $\varphi \in \text{Hom}(\Omega_d^{\text{Spin}}(BG), U(1))$ . We will see in a moment that elements of this group classify a particular set of simple topological field theories, under which definition falls also the  $\eta$  invariant<sup>10</sup>, and in Section 3.4 we will show that such indeterminacy is something characteristic of them. For example, in this particular instance, its presence is related to the phase ambiguity associated to the Hilbert space of the  $(d+1)$   $\eta$  invariant employed in (2.59).

### Deformation classes

Before turning into the description of the generic case, the analysis of the  $\eta$ -invariant can still give a physical insight for certain aspects worth to discuss.

We figured that for a theory to be considered anomaly-free, the condition (2.49) must be satisfied. However, in the instance that this does not hold, one may still wonder whether it is possible to have a regularization of the partition function  $Z[X, A]$  by adding to it a *smooth* term  $\exp(2\pi i \Upsilon[Y_i])$  depending on the extension  $Y_i$  (so  $\partial Y_i = X$ ), such that for any  $Y$  with  $\partial Y = \emptyset$

$$\exp(-2\pi i \eta[Y]) \exp(2\pi i \Upsilon[Y]) = 1. \quad (2.60)$$

This in turn would provide a new regularization of the partition function, which would then be perfectly anomaly-free and well defined independently of the choice of extension  $Y_i$ . However, terms of these kinds must have a particular form. Indeed, they cannot spoil properties of the conserved currents and the energy-momentum tensor of the theory on  $X$ . For this to be satisfied,  $\Upsilon[Y_i]$  must be<sup>11</sup> the integral of a local exact form expressed in terms of the background fields extended on  $Y_i$ . Not only this, but it must be a  $d+1$  topological (cobordism) invariant as well due to its relation with  $\eta$ . Therefore it must describe a topological field theory (TQFT) with the same properties of  $\eta$ . We can infer more: if such a TQFT exists, then the family of  $d+1$  TQFTs associated to

$$\exp(-2\pi i \eta[Y_i]) \exp(2\pi i t \Upsilon[Y_i]), \quad t \in [0, 1], \quad (2.61)$$

<sup>10</sup>Of course, in the hypothesis of no perturbative anomalies. Note also that the  $\eta$ -invariant we use to define a partition function lives in one dimension higher than  $\varphi$ .

<sup>11</sup>Consider for example  $\delta\Upsilon/\delta A$ : its addition will produce a correction to the effective potential and thus to the current  $\langle j \rangle$  associated to the corresponding symmetry. Therefore, for consistency this must not affect the properties of  $\langle j \rangle$  of being local, gauge invariant and preserved along  $X$ , from which the conclusion follows.

describes a 1-parameter deformation of the  $\eta$ -invariant to the trivial theory in the moduli space of TQFTs. This tells us an important property about the anomaly: it is not a property associated to the TQFT described by the  $\eta$ -invariant, but rather of its deformation class.

## 2.2.2 The modern perspective

At this point, we can generalize the lessons offered by the analysis of massless fermions by looking at the crucial points that characterized such problem and its solution.

The claim is that a similar procedure allows us to see *any* anomalous theory  $\mathfrak{E}$ , be it bosonic or fermionic, as the description of the edge modes of some higher,  $d+1$  dimensional bulk theory  $\mathfrak{B}$  with particular properties [7, 8, 54–56]. Of course, these must reproduce the same critical features that allowed us to arrive to the conclusion for the description of massless fermions in the first place.

- The first thing to notice is that the  $d+1$  bulk should describe *gapped* phases of quantum systems with the same symmetry  $G$  of the boundary theory  $\mathfrak{E}$ . The gapness requirement is necessary so that the Hilbert space of the bulk theory on any closed spatial manifold  $X$  is one-dimensional at large distances,  $\dim \mathcal{H}_{\mathfrak{B}}(X) = 1$ . Indeed, this is exactly what happens in the large mass limit  $|m| \rightarrow \infty$  one had to consider to arrive to the result in Section 2.2.1. In general this one-dimensional Hilbert space will be exactly the space which the anomalous  $d$ -dimensional partition function lives in, corresponding to the bundle  $\mathfrak{D}(X)$ .
- The second requirement is that, once  $\mathfrak{B}$  is put on a manifold  $Y$  with boundary  $\partial Y = X$ , on this it cannot describe a trivial gapped phase with a single ground state, but instead there must appear a nontrivial theory, which is identified with  $\mathfrak{E}$ .
- It is important that the evaluation of the partition function of these theories over closed manifolds  $Y$  with any background field  $\mathcal{F}(Y)$  is equal to a phase, which indeed dictates the anomaly of  $\mathfrak{E}$ ,

$$Z_{\mathfrak{B}}[Y, \mathcal{F}(Y)] = \exp(2\pi i \alpha[Y, \mathcal{F}(Y)]) \in U(1). \quad (2.62)$$

- Finally, such theories are expected to be topological in absence of perturbative anomalies, as it happens for the  $\eta$  invariant.

Note that (2.62) is very restrictive: it tells us that such class of theories is equipped with a product  $\otimes : (\mathfrak{B}_1, \mathfrak{B}_2) \rightarrow \mathfrak{B}_1 \otimes \mathfrak{B}_2$  that comes from their physical stacking with no additional interaction. In terms of Hilbert spaces and partition functions

$$\begin{aligned} \mathcal{H}_{\mathfrak{B}_1 \otimes \mathfrak{B}_2}(X) &:= \mathcal{H}_{\mathfrak{B}_1}(X) \otimes \mathcal{H}_{\mathfrak{B}_2}(X), \\ Z_{\mathfrak{B}_1 \otimes \mathfrak{B}_2}[Y, \mathcal{F}_1 \otimes \mathcal{F}_2(Y)] &:= Z_{\mathfrak{B}_1}[Y, \mathcal{F}_1(Y)] Z_{\mathfrak{B}_2}[Y, \mathcal{F}_2(Y)]. \end{aligned} \quad (2.63)$$

This operation defines a group of *invertible* theories  $\text{Inv}_{d+1}^{\text{Spin}}(BG)$ , as for any theory  $\mathfrak{B}$  there exists its inverse  $\mathfrak{B}^{-1}$ , defined as the theory with conjugate partition function. Let us now try to understand the structure of it.

Consider a theory equipped with spin structure and a principal  $G$ -bundle  $A : Y \rightarrow BG$ . The partition function of any  $\mathfrak{B}$  with such data must be expressed as

$$Z_{\mathfrak{B}}[Y, A] = \exp(2\pi i \alpha[A : Y \rightarrow BG]). \quad (2.64)$$

In particular, if  $Y = \partial W$  and the theory is properly extended on  $W$ , there should be a local closed  $(d+2)$  form  $\Phi_{d+2}$  function of the extension  $A : W \rightarrow BG$ , so that

$$\exp(2\pi i \alpha[A : Y \rightarrow BG]) = \exp\left(2\pi i \int_W \Phi_{d+2}\right). \quad (2.65)$$

The data we have at our disposal to write such a term is given by characteristic classes associated to the tangent and principal  $G$  bundles of the theory. Therefore,  $\Phi_{d+2}$  should be the pullback of some (linear combination of) cohomology classes  $H^{d+2}(B(\text{Spin} \times G); \mathbb{Q})$ . At the same time, in order for (2.64) to not depend on the extension  $W$  chosen, so that  $\mathfrak{B}$  is a well-defined theory by itself, one needs to require that for *closed*  $\hat{W}$  it holds

$$\int_{\hat{W}} \Phi_{d+2} \in \mathbb{Z}. \quad (2.66)$$

This means that  $\int \Phi_{d+2}$  must also define an element of  $\text{Hom}(\Omega_{d+2}^{\text{Spin}}(BG), \mathbb{Z})$ .

Note now two things about our discussion: first, the instance we just described correspond to the presence of perturbative anomalies for the corresponding boundary theory  $\mathfrak{E}$ . Indeed, identifying  $Y$  with the manifold associated to an anomalous gauge transformation, the existence of an extension  $W$  is equivalent to describing the anomaly via the descent equation. Therefore,  $\Phi_{d+2}$  is the anomaly polynomial of (2.20). Second, note that once the class in  $\text{Hom}(\Omega_{d+2}^{\text{Spin}}(BG), \mathbb{Z})$  (corresponding to  $\int \Phi_{d+2}$ ) is fixed, it correctly defines an element (i.e. the anomaly polynomial) in  $H^{d+2}(B(\text{Spin} \times G); \mathbb{Q})$  via the injection<sup>12</sup>

$$\text{Hom}\left(\Omega_{d+2}^{\text{Spin}}(BG), \mathbb{Z}\right) \rightarrow \text{Hom}\left(\Omega_{d+2}^{\text{Spin}}(BG), \mathbb{Z}\right) \otimes \mathbb{Q} \cong H^{d+2}(B(\text{Spin} \times G); \mathbb{Q}). \quad (2.68)$$

Let's now consider the case when  $\Phi_{d+2}$  vanish. If this is the case, then (2.64) can only depend on the bordism class of  $Y$  and therefore must describe an element

$$\nu \in \text{Hom}\left(\Omega_{d+1}^{\text{Spin}}(BG), U(1)\right). \quad (2.69)$$

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<sup>12</sup>We have  $\Omega_{\bullet}^{\text{Spin}}(\text{pt}) \otimes \mathbb{Q} \cong H^{\bullet}(B\text{Spin}; \mathbb{Q})$  [57]. Moreover, the short exact sequence of abelian groups

$$1 \longrightarrow \Omega_{\bullet}^{\text{Spin}}(\text{pt}) \longrightarrow \Omega_{\bullet}^{\text{Spin}}(BG) \longrightarrow \tilde{\Omega}_{\bullet}^{\text{Spin}}(BG) \equiv \frac{\Omega_{\bullet}^{\text{Spin}}(BG)}{\Omega_{\bullet}^{\text{Spin}}(\text{pt})} \longrightarrow 1 \quad (2.67)$$

is split.

To be more precise, recall that we are interested in looking for deformation classes of such theories, as otherwise we would be able to find a redefinition of our original anomalous theory which deletes its ambiguity. These deformation classes are more correctly described by its torsion subgroup, namely  $\text{Hom}(\text{Tor } \Omega_{d+1}^{\text{Spin}}(BG), U(1))$ , which can then be identified with the 1-st derived functor of  $\text{Hom}(\Omega_{d+1}^{\text{Spin}}(BG), \mathbb{Z})$ , i.e. the group [57]

$$\text{Hom}\left(\text{Tor } \Omega_{d+1}^{\text{Spin}}(BG), U(1)\right) \cong \text{Ext}^1\left(\Omega_{d+1}^{\text{Spin}}(BG), \mathbb{Z}\right). \quad (2.70)$$

All in all, the group  $\text{Inv}_{\text{Spin}}^{d+1}(BG)$  must sit in the short exact sequence

$$1 \rightarrow \text{Ext}^1\left(\Omega_{d+1}^{\text{Spin}}(BG), \mathbb{Z}\right) \rightarrow \text{Inv}_{\text{Spin}}^{d+1}(BG) \rightarrow \text{Hom}\left(\Omega_{d+2}^{\text{Spin}}(BG), \mathbb{Z}\right) \rightarrow 1. \quad (2.71)$$

Note now that the bordism  $\Omega_{\bullet}^{\text{Spin}}(BG)$  describes a generalized homology theory of  $BG$ . For any generalized homology theory  $E_{\bullet}(X)$  there is the Anderson dual cohomology theory  $H_{I\mathbb{Z}}^{\bullet}(X)$  which satisfies

$$1 \rightarrow \text{Ext}^1(E_{d+1}(X)) \rightarrow H_{I\mathbb{Z}}^{d+2}(X) \rightarrow \text{Hom}(E_{d+2}(X), \mathbb{Z}) \rightarrow 1. \quad (2.72)$$

For  $X = BG$ , (2.71) describes the same exact short exact sequence. Therefore we can understand the group of invertible theories of dimension  $d + 1$  as the generalized cohomology group of  $BG$  of degree  $d + 1$ , defined by (2.71). For a light review of the relevant notions necessary to understand the statement above, see Appendix A. Note also that such short exact sequences are always split, so that the total group of invertible theories is

$$\text{Inv}_{\text{Spin}}^{d+1}(BG) \cong \text{Hom}\left(\Omega_{d+1}^{\text{Spin}}(BG), U(1)\right) \oplus \text{Hom}\left(\Omega_{d+2}^{\text{Spin}}(BG), \mathbb{Z}\right). \quad (2.73)$$

However, it is also important that such split is *non canonical*. We already made implicitly use of this when we discussed the anomaly interplay between global and perturbative anomalies in Section 2.1.2. In fact, if this were not the case, we would not have been able to uncover global anomalies from the perturbative ones via an injection  $\iota : G \hookrightarrow \hat{G}$ . We will see that this property will be very useful also later on for our computations.

Let us note that due to this split, asking about the presence of global anomalies is by no mean constrained to the requirement of having perturbative anomalies to vanish, as the two describes distinct components in the group of  $d + 1$  invertible theories, which we ultimately identify as the group of anomalies in  $d$  dimensions. Therefore, asking if a theory exhibits a global anomaly in the presence of perturbative anomalies is a sensible thing to do, though the non-canonicity of the isomorphism (2.73) may require an ad-hoc analysis for different symmetry groups. Again, this will be important later on.

The statement (2.73) we presented is based on the analysis first done in [9, 58], where indeed invertible field theories are identified<sup>13</sup> with homotopy classes of maps

$$\text{Inv}_{\bullet}^{\text{Spin}}(BG) \cong [MT(\text{Spin} \times G), \Sigma^{\bullet+1}I\mathbb{Z}]. \quad (2.74)$$

Again, see Appendix A for details on the notation. What is notable is that via this relation it is possible to show that indeed the group component of anomalies associated to global anomalies, i.e.  $\text{Hom}(\Omega_{d+1}^{\text{Spin}}(BG), U(1))$ , can be identified with deformation classes of *invertible TQFTs* (iTQFTs). The same statement can also be proved via more accessible terms as done in [10]. On a side note, the same formalism allows a description of the relation between anomalous theories and invertible theories via the formalization of the concept of *Relative Field Theory*, see [55, 59]. For the time being, we are content with the physical meaning of the statements (2.69) and (2.73), which we will make heavy use of in our analysis. In particular, since in our work we will mostly be interested in the study of global anomalies,  $\text{Hom}(\Omega_{d+1}^{\text{Spin}}(BG), U(1))$  is the group we will mostly be interested to. Finally, it is worth to point out that these particular set of topological theories are interesting not only from the anomalies point of view, but also from a more general condensed matter framework. Indeed, such iTQFTs turn out to describe the continuum limit of so-called *Symmetry Protected Topological phases* (SPTs). In the next chapter we will review some details about them, both from their UV lattice description and their IR continuum-limit, to better understand their connection with anomalies. Taking the longer route and describing them also from the lattice description will be very useful, as it will provide a way to understand the physical origin of the different cohomological layers that build up the total invariants (2.69).

### 2.2.3 Not only fermions: Dijkgraaf-Witten theories

Our discussion so far distinguished only two kinds of anomalies: perturbative and non-perturbative. However, it is actually possible to refine such classification and differentiate between *fermionic* and *bosonic* anomalies. Morally, the former kind of anomalies is given by those which are present simply because of the fermionic nature of the theory, while the latter describes an ambiguity of the theory which is still present in case one forgets about the spin structure, i.e. by looking only at the bosonic degrees of freedom.

An important class of bosonic iTQFTs which we will make heavy use of is given by (ungauged) Dijkgraaf-Witten (DW) theories [11]. These theories can be defined in any dimension  $d$  and symmetry  $G$  and their inequivalent classes are represented by

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<sup>13</sup>In [9, 58] such invertible field theories are reflection positive and extended, which are hypothesis that we are implicitly assuming too. In particular, reflection positive is the equivalent well-known notion of unitarity in Euclidean spacetime. The extended property signals instead the requirement of locality of the theory. In the topological case, via the Atiyah-Siegel axiomatic definition of a TQFT (see Section 3.4), such notion is described by requiring the TQFT to be a (symmetric monoidal) functor between two  $(\infty, n)$ -categories.

elements in the cohomology group  $H^d(BG, U(1))$ . Being these theories topological *and* bosonic in nature, it is natural to expect that they are associated to global bosonic anomalies and that there is a map from the spin case that identify them. Indeed, such a map from the spin-iTQFTs classified by  $\text{Hom}(\Omega_d^{\text{Spin}}(BG), U(1))$  exists and is defined as

$$\begin{aligned} \pi_{\text{DW}}^{\text{Spin}} : \Omega_d^{\text{Spin}}(BG) &\rightarrow H_n(BG, \mathbb{Z}), \\ [(X, f)] &\mapsto f_*[X] \end{aligned} \quad (2.75)$$

after making use of the relation  $H^n(BG, U(1)) \cong \text{Hom}(H_n(BG, \mathbb{Z}), U(1))$ .

However, (2.75) in general is neither injective nor surjective, so in principle one must proceed with care by identifying bosonic anomalies with Dijkgraaf-Witten like theories.

In light of our previous discussion, bosonic (global) anomalies are more naturally identified by elements of the cobordism groups  $\text{Hom}(\Omega_{d+1}^{SO/O}(BG), U(1))$ . Indeed, forgetting about the presence of the spin tangential structure, but only of its bosonic projection, gives the surjective map

$$(\rho_d)_* : \Omega_d^{\text{Spin}}(BG) \rightarrow \Omega_d^{SO}(BG), \quad (2.76)$$

which dual describes bosonic anomalies. The relation between DW-like theories and bosonic iTQFTs can be clarified by noticing that (2.75) factorizes via

$$\begin{array}{ccc} \Omega_d^{\text{Spin}}(BG) & \xrightarrow{(\rho_d)_*} & \Omega_d^{SO}(BG) \\ & \searrow \pi_{\text{DW}}^{\text{Spin}} & \downarrow \pi_{\text{DW}}^{SO} \\ & & H_d(BG, \mathbb{Z}). \end{array} \quad (2.77)$$

The more appropriate question to ask is then *when* bosonic anomalies are reproduced by Dijkgraaf-Witten theories or not, namely when  $\pi_{\text{DW}}^{SO}$  is injective and/or surjective. This can be addressed by the following considerations [8, 60]:

- First, injectivity may be lost if there are distinguished phases which are instead identified via the DW-classification. This is the case when the iTQFTs are expressed in terms of Stiefel-Whitney (or even Potryagin) classes, which are treated like  $\mathbb{Z}_2$  gauge fields by Dijkgraaf-Witten. Since in  $d \leq 3$  we have  $w_1 = 0$ ,  $w_2 = w_1^2$  and  $w_3 = 0$ , such possibility is only present for  $d \geq 4$ .
- Surjectivity of the map may instead fails if there are equivalence classes in the group cohomology which pullback vanish once they are evaluated over an orientable manifold. It can be proved that such instance can only happen in  $d \geq 7$  [61].

Therefore, we see that in  $d \leq 3$  the map  $\pi_{\text{DW}}^{SO}$  is an isomorphism and we can treat the two classifications indifferently. Since this will be the framework we will mostly work in, in the following we will not distinguish between the two groups.

# Chapter 3

## Invertible Topological Field Theories

In the previous chapter we discussed the relation between theories with global anomalies in  $d$  dimensions and  $D = d + 1$  dimensional iTQFTs through anomaly inflow. In order to gain information on anomalies, it is thus clear that one needs to better understand how to classify the TQFTs corresponding to the  $D$  bulk.

As anticipated, such iTQFTs have been extensively studied from the condensed matter point of view, as they describe the continuum limit of so-called *Symmetry Protected Topological phases*. To be more precise, this is not always the case. Indeed, the precise definition of these phases of matter is given as the phases that describe short-range entangled (SRE) states, of which existence is protected by nontrivial global symmetry  $G$  [62]. More specifically, this means they describe local, gapped systems that can be deformed to a trivial one (and so into each other) via local, unitary transformations only at the price of breaking the symmetry  $G$ . However, the axiomatic setup that describes iTQFTs and that capture the cobordism groups classification is able to describe the continuum limit also of another kind of topological phases, i.e. the long range entangled (LRE) states. These are kind of states that cannot be transformed into direct product states (and so into each other) via local and unitary transformation even at the price of breaking the symmetry  $G$ . Therefore, their classification is captured by the cobordism groups  $\text{Hom}(\Omega_{d+1}^{\text{Spin}}(\text{pt}), U(1))$ , as its non-triviality equals the presence of phases already distinguished without the addition of any symmetry. Therefore, when we identify iTQFTs classified by cobordism groups with SPTs, one should be careful and remember that the two are equivalent up to modding out<sup>1</sup>  $\text{Hom}(\Omega_{d+1}^{\text{Spin}}(\text{pt}), U(1))$ . Luckily, in  $D = 3$  we have  $\Omega_3^{\text{Spin}}(\text{pt}) = \emptyset$ , so that we don't have to worry about such technicality.

In this chapter we will thus analyze constructions of fermionic SPTs, starting from their lattice description and we will gradually move up to their classification

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<sup>1</sup>To this regards, it is interesting to note how in principle these states needs to be modded out in  $D$  dimensions, but can instead contribute to lower dimensional defects that classifies different  $D$ -dimensional SPTs; see Section 3.3.

in terms of cobordism groups, which emerges from the low-energy point of view, as it will provide important informations on the structure of the cobordism groups we will need to work with for the classification and analysis of anomalies.

## 3.1 The Gu-Wen construction of fermionic SPTs

The idea behind Gu and Wen's construction of fermionic SPTs is to make use of a group cohomological lattice description of bosonic phases of matter and eventually dress the discrete lattice used to describe them with appropriate fermionic degrees of freedom. This can be done in a few steps, which we will briefly explain now.

### 3.1.1 Bosonic layer

Let's consider a gapped theory with a symmetry  $G$  with a topological dependence on its actions and suppose we want to describe its low-energy limit. If the symmetry is not broken in the IR, the theory will be trivial with a single vacuum state unless its topological properties make it flow to a topological field theory. Indeed, its original topological dependence will be robust under RG flow and will be eventually captured by some topological term  $S_{\text{top}}$ . In the case of continuous  $G$ , we know two standard topological contributions to  $S_{\text{top}}$ : the Wess-Zumino-Witten (WZW) and the  $\theta$  terms. Both are described by some smooth map from the spacetime manifold  $Y$  to either the symmetry group  $G$  or  $BG$ ,  $\tilde{g}_\theta : Y \rightarrow G$  and  $\tilde{g}_{\text{WZW}} : Y \rightarrow BG$ .

However, in the case of SPTs, one is interested in describing disordered phases of matter, all while still preserving the symmetry of the theory. This means that the fields responsible for the topological dependence must strongly fluctuate at the low-energy limit and we cannot expect their behaviour to be captured by some smooth, continuum-like description. Therefore, the standard topological terms we know are of no use and one needs to resort to a discretization of spacetime in order to be able to describe such topological terms. In this case the smooth requirement of  $S_{\text{top}}$  in the low-energy description drops and there is hope to be able to find a proper description of the phenomena.

Therefore, let's consider a triangulated  $D$ -manifold  $Y$ , with fundamental class in  $H_D(Y; \mathbb{Z})$  of the form

$$[Y] = \sum_i \varepsilon_i \sigma_i(\Delta^D) \in H_D(Y; \mathbb{Z}). \quad (3.1)$$

Here  $\sigma_i : \Delta^D \rightarrow Y$  are  $D$ -dimensional simplices labelled by an index  $i$  and  $\varepsilon_i = \pm 1$  corresponds to their orientations. We consider the manifold  $Y$  to be equipped with a branching structure [62]. This means that all its 1-simplices will be oriented such that its 2-simplices do not form closed loops. This branching structure allows to define a global ordering of the vertices that compose the 0-skeleton of  $Y$ , compatibly with its orientation. Moreover, such global ordering  $>$  will induce a local ordering

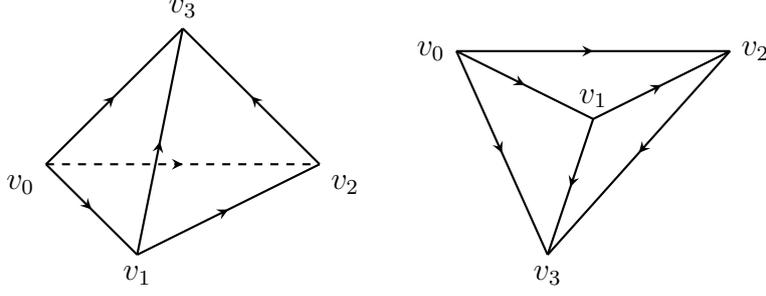


Figure 3.1: The two possible orientations of a 3-simplex and the corresponding local ordering of its vertices. Left: a simplex with orientation  $\varepsilon = +1$ . Right: a simplex with orientation  $\varepsilon = -1$ .

in the vertices that compose each  $D$ -simplex  $\sigma_i$ , such that generally these will be of the form  $\sigma_i = [v_0 v_1 \dots v_D]$ , where  $v_j > v_k$  for  $j > k$ . For a single simplex, it is easy to figure that the vertex  $v_j$  will have  $D - j$  outgoing edges and that there are two possible orientations of each  $\sigma_i$  which will correspond to  $\varepsilon_i = \pm 1$ , see Figure 3.1.

In a discretized spacetime the topological term should appear as a discrete analogue of the map  $\tilde{g}_\theta : Y \rightarrow G$ . We capture this by decorating each vertex  $v_j$  with a degree of freedom  $g_j \in G$ . In the continuum limit this would correspond to the evaluation of  $\tilde{g}_\theta|_{v_j}$ . Given this assumption, the action contribution restricted to a single simplex  $\sigma_i$  would be equivalent to

$$\exp(-S_{\text{top}}|_{\sigma_i}) = \exp\left(-\int_{\sigma_i} \mathcal{L}_{\text{top}}(g_0, g_1, \dots, g_D)\right) = \hat{\nu}^{\varepsilon_i}(g_0, g_1, \dots, g_D). \quad (3.2)$$

Here  $\hat{\nu}$  is a homogeneous function  $\hat{\nu} : G^{D+1} \rightarrow U(1)$ , as the action must satisfy the symmetry requirement of the theory

$$\hat{\nu}(gg_0, gg_1, \dots, gg_D) = \hat{\nu}(g_0, g_1, \dots, g_D), \quad \forall g \in G. \quad (3.3)$$

Such redundancy can be eliminated via the gauge fixing

$$\nu(g_{01}, g_{12}, \dots, g_{(D-1)D}) := \hat{\nu}(1, g_{01}, g_{01}g_{12}, \dots, g_{01} \dots g_{(D-1)D}), \quad g_{jk} := g_j^{-1}g_k, \quad (3.4)$$

where  $g_{jk}$  is now interpreted as a group element associated to the simplex  $[v_j v_k]$ , which describes the difference between the degrees of freedom in  $v_j$  and  $v_k$ . Note that we can see the function  $\nu$  as a particular cochain in group cohomology,  $\nu \in \mathcal{C}^D(G; U(1))$  and  $\hat{\nu}$  its homogeneous representation. In the following, we will keep using the hat to denote homogeneous representation of cocycles and cochains, while reserving the symbols without it to the standard, non-homogeneous, representation.

The total partition function will then be

$$Z_{\text{bSPT}}[Y, A, \nu] = \prod_i \nu^{\varepsilon_i}(g_{01}, g_{12}, \dots, g_{(D-1)D}). \quad (3.5)$$

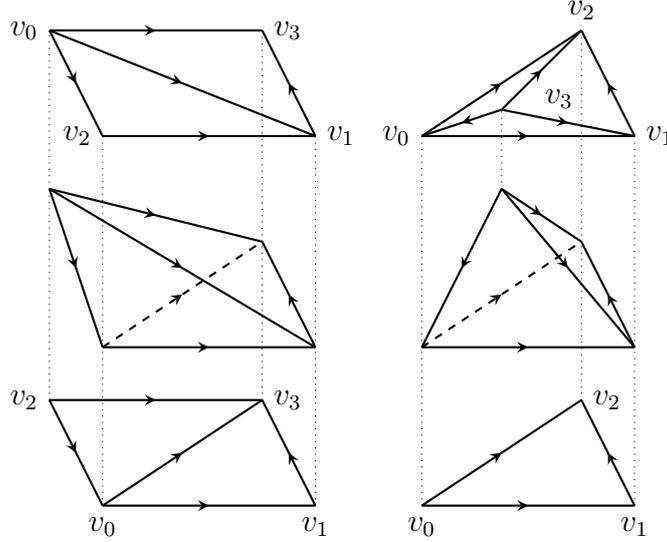


Figure 3.2: The  $D$ -dimensional Pachner moves can be seen as the boundary of a  $(D + 1)$ -simplex. In this case, the two 2-dimensional Pachner moves  $2 \rightarrow 2$  and  $1 \rightarrow 3$  are the boundary of a 3-simplex. Note that the orientation of the 3-simplex is naturally induced via the orientation of its faces, given by the branching structure of the  $D$ -simplices.

Note that such description works not only for continuous symmetries, but for discrete ones too.

For the action to be a topological invariant, one must show that its evaluation does not change for equivalent triangulations of  $Y$ , so  $\nu$  must be invariant under Pachner moves [63]. Recall that any Pachner move in  $d$  dimension describes the boundary of a  $(d + 1)$ -simplex, see Figure 3.2. Therefore, one needs the requirement

$$\delta\nu(a_1, \dots, a_D) = 1, \quad (3.6)$$

where

$$\delta\nu(a_1, \dots, a_{D+1}) := \nu(a_2, \dots, a_{D+1})\nu^{(-1)^{D+1}}(a_1, \dots, a_D) \prod_{j=1}^D \nu^{(-1)^j}(a_1, \dots, a_j a_{j+1}, \dots, a_{D+1}) \quad (3.7)$$

is the group cohomology coboundary operation<sup>2</sup>. Moreover, the addition to counterterms in (3.2) amounts to a redefinition up to coboundary terms associated to the

<sup>2</sup>Equation (3.7) describes the coboundary operation for the standard representation of cochains. In terms of homogeneous representative, the coboundary representation can be rewritten as

$$\delta\nu(a_0, \dots, a_{D+1}) := \prod_{i=0}^{D+1} \nu^{(-1)^i}(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_{D+1}). \quad (3.8)$$

$(D - 1)$  simplices that compose each  $\sigma_i$ ,

$$\nu(a_1, \dots, a_D) \rightarrow \nu(a_1, \dots, a_D) \delta\beta(a_1, \dots, a_D). \quad (3.9)$$

Note that these do not change (3.5), as each  $(D-1)$  contribution is summed over twice from two  $D$  simplices with opposite orientation and thus is canceled out. Therefore, it follows that inequivalent topological terms of this kind are classified by elements  $\nu \in \mathcal{H}^D(G; U(1))$ .

Now that the physical construction of this kind of topological term is clear, let's understand it from a more geometrical point of view. Recall that the group cohomology of  $G$  can be identified with the topological cohomology of the classifying space  $BG$ ,  $\varphi : \mathcal{H}^\bullet(G; \mathbb{Z}) \cong H^\bullet(BG; \mathbb{Z})$ . Naturally, this relation holds for  $U(1)$ -valued cohomology as well. This gives us a way to re-interpret (3.2): for a fixed  $G$ -bundle defined on  $Y$  and identified by  $A : Y \rightarrow BG$ , we can create different topological terms on  $Y$  classified by  $\nu \in H^D(BG; U(1))$  and evaluate them via the pullback  $A^*$ . Their contribution to the topological action for a single simplex  $\sigma_i = [v_0 \dots v_D]$  then will be

$$A^*\nu([v_0 \dots v_D]) = \nu(A_*([v_0 \dots v_D])) \stackrel{\varphi}{=} \nu(g_{01}, \dots, g_{(D-1)D}), \quad (3.10)$$

where it is now clear that  $g_{ij} = A_*([v_i v_j])$  are the pushforward of the 1-simplices of  $Y$  in  $BG$ . In other words, what we just presented is a constructive way to represent the (ungauged) Dijkgraaf-Witten theories discussed in Section 2.2.3, which we now understand to be associated to the group cohomology classification of bosonic SPTs.

### 3.1.2 Grassmanian tensors on simplices

So far we rewrote the contribution to the partition function given by bosonic degrees of freedom. The next step is to naturally account for fermions living on the lattice. In  $D = d + 1 = 3$ , such degrees of freedom will be organized into two layers. Here we present the construction of the first of the two, which holds generally in any dimension  $D$ .

To do so, we are going to present the Gu-Wen's construction [64] of fermionic SPTs by dressing the discrete spacetime with Grassmannian-like variables which describe fermionic degrees of freedom, as it is customary in lattice constructions. The final SPTs should then be a slight modification of the original bosonic ones described in Section 3.1.1, with some minor modifications that should factorize and allow to properly take into account new effects. Morally it is expected

$$Z_{\text{ferm}}[Y, A, \nu, \Theta] \sim Z_{\text{bSPT}}[Y, A, \nu] \zeta[Y, \Theta], \quad (3.11)$$

where  $\zeta[Y, \Theta]$  is the aforementioned change, function of some (temporarily unknown) variables  $\Theta$ .

To understand how we may be able to properly define some Grassmanian-odd degrees of freedom in our construction, there are a few hints that one can make use of. First, recall that the presence of fermions should be accompanied by the presence

of a spin structure on the manifold  $Y$ . Via the *bosonization* procedure, which we will review later, this dependence should be mimicked in an equivalent description via some  $(D - 2)$ -form  $\mathbb{Z}_2$  symmetry, then supported by a  $\mathbb{Z}_2$   $(D - 1)$  gauge field on  $(D - 1)$  simplices of the triangulation of  $Y$ . Second, if fermionic degrees of freedom are represented by Grassmanian numbers, one cannot simply dress  $D$ -simplices with such data like it was done for the bosonic case. Indeed, the partition function must always be a Grassmanian-even number and Grassmanian-odd numbers associated to  $D$ -simplices would not be compatible with its invariance requirement under change of triangularizations. Therefore, the natural step for the creation of fSPTs is to associate them to lower  $(D - 1)$ -dimensional simplices.

In fact, Gu and Wen's idea is to attach to  $(D - 1)$ -faces  $[v_{i_1} \dots v_{i_D}]$  of each  $D$ -simplex  $[v_0 \dots v_D]$  of  $Y$  sets of Grassman-like variables  $\theta_{i_1 \dots i_D}$ ,  $\bar{\theta}_{i_1 \dots i_D}$ . Here  $\theta_{i_1 \dots i_D}$  ( $\bar{\theta}_{i_1 \dots i_D}$ ) will be associated to  $[v_{i_1} \dots v_{i_D}]$  if this has a positive (negative) orientation in  $[v_0 \dots v_D]$ . Note also that each simplex  $[v_{i_1} \dots v_{i_D}]$  will describe a common face of two different  $D$ -simplices, which thus will give a total contribution  $\theta_{i_1 \dots i_D} \bar{\theta}_{i_1 \dots i_D}$ . Defining for each  $(D - 1)$ -simplex  $[v_{i_1} \dots v_{i_D}]$  the quantity  $\hat{n}(g_{i_1}, \dots, g_{i_D}) = 0, 1$  as the occupation number of these fermionic degrees of freedom, namely  $\hat{n}(g_{i_1}, \dots, g_{i_D}) = 0$  if the degree is present and  $\hat{n}(g_{i_1}, \dots, g_{i_D}) = 1$  if not, the contribution in the path integral associated to  $[v_{i_1} \dots v_{i_D}]$  is eventually captured by

$$\theta_{i_1 \dots i_D}^{\hat{n}(g_{i_1} \dots g_{i_D})} \bar{\theta}_{i_1 \dots i_D}^{\hat{n}(g_{i_1}, \dots, g_{i_D})}, \quad \hat{n}(g_{i_1}, \dots, g_{i_D}) \in \mathbb{Z}_2. \quad (3.12)$$

Here  $\hat{n}$  is again a homogeneous cochain, like  $\hat{v}$  for the bosonic layer. Per usual, we identify with  $n(a_1, \dots, a_{D-1})$  its non-homogeneous counterpart. Of course, these variable needs to be integrated over via a proper Grassmanian measure, defined by

$$\int \prod_{[v_{i_1} \dots v_{i_D}]} d\theta_{i_1 \dots i_D}^{\hat{n}(g_{i_1} \dots g_{i_D})} d\bar{\theta}_{i_1 \dots i_D}^{\hat{n}(g_{i_1}, \dots, g_{i_D})}. \quad (3.13)$$

The contribution to the partition function due to the evaluation of each  $D$ -simplex  $\sigma_i$  will then be proportional to

$$\mathcal{V}_i \propto \hat{v}^{\varepsilon_i}(g_0, \dots, g_D) \prod_{[v_{i_1} \dots v_{i_D}] \in \mathcal{I}} \theta_{i_1 \dots i_D}^{\hat{n}(g_{i_1}, \dots, g_{i_D})} \prod_{[v_{i_1} \dots v_{i_D}] \in \mathcal{J}} \bar{\theta}_{i_1 \dots i_D}^{\hat{n}(g_{i_1}, \dots, g_{i_D})}, \quad (3.14)$$

such that the fermionic integral can be conventionally fixed as

$$\Sigma[Y, n] := \int \prod_{[v_{i_1} \dots v_{i_D}]} d\theta_{i_1 \dots i_D}^{\hat{n}(g_{i_1} \dots g_{i_D})} d\bar{\theta}_{i_1 \dots i_D}^{\hat{n}(g_{i_1}, \dots, g_{i_D})} \prod_{[v_{i_1} \dots v_{i_D}] \in \mathcal{I}} \theta_{i_1 \dots i_D}^{\hat{n}(g_{i_1}, \dots, g_{i_D})} \prod_{[v_{i_1} \dots v_{i_D}] \in \mathcal{J}} \bar{\theta}_{i_1 \dots i_D}^{\hat{n}(g_{i_1}, \dots, g_{i_D})}. \quad (3.15)$$

Here  $\mathcal{I}$  ( $\mathcal{J}$ ) are shorthand for the set of  $(D - 1)$  faces  $[v_{i_1} \dots v_{i_D}]$  of  $\sigma_i = [v_0 \dots v_D]$  with equal (opposite) orientation with respect to the one induced by  $\sigma_i$  if  $\varepsilon_i = +1$ , see Figure 3.3. In the case of negative orientation  $\varepsilon_i = -1$ , the two subsets are instead swapped. Note also that the terms must appear in a fixed order for each

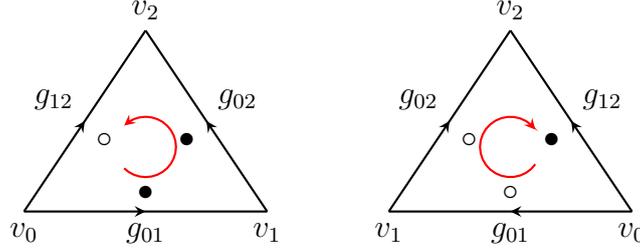


Figure 3.3: Fermionic degrees of freedom associated to 1-faces of a 2-simplex with positive orientation (left) or negative (right). The Grassmann variables here are conventionally represented by  $\circ$  for  $\theta$  and  $\bullet$  for  $\bar{\theta}$ .

$D$ -simplex  $\sigma_i$ , which is fixed by convention as in [65, 66]. For example, in  $D = 2$  for  $\sigma_i = [v_0 v_1 v_2]$  with  $\varepsilon_i = +1$ , then

$$\prod_{[v_{i_1} \dots v_{i_D}] \in \mathcal{I}} \theta_{i_1 \dots i_D}^{\hat{n}(g_{i_1}, \dots, g_{i_D})} \prod_{[v_{i_1} \dots v_{i_D}] \in \mathcal{J}} \bar{\theta}_{i_1 \dots i_D}^{\hat{n}(g_{i_1}, \dots, g_{i_D})} = \theta_{12}^{\hat{n}(g_1, g_2)} \theta_{01}^{\hat{n}(g_0, g_1)} \bar{\theta}_{02}^{\hat{n}(g_0, g_2)}. \quad (3.16)$$

This integration term  $\Sigma(n)$  will be the part of  $\zeta[Y, \Theta]$  written in terms of Grassmann-odd numbers. Therefore, to ensure that  $Z_{\text{ferm}}$  is a Grassmann-even quantity, one must require that  $n$  is a  $(D - 1)$  group  $\mathbb{Z}_2$  cocycle,

$$\delta n = 0. \quad (3.17)$$

Its pullback  $A^*n \in H^{D-1}(Y; \mathbb{Z}_2)$  via  $A : Y \rightarrow BG$  will then be one of the new variables captured by  $\Theta$ . Moreover, by requiring for  $Z_{\text{ferm}}$  to be invariant under Pachner moves, also two other conditions needs to be satisfied [64]:

- First, the constraint (3.6) needs to be changed into

$$\delta \nu = (-1)^{\text{Sq}^2(n)}, \quad (3.18)$$

where  $\text{Sq}^2 : H^\bullet(-; \mathbb{Z}_2) \rightarrow H^{\bullet+2}(-; \mathbb{Z}_2)$  is the Steenrod square operation, which in  $D = 3$  is equal to the standard cup product between 2-cocycles<sup>3</sup>. In order for (3.18) to hold, it is necessary that

$$n \in SH^{D-1}(BG; \mathbb{Z}_2) := \left\{ x \in H^{D-1}(BG; \mathbb{Z}_2) \mid \frac{1}{2} \text{Sq}^2(x) \in B^{D+1}(BG; \mathbb{R}/\mathbb{Z}) \right\}. \quad (3.20)$$

Such twist is needed to account for the change of arrangement that the fermionic degrees of freedom undergo under change of triangularization.

<sup>3</sup>In particular, we have that

$$\text{Sq}^q(a) := a \cup_{p-q} a, \quad \forall a \in H^p(-, \mathbb{Z}_2), \quad (3.19)$$

where  $\cup_i$  is the standard  $i$ -cup product [67].

- Second, it turns out that even after accounting for (3.18), one needs to introduce another contribution to  $\zeta$ , related to another function  $\hat{m} : G^{D-1} \rightarrow \mathbb{Z}_2$ , which is not homogeneous in  $G$  and that satisfies  $\delta\hat{m} = \hat{n}$ . Note that non-trivial solutions to this equations can only exists using the homogeneous representation  $\hat{n}$  of the  $(D - 1)$ -cocycle (3.17), by requiring  $\hat{m}$  to not be homogeneous when  $\hat{n}$  is not a coboundary. The total contribution due to the addition of such term equals to

$$\gamma[Y, S, \hat{m}] := \prod_{x \in S} (-1)^{\hat{m}(x)}, \quad (3.21)$$

where  $S$  is a particular subset of  $(D - 2)$ -simplices, see [64] for general details. In  $D = 3$  this set correspond to all the 1-simplices of  $Y$  plus all the edges  $[v_0v_2]$  of 3-simplices with positive orientation and  $[v_1v_3]$  for the negative ones.

Therefore, after writing explicitly the dependences that before were collected by  $\Theta$ , the total spin contribution to the partition function is

$$\zeta[Y, A^*n, S, \hat{m}] := \Sigma[Y, n] \gamma[Y, S, \hat{m}]. \quad (3.22)$$

Note that this term is not triangulation-invariant per se, but rather its change is matched by the same change of phase that  $Z_{\text{bSPT}}[Y, A, \nu]$  has, given by the fact that for a non-trivial  $n \in SH^{D-1}(BG; \mathbb{Z}_2)$  then  $\nu$  is not a cocycle anymore. In particular, if we model a change of triangulation  $T[Y] \rightarrow T'[Y]$  as the triangularized manifold  $Y \times [0, 1]$ , such that this matches with  $T$  at  $Y \times (0)$  and with  $T'$  at  $Y \times (1)$ , its total change is equal to

$$\zeta[T[Y], A^*n, S, \hat{m}] \rightarrow \zeta[T'[Y], A^*n, S, \hat{m}] = \zeta[T[Y], A^*n, S, \hat{m}] (-1)^{\int_{Y \times [0,1]} \text{Sq}^2(A^*n)}. \quad (3.23)$$

The resulting class of theories, defined by

$$Z_{\text{ferm}}[Y, A, \nu, n, S, \hat{m}] := Z_{\text{bSPT}}[Y, A, \nu] \zeta[Y, A^*n, S, \hat{m}], \quad (3.24)$$

are often called *supercohomology*-like, based on the fact that they are extensions of the previous cohomological classification of bSPTs via a sort of  $\mathbb{Z}_2$  grading.

### 3.1.3 Encoding of the spin structure

So far we have presented a construction for fermionic SPTs from the lattice point of view, making explicit use of fermionic degrees of freedom associated to simplices of a triangulated manifold. In order to connect such constructions to anomalies in  $d = 2$ , we are eventually interested in the continuum description of these phases, which in principle should be captured by spin-iTQFTs.

The group of Gu-Wen fSPTs  $\mathfrak{F}(G)$  (i.e. fSPTs that have a partition function of the form (3.24)) is described by a short exact sequence

$$1 \rightarrow SH^{D-1}(BG; \mathbb{Z}_2) \rightarrow \mathfrak{F}(G) \rightarrow H^D(BG; U(1)) \rightarrow 1 \quad (3.25)$$

and one would eventually be able to connect such group with the cobordism classification, which in the continuum limit is defined for manifolds equipped not only with a  $G$  bundle, but also with a spin structure. While the presence of the former is evident in the construction (3.25), the presence of the latter is less clear.

In order to show that indeed (3.25) describes fermionic SPTs which continuum limit is defined on spin-manifold, it is possible to rewrite the integral  $\zeta[Y, A^*n, S, \hat{m}]$  to make manifest such dependence. This has been obtained by Kapustin and Gaiotto [68]. The crucial point is to rewrite the term  $\gamma[Y, S, \hat{m}]$  in terms of  $\hat{n}$ , by evaluating it over some extension  $E$  of  $S$  such that  $\partial E = S$ . Naturally, this is not always possible, but if so, the new term will have a dependence on the choice of representative  $E$ , call it  $\Gamma[Y, \beta, E]$ . This is defined as

$$\Gamma[Y, A^*n, E] := \prod_{e \in E} (-1)^{A^*n(e)}. \quad (3.26)$$

The choice of  $E$  is then interpreted as the choice of a spin structure for  $Y$ . Albeit the rigorous proof of this statement is not available in every dimension  $D$ , this can be seen explicitly in the low dimensional cases, particularly for our case of interest  $D = 3$ . To show it, let's consider a barycentric triangulation  $\mathcal{BT}$  induced by a generic triangulation  $T$  of the 3-manifold  $Y$  of interest. Via invariance under Pachner moves of the total partition function, it is always possible to reduce the discussion to such case. A branching structure on  $\mathcal{BT}$  will be naturally induced by the branching structure on  $T$ , by simply ordering the additional 1-simplices from the barycenter of fewer vertices to the barycenter of more vertices. In  $\mathcal{BT}$ , the additional contributions of 1-simplices  $[v_0v_2]$  ( $[v_1v_3]$ ) for positive (negative) tetrahedrons cancel out, so the set  $S$  effectively reduces to the sum of all 1-simplices of  $\mathcal{BT}[Y]$ , which is well-known to be the standard representative of the Poincaré dual of the Stiefel-Whitney class  $w_2 \in H^2(Y; \mathbb{Z}_2)$  [69]. Therefore, the dependence of the fermionic partition function on the unphysical variable  $\hat{m}$  is deleted and translated in terms of  $n \in SH^{D-1}(BG; \mathbb{Z}_2)$  (and ultimately  $A^*n$ ) via the redefinition (3.26) iff the manifold  $Y$  is spin, i.e.  $w_2$  is trivial. In this case, as it is well-known from obstruction theory, the choice of a representative  $E$  such that  $\partial E = S = \text{PD}(w_2)$  is equivalent to a choice of a cochain representative  $s$  such that  $\delta s = w_2$ , which matches with the choice of a spin structure on  $Y$ .

## 3.2 Bosonization and extended supercohomology

It turns out that the classification provided by Gu and Wen is not the most general one, but, for example, in  $D = 3$  there is actually an additional layer of fermionic degrees of freedom that can play a role in the classification of fermionic SPTs. We will give a geometrical explanation of these in cobordism terms in the next section, for the moment being we try to provide a physical understanding for their introduction, based on what was learned in the previous one. While our exposition will be mostly from the partition function point of view, an equivalent description in the fixed-point

Hamiltonian and wave function picture is offered in [65, 70]. The perk of using the first of the two is that it will become evident how the process can be carried out and extended in general setup. Our exposition is based on discussions from [68, 70–72].

Working with bosonic degrees of freedom is usually easier than working with fermionic ones. Therefore, one may be tempted to construct fermionic SPTs through a fermionization procedure. This means that the partition function of a fermionic SPT should look like

$$Z_{\text{ferm}}[Y, s] = \frac{1}{\sqrt{H^{D-1}(Y; \mathbb{Z}_2)}} \sum_{\beta \in H^{D-1}(Y; \mathbb{Z}_2)} Z_{\text{bos}}[Y, \beta] w[Y, \beta, s], \quad (3.27)$$

where  $s$  is the spin structure on  $Y$  and  $w[Y, \beta, s]$  a to-be-understood weight that couples  $s$  with a  $\mathbb{Z}_2$   $(D-2)$ -form symmetry  $\beta$  that is present in the bosonic counterpart theory,  $Z_{\text{bos}}$ . The requirement is then that the *final* bosonic degrees of freedom of  $Z_{\text{ferm}}$  should resemble bosonic SPTs, like we discussed in the previous section, while in principle  $Z_{\text{bos}} \neq Z_{\text{bSPT}}$ .

As anticipated,  $Z_{\text{bos}}[Y, \beta]$  should be equipped with a  $\mathbb{Z}_2$   $(D-2)$ -form symmetry, described by a cocycle  $\beta \in H^{D-1}(Y; \mathbb{Z}_2)$ . Indeed, recall that in a bosonic setup the gauging of a  $p$ -symmetry produces a  $(D-p-2)$ -symmetry in the gauged theory [73, 74]. The same should hold also in the fermionic case, with the obvious difference that the 0-form symmetry is replaced by the spin structure  $s$  of  $Y$ , which is a torsor over  $H^1(Y; \mathbb{Z}_2)$  rather than being  $H^1(Y; \mathbb{Z}_2)$  itself. Similarly,  $w[Y, \beta, s]$  should be the corresponding weight in the gauging procedure in (3.27) and its behaviour is what should guarantee that the presence of a  $(D-2)$   $\mathbb{Z}_2$ -gauge field in  $Z_{\text{bos}}$  translates into a description of *fermionic* degrees of freedom in  $Z_{\text{ferm}}$ . In particular, via Poincaré duality it is natural to interpret  $\text{PD}(\beta)$  as the worldline of a probe fermion and correspondingly the insertion of  $w$  as equivalent to the the insertion of its Wilson loop, which thus introduce fermionic degrees of freedom in the description of the “new” theory  $Z_{\text{ferm}}$  from the “old” one  $Z_{\text{bos}}$ . This description has a very close relation to the notion of fermionic anyon condensation [75].

The inverse of the map (3.27) is called *first bosonization*<sup>4</sup> and allows us to go freely from  $Z_{\text{ferm}}$  to  $Z_{\text{bos}}$  and viceversa [71]. It is defined as

$$Z_{\text{bos}}[Y, \beta] := \frac{1}{\sqrt{H^{D-1}(Y; \mathbb{Z}_2)}} \sum_{\lambda \in H^1(Y; \mathbb{Z})} Z_{\text{ferm}}[Y, s + \lambda] w^{-1}[Y, \beta, s + \lambda], \quad (3.29)$$

---

<sup>4</sup>This kind of bosonization procedure is invertible and is what allows us to the define the fermionization (3.27) in the first place. Therefore, it follows that the same amount of information is captured by the fermionic theory and its bosonic shadow, albeit written in different terms. This is opposed to the standard 0<sup>th</sup> bosonization, where the bosonic shadow of the fermionic theory is defined as a simple sum over the spin structures  $s$ :

$$Z_{\text{bos}}[Y] := \sum_{\lambda \in H^1(Y; \mathbb{Z}_2)} Z_{\text{ferm}}[Y, s + \lambda], \quad (3.28)$$

Indeed, this kind of bosonization in general has no inverse and no unambiguous fermionization procedure can arise from it.

where in general  $w$  is not expected to be written as a standard action, but for which one can make use of the properties

$$w[Y, \beta, s + \lambda] = w[Y, \beta, s](-1)^{\int_Y \lambda \cup \beta}, \quad (3.30)$$

$$w[Y, \beta, s]w^{-1}[Y, \beta, s'] = (-1)^{\int_Y (s-s') \cup \beta}, \quad (3.31)$$

$$\sum_{\beta} w[Y, \beta, s]w^{-1}[Y, \beta, s'] = |H^{D-1}(Y; \mathbb{Z}_2)|\delta(s - s'). \quad (3.32)$$

The reason one may be tempted to take the fermionization route to describe fSPTs is that, due to its nice properties, we are prone to identify  $\zeta$  in (3.22) with the Wilson loop insertion term  $w$ , which a priori is a different concept. Indeed, the two have the exact same structure and variables dependence one may look for, with the only constraint that in (3.22)  $\beta = A^*n$ , where  $A : Y \rightarrow BG$  is the background gauge field of the symmetry  $G$ . Therefore, we make the choice to interpret  $w$  as the generalization of the term appearing in Gu-Wen's phases and we expect to be eventually able to recover its expression in the particular case  $\beta = A^*n$  as discussed, i.e.  $w[Y, A^*n, s] = \zeta[Y, A^*n, s]$ .

It is now instructive to work out some details regarding the fermionization procedure while forgetting about the presence of the symmetry  $G$  by turning off  $A$ . Therefore, let's try to carry out the bosonization/fermionization between the trivial fermionic phase and its bosonic shadow. In this case it is easy to find that

$$Z_{\text{ferm}}^0[Y, s] = 1 \quad \rightarrow \quad Z_{\text{bos}}^0[Y, \beta] = \delta(\beta)w^{-1}[Y, \delta\lambda, s], \quad (3.33)$$

where  $\delta(\beta)$  implies that  $Z_{\text{bos}}$  vanish when  $[\beta] \neq [\emptyset] \in H^{D-1}(Y; \mathbb{Z}_2)$ . If so,  $\lambda$  is the cochain such that  $\delta\lambda = \beta$ . In particular, note that we left open the possibility that

$$w[Y, \delta\lambda, s] \neq 0. \quad (3.34)$$

This is an important condition: in fact, if our association of  $w$  with the Gu-Wen fermionic path integral contribution is correct, then it is natural to expect that the invariance under gauge transformation of  $Z_{\text{ferm}}$  can only be preserved if  $Z_{\text{bos}}$  has a particular anomaly, such that it matches the one of  $w$  itself.

After turning on  $A$ , the straightforward (and minimal) step to promote the theory to a  $G$ -equivariant model is done by replacing  $\beta \mapsto \beta + A^*n$ , where  $n \in H^{D-1}(BG; \mathbb{Z}_2)$ , so that<sup>5</sup>

$$Z_{\text{bos}}[Y, \beta, A] = Z_{\text{bos}}^0[Y, \beta + A^*n]. \quad (3.35)$$

---

<sup>5</sup>Although it is not obvious from the following, the bosonic theory described by (3.35) is a  $G$ -equivariant  $\mathbb{Z}_2$  gauge theory, see [72].

The fermionic theory that corresponds to this is

$$\begin{aligned} Z_{\text{ferm}}[Y, A, \nu, n, s] &= \frac{1}{\sqrt{H^{D-1}(Y; \mathbb{Z}_2)}} \sum_{\beta \in H^{D-1}(Y; \mathbb{Z}_2)} Z_{\text{bos}}[Y, \beta, A] w[Y, \beta, s] \\ &= Z_{\text{bSPT}}[Y, A, \nu] \zeta[Y, A^*n, s]. \end{aligned} \quad (3.36)$$

Here the term  $Z_{\text{bSPT}}$  replaces the non-vanishing term of (3.33) when  $[\beta] = [\emptyset]$ . Indeed, we know we need to have a term that cures the anomaly of  $w$  to get a well-defined  $Z_{\text{ferm}}$ . However, since now  $\beta = A^*n$  then  $w[Y, \beta, s] = \zeta[Y, A^*n, s]$  and we can infer that the addition of  $Z_{\text{bSPTs}}$  is necessary to cure the final expression. Thus, we just arrived again to the Gu and Wen's phases (3.24). Note that now, through this description the fermionic loop can be properly understood as the support of a fermion probe charged under the symmetry  $G$ . Moreover, this results allows to see that in full generality it is always necessary that a bosonic theory found via first bosonization is equipped with an anomalous  $(D-1)$  form symmetry  $\beta$  with anomaly under  $\beta \mapsto \beta + \delta\lambda$  equal to

$$(-1)^{\text{Sq}^2(\lambda)}. \quad (3.37)$$

In the Gu-Wen phases it only happens that this is reproduced via (3.18), when  $\beta = A^*n$ .

Having understood how Gu-Wen phases can be reproduced via fermionization and bosonization, we can now wonder how and if it is possible to enlarge the classification of fSPTs. The generalization is suggested by looking at how we went from the 0<sup>th</sup> bosonization to 1<sup>st</sup> bosonization (see footnote 4), with the new tool between the two being the introduction of a fermionic Wilson loop contribution  $w$  which makes the map invertible. The next step is then to ask whether it is possible to change the bosonization relations by introducing higher dimensional fermionic probes. In principle we know of the existence of such probes, as the higher dimensional equivalent of fermionic loops is described by Kitaev wires [76],  $p + ip$  superconductors and so on<sup>6</sup>. At this point we can specialize to our case of interest and set  $D = 3$ , where the only higher dimensional probe that we need to focus on is given by Kitaev wires, which worldsheets are surfaces Poincaré dual to elements  $\alpha \in Z^1(Y; \mathbb{Z}_2)$ .

By introducing these new kind of degrees of freedom we are actually defining a new notion of 2<sup>nd</sup> degree bosonization and fermionization, which is a straightforward generalization of (3.27) as being defined (up to an overall factor) as

$$Z_{\text{ferm}}[Y, s] = \sum_{\beta, \alpha} Z_{\text{bos}}[Y, \beta, \alpha] w[Y, \beta, \alpha, s]. \quad (3.38)$$

For consistency it is required  $w[Y, \beta, \alpha = 0, s] = w[Y, \beta, s]$ , thus reducing to the first fermionization (3.27). Moreover, we require again the condition

$$\delta\beta = 0. \quad (3.39)$$

---

<sup>6</sup>In light of our future discussion, note that all of these describe LRE states.

For the moment we will take this as reasonable, albeit in principle one may wonder why  $\delta\beta$  is not twisted by the presence of  $\alpha$ , much like it happens for  $\nu$  when turning on  $\beta$ . We will try to justify it at the end of the section, although the real origin of such constraints will be more clear in light of the discussion in Section 3.3. At this point we can then proceed by analogy and promote the trivial theory to a  $G$ -equivariant one by shifting  $\beta \mapsto \beta + A^*n$  and  $\alpha \mapsto \alpha + A^*a$  and arrive to a class of fermionic SPTs characterized by

$$Z_{\text{ferm}}[Y, A, \nu, n, a, s] = Z_{\text{bSPT}}[Y, A, \nu, n]w[Y, A^*n, A^*a, s]. \quad (3.40)$$

In particular, from (3.39) it follows that the addition of the Kitaev worldsheets in 3 dimensions is uncorrelated<sup>7</sup> to the presence of constraints from the Gu-Wen phases, so that no new twisting appears in the conditions (3.18) necessary to guarantee the gauge-invariance of  $Z_{\text{ferm}}$ . Therefore, all the conditions on group-cochains to classify fSPTs can be restated as a triple of elements  $(\nu, p, a) \in C^3(BG; \mathbb{R}/\mathbb{Z}) \times Z^2(BG; \mathbb{Z}_2) \times Z^1(BG; \mathbb{Z}_2)$  satisfying

$$\delta\nu = \frac{1}{2}p \cup p, \quad \delta p = 0, \quad \delta a = 0, \quad (3.41)$$

where we swapped to the additive notation of  $U(1) \cong \mathbb{R}/\mathbb{Z}$  for future convenience. Note that we have not yet discussed the group structure that comes with stacking of such fSPTs. This is highly non-trivial and will be argument of discussion in the next section.

Now that we stated how in principle one can decorate new fSPTs via addition of Kitaev wires, we try to justify the condition (3.39). In particular, recall that the bosonic shadows of Gu-Wen phases are described by  $G$ -equivariant  $\mathbb{Z}_2$  gauge theories. These systems can be equivalently stated as the description of the  $G$  equivariant toric code, where the  $G$  symmetry acts on quasi-particles in a way determined by  $\nu$  and  $p$  [72]. However, at low energy the toric code is known for having an additional  $\mathbb{Z}_2$  symmetry which exchanges the  $e$  and  $m$  quasi-particles. It turns out that in the  $G$ -equivariant case this symmetry can be modeled by the presence of a homomorphism  $a : G \rightarrow \mathbb{Z}_2$  which tells us which elements of  $G$  exchange the excitations  $e$  and  $m$  of the model. With this knowledge, we can understand the triple of elements (3.41) as a set of physical data associated to the defects of the  $G$  symmetry present in the theory. Let's start with discussing  $a$  first. As stated, a nonzero  $a(g)$  means that the element  $g$  acts as particle-vortex symmetry, so we can interpret it as if an insertion of a flux  $\hat{g}$  of the background gauge field carries a Majorana zero mode. Obviously, the homomorphism condition of  $a$  comes from the fact that fusing defects  $\hat{g}$  and  $\hat{h}$  produces the defect  $\widehat{gh}$  and the number of Majorana zero modes must be conserved, i.e.

$$a(g) + a(g) = a(gh). \quad (3.42)$$

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<sup>7</sup>More precisely, it is *almost* uncorrelated. Indeed, it does not affect the classification problem of phases but *do* affect the group product definition associated to stacking of different fSPTs. See again Section 3.3 for details.

In the case  $a = 0$ , one can then proceed to interpret  $p \in Z^2(BG; \mathbb{Z}_2)$  as assigning the fermionic parity  $p(g, h)$  to the junction between  $\hat{g}$ ,  $\hat{h}$  and  $\widehat{gh}$ . The requirement for this to be invariant under Pachner moves then guarantees again  $\delta p = 0$ . Moreover, in the case of  $a \neq 0$  the situation is not much different, since at the junction of three domain walls we have an even number of Majorana zero modes, as assured by condition (3.42). Therefore, we can infer that the guess (3.39) was correct. Finally, if one want to complete this description of the triple of elements (3.41), the cochain  $\nu$  should be associated to the amplitude assigned to a point-like junction of four domain wall worldsheets. Note that this precisely the dual description of what we discussed in Section 3.1.1, as point-like junctions of defects correctly represent the 0-dimensional dual cell structure to 3-dimensional simplices. Thus, it is not difficult to convince ourselves that (3.18) must hold also in this dual description for consistency, so that eventually one is able to recover the full expression (3.41).

### 3.3 From cohomology to cobordism: AHSS

At this point we understand how SPTs and their cohomological layers describe invertible  $(d + 1)$ -dimensional bulk theories associated to anomalous  $d$ -dimensional ones. However, the techniques presented deal mostly with lattice construction and partly with some local QFT effective action description. Instead, historically the classification of invertible theories related to the cobordism group  $\text{Hom}(\Omega_{d+1}^{\text{Spin}}(BG), U(1))$  is based on the assumption that they can all be properly captured via the abstract axiomatic definition of iTQFTs. However, this cobordism proposal is in principle not constructive: there is no straightforward way to reproduce the homotopy theory definition into a construction of phases of matter based on lattice. To be pessimistic, it may even be possible that some invertible homotopy TFTs may simply not be realizable as local lattice systems or even as local quantum field theories. As this would undoubtedly affect the classification of anomalies, it is important to understand if we can say more on this regard. Therefore, it is important to understand whether there exists some sort of connection between the cohomological layers construction with the cobordism groups.

We already presented evidence of this hypothetical relation in the case of bosonic theories, as we already discussed that in  $D = 3$  we have  $\text{Hom}(\Omega_3^{SO}(BG), U(1)) \cong H^3(BG, U(1))$ . Therefore, if the statement is really true, one should eventually be able to find such an isomorphism also in the spin case, or even for more general structures.

Note that the condensed matter systems discussed so far are limited to the presence of a symmetry  $G$  and a spin structure; the presence of the latter being elucidated thanks to Kapustin and Gaiotto's work. Therefore, we will limit ourselves to present such a relation for cobordism groups  $\text{Hom}(\Omega_3^{\text{Spin}}(BG), U(1))$ .

Based on what has been presented so far, the data that dress the defects for fermionic SPTs in  $D = 3$  can be summarized in a triple of elements  $(\nu, p, a) \in$

$C^3(BG; U(1)) \times Z^2(BG; \mathbb{Z}_2) \times Z^1(BG; \mathbb{Z}_2)$  that satisfies (3.41).

It turns out that such triples of elements identify a group  $\mathfrak{G}(G)$  with a non-trivial group product, which can be restated as the following [Theorem 1.1, [77]]:

**Theorem 3.43.** *The spaces*

$$\begin{aligned} \mathcal{C}(G) &:= C^1(BG; \mathbb{Z}_2) \times C^0(BG; \mathbb{Z}_2), \\ \mathcal{C}'(G) &:= C^3(BG; \mathbb{R}/\mathbb{Z}) \times Z^2(BG; \mathbb{Z}_2) \times Z^1(BG; \mathbb{Z}_2) \end{aligned} \quad (3.44)$$

admit the group structures defined by

$$\begin{aligned} \mathcal{C}(G) : \quad (t, x) \cdot (s, y) &= (t + s + x \cup \delta y, x + y), \\ \mathcal{C}'(G) : \quad (\nu, p, a) \cdot (\mu, q, b) &= (\eta, p + q + a \cup b, a + b), \end{aligned} \quad (3.45)$$

where

$$\eta = \nu + \mu + \frac{1}{2} [p \cup_1 q + (p + q) \cup_1 (a \cup b) + a \cup (a \cup_1 b) \cup b] + \frac{1}{4} A \cup B \cup B. \quad (3.46)$$

Here  $A, B$  denotes respectively the lifts of  $a, b$  via the map  $1/n : \mathbb{Z}_n \rightarrow \mathbb{R}/\mathbb{Z}$ . The group  $\mathfrak{G}(G)$  is then defined as

$$\mathfrak{G}(G) := \frac{\text{Ker } \mathcal{D}'}{\text{Im } \mathcal{D}}, \quad (3.47)$$

where

$$\begin{aligned} \mathcal{D} : \quad \mathcal{C}(G) &\rightarrow \mathcal{C}'(G), & \mathcal{D}' : \quad \mathcal{C}'(G) &\rightarrow C^4(BG; \mathbb{R}/\mathbb{Z}). \\ (t, x) &\mapsto \left(\frac{1}{2}t \cup \delta t, \delta t, \delta a\right) & (w, p, a) &\mapsto \delta w + \frac{1}{2}a \cup a \end{aligned} \quad (3.48)$$

Moreover, it holds

$$\mathfrak{G}(G) \cong \text{Hom} \left( \Omega_3^{\text{Spin}}(BG), U(1) \right). \quad (3.49)$$

Therefore, the classification given by a tower of cohomological groups indeed matches to the cobordism one in 3 dimensions. This will be very useful, and we will make use of it in the next chapter in order to switch to the most convenient representation based on the computations we have to do.

Although at first the existence of the isomorphism (3.49) may sound surprising, there is a deep mathematical meaning, which is explained via the *Atiyah-Hirzebruch Spectral Sequence* (AHSS). This mathematical device allows to compute bordism groups  $\Omega_D^{\text{Spin}}(BG)$  in terms of homology groups

$$H_j \left( BG, \Omega_k^{\text{Spin}}(\text{pt}) \right), \quad j + k = D. \quad (3.50)$$

The same procedure can be also carried out for the cobordism groups via the exact functor  $\text{Hom}(-, U(1))$ .

As a quick summary, for a fixed symmetry group  $G$ , by considering the fibration

$$\text{pt} \longrightarrow BG \longrightarrow BG, \quad (3.51)$$

the technique provides us a particular filtration that builds up the whole group  $\Omega_d^{\text{Spin}}(BG)$ :

$$0 = F_{-1}\Omega_d^{\text{Spin}}(BG) \subset F_0\Omega_d^{\text{Spin}}(BG) \subset \cdots \subset F_d\Omega_d^{\text{Spin}}(BG) = \Omega_d^{\text{Spin}}(BG), \quad (3.52)$$

where  $F_j\Omega_d^{\text{Spin}}(BG) = \text{Im}(\Omega_d^{\text{Spin}}(BG^{(j)}) \hookrightarrow \Omega_d^{\text{Spin}}(BG))$ , with  $BG^{(j)}$  the  $j$ -skeleton of  $BG$ .

In particular, the AHSS describes a set of  $\mathbb{Z}_+^2$  graded ‘‘pages’’ of groups  $E_{p,q}^r$  labelled by  $r \in \mathbb{Z}_+$  and related to each other such that the  $\infty$ -page  $E_{p,q}^\infty$  is equal to the groups

$$E_{k,d-k}^\infty = \frac{F_k\Omega_d^{\text{Spin}}(BG)}{F_{k-1}\Omega_d^{\text{Spin}}(BG)}. \quad (3.53)$$

The relations between the  $r^{\text{th}}$  and  $(r+1)^{\text{th}}$  pages is given by the presence of differentials  $d_r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$  such that  $d_r^2 = 0$  and

$$E_{p,q}^{r+1} := \frac{\ker d_r : E_{p,q}^r \longrightarrow E_{p-r,q+r-1}^r}{\text{im } d_r : E_{p+r,q-r+1}^r \longrightarrow E_{p,q}^r}. \quad (3.54)$$

In short, the  $(r+1)^{\text{th}}$  page of groups describes the homology groups of the  $r^{\text{th}}$  page. Another important fact is that the second page of groups is actually described by

$$E_{p,q}^2 = H_p(BG; \Omega_q^{\text{Spin}}(\text{pt})). \quad (3.55)$$

Therefore, once we know the values of the spin bordism groups  $\Omega_d^{\text{Spin}}(\text{pt})$  we can start to find the various group quotients that make up the whole  $\Omega_d^{\text{Spin}}(BG)$  by looking at the page  $E_{p,q}^2$  and gradually move up to  $E_{p,q}^\infty$ . Fortunately, in the AHSS it can be shown that eventually the sequence converges at finite  $r$ , so that we only need to look at a finite number of terms to figure out the whole group. For details on these and other issues, useful references are [78–80]. Of course, the computation in general is still far from trivial, as one needs to know the expression of the differentials  $d_r$  in order to move from one page to another. However, in general it is possible to figure these out via complementary knowledge, so that one is still able to compute the group they are interested in. This holds especially for low dimensional groups, like in our case, where very few details enter the game and  $\Omega_d^{\text{Spin}}(BG)$  can be figured out with relative ease.

The spin bordism groups we need for our computations are known [8, 81] and are:

$$\Omega_0^{\text{Spin}}(\text{pt}) = \mathbb{Z}, \quad \Omega_1^{\text{Spin}}(\text{pt}) = \mathbb{Z}_2, \quad \Omega_2^{\text{Spin}}(\text{pt}) = \mathbb{Z}_2, \quad \Omega_3^{\text{Spin}}(\text{pt}) = 1, \quad \dots \quad (3.56)$$

By applying the exact functor  $\text{Hom}(-, U(1))$  the spectral sequence  $E_{p,q}^r$  then gets translated into a new sequence  $E_r^{p,q}$ , where similarly to before

$$E_2^{p,q} = H^p(BG, \text{Hom}(\Omega_d^{\text{Spin}}, U(1))), \quad d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}, \quad (3.57)$$

and

$$E_\infty^{d-k, k} = \frac{F^k \text{Hom}(\Omega_d^{\text{Spin}}(BG), U(1))}{F^{k+1} \text{Hom}(\Omega_d^{\text{Spin}}(BG), U(1))}, \quad (3.58)$$

where  $F^k \text{Hom}(\Omega_d^{\text{Spin}}(BG), U(1)) \equiv F^k(BG)$  is again a filtration of  $F^0(BG) = \text{Hom}(\Omega_d^{\text{Spin}}(BG), U(1))$ .

Since we are interested to cobordism groups in  $d = 3$ , we must figure out the value of  $E_\infty^{k, 3-k}$ . The second page of the spectral sequence is the one represented in Figure 3.4. In particular, we see that the only possibly non-trivial differentials  $d_2$  are the maps  $d_2 : H^1(BG; \mathbb{Z}_2) \rightarrow H^3(BG; \mathbb{Z}_2)$  and  $d_2 : H^2(BG; \mathbb{Z}_2) \rightarrow H^4(BG; \mathbb{Z}_2)$ . It turns out that there is no natural operation of the first form<sup>8</sup>, so that the first differential vanish and  $E_2^{1,2} = E_\infty^{1,2}$ , like what happens for  $E_2^{3,0} = E_\infty^{3,0}$ . The only non-trivial differential is then

$$\begin{aligned} d_2 : H^2(BG; U(1)) &\rightarrow H^4(BG; U(1)). \\ p &\mapsto \frac{1}{2}p \cup p \end{aligned} \quad (3.59)$$

Such expression can be found by considering that the only possible operation in this case corresponds to the first  $k$ -invariant of the spectra  $M\text{Spin}$  [82] and is a combination of the Steenrod square  $\text{Sq}^2 : H^2(BG; \mathbb{Z}_2) \rightarrow H^4(BG; \mathbb{Z}_2)$ , which reduces the standard cup product, and the lift  $H^4(BG; \mathbb{Z}_2) \rightarrow H^4(BG; U(1))$ . Therefore, we have been able to figure out the structure of degree 3 of the page  $E_3 = E_\infty$ , which tells us

$$\text{Hom}(\Omega_d^{\text{Spin}}(BG), U(1)) = F^0(BG) \supset F^1(BG) \supset F^2(BG) \supset 0, \quad (3.60)$$

where

$$F^2(BG) = H^3(BG; U(1)), \quad \frac{F^1(BG)}{F^2(BG)} = SH^2(BG; \mathbb{Z}_2), \quad \frac{F^0(BG)}{F^1(BG)} = H^1(BG; \mathbb{Z}_2). \quad (3.61)$$

What Theorem 3.43 tells us then is the complete structure of the quotients (3.61) described by the cohomological layers.

At this point, with the informations given by AHSS we can reinterpret the construction of Section 3.2. Via the bosonization map we learned that the various cohomological layers correspond to the addition of various fermionic degrees of freedom with increasing dimensionality. This is *exactly* what we see happenings in the AHSS computation. Therefore, we understand how the higher layers

---

<sup>8</sup>The only natural operation would be  $\text{Sq}^2 : H^i(-; \mathbb{Z}_2) \rightarrow H^{i+2}(-; \mathbb{Z}_2)$ , which however vanish when  $i < 2$ .

|   |     |                         |                         |                         |                 |
|---|-----|-------------------------|-------------------------|-------------------------|-----------------|
| 3 | 1   | 1                       | 1                       | 1                       | 1               |
| 2 | ... | $H^1(BG; \mathbb{Z}_2)$ | ...                     | ...                     | ...             |
| 1 | ... | ...                     | $H^2(BG; \mathbb{Z}_2)$ | $H^3(BG; \mathbb{Z}_2)$ | ...             |
| 0 | ... | ...                     | ...                     | $H^3(BG; U(1))$         | $H^4(BG; U(1))$ |
|   | 0   | 1                       | 2                       | 3                       | 4               |

Figure 3.4: The second page  $E_2^{p,q}$  for the AHSS associated to the fibration (3.51).

$F^{1,2}(BG)$  correspond to the insertion of fermionic probes of the proper dimension, supporting LRE fermionic states identified by elements in  $\Omega_{1,2}^{\text{Spin}}(\text{pt}) \cong \mathbb{Z}_2$  [71]. Note also that now we can easily interpret (3.39) as the vanishing<sup>9</sup> of the differential  $d_2 : H^1(BG; \mathbb{Z}_2) \rightarrow H^3(BG; \mathbb{Z}_2)$ . Another interesting point of view on this regard can be also seen by describing invertible field theories by means of their topological defects, in which case this construction appears once again in a natural way; for more details see [82].

### 3.4 Spin-iTQFTs and the cobordism classification

In this section we formalize the definition of iTQFTs we employ for the analysis of anomalies and the construction of the related Hilbert space, which will be of great importance for computations. The definition of TQFT presented here regards the non-extended case, with some minor details that account for its higher categorical extension. Despite this slight different choice of axioms, the isomorphism between the group of  $d$ -dimensional iTQFTs and  $d$ -cobordism groups still holds, albeit at the level of isomorphism classes rather than deformation classes; see [10]. So far we worked with manifolds equipped with a spin structure and a  $G$ -bundle, without recalling the details of the definition of the former; here we give such details and extend the construction.

For any QFT, to be defined it is necessary first to specify its dimension  $d$  and its symmetry  $H_d$ . The group  $H_d$  includes the symmetry of the internal degrees of freedom as well as the spacetime symmetries via the group homomorphism

$$\rho_d : H_d \rightarrow O(d). \quad (3.62)$$

The map  $\rho_d$  is not necessarily surjective, as it may be<sup>10</sup>  $\rho_d(H_d) = SO(d)$  if the theory does not have time reflection symmetry. In general, the internal symmetry group  $G^f$  is defined as the kernel of (3.62) and it helps to identify  $H_d$  as its extension via the short exact sequence

$$0 \longrightarrow G^f \longrightarrow H_d \longrightarrow \rho_d(H_d) \longrightarrow 0. \quad (3.63)$$

<sup>9</sup>Not only this: we see in fact that if we were in higher dimensions, such constraint would need to be modified and its *a priori* physical justification would not be naive.

<sup>10</sup>We do not consider the case when  $\rho_d(H_d) = \emptyset$ .

In absence of time-reversal symmetry,  $H_d$  is of the form

$$H_d = \text{Spin}(d) \times_{\mathbb{Z}_2^f} G^f = \frac{\text{Spin}(d) \times G^f}{\langle((-1)^F, k_0)\rangle}, \quad (3.64)$$

where  $k_0 \in G^f$  is an order 2 element,  $k_0^2 = \mathbb{I}$ ,  $(-1)^F$  is the fermionic parity, i.e. the non-trivial element in the center of  $\text{Spin}(d)$  and  $\langle((-1)^F, k_0)\rangle$  is the group generated by the diagonal element  $((-1)^F, k_0) \in \text{Spin} \times G^f$ . For example, for  $G^f = \mathbb{Z}_2 \times G$  and  $k_0 = (-1, \mathbb{I})$  we have  $H_d = \text{Spin}(d) \times G$ , which is the general expression for the type of symmetry we supposed to have so far. Whenever we have  $k_0 \neq \mathbb{I}$ , the theory is fermionic and the symmetry group  $G^f$  stands for the internal symmetry of the fermionic degrees of freedom. The internal symmetry for bosonic degrees of freedom will be instead denoted by  $G$  and identified by  $G := G^f / \mathbb{Z}_2^f$ , where  $\mathbb{Z}_2^f = \langle k_0 \rangle = \langle (-1)^F \rangle$ . An important example when the fermionic symmetry group does not factorize like  $G^f \neq \mathbb{Z}_2 \times G$  is when  $G^f = U(1)$ , in which case  $H_d = \text{Spin}^c(d)$ . In our work we will also analyze discrete subgroups of it, when  $G^f = \mathbb{Z}_{2^l}$ . If instead  $k_0 = \mathbb{I}$ , the theory will be purely bosonic, with symmetry  $H_d = SO(d) \times G$ .

To make the theory *topological*, instead of the compact Lie group (3.64) one should more properly consider the corresponding non-compact group

$$H_d = \widetilde{\text{GL}}_+(d, \mathbb{R}) \times_{\mathbb{Z}_2^f} G^f \quad (3.65)$$

where  $\widetilde{\text{GL}}_+(d, \mathbb{R})$  is the double cover of  $\text{GL}_+(d, \mathbb{R})$ , the group of general linear transformations of  $\mathbb{R}^d$  preserving orientation (i.e. given by matrices with positive determinant). Note that the group  $\text{GL}_+(d, \mathbb{R})$  deformation retracts to  $SO(d)$ , while  $\widetilde{\text{GL}}_+(d, \mathbb{R})$  deformation retracts to  $\text{Spin}(d)$ , so that the following diagram commutes:

$$\begin{array}{ccccc} & & \text{Spin}(d) & \longrightarrow & SO(d) \\ & \nearrow & \uparrow & & \uparrow \\ 1 & \longrightarrow & \mathbb{Z}_2^f & & 1 \\ & \searrow & \downarrow & & \downarrow \\ & & \widetilde{\text{GL}}_+(d) & \longrightarrow & \text{GL}_+(d) \end{array} \quad (3.66)$$

From the deformation retract property, unless strictly necessary we can often think of  $H_d$  as of the form (3.64), as most properties still follow.

In the case with time-reversal symmetry  $\mathbf{T}$  the  $\text{Spin}(d)$  group appearing above and below should be replaced by  $\text{Pin}^\pm(d)$  group, depending on whether  $\mathbf{T}^2 = 1$  or  $\mathbf{T}^2 = (-1)^F$ , the  $SO(d)$  group replaced with  $O(d)$ , and  $\text{GL}_+(d, \mathbb{R})$  replaced with  $\text{GL}(d, \mathbb{R})$ .

An important concept for the generalization of the definition of TQFTs in the extended case is that its symmetry  $H_d$  should be stable in the homotopy sense over

$\mathrm{GL}_+(d)$  (or  $\mathrm{GL}(d)$ ). This implies that we should have an inclusion map  $i_d : H_d \hookrightarrow H_{d+1}$  that together with  $\rho_d$  and the natural inclusion

$$\begin{aligned} j_d : \mathrm{GL}_+(d) &\hookrightarrow \mathrm{GL}_+(d+1), \\ R &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \end{aligned} \quad (3.67)$$

provide the commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_{d-1} & \xrightarrow{i_{d-1}} & H_d & \xrightarrow{i_d} & H_{d+1} & \longrightarrow & \dots \\ & & \downarrow \rho_{d-1} & & \downarrow \rho_d & & \downarrow \rho_{d+1} & & \\ \dots & \longrightarrow & \mathrm{GL}_+(d-1) & \xrightarrow{j_{d-1}} & \mathrm{GL}_+(d) & \xrightarrow{j_d} & \mathrm{GL}_+(d+1) & \longrightarrow & \dots \end{array} \quad (3.68)$$

In this case, a theory with symmetry  $H_d$  in any  $d$  dimension is said to be of *H symmetry type*, where

$$H = \mathrm{colim}_{n \rightarrow \infty} H_n. \quad (3.69)$$

The case of (3.64) where  $G^f$  is fixed for any  $d$  falls into this description.

The next step is understanding the data that manifolds must be equipped with in order to support TQFTs with a particular symmetry type  $H$ . This data is called *H structure* and is defined by the following:

**Definition 3.70.** *An  $H_d$ -manifold is a triple  $(X, P, \vartheta)$ , where:*

1.  $X$  is an oriented  $d$ -manifold;
2.  $P \rightarrow X$  is a principal  $H_d$  bundle;
3.  $\vartheta$  is a bundle map from  $P$  into  $F_+(TX)$ , the  $\mathrm{GL}_+(d)$  principal bundle of oriented frames in the tangent bundle  $TX$ :

$$\begin{array}{ccccc} H_d & \longrightarrow & P & \xrightarrow{\vartheta} & F_+(TX) & \longleftarrow & \mathrm{GL}_+(d, \mathbb{R}) \\ & & \downarrow & \swarrow & & & \\ & & X & & & & \end{array} ; \quad (3.71)$$

4. The bundle map is required to commute with the corresponding group actions on the fibers:

$$\vartheta(g \cdot p) = \rho_d(g) \cdot \vartheta(p), \quad \forall p \in P, g \in H_d. \quad (3.72)$$

Sometimes, for the sake of brevity, we will denote an  $H_d$ -manifold simply as  $X$  (instead of  $(X, P, \vartheta)$ ), keeping in mind that it comes equipped with a particular choice of  $H_d$ -structure. Equivalently, when  $G^f = \mathbb{Z}_2^f \times G$  we will often say that the theory has a spin structure (and a  $G$ -bundle), while for  $G^f = \mathbb{Z}_{2^{l+1}}^f$  the theory is said to be a  $\mathrm{Spin}\text{-}\mathbb{Z}_{2^{l+1}}^f$  type.

Note that the original Definition 2.56 of bordism groups  $\Omega_d^{\text{Spin}}(BG)$  is easily generalized for any  $H$  symmetry type and not only for  $H = \text{Spin} \times G$ . We denote these general bordism groups, which are again abelian, via the notation  $\Omega_d^H$ . These kinds of groups can be categorified and give origin to the bordism categories  $\mathbf{Bord}_{\langle d-1, d \rangle}(H)$ , where the elements are  $H_{d-1}$ -manifolds and morphisms  $\text{Hom}(X_1, X_2)$  are  $H_d$ -bordisms  $Y : X_1 \rightarrow X_2$ , i.e.  $Y$  is such that  $\partial Y \cong X_1 \sqcup X_2$ . In particular, note that there is always the identity map  $I_X := X \times [0, 1] : X \rightarrow X$  and that if  $[X_1] \neq [X_2]$  in  $\Omega_{d-1}^H$ , then  $\text{Hom}(X_1, X_2) = \emptyset$ . Moreover, the group multiplication  $\sqcup$  induces a symmetric monoidal structure defined by  $X_1 \otimes X_2 := X_1 \sqcup X_2$ .

Another important property of  $H_d$  manifolds  $(X, P, \vartheta)$  is that we can define their opposite structure  $(X, \bar{P}, \bar{\vartheta})$ , a concept that was already anticipated in Section 2.2. To do so, one first needs to define the principal  $H_{d+1}$ -bundle  $P \times_{i_d} H_{d+1} := (P \times H_{d+1}) / \sim_{i_d}$  via the identification  $(p \cdot h, \mathbb{I}) \sim_{i_d} (p, i_d(h))$  for any  $p \in P$  and  $h \in H_d$ , as well as the map

$$\begin{aligned} \Theta : P \times_{i_d} H_{d+1} &\longrightarrow F_+(\mathbb{R} \oplus TX) \\ (p, k) &\longmapsto (1, \vartheta(p)) \cdot \rho_{d+1}(k) \end{aligned} \quad (3.73)$$

which can be checked to be well-defined. With the canonical forgetful map  $r : F_+(\mathbb{R} \oplus TX) \rightarrow F_+(TX)$ , the opposite structure  $(X, \bar{P}, \bar{\vartheta})$  is then defined by

$$\bar{\vartheta} := r \circ \Theta, \quad \bar{P} := \Theta^{-1}(-1, x), \quad \forall x \in F_+(TX). \quad (3.74)$$

Note that in the case  $H = SO$ , the opposite structure correspond to a change of orientation of the manifold. More in general, the meaning of an  $H$  opposite structure can be easily understood if one consider  $X$  to be a boundary component of a  $H_{d+1}$  manifold  $Y$ . In this case, the addition of  $\mathbb{R}$  in the vector space of which we consider the framings,  $\mathbb{R} \oplus TX$ , can be seen as the addition of the normal bundle  $NX$  of  $Y$  with a choice of direction, seen embedded into  $Y$ . In this case, the opposite structure is equivalent to reversing such direction.

Sometimes, for the sake of brevity, we will denote an  $H_d$ -manifold simply as  $X$  (instead of  $(X, P, \vartheta)$ ), keeping in mind that it comes equipped with a particular choice of  $H_d$ -structure. The same manifold with the opposite  $H_d$  structure will also denoted as  $\bar{X}$ .

### 3.4.1 Axioms of TQFTs and anomalous phases

With these necessary preliminaries, we can make use of the functorial definition of a TQFT [9, 37]. This section is based also on additional formalization of the concepts presented in the original work [1].

**Definition 3.75.** *A  $d$ -dimensional (Euclidean) TQFT is a symmetric monoidal functor*

$$\mathcal{Z} : \mathbf{Bord}_{\langle d-1, d \rangle}(H) \longrightarrow \mathbf{sVect}_{\mathbb{C}}, \quad (3.76)$$

where  $\mathbf{Bord}_{\langle d-1, d \rangle}(H)$  is the categorification of the abelian group  $\Omega_{d-1}^H$  and  $\mathbf{sVect}_{\mathbb{C}}$  is the category of  $\mathbb{Z}_2$ -graded complex vector spaces, equipped with the standard tensor

product. Moreover, the theory is unitary if  $\mathcal{Z}$  is equivariant with respect to the involution pair of functors  $(\beta_B, \beta_V)$ , defined by

$$\begin{cases} \beta_B(X) := \overline{X}, & \forall X \in \text{Obj}(\mathbf{Bord}_{(d-1,d)}(H)), \\ \beta_B(Y) := \overline{Y}, & \forall Y \in \text{Hom}(\mathbf{Bord}_{(d-1,d)}(H)), \end{cases} \quad (3.77)$$

and

$$\begin{cases} \beta_V(\mathcal{H}) := \overline{\mathcal{H}}, & \forall \mathcal{H} \in \text{Obj}(\mathbf{sVect}_{\mathbb{C}}), \\ \beta_V(\phi) := \phi^\dagger, & \forall \phi \in \text{Hom}(\mathbf{sVect}_{\mathbb{C}}). \end{cases} \quad (3.78)$$

The TQFT is said to be invertible if for each  $\mathcal{H}(X) \equiv \mathcal{Z}(X)$  we have

$$\dim \mathcal{H}(X) = 1, \quad \forall X \in \text{Obj}(\mathbf{Bord}_{(d-1,d)}(H)). \quad (3.79)$$

The requirement of equivariancy under (3.77)-(3.78) is how we guarantee reflection positivity of the theory, which in Euclidean is equivalent to the notion of unitarity.

A particular set of bordisms are *isomorphisms* between pairs of  $H_d$ -manifolds  $(X, P, \vartheta)$  and  $(X', P', \vartheta')$ , which are defined as smooth bundle maps  $\tilde{\psi}$  covering a diffeomorphism  $\psi$  such that the following diagram is commutative:

$$\begin{array}{ccc} F_+(TY) & \xrightarrow{\psi_*} & F_+(TY') \\ \downarrow & & \downarrow \\ \vartheta \left( \begin{array}{ccc} & & \\ & \psi & \\ & & \end{array} \right) & & \vartheta' \\ \uparrow & & \uparrow \\ P & \xrightarrow{\tilde{\psi}} & P' \end{array} \quad (3.80)$$

where  $\psi_*$  is the pushforward (differential) of  $\psi$ .

A TQFT then associates an isomorphism  $\hat{\psi}$  between the pair of the corresponding vector spaces:

$$\hat{\psi} : \mathcal{H}(Y, P, \vartheta) \longrightarrow \mathcal{H}(Y', P', \vartheta') \quad (3.81)$$

which is functorial with respect to composition:  $\widehat{\tilde{\psi} \circ \tilde{\phi}} = \hat{\psi} \circ \hat{\phi}$ . More generally, as we saw a TQFT associates a linear map to a bordism between a pair of  $H_d$  manifolds: an  $H_{d+1}$ -manifold equipped with isomorphism of its boundary (considered as an  $H_d$ -manifold) to the disjoint union of this pair, with orientation of one of the  $H_d$ -manifold flipped. The bordisms are considered modulo isomorphisms of  $H_{d+1}$ -manifolds identical at the boundary. The special case (3.81) is then realized by taking the bordism to be  $Y \times [0, 1]$  with the product  $H_{d+1}$  structure and the boundary isomorphisms given by the identity and  $\tilde{\psi}$ .

In practice, for concrete calculations, it may be not convenient to work with all possible triples  $(X, P, \vartheta)$ . Instead, let us fix one particular oriented  $d$ -manifold  $X$  in its orientation-preserving diffeomorphism class. For this fixed manifold consider

equivalence relation between the triples given by the isomorphisms acting identically on  $X$ , i.e. the bundle isomorphism  $\mu$  such that the following diagram is commutative:

$$\begin{array}{ccc}
 & F_+(TX) & \\
 \vartheta \nearrow & \downarrow & \nwarrow \vartheta' \\
 P & X & P' \\
 \mu \searrow & & \nearrow
 \end{array} . \quad (3.82)$$

Denote the set of equivalence classes as  $\text{Spin}_{G^f}(X)$ . For each class  $a \in \text{Spin}_{G^f}(X)$  we then fix a particular representative bundle  $P_a \rightarrow X$  and a map  $\vartheta_a : P_a \rightarrow F_+(TX)$ . A TQFT then provides us with a family of vector spaces

$$\mathcal{H}_a := \mathcal{H}(X, P_a, \vartheta_a), \quad a \in \text{Spin}_{G^f}(X). \quad (3.83)$$

The equivalence class  $a$  can be understood as the choice of the background  $\text{Spin}(d) \times_{\mathbb{Z}_2^f} G^f$ -structure for some fixed metric on  $X$ . When the group  $G^f$  is finite, there is a finite number of equivalence classes. Moreover, if the internal symmetry is of the form  $G^f = \mathbb{Z}_2^f \times G$ , and  $G$  is a discrete abelian group, one can understand it as a pair  $a = (s, a') \in \text{Spin}_{G^f}(X)$  where  $a' \in H^1(X, G)$  is the background  $G$  gauge field and  $s \in \text{Spin}(X)$  is a choice of spin structure on  $X$ .

By  $\psi^{-1*}a$  let us denote the pullback of the structure  $a$  with respect to the inverse of the diffeomorphism  $\psi : X \rightarrow X$ . Namely, it is the equivalence class of the triple  $(X, \psi^{-1*}P_a, \psi_*\vartheta_a\psi^*)$ . We then restrict our attention to the isomorphisms between the finite set of triples  $(X, P_a, \vartheta_a)$ :

$$\begin{array}{ccc}
 F_+X & \xrightarrow{\psi_*} & F_+X \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{\psi} & X \\
 \uparrow & & \uparrow \\
 P_a & \xrightarrow{\tilde{\psi}} & P_{\psi^{-1*}a}
 \end{array} \quad \vartheta_a \left( \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) \vartheta_{\psi^{-1*}a}. \quad (3.84)$$

A TQFT associates to them maps

$$\hat{\psi} : \mathcal{H}_a \longrightarrow \mathcal{H}_{\psi^{-1*}a}. \quad (3.85)$$

The maps only depend on the homotopy classes of the maps  $\psi$  as the theory is topological. This is because one can relate the homotopic maps by an isomorphism inside the cylinder  $X \times [0, 1]$ .

For a fixed diffeomorphism  $\psi : X \rightarrow X$  the covering bundle map  $\psi$  is fixed up to compositions automorphisms of  $H_d$ -structures:

$$\begin{array}{ccc}
 & F_+X & \\
 \theta_a \nearrow & \downarrow & \nwarrow \theta_a \\
 P_a & X & P_a \\
 \mu \longrightarrow & & \longleftarrow
 \end{array}
 \tag{3.86}$$

which are finite number of, assuming again that  $G$  is finite. They similarly define the action

$$\hat{\mu} : \mathcal{H}_a \longrightarrow \mathcal{H}_a. \tag{3.87}$$

All the isomorphisms (3.84), considered up to homotopy, form a group, which we denote by  $\widetilde{\text{MCG}}(X)$ . If  $X$  is connected, the automorphisms (3.87) form a subgroup group isomorphic to  $G^f$ . The total group  $\widetilde{\text{MCG}}(X)$  is then an extension of  $\text{MCG}(X)$ , the orientation preserving mapping class group of  $X$ , by  $G^f$ :

$$1 \longrightarrow G^f \xrightarrow{i} \widetilde{\text{MCG}}(X) \xrightarrow{\pi} \text{MCG}(X) \longrightarrow 1. \tag{3.88}$$

When  $G^f$  is abelian, the extension is central. Elements of  $\text{MCG}(X)$  act on the space of isomorphism classes of structures  $\text{Spin}_{G^f}(X)$ :

$$\text{MCG}(X) \times \text{Spin}_{G^f}(X) \longrightarrow \text{Spin}_{G^f}(X), \tag{3.89}$$

$$([\psi], a) \longrightarrow [\psi] \cdot a \equiv \psi^{-1*}a,$$

where  $\psi : X \rightarrow X$  is a diffeomorphism representing an element  $[\psi] \in \text{MCG}(X)$ . The full  $\widetilde{\text{MCG}}(X)$  then also acts on  $\text{Spin}_{G^f}(X)$  through the projection  $\pi$ .

The partition function  $Z(X, g, a)$  of the boundary  $d$ -dimensional QFT can be understood as an element of  $\mathcal{H}_a$ , which is one-dimensional. To assign to it a particular number one still needs to choose a basis element  $e_a$  in the vector space, so that

$$Z(X, g, a)e_a \in \mathcal{H}_a. \tag{3.90}$$

The linear maps (3.85) then act on the basis elements as follows:

$$\widehat{\psi} : e_a \longmapsto C_{[\tilde{\psi}]}(a) e_{[\psi] \cdot a} \tag{3.91}$$

where  $[\tilde{\psi}] \in \widetilde{\text{MCG}}(X)$  and  $[\psi] \equiv \pi([\tilde{\psi}]) \in \text{MCG}(X)$ . We can assume that  $C_{[\tilde{\psi}]}(a) \in U(1) \subset \mathbb{C}$  because of unitarity. The coefficients must satisfy the following cocycle condition:

$$C_{[\tilde{\psi}][\tilde{\phi}]}(a) = C_{[\tilde{\psi}]}([\phi] \cdot a) C_{[\tilde{\phi}]}(a). \tag{3.92}$$

Moreover, the change of basis  $e_a \rightsquigarrow \alpha(a)e_a$  results in the following redefinition of the coefficients  $C$ :

$$C_{[\tilde{\psi}]}(a) \rightsquigarrow C_{[\tilde{\psi}]}(a) \frac{\alpha([\tilde{\psi}] \cdot a)}{\alpha(a)}. \quad (3.93)$$

The coefficients  $C$ , regarded as a function<sup>11</sup>

$$C : \widetilde{\text{MCG}}(X) \longrightarrow U(1)^{\text{Spin}_{Gf}(X)} \quad (3.94)$$

modulo redefinitions (3.93), define an element in the first group cohomology of  $\widetilde{\text{MCG}}(X)$ :

$$[C] \in \mathcal{H}^1(\widetilde{\text{MCG}}(X), U(1)^{\text{Spin}_{Gf}(X)}). \quad (3.95)$$

Here  $U(1)^{\text{Spin}_{Gf}(X)}$  is defined as a module of  $\widetilde{\text{MCG}}(X)$  with the action determined by the action on  $\text{Spin}_{Gf}(X)$  defined above. Equivalently, one can consider  $C$  as a representation of the action groupoid  $\widetilde{\text{MCG}}(X) \ltimes \text{Spin}_{Gf}(X)$ . In its categorical description, the objects are elements  $a \in \text{Spin}_{Gf}(X)$  and the arrows (i.e. morphisms) between the pair of objects  $(a_1, a_2)$  are elements  $[\tilde{\psi}] \in \widetilde{\text{MCG}}(X)$  such that  $a_2 = [\tilde{\psi}] \cdot a_1$ . Recall that in the groupoid representation, to each object one associates a vector space and to each arrow a linear map. Therefore (3.85) provides a representation of the groupoid where all vector spaces are one-dimensional.

Representations of a grupoid are known to be described in terms of individual representations of its connected components. For each connected component, up to isomorphisms, there is a one-to-one correspondence between groupoid representations and representation of the group of closed loops in the groupoid starting and ending at some fixed ‘‘base’’ object  $a_*$  (does not matter which). The correspondence is realized as follows. First, it is clear that a groupoid representation provides a representation of the group of loops. To construct a representation of a connected groupoid from the representation of the loop group one can proceed as follows. For each object  $a$  in the groupoid choose an arrow  $\ell_a$  to the base object  $a_*$ . Then, take  $\mathcal{H}_a := \mathcal{H}_{a_*}$  and to an arrow  $g : a_1 \rightarrow a_2$  assign a linear map which was assigned to the loop  $\ell_{a_2} \circ g \circ \ell_{a_1}^{-1}$ .

In our setup, the connected components of the groupoid correspond to the orbits of  $\widetilde{\text{MCG}}(X)$  action on  $\text{Spin}_G(X)$ . Let  $a_i \in \text{Spin}_{Gf}(X)$  be some representatives of the orbits and  $\widetilde{\text{MCG}}_i := \text{Stab}_{a_i}(\widetilde{\text{MCG}}(X))$  be their stabilizer subgroups in  $\widetilde{\text{MCG}}(X)$ .

<sup>11</sup>Due to unitarity of TQFT, without loss of generality we can assume that  $C$  and  $\alpha$  are valued in  $U(1) \subset \mathbb{C}^*$ .

We then have the following decomposition in terms of the product over the orbits:

$$\begin{aligned} \mathcal{H}^1(\widetilde{\text{MCG}}(X), U(1)^{\text{Spin}_{G^f}(X)}) &\cong \\ \mathcal{H}^1(\widetilde{\text{MCG}}(X) \times \text{Spin}_{G^f}(X), U(1)) &\cong \\ \prod_i \mathcal{H}^1(\widetilde{\text{MCG}}_i, U(1)). \end{aligned} \quad (3.96)$$

Note that although the right hand side of (3.96) is expressed in terms of group cohomology with trivial action on the coefficients (unlike the left hand side), in practice it is often easier to describe the group structure of full group  $\widetilde{\text{MCG}}(X)$  than of its stabilizer subgroups  $\widetilde{\text{MCG}}_i$ . Note that representations of this groupoid can be also understood as line bundles on the corresponding stack  $\text{Spin}_{G^f}(X)/\text{MCG}(X)$ .

The phases  $C_{[\tilde{\psi}]}(a)$  determine the anomalous phases that appear in the partition function under the large gauge transformations and diffeomorphisms. If the automorphisms of the  $H_d$  structure (i.e. the elements of  $\widetilde{\text{MCG}}(X)$  that project to a trivial element of  $\text{MCG}(X)$ ) act non-trivially on  $\mathcal{H}_a$ , this means that the partition function  $Z(X, g, a)$  vanishes, unless one inserts some observables transforming non-trivially under the global symmetries, so that the anomalous phase is compensated. Otherwise, the partition function transforms as follows under a large diffeomorphism  $\psi$ :

$$Z(X, \psi^{-1*}g, \psi^{-1*}a) = C_{[\tilde{\psi}]}(a)Z(X, g, a). \quad (3.97)$$

The elements (3.95) in the group cohomology capture the robust information about the anomaly, invariant under redefinitions of the partition function.

### 3.4.2 iTQFTs Hilbert space from cobordism invariants

Having described the construction of iTQFTs, it is now important to discuss how these can be related to bordism invariants. The precise statement about the relation of the cobordism group  $\text{Hom}(\Omega_{d+1}^{\text{Spin} \times_{\mathbb{Z}_2^f} G^f}, U(1))$  and the classification of  $G^f$  anomalies in  $d$  dimensions can be stated as the fact that deformation classes of iTQFTs, which form the torsion subgroup in the group of all  $(d+1)$ -dimensional TQFTs, are classified by the Pontryagin dual of the torsion subgroup of the bordism group of  $(d+1)$ -manifolds with  $\text{Spin} \times_{\mathbb{Z}_2^f} G^f$  structure [8–10]:

$$\text{Tor}\{\text{ref.-pos. } (d+1)\text{-dim iTQFTs}\} / \sim_{\text{def}} \cong \text{Hom}(\text{Tor } \Omega_{d+1}^{\text{Spin} \times_{\mathbb{Z}_2^f} G^f}, U(1)). \quad (3.98)$$

In the case when  $G^f = \mathbb{Z}_2^f \times G$ , the relevant bordism group can be understood as the spin bordism group of the classifying space of  $G$  which we focused on so far:

$$\Omega_{d+1}^{\text{Spin} \times_{\mathbb{Z}_2^f} G^f} \cong \Omega_{d+1}^{\text{Spin}}(BG). \quad (3.99)$$

The elements of the group in the right-hand side of (3.98) can be understood as bordism invariants.

The classifying group on the right-hand side of (3.98) is canonically isomorphic to the group of connected components of the Pontryagin dual of the full bordism group<sup>12</sup>:

$$\mathrm{Hom}(\mathrm{Tor} \Omega_{d+1}^{G^f \times_{\mathbb{Z}_2^f} \mathrm{Spin}}, U(1)) \cong \pi_0 \mathrm{Hom}(\Omega_{d+1}^{G^f \times_{\mathbb{Z}_2^f} \mathrm{Spin}}, U(1)). \quad (3.101)$$

What we discuss now (mostly following [10]) is how given a fixed  $U(1)$ -valued bordism invariant

$$\nu \in \mathrm{Hom}(\Omega_{d+1}^{G^f \times_{\mathbb{Z}_2^f} \mathrm{Spin}}, U(1)) \quad (3.102)$$

one can construct an invertible  $(d+1)$ -dimensional TQFT with internal symmetry  $G$  in the corresponding deformation class.

The bordism groups  $\Omega_{d+1}^{\mathrm{Spin} \times_{\mathbb{Z}_2^f} G^f}$  (as well as their  $\mathrm{pin}^\pm$  versions), together with the corresponding invariants, have been calculated for various choices of the global symmetry group  $G^f$  [9, 60, 83–91], in part motivated by the classification statement above. Considering a TQFT corresponding to an element of the group in the right hand side of (3.98) on a fixed  $d$ -manifold  $X$  provides a homomorphism of abelian groups:

$$\mathrm{Hom}(\Omega_{d+1}^{G^f \times_{\mathbb{Z}_2^f} \mathrm{Spin}}, U(1)) \longrightarrow \mathcal{H}^1(\widetilde{\mathrm{MCG}}(X), U(1)^{\mathrm{Spin}_{G^f}(X)}). \quad (3.103)$$

An explicit functorial description of an invertible TQFT corresponding to a particular group element in the right hand side of (3.98) can be obtained by the method described in [10]. This is a version of the “universal construction” used in [92] to extend invariants of closed manifolds to a TQFT. Below we briefly review the main points of the construction, omitting some subtle details (in particular details related to ordering of manifolds in disjoint union and orientation reversal) for the moment. The reader is welcome to follow the systematic discussion in [10]. Such details will be however relevant for concrete calculations later on.

The first step of the construction is to fix a representative  $H_d$ -manifold  $(X_\alpha, P_\alpha, \theta_\alpha)$  for each element  $\alpha \in \Omega_d^{\mathrm{Spin} \times_{\mathbb{Z}_2^f} G^f}$  of the bordism group in one less dimension. Then a Hilbert space associated to an arbitrary  $H_d$ -manifold  $(X, P, \theta)$  is defined as a quotient of an infinite dimensional vector space generated by all possible  $(d+1)$ -dimensional bordisms from it to the fixed representative  $(X_\alpha, P_\alpha, \theta_\alpha)$  in its bordism class  $\alpha = [(X, P, \theta)] \in \Omega_d^{G^f \times_{\mathbb{Z}_2^f} \mathrm{Spin}}$ :

$$\mathcal{H}(X, P, \theta) := \frac{\bigoplus_{Y \in \{\mathrm{Bordism}(X_\alpha, P_\alpha, \theta_\alpha) \rightarrow (X, P, \theta)\}} |Y\rangle}{|Y_1\rangle \sim \nu([Y_1 \sqcup \overline{Y_2}]) |Y_2\rangle} \quad (3.104)$$

<sup>12</sup>Note that

$$\mathrm{Tor} \Omega_{d+1}^{\mathrm{Spin} \times_{\mathbb{Z}_2^f} G^f} \cong \Omega_{d+1}^{\mathrm{Spin} \times_{\mathbb{Z}_2^f} G^f} \quad (3.100)$$

when  $G^f$  is discrete and  $d \not\equiv -1 \pmod{4}$ .

where  $[Y_1 \sqcup \bar{Y}_2]$  is the class a closed  $H_{d+1}$ -manifold obtained by gluing together  $Y_1$  and  $\bar{Y}_2$  along the common boundary  $(X, P, \theta) \sqcup (X_\alpha, P, \theta)$ , where bar denotes reversing of orientation. The equivalence relation used in the quotient identifies any two bordisms up to a phase given by the value of the invariant  $\nu$  evaluated on the closed manifold obtained by such a gluing. It follows that the resulting vector space is one-dimensional. A basis vector can be fixed by choosing a particular bordism from  $(X_\alpha, P_\alpha, \theta_\alpha)$  to  $(X, P, \theta)$ . Note that choosing a different representative  $(X'_\alpha, P'_\alpha, \theta'_\alpha)$  in the same bordism class  $\alpha$  would result in an isomorphic vector space. An explicit isomorphism can be constructed by choosing a particular bordism  $(X'_\alpha, P'_\alpha, \theta'_\alpha) \rightarrow (X_\alpha, P_\alpha, \theta_\alpha)$ .

Using the definition (3.104), the TQFT action of a bordism  $W : (X, P, \theta) \rightarrow (X', P', \theta')$  on the Hilbert spaces is simply given by the composition of bordisms:

$$\begin{aligned} \hat{W} : \mathcal{H}(X, P, \theta) &\longrightarrow \mathcal{H}(X', P', \theta') \\ |Y\rangle &\longmapsto |W \circ Y\rangle \end{aligned} \quad (3.105)$$

which is well defined, as the bordism classes of  $(X, P, \theta)$  and  $(X', P', \theta')$  are necessarily the same, so they have the same representative  $(X_\alpha, P_\alpha, \theta_\alpha)$ .

Consider then the setup described in Section 3.4.1. Let us realize the basis elements  $e_a$  in terms of particular reference bordisms  $Y_a$ :

$$e_a := |Y_a\rangle \in \mathcal{H}_a \equiv \mathcal{H}(X, P_a, \theta_a), \quad Y_a : (X_\alpha, P_\alpha, \theta_\alpha) \rightarrow (X, P_a, \theta_a) \quad (3.106)$$

where  $\alpha = [(X_a, P_a, \theta_a)]$ . Note that in principle one can always choose  $(X_\alpha, P_\alpha, \theta_\alpha) = (X, P_b, \theta_b)$  for some  $b$  in the same orbit the mapping class group action as  $a$ . By considering an isomorphism  $\tilde{\psi}$  in (3.84) as a bordism

$$\tilde{\psi} : (X, P_a, \theta_a) \longrightarrow (X, P_{\psi^{-1}a}, \theta_{\psi^{-1}a}) \quad (3.107)$$

it then follows that

$$C_{[\tilde{\psi}]}(a) = \nu((\tilde{\psi} \circ Y_a) \sqcup \overline{Y_{\psi^{-1}a}}) \quad (3.108)$$

where the union means gluing of the bordisms along the common boundary  $(X, P_{\psi^{-1}a}, \theta_{\psi^{-1}a}) \sqcup (X_\alpha, P_\alpha, \theta_\alpha)$ . Note that a change of various choices made above (such as choice of representative manifolds in the bordism classes and the choice of basis bordisms) will result in the redefinition of the coefficients  $C$  of the form (3.93). This will not change the cohomology class  $[C]$  in (3.95). Therefore the formula (3.108) provides us with the map (3.103). The bosonic version of such a map (in the case of  $d = 2$  and  $X = \mathbb{T}^2$ , a 2-torus), when the left hand side is replaced by group cohomology  $H^3(BG, U(1))$ , was considered in [12–14] (see also [11, 93–95]). The generalization to the fermionic case that we consider also allows the construction of the representation  $C$  using more general closed manifolds, not just mapping tori. As we will see, this can sometimes simplify calculations.

# Chapter 4

## The case of $\mathbb{Z}_2^f \times \mathbb{Z}_2$ symmetry

In this and the next chapters we discuss the case of 2-dimensional fermionic QFTs with a discrete  $G^f$  global symmetry as a working example of the general framework presented above. In particular, we are going to focus on the presence of anomalous phases that anomalous theories may have under large transformations once they are evaluated over a torus  $X = \mathbb{T}^2$ . This means that we will consider how theories behave under modular transformations depending on the anomaly that characterizes their symmetry group. In order to do so we study explicitly their relation with 3-dimensional invertible spin-TQFTs living on appropriate mapping tori representing different classes of  $\Omega_3^{\text{Spin} \times_{\mathbb{Z}_2^f} G^f}$ .

We start our analysis by discussing in full details the case of  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2$ . This will serve us as an important example to get used to the relevant techniques. Indeed, all the important features of the cobordism classification are already present here and their understanding in a particular, simple enough setup will be of great help in pointing out how to develop techniques which may help to deal with the more complicated cases. The analysis presented here is based on the work [1].

Let us start first by discussing what are the full mapping class group  $\widetilde{\text{MCG}}(\mathbb{T}^2)$  and  $\mathcal{H}^1(\widetilde{\text{MCG}}(\mathbb{T}^2), U(1)^{\text{Spin}_{G^f}(\mathbb{T}^2)})$  in this case. The first is the fermionic central extension of  $\text{MCG}(\mathbb{T}^2) = \text{SL}(2, \mathbb{Z}) = \langle S, T | S^4 = 1, (ST)^3 = S^2 \rangle$  by  $G^f = \mathbb{Z}_2 \times \mathbb{Z}_2^f$ . This means that the relations between  $S$  and  $T$  now close up to some elements of  $G$ . A redefinition of  $S$  or  $T$  always allow to fix the relation  $(ST)^3 = S^2$ , while in principle the first equation can instead change. By the requirement of acting non-trivially with fermionic parity on  $\text{SL}(2, \mathbb{Z})$ , the relation  $S^4 = 1$  will have to be properly modified. Therefore the full mapping class group will be the unique, up to an isomorphism, non-trivial extension  $\widetilde{\text{MCG}}(\mathbb{T}^2) \cong \widetilde{\text{SL}}(2, \mathbb{Z}) \times \mathbb{Z}_2$ , where

$$\widetilde{\text{SL}}(2, \mathbb{Z}) =$$

$$\langle S, T, (-1)^{\mathcal{F}} | S^4 = (-1)^{\mathcal{F}}, (ST)^3 = S^2, ((-1)^{\mathcal{F}})^2 = 1, (-1)^{\mathcal{F}} T = T(-1)^{\mathcal{F}} \rangle. \quad (4.1)$$

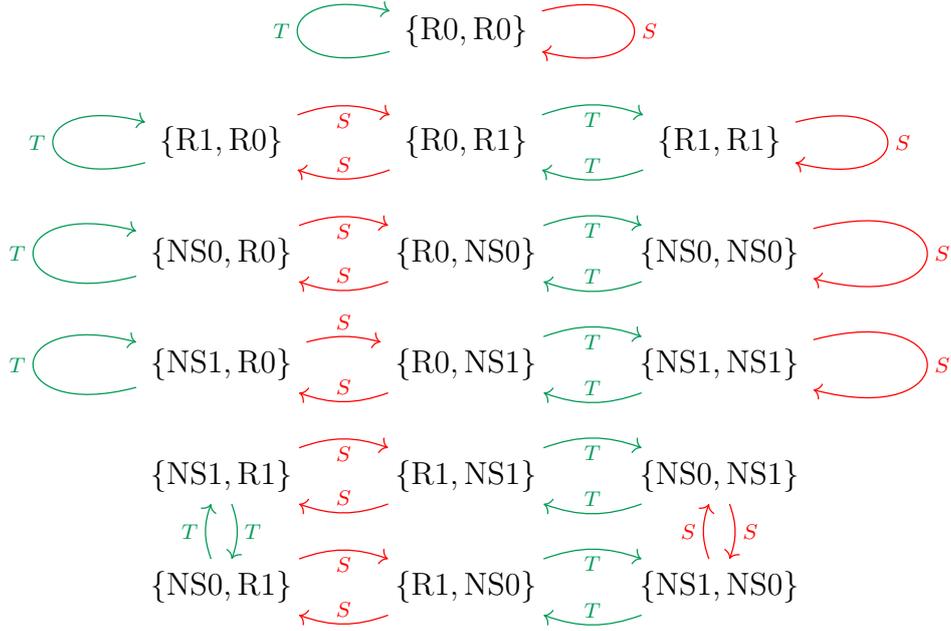


Figure 4.1: The structure of the action groupoid  $\widetilde{\text{SL}}(2, \mathbb{Z}) \times \text{Spin}_G(\mathbb{T}^2)$  for  $G = \mathbb{Z}^2$ . The  $(-1)^{\mathcal{F}}$  and  $(-1)^{\mathcal{Q}}$  generators always act from a groupoid object to itself and are not shown explicitly.

The group  $\widetilde{\text{SL}}(2, \mathbb{Z})$  is a non-trivial extension of the modular group  $\text{SL}(2, \mathbb{Z})$  by  $\mathbb{Z}_2$ . It is often referred to as *metaplectic* group and often also denoted as  $\text{Mp}_1(\mathbb{Z})$ . For a fermionic theory we expect the modular transformations to be a representation of (4.1) acting on the partition functions, with  $(-1)^{\mathcal{F}}$  playing the role of the non-trivial automorphism of the spin bundle over  $\mathbb{T}^2$ , or, equivalently, the fermion parity operator of the 3d bulk TQFT considered on  $\mathbb{T}^2$  “spatial” slices. We will denote the generator of the  $\mathbb{Z}_2$  factor in  $\widetilde{\text{MCG}}(\mathbb{T}^2)$  which is not included in  $\widetilde{\text{SL}}(2, \mathbb{Z})$  as  $(-1)^{\mathcal{Q}}$ . It has a meaning of the non-trivial automorphism of  $G = \mathbb{Z}_2$  principal bundle over  $\mathbb{T}^2$ , or, equivalently, the  $\mathbb{Z}_2$  global symmetry charge operator of the 3d TQFT.

For what regards the cohomology group  $\mathcal{H}^1(\widetilde{\text{MCG}}(\mathbb{T}^2); U(1)^{\text{Spin}_{Gf}(\mathbb{T}^2)})$ , we can proceed with a direct computation. Note that the cocycle equation (3.92) defines relations only between cocycles evaluated at elements  $a \in \text{Spin}_{Gf}(\mathbb{T}^2) = \text{Spin}(\mathbb{T}^2) \times H^1(\mathbb{T}^2; \mathbb{Z}_2)$  in the same orbits of  $\text{SL}(2, \mathbb{Z})$ . These are the set of couples of spin periodicity conditions NS, R (Neveu-Schwarz and Ramond, or, equivalently, anti-periodic and periodic respectively) and  $0, 1 \in \mathbb{Z}_2$  holonomies defined over a basis of  $H^1(\mathbb{T}^2; \mathbb{Z}_2)$ . Therefore, in our case of interest there are three kinds of orbits, which are composed by one, three and six elements respectively. The total cohomology group is then the direct product of the cohomology groups generated by each orbit. The action groupoid is displayed in Figure 4.1.

The orbit of a single element is given by the cocycles evaluated at  $\{R0, R0\}$  and its contribution to (3.96) is  $\mathbb{Z}_{24} \times \mathbb{Z}_2$ .

The orbits composed by three elements are instead  $\{\{R0, NS0\}, \{NS0, R0\}, \{NS0, NS0\}\}$ ,  $\{\{R0, R1\}, \{R1, R0\}, \{R1, R1\}\}$  and  $\{\{R0, NS1\}, \{NS1, R0\}, \{NS1, NS1\}\}$ , with a contribution to the total group equal to  $U(1) \times \mathbb{Z}_8 \times \mathbb{Z}_2$  for each one.

Finally, the six elements left generates  $U(1)^2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$ , so that the total group is

$$\mathcal{H}^1(\widetilde{\text{MCG}}(\mathbb{T}^2); U(1)^{\text{Spin}_G(X)}) \cong U(1)^5 \times \mathbb{Z}_8^4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2^5. \quad (4.2)$$

We anticipate here that the anomalous phases captured by the modular transformations lie in the diagonal subgroup  $\mathbb{Z}_8$  for the two orbits of three elements with non-trivial  $\mathbb{Z}_2$  holonomies and the  $U(1)$  subgroup for the 6-dimensional orbit.

## 4.1 Modular matrices from cobordism invariants

With these premises, we can now turn to study the modular transformations of the theories under discussion. In order to do so, we first review what are the cobordism invariants that one can construct for closed low dimensional manifolds equipped with a spin or  $\text{pin}^-$  structure (see e.g. [60, 84, 96] for details). With this knowledge one is able to determine such transformations simply by knowing the anomaly of a theory and without further knowledge of its field content.

### 4.1.1 Invariants of 1-manifolds

The only connected and compact 1-dimensional manifold is  $S^1$ , which we can equip with two different spin structures. They correspond to the two possible  $\mathbb{Z}_2$  bundles covering the trivial  $SO(1)$ -bundle over  $S^1$ . More precisely, if such a bundle is non-trivial, then we denote the spin-manifold as  $S_{\text{NS}}^1$  and  $S_{\text{R}}^1$  otherwise. Only  $S_{\text{NS}}^1$  can be seen as the boundary of a disk, so that the spin structure on the former is the restriction of the unique spin structure admitted on the latter. The case  $S_{\text{R}}^1$ , which is equivalent to having periodic boundary conditions on the circle, is instead the generator of  $\Omega_1^{\text{Spin}}(\text{pt}) \cong \mathbb{Z}_2$ . We will denote with  $\eta$  the non-trivial  $\mathbb{Z}_2$ -valued bordism invariant defined by

$$\eta(M) = \begin{cases} 0 & M = S_{\text{NS}}^1, \\ 1, & M = S_{\text{R}}^1, \\ \sum_i \eta(N_i), & M = \sqcup_i N_i. \end{cases} \quad (4.3)$$

### 4.1.2 Invariants of 2-manifolds

An oriented 2-manifold  $X$  with non-trivial genus can always be written as the connected sum of an arbitrary number of tori. The spin structures on the torus are canonically identified with  $H^1(\mathbb{T}^2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ , corresponding to the periodicity

conditions on the two non-trivial generating 1-cycles. From the argument above one can see that  $\mathbb{T}^2 = \partial(D^2 \times S^1)$  as a spin-manifold if at least one of the two generators has an antiperiodic boundary condition, while the trivial element  $(0, 0)$  is the only one which does not bound. Extending the reasoning to surfaces of generic genus, it follows that  $\Omega_2^{\text{Spin}}(\text{pt}) \cong \mathbb{Z}_2$  as well.

The bordism invariant which tells us in which element of  $\Omega_2^{\text{Spin}}(\text{pt})$  our manifold falls in is the Arf invariant. To build it we recall that spin structures on an oriented 2-manifold  $X$  are in one-to-one correspondence with quadratic forms

$$\tilde{q} : H^1(X; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2 \quad (4.4)$$

such that

$$\tilde{q}(a + b) - \tilde{q}(a) - \tilde{q}(b) = \int_X a \cup b. \quad (4.5)$$

Geometrically such form is equivalent to evaluating  $\eta$  over the 1-manifolds Poincaré duals to the 1-cycles. From this we can define the Arf invariant as

$$\begin{aligned} \text{Arf} : \Omega_2^{\text{Spin}}(\text{pt}) &\longrightarrow \mathbb{Z}_2 \\ [X] &\longmapsto \text{Arf}(X) := \sum_{i=1}^g \tilde{q}(\tilde{a}_i) \tilde{q}(\tilde{b}_i) \pmod{2}, \end{aligned} \quad (4.6)$$

where  $\{\tilde{a}_i, \tilde{b}_i\}_{i=1}^g$  is a symplectic basis for  $H^1(X; \mathbb{Z}_2)$  and  $X$  the chosen representative of the bordism class.

Next we are interested in non-oriented closed surfaces. Although they can admit also a  $\text{pin}^+$  structure, they always admit a  $\text{pin}^-$  structure, which we will focus on. A generic non-oriented surface  $X$  can be written as a direct sum of tori and projective spaces  $\mathbb{R}P^2$ . The cup product defines again a non-degenerate symmetric bilinear form on  $H^1(X; \mathbb{Z}_2)$ , allowing us to define, in similarity with the oriented case, the quadratic enhancement

$$q : H^1(X; \mathbb{Z}_2) \rightarrow \mathbb{Z}_4, \quad (4.7)$$

that satisfies

$$q(a, b) - q(a) - q(b) = 2 \int_X a \cup b. \quad (4.8)$$

Such quadratic enhancements on  $X$  are in one-to-one correspondence with the set of its  $\text{pin}^-$  structures. One can also associate to any enhancement the Arf-Brown-Kervaire (ABK) invariant, defined by the Gaussian sum<sup>1</sup>

$$e^{i\pi \text{ABK}(X)/4} := \frac{1}{\sqrt{|H^1(X; \mathbb{Z}_2)|}} \sum_{a \in H^1(X; \mathbb{Z}_2)} e^{i\pi q(a)/2}, \quad (4.9)$$

which in turns provides a  $\text{Pin}^-$ -bordism invariant as well. Thus we can think of it as an isomorphism labelling the classes in  $\Omega_2^{\text{Pin}^-}(\text{pt})$ , i.e.  $\text{ABK} : \Omega_2^{\text{Pin}^-}(\text{pt}) \rightarrow \mathbb{Z}_8$ .

Note that in the case of orientable surfaces, the two enhancements and invariants match as one should expect,  $q = 2\tilde{q} \pmod{4}$  and  $\text{ABK}(X) = 4\text{Arf}(X) \pmod{8}$ .

<sup>1</sup>Here  $X$  represents a manifold together with a spin structure, which is usually clear by the context. Whenever this is not the case, we use the notation  $\text{ABK}(X, s)$  to specify the spin structure  $s$  of the manifold  $X$ .

### 4.1.3 Invariants of 3-manifolds

Up to some differences, in the case of a 3-dimensional manifold  $Y$  one can proceed by analogy and make use of the intersection form to build bordism invariants. When  $d = 3$ , this instead defines a symmetric trilinear product which allows to define also the bilinear form

$$\begin{aligned} \lambda: H^1(Y; \mathbb{Z}_2) \times H^1(Y; \mathbb{Z}_2) &\longrightarrow \mathbb{Z}_2. \\ (a, b) &\longmapsto \lambda(a, b) := \int_Y a \cup a \cup b \end{aligned} \quad (4.10)$$

One can think of  $\lambda$  in the following way: given smooth surfaces representing<sup>2</sup> the Poincaré duals of  $a$  and  $b$ ,  $\text{PD}(a)$  and  $\text{PD}(b)$ , their intersections will be a disjoint union of embedded circles. Restricting the normal bundle of  $\text{PD}(a)$  on these,  $\lambda(a, b)$  counts the number of them mod 2 on which such a bundle is non-trivial.

If the manifold  $Y$  is actually equipped with a spin structure  $s$ , then it is possible to enhance  $\lambda$  to a function

$$\delta_s : H^1(Y; \mathbb{Z}_2) \times H^1(Y; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_4. \quad (4.11)$$

Indeed, let  $f : \Sigma \hookrightarrow Y$  be an embedded surface which describes the Poincaré dual of  $a \in H^1(Y; \mathbb{Z}_2)$ , i.e.  $f(\Sigma) = \text{PD}(a)$ . Since  $Y$  is orientable, it is possible to show that the pullback of the tangent bundle is of the form  $f^*TY \cong T\Sigma \oplus \det T\Sigma$  and therefore, by means of  $s$ , in bijection with the  $\text{Pin}^-$  structures defined on  $\Sigma$ .

At this point consider a generic smooth embedding  $\iota : S^1 \hookrightarrow Y$ . Using the canonical spin structure  $S_{\text{NS}}^1$  and the spin structure on  $Y$  it is possible to define a spin structure on the normal bundle  $N_{Y|\iota(S^1)}$  as well. The framings of  $N_{Y|\iota(S^1)}$  which are compatible with this choice are called *even framings*, while the others are called *odd*.

There is a unique quadratic enhancement  $q_{\text{PD}(a)}$  on a representative  $\Sigma$  of  $\text{PD}(a)$  with the following property: for any  $\tilde{b} \in H^1(\Sigma; \mathbb{Z}_2)$ ,  $q_{\text{PD}(a)}(\tilde{b})$  equals the number mod 4 of left (positive) half turns which the restriction  $N_{Y|\text{PD}(a)}|_{\iota(\tilde{b})}$  does with respect to any *even* framing in  $Y$ <sup>3</sup> [96]. Then the definition of (4.11) follows simply as

$$\delta_s(a, b) := q_{\text{PD}(a)}(\tilde{b}), \quad (4.12)$$

with the obvious restriction  $\tilde{b} = b|_{\text{PD}(a)}$ . Geometrically it is clear why  $\delta_s$  is the enhancement of  $\lambda$  for  $s$  fixed.

Since the  $\text{pin}^-$  structures of  $\text{PD}(a)$  are classified by its ABK invariant, for any spin structure  $s$  one can define without additional effort also the invariant

$$\begin{aligned} \beta_s: H^1(Y, \mathbb{Z}_2) &\longrightarrow \mathbb{Z}_8 \\ a &\longmapsto \beta_s(a) := \text{ABK}(\text{PD}(a), s|_{\text{PD}(a)}). \end{aligned} \quad (4.13)$$

<sup>2</sup>It is known that for a general  $d$ -manifold  $X$  the elements of  $H_{d-1}(X; \mathbb{Z}_2)$  can always be represented by *smooth* codimension one submanifolds.

<sup>3</sup>Note that this definition is independent on which direction we are moving along the circle.

From the definition it follows that

$$\beta_s(a + b) = \beta_s(a) + \beta_s(b) + 2\delta_s(a, b). \quad (4.14)$$

Moreover, if one changes the spin structure acting on it with  $c \in H^1(Y, \mathbb{Z}_2)$ , then we have the relation

$$\beta_{s+c}(a) = \beta_s(a) + \delta_s(a, c). \quad (4.15)$$

Finally we remark that unlike the lower dimensional cases, the spin bordism group  $\Omega_3^{\text{Spin}}(\text{pt})$  is empty, since it can be shown that any spin 3-manifold bounds a spin 4-manifold [97]. Therefore the only invariants one can build arises from lower dimensional submanifolds, as we have just shown.

#### 4.1.4 Mapping manifolds and modular matrices

As already mentioned, generic bordism groups  $\Omega_n^{\text{Spin}}(BG)$  can be computed by well known methods, for example via the Atiyah-Hirzebruch spectral sequence discussed in Section 3.3 or the Adams spectral sequence [9, 60, 77, 83–88, 98]. However, the case  $G = \mathbb{Z}_2$  is simple enough that one can understand its structure geometrically for  $d = 2, 3$  by the basic tools we presented<sup>4</sup>.

Let us focus on the 2-dimensional case first. After choosing a proper spin structure  $s$  on a surface  $X$ , the only thing one has left to fix is the map  $g : X \rightarrow B\mathbb{Z}_2$ , which can be equivalently seen as an element  $a_g \in H^1(X; \mathbb{Z}_2)$ . Therefore, for labelling elements in  $\Omega_2^{\text{Spin}}(B\mathbb{Z}_2)$  we can build only one additional invariant besides the ones of the previous section, i.e.  $\eta_g := \tilde{q}(a_g)$ . This means that  $\Omega_2^{\text{Spin}}(B\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ , where the isomorphism is given by the couple  $(\text{Arf}, \eta_g)$ .

By the same reasoning, one can convince themselves that the only possible invariant which can label elements of  $\Omega_3^{\text{Spin}}(B\mathbb{Z}_2)$  is  $\beta_s(a_g)$ , where now  $g : Y \rightarrow B\mathbb{Z}_2$ . Thus  $\Omega_3^{\text{Spin}}(B\mathbb{Z}_2)$  is classified by the possible  $\text{pin}^-$  structures that can be defined on  $\text{PD}(a_g)$  and therefore is isomorphic to  $\Omega_2^{\text{Pin}^-}(\text{pt}) \cong \mathbb{Z}_8$ . The partition function of the TQFT corresponding to the element  $\nu \in \mathbb{Z}_8 \cong \text{Hom}(\Omega_3^{\text{Spin}}(B\mathbb{Z}_2), U(1))$  on  $Y$  is given by

$$\exp\left(\frac{i\pi\beta_s(a_g)}{4}\right). \quad (4.16)$$

In order to apply this general knowledge to the study of modular transformations, we now consider the 2-manifold  $X = \mathbb{T}^2$ . As previously mentioned, we denote  $\{s_0g_0, s_1g_1\}$  the data that define  $X$  as a manifold with extra structure, so that it represents an element in  $\Omega_2^{\text{Spin}}(B\mathbb{Z}_2)$ . Here  $s_0, s_1 \in \{\text{R}, \text{NS}\}$  are the periodicity conditions along the time and space direction respectively (before acting with the  $\mathbb{Z}_2$  gauge field), while  $g_i \in \{0, 1\}$  are the  $\mathbb{Z}_2$  holonomies. With the notation  $Z_{s_0g_0}^{s_1g_1}(\tau, \bar{\tau})$  we

<sup>4</sup>By a simple generalization one can actually determine in this way the group for the generic case  $G = \mathbb{Z}_2^k$  as well [60, 99, 100].

refer to the corresponding partition functions<sup>5</sup> with appropriate insertions of  $(-1)^{F,Q}$ , the 2-dimensional fermion parity and  $G = \mathbb{Z}_2$  global symmetry charge operators<sup>6</sup>. E.g. in the case of a CFT

$$Z_{s_1 g_1}^{\text{NS}g_0}(\tau, \bar{\tau}) = \text{Tr}_{\mathcal{H}_{s_1 g_1}} \left[ (-1)^{g_0 Q} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \right], \quad (4.17)$$

$$Z_{s_1 g_1}^{\text{R}g_0}(\tau, \bar{\tau}) = \text{Tr}_{\mathcal{H}_{s_1 g_1}} \left[ (-1)^F (-1)^{g_0 Q} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \right], \quad (4.18)$$

where  $\mathcal{H}_{\text{NS/R}0}$  and  $\mathcal{H}_{\text{NS/R}1}$  are respectively the untwisted and twisted Hilbert spaces with NS or R periodicity conditions and  $q = \exp(2\pi i\tau)$ .

The modular transformations will mix only partition functions of tori which lie in the same classes of  $\Omega_2^{\text{Spin}}(B\mathbb{Z}_2)$ . Therefore we put these together as entries of some vectors  $(\mathbf{Z}_{(\text{Arf}, \eta_g)})_i$ , so that their transformations are described by the corresponding matrices  $S_{(\text{Arf}, \eta_g)}$ ,  $T_{(\text{Arf}, \eta_g)}$ :

$$\begin{aligned} \mathbf{Z}_{(0,0)}^T &= (Z_{\text{NS}0}^{\text{R}0}, Z_{\text{R}0}^{\text{NS}0}, Z_{\text{NS}0}^{\text{NS}0}, Z_{\text{NS}1}^{\text{NS}0}, Z_{\text{NS}0}^{\text{R}1}, Z_{\text{R}1}^{\text{NS}0}, Z_{\text{NS}0}^{\text{NS}1}, Z_{\text{NS}1}^{\text{R}1}, Z_{\text{R}1}^{\text{NS}1}), \\ \mathbf{Z}_{(1,0)}^T &= (Z_{\text{R}0}^{\text{R}0}), \\ \mathbf{Z}_{(0,1)}^T &= (Z_{\text{NS}1}^{\text{R}0}, Z_{\text{R}0}^{\text{NS}1}, Z_{\text{NS}1}^{\text{NS}1}), \\ \mathbf{Z}_{(1,1)}^T &= (Z_{\text{R}1}^{\text{R}0}, Z_{\text{R}0}^{\text{R}1}, Z_{\text{R}1}^{\text{R}1}). \end{aligned} \quad (4.19)$$

In order to determine how the modular group acts we must first choose the reference frame  $(X_\alpha, P_\alpha, \theta_\alpha)$  for each bordism class, call it  $\tilde{X}_{(\text{Arf}, \eta_g)}$ , and the basis elements  $e_a$  for the Hilbert space associated to each tori in (4.19):

- For the trivial class the canonical choice is given by  $\tilde{X}_{(0,0)} = \emptyset$ , while the manifolds defining the basis elements  $e_a$  are solid tori  $Y_a \cong D^2 \times S^1$  with  $\partial Y_a = X_a$  and the radial direction given by the continuous contraction of one direction<sup>7</sup> NS0 of  $X_a$  to a point;
- For  $(1, 0)$  there is no choice to do,  $\tilde{X}_{(1,0)} = \{\text{R}0, \text{R}0\}$  and  $Y_{\{\text{R}0, \text{R}0\}} = \text{id}_{\{\text{R}0, \text{R}0\}}$ ;
- For  $(0, 1)$  we choose  $\tilde{X}_{(0,1)} = \{\text{R}0, \text{NS}1\}$ . The degrees of freedom we are left

<sup>5</sup>In the case of an anomalous theory with  $\nu = 1 \pmod 2$  and depending on the Hilbert space considered, it holds  $[(-1)^F, (-1)^Q] \neq 0$ , so one has to proceed with care defining  $Z_{s_0 g_0}^{\text{NS}1}$ . See Section 4.3 for further discussion.

<sup>6</sup>Recall that we denote their 3d bulk counterparts as  $(-1)^{\mathcal{F}, \mathcal{Q}}$ .

<sup>7</sup>There are multiple directions which satisfy this constraint. See Appendix B for more details on how we set this choice.

with allows us to choose<sup>8</sup>

$$Y_{\{\text{NS1}, \text{R0}\}} = \mathcal{S}_{\{\text{NS1}, \text{R0}\}}^{\{\text{R0}, \text{NS1}\}} : \{\text{R0}, \text{NS1}\} \xrightarrow{S} \{\text{NS1}, \text{R0}\}, \quad (4.20)$$

$$Y_{\{\text{NS1}, \text{NS1}\}} = \mathcal{T}_{\{\text{NS1}, \text{NS1}\}}^{\{\text{R0}, \text{NS1}\}} : \{\text{R0}, \text{NS1}\} \xrightarrow{T} \{\text{NS1}, \text{NS1}\}; \quad (4.21)$$

- Analogously to (0, 1) up to the change  $\text{NS1} \mapsto \text{R1}$ , for the class (1, 1) we choose  $\tilde{X}_{(1,1)} = \{\text{R0}, \text{R1}\}$  and

$$Y_{\{\text{R1}, \text{R0}\}} = \mathcal{S}_{\{\text{R1}, \text{R0}\}}^{\{\text{R0}, \text{R1}\}} : \{\text{R0}, \text{R1}\} \xrightarrow{S} \{\text{R1}, \text{R0}\}, \quad (4.22)$$

$$Y_{\{\text{R1}, \text{R1}\}} = \mathcal{T}_{\{\text{R1}, \text{R1}\}}^{\{\text{R0}, \text{R1}\}} : \{\text{R0}, \text{R1}\} \xrightarrow{T} \{\text{R1}, \text{R1}\}. \quad (4.23)$$

Now the action of  $\widetilde{\text{MCG}}(X)$  is fixed, with the representation of its generators corresponding to the evaluation of the appropriate cobordism invariant  $\nu \in \mathbb{Z}_8$  for the closed manifolds determined by (3.108).

In order to understand how these closed manifolds are built, one needs to proceed with care since the theories in discussion have a dependence on the spin structure. Indeed, for a given isomorphism  $\tilde{\psi} : X_a \rightarrow X_b$  and fixed basis on the respective Hilbert spaces by  $Y_a : X_\alpha \rightarrow X_a$ ,  $Y_b : X_\alpha \rightarrow X_b$ , the building blocks of the closed manifold are the bordisms  $\tilde{\psi} \circ Y_a : X_\alpha \rightarrow X_b$  and  $(\overline{Y_{\psi^{-1} * a}}) : \overline{X_b} \rightarrow \overline{X_\alpha}$ . However, one still needs to connect these manifolds by two additional bordisms which properly reverse the structure of them. This operation is given by the *evaluation* and *coevaluation* bordisms, both defined for any  $X$  as the manifold  $I_X := [0, 1] \times X$ , but with different directions:

$$e_X := I_X : X \sqcup \overline{X} \rightarrow \emptyset, \quad (4.24)$$

$$c_X := I_X : \emptyset \rightarrow \overline{X} \sqcup X. \quad (4.25)$$

With these tools the closed manifold corresponding to isomorphism  $\tilde{\psi}$  can be defined more precisely by

$$\mathcal{M}(\tilde{\psi}) := e_{X_\alpha} \circ (\tilde{\psi} \circ Y_a \sqcup (\overline{Y_{\psi^{-1} * a}})) \circ c_{\overline{X_b}}, \quad (4.26)$$

where the presence of  $e_{X_\alpha}$  and  $c_{\overline{X_b}}$  are usually omitted. In our setup, for non-trivial classes  $[X_\alpha] \in \Omega_2^{\text{Spin}}(B\mathbb{Z}_2)$  these closed manifolds are mapping tori of  $\mathbb{T}^2$ , that is fibrations over  $S^1$  with fibers isomorphic to  $\mathbb{T}^2$ . For a trivial classes these 3-manifolds are isomorphic to lens-spaces, i.e. manifolds of the form  $(D^2 \times S^1) \cup (D^2 \times S^1)$  where the gluing along the common  $\mathbb{T}^2$  boundary is done using a non-trivial automorphism.

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<sup>8</sup>Actually for these last two classes one still has to choose a proper resolution of the Poincaré dual of  $a_g \in H^1(X, \mathbb{Z}_2)$ . See Appendix B for our convention.

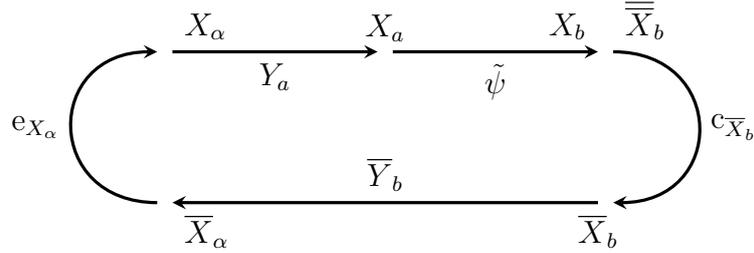


Figure 4.2: The structure of the closed manifolds (4.26) (mapping tori for non-trivial classes in  $\Omega_2^{\text{Spin}}(B\mathbb{Z}_2)$ ), from which is clear one has to remember the identification  $\overline{\overline{X}} \cong X$  via  $(-1)^{\mathcal{F}}$ .

If the theory have a dependence on the spin structure the subtlety we mentioned arise in the identification  $\overline{\overline{X}}_b \cong X_b$  which appears in the definition of  $c_{\overline{\overline{X}}_b}$ . Indeed, by the requirement of unitarity it follows that such isomorphism is defined via action of  $(-1)^{\mathcal{F}}$  on one of the two manifolds (see [10] for more details). Once this is properly taken into account for the description of the mapping tori, the evaluation of  $\nu$  for (4.26) defines the action of the modular transformations, which we are now ready to study.

To illustrate how this framework determines them, here we present the simplest case out of the action of the three generators of  $\widetilde{\text{SL}}(2, \mathbb{Z})$ , namely  $(-1)^{\mathcal{F}}$ . Instead, we omit the detailed computations of the  $S$  and  $T$  matrices, leaving them to Appendix B. Fermionic parity maps each element  $\{s_0g_0, s_1g_1\}$  to itself, which means that on the vectors of partition functions (4.19) it will act diagonally, with entries  $(-1)_a^{\mathcal{F}}$ . Indeed, for a generic  $d$ -manifold  $X_a$  one has  $\mathcal{M}((-1)_a^{\mathcal{F}}) = X_a \times S_{\mathbb{R}}^1$ , where  $S^1$  is equipped with a periodic spin structure. This immediately tells something more, i.e. that  $(-1)^{\mathcal{F}}$  is a bordism invariant independently of the  $H_d$ -structure we might have on  $d$ -manifolds. In fact, if two elements  $X_a, X_b$  lie in the same class of  $\Omega_d^H$  then  $X_a \times S_{\mathbb{R}}^1$  and  $X_b \times S_{\mathbb{R}}^1$  lie in the same class of  $\Omega_{d+1}^H$  as well, because any bordism  $Y : X_a \rightarrow X_b$  can be extended to  $Y \times S_{\mathbb{R}}^1 : X_a \times S_{\mathbb{R}}^1 \rightarrow X_b \times S_{\mathbb{R}}^1$ . Since the action of  $(-1)^{\mathcal{F}}$  is given by the evaluation of a cobordism invariant  $\nu \in \Omega_{d+1}^H$ , then this must hold.

Thus for our needs we simply evaluate the invariant  $\nu \in \Omega_{\text{Spin}}^3(B\mathbb{Z}_2)$  for the reference manifold in each bordism class  $(\text{Arf}, \eta_g)$ . In the classes  $(0, 0)$  and  $(0, 1)$  it follows that  $(-1)_{(0,0)}^{\mathcal{F}} = (-1)_{(1,0)}^{\mathcal{F}} = \text{id}$ . Indeed, while for  $(0, 0)$  the result is straightforward, the torus in  $(1, 0)$  has trivial  $\mathbb{Z}_2$  holonomies and therefore there is no  $\text{pin}^-$ -surface associated with a non-trivial ABK invariant. The two more interesting cases are  $(0/1, 1)$ , of which the corresponding mapping manifolds are reported in Figure 4.3.

From the figure it is clear that the  $\text{pin}^-$  surface  $\Sigma_{(0/1,1)}^{(-1)^{\mathcal{F}}}$  in question is simply a torus with a symplectic basis of  $H^1(\Sigma_{(0/1,1)}^{(-1)^{\mathcal{F}}}, \mathbb{Z}_2)$  given by the two R0 directions

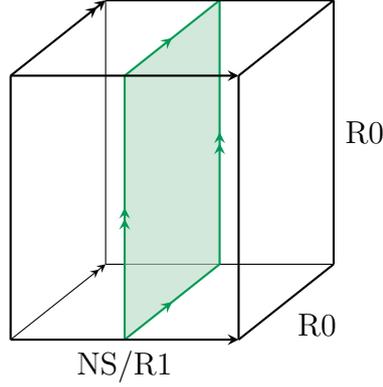


Figure 4.3: The mapping tori describing the action of  $(-1)^{\mathcal{F}}$  on the bordism classes  $(0/1, 1)$  with the  $\text{pin}^-$  surface  $\Sigma_{(0/1,1)}^{(-1)^{\mathcal{F}}}$  highlighted in green. The horizontal slices correspond to the tori  $\{R0, NS/R1\}$  while the vertical direction corresponds to  $S_{\mathbb{R}}^1$ , with its direction being from bottom to top.

depicted. Therefore we have

$$(-1)_{(0/1,1)}^{\mathcal{F}} \equiv \nu(\mathcal{M}((-1)_{(0/1,1)}^{\mathcal{F}})) = e^{\pi i \nu \text{Arf}(\Sigma_{(0/1,1)}^{(-1)^{\mathcal{F}}})} = (-1)^{\nu} \cdot \text{id}. \quad (4.27)$$

For the  $S$  and  $T$  matrices one can proceed in an analogous way, albeit the mapping tori become more elaborated and the computation of their bordism classes more involved. As already mentioned, we refer the interested reader to Appendix B for their computation. Here we simply report in Figures 4.4-4.6 the tori associated to their non-trivial entries, together with the  $\text{pin}^-$  surfaces of which the ABK invariant determines the bordism classes.

In the basis fixed by (4.19) the final result is

$$S_{(0,0)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad T_{(0,0)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-i\frac{\pi\nu}{4}} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\frac{\pi\nu}{4}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (4.28)$$

$$S_{(0,1)} = \begin{pmatrix} 0 & (-i)^{\nu} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & e^{-i\frac{\pi}{4}\nu} \end{pmatrix}, \quad T_{(0,1)} = \begin{pmatrix} 0 & 0 & e^{i\frac{\pi\nu}{4}} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (4.29)$$

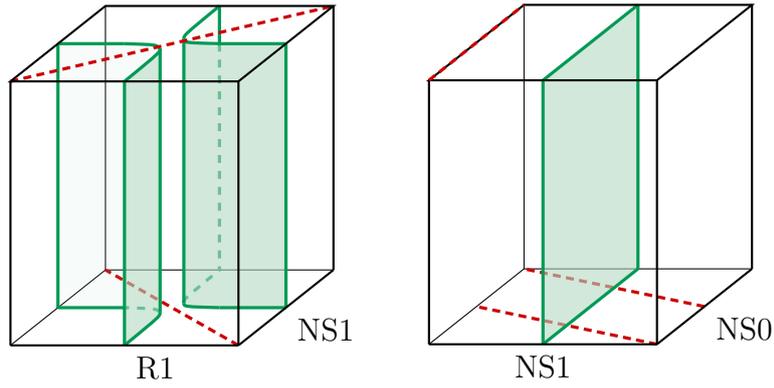


Figure 4.4: The closed 3-manifolds (lens spaces) associated to non-trivial entries  $(T_{(0,0)}^9)_6$  and  $(T_{(0,0)}^4)_8$ . The vertical direction represents the bordism from  $\{s_0g_0, s_1g_1\}$  on the bottom to  $\{(s_0 + s_1)(g_0 + g_1), s_1g_1\}$  on the top. The red dashed lines represent instead the directions which get contracted to single points.

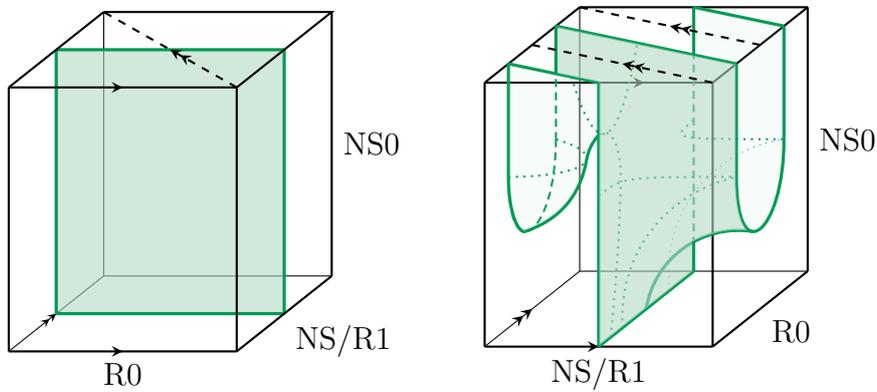


Figure 4.5: The mapping tori associated to  $(T_{(0/1,1)}^3)_1$  and  $(T_{(0/1,1)}^2)_2$ . Here the torus on the bottom and the one on the top are identified. The dashed arrows represent how the generators of the bottom are transposed to the top.

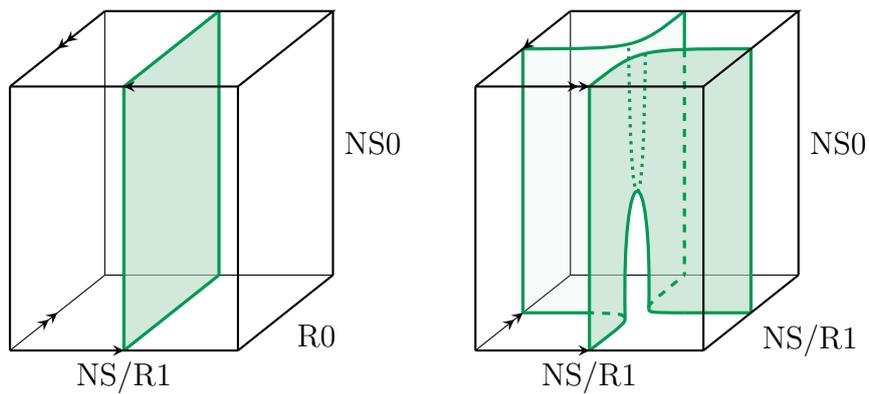


Figure 4.6: Mapping tori associated to the non-trivial entries  $(S_{(0/1,1)}^2)_1$  and  $(S_{(0/1,1)}^3)_3$ . The vertical direction has to be read from the bottom to the top as before.

$$S_{(1,1)} = \begin{pmatrix} 0 & i^\nu & 0 \\ 1 & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi\nu}{4}} \end{pmatrix}, \quad T_{(1,1)} = \begin{pmatrix} 0 & 0 & e^{-i\frac{\pi\nu}{4}} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (4.30)$$

For the class  $(1,0)$  instead it follows immediately that the modular transformations act trivially on the single element  $\{R0, R0\}$ . As one can immediately check, the matrices presented indeed satisfy the relations (4.1).

Finally, to see the full  $\widehat{\text{MCG}}(X)$  representation we need to compute the action of  $(-1)^\mathcal{Q}$  as well. With the previous results one can determine it for free. In fact,  $(-1)^\mathcal{Q}$  will act diagonally for a generic torus  $\{s_0g_0, s_1g_1\}$  and the 3-manifold describing it is again a 3-torus, with the vertical direction now given by NS1. Moreover  $(-1)^\mathcal{Q}$  commutes with  $\widetilde{\text{SL}}(2, \mathbb{Z})$ , so it is enough to compute its value for a single element in each orbit of it, as one can see by applying  $S$  and  $T$  to the result found. All the orbits have at least one element with trivial  $g_0 = 0$ , so we can restrict to such cases. If we denote with  $\{s_0g_0, s_1g_1, s'g'\}$  the spin periodicity and holonomies of a  $\mathbb{T}^3$  associated to the bordisms  $\text{id}/(-1)^\mathcal{F} : \{s_0g_0, s_1g_1\} \xrightarrow{\text{id}/(-1)^\mathcal{F}} \{s_0g_0, s_1g_1\}$ , then we know that  $g' = 0$  and  $s' = \text{NS/R}$  for  $\text{id}/(-1)^\mathcal{F}$ . Therefore a 3-torus  $\{s_00, s_1g_1, \text{NS1}\}$  that describes a  $(-1)^\mathcal{Q}$  action describes also the action of  $\text{id}$  or  $(-1)^\mathcal{F}$  for  $\{s_1g_1, \text{NS1}\}$ , depending on the value of  $s_0$ . This means  $(-1)^\mathcal{Q}$  acts non-trivially only if the mapping torus describes  $(-1)_{(0/1,1)}^\mathcal{F}$  as well, which implies that necessarily  $s_0 = \text{R}$  and  $s_1g_1 = \text{NS1}, \text{R1}$ . From this it follows that the action of  $(-1)^\mathcal{Q}$  is exactly the same of  $(-1)^\mathcal{F}$  and, not surprisingly, a bordism invariant, i.e.

$$(-1)_{(0,0)}^\mathcal{Q} = (-1)_{(1,0)}^\mathcal{Q} = \text{id}, \quad (-1)_{(0,1)}^\mathcal{Q} = (-1)_{(1,1)}^\mathcal{Q} = (-1)^\nu \cdot \text{id}. \quad (4.31)$$

## 4.2 Defect rules

As was used in the previous section, the  $\mathbb{Z}_8$  valued cobordism invariant generating the group 3d iTQFTs, on a given closed spin 3-manifold  $Y$  and  $a \in H^1(Y, \mathbb{Z}_2)$ , can be realized as

$$\beta_s(a) = \text{ABK}(\text{PD}(a), s|_{\text{PD}(a)}), \quad (4.32)$$

where the right hand side is the Arf-Brown-Kervaire (ABK) invariant of a 2d smooth surface representing the Poincaré dual to  $a$  in  $Y$ , with  $\text{pin}^-$  structure induced from the ambient spin structure (see [84] for details). Physically such surface can be understood as the support of the  $\mathbb{Z}_2$  global symmetry charge operator. On an arbitrary bordism the 3d  $\text{Spin} \times \mathbb{Z}_2$ -TQFT can be described in terms of a 2d  $\text{pin}^-$  iTQFT supported on such codimension one defects inside the bordism, possibly ending on codimension one defects in the 2d boundaries. The 2d  $\text{pin}^-$  iTQFT has an action given by a multiple of the ABK invariant and physically corresponds to a stack of Kitaev spin chains.

When the anomalous 2d QFT is considered on the boundary of the spacetime of the 3d TQFT, the boundaries of the  $\mathbb{Z}_2$  bulk charge operator are identified with the  $\mathbb{Z}_2$  charge operators in the 2d QFT.

A 3d cylinder  $X \times [0, 1]$  with a non-trivial surface defect inside can be understood as sequence of “moves” applied to the line defects in  $X$  as one goes along the “time” direction  $[0, 1]$ . The calculation of the matrix elements of modular transformations that was reviewed in the previous section (with technical details in Appendix B) then can be reformulated in terms of certain rules on the changes of the charge defects in 2d, similar to the rules used in [19, 21, 101] in the bosonic case.

Because of the fermionic nature of the 2d theory we decorate the topological  $\mathbb{Z}_2$ -charge line operators with additional information: a lift of the tangent vector to a  $\text{Spin}(2)$ , a double cover of  $SO(2)$  with respect to some fixed reference frame. That is, at each point of the line one has to specify a lift of the angle of the slope, defined modulo  $2\pi$ , to a value  $\pmod{4\pi}$  in a continuous manner. Below we will explicitly indicate such lifts in the diagrams near the relevant points.

Consider first configuration of such line operators on a torus with periodic boundary condition on fermions. This means that the transition functions for the  $\text{Spin}(2)$  bundle are trivial. For each connected component of the line defect it is then sufficient to fix a lift of the slope to  $\text{Spin}(2)$  at a single point. As we will see, in this case it is enough to consider the following single basic move:

$$\begin{array}{c} \text{---} \overset{-\frac{\pi}{2}}{\curvearrowright} \text{---} \\ \text{---} \underset{-\frac{\pi}{2}}{\curvearrowleft} \text{---} \end{array} \sim \alpha \cdot \begin{array}{c} \text{---} \overset{\frac{\pi}{2}}{\curvearrowleft} \text{---} \\ \text{---} \underset{\frac{\pi}{2}}{\curvearrowright} \text{---} \end{array} \quad (4.33)$$

where  $\alpha$  is certain phase. The endpoints of the defects in the diagrams are pairwise identified. By applying this basic rule twice, we get the following rule for changing the orientation of the defect:

$$\text{---} \overset{0}{\rightarrow} \text{---} \sim \alpha^2 \cdot \text{---} \overset{\pi}{\leftarrow} \text{---} \quad (4.34)$$

from which it follows that

$$\text{---} \overset{0}{\rightarrow} \text{---} \sim \alpha^4 \cdot \text{---} \overset{2\pi}{\rightarrow} \text{---} \quad (4.35)$$

And, in particular

$$\text{---} \overset{0}{\rightarrow} \text{---} \sim \alpha^8 \cdot \text{---} \overset{4\pi \sim 0}{\rightarrow} \text{---} \quad (4.36)$$

so that  $\alpha^8 = 1$ . Therefore, for self-consistency  $\alpha$  is required to be an 8th root of unity. This is in agreement with classification of the anomalies by  $\mathbb{Z}_8$ .

The basis elements then correspond to particular configuration of the defects on  $T^2$ . As in the previous section, we can choose them as the results of the application of  $\text{id}, S, T$  transforms to some particular configuration, as shown in Figure 4.7. The

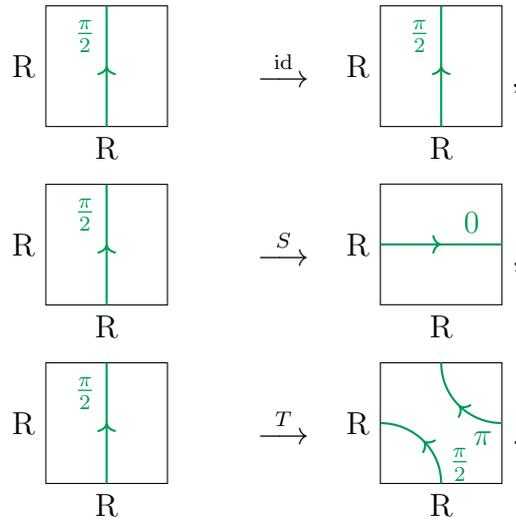


Figure 4.7: The choice of basis configurations of  $\mathbb{Z}_2$  charge defects on  $\mathbb{T}^2$  with periodic-periodic spin structure.

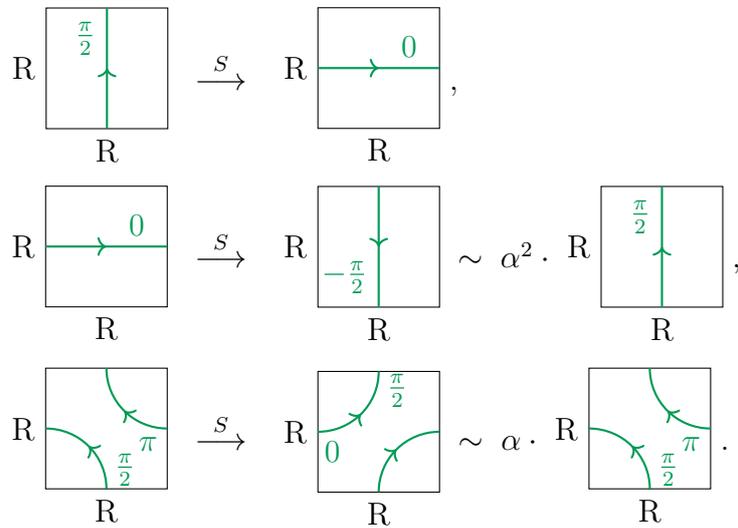


Figure 4.8: The action of  $S$  transformation on the basis elements chosen in Figure 4.7.

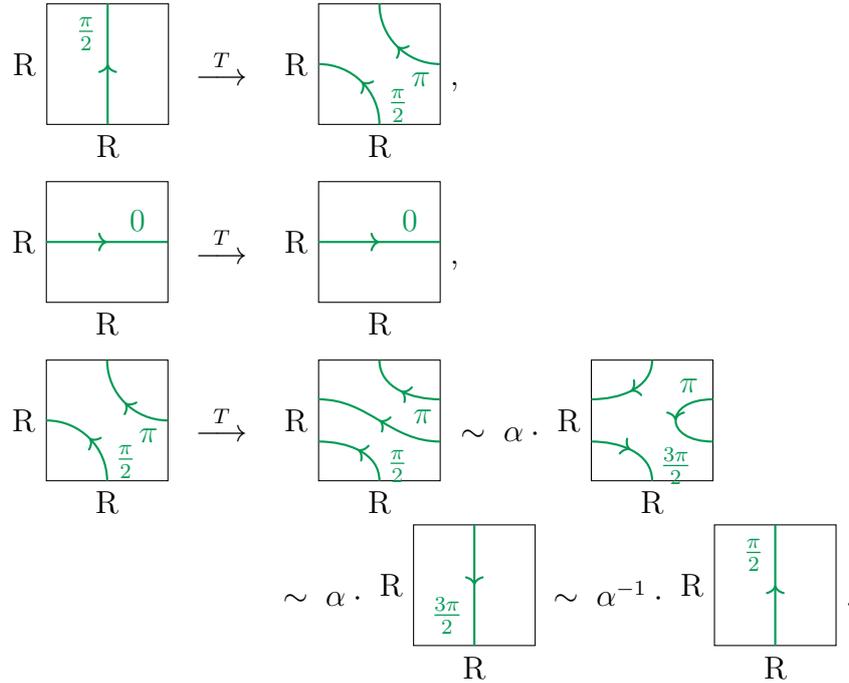


Figure 4.9: The action of  $T$  transformation on the basis elements chosen in Figure 4.7.

result of the application of the  $S$  and  $T$  transformation to the basis elements is shown in Figures 4.8 and 4.9.

In terms of the basis elements in Figure 4.7 the corresponding matrices read as follows:

$$S = \begin{pmatrix} 0 & \alpha^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & \alpha^{-1} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (4.37)$$

They satisfy the conditions

$$(ST)^3 = S^2, \quad S^4 = \alpha^4 \cdot \text{id}. \quad (4.38)$$

By comparing with the calculations in the previous section we fix  $\alpha = e^{i\frac{\pi}{4}\nu}$ . Note that so far we have considered the bordism class  $(1, 1)$ , where all the boundary conditions are periodic.

Now turn to the case when at least one of the boundary conditions on  $T^2$  is antiperiodic, that is the spin-structure is different from the one considered before. Note that, in general, although the set of spin-structures on a manifold  $X$  is not canonically isomorphic to  $H^1(X; \mathbb{Z}_2)$ , they form a torsor over this group. This means the differences between spin structures do canonically correspond to elements of  $H^1(X; \mathbb{Z}_2)$ . Physically this means that a change of spin structure corresponding to

an element  $b \in H^1(X; \mathbb{Z}_2)$  can be described by insertion of a  $\mathbb{Z}_2^f$ -charge operator (i.e. “ $(-1)^F$ ”) supported on a codimension-1 manifold representing Poincaré dual of  $b$ .

The change of the invariant (4.32) on a closed 3-manifold  $Y$  with respect to the change of the spin structure by  $b \in H^1(Y; \mathbb{Z}_2)$  has natural description in terms of such codimension-1 submanifold  $\text{PD}(b)$  (see [84] for details):

$$\beta_{s+b}(a) - \beta_s(a) = 2q_{\text{PD}(a)}(b|_{\text{PD}(a)}) \pmod 8 \quad (4.39)$$

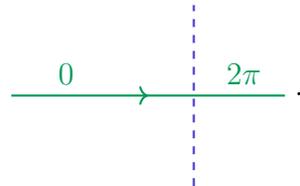
where

$$q_{\text{PD}(a)} : H^1(\text{PD}(a); \mathbb{Z}_2) \longrightarrow \mathbb{Z}_4 \quad (4.40)$$

is the quadratic enhancement of the mod 2 intersection pairing on the  $\text{pin}^-$ -surface  $\text{PD}(a)$  (as before, the  $\text{pin}^-$  structure is induced from the ambient spin structure  $s$ ). This relation has already appeared in the previous section. The value of (4.39) then has a geometric meaning of counting the number of “half-twists” modulo 4 along the intersection  $\text{PD}(a) \cap \text{PD}(b)$ . With this interpretation it becomes explicitly symmetric under exchange of  $a$  and  $b$ . Physically this means that the partition function of the 3d TQFT gets an extra contribution from the intersections of  $(-1)^F$ -defects and more usual  $(-1)^Q$  charge defects. Namely, each closed orientation-preserving loop in the intersection contributes  $(\pm 1)^\nu$ , depending whether the induced spin-structure is even or odd. Each closed orientation-reversing loop contributes  $(\pm i)^\nu$ .

As in the case of  $\mathbb{Z}_2$ -charge defects, the bulk  $(-1)^F$  surface defects end on  $(-1)^F$ -line defects of the boundary anomalous QFT. Therefore the case of general spin structure on the torus can be understood in terms of moves on configurations of two types of defects:  $\mathbb{Z}_2$ -charge lines and  $(-1)^F$  lines that are inserted on a torus with periodic-periodic spin structure.

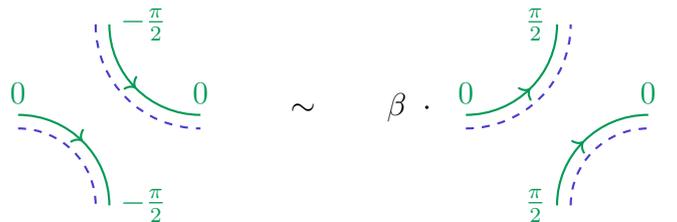
Let us depict  $(-1)^F$ -line operators by dashed blue lines. Note that the  $\text{Spin}(2)$  lift of the slope angle of  $\mathbb{Z}_2$ -charge lines is changed at the place of intersection. Thus the nontrivial  $\text{Spin}(2)$  transition function should be applied at the locus of the  $(-1)^F$  operator. For example:



$$\quad (4.41)$$

Therefore, if a  $\mathbb{Z}_2$ -charge line has odd number of intersections with the  $(-1)^F$  line, the lift is not globally well defined, and the corresponding decorations can be omitted.

We will use the following three additional moves:



$$\quad (4.42)$$

$$\sim \gamma \cdot \text{[diagram]}, \quad (4.43)$$

$$\sim \delta \cdot \text{[diagram]}. \quad (4.44)$$

The self-consistency requires that  $\beta^8 = 1$  and  $\delta^2 = 1$ . Note that unlike in the case without the  $(-1)^F$  line, the move (4.44) cannot be obtained by applying the move (4.43) twice. Also, since there is no pure fermion parity anomaly, there are no non-trivial rules for the moves involving only  $(-1)^F$  lines. In particular:

$$\sim \text{[diagram]}. \quad (4.45)$$

Consider now the action of  $S$  and  $T$  transformations on the sector corresponding to  $(0, 1) \in \Omega_2^{\text{Spin}}(B\mathbb{Z}_2) \cong \mathbb{Z}_2^2$  class in the 2-dimensional bordism group. That is the case when the torus has even spin structure, but the spin structure induced on the  $\mathbb{Z}_2$ -charge line is odd. Figure 4.10 displays the basis similar to the one in Figure 4.7. By replacing the defect lines in Figures 4.8 and 4.9 with double lines we then get the following matrix elements for these three basis elements:

$$S = \begin{pmatrix} 0 & \beta^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & \beta^{-1} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (4.46)$$

The matrices satisfy the similar conditions:

$$(ST)^3 = S^2, \quad S^4 = \beta^4 \cdot \text{id}. \quad (4.47)$$

By comparing it with the bordism invariant calculation we have  $\beta = e^{-i\frac{\pi}{4}\nu}$ .

Finally we consider the trivial class in the 2-dimensional bordism group  $\Omega_2^{\text{Spin}}(B\mathbb{Z}_2) \cong \mathbb{Z}_2^2$ . Excluding configurations without  $\mathbb{Z}_2$ -charge defects, we can choose the basis elements as displayed in Figure 4.11. The  $S$  and  $T$  transformations then act as shown Figures 4.12 and 4.13.

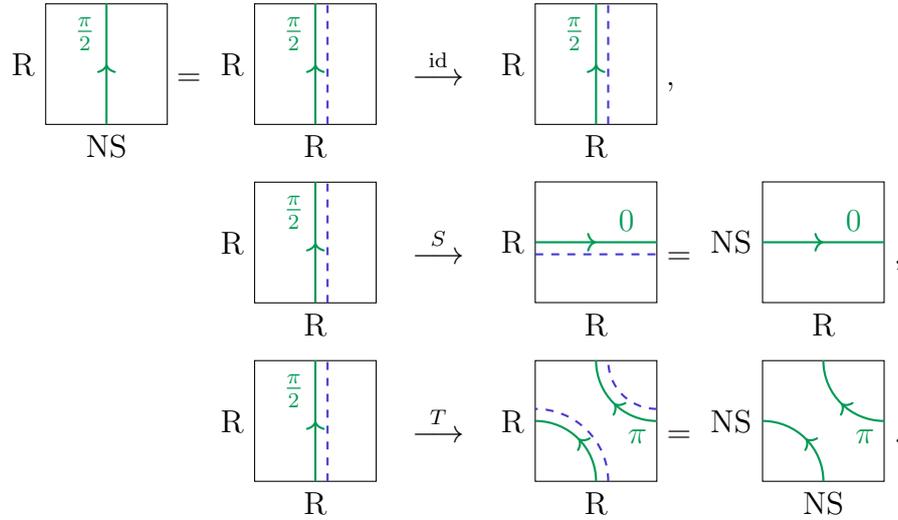


Figure 4.10: The choice of basis configurations of line defects for the case of  $(0, 1) \in \Omega_2^{\text{Spin}}(B\mathbb{Z}_2)$  class.

The corresponding matrix elements are then as follows:

$$S = \begin{pmatrix} 0 & \delta & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & \delta/\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.48)$$

For any  $\gamma$  and  $\delta$  (such that  $\delta^2 = 1$ ) the matrices satisfy:

$$(ST)^3 = S^2, \quad S^4 = \text{id}. \quad (4.49)$$

By comparing it with the calculation of cobordism invariants in the previous section we have  $\delta = 1$ ,  $\gamma = e^{\frac{\pi i \nu}{4}}$ .

### 4.3 Interpretation in terms of the Hilbert space on a circle

In [17] it was analyzed how the anomaly of  $\mathbb{Z}_2^f \times \mathbb{Z}_2$  symmetry exhibits itself on the Hilbert space of a 2-dimensional theory. The authors made analysis by considering a basic theory that realizes such anomaly, namely a theory of  $\nu$  Majorana fermions, where only the left-moving ones are non-trivially charged with respect to  $Q$ . In this section we demonstrate how it is in agreement with description of the anomalies

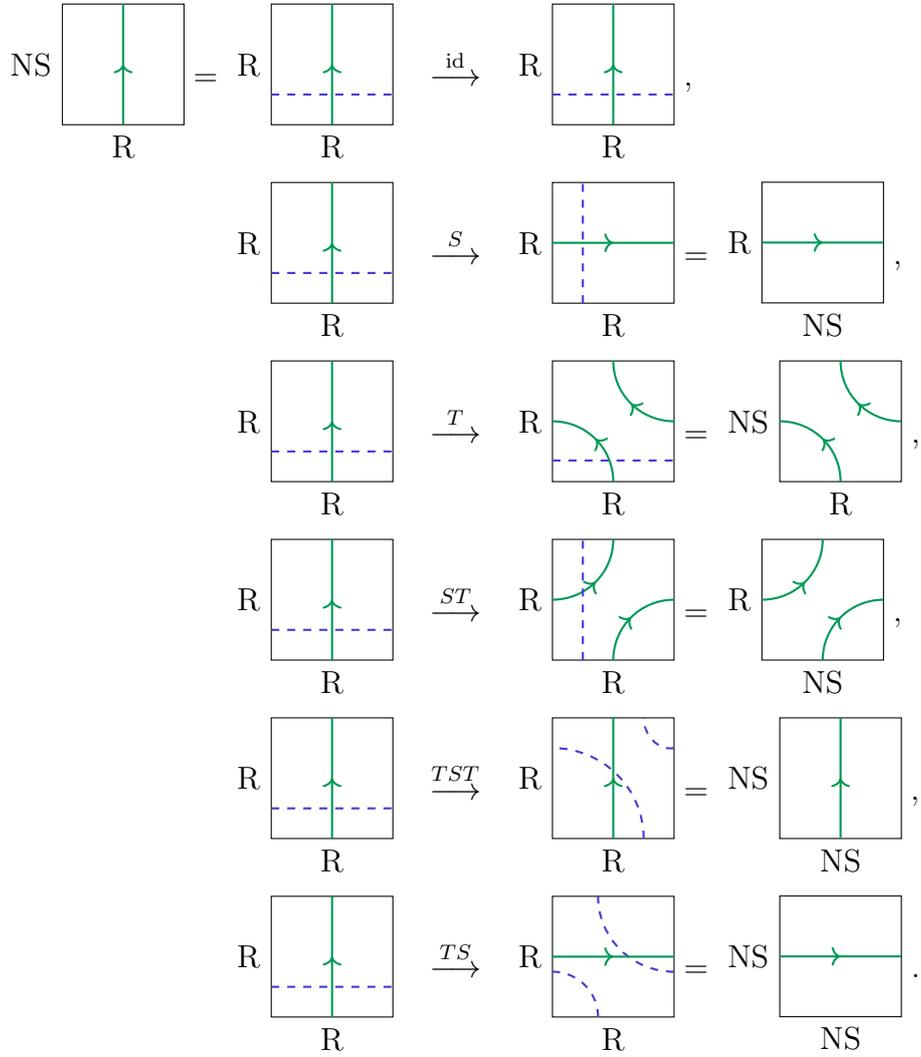


Figure 4.11: The choice of basis for the trivial class in  $\Omega_2^{\text{Spin}}(B\mathbb{Z}_2) \cong \mathbb{Z}_2^2$ , excluding configurations without  $\mathbb{Z}_2$ -charge defects.

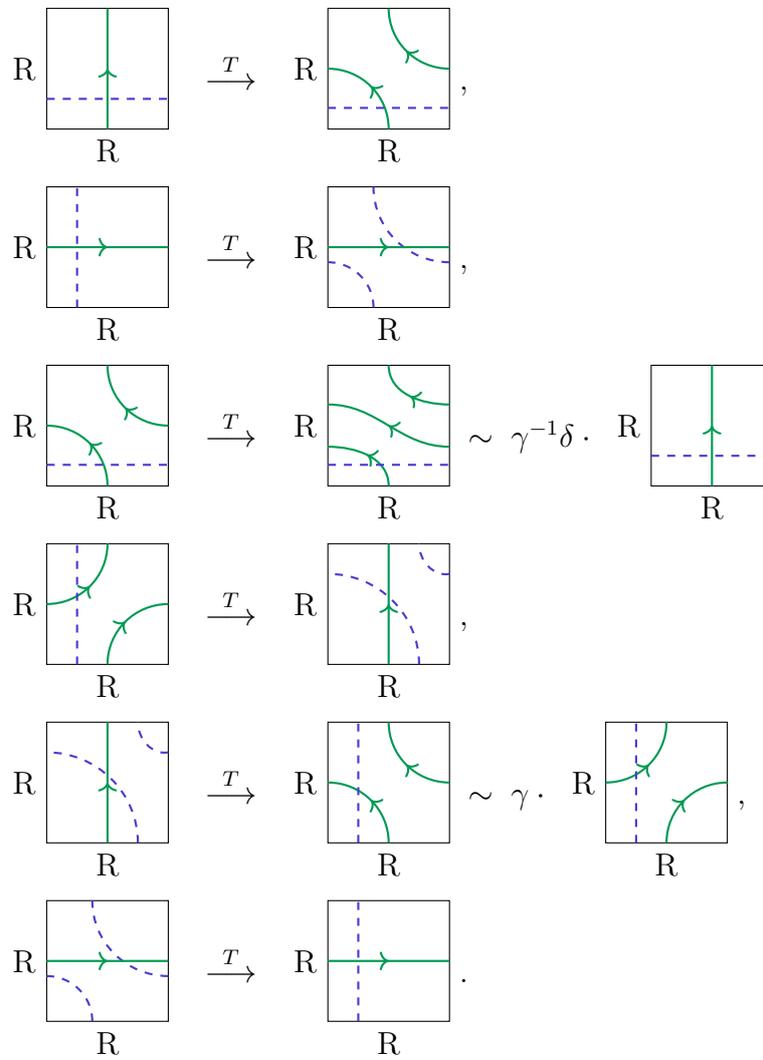


Figure 4.12: The action of  $T$  transformation on the basis elements chosen in Figure 4.11.

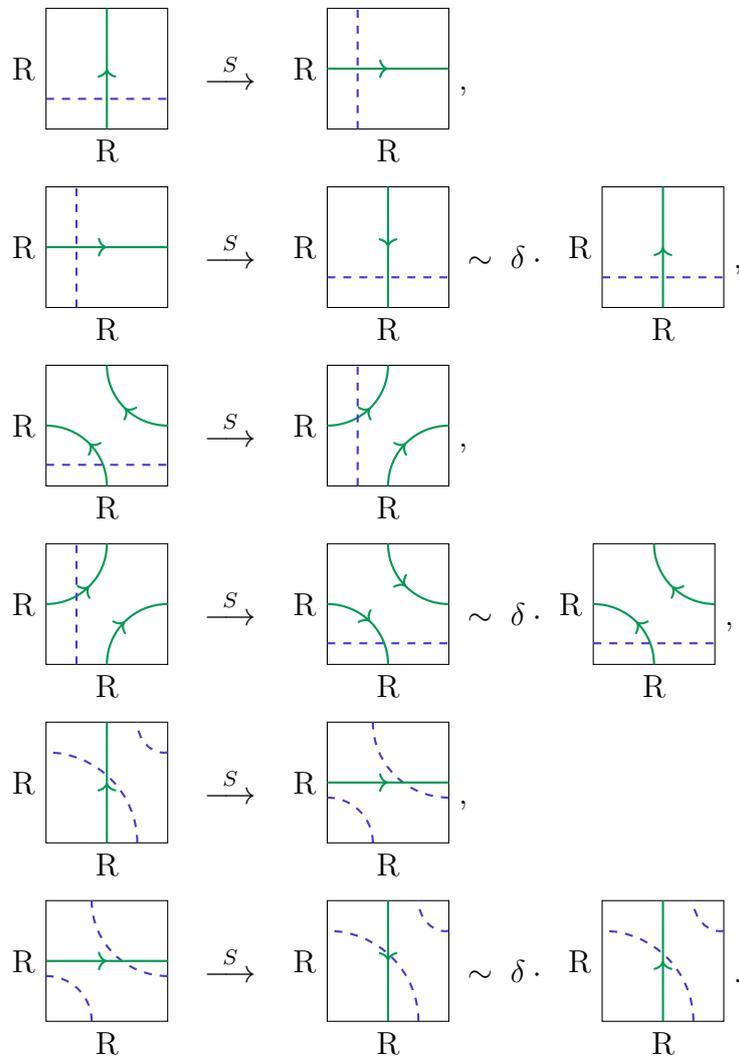


Figure 4.13: The action of  $S$  transformation on the basis elements chosen in Figure 4.11.

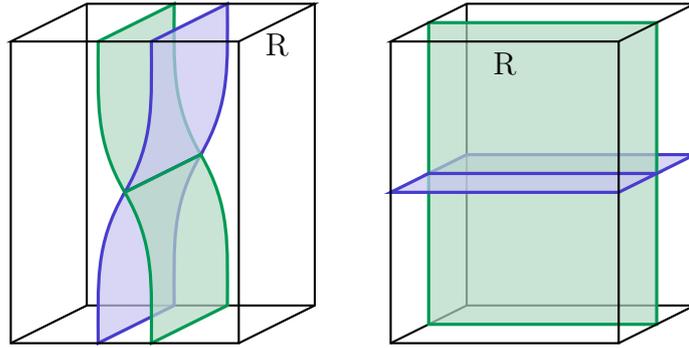


Figure 4.14: Left: illustration of anti-commutativity of  $(-1)^F$  and  $(-1)^Q$  operators in the Ramond sector of the 2d boundary QFT for  $\nu = 1 \pmod 2$  in terms of the bulk TQFT. Right: illustration of sign ambiguity of the trace over the Ramond or NS twisted Hilbert spaces with  $(-1)^F$  inserted. In both cases the spacetime of the boundary theory lies in the horizontal plane, with the time going from left to right. The intersection of  $(-1)^F$  and  $(-1)^Q$  surface defects in the bulk contributes is an orientation-preserving loop with odd induced spin structure. Therefore it contributes  $(-1)^\nu$  (see discussion below (4.39)).

using the bulk iTQFT or defect rules described above. As those descriptions are universal, there is no ambiguity about which features are generic and which could be specific for a particular two-dimensional theory.

In the geometric description the action of the operators  $(-1)^Q$  and  $(-1)^F$  on the Hilbert space is realized by the corresponding defect lines going along the spatial circle. Consider first the Hilbert space of the 2d theory in Ramond untwisted sector. A continuous process exchanging the two operators can be realized by the configuration of the surface defects in the bulk TQFT as depicted on the left side of Figure 4.14. The surfaces intersect along a circle with induced periodic spin structure, which contributes  $(-1)^\nu$ . This implies that for  $\nu = 1 \pmod 2$  the operators anticommute:

$$(-1)^F (-1)^Q = (-1)^\nu \cdot (-1)^Q (-1)^F \quad \text{on } \mathcal{H}_{R0}. \quad (4.50)$$

This indeed agrees with the previous analysis in the literature [17, 71, 102].

This phenomenon can be also seen from the rules on the defect moves considered in the Section 4.2. When the  $(-1)^Q$  defect is passed through the  $(-1)^F$  defect, the lift of the slope angle to  $\text{Spin}(2)$  shifts by  $2\pi$  (cf. (4.41)):

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \xrightarrow{0} \sim \begin{array}{c} \xrightarrow{2\pi} \\ \text{---} \text{---} \end{array} \sim (-1)^\nu \cdot \begin{array}{c} \xrightarrow{0} \\ \text{---} \text{---} \end{array} \quad (4.51)$$

where in the last relation we applied the rule (4.35) and used that  $\alpha^4 = (-1)^\nu$ .

In the Neveu-Schwarz untwisted sector  $\mathcal{H}_{NS0}$  the operators commute for any  $\nu$ . The corresponding configuration of the surface defects in the bulk TQFT will be as the left part of Figure 4.14, but the induced spin-structure on the intersection will

be anti-periodic instead, so it will contribute  $1^\nu = 1$ . In terms of the defect rules, the NS sector has a  $(-1)^F$  line inserted along the time direction, so the  $(-1)^Q$  line will intersect once with it and will have no globally defined lift of the slope angle to  $\text{Spin}(2)$ . Therefore moving it through the  $(-1)^F$  defect does not change it:

(4.52)

On the other hand, the twisted Ramond or Neveu-Schwarz sector has a  $(-1)^Q$  line inserted along the time direction, for which, at a given time slice, one has to choose the slope angle to  $\text{Spin}(2)$ . If the  $(-1)^Q$  line does not intersect a  $(-1)^F$  line, the change of the lift results in the extra phase  $\alpha^4 = (-1)^\nu$ . This in particular results in the ambiguity of the partition function with periodic boundary condition traced over the Hilbert space with  $(-1)^F$  inserted for  $\nu = 1 \pmod 2$ . Another way to see the sign ambiguity is through the configuration of the surface defects in the bulk TQFT depicted on the right part of Figure 4.14. The  $(-1)^F$  defect corresponds to an automorphism of spin structure of the spacetime of the 2d QFT. It intersects with  $(-1)^Q$  defect along a circle with periodic induced spin structure and thus gives an extra  $(-1)^\nu$  phase.

Suppose one chooses to fix a particular lift to  $\text{Spin}(2)$  for the  $(-1)^Q$  defect inserted along the time direction in 2d and a Hilbert space to it. Then the  $(-1)^F$  operator will not act within this Hilbert space, but will transform it to a different one:

(4.53)

This is again in the agreement with the analysis of [17] for free fermions.

## 4.4 Surgery approach

In this section we will present yet another approach to calculating the elements of the modular  $S$  and  $T$  matrices. This approach is probably the least geometrical and physical, but allows a rather straightforward and algorithmic calculation, due to its combinatorial nature. Also, because of its simplicity, this is the approach that we will follow when we consider generalization to other groups in Section 5.

As we have already seen, the calculation of matrix elements always boils down to calculation of the values of the invariant  $\beta_s(a)$  on closed 3-dimensional manifolds.

This invariant is known to be directly related to the mod 16 valued Rokhlin invariant (also known as  $\mu$ -invariant) of spin manifolds [84] (see also [60] for a physics interpretation of this relation):

$$\beta_s(a) = \frac{\mu_s - \mu_{s+a}}{2} \pmod{8}. \quad (4.54)$$

Note that the difference of the values of the Rokhlin invariant for different choices of spin structure is always even.

The Rokhlin invariant  $\mu_s$  of the 3-manifold  $Y$  with spin structure  $s \in \text{Spin}(Y)$  is defined as the signature mod 16 of a spin 4-manifold  $U$  which has  $Y$  as its boundary, with  $s$  induced from the spin structure of the 4-manifold:

$$\mu_s := \sigma(U) \pmod{16}. \quad (4.55)$$

There is however also a formula for the value of the Rokhlin invariant in terms of Dehn surgery representation of 3-manifold [103] which we will review below.

First we recall that any closed oriented manifold can be obtained by a Dehn surgery on a framed link  $\mathcal{L}$  in a 3-sphere  $S^3$ . The framing can be geometrically understood as a choice of a non-vanishing normal vector field on each link component, considered up to isotopy (continuous deformation that respects the non-vanishing condition). Combinatorially, the framing can be described by assigning an integer number  $p_I \in \mathbb{Z}$  to each link component  $\mathcal{L}_I \subset \mathcal{L}$  ( $I = 1, \dots, V$ ) – the self-linking number. The self-linking number is the linking number between the link component and its push-off towards the framing vector field. The surgery operation amounts to removing tubular neighborhoods for each link component and then gluing back solid tori so that their meridians (contractible cycles) are mapped to the curves traced by the framing vectors on the boundaries of tubular neighborhoods. The results of such operation is usually denoted by  $S^3(\mathcal{L})$ .

Any 3-manifold is spin and the choice of spin structure has a natural description in terms of the surgery representation of 3-manifold. Namely, the spin-structures are in canonical one-to-one correspondence with *characteristic sublinks* of the framed link representing a 3-manifold. A characteristic sublink  $\mathcal{C}$  is any sublink  $\mathcal{C} \subset \mathcal{L}$  that satisfies:

$$\ell k(\mathcal{C}, \mathcal{L}_I) = \ell k(\mathcal{L}_I, \mathcal{L}_I) \equiv p_I \pmod{2}, \quad \forall I \quad (4.56)$$

where  $\ell k$  denotes the linking number<sup>9</sup>. To make this condition even more explicit, consider the  $V \times V$  linking matrix  $B$  with components

$$B_{IJ} := \ell k(\mathcal{L}_I, \mathcal{L}_J). \quad (4.57)$$

Then characteristic sublinks (and, therefore, spin structures on the corresponding 3-manifold  $S^3(\mathcal{L})$ ) are in one-to-one correspondence with mod 2 vectors  $s \in \mathbb{Z}_2^V$  such that

$$\sum_J B_{IJ} s_J = B_{II} \pmod{2}, \quad \forall I. \quad (4.58)$$

---

<sup>9</sup>Recall that  $\ell k(\mathcal{A}, \mathcal{B})$  for a pair of oriented links  $\mathcal{A}, \mathcal{B}$  can be defined as the algebraic number of intersection points in  $S \cap \mathcal{B}$ , where  $S$  is any surface such that  $\partial S = \mathcal{A}$ .

The characteristic sublink corresponding to given  $s$  is the union of components of  $\mathcal{L}$  for which  $s_I = 1 \pmod 2$ :

$$\mathcal{C} = \bigsqcup_{I: s_I=1 \pmod 2} \mathcal{L}_I. \quad (4.59)$$

The Rokhlin invariant of the 3-manifold  $Y = S^3(\mathcal{L})$  is then given by the following formula in terms of the surgery data [103]:

$$\mu_s(Y) = \sigma(B) - lk(\mathcal{C}_s, \mathcal{C}_s) + 8 \text{Arf}(\mathcal{C}_s) \pmod{16} \quad (4.60)$$

where  $\mathcal{C}_s$  is the characteristic sublink corresponding to spin-structure  $s \in \text{Spin}(Y)$ ,  $\sigma(B)$  is the signature of the linking matrix  $B$ , and  $\text{Arf}(\mathcal{C}_s)$  is the mod 2 valued Arf invariant of link  $\mathcal{C}_s$ . Note that the Arf invariant is only defined for *proper* links, that is links such that the sum of the linking numbers of any component with all the other components is even. For any characteristic sublink  $\mathcal{C}_s$  this property is satisfied automatically due to (4.56).

Similarly to spin structures, the elements  $a \in H^1(Y; \mathbb{Z}_2)$  can be described combinatorially in terms of surgery data as mod 2 vectors  $a \in \mathbb{Z}_2^V$  satisfying<sup>10</sup>

$$\sum_J B_{IJ} a_J = 0 \pmod 2, \quad \forall I. \quad (4.61)$$

This realization is in agreement with the action of elements of  $H^1(Y, \mathbb{Z}_2)$  on  $\text{Spin}(Y)$  by  $s \mapsto s + a$ . Combining (4.54) with (4.60) we have the following explicit formula for the  $\beta$ -invariant of  $Y = S^3(\mathcal{L})$  in terms of surgery data:

$$\beta_s(a) = \frac{(s+a)^T B (s+a) - s^T B s}{2} + 4(\text{Arf}(\mathcal{C}_{s+a}) - \text{Arf}(\mathcal{C}_s)) \pmod 8, \quad (4.62)$$

Note that here  $s+a$  is to be intended as an element of  $\mathbb{Z}_2$ . The Arf invariant of any link  $\mathcal{L}$  can be computed combinatorially using its relation to the Jones polynomial  $V_{\mathcal{L}}(q)$  (unframed version of, with normalization  $V_{\text{unknot}}(q) = 1$ ) at  $q = i$  (with  $q^{1/2} = e^{\frac{\pi i}{4}}$ ) [104, 105]:

$$V_{\mathcal{L}}(i) = \begin{cases} (-\sqrt{2})^{\#(\mathcal{L})-1} (-1)^{\text{Arf}(\mathcal{L})}, & \mathcal{L} \text{ is proper,} \\ 0, & \mathcal{L} \text{ is not proper,} \end{cases} \quad (4.63)$$

where  $\#(\mathcal{L})$  is the number of components of  $\mathcal{L}$ . The Jones polynomial  $V_{\mathcal{L}}(q)$  can be calculated by applying skein relations to the link. Moreover, its values for common links are readily available in the literature. Altogether this provides us with an explicit way to calculate the values of  $\beta_s$  for any closed spin 3-manifold.

As we have already seen in order to determine the modular matrices it is enough to determined the values of  $\beta_s$  for mapping tori  $MT(\phi)$  of  $\mathbb{T}^2$  for certain elements  $\phi \in \text{SL}(2, \mathbb{Z})$  and also manifolds of the form<sup>11</sup>  $(D^2 \times S^1) \cup_{\phi} (D^2 \times S^1)$ . These

<sup>10</sup>The same condition holds for the description of any group  $H^1(Y; \mathbb{Z}_n)$  by changing the conditions from being valid for  $\pmod 2$  to  $\pmod n$ .

<sup>11</sup>In principle one could use only mapping tori by choosing the representative of the trivial class in  $\Omega_2^{\text{Spin}}(B\mathbb{Z}_2)$  to be also a 2-torus instead of an empty space.

3-manifolds have canonical surgery realization given by a word presentation of an element  $\phi \in \text{SL}(2, \mathbb{Z})$  in terms of generators  $T$  and  $S$ .

Consider first the case  $Y = (D^2 \times S^1) \cup_\phi (D^2 \times S^1)$ . Without loss of generality one can assume that  $\phi$  is of the form  $\phi = ST^{p_V} \dots ST^{p_2} ST^{p_1} S$ . The manifold is realized by surgery on the following link:

$$\mathcal{L} = \begin{array}{c} p_1 \quad p_2 \quad \dots \quad p_V \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \quad (4.64)$$

where  $p_I$  are framings. The linking matrix is the following:

$$B = \begin{pmatrix} p_1 & -1 & 0 & \dots & 0 \\ -1 & p_2 & -1 & 0 & \dots \\ 0 & -1 & p_3 & -1 & \dots \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & -1 & p_V \end{pmatrix}. \quad (4.65)$$

Due to its form it is clear that a solution of (4.58) is completely fixed by the value of  $s_1 \in \mathbb{Z}_2$ , which can be arbitrary. The value  $s_1 = 1$  ( $s_1 = 0$ ) corresponds to periodic (antiperiodic) spin structure along the non-contractible circle on  $\mathbb{T}^2 = \partial(D^2 \times S^1)$  on which  $\phi \in \text{SL}(2, \mathbb{Z})$  acts. Note that in this case the corresponding characteristic sublink  $\mathcal{C}_s$  is always a disjoint union of unlinked unknots. Its Arf invariant is always trivial:  $\text{Arf}(\mathcal{C}_s) = 0 \pmod{2}$ .

Consider now the case  $Y = \mathcal{M}(\phi)$ . Again, without a loss of generality we can assume that  $\phi$  is of the form  $\phi = ST^{p_{V-1}} \dots ST^{p_2} ST^{p_1}$ . This manifold can be similarly realized by surgery on a framed link  $\mathcal{L}$  which can be described as follows. Instead of the linear “chain” of unknots (framed according to the powers of  $T$ ) we now have a circular chain passing through one extra unknot with framing zero:

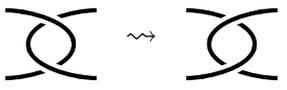
$$\mathcal{L} = \begin{array}{c} 0 \\ p_3 \\ \dots \\ p_2 \\ \dots \\ p_1 \\ \dots \\ p_{V-1} \\ \dots \\ p_{V-2} \end{array} \quad (4.66)$$

The linking matrix now has the following form:

$$B = \begin{pmatrix} p_1 & -1 & 0 & \dots & -1 & 0 \\ -1 & p_2 & -1 & 0 & \dots & 0 \\ 0 & -1 & p_3 & -1 & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ -1 & \dots & 0 & -1 & p_{V-1} & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.67)$$

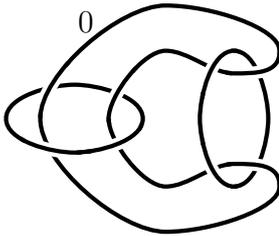
The solution to the characteristic sublink condition (4.58) is completely fixed by the values of  $s_1, s_2$  and  $s_V$ . The values of  $s_{1,2}$  correspond to the spin structures on the meridian and longitude of the 2-torus on which  $\phi \in \text{SL}(2, \mathbb{Z})$  acts. As before, the value 1 (0) corresponds to periodic (anti-periodic) condition on spinors. The value of  $s_V$  is always arbitrary and independent of other values  $s_I$ . It corresponds to the choice of spin structure on the base circle of the mapping torus, if the latter is considered as a fibration over  $S^1$  with torus in the fibers.

Note that in both cases one can consider more general words that also contain  $S^{-1}$  (without expressing it through  $S$  and  $T$ , e.g.  $S^{-1} = ST^0ST^0S$ ) by flipping the links as follows:

$$\begin{aligned} S &\rightsquigarrow S^{-1}, \\ \text{Diagram 1} &\rightsquigarrow \text{Diagram 2}, \\ B_{I,I+1} = -1 &\rightsquigarrow B_{I,I+1} = 1. \end{aligned} \quad (4.68)$$


Of course, making this replacement at any place in the linear chain of unknots (4.64) will always result in an equivalent link. This, however, is not the case for a circular chain (4.66).

As an example of using this method, consider the mapping torus of  $\phi = T^2 \in \text{SL}(2, \mathbb{Z})$ , which is needed to determine some of the matrix elements of the  $T$  modular matrix. To represent  $Y = \mathcal{M}(\phi)$  by surgery we write it in the form  $\phi = ST^2S^{-1}T^0$ , so

$$\mathcal{L} = \begin{array}{c} \text{Diagram of Borromean rings} \\ \begin{array}{ccc} & 0 & \\ 0 & & \\ & & 2 \end{array} \end{array}. \quad (4.69)$$


This 3-component link is commonly known as *Borromean rings* and has a property that if either of the components is removed it becomes an unlink. The linking matrix is the following:

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.70)$$

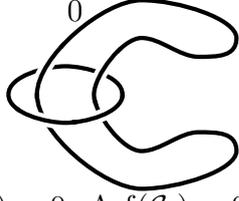
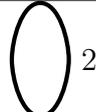
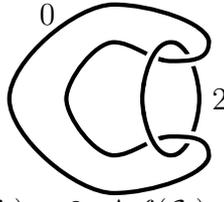
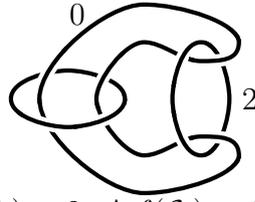
| $(s_1, s_2)$ | $s_3 = 0$  | $s_3 = 1$   |
|--------------|--|---|
| $(0, 0)$     | $\mathcal{C}_s = \emptyset$<br>$\ell k(\mathcal{C}_s, \mathcal{C}_s) = 0, \text{Arf}(\mathcal{C}_s) = 0,$<br>$\mu_s = 1 \pmod{16}$   | $\mathcal{C}_s = 0$ <br>$\ell k(\mathcal{C}_s, \mathcal{C}_s) = 0, \text{Arf}(\mathcal{C}_s) = 0,$<br>$\mu_s = 1 \pmod{16}$  |
| $(1, 0)$     | $\mathcal{C}_s = 0$ <br>$\ell k(\mathcal{C}_s, \mathcal{C}_s) = 0, \text{Arf}(\mathcal{C}_s) = 0,$<br>$\mu_s = 1 \pmod{16}$   | $\mathcal{C}_s = 0$ <br>$\ell k(\mathcal{C}_s, \mathcal{C}_s) = 0, \text{Arf}(\mathcal{C}_s) = 0,$<br>$\mu_s = 1 \pmod{16}$  |
| $(0, 1)$     | $\mathcal{C}_s = 0$ <br>$\ell k(\mathcal{C}_s, \mathcal{C}_s) = 2, \text{Arf}(\mathcal{C}_s) = 0,$<br>$\mu_s = -1 \pmod{16}$  | $\mathcal{C}_s = 0$ <br>$\ell k(\mathcal{C}_s, \mathcal{C}_s) = 2, \text{Arf}(\mathcal{C}_s) = 0,$<br>$\mu_s = -1 \pmod{16}$ |
| $(1, 1)$     | $\mathcal{C}_s = 0$ <br>$\ell k(\mathcal{C}_s, \mathcal{C}_s) = 2, \text{Arf}(\mathcal{C}_s) = 0,$<br>$\mu_s = -1 \pmod{16}$ | $\mathcal{C}_s = 0$ <br>$\ell k(\mathcal{C}_s, \mathcal{C}_s) = 2, \text{Arf}(\mathcal{C}_s) = 1,$<br>$\mu_s = 7 \pmod{16}$  |

Table 4.1: Different characteristic sublinks of the link realizing the mapping torus of  $T^2 \in \text{SL}(2, \mathbb{Z})$  and the corresponding values of the Rokhlin invariant. We used the fact that  $\sigma(B) = 1$  which gives a spin-structure independent contribution to  $\mu_s$ .

Therefore any sublink of  $\mathcal{L}$  is a characteristic sublink, which is consistent with the fact that there are  $|H^1(Y, \mathbb{Z}_2)| = 8$  different spin structures on  $Y = S^3(\mathcal{L})$ . Table 4.1 shows different choices of characteristic sublinks  $\mathcal{C}_s$  corresponding to different spin structures  $s \in \text{Spin}(Y)$  and the corresponding values of the Rokhlin invariant  $\mu_s$ . Note that the Arf invariant is non-trivial only for the complete Borromean rings.

This table can be used to determine all the matrix elements of the diagonal matrix  $T^2$  (which, in turn, determines off-diagonal elements of the matrix  $T$  after fixing the basis). For example:

$$T^2|_{\{R_0, R_1\}} = e^{\frac{\pi i \nu}{4} \beta_{(1,1,0)}((0,1,0))} = e^{\frac{\pi i \nu}{8} (\mu_{(1,1,0)} - \mu_{(1,0,0)})} = e^{-\frac{\pi i \nu}{4}} \quad (4.71)$$

which is in agreement with the geometric calculation in Section 4.1. One can easily determine other matrix elements of  $S$  and  $T$  in a similar way.

# Chapter 5

## Generalization to other symmetry groups in $d = 2$

In the previous chapter we computed via different methods the map (3.103) for theories with a  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2$  symmetry. However, as we saw, by standard means the explicit computation of anomalous phases can be rather involved already in the simplest of the setups. Therefore, it is important to understand whether it is possible to develop further the complementary approaches used so far so that we may be able to compute the map in a systematic fashion for the general case. In particular, we will focus our effort on generalizing the approach based on surgery of Section 4.4, as this will turn out to be not only very efficient, but also not heavily reliant on geometrical knowledge which may depend on the particular setup considered.

Another important question which is worth to study is trying to understand in a systematic way which kind of anomalies can be reproduced by which kind of free fermions content. Indeed, in our presentation we mostly focused on how anomalies are classified, but the classical tools we have at our disposal make their actual computation complicated already for free fermions, due to their relation with the  $\eta$  invariant, which in general may be difficult to compute. Therefore, one may also wonder if it is possible to gain knowledge on this topic as well via some different, complementary point of view on the subject. To answer this, we must be able to efficiently describe the map

$$\{\text{representations}\} \longrightarrow \{\text{invertible TQFTs}\}. \quad (5.1)$$

By merging the two maps together, we should be able to describe for any symmetry group  $G$  a whole set of maps

$$\{\text{representations}\} \longrightarrow \{\text{invertible TQFTs}\} \longrightarrow \{\text{anomalous phases}\}. \quad (5.2)$$

In this chapter we will outline a method to explicitly determine the maps in the sequence (5.2) for the case of two-dimensional theories on a 2-torus with arbitrary finite group symmetry, based on joint results of the works [1, 2].

## 5.1 Naturality property

In the setup of our consideration the sequence (5.2) reads explicitly as follows:

$$RO^f(G^f) \longrightarrow \text{Hom}(\Omega_3^{\text{Spin} \times_{\mathbb{Z}_2^f} G^f}, U(1)) \longrightarrow H^1(\widetilde{\text{MCG}}(\mathbb{T}^2), U(1)^{\text{Spin}_{G^f}(\mathbb{T}^2)}). \quad (5.3)$$

Here  $RO^f(G^f)$  is defined as the subgroup of  $RO(G^f)$ , i.e. the free group generated by real irreps of  $G^f$ , that is generated by the image of the non-trivial representation  $\llbracket 1 \rrbracket$  of  $\mathbb{Z}_2^f$  via the pullback of the inclusion  $\xi : \mathbb{Z}_2^f \hookrightarrow G^f$ , i.e.

$$RO^f(G^f) := (\xi^*)^{-1}(\mathbb{Z} \llbracket 1 \rrbracket). \quad (5.4)$$

In the simple case  $G^f = \mathbb{Z}_2^f \times G$  we have  $RO^f(G^f) \cong RO(G)$ . Note that in the second place in the sequence we only consider the subgroup of the total anomaly group corresponding to non-perturbative anomalies. That is we assume that perturbative anomalies (which can be only purely gravitational when  $G^f$  is discrete and which for  $G^f = \mathbb{Z}_2^f \times G$  are formally classified by  $\text{Hom}(\Omega_4^{\text{Spin}}(\text{pt}), U(1)) \cong \mathbb{Z}$ ) are absent. The first map is physically realized as follows. The elements of  $RO^f(G^f)$ , by definition, are formal differences  $\rho_1 - \rho_2$  of (isomorphism classes of) representations. We then first consider the theory of free right-moving (or anti-holomorphic) Majorana-Weyl fermions in representation  $\rho_1$  and left-moving (or holomorphic) Majorana-Weyl fermions in representation  $\rho_2$ . We then add to it an appropriate number of left- or right-moving free fermions in the trivial representation to ensure that the chiral central charge of the theory is zero. That is we always add or subtract copies of a trivial representation to  $\rho_1 - \rho_2$  to make the total virtual dimension zero. The anomaly of the resulting theory is then the value of the first map of the sequence.

Let's focus for the moment on the case  $G^f = \mathbb{Z}_2^f \times G$ . Mathematically the first map can be understood as a spin-cobordism generalization of a characteristic class map  $RU(G) \rightarrow H^4(BG, \mathbb{Z}) \cong H^3(BG, U(1))$  [106–108] (here  $RU(G)$  is the complex representation ring). A real representation  $\rho$  of dimension  $n$  can be understood as a homomorphism  $\rho : G \rightarrow O(n)$ . The value of its image in  $\text{Hom}(\Omega_3^{\text{Spin}}(BG), U(1))$  then can be understood as a pullback under  $\rho$  of a certain “universal characteristic class” in<sup>1</sup>  $\text{Hom}(\Omega_3^{\text{Spin}}(BO), U(1)) \oplus \text{Hom}(\Omega_4^{\text{Spin}}(BO), \mathbb{Z})$  that vanishes under the forgetful map to  $\text{Hom}(\Omega_4^{\text{Spin}}(\text{pt}), \mathbb{Z})$ . This latter description is however hard to use in practice to evaluate the map for a given representation and we will use a different approach.

The second map is just the standard map (3.103) described in Section 3.4.1. The evaluation of this map for a given element of the anomaly group boils down to calculation of the value of the corresponding spin-bordism invariant on certain closed spin 3-manifolds.

All three groups in the sequence (5.3) can now be understood as values of contravariant functors from the category of groups to the category of abelian groups. The

---

<sup>1</sup>Remember that the splitting of this group (Anderson dual of spin-bordism of  $BO$ ) into direct sum is non-canonical.

sequence (5.3) is then a sequence of natural transformations between the corresponding functors, evaluated on an object  $G$ . Therefore, for any group homomorphism

$$G \xrightarrow{f} G' \quad (5.5)$$

we have induced maps (pullbacks) for the corresponding abelian groups in (5.3), so that they fit into the following commutative diagram:

$$\begin{array}{ccc} RO(G) & \xleftarrow{f^*} & RO(G') \\ \downarrow & & \downarrow \\ \text{Hom}(\Omega_3^{\text{Spin}}(BG), U(1)) & \xleftarrow{f^*} & \text{Hom}(\Omega_3^{\text{Spin}}(BG'), U(1)) \\ \downarrow & & \downarrow \\ H^1((\widetilde{\text{SL}}(2, \mathbb{Z}) \times G) \rtimes \text{Spin}_G(\mathbb{T}^2), U(1)) & \xleftarrow{f^*} & H^1((\widetilde{\text{SL}}(2, \mathbb{Z}) \times G') \rtimes \text{Spin}_{G'}(\mathbb{T}^2), U(1)). \end{array} \quad (5.6)$$

Let us clarify how the horizontal maps in this diagram are defined. The first horizontal map in the diagram is the standard pullback of representations.

For the second map, a homomorphism  $f : G \rightarrow G'$  first induces a map  $BG \rightarrow BG'$  between the corresponding classifying space. This map then induces the pushforward of the corresponding spin-bordism group  $\Omega_3^{\text{Spin}}(BG) \rightarrow \Omega_3^{\text{Spin}}(BG')$ . Finally, taking the Pontryagin dual  $\text{Hom}(-, U(1))$  to both of this group produces the dual map  $f^*$  in the second line of the diagram (5.6). More explicitly, one can describe this map in terms of supercohomology representation of Pontryagin duals to the bordism groups [65, 68, 77, 109, 110]. In this approach the elements of the anomaly group  $\Omega_3^{\text{Spin}}(BG)$  are given by equivalence classes of a collection of functions on the products of copies of  $G$  satisfying certain generalized cocycle conditions, as we already saw. These functions are then pulled back under  $f : G \rightarrow G'$ .

Physically the second horizontal map in (5.6) can be understood as follows. Consider a theory with symmetry  $G'$  that has an anomaly corresponding to a certain element of  $\text{Hom}(\Omega_3^{\text{Spin}}(BG'), U(1))$ . By using the homomorphism  $f : G \rightarrow G'$  one can consider this as a theory with symmetry  $G$  instead<sup>2</sup> (i.e.  $g \in G$  acts on the operators of the theory as  $f(g) \in G'$ ). Its anomaly then gives an element in  $\text{Hom}(\Omega_3^{\text{Spin}}(BG), U(1))$ .

To define the third horizontal map in (5.6) first note that explicitly we have

$$\text{Spin}_G(\mathbb{T}^2) = \text{Spin}(\mathbb{T}^2) \times \text{Hom}(\pi_1(\mathbb{T}^2), G) \cong \text{Spin}(\mathbb{T}^2) \times \text{Hom}(\mathbb{Z}^2, G), \quad (5.7)$$

where the last isomorphism is achieved by choosing a basis in  $\pi_1(\mathbb{T}^2) \cong H_1(\mathbb{T}^2, \mathbb{Z})$ . The elements of  $\text{Hom}(\mathbb{Z}^2, G)$  can be understood as pairs of commuting elements in  $G$ . We have a natural induced pushforward map

$$f_* : \text{Spin}_G(\mathbb{T}^2) \longrightarrow \text{Spin}_{G'}(\mathbb{T}^2) \quad (5.8)$$

<sup>2</sup>Here we have no requirement of action being faithful.

that acts on elements of  $\text{Hom}(\mathbb{Z}^2, G)$  by composition with  $f$ . Recall that an element of  $H^1((\widetilde{\text{SL}}(2, \mathbb{Z}) \times G') \times \text{Spin}_{G'}(\mathbb{T}^2), U(1))$  can be represented by a representation  $\rho$  of the action groupoid  $(\widetilde{\text{SL}}(2, \mathbb{Z}) \times G') \times \text{Spin}_{G'}(\mathbb{T}^2)$ , that is a collection of one-dimensional complex vector spaces  $\mathcal{H}_a$  for each  $a \in \text{Spin}_{G'}(\mathbb{T}^2)$  and linear maps

$$\rho(g') : \mathcal{H}_{a'} \longrightarrow \mathcal{H}_{g' \cdot a'} \quad (5.9)$$

for each  $g' \in G'$  which satisfy the 1-cocycle condition  $\rho(g_1 g_2) = \rho(g_1) \circ \rho(g_2)$ . To construct a pullback of this representation under  $f$ , one takes  $f^* \mathcal{H}_a := \mathcal{H}_{f_*(a)}$  for each  $a \in \text{Spin}_G(\mathbb{T}^2)$  and the maps

$$(f^* \rho)(g) := \rho(f(g)) : f^* \mathcal{H}_a \equiv \mathcal{H}_{f_*(a)} \longrightarrow \mathcal{H}_{f(g) \cdot f_*(a)} = \mathcal{H}_{f_*(g \cdot a)} \equiv f^* \mathcal{H}_{g \cdot a}. \quad (5.10)$$

As we will see, the commutativity of the diagram (5.6) for any group homomorphism  $f : G \rightarrow G'$  is a very constraining property that can be used to essentially fix the vertical maps for any  $G$  from the knowledge of these maps for some basic cases.

Before presenting the details of such computations, let us also comment on the twisted case, when  $G^f \neq \mathbb{Z}_2^f \times G$ . In this case one has to be careful about the presence of gravitational anomalies. Indeed, while (5.3) still holds, as it is always possible to add free matter content that removes perturbative gravitational anomalies, it is instead not true anymore that the functorial properties described above holds at the level of global anomalies. In fact, the precise statement involves the total anomaly groups  $\text{Hom}(\Omega_3^{\text{Spin} \times_{\mathbb{Z}_2^f} G^f}, U(1)) \oplus \text{Hom}(\Omega_4^{\text{Spin} \times_{\mathbb{Z}_2^f} G^f}, \mathbb{Z})$  and one must be careful if by considering different kinds of groups  $G^f$  and  $G^{f'}$  there is a non-trivial anomaly interplay between gravitational anomalies of a group with global ones of the other, similarly as what we already described in Section 2.1.2. An important example of this is when we are analyzing anomalies between the groups  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_n$  and  $G^{f'} = \mathbb{Z}_{2n}^f$ . Generally, the argument we just explained holds when considering homomorphisms of groups of the form

$$G^f = \mathbb{Z}_2^f \times_{\mathbb{Z}_2^f} G_0 \times G \xrightarrow{f} \mathbb{Z}_2^f \times_{\mathbb{Z}_2^f} G_0 \times G' = G^{f'}, \quad (5.11)$$

where the difference between  $G^f$  and  $G^{f'}$  is not tied to the fermionic parity  $\mathbb{Z}_2^f$ .

With this premise, in the next Sections 5.2 and 5.3 we focus on the description of the class of models with  $G^f = \mathbb{Z}_2^f \times G$ , so that we can freely discuss the relation of global anomalies with different groups without having to worry about the presence of interplay with perturbative anomalies. This will be the fundamental tool that will help us to work out the surgery expression for the evaluation of bordism invariants related to anomalous transformations on  $\mathbb{T}^2$ . In Section 5.4 we will instead provide the prototypical example for the analysis of global anomalies of theories with twisted symmetry structure, namely when  $G^f = \mathbb{Z}_{2^{l+1}}^f$ . While we will see that in this case a more careful analysis is necessary, we will still be able to reach the same kind of surgery description of invariants provided for  $G^f = \mathbb{Z}_2^f \times G$ .

## 5.2 The map from the representation ring to the anomaly group

As was already mentioned, the ‘‘Chern character’’ map

$$\mathbf{ch}_2 : RO(G) \longrightarrow \text{Hom}(\Omega_3^{\text{Spin}}(BG), U(1)) \quad (5.12)$$

can be understood as a spin-bordism analog of the maps  $c_i : RU(G) \rightarrow H^{2i}(BG, \mathbb{Z})$  and  $w_i : RO(G) \rightarrow H^i(BG, \mathbb{Z}_2)$  given respectively by Chern and Stiefel-Whitney classes of linear representations of  $G$  [106–108]. This in part motivates the choice of the notation and terminology, as the map  $\mathbf{ch}_2$  reduces to the map  $2\text{ch}_2 = c_1^2 - 2c_2 : RU(G) \rightarrow H^4(BG, \mathbb{Z}) \cong H^3(BG, U(1)) \subset \text{Hom}(\Omega_3^{\text{Spin}}(BG), U(1))$  when the corresponding anomaly is bosonic and realized by taking a *double*<sup>3</sup> of *complex* Weyl fermions in the same representation. I.e. the following diagram is commutative:

$$\begin{array}{ccc} RU(G) & \xrightarrow{2\text{ch}_2=c_1^2-2c_2} & H^3(BG, U(1)) \\ \downarrow (\cdot)_{\mathbb{R}}^{\oplus 2} & & \downarrow \\ RO(G) & \xrightarrow{\mathbf{ch}_2} & \text{Hom}(\Omega_3^{\text{Spin}}(BG), U(1)) \end{array} \quad (5.13)$$

where the left vertical map is realized by taking two copies of a complex representation, considered as a real representation (e.g. one-dimensional complex representation is mapped to a four-dimensional real one). The right vertical map physically corresponds to embedding the group of bosonic anomalies into the group of fermionic anomalies<sup>4</sup>. Mathematically it is realized as the dual to the forgetful map  $\Omega_3^{\text{Spin}}(BG) \rightarrow H_3(BG, \mathbb{Z})$ .

In this cases it is known that one can define these classes axiomatically, by specifying their properties which determine them uniquely. Although we do not prove this, we would like to claim that one can axiomatically define  $\mathbf{ch}_2$  in a similar way. Namely we consider the following three axioms (which are in parallel to the axioms for Chern and Stiefel-Whitney characteristic classes):

1. Linearity:

$$\mathbf{ch}_2(\rho_1 \oplus \rho_2) = \mathbf{ch}_2(\rho_1) + \mathbf{ch}_2(\rho_2). \quad (5.14)$$

2. Naturality: for any group homomorphism  $f : G \rightarrow G'$

$$f^* \mathbf{ch}_2(\rho) = \mathbf{ch}_2(f^* \rho) \quad (5.15)$$

i.e. commutativity of the upper square in the diagram (5.6).

---

<sup>3</sup>Note that  $\text{ch}_2 = c_1^2/2 - c_2$  in general does not give a well-defined element in integral cohomology because of  $1/2$  factor.

<sup>4</sup>Note that in 3 dimensions this map is injective, so it can be understood as an embedding, but this is not true in general

3. Normalization: for an abelian  $G$  the value of  $\mathfrak{ch}_2$  is given according to Tables 5.1, 5.2, and 5.3 (see Section 5.2.1 for details).

We conjecture that these properties uniquely fix  $\mathfrak{ch}_2$ . In the Section 5.2.2 we give several examples on how one can use this axioms to evaluate this map in the case of a non-abelian  $G$ . Let us note that in [111] the similar naturality property was used for the homomorphism  $\mathbb{Z}_2 \rightarrow U(1)$  (in this case the target group  $U(1)$  is continuous and has non-trivial perturbative anomaly group).

### 5.2.1 Abelian groups

Consider a general abelian group  $G$ , written in the form

$$G = \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \dots \times \mathbb{Z}_{p_N^{r_N}} \quad (5.16)$$

where  $p_i$  are prime numbers and  $r_i \geq 1$ . From the results of [60, 65, 109, 110] one can conclude that the anomaly group has the following form:

$$\text{Hom}(\Omega_3^{\text{Spin}}(BG), U(1)) \cong \prod_i A_{p_i; r_i} \times \prod_{\substack{i < j \\ p_i = p_j}} B_{p_i; r_i, r_j} \times \prod_{\substack{i < j < k \\ p_i = p_j = p_k}} C_{p_i; r_i, r_j, r_k} \quad (5.17)$$

where

$$A_{p;r} = \begin{cases} \mathbb{Z}_8, & p = 2, r = 1, \\ \mathbb{Z}_2 \times \mathbb{Z}_{2^{r+1}}, & p = 2, r > 1, \\ \mathbb{Z}_{p^r}, & p > 2, \end{cases} \quad (5.18)$$

$$B_{p;r_1, r_2} = \begin{cases} \mathbb{Z}_4, & p = 2, r_1 = r_2 = 1, \\ \mathbb{Z}_2 \times \mathbb{Z}_{2^{\min(r_1, r_2)}}, & p = 2, \max(r_1, r_2) > 1, \\ \mathbb{Z}_{p^{\min(r_1, r_2)}}, & p > 2, \end{cases} \quad (5.19)$$

$$C_{p;r_1, r_2, r_3} = \mathbb{Z}_{p^{\min(r_1, r_2, r_3)}}. \quad (5.20)$$

The decomposition into factors (5.17) is determined by considering the pullbacks of the projections of  $G$  on the factors of the form  $\mathbb{Z}_{p_i^{r_i}}, \mathbb{Z}_{p_i^{r_i}} \times \mathbb{Z}_{p_j^{r_j}}, \mathbb{Z}_{p_i^{r_i}} \times \mathbb{Z}_{p_j^{r_j}} \times \mathbb{Z}_{p_k^{r_k}}$ . In particular the subgroup  $A_{p_i; r_i}$  in the decomposition is the image of the anomaly group  $\text{Hom}(\Omega_3^{\text{Spin}}(B\mathbb{Z}_{p_i^{r_i}}))$  under the projection  $G \rightarrow \mathbb{Z}_{p_i^{r_i}}$ . For comparison, the group of bosonic anomalies is given by

$$H^3(BG, U(1)) \cong \prod_i \mathbb{Z}_{p_i^{r_i}} \times \prod_{\substack{i < j \\ p_i = p_j}} \mathbb{Z}_{p_i^{\min(r_i, r_j)}} \times \prod_{\substack{i < j < k \\ p_i = p_j = p_k}} \mathbb{Z}_{p_i^{\min(r_i, r_j, r_k)}} \quad (5.21)$$

To describe the map (5.12) for a general abelian  $G$  it is then enough to describe the maps

$$\begin{aligned} RO(\mathbb{Z}_{p^r}) &\longrightarrow A_{p;r} \\ RO(\mathbb{Z}_{p^{r_1}} \times \mathbb{Z}_{p^{r_2}}) &\longrightarrow B_{p;r_1, r_2} \\ RO(\mathbb{Z}_{p^{r_1}} \times \mathbb{Z}_{p^{r_2}} \times \mathbb{Z}_{p^{r_3}}) &\longrightarrow C_{p;r_1, r_2, r_3} \end{aligned} \quad (5.22)$$

for arbitrary  $p$  and  $r_i$ . Without loss of generality we can assume that  $r_1 \leq r_2 \leq r_3$ .

The representation theory of abelian groups is simple. Consider first the case of a single cyclic group  $\mathbb{Z}_n$ . Consider first complex 1-dimensional representations  $\llbracket q \rrbracket$  of a given “mod  $n$  charge”  $q$  which is the pullback of the standard charge  $q$  representation of  $U(1)$  under the embedding  $\mathbb{Z}_n \subset U(1)$ . That is the generator of  $\mathbb{Z}_n$  acts as multiplication by  $e^{\frac{2\pi i q}{n}}$ . This representation can be considered as a 2-dimensional real representation of charge  $\min\{q, n - q\}$ . As a real representation, it is irreducible unless  $q = 0$  or  $q = n/2$ . The second case is only possible if  $n$  is even. In both of these cases  $\llbracket q \rrbracket$  decomposes into direct sum of two copies of an irreducible 1-dimensional representation, which we will denote using single brackets instead:

$$\llbracket 0 \rrbracket = [0] \oplus [0], \quad (5.23)$$

$$\llbracket n/2 \rrbracket = [n/2] \oplus [n/2]. \quad (5.24)$$

The representations  $[0]$  and  $[n/2]$  are also known as *trivial* and *sign* representation respectively. Whenever we are going to consider a 2-dimensional irrep different than (5.23) and (5.24), we are going to denote it with the correspondent complex representation  $\llbracket q \rrbracket$ , where  $0 < q \leq n/2$ . There are no other irreducible representations over  $\mathbb{R}$ .

Similarly, for a product  $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \dots$  we have the complex 1-dimensional representations

$$\llbracket q_1, q_2, \dots \rrbracket = \llbracket q_1 \rrbracket \otimes_{\mathbb{C}} \llbracket q_2 \rrbracket \otimes_{\mathbb{C}} \dots, \quad \text{where } q_i \in \mathbb{Z}_{n_i}, \quad (5.25)$$

which are irreducible over  $\mathbb{R}$  for generic charges  $q_i$ , and reduce to two copies of a real 1-dimensional representation

$$[q_1, q_2, \dots] = [q_1] \otimes_{\mathbb{R}} [q_2] \otimes_{\mathbb{R}} \dots \quad (5.26)$$

when  $q_i = 0$  or  $n_i/2$  for each  $i$ .

The values of the first two maps in (5.22) then can be determined, *up to an automorphism* in the target groups, using the following facts:

1. The representation  $\llbracket q \rrbracket$  of  $\mathbb{Z}_{p^r}$  is the pullback of the representation with charge  $q$  of  $U(1)$  under the standard inclusion. The group of fermionic anomalies for  $U(1)$  is classified by  $\mathbb{Z}$  (the generator corresponds to the anomaly polynomial  $c_1^2/2$ ) and the corresponding value is well known to be  $q^2$ . The image under the pullback of the anomaly group is the subgroup  $\mathbb{Z}_{2^{r+1}}$  for  $p = 2$  and  $\mathbb{Z}_{p^r}$  for odd  $p$  in  $A_{p;r}$ .
2. Similarly, the representation  $\llbracket q_1, q_2 \rrbracket$  of  $\mathbb{Z}_{p^{r_1}} \times \mathbb{Z}_{p^{r_2}}$  is the pullback of representation with charges  $(q_1, q_2)$  of  $U(1)^{(1)} \times U(1)^{(2)}$  under the standard inclusion. We introduced the extra superscripts (1) and (2) to distinguish the  $U(1)$  subgroups in the discussion below. The group of mixed fermionic anomalies between two copies of  $U(1)$  is classified by  $\mathbb{Z}$  (the generator corresponds to the anomaly

polynomial  $c_1^{(1)}c_1^{(2)}$ ) and the corresponding value is well known to be  $q_1q_2$ . The image under the pullback of the mixed anomaly group is the subgroup  $\mathbb{Z}_{p^{\min(r_1, r_2)}}$  in  $B_{p; r_2, r_2}$ .

3. From the supercohomology description of the anomaly group for a finite  $G$  we have

$$\mathrm{Hom}(\Omega_3^{\mathrm{Spin}}(BG), U(1)) \stackrel{\mathrm{set}}{\cong} H^3(BG, U(1)) \times SH^2(BG, \mathbb{Z}_2) \times H^1(BG, \mathbb{Z}_2), \quad (5.27)$$

where there is a canonical inclusion homomorphism

$$H^3(BG, U(1)) \longrightarrow \mathrm{Hom}(\Omega_3^{\mathrm{Spin}}(BG), U(1)) \quad (5.28)$$

and a projection homomorphism

$$\pi : \mathrm{Hom}(\Omega_3^{\mathrm{Spin}}(BG), U(1)) \longrightarrow H^1(BG, \mathbb{Z}_2) \quad (5.29)$$

such that  $\pi \mathbf{ch}_2(\rho) = w_1(\rho)$  where  $w_1$  is the first Stiefel-Whitney class of the representation  $\rho$  [112]. The latter is realized by the 1-cocycle  $\det \rho$ , where  $\rho$  is considered as a map  $G \rightarrow O(n)$  for some  $n$  and  $\det : O(n) \rightarrow \mathbb{Z}_2$ . Moreover, the component of  $\mathbf{ch}_2(\rho)$  in  $SH^2(BG, \mathbb{Z}_2)$  is given by the second Stiefel-Whitney class  $w_2(\rho)$ , possibly up to  $w_1(\rho)^2$ .

The result is presented in the Tables 5.1 and 5.2. The freedom related to automorphisms can be fixed by assigning particular bordism invariants to the generators of the anomaly groups. This will be done in the Section 5.3.2. However, let us note that for the questions like anomaly cancellations for the theory of free fermions it is sufficient to consider the map to the anomaly group up to automorphisms (since zero element is invariant).

| $(p; r)$     | $RO(\mathbb{Z}_{p^r}) \longrightarrow A_{p;r}$   | $A_{p;r}$                                  |
|--------------|--|--|
| $(p > 2; r)$ | $\llbracket q \rrbracket \longmapsto q^2$  | $\mathbb{Z}_{p^r}$                         |
| $(2; r > 1)$ | $\begin{array}{l} \llbracket q \rrbracket \longmapsto (0, q^2) \\ [2^{r-1}] \longmapsto (1, 2^{2r-3}) \end{array}$ | $\mathbb{Z}_2 \times \mathbb{Z}_{2^{r+1}}$ |
| $(2; 1)$     | $[1] \longmapsto 1$  | $\mathbb{Z}_8$                             |

Table 5.1: The values of the map  $RO(\mathbb{Z}_{p^r}) \rightarrow A_{p;r}$  in terms of the generators. The trivial representation  $[0]$  is always mapped to zero, so it is not written.

As an illustration of the use of the properties listed above to fixing the maps, we explain in detail a couple of cases.

As a first example, consider the second line in Table 5.1 for  $r = 2$ . The symmetry group is  $\mathbb{Z}_4$  and the anomaly group is  $\mathbb{Z}_2 \times \mathbb{Z}_8$ . The image of the  $U(1)$  anomaly group is the  $\mathbb{Z}_8$  subgroup. The projection on  $\mathbb{Z}_2$  factor can be identified with the projection on

| $(p; r_1, r_2), r_1 \leq r_2$ | $RO(\mathbb{Z}_{p^{r_1}} \times \mathbb{Z}_{p^{r_2}}) \longrightarrow B_{p; r_1, r_2}$                               | $B_{p; r_1, r_2}$                          |
|-------------------------------|--|--|
| $(p > 2; r_1, r_2)$           | $\llbracket q_1, q_2 \rrbracket \longmapsto q_1 q_2$   | $\mathbb{Z}_{p^{r_1}}$                     |
| $(2; 1, 1)$                   | $\llbracket 1, 1 \rrbracket \longmapsto 1$   | $\mathbb{Z}_4$                             |
| $(2; r_1, r_2 > 1)$           | $\llbracket q_1, q_2 \rrbracket \longmapsto (0, q_1 q_2)$<br>$[2^{r_1-1}, 2^{r_2-1}] \longmapsto (1, 2^{r_1+r_2-3})$ | $\mathbb{Z}_2 \times \mathbb{Z}_{2^{r_1}}$ |

Table 5.2: The values of the map  $RO(\mathbb{Z}_{p^{r_1}} \times \mathbb{Z}_{p^{r_2}}) \rightarrow B_{p; r_1, r_2}$  in terms of the generators. The representations that contain a zero charge are always mapped to zero in  $B_{p; r_1, r_2}$ , so they are not written.

| $(p; r_1, r_2, r_3), r_1 \leq r_2 \leq r_3$ | $RO(\mathbb{Z}_{p^{r_1}} \times \mathbb{Z}_{p^{r_2}} \times \mathbb{Z}_{p^{r_3}}) \longrightarrow C_{p; r_1, r_2, r_3}$ | $C_{p; r_1, r_2, r_3}$ |
|---|---|------------------------|
| $(p > 2, r_1, r_2, r_3)$                    | $\llbracket q_1, q_2, q_3 \rrbracket \longmapsto 0$   | $\mathbb{Z}_{p^{r_1}}$ |
| $(p = 2, r_1, r_2, r_3)$                    | $\llbracket q_1, q_2, q_3 \rrbracket \longmapsto 0$<br>$[2^{r_1-1}, 2^{r_2-1}, 2^{r_3-1}] \longmapsto 2^{r_1-1}$        | $\mathbb{Z}_{2^{r_1}}$ |

Table 5.3: The values of the map  $RO(\mathbb{Z}_{p^{r_1}} \times \mathbb{Z}_{p^{r_2}} \times \mathbb{Z}_{p^{r_3}}) \rightarrow C_{p; r_1, r_2, r_3}$  in terms of the generators. The representations that contain a zero charge are always mapped to zero in  $C_{p; r_1, r_2, r_3}$ , so they are not written.

$H^1(B\mathbb{Z}_4, \mathbb{Z}_2) \cong \mathbb{Z}_2$  in the supercohomology description. Consider the representation  $\llbracket q \rrbracket$ . It can be lifted to a representation of  $U(1)$  and has  $w_1(\llbracket q \rrbracket) = 0 \in H^1(B\mathbb{Z}_4, \mathbb{Z}_2)$ . Therefore its image is  $(0, q^2)$ . Next consider the real 1-dimensional representation  $[2]$ . Its image is necessarily of the form  $(1, a)$  where  $2a = 4 \pmod{8}$ . This is because  $w_1([2]) = 1 \in H^1(B\mathbb{Z}_4, \mathbb{Z}_2)$  and  $[2] \oplus [2] = \llbracket 2 \rrbracket$ . There are two choices satisfying this condition:  $a = 2 \pmod{8}$  or  $a = -2 \pmod{8}$ . However they are related by an automorphism  $\mathbb{Z}_2 \times \mathbb{Z}_8 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_8$  where  $(c, d) \mapsto (c, d + 4c)$ .

As a second example, consider the third line in Table 5.2 for  $r_2 = 2$  and  $r_1 = 1$ . The symmetry group is  $G = \mathbb{Z}_2 \times \mathbb{Z}_4$  and the anomaly group is  $A_{2;1} \times A_{2;2} \times B_{2;1,2}$ , where  $A_{2;1} = \mathbb{Z}_8$ ,  $A_{2;2} = \mathbb{Z}_4 \times \mathbb{Z}_2$  are the anomaly group of the individual factors  $\mathbb{Z}_2$  and  $\mathbb{Z}_4$  in  $G$ , and  $B_{2;1,2} = \mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(2)}$  are the group of ‘‘mixed’’ anomalies. We introduced the extra subscripts (1) and (2) to distinguish the subgroups in the discussion below. In particular we take  $\mathbb{Z}_2^{(2)}$  to be the image of the pullback of the mixed anomaly group for  $U(1) \times U(1)$  under the inclusion  $\mathbb{Z}_2 \times \mathbb{Z}_4 \subset U(1) \times U(1)$ . This in particular implies that  $\mathbf{ch}_2(\llbracket q_1, q_2 \rrbracket) = (0, q_1 q_2) \in B_{2;1,2}$ , which belongs to the subgroup of bosonic anomalies. Next, note that  $H^1(BG, \mathbb{Z}_2) \cong \mathbb{Z}_2^2$  is generated by  $w_1([1, 0])$  and  $w_1([0, 2])$ , while  $SH^2(BG, \mathbb{Z}_2) \cong H^2(BG, \mathbb{Z}_2) \cong \mathbb{Z}_2^3$  is generated by  $w_2(\llbracket 1, 0 \rrbracket) = w_1([1, 0])^2$ ,  $w_2(\llbracket 0, 1 \rrbracket)$  and  $w_1([1, 0])w_1([0, 2])$ . Note that  $\mathbf{ch}_2[1, 0]$ ,  $\mathbf{ch}_2[0, 2]$  and  $\mathbf{ch}_2\llbracket 0, 1 \rrbracket$  generate the subgroup  $A_{2;1} \times A_{2;2}$  under addition, which corresponds to taking direct sum of representations. It is left to determine the value

of  $\mathfrak{ch}_2[1, 2]$ . We have  $w_1([1, 2]) = w_1([1, 0]) + w_1([0, 2])$  and  $w_2([1, 2]) = 0$ . We can see that these values of  $(w_1, w_2)$  are non-trivial and never realized by any direct sum of representations  $[1, 0]$ ,  $[0, 2]$  and  $[[0, 1]]$ . This implies that  $\mathfrak{ch}_2[1, 2]$  is a non-trivial element not belonging to the  $A_{2;1} \times A_{2;2}$  subgroup. We also know that it is an order 2 element, since  $2\mathfrak{ch}_2[1, 2] = \mathfrak{ch}_2[[1, 2]] = (0, 2) \equiv (0, 0) \in B_{2;1}$ . It also does not lie in  $\mathbb{Z}_2^{(2)}$  subgroup since  $[1, 2]$  cannot be lifted to a complex representation. Therefore, up an automorphism of  $A_{2;1} \times A_{2;2} \times B_{2;2,1}$  we can assume that  $\mathfrak{ch}_2[1, 2] = (1, 1) \in B_{2;1}$ .

The arguments for the other cases in Tables 5.1 and 5.2 are similar. The last map in (5.22) will be determined in Section 5.3.2 using different arguments. The result is presented in Table 5.3.

## 5.2.2 Non-abelian examples

In this section we will show how one can use the axiomatic properties of the  $\mathfrak{ch}_2$  map to determine its values for some small non-abelian groups, namely when  $G = S_3, S_4, D_8$ . In the following we are going to use the notation  $\mathfrak{ch}_2^G$  to specify the evaluation of  $\mathfrak{ch}_2$  for a particular group  $G$ .

### Case 1: $G = S_3$

We start our analysis from  $S_3$ , the group of permutation of 3 elements, or, equivalently the dihedral group of order 6. It can also be realized as a semiproduct  $G = \mathbb{Z}_2 \ltimes \mathbb{Z}_3$ , where the generator of  $\mathbb{Z}_2$  acts on the elements of  $\mathbb{Z}_3$  by inversion. This group has three real irreducible representations, which we will denote as  $\rho_{0,a,b}$  (see Appendix C for the character table). First,  $\rho_0$  is the 1-dimensional *trivial* representation. Second,  $\rho_a$  is the 1-dimensional sign representation, such that  $\rho_a(g, h) = (-1)^g$ , where  $(g, h) \in \mathbb{Z}_2 \times \mathbb{Z}_3$ . Third one,  $\rho_b$  is the *standard* 2-dimensional representation realized by considering  $S_3$  as the subgroup of  $O(2)$  preserving an equilateral triangle.

Consider the following three homomorphisms to and from  $S_3$ :

$$\begin{array}{ccccc}
 & & \text{id} & & \\
 & & \text{-----} & & \\
 \mathbb{Z}_2 & \xrightarrow{i} & S_3 = \mathbb{Z}_2 \ltimes \mathbb{Z}_3 & \xrightarrow{\pi} & \mathbb{Z}_2 \\
 & \nearrow j & & & \\
 \mathbb{Z}_3 & & & & 
 \end{array} \tag{5.30}$$

The maps  $i$  and  $j$  are the standard inclusions in the semi-direct product and the map  $\pi$  is the projection on the quotient over  $\mathbb{Z}_3$  normal subgroup. Note that we have  $\pi \circ i = \text{id}$ . The maps have the following pullbacks between the corresponding

anomaly groups:

$$\begin{array}{ccc}
 & \text{id} & \\
 & \text{---} & \\
 \mathbb{Z}_8 & \xleftarrow{\hat{i}^*} \text{Hom}(\Omega_3^{\text{Spin}}(BS_3), U(1)) \xleftarrow{\hat{\pi}^*} & \mathbb{Z}_8 \\
 & \searrow \hat{j}^* & \\
 \mathbb{Z}_3 & & 
 \end{array} \tag{5.31}$$

The cohomology groups relevant for supercohomology description of  $\text{Hom}(\Omega_3^{\text{Spin}}(BS_3), U(1))$  are  $H^3(BS_3, U(1)) \cong H^4(BS_3, \mathbb{Z}) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ ,  $H^2(BS_3, \mathbb{Z}_2) \cong \mathbb{Z}_2$ ,  $H^1(BS_3, \mathbb{Z}_2) \cong \mathbb{Z}_2$ . Therefore, without determining specifics of the supercohomology structure, we know that the anomaly group must be of order  $2^3 \cdot 3$  or  $2^2 \cdot 3$ . The commutativity of the diagram (5.31) immediately implies that  $\pi^*$  is injective, therefore we necessarily have

$$\text{Hom}(\Omega_3^{\text{Spin}}(BS_3), U(1)) \cong \mathbb{Z}_8 \times \mathbb{Z}_3. \tag{5.32}$$

By means of the commutative diagram

$$\begin{array}{ccccc}
 RO(\mathbb{Z}_3) & \xleftarrow{i_b^*} & RO(S_3) & \xleftarrow{\pi^*} & RO(\mathbb{Z}_2) \\
 \downarrow \text{ch}_2^{\mathbb{Z}_3} & & \downarrow \text{ch}_2^{S_3} & & \downarrow \text{ch}_2^{\mathbb{Z}_2} \\
 \mathbb{Z}_3 & \xleftarrow{\hat{i}_b^*} & \mathbb{Z}_8 \times \mathbb{Z}_3 & \xleftarrow{\hat{\pi}^*} & \mathbb{Z}_8
 \end{array}, \tag{5.33}$$

one can also check that the short sequence given by  $\hat{\pi}^*$  and  $\hat{j}^*$  is in fact exact and split. Therefore we can simply define the elements  $(p, q) \in \mathbb{Z}_8 \times \mathbb{Z}_3 \cong \text{Hom}(\Omega_2^{\text{Spin}}(BS_3), U(1))$  by the maps

$$\hat{\pi}^*(p) = (p, 0), \quad \hat{i}_a^*(p, q) = p, \quad \hat{i}_b^*(p, q) = q. \tag{5.34}$$

Finally, the corresponding pullbacks of the representations are the following:

$$\begin{array}{ccc}
 RO(\mathbb{Z}_2) & \xleftarrow{i^*} & RO(S_3), \\
 [0] & \longleftarrow & \rho_0, \\
 [1] & \longleftarrow & \rho_a, \\
 [0] \oplus [1] & \longleftarrow & \rho_b,
 \end{array} \tag{5.35}$$

$$\begin{array}{ccc}
 RO(\mathbb{Z}_3) & \xleftarrow{j^*} & RO(S_3), \\
 [0] & \longleftarrow & \rho_0, \\
 [0] & \longleftarrow & \rho_a, \\
 [[1]] & \longleftarrow & \rho_b,
 \end{array} \tag{5.36}$$

$$\begin{array}{ccc}
 RO(S_3) & \xleftarrow{\pi^*} & RO(\mathbb{Z}_2), \\
 \rho_0 & \longleftarrow & [0], \\
 \rho_a & \longleftarrow & [1].
 \end{array} \tag{5.37}$$

From this and the Tables 5.1 and 5.2 it is easy to conclude that:

$$\begin{aligned}
 RO(S_3) &\xrightarrow{\text{ch}_2^{S_3}} \text{Hom}(\Omega_3^{\text{Spin}}(BS_3), U(1)) \cong \mathbb{Z}_8 \times \mathbb{Z}_3, \\
 \rho_0 &\mapsto (0, 0), \\
 \rho_a &\mapsto (1, 0), \\
 \rho_b &\mapsto (1, 1).
 \end{aligned} \tag{5.38}$$

**Case 2:**  $G = S_4$

As before, we start by looking at the relevant homomorphisms from and into  $S_4$ :

$$\begin{array}{ccccc}
 & & \phi & & \\
 & & \text{---} & & \\
 \mathbb{Z}_2 & \xrightarrow{i_{1,2,d}} & \mathbb{Z}_2 \times \mathbb{Z}_2 & & \mathbb{Z}_2 \\
 \downarrow i_4 & & \downarrow j_1 & \nearrow \pi_4 & \uparrow \tilde{\pi} \\
 \mathbb{Z}_4 & \xrightarrow{j_2} & S_4 & & \\
 \nearrow j_4 & & \uparrow j_3 & \searrow \pi_1 & \\
 A_4 & & S_3 & \xrightarrow{\text{id}} & S_3
 \end{array} . \tag{5.39}$$

The maps here are defined by

$$i_1(1) = (1, 0), \quad i_2(1) = (0, 1), \quad i_d(1) = (1, 1), \tag{5.40}$$

$$i_4(1) = 2, \quad j_1(n, m) = (ab)^n (cd)^m, \quad j_2(n) = (acbd)^n, \tag{5.41}$$

while  $j_{3,4}$  are the natural inclusions. The projections are defined by the property  $\ker \pi_i = \text{Im } j_i$ . The diagram is commutative if we drop  $i_{1,2}$ , so that we are left with  $i_d$ . In this case  $\phi = \tau_\emptyset$ . Instead, if we consider the maps  $i_{1,2}$  and drop  $i_d, i_4$  then the commutativity of the diagram is restored with  $\phi = \text{id}$ .

From the super-cohomology description of the cobordism groups we are able to constrain the order of  $\text{Hom}(\Omega_3^{\text{Spin}}(BS_4), U(1))$ . In particular we have that

$$H^3(BS_4; U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3, \quad H^2(BS_4; \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2, \quad H^1(BS_4; \mathbb{Z}_2) = \mathbb{Z}_2, \tag{5.42}$$

so  $2^4 \cdot 3 \leq |\text{Hom}(\Omega_3^{\text{Spin}}(BS_4), U(1))| \leq 2^6 \cdot 3$ . The order of the group can be actually computed by means of (5.39). The first thing to notice is that by functoriality  $\pi_1 \circ j_3 = \text{id}_{S_3}$  implies  $\hat{\pi}_1^* : \mathbb{Z}_8 \times \mathbb{Z}_3 \rightarrow \text{Hom}(\Omega_3^{\text{Spin}}(BS_4), U(1))$  is injective and thus  $\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \subseteq \text{Hom}(\Omega_3^{\text{Spin}}(BS_4), U(1))$ .

Next we focus on the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  and  $\mathbb{Z}_4$  subgroups of  $S_4$ . By looking at the character tables (see again Appendix C) one finds that  $j_1^* : RO(S_4) \rightarrow RO(\mathbb{Z}_2 \times \mathbb{Z}_2)$  is defined by

$$\begin{aligned}
 j_1^*(\rho_a) &= [1, 1], & j_1^*(\rho_b) &= [0, 0] + [1, 1], \\
 j_1^*(\rho_c) &= [0, 1] + [1, 0] + [0, 0], & j_1^*(\rho_d) &= [0, 1] + [1, 0] + [1, 1].
 \end{aligned} \tag{5.43}$$

It follows also that the map  $j_2^* : RO(S_4) \rightarrow RO(\mathbb{Z}_4)$  is surjective, as it maps

$$\begin{aligned} j_2^*(\rho_a) &= [2], & j_2^*(\rho_b) &= [0] + [2], \\ j_2^*(\rho_c) &= [2] + [1], & j_2^*(\rho_d) &= [0] + [1]. \end{aligned} \quad (5.44)$$

The maps  $\mathbf{ch}_2^{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4}$  are defined by

$$\begin{aligned} \mathbf{ch}_2^{\mathbb{Z}_2 \times \mathbb{Z}_2} : RO(\mathbb{Z}_2 \times \mathbb{Z}_2) &\rightarrow \mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_4, & \mathbf{ch}_2^{\mathbb{Z}_4} : RO(\mathbb{Z}_4) &\rightarrow \mathbb{Z}_2 \times \mathbb{Z}_8. \\ [1, 0] &\mapsto (1, 0, 0) & [1] &\mapsto (0, 1) \\ [0, 1] &\mapsto (0, 1, 0) & [2] &\mapsto (1, 2) \\ [1, 1] &\mapsto (1, 1, 1) & & \end{aligned} \quad (5.45)$$

Recall that in both cases we have at our disposal the commutative diagram

$$\begin{array}{ccc} RO(G_i) & \xleftarrow{j_i^*} & RO(S_4) \\ \downarrow \mathbf{ch}_2^{G_i} & & \downarrow \mathbf{ch}_2^{S_4} \\ \text{Hom}(\Omega_3^{\text{Spin}}(BG_i), U(1)) & \xleftarrow{\hat{j}_i^*} & \text{Hom}(\Omega_3^{\text{Spin}}(BS_4), U(1)) \end{array}, \quad (5.46)$$

where  $G_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$  and  $G_2 = \mathbb{Z}_4$ . By denoting  $f_i := \hat{j}_i^* \circ \mathbf{ch}_2^{S_4} = \mathbf{ch}_2^{G_i} \circ j_i^*$ , we have

$$f_1 = (n_a + n_b + n_c + 2n_d \pmod 8, n_a + n_b + n_c + 2n_d \pmod 8, n_a + n_b + n_d \pmod 4), \quad (5.47)$$

$$f_2 = (n_a + n_b + n_c \pmod 2, 2(n_a + n_b + n_c) + n_c + n_d \pmod 8). \quad (5.48)$$

Here  $n_\rho$  represent the multiplicity of the generators  $[[\rho]]$  of  $RO(S_4)$ . It follows that  $\hat{j}_2^*$  is surjective and that  $\text{Im}(\hat{j}_1^*) \supseteq \text{Im}(f_1) = \mathbb{Z}_8 \times \mathbb{Z}_4$ .

Next we focus on the kernels of  $\hat{j}_{1,2}^*$ . We know that they satisfy

$$|\ker \hat{j}_i^*| \geq |\mathbf{ch}_2^{S_4}(\ker f_i)| \geq |f_k(\ker f_i)|, \quad i \neq k. \quad (5.49)$$

and from the super-cohomology description we already know that  $|\ker \hat{j}_k^*| \leq 2^k$ . We have

$$\ker f_1 : \begin{cases} n_a + n_b + n_d = 4N_1, \\ n_c + n_d = 8M_1 + 4N_1, \end{cases} \quad \ker f_2 : \begin{cases} n_a + n_b + n_c = 2N_2, \\ n_c + n_d = 8M_2 + 4N_2. \end{cases} \quad (5.50)$$

Therefore

$$f_1(\ker f_2) = (2(N_2 + n_d) \pmod 8, 2(N_2 + n_d) \pmod 4) \cong \mathbb{Z}_4,$$

$$f_2(\ker f_1) = (0 \pmod 2, 4(N_1 + n_c) \pmod 8) \cong \mathbb{Z}_2.$$

This means there are two exact sequences

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \mathcal{E}(\text{Hom}(\Omega_3^{\text{Spin}}(BS_4), U(1))) \rightarrow \mathbb{Z}_8 \times \mathbb{Z}_4 \rightarrow 1,$$

$$1 \rightarrow \mathbb{Z}_4 \rightarrow \mathcal{E}(\text{Hom}(\Omega_3^{\text{Spin}}(BS_4), U(1))) \rightarrow \mathbb{Z}_8 \times \mathbb{Z}_2 \rightarrow 1,$$

where  $\mathcal{E}(\cdot)$  denotes the 2-torsion subgroup. This implies that the map  $\mathfrak{ch}_2^{S_4}$  is surjective and that the cobordism group is one between  $\mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_3$ ,  $\mathbb{Z}_{16} \times \mathbb{Z}_4 \times \mathbb{Z}_3$  and  $\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ . For the computation of the spin selection rule it is enough to be able to describe it from a set point of view, under which we can identify it by

$$\text{Hom}(\Omega_3^{\text{Spin}}(BS_4), U(1)) \stackrel{\text{set}}{\cong} \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3. \quad (5.51)$$

To describe its elements we will use the 4-tuple  $(p, q, r, t) \in \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ . The first two terms can be identified with the generators of  $\text{Im} f_1$ , while  $r$  and  $t$  describe respectively the even and odd torsion part of its kernel. In particular, we have

$$f_1^{-1}(p, q) = \begin{cases} n_a + n_b + n_d = 4N_1 + q, \\ n_c + n_d = 8M_1 - 4N_1 + p - q, \end{cases} \quad (5.52)$$

$$f_2(f_1^{-1}(p, q)) = (p \pmod 2, 3q - p + 4(n_c + N_1) \pmod 8). \quad (5.53)$$

Therefore from the set point of view<sup>5</sup> we see that  $\hat{j}_2^*(p, q, r, t) = (p \pmod 2, 3q - p + 4r \pmod 8)$ . Following the same line of thought and looking at the commutative diagram (5.39) we get the following set of maps between cobordism groups<sup>6</sup>

$$\begin{array}{ccccc} & & \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3^{(p,q,r,t)} & & \\ & \swarrow \hat{j}_3^* & \downarrow \hat{j}_1^* & \searrow \hat{j}_2^* & \\ \mathbb{Z}_8 \times \mathbb{Z}_3^{(p,t)} & & \mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_4^{(p,p,q)} & & \mathbb{Z}_2 \times \mathbb{Z}_8^{(p,3q-p+4r)} \cdot \\ & \downarrow \hat{i}_b^* & \downarrow \hat{i}_{1,2}^* & \searrow \hat{i}_d^* & \downarrow \hat{i}_4^* \\ & \mathbb{Z}_3^{(t)} & \mathbb{Z}_8^{(p)} & & \mathbb{Z}_8^{(2p-2q)} \end{array} \quad (5.55)$$

Here the elements in the parenthesis specify elements in the corresponding groups.

<sup>5</sup>Here the maps  $\hat{j}_i^*$  are to be intended only as maps between sets, since we are describing  $\text{Hom}(\Omega_3^{\text{Spin}}(BS_4), U(1))$  from that point of view.

<sup>6</sup>Via the commutative diagram associated to  $i_d : \mathbb{Z}_2^{(d)} \hookrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$  and  $i_4 : \mathbb{Z}_2^{(d)} \hookrightarrow \mathbb{Z}_4$  one finds that

$$\begin{array}{ccc} \hat{i}_d^* : \mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_4 & \rightarrow & \mathbb{Z}_8 \\ (a, b, c) & \mapsto & a + b - 2c \pmod 8 \end{array}, \quad \begin{array}{ccc} \hat{i}_4^* : \mathbb{Z}_2 \times \mathbb{Z}_8 & \rightarrow & \mathbb{Z}_8 \\ (a, b) & \mapsto & 2b - 4a \pmod 8 \end{array}. \quad (5.54)$$

Finally, let us note that albeit we do not find the full group structure, it is possible to compute via Adams spectral sequence [113] that

$$\mathrm{Hom}(\Omega_3^{\mathrm{Spin}}(BS_4), U(1)) \cong \mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_3. \quad (5.56)$$

However, for our purpose it is not necessary to figure out the structure of the group, as the map  $\mathfrak{ch}_2^{S_4}$  can still be defined regardless. We will also see that the information we gained by our analysis will be still enough to find in Section 6 dynamical constraints that anomalies impose on theories.

**Case 5:**  $G^f = \mathbb{Z}_2^f \times D_8$

Like before, we start again by looking at the super-cohomology description in order to gain knowledge on the order of  $\mathrm{Hom}(\Omega_3^{\mathrm{Spin}}(BD_8), U(1))$ . We have

$$H^3(BD_8; U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, \quad H^2(BD_8; \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \quad H^1(BD_8; \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

This means that  $2^6 \leq |\mathrm{Hom}(\Omega_3^{\mathrm{Spin}}(BD_8), (1))| \leq 2^9$ . We start by considering a set of different homomorphisms all described by the same sequence

$$\mathbb{Z}_2 \xrightarrow{i_x} \mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{j_y} D_8 \xrightarrow{\pi_z} \mathbb{Z}_2. \quad (5.57)$$

Here  $i_x$  and  $j_y$  are defined by

$$i_x(1) = x, \quad j_y(1, 0) = (y, 1), \quad j_y(0, 1) = (2, 0), \quad (5.58)$$

where we are using the notation  $(n, m) \in D_8 \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_2$ . Instead,  $\pi_z : D_8 \rightarrow \mathbb{Z}_2$  defines the homomorphism with  $\ker \pi_z = \langle (z, 1), (2, 0) \rangle$ . The inequivalent set of choices of sequences (5.57) are given by the possible values  $y, z = 0, 1$ .

By definition  $\pi_y \circ j_y \circ i_{(1,0)} = \tau_\emptyset$ , while  $\pi_z \circ j_y \circ i_{(1,0)} = \mathrm{id}$  for  $z \neq y$ . Using the latter condition we get that the maps  $\hat{i}_{(1,0)}^* \circ \hat{j}_x^*$  are surjective, while  $\hat{\pi}_z^*$  are injective. Moreover  $\mathbb{Z}_8^{(x)} \equiv \mathrm{Im}(\hat{\pi}_x^*) \subseteq \ker(\hat{i}_{(1,0)}^* \circ \hat{j}_x^*)$  and  $\mathbb{Z}_8^{(y)} \subseteq \mathrm{Im}(\hat{i}_{(1,0)}^* \circ \hat{j}_y^*)$ , so the two  $\mathbb{Z}_8^{(0)}$  and  $\mathbb{Z}_8^{(1)}$  components are disjoint.

More precisely consider now the commutative diagram related to (5.57). We have

$$\begin{array}{ccccc} & & \xleftarrow{\tau_\emptyset} & & \\ & & \text{---} & & \\ RO(\mathbb{Z}_2 \times \mathbb{Z}_2) & \xleftarrow{j_x^*} & RO(D_8) & \xleftarrow{\pi_x^*} & RO(\mathbb{Z}_2) \\ & \downarrow \mathfrak{ch}_2^{\mathbb{Z}_2 \times \mathbb{Z}_2} & \downarrow \mathfrak{ch}_2^{D_8} & & \downarrow \mathfrak{ch}_2^{\mathbb{Z}_2} \\ \mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_4 & \xleftarrow{\hat{j}_x^*} & \mathrm{Hom}(\Omega_3^{\mathrm{Spin}}(BS_4), U(1)) & \xleftarrow{\hat{\pi}_x^*} & \mathbb{Z}_8 \\ & \text{---} & \text{---} & \text{---} & \\ & & \xleftarrow{\tau_\emptyset} & & \end{array} \quad (5.59)$$

Since  $\mathfrak{ch}_2^{\mathbb{Z}_2}$  is surjective, then  $\hat{j}_x^* \circ \hat{\pi}_x^* = \mathrm{tr}$  and  $\mathrm{Im}(\hat{\pi}_x^*) \subseteq \ker \hat{j}_x^*$ . Moreover  $\mathrm{Im}(\hat{j}_x^*) \supseteq \mathrm{Im}(\hat{j}_x^* \circ \mathfrak{ch}_2^{D_8}) = \mathrm{Im}(\mathfrak{ch}_2^{\mathbb{Z}_2 \times \mathbb{Z}_2} \circ j_x^*)$ .

To figure out how the maps  $j_x^*$  are defined we need the character table of  $D_8$ , see Table C.3. We have

$$\begin{array}{ll}
 j_0^* : RO(D_8) & \rightarrow RO(\mathbb{Z}_2^{(0)} \times \mathbb{Z}_2), & j_1^* : RO(D_8) & \rightarrow RO(\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2). \\
 \rho_a & \mapsto [0, 0] & \rho_a & \mapsto [1, 0] \\
 \rho_b & \mapsto [1, 0] & \rho_b & \mapsto [1, 0] \\
 \rho_c & \mapsto [1, 0] & \rho_c & \mapsto [0, 0] \\
 \rho_d & \mapsto [0, 1] + [1, 1] & \rho_d & \mapsto [0, 1] + [1, 1]
 \end{array} \tag{5.60}$$

This means that defining  $f_i := \mathfrak{ch}_2^{\mathbb{Z}_2 \times \mathbb{Z}_2} \circ j_i^*$ , then

$$f_0 = (n_b + n_c + n_d \pmod{8}, 2n_d \pmod{8}, n_d \pmod{4}), \quad \text{Im}(f_0) \cong \mathbb{Z}_8 \times \mathbb{Z}_4, \tag{5.61}$$

$$f_1 = (n_a + n_b + n_d \pmod{8}, 2n_d \pmod{8}, n_d \pmod{4}), \quad \text{Im}(f_1) \cong \mathbb{Z}_8 \times \mathbb{Z}_4. \tag{5.62}$$

In particular

$$f_1(\ker f_0) = (n_a - n_c \pmod{8}, 0 \pmod{8}, 0 \pmod{4}) \cong \mathbb{Z}_8. \tag{5.63}$$

We then consider the inclusion map  $j_2 : \mathbb{Z}_4 \hookrightarrow D_8$  finding

$$\begin{array}{ll}
 j_2^* : RO(D_8) & \rightarrow RO(\mathbb{Z}_4) \\
 \rho_a & \mapsto [2] \\
 \rho_b & \mapsto [0] \\
 \rho_c & \mapsto [2] \\
 \rho_d & \mapsto [[1]]
 \end{array} . \tag{5.64}$$

Then the image of  $f_2 := \mathfrak{ch}_2^{\mathbb{Z}_4} \circ j_2^*$  is such that

$$f_2 = (n_a + n_c \pmod{2}, 2(n_a + n_c) + n_d \pmod{8}) \tag{5.65}$$

$$f_2(\ker f_0 \cap \ker f_1) = (0 \pmod{2}, 4(N + n_b) \pmod{8}) \cong \mathbb{Z}_2, \tag{5.66}$$

where  $n_d = 4N$ . Therefore

$$|\ker \hat{\iota}_0^*| \geq |\mathfrak{ch}_2^{D_8}(\ker f_0)| = |f_1(\ker f_0)| \cdot |\ker f_1|_{\ker f_0} \geq |f_1(\ker f_0)| \cdot |f_2(\ker f_0 \cap \ker f_1)| = 2^4. \tag{5.67}$$

But we already know that it cannot be higher than  $2^4$ . This means that the inequalities are saturated and from the set-point of view we have<sup>7</sup>

$$\text{Hom}(\Omega_3^{\text{Spin}}(BD_8), U(1)) \stackrel{\text{set}}{\cong} \mathbb{Z}_8^{(0)} \times \mathbb{Z}_8^{(1)} \times \mathbb{Z}_4 \times \mathbb{Z}_2. \tag{5.68}$$

---

<sup>7</sup>In this case the group can be shown to be  $\text{Hom}(\Omega_3^{\text{Spin}}(BD_8), U(1)) \cong \mathbb{Z}_8^3$  via Adams spectral sequence. The author thanks Arun Debray for private communication on this regard.

We denote the elements as  $(p, q, r, t)$ . In this case the diagram of the cobordism subgroups is<sup>8</sup>

$$\begin{array}{ccccc}
 & & \mathbb{Z}_8^{(0)} \times \mathbb{Z}_8^{(1)} \times \mathbb{Z}_4 \times \mathbb{Z}_2^{(p,q,r,t)} & & \\
 & \hat{j}_0^* \swarrow & \downarrow \hat{j}_2^* & \searrow \hat{j}_1^* & \\
 \mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_4^{(p+r, 2r, r)} & & \mathbb{Z}_2 \times \mathbb{Z}_8^{(p+q, 2(p+q)+r+4t)} & & \mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_4^{(q+r, 2r, r)} \\
 \downarrow i_{(1,0)}^* & \hat{i}_{(0,1)}^* \swarrow & \downarrow \hat{i}_2^* & \searrow \hat{i}_{(0,1)}^* & \downarrow i_{(1,0)}^* \\
 \mathbb{Z}_8^{(0)(p+r)} & & \mathbb{Z}_8^{(2r)} & & \mathbb{Z}_8^{(1)(q+r)}
 \end{array} \quad (5.69)$$

We took as definition of  $(p, q, r, t) = (1, 0, 0, 0)$  and  $(p, q, r, t) = (0, 0, 1, 0)$  the generators of  $\text{Im}(j_0^*) \cong \mathbb{Z}_8 \times \mathbb{Z}_4$ , i.e.  $j^*(p, q, r, t) = (p+r \bmod 8, 2r \bmod 8, r \bmod 4)$ . The subgroup  $\mathbb{Z}_8 \subset \text{Im}(j_1^*)$  is instead generated by  $q$ , while  $t$  defines the remaining subgroup (5.66).

## 5.3 The map from the anomaly group to the group of anomalous phases

Considering again an arbitrary finite group  $G$ , we now expand on the evaluation of the second map in the sequence (5.3).

### 5.3.1 Reducing to abelian groups

Suppose first we are interested to determine only the anomalous phases for modular transformations, but not the large gauge transformations. This means we consider only the following subgroupoid:

$$\widetilde{\text{SL}}(2, \mathbb{Z}) \ltimes \text{Spin}_{Gf}(\mathbb{T}^2) \quad \hookrightarrow \quad (\widetilde{\text{SL}}(2, \mathbb{Z}) \times G) \ltimes \text{Spin}_{Gf}(\mathbb{T}^2) \quad (5.70)$$

This is the action of the subgroup  $\widetilde{\text{SL}}(2, \mathbb{Z})$  acting on the same set  $\text{Spin}_G(\mathbb{T}^2)$ . This inclusion induces the forgetful map on the groups classifying 1-dimensional representations of the groupoids:

$$H^1((\widetilde{\text{SL}}(2, \mathbb{Z}) \times G) \ltimes \text{Spin}_{Gf}(\mathbb{T}^2), U(1)) \longrightarrow H^1(\widetilde{\text{SL}}(2, \mathbb{Z}) \ltimes \text{Spin}_{Gf}(\mathbb{T}^2), U(1)). \quad (5.71)$$

As was already mentioned before, to describe a representation of the groupoid one can consider its connected components independently. A connected component of  $\widetilde{\text{SL}}(2, \mathbb{Z}) \ltimes \text{Spin}_{Gf}(\mathbb{T}^2)$  contains elements of  $\text{Spin}_{Gf}(\mathbb{T}^2) = \text{Spin}(\mathbb{T}^2) \times \text{Hom}(\mathbb{Z}^2, G)$  of the form  $(s, (g_1^a g_2^b, g_1^c g_2^d))$  where  $s \in \text{Spin}(\mathbb{T}^2)$ ,  $ab - cd = 1$ , and  $g_1, g_2$  are a pair of fixed commuting elements of  $G$ . Let  $G' \subset G$  be the abelian subgroup of  $G'$

<sup>8</sup>Here  $\hat{i}_2^*$  is the pullback associated to  $i_2 : \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_4$ .

generated by  $g_1$  and  $g_2$ . It follows that  $G'$  is an abelian group with at most two *independent* generators. The considered connected component then can be identified with a connected component of the groupoid  $\widetilde{\text{SL}}(2, \mathbb{Z}) \times \text{Spin}_{G'}(\mathbb{T}^2)$ . The restriction to this connected component of the representation corresponding to a given anomaly of the symmetry  $G$  then can be determined by considering the corresponding anomaly for the subgroup  $G'$  and determining the representation in this case. Namely, one can use the commutativity of the following diagram:

$$\begin{array}{ccc}
 \text{Hom}(\Omega_3^{\text{Spin}}(BG), U(1)) & \longrightarrow & \text{Hom}(\Omega_3^{\text{Spin}}(BG'), U(1)) \\
 \downarrow & & \downarrow \\
 H^1(\widetilde{\text{SL}}(2, \mathbb{Z}) \times \text{Spin}_{G'}(\mathbb{T}^2), U(1)) & \longrightarrow & H^1(\widetilde{\text{SL}}(2, \mathbb{Z}) \times \text{Spin}_{G'}(\mathbb{T}^2), U(1))
 \end{array} \tag{5.72}$$

combined with the fact the bottom horizontal map becomes an isomorphism when restricted on the considered connected components, so this restriction can be inverted. Therefore the problem of finding anomalous modular transformations can be reduced to the case of abelian symmetry group. Note that however one in general will need to consider different possibly non-isomorphic subgroups  $G'$  for a given  $G$ .

To determine in addition the action of  $G$  on the elements of  $\text{Spin}_G(\mathbb{T}^2)$  it is enough to evaluate the cobordism invariants on a 3-torus  $\mathbb{T}^3$  in a non-trivial background  $G$  gauge field. Since the space  $\text{Hom}(\pi_1(\mathbb{T}^3), G) \cong \text{Hom}(\mathbb{Z}^3, G)$  consists of commuting triples of elements of  $G$ , for this purpose it is enough to consider the abelian subgroups with at most three independent generators.

### 5.3.2 Abelian SPTs on closed 3-manifolds via surgery

Having reduced the (simplified version of) problem of determining the anomalous phases to the case of abelian groups with at most two independent generators, in this section we describe the method of computing the corresponding spin-bordism invariants on mapping tori for this class of groups. In fact, we will provide a method of calculating the invariants on arbitrary closed 3-manifold  $Y$  using its surgery representation. The mapping tori can be realized by surgeries of particular type, already described in Section 4.4.

First we assign particular spin-bordism invariants to each generator of the groups  $A_{p;r}$  and  $B_{p;r_1, r_2}$  in the factorization (5.17) of the anomaly group for an abelian  $G$ . The assignment is presented in Tables 5.4 and 5.5. In principle we do not need to consider  $C_{p;r_1, r_2, r_3}$  for the purpose of determining modular transformation, since, as explained in the previous section we can assume the symmetry group has at most two generators. We however still consider it for completeness and for the purpose of fixing the maps from the representation ring in Table 5.3. In this case the corresponding invariants are purely bosonic (i.e. do not depend on spin structure and can be described in terms of Dijkgraaf-Witten action for certain subgroup of  $H^3(BG, U(1))$ ) and well known. They are summarized in Table 5.6.

| $(p; r)$     | bordism invariants                               | $A_{p;r}$                                  |
|--------------|--|--|
| $(p > 2; r)$ | $1 : \ell k(a, a)$                               | $\mathbb{Z}_{p^r}$                         |
| $(2; r > 1)$ | $(0, 1) : \gamma_s(a)$<br>$(1, 0) : \omega_s(a)$ | $\mathbb{Z}_2 \times \mathbb{Z}_{2^{r+1}}$ |
| $(2; 1)$     | $1 : \beta_s(a)$                                 | $\mathbb{Z}_8$                             |

Table 5.4: The correspondence between the generators of  $A_{p;r}$  and particular bordism invariants, considered on a closed manifold  $Y$  with spin structure  $s \in \text{Spin}(Y)$ , and depending on the background gauge field  $a \in H^1(Y, \mathbb{Z}_{p^r})$ .

| $(p; r_1, r_2), r_1 \leq r_2$ | bordism invariants  | $B_{p;r_1, r_2}$                           |
|-------------------------------|---|--|
| $(p > 2; r_1, r_2)$           | $1 : \ell k(a, b \bmod p^{r_1})$  | $\mathbb{Z}_{p^{r_1}}$                     |
| $(2; 1, 1)$                   | $1 : \delta_s(a, b)$  | $\mathbb{Z}_4$                             |
| $(2; r_1, r_2 > 1)$           | $(1, 0) : \varepsilon_s(a, b)$<br>$(0, 1) : \ell k(a, b \bmod 2^{r_1})$ | $\mathbb{Z}_2 \times \mathbb{Z}_{2^{r_1}}$ |

Table 5.5: The correspondence between the generators of  $B_{p;r_1, r_2}$  and particular bordism invariants, considered on a closed manifold  $Y$  with spin structure  $s \in \text{Spin}(Y)$ , and depending on the background gauge fields  $a \in H^1(Y, \mathbb{Z}_{p^{r_1}})$  and  $b \in H^1(Y, \mathbb{Z}_{p^{r_2}})$ .

| $(p; r_1, r_2, r_3), r_1 \leq r_2 \leq r_3$ | bordism invariants   | $C_{p;r_1, r_2, r_3}$  |
|---|--|------------------------|
| $(p; r_1, r_2, r_3)$                        | $1 : \int_Y a \cup (b \bmod p^{r_1}) \cup (c \bmod p^{r_1})$ | $\mathbb{Z}_{p^{r_1}}$ |

Table 5.6: The correspondence between the generators of  $C_{p;r_1, r_2, r_3}$  and particular bordism invariants, considered on a closed manifold  $Y$  with spin structure  $s \in \text{Spin}(Y)$ , and depending on the background gauge fields  $a \in H^1(Y, \mathbb{Z}_{p^{r_1}})$ ,  $b \in H^1(Y, \mathbb{Z}_{p^{r_2}})$ , and  $c \in H^1(Y, \mathbb{Z}_{p^{r_3}})$ .

As we will see shortly, there are relations between different invariants in the Tables 5.4 and 5.5. In particular, all invariants can be expressed using only  $\ell k$ ,  $\gamma_s$  and  $\beta_s$ . We will also insure that the assignment of the invariants to the particular generators is consistent with the relations and the maps from the representations rings considered in Section 5.2.1.

The invariant  $\beta_s$ , has been already described and used earlier in detail. In particular a formula for it in terms of surgery realization of  $Y$  was given in (4.62).

The invariants

$$\ell k : H^1(Y, \mathbb{Z}_n) \otimes H^1(Y, \mathbb{Z}_n) \longrightarrow \mathbb{Z}_n \quad (5.73)$$

and its quadratic spin-refinement

$$\gamma_s : H^1(Y, \mathbb{Z}_n) \longrightarrow \mathbb{Z}_{2n} \quad (5.74)$$

can be expressed using the standard linking pairing and its spin-refinement [84] which are defined on  $\text{Tor } H_1(Y, \mathbb{Z})$  and valued in  $\mathbb{Q}/\mathbb{Z}$ . The relation is obtained using the isomorphisms  $H^1(Y, \mathbb{Z}_n) \cong \text{Hom}(H_1(Y, \mathbb{Z}), \mathbb{Z}_n) \subset \text{Hom}(H_1(Y, \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ . These invariants can also be understood as the restriction of the abelian Chern-Simons action and spin-version under the inclusion  $\mathbb{Z}_n \subset U(1)$  [60]. Note that we also have  $\ell k(a, b) = \int_Y a \cup \mathcal{B}b$  where  $\mathcal{B} : H^1(Y, \mathbb{Z}_p) \rightarrow H^2(Y, \mathbb{Z}_p)$  is the Bockstein map. The refinement  $\ell k$  from  $\gamma_s$  by using the relation

$$\gamma_s(a + b) = \gamma_s(a) + \gamma_s(b) + 2\ell k(a, b). \quad (5.75)$$

In terms of the surgery representation (see Section 4.4 for conventions) of the spin 3-manifold  $Y$  we have the following explicit formulas:

$$\ell k(a, b) = \frac{a^T B b}{n} \pmod{n}, \quad (5.76)$$

$$\gamma_s(a) = \frac{a^T B a}{n} + s^T B a \pmod{2n}, \quad (5.77)$$

Here, as before, an element  $a \in H^1(Y, \mathbb{Z}_n)$  is represented by a mod  $n$  vector  $a \in \mathbb{Z}_n^V$  satisfying  $Ba = 0 \pmod{n}$ . Note that we have

$$\beta_s(a) = \gamma_s(a) \pmod{4} \quad (5.78)$$

for  $a \in H^1(Y, \mathbb{Z}_2)$ .

To obtain our first relations between the invariants consider the following sequence of maps between the symmetry groups for  $r > 1$ :

$$\mathbb{Z}_2 \xrightarrow{2^{r-1}} \mathbb{Z}_{2^r} \xrightarrow{\pmod{2}} \mathbb{Z}_2 \quad (5.79)$$

The pullbacks on the representation rings read:

$$\begin{array}{ccccc} RO(\mathbb{Z}_2) & \longleftarrow & RO(\mathbb{Z}_{2^r}) & \longleftarrow & RO(\mathbb{Z}_2) \\ [0] & \longleftarrow & [2^{r-1}] & \longleftarrow & [1] \\ 2[q \pmod{2}] & \longleftarrow & [[q]] & & \end{array} \quad (5.80)$$

Using the Table 5.4 we then obtain the following maps between the corresponding anomaly groups:

$$\begin{array}{ccccc} \mathbb{Z}_8 & \longleftarrow & \mathbb{Z}_2 \times \mathbb{Z}_{2^{r+1}} & \longleftarrow & \mathbb{Z}_8 \\ & & (1, 2^{2r-3}) & \longleftarrow & 1 \\ 2 & \longleftarrow & (0, 1) & & \\ -2^{2r-2} & \longleftarrow & (1, 0) & & \end{array} \quad (5.81)$$

From this we obtain the following relations involving the invariant

$$\omega_s : H^1(Y, \mathbb{Z}_{2^r}) \longrightarrow \mathbb{Z}_2 \quad (5.82)$$

and the invariants  $\beta_s, \gamma_s$ :

$$\beta_s(a) = \frac{1}{2^{r-1}} \cdot \gamma_s(2^{r-1}a) \pmod{4}, \quad a \in H^1(Y, \mathbb{Z}_2), \quad (5.83)$$

and

$$\beta_s(a \pmod{2}) = 4\omega_s(a) + 2^{r-1}\gamma_s(a) \pmod{8}, \quad a \in H^1(Y, \mathbb{Z}_{2^r}). \quad (5.84)$$

The relation (5.83) is indeed consistent with the surgery formula (4.62) and (5.77) for  $\beta_s$  and  $\gamma_s$  respectively. The relation (5.83) can be used to actually *define*  $\omega_s$  via  $\beta_s$  and  $\gamma_s$ :

$$\omega_s(a) = \frac{\beta_s(a \pmod{2}) - 2^{r-1}\gamma_s(a)}{4} \pmod{2}, \quad a \in H^1(Y, \mathbb{Z}_{2^r}). \quad (5.85)$$

Similarly, considering the pullback of the sum map

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{+} \mathbb{Z}_2 \quad (5.86)$$

between the symmetry groups we obtain another relation:

$$\beta_s(a + b) = \beta_s(a) + \beta_s(b) + 2\delta_s(a, b), \quad a, b \in H^1(Y, \mathbb{Z}_2). \quad (5.87)$$

It can be used to express  $\delta_s$  via  $\beta_s$ :

$$\delta_s(a, b) = \frac{\beta_s(a + b) - \beta_s(a) - \beta_s(b)}{2}, \quad a, b \in H^1(Y, \mathbb{Z}_2). \quad (5.88)$$

One can conclude that this  $\delta_s$  is the same as  $\delta_s$  invariant already reviewed in Section 4 from a more geometric point of view. This is in agreement with the description of the spin-bordism invariants in [60].

More generally, one can consider the pullback of the map

$$\mathbb{Z}_{2^{r_1}} \times \mathbb{Z}_{2^{r_2}} \xrightarrow{(\cdot)+(\cdot \pmod{2^{r_1}})} \mathbb{Z}_{2^{r_1}} \quad (5.89)$$

for  $1 \leq r_1 \leq r_2 > 1$  to obtain the expressions for the invariant

$$\varepsilon_s : H^1(Y, \mathbb{Z}_{2^{r_1}}) \times H^1(Y, \mathbb{Z}_{2^{r_2}}) \longrightarrow \mathbb{Z}_2 \quad (5.90)$$

via already known invariants. When  $1 < r_1 \leq r_2$  we have

$$\varepsilon_s(a, b) = \omega_s(a + (b \pmod{2^{r_1}})) + \omega_s(a) + \omega_s(b) \pmod{2}, \quad (5.91)$$

And for  $1 = r_1 < r_2 = r$ :

$$\varepsilon_s(a, b) = \omega_s(b) + \frac{\beta_s(a + (b \pmod{2})) - \beta_s(a) - 2^{r-1}\gamma_s(b) - 2^r \ell k(a, b \pmod{2})}{4} \pmod{2}. \quad (5.92)$$

This provides the expression for all the spin-bordism invariants via the invariants  $\beta_s$ ,  $\gamma_s$  and  $\ell k$  (the latter can be also expressed via  $\gamma_s$ ), for which the explicit expressions in terms of surgery representation are given in (4.62), (5.77) and (5.76).

We have found relations between different invariants corresponding to the subgroups  $A_{p;r}$ ,  $B_{p;r_1,r_2}$  using the homomorphism between different symmetry groups and the maps from the representation rings listed in Tables 5.1 and 5.2. For the subgroups  $C_{p;r_1,r_2,r_3}$  we already know the invariants (listed in Table 5.3) so we can reverse the logic and use the homomorphisms between different symmetry groups to determine the map from the representation ring. Namely, one can consider the homomorphisms of the form

$$\mathbb{Z}_{2^{r_1}} \times \mathbb{Z}_{2^{r_2}} \times \mathbb{Z}_{2^{r_3}} \xrightarrow{q_1(\cdot)+q_2(\cdot \bmod 2^{r_1})+q_3(\cdot \bmod 2^{r_1})} \mathbb{Z}_{2^{r_1}} \quad (5.93)$$

for  $r_1 \leq r_2 \leq r_3$  and some integer  $q_i$ , and use the relation (5.75) together with the known relation

$$\begin{aligned} \beta_s(a+b+c) &= \beta_s(a) + \beta_s(b) + \beta_s(c) + 2\delta_s(a,b) + 2\delta_s(b,c) + 2\delta_s(a,c) \\ &\quad + 4 \int_Y a \cup b \cup c \quad (5.94) \end{aligned}$$

for  $a, b, c \in H^1(Y, \mathbb{Z}_2)$  to arrive at the result listed in Table 5.3.

Finally let us note that the invariants  $\omega_s(a)$  and  $\varepsilon_s(a, b)$  can be given rather simple a geometrical interpretations (similar to the ones of  $\beta_s(a)$  and  $\delta_s(a, b)$ ) when  $a, b$  can be lifted to classes in integral cohomology  $H^1(Y, \mathbb{Z})$  [60].

## 5.4 Twisted SPTs on closed 3-manifolds via surgery

In order to study the more complex case where  $G^f$  has a non-trivial twist, the starting point is focusing on the case of  $G^f$  abelian. Here we are going to work out the prototypical example, namely when  $G^f = \mathbb{Z}_2^f \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}} \cong \mathbb{Z}_{2^{l+1}}^f$ . Its bordism group will indeed describe the 2-torsion part of any bordism group with  $G^f = \mathbb{Z}_2^f \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2n}$ , which is its most interesting component [60]. In fact, by decomposing  $n = 2^l \cdot k$  with  $k$  odd, it holds  $\Omega_3^{\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}}} = \Omega_3^{\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}}} (B\mathbb{Z}_k)$ . We start by the known results [60, 114]

$$\Omega_3^{\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}}} \cong \mathbb{Z}_{2^{l-1}}, \quad \Omega_4^{\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}}} \cong \mathbb{Z}. \quad (5.95)$$

The question then is which kind of invariant generates  $\text{Hom}(\Omega_3^{\text{Spin}-\mathbb{Z}_2^f}, U(1)) \cong \mathbb{Z}_{2^{l-1}}$ . To address it we are going to make use of the anomaly interplay given by the homomorphism  $f : \mathbb{Z}_{2^{l+1}}^f \hookrightarrow U(1)$ , where we recall that  $G^f = U(1)$  describes

theories with a  $\text{Spin}^c$  structures. We will need to consider the total anomaly group  $H_{I\mathbb{Z}}^4(MTH)$  for both cases, which sits in the short exact sequence

$$1 \longrightarrow \text{Hom}(\text{Tor } \Omega_3^H, U(1)) \longrightarrow H_{I\mathbb{Z}}^4(MTH) \longrightarrow \text{Hom}(\Omega_4^H, \mathbb{Z}) \longrightarrow 1. \quad (5.96)$$

Then, as we already discussed, the functorial properties are such that the homomorphism  $f$  induces a commutative diagram between the two different short exact sequences corresponding to  $H = \text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}}$  and  $H' = \text{Spin}^c$ .

Since it is possible to compute and find that [89]

$$\text{Hom}(\Omega_3^{\text{Spin}^c}, U(1)) = 1, \quad \text{Hom}(\Omega_4^{\text{Spin}^c}, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}, \quad (5.97)$$

we can write the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}_{2^{l-1}} & \longrightarrow & H_{I\mathbb{Z}}^4(MT(\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}})) & \longrightarrow & \mathbb{Z} \longrightarrow 1 \\ & & \uparrow & & \uparrow \tilde{f}^* & & \uparrow \\ 1 & \longrightarrow & 1 & \longrightarrow & H_{I\mathbb{Z}}^4(MT\text{Spin}^c) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 1 \end{array}. \quad (5.98)$$

To see how  $\text{Spin}^c$  structure invariants in  $d = 4$  are translated into  $\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}}$  ones in  $d = 3$ , it is necessary to uncover the torsion part of the map  $\tilde{f}^* : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}_{2^{l-1}} \oplus \mathbb{Z}$ .

Consider now the series of manifolds quotients of the unit sphere bundle of the Whitney sum of the tensor square of the complex Hopf line bundle  $H$  over  $\mathbb{C}\mathbb{P}^1$ ,

$$Y(2^l; \llbracket a \rrbracket) := \text{Sph}(\mathbb{H} \otimes \mathbb{H}) / \llbracket a \rrbracket. \quad (5.99)$$

Here  $l$  and  $a$  are both integers,  $\llbracket a \rrbracket$  is the representation  $\llbracket a \rrbracket : \lambda \mapsto \lambda^a$  of  $\mathbb{Z}_{2^l}$  in  $U(1)$  and its action on the associated unit sphere bundle is fixed-point free [115]. It is possible to show [116, 117] that these manifolds a natural  $\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}}$  structure, i.e.  $Y(2^l; \llbracket a \rrbracket) \in \Omega_3^{\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}}}$ . Moreover, in this case the  $\eta$  invariant gives an homomorphism

$$\eta(\rho) : \Omega_3^{\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}}} \rightarrow \mathbb{R}/\mathbb{Z}, \quad (5.100)$$

for any  $\rho \in RU_0(\mathbb{Z}_{2^l})$ , where  $RU_0(\mathbb{Z}_{2^l})$  is the augmentation ideal of  $RU(\mathbb{Z}_{2^l})$  [117]. We recall that

$$RU(\mathbb{Z}_{2^l}) = \bigoplus_{q=0}^{2^l-1} \mathbb{Z} \cdot \llbracket q \rrbracket, \quad (5.101)$$

with  $\llbracket q \rrbracket : \lambda \mapsto \lambda^q$  the 1-dimensional complex irreps of  $\mathbb{Z}_{2^l}$ , while

$$RU_0(\mathbb{Z}_{2^l}) = \left\{ \rho = \sum_{q=0}^{2^l-1} b_q \llbracket q \rrbracket \in RU(\mathbb{Z}_{2^l}) \mid \sum_{q=0}^{2^l-1} b_q = 0 \right\}. \quad (5.102)$$

An alternative and useful description of (5.102) is given by

$$RU_0(\mathbb{Z}_{2^l}) \cong ([0] - \llbracket 1 \rrbracket) \cdot \sum_{k=0}^{2^l-2} c_k \llbracket k \rrbracket, \quad (5.103)$$

where, after fixing  $b_{2^l-1}$  using the constraint of (5.102) one has the relation

$$b_q = \sum_{k=0}^{2^l-2} A_{qk} c_k, \quad A = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & & \ddots & \\ & & & -1 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & \ddots & \\ 1 & 1 & 1 & 1 \end{pmatrix}. \quad (5.104)$$

For this particular class of manifolds the  $\eta$  invariant can be computed explicitly via the formula [117]

$$\eta(Y(2^l; \llbracket a \rrbracket))(\rho) = \frac{1}{2^l} \sum_{\lambda \in \mathbb{Z}_{2^l}, \lambda \neq 1} \text{Tr}(\rho(\lambda)) \frac{(1 + \lambda^a)}{(1 - \lambda^a)^2} \lambda^{(a+1)/2} \pmod{\mathbb{Z}}. \quad (5.105)$$

Therefore  $Y(2^l; \llbracket 1 \rrbracket)$  is a generator of  $\Omega_3^{\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}}}$ , since<sup>9</sup>

$$\eta(Y(2^l; \llbracket 1 \rrbracket))(\llbracket [0] - \llbracket 1 \rrbracket \rrbracket \cdot \llbracket [s] \rrbracket) = \frac{1}{2^l} \sum_{\lambda \in \mathbb{Z}_{2^l}, \lambda \neq 1} \frac{\lambda^{s+1} + \lambda^{s+2}}{1 - \lambda} = \frac{s+1}{2^{l-1}} \pmod{\mathbb{Z}}. \quad (5.106)$$

For  $\rho = (\llbracket [0] - \llbracket 1 \rrbracket \rrbracket) \cdot \sum_q c_q \llbracket [q] \rrbracket$  we arrive to

$$\eta(Y(2^l; \llbracket 1 \rrbracket))(\rho) = \sum_{q=0}^{2^l-2} \frac{c_q(q+1)}{2^{l-2}} \pmod{\mathbb{Z}} = - \sum_{q=0}^{2^l-2} b_q \frac{q(q+1)}{2} \pmod{\mathbb{Z}}. \quad (5.107)$$

Note that  $\llbracket [q] \rrbracket$  here denotes the irreps of charge  $q$  of  $\mathbb{Z}_{2^l}$ . The relation between the charges of this  $\mathbb{Z}_{2^l}$  and  $\mathbb{Z}_{2^{l+1}}$  is [87]

$$\tilde{q} = 2q + 1 \pmod{2^{l+1}}, \quad (5.108)$$

where  $\tilde{q}$  is instead the charge of  $\mathbb{Z}_{2^{l+1}}$ , consistently with the fact that fermions are characterized by odd charges under  $\mathbb{Z}_{2^{l+1}}$ .

Formula (5.107) tells us the anomaly coefficient for the torsion part of  $H_{\mathbb{Z}}^4(MT(\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}}))$ . Now we focus on the coefficients for the anomaly group of  $\text{Spin}^c$ . Being its anomalies perturbative, we can determine them by computing the anomaly polynomial for a  $d = 4$  manifold  $W$ . If  $W$  is equipped with a line bundle  $\mathfrak{v}$  corresponding to a  $U(1)$  gauge field and tangent space  $TW$  then its value is

$$\Phi_4 = \frac{1}{2} c_1^2(\mathfrak{v}) - \frac{1}{24} p_1(TW) \overset{\text{Spin}^c}{\rightsquigarrow} \frac{1}{8} c_1^2(\mathfrak{s}) - \frac{1}{8} \sigma(TW), \quad (5.109)$$

<sup>9</sup>Use the identity

$$\sum_{\lambda \in \mathbb{Z}_{2^l}, \lambda \neq 1} \frac{\lambda^n}{1 - \lambda} = - \sum_{\lambda \in \mathbb{Z}_{2^l}, \lambda \neq 1} \sum_{k=0}^{n-1} \lambda^k + \frac{1}{2} \sum_{\lambda \in \mathbb{Z}_{2^l}, \lambda \neq 1} \left( \frac{1}{1 - \lambda} + \frac{1}{1 - \bar{\lambda}} \right) = n + 2^{l-1} - \frac{1}{2} \pmod{2^l}.$$

where  $\sigma(TW)$  is the signature of  $W$ , related to the Pontryagin class by  $\sigma(TW) = p_1(TW)/3$ . The second expression holds if  $W$  is a  $\text{Spin}^c$  manifold with a determinant line bundle  $\mathfrak{s}$ . This can be found by formally considering  $\mathfrak{v}$  the virtual bundle that satisfy  $\mathfrak{v}^{\otimes 2} = \mathfrak{s}$ , together with the properties of the Chern classes under tensor product. Since the APS index theorem guarantees that (5.109) is always an integer, we arrive to<sup>10</sup>

$$\text{Hom}(\Omega_4^{\text{Spin}^c}, \mathbb{Z}) \cong \mathbb{Z}c_1^2(\mathfrak{s}) \oplus \mathbb{Z}\frac{c_1^2(\mathfrak{s}) - \sigma(TW)}{8}. \quad (5.110)$$

Suppose now that our 4-manifold  $W$  is fixed. Then a set of fermions  $\{\psi_i\}_{i \in \mathcal{I}}$  with charges  $\tilde{q}_i$  represent an element

$$\omega = \left( \sum_i \frac{\tilde{q}_i^2 - 1}{8}, 1 \right) = \left( \sum_i \frac{q_i(q_i + 1)}{2}, 1 \right). \quad (5.111)$$

Note that up to a sign the first entry is exactly the value (5.107) of the  $\eta$  invariant for the generators of  $\Omega_3^{\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2l+1}}$ , thus showing us what is the torsion part of the map  $\tilde{f}^*$  in (5.98). Therefore we know that the bordism invariant that describes  $\text{Hom}(\Omega_3^{\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2l+1}}, U(1)) \cong \mathbb{Z}_{2^{l-1}}$  arises from the bordism invariant  $c_1^2(\mathfrak{s})$  of  $\text{Spin}^c$  4-manifolds.

To determine how the Chern class induces an invariant in 3 dimensions, we now consider the case of a 4-manifold  $W$  with non-trivial boundary  $\partial W = Y$ . We suppose  $W$  to be simply connected and  $Y$  a rational homology sphere, so in particular  $H_2(W; \mathbb{Z})$  will be torsion-free and isomorphic to  $\mathbb{Z}^L$  for some  $L$ . The linking matrix  $B$  will be invertible over  $\mathbb{Q}$  and  $H_1(Y; \mathbb{Z}) \cong \text{Tor } H_1(Y; \mathbb{Z})$ , while  $Y$  will be represented via Dehn surgery on  $S^3$  by a link  $\mathcal{L}$  with  $L$  components,  $Y = S^3(\mathcal{L})$ . Under the appropriate basis the linking form  $B$  of  $\mathcal{L}$  describes the intersection form  $\cap : H_2(W; \mathbb{Z}) \times H_2(W; \mathbb{Z}) \rightarrow \mathbb{Z}$  and induces the cohomological pairing<sup>11</sup>

$$\begin{aligned} B^{-1} : H^2(W; \mathbb{Z}) \otimes H^2(W; \mathbb{Z}) &\rightarrow \mathbb{Q}/\mathbb{Z}, \\ \alpha \otimes \beta &\mapsto \alpha^T B^{-1} \beta \end{aligned} \quad (5.112)$$

so that  $c_1^2(\mathfrak{s}) = c^T B^{-1} c$  for<sup>12</sup>  $c_1 = c \in \mathbb{Z}^L \cong H^2(W; \mathbb{Z})$ . Actually (5.112) induces also a pairing for the homology class  $H_1(Y; \mathbb{Z})$  by recalling the long exact sequence

$$\dots \rightarrow H_2(W; \mathbb{Z}) \xrightarrow{B} H_2(W, X; \mathbb{Z}) \xrightarrow{\partial} H_1(Y; \mathbb{Z}) \rightarrow H_1(W; \mathbb{Z}) \cong 1, \quad (5.113)$$

<sup>10</sup>From a more geometrical point of view, one can use as generators of  $\text{Hom}(\Omega_4^{\text{Spin}^c}, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$  the pair of maps  $\varphi_1 = \sigma$ ,  $\varphi_2 = (\sigma - \Sigma \cdot \Sigma)/8$ . Here  $c_1^2(\mathfrak{s}) = \Sigma \cdot \Sigma$ , with  $\Sigma$  a characteristic surface determined by the bundle  $\mathfrak{s}$ . It is possible to show [118] that in this base a good set of generators  $(W, \Sigma)$  of  $\Omega_4^{\text{Spin}^c}$  is given by  $(W, \Sigma)_1 = (\mathbb{C}\mathbb{P}^2, \mathbb{C}\mathbb{P}^1)$  and  $(W, \Sigma)_2 = (\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}, \#_3 \mathbb{C}\mathbb{P}^1 \# \overline{\mathbb{C}\mathbb{P}^1})$ , as  $\varphi_i((W, \Sigma)_j) = \delta_{ij}$ .

<sup>11</sup>Since  $H_2(W; \mathbb{Z})$  is torsionless one can actually think of it in terms of de Rahm cohomology, where the pairing is given by the wedge product.

<sup>12</sup>In the case of a manifold  $W$  with boundary one might ask whether  $c_1$  defines an element in  $H^2(W, Y; \mathbb{Z})$  rather than  $H^2(W; \mathbb{Z})$ . From the physical point of view it is clear that the latter is the correct one, as an element in the relative cohomology would imply a trivial theory on the boundary  $X$ .

and the Poincaré-Lefschetz duality  $H^2(W; \mathbb{Z}) \cong H_2(W, Y; \mathbb{Z})$ . As a result

$$c_1^2(\mathfrak{s}) = c^T B^{-1} c \pmod{\mathbb{Z}} \mapsto \partial(c)^T B^{-1} \partial(c) \pmod{\mathbb{Z}} = \tilde{c}^T B \tilde{c} \pmod{\mathbb{Z}}. \quad (5.114)$$

Here  $\partial(c) \in H_1(Y; \mathbb{Z}) \cong \text{coker} B$ , while  $\tilde{c} \in H^1(X; \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}(H_1(Y; \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$  is its image under the isomorphism given by the linking form

$$\begin{aligned} B^{-1} : H_1(Y; \mathbb{Z}) &\rightarrow \text{Hom}(H_1(Y; \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \\ x &\mapsto x^T B^{-1} \end{aligned} \quad (5.115)$$

Now one needs to understand which kind of element  $\tilde{c}$  is determined by a  $\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}}$  structure on  $Y$ . First we know there is a commutative diagram of tangential structures given by

$$\begin{array}{ccc} \text{Spin}(3) \times \mathbb{Z}_{2^{l+1}} & \xrightarrow{\varphi_2} & \text{Spin}(3) \times_{\mathbb{Z}_2} \mathbb{Z}_{2^{l+1}} \\ \downarrow \psi_1 & & \downarrow \psi_2 \\ \text{SO}(3) \times \mathbb{Z}_{2^{l+1}} & \xrightarrow{\varphi_1} & \text{SO}(3) \times \mathbb{Z}_{2^l} \end{array} \quad (5.116)$$

We already know from (4.58) and (4.61) that a spin structure and a  $\mathbb{Z}_{2^{l+1}}$  gauge field on  $Y$  define a couple  $(s, a) \in \text{Spin}(Y) \times H^1(Y; \mathbb{Z}_{2^{l+1}})$  written in terms of surgery data. Then we define the maps

$$\begin{aligned} \psi_1 : \text{Spin}(Y) \times H^1(X; \mathbb{Z}_{2^{l+1}}) &\rightarrow H^1(Y; \mathbb{Z}_{2^{l+1}}), \\ (s, a) &\mapsto a \\ \varphi_1 : H^1(Y; \mathbb{Z}_{2^{l+1}}) &\rightarrow H^1(Y; \mathbb{Z}_{2^l}). \\ a &\mapsto \tilde{a} = a \pmod{2^l} \end{aligned} \quad (5.117)$$

The set of  $\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}}$  structures is instead described by

$$\text{Spin}_{\mathbb{Z}_{2^{l+1}}}(Y) = \left\{ b \in \mathbb{Z}_{2^{l+1}}^L \left| \sum_J B_{IJ} b_J = 2^l B_{II} \pmod{2^{l+1}} \right. \right\}, \quad (5.118)$$

while the maps  $\psi_2, \varphi_2$  are

$$\begin{aligned} \varphi_2 : \text{Spin}(Y) \times H^1(Y; \mathbb{Z}_{2^{l+1}}) &\rightarrow \text{Spin}_{\mathbb{Z}_{2^{l+1}}}(Y), \\ (s, a) &\mapsto a + 2^l s \\ \psi_2 : \text{Spin}_{\mathbb{Z}_{2^{l+1}}}(Y) &\rightarrow H^1(X; \mathbb{Z}_{2^l}). \\ b &\mapsto \tilde{a} = b \pmod{2^l} \end{aligned} \quad (5.119)$$

Note that we have the property  $\varphi_2(s + \delta, a - 2^l \delta) = \varphi_2(s, a)$  for any  $\delta \in H^1(Y; \mathbb{Z}_2)$ . Via  $\psi_2$  we know that there exists a  $\mathbb{Z}_{2^l}$  bundle for any  $\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}}^f$  structure, which under the inclusion map describes an element

$$\begin{aligned} H^1(Y; \mathbb{Z}_{2^l}) &\rightarrow H^1(Y; \mathbb{Q}/\mathbb{Z}). \\ \tilde{a} &\mapsto \frac{1}{2^l} \tilde{a} \end{aligned} \quad (5.120)$$

Thus, by restricting the tangential structure of  $Y$  to be described by  $b \in \text{Spin}_{\mathbb{Z}_{2^{l+1}}}(Y)$ , we get the surgery description of the  $\eta$  invariant

$$\tilde{c}^T B \tilde{c} = \frac{\tilde{a}^T B \tilde{a}}{2^{2l}} \pmod{\mathbb{Z}} \equiv \frac{1}{2^{l-1}} \tilde{\ell}k(\tilde{a}, \tilde{a}), \quad (5.121)$$

where  $\tilde{a} = b \pmod{2^l}$  and

$$\tilde{\ell}k(\tilde{a}, \tilde{a}) := \frac{\tilde{a}^T B \tilde{a}}{2^{l+1}} \pmod{2^{l-1}}. \quad (5.122)$$

Given a couple  $(s, a)$  such that  $\psi_2 \circ \varphi_2(s, a) = \tilde{a}$  and substituting<sup>13</sup>

$$\tilde{a} = 2^l \delta - a, \quad \delta \in H^1(Y; \mathbb{Z}_2), \quad (5.123)$$

it is easy to prove that  $\tilde{\ell}k(\psi_2 \circ \varphi_2(s, a)) = \tilde{\ell}k(\psi_2 \circ \varphi_2(s + \delta, a - 2^l \delta))$  and that  $\tilde{\ell}k$  is always an integer, so that such invariant is well defined in terms of the  $\text{Spin} \times_{\mathbb{Z}_2^f} \mathbb{Z}_{2^{l+1}}^f$  structures (5.118) and is precisely the generator of  $\text{Hom}(\Omega_3^{\text{Spin}-\mathbb{Z}_{2^{l+1}}^f}, U(1))$ . Note that the commutativity of (5.116) tells us that for the purpose of computing  $\tilde{\ell}k$  one can formally think of  $b$  as a  $\mathbb{Z}_{2^{l+1}}$  gauge field and  $\tilde{a}$  as its  $\pmod{2^l}$  reduction.

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<sup>13</sup>Here the identity is to be understood under the lifting  $\mathbb{Z}_n \rightarrow \mathbb{Z}$ .

# Chapter 6

## Lightest operators constraints from modular consistency

The results of the previous chapters describe how a generic theory with fermions transforms under modular transformations given the value of its anomaly. As the next step, it is interesting to understand if it is possible to make use of this knowledge in order to figure out dynamical constraints that theories must satisfy simply because of the presence of anomalies. If such statement exists, it should be fully general and should not rely on the details of theories other than the presence of the symmetry  $G^f$  itself and its anomaly. To this regard, the robustness of 't Hooft anomalies under RG flow turns out to be very useful, as this means it should hold in particular for the IR description of any theory. We know that a generic UV theory can flow in the IR to one out of three possibilities in presence of a non-trivial anomaly  $\mathcal{A}_{UV}$ . First, it can flow to a theory with spontaneous symmetry breaking, while second, it can flow to a non-trivial TQFT that matches  $\mathcal{A}_{UV}$ . Finally, the most interesting case for us are theories that flow to CFTs such that  $\mathcal{A}_{IR} = \mathcal{A}_{UV}$ . For this last class of theories we will be able to find some spectrum constraints thanks to the setup that the CFT formalism provides. Let us notice that these kind of constraints for non-zero values of the anomalies do *not* depend on accidental symmetries that theories might have in the IR. Indeed, by IR matching, a symmetry  $G$  with non-trivial value of the anomaly must necessarily be present in the UV as well,  $G \subseteq G_{UV}$ .

One of the many things interesting to study in this setup is to infer the existence or not of relevant or marginal  $G_{UV}$ -invariant operators for candidate low-energy CFTs. Indeed, the absence of such operators would mean that the CFT in analysis is a stable fixed point under RG flow and thus a satisfactory description of low-energy degrees of freedom of any system that approaches it under RG flow. In the particular example of  $d = 2$ , we know a vast class of systems with fractional values of the central charge  $c$  that are potentially interesting from this point of view, many of which even exhibit a dependence on a spin structure, namely the WZW models. However, understanding how relevant/marginal perturbations exist in general terms is a difficult task, see for example [119]. Therefore an alternative approach that quickly offers a way to gather

this kind of information is appreciable. One way to try to answer this question is via *modular bootstrap*. The consistency constraints that modular transformations impose on a CFT have indeed already been proved to be effective in the bosonic case [18, 19, 21, 120–131]. For this reason, it is worth to analyze what else can we learn from it in the case of fermionic systems, focusing on the dependency of the bounds on the presence of global anomalies. This is precisely the goal of the present chapter, namely we are going to compute a set of upper bounds on the lightest symmetry-preserving scalar operators for various sets of fermionic theories with different discrete symmetry groups  $G^f$  and anomalies. Following the same philosophy of the previous section, we are going to first analyze in better details the case of  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2$  and finally present the results for selected cases with different symmetries and anomalies.

Finally, let us mention here that fermionic theories in principle can always be studied in terms of bosonic systems by means of the bosonization and fermionization maps we already described in Section 3.2. However, one thing we specifically neglected in our previous discussion is that the resulting bosonic theory that emerges after  $\mathbb{Z}_2^f$  gauging often exhibits non-trivial traits, like the presence of non-invertible symmetries [132], and their relation with the anomaly of the original fermionic description might be subtle. Therefore studying fermionic theories without relying on their bosonization is more natural from this point of view, which indeed will be our approach.

In the following we start by presenting the general setup needed in order to apply techniques of modular bootstrap and then proceed to show the bounds implied by modular crossing equations.

## 6.1 General setup

For our working case we will consider unitary theories with no gravitational anomalies, which means we are going to assume central charges  $c = c_R = c_L \geq 1$ .

By assuming a discrete symmetry group  $G^f$ , the transformation rules of the fields are implemented in the theory by the presence of topological defect lines (TDLs). As already discussed, these defects  $\hat{g}$  are such that sweeping them past some local operator, be it  $\phi(x)$ , then they are transformed into  $\phi'(x) = g \cdot \phi(x)$ , where  $g \in G^f$ . Their presence is particularly useful for the definition of the twisted Hilbert spaces. Indeed, considering partition functions on a torus with a TDL  $\hat{g}$  parallel to the space direction<sup>1</sup> is equivalent to twisting the boundary condition on the time direction by

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<sup>1</sup>We are working in the Euclidean so the notion of time and space direction is arbitrary. We will work by pretending that the direction parallel to the defect line of (6.1) is the space direction.

the action of  $g$ . In pictorial terms

$$Z^g(\tau, \bar{\tau}) := \text{Tr}_{\mathcal{H}_{\mathbb{I}}}[\hat{g} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24}] \sim g \begin{array}{c} \square \\ \hline \xrightarrow{\quad} \\ \hline \square \\ \mathbb{I} \end{array} . \quad (6.1)$$

Instead, the partition function defined over the twisted Hilbert spaces  $\mathcal{H}_g$  corresponds to the insertion of a line defect along the time direction:

$$Z_g(\tau, \bar{\tau}) := \text{Tr}_{\mathcal{H}_g}[q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24}] \sim \mathbb{I} \begin{array}{c} \square \\ \uparrow \\ \square \\ g \end{array} . \quad (6.2)$$

Here one clarification is in order. Since we are focusing on the case of spin-theories, we choose as our convention that the untwisted boundary condition defined along the space and time directions and associated to  $\mathbb{I} \in G^f$  is assumed to be NS, namely anti-periodic<sup>2</sup>. Thus for us in any Hilbert space  $\mathcal{H}_{g \neq \mathbb{I}}$  the fermions exhibit NS boundary condition *plus* a twist by  $g \in G^f$ . In particular, an R boundary condition correspond to NS plus an additional twist by  $(-1)^F$ .

TDLs describe the  $G^f$  action over the untwisted Hilbert space  $\mathcal{H}_{\mathbb{I}}$ , which admits the natural grading

$$\mathcal{H}_{\mathbb{I}} = \bigoplus_{\Gamma \text{ irreps}} \mathcal{H}_{\mathbb{I}}^{(\Gamma)}. \quad (6.3)$$

Therefore the untwisted partition function can be expressed as a sum

$$Z(\tau, \bar{\tau}) = \text{Tr}_{\mathcal{H}_{\mathbb{I}}}[q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24}] = \text{Tr}_{\mathcal{H}_{\mathbb{I}}} \left[ \sum_{\Gamma} \Pi_{\Gamma} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \right] \equiv \sum_{\Gamma} Z^{\Gamma}(\tau, \bar{\tau}), \quad (6.4)$$

where  $\Pi_{\Gamma}$  are the projectors over the sectors  $\mathcal{H}_{\mathbb{I}}^{(\Gamma)}$ , defined by

$$\Pi_{\Gamma} = \frac{d_{\Gamma}}{|G^f|} \sum_{g \in G^f} \xi_{\Gamma}^*(g) \hat{g}. \quad (6.5)$$

---

<sup>2</sup>Note that this convention is different than the one adopted in Section 4.2. However, this is not contradictory as the choice of convention is equivalent to the choice of background spin structure: in general there is no canonical choice and no results will depend on it. In particular, for the description of topological defects alone, the choice of a R background on 1-cycles appears more natural. Instead, while working with partition functions, the natural choice of boundary conditions is given by NS, which is why we are changing conventions here.

Here  $d_\Gamma$  and  $\xi_\Gamma$  are the dimension and the character of the irrep  $\Gamma$ . Using (6.1) the sectors  $Z^\Gamma$  that compose  $Z(\tau, \bar{\tau})$  are

$$Z^\Gamma(\tau, \bar{\tau}) = \frac{d_\Gamma}{|G^f|} \sum_{C_{G^f}} \xi_\Gamma^*(C_{G^f}) Z^{C_{G^f}}(\tau, \bar{\tau}), \quad Z^{C_{G^f}}(\tau, \bar{\tau}) := \sum_{g \in C_{G^f}} Z^g(\tau, \bar{\tau}), \quad (6.6)$$

where  $\{C_{G^f}\}$  are the conjugacy classes of  $G^f$ .

The modular bootstrap technique will allow us to uncover bounds on various sector of the theory. In our case of interests, these sectors will corresponds to the sectors  $\mathcal{H}_{\mathbb{I}}^{(\Gamma)}$  of the untwisted Hilbert space. In particular, possible deformations of the theories must be described via operator-state correspondence by states in  $\mathcal{H}_{\mathbb{I}}^{(\Gamma_0)}$ , where  $\Gamma_0$  is the trivial irrep of  $G^f$ . Therefore, in the following we will be particularly interested in working with the various  $Z^\Gamma$ .

### 6.1.1 General $S$ transformations

In Chapter 4 we discussed the modular transformation for the symmetry case  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2$ . However, we want to compute now the same kind of results for more general groups via the techniques we have developed so far, so that they can be used to uncover bounds on spectra of theories.

Our final goal is to study constraints on the sector of the trivial irrep  $\Gamma_0$  on  $\mathcal{H}_{\mathbb{I}}$ , where the symmetry-preserving scalar operators lie. Thus, in order to make use of the modular bootstrap technique one needs to consider all the defect partition functions  $Z^g$  and  $Z_g$ , since under  $S$  transformation

$$S[Z^g](\tau, \bar{\tau}) \equiv Z^g(-1/\tau, -1/\bar{\tau}) = e^{2\pi i \vartheta_S(g)} Z_g(\tau, \bar{\tau}). \quad (6.7)$$

A priori, the angle  $\vartheta_S$  can be non-zero for non-trivial anomalies. However, in the cases we are going to consider we can fix it to  $\vartheta_S(g) = 0$  for any  $g \in G^f$ . To motivate it, consider initially the basic case with  $G^f = \mathbb{Z}_2^f \times G$  and then the general one with  $G^f = \mathbb{Z}_2^f \times_{\mathbb{Z}_2^f} G$ .

In the first case, one can immediately note that the partition functions we are interested in always exhibit either a time or space direction with boundary condition<sup>3</sup> NSI. This means that we can always contract such direction to a point in order to create a solid torus  $\mathcal{W}_{\mu, \beta}$  such that  $\partial \mathcal{W}_{\mu, \beta} = \mathbb{T}_{\mu, \beta}^2$ . Here the subscripts  $\mu, \beta$  represent the second direction on  $\mathbb{T}_{\mu, g}^2$  ( $\mu = 0$  the time one and  $\mu = 1$  the space one) and its boundary condition  $\beta$ . This means that  $\mathbb{T}_{\mu, \beta}^2$  always lies in the trivial bordism class  $[\emptyset] \in \Omega_2^{\text{Spin}}(BG)$ . Moreover, it is clear that when  $\beta \neq \text{NSI}$ , then under  $S$  the bordism  $\mathcal{S} : \mathbb{T}_{\mu, \beta}^2 \mapsto \mathbb{T}_{\mu+1 \pmod{2}, \beta}^2$  is such that

$$\mathcal{S} \circ \mathcal{W}_{\mu, \beta} = \mathcal{W}_{\mu+1 \pmod{2}, \beta}. \quad (6.8)$$

<sup>3</sup>Here when  $G^f = \mathbb{Z}_2^f \times G$  we denote by NSh (Rh) the boundary condition where fermions are antiperiodic (periodic) and further twisted by  $h \in G$ .

Thus by choosing these manifolds as the proper base of the Hilbert spaces describing the partition functions, we can always set  $\vartheta_S(g \neq \mathbb{I}) = 0$ . The only slightly different case is when  $g = \mathbb{I}$ , since a priori  $[(\mathcal{S} \circ \mathcal{W}_{0,\text{NSI}}) \sqcup \overline{\mathcal{W}_{1,\text{NSI}}}] \neq [\emptyset]$  even if  $\mathbb{T}_{0,\text{NSI}}^2 = \mathbb{T}_{1,\text{NSI}}^2$ . However, since more accurately  $(\mathcal{S} \circ \mathcal{W}_{0,\text{NSI}}) \sqcup \overline{\mathcal{W}_{1,\text{NSI}}} \in \Omega_3^{\text{Spin}}(\text{pt}) \subset \Omega_3^{\text{Spin}}(BG)$  and  $\Omega_3^{\text{Spin}}(\text{pt}) = 1$ , we can set again  $\vartheta_S(\mathbb{I}) = 0$ . Note that this choice of base for generic  $G^f$  matches with the one we used in the particular case  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2$ .

In the twisted case  $G^f = \mathbb{Z}_2^f \times_{\mathbb{Z}_2^f} G$  we can use the knowledge we have from the argument above. Indeed, we have always a map

$$\sigma : \text{Spin}(2) \times G \rightarrow \text{Spin}(2) \times_{\mathbb{Z}_2^f} G \quad (6.9)$$

which induces the bordism map

$$\hat{\sigma}_* : \Omega_2^{\text{Spin}}(BG) \rightarrow \Omega_2^{\text{Spin} \times_{\mathbb{Z}_2^f} G}. \quad (6.10)$$

This means that the torus with a NSI boundary condition in one of the two directions and viewed as an element of  $\Omega_2^{\text{Spin} \times_{\mathbb{Z}_2^f} G}$  admits in its fiber under  $\hat{\sigma}_*$  the trivial bordism class  $[\emptyset] \in \Omega_2^{\text{Spin}}(BG)$ . Being (6.10) a homomorphism, it follows that the angle  $\vartheta_S(g)$  can be set again to zero.

Therefore for the cases we are going to consider it is always possible to choose an appropriate bases of the Hilbert spaces of the partition functions such that all the information of the global anomalies can be recasted into the  $T$  transformation properties and, ultimately, the spin selection rule for  $\mathcal{H}_{g \neq \mathbb{I}}$ .

### 6.1.2 General $T$ transformations and spin selection rules

We now proceed to compute the spin selection rule for the twisted Hilbert space as a function of the anomaly of  $G^f$ . As before, here we work out the case of a generic  $G^f$  as an extension of the results already found for  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2$ . We start first considering the case  $G^f = \mathbb{Z}_2^f \times G$  since the second case will follow analogously. Under a  $T$  transformation the torus associated to  $\mathcal{H}_g$  will be mapped as

$$\mathcal{T} : \mathbb{I} \begin{array}{c} \square \\ \uparrow \\ g \end{array} \mapsto (-1)^F g \begin{array}{c} \square \\ \curvearrowright \\ g \end{array}. \quad (6.11)$$

This means that generally, considering the smallest integer  $n_g$  such that  $((-1)^F g)^{n_g} = \mathbb{I}$ , then  $T^{n_g}$  is an automorphism on the Hilbert space related to the partition function  $Z_g$ , described by a phase  $e^{2\pi i \vartheta_T(g)}$ . Moreover, we know that for a Virasoro module associated to a primary of weight  $(h, \bar{h})$  the  $T$  transformation multiply it by a phase given by the spin  $s = h - \bar{h}$ ,

$$q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} |h, \bar{h}\rangle \xrightarrow{T} e^{2\pi i s} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} |h, \bar{h}\rangle. \quad (6.12)$$

Therefore, one gets that the spin selection rule for  $\mathcal{H}_g$  is given by

$$T^{n_g}[Z_g](\tau, \bar{\tau}) = e^{2\pi i \vartheta_T(g)} Z_g(\tau, \bar{\tau}) \quad \leftrightarrow \quad \{s\}_{\mathcal{H}_g} \subseteq \frac{\vartheta_T(g)}{n_g} + \frac{1}{n_g} \mathbb{Z}. \quad (6.13)$$

For  $\mathcal{H}_{\mathbb{I}}$  we are more specifically interested to the spin rule for each of the various sectors  $\mathcal{H}_{\mathbb{I}}^{(\Gamma)}$ . In this case we have that under  $T$  any TDL  $\hat{g}$  parallel to the time direction is mapped into  $(-1)^F \hat{g}$  and, since  $\mathbb{Z}_2^f$  sits in the center of  $G^f$ , that

$$T[Z^\Gamma](\tau, \bar{\tau}) = \frac{d_\Gamma}{|G^f|} \sum_{g \in G^f} \xi_\Gamma^*(g(-1)^F) \frac{\xi_\Gamma^*((-1)^F)}{\xi_\Gamma^*(1)} Z^{(-1)^F g}(\tau, \bar{\tau}) = \pm Z^\Gamma(\tau, \bar{\tau}). \quad (6.14)$$

Thus the spin selection rule is not affected by the anomaly of  $G^f$ , but is simply given by

$$\{s\}_{\mathcal{H}_{\mathbb{I}}^{(\Gamma)}} \subseteq \frac{\chi_\Gamma(1) - \chi_\Gamma((-1)^F)}{4\chi_\Gamma(1)} + \mathbb{Z}. \quad (6.15)$$

The 3-manifolds

$$\mathcal{M}(T^{n_g}) = (\mathcal{T}^{n_g} \circ \mathcal{W}_{1,g}) \sqcup \overline{\mathcal{W}_{1,g}} \quad (6.16)$$

associated to the phases (6.13) admit a simple description in terms of an unknot with framing number  $n_g$ , so that its linking matrix is simply equal to  $B = (n_g)$ . Moreover, the vectors  $s$ ,  $a$ ,  $b$  that define the various possible structures (4.58), (4.61) and (5.118) of  $\mathcal{M}(T^{n_g})$  are fixed by the element  $g$ . Finally, we note that the group  $G'$  that describes the background gauge fields generated by the modular transformation in this case is just  $\langle g \rangle = \mathbb{Z}_{n_g}$ . Therefore the value of  $\vartheta_T(g)$  is completely fixed via the commutative diagram (5.72).

We now proceed to compute explicitly the spin selection rules for a few selected cases, namely when

$$G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2, \mathbb{Z}_2^f \times \mathbb{Z}_4, \mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2^f \times S_3, \mathbb{Z}_2^f \times S_4, \mathbb{Z}_2^f \times D_8, \mathbb{Z}_8^f. \quad (6.17)$$

**Case 1:**  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2$

By confronting (6.13) with  $T$  and  $T^2$  from (4.29) and (4.28) we can infer the spins allowed in the various Hilbert spaces are

$$\{s\}_{\mathcal{H}_{\text{NSO}}} \in \mathbb{Z}/2, \quad \{s\}_{\mathcal{H}_{\text{R0}}} \in \mathbb{Z}, \quad \{s\}_{\mathcal{H}_{\text{NS1}}} \in \nu/16 + \mathbb{Z}/2, \quad \{s\}_{\mathcal{H}_{\text{R1}}} \in -\nu/16 + \mathbb{Z}/2. \quad (6.18)$$

In particular (6.14) tells us that<sup>4</sup>

$$\{s\}_{\mathcal{H}_{\text{NS0}}^{(+,+)}} \in \mathbb{Z}, \quad \{s\}_{\mathcal{H}_{\text{NS0}}^{(-,+)}} \in 1/2 + \mathbb{Z}, \quad \{s\}_{\mathcal{H}_{\text{NS0}}^{(+,-)}} \in \mathbb{Z}, \quad \{s\}_{\mathcal{H}_{\text{NS0}}^{(-,-)}} \in 1/2 + \mathbb{Z}. \quad (6.19)$$

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<sup>4</sup>Here our notation is that  $(F, Q)$  denote the sectors of states with such charges for  $\mathbb{Z}_2^f \times \mathbb{Z}_2$ .

**Case 2:**  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_4$

In this case we that the invariants that describe  $\text{Hom}(\Omega_3^{\text{Spin}}(B\mathbb{Z}_4), U(1))$  are

$$\gamma_s(a) = \frac{a^T B a}{4} + s^T B a \pmod{8}, \quad (6.20)$$

$$\omega_s(a) = \frac{1}{4}\beta_s(a \pmod{2}) - \frac{1}{2}\gamma_s(a) \pmod{2}, \quad (6.21)$$

where

$$\beta_s(a) = \frac{(s+a)^T B (s+a) - s^T B s}{2} + 4(\text{Arf}(\mathcal{C}_{s+a}) + \text{Arf}(\mathcal{C}_s)) \pmod{8}. \quad (6.22)$$

Note that the Arf invariant for the links that represent  $(T^n)$  are always zero for any  $n$ , so we can drop its term. Moreover, here  $(s+a)$  is intended to take values in  $\mathbb{Z}_2$ . The generic phase  $\vartheta_T(g)$  will be equal to

$$\vartheta_T(g) = \frac{\nu_\omega}{2}\omega_s(a) + \frac{\nu_\gamma}{8}\gamma_s(a), \quad (6.23)$$

where  $(\nu_\omega, \nu_\gamma) \in \mathbb{Z}_2 \times \mathbb{Z}_8$ . The resulting spin selection rules are reported in Figure 6.2.

**Case 3:**  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Let differentiate the two  $\mathbb{Z}_2^{(i)}$  subgroups by the index  $i = 1, 2$ . Then the cobordism group is described by the triple  $(\nu_1, \nu_2, \nu_\delta) \in \mathbb{Z}_8^{(1)} \times \mathbb{Z}_8^{(2)} \times \mathbb{Z}_4$ . The generators are given by the invariants  $\beta_s(a_i)$  and  $\delta(a_1, a_2)$ , where

$$\delta(a_1, a_2) = \frac{\beta_s(a_1 + a_2) - \beta_s(a_1) - \beta_s(a_2)}{2} \pmod{4} \quad (6.24)$$

and  $a_i \in \mathbb{Z}_2^{(i)}$ . The spin selection rules can be found in Table 6.1.

**Case 4:**  $G^f = \mathbb{Z}_2^f \times S_3$

For this and the next cases we are going to use the notation employed in Section 5.2.2. In particular, for this case having figured out the maps  $\hat{i}_{a,b}^*$ , the spin selection rules follow easily, see Table 6.3. Note that the spin selection rule is defined uniquely for elements in the same conjugacy class in  $G^f$ , as one expects from consistency.

**Case 5:**  $G^f = \mathbb{Z}_2^f \times S_4$

As before, see Section 5.2.2 for our choice of notation. In particular, we make use of the standard representation of elements of  $S_4$  via cycles. The results of the spin selection rules can be found in Table 6.4.

| $(s, a_1, a_2)$ | $\vartheta_T(g)$                    | $\{s\}\mathcal{H}_{(s,a_1,a_2)}$                     |
|-----------------|-------------------------------------|--|
| $(0, 0, 0)$     | 0                                   | $\mathbb{Z}/2$                                       |
| $(0, 0, 1)$     | $\nu_2/8$                           | $\nu_2/16 + \mathbb{Z}/2$                            |
| $(0, 1, 0)$     | $\nu_1/8$                           | $\nu_1/16 + \mathbb{Z}/2$                            |
| $(0, 1, 1)$     | $\nu_1/8 + \nu_2/8 - \nu_\delta/4$  | $\nu_1/16 + \nu_2/16 - \nu_\delta/8 + \mathbb{Z}/2$  |
| $(1, 0, 0)$     | 0                                   | $\mathbb{Z}$   |
| $(1, 0, 1)$     | $-\nu_2/8$                          | $-\nu_2/16 + \mathbb{Z}/2$                           |
| $(1, 1, 0)$     | $-\nu_1/8$                          | $-\nu_1/16 + \mathbb{Z}/2$                           |
| $(1, 1, 1)$     | $-\nu_1/8 - \nu_2/8 + \nu_\delta/4$ | $-\nu_1/16 - \nu_2/16 + \nu_\delta/8 + \mathbb{Z}/2$ |

 Table 6.1: Spin selection rule for  $G^f = \mathbb{Z}_f^2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ .

| $(s, a)$ | $\vartheta_T(g)$               | $\{s\}\mathcal{H}_{(s,a)}$                     |
|----------|--------------------------------|--|
| $(0, 0)$ | 0                              | $\mathbb{Z}/2$                                 |
| $(0, 1)$ | $\nu_\gamma/8$                 | $\nu_\gamma/32 + \mathbb{Z}/4$                 |
| $(0, 2)$ | $\nu_\omega/2 + \nu_\gamma/4$  | $\nu_\omega/4 + \nu_\gamma/8 + \mathbb{Z}/2$   |
| $(0, 3)$ | $\nu_\gamma/8$                 | $\nu_\gamma/32 + \mathbb{Z}/4$                 |
| $(1, 0)$ | 0                              | $\mathbb{Z}$                                   |
| $(1, 1)$ | $\nu_\omega/2 + 5\nu_\gamma/8$ | $\nu_\omega/8 + 5\nu_\gamma/32 + \mathbb{Z}/4$ |
| $(1, 2)$ | $\nu_\omega/2 + 3\nu_\gamma/4$ | $\nu_\omega/4 + 3\nu_\gamma/8 + \mathbb{Z}/2$  |
| $(1, 3)$ | $\nu_\omega/2 + 5\nu_\gamma/8$ | $\nu_\omega/8 + 5\nu_\gamma/32 + \mathbb{Z}/4$ |

 Table 6.2: Spin selection rule for  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_4$ .

| $\{(s, g)\}$     | $\vartheta_T(g)$ | $\{s\}\mathcal{H}_{(s,g)}$ |
|------------------|------------------|----------------------------|
| $\{(0, ( ))\}$   | 0                | $\mathbb{Z}/2$             |
| $\{(0, (ab))\}$  | $p/8$            | $p/16 + \mathbb{Z}/2$      |
| $\{(0, (abc))\}$ | $-q/3$           | $-q/18 + \mathbb{Z}/6$     |
| $\{(1, ( ))\}$   | 0                | $\mathbb{Z}$               |
| $\{(1, (ab))\}$  | $-p/8$           | $-p/16 + \mathbb{Z}/2$     |
| $\{(1, (abc))\}$ | $q/3$            | $q/9 + \mathbb{Z}/3$       |

 Table 6.3: Spin selection rule for  $G^f = \mathbb{Z}_2^f \times S_3$ . The couples  $\{(s, g)\}$  represent the various conjugacy classes of  $G^f$ . The elements of  $S_3$  here are represented by the standard cycle notation used to describe permutation of three elements.

| $\{(s, g)\}$        | $\vartheta_T(g)$   | $\{s\}_{\mathcal{H}(s,g)}$         |
|---------------------|--------------------|------------------------------------|
| $\{(0, ( ))\}$      | 0                  | $\mathbb{Z}/2$                     |
| $\{(0, (ab))\}$     | $p/8$              | $p/16 + \mathbb{Z}/2$              |
| $\{(0, (ab)(cd))\}$ | $(p - q)/4$        | $(p - q)/8 + \mathbb{Z}/2$         |
| $\{(0, (abcd))\}$   | $(3q - p)/8 + r/2$ | $(3q - p)/32 + r/8 + \mathbb{Z}/4$ |
| $\{(0, (abc))\}$    | $-t/3$             | $-t/18 + \mathbb{Z}/6$             |
| $\{(1, ( ))\}$      | 0                  | $\mathbb{Z}$                       |
| $\{(1, (ab))\}$     | $-p/8$             | $-p/16 + \mathbb{Z}/2$             |
| $\{(1, (ab)(cd))\}$ | $(q - p)/4$        | $(q - p)/8 + \mathbb{Z}/2$         |
| $\{(1, (abcd))\}$   | $-(p + q)/8 + r/2$ | $-(p + q)/32 + r/8 + \mathbb{Z}/4$ |
| $\{(1, (abc))\}$    | $t/3$              | $t/9 + \mathbb{Z}/3$               |

Table 6.4: Spin selection rule for  $G^f = \mathbb{Z}_2^f \times S_4$ . The convention here is the same as for  $G^f = \mathbb{Z}_2^f \times S_3$ .

| $\{(s, g)\}$      | $\vartheta_T(g)$         | $\{s\}_{\mathcal{H}(s,g)}$                |
|-------------------|--------------------------|---|
| $\{(0, (0, 0))\}$ | 0                        | $\mathbb{Z}/2$                            |
| $\{(0, (1, 0))\}$ | $-(p + q)/4 + r/8 + t/2$ | $-(p + q)/16 + r/32 + t/8 + \mathbb{Z}/4$ |
| $\{(0, (2, 0))\}$ | $r/4$                    | $r/8 + \mathbb{Z}/2$                      |
| $\{(0, (0, 1))\}$ | $(p + r)/8$              | $(p + r)/16 + \mathbb{Z}/2$               |
| $\{(0, (1, 1))\}$ | $(q + r)/8$              | $(q + r)/16 + \mathbb{Z}/2$               |
| $\{(1, (0, 0))\}$ | 0                        | $\mathbb{Z}$                              |
| $\{(1, (1, 0))\}$ | $(p + q)/4 + 5r/8 + t/2$ | $(p + q)/16 + 5r/32 + t/8 + \mathbb{Z}/4$ |
| $\{(1, (2, 0))\}$ | $-r/4$                   | $-r/8 + \mathbb{Z}/2$                     |
| $\{(1, (0, 1))\}$ | $-(p + r)/8$             | $-(p + r)/16 + \mathbb{Z}/2$              |
| $\{(1, (1, 1))\}$ | $-(q + r)/8$             | $-(q + r)/16 + \mathbb{Z}/2$              |

Table 6.5: Spin selection rule for  $G^f = \mathbb{Z}_2^f \times D_8$ . Here the conjugacy classes are represented for convenience by a single representative.

**Case 6:**  $G^f = \mathbb{Z}_2^f \times D_8$

As before, see Section 5.2.2 for our choice of notation. In particular, we make use of the standard representation of elements of  $D_8 \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_4$  via semidirect product. The results of the spin selection rules can be found in Table 6.5.

**Case 7:**  $G^f = \mathbb{Z}_8^f$

In this case the computation of the spin selection rule is straightforward. Here we switch to the additive notation for  $\mathbb{Z}_8^f$ , so that the Hilbert space  $\mathcal{H}_1$  now corresponds to  $\mathcal{H}_0$  and the NS boundary condition to  $0 \in \mathbb{Z}_8^f$ . It follows that the Hilbert spaces  $\mathcal{H}_n$  are associated to the tori with time boundary condition equal to NS and space boundary condition equal to NS plus a twist by  $n \pmod 8$ . In this case the element  $b$  of (5.118) that identify the Spin- $\mathbb{Z}_8^f$  structure of the correspondent mapping torus

| $n$     | $B$ | $\vartheta_T(n)$ | $\{s\}_{\mathcal{H}_n}$ |
|---------|-----|------------------|-------------------------|
| 0       | 2   | 0                | $\mathbb{Z}/2$          |
| 1 mod 2 | 8   | $\nu/2$          | $\nu/16 + \mathbb{Z}/8$ |
| 2 mod 4 | 4   | 0                | $\mathbb{Z}/4$          |
| 4 mod 8 | 1   | 0                | $\mathbb{Z}$            |

Table 6.6: Spin selection rule for  $G^f = \mathbb{Z}_8^f$ . Here for completeness we reported also the value of the linking matrix  $B = (n_g)$ .

is simply  $b = n \pmod 8$ . The spin selection rule for  $\mathcal{H}_n$  follows easily and is given in Table 6.6.

## 6.2 Modular bootstrap

Having determined the modular transformations and the spin selection rules that they imply on the CFT matter content, we can proceed to apply such informations to work out the bounds we anticipated.

The modular bootstrap technique that we are going to employ is the so-called linear functional method. For its applicability we first need to find a set of partition functions (or combinations of them) that admit a positive expansion over some basis. Since we are mostly interested in studying symmetry-preserving scalar operators on  $\mathcal{H}_1$ , a natural choice is given by starting to consider the sectors  $Z^\Gamma$ . In particular we will be interested in determining bounds for  $Z^{\Gamma_0}$ . As we will explain in a moment, in order to apply our bootstrap method we need to complete this set of partition functions so that it is closed under  $S$  transformation. Since all the  $Z^\Gamma$  admit an expansions in terms of the various  $Z^{C_{G^f}}$  and that under  $S$  transformation we can safely have

$$Z^{C_{G^f}} \xrightarrow{S} Z_{C_{G^f}} = \sum_{g \in C_{G^f}} Z_g, \quad (6.25)$$

we can use  $\{Z_{C_{G^f}}\}$  to complete such set. By labelling with some index  $i = 0, 1, \dots, n$  the irreps  $\Gamma_i$  and conjugacy classes  $C_i$  so that  $i = 0$  correspond to the trivial ones, we define a  $(2n - 1)$ -dimensional *partition vector*

$$\mathbf{Z}_{G^f}^T = (Z^{\Gamma_0}, Z^{\Gamma_1}, \dots, Z^{\Gamma_n}, Z_{C_1}, \dots, Z_{C_n}). \quad (6.26)$$

The  $S$  modular transformations can then be recasted into the form (restoring the dependence of the partition functions on  $\tau$ )

$$\mathbf{Z}_{G^f}(-1/\tau, -1/\bar{\tau}) - S\mathbf{Z}_{G^f}(\tau, \bar{\tau}) = 0, \quad (6.27)$$

where the matrix  $S$  can be easily found from (6.25).

Our choice of the partition vector  $\mathbf{Z}_{G^f}$  always admits a positive expansion in terms of Virasoro characters

$$\chi_0(\tau) = (1 - q) \frac{q^{-\frac{c-1}{24}}}{\eta(\tau)}, \quad \chi_{h>0}(\tau) = \frac{q^{h-\frac{c-1}{24}}}{\eta(\tau)}, \quad (6.28)$$

where  $q = e^{2\pi i\tau}$ . Indeed, recall that any TDL  $\hat{g}$  commutes with the stress tensor and any  $\mathcal{H}_g$  can be organized into representations of the Virasoro algebra with definite conformal weights  $(h, \bar{h})$ . In particular, for  $\mathcal{H}_{\mathbb{I}}$  we can express

$$Z(\tau, \bar{\tau}) = \sum_{\Gamma, h, \bar{h}} n_{\Gamma, h, \bar{h}} \chi_h(\tau) \bar{\chi}_{\bar{h}}(\bar{\tau}), \quad n_{\Gamma, h, \bar{h}} \in \mathbb{Z}_+ d_{\Gamma}, \quad (6.29)$$

so that for any  $\Gamma$

$$Z^{\Gamma}(\tau, \bar{\tau}) = \sum_{h, \bar{h}} n_{\Gamma, h, \bar{h}} \chi_h(\tau) \bar{\chi}_{\bar{h}}(\bar{\tau}). \quad (6.30)$$

Note that both  $Z^{\Gamma}$  and  $Z_{C_{G^f}}$  admits a positive expansion in terms of Virasoro characters. This is a crucial property that will allow us to make use of the mentioned linear functional method. It will be useful in the following to denote with  $\mathcal{H}_i$  the Hilbert spaces corresponding to the entries  $(\mathbf{Z}_{G^f})_i$ .

Note also that by construction we are reducing the dimensionality of the problem, as we are only considering the modular transformations between a subset of all the possible partition functions with various defects insertions. However, this poses no problem and the bounds produced for our case of interest will still be as strict as possible. As an example, take  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2$ . In this case the complete modular crossing equation is defined by the 16-dimensional matrix  $\tilde{S} = S_{(0,0)} \oplus S_{(0,1)} \oplus S_{(1,0)} \oplus S_{(1,1)}$ , i.e.

$$\tilde{\mathbf{Z}}(-1/\tau, -1/\bar{\tau}) = \tilde{S} \tilde{\mathbf{Z}}(\tau, \bar{\tau}), \quad (6.31)$$

where  $\tilde{\mathbf{Z}}(\tau, \bar{\tau}) = \mathbf{Z}_{(0,0)} \oplus \mathbf{Z}_{(0,1)} \oplus \mathbf{Z}_{(1,0)} \oplus \mathbf{Z}_{(1,1)}$  (see (4.19)). The reduction to the 7-dimensional basis (6.26) is then given by the modular crossing matrix

$$S = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}. \quad (6.32)$$

It turns out that, considering  $R$ , the reduction matrix that satisfies  $\mathbf{Z}_{G^f}(\tau, \bar{\tau}) = R \tilde{\mathbf{Z}}(\tau, \bar{\tau})$ , then one can always lift a solution of the modular crossing equation for  $\mathbf{Z}_{G^f}$  to the full system via the lift matrix  $L = R^T (R R^T)^{-1}$  and there is no loss on the strength of the bounds [21]. For the other symmetry cases in exam we report the resulting reduced matrices in Appendix E.

### 6.2.1 The linear functional method

In order to find bounds on the spectrum, we will apply the linear functional method to the modular crossing equation defined on the basis  $\mathbf{Z}_{G^f}$ .

From now on it will be useful to remember that we can divide any partition function as a sum of three kinds of combinations of Virasoro characters. These are

1. The vacuum, the unique character that we are going to assume to exist with  $h = \bar{h} = 0$ , living in  $\mathcal{H}_{\mathbb{I}}$ ;
2. The conserved currents, namely combinations of characters with either  $h = 0$  or  $\bar{h} = 0$ . These we assume to be a priori present in all sectors;
3. The non-degenerate (ND) primaries, the characters with both  $h, \bar{h} > 0$ , present in all sectors as well.

With this premise, we quickly recall the main idea of the technique and leave the details for Appendix D. The single entries of the modular crossing equation (6.27) is

$$\delta_i^j(\mathbf{Z}_{G^f})_j(-1/\tau, -1/\bar{\tau}) - S_i^j(\mathbf{Z}_{G^f})_j(\tau, \bar{\tau}) = 0. \quad (6.33)$$

After applying a linear functional to it the identity will still hold. If, by taking a putative spectrum of Virasoro characters that may compose each  $(\mathbf{Z}_{G^f})_i$ , there exists instead a functional  $\alpha$  that returns

$$\alpha[\delta_i^j(\mathbf{Z}_{G^f})_j(-1/\tau, -1/\bar{\tau})] - \alpha[S_i^j(\mathbf{Z}_{G^f})_j(\tau, \bar{\tau})] > 0, \quad \forall i, \quad (6.34)$$

then such a spectrum is not a consistent one and thus is ruled out. In practice, instead of choosing some particular spectra, it is much more effective to start from a generic sum of Virasoro characters and then look for bounds from above on the scaling dimension  $\Delta = h + \bar{h}$  of the lightest operators in the sectors we decide to analyze. In particular, in our analysis we will focus on two possible bounds:

1. The maximal gap  $\Delta_{\text{gap}}^j$  in the scaling dimension of the lightest non-degenerate primary in a given  $\mathcal{H}_j$ .
2. The maximal gap  $\Delta_{\text{scal}}^j$  in the scaling dimension of the lightest scalar primary in a given  $\mathcal{H}_j$ . Note that this bound make sense only in the sectors where  $s = 0$  is an allowed value of the spin.

Since the search for a functional  $\alpha$  is numerical, it is convenient to work with the reduced partition functions

$$\hat{Z}_i(\tau, \bar{\tau}) := |\tau|^{1/2} |\eta(\tau)|^2 (\mathbf{Z}_{G^f})_i(\tau, \bar{\tau}), \quad (6.35)$$

so that the Virasoro characters one has to deal with are the reduced versions

$$\hat{\chi}_0(\tau) = \tau^{1/4} (1 - q) q^{-\frac{c-1}{24}}, \quad \hat{\chi}_h(\tau) = \tau^{1/4} q^{h - \frac{c-1}{24}}. \quad (6.36)$$

Indeed under  $S$  the combination  $|\tau|^{1/2} |\eta(\tau)|^2$  is invariant and will not change the results.

### 6.3 Bounds on operators in $\mathcal{H}_{\mathbb{I}}$

We now investigate what are the generic bounds present on the spectrum of  $\mathcal{H}_{\mathbb{I}}$ . As anticipated, in order to make full use of the constraints that the modular crossing equation and the linear functional method offer to us, a numerical study is needed. In order to do so, the analysis has been carried out with the use of the SDPB package [133, 134] for CFTs with value of central charge in the range  $1 \leq c \leq 10$ .

In particular, we present bounds on the spectra for all the sector  $\mathcal{H}_{\text{NS0}}^{(F,Q)}$  for  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2$ , while for the remaining cases we will focus on  $\mathcal{H}_{\mathbb{I}}^{(\Gamma_0)}$ .

#### 6.3.1 The case of $\mathbb{Z}_2^f \times \mathbb{Z}_2$ symmetry

This particular symmetry provides the simplest example where all the three cohomological layers that define  $\text{Hom}(\Omega_3^{\text{Spin}}(BG), U(1))$  are non-zero. In particular, the value of anomaly can be recast as

$$\nu = 4w + 2p + a \pmod{8}, \quad (6.37)$$

where  $w$ ,  $p$  and  $a$  are elements of  $H^3(B\mathbb{Z}_2, U(1))$ ,  $SH^2(B\mathbb{Z}_2, \mathbb{Z}_2)$  and  $H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$ . Here  $H^3(B\mathbb{Z}_2, U(1)) \cong H^2(B\mathbb{Z}_2, \mathbb{Z}_2) \cong SH^2(B\mathbb{Z}_2, \mathbb{Z}_2) \cong H^1(B\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$ .

It is interesting to see how spectra of fermionic theories are affected by the presence of each layer, so we will study in greater details this case to uncover possible interesting layer behaviour. Therefore, we will focus our attention on theories where a single layer of anomaly is present, namely when  $\nu = 0, 1, 2, 4 \pmod{8}$ . Note that since we are looking at quantities that are not sensitive to the swap  $s \mapsto -s$ , a theory with anomaly  $\nu$  and its complex conjugate with anomaly  $\bar{\nu} = -\nu \pmod{8}$  will produce the same bounds  $\Delta_{\text{gap}}^j$  and  $\Delta_{\text{scal}}^j$ . Thus our results will hold also for  $\nu = 6, 7 \pmod{8}$ .

One last comment is in order. Usually for the  $\nu = 1 \pmod{2}$  case is argued that no proper Hilbert space  $\mathcal{H}_{\text{NS0}}$  exists, due to the partition functions being proportional to the formal  $\dim Cl(1) = \sqrt{2}$  [17]. However, one can still assume the case when the partition function of the CFT is defined as  $\sqrt{2}$  times the trace over a well-defined Hilbert space. In our analysis for  $\nu = 1 \pmod{8}$  we will precisely explore the space of fermionic CFTs that satisfy such hypothesis, which is still a valid and sensible thing to do.

#### The starting point: $c = 1$

The first thing that is instructive to do is to ask for which sectors we actually expect the presence of a bound. In this regard it is illuminating to look at CFTs with  $c = 1$ , for which we have two main interesting examples at our disposal, namely the free compact boson and the Dirac fermion.

For bosonic theories with  $\mathbb{Z}_2$  non-anomalous symmetry, it is known that no bound is expected in either the charged sector<sup>5</sup>  $\mathcal{H}_0^{-Q}$  of the untwisted Hilbert space and in the twisted Hilbert space,  $\mathcal{H}_1$ . We briefly recall here the main argument of [19]. By stacking a free compact boson of radius  $R$  with a generic non-anomalous CFT uncharged with respect to one of the two non-anomalous  $\mathbb{Z}_2$  subgroups of  $U(1)_n$  and  $U(1)_w$  (symmetries corresponding to the winding and momentum quantum numbers), one can always produce a non-anomalous CFT with  $c > 1$  and arbitrarily high lightest non-degenerate state on  $\mathcal{H}_0^{-Q}$  (and  $\mathcal{H}_1$ ) by taking the radius to be arbitrarily large. By the same argument, if the uncharged CFT is fermionic, then we see that for  $\nu = 0 \pmod 8$  there is no bound for both  $\mathcal{H}_{\text{NS}0}^{\pm F-Q}$ .

The next interesting example is given by a free massless Dirac fermion. Consider two Majorana fermions with holomorphic sectors  $\psi_a$ ,  $a = 1, 2$ , described by

$$\mathcal{L} = \sum_{a=1}^2 \psi_a \bar{\partial} \psi_a + \bar{\psi}_a \partial \bar{\psi}_a \quad (6.38)$$

and arrange them as the single complex fermion<sup>6</sup>  $\Psi = \psi_1 + i\psi_2$ . The theory presents the chiral and non-chiral fermionic parities acting as

$$(-1)^F : \quad \Psi \rightarrow -\Psi, \quad \bar{\Psi} \rightarrow -\bar{\Psi}, \quad (6.39)$$

$$(-1)^{FL} : \quad \Psi \rightarrow -\Psi, \quad \bar{\Psi} \rightarrow \bar{\Psi}. \quad (6.40)$$

By looking at the spin selection rule of the twisted sectors of this theory, one can find that this presents an anomaly  $\nu = 2$  (see Appendix D).

Unlike before, a priori now there is no natural parameter that can drive the bound for some sectors to infinity. However, one has still a sensible procedure to take into account. Indeed, note that the system has the more generic global symmetry  $U(1) \times \overline{U(1)}$ , where each component acts only on one of the sectors, and gauging some diagonal subgroup  $\mathbb{Z}_p$  with  $p$  odd still preserves the global symmetry  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2$ . In fact, there is no mixed anomaly between fermionic parity and  $\mathbb{Z}_p$ , since  $\text{Hom}(\Omega_3^{\text{Spin}}(B\mathbb{Z}_p), U(1)) \cong H^3(B\mathbb{Z}_p, U(1))$ . Moreover, by construction  $\Psi$  and  $\Psi^*$  have opposite charges, so it follows that  $\mathbb{Z}_p$  is anomaly-free and can be gauged.

A generic state on the holomorphic sector for the NS untwisted Hilbert space of the gauged theory will be of the form

$$\prod_{n=1}^N \Psi_{-k_n^+} \prod_{m=1}^M \Psi_{-k_m^-}^* \prod_{\bar{n}=1}^{\bar{N}} \bar{\Psi}_{-\bar{k}_{\bar{n}}^+} \prod_{\bar{m}=1}^{\bar{M}} \bar{\Psi}_{-\bar{k}_{\bar{m}}^-}^* |0\rangle, \quad (6.41)$$

where

$$N - M + \bar{N} - \bar{M} = 0 \pmod p, \quad k_i^\pm, \bar{k}_{\bar{i}}^\pm = \frac{1}{2} \pm \frac{1}{p} \pmod 1. \quad (6.42)$$

<sup>5</sup>Here for convenience we slightly change our notation:  $\mathcal{H}^{(\pm, \pm)} \mapsto \mathcal{H}^{\pm F \pm Q}$ , while  $\mathcal{H}^{\pm Q} = \mathcal{H}^{(+, \pm)} \oplus \mathcal{H}^{(-, \pm)}$  and  $\mathcal{H}^{\pm F} = \mathcal{H}^{(\pm, +)} \oplus \mathcal{H}^{(\pm, -)}$ .

<sup>6</sup>Here  $\bar{\Psi}$  will denote the antiholomorphic counterpart of  $\Psi$  that composes the complex fermion. Instead, we will denote with  $\Psi^* = \psi_1 - i\psi_2$  the conjugate field.

By requiring for a state to be charged under fermionic parity one needs to refine the first constraint

$$N - M + \bar{N} - \bar{M} = sp, \quad s \in 2\mathbb{Z} + 1. \quad (6.43)$$

Without loss of generality, we can assume  $s > 0$ . Let us focus on the contribution from the set  $\Psi_{-k_n^+}$ : the states with the lowest possible scaling dimension are defined by towers of operators with increasing dimension

$$k_n^+ = \frac{1}{2} + \frac{1}{p} + n - 1, \quad 1 \leq n \leq N. \quad (6.44)$$

Applying the same logic to the other sets, one find that the lowest non-degenerate states are characterized by

$$\Delta = \frac{N^2 + M^2 + \bar{M}^2 + \bar{N}^2}{2} + \frac{N - M + \bar{N} - \bar{M}}{p} \quad (6.45)$$

$$\geq \frac{N + M + \bar{N} + \bar{M}}{2} + s \quad (6.46)$$

$$\geq \frac{sp}{2}. \quad (6.47)$$

Therefore, by gauging  $\mathbb{Z}_p$  with  $p$  arbitrarily large, we can build a fermionic CFT with  $c = 1$ ,  $\nu = 2$  and no bound on<sup>7</sup>  $\mathcal{H}_{\text{NS0}}^{-F\pm Q}$ . By using the same logic already explained in the bosonic case, the stacking of such theory with a bosonic theory produce generic theories with  $c > 1$  and anomaly  $\nu = 2 \pmod{4}$  with no bounds on the sectors  $\mathcal{H}_{\text{NS0}}^{-F\pm Q}$ .

We are now ready to discuss the numerical results and see what are the bounds for the sectors we are left with to analyze.

### Bounds on $\mathbb{Z}_2$ even operators

The results obtained from the analysis of the various sectors are reported in Figures 6.1-6.3. At certain values of the central charge the bounds are almost saturated by known theories, most of which are different stacking combinations of free Majorana fermions. We marked these points with black dots in the figures. We summarize in the following all our results, while we refer the reader to Appendix D for details on the limit cases.

- As a consistency check of our numerical method, one can see that all the bounds for fermionic theories with  $\mathbb{Z}_2$  symmetries are higher than the bounds of bosonic ones found by [19];

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<sup>7</sup>Note that the conditions  $N + M = 0, 1 \pmod{2}$ , necessary for considering operators specifically in the uncharged and charged sectors with respect to  $(-1)^Q$ , do not change the argument. Therefore one can apply it to reach the same conclusion for both  $\mathcal{H}_{\text{NS0}}^{-F\pm Q}$ .

- Additionally to our previous discussion, our findings suggest that there is no actual bound for the sectors  $\mathcal{H}_{\text{NSO}}^{-F\pm Q}$  for any value of the anomaly. This hints at the possibility of a generalization of the argument of section 6.3.1 which is not sensible to the anomaly of the theory;
- At  $c = 1$  all (finite) bounds are almost saturated by known theories. In the case  $\nu = 0, 1, 2 \pmod{8}$  these are stacks of free fermions, while for  $\nu = 4 \pmod{8}$  the ND bound has as limiting case the bosonic WZW model  $su(2)_1$ , which already saturated the bound for bosonic CFTs with  $\mathbb{Z}_2$  anomaly [19];
- For  $c > 1$  the bounds on scalars on theories with  $\nu = 0, 4 \pmod{8}$  are almost as strict as bounds for generic non-degenerate primary states, suggesting that indeed the two coincide. In fact, this is the case for the free fermion theories with  $\nu = 0 \pmod{8}$ ,  $c = 4$  and  $\nu = 4 \pmod{8}$ ,  $c = 2, 6$ ;
- The bounds on non-degenerate primaries coincide for anomalies  $\nu = 1, 2 \pmod{8}$ , so that they can not be used as a way to tell the two cases apart;
- Most interestingly, relevant operators always allowed in the following intervals of the central charge

| $\nu$        | $c$ interval                    |
|--------------|---------------------------------|
| $0 \pmod{8}$ | $1 \leq c \lesssim 6.63$        |
| $1 \pmod{8}$ | $1 \leq c \lesssim 5.95$        |
| $2 \pmod{8}$ | $1.09 \lesssim c \lesssim 4.93$ |
| $4 \pmod{8}$ | $2.05 \lesssim c \lesssim 5.73$ |

Such limits can possibly be extended considering functionals with higher order of derivatives.

The limits of marginal operators are instead almost saturated by free fermions with central charge  $c = 4 \pm \nu/2$ .

### Bounds on $\mathbb{Z}_2$ odd operators

The behaviour of the bounds for charged operators is milder than for the even ones: we briefly summarize below the notable things that the numerical analysis highlighted. We report the full results in Figures 6.4-6.6.

- Our findings are again coherent with the results founds from the analysis of the bosonic CFTs, i.e. they are always higher;
- Interestingly enough, the same behaviour of before is found for the operators charged under  $(-1)^{\mathcal{F}}$  and  $(-1)^{\mathcal{Q}}$  as well. This indeed seems to strength the hypothesis of before;

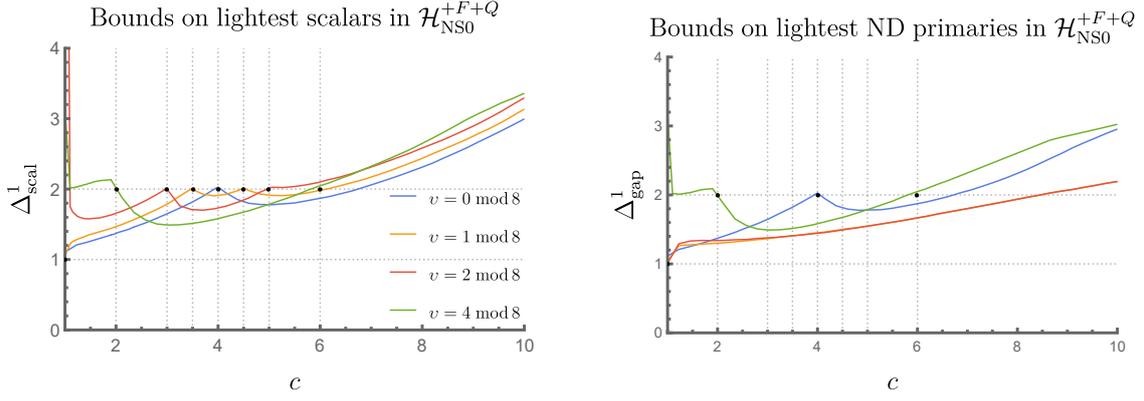


Figure 6.1: Upper bounds in the sector  $\mathcal{H}_{\text{NSO}}^{+F+Q}$  for values of anomaly  $\nu = 0, 1, 2, 4 \pmod{8}$ . Bounds on the lightest non-degenerate primaries are represented by the same colored lines used to denote the bounds for the lightest scalars. The black dots mark the free fermion theories which almost saturate the bounds at the kinks we found.

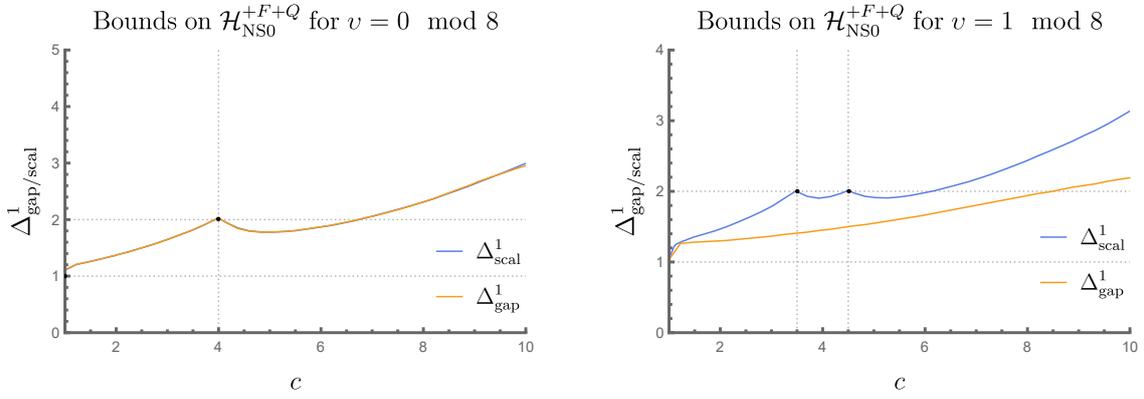


Figure 6.2: Left: upper bounds for lightest scalars and non-degenerate primaries confronted for theories with  $\nu = 0 \pmod{8}$ . Right: upper bounds for lightest scalars and non-degenerate primaries confronted for theories with  $\nu = 1 \pmod{8}$ .

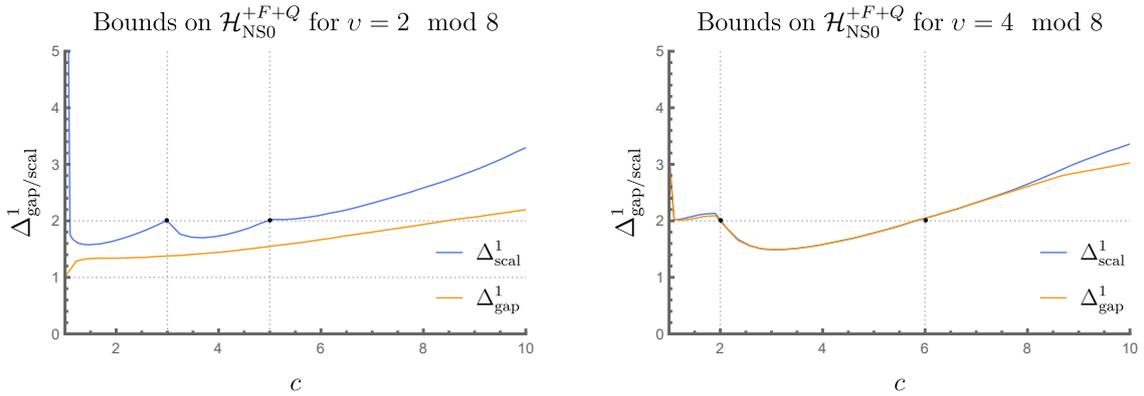


Figure 6.3: Left: upper bounds for lightest scalars and non-degenerate primaries confronted for theories with  $\nu = 2 \pmod{8}$ . Right: upper bounds for lightest scalars and non-degenerate primaries confronted for theories with  $\nu = 4 \pmod{8}$ .

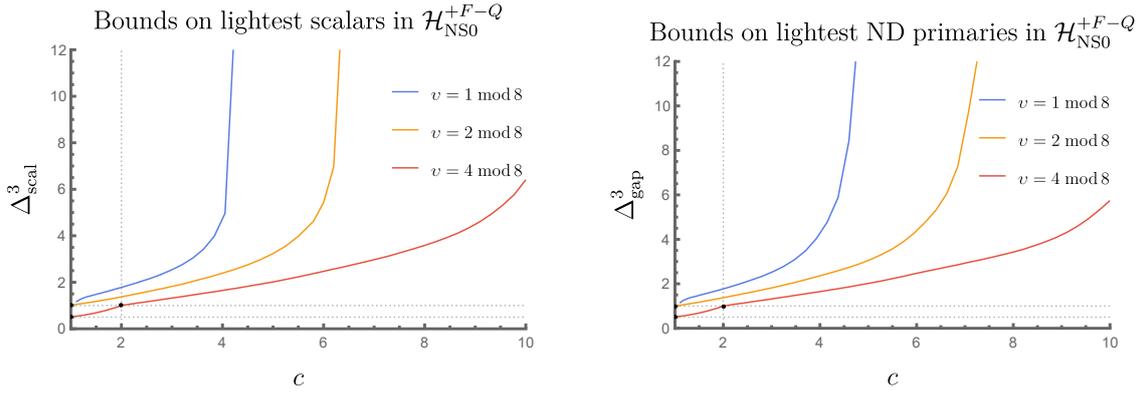


Figure 6.4: Upper bounds in the sector  $\mathcal{H}_{\text{NS0}}^{+F-Q}$  for values of anomaly  $\nu = 1, 2, 4 \pmod 8$ . Bounds on the lightest non-degenerate primaries are represented by the same colored lines used to denote the bounds for the lightest scalars. The black dots mark the free fermion theories which almost saturate the bounds at the kinks we found.

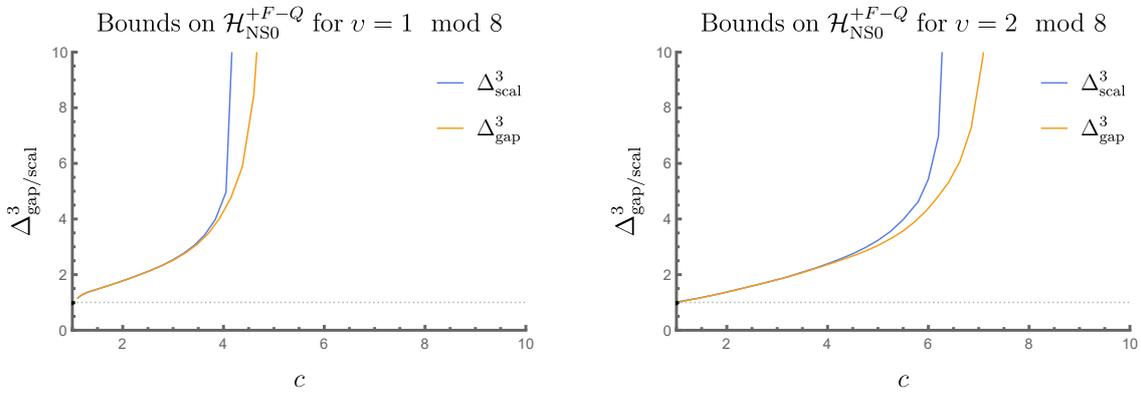


Figure 6.5: Left: upper bounds for lightest scalars and non-degenerate primaries confronted for theories with  $\nu = 1 \pmod 8$ . Right: upper bounds for lightest scalars and non-degenerate primaries confronted for theories with  $\nu = 2 \pmod 8$ .

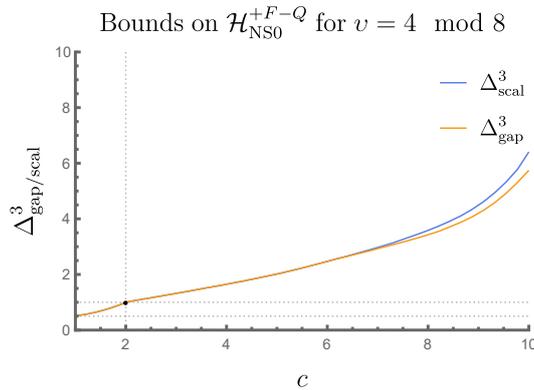


Figure 6.6: Upper bounds for lightest scalars and non-degenerate primaries confronted for theories with anomalies  $\nu = 4 \pmod 8$ .

- The bounds present a single kink for  $c > 1$ , namely a stack of free fermions with  $c = 2$  and anomaly  $\nu = 4 \pmod{8}$ ;
- The other limit cases are again at  $c = 1$ , which reproduces like before known theories;
- The overlap of the bounds for generic non-degenerate states and scalars is present for all values of the anomalies analyzed, with a divergence between the two only for higher values of  $c$ .

### 6.3.2 The remaining cases: bounds on $\mathcal{H}_{\mathbb{I}}^{(\Gamma_0)}$

Here we present the bounds found for the other groups for which we examined the modular transformations so far. In particular, we are going to focus on bounds (from above) on symmetry-preserving scalar operators. We know that via the operator-state correspondence these are associated to states in the Hilbert space  $\mathcal{H}_{\mathbb{I}}^{(\Gamma_0)}$  with  $s = 0$ . Therefore, the linear functional method and the following numerical analysis will be applied restricted to states of this kind.

Since both the order of the group of anomalies and the dimensionality of the modular crossing equations increase drastically by increasing the order of the original group  $G^f$ , we are unfortunately limited in our analysis to select a few, potentially instructive, cases for each of them. However, we note that up to outer automorphisms of  $G^f$ , our choices describe various possible anomaly combinations. This means that the bounds we found for a selected value  $\nu \in \text{Hom}(\Omega_3^{\text{Spin} \times_{\mathbb{Z}_2^f} G^f}, U(1))$  are valid also for any  $\tilde{\nu}$  such that

$$\tilde{\nu} = \hat{\phi}^*(\nu), \quad \phi \in \text{Aut}(G^f). \quad (6.48)$$

Moreover, the bounds for any anomaly  $\nu$  are also equivalent to the bounds associated to  $-\nu$ . Indeed, for any theory with anomaly  $\nu$  we can associate the conjugate theory with complex conjugate partition function. This will have anomaly  $-\nu$ , implying that the numerical bounds for  $\nu$  and  $-\nu$  must be the same.

In the upcoming analysis several kinks appear, suggesting the presence of particular spin-theories which saturate the bounds found. For most of these kinks the range of values of  $(c, \Delta)$  that they represent imply that their description cannot be found in terms of free fermion theories (where one expects  $c, \Delta \in \mathbb{Z}/2$ ). Thus, the most promising candidates of models that might describe them are given by gauged/ungauged WZW models with a dependence on the spin-structure. However, a first analysis shows that between the simplest models with a spin-structure dependence, e.g. of the kind  $SO(N)_{2k+1}$ , no particular candidate seems to be satisfying. Therefore, being these kinks beyond the critical interesting value  $\Delta_{\text{marg}} \equiv 2$  which would hint to the presence of particular relevant/marginal operators and for the absence of further motivations, we do not provide their description.

**Case 1:**  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_4$

Here the values of the anomaly that were studied are

$$(\nu_\omega, \nu_\gamma) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\} \subset \mathbb{Z}_2 \times \mathbb{Z}_8. \quad (6.49)$$

We found smooth bounds when the anomaly is non-trivial, while for the trivial case two kinks appear around the values

$$c = 2.72 \pm 0.08, \quad \Delta = 2.65 \pm 0.02, \quad (6.50)$$

$$c = 5.50 \pm 0.08, \quad \Delta = 3.10 \pm 0.02. \quad (6.51)$$

Note that our bounds for  $c > 1$  are always higher than the critical value  $\Delta_{\text{mar}}$  for marginal operators, see Figures 6.7 and 6.8.

**Case 2:**  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_2$

This is the only symmetry group that we analyzed and for which we have been able to find a range of central charges where it is implied the existence of relevant/marginal operators, see Figures 6.9 and 6.10. We focused on anomalies with values

$$(\nu_1, \nu_2, \nu_\delta) \in \{(0, 0, 0), (0, 0, 1), (1, 0, 1), (1, 1, 0)\} \subset \mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_4. \quad (6.52)$$

The ranges of central charges which admits  $\Delta \leq 2$  are

$$(\nu_1, \nu_2, \nu_\delta) = (0, 0, 0) \quad 1.00 \leq c \lesssim 3.55, \quad (6.53)$$

$$(\nu_1, \nu_2, \nu_\delta) = (0, 0, 1) \quad 1.23 \lesssim c \lesssim 2.90, \quad (6.54)$$

$$(\nu_1, \nu_2, \nu_\delta) = (1, 0, 1) \quad 1.00 \leq c \lesssim 3.17, \quad (6.55)$$

$$(\nu_1, \nu_2, \nu_\delta) = (1, 1, 0) \quad 1.23 \lesssim c \lesssim 2.68. \quad (6.56)$$

We found also the presence of a sharp kink in the non-anomalous case at

$$c = 5.61 \pm 0.08, \quad \Delta = 3.05 \pm 0.02. \quad (6.57)$$

**Case 3:**  $G^f = \mathbb{Z}_2^f \times S_3$

In this case the values of anomalies studied are

$$(p, q) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\} \subset \mathbb{Z}_8 \times \mathbb{Z}_3. \quad (6.58)$$

No particular kink emerged in the analysis, but only few hills that might be worth to further analyze for the interested reader, see Figures 6.11 and 6.12.

**Case 4:**  $G^f = \mathbb{Z}_2^f \times S_4$

In this case we decided to focus on the anomalies that generate the anomaly group, namely

$$(p, q, r, t) \in \{(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (1, 0, 0, 0)\} \subset \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3. \quad (6.59)$$

Interestingly, we have not found any significant bound of the lightest operator for any of these cases.

**Case 5:**  $G^f = \mathbb{Z}_2^f \times D_8$

The cases we focused on are

$$(p, q, r, t) \in \{(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (1, 1, 0, 0)\} \subset \mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2. \quad (6.60)$$

Here for the following values of the central charge and anomaly

$$(p, q, r, t) = (0, 0, 0, 1), \quad c \approx 2.15, \quad (6.61)$$

$$(p, q, r, t) = (1, 1, 0, 0), \quad c \approx 2.3, \quad (6.62)$$

the bound approaches the critical value  $\Delta_{\text{mar}}$ , see Figures 6.13 and 6.14. Moreover, a set of kinks appear for

$$(p, q, r, t) = (0, 0, 0, 0) \quad c = 7.00 \pm 0.08, \quad \Delta = 4.45 \pm 0.02, \quad (6.63)$$

$$(p, q, r, t) = (1, 1, 0, 0) \quad c = 5.7 \pm 0.1, \quad \Delta = 5.4 \pm 0.04. \quad (6.64)$$

**Case 6:**  $G^f = \mathbb{Z}_8^f$

No sharp kinks seem to appear for this symmetry group. Two slightly pronounced couple of valleys at

$$\nu = 0, \quad c \approx 4.0, 6.5, \quad (6.65)$$

$$\nu = 1, \quad c \approx 3.5, 6.0, \quad (6.66)$$

are close to  $\Delta_{\text{mar}}$ , suggesting that for functionals of higher order these points might actually describe some reversed kinks, see 6.15. To this regard, a quick computation tells us that for a set of 4 complex free fermions  $\Psi_i, \Psi_i^*$  associated to the representation  $4[1] \in RO^f(\mathbb{Z}_8^f)$  on the left sector and  $4[3]$  on the right one, the lightest  $\mathbb{Z}_8^f$ -symmetry preserving scalar operators are associated to the states

$$\Psi_i \Psi_i^* \bar{\Psi}_i \bar{\Psi}_i^* |0\rangle, \quad (6.67)$$

thus describing a point  $(c, \Delta) = (4, 2)$  on the plot. However, no other similar descriptions have been founds for the other supposed reversed kinks (6.65) and (6.66).

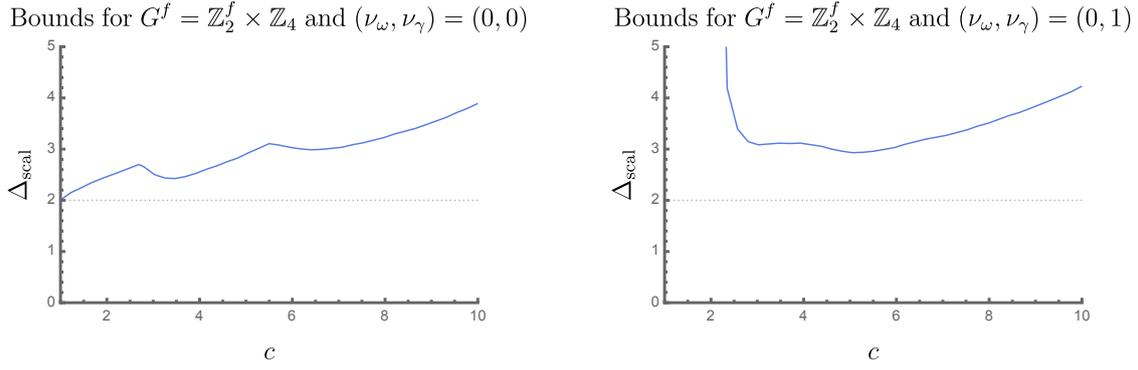


Figure 6.7: Upper bounds on the lightest  $\mathbb{Z}_2^f \times \mathbb{Z}_4$ -preserving scalar operators for anomalies  $(\nu_\omega, \nu_\gamma) = (0, 0)$  (left) and  $(\nu_\omega, \nu_\gamma) = (0, 1)$  (right).

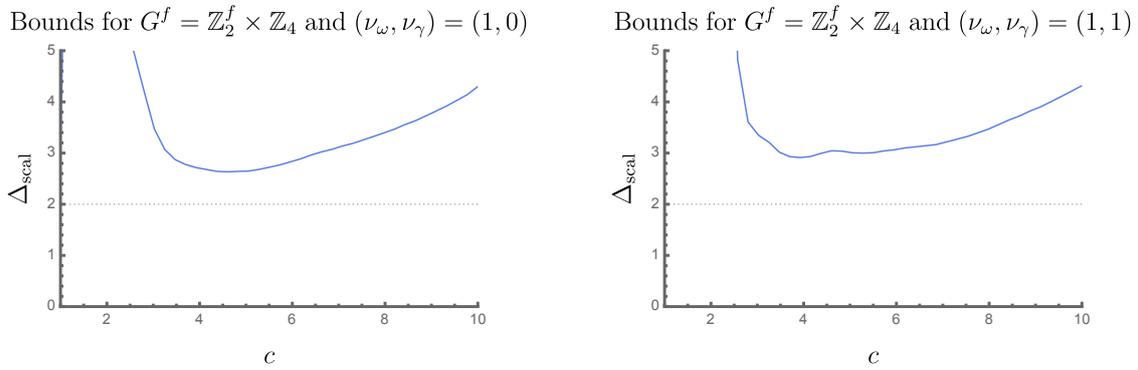


Figure 6.8: Upper bounds on the lightest  $\mathbb{Z}_2^f \times \mathbb{Z}_4$ -preserving scalar operators for anomalies  $(\nu_\omega, \nu_\gamma) = (1, 0)$  (left) and  $(\nu_\omega, \nu_\gamma) = (1, 1)$  (right).

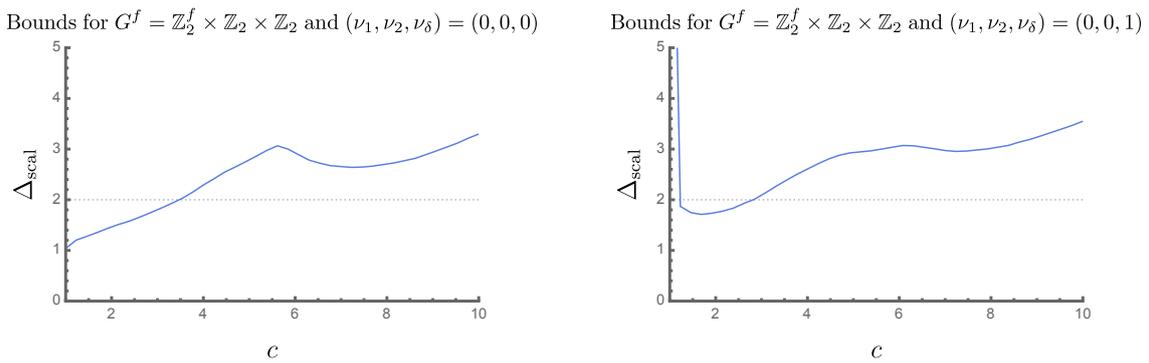


Figure 6.9: Upper bounds on the lightest  $\mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_2$ -preserving scalar operators for anomalies  $(\nu_1, \nu_2, \nu_\delta) = (0, 0, 0)$  (left) and  $(\nu_1, \nu_2, \nu_\delta) = (0, 0, 1)$  (right).

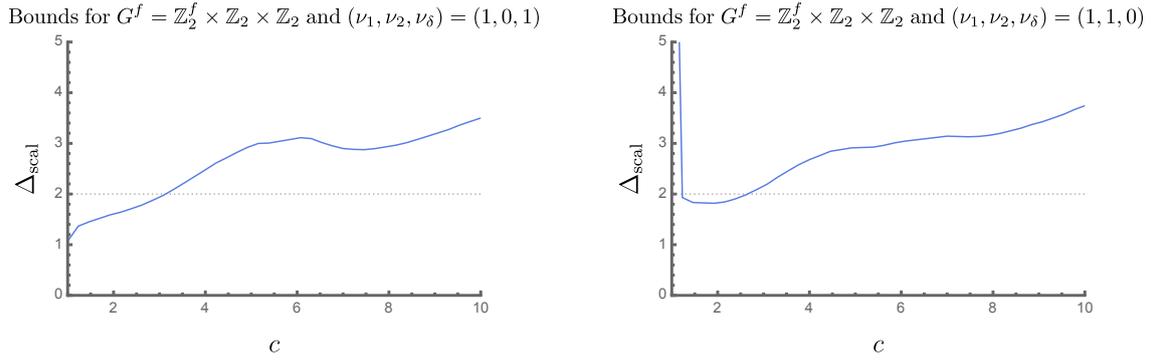


Figure 6.10: Upper bounds on the lightest  $\mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_2$ -preserving scalar operators for anomalies  $(\nu_1, \nu_2, \nu_\delta) = (1, 0, 1)$  (left) and  $(\nu_1, \nu_2, \nu_\delta) = (1, 1, 0)$  (right).

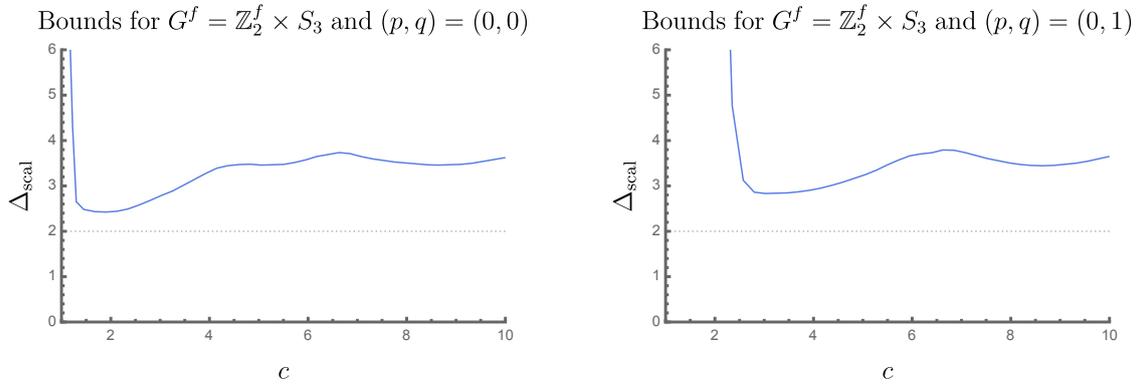


Figure 6.11: Upper bounds on the lightest  $\mathbb{Z}_2^f \times S_3$ -preserving scalar operators for anomalies  $(p, q) = (0, 0)$  (left) and  $(p, q) = (0, 1)$  (right).

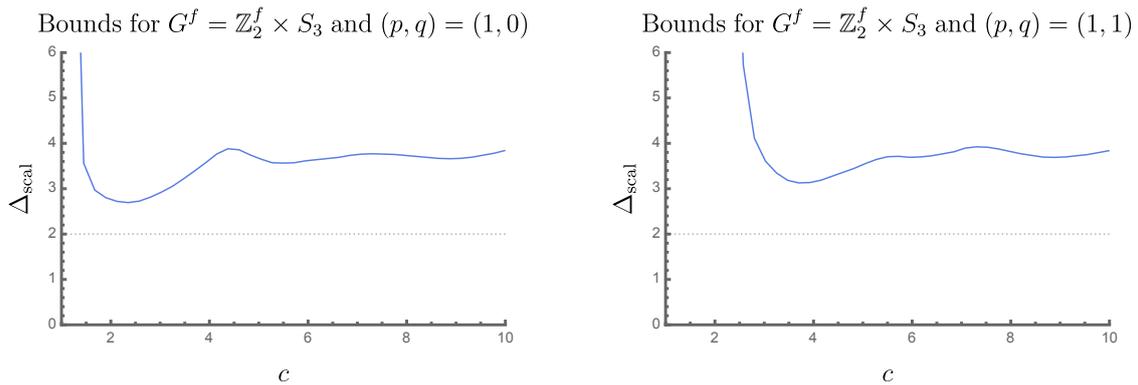


Figure 6.12: Upper bounds on the lightest  $\mathbb{Z}_2^f \times S_3$ -preserving scalar operators for anomalies  $(p, q) = (1, 0)$  (left) and  $(p, q) = (1, 1)$  (right).

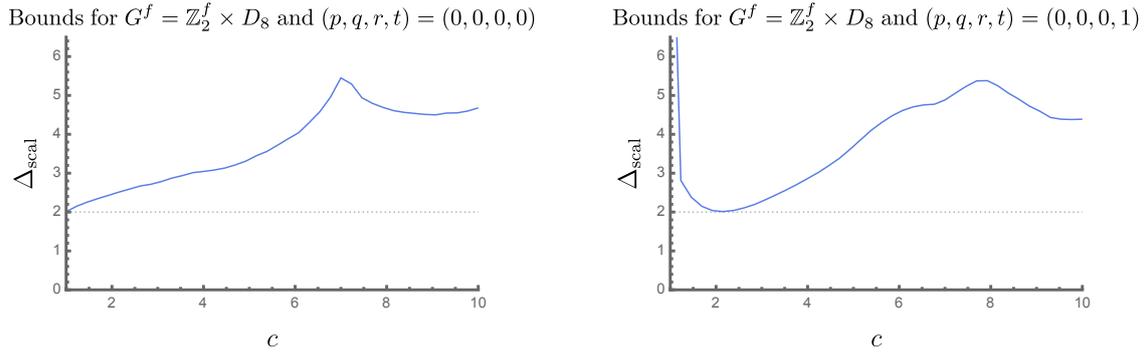


Figure 6.13: Upper bounds on the lightest  $\mathbb{Z}_2^f \times D_8$ -preserving scalar operators for anomalies  $(p, q, r, t) = (0, 0, 0, 0)$  (left) and  $(p, q, r, t) = (0, 0, 0, 1)$  (right).

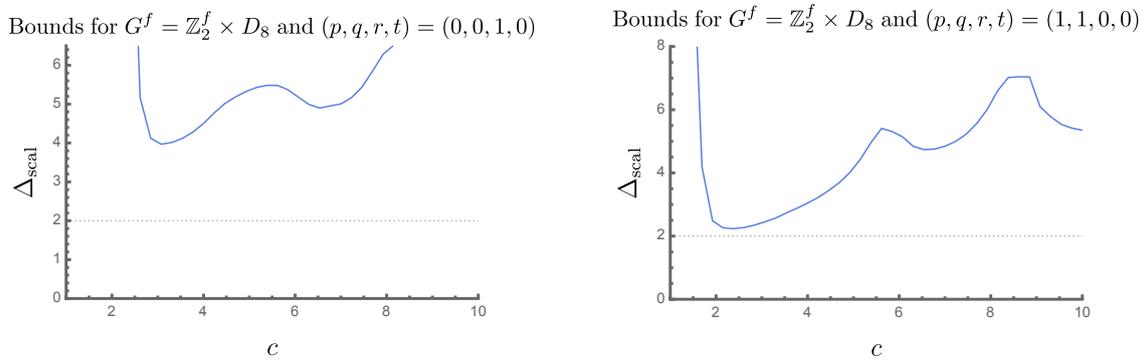


Figure 6.14: Upper bounds on the lightest  $\mathbb{Z}_2^f \times D_8$ -preserving scalar operators for anomalies  $(p, q, r, t) = (0, 0, 1, 0)$  (left) and  $(p, q, r, t) = (1, 1, 0, 0)$  (right).

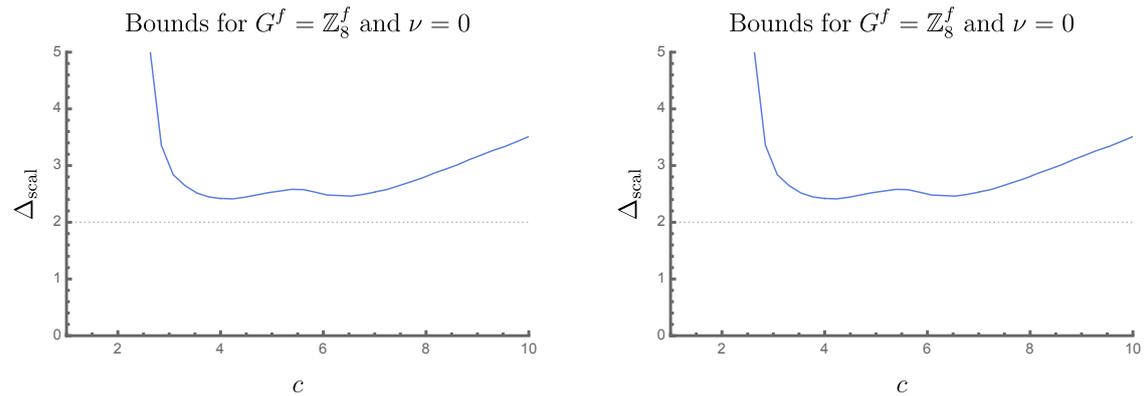


Figure 6.15: Upper bounds on the lightest  $\mathbb{Z}_8^f$ -preserving scalar operators for anomalies  $\nu = 0$  (left) and  $\nu = 1$  (right).

# Appendix A

## Background notions

In this Appendix we quickly review the necessary notions to understand the spectral representation of invertible field theories and other mathematical concepts around it. Our presentation will be very limited with no proof of the statements provided. Therefore, we refer for a more in depth exposition of the topics to the references [9, 79, 86, 89] from which this Appendix is based on.

As mentioned in Chapter 2, cobordism theory can be seen as a generalized cohomology theory. To expand this concept, let us start from the standard notions of homology and cohomology theories. We know that an homology theory can be described by a covariant functor  $E_* : \mathbf{Top}_* \rightarrow \mathbf{Ab}$  from the category  $\mathbf{Top}_*$  of based topological spaces with continuous maps that preserve the base points, to the category  $\mathbf{Ab}$  of graded abelian groups. Analogously, a cohomology theory is described by a correspondent contravariant functor  $E^* : \mathbf{Top}_*^{\text{op}} \rightarrow \mathbf{Ab}$ , where  $\mathbf{Top}_*^{\text{op}}$  is the opposite category of  $\mathbf{Top}_*$ .

Thanks to Brown's representability theorem it is possible to represent a cohomology theory  $E^*(-)$  via a sequence of representing spaces  $E_n$ , such that  $E^n(X) \cong [X, E_n]$ . Moreover, the reduced suspension and based loop space functors, denoted respectively as  $\Sigma(-)$  and  $\Omega(-)$ , are such that

$$[\Sigma X, E_n] \cong [X, \Omega E_n], \quad [X, E_n] \cong E^n(X) \cong E^{n+1}(\Sigma X) \cong [X, \Omega E_{n+1}]. \quad (\text{A.1})$$

In particular, via Yoneda Lemma the last isomorphism corresponds to a weak equivalence  $\omega_n : E_n \xrightarrow{\sim} \Omega E_{n+1}$ . From this relations, it appears clear that the notion of a (co)homology theory is deeply connected to some  $\mathbb{Z}$ -graded set of spaces. We now make this concept more precise. We start by defining the notion of *prespectrum* and *spectrum*.

**Definition A.2.** *A prespectrum is a sequence of pointed spaces  $X_n \in \mathbf{Top}_*$  for  $n \geq 0$  and continuous maps  $\sigma_n : \Sigma X_n \rightarrow X_{n+1}$ , which equivalently correspond to the set of adjoint maps  $\omega_n : X_n \rightarrow \Omega X_{n+1}$ . A prespectrum is then called spectrum if  $\omega_n$  are homeomorphisms. The categories of such objects are denoted respectively as  $\mathbf{PreSp}$  and  $\mathbf{Sp}$ .*

To any topological space  $X$  we can then associate a particular prespectrum, defined as  $\Sigma^\infty X$ , where  $X_n := \Sigma^n X$ . Moreover, this can be composed with the *spectrification functor*  $L : \mathbf{PreSp} \rightarrow \mathbf{Sp}$ , which is the left adjoint of the forgetful functor  $U : \mathbf{PreSp} \rightarrow \mathbf{Sp}$  (see [86]) so that we are able to get a proper spectrum out of any topological space  $X$ . We will implicitly consider this composition to be the case whenever we talk about a suspension spectrum. Where clear, we will also implicitly suppress the notation  $\Sigma^\infty$  and identify with  $X$  both a topological space and its suspension spectrum. Note also that for a spectrum  $K$  and topological space  $X$ , we can naturally define a new spectrum  $X \wedge K$ , which  $n$ -th space is defined as  $(X \wedge K)_n := X \wedge K_n$

With these definitions of spectra at hand, let's now look at other, seemingly unrelated, constructions. In particular, let us start by considering a  $n$ -dimensional real vector bundle  $\nu : V \rightarrow X$  on a topological space  $X$ . From it we can define the associated  $n$ -sphere bundle  $\text{Sph}(\nu) : \text{Sph}(V) \rightarrow X$ , where the fibers are the one-point compactifications of the fibers of  $\nu$ . Note that we make use of this construction also in Section 5.4. If we consider the section  $s : X \rightarrow \text{Sph}(V)$  which maps each point  $x \in X$  to the point at infinity in the fiber  $\text{Sph}(V)_x$  (i.e. the point added from  $V_x$  to  $\text{Sph}(V)_x$ ), we can then define the *Thom space* as

$$\text{Thom}(X; V) := \text{Sph}(V)/s(X). \quad (\text{A.3})$$

These spaces satisfy

$$\text{Thom}(X \times Y; V) = \text{Thom}(X; V) \wedge \text{Thom}(Y; W), \quad (\text{A.4})$$

$$\text{Thom}(X; V \oplus \mathbb{R}^n) = \Sigma^n \text{Thom}(X; V), \quad (\text{A.5})$$

$$\text{Thom}(X; \mathbb{R}^n) = \Sigma^n X_+, \quad (\text{A.6})$$

where  $X_+$  denotes the disjoint union of  $X$  and a point.

Consider now a map<sup>1</sup>  $V : X \rightarrow BO$ . This defines a sequence of maps  $V_n : X_n \rightarrow BO(n)$  by the homotopy pullback squares

$$\begin{array}{ccc} X_n & \longrightarrow & X \\ \downarrow V_n & & \downarrow V \\ BO(n) & \longrightarrow & BO \end{array}$$

By construction  $V_n : X_n \rightarrow BO(n)$  classifies a vector bundle of dimension  $n$  over  $X_n$ . Moreover, the pullback of  $V_{n+1} : X_{n+1} \rightarrow BO(n+1)$  into  $X_n$  induces the map of Thom spaces

$$\Sigma \text{Thom}(X_n; V_n) = \text{Thom}(X_n; V_n \oplus \mathbb{R}) \rightarrow \text{Thom}(X_{n+1}; V_{n+1}), \quad (\text{A.7})$$

which then allows to define the spectrum  $\text{Thom}(X; V)$ , which  $n$ -th space is  $\text{Thom}(X_n; V_n)$ . If  $X = BH$ , the associated Thom spectrum is denoted as  $MH$ .

---

<sup>1</sup>Here  $BO$  is defined as the colimit of  $BO(n)$ , see (3.69).

Note now that also at the spectra level we have that for  $W = P \oplus \mathbb{R}^n$  then

$$\mathrm{Thom}(X; W) = \Sigma^n \mathrm{Thom}(X; P). \quad (\text{A.8})$$

This motivates the definition of a Thom spectra also for virtual bundles. Recall that a virtual bundle  $W$  on  $X$  is a formal difference  $W = P - Q$  of vector bundles  $P$  and  $Q$  over  $X$ , such that  $\dim W := \dim P - \dim Q$ . In particular, if  $\dim W < 0$ , we can define  $\mathrm{Thom}(X; W)$  by choosing a bundle  $Q^\perp$  such that  $Q \oplus Q^\perp \cong \mathbb{R}^k$  for some  $k$  via<sup>2</sup>

$$\mathrm{Thom}(X; W) := \Sigma^{-k} \mathrm{Thom}(X; P \oplus Q^\perp). \quad (\text{A.9})$$

In the particular case of before when  $X = BH$ , the spectrum associated to  $W = -V$  is the *Madsen-Tillmann spectrum* of  $H$  and is denoted  $MTH$ .

At this point we can join all the notion introduced so far. We first introduce the notion of *generalized homology*.

**Definition A.10.** *Let  $K$  be a spectrum and  $X$  a topological space. We define the (unreduced) homology with coefficients in the spectrum  $K$  to be the functor taking the space  $X$  to the abelian groups*

$$H_d(X; K) := \pi_d(X_+ \wedge K). \quad (\text{A.11})$$

Here  $\pi_d$  denotes the stable homotopy groups, defined as

$$\pi_d(X) := \mathrm{colim}_{n \rightarrow \infty} \pi_{d+n} X_n. \quad (\text{A.12})$$

It turns out that the bordism groups  $\Omega_d^H$  of manifolds with  $H$ -structure define a generalized homology theory, where the spectra  $K$  is identified with the Madsen-Tillmann spectra  $MTH$  we introduced [79], i.e.

$$\Omega_d^H(X) \equiv H_d(X; MTH) = \pi_d(X_+ \wedge MTH). \quad (\text{A.13})$$

Let us now consider the functor

$$\begin{aligned} I_A^\bullet(X) : \mathrm{Top}_* &\rightarrow \mathrm{Ab}, \\ X &\mapsto \mathrm{Hom}_{\mathbb{Z}}(\pi_\bullet(\Sigma^\infty X), A) \end{aligned} \quad (\text{A.14})$$

which defines a cohomology theory represented by a spectrum  $I_A$  for any injective abelian group  $A$ . For example, for  $A = \mathbb{Q}$  we get

$$I_{\mathbb{Q}}^\bullet(X) \cong \tilde{H}^\bullet(X; \mathbb{Q}). \quad (\text{A.15})$$

Since also  $\mathbb{Q}/\mathbb{Z}$  is an injective abelian group, one can consider the map  $\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$ , which thanks to Yoneda Lemma provides a map of spectra  $I_{\mathbb{Q}} \rightarrow I_{\mathbb{Q}/\mathbb{Z}}$ . From this, we can define the *Anderson dual spectrum*  $I_{\mathbb{Z}}$  via the exact triangle

$$I_{\mathbb{Z}} \rightarrow I_{\mathbb{Q}} \rightarrow I_{\mathbb{Q}/\mathbb{Z}} \rightarrow \Sigma I_{\mathbb{Z}}. \quad (\text{A.16})$$

---

<sup>2</sup>Up to subtleties, here  $\Sigma^{-1}(-) = \Omega(-)$ ; see [86].

This exact triangle induces a long exact sequence on cohomology given by

$$\cdots \rightarrow I_{\mathbb{Q}}^{\bullet-1}(X) \rightarrow I_{\mathbb{Q}/\mathbb{Z}}^{\bullet-1}(X) \rightarrow I_{\mathbb{Z}}^{\bullet}(X) \rightarrow \dots \quad (\text{A.17})$$

Finally, if  $\pi_{\bullet}(\Sigma^{\infty} X)$  (the *stable* homotopy groups of the spectra associated to  $X$ ) are finitely generated, (A.17) induces the long exact sequence

$$1 \longrightarrow \text{Ext}^1(\pi_{\bullet}(X), \mathbb{Z}) \longrightarrow I_{\mathbb{Z}}^{\bullet+1}(X) \longrightarrow \text{Hom}(\pi_{\bullet+1}(X), \mathbb{Z}) \longrightarrow 1, \quad (\text{A.18})$$

which is precisely (2.71) for  $\Sigma^{\infty} X \mapsto MT\text{Spin} \wedge X_+$  and  $X = BG$ , so that actually  $\Sigma^{\infty} X \mapsto MTH = MT(\text{Spin} \times G)$ .

# Appendix B

## $S$ and $T$ matrices for $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2$

Here we report the computation for the  $S$  and  $T$  matrices.

Let us start from the class  $(0,0)$  and explain what are the NS0 directions we chose to contract in order to define the basis elements of  $Z_i^{(0,0)}$ . Under  $S$  the only element of this vector mapped to itself is  $Z_3^{(0,0)} = Z_{\text{NS0}}^{\text{NS0}}$ , while the others are mapped to themselves under  $S^2$ . This means that we can choose deliberately the direction of contraction for 4 out of these 8 tori in such a way that they are mapped to themselves under  $S^2$  with no additional phase. The basis of the 4 remaining ones will follow by using as reference bordism the  $S$  transformation itself. As a consequence, we are able to fix all the non-zero entries of  $S_{(0,0)}$  to 1 with the exception of  $\{\text{NS0}, \text{NS0}\}$ . However this particular case has no  $\mathbb{Z}_2$  holonomies and no  $\text{pin}^-$  surface. Therefore  $(S_{(0,0)})_3^3 = 1$  as well, which completes  $S_{(0,0)}$ . The basis we chose that satisfy this property is

$$\text{NS0} \begin{array}{|c|} \hline \square \\ \hline \text{---} \\ \hline \end{array}, \quad (\text{B.1})$$

NS0

$$\begin{array}{cccc} \text{R0} \begin{array}{|c|} \hline \square \\ \hline \text{---} \\ \hline \end{array} & \text{NS0} \begin{array}{|c|} \hline \square \\ \hline \text{---} \\ \hline \end{array} & \text{NS0} \begin{array}{|c|} \hline \square \\ \hline \text{---} \\ \hline \end{array} & \text{R1} \begin{array}{|c|} \hline \square \\ \hline \text{---} \\ \hline \end{array}, \\ \text{NS0} & \text{NS1} & \text{R1} & \text{NS1} \end{array}, \quad (\text{B.2})$$

$$\begin{array}{cccc} \text{NS0} \begin{array}{|c|} \hline \square \\ \hline \text{---} \\ \hline \end{array} & \text{NS1} \begin{array}{|c|} \hline \square \\ \hline \text{---} \\ \hline \end{array} & \text{R1} \begin{array}{|c|} \hline \square \\ \hline \text{---} \\ \hline \end{array} & \text{NS1} \begin{array}{|c|} \hline \square \\ \hline \text{---} \\ \hline \end{array}, \\ \text{R0} & \text{NS0} & \text{NS0} & \text{R1} \end{array}, \quad (\text{B.3})$$

where the red dashed lines represent the direction of each torus contracted to a point in the bounding solid 3-torus.

Next we turn to the computation of  $T_{(0,0)}$ . One can see that for the set (B.1)-(B.3) each element of basis  $e_a$  ends up to the proper basis element  $e_{T.a}$  with the exception of two cases, namely for  $(T_{(0,0)})_6^9$  and  $(T_{(0,0)})_8^4$ .

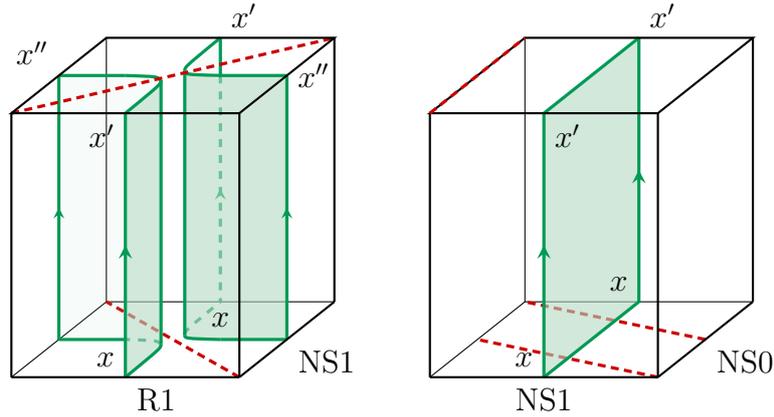


Figure B.1: Left: a mapping torus for which evaluation of  $\nu$  gives the element  $(T_{(0,0)})_6^9$ . Right: a mapping torus for which evaluation of  $\nu$  gives the element  $(T_{(0,0)})_8^4$ .

Let us start from the first. By applying equation (4.26) it follows that the bounding 3-manifold  $Y = MT((T_{(0,0)})_6^9)$  is given by joining the two solid tori:

$$\text{NS1} \begin{array}{c} \square \\ \text{R1} \end{array} \sqcup \overline{\left( \begin{array}{c} \square \\ \text{R1} \end{array} \text{NS1} \right)}. \quad (\text{B.4})$$

At this point one needs to find a surface  $\text{PD}(a_g)$  associated to the 1-cocycle  $a_g \in H^1(Y, \mathbb{Z}_2)$ , which is determined by appropriately extending over the 3<sup>rd</sup> direction the curve Poincaré dual of the restriction  $a_g|_{\{\text{NS1}, \text{R1}\}}$ . This curve  $\text{PD}(a_g|_{\{\text{NS1}, \text{R1}\}})$  is found by a  $T$  transformation of  $\text{PD}(a_g|_{\{\text{NS0}, \text{R1}\}})$ , i.e.

$$\text{NS0} \begin{array}{c} \square \\ \text{R1} \end{array} \xrightarrow{T} \text{NS1} \begin{array}{c} \square \\ \text{R1} \end{array}. \quad (\text{B.5})$$

Therefore the 3-manifold is the one on the left in Figure B.1. One can reach analogous conclusions for  $(T_{(0,0)})_8^4$  and find the surface on the right of Figure B.1.

The closed 3-manifold in discussion is (in both cases)  $\mathbb{RP}^3$ . Since  $\mathbb{RP}^2$  is not orientable, there is no possibility to draw the full  $\text{pin}^-$  surface, but only some open subsets of it. To understand which kind of surface we are talking about, we need to inspect how it twists once we approach the radial directions of the tori that compose the full 3-manifold. We start from the first of (B.5). We can draw the solid torus as on the left of Figure B.2, where the symplectic basis depicted of the 2-torus sections changed, so that one of the generators is the NS0 direction which is being contracted to a point. In the figure the radial direction is represented by the two near-vertical lines, which are identified. It is clear that in this case there is no twist and thus this

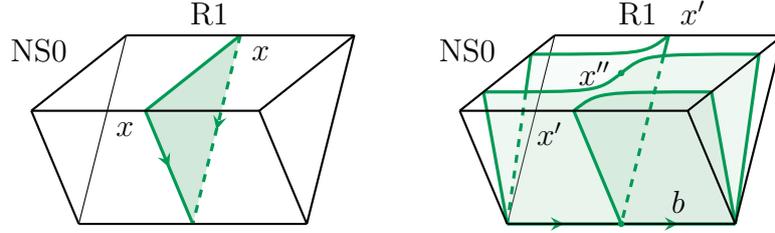


Figure B.2: Representation of the two solid tori describing the manifold (B.4). On the torus on the right is represented the generator  $b$  of  $H^1(\mathbb{RP}^2, \mathbb{Z}_2)$ , given by the contraction of the path going from  $x'$  to  $x''$  to the core.

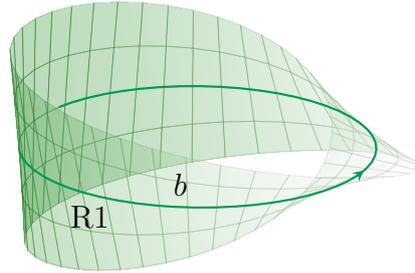


Figure B.3: The Möbius strip embedded into the solid torus depicted on the right side of Figure B.2.

particular subset of the surface can be represented as a disk. By repeating the same reasoning with the second solid torus one arrives to the different situation represented on the right of Figure B.2. Here there is a clear twist of the surface so this part of its 2-skeleton is represented by a Möbius band with core circle  $\mathbb{RP}^1$ . This means that the surface we are working with is  $\mathbb{RP}^2$ , for which ABK is determined by the value  $q(b)$  of its generator  $b$ . To compute it, we look more carefully at the contraction of  $b$  to the core of this solid torus. We can consider as even framing along  $b$  a framing which does a full  $2\pi$  rotation while going around the loop. As one can see from Figure B.3, the normal bundle on the surface does a single negative half twist with respect to such framing, so that

$$q(b) = -1 \quad \Rightarrow \quad \text{ABK} = 7 \pmod{8}. \quad (\text{B.6})$$

By repeating the same reasoning we see that  $\mathbb{RP}^2$  for  $(T_{(0,0)})_8^4$  has instead  $\text{ABK} = 1 \pmod{8}$ . Indeed the procedure is almost the same, with the main difference being that the core of the torus now has NS periodicity, which is equivalent to changing the quadratic enhancement of the generator as  $q(b) \rightarrow q(b) + 2 \pmod{4}$ .

Next we continue with the computation of  $S_{(0/1,1)}$ . Since these classes behave almost identically with the exception of changing the periodicity  $\text{NS1} \leftrightarrow \text{R1}$ , we

will use from now on a special notation to identify the two simultaneously. Depending on the class we are working with, the surfaces  $\{X_1, X_2, X_3\}$  will stand for  $\{\{R0, NS1\}, \{NS1, R0\}, \{NS1, NS1\}\}$  or  $\{\{R0, R1\}, \{R1, R0\}, \{R1, R1\}\}$  for  $(0, 1)$  and  $(1, 1)$  respectively. The  $\text{pin}^-$ -surfaces for which ABK invariant determines the value of the matrix entries  $(S_{(0/1,1)})_j^i$  and  $(T_{(0/1,1)})_j^i$  will instead be denoted as  $\Sigma_{ij}^{S/T}$ .

The first entry one has to compute is  $(S_{(0/1,1)})_2^1$ . The mapping torus  $MT((S_{(0/1,1)})_2^1)$  associated to it is given by applying  $S$  twice to  $X_1$ , which is our reference 2-manifold, and then identifying the boundaries  $X_1 \times \{0\} \sim X_1 \times \{1\}$  (remembering of the additional  $(-1)^{\mathcal{F}}$  action); see Figure B.4. By the same argument of before one can see that the  $\text{pin}^-$  surface we are interested in is just a curve parallel to the time direction of  $X_1$  spanned along the vertical direction. Therefore the surface  $\Sigma_{12}^S$  in question is a Klein bottle  $K = \mathbb{RP}_b^2 \# \mathbb{RP}_{c=a+b}^2$ . Here  $b$  and  $c$  denote the generators of  $H_1(K, \mathbb{Z}_2)$  related to each component of its connected sum.

After identifying the surface, we have to fix the transition function from the top slice  $X_1 \times \{1\}$  to the bottom one  $X_1 \times \{0\}$ . In this case the two canonical basis are related by

$$(\partial'_\chi \ \partial'_\theta) = (\partial_\chi \ \partial_\theta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^T, \quad \begin{pmatrix} \theta' \\ \chi' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \theta \\ \chi \end{pmatrix}. \quad (\text{B.7})$$

Here we denoted  $(\theta, \chi)$  and  $(\theta', \chi')$  the canonical coordinates on the tori  $X_1 \times \{0\}$  and  $X_1 \times \{1\}$  respectively. They are determined by identifying  $(\partial_\theta, \partial_\chi) = (\omega_1, \omega_2)$  and  $(\partial'_\theta, \partial'_\chi) = (\omega'_1, \omega'_2)$ , where  $\omega_1$  and  $\omega_2$  are the lattice generators that define the modular parameter  $\tau = \omega_2/\omega_1$ .

Therefore if we start from a vector  $v \in T_p MT((S_{(1,0/1)})_2^1)$  for any point  $p \in X_1 \times \{0\} \cong X_1 \times \{1\}$  and write it in the canonical basis of  $X_1 \times \{1\}$ , then its entries in the canonical basis used for  $X_1 \times \{0\}$  are given by applying the transition function  $\mathbb{T}_{S^2} = \mathbb{T}_S \mathbb{T}_S = R_3(\pi/2)R_3(\pi/2)$ ,

$$\begin{pmatrix} v_\theta \\ v_\chi \\ v_z \end{pmatrix} = R_3(\pi/2)R_3(\pi/2) \begin{pmatrix} v'_\theta \\ v'_\chi \\ v'_z \end{pmatrix}. \quad (\text{B.8})$$

Here  $z$  denotes the third direction (i.e. the vertical  $\partial_z$ , with positive sign going from bottom to top in Figure B.4) and  $R_i(\alpha)$  denotes the rotation around  $i$ -th axis by angle  $\alpha$ .

The next step is choosing a lift for the transition function to  $\text{Spin}(3) \cong SU(2)$ . Obviously we have

$$\mathbb{T}_S = R_3(\pi/2) \xrightarrow{\text{lift}} \widetilde{\mathbb{T}}_S^\pm = \pm e^{-i\frac{\pi}{4}\sigma_3}. \quad (\text{B.9})$$

Since it will be useful also for the next computations<sup>1</sup>, we choose from now on to use

<sup>1</sup>Note: the choice of the lift do not actually play any significant role as long as one is consistent in using it. A different choice of the lift for  $S$  transformation is equivalent to the redefinition  $S \rightsquigarrow S(-1)^{\mathcal{F}}$  of the generator  $S$  in the metaplectic group (4.1), and similarly for  $T$  transformation. Such redefinitions in general change the group relations. We will check a posteriori that our choice of the lifts is consistent with the relations presented in (4.1).

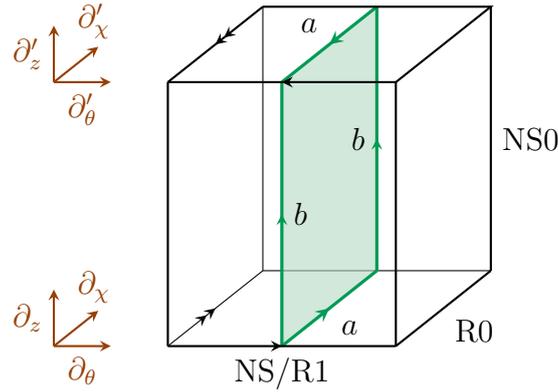


Figure B.4: Mapping torus  $MT((S_{(0/1,1)})_2^1)$ : the bottom slice is  $X_1 \times \{0\}$ , while the top one is the image of  $S^2$ , i.e.  $X_1 \times \{1\}$ . The figure also displays cycles  $a$  and  $b$  along which one moves to determine the ABK invariant of the Klein bottle.

the lift  $\tilde{\mathbb{T}}_S := \tilde{\mathbb{T}}_S^+$ .

Lastly, for  $(0, 1)$  the transition functions in  $\text{Spin}(2)$  for the identifications of the points  $(\theta + 1, \chi) \sim (\theta, \chi)$  and  $(\theta, \chi + 1) \sim (\theta, \chi)$  are respectively  $-\text{id}$  and  $+\text{id}$ , which are properly lifted to  $\mp \text{id} \in \text{Spin}(3) \cong \text{SU}(2)$  for the mapping torus. For  $(1, 1)$  we do not have to keep track of this since it is always  $+\text{id}$ .

At this point we can compute the value of the enhancement  $q$  for the 1-cycles  $a$  and  $b$ .

- For the cycle  $a$  a framing is given by doing a  $2\pi n$  rotation while going from the point  $x \in X_1 \times \{0\}$  with coordinates  $(\theta, \chi) = (1/2, 0)$  to itself along the direction depicted. This means that a vector  $v \in T_x MT((S_{(0/1,1)})_2^1)$  goes back to itself via

$$v' = R_2(2\pi n)v = v. \quad (\text{B.10})$$

The lift in  $\text{SU}(2)$  is  $(-\text{id})^n$ , which implies that  $n = 1$  is needed for the framing to be even. In this way the normal bundle is doing two negative half twists around the framing chosen and  $q(a) = 2 \pmod 4$ .

- For the cycle  $b$  we start again from the point  $x$ . Going along  $b$  the generic vector does the transformation

$$v' = \underbrace{R_3(\pi/2)^2}_{\mathbb{T}_{S^2}} R_3((2n + 1)\pi)v. \quad (\text{B.11})$$

The ending point is  $x'$ , with coordinates  $(\theta', \chi') = (1/2, 0)$ , equivalent to  $(\theta, \chi) = (-1/2, 0)$ . Thus to be back at  $x$  one has an additional  $-\text{id}$  transformation to apply in the case of  $\{R0, \text{NS}1\}$ . We will keep track of it by writing it inside square brackets.

In  $SU(2)$  this means that the transformation is

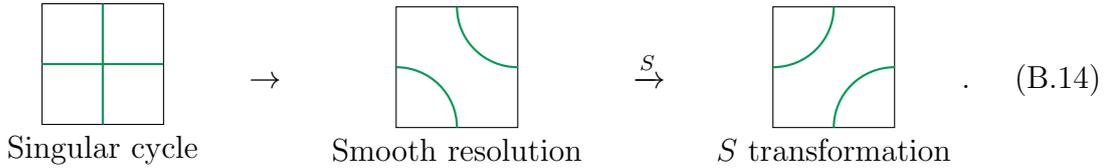
$$[-\text{id}] \underbrace{(-\text{id})}_{(-1)^{\mathcal{F}}} \underbrace{e^{-i\frac{\pi}{2}\sigma_3}}_{\tilde{\tau}_{S^2}} e^{-i(2n+1)\frac{\pi}{2}\sigma_3} = (-\text{id})^{n[+1]}. \quad (\text{B.12})$$

Consider now the two different classes. For  $(1, 1)$  the even framing is found for  $n = 1$ . In this case the framing is then defined by doing three *positive*  $\pi$  rotations, which means the normal bundle is doing three *negative* half rotations and  $q(b) = -3 \pmod 4 = 1 \pmod 4$ . For  $(0, 1)$  instead we have  $n = 0$  and accordingly  $q(b) = 3 \pmod 4$ .

Now we can use (4.8) and find that  $q(a) = q(b)$ . Therefore we arrive to the values

$$(S_{(1,1)})_2^1 = i^\nu, \quad (S_{(0,1)})_2^1 = (-i)^\nu. \quad (\text{B.13})$$

The next matrix element we are going to compute is  $(S_{(0/1,1)})_3^3$ . In this case the manifold describes a single  $S$  transformation of  $X_3$ . Proceeding as usual, the  $\text{pin}^-$  surface is a smooth curve Poincaré dual to the  $\mathbb{Z}_2$  holonomies spanned along the vertical direction. In this case one has to pay attention when choosing such curve. Indeed, this amounts to choosing a resolution of its singular representative, which in general will not be invariant under  $S$ . From consistency with (4.21), (4.23) we have



The diagram consists of three square boxes connected by arrows. The first box, labeled 'Singular cycle', shows a square with a green cross in the center. An arrow points to the second box, labeled 'Smooth resolution', which shows a square with four green quarter-circles in the corners, meeting at the center. A second arrow, labeled with a script 'S', points to the third box, labeled 'S transformation', which shows a square with four green quarter-circles in the corners, but they are now separated from the center, representing a resolution of the singular cycle after an S transformation. The equation number (B.14) is on the right.

With this fixed, it follows that  $\text{PD}(a_g)$  is the saddle surface in Figure B.5, represented by the string of 1-chains

$$acb^{-1}db^{-1}cad \sim eefh, \quad (\text{B.15})$$

where  $e = cb^{-1}db^{-1}$ ,  $f = bd^{-1}$  and  $h = da$ . Thus  $\Sigma_{33}^S = \mathbb{RP}_e^2 \# \mathbb{RP}_f^2 \# \mathbb{RP}_h^2$ , where again the indices represent the generators of the first  $\mathbb{Z}_2$  homology group of each  $\mathbb{RP}^2$  component. By knowing the value of  $q$  for three independent elements in  $H^1(\Sigma_{33}^S, \mathbb{Z}_2)$  we can determine the value of its ABK invariant.

In this case the change of coordinates in  $\mathbb{T}^2 \times \{0\} \sim \mathbb{T}^2 \times \{1\}$  is

$$\begin{pmatrix} \partial'_\chi & \partial'_\theta \end{pmatrix} = \begin{pmatrix} \partial_\chi & \partial_\theta \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^T, \quad \begin{pmatrix} \theta' \\ \chi' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \chi \end{pmatrix}, \quad (\text{B.16})$$

while the transition functions in  $SU(2)$  for the identifications  $(\theta + 1, \chi) \sim (\theta, \chi)$  and  $(\theta, \chi + 1) \sim (\theta, \chi)$  are both  $-\text{id}$  for  $(0, 1)$  and  $+\text{id}$  for  $(1, 1)$ .

We now compute the value of  $q$  for the cycles  $h$ ,  $f$  and  $e + f$ .

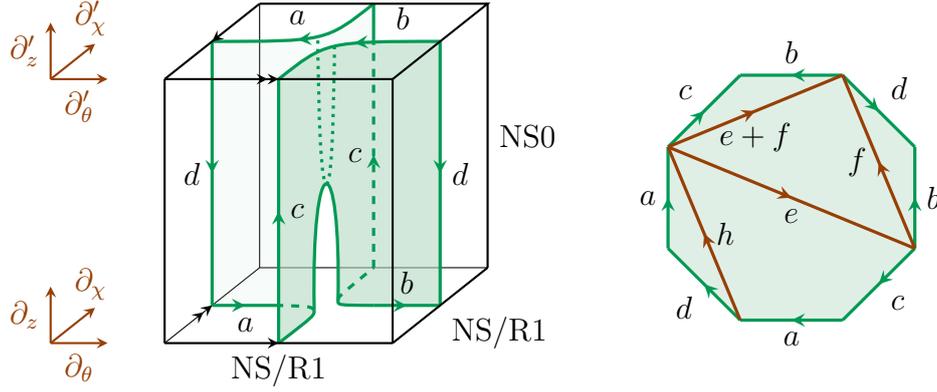


Figure B.5: Left: the mapping torus  $MT((S_{(0/1,1)})_3^3)$  with the embedded surface  $\Sigma_{33}^S$ . It is understood we are using the same convention of the other figures. Right: the polygon representation of  $\Sigma_{33}^S$  and its generators.

- We can move along the cycle  $h$  in 4 steps. First we start at  $x$  with coordinates  $(\theta, \chi) = (1/2, 0)$  and arrive to the point  $y$  with  $(\theta, \chi) = (0, 1/2)$  by moving along  $a^{-1}$  and doing a  $\pi/2$  rotation around the 3<sup>rd</sup> axis  $\partial_z$ . Then at  $y$  the cycle can be smoothed so that by following it we do a rotation of  $\pi/2$  around  $\partial_\chi$ . We then go to  $x'$  with coordinates  $(\theta', \chi') = (0, 1/2)$  along  $d^{-1}$  by doing a  $(2n+1)\pi$  rotation around  $\partial_z$ . Finally, we do a  $\pi/2$  rotation around  $\partial'_\chi$  before applying the transition function. Since  $x' = (\theta = -1/2, \chi = 0)$ , in order to go completely back to the starting basis of the spin bundle on  $x$  there is an additional  $[-\text{id}] \in SU(2)$  for the manifold  $\{\text{NS1}, \text{NS1}\}$ .

The total transformation for a vector  $v$  is

$$v' = \underbrace{R_3(\pi/2) R_2(\pi/2) R_3((2n+1)\pi) R_2(\pi/2) R_3(\pi/2)}_{\mathbb{T}_S} v = v, \quad (\text{B.17})$$

so that the requirement of an even framing is equivalent to the condition

$$[-\text{id}] \underbrace{(-\text{id})}_{(-1)^{\mathcal{F}}} \underbrace{e^{-i\frac{\pi}{4}\sigma_3} e^{-i\frac{\pi}{4}\sigma_2} e^{-i\frac{2n+1}{2}\pi\sigma_3} e^{-i\frac{\pi}{4}\sigma_2} e^{-i\frac{\pi}{4}\sigma_3}}_{\mathbb{T}_S} = (-\text{id})^{n[+1]} \stackrel{!}{=} -\text{id}. \quad (\text{B.18})$$

- Dividing in similar steps the framing of cycle  $f$ , a generic  $v \in T_x MT((S_{0/1,1})_2^2)$  transforms as

$$v' = \underbrace{R_3(\pi/2) R_2(-\pi/2) R_3((2n+1)\pi) R_2(-\pi/2) R_3(\pi/2)}_{\mathbb{T}_S} v = v. \quad (\text{B.19})$$

With the aid of Figure B.5 it is clear that by starting at  $x$  considered with coordinates<sup>2</sup>  $(\theta = 1/2, \chi = 1)$ , then we arrive at  $x'$  with coordinates  $(\theta', \chi') =$

<sup>2</sup>Here and in the following we will sometimes rewrite the same points with different coordinates,

$(1, 1/2) \sim (\theta, \chi) = (-1/2, 1)$ . Thus the lift of the full rotation in  $\text{Spin}(3)$  gets again an additional  $[-\text{id}]$  in the case  $\{\text{NS1}, \text{NS1}\}$ . Therefore the even framing condition for  $f$  is

$$[-\text{id}] \underbrace{(-\text{id})}_{(-1)^{\mathcal{F}}} \underbrace{e^{-i\frac{\pi}{4}\sigma_3}}_{\tilde{\Gamma}_S} e^{i\frac{\pi}{4}\sigma_2} e^{-i\frac{2n+1}{2}\pi\sigma_3} e^{i\frac{\pi}{4}\sigma_2} e^{-i\frac{\pi}{4}\sigma_3} = (-\text{id})^{n+1} \stackrel{!}{=} -\text{id}. \quad (\text{B.20})$$

– Finally we look at the cycle  $e + f \sim cb^{-1}$ . Starting at  $x$  with coordinates  $(\theta, \chi) = (1/2, 0)$  a vector  $v$  transforms as

$$v' = \underbrace{R_3(\pi/2) R_2(-\pi/2) R_3(-\pi/2) R_1(-\pi/2) R_3(2\pi n)}_{\Gamma_S} = v. \quad (\text{B.21})$$

The arrival point of this cycle now is  $x'$  with coordinates  $(\theta', \chi') = (1, 1/2) \sim (\theta, \chi) = (-1/2, 1)$ , so there is no additional sign to keep track of. Therefore in  $\text{Spin}(3)$  we have

$$\underbrace{(-\text{id})}_{(-1)^{\mathcal{F}}} \underbrace{e^{-i\frac{\pi}{4}\sigma_3}}_{\tilde{\Gamma}_S} e^{i\frac{\pi}{4}\sigma_2} e^{i\frac{\pi}{4}\sigma_3} e^{-i\frac{\pi}{4}\sigma_1} e^{i\pi n\sigma_3} = (-\text{id})^{n+1} \stackrel{!}{=} (-\text{id}). \quad (\text{B.22})$$

From these results we see that for the class  $(1, 1)$  one must set  $n = 1$  for the loops  $h$  and  $f$ . Thus, in both cases the normal bundle does three negative half twists with respect to the framing chosen, or, in other words,  $q(f) = q(h) = -3 \pmod{4} = 1 \pmod{4}$ . Instead,  $n = 0$  is necessary for  $e + f$ , so that on this cycle the normal bundle does not do any twist and  $q(e) = q(f) + 2 \pmod{4}$ .

By the same logic, for the class  $(0, 1)$  one must impose  $n = 0$  for all the loops  $h$ ,  $f$  and  $e + f$ . This means that in this case the enhancement has the values  $q(h) = q(f) = 3 \pmod{4}$  and  $q(e) = q(f) + 2 \pmod{4}$ .

Therefore the final result is

$$(S_{(0,1)})_2^2 = e^{-i\frac{\pi}{4}\nu}, \quad (S_{(1,1)})_2^2 = e^{i\frac{\pi}{4}\nu}. \quad (\text{B.23})$$

The last two mapping tori we need to look at are the ones related to the entries  $(T_{(0/1,1)})_2^2$  and  $(T_{(0/1,1)})_1^3$ . By applying (4.26) it follows that the manifold describing the first of these two entries is given by the  $\mathcal{T}$ -bordism of  $X_2$  with the identification of the boundaries  $X_2 \times \{0\} \sim X_2 \times \{1\}$  like in the previous cases. We can see such manifold in Figure B.6, where we depicted also  $\Sigma_{22}^T$ , found with the same reasoning of before. In this case the surface is just a torus represented by the string of 1-cycles

$$acb^{-1}a^{-1}c^{-1}b \sim e^{-1}ded^{-1}, \quad (\text{B.24})$$

where  $d = a^{-1}c^{-1}$  and  $e = bc^{-1}$  are a symplectic basis of  $H^1(\Sigma_{(0/1,1)}^T, \mathbb{Z}_2)$ . The

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making use of the identification  $(\theta, \chi) \sim (\theta + n, \chi + m)$  for  $n, m \in \mathbb{Z}$ . As long as one is consistent, this has no effect other than making easier the visualization of the path by looking at the mapping tori.

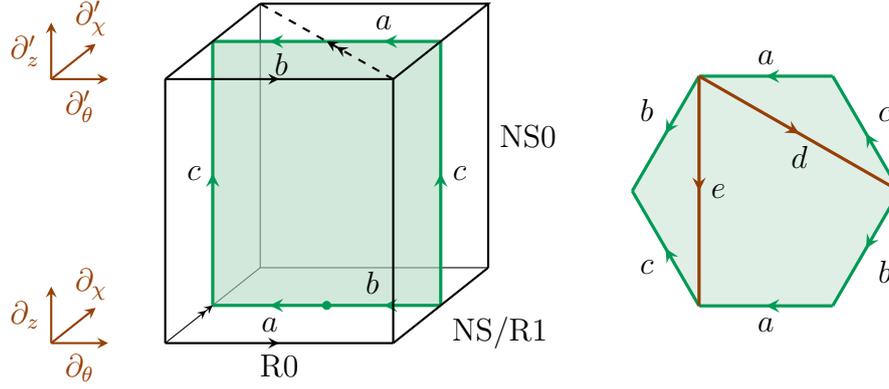


Figure B.6: Left: the mapping torus  $MT((T_{(0/1,1)}^2)_2)$  with the embedded surface  $\Sigma_{22}^T$ . Right: the plain representation of  $\Sigma_{22}^T$  and its generators.

next step is defining the transition function which changes the canonical basis of the tangent space given by the coordinates used in the top slice  $X_2 \times \{1\}$  to the ones of the bottom slice  $X_2 \times \{0\}$ . Here the transition function will be lifted to elements of  $\widetilde{\text{SL}}(3, \mathbb{R})$ , the double cover of  $\text{SL}(3, \mathbb{R})$ . Under a  $T$  transformation

$$\begin{pmatrix} \partial'_x & \partial'_\theta \end{pmatrix} = \begin{pmatrix} \partial_x & \partial_\theta \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^T, \quad \begin{pmatrix} \theta' \\ \chi' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \theta \\ \chi \end{pmatrix}. \quad (\text{B.25})$$

If we start from a vector  $v \in T_p MT((T_{(0/1,1)}^2)_2)$  with  $p \in X_2 \times \{0\} \simeq X_2 \times \{1\}$  and write it in the canonical basis of  $X_2 \times \{1\}$ , then by applying  $\mathbb{T}_T = H(1)$ , where

$$H(t) := \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{B.26})$$

we have:

$$\begin{pmatrix} v_\theta \\ v_\chi \\ v_z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v'_\theta \\ v'_\chi \\ v'_z \end{pmatrix}. \quad (\text{B.27})$$

For the lift  $\mathbb{T}_T \rightarrow \tilde{\mathbb{T}}_T \in \widetilde{\text{SL}}(3, \mathbb{R})$  one notes that  $\text{SL}(3, \mathbb{R})$  can be continuously retracted to  $\text{SO}(3)$  via the Gram-Schmidt procedure. If  $\gamma : \text{SL}(3, \mathbb{R}) \rightarrow \text{SO}(3)$  is such retraction, then its lifting  $\tilde{\gamma}$  has to make the following diagram commute:

$$\begin{array}{ccc} \widetilde{\text{SL}}(3, \mathbb{R}) & \xrightarrow{\tilde{\gamma}} & \text{SU}(2) \\ \pi \downarrow & & \downarrow \pi \\ \text{SL}(3, \mathbb{R}) & \xrightarrow{\gamma} & \text{SO}(3) \end{array} \quad (\text{B.28})$$

Here  $\pi$  denotes the projections. It follows that  $\tilde{\gamma}$  maps the center  $\{\pm \text{id}\}$  to itself<sup>3</sup>. From the fact that  $\gamma(H(t)) = \text{id} \forall t \in \mathbb{R}$ , then the lifting of the transition function can be  $\tilde{\Gamma}_T = \tilde{H}_\pm(1)$ , defined by the property  $\tilde{\gamma}(\tilde{H}_\pm(1)) = \pm \text{id} \in SU(2)$ . For consistency with the choice made for  $S$ , we choose<sup>4</sup> the lift  $\tilde{\Gamma}_T = \tilde{H}_+(1)$ .

We can now turn to computing the value of  $q$  for the cycles  $d$  and  $e$ .

- For the loop  $e$  we start from  $x$ , which has the coordinates  $(\theta, \chi) = (0, 1/2)$ . We then go to a point  $p$  in the middle of the segment  $c$  connecting  $x$  to  $y = (\theta' = 0, \chi' = 1/2)$  while doing a  $2\pi n$  rotation along the direction  $\partial_z$ . From here we reach  $y$  by doing a continuous transformation  $H(-t)$  with  $t \in [0, 1]$ . At this point the cycle can be smoothed so that by following it we do a  $\pi$  rotation along the  $\partial'_x$ . Then  $x' = (\theta' = 1/2, \chi' = 1/2)$  is reached without doing any rotation to the reference frame. Applying the transition function we are back to  $x$  with the original basis of the tangent space. So taking everything into account we see that a vector  $v$  goes to itself by

$$v' = \underbrace{H(1)}_{\tilde{\Gamma}_T} R_2(-\pi) R_2(\pi) H(-1) R_3(2\pi n) v = v. \quad (\text{B.29})$$

The lift of this transformation is<sup>5</sup>

$$\underbrace{(-\text{id})}_{(-1)^{\mathcal{F}}} \underbrace{\tilde{H}_+(1)}_{\tilde{\Gamma}_T} e^{i\frac{\pi}{2}\sigma_2} e^{-i\frac{\pi}{2}\sigma_2} \tilde{H}(-1) e^{-in\pi} \xrightarrow{\tilde{\gamma}} \underbrace{\tilde{\gamma}(\tilde{H}_-(1))}_{\tilde{\gamma}((-1)^{\mathcal{F}}\tilde{\Gamma}_T)} \tilde{\gamma}(\tilde{H}(-1)(-\text{id})^n) = (-\text{id})^{n+1}. \quad (\text{B.30})$$

For the deformation contraction one has to group together all the components that define the transition function between the two bases of the tangent space of  $X_2$ , so in this case  $(-1)^{\mathcal{F}}$  and  $\tilde{\Gamma}_T$ , and the transformations done by moving along the loop. The equality on the right side follows by remembering that  $\tilde{H}(-1)$  is the ending point of a continuous lift  $\tilde{H}(-t)$  with  $t \in [0, 1]$ ,  $\tilde{H}(0) = \text{id}$  and using also the fact that for any element  $A \in \widetilde{\text{SL}}(3, \mathbb{R})$  it holds  $\tilde{\gamma}(AB) = \tilde{\gamma}(A)\tilde{\gamma}(B)$  if  $B \in Z(\widetilde{\text{SL}}(3, \mathbb{R}))$ . At this point we can simply set  $n = 0$  to have an even framing, which means the normal bundle does not twist with respect to it and  $q(e) = 0 \pmod{4}$ .

- With the same approach along the cycle  $c$  any vector  $v$  rotates by

$$\underbrace{H(1)}_{\tilde{\Gamma}_T} R_2(\pi) R_2(-\pi) H(-1) R_3(2\pi n) = \text{id}. \quad (\text{B.31})$$

---

<sup>3</sup>The commutativity of the diagram would still be satisfied if the center of  $\widetilde{\text{SL}}(3, \mathbb{R})$  were mapped to the identity in  $SU(2)$ . However, one needs to require also surjectivity, otherwise  $\tilde{\gamma}(\widetilde{\text{SL}}(3, \mathbb{R})) = SO(3)$ .

<sup>4</sup>The other choice is still legitimate, but would correspond to a redefinition  $T \rightsquigarrow T(-1)^{\mathcal{F}}$ .

<sup>5</sup>In  $\widetilde{\text{SL}}(3, \mathbb{R})$  we formally represent elements of  $SU(2) \subset \widetilde{\text{SL}}(3, \mathbb{R})$  by their matrix representation in the fundamental of  $SU(2)$ .

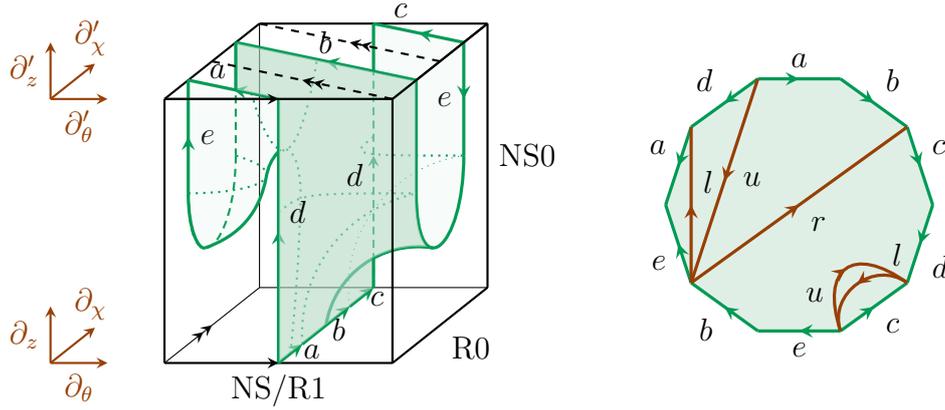


Figure B.7: Left: the mapping torus  $MT((T_{(0/1,1)}^T)_1^3)$  with the embedded surface  $\Sigma_{31}^T$ . Right: the polygon representation of  $\Sigma_{31}^T$  and its generators.

The deformation contraction of its lift gives again the same element of (B.30), so we can conclude that  $q(d) = q(e)$ .

This means that for both classes  $(0/1, 1)$  we have

$$(T_{(0,1)})_2^2 = (T_{(1,1)})_2^2 = 1. \quad (\text{B.32})$$

Finally, we are left with the computation of  $(T_{(0/1,1)}^T)_1^3$ . In this case the 3-manifold is given by identifying the two boundaries  $X_1$  of the  $T^2$ -bordism  $\mathcal{T}_{\{R0, NS/R1\}}^{\{R0, NS/R1\}} : \{R0, NS/R1\} \rightarrow \{R0, NS/R1\}$ . The surface  $\Sigma_{13}^T$  is found by looking for a smooth interpolation between its projection on  $X_1 \times \{0\}$  and  $X_1 \times \{1\}$ . A schematic representation is given by

$$\begin{array}{ccccccc}
 \begin{array}{|c|} \hline \square \\ \hline \end{array} & \rightarrow & \begin{array}{|c|} \hline \square \\ \hline \end{array} & \rightarrow & \begin{array}{|c|} \hline \square \\ \hline \end{array} & \rightarrow & \begin{array}{|c|} \hline \square \\ \hline \end{array} \\
 X_1 \times \{0\} & & X_1 \times \{t\} & & X_1 \times \{t'\} & & X_1 \times \{1\}
 \end{array} \quad (\text{B.33})$$

with  $0 < t < t' < 1$ , while in Figure B.7 we can see its embedding in the 3-manifold. As usual we can use some chains to write the string presentation of the surface, which in this case is

$$abcdc^{-1}ebeat^{-1}d^{-1} \sim rruull, \quad (\text{B.34})$$

with  $r = abea^{-1}d^{-1}$ ,  $u = dae^{-1}c$  and  $l = c^{-1}ea^{-1}$ . Therefore  $\Sigma_{13}^T = \mathbb{RP}_r^2 \# \mathbb{RP}_u^2 \# \mathbb{RP}_l^2$ . To find the value of its ABK invariant we compute the enhancement  $q$  for  $l$ ,  $l + u$  and  $l + u + r$ . For our purpose it comes at hand to consider the portions of these cycles on  $X_1 \times \{1\}$  as if they were instead on the slice  $X_1 \times \{t'\}$ . This does not pose any problem, because the value of  $q$  is robust under any continuous deformation of the surface.

- We begin with  $l \sim ae^{-1}c$ , which we divide as usual in various pieces. From the starting point  $p$  on  $X_1 \times \{1\}$  with coordinates  $(\theta', \chi') = (1/2, 0)$  we go to  $q = (\theta' = 0, \chi' = 1/4)$  while doing a  $\pi/2$  rotation around  $\partial_z$ . At  $q$  we smooth the cycle and do a  $-\pi/2$  rotation around  $\partial'_\chi$ . From  $q$  to  $r = (\theta' = 0, \chi' = 3/4)$  along  $e^{-1}$  we do first a  $\pi$  rotation around the 1<sup>st</sup> axis describing a u-turn, followed by a  $(2n+1)\pi$  rotation around  $\partial_z$ . Like before, at  $r$  we have to do a second  $-\pi/2$  rotation around  $\partial'_\chi$  and, while going back to  $p$ , one last  $-\pi/2$  rotation around  $\partial'_z$ . This means that  $v \in T_p MT((T_{0/1,1})_1^3)$  transforms as

$$v' = R_3(-\pi/2)R_2(-\pi/2)R_3((2n+1)\pi)R_1(\pi)R_2(-\pi/2)R_3(\pi/2)v = v. \quad (\text{B.35})$$

The lift is given by

$$[-\text{id}]e^{+i\frac{\pi}{4}\sigma_3}e^{i\frac{\pi}{4}\sigma_2}e^{-i\frac{(2n+1)\pi}{2}\sigma_3}e^{-i\frac{\pi}{2}\sigma_1}e^{i\frac{\pi}{4}\sigma_2}e^{-i\frac{\pi}{4}\sigma_3} = (-\text{id})^{n[+1]}, \quad (\text{B.36})$$

where the first  $[-\text{id}]$  is present only for the bordism class  $(0, 1)$ . This means that  $n = 1$  and  $q(l) = -3 \pmod 4 = 1 \pmod 4$  for  $(1, 1)$  and  $n = 0$ ,  $q(l) = 3 \pmod 4$  for  $(0, 1)$ .

- For the cycle  $l+u+r \sim e^{-1}b^{-1}$  we start from  $q = (\theta' = 0, \chi' = 1/4)$  on  $X_1 \times \{1\}$ . Here running along  $e^{-1}$  we do the same rotations as before. The difference in the computation is that once we arrive at the point  $r$  we have instead to do a  $\pi/2$  rotation around  $\partial'_\chi$ , then go back to  $q$  and conclude repeating doing a second  $\pi/2$  rotation. The final result is that from  $v \in T_q MT((T_{0/1,1})_1^3)$  we arrive to

$$v' = R_2(\pi/2)R_2(\pi/2)R_3((2n+1)\pi)R_1(\pi)v = v. \quad (\text{B.37})$$

This is essentially the exact expression of (B.35) up to changing the orientation of the rotations around the 2<sup>nd</sup> axis, so it is no surprise that the lift in  $SU(2)$  gives  $(-\text{id})^{n+1[+1]}$ . Therefore  $q(l+u+r) = 3 \pmod 4$  for  $(1, 1)$  and  $q(l+u+r) = 1 \pmod 4$  for  $(0, 1)$ .

- Finally it is the turn of  $l+u \sim d$ . In this case going from the  $X_1 \times \{0\}$  to  $X_1 \times \{1\}$  we do a  $2\pi n$  rotation around  $\partial_z$ , followed by a continuous transformation  $H(-2t)$ ,  $t \in [0, 1]$ . Thus a vector transforms as

$$v' = \underbrace{H(1)H(1)}_{\Gamma_{T^2}} H(-2)R_3(2\pi n)v = v. \quad (\text{B.38})$$

The lift of the transformation in  $\widetilde{SL}(3, \mathbb{R})$  is then

$$\underbrace{(-\text{id})}_{(-1)^{\mathcal{F}}} \widetilde{H}_+(1)\widetilde{H}_+(1)\widetilde{H}(-2)e^{-i\pi n\sigma_3} = (-\text{id})^{n+1}. \quad (\text{B.39})$$

We conclude that  $n = 0$  and  $q(l+u) = 0 \pmod 4$ .

With these explicit computations one can arrive to the last results that determine the  $T$  matrix, i.e.

$$(T_{(0,1)})_1^3 = e^{i\frac{\pi}{4}\nu}, \quad (T_{(1,1)})_1^3 = e^{-i\frac{\pi}{4}\nu}. \quad (\text{B.40})$$

# Appendix C

## Real character tables of groups

Here we report the *real*<sup>1</sup> character table of the non-abelian groups needed for our analysis, see Tables C.1, C.2 and C.3. In these cases we have  $RO(G) \cong RU(G)$ .

Recall that for groups of the form  $G^f = \mathbb{Z}_2^f \times G$  the character table of  $RO(G^f) \cong RU(G^f)$  is just given by the tensor product with the character table of  $G$  and  $\mathbb{Z}_2^f$ . In our conventions, the irreps  $\Gamma = (\pm, \gamma)$  of  $G^f = \mathbb{Z}_2^f \times G$  are given by the tensor product  $[\pm] \otimes \gamma$  of irreps  $\gamma$  of  $G$  with the two irreps of  $\mathbb{Z}_2^f$ , that we will differentiate from the others by denoting them  $[+]$  (the trivial representation) and  $[-]$  (the sign representation).

| $S_3$    | $\{()\}_1$ | $\{(ab)\}_3$ | $\{(abc)\}_2$ |
|----------|------------|--------------|---------------|
| $\rho_0$ | 1          | 1            | 1             |
| $\rho_a$ | 1          | -1           | 1             |
| $\rho_b$ | 2          | 0            | -1            |

Table C.1: Character table for the generators of  $RO(S_3) \cong RU(S_3)$ . Here the subscripts denote the size of the conjugacy classes.

| $S_4$    | $\{()\}_1$ | $\{(ab)(cd)\}_3$ | $\{(abcd)\}_6$ | $\{(abc)\}_8$ | $\{(ab)\}_6$ |
|----------|------------|------------------|----------------|---------------|--------------|
| $\rho_0$ | 1          | 1                | 1              | 1             | 1            |
| $\rho_a$ | 1          | 1                | -1             | 1             | -1           |
| $\rho_b$ | 2          | 2                | 0              | -1            | 0            |
| $\rho_c$ | 3          | -1               | -1             | 0             | 1            |
| $\rho_d$ | 3          | -1               | 1              | 0             | -1           |

Table C.2: Character table for the generators of  $RO(S_4) \cong RU(S_4)$ . Here the subscripts denote the size of the conjugacy classes.

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<sup>1</sup>For finite groups  $G^f$  there are no indecomposable but reducible representations to worry about, since  $\text{char}(\mathbb{R}) \nmid |G^f|$ .

| $D_8$    | $\{(0, 0)\}$ | $\{(2, 0)\}$ | $\{(1, 0), (3, 0)\}$ | $\{(0, 1), (2, 1)\}$ | $\{(1, 1), (3, 1)\}$ |
|----------|--------------|--------------|----------------------|----------------------|----------------------|
| $\rho_0$ | 1            | 1            | 1                    | 1                    | 1                    |
| $\rho_a$ | 1            | 1            | -1                   | 1                    | -1                   |
| $\rho_b$ | 1            | 1            | 1                    | -1                   | -1                   |
| $\rho_c$ | 1            | 1            | -1                   | -1                   | 1                    |
| $\rho_d$ | 2            | -2           | 0                    | 0                    | 0                    |

Table C.3: Character table for the generators of  $RO(D_8) \cong RU(D_8)$ .

# Appendix D

## Details on modular bootstrap

Here we review in detail the linear functional method used to find the bounds presented in section 6.3.1, based on the original papers [18, 121]. We also present the free fermion CFTs that almost saturate the bounds found for the kinks of Figures 6.1, 6.4.

In order to apply the linear functional method to (6.33) one needs to expand the partition functions in terms of Virasoro characters:

$$\sum_{h,\bar{h},j} n_{h,\bar{h},j} \left( \delta_i^j \chi_h(-1/\tau) \bar{\chi}_{\bar{h}}(-1/\bar{\tau}) - S_i^j \chi_h(\tau) \bar{\chi}_{\bar{h}}(\bar{\tau}) \right) = 0, \quad \forall i. \quad (\text{D.1})$$

Recall that in the proper basis the degeneracies  $n_{h,\bar{h},j}$  are positive. Then, by defining  $\vec{M}^j$  the vectors with entries

$$M_i^j = \delta_i^j \chi_h(-1/\tau) \bar{\chi}_{\bar{h}}(-1/\bar{\tau}) - S_i^j \chi_h(\tau) \bar{\chi}_{\bar{h}}(\bar{\tau}), \quad (\text{D.2})$$

we simply look for a functional  $\alpha$  which returns a real function of  $\Delta$  and  $s$  that satisfies

$$\begin{cases} \alpha[\vec{M}^1](0, 0) > 0, \\ \alpha[\vec{M}^j](|s|, s) \geq 0, & \forall j, s \neq 0, \\ \alpha[\vec{M}^j](\Delta, s) \geq 0, & \forall j, s, \Delta > \Delta_j^*. \end{cases} \quad (\text{D.3})$$

Here  $s$  is supposed to be any admitted value of the spins on  $\mathcal{H}_j$ , with the exception of  $s = 0$  for the second condition if it is a priori allowed in the corresponding  $\mathcal{H}_j$ . Instead  $\Delta_j^*$  will vary depending on the bound we want to find.

The first condition asks for the functional to be strictly positive when evaluated on the vacuum, so that by applying it on the modular crossing equation always guarantees to rule out the corresponding spectra.

In our analysis we will consider a derivative basis for it around the point  $\tau = i$ . By introducing the variable  $z$  such that  $\tau = i \exp z$  we can then expand it as follows:

$$\alpha[\vec{M}^j](\Delta, s) = \sum_{n,m,i} \gamma_{n,m,i} \partial_z^n \partial_{\bar{z}}^m M_i^j |_{z=\bar{z}=0}. \quad (\text{D.4})$$

We now turn into explaining what are the values  $\Delta_j^*$  for which we try to find a functional  $\alpha$  that satisfies (D.3):

1. In order to find the lightest non-degenerate primary on the spectrum of  $\mathcal{H}_{j_0}$ , the definition is

$$\Delta_j^* = \begin{cases} \max(\Delta_{\text{gap}}^{j_0}, |s|), & \text{if } j = j_0, \\ |s|, & \text{otherwise.} \end{cases} \quad (\text{D.5})$$

2. In order to find the same bound, but for scalar primaries, we have

$$\Delta_j^* = \begin{cases} \Delta_{\text{scal}}^{j_0}, & \text{if } j = j_0, s = 0, \\ |s|, & \text{otherwise.} \end{cases} \quad (\text{D.6})$$

We mention here that numerically it is convenient to allow for the possibility of having the limit  $\Delta_{\text{gap/scal}}^j \rightarrow |s|$  for non-degenerate Virasoro characters. In this case the contribution to the partition function (assuming for example  $s > 0$ ) is

$$\lim_{\Delta \rightarrow s} \chi_{\frac{\Delta+s}{2}}(\tau) \bar{\chi}_{\frac{\Delta-s}{2}}(\bar{\tau}) = \chi_s(\tau) (\bar{\chi}_0(\bar{\tau}) + \bar{\chi}_1(\bar{\tau})), \quad (\text{D.7})$$

i.e. it is equivalent to the contribution of a primary with  $\Delta = s + 1$  and a conserved current. We call this particular limit case a non-degenerate Virasoro character of generic type.

With these premises, the algorithm that determines the bound on the primaries is simple: one starts with some fixed value of  $\Delta_{\text{gap/scal}}^{j_0}$  and then searches for a functional that has the properties discussed. Whether one exists or not then determines if the bound considered can be lowered or raised. The procedure is then repeated until one reaches the wanted precision.

To numerically implement this search, we must truncate the basis of the linear functional up to some derivative order  $\Lambda$ , i.e.  $n + m \leq \Lambda$ . For us the choice will be set at  $\Lambda = 10$ .

Moreover, we note that the usual procedure to utilize parity invariant partition functions is of no use here, with the exception of when  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_2$  and  $\nu = 0, 4 \pmod{8}$ . Indeed, these are the only values for which the spin selection rules for the sectors  $\mathcal{H}_i$  are symmetric under the reflection  $s \mapsto -s$ . Thus, if for other values of  $\nu$  we were to restrict to parity invariant partition functions, the bounds we would find would not be as strict as possible, since we would mix  $\tilde{Z}_i$  and  $\tilde{\tilde{Z}}_i$ , that have different spin selection rules. This means that computationally we are able to reduce the basis of functionals to a symmetric one only for the aforementioned cases  $\nu = 0, 4 \pmod{8}$ .

Finally, we recall that by performing the computations with the SDPB solver, while we are actually able to consider for each  $i$  a continuum spectrum of operators with  $\Delta > \Delta_i^*$ , we can instead enforce the conditions (D.3) only up to some value of spin  $s \leq s_{\text{max}}$ . Albeit a priori this might be a significant problem, one usually finds that at fixed  $\Lambda$  for a reasonable value of  $s_{\text{max}}$  numerical stability is reached and the bounds converge within some error  $\Delta_\delta$ . Of course  $s_{\text{max}}$  will vary depending

on the precision we want to meet, but generally is found to be of the same order of magnitude of  $\Lambda$ . In our case we decided to set  $\Delta_\delta = 0.01$ , for which numerical stability is reached by considering values of the spins up to  $s_{\max} = 40$ .

We report also the other relevant parameters with which the numerical analysis has been performed, i.e.

|                              |            |
|------------------------------|------------|
| precision                    | 700        |
| primalErrorThreshold         | $10^{-30}$ |
| dualErrorThreshold           | $10^{-30}$ |
| maxComplementarity           | $10^{100}$ |
| feasibleCenteringParameter   | 0.1        |
| infeasibleCenteringParameter | 0.3        |
| stepLengthReduction          | 0.7        |

## D.1 Free fermions kinks

Here we recall some basic facts about stacks of free fermion CFTs, in order to show that these theories saturate the bounds at the kinks found from the numerical analysis of section 6.3.1.

A generic stack of Majorana fermions

$$\mathcal{L} = \sum_{a=1}^n \psi_a \bar{\partial} \psi_a + \bar{\psi}_a \partial \bar{\psi}_a \quad (\text{D.8})$$

describes a spin-CFT with value of central charge  $c = n/2$ . For us it is sufficient to consider fermions on a torus with periodicity conditions

$$\psi(\theta + 1, \chi) = e^{2\pi i \alpha} \psi(\theta, \chi), \quad \psi(\theta, \chi + 1) = e^{2\pi i \beta} \psi(\theta, \chi), \quad (\text{D.9})$$

where  $\alpha, \beta = 0, 1/2$  are the sum mod 1 of the periodicity conditions defined by the spin structure and the  $\mathbb{Z}_2$  global symmetry. We can treat these together as an element in  $H^1(\mathbb{T}^2, \mathbb{Z}_2)$ . Then the holomorphic partition function contribution of each Majorana fermion is of the form<sup>1</sup>

$$\begin{aligned} 0 \begin{array}{|c|} \hline \square \\ \hline 0 \end{array} &= \sqrt{\frac{\vartheta_1(\tau)}{\eta(\tau)}}, & 0 \begin{array}{|c|} \hline \square \\ \hline 1 \end{array} &= \sqrt{\frac{\vartheta_4(\tau)}{\eta(\tau)}}, \\ 1 \begin{array}{|c|} \hline \square \\ \hline 0 \end{array} &= \sqrt{\frac{\vartheta_2(\tau)}{\eta(\tau)}}, & 1 \begin{array}{|c|} \hline \square \\ \hline 1 \end{array} &= \sqrt{\frac{\vartheta_3(\tau)}{\eta(\tau)}}, \end{aligned} \quad (\text{D.10})$$

---

<sup>1</sup>Note that the square root of Jacobi functions presents branch-cuts, so after modular transformation one has to pay attention on how to move between different sheets. This behaviour takes place only when we have unpaired Majorana fermions, which, as we will see, usually contribute to the total anomaly by a 1 mod 8 factor. Indeed, this precisely reflects the fact that  $\{(-1)^F, (-1)^Q\} = 0$  on  $\mathcal{H}_{R0}$  when  $\nu = 1 \pmod 2$ .

where  $\vartheta_i(\tau)$  are the Jacobi Theta functions. We remember they satisfy the following identities:

$$\begin{aligned}
 \vartheta_1(\tau) &= 0, \\
 \frac{\vartheta_2(-1/\tau)}{\eta(-1/\tau)} &= \frac{\vartheta_4(\tau)}{\eta(\tau)}, & \frac{\vartheta_2(\tau+1)}{\eta(\tau+1)} &= e^{i\pi/6} \frac{\vartheta_2(\tau)}{\eta(\tau)}, \\
 \frac{\vartheta_3(-1/\tau)}{\eta(-1/\tau)} &= \frac{\vartheta_3(\tau)}{\eta(\tau)}, & \frac{\vartheta_3(\tau+1)}{\eta(\tau+1)} &= e^{-i\pi/12} \frac{\vartheta_4(\tau)}{\eta(\tau)}, \\
 \frac{\vartheta_4(-1/\tau)}{\eta(-1/\tau)} &= \frac{\vartheta_2(\tau)}{\eta(\tau)}, & \frac{\vartheta_4(\tau+1)}{\eta(\tau+1)} &= e^{-i\pi/12} \frac{\vartheta_3(\tau)}{\eta(\tau)}.
 \end{aligned} \tag{D.11}$$

If one considers a single Majorana fermion charged under both  $(-1)^F$  and  $(-1)^Q = (-1)^{F_L}$ , its total contribution to the partition function in  $\mathcal{H}_{\text{NS1}}$  will be

$$\text{NS0} \begin{array}{|c|} \hline \square \\ \hline \text{NS1} \end{array} = 1 \begin{array}{|c|} \hline \square \\ \hline 0 \end{array} \cdot \overline{1 \begin{array}{|c|} \hline \square \\ \hline 1 \end{array}} \xrightarrow{T^2} e^{i\pi/4} 1 \begin{array}{|c|} \hline \square \\ \hline 0 \end{array} \cdot \overline{1 \begin{array}{|c|} \hline \square \\ \hline 1 \end{array}}, \tag{D.12}$$

signaling that under stacking it increases the anomaly  $\nu \mapsto \nu + 1 \pmod 8$ . Instead, by considering it to be charged under  $(-1)^Q = (-1)^{F_R}$  rather than  $(-1)^{F_L}$ , one gets a decrease in the anomaly  $\nu \mapsto \nu - 1 \pmod 8$ . Therefore, for a free fermion theory with  $c = n/2$  we can have arbitrary anomaly  $\nu = \tilde{\nu}_L - \tilde{\nu}_R \pmod 8$ , where  $\tilde{\nu}_{L,R} \leq n$  are the number of left/right-moving fermions charged under  $(-1)^Q$ .

We start to look at the case  $c = 1$ . This means the possible anomalies are  $\nu = 0, \pm 1, \pm 2 \pmod 8$ . The partition functions for  $\nu = 0, 1, 2$  are the following:

$$\nu = 0 : \begin{cases} Z_{\text{NS0}}^{+F+Q} = \frac{1}{2} \frac{|\vartheta_3(\tau)|^2 + |\vartheta_4(\tau)|^2}{|\eta(\tau)|^2}, \\ Z_{\text{NS0}}^{+F-Q} = 0, \end{cases} \tag{D.13}$$

$$\nu = 1 : \begin{cases} Z_{\text{NS0}}^{+F+Q} = \frac{1}{4} \frac{(\sqrt{\vartheta_3(\tau)} + \sqrt{\vartheta_4(\tau)})(\overline{\vartheta_3(\tau)}\sqrt{\vartheta_3(\tau)} + \overline{\vartheta_4(\tau)}\sqrt{\vartheta_4(\tau)})}{|\eta(\tau)|^2}, \\ Z_{\text{NS0}}^{+F-Q} = \frac{1}{4} \frac{(\sqrt{\vartheta_3(\tau)} - \sqrt{\vartheta_4(\tau)})(\overline{\vartheta_3(\tau)}\sqrt{\vartheta_3(\tau)} - \overline{\vartheta_4(\tau)}\sqrt{\vartheta_4(\tau)})}{|\eta(\tau)|^2}, \end{cases} \tag{D.14}$$

$$\nu = 2 : \begin{cases} Z_{\text{NS0}}^{+F+Q} = \frac{1}{4} \left| \frac{\vartheta_3(\tau) + \vartheta_4(\tau)}{\eta(\tau)} \right|^2, \\ Z_{\text{NS0}}^{+F-Q} = \frac{1}{4} \left| \frac{\vartheta_3(\tau) - \vartheta_4(\tau)}{\eta(\tau)} \right|^2. \end{cases} \tag{D.15}$$

The partition functions for  $\nu = -1, -2$  cases are complex conjugated partition functions for  $\nu = 1, 2$  respectively.

Expanding them in  $q$  we learn that the lightest primary states are:

| $\nu$   | $\mathcal{H}_{\text{NS0}}^{+F+Q}$ | $\mathcal{H}_{\text{NS0}}^{+F-Q}$ |
|---------|-----------------------------------|-----------------------------------|
| 0       | $(1/2, 1/2)$                      | $\times$                          |
| $\pm 1$ | $(1/2, 1/2)$                      | $(1/2, 1/2)$                      |
| $\pm 2$ | $(1, 0), (0, 1)$                  | $(1/2, 1/2)$                      |

As anticipated, these values almost saturate the bounds found from the numerical analysis presented in Figures 6.1 and 6.4. Note that for  $\nu = 2$  we have a conserved current state. However, this is not unexpected, as its mix with the vacuum  $(h, \bar{h}) = (0, 0)$  defines a non-degenerate Virasoro character of generic type.

Next we focus on the kinks we found at  $c = n/2 = 4 \pm \nu/2$  for  $\mathcal{H}_{\text{NS0}}^{+F+Q}$ . These are simply a set of  $n = 8 \pm \nu$  Majorana fermions charged under  $(-1)^Q = (-1)^{F_L}$  for  $n = 8 + \nu$  and  $(-1)^Q = (-1)^{F_R}$  for  $n = 8 - \nu$ . For example, in light of what we said, the partition function for  $c = 4 + \nu/2$  is just

$$Z_{\text{NS0}}^{+F+Q} = \frac{1}{4} \left| \frac{\vartheta_3^c(\tau) + \vartheta_4^c(\tau)}{\eta(\tau)} \right|^2 \quad (\text{D.16})$$

and its expansion confirms that the lightest scalar is indeed a marginal operator with  $\Delta = 2$ .

Finally, we note that the set of  $n = 4$  real fermions charged under  $(-1)^Q = (-1)^{F_L}$  describes also the kink at  $c = 2$  for  $\nu = 4 \pmod{8}$ . In fact in this case the partition function is

$$Z_{\text{NS0}}^{+F-Q} = \frac{1}{4} \left| \frac{\vartheta_3^2(\tau) - \vartheta_4^2(\tau)}{\eta(\tau)} \right|^2, \quad (\text{D.17})$$

so the lightest non-degenerate primary has again  $(h, \bar{h}) = (1/2, 1/2)$ .

# Appendix E

## Reduced S matrices

In the following we report the  $S_{\text{red}}$  matrix and the reduced partition vector  $\mathbf{Z}_{G^f}^{\text{red}}$  of the reduced systems used in the numerical computation. We are going to represent them in the form

$$(S_{\text{red}} | \mathbf{Z}_{G^f}^{\text{red}}). \quad (\text{E.1})$$

Here we adopt the following notation for the sets  $\{Z_{C_{G^f}}\}$ :

- If  $G^f = \mathbb{Z}_2^f \times G$ , the conjugacy classes correspond to single elements of  $G^f$ . Thus we identify respectively with  $Z_{+,n}$  and  $Z_{-,n}$  the elements  $(0, n)$  and  $(1, n)$ .
- If  $G^f = \mathbb{Z}_2^f \times G$  with  $G$  non-abelian, the conjugacy classes of  $G^f$  are  $\{(0, C_G)\}$  and  $\{(1, C_G)\}$ . Here we are going to adopt the notation  $Z_{+,i}$  for the first and  $Z_{-,i}$  for the latter, where  $i = 0, 1, 2, \dots$  denote the  $(i - 1)$ -esimal conjugacy class represented in the Tables of Appendix C.
- For  $G^f = \mathbb{Z}_8^f$ ,  $Z^{[n]}$  corresponds to the trace over  $\mathcal{H}_0$  with the insertion of the projector of charge  $n \pmod 8$ .

**Case 2:**  $G^f = \mathbb{Z}_2^f \times \mathbb{Z}_4$

$$\left( \begin{array}{cccccccccccc} \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \middle| \begin{array}{l} Z^{(+,[0])} \\ Z^{(-,[0])} \\ Z^{(+,[2])} \\ Z^{(-,[2])} \\ Z^{(+,[1])} + Z^{(+,[3])} \\ Z^{(-,[1])} + Z^{(-,[3])} \\ Z_{-,0} \\ Z_{+,2} \\ Z_{-,2} \\ Z_{+,1} + Z_{+,3} \\ Z_{-,1} + Z_{-,3} \end{array} \right) \quad (\text{E.2})$$







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